Boundary Value Problems of Second Order Sublinear ODEs

by

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Abstract

In this work, boundary value problems (BVPs) of Emden-Fowler equations (equations which usually arise from models in the study of gaseous dynamics and pseudo-plastic fluids) are studied with keen interest in the case $0 < \lambda < 1$ with non-zero boundary conditions imposed.

Motivated by earlier works of authors like Zhang [54], we investigate the problem under non-zero boundary data.

The approach used is to transform the BVP to an equivalent integral operator so that the problem of existence of solutions reduces to seeking fixed points to a Hammerstein integral operator upon successful construction of lower and upper solutions to the given problem.

The uniqueness of $C^1[0, 1]$ solutions is as well studied.

Our work generalizes some earlier results in the literature.

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1 Introduction

Over the years, there have been a surge in the study of second order nonlinear ordinary differential equations (ODEs) of the form:

$$x'' + f(t,x) = 0, \ x(t) > 0, \ t \in (0,1) \tag{1.0}$$

see ([42], [35]) subject to different boundary conditions based on interest.

A special case of equation (1.0) can be transformed to an ODE of the Lane-Emden type which in turn reduces (with the help of a change of variable) to the Emden-Fowler type [35].

The following equation:

$$x'' + p(t)x^{\lambda} = 0 \tag{1.1}$$

is called the generalized Emden-Fowler equation where: $p(t) \in C(0,1), \ \lambda \in \mathbf{R}, \ p(t) > 0$. In this work, we study equation (1.1) in the case $0 < \lambda < 1$.

Equations of the form (1.1) above usually arise from models in the study of pseudo-plastic fluids, gaseous dynamics and fluid mechanics ([52], [14], [50]). In equation (1.1), x represents the pressure of the gas, t the radius of the ball and λ called polytropic index.
The interest of researchers has been to state conditions (necessary and sufficient) for the existence of solutions to equation (1.1), subject to certain boundary conditions as well as study the uniqueness of such solutions.

Using the method of upper and lower solutions, Zhang[54] studied equation (1.1) extensively in the sublinear case while Taliaferro [48] studied (1.1) using the shooting method subject to the boundary condition:

$$x(0) = x(1) = 0 \tag{1.2}$$

While in their work, Zhang[54], Taliaferro[48] studied (1.1) with zero boundary values using suitable methods stated above, we study (1.1) using fixed point approach. Necessary and sufficient conditions for the equation (1.1) to have positive solutions was studied with nonnegative boundary conditions imposed thereby generalizing the work of Zhang[54].

The uniqueness of a $C^1[0,1]$ positive solution to (1.1) subject to nonnegative boundary conditions was as well studied.

The equation (1.1) shall be the center of interest in our work and as such, we owe it a brief historical background.
Lord Kelvin had put forward that the gaseous cloud is under convective equilibrium ([52], [49]). With interest in the study of the equilibrium of the mass of such gases, the following equation was reviewed by Lane and was later named after him as Lane-Emden equation:

$$\frac{1}{t^2}\frac{d}{dt}(t^2\frac{dx}{dt}) + x^n = 0 \tag{1.3}$$

Results on the Lane-Emden equation (1.3) above subject to the initial conditions:

$$x(0) = 1, x'(0) = 0$$

can be found in Ritter[45]. For more work on the equation, see [52].
The more general equation:

$$\frac{d}{dt}(x^p\frac{dx}{dt}) + t^\alpha x^\lambda = 0 \tag{1.4}$$

where p, α, λ are real numbers is called the Emden-Fowler equation whose study also follows from physical phenomena like the the study of fluid mechanics in Physics [14].

In his work, R. Emden [18] had studied the Lane-Emden equation (1.3) by a transformaton of (1.3) into a quadratic system (which represents a system of first order nonlinear autonomous systems) whose results were gotten through phase plane analysis [52].
This technique was later used to study the more general equation (1.4) (see Coppel [15] and references therein).
The equation (1.4) subjected to the boundary condition:

$$x(0) = x(1) = 0$$

has been studied by many authors with different conditions imposed on λ and p.

If $\lambda > 0$ and $p \in C[0,1]$, the problem is called nonsingular. For the case $\lambda < 0$, with $p(t)$ not necessarily continuous at the boundary of $(0,1)$ and the case where $p(t)$ is allowed to be unbounded, the problem is called singular. The cases $\lambda > 1$ and $0 < \lambda < 1$, the problem is called superlinear and sublinear respectively.

Researchers over the years have sought ways to establish the existence and uniqueness of solutions (primarily positive solutions) to the respective cases above. While Zhang[54] studied the singular sublinear case, Wong[52], Taliaferro[48] studied extensively the singular case. Although not much was earlier said about the superlinear case in the literature, for a treatment on this case, see Guedda[23] who obtained multiple positive solutions to the problem in this case. see also Jiang[32].

We point out here that prior to the work by Zhang[54] in which the method of upper and lower solutions was used to treat the problem (1.1),(1.2), works on sublinear cases have been treated with the topological degree method or the shooting method. Since then, the method of upper and lower solutions has remained indispensable in the study of the sublinear case.

In equation (1.1) above, $f(t,x) = p(t)x^{\lambda}$ is nonnegative. To consider a more general $f(t,x)$, the following equation has been studied.

$$x'' + p(t)|x|^{\lambda} sgn x = 0, t \geqslant 0. \tag{1.5}$$

subject to the boundary condition $x(0) = r_1, x'(0) = r_2$. (see [33]).

We will give a summary of some of the results from the study of (1.5) in the literature. For more details, see ([33], [28], and [52]).
Pending then, we define some terminologies as used in the work.

The main tools used in the work are the Arzela-Ascoli theorem, the Schauder fixed point theorem, construction of upper and lower solutions to the equation (1.1) with nonnegative boundary conditions as well as defining an appropriate operator $\mathbf{T}$(on a family, $\mathbf{X}$ of functions in which $\mathbf{T}$ is shown to have a fixed point) whose fixed point is a solution to the problem (4.0).

Chapter 1 of this Thesis summarizes the goal of our work. In Chapter 2, we give the definitions of terminologies used as well as state the basic tool employed in the proof of our results. While Chapter 3 was devoted to the study of Boundary Value Problems of second order, a short collection of basic results for two point boundary value problems, as well as known results on BVPs of Emden-Fowler type, Chapter 5 gave general results on the theory and study of Emden-Fowler equations. Our main results are contained in Chapter 4.

2 Definitions

We pursue now the study of the generalized Emden-Fowler equation starting with the definitions of terminologies used in the work.

2.1 Solution/Positive Solution

By a solution of a boundary value problem BVP, we mean a function x satisfying the given ODE together with the boundary conditions imposed.

The solution is called positive if $x(t) > 0$ for all t, except at the endpoints.

2.2 Singular / Non-Singular BVP

For an interval I, we denote by $C(I)$ the space of all continuous real valued functions on I. $C^1(I)$ denotes the space of all continuously differentiable functions on I.

For the generalized Emden-Fowler equation (1.1) above with $p(t) \in C[0,1]$, $p(t) \geq 0$ for $t \in (0,1), \lambda \in \mathbf{R}$.

If $p(t) \in C[0,1]$ *and* $\lambda > 0$, then we call equation (1.1) nonsingular. However, if $p(t)$ is not continuous at the endpoints of (0,1) (including the case where p(t) is unbounded on (0,1) or $\lambda < 0$) the equation is called singular [54].

We discuss briefly two tools used in obtaining existence results to Boundary Value problems.

2.3 Upper and Lower Solution

A function $y \in C^2(0,1) \cap C[0,1]$ is called a *lower solution* of (1.1), (1.2) if

$$y''(t) + p(t)y^\lambda(t) \geq 0 \text{ for } t \in (0,1)$$

and

$$y(0) = y(1) = 0.$$

On the other hand, a function

$$y \in C^2(0,1) \cap C[0,1]$$

is called an *upper solution* of (1.1), (1.2) if

$$y''(t) + p(t)y^{\lambda}(t) \leq 0 \text{ for } t \in (0,1) \text{ and } y(0) = y(1) = 0.$$

If the function y satisfies the boundary conditions (1.2) and for $t \in (0,1)$, $y''(t)+p(t)y^{\lambda}(t) > 0$, then y is a strict lower solution. On the other hand, if y satisfies the boundary condition (1.2) and for $t \in (0,1)$, $y''(t)+p(t)y^{\lambda}(t) < 0$, then y is a strict upper solution.

On a historical note, though Dragoni[16] had showcased this method in his 1931 paper, it is credited to Picard[44] see [24]. The next tool is given below:

2.4 Nagumo Condition [6]

Let J be an interval. $f \in C[J \times R \times R; R]$ and $\alpha, \beta \in C[J, R]$ with $\alpha(t) \leq \beta(t)$ on J. Suppose that for $t \in J$, $\alpha(t) \leq x \leq \beta(t)$ and $x' \in R$,

$$|f(t,x,x')| \leq h(|x'|) \tag{2.1}$$

where $h \in C[R, (0, \infty)]$. If

$$\int_{\lambda}^{\infty} \frac{sds}{h(s)} > max_{t \in J}\beta(t) - min_{t \in J}\alpha(t), \tag{2.2}$$

where λ is a constant defined by

$$\lambda(b-a) = max[|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|],$$

then we say that f satisfies Nagumo's condition on J with respect to α, β. It is a condition imposed on the function f so it does not grow too fast with respect to the derivative x'. This notion was introduced by Nagumo in 1937. See ([40], [24]). Nagumo[39] later showed that the existence of upper and lower solutions to a BVP alone does not really guarantee the existence of solutions to the problem. However, if a suitable Nagumo condition with respect to the lower and upper solutions is imposed, we can be sure of the existence of solutions. For negative examples where lower and upper solutions to a BVP were obtained but solutions nonexistent in the absence of a Nagumo condition, see Habets[24].

2.5 $C[0,1]$ and $C^1[0,1]$ solution

A function $x(t) \in C^2(0,1) \cap C[0,1]$ that satisfies equation (1.1) on (0,1) is called a $C[0,1]$ solution of (1.1).
If in addition $x(t) \in C^1[0,1] \cap C^2(0,1)$, we call it a $C^1[0,1]$ solution of (1.1). Put differently, $x(t)$ is a $C^1[0,1]$ solution if it is a $C[0,1]$ solution and $x'(0+)$ and $x'(1-)$ both exist.

2.6 Fixed Point

Let $\mathbf{X} \neq \emptyset$ be a normed space. Let $T : \mathbf{X} \to \mathbf{X}$ be a map. Then a point $x \in \mathbf{X}$ is called a fixed point of T if $Tx = x$.
In the Theory of Differential Equations, the concept of fixed point and fixed point theorems are indispensable as they are deployed to show the existence of solutions to given problems.
In common practice, given a BVP to establish the existence of solutions one transforms the given BVP as an equivalent fixed point problem to a Hammerstein integral operator:

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s)) ds$$

primarily in the function space $C[0,1]$. So that the problem of existence of solutions reduces to seeking fixed points to the operator, T so defined.
With many Fixed Point Theorems known, we will only state below the Schauder Fixed Point Theorem applied in Chapter Four.

2.7 Schauder Fixed Point Theorem

Let Y be a closed bounded convex subset of a normed space X. Let $T : Y \to X$ be a compact map such that $T(Y) \subseteq Y$. Then there is a point $x \in Y$ such that $Tx = x$.

2.8 Continuation.

Let x be a solution to a given IVP on an interval J. By a continuation of x we mean an extension $\bar{x}$ of x to a larger interval J_0 such that $J \subset J_0$. When no such continuation exists, we say x is noncontinuable. If no extension exists to the left (right) of J, we say x is noncontinuable to the left (right).

3 BVPs of Second Order ODEs

The ODE:

$$x'' + f(t, x, x') = 0, t \in (0, 1) \tag{3.1}$$

can be subjected to different boundary conditions. Examples are:

(a) The Dirichlet Boundary Condition:
$x(0) = r_1, \quad x(1) = r_2$

(b) Neumann Boundary Condition:
$x'(0) = r_1, \quad x'(1) = r_2$

(c) Robin Boundary Condition:
$\alpha_1 x(0) + \beta_1 x'(0) = r_1, \quad \alpha_2 x(1) + \beta_2 x'(1) = r_2$
where r_1 and r_2 are constants.

In our work, we will pay attention to the Dirichlet Boundary Condition. To this end, we give some classical results on the equation (3.1) under the Dirichlet Boundary Condition.

3.1 Two Point BVPs of Nonlinear Equations

Interest of numerous researchers has been to study the equation (3.1) with the boundary condition (a) above. They have given conditions for the existence of solutions to such problems with different conditions imposed on the function f. We collect a few of some classical results concerning the problem starting of with the result of Lasota and Opial[37] for the existence of solutions to general BVPs of the form (3.1), (a) though depending on the uniqueness of IVPs to (3.1).

Theorem 3.1.1. [37]
In the equation (3.1) suppose that
(i) the function $f(t, x, x')$ is continuous on $[a, b] \times \mathbf{R}^2$,
(ii) the solutions to IVPs for the equation (3.1) are unique and extendable throughout the interval [a,b],
(iii) for any $a < t_1 < t_2 < b$ and any $r_1, r_2 \in \mathbf{R}$ the solution of the boundary

value problem:

$$x'' = f(t, x, x') \tag{3.2}$$
$$x(t_1) = r_1, x(t_2) = r_2 \tag{3.3}$$

when it exists is unique. Then all boundary value problems of the type (3.2), (3.3) have solutions.

The result of Lasota and Opial [37] requires the uniqueness of solutions of IVPs for (3.1). This condition was relaxed by other authors to improve Theorem 3.1.1. We give some of those results below.

Theorem 3.1.2. Schrader [46]
Let f be defined and continuous. Suppose that;
(i) For any $I \subset (a, b)$ solutions of the boundary value problem (3.1),(a) when they exist are unique
(ii) Every solution x of an IVP for (3.1) may be extended to the whole interval [a,b].
Then every boundary value problem on any proper subinterval of [a,b] has a solution.

Dropping the assumption of uniqueness of solutions of IVPs in Theorem 3.1.1 above, Schrader[46] showed that the result extends to third order equations. Though extensions to higher orders have been obtained, we will not give such results here as we confine our interest to equations of second order.
Habets and Zanolin [25] considered the scalar problem:

$$x'' = f(t, x), \quad x(0) = r_1, x(1) = r_2 \tag{3.4}$$

and proved existence result in integral form using the concept of upper and lower solutions.

Theorem 3.1.3 [6]
Let $f : \mathcal{D} \to \mathbf{R}$ be continuous and $\mathcal{D} \subset (0, 1) \times \mathbf{R}$. Let α, β be upper and lower solutions to problem (3.4) respectively. Define:

$$\mathcal{D}_\alpha^\beta := \{(t, y) \in (0, 1) \times \mathbf{R} : \alpha(t) \le y \le \beta(t)\}.$$

Suppose:
(i) $\alpha(t) \leq \beta(t)$, for all $t \in [0,1]$
(ii)$\mathcal{D}_\alpha^\beta \subset \mathcal{D}$
(iii) there is a function $h \in C((0,1), \mathbf{R}^+)$ such that $|f(t,y)| \leq h(t)$, for all $(t,y) \in \mathcal{D}_\alpha^\beta$ and

$$\int_0^1 s(1-s)h(s)ds < \infty$$

then the problem (3.4) has at least one solution $\tilde{x}$ such that
$\alpha(t) \leq \tilde{x}(t) \leq \beta(t)$ for all $t \in (0,1)$.

With this said, we give some of the earliest uniqueness results for the problem (3.1), (a). The first will be a result whose proof relies primarily on the well known Contraction Mapping Principle.

Theorem 3.1.4 [6]
Let f be continuous on $\mathbf{J} \times \mathbf{R} \times \mathbf{R}$ and for $(t, x_1, y_1), (t, x_2, y_2) \in \mathbf{J} \times \mathbf{R} \times \mathbf{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq K|x_1 - x_2| + L|y_1 - y_2|$$

where $K, L > 0$ are constants such that

$$K[(t_2 - t_1)^2/8] + L[(t_2 - t_1)^2/2] < 1.$$

Then the problem (3.1), (a) has a unique solution.

For the largest possible interval of validity of the above result, see [6].

We pursue next a uniqueness result for two point boundary value problems which is weaker than a well known result for second order equations. Consider the differential system:

$$x'' = f(t, x), \tag{3.5}$$

$$x_1(t_1) = c_1,\ x_1(t_2) = c_2 \tag{3.6}$$

with f continuous on $\mathbf{R} \times \mathbf{R}^2$
Theorem 3.1.5 [6]
Let $\mathbf{J} = (a, b], -\infty \leq a < \infty$, and $\mathbf{J}^o = (a, b)$. Assume:

(i)$f_1(t, x_1, x_2)$ is an increasing function of x_2 for fixed (t, x_1) satisfying $f_1(t, x_1, x_2) \to \pm\infty$ as $x_2 \to \pm\infty$ uniformly on compact sets in $(a, b) \times \mathbf{R}$;

(ii) all solutions of (3.5) exist on J;

(iii) there exists at least one solution of (3.5),(3.6) for all $t_1, t_2 \in \mathbf{J}^0$ and all c_1, c_2.
Then the BVP (3.5),(3.6) has exactly one solution if $t_1 \in \mathbf{J}^0, t_2 \in \mathbf{J}$.

From the Theorem (3.1.5) above, setting

$$\begin{gathered} x_1' = x_2, x_2' = f(t, x_1, x_2) \\ x_1(t_1) = c_1, x_1(t_2) = c_2, \end{gathered}$$

then we have the following corollary.

Corrolary 3.1.6 [6]
Consider the equation (3.1) with the boundary condition,

$$x(t_1) = c_1, \quad x(t_2) = t_2 = c_2, \ t_1 \in (a, b), t_2 \in (a, b] \tag{3.7}$$

where $f \in C[J \times R^2, R], J = (a, b]$.
Suppose;

(i) all solutions of (3.1) exist on (a,b],

(ii) there exists at most one solution of (3.1), (3.7), for $t_1, t_2 \in (a, b], c_1, c_2 \in \mathbf{R}$.
Then there exists exactly one solution of (3.1),(3.7).

Not necessarily requiring the uniqueness condition of the BVP (3.1),(3.6), for all $t_1, t_2 \in [a, b+\epsilon)$ *or* $(t_1, t_2 \in (a-\epsilon, b])$, the following weaker uniqueness result was given and proved by [6].

Theorem 3.1.7 [6]
Let $f \in C([a, b+\epsilon) \times \mathbf{R}^2, \mathbf{R}]$ for some $\epsilon > 0$ and suppose that all solutions of initial value problems for (3.5) exists on $[a, b+\epsilon)$. Assume that for a fixed c_1 there exists at most one solution on $[a, t_2]$ of the BVP (3.5) and

$x(a) = c_1,\ x(t_2) = c_2$ for all $c_2 \in \mathbf{R}$ and $t_2 \in (b - \epsilon, b + \epsilon)$. Then there exists exactly one solution of (3.5) and $x(a) = c_1, x(b) = c_2$ for each $c_2 \in \mathbf{R}$.

3.2 Positive Solutions

A function $y(t) \in C[0,1] \cap C^2(0,1)$ satisfying the equations (3.1),(a) for which $y(t) > 0$ for all $t \in (0,1)$ is called a positive $C[0,1]$ solution of (3.1),(a). A positive $C^1[0,1]$ solution is similarly defined.

O'Regan[43] studied the existence of positive solutions to equations of the form (3.1),(a) with the nonlinear term f being singular and the nonsingular case and obtained the following result.

Consider the following BVP:

$$x'' + \psi(t) f(t, x, x') = 0, \quad 0 < t < 1 \tag{3.8}$$

$$x(0) = a, x(1) = b, \quad a, b > 0 \tag{3.9}$$

where:

$$\psi \textit{ is continuous and positive on } (0,1) \textit{ with } \int_0^1 \psi(s) ds < \infty. \tag{3.10}$$

suppose that the function f satisfies the following conditions:
(i) f is continuous on $[0,1] \times (0,\infty) \times (-\infty,\infty)$ with $lim_{x \to 0^+} f(t,x,p) = \infty$ for each $(t,p) \in [0,1] \times (-\infty,\infty)$
(ii) $0 < f(t,x,p) < g(x)\phi(|p|)$ on $[0,1] \times (0,\infty) \times (-\infty,\infty)$, where g is continuous and nonincreasing on $(0,\infty)$, $\phi > 0$ is continuous and nonincreasing on $[0,\infty)$.

Theorem 3.2.1 [43] Suppose the conditions (3.10), (i) and (ii) above are satisfied. Then there exists a $C^1[0,1] \cap C^2(0,1)$ solution of (3.8),(3.9).

They also proved the existence of positive $C[0,1]$ solutions to (3.8),(3.9) with f independent of x'.

Their proof uses the topological degree method. (One of the methods for studying singular problems).

On his part, Zhang[53] imposed a sublinear hypothesis on f and with the use of upper and lower solutions technique gave necessary and sufficient conditions for the existence of positive solutions given below.

(H) $f(t,x) \in C((0,1)\times(0,\infty),[0,\infty)), f(t,1) \neq 0 \; for \; t \in (0,1)$, and there exist constants $\lambda, \mu, N, M \; with \; (-\infty < \lambda \leq \mu < 1, \; and \; 0 < N \leq 1 \leq M)$, such that for $t \in (0,1)$ and $x \in (0,\infty)$,

$$c^{\mu} f(t,x) \leq f(t,cx) \leq c^{\mu} f(t,x), \; if \; 0 \leq c < N \tag{3.11}$$

$$c^{\lambda} f(t,x) \leq f(t,cx) \leq c^{\lambda} f(t,x), \; if \; c \geq M. \tag{3.12}$$

With these assumptions, he got the following results for the BVP:

$$x'' = f(t,x), \qquad x(0) = x(1) = 0. \tag{3.13}$$

Typical examples of f satisfying condition **(H)** above are functions of the form

$$f(t,x) = \textstyle\sum_{i=1}^{n} p_i(t) x^{\lambda_i}$$

with $0 < \lambda_i < 1$ (i = 1,2,...n). See[53].

Theorem 3.2.2 [53] Suppose the hypothesis **(H)** holds. Then a necessary and sufficient condition for (3.13) to have a $C[0,1]$ positive solution is that

$$\textstyle\int_0^1 t(1-t) f(t,1) dt < \infty.$$

Theorem 3.2.3 [53] Suppose **(H)** holds. Then a necessary and sufficient condition for (3.13) to have a $C^1[0,1]$ solution is that

$$\textstyle\int_0^1 f(t, t(1-t)) dt < \infty.$$

See also accompanying references for similar conditions to these.
Zhang[53] also gave the following uniqueness result for the problem (3.13).

Theorem 3.2.4 [53] Suppose that
(I) $f(t,x) \in C((0,1)\times(0,\infty),[0,\infty))$, and $f(t,x)/x$ is strictly decreasing in

x for all $t \in (0,1)$.
Then if problem (3.13) has a $C^1[0,1]$ positive solution, it will admit no other positive solutions.
Useful applications of the results above to the study of Dirichlet problems of semilinear elliptic partial differential equations can as well be found in [53].

3.3 BVPs of Emden-Fowler equations

The equation (1.1) is called the generalized Emden-Fowler equation the study of which has attracted the attention of many researchers.
If $\lambda > 0$ and $p \in C[0,1]$, the problem is called nonsingular. For the case $\lambda < 0$, with $p(t)$ not necessarily continuous at the boundary of $(0,1)$ and the case where $p(t)$ is allowed to be unbounded, the problem is called singular. See Zhang[54].
The cases $\lambda > 1$ and $0 < \lambda < 1$, the problem is called superlinear and sublinear respectively.

Using upper and lower solution technique, Zhang[54] obtained the following existence and uniqueness results.
Assuming the following hypothesis holds.

(N). $p(t) \in C(0,1), p(t) \geq 0, t \in (0,1),\ 0 < \lambda < 1.$

Theorem 3.3.1 [54].
Suppose the hypothesis **(N)** above holds. Then a necessary and sufficient condition for the problem (1.1),(1.2) to have a $C[0,1]$ positive solution is that

$$0 < \int_0^1 t(1-t)p(t)dt < \infty.$$

Theorem 3.3.2 [54].
Suppose the hypothesis **(N)** holds. Then a necessary and sufficient condition for the problem (1.1), (1.2) to have a $C^1[0,1]$ positive solution is that

$$0 < \int_0^1 t^{\lambda}(1-t)^{\lambda}p(t)dt < \infty.$$

We will adopt the method of proof for the necessity part to show that the two theorems above hold true for the boundary conditions $x(0) = a, x(1) = b.$ *with* $a, b \geq 0$. However, use fixed point results to establish the sufficiency parts.

Theorem 3.3.3 [54].
Suppose the hypothesis **(N)** holds. Any $C^1[0,1]$ positive solution of the problem (1.1), (1.2) is unique.

Using the shooting method, Taliaferro[48] showed that for the case where $\lambda < 0$ the Theorem (3.3.1) holds true.

(T). Let $p(t) > 0, and\ p(t) \in C(0,1)\ \lambda < 0$.

Theorem 3.3.4 [48].
Suppose the hypothesis **(T)** holds. Then a necessary and sufficient condition for the BVP (1.1), (1.2) to have a C[0,1] positive solution is that

$$0 < \int_0^1 t(1-t)p(t)dt < \infty.$$

Lately, the problem (1.1),(1.2) has been investigated to seek conditions upon which there are positive solutions in Holder spaces. The following result was obtained. For more work along this line, see the accompanying references.

Theorem 3.3.5 [9]
Suppose the hypothesis **(N)** holds. If the problem (1.1),(1.2) has a positive solution $x \in C^{0,\alpha}[0,1] \cap C^2(0,1)\ for\ some\ \alpha \in (0,1)$, then

$$\int_0^1 (t(1-t))^{1-\alpha(1-\lambda)}(\psi(t)\psi(1-t))p(t)dt < \infty,$$

for every concave function $\psi \in C[0,1]$ such that $\psi(0) = 0, \psi(t) > 0$ for $t > 0$ and $\int_0^1 \frac{\psi(t)}{t}dt < \infty$.

4 Nonnegative Boundary Conditions and Solutions

. Let us consider the problem

$$\begin{aligned} x''(t) &= -p(t)x^{\lambda}(t) \; 0 < \lambda < 1 \\ x(0) &= a, x(1) = b \; a, b \geq 0 \\ p(t) &> 0 \; and \; p(t) \in C(0,1) \end{aligned} \tag{4.0}$$

Researchers over time have sought different methods to prove existence and uniqueness of solutions to the problem (4.0) above, though subject to different boundary conditions as well as different conditions imposed on either λ or the behavior of the function $p(t)$ on an interval of interest as shown in previous chapters.

We start by giving a necessary and sufficient condition for the problem (4.0) to have a C[0,1] positive solution.
Our result will be an extension of some earlier results ([4], [54]) on the study of (4.0).

Theorem 4.1.
The problem (4.0) has a C[0,1] positive solution if and only if:

$$0 < \int_0^1 t(1-t)p(t)dt < \infty$$

Proof Let $x(t)$ be a positive solution of the problem (4.0). Because the solution x is concave down, we study (4.0) under the following cases:

Case I:
Assume that the solution x is decreasing on (0,1).
Then, there is some point $t_0 \in (0,1)$ such that:

$$x'(t_0) = b - a$$

So that for $t \in [t_0, 1)$, we have that

$$-\int_{t_0}^{t} x''(s)ds = \int_{t_0}^{t} p(s)x^{\lambda}(s)ds.$$

That is,

$$\int_{t_0}^{t} p(s)x^{\lambda}(s)ds = -x'(t) + (b-a) \leq -x'(t) \ since\ b \leq a.$$

Multiplying both sides by $x^{-\lambda}(t)$ and integrating on $[t_0, 1)$, we have :

$$\begin{aligned}\int_{t_0}^{1} x^{-\lambda}(t)\int_{t_0}^{t} p(s)x^{\lambda}(s)dsdt &= -\int_{t_0}^{1} x^{-\lambda}(t)x'(t)dt + \int_{t_0}^{1}(b-a)x^{-\lambda}(t)dt \\ &\leq -\frac{x^{1-\lambda}(1)}{1-\lambda} + \frac{x^{1-\lambda}(t_0)}{1-\lambda} < \infty.\end{aligned}$$

Let $t \in [t_0, 1)$, then:

$$x^{\lambda}(t)\int_{t_0}^{t} p(s)ds \leq \int_{t_0}^{t} p(s)x^{\lambda}(s)ds.$$

So that,

$$\int_{t_0}^{1}(1-s)p(s)ds = \int_{t_0}^{1}\int_{t_0}^{t} p(s)dsdt \leq \int_{t_0}^{1} x^{-\lambda}(t)\int_{t_0}^{t} p(s)x^{\lambda}(s)dsdt < \infty.$$

Next, we establish the convergence of the integral; $\int_0^{t_0} sp(s)ds$.
From (4.0) again,we have that:

$$\int_{t}^{t_0} p(s)x^{\lambda}(s)ds = x'(t) - (b-a).$$

so that for $t \in [0, t_0)$,

$$x'(t) - (b-a) = \int_t^{t_0} p(s)x^{\lambda}(s)ds \geq x^{\lambda}(t_0)\int_t^{t_0} p(s)ds.$$

Thus,

$$\infty > x(t_0) - a - (b-a)t_0 = \int_0^{t_0}(x'(t) - (b-a))dt \geq x^{\lambda}(t_0)\int_0^{t_0}\int_t^{t_0} p(s)dsdt$$

That is,

$$x^{\lambda}(t_0)\int_0^{t_0} sp(s)ds = x^{\lambda}(t_0)\int_0^{t_0}\int_t^{t_0} p(s)dsdt < \infty.$$

Thus, we obtain the convergence of the integral $\int_0^{t_0} sp(s)ds$ as required.
We use the following results shown above:

$$\int_{t_0}^{1} (1-s)p(s)ds < \infty \tag{3.1}$$

and

$$\int_{0}^{t_0} sp(s)ds < \infty \tag{3.2}$$

to establish the necessity condition

$$\begin{aligned}\int_0^1 t(1-t)p(t)dt &= \int_0^{t_0} t(1-t)p(t)dt + \int_{t_0}^1 t(1-t)p(t)dt \\ &\leq \int_0^{t_0} tp(t)dt + \int_{t_0}^1 (1-t)p(t)dt < \infty\end{aligned}$$

Thus, the necessity condition holds for this case.

Case II:
Assume that the solution x is increasing on (0,1)
Like in case (I), there exists some $t_0 \in (0,1)$ such that $x'(t_0) = b - a$
We will parrot the procedure of case (I) above to show that under this assumption, the necessity condition still holds true.
Let $t \in [0, t_0)$, then from (4.0),

$$\int_t^{t_0} p(s)x^{\lambda}(s)ds = x'(t) - (b-a) < x'(t) \ since\ b \geq a.$$

Thus,

$$\int_0^{t_0} x^{-\lambda}(t) \int_t^{t_0} p(s)x^{\lambda}(s)dsdt = \int_0^{t_0} x^{-\lambda}(t)x'(t)dt \leq \frac{1}{1-\lambda}[x^{1-\lambda}(t_0) - x^{1-\lambda}(0)] < \infty$$

Under the assumption on the solution x in this case;

$$0 \leq x^{\lambda}(t)\int_t^{t_0} p(s)ds \leq \int_t^{t_0} x^{\lambda}(s)p(s)ds.$$

Hence,

$$0 \leq \int_0^{t_0} sp(s)ds = \int_0^{t_0} dt \int_t^{t_0} p(s)ds \leq \int_0^{t_0} x^{-\lambda}(t) \int_t^{t_0} p(s)x^{\lambda}(s)dsdt < \infty.$$

Furthermore, for $t \in [t_0, 1)$,

$$\int_{t_0}^{t} p(s)x^{\lambda}(s)ds = (b-a) - x'(t).$$

$$x^{\lambda}(t_0) \int_{t_0}^{t} p(s)ds \leq \int_{t_0}^{t} p(s)x^{\lambda}(s)ds = (b-a) - x'(t).$$

Therefore,

$$0 \leq x^{\lambda}(t_0) \int_{t_0}^{1} (1-s)p(s)ds = x^{\lambda}(t_0) \int_{t_0}^{1} \int_{t_0}^{t} p(s)dsdt \leq (b-a)(1-t_0) - b + x(t_0) < \infty.$$

Hence,

$$0 \leq \int_{t_0}^{1} (1-s)p(s)ds < \infty.$$

Like in the previous case, the convergence of the integrals

$$\int_0^{t_0} sp(s)ds \textit{ and } \int_{t_0}^{1} (1-s)p(s)ds$$

guarantees that

$$0 < \int_0^1 t(1-t)p(t)dt < \infty$$

as required.

Case III;

If $x(t)$ is neither increasing nor decreasing on (0,1), there is some $t_0 \in (0,1)$ such that $x'(t_0) = 0$.
Let $t \in [0, t_0)$ then;

$$\int_0^{t_0} x^{-\lambda}(t) \int_t^{t_0} p(s)x^{\lambda}(s)dsdt = \int_0^{t_0} x'(t)x^{-\lambda}(t)dt$$

$$0 \leq \int_0^{t_0} x^{-\lambda}(t) \int_t^{t_0} p(s)x^{\lambda}(s)dsdt = \frac{1}{1-\lambda}[x^{1-\lambda}(t_0) - a^{1-\lambda}]$$

Clearly, $x^{\lambda}(t)\int_t^{t_0} p(s)ds \leq \int_t^{t_0} x^{\lambda}(s)p(s)ds$.
Hence,

$$0 \leq \int_0^{t_0} sp(s)ds = \int_0^{t_0} dt \int_t^{t_0} p(s)ds \leq \int_0^{t_0} x^{-\lambda}(t) \int_t^{t_0} x^{\lambda}(s)p(s)dsdt < \infty.$$

On the other hand, let $t \in [t_0, 1)$, then;

$$\int_{t_0}^{t} p(s)x^{\lambda}(s)ds = -\int_{t_0}^{t} x''(s)ds = -x'(t).$$

So that,

$$\int_{t_0}^{1} x^{-\lambda}(t) \int_{t_0}^{t} p(s)x^{\lambda}(s)dsdt = -\int_{t_0}^{1} x'(t)x^{-\lambda}(t)dt = \frac{1}{1-\lambda}[x^{1-\lambda}(t_0) - b^{1-\lambda}] < \infty.$$

In addition, for $t \in [t_0, 1)$,

$$x^{\lambda}(t) \int_{t_0}^{t} p(s)ds \leq \int_{t_0}^{t} p(s)x^{\lambda}(s)ds.$$

$$0 \leq \int_{t_0}^{1} (1-s)p(s)ds = \int_{t_0}^{1} dt \int_{t_0}^{t} p(s)ds \leq \int_{t_0}^{1} x^{-\lambda}(t) \int_{t_0}^{t} p(s)x^{\lambda}(s)dsdt < \infty.$$

Thus, we establish the non negativity and convergence of the integrals:

$$\int_0^{t_0} sp(s)ds \ and \int_{t_0}^{1} (1-s)p(s)ds,$$

which in turn guarantee that;

$$0 \leq \int_0^1 t(1-t)p(t)dt < \infty.$$

We claim that the left inequality is strict. For suppose not, then;

$$\int_0^1 t(1-t)p(t)dt = 0.$$

This would force $p(t) = 0 \ \forall t \in (0, 1)$, contradicting the assumption that $p(t) > 0$.

The three cases studied above show that the necessity condition holds true. Next, we prove the sufficiency part.

Sufficiency

Let

$$0 < \int_0^1 t(1-t)p(t)dt < \infty$$
$$p(t) \in C(0,1), p(t) > 0, t \in (0,1).$$

We now show the existence of a positive C[0,1] solution to (4.0).
We note that, if $x(t)$ is a positive solution of (4.0), then

$$x(t) = a + (b-a)t + (1-t)\int_0^t sp(s)x^\lambda(s)ds + t\int_t^1 (1-s)p(s)x^\lambda(s)ds.$$

In view of Zhang[54], we construct lower and upper solutions to (4.0) as follows
Define:

$$\phi : t \mapsto \phi(t) \ by :$$

$$\phi(t) = a + (b-a)t + L[(1-t)\int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds + t\int_t^1 s^\lambda(1-s)^{1+\lambda}p(s)ds]$$

where;

$$L = [\int_0^1 s^{1+\lambda}(1-s)^{1+\lambda}p(s)ds]^{\frac{\lambda}{1-\lambda}}$$

Then $\phi(0) = a,\ \phi(1) = b$ and for $t \in (0,1)$

$$\phi''(t) + p(t)\phi^\lambda(t) = -Lt^\lambda(1-t)^\lambda p(t) + p(t)\phi^\lambda(t)$$

$$\geqslant -Lt^\lambda(1-t)^\lambda p(t) + p(t)L^\lambda[(1-t)\int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds + t\int_t^1 s^\lambda(1-s)^{1+\lambda}p(s)ds]^\lambda$$

$$= L^\lambda t^\lambda(1-t)^\lambda p(t)\left[\frac{(1-t)\int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds + t\int_t^1 s^\lambda(1-s)^{1+\lambda}p(s)ds]^\lambda}{t^\lambda(1-t)^\lambda} - L^{1-\lambda}\right] \quad \textbf{(4.1)}$$

Also, we can see that

$$t^{\lambda}(1-t)^{\lambda}L^{1-\lambda} \leq \big[(1-t)\int_0^t s^{1+\lambda}(1-s)^{\lambda}p(s)ds + t\int_t^1 (1-s)^{1+\lambda}s^{\lambda}p(s)ds\big]^{\lambda}.$$

Hence;

$$\phi''(t) + p(t)\phi^{\lambda}(t) \geqslant 0.$$

so that the function ϕ so defined is a lower solution to problem (4.0).
One observes that $\phi \in C[0,1] \cap C^2(0,1)$ and $\phi(0) = a, \phi(1) = b$.
Furthermore, define

$$\psi : t \mapsto \psi(t)$$

by:

$$\psi(t) = a + (b-a)t + U[(1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds]$$

where U is a constant large enough.
Then; $\psi(0) = a, \psi(1) = b$ and $\psi(t) \in C[0,1] \bigcap C^2(0,1)$

$$\psi''(t) + p(t)\psi^{\lambda}(t) = -Up(t) + p(t)[a + (b-a)t + Ug]^{\lambda}$$
$$where\ g = (1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds.$$

$$\begin{aligned}\psi''(t) + p(t)\psi^{\lambda}(t) &\leq -Up(t) + p(t)[max(a,b) + Ug]^{\lambda}\\ &= p(t)U[-1 + \frac{1}{U}(max(a,b) + Ug)^{\lambda}].\end{aligned}$$

so that,

$$\psi''(t) + p(t)\psi^{\lambda}(t) \leq 0\ where\ U\ is\ sufficiently\ large,$$

since

$$lim_{U\to\infty}\frac{1}{U}(max(a,b) + Ug)^{\lambda} = 0.$$

Also, taking U large enough to exceed L, we see that;

$$\psi(t) - \phi(t) = U[(1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds]$$

$$-L[(1-t)\int_0^t s^{1+\lambda}(1-s)^{\lambda}p(s)ds + t\int_t^1 s^{\lambda}(1-s)^{1+\lambda}p(s)ds \geqslant 0,$$

verifying that $\psi(t) \geqslant \phi(t)$.

Hence $\phi(t)$ *and* $\psi(t)$ so defined give lower and upper solutions respectively to (4.0)
In fact, the functions ϕ and ψ are strict lower solutions and strict upper solutions respectively. To see this

$$\phi''(t) + p(t)\phi^{\lambda}(t) = -Lt^{\lambda}(1-t)^{\lambda}p(t) + p(t)\phi^{\lambda}(t)$$

$$\geq L^{\lambda}t^{\lambda}(1-t)^{\lambda}p(t)\left[\frac{(1-t)\int_0^t s^{1+\lambda}(1-s)^{\lambda}p(s)ds + t\int_t^1 s^{\lambda}(1-s)^{1+\lambda}p(s)ds]^{\lambda}}{t^{\lambda}(1-t)^{\lambda}} - L^{1-\lambda}\right] \quad \textbf{(4.1)}$$

So we only show that from **(4.1)**, $\tau(t) - \eta(t) > 0$. Where:

$$\tau(t) = (1-t)\int_0^t s^{1+\lambda}(1-s)^{\lambda}p(s)ds + t\int_t^1 s^{\lambda}(1-s)^{1+\lambda}p(s)ds.$$

$$\eta(t) = (1-t)\int_0^t s^{1+\lambda}(1-s)^{1+\lambda}p(s)ds + t\int_t^1 s^{1+\lambda}(1-s)^{1+\lambda}p(s)ds.$$

Clearly,

$$t(1-t)L^{1-\lambda} \leq \eta(t).$$

In fact,

$$\tau(t) - \eta(t) = (1-t)\int_0^t s^{2+\lambda}(1-s)^{\lambda}p(s)ds + t\int_t^1 s^{\lambda}(1-s)^{\lambda+2}p(s)ds > 0.$$

Thus, ϕ is a strict lower solution.
On the other hand;

$$\psi''(t) + p(t)\psi^{\lambda}(t) \leq p(t)U^{\lambda}[-1 + \frac{1}{U}(max(a,b) + Ug)^{\lambda}] < 0.$$

If we choose $U > \frac{max(a,b)}{1-C}$, $U > 0$ where: $|g(t)| < C$.

For $t \in [0,1]$, define:

$$f(t,x) = \begin{cases} p(t)\psi^{\lambda}(t) & x > \psi(t) \\ p(t)x^{\lambda}(t) & \phi(t) \leq x \leq \psi(t) \\ p(t)\phi^{\lambda}(t) & x < \phi(t). \end{cases}$$

We will show the existence of a C[0,1] solution to the BVP

$$x'' + f(t,x) = 0 \; x(0) = a, x(1) = b. \qquad (4.2).$$

and prove that the solution to (4.2) is indeed a positive C[0,1] solution to (4.0).
To do this, Let

$$\mathbf{X} = \{x \in C[0,1] : \phi(t) \leq x(t) \leq \psi(t), 0 \leq t \leq 1\}$$

endowed with the norm:

$$||x|| = sup_{t\in[0,1]}|x(t)|.$$

Define:

$$T : \mathbf{X} \mapsto \mathbf{X}$$

by:

$$Tx(t) = a + (b-a)t + (1-t)\int_0^t sf(s,x(s))ds + t\int_t^1 (1-s)f(s,x(s))ds \; t \in [0,1].$$

Then it is clear that $Tx(t)$ is continuous and well defined on [0,1].

$$For\ t \in (0,1),\ x(t) \in \mathbf{X},\ we\ have:$$

$$f(t,\phi(t)) \leq f(t,x(t)) \leq f(t,\psi(t)).$$

So that $T(\phi) \leq T(x) \leq T(\psi)$.
We want to show that indeed $T(x) \in \mathbf{X}$. To do this, we show that

$$\phi \leq T(\phi) \leq T(\psi) \leq \psi$$

For $\phi \le T(\phi)$: Suppose not. Then there exists some $t_0 \in (0,1)$ *such that*

$$\phi(t_0) > T(\phi)(t_0)$$

Without loss of generality, we assume

$$max_{t\in[0,1]}(\phi(t) - T(\phi)(t)) = \phi(t_0) - T(\phi)(t_0)$$
$$Then \;\; \phi''(t_0) - T(\phi)''(t_0) = 0.$$

But
$T(\phi)''(t_0) = f(t_0, \phi(t_0)) = p(t_0)\phi^\lambda(t_0)$ *and* $\phi''(t_0) > p(t_0)\phi^\lambda(t_0)$.
This implies that $\phi''(t_0) - (T\phi)''(t_0) > 0$, a contradiction. This thus shows that $\phi(t) \le T(\phi)(t)$ for all $t \in [0,1]$.
Similarly, $T(\psi)(t) \le \psi(t)$ for all $t \in [0,1]$.
So, if $\phi \le x \le \psi$ and $x \in C[0,1]$, we have

$$\phi \le T(\phi) \le T(x) \le T(\psi) \le \psi.$$

Thus, $T(x) \in X \; if \; x \in X$.
We observe that

$$\begin{aligned}
\psi(t) &= a + (b-a)t + U[(1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds] \\
&\le max(a,b) + U[\int_0^t s(1-s)p(s)ds + \int_t^1 s(1-s)p(s)ds] \\
&= max(a,b) + U[\int_0^1 s(1-s)p(s)ds] \\
&\le K \; for \; some \; K > 0.
\end{aligned}$$

For $x(t) \in \mathbf{X}$, we have

$$\begin{aligned}
|\frac{d}{dt}Tx(t)| = |(b-a) + \frac{d}{dt}(1-t)\int_0^t f(s,x(s))ds + \frac{d}{dt}t\int_t^1 (1-s)f(s,x(s))ds| \; t \in (0,1) \\
\le (b-a) + \int_0^t sf(s,\psi(s))ds + \int_t^1 (1-s)f(s,\psi(s))ds. \\
= (b-a) + \int_0^t sp(s)\psi^\lambda(s)ds + \int_t^1 (1-s)p(s)\psi^\lambda(s)ds \\
\le (b-a) + \int_0^t Ksp(s)ds + \int_t^1 K(1-s)p(s)ds.
\end{aligned}$$

Let $h(t) = \int_0^t Ksp(s)ds + \int_t^1 K(1-s)p(s)ds$.
Then maximum point is at $t = \frac{1}{2}$.
So that

$$|\frac{d}{dt}Tx(t)| \leq (b-a) + 2\int_0^{\frac{1}{2}} s(1-s)p(s)Kds + 2\int_{\frac{1}{2}}^1 s(1-s)p(s)Kds$$
$$= (b-a) + 2K\int_0^1 s(1-s)p(s)ds = K_0.$$

for all $x \in \mathbf{X}$ *and* $t \in (0,1)$.

This shows that $T(\mathbf{X})$ is uniformly bounded and equicontinuous. Thus, by the Arzela-Ascoli theorem, $T(\mathbf{X})$ is Precompact.
Since $T(\mathbf{X})$ is precompact, Schauder fixed point theorem guarantees that T has a fixed point in **(X)**.
The fixed point of **T** is given by:

$$x(t) = a + (b-a)t + (1-t)\int_0^t sf(s, x(s))ds + t\int_t^1 (1-s)f(s, x(s))ds$$

which is clearly a C[0,1] solution of (4.2).
We claim that this solution is a positive C[0,1] solution of (4.0). For this we show

$$\phi(t) \leq x(t0 \leq \psi(t) \ t \in [0,1].$$

Suppose $x(t) > \psi(t)$ for some t, then there is some point $t_0 \in (0,1)$ such that;

$$(x-\psi)'(t_0) = 0 \ and \ x''(t_0) - \psi''(t_0) \leq 0.$$

Then from (4.2),

$$\psi''(t_0) + p(t)\psi^{\lambda}(t_0) \geq 0.$$

contradicting ψ being a strict upper solution.
In addition, suppose $x < \phi$ at some t, then there is some $t_0 \in (0,1)$ such that;

$$(\phi - x)'(t_0) = 0, \ and \ (\phi - x)''(t_0) \geq 0.$$

But this from (4.2) contradicts that ϕ is a strict lower solution as seen in the previous case.

Contradiction reached in both cases thus guarantee that the solution so affirmed from the Schauder fixed point theorem is a solution of the problem (4.0) as claimed. That the solution is positive is readily seen.
This completes the proof of Theorem (4.1)

Theorem 4.2
The problem (4.0) has a positive $C^1[0,1]$ solution if and only if

$$0 < \int_0^1 t^\lambda (1-t)^\lambda p(t)dt < \infty.$$

Proof
$\Longrightarrow$
Let $x(t)$ be a positive $C^1[0,1]$ solution of (4.0).
Since $t^\lambda(1-t)^\lambda p(t) \geq 0$

$$\int_0^1 t^\lambda (1-t)^\lambda p(t)dt \geqslant 0.$$

If $\int_0^1 t^\lambda (1-t)^\lambda p(t)dt = 0$, then, $p(t) = 0,\ t \in (0,1)$.
This would lead to a contradiction on the condition that $p(t) > 0$.
Thus,

$$0 < \int_0^1 t^\lambda (1-t)^\lambda p(t)dt$$

Take $k \in \mathbf{R}$ such that;

$$x(t) \geqslant kt(1-t) + a + (b-a)t$$

See Zhang [55] for the existence of such k. Then,

$$x(t) \geqslant kt(1-t).$$

From (4.0),

$$-\int_0^1 x''(t)dt = \int_0^1 p(t)x^\lambda(t)dt \geqslant \int_0^1 p(t)t^\lambda(1-t)^\lambda k^\lambda dt$$

$$k^\lambda \int_0^1 p(t)t^\lambda(1-t)^\lambda dt \leqslant x'(0) - x'(1).$$

Thus

$$\int_0^1 p(t)t^\lambda(1-t)^\lambda dt \leqslant k^\lambda[x'(0)-x'(1)] < \infty$$

since by the assumption, $x'(0)$ *and* $x'(1)$ both exist. So that;

$$0 < \int_0^1 p(t)t^\lambda(1-t)^\lambda dt < \infty$$

and the proof of the necessity is complete.

Sufficiency

Suppose

$$0 < \int_0^1 p(t)t^\lambda(1-t)^\lambda dt < \infty.$$

We show the existence of a $C^1[0,1]$ positive solution x of (4.0).
Let

$$\phi(t) = a + (b-a)t + L[(1-t)\int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds + t\int_t^1 s^\lambda(1-s)^{1+\lambda}p(s)ds]$$

where:

$$L = [\int_0^1 s^{1+\lambda}(1-s)^{1+\lambda}p(s)ds]^{\frac{\lambda}{1-\lambda}}$$

Then; $\phi(0) = a, \phi(1) = b$

$$\phi''(t) + p(t)\phi^\lambda(t) = -Lt^\lambda(1-t)^\lambda p(t) + p(t)\phi^\lambda(t).$$

So $\phi \in C^2(0,1)$. Also;

$$\phi'(t) = (b-a) + L[\int_t^1 s^\lambda(1-s)^{1+\lambda}p(s)ds - \int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds]$$

$$\text{as } t \to 0^+, \phi'(t) \to (b-a) + L\int_0^1 s^\lambda(1-s)^{1+\lambda}p(s)ds$$

$$\text{as } t \to 1^-,\ \phi'(t) \to (b-a) - L\int_0^1 s^{1+\lambda}(1-s)^\lambda p(s)ds$$

So that $\phi \in C^1[0,1]$ Also,

$$\phi''(t) + p(t)\phi^\lambda(t) = -Lp(t)(1-t)^\lambda t^\lambda + p(t)\phi^\lambda(t) > 0$$

(This has been shown earlier in page 21). So that ϕ is a strict lower solution of (4.0).
Define:

$$\psi(t) = a + (b-a)t + U[(1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds]$$

for some constant U large enough. Then; $\psi(0) = a, \psi(1) = b$ and $\psi(t) \in C[0,1]\bigcap C^2(0,1) \quad and$

$$\psi''(t) + p(t)\psi^\lambda(t) = -Up(t) + p(t)[a + (b-a)t + Ug]^\lambda$$
$$where\ g(t) = (1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds.$$

$$\begin{aligned}\psi''(t) + p(t)\psi^\lambda(t) &\leq -Up(t) + p(t)[max(a,b) + Ug]^\lambda \\ &= p(t)U[-1 + \frac{1}{U}(max(a,b) + Ug)^\lambda]\end{aligned}$$

But,$lim_U \to \infty\ \frac{1}{U}(max(a,b) + Ug)^\lambda = 0$ so that,

$$\psi''(t) + p(t)\psi^\lambda(t) < 0$$

So that ψ is a strict upper solution of (4.0) Also, taking $U \geqslant L$;

$$\begin{aligned}\psi(t) - \phi(t) = U\big[(1-t)\int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds + t\int_t^1 (1-s)^{1+\lambda}s^\lambda p(s)ds\big] \\ - L\big[(1-t)\int_0^t s^{1+\lambda}(1-s)^\lambda p(s)ds + t\int_t^1 s^\lambda(1-s)^{1+\lambda}p(s)ds\big] \geqslant 0\end{aligned}$$

For $t \in [0,1]$, define:

$$f(t,x) = \begin{cases} p(t)\psi^\lambda(t) & x > \psi(t) \\ p(t)x^\lambda(t) & \phi(t) \leq x \leq \psi(t) \\ p(t)\phi^\lambda(t) & x < \phi(t) \end{cases}$$

Like we did in the $C[0,1]$ case,we will show the existence of a solution to the BVP:

$$x'' + f(t,x) = 0 \qquad x(0) = a,\ x(1) = b \qquad \textbf{(4.4)}.$$

and prove that the solution to (4.4) is a C^1 positive solution to (4.0).
To do this, Let;

$$\mathbf{X} = \{x \in C[0,1] : \phi(t) \le x(t) \le \psi(t), \quad 0 \le t \le 1\}.$$

endowed with the norm:

$$||x|| = sup_{t\in[0,1]}|x(t)|.$$

Define:

$$T : \mathbf{X} \mapsto \mathbf{X}\ by :$$

$$Tx(t) = a + (b-a)t + (1-t)\int_0^t sf(s,x(s))ds + t\int_t^1 (1-s)f(s,x(s))ds$$

Then it is clear that T is continuous and well defined.

For $t \in (0,1)$ and $x \in \mathbf{X}$,

$$\begin{aligned}
|\frac{d}{dt}Tx(t)| &= |(b-a) + \frac{d}{dt}(1-t)\int_0^t sp(s)x^\lambda(s)ds + \frac{d}{dt}t\int_t^1 (1-s)p(s)x^\lambda(s)ds|\ t \in (0,1) \\
&\le |(b-a)| + \int_0^t sp(s)x^\lambda(s)ds + \int_t^1 (1-s)p(s)x^\lambda(s)ds \\
&\le |(b-a)| + \int_0^t sp(s)\psi^\lambda(s)ds + \int_t^1 (1-s)p(s)\psi^\lambda(s)ds \\
&\le |(b-a)| + \int_0^t sp(s)Kds + \int_t^1 (1-s)p(s)Kds \\
& |(b-a)| + 2\int_0^{\frac{1}{2}} s(1-s)p(s)Kds + 2\int_{\frac{1}{2}}^1 s(1-s)p(s)Kds \\
&= |(b-a)| + 2K\int_0^1 s(1-s)p(s)ds \\
&\le |(b-a)| + 2K\int_0^1 s^\lambda(1-s)^\lambda p(s)ds \\
&= K_1 \quad for\ some\ constant\ K_1.
\end{aligned}$$

This shows that $T(\mathbf{X})$ is uniformly bounded and equicontinuous. Thus, by the Arzela-Ascoli theorem, $T(\mathbf{X})$ is Precompact.
Applying Schauder fixed point theorem, T has a fixed point in $(\mathbf{X})$.
We claim again that any fixed point of T in $\mathbf{X}$ is a $C^1[0,1]$ solution to problem (4.0).
Let $x(t) > \psi(t)$ for some t. Then there is some $t_0 \in (0,1)$ such that $(x' - \psi')(t_0) = 0$, and $(x'' - \psi'')(t_0) \leq 0$.
This would force:

$$\psi''(t_0) + p(t)\psi^{\lambda}(t_0) \geq 0.$$

But this would imply that ψ(t) is not a strict upper solution. A contradiction.
Similarly,
If $x(t) < \phi(t)$ for some t, then there is some point $t_0 \in (0,1)$ such that:

$$(x' - \phi')(t_0) = 0, and\ (x'' - \phi'')(t_0) \geq 0.$$

This would lead to:

$$\phi''(t_0) + p(t)\phi^{\lambda}(t_0) \leq 0.$$

A contradiction that ϕ is a strict lower solution of (4.0).

The contradiction reached in the two cases above shows that: $\phi(t) \leq x \leq \psi(t)$.
So that x is a solution of problem (4.0).

We will show next that with the assumed conditions, the solution x is a $C^1[0,1]$ solution. But this follows readily because
$x'(t)$ is monotone decreasing on (0,1),

$$\lim_{0^+} \tfrac{x(t)-a}{t} \ = \lim_{\xi \to 0^+} x'(\xi) for\ 0 < \xi < t$$

is either finite or ∞.

$$\lim_{t \to 0^+} \frac{\phi(t) - a}{t} \leq \lim_{t \to 0^+} \frac{x(t) - a}{t} \leq \lim_{t \to 0^+} \frac{\psi(t) - a}{t}.$$

That is, $P \leq \lim_{t\to 0^+} \frac{x(t)-a}{t} \leq Q$ for some constants P, Q.

The existence of P, Q above guarantee that $\lim_{t\to 0^+} \frac{x(t)-a}{t}$ is finite as required.
Also

$$\lim_{t\to 1^-} \frac{\phi(t)-b}{t} \leq \lim_{t\to 1^-} \frac{x(t)-b}{t} \leq \lim_{t\to 1^-} \frac{\psi(t)-b}{t}.$$

That is $R \leq \lim_{t\to 1^-} \frac{x(t)-b}{t} \leq S$ for some constants R, S. A similar argument like in the previous case establish the existence of $\lim_{t\to 1^-} \frac{x(t)-b}{t}$.

Thus, the solution x is a $C^1[0,1]$ solution of (4.0) as required.
This completes the proof of **Theorem 4.2**.

Remark The constants P, Q, R, S follow from **(4.3)** and the observation on $\psi(t)$ in page 25.

Theorem 4.3.
Any $C^1[0,1]$ solution of problem (4.0) is unique.

Proof:
Let x, y be two positive $C^1[0,1]$ solutions of (4.0). See ([15], [48], [45]).
Our method of proof here is to show that x intersects y on any subinterval of (0,1).
Suppose this is not the case, then we have an interval, $(t_1, t_2) \subset (0,1)$ such that:

$\forall t \in (t_1, t_2),\ x(t) > y(t),\ x'(t_1) \geq y'(t_1),\ x'(t_2) \leq y'(t_2)$ and $x(t_1) = y(t_1) = a_1,\ x(t_2) = y(t_2) = b_1$ for some constants a_1, b_1.

Now, define the Wronskian of $x,\ y$:

$$\mathbf{W}(t) = x(t)y'(t) - x'(t)y(t).$$

Then

$$\mathbf{W}'(t) = x(t)y''(t) - x''(t)y(t) = xyp(t)[x^{\lambda-1} - y^{\lambda-1}] < 0.$$

Thus, $\mathbf{W}(t)$ is strictly decreasing.
Now,

$$0 \leq x(t_2)y'(t_2) - y(t_2)x'(t_2) = b_1[y'(t_2) - x'(t_2)] < a_1[y'(t_1) - x'(t_1)] \leq 0.$$

A contradiction. Thus, we must have some point $t_3 \in (t_1, t_2)$ such that x intersects y at t_3.
Next, we show that x and y will again intersect in (t_3, t_2) so that continuing we would show that x intersects y on every subinterval of (0,1).
The argument is just like in the previous case. Suppose for contradiction that $x(t) > y(t)$ $\forall t \in (t_3, t_2)$, then we have: $x'(t_2) \leq y'(t_2),\ x'(t_3) \geq y'(t_3)$ and the monotonicity of the Wronskian again gives:

$$0 \leq x(t_2)y'(t_2) - y(t_2)x'(t_2) = b_1[y'(t_2) - x'(t_2)] < a_1[y'(t_3) - x'(t_3)] \leq 0$$

Again, a contradiction.
This is true for all intervals $(t_1, t_2) \subset [0, 1]$. So $x(t) = y(t)$ on a dense subset of [0,1]. Since $x(t),\ y(t)$ are continuous and equal on a dense subset of $(0, 1)$, $x(t) = y(t)$ on $(0, 1)$. Therefore, the $C^1[0, 1]$ solution of (4.0) is indeed unique.

5 Other Elements of Emden-Fowler equations

Consider the following boundary value problem:

$$x'' = f(t, x, x'),\ x(0) = a,\ x(1) = b,\ f \in C([0, 1] \times R \times R, R), \quad (5.0)$$

interest is always to establish the existence of solutions to the BVP above. To do this, one rewrites the given BVP into an integral equation of the form:

$$Tx(t) = \int_0^1 G(t, s)h(s, x(s))ds$$

on the function space $C[0, 1]$. The function $G(t, s)$ is called the Green's function to the BVP.
We used this approach to show the existence of positive solutions to the generalized Emden-Fowler equation. Our work is a generalization of some earlier results in the literature. ([40], [48], [54]) and references therein.

Definition 5.1

Let X be a normed space. $T : X \to X$ a map defined on X. Then, a point $x \in X$ is called a *fixed point* of T if $Tx = x$.
The generalized Emden-Fowler equation:

$$x'' + p(t)x^{\lambda} = 0,\ p(t) \in C(0, 1)$$

has been studied by many Researchers ([48], [54], [52]) subject to the boundary condition

$$x(0) = x(1) = 0.$$

In the references above, $p(t)$ was assumed nonnegative so that x is concave down.
Some other authors see ([16], [13]) have tried to examine the above equation with $f(t, x)$ not necessarily positive.
To this end, they considered the more realistic problem:

$$x'' + p(t)|x|^{\lambda} sgnx = 0 \quad x(0) = r_1, \quad x' = r_2 \qquad \textbf{(5.1)}$$

on the interval $I = [0, \infty)$.

Other aspects of solutions of (1.1) has also been investigated in the literature and results concerning the existence of solutions, uniqueness of solutions, continuation of solutions, assymptotic solutions, boundedness and stability of solutions, oscillation and nonoscilation, conditions for the solution to be in the Holder space as well as conditions for the solution to be in Sobolev space may be found in [12], [28], [33-34].
We summarize some important results in this chapter for further investigations in the future.

5.1 Uniqueness and Continuation

From equation (1.1), it is clear that if $p(t)$ is continuous for given initial values, we are sure of the existence of local solutions for the case $\lambda > 1$. That the solution is unique is quickly seen as $f(t, x)$ is Locally Lipschitz. In his work, Belohorec[4] showed that in the case $0 < \lambda < 1$, all solutions are continuable. This result and the observation stated above then leads us to query the conditions for the continuation of solutions in the superlinear case as well as the conditions for the uniqueness of solutions in the sublinear case. We will give a summary of most of related results in the literature. Results in the sublinear case are from ([12], [28]). Coffman[12] gave a uniqueness result based on the boundary values as well as a uniqueness result which uses $\log p(t)$ having a finite lower variation on compact subintervals of $(0, \infty)$, the result of Heidel[15] was based on $p(t)$ being of bounded variation on $(0, \infty)$.

We begin with some results about the uniqueness of solutions in the sublinear case and continuability of solutions in the superlinear case.

SUBLINEAR CASE:

Theorem 5.1.1(Coffman and Wong [12])
Let $0 < \lambda < 1, r_1 = 0, r_2 \neq 0$, then the IVP (5.1) has a unique local solution.

Theorem 5.1.2(Coffman and Wong [12])
Let $0 < \lambda < 1$ and $p(t)$ be positive and $\log p(t)$ have finite lower variation on every compact subinterval of $[0, \infty)$, then the zero solution of the IVP (5.1) is unique.

Theorem 5.1.3(Heidel [28])
Let $0 < \lambda < 1$, and $p(t)$ be positive and locally of bounded variation on

$(0, \infty)$. Then the zero solution of the problem (5.1) is unique.

SUPERLINEAR CASE:
Here are some results concerning continuation.

Theorem 5.1.4(Coffman and Wong [12])
Let $\lambda > 1$ and $p(t)$ be positive and $\log p(t)$ have finite upper variation on every compact subinterval $[0, \infty)$, then every solution of the problem (5.1) is continuable on $[0, \infty)$.

Theorem 5.1.5(Coffman and Ulrich [13])
Let $\lambda > 1$ and $p(t)$ be positive and locally of bounded variation on $[0, \infty)$. Then every solution of (5.1) is continuable on $[0, \infty)$.

One then wonders conditions for which there would be non-continuable solutions and non-unique solutions.

An example for a non-unique solution was given by Heidel [15]. While Coffman[12] gave an example of a noncontinuable solution.It has also been shown (see Kiguradze [33]) that the non-negativity condition cannot be dropped to allow for continuation of solutions. He showed solutions which are non-continuable when the non-negativity condition was dropped. For an illustrative example, see Heidel[28].

5.2 Oscillatory and Nonoscillatory Results

. **Definition 5.2**
Let $x(t)$ be a nontrivial solution of (1.1). Then, $x(t)$ is called *oscillatory* if for any $k > 0$, there is $t \geq k$ such that $x(t) = 0$. Put differently, $x(t)$ has arbitrarily large zeros. On the other hand, $x(t)$ is called *nonoscillatory* if it is not oscillatory.
We try now to gather various oscillatory and nonoscillatory results in the literature on the equation (5.1) and the Generalized Emden-Fowler equation both in the sublinear and superlinear cases.
First, we state a result which tells us the form which any solution to the generalized Emden-Fowler Equation takes. In this result, we assume $p(t)$ only continuous not necessarily nonnegative. Here, (1.10 was studied on the interval $[0, \infty)$.

Theorem 5.2.1 [4]
Let $p(t)$ be continuous on $[t_0, \infty)$ and let:

$$\int^{\infty} t^{\lambda}|p(t)|dt < \infty.$$

Then, for every solution $x(t)$ to the generalized Emden-Fowler equation, there exists a constant k such that $lim_{t\to\infty}x'(t) = k$. If in addition;

$$\int^{\infty} t^{1+\lambda}|p(t)|dt < \infty,$$

then any solution of (1.1) is of the form:

$$x(t) = k_2(t) + k_1 = o(1),$$

where k_1, k_2 are suitable constants and "o" stands for *little oh*.
For the sublinear case, the existence of oscillatory and nonoscillatory solution to the equation (1.1) is contained in the following result with the condition of nonegativity imposed on $p(t)$.

Theorem 5.2.2[4].
Let $p(t)$ be continuous and positive on the interval $[t_0, \infty)$ $(t_0 > 0)$. Also, let:

$$p(t)t^{(3+\lambda)/2}$$

be nonincreasing and bounded from below with a positive constant k. Then equation (1.1) has both nontrivial oscillatory and nonoscillatory solutions.

Since the theorem above shows the possibility of both nontrivial oscillatory and nonoscillatory solutions to (1.1), under what conditions in the sublinear case are we guaranteed that the solution is nonoscillatory. Belohorec[4] has this in the next theorem. We will as well give a necessary and sufficient condition for which the equation (1.1) is oscillatory for $0 < \lambda < 1$.

Theorem 5.2.3 [4].
Let the function $p(t)$ be positive and continuous in the interval $[t_0, \infty)$ $(t_0 > 0)$. Let there exist a number $\beta, 0 < \beta < (1 - \lambda)/2)$ such that the function :

$$p(t)t^{(3+\beta)/2+\beta}$$

is nondecreasing and bounded from above by a positive constant k. Then all solutions of (1.1), besides the trivial one, are nonoscillatory.

If all regular solutions of the equation (1.1) are oscillatory, then we call the equation (1.1) Oscillatory. In the same vein, if all regular solutions of (1.1) are nonoscillatory, then we call (1.1) nonoscillatory [52].
A necessary and sufficient condition for equation (1.1) to be Oscillatory under the sublinearity assumption is next presented:

Theorem 5.2.4 ([5] [52])
Let $0 < \lambda < 1$. Then the equation (1.1) is oscillatory if and only if:

$$\int_0^\infty t^\lambda p(t)dt = \infty.$$

Theorem 5.2.5 ([52],[4]) Let $0 < \lambda < 1$. Let

$$t^{(\lambda+3)/2}p(t)$$

be nonincreasing for all large values of t and be bounded below. Then equation (1.1) has oscillatory solutions.
A similar result was stated and proved by [29] in which the assumption of nonincreasing and being bounded below were reversed.

Theorem 5.2.6 ([52], [29])
Let $0 < \lambda < 1$. Let

$$t^{(\lambda+3)/2}p(t)$$

be nondecreasing and bounded above. Then (1.1) has oscillatory solutions.
The next result will show that the boundedness condition given by [29] was not necessary for the equation (1.1) to have oscillatory solutions.

Theorem 5.2.7 ([11], [52]).
Let $0 < \lambda < 1$. Let;

$$t^{(\lambda+3)/2}p(t)$$

be nondecreasing for large values of t. Then (1.1) has oscillatory solutions.

More general results can be found in [11].
An integral condition for the existence of oscillatory solution is given below[3].

Theorem 5.2.8 [30], [52].
Let $\psi = p^{-1/(3+\lambda)}$ and assume that
(i) $\int_0^\infty 1/\psi^2 = \infty$ and
(ii) $\int_0^\infty |\psi\psi''| < \infty$.
Then (1.1) has Oscillatory solutions.

Next we give some nonoscilllatory results on the sublinear case.

The following are some integral conditions for nonoscillatory solutions of problem (1.1).

Theorem 5.2.9 [27].
Let $0 < \lambda < 1$, if $p(t)$ is nonincrerasing and;

$$\int_0^\infty tp(t)dt < \infty.$$

Then (1.1) is nonoscillatory.

An analogue of the above theorem in the superlinear case is given below:

Theorem 5.2.10 [2].
Let $\lambda > 1$. Let $p(t)$ be nonincreasing and;

$$\int_0^\infty t^\lambda p(t)dt < \infty.$$

Then (1.1) is nonoscillatory.
An analogue of $Theorem 5.2.4$ above in the superlinear case is given below:
Theorem 5.2.11 [2]
Let $\lambda > 1$. A necessary and sufficient condition for all solutions of (1.1) to be oscillatory is that:

$$\int_0^\infty tp(t)dt = \infty.$$

The following give results which are the superlinear versions of the results stated above:

Theorem 5.2.12 [12].

Let $\lambda > 1$. Let $t^{(\lambda+3)/2}p(t)$ be nondecreasing for all large values of t. Then every solution of (1.1) with a zero must be oscillatory.

Theorem 5.2.13 [41]
Let $\lambda > 1$. Let

$$(t \log t)^{(\lambda+3)/2} p(t)$$

be nonincreasing for all large values of t. Then (1.1) is nonoscillatory.

Theorem 5.2.14 [34].
Let $\lambda > 1$. Let

$$t^{(\lambda+3)/2+\epsilon p(t)}$$

is nondecreasing for some $\epsilon > 0$ and for all large values of t. Then (1.1) is nonoscillatory.

Belohorec[4] gave a condition for which nonoscillatory solutions of equation(1.1) are bounded in the following result.
Theorem 5.2.15 [4].

Let $0 < \lambda < 1$. $p(t)$ be nonnegative, continuous in the interval $[t_0, \infty)$ and;

$$\int^{\infty} tp(t)dt < \infty.$$

Then any nonoscillatory solution of (1.1) is either bounded or of the form $x(t)ct(c \neq 0)$.

Erbe[19] introduced the following notation from which he gave nonoscillatory results which relaxes the nonincreasing assumption imposed on $p(t)$ by many authors above.

$$P_+(t) = \exp\left(\int_{t_0}^{t} \frac{dp_+(s)}{p(s)}\right), \qquad t_0 \leq s < \infty$$

$$P_-(t) = \exp\left(\int_{t_0}^{t} \frac{dp_-(s)}{p(s)}\right), \qquad t_0 \leq s < \infty$$

where we have the Jordan representation of $p(t)$ as: $p(t) = p_+(t) - p_-(t)$. From which the following identity is immediate.

$$\tfrac{P_+(t)}{P_-(t)} \equiv \tfrac{p(t)}{p(t_0)} \qquad\qquad t_0 \leq s < \infty$$

We now state the results.

Theorem 5.2.16 [19]
Let $\lambda > 1$. Then the equation (1.1) is nonoscillatory if any of the following conditions hold:

$$(a) \quad \int_{t_0}^{\infty} t^{\lambda-1} p(t)(P_+(t))^{(\lambda-1)/2} dt < \infty \ and$$

$$lim_{t\to\infty} t \int_{t}^{\infty} t^{\lambda-1} p(t)(P_+(t))^{(\lambda-1)/2} dt = 0,$$

$$(b) \quad \int_{t_0}^{\infty} (p(t))^{2/\lambda+1}(P_+(t))^{(\lambda-1)/(\lambda+1)} dt < \infty, \ and$$

$$lim_{t\to\infty} t \int_{t}^{\infty} (p(t))^{2/(\lambda+1}(P_+(t))^{(\lambda-1)/(\lambda+1)} dt = 0,$$

$$(c) \quad \int_{t_0}^{\infty} p(t)(P_-(t))^{(\lambda-1)/(\lambda+1)} dt < \infty, \ and$$

$$lim_{t\to\infty} \ t \int_{t}^{\infty} p(t)(P_-(t))^{(\lambda-1)/(\lambda+1)} dt = 0,$$

$$(d) \quad \int_{t_0}^{\infty} t^{\lambda} p(t) dt < \infty, \ and$$

$$lim_{t\to\infty} (P_+(t))^{(\lambda-1)/2} \int_{t}^{\infty} t^{\lambda} p(t) dt = 0,$$

$$(e) \quad \int_{t_0}^{\infty} (p(t))^{1/(\lambda+1)}(P_+(t))^{\lambda/(\lambda+1)} dt < \infty,$$

$$(f) \quad lim_{t\to\infty} \ t^{\lambda+1} p(t)(P_+(t))^{(\lambda-1)/2} = 0,$$

$$(g) \quad lim_{t\to\infty} t^2 p(t)(P_-(t))^{(\lambda-1)/(\lambda+1)} = 0.$$

Remark In proving Theorem 5.2.16, [19] showed that the Theorem would remain valid if $\lambda = 1$. It is not as well difficult to see that in theorem 2.2.16, the conditions $(a), (b)$ are equivalent. The equivalences of the conditions (c) and (d) as well as conditions (e) and (f) are readily seen as well.

For $p > 0$, the following corollary was proved for the above theorem.

Corollary 5.2.17 [19]
Let $p(t)$ be locally of bounded variation on $[t_0, \infty)$ $(t_0 \geq 0)$. If

$$\int_{t_0}^{\infty}(p(t))^{\frac{1}{2}}(P_+(t))^{\frac{1}{2}}dt < \infty,$$

then the equation (1.1) is nonoscillatory.

For equivalent results and proofs, see [19]. However, we give a collection of results by [19] which are sublinear analogues of Theorem 2.2.16 above.

Theorem 5.2.18 [19]
Let $0 < \lambda < 1$. Then the equation (1.1) is nonoscillatory in case any of the following holds:

$$(a) \quad \int_{t_0}^{\infty} t^{\lambda}p(t)dt < \infty, \ and$$

$$lim_{t\to\infty}\ (p(t))^{(\lambda-1)/2}(P_+(t)^{(1-\lambda)/2}\int_{t}^{\infty} t^{\lambda}p(t)dt = 0,$$

$$(b) \quad \int_{t_0}^{\infty} t^{\lambda}p(t)dt < \infty, \ and$$

$$lim_{t\to\infty}(P_-(t))^{(1-\lambda)/2}\int_{t}^{\infty} t^{\lambda}p(t)dt = 0,$$

$$(c) \quad \int_{t_0}^{\infty}(p(t))^{1/(\lambda+1)}(P_+(t))^{\lambda/(\lambda+1)}dt < \infty \ and$$

$$lim_{t\to\infty}(P_-(t))^{1-\lambda/2(\lambda+1)}(P_+(t))^{-\lambda/(\lambda+1)} \times \int_{t}^{\infty}(p(t))^{1/(\lambda+1)}(P_+(t))^{\lambda/(\lambda+1)}dt = 0,$$

$$(d) \quad \int_{t_0}^{\infty}(p(t))^{1/(\lambda+1)}(P_+(t))^{\lambda/(\lambda+1)}dt < \infty, \ and$$

$$lim_{t\to\infty}(p(t))^{(\lambda-1)/2(1+\lambda)}(P_+(t))^{(1-3\lambda)/2(1+\lambda)} \times \int_{t}^{\infty}(p(t))^{1/(\lambda+1)}(P_+(t))^{\lambda/(\lambda+1)}dt = 0,$$

$$(e) \quad lim_{t\to\infty}t^2p(t)(P_+(t))^{(1-\lambda)/(1+\lambda)} = 0,$$

$$(f) \quad lim_{t\to\infty}t^{\lambda+1}p(t)(P_-(t))^{(1-\lambda)/2} = 0.$$

Remark
Again, the theorem above holds true in the linear case where $\lambda = 1$. Like in

the previous case, the conditions (a) and (b) are equivalent. Also, just like conditions (c) and (d), conditions (e) and (f) are equivalent.

Recently, Cecchi[8], Grace[21], Graef and Saker[22] have extended earlier oscillatory results to third order Emden-Fowler differential equations. For much recent nonoscillatory results see [51], [36]. We turn our attention next to assymptotic results for solutions of the generalized Emden-Fowler equation.

5.3 Assymtotic Solutions

Definition 5.3
(a) **Bounded Nonoscillatory Solution**:
Let $x(t)$ be a solution of (1.1). If:

$$lim_{t\to\infty} = a \neq 0$$

then x is a bounded nonoscillatory solution of (1.1).
(b) **Assymptotically linear solution**:
A solution $x(t)$ of (1.1) such that

$$lim_{t\to\infty}\frac{x(t)}{t} = 0$$

is called an assymptotically linear solution of (1.1).

A natural question that comes to mind is if such solutions actually exist. The answer is in the affirmative as revealed by [20]. The work of [20] led reasearchers to seek conditions for which such solutions are guaranteed. Such conditions were revealed by [2] for $\lambda > 1$. For the sublinear case, see [49]. A combination of both results is the following[52].

Theorem 5.3.1 [52]
Let $\lambda > 0$. then the equation (1.1) has a bounded nonoscillatory solution if and only if:

$$\int_0^\infty tp(t)dt < \infty.$$

Similarly, a nenessary and sufficient condition for the equation (1.1) to have assymptotically linear solution was reported as early as 1962. See [52].

Theorem 5.3.2 Let $\lambda > 0$. Then a necessary and sufficient condition for (1.1) to have assymptotically linear solution is that:

$$\int_0^\infty t^\lambda p(t)dt < \infty.$$

Remark
The study of such solutions then attracted the interest of researchers after [3] revealed that whenever a bounded nonoscillatory solution or an assymptotically linear solution is shown to exist, then all solutions to (1.1) must as well be either bounded or assymptotically linear. Though this result was for the superlinear case, the sublinear analogue was later given by [4].

Theorem 5.3.3 [38].
Let $\lambda > 1$. If (1.1) has an assymptotically linear solution, then every nonoscillatory solution is either bounded or satisfies: $lim_{t\to\infty}\frac{x(t)}{t} = 0$.

Theorem 5.3.4 [3] .
Let $0 < \lambda < 1$. If (1.1) has a bounded non oscillatory solution, then every nonoscillatory solution is either bounded or satisfies:$lim_{t\to\infty}\frac{x(t)}{t} = 0$.

More development and assymptotic results can be found in [52]. For more recent treatment on assymptotic solutions, see Dulina[17].

5.4 $H_0^1(0,1)$ *and* $C^{0,\alpha}[0,1] \cap C^2(0,1)$ *solutions*

Recently, the Generalized Emden-Fowler equation has been studied in the space of Holder's continuous functions as well as the Sobolev space. Conditions for solutions to be in $H_0^1(0,1)$ can be found in [10].
Chaparova[9] has the following result for solutions in Holder spaces.

Theorem 5.4.1 [9].
Let $p(t) \geq 0,\ on\ (0,1)\, p(t) \in C(0,1),\ and\ 0 < \lambda < 1$.
If the problem (1.1), (1.2) has a positive solution $x \in C^{0,\alpha}[0,1] \cap C^2(0,1)$ for some $\alpha \in (0,1)$ then

$$\int_0^1 (t(1-t))^{1-\alpha(1-\lambda)}(\psi(t)\psi(1-t))p(t)dt < \infty$$

for every concave function $\psi \in C[0,1]$ such that $\psi(0) = 0,\ \psi(t) > 0$ for $t > 0$, and $\int_0^1 \frac{\psi(t)}{t}dt < \infty$.

See also [9] for an example of such function ψ.
For two independent conditions for which solutions of (1.1), (1.2) are in sobolev spaces see[10].

6 Conclusion

Treating BVPs which describe real-life problems numerically tends to be time consuming. In gas dynamics for instance, a numerical approach to the study of equation (1.1) tends to be difficult because of the stiffness of the system occasioned by drastic changes in the flow behavior of the gases through a nozzle.

For accuracy, we study equation (1.1) with nonnegative boundary data analytically, obtained necessary and sufficient conditions for the existence of positive solutions and also showed the uniqueness of positive $C^1[0,1]$ solutions.

The study of solutions in Holder space for the superlinear case is left for future studies.

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