The Pure Expectations Theory of the Term Structure of Interest Rates: Arbitrage Opportunities Between Futures and Cash Markets for United States Treasury Bills.

A Thesis Submitted To The
Graduate Faculty of Arts and Sciences
In Partial Fulfillment Of The
Requirements For The Degree of
Master of Arts

Department of Economics

by

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THE PURE EXPECTATIONS THEORY OF THE TERM STRUCTURE OF INTEREST RATES: ARBITRAGE OPPORTUNITIES BETWEEN FUTURES AND CASH MARKETS FOR UNITED STATES TREASURY BILLS

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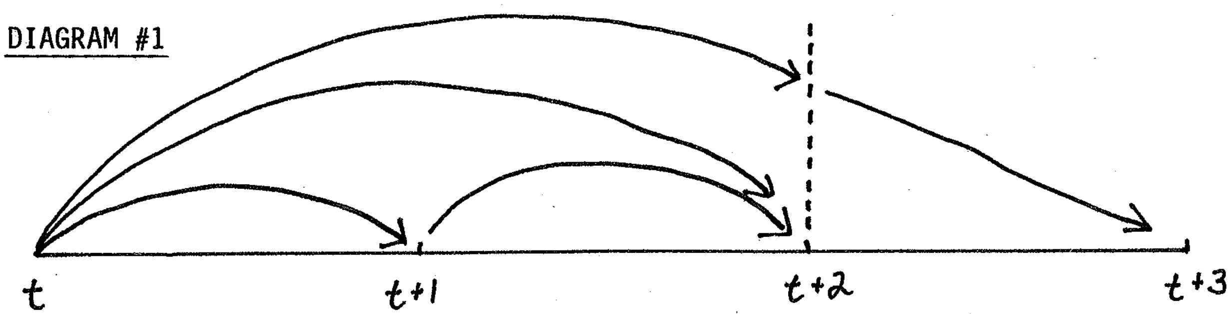
CHAPTER #1
INTRODUCTION

The term structure of interest rates is the description of the various returns available to an investor who wishes to make a loan of his capital for various durations. The return paid upon the capital, or the interest rate paid when the loan is made in monetary units, depends upon the size of the loan and the length of time the investor must wait for repayment. In a modern financial environment, the size of the loan is often dictated by institutional considerations with certain economies of scale usually resulting in greater returns being paid to larger loans. The remaining determining factor of the return is, therefore, the time until repayment or duration of the loan. It is this factor that is the focus of term structure research.

With the size of the investment fixed, the term structure of interest rates refers to the level of interest rates as a function of time or duration of the loan. The investor chooses the length of time he wishes to loan his capital and the rate of interest is determined by market factors. The length of time that the capital is invested is divided into investment periods. These may be as short or as long as convenient for measurement but the usual division is into quarterly periods. The investor may wish to loan out his capital for one investment period or many periods depending upon his inclination but regardless of his choice, the length of time he wishes to have his capital invested is his holding period. Thus, with a known holding period, the investor studies the term structure of interest rates, which describes the rate of interest as a function of time, and attempts to maximize his return over his holding period.

Since an investor's holding period may span several possible investment periods, the investor can achieve his return in several possible ways. First of all, he may match perfectly his investment period and his holding period by contracting his loan for the same duration as his holding period. This would be the easiest route and would only require that the investor consult the interest rates quoted to match his holding period. If the holding period was sufficiently long that funds could be invested in more than one period, then the investor could invest the capital in a series of one period loans which in total would match his holding period. With this strategy, the investor would make a one period loan, then, at the end of the period, reinvest the principal and the interest in a second one period loan. This procedure would be repeated, continuously compounding the interest until the entire holding period had been spanned. The third possibility would be to invest the capital for a number of periods which exceeded the planned holding period. At the completion of the holding period, the loan then could be resold as a loan of shorter duration, the invested capital realized, and any profit or loss on the transaction would be the investor's return on capital invested. Since all three of these choices are available to the investor, then it would seem appropriate that he would choose the strategy that yields the greatest return on his capital invested over the holding period. Likewise, users of capital would apply similar reasoning in their decisions on accepting the loan.

DIAGRAM #1



An example of the three possible strategies is depicted in Diagram #1. Here an investor with a two period holding period faces three possible strategies. Starting at time "t", the investor wishes to place his capital such that he maximizes its future value at time "t+2", the termination of his holding period. The first and simplest method would be to make a two period loan at time "t" which matures at time "t+2". This loan which spans two investment periods perfectly matches his planned holding period. A second method would be to make two one period loans. The first loan matures at "t+1" and then, reinvesting his capital and interest, he can make a second one period loan which matures at "t+2". As a third alternative, he could make a three period loan which matures at time "t+3". Since the investor only has a two period holding period, he plans to sell his three period loan at time "t+2" as a one period loan. At that time, the original loan will have one period left to maturity. This is shown in the diagram as the dotted line at time "t+2". In each of these strategies, the investor regains his capital as well as any interest at time "t+2".

This paradigm was developed into a general theory by John R. Hicks (Hicks, 1939). He proposed that given a known holding period and the rate of interest on a one period loan, the investor calculated the future value of his capital (FV) by multiplying its present value (PV) by one plus the current one period interest rate (r).

$$FV = PV \times (1+r)$$

For a multi-period investment with all the one period interest rates known, then, the future value is equal to the product of the present value and the series of one period rates. To show this, subscripts are added for each of the investment periods 1, 2, 3, ... n.

$$FV = PV \times (1+r_1) \times (1+r_2) \times (1+r_3) \times \ldots (1+r_n)$$

The yield of the investment over the holding period is the ratio of the future value to the present value which is equal to the product of one plus the interest rate for each investment period. Thus, if we let "R" represent the yield on a "n" period investment:

$$R = \frac{FV}{PV} = (1+r_1)(1+r_2)(1+r_3) \ldots (1+r_n)$$

If all the one period interest rates are known, the investor can compare the yield on one and multi-period investments. By making this comparison, the investor can determine which of the three types of strategies will create the maximum future value in relation to the present value of his investment.

Unfortunately, the investor situated at time "t" does not know the one period interest rates in the future periods. He knows the one period rate interest rate "r_1", and from the term structure, he knows the multi-period rates which terminate at "t+2, t+3", through "t+n", but he does not know the individual one period rates "r_2, r_3", through "r_n". He may be of the opinion that one period rates will in the future rise, fall, or remain the same. Based on these assumptions, he can estimate his best possible strategy but the certainty of the outcome will be limited to the accuracy of his expectations about the future one period rate. This form of term structure theory is therefore called "Expectations Theory" because it is based on the investor's expectations of interest rates which will prevail in the future.

The obvious problem with this theory is that the investors' expectations of future one period rates are hard to measure. One attempt to measure the expected rates was made by Friedman (1979). He used survey data from a group of professional investors to construct a series of expected one period rates extending four periods into the future. These expected one period rates were tested for equality with implied forward one period rates. The results of this test proved inconclusive as it could not be determined whether the investors failed to act in the predicted manner or whether they reported poor or biased replies to the survey of their expectations.

Other tests have presupposed that the investors' expectations of future one period rates are formed from their experience of past one period rates. The investor has a known holding period but he has no expectation of what the one period rates will be in the future. Therefore, if he alters his investment plans in any way that does not perfectly match his investment period with his holding period, he accepts a risk of receiving a smaller or larger return on his investment. Since it is generally assumed that the investor is risk averse then he will require a greater return in order to induce him away from a perfectly matched investment. Models of term structure presented by Meiseleman and by Malkiel were based on this premise.

While their results proported to support the Expectations Theory, the independent variable in these tests was in fact, the difference between present and past short term rates while the dependent variable was the difference between spot and forward rates. Since errors in predicting past short term rates were used to forecast forward rates, the test is really a measure of efficiency in the short term bond market. Rather than saying anything about expected forward rates, this method tests the links between past rates and present rates.

Alternative theories were then developed which required the addition of premiums to equate expected future rates with the implied forward rates. These theories imposed several assumptions about individual investors' behaviours in order to explain the shape of the yield curve. Risk aversion, transactions costs, and fixed holding periods were utilized to explain the differences in yield over term to maturity.

Most notably, Modigliani and Sutch proposed that investors were uncertain in their forecasts of future rates, faced positive transactions costs, had a preferred holding period, and were risk averse. Expected future rates are seen as a function of past rates. To express the uncertainty and costs of making a forward transaction, a liquidity premium was added which varied in proportion to the term to maturity. The forecasting equation was then fitted mathematically to the existing data using an Almon polynomial. While this procedure worked well for the initial data set, it was found difficult to replicate without changing the degree of the polynomial and the lag the equation was tested over. The explanational power of the theory was based on the mathematical fitting technique of the Almon procedure.

From this it can be seen that the importance of the Pure Expectations Theory, is its basis on future short term rates and its efficiency through the exclusion of any other extraneous variables. By predicting a simple equality of future and implied forward rates, the Expectations Theory has a simple elegance which is rich in its explanatory power and concise in its forecasts. With a wide range of future one period rates known to the investor, he can quickly and easily construct a required multi-period rate which would perfectly match his holding period.

The introduction of a Financial Futures market provided the investor with the information necessary on future short term rates to complete his decision-making process. While futures rates may not be accurate predictors of future one period rates, they are certain predictors of one period rate in the future that can be contracted for immediately. There is no allowance for risk because he knows with certainty what his future one period rate will be. This is the informat~~or~~ion that the investor requires at time "t" to construct his multi-period loan.

This development in financial markets provides an opportunity to test the Expectations Theory with a greater reliability than was available earlier. The futures market provided certain futures rates that can be contracted for in the present time period. This eliminates the necessity of making assumptions about investors' expectations of future rates and vastly increases the power of any test of the theory.

CHAPTER #2
REVIEW OF LITERATURE

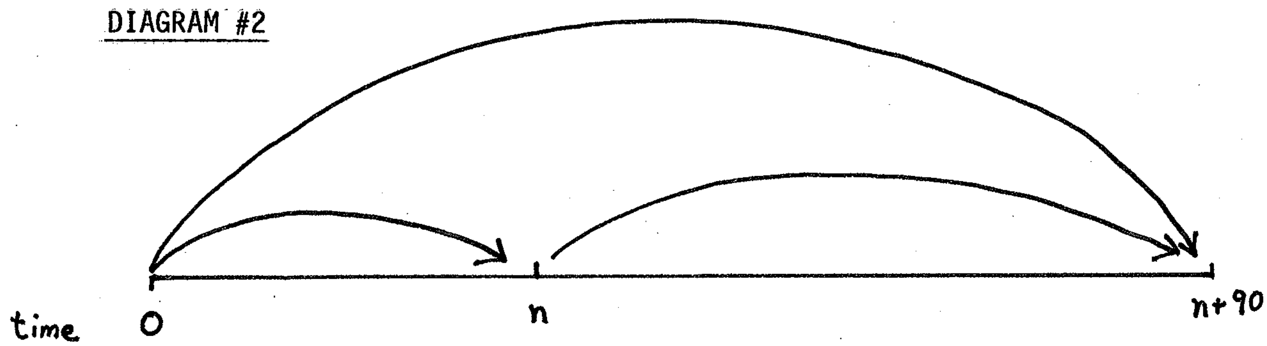
Studies of the relationship between futures contracts and forward rates have, in the past, attempted to verify the Pure Expectations Theory of equality between the implied forward rates from the cash market and the rates available on futures contracts for the same instrument over the same time period. Because the comparison was made between instruments, which are homogeneous in terms of liquidity, risk, term, and time to delivery, it was postulated that the rate of return on both investments should be equal. In all of the studies, an implied forward rate was calculated from the bid and ask price quotes in the cash market. This rate was then compared to the yield on a corresponding futures contract. While the findings of these studies varied in the results found for different time series data, the common conclusion was that futures rates most often exceeded the implied rate from the cash market and that the size of the premium was directly proportional to the time to delivery of the futures contract. These results contradicted the theory developed by Hicks which suggested that since the futures rates guaranteed a one period rate in the future, they would be equivalent to the implied forward rate from the cash market.

Various reasons have been presented to explain the premium of future over implied forward rates. Transactions costs such as commissions and margin costs add an extra expense to trading in the futures market. In order to compensate the purchaser of the futures contract for the extra cost, the price must be lowered and thus, the yield quoted in the market would rise. Also, liquidity in the longer term futures contracts is limited and this may hinder the pricing efficiency of these contracts. For risk averse investors, the possibility of default on delivery of a futures contract, even though this has never happened and delivery is guaranteed by the clearing corporation, might lead them to look for a premium to induce them into the market. Other added but hidden costs such as the taxation differences between futures and cash contracts might also effect pricing decisions and thus, the stated yield on the futures contracts. Institutions wishing to trade in the futures market also face added costs for accounting and information gathering. While in the long run these costs would be negligible, they do imply a real added cost that an institutional trader would face. Thus, the differences in rate of return between the futures and implied forward rate could result from structural costs and barriers which add to the costs of trading the futures contracts.

The alternative possibility that these studies present is that they could have incorrectly calculated the yields from the alternative arbitrage strategies. By not allowing for the differences in cash returned per dollar invested, these studies may have overstated the yield from the futures contract. The yield from different arbitrage strategies is dependent upon the price of the securities traded and the amount of money that has to be deposited to guarantee the completion of the transaction. This factor complicates the calculation of yield on treasury bill transactions because treasury bills are traded on a discount basis. Unlike bonds which are priced at face value and pay interest, treasury bills are sold at a fraction of their face value but are redeemed for their full face value upon maturity. Thus, the price of a treasury bill is inversely related to both its term and its yield. This facet is very important in the calculation of the final yield because the initial capital outlay can be significantly less than the total face value of the transaction. By not allowing for this difference, it is very easy to overstate the yield from the futures contract.

Expectations theories predicted the equality of forward rates and future rates but at the time that they were first developed, futures were only traded in basic commodities. Therefore, the theory was limited to an academic posulate because of its untestable nature. The introduction of United States Treasury Bill Futures Contracts by the Chicago Board of Trade in 1976 opened the door for academic research into Pure Expectations Theory (PET). The first study was conducted by William Poole in the summer of 1977. Poole tested the arbitrage possibilities of holding either a single treasury bill with more than ninety-one days to maturity or a combination of a short term treasury bill and a futures contract for delivery of a ninety day treasury bill at the time the shorter bill matures. These two strategies are constructed so that the holding periods are equal. This is shown in Diagram #2.

DIAGRAM #2



In examining the data from November 1976 through June 1977, Poole found that the futures contract nearest to delivery closely followed the yields implied by the cash market and that opportunities for profitable arbitrage between the two markets rarely exist. He did not test futures contracts further from delivery and thus did not show a premium developing as the delivery date moved farther away.

Poole examines the arbitrage possibilities by testing to see if a long term cash bill can be sold for more than it would cost to purchase a short term cash bill and a second short term cash bill, contracted for in the futures market, such that the two investment vehicles mature on the same day. In Diagram #1, this would mean comparing the price of a single bill, bought at time "t = 0" and maturing at time "n + 90", to the price of two bills, the first bought at "t = 0" and maturing at time "t = n", and the second contracted for at time "t = 0", with delivery taken at time "t = n" and maturing at time "n + 90". The price that the long term cash bill can be sold for is the dealer bid price:

$$P^b_{n+90,0} = 100 - \frac{n + 90}{360} R^b_{n+90,0}$$

where: $P^b_{n+90,0}$ is the price a dealer will pay for a cash bill at time "0" with "n + 90" days to maturity.

$R^b_{n+90,0}$ is the annual rate of return in percentage form on the bid price.

The dealer bid price is compared to the cost of purchasing two shorter term cash bills. Working backwards from time "n + 90", Poole first calculates the amount "q_n" which will be required to purchase the futures contract and take delivery of the ninety-~~one~~ day bill at time "n". This amount is:

$$q_n = 100 - \frac{90}{360} F_{n,0}$$

Where "Fn,0" is the annual percentage rate of return on a ninety-~~one~~ day bill contracted for at time "0", with delivery taken at time "n".

Since treasury bills are purchased at a discount to mature at par, only a fractional amount of treasury bills need be purchased to generate amount "q_n" at time "n". The amount "q_n" is based on the price of one hundred dollars in face value of bills, therefore, the amount which must initially be purchased is:

$$Q_n = \frac{q_n}{100}$$

Where "Qn" is the quantity of bills purchased in one hundred dollar units.

Thus, less than the full face value of a futures contract of one million dollars is required from the maturity of the first treasury bill. Also, in purchasing the futures contract, a margin requirement of $1,500 per million is deposited. Since this amount is returned at the time of delivery, the net cash requirement per $100 of bills at time "n" is:

$$Q_n = \frac{q_n - 0.15}{100}$$

Now a cash bill must be purchased at time "0" which will generate the amount "Q_n" with its maturity at time "n". The price of this bill per one hundred dollars of face value is the asking price in the cash market denoted as "$P_{n,0}$". Commissions and margins of \$60. and \$1,500. per contract will also have to be paid at time "0". Since these are based on a million dollar contract, the cost per hundred will be \$0.006 and \$0.15 respectively. Thus, the amount paid out at time "0" will be:

$$P^{*}_{n,0} = P^{a}_{n,0} + 0.15 + 0.006$$

But since only a fraction of a cash bill must be purchased, the total price will be:

$$P_{n,0} = 100\ Q_n \cdot P^{*}_{n,0}$$

For this amount, the investor buys an "n" day cash treasury bill and pays commissions and margin costs on a futures contract. At time "n", the cash bill matures in the amount "Q_n" which is required to purchase the futures contract. This contract gives delivery of a treasury bill which matures in ninety days and will be redeemed at full face value.

To examine the arbitrage possibilities, the price of the two short term bills "$P_{n,0}$" is compared to a single long term cash bill "$P_{n+90,0}$".

A profitable arbitrage situation exists if the long cash bill can be sold for more than the cost of a shorter cash bill and a futures contract.

From the equations above, the cost of the shorter cash bill is "$P_{n,0}$" and the cost of the futures contract is "qn". Breaking these two costs into their components, we can find by substitution that the cost of a bill of longer than ninety days to maturity can be described as:

$$P^{b}_{n+90,0} = \left(\frac{q_{n,0} - 0.15}{100}\right)\left(P^{a}_{n,0} + 0.15 + 0.006\right)$$

Here the right-hand side of the inequality shows the cost of the futures contract and the shorter term cash bill. The first part of the right-hand side is the cost of the futures contract. The price of the futures contract is "q_n". From this amount, a margin cost of fifteen basis points is subtracted. This amount is divided by one hundred to find the unit cost per hundred dollars of treasury bills contracted for in the futures market. The second part of the right-hand side is the total cost of the shorter term cash bill. The price "Pn,0" is the dealer ask price for the "n" day bill at time "0". To this price, margin costs of fifteen basis points and commission costs of six-tenths of a basis point are added. Thus the right-hand side of the expression shows the total cost of the futures contract and the shorter term cash bill.

From here, Poole develops the upper and lower arbitrage conditions but switches to yield terms:

$$(1)\quad F_{n,0} > \left(1 - \frac{n}{36000} R^{a}_{n,0}\right)^{-1} \left(\frac{n+91}{91} R^{b}_{n+90,0} - \frac{n}{90} R^{a}_{n,0} + \frac{360}{90}(0.006) + \frac{n}{90}(0.0015\, R^{a}_{n,0})\right)$$

$$(2)\quad F_{n,0} < \left(1 - \frac{n}{36000} R^{b}_{n,0}\right)^{-1} \left(\frac{n+90}{90} R^{a}_{n+90,0} - \frac{n}{90} R^{a}_{n,0} - \frac{360}{90}(0.006) - \frac{n}{90}(0.0015\, R^{b}_{n,0})\right)$$

Here expression (1) defines the upper critical point and expression (2) defines the lower critical point for profitable arbitrage. If the future rate "Fn,0" is greater than the expression on the right-hand side of inequality (1) or less than (2), then it will be profitable to trade the securities. For the upper limit, it would be profitable to sell a "n+90" day bill at time "0" and invest the proceeds in a "n" day bill which could be used to take delivery of an "90" day bill at time "n" with the total maturing at time "n+90". Alternatively, for the lower arbitrage point (2), the long bill could be purchased and the "n" day bill and the future contract sold.

In testing these two arbitrage conditions, Poole found that rarely did the futures rate "Fn,0" fall outside the upper or lower limits. Thus, he concluded that there is an equality between the implied forward rate and the futures rate. Since arbitrage opportunities rarely exist and since they are small in magnitude and do not persist for long, then investors act to arbitrage the inefficiencies in the market and bring the rates back into equilibrium. The small differences that persist between the implied forward rate and the futures rate are attributable to the transactions costs involved in trading the futures market. Thus, the futures rate is an unbiased expectation of the forward rate of interest "n" days into the future.

William Poole's findings that the yields on futures contracts correspond to the implied forward rates were quickly countered by Lang and Rasche (1978). They tested contracts of longer duration and found that Poole's findings were applicable only to futures contracts nearest to delivery. By allowing for slight differences in delivery dates and by utilizing treasury note prices for cash securities longer than one year to maturity, Lang and Rasche tested contracts up to two years away from delivery. The results showed that over the period March 1976 through March 1978, premiums existed on future contract yields over the corresponding implied forward rate. Also, while the premium was very small for the contract closest to delivery, the premium grew with contracts further from their delivery date. Testing time series data, it was also shown that the premium did not disappear with the expansion of trading in the Treasury Bill Futures Market.

These findings confirmed the idea that futures rates were inefficient estimates of expected forward rates because of the transactions costs and risks involved in trading futures contracts. It was thought that because of the costs of margins and commissions that the interest rates on futures contracts before transactions costs would automatically exceed the implied forward rate. By including a fudge factor for a liquidity premium to cover the default risks of making a forward contract, this simply put a further positive bias on the futures rate. As the level of uncertainty was expected to increase with time to delivery then, this liquidity premium would be directly proportional to the time to delivery of the futures contract. Thus Lang and Rasche confirmed the theory that the futures rates were positively bias estimates of the expected forward rate.

Reviewing these methods, however, a conceptual error can be spotted in their formula for calculating the arbitrage involved in establishing the implied forward rate. Lang and Rasche utilized both treasury bill and treasury note instruments in studying yields up to two years to maturity. To form a continuous data set, they converted the treasury bill yields from a 360-day basis to a 365-day bond basis. This conversion is valid when comparing the yields but it overlooks one of the prime considerations when establishing the arbitrage possibilities and that is the price. Since the arbitrageur is interested in trading the securities, his primary focus is on price rather than yield. The arbitrageur buys in one market and sells in the other hoping to make a profit on the price difference. This difference is especially important when discount instruments are involved in the transactions since they do not earn interest but rather appreciate in price over time.

This conceptual difference is evident when comparing the methods used by Lange and Rasche with those used by Poole. In Poole's formula, the price of a long term treasury bill was compared to the prices of two shorter term bills that would mature to the same future value as the longer term bill. Because the second bill was purchased in discount form, less than one trading unit of the first bill had to be purchased. Thus in the two time periods, unequal amounts of capital was invested. A smaller amount of capital was invested in the first treasury bill than in the second.

Lang and Rasche relied on Richard Roll's (1970) formula for interest paying instruments to calculate the implied forward rate. This formula assumes that the capital invested is the same in all periods and that the implied forward rate is the weighted average of the long and the short rates currently available in the cash market.

$$R_f = \frac{n \cdot R_n - R_{n+90}}{90} \cdot (n+90)$$

R_f = implied forward rate of time "0"

R_n = cash rate on "n" day bill

R_{n+90} = cash rate on "n+90" day bill

The Richard Roll formula assumes that the same amount of money is invested over the entire holding period. The long rate for the "n+90" day bill is assumed to be the weighted average of the two shorter bills. The implied forward rate is thus the weighted difference between the long and short cash bills.

This formula makes no allowance for the prices of the two discount cash bills. Since the bills are sold at a discount from face value to mature at par, the longer term bill generally sells for a lower price. The short term cash bill matures to par at some time over the holding period and the proceeds are then re-invested in a second bill. To equate the yields, the higher price of the short term bill and the compunding effect at the roll-over point have to be taken into account. The simplest method to do this is to invest equal amounts in each bill at the start of the holding period. Because the price of the shorter bill is ~~less~~ greater than the price of the longer bill, a smaller amount, in terms of face value, must be purchased of the shorter term bill.

The effect of purchasing like amounts of both the long and short cash bills can be shown on the Richard Roll formula:

$$R_f = \frac{(n+90) \cdot R_{n+90} - n \cdot R_n}{90}$$

for a discount bill: $R = \frac{D}{F} \frac{360}{n}$

where: D = discount in dollars

F = face value in dollars

R = annual rate of return

Substituting into the Richard Roll formula:

$$\frac{D_f}{F_F}\,\frac{360}{n} = \frac{n+90\left(\frac{D_L}{F_F}\,\frac{360}{n+90}\right) - n\left(\frac{D_s}{F_s}\,\frac{360}{n}\right)}{90}$$

with n = 90

$$\frac{360\,D_f}{F_F} = 180\left(\frac{D_L}{F_L}\,\frac{360}{180}\right) - 90\left(\frac{D_s}{F_s}\,\frac{360}{90}\right)$$

reduces to $\frac{D_f}{F_f} = \frac{D_L}{F_L} - \frac{D_s}{F_s}$

if $F_L = F_s = F_F = 100$ then,

$$D_f = D_L - D_s$$

however $F_L = F_F = 100$ but $F_s < 100$

in fact the amount required at the roll-over time is the amount required to purchase the second short term bill P_F. Thus:

$$F_s = 100\left(\frac{P_F}{100}\right)$$

since the bills have a face value of one hundred. The ratio of discounts to face value of the bills is therefore:

$$\frac{D_s}{100\left(\frac{P_F}{100}\right)} = \frac{D_L}{100} - \frac{D_F}{100}$$

reducing and solving for the forward bill discount D_F

$$D_F = D_L - \frac{P_F}{100}\,D_s$$

To change this back into a yield formula for the forward rate R_f, we use:

$$D = RF\,\frac{t}{360}$$

Substituting this into the discount formula:

$$R_f \; F_f \; \frac{90}{360} = R_L \; F_L \; \frac{180}{360} - \left(\frac{P_F}{100}\right) R_s \; F_s \; \frac{90}{360}$$

$$R_f = \frac{180 \; R_L - \left(\frac{P_F}{100}\right) \cdot 90 \cdot R_s}{90}$$

From this it can be seen that since "P_F" is always less than one hundred, the weighting of the short term bill is reduced when calculating the average yield. This increases the implied forward rate. This is logical since a smaller amount of money is invested in the short term bill which must grow to the same amount by the end of the holding period.

The Richard Roll formula understates the implied forward rate. By taking the weighted average of the two short term bill yields, no allowance is made either for compounding or for purchasing a lesser amount of the first short term cash bill. This explains the findings of Lang and Rasche that the future rate exceeded the implied forward rate. The yields calculated for the implied forward rate were understated because of the improper weighting of their formula.

Supporting the Lang and Rasche findings, Capozza and Cornell (1978) also found large deviations between returns from futures contracts and the implied forward rate and they also verified that these deviations increased systematically with the time to maturity of the futures contract. Testing data over the period March 1976 through June 1978, they also confirmed that the premium on futures rates remained constant with respect to time to delivery over the test period.

The reasons for these findings was once again the method of calculating the implied forward rate, or rather in their test, the implied long rate given a spot and futures rate:

$$r = \frac{r_n^s\ (n) + {}_n r_m^f\ (m)}{n + m}$$

where r = implied long rate

r_n^s = yield of a "n" day bill in spot market

${}_n r_m^f$ = in the futures rate, the rate on a "n" day bill, "n" days in the future

As with the Lang and Rasche formula, this calculation, because it is based on yields rather than the prices of discount instruments, puts a downward bias on the implied long cash rate. The findings that the futures rate exceeded the implied forward rate was therefore not valid. This finding can be explained by the formula utilized to construct the implied forward rate.

Recognizing the shortcoming in current term structure research, Kane (1980) addressed the problem of establishing the equilibrium prices of alternative ways of making two period investments. In examining the options open to the investor, Kane studied seven possible methods of spanning a two period investment horizon. This opportunity to arbitrage provided a method for reformulating existing theories of term structure in terms of bond prices rather than yields. Under Pure Expectations Theory with access to arbitrage possibilities, Kane postulated that the seven strategies would be subject to one price.

In Kane's approach, traditional term structure theories base their explanations on the yield side of the pricing identity:

$$P_{n,t} = 1/(1 + R_{n,t})^n$$

where "$P_{n,t}$" is the unit price of an "n" period security at time "t" which is found by discounting "R_{nt}", the yield to maturity of an "n" period bond, by the future value of a dollar at the maturity date. In his interpretation, Kane treats prices as the independent factor while the yields are the resulting by-product. Calculating an implied forward rate, the price of a two period bond is the product of the two one period prices.

$$P_{2t} = P_{1t} \left(P^F_{1(t,t+1)} \right)$$

where P_{2t} is the price of a two period bond at time "t"

P_{1t} is the price of a one period bond at time "t"

$P^F_{1(t,t+1)}$ is the price of a future one period bond contracted for at time "t" for delivery at time "t+1"

The forward price of bills at time "t+1" can be determined from the term structure at time "t" as:

$$P^F_{1(t,t+1)} \equiv P_{2t}/P_{1t}$$

Viewing the two period price "P_{2t}" as the product of the current one period price and the future one period price, allows for the cash flow implications of discount securities. The right-hand side of the equation is the product of price and quantity of securities purchased at time "t" which will produce one dollar of income at maturity. Thus, the price of the two period bill is shown as the discounted present value of the dollar paid at maturity. Note that this is the same as William Poole's formula except that no allowance is made for commissions or margin costs.

While Kane was able to develop this technique, he did not test his equation against the current data from the futures and cash markets. The balance of this thesis will be devoted to testing this interpretation of Pure Expectations Theory utilizing the prices quoted in the cash and futures markets rather than their respective yields.

CHAPTER #3

THEORY

The introduction of financial futures has created the opportunity to empirically test the Pure Expectations Theory of the term structure of interest rates. By examining the interest rates available in cash and futures markets, it can be seen whether or not the interest rate available on a multi-period investment is a function of a series of one period rates contracted for delivery in the future. As we have seen, the PET theory predicts that arbitrage will cause the rates between cash and futures markets to tend towards an equilibrium.

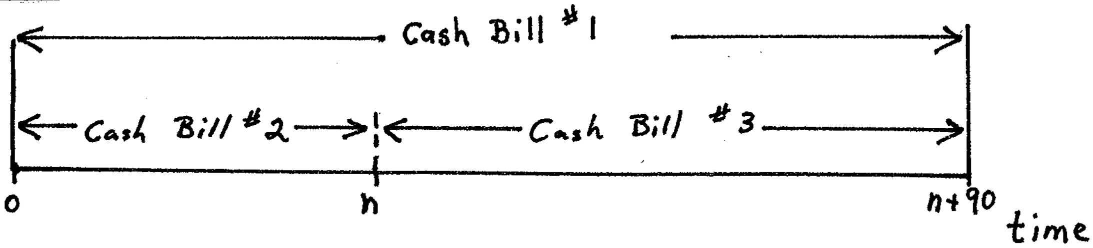
If the Expectations Theory is valid, if both the cash Treasury Bill Market and the Treasury Bill Futures Market are efficient, and if the investor has unrestricted access to both markets, then the implied forward rate of interest from the cash market must equal the forward rate available in the futures market. Arbitrage between the two markets would tend to equate the relative prices such that the rates of return available from competing strategies would be equal. The implied forward rate would equal the actual forward rate in the futures market.

In this chapter, another method of calculating the implied forward rate will be introduced. Since this method is widely used within the financial industry to determine the arbitrage conditions, it will be utilized here to test the Pure Expectations Theory. Also, the effect of commission and margin costs will be discussed with reasons on why they should be included or excluded when calculating the implied forward rate. The Industry formula will then be compared to the alternative formulas presented in Chapter 2 and the results of each will be predicted. Finally, a testable hypothesis on the Pure Expectations Theory will be presented along with a method of testing the hypothesis.

Since many financial institutions currently actively trade their portfolios of short term securities, methods for calculating the implied forward rate are widely known. Within the securities industry, the method developed by Marcia Stigum is most widely accepted. As with Poole's formula, this method is based on the prices of the securities. Since this formula is in common usage, it has been included in this paper as a further test of the equality between future and implied forward rates.

Stigum uses the same method as Poole except that no allowance is made for commission or margin costs. She starts with the same reminder that treasury bills are discount securities which do not earn interest but rather are purchased at a discount and mature at face value. Therefore, at the end of the investment period, the face value of the bills must be equal. To find the implied forward rate, Stigum examines three treasury bills as shown in Diagram No. 3. Here, cash bill number one spans the entire holding period. Cash bill number two is a shorter term bill which covers the first portion of the holding period. Finally, cash bill number three covers the final portion of the holding period. For our purposes, cash bill number three will be limited to a ninety day treasury bill which can be contracted for delivery in the futures market.

DIAGRAM #3



If we let:

Subscript 1 denote terms applicable to Cash Bill 1
Subscript 2 denote terms applicable to Cash Bill 2
Subscript 3 denote terms applicable to Cash Bill 3

and if we let:

F = face value at maturity
D = discount from face value in dollars
d = the annual rate of return in decimal form
t = days to maturity

then at the end of the investment period:

$$F_1 = F_3$$

since at the end of the investment period the face values must be equal. Since "F_2" is reinvested to purchase Cash Bill 3, the discount in cash paid for Cash Bill 3 will be:

$$D_3 = F_1 - F_2$$

Substituting this into the discount formula,

$$d_3 = \frac{D_3}{F_3} \left(\frac{360}{t_3}\right)$$

we have:

$$d_3 = \left(\frac{F_1 - F_2}{F_1}\right) \frac{360}{t_3}$$

The discount, D, is calculated as,

$$D = F \frac{td}{360}$$

If we assume an investment of 1 then the value of F_1 at maturity is:

$$\begin{aligned} F_1 &= 1 + D_1 \\ &= 1 + F_1 \frac{td}{360} \end{aligned}$$

which reduces to:

$$F_1 = \frac{1}{1 - \frac{t_1 d_1}{360}}$$

The value of the investment with the maturity of the first Cash Bill 2 can be calculated similarly as:

$$F_2 = \frac{1}{1 - \frac{t_2 d_2}{360}}$$

These can be substituted back into expression for d_3 to get:

$$d_3 = \left[1 - \frac{\left(1 - \frac{d_1 t_1}{360}\right)}{\left(1 - \frac{d_2 t_2}{360}\right)} \right] \frac{360}{t_3}$$

which solves for the implied Cash Bill 3 rate in terms of "d_1, t_1, d_2" and "t_2".

Thus, the implied forward yield on a treasury bill can be determined from the yields and durations of bills sold in the cash market.

If the Pure Expectations Theory is correct, then the implied forward rate d_3 would be equal to the rate obtainable on the appropriate futures contract. The investor, knowing the yield available on a futures contract, would observe the yields available in the cash market and choose the investment strategy which would result in the greatest return on his capital. With a broad and rapid dissemination of the information on both markets, it is expected that the two markets would operate efficiently. This would result in an equality between the futures rate and the implied forward rate. Any inequality would result in arbitrage between the two markets until the yields returned to an equilibrium.

The Industry formula does not allow for commissions or margin costs. This is because the arbitrage is carried out by hedgers in the future market rather than speculators. As hedgers, these investors are interested in locking in a rate at a future date rather than speculating on a price change. Because they already own a treasury bill as part of the hedging strategy they need not post additional margin against their futures position. Commissions have been ignored because they are first of all negotiable and arbitrary within the industry, as well as because their effect would be very small on the final outcome. The suggested commission rate is \$60.00 per million dollar contract. As seen in Poole's formula, this amounts to \$0.006 per hundred dollars in face value at maturity. From the discount formula:

$$d = \frac{D}{F} \left(\frac{360}{t}\right)$$

the net effect on the yield would be:

$$d = \frac{.006}{100} \left(\frac{360}{90}\right) = .0002 = 0.02\%$$

or two basis points on a ninety day bill. The cost in yield terms does grow, however, as the term decreases. At its minimum term of one day, the cost in yield grows to:

$$d = \frac{.006}{100} \left(\frac{360}{1}\right) = 0.0216 = 2.16\%$$

While this may cause an error in the Industry's formula for bills very near maturity, the level of the error cannot be forecasted accurately because of negotiated commissions. For this reason, the commission payment is omitted from the general formula.

To test the Expectations Theory, we test for the equality of the implied forward rate and the rate available in the futures market. The null hypothesis is that for every observation taken individually, the implied forward rate from the cash market is equal to the rate available in the futures market. From the foregoing equations:

$$HO : d_3 = d_f$$

where "d_f" is the discount yield on a futures contract which has the same characteristics as Cash Bill 3.

If this hypothesis is correct, then information from the futures market can be utilized to establish the implied forward rate. As described by the Pure Expectations Theory, the present long term rate is dependent upon the current short term rate in the cash market and the future short term rate that can be contracted for in the futures market.

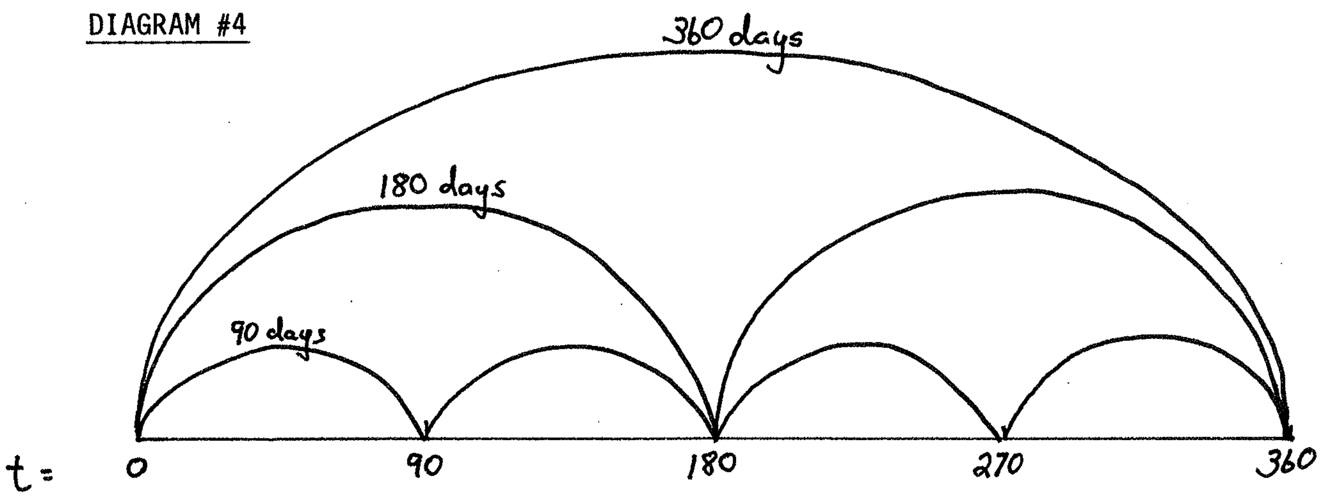
Method:

Treasury bills are traded in a secondary cash market on a daily basis. Since they are originally issued with maturities of 90, 180, and 360 days, as each day passes they can be traded with shorter maturities in the secondary market. Thus, at the original issue date, a holding period can be established which matches the original term on each bill. Every day after the original issue, a term can be established in the secondary market which is one day shorter. Since the term of both the long and the short cash bill decay at the same rate, the forward cash bill maintains a constant maturity.

This creates an opportunity to match the forward cash bill to a futures contract for a three-month bill deliverable on the date the short cash bill matures. The bill which is received upon taking delivery of the futures contract will mature at the same time the long cash bill matures. Thus, the investor has an arbitrage opportunity identical to that described by the Expectations Theory and shown in Diagram #1.

Since long cash bills could be purchased in a range of maturities up to 360 days, then at least three investment periods could be tested simultaneously. This is shown in Diagram #~~2~~4. The 360-day investment period could be obtained by purchasing a 360-day treasury bill or the combination of a 270-day bill and a futures contract deliverable in nine months' time. Short investment periods are bridged in a similar fashion.

DIAGRAM #4



As the investment period decays, the term of the first cash bill becomes shorter but the term of the second bill remains the same. Over the term of the first cash bill, a series of implied discount rates can be generated and compared to the discount rate on the futures contract.

To test the null hypothesis that the implied forward rate does equal the futures rate, four equations will be used to calculate the implied forward rate. First, the equation used by Lang and Rasche:

(1) $$d_3 = \frac{d_1 t_1 - d_2 t_2}{t_1 - t_2}$$

where: d_3 = implied forward rate

d_1 = rate on long cash bill

d_2 = rate on short cash bill

t_1 = term of long cash bill

t_2 = term of short cash bill

Secondly, the equation formulated by Poole:

(2) $$d_3 = 100 - \left(\frac{P_2}{P1} + .0156\right)\left(\frac{360}{t_1 - t_2}\right) + .015$$

where: P_2 is the price of the long cash bill

P_1 is the price of the short cash bill

.0156 is the combined cost of margin and commissions

.015 is the margin cost.

Thirdly, the Industry formula developed by Marcia Stigum:

(3) $$d_3 = \left(100 - \frac{P_1}{P_2}\right)\left(\frac{360}{t_1 - t_2}\right)$$

Finally, the Industry formula with commission costs.

$$(4) \qquad d_3 = 100 - \left(\frac{P_2}{P_1} + .0006\right)\left(\frac{360}{t_1 - t_2}\right)$$

It is expected that the Lang and Rasche formula will understate the implied forward rate because it does not allow for the discount paid for the short term cash bill (see Appendix "A"). By investing a smaller amount of money for the initial bill, the Poole and Stigum formulas will more closely approximate the actual yield on the implied cash bill. For this reason, it is expected that, if the null hypothesis is accepted, it will be accepted with the greatest confidence using either the Poole or Stigum formula.

APPENDIX "A"

To show the differences in yield and price based calculations of the implied forward rate under both high and low interest rate levels as well as long and short holding periods, the results of both Richard Roll's formula and the formula developed for this paper are compared. In the examples below, Case (a) uses Richard Roll's formula:

Case (a):
$$_{n-91}F_{91} = \frac{n R_n - (n-91)\ R_{n-91}}{91}$$

and Case (b) uses the Industry Formula:

Case (b):
$$d_3 = \left[1 - \frac{\left(1 - \frac{d_1 t_1}{360}\right)}{\left(1 - \frac{d_2 t_2}{360}\right)} \right] \frac{360}{t_3}$$

In all cases, the implied forward rate calculated on the yield basis is less than the implied forward rate calculated on the price basis.

Example (1): High Interest Rate Environment

$R_{90} = 14.75 \qquad R_{180} = 15.00$

(a) $$\frac{180(15.00) - 90(14.75)}{90} = 15.25$$

(b) $$\left[1 - \frac{\left(1 - \frac{.15(180)}{360}\right)}{\left(1 - \frac{.1475(90)}{360}\right)} \right] \frac{360}{90} = 15.83$$

Example (2): Low Interest Rate Environment

$R_{90} = 6.00 \qquad R_{180} = 6.15$

(a) $$\frac{(6.15)180 - 90(6.00)}{90} = 6.30$$

(b) $$\left[1 - \frac{\left(1 - \frac{.0615(180)}{360}\right)}{\left(1 - \frac{.06(90)}{360}\right)} \right] \frac{360}{90} = 6.40$$

Example (3): Long Holding Period

$R_{90} = 6.00 \qquad R_{300} = 6.25$

$t_{90} = 90 \text{ days} \qquad t_{300} = 270 \text{ days}$

(a) $$\frac{270(6.25) - 90(6.00)}{180} = 6.38$$

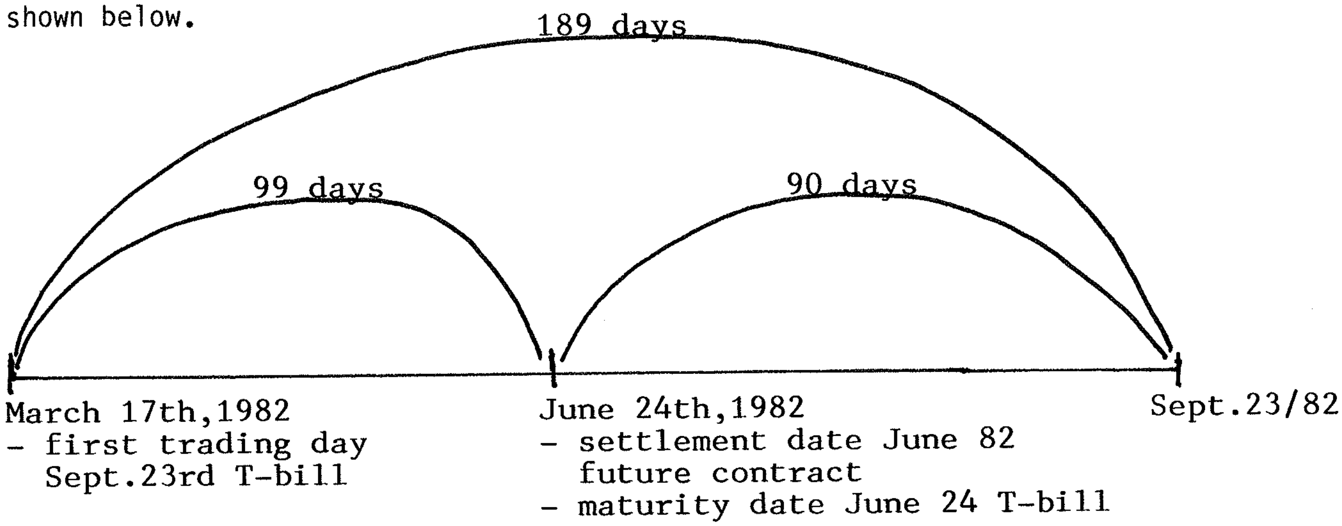
(b) $$\left[1 - \frac{\left(1 - \frac{.0625(270)}{360}\right)}{\left(1 - \frac{.06(90)}{360}\right)} \right] \frac{360}{180} = 6.47$$

CHAPTER #4
DESCRIPTION OF THE DATA

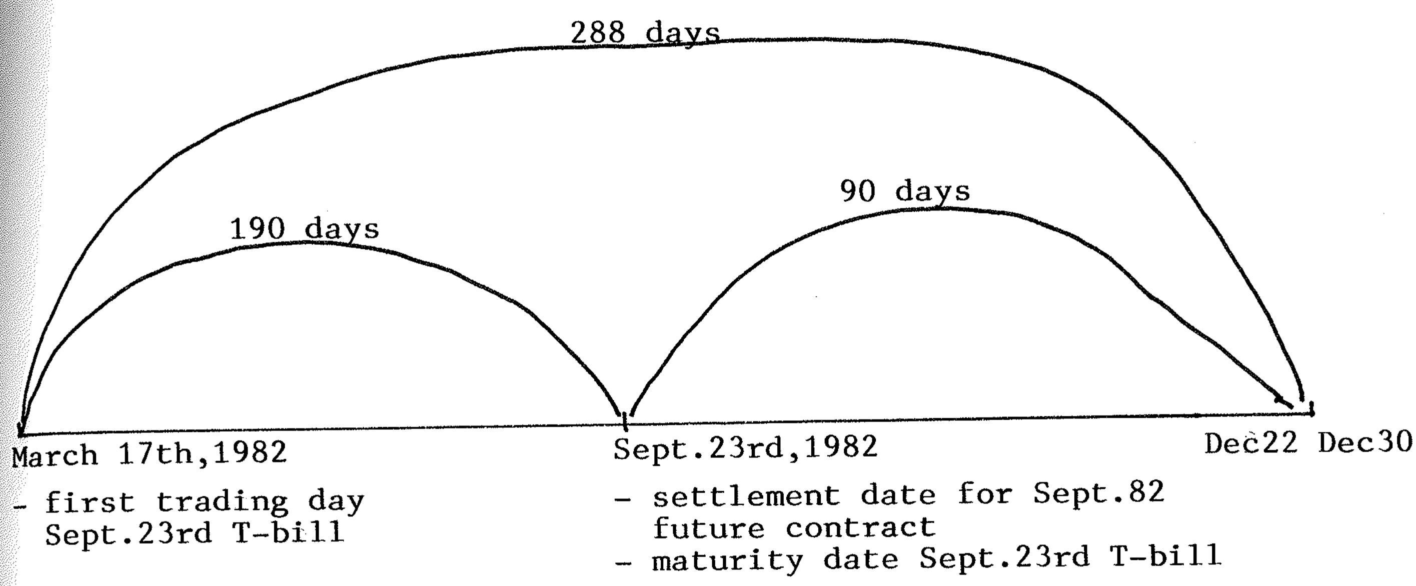
To test the Expectations Hypothesis, data was gathered on three treasury bill futures contracts and treasury bills trading in the secondary cash market. Over the time period from March 17th, 1982 through June 29th, 1982, the settlement prices of three treasury bill futures contracts dated for June 1982, September 1982, and December 1982, were collected. These contracts were chosen because they were easily matched to cash treasury bill maturities, provided a range of holding periods, and because they provided an investment period of less than one year. This was important because the futures contracts delivered discount treasury bills which were being compared to longer term discounted treasury bills which have maturities of less than one year. By keeping all of the investment vehicles equivalent and in discount form, no allowance had to be made for interest payments.

The cash treasury bills were chosen such that the maturities of the short term bills would coincide with the delivery dates of the futures contracts. The longer term bills were chosen such that their maturities would coincide with the maturity dates of the bills contracted for in the futures contract. In this way, a trader could either take delivery of the futures contract if he held the contract long or he could make delivery of his long term treasury bill if he held the futures contract short. Because maturity dates and delivery dates did not always coincide, the cash bill was chosen which could be easily and realistically held by an investor wishing to complete the required arbitrage.

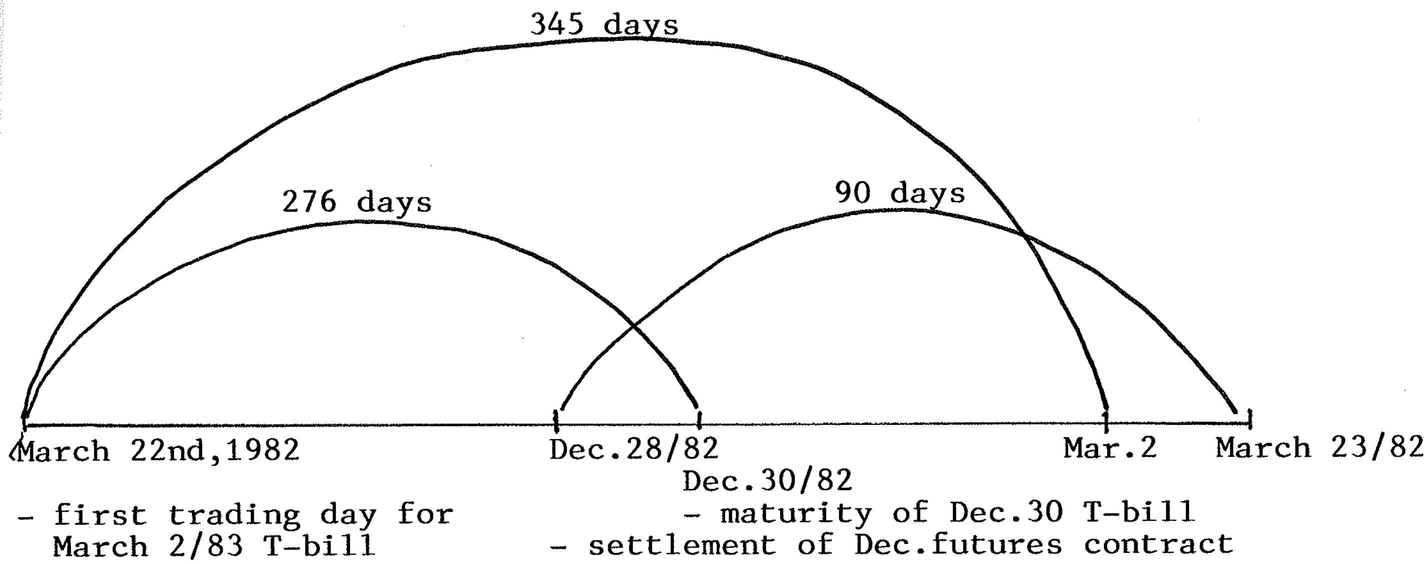
For the June 1982 contract, the settlement date for delivery of a ninety day treasury bill was June 24th, 1982. This bill would mature on September 23rd, 1982. Constructing the arbitrage situation, the treasury bills maturing on September 23rd, 1982 and June 24th, 1982 were utilized. The starting date at March 17th, 1982 was chosen because it was the first day that the September 23rd treasury bill was traded in the cash market. These bills allowed for a perfect matching of delivery and maturity dates as shown below.



The September contract requires the delivery of a ninety-day treasury bill on September 23rd, 1982. This treasury bill will mature on December 22nd of the same year. In the cash market, treasury bills could be purchased on March 17th which matured on September 23rd and December 30th. While the long cash bill exceeds the term of the bill required for the futures contract, it is deliverable against settlement of the contract. Thus, while the maturity of the cash treasury bills does not match perfectly the arbitrage requirements, the transaction is still possible. For the September contract, the diagram below depicts the arbitrage strategy:



In March of 1982, the December 1982 treasury bill futures contract was not perfectly reconcilable to any treasury bill traded in the cash market. The December futures contract provided for the delivery of a ninety-day treasury bill on December 23rd, 1982. This bill would mature on March 23rd, 1983. The nearest long term treasury bill issued in March of 1982 matured on March 3rd, 1983. This maturity is obviously 20 days shorter in term than the treasury bill delivered under the December futures contract. Thus, a perfect matching of maturities was unavailable for the December contract in March of 1982. As the year progressed, new issues of treasury bills would provide a better matching of maturities. The difference of the shorter term to the final yield would, however, be slight as the duration of the investment period is quite long when measured in days. Thus, the December arbitrage has the form:



All of the daily price quotes were obtained from the Wall Street Journal. The futures prices were the settlement price at the close of the market. The cash prices were the average bid and asked price at the close of the cash market. It should be noted that these two data series are not strictly independent as the futures market closes at 2:00 p.m. daily while the cash market remains open until 5:00 p.m. Thus, there is a three-hour time difference between each of the pairs of observations.

When required, the yields in discount form were converted to bond form using the formula:

$$Y\ bond = \frac{365 \times Y\ discount}{360 - (Y\ discount \times TM)}$$

where: Y bond = yield in bond basis in decimal form

Y discount = yield in discount basis in decimal form

TM = days to maturity

To allow delivery of the cash treasury bill against the future contract where the term of the bills do not match the futures exchange, makes allowances for the price paid on settlement of the futures contract. The formula used to correct for differences in term is:

$$P_F = 100 - \frac{tsm - 90}{360} F_R$$

where: P_F = settlement price of futures contract

F_R = yield on futures contract at delivery

tsm = time to maturity of delivered bill

Because none of the cash bills in the three contract months tested matched the term of the futures contract, this formula was used for all contracts.

CHAPTER #5
RESULTS

The four methods for calculating the implied forward rate were utilized to test for equality between the implied forward rate and the rate available from a futures contract. For each observation over the time series, the implied forward rate was calculated and both the difference between each rate and the difference between the means of the rates were calculated. From these statistical tests were employed to test for equality between the implied forward and future rates.

Tables 1 through 4 present the summary statistics. The Industry formula is presented in Table 1. Here no allowance is made for transactions costs such as commissions and margin costs which would be incurred in purchasing the futures contract. Table 2 uses the same Industry formula but allows for commission costs of $60.00 for purchasing the future contract. Table 3 presents the results using the Poole formula to calculate the implied forward rate. This formula incorporates both commissions and margin costs in calculating the implied rate. Finally, Table 4 shows the results utilizing the formula developed by Lang and Rasche. Here the average yield over the investment period is taken as the implied forward rate without allowing for the compounding effect at the roll-over between cash and future bills.

The future rate shown on Tables 1, 2 and 3 are the banker's discount rate based on the settlement prices in the futures market. To compare the rates using the Lang and Rasche formula, the future yield was converted from banker's discount form to the 365 day bond yield basis. For this reason, the future rates shown on Table 4 appear higher than the future rates on Tables 1, 2 and 3.

To test the null hypothesis that the implied forward rate is equal to the futures rate, a comparison of the means of each sample was made. Since the population variance is unknown, a two sample "t" test was used to test for significance in the results. For this test, independent random samples of the two populations are required and since the population variances are unknown, a large sample size is necessary to invoke the central limit theorem. The mean of each sample and their variances are calculated. The null hypothesis "$d_3 = d_f$" is then tested using the pooled sample variance and the "t" statistic with "$2_n - 2$" degrees of freedom. Thus, the mean difference "$\overline{D} = d_3 - d_f = 0$",

$$t = \frac{\overline{D}}{\dfrac{(n_1-1)s_3^2 + (n_2-1)s_t^2}{2n-2} \qquad \dfrac{1}{n_1} + \dfrac{1}{n_2}}$$

where: d_3 is the mean implied forward rate of return

d_f is the mean future rate of return

$\overline{D}$ is the mean difference between the rates

s_3^2 is the variance of the mean of the implied forward rate

s_f^2 is the variance of the mean of the future rate

$n_1 + n_2$ are the sample sizes

In utilizing the two sample "t" tests to compare the means of the implied forward rate with that of the futures contracts, it is first necessary to test to see if the samples have equal variance. To do this, an "F" statistic is calculated to compare the ratio of the variances. This statistic is shown on Tables 1 through 4. The "F" statistic calculated is shown to be significant for the June contract using the first three formulas but not for the Lang and Rasche method of calculating the implied forward rate. This invalidates the usage of the two sample "t" test for the June contract because it violates the assumption of equal variance between the samples. The reason the variances are unequal can be inferred by examining the correlation matrix for the cash treasury bills. The implied forward rate for June is calculated from the cash bills maturing September 23rd and June 24th and since these two cash bills are negatively correlated, then a variable derived from both these bills should have a smaller variance than the June future rate. This is indeed the case as can be seen from examining the variances of the future and implied forward rate for the June contract.

Besides unequal variance for the June contract, the samples are also not totally independent. A problem exists in the available data in that the only prices quoted for the two markets are the closing prices of the day. The markets do not close at the same time, the futures market at 2:00 p.m. while the cash market closes at 5:00 p.m., therefore, there is a time factor involved which may effect the sample. Often news relating to credit markets is released after the 4:00 p.m. close of the stock markets but prior to the close of the money markets. These shocks could cause pronounced and short term changes in short term bill prices which would result in the increased variance observed for the cash treasury bills.

One method of overcoming this problem is to pair the data for each observation and then use the difference between the two for each observation as the test statistic. By pairing each observation, the effect of the time difference is eliminated and the variance becomes statistically more efficient (see Appendix). The test statistic becomes:

$$t = \frac{\overline{Y}}{\frac{s^2}{n}}$$

where: $\overline{Y}$ = the mean of the difference of the implied forward rate - futures rate for each observation

The degrees of freedom for this test is "n" instead of "n_1+n_2 - 2". However, since a large sample was used, both samples are assumed to be approaching a normal distribution and very little statistical efficiency is lost due to degrees of freedom.

The null hypothesis of equality between the implied and future rates was tested for all four formulas using both the paired and non-paired "t" test. The weaker non-paired test rejected the null hypothesis except for three cases. In the tests of the September contr~~ol~~act, the null hypothesis is accepted for the Industry formula with commission costs and for the Poole formula. The December contract is rejected in all cases. For the June contract, the non-paired test leads us to accept the null hypothesis for the Lang and Rasche formula. The statistically more efficient paired "t" test yields a more conclusive result and leads to the rejection of the null hypothesis in every test.

From the results, we can see that as the costs of trading increased, the implied forward rate increased. This is because the cost of purchasing the future contract increased the cost of the transactions in the cash market and thus increased the yield necessary to equate the instruments for the second half of the holding period. The penalty imposed by these costs is reduced as the term of the transaction increases. The sign of the difference in rates is what would be expected as the future rate exceeds the implied rate without transactions costs, but as these costs are added, the size of the difference in rates diminishes.

The results for the Lang and Rasche formula also agree with our a priori assumptions. Since the implied forward rate is calculated as the weighted average of long and short rates with no allowance for price difference, then the difference in yield terms increases as the time to delivery of the futures contract increases. There is very little difference in the contract closest to delivery but for contracts further from delivery, the futures rate exceeds the implied forward rate.

Finally, charts of the future and implied forward rates for each of the three contract months provide an insight into the relationships between the rates which is not developed by a comparison of the means. In all three contracts, but most notably in the June and September contract, large differences existed over the first four weeks of observation. Between March 17th and mid April of 1982, large arbitrage profits existed that were not erased for weeks. In the June and September contracts, the difference between the implied rate and the future rate often exceeded seventy-five basis points. At the same time, the arbitrage profit between the December bills approached three hundred basis points. However by mid April, these arbitrage possibilities had been closed and few opportunities for arbitrage appeared again until the final days of the sample in June of 1982.

Thus while opportunities for arbitrage existed during the sample period, their occurrence was not evenly distributed throughout the period. The exception to this rule is the December contract which showed consistently large spreads between the future and implied forward rate. An explanation of this phenomenona can be found in the chart of the cash ninety day treasury bill. The sample period seems to coincide with a period of relatively level short term rates immediately preceded and followed by large changes in the level of short term interest rates. The large changes in the level of interest rates may have had an effect on the existence of arbitrage possibilities between the markets but unfortunately, the sample taken does not span the months in which the changes occurred.

TABLE 1

INDUSTRY FORMULA

June Contract	Mean	Standard Deviation	Variance
June 1982 future	12.143	.53393	.28509
June implied forward	11.834	.36474	.13303
Difference (implied-future)	-0.309	.29361	.08621

$n = 68$ $F = 2.14$ $R^2_{if} = 0.02435$

Paired Data $\bar{Y} = -0.309$ Non-paired Data $\bar{D} = -0.309$

$S\bar{y} = 0.03561$ $S\bar{d} = 0.078997$

$t_c = -8.68$ $t_c = 3.91$

September Contract	Mean	Standard Deviation	Variance
September 1982 future	11.592	.52005	.27045
September implied forward	11.418	.45335	.20553
Difference (implied-future)	-0.174	.44419	.19730

$n = 73$ $F = 1.32$ $R^2_{if} = 0.59100$

Paired Data $\bar{Y} = -0.174$ Non-paired Data $\bar{D} = -0.174$

$S\bar{y} = 0.05199$ $S\bar{d} = 0.08131$

$t_c = -3.35$ $t_c = 2.14$

December Contract	Mean	Standard Deviation	Variance
December 1982 future	12.811	.50215	.25215
December implied forward	10.779	.54991	.30240
Difference (implied-future)	-2.032	.64130	.41127

$n = 70$ $F = 1.20$ $R^2_{if} = 0.08002$

Paired Data $\bar{Y} = -2.032$ Non-paired Data $\bar{D} = -2.032$

$S\bar{y} = 0.07665$ $S\bar{d} = 0.08965$

$t_c = -26.51$ $t_c = 22.67$

TABLE 2

INDUSTRY FORMULA WITH COMMISSIONS

June Contract	Mean	Standard Deviation	Variance
June 1982 future	12.143	.53393	.28509
June implied forward	11.858	.36484	.13311
Difference (implied-future)	-0.286	.29351	.08615

$n = 68$ $\quad F = 2.14$ $\quad R^2_{if} = 0.02404$

Paired Data			Non-paired Data		
$\overline{Y}$	=	-0.286	$\overline{D}$	=	-0.286
$S\overline{y}$	=	0.03559	$S\overline{d}$	=	0.7901
t_c	=	-8.04	t_c	=	3.61

September Contract	Mean	Standard Deviation	Variance
September 1982 future	11.592	.52005	.27045
September implied forward	11.440	.45320	.20539
Difference (implied-future)	-0.152	.44397	.19711

$n = 73$ $\quad F = 1.32$ $\quad R^2_{if} = 0.59132$

Paired Data			Non-paired Data		
$\overline{Y}$	=	-0.152	$\overline{D}$	=	-0.152
$S\overline{y}$	=	0.05196	$S\overline{d}$	=	0.08129
t_c	=	-2.93	t_c	=	1.87

December Contract	Mean	Standard Deviation	Variance
1982 future	12.811	.50215	.25215
implied forward	10.816	.54965	.30212
Difference (implied-future)	-1.995	.64100	.41089

$n = 70$ $\quad F = 1.42$ $\quad R^2_{if} = -0.08001$

Paired Data			Non-paired Data		
$\overline{Y}$	=	-1.995	$\overline{D}$	=	-1.995
$S\overline{y}$	=	0.07661	$S\overline{d}$	=	0.08963
t_c	=	-26.04	t_c	=	22.26

TABLE 3

POOLE FORMULA

June Contract	Mean	Standard Deviation	Variance
June 1982 future	12.143	.53393	.28509
June implied forward	11.849	.36758	.13512
Difference (implied-future)	-0.294	.29115	.08477

$n = 68$ $F = 2.11$ $R^2_{if} = -0.01629$

Paired Data $\overline{Y} = -0.294$ Non-paired Data $\overline{D} = -0.294$

$S\overline{y} = 0.03531$ $S\overline{d} = 0.07912$

$t_c = -8.33$ $t_c = 3.71$

September Contract	Mean	Standard Deviation	Variance
September 1982 future	11.592	.52005	.27045
September implied forward	11.448	.44954	.20209
Difference (implied-future)	-0.144	.43861	.19238

$n = 73$ $F = 1.34$ $R^2_{if} = 0.59920$

Paired Data $\overline{Y} = -0.144$ Non-paired Data $\overline{D} = -0.144$

$S\overline{y} = 0.05134$ $S\overline{d} = 0.08101$

$t_c = -2.81$ $t_c = 1.78$

December Contract	Mean	Standard Deviation	Variance
December 1982 future	12.811	.50215	.25215
December implied forward	10.867	.54331	.29518
Difference (implied-future)	-1.944	.63363	.40149

$n = 70$ $F = 1.17$ $R^2_{if} = -0.07982$

Paired Data $\overline{Y} = -1.944$ Non-paired Data $\overline{D} = -1.944$

$S\overline{y} = 0.07573$ $S\overline{d} = 0.08906$

$t_c = -25.67$ $t_c = 21.83$

TABLE 4

LANG & RASCHE FORMULA

June Contract	Mean	Standard Deviation	Variance
June 1982 future	12.698	.57598	.33175
June implied forward	12.588	.48110	.23146
Difference (implied-future)	-0.110	.28425	.08080

$n = 68$ $F = 1.43$ $R^2_{if} = 0.11645$

Paired Data		Non-paired Data	
$\overline{Y}$ =	-0.110	$\overline{D}$ =	-0.110
$S\overline{y}$ =	0.03447	$S\overline{d}$ =	0.11015
t_c =	-3.19	t_c =	-0.9987

September Contract	Mean	Standard Deviation	Variance
September 1982 future	12.105	.55911	.31261
September implied forward	12.516	.45720	.20904
Difference (implied-future)	0.412	.37719	.14227

$n = 73$ $F = 1.50$ $R^2_{if} = 0.74204$

Paired Data		Non-paired Data	
$\overline{Y}$ =	0.412	$\overline{D}$ =	0.412
$S\overline{y}$ =	0.04415	$S\overline{d}$ =	0.08512
t_c =	9.33	t_c =	-4.83

December Contract	Mean	Standard Deviation	Variance
December 1982 future	13.419	.54331	.29519
December implied forward	12.029	.56208	.31594
Difference (implied-future)	-1.390	.63267	.40027

$n = 70$ $F = 1.07$ $R^2_{if} = -0.07660$

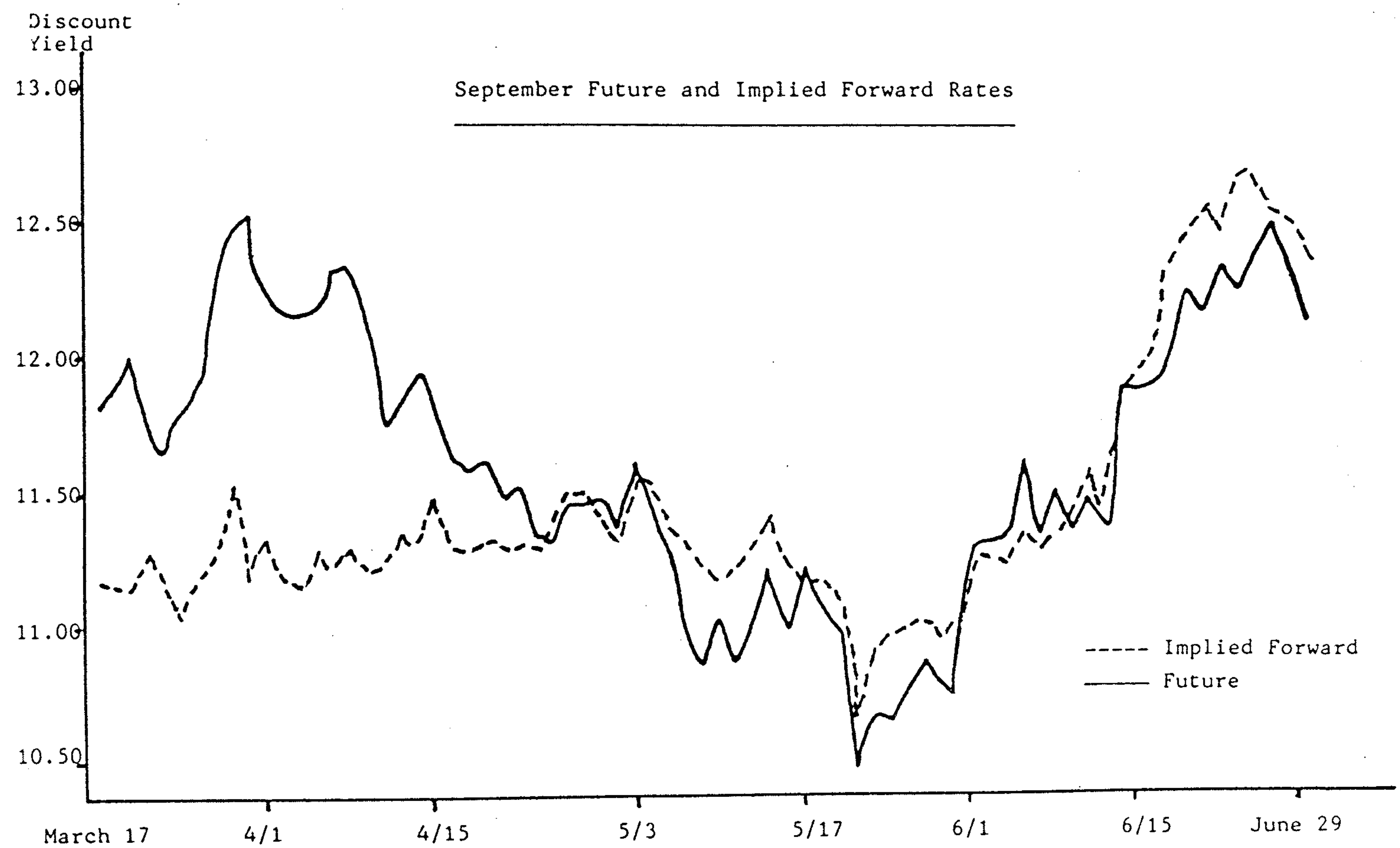
Paired Data		Non-paired Data	
$\overline{Y}$ =	-1.390	$\overline{D}$ =	-1.390
$S\overline{y}$ =	0.07562	$S\overline{d}$ =	0.09411
t_c =	-18.38	t_c =	-14.77

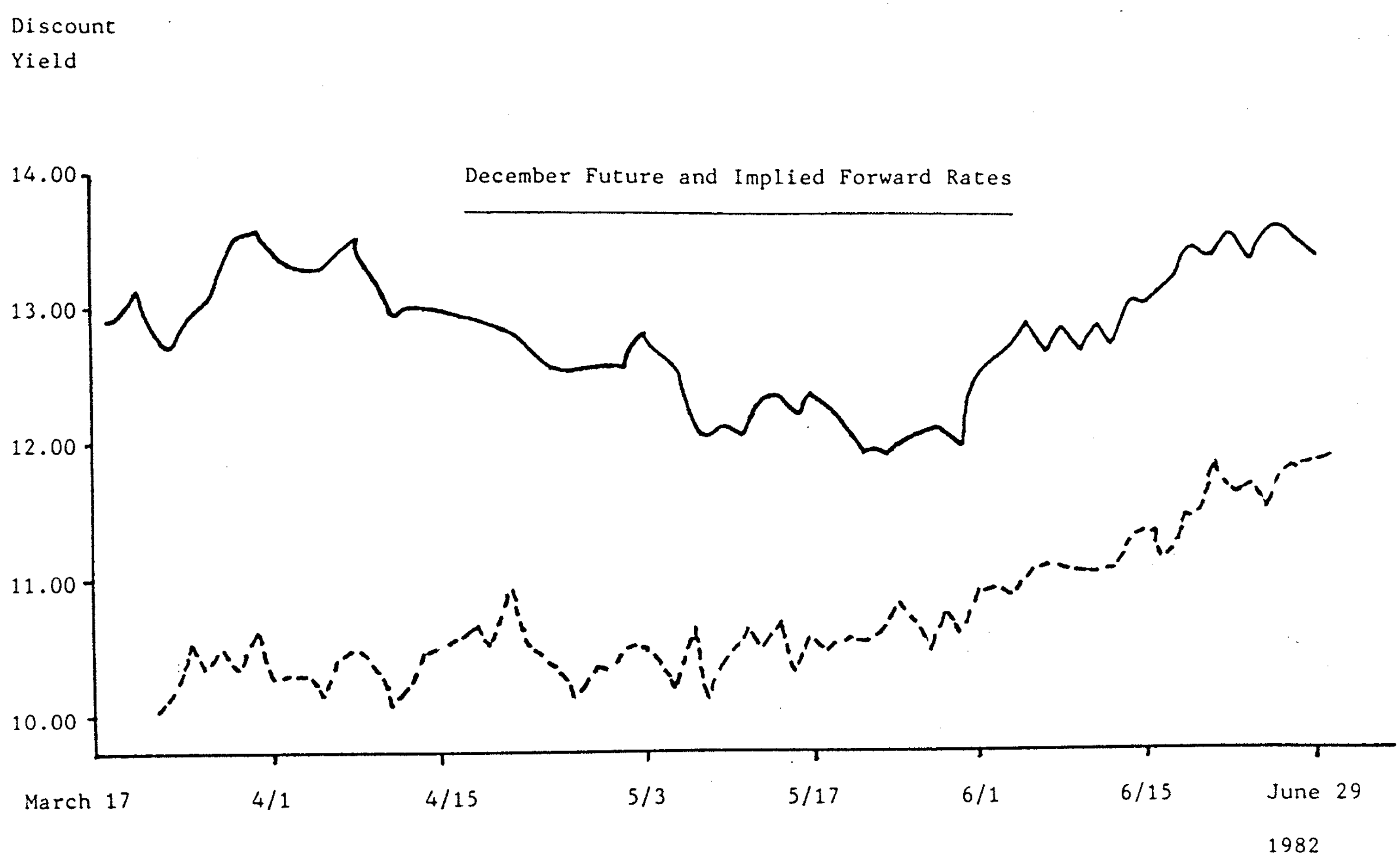
TABLE 5

CORRELATION MATRIX

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Cash Bills:										
1. December 30th	1.00									
2. September 23rd	0.91	1.00								
3. June 24th	-0.55	-0.34	1.00							
4. March 2nd	0.17	0.06	-0.16	1.00						
Future Contracts:										
5. December 82	0.86	0.87	-0.29	0.03	1.00					
6. September 82	0.86	0.90	-0.29	0.00	0.99	1.00				
7. June 82	-0.42	-0.25	0.87	-0.15	-0.18	-0.16	1.00			
Implied Forward Rates										
8. December	0.04	-0.05	-0.09	0.99	-0.08	-0.11	-0.10	1.00		
9. September	0.88	0.60	-0.68	0.27	0.63	0.59	-0.52	0.16	1.00	
10. June	0.80	0.92	-0.16	0.02	0.81	0.84	-0.02	-0.09	0.48	1.00

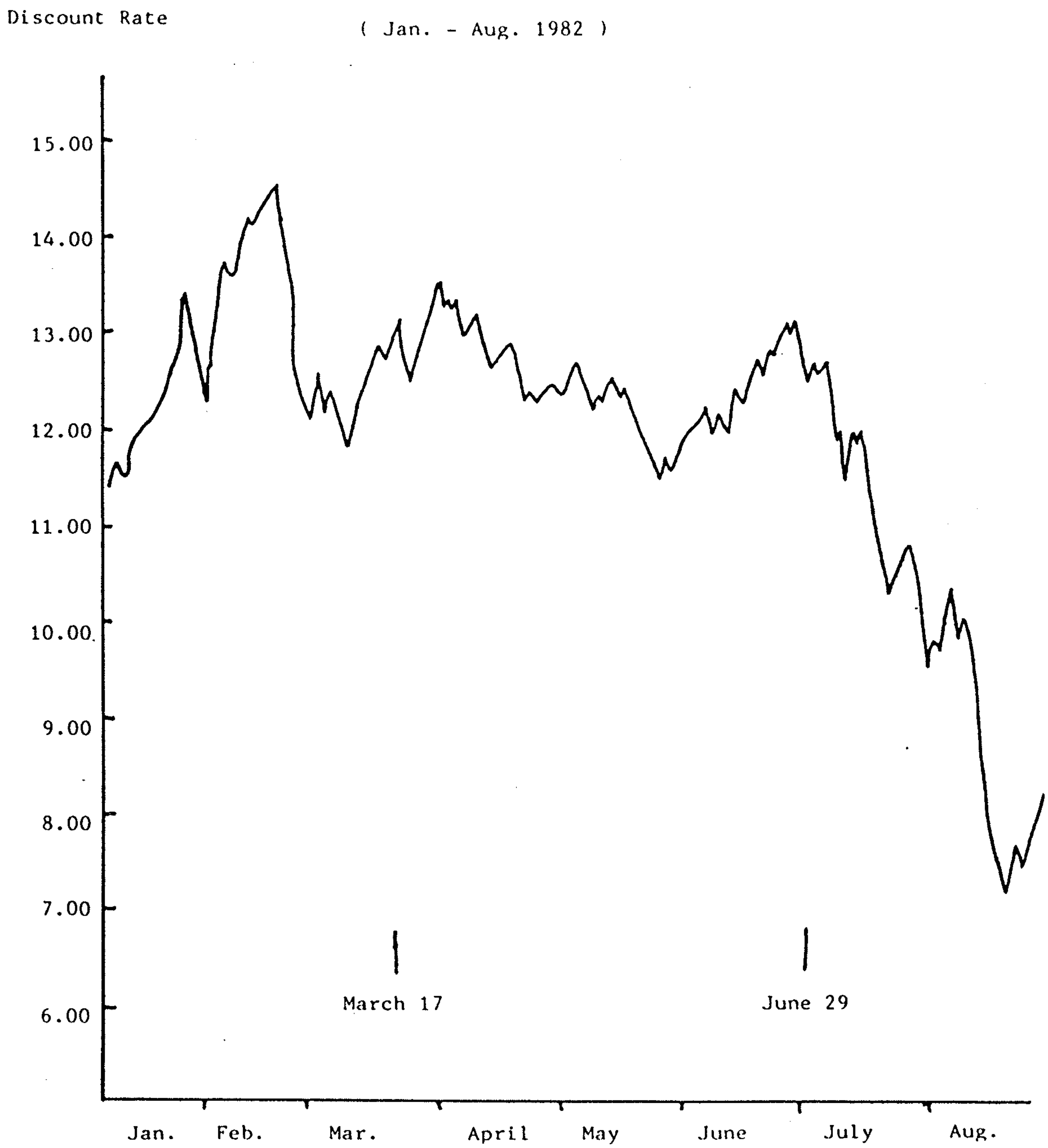
Discount
Yield
June Future and Implied Forward Rates
13.50
13.00
12.50
12.00
11.50
11.00
March 17
4/1
4/15
5/3
5/17
6/1
6/15
June 29
1982


Discount Yield
13.00
12.50
12.00
11.50
11.00
10.50
September Future and Implied Forward Rates
----- Implied Forward
——— Future
March 17
4/1
4/15
5/3
5/17
6/1
6/15
June 29


Discount
Yield
14.00
13.00
12.00
11.00
10.00
December Future and Implied Forward Rates
March 17
4/1
4/15
5/3
5/17
6/1
6/15
June 29
1982

United States 90 Day Treasury Bills

(Jan. - Aug. 1982)



APPENDIX:

First of all, it can be shown that the mean difference is the same for both the paired and non-paired sample. For the non-paired sample:

$$\begin{aligned}\bar{D} &= \bar{d}_3 - \bar{d}_f \\ &= \frac{1}{n}\sum_{j=1}^{n} d_{3j} - \frac{1}{n}\sum_{j=1}^{n} d_{fj} \\ &= \frac{1}{n}\sum_{j=1}^{n}\left(d_{3j} - d_{fj}\right)\end{aligned}$$

and since $Y_j = (d_{3j} - d_{fj})$

$$= \frac{1}{n}\sum_{j=1}^{n}\left(Y_j\right) = \bar{Y}$$

thus the mean difference for each method is equal.

Now, the standard error for the mean difference can be calculated in two ways. For the non-paired method:

$$s^2 = \frac{(n_1-1)s_3^2 + (n-1)s_F^2}{(n-1)+(n-1)}$$

$$= \frac{\Sigma d_3^2 + \frac{(\Sigma d_3)^2}{n} - \Sigma d_f^2 + \frac{(\Sigma d_f)^2}{n}}{(n-1)+(n-1)}$$

$$= \frac{\Sigma d_3^2 + \Sigma d_f^2 - \frac{1}{n}\left[(\Sigma d_3)^2 + (\Sigma d_f)^2\right]}{2(n-1)}$$

Now let s_1^2 denote the variance of the difference between the means:

$$s_1^2 = s^2\left(\frac{1}{n} + \frac{1}{n}\right) = \frac{2 s^2}{n}$$

$$= \frac{\Sigma d_3^2 + \Sigma d_f^2 - \frac{1}{n}\left[(\Sigma d_3)^2 + (\Sigma d_f)^2\right]}{(n-1)}$$

Using the paired method, let "s_2^2" denote the variance of the mean difference between the means:

$$s_2^2 = \frac{s^2}{n}$$

and

$$s_2^2 = \frac{\Sigma (y-\bar{y})^2}{n-1} = \frac{\Sigma y^2 - \frac{1}{n}(\Sigma \bar{y})^2}{n-1}$$

$$= \frac{1}{n-1}\left[\Sigma (d_3 - d_f)^2 - \frac{1}{n}\left(\Sigma (d_3 - d_f)\right)^2\right]$$

which can be reduced to:

$$s_2^2 = \frac{1}{n(n-1)}\left[\Sigma d_3^2 + \Sigma d_f^2 - \frac{1}{n}(\Sigma d_3)^2 - \frac{1}{n}(\Sigma d_f)^2 - 2\left\{\Sigma d_3 d_f - \frac{1}{n}(\Sigma d_3)(\Sigma d_f)\right\}\right]$$

$$= s_1^2 - \frac{2}{n(n-1)} \Sigma (d_3 - \bar{d}_3)\cdot(d_f - \bar{d}_f)$$

$$= s_1^2 - \frac{2}{n} \frac{\Sigma (d_3 - \bar{d}_3)(d_f - \bar{d}_f)}{\sqrt{\Sigma (d_3 - \bar{d}_3)^2 \, \Sigma (d_f - \bar{d}_f)^2}} \cdot \sqrt{\frac{\Sigma (d_3 - \bar{d}_3)^2}{n-1} \quad \frac{\Sigma (d_f - \bar{d}_f)^2}{n-1}}$$

$$= s_1^2 - \frac{2}{n} \cdot r_{d_3 d_f} \cdot s_3 \cdot s_f$$

where "s_j" and "s_f" are the sample variances of the implied forward rate and the futures rate respectively and where "r" is the coefficient of correlation between the implied forward rate and the future rate.

From this, it can be seen that where "r_{if}" is positive, "s_2^2" will be less than "$\cancel{D}\,s_1^2$". Thus, the paired method has a smaller sample variance and is therefore, statistically more efficient.

CHAPTER #6

CONCLUSIONS

The results shown in the previous chapter replicate the findings of Lang and Rasche and contradict the findings of Poole. Using the paired sample "t" test, it can be shown that the implied forward rate and the futures rate are significantly different in all three contract months. The difference does, however, decrease as the contract nears its delivery date. Also the difference between the rates does seem to increase during periods of rapidly rising or falling levels of interest rates. Thus Poole's findings of equality between the rates could have been dependent upon the specific time period he studied.

The results of all four formulas show that the futures rate exceeds the implied forward rate in all contract months. The size of the difference increases as the length of time to delivery of the future contract increases. This replicates the results of Lang and Rasche even though their formula is shown to be flawed. Also Poole stated that while the implied forward rate and the future rate were statistically equivalent, the futures rate tended to exceed the implied forward rate.

From the Pure Expectations Theory of interest rates, it was hypothesized that the future and forward rates would be equal. With free access to the financial markets, the prices of treasury bills in the futures market should equal the price of treasury bills implied by the yield curve. If the prices were not equal, then arbitragers could enter the market and through a simultaneous transaction of buying and selling between the two markets trading the implied and future treasury bill. The arbitragers would continue this trading, each time realizing a risk-free ~~project~~ profit, until the prices were equated between the markets.

The results for each test of the four formulas for calculating the implied forward rate did not support this Pure Expectations Hypothesis. In each case, significant differences between the implied forward rate and the futures rate were found. This implies that the differences would have allowed arbitragers to enter the market on a regular basis and make a risk-free profit trading between the cash and futures market.

The Industry formula presents the simplest case because no allowance is made for transactions costs. Here it can be seen that the arbitrage profit on the difference in rates is on average thirty-one basis points for the June contract, seventeen basis points for the September, and two hundred and three basis points for the December contract. Since a basis point in price difference on a ninety day bill is worth twenty-five dollars, the average profit is between four hundred and fifty and five thousand dollars per contract.

Incorporating commission costs of sixty dollars per contract only reduces the profit slightly. The same can be said for the fifteen hundred dollar margin requirement which may be required to purchase the future contract. Neither of these costs significantly reduce the arbitrage profit. Commissions in the cash market are incorporated into the price paid for the cash treasury bill. Since dealers act as principles, the bid and ask prices quoted include the commissions costs. By utilizing the average of bid-ask price spreads these costs were included in the price of cash bill whether the arbitrager was buying or selling a cash bill. Thus while transactions costs would be paid by the arbitrager these do not significantly reduce his profit from the transaction.

The three graphs of the June, September and December contracts presented in Chapter 5 show that the arbitrage profits are neither uniform or evenly distributed over the period studied. For the June and September contracts specifically, the largest differences are observed at the beginning and near the end of the period. From the graph of cash bill yields throughout the first half of 1982, it can be suggested that the differences could be the result of large changes in the level of short term rates. It may take a time lag for the arbitragers to bring the markets into equilibrium. This suggestion is, however, questionable given the speed and breadth of information dispersal on treasury bill markets. Also since the arbitrage is covered and risk-free the size of the participation is essentially limitless. Therefore, the markets would be expected to absorb price differences very rapidly.

The arbitrage profits are relatively easy for the trader to calculate and the transactions are relatively simple to execute. Using the average rates from the Industry formula, the steps necessary to complete the arbitrage can be followed to verify the profits. For the June contract, the average future yield was 12.14% and the implied forward rate was 11.83%, with a difference between the two yields of -0.309%. At twenty-five dollars per basis points, the arbitrage profit should have been around \$775.00. To test the Industry formula, a transaction is calculated using the prices from

June 9th, 1982. On that day, the June 1982 future was priced to yield 11.73%, the corresponding rates in the cash market on the June 24th and September 24th treasury bills were 12.539% and 11.529% respectively. The implied forward rate was calculated to be 11.422% resulting in a difference of -0.309% between the implied forward and the future rate. Below, the transactions and prices of this arbitrage are presented in a cash flow format.

Date: June 9th, 1982

Treasury Bill	Yield	Price	Date to Maturity
September 23rd, 1982	11.529	96.60535	106
June 24th, 1982	12.539	99.47754	16
June 1982 future	11.73	97.03490	90

June 9th, 1982

1) Sell: 100 September 24th, 1982 @ 96.60535 96.60535

2) Buy: $\frac{\text{September 24th, 1982}}{\text{June 24th, 1982}}$ x June 14th, 1982

= .97112 x 99.47754 - 96.60535

Ø

3) Buy: June 1982 future to yield 11.73%

June 24th, 1982

1) June 24th bill matures		
.97112 x 100		97.11200
2) Take delivery of June future		
100 @ 97.0349	-	97.0349
Balance in Account	=	0.077783

September 23rd, 1982

1) June future matures	-	100.00
2) Close out short position, September 23rd bill		100.00
		Ø
Balance in Account from June 24th, 1982	=	0.077783

Therefore, profit from transaction is 0.077783 per $100.00.

- on 1,000,000, the profit would be $\frac{1,000,000}{100} \times 0.077783$

$= \$777.83$

- in terms of basis points $\frac{777.83}{25} =$ 31 basis points

Thus, the arbitrage would have yielded a profit of $777.83 or the equivalent of thirty-one basis points, as predicted by the Industry formula.

Arbitrage on the September treasury bill contract yielded lower average profits of only seventeen basis points. As an example of the arbitrage technique, the transactions for April 22nd, 1982 are followed using the same cash flow method.

Date: April 22nd, 1982

Treasury Bill	Yield	Price	Date to Maturity
December 30th, 1982	11.399	92.02070	252
September 23rd, 1982	11.809	94.94837	154
September 1982 future	11.507	96.86754	90

April 22nd, 1982

1) Sell: 100 December 30th, 1982 @ 92.02070   92.02070

2) Buy: $\frac{\text{December 30th, 1982}}{\text{September 23rd, 1982}}$ x September 23rd, 1982

$= .96916 \times 94.94837$ $- \underline{92.02070}$

Ø

3) Buy: September 1982 future to yield 11.507%

September 23rd, 1982

1)	September 23rd bill matures		
	.96916 x 100		96.91600
2)	Take delivery of September future		
	100 @ 96.86754	-	96.86754
	Balance in Account	=	0.04846

December 30th, 1982

1)	December future matures		100.00
2)	Close out short position, December 30th bill	-	(100.00)
			Ø
	Balance in Account from September 23rd, 1982	=	0.04846

Therefore, profit from transaction is \$0.04846 per \$100.00.

- on \$1,000,000, the profit would be $\frac{1,000,000}{100} \times 0.04846$

  $= \$484.60$

- in terms of basis points $\frac{484.60}{25} =$ 19 basis points

The transaction used to complete the arbitrage on the December bill is more complicated because the maturities do not match. However, the ability to trade the future bill against a forward bill does provide an opportunity to profit from the price differences between the markets. The average gain from the arbitrage was two hundred and three basis points. This presents an average profit of over five thousand dollars to lure the trader to this slightly uncovered arbitrage. The prices available on March 26th, 1982 are used to demonstrate the arbitrage.

Date: March 26th, 1982

Treasury Bill	Yield	Price	Days to Maturity
March 2nd, 1983	10.963	89.49379	345
December 30th, 1982	11.362	91.06821	283
December 1982 future	12.85	96.78750	90

March 26th, 1982

1) Sell: 100 March 2nd, 1983 @ 89.49379 — 89.49379

2) Buy: $\frac{\text{March 2nd, 1983}}{\text{December 30th, 1982}}$ x December 30th, 1982

$$= .9827116 \times 91.06821 \qquad - \underline{89.49379}$$

Ø

3) Buy: December 1982 future

December 30th, 1982

1) December 30th bill matures		98.27116
2) Take delivery of December future		
100 @ 96.78750	-	96.78750
Cash Balance	=	1.48366

March 2nd, 1983

1) Close out short position, March 2nd bill	-	(100.00)
Balance in Account	=	(98.51634)
2) Sell December future @ 98.51634		98.51634
Balance in Account	=	Ø

Therefore, the arbitrage is even if the December future bill is sold on March 2nd, 1983 for $98.51 to yield 19.08% for the remaining twenty-eight days. To gain the estimated five thousand dollars, the arbitrager would sell the future contract at the same yield he purchased it at, namely 12.85%. He would realize a profit as long as the twenty-eight day rate did not exceed 19.08% on March 2nd, !983.

From these three examples, it can be seen that the arbitrage profit is correctly predicted by the formulas. Also that since the arbitrage is covered in all but the December contract, the transaction is essentially risk-free. In all cases, a sizeable profit can be made which would far exceed the commissions and margin costs.

These findings contradict the Pure Expectations Theory of the term structure of interest rates. The multi-period interest rate is shown not to be the sum of the present and one period forward rate available from a futures market. The forward rate implied by cash bill rates is less than the equivalent future rate. Therefore, an investor can profit from selling a multi-period cash bill and lending the proceeds at one period rates in the futures market.

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