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Asymptotic Expansions of Some Canonical Diffraction Integrals

by

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A thesis
submitted to the Faculty of Graduate Studies
in partial fulfilment of the requirements for the degree
of Doctor of Philosophy

Department of Mathematics and Astronomy

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Abstract

We examine the asymptotic behaviour of two "generalized Airy functions",

$$(1) \qquad P(x, y) = \int_{-\infty}^{+\infty} \exp[\, i(t^4/4 + xt^2/2 + yt)]\, dt$$

$$(2) \qquad Q(x, y, z) = \int_{-\infty}^{+\infty} \exp[\, i(t^5/5 + xt^3/3 + yt^2/2 + zt)]\, dt,$$

for large values of their parameters. In the case of (1), the so-called Pearcey function, we develop an asymptotic expansion of $P(x,y)$ which remains uniformly valid near the caustic $4x^3 + 27y^2 = 0$.

For the latter integral, we discuss its caustic, and obtain asymptotic expansions which are uniformly valid near the caustic as $|x| + |y| + |z| \to +\infty$.

In obtaining the asymptotics of P and Q, we make use of both cubic and quartic changes of variables. While the use of a cubic change of variables is well-known, the quartic transformation,

$$(3) \qquad f(t) = z^4/4 - \zeta z^2/2 + \eta z + \theta,$$

used in the uniform expansion of (2), presents difficulties not found in the cubic transformation: first, the critical points of the right hand side of (3) are complicated functions of the coefficients ζ and η; second, the coefficients ζ, η, and θ are not easily expressed in terms of the known function f at its critical points.

Both of these problems are completely resolved in the course of calculating the uniform asymptotics of (2).

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Finally, I couldn't have written this small book without the caring and attention of some important people: my close friend Andreas Gülzow, my parents (who no longer ask what I do), and my friend and loving wife of five years, Laurie.

Before releasing you to asymptotics, I want you to know this dissertation was nearly submitted with the subtitle of "A Colouring Book in Analysis".

Chapter One: The Asymptotics of Oscillatory Integrals

In diverse fields of physics, one encounters integrals of the form

$$(1.1) \qquad I(\lambda) = \int_a^b f(x)\, e^{i\lambda\phi(x)} dx \quad ,$$

where f and ϕ are generally assumed to possess a number of properties, frequently including analyticity, and λ is some real parameter representing some physical variable of interest (frequency of light, the reciprocal of Planck's constant, etc.). For example, a number of authors have used integrals of the form (1.1) in geometric optics **[Ups]**, scattering theory **[Con1, Con2]**, quantum mechanics **[Mas]**, the theory of nonlinear waves **[Hab]**, and in electromagnetics **[Pea]**.

In most cases, integrals of the form (1.1) are difficult or impossible to evaluate in closed form, and so indirect or numerical methods are required in their study. When we examine (1.1) with the parameter λ very large, the tools of asymptotics become available for the purpose of approximation. The large-λ limits are not only mathematical contrivances: physical interpretations of a large λ in (1.1) correspond to short wavelength limits (geometric optics) or semi-classical limits (quantum mechanics), among others.

In the case where the phase function of (1.1), $\phi(x)$, is strictly a function of x, we can avail ourselves of the numerous variants of the method of stationary phase for determining the large-λ behaviour of (1.1) under fairly broad hypotheses on the functions f(x) and $\phi(x)$. In this regard, see **[Olv2]** for an excellent account of stationary phase.

The method of stationary phase provides us with an elegant means of determining the principal contribution to the value of (1.1) for large λ. For the sake of simplicity, we assume that f and ϕ are smooth real-valued functions, and that the interval of integration is the real axis, the integral converging for all sufficiently large positive λ. Suppose further that ϕ has only finitely many critical points (also referred to as *saddle points*), say $x_1, x_2, \dots, x_n$, and that $\phi''(x_j)$ is nonzero for $j = 1, 2, \dots, n$. Then it is well-known that $I(\lambda)$ possesses the large-λ behaviour

$$(1.2) \qquad I(\lambda) \sim \sum_{j=1}^{n} f(x_j)\, e^{i\lambda\phi(x_j) + i\frac{\pi}{4}\operatorname{sgn}(\phi''(x_j))} \sqrt{\frac{2\pi}{|\lambda\phi''(x_j)|}} .$$

Situations in which a critical point is of higher order, i.e. , $\phi''(x_j) = 0$ for some j, are easily handled by a straightforward modification; see, for example, [**Olv**2].

If ϕ depends on a number of auxiliary parameters, the problem of determining the large-λ behaviour of (1.1) becomes much more involved. For the purposes of illustration, assume that $\phi = \phi(x; \alpha)$ has two distinct critical points x_1 and x_2 for $\alpha \neq 0$, and that as α tends to zero, x_1 and x_2 tend to a common value x_0, i.e., we have $\phi''(x_1), \phi''(x_2) \neq 0$ for $\alpha \neq 0$, and $\phi''(x_0) = 0$, $\phi'''(x_0) \neq 0$. Thus

$$(1.3) \qquad I(\lambda) \sim \sum_{j=1}^{2} f(x_j)\, e^{i\lambda\phi(x_j;\alpha) + i\frac{\pi}{4}\operatorname{sgn}(\phi''(x_j;\alpha))} \sqrt{\frac{2\pi}{|\lambda\phi''(x_j;\alpha)|}}$$

for nonzero α, and

$$(1.4) \qquad I(\lambda) \sim f(x_0)\, e^{i\lambda\phi(x_0;0)} 2^{1/3} 3^{-1/6} \Gamma(1/3) (\lambda|\phi'''(x_0;0)|)^{-1/3}$$

for $\alpha = 0$. The behaviour of these two approximations differ radically; indeed, as $\alpha \to 0$, (1.3) becomes singular. Furthermore, in the transition to $\alpha = 0$, there is a discontinuous change in the order of λ used in the approximations (1.3) and (1.4), from $\lambda^{-1/2}$ to $\lambda^{-1/3}$. This occurs regardless of the fact the function $I(\lambda)$ may be analytic in both λ and α. The problem, then, is one of constructing an asymptotic approximation of $I(\lambda)$ which remains valid in an interval around $\alpha = 0$. Such an approximation is said to be *uniform* ; a formal definition follows; cf. [**Olv**1, p. 25 - 26].

Let $\boldsymbol{\Lambda}$ be a set with λ_0 as an accumulation point of $\boldsymbol{\Lambda}$ (possibly at infinity). A sequence of functions $\{\phi_n(\lambda) \mid n = 0, 1, 2, \dots\}$ form an *asymptotic scale* if, for any n, we have $\phi_{n+1}(\lambda) = o(\phi_n(\lambda))$ as $\lambda \to \lambda_0$. A formal sum $\sum f_n$ is said to be a *(generalized) asymptotic expansion* of $f(\lambda)$ with respect to the asymptotic scale $\{\phi_n\}$ if

$$(1.5) \qquad f(\lambda) - \sum_{j=0}^{N-1} f_j(\lambda) = O(\phi_N(\lambda))$$

for each N = 1, 2, ... If $f(\lambda)$, $f_j(\lambda)$ and possibly $\phi_j(\lambda)$ are also functions of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ in a set $\boldsymbol{\Omega}$, and the O- and o- symbols above hold uniformly in $\boldsymbol{\alpha} \in \boldsymbol{\Omega}$, then we say that the resulting asymptotic expansion is *uniform* with respect to the parameters $\boldsymbol{\alpha}$.

Prior to 1957, little was known regarding how one might obtain, directly from (1.1), an asymptotic approximation for $I(\lambda)$ which was uniformly valid for all α in a neighborhood of the origin.

In contrast with the asymptotic theory of integrals, the corresponding theory for ordinary differential equations was more complete: for, when the function $I(\lambda)$ could be shown to satisfy a second order linear differential equation, not only could one obtain asymptotic approximations of $I(\lambda)$ (using the so-called Liouville-Green approximation; see **[Olv1**, Ch. 6]), but if $I(\lambda)$ depended on parameters $\boldsymbol{\alpha}$, then an extensive body of theory was available for developing asymptotic expansions uniform in $\boldsymbol{\alpha}$; see **[Olv1, Lan]**.

At first sight, it might appear irrelevant whether we use either integral-theoretic or differential equation-theoretic methods for developing uniform asymptotic expansions of $I(\lambda)$ in the case where two or more saddle points coalesce. However, there are many "classical" special functions of mathematical physics which cannot be expressed as solutions to linear differential equations, although relatively simple integral expressions are available. The gamma function provides an excellent example of a function amenable to integral-theoretic methods for asymptotics, although it satisfies no linear ordinary differential equation (with rational coefficients).

Thus, a complete asymptotic theory of special functions requires both approaches to developing expansions. In the case of working directly with (1.1), the first major innovation for obtaining uniform expansions (with ϕ depending on one parameter) resulted from a collaboration of three mathematicians, Chester, Friedman and Ursell, in what is now regarded as a seminal paper **[CFU]**. Their principal idea was straightforward: since the "simplest" phase function exhibiting two coalescing saddle points (ie., critical points) is a cubic polynomial, a change of variables of the form

$$\phi(x; \alpha) = z^3/3 - \zeta z + \eta \tag{1.6}$$

was introduced. Under suitable restrictions on f and ϕ (for example, that of analyticity), Chester *et al* were able to show that the mapping (1.6) defines a local uniformly analytic one-to-one change of variables from x to z, which provides for a reduction of (1.1) to an "Airy" type integral:

$$I(\lambda; \alpha) = \int_C g_0(z; \alpha)\, e^{i\lambda(z^3/3 - \zeta z + \eta)}\, dz, \tag{1.7}$$

where C is some integration path, $g_0(z; \alpha) = f(x)(dx/dz)$, and the equality in (1.7) might need to be replaced by "$\sim$" if the change of variables is indeed 'local'.

Using a 'two-point expansion', it was found, in the case where $\phi(x; \alpha)$ had two coalescing saddles, that

$$I(\lambda; \alpha) \sim 2\pi\lambda^{-1/3} \exp(i\lambda[\phi(x_1 ; \alpha) + \phi(x_2 ; \alpha)]/2) \cdot \\ [p_0(\alpha)Ai(-\lambda^{2/3}\zeta) + q_0(\alpha)\lambda^{-1/3}Ai'(-\lambda^{2/3}\zeta)] \tag{1.8}$$

as $\lambda \to +\infty$, uniformly valid for all α in a neighborhood of the origin. Here, the coefficients p_0 and q_0 are continuous in α. This result, which plays a central rôle in the next chapter, led to a fount of work devoted to integral-theoretic methods for developing uniform expansions of integrals.

Shortly after this development, in the late '50s and early '60s, a number of mathematicians, including Norman Levinson, studied the type of change of variables used by Chester *et al*, thereby laying the groundwork for a complete theory of uniform expansions of integrals with an arbitrary (but finite) number of coalescing saddle points of ϕ depending on a number of parameters. Accompanying this activity was the rise of catastrophe theory, which would permit the extension of results from the analytic to the C^∞- category.

The asymptotic theory that emerged was first put forward (in flawed form) by Norman Bleistein in 1967 (see [**Ble1**]), and later, by F. Ursell in 1972 (who strengthened Bleistein's result; see [**Urs**, p.64]). Briefly, the theory developed provided for uniform asymptotic expansions of integrals with phase functions $\phi(x\,;\alpha_1, \dots, \alpha_n)$ having m coalescing saddle points, confluence being achieved as the parameters $\alpha_1, \alpha_2, \dots, \alpha_n$ all tended to some fixed value, say 0. The resulting expansion contained, as coefficients, functions akin to the Airy function.

To illustrate, let $\phi(x\,;\alpha_1, \dots, \alpha_n)$ be analytic in x, with m saddles that coalesce as $(\alpha_1, \dots, \alpha_n) \to (0, \dots, 0)$. Introduce a change of variables determined by

$$\phi(x;\alpha_1, \dots, \alpha_n) = z^{m+1}/(m+1) + \theta_{m-1} z^{m-1}/(m-1) + \dots + \theta_1 z + \theta_0. \tag{1.9}$$

From [**Urs**, p.52] or [**Lev**], (1.9) has a local, uniformly analytic 1 - 1 solution $z = z(x)$, with the coefficients θ_i depending only on the parameters $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Set $\boldsymbol{\theta} = (\theta_0, \dots, \theta_{m-1})$. Then the integral (1.1) becomes

$$I(\lambda;\boldsymbol{\alpha}) = \int_{-\infty}^{+\infty} g_0(z;\boldsymbol{\theta})\, e^{i\lambda A_m(z;\boldsymbol{\theta})}\, dz \tag{1.10}$$

where, for ease of discussion, we have selected the real line as the integration contour, and have written "=" for "∼". The function g_0 appearing is the function $f(x(z))\cdot(dx/dz)$. The new phase function, $A_m(z;\boldsymbol{\theta})$, is the right hand side of (1.9).

Following Bleistein [**Ble1**], we introduce the function sequences $\{p_{k\,s}\}$, $\{g_k\}$, $\{h_k\}$ defined by

$$\begin{aligned} g_k(z;\boldsymbol{\theta}) &= p_{k\,0} + p_{k\,1} z + \dots + p_{k\,m-1} z^{m-1} + A_m'(z;\boldsymbol{\theta})\, h_k(z;\boldsymbol{\theta}) \\ g_{k+1}(z;\boldsymbol{\theta}) &= \partial h_k(z;\boldsymbol{\theta})/\partial z, \quad k = 0, 1, 2, \dots, \end{aligned} \tag{1.11}$$

where we have set $A_m'(z;\boldsymbol{\theta}) = \partial A_m(z;\boldsymbol{\theta})/\partial z$. By repeated partial integration, we obtain an expansion in descending fractional powers of λ

with coefficients $p_{ks}F_s$, where the functions F_s are given by

$$(1.12) \qquad F_s(\lambda;\theta) = \int_{-\infty}^{+\infty} z^s e^{i\lambda A_m(z;\theta)}\, dz$$

for $s = 0, 1, \dots, m-1$. F_s can be expressed in terms of the special function

$$(1.13) \qquad Y_s(\theta) = F_s(1; \theta),$$

in view of the relation

$$F_s(\lambda; \theta) = \lambda^{-(s+1)/(m+1)} F_s(1; \lambda\theta_0, \lambda^{m/(m+1)}\theta_1, \dots, \lambda^{2/(m+1)}\theta_{m-1}).$$

We see from (1.8) that we can regard the functions Y_s as generalizations of Airy functions. Indeed, the Airy function Ai(t) is the function $Y_0(u,t)e^{-i\alpha}/2\pi$.

It is the asymptotic behaviour of the functions in (1.13) that the subsequent chapters examine. Their importance is assured by the rôle they play in the uniform asymptotic theory of integrals with several coalescing saddles. For this reason, these integrals warrant further examination. As Olver remarks about a related integral:

> $$(2.9) \qquad \int_{\infty e^{-\pi i/4}}^{\infty e^{\pi i/4}} \tau^j \exp z(\tau^4/4 - a\tau^2 - b\tau)\; d\tau\,, \; j = 0, 1, 2$$
>
> Analogous problems also arise in connection with methods for approximating integrals. For example, we know very little about the functions (2.9) that are used as uniform approximants in Ursell's method for integrals with three coalescing saddle points. **[Olv3**, p.133]

The situation involving these integrals has improved somewhat, spurred on by extensive numerical work led by Connors; see **[Con1]** and **[Con2]**. Connors and his colleagues have successfully generated tables of values for integrals related to $Y_0(x_1, x_2)$ and $Y_0(x_1, x_2, x_3)$. However, their work covers only a small range of values for the x_i. We extend this by developing uniform asymptotic approximations for the two simplest "canonical diffraction integrals" Y_s, apart from the well-known Airy functions.

Chapter Two develops a uniform asymptotic expansion of the Pearcey integral

$$(1.14) \qquad P(x\,, y) = e^{-i\theta_0}\, Y_0(\theta_0\,, y, x)$$

used by Pearcey in his investigation of electromagnetic fields **[Pea]**, and first studied by Brillouin at the turn of the century **[Bri]**. The asymptotic expansion to be obtained is uniformly valid near the "caustic" associated with the phase function.

(The "caustic" associated with the phase function $\phi = \phi(x; \alpha_1, \ldots, \alpha_n)$ of an oscillatory integral (1.1) is, for the functions we consider, the set of parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ for which ϕ has critical points of order ≥ 2; more compactly, the caustic is the set

$$\Sigma(\phi) = \{\boldsymbol{\alpha} \in \mathbb{R}^n \mid \exists\, x_0 \in \mathbb{R} \text{ for which } \phi'(x_0; \boldsymbol{\alpha}) = \phi''(x_0; \boldsymbol{\alpha}) = 0\}.$$

Caustics are of central importance in geometric optics, although our work makes no use of their geometric structure.)

The next two chapters are devoted to the more interesting (and more difficult) problem of developing the uniform asymptotics of a function related to $Y_0(\theta_0\,, \theta_1\,, \theta_2\,, \theta_3)$:

$$(1.15) \qquad Q(x\,, y\,, z) = e^{-i\theta_0}\, Y_0(\theta_0\,, z\,, y\,, x).$$

The phase function of $Q(x\,, y\,, z)$ is a quintic polynomial, so the saddles of Q are determined by a quartic. A novel feature of chapters Three and Four is the use of formulae developed by Greenhill **[Gre1]** for the zeroes of quartic polynomials. These expressions involve the use of Weierstrass' elliptic functions and the trigonometric solution for zeroes of an associated cubic (see Chapter III, S 2 and 3).

Before continuing with the treatment of (1.14), we note a convention used repeatedly throughout the subsequent chapters: fractional powers and radicals are always taken to have their principal values unless otherwise noted. This convention is put aside only when applying the formulae of Greenhill, where appropriate branches must be chosen. Whenever this occurs, we take pains to alert the reader as to what is being done.

We also note that in subsequent chapters, we will always use full integration contours, even though the asymptotics of the integrals Y_0 can be obtained through work involving only portions of the integration contour near the saddle points. (Recall the comment following (1.10); when the change of variables (1.9) is purely local, the equality in (1.10) must be replaced by $\sim$.) We elect to use the full integration path in the hopes that should a theory of error bounds for this uniform asymptotic theory emerge, it could be readily applied to our results. By using portions of the integration path, we introduce exponentially negligible errors that cannot be easily managed after the application of (1.9).

We begin with the Pearcey integral.

Chapter Two: The Expansion of the Pearcey Integral

1. Introduction

The Pearcey function of two real variables x and y is defined by

$$(1.1) \qquad P(x\,,y) = \int_{-\infty}^{+\infty} \exp\left\{i(t^4/4 + xt^2/2 + yt)\right\} dt.$$

As mentioned in chapter I, it is one of a class of 'generalized Airy functions'; see (1.1.13). The caustic associated with the phase of (1.1) is comparatively simple: if $(t^4/4 + xt^2/2 + yt)' = 0$ and $(t^4/4 + xt^2/2 + yt)'' = 0$, then the latter equation implies $x = -3t^2$ which, when substituted into the former, yields $y = 2t^3$. Cubing the relation involving x and squaring that for y implies $4x^3 + 27y^2 = 0$. Hence, the caustic associated with the Pearcey function is the so - called 'cusp'. When $4x^3 + 27y^2 < 0$, the point (x, y) is said to lie "inside" the caustic, and those points for which $4x^3 + 27y^2 > 0$ are said to lie "outside" the caustic.

Maslov and Fedoryuk encountered the Pearcey function while investigating semi-classical limits in quantum mechanics. Of (1.1), they note that it is not regarded as a special function, and go on to suggest that it *should* be; see [**Mas**, p. 172].

As we saw in chapter I, the Airy function plays an important rôle in the theory of uniform asymptotic expansions of integrals with two coalescing saddle points [**CFU**], and the functions P(x, y), $\partial P(x, y)/\partial x$, $\partial P(x,y)/\partial y$ play a corresponding rôle in the uniform asymptotic theory of integrals with three coalescing saddle points. It is with such functions that Ursell illustrated his asymptotic theory in [**Urs**].

If we apply the method of stationary phase to (1.1), we find that the asymptotic behaviour of P depends on one or three stationary points, provided one of the variables stays fixed. If we denote the phase function in the integrand of (1.1) by $\phi(t;x,y)$ and let $\delta = 4x^3 + 27y^2$, then $\phi_t = 0$ gives three stationary points if $\delta < 0$, and only one if $\delta > 0$.

However, as $\delta \to 0$, we have some (or all) stationary points coalescing and in transition from negative to positive δ, it is not clear how P behaves. Therefore, it is the large negative x-behaviour of (1.1) for (x, y) near the caustic $\delta = 0$ that concerns us (if x is positive, then δ is always positive).

If $x \to -\infty$, then for δ to remain small, we must have $|y| \to +\infty$. Since $P(x,y)$ is clearly an even function of y, we may restrict ourselves to the case $y > 0$; furthermore, we shall find it to be notationally convenient to replace x by -x in (1.1) and examine $x \to +\infty$. With these conventions, $\delta = 0$ implies $y = 2x^{3/2}/\sqrt{27}$. This suggests that we examine $P(-x, y)$ with $y = \mu x^{3/2}$; when $\mu = 2/\sqrt{27}$, we are right at the caustic. With this relation between x and y, we find that as x and y tend to infinity along $y = \mu x^{3/2}$, μ fixed,

$$(1.2)\qquad P(-x, \mu x^{3/2}) \sim \sum_{j=1}^{3} e^{ix^2 f(t_j;\mu) + \pi i/4 \cdot \mathrm{sgn}(3t_j^2 - 1)} \sqrt{\frac{2\pi}{x|3t_j^2 - 1|}} .$$

Here, $0 \le \mu < 2/\sqrt{27}$, $f(t;\mu) = t^4/4 - t^2/2 + \mu t$, and the t_j are the real roots of $f_t = 0$ (cf. § 2 for a description of the t_j). At the caustic $\mu = 2/\sqrt{27}$, we find that

$$(1.3)\qquad P(-x, 2x^{3/2}/\sqrt{27}) \sim \sqrt{\frac{2\pi}{3x}}\, e^{i(\pi/4 - 2x^2/3)} + \frac{\Gamma(1/3)}{3^{1/3} x^{1/6}} e^{ix^2/12} - \frac{i\Gamma(2/3)}{2\cdot 3^{2/3} x^{5/6}} e^{ix^2/12}$$

and for $\mu > 2/\sqrt{27}$,

$$(1.4)\qquad P(-x, \mu x^{3/2}) \sim \sqrt{\frac{2\pi}{x(3t_0^2 - 1)}}\, e^{\pi i/4 + ix^2 f(t_0;\mu)} ,$$

where t_0 denotes the only real zero of f_t.

Note that the approximations in (1.2) - (1.4) are of radically differing characters. As $\mu \to 2/\sqrt{27}^-$, t_2 and $t_3 \to 1/\sqrt{3}$ and hence the approximation (1.2) becomes singular; see § 2. The important observation to make, at this point, is that as $\delta \to 0$ (i.e., $\mu \to 2/\sqrt{27}$) only two stationary points are coalescing. To exploit this phenomenon, we retain μ as a uniformity parameter and rewrite (1.1) as a sum of two contour integrals, one of which has exactly two relevant, coalescing, saddle points. This allows us to apply a cubic transformation introduced by Chester, Friedman and Ursell [CFU], and to construct a uniform asymptotic expansion of (1.1) as $x \to -\infty$ with δ varying in an interval containing 0 (i.e., μ in an interval containing $2/\sqrt{27}$). The expansion we present is in fact valid for certain complex values of the arguments when $P(x, y)$ is extended to $\mathbb{C} \times \mathbb{C}$ by analytic continuation.

At the time of writing, we became aware of a recent publication by Stamnes and Spjelkavik [Sta] who also noted that only two stationary points coalesce, but their subsequent derivations of asymptotic expansions of the Pearcey function are purely formal. For a brief discussion of their arguments, see § 6.

2. Alternate Representation

An application of Jordan's inequality shows that for real x and y, the path of integration in (1.1) may be rotated onto the contour Γ in the complex t-plane, where Γ is the straight line through the origin making an angle of $\pi/8$ with the positive real axis. With this integral representation we can continue $P(x,y)$ analytically to an entire function in $\mathbb{C} \times \mathbb{C}$.

Let

$$\mu = 2/\sqrt{27} - \alpha \tag{2.1}$$

and set $y = \mu x^{3/2}$. With $x > 0$, the change of variable $t \to x^{1/2}t$ gives

$$P(-x,\mu x^{3/2}) = x^{1/2}\int_\Gamma e^{ix^2(t^4/4 - t^2/2 + \mu t)}\,dt\ . \tag{2.2}$$

Since this integral converges for $\mu \in \mathbb{C}$ and for x with $|\arg x| < \pi/4$, subsequent work is valid for these values of x and μ.

By Cauchy's theorem, we may write

$$P(-x, \mu x^{3/2}) = x^{1/2}\int_{\Gamma_1} e^{ix^2(t^4/4 - t^2/2 + \mu t)}dt + x^{1/2}\int_{\Gamma_2} e^{ix^2(t^4/4 - t^2/2 + \mu t)}dt \tag{2.3}$$

where Γ_1 is the contour beginning at $\infty e^{5\pi i/8}$ and ending at $\infty e^{\pi i/8}$, and Γ_2 is the contour beginning at $\infty e^{9\pi i/8}$ and ending at $\infty e^{5\pi i/8}$ (see figure 2.1).

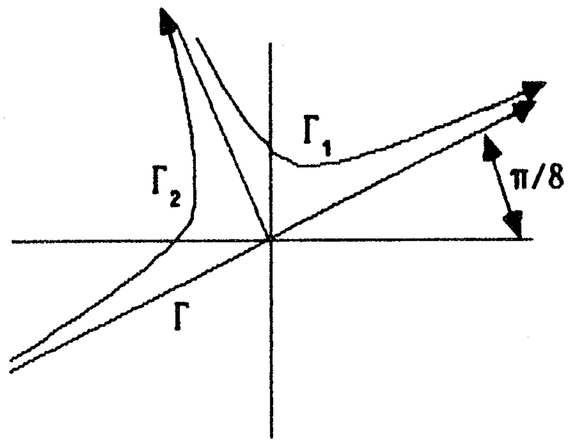


Figure 2.1. The integration contours Γ, Γ_1, and Γ_2.

For i = 1, 2, set

$$(2.4) \qquad P_i(\lambda\,;\mu) = \int_{\Gamma_i} e^{i\lambda f(t;\,\mu)}\,dt \quad ,$$

where, as in § 1, we have put

$$(2.5) \qquad f(t;\mu) = t^4/4 - t^2/2 + \mu t.$$

P_i is analytic for all μ and all λ with Re $\lambda > 0$.

Since $f_t(t;\mu) = 0$ has the solutions

$$t_1(\mu) = -2[\sin(\pi/3 + \phi)]/\sqrt{3}$$

$$t_2(\mu) = 2[\sin \phi]/\sqrt{3}$$

$$t_3(\mu) = 2[\sin(\pi/3 - \phi)]/\sqrt{3}$$

for $\Delta = -4 + 27\mu^2 < 0$, with ϕ given by

$$3\phi = \arcsin(27^{1/2}\mu/2), \quad |\phi| \le \pi/6$$

(see [**Miu**]), we see that as $\Delta \to 0^-$ (i.e., $\mu \to 2/\sqrt{27}^-$), $t_1 \to -2/\sqrt{3}$, $t_2 \to 1/\sqrt{3}$, $t_3 \to 1/\sqrt{3}$. Thus the two positive roots of $f_t = 0$ coalesce as $\mu \to 2/\sqrt{27}$, but remain well-separated from the negative root.

If $\Delta > 0$, we can invoke Cardan's formulae (or use the trigonometric solution above with ϕ complex) to observe the same phenomenon, only this time, as $\Delta \to 0^+$, the complex conjugate pair of roots of $f_t = 0$ coalesce to $1/\sqrt{3}$ with the real root remaining isolated.

Also, by examining $\sin\phi$ and $\sin(\pi/3 - \phi)$ for $\phi \in]\,0, \pi/6\,[$, we find that $t_3(\mu) > 1/\sqrt{3} > t_2(\mu)$ for $\mu < 2/\sqrt{27}$.

With this background, we shall obtain the large-λ behaviour of the $P_i(\lambda;\mu)$ as $\lambda \to \infty$. This in turn will yield the desired expansion of (2.3) as $x \to \infty$ uniformly for $\mu \in [\mu_0, \mu_1]$ where $0 < \mu_0 < 2/\sqrt{27} < \mu_1$.

3. Uniform Expansion of P_1

Since P_1 has two relevant coalescing saddle points, we invoke a cubic transformation as is done in [**CFU**] or [**Ble**2, § 9.2]. For convenience, we shall choose α, given in (2.1), as our uniformity parameter since the saddles coincide when α vanishes.

As the saddles coalesce at $t = 1/\sqrt{3}$, we develop f as follows:

$$\begin{aligned} f(t;\mu) &= -\alpha/\sqrt{3} + 1/12 - \alpha z + z^3/\sqrt{3} + z^4/4 \\ &= g(z;\alpha) - \alpha/\sqrt{3} + 1/12 \end{aligned}$$

where we have set $z = t - 1/\sqrt{3}$. Thus

$$P_1(\lambda;\mu) = e^{-i\lambda(\alpha/\sqrt{3} - 1/12)} \int_{\Gamma'_1} e^{i\lambda g(z;\alpha)}\,dz$$

where Γ'_1 is the translate of Γ_1 by $1/\sqrt{3}$. Notice that the zeroes of $g_z(z;\alpha)$

are given by

$$z_1(\alpha) = -[2\sin(\phi + \pi/3) + 1]/\sqrt{3}$$
$$z_2(\alpha) = [2\sin\phi - 1]/\sqrt{3}$$
$$z_3(\alpha) = [2\sin(\pi/3 - \phi) - 1]/\sqrt{3},$$

where

$$3\phi = \arcsin(1 - 27^{1/2}\alpha/2)\,,\ |\phi| \le \pi/6.$$

As in § 2, by examining $\sin\phi$ and $\sin(\pi/3 - \phi)$, we have $z_1(\alpha) < -1 < z_2(\alpha) < 0 < z_3(\alpha)$ for $\alpha > 0$ sufficiently small. When $\alpha < 0$, z_2 and z_3 are complex conjugates. In either case, $\alpha \to 0$ implies $z_2(\alpha), z_3(\alpha) \to 0$.

Now we introduce the change of variables

$$g(z;\alpha) = u^3/3 - \zeta u + \eta, \tag{3.1}$$

where $\zeta = \zeta(\alpha)$ and $\eta = \eta(\alpha)$ are to be determined. In order that (3.1) defines an analytic function $u(z)$ near z_2 and z_3, we require that z_2 and z_3 correspond to $-\zeta^{1/2}$ and $\zeta^{1/2}$ respectively. Accordingly, we find that

$$\zeta^{3/2} = 3[g(z_2;\alpha) - g(z_3;\alpha)]/4 \tag{3.2}$$

and

$$\eta = [g(z_2;\alpha) + g(z_3;\alpha)]/2. \tag{3.3}$$

(3.1) may be solved explicitly to give three possible candidates for our change of variables: for $\zeta \neq 0$,

$$u_1(z;\alpha) = -2\zeta^{1/2}\sin(\pi/3 + \psi)$$
$$u_2(z;\alpha) = 2\zeta^{1/2}\sin\psi$$
$$u_3(z;\alpha) = 2\zeta^{1/2}\sin(\pi/3 - \psi),$$

with

$$\sin 3\psi = 3[\eta - g(z;\alpha)]/2\zeta^{3/2};$$

for $\zeta = 0$, we have $u = (3[g(z) - \eta])^{1/3}$ and again, there are three branches. Please note that we frequently omit mentioning explicitly the dependence of g on α.

From (3.2) and (3.3), we have

$$\sin 3\psi = 3[g(z_3) - g(z_2)]/4\zeta^{3/2} = -1$$

when $z = z_2$, and when $z = z_3$ we get

$$\sin 3\psi = 3[g(z_2) - g(z_3)]/4\zeta^{3/2} = +1.$$

Thus, we have

$$\begin{array}{ll}
u_1(z_2;\alpha) = -\zeta^{1/2} & u_1(z_3;\alpha) = -2\zeta^{1/2} \\
u_2(z_2;\alpha) = -\zeta^{1/2} & u_2(z_3;\alpha) = \zeta^{1/2} \\
u_3(z_2;\alpha) = 2\zeta^{1/2} & u_3(z_3;\alpha) = \zeta^{1/2}.
\end{array}$$

Therefore, $u_2(z;\alpha)$ is the desired uniformly analytic solution to (3.1). We now set $u(z;\alpha) = u_2(z;\alpha)$.

Before continuing further, let us briefly examine the nature of the mapping (3.1). From Chester *et al* [CFU], we know that $u(z;\alpha)$ is uniformly analytic and one-to-one near $z = 0$. Much more, however, can be said. In the following we will show that the mapping (3.1) in the present case is in fact one-to-one and analytic along the contour Γ_1'.

In the subsequent analysis, assume $\alpha > 0$. The mapping between the z- and u-planes can be most easily studied by introducing intermediate variables Z, ψ given by

$$\begin{aligned}
Z &= g(z;\alpha) - \eta \\
\sin 3\psi &= -3Z/2\zeta^{3/2} \\
u &= 2\zeta^{1/2}\sin\psi;
\end{aligned}$$

see, for example, Copson **[Cop1]**. We begin by decomposing the z-plane into four regions, each of which maps onto the Z-plane. A useful device in this process is the determination of all curves which g sends to the real Z-axis. We adopt the notational convention of denoting that part of a region Ω which has positive imaginary part by Ω^+.

The mapping $Z = g(z;\alpha) - \eta$ clearly takes the real line in the z-plane to the real line in the Z-plane in the following manner:

$$
(3.4)\qquad
\begin{aligned}
&Z([z_3, +\infty[) = [-2\zeta^{3/2}, +\infty[\\
&Z([z_2, z_3]) = [-2\zeta^{3/2}/3, 2\zeta^{3/2}/3]\\
&Z([z_1, z_2]) = [g(z_1) - \eta, 2\zeta^{3/2}/3]\\
&Z(]-\infty, z_1]) = [g(z_1) - \eta, +\infty[,
\end{aligned}
$$

the mapping Z being one-to-one on each interval. The remaining curves are the nontrivial solutions of

$$
(3.5)\qquad \mathrm{Im}(g(z;\alpha) - \eta) = 0 = \mathrm{Im}\, g(z;\alpha) = \mathrm{Im}(g(z;\alpha) - g(z_k;\alpha)),\quad k = 1,2,3,
$$

since each critical point z_k being real implies that $g(z_k;\alpha)$ is real. Note that since the left half of (3.5) is independent of k, the solution curves arising from the right half of (3.5) are the same for each k.

Develop g about the critical point $z = z_k$. Then (3.5) becomes

$$
\mathrm{Im}[\, g''(z_k;\alpha)t^2/2 + g'''(z_k;\alpha)t^3/6 + t^4/4] = 0,
$$

where we have set $t = z - z_k = \sigma + i\tau$, and suppressed dependence on k. Thus

$$
0 = (2\sqrt{3}z_k + 3z_k^2)\sigma\tau + ((2\sqrt{3} + 6z_k)/6)(3\sigma^2\tau - \tau^3) + (\sigma^3\tau - \sigma\tau^3)
$$

or, since we have accounted for $\tau = 0$ in (3.4),

$$
0 = (2\sqrt{3}z_k + 3z_k^2)\sigma + ((2\sqrt{3} + 6z_k)/2)\sigma^2 + \sigma^3 - ((2\sqrt{3} + 6z_k)/6 + \sigma)\tau^2
$$

yielding

$$\tau = \pm\sqrt{\frac{(2\sqrt{3}z_k + 3z_k^2)\sigma + \left(\frac{2\sqrt{3} + 6z_k}{2}\right)\sigma^2 + \sigma^3}{\frac{2\sqrt{3} + 6z_k}{6} + \sigma}}, \tag{3.6}$$

wherever the quantity inside the surd is non-negative.

Since $\sigma = \tau = 0$ satisfies (3.6) and $\sigma + i\tau = z - z_k$, each z_k lies on at least one of the curves determined by (3.6). Equation (3.6) gives rise to three curves; two of these curves have the 45^0 lines through the origin as asymptotes, and the other (the one containing z_2) has the vertical line $\sigma = -(2\sqrt{3} + 6z_k)/6$ as its asymptote. These curves are displayed in figure 2.2.

The curves that do not lie along the real z-axis partition the z-plane into four regions, R_1, R_2, R_3, and R_4, each of which maps in a one-to-one fashion onto the Z-plane (on the boundaries, the map g may be two-to-one) (see figure 2.2).

Consider R_1^+ and the map $z \to g(z; \alpha) - \eta$. From (3.4) we know that $[z_3, +\infty[$ maps onto $[-2\zeta^{3/2}/3, +\infty[$, and the curve bounding the upper extent of R_1^+ is mapped to $]-\infty, -2\zeta^{3/2}/3]$. Thus we see that R_1^+ maps to the upper half of the Z-plane. As well, the steepest descent curve of $ig(z; \alpha)$ from $z = z_3$ to $\infty e^{\pi i/8}$ lies within R_1^+ and maps to a vertical line in the Z-plane running through $Z=-2\zeta^{3/2}/3$. In figures 2.3 to 2.5, shaded regions map to shaded regions, dotted lines represent steepest descent curves, and in each figure, letters that correspond represent points that

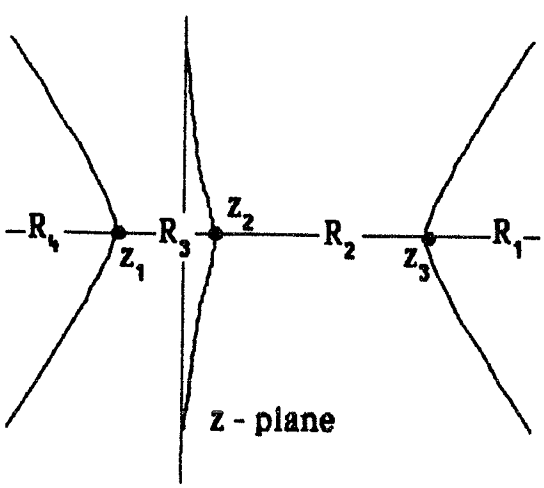


Figure 2.2. Curves that map to the real Z-axis (solid curves) and the regions R_1, R_2, R_3, and R_4.

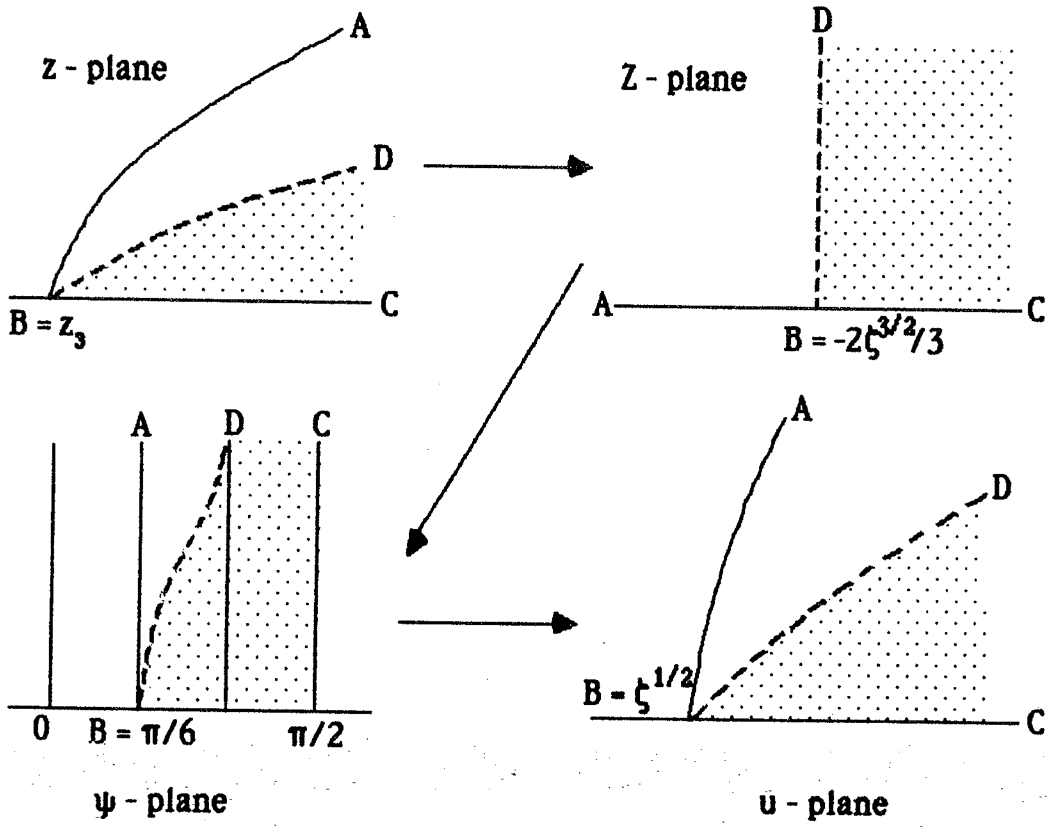


Figure 2.3. Effect of the mappings $Z = g(z; \alpha) - \eta$, $Z = -2\zeta^{3/2}(\sin 3\psi)/3$, $u = 2\zeta^{1/2} \sin \psi$ on R_1^+.

correspond under indicated map (see figure 2.3).

Under the mapping $\sin 3\psi = -3Z/2\zeta^{3/2}$, we note the following: $Z = -2\zeta^{3/2}/3$ is mapped to $\psi = \pi/6$; with $\psi \in [\pi/6, \pi/2]$, Z is real and in $[-2\zeta^{3/2}/3, 2\zeta^{3/2}/3]$, and the rays $\pi/6 + i\tau$, $\pi/2 + i\tau$, $\tau \geq 0$ map to the segments $]-\infty, -2\zeta^{3/2}/3]$ and $[2\zeta^{3/2}/3, +\infty[$ in the Z-plane, respectively. Morever, a straightforward calculation reveals that the steepest descent curve lying in the Z-plane maps to the curve $\gamma=(1/3)\log((1-\cos 3\beta)/\sin 3\beta)$, $\psi = \beta + i\gamma$, $\gamma \geq 0$, $\beta \leq \pi/3$, in the ψ-plane (see figure 2.3).

The map $u = 2\zeta^{1/2}\sin\psi$ takes the interval $[\pi/6, \pi/2]$ in the ψ-plane to $[\zeta^{1/2}, 2\zeta^{1/2}]$ in the u-plane, and maps $\pi/2 + i\tau$, $\tau \geq 0$ onto $[2\zeta^{1/2}, +\infty[$. The image of the ψ-plane ray $\pi/6 + i\tau$, $\tau \geq 0$, is the curve

$$u = \zeta^{1/2}[\cosh \tau + i\sqrt{3}\sinh \tau],$$

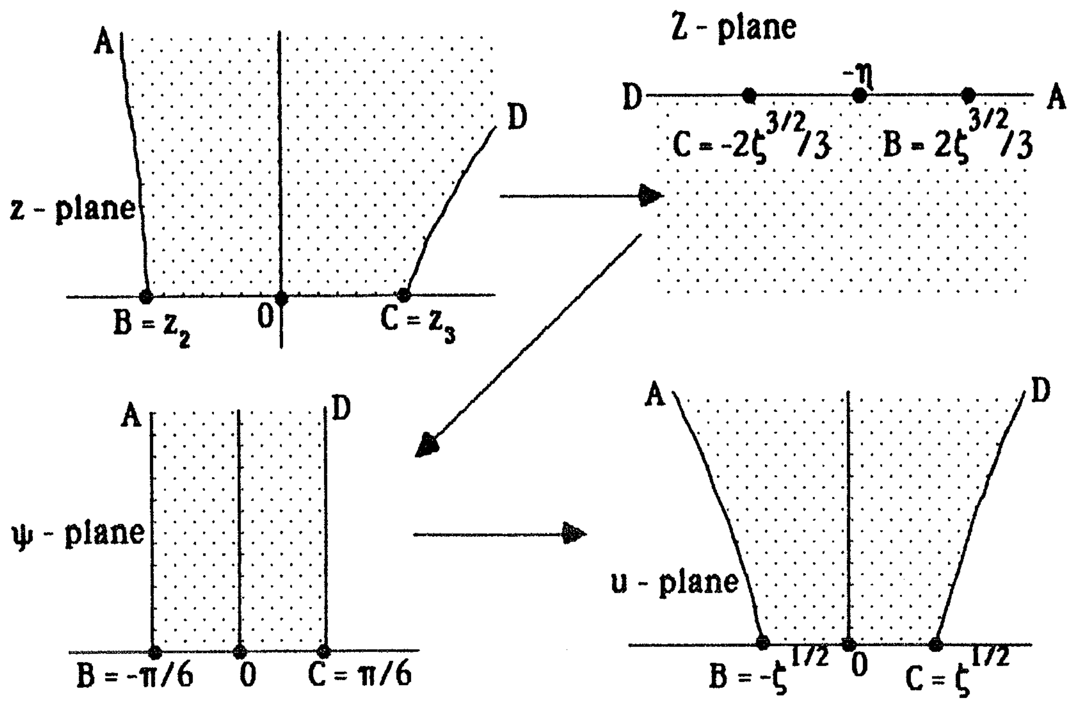


Figure 2.4. Effect of the sequence of maps $z \to Z \to \psi \to u$ applied to R_2^+.

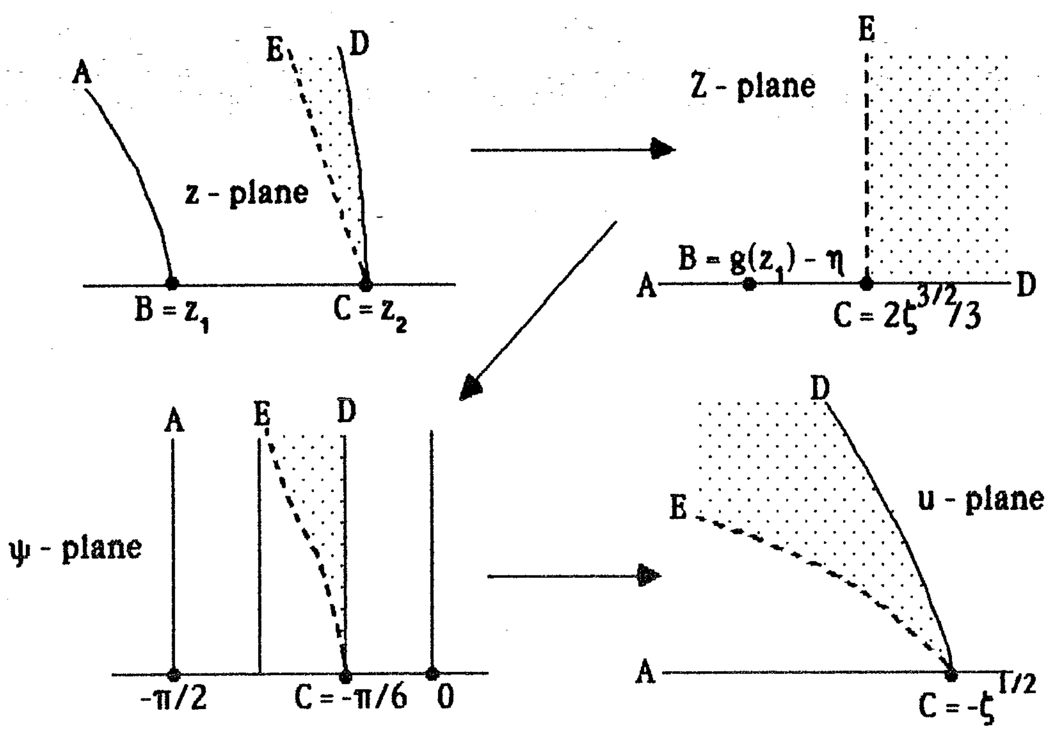


Figure 2.5. Effect of the sequence of maps $z \to Z \to \psi \to u$ applied to R_3^+.

which we see as being the first quadrant branch of the hyperbola $(\mathrm{Re}\,u)^2-(\mathrm{Im}\,u)^2/3 = \zeta$ (to see this, put $\mathrm{Re}\ u = \zeta^{1/2}\cosh\tau$, $\mathrm{Im}\ u = \sqrt{3}\zeta^{1/2}\sinh\tau$ and employ the identity $\cosh^2\tau - \sinh^2\tau = 1$). The steepest descent curve in the ψ-plane maps to the steepest descent curve of $i(u^3/3 - \zeta u + \eta)$ beginning at $u=\zeta^{1/2}$ and ending at $\infty e^{\pi i/6}$. This is most easily seen by examining $Z = u^3/3 - \zeta u + \eta$ directly; see figure 2.3.

In a completely similar fashion, we find the images of R_2^+ and R_3^+ under the sequence of transformations $z \rightarrow Z \rightarrow \psi \rightarrow u$. The effect of these maps are represented in figures 2.4 and 2.5.

Collecting these maps together gives a conformal map from $R_1^+ \cup R_2^+ \cup R_3^+$ to the upper half of the u-plane which is one-to-one, although the intermediate transformations fail to be one-to-one when applied to all of $R_1^+ \cup R_2^+ \cup R_3^+$.

In the event that $\alpha < 0$, we would proceed as before, this time using (3.5) with $k = 1$ since (3.6) holds only for the real saddle z_1. Notice that $\mathrm{Im}\ \eta = 0$ for $\alpha < 0$.

If we take Γ_1' to be the steepest descent curve beginning at $\infty e^{5\pi i/8}$ and ending at $z = z_2$, followed by the straight line segment $[z_2, z_3]$ and thereafter the steepest descent curve from $z = z_3$ to $\infty e^{\pi i/8}$, then $u(z;\alpha)$ maps Γ_1' onto the curve formed by the steepest descent curves from $-\zeta^{1/2}$ to $\infty e^{5\pi i/6}$, and $+\zeta^{1/2}$ to $\infty e^{\pi i/6}$, along with the line segment $[-\zeta^{1/2}, \zeta^{1/2}]$, suitably oriented. Call the image of Γ_1' in the u-plane C. We may now write

$$e^{i\lambda(\alpha/\sqrt{3} - 1/12)} P_1(\lambda;\mu) = \int_C e^{i\lambda(u^3/3 - \zeta u + \eta)} g_0(u;\alpha)\, du,$$

where we have put $g_0(u;\alpha) = dz/du$. Define, as in [Ble2], the function sequences $\{g_n\}$, $\{h_n\}$, $\{p_n\}$ and $\{q_n\}$ by

$$
\begin{aligned}
& p_n(\alpha) = [g_n(\zeta^{1/2};\alpha) + g_n(-\zeta^{1/2};\alpha)]/2 \\
(3.7) \qquad & q_n(\alpha) = [g_n(\zeta^{1/2};\alpha) - g_n(-\zeta^{1/2};\alpha)]/2\zeta^{1/2} \\
& g_n(u;\alpha) = p_n(\alpha) + q_n(\alpha)u + (u^2-\zeta)h_n(u;\alpha)
\end{aligned}
$$

$$g_{n+1}(u;\alpha) = \partial h_n(u;\alpha)/\partial u, \qquad n = 0, 1, 2, \ldots .$$

Then, by successive substitution and partial integration, we obtain

$$e^{i\lambda(\alpha/\sqrt{3} - 1/12)} P_1(\lambda;\mu) \sim e^{i\lambda\eta} \sum_{n=0}^{\infty} \left(\frac{i}{\lambda}\right)^n [p_n(\alpha) F(\lambda;\zeta) + q_n(\alpha) G(\lambda;\zeta)]$$

as $\lambda \to \infty$, uniformly in ζ. Here,

$$F(\lambda;\zeta) = \int_C e^{i\lambda(u^3/3 - \zeta u)} du \tag{3.8}$$

$$G(\lambda;\zeta) = \int_C u e^{i\lambda(u^3/3 - \zeta u)} du, \tag{3.9}$$

both integrals converging absolutely and uniformly for Re $\lambda > 0$, $\zeta \in \mathbb{C}$. By a simple change of variables, we see that in Re $\lambda > 0$,

$$F(\lambda;\zeta) = 2\pi\lambda^{-1/3} \mathrm{Ai}(-\lambda^{2/3}\zeta)$$

$$G(\lambda;\zeta) = -2\pi i \lambda^{-2/3} \mathrm{Ai}'(-\lambda^{2/3}\zeta).$$

Hence,

$$e^{i\lambda(\alpha/\sqrt{3} - 1/12)} P_1(\lambda; 2/\sqrt{27} - \alpha) \sim$$

$$\frac{2\pi}{\lambda^{1/3}} e^{i\lambda\eta} \sum_{n=0}^{\infty} [p_n(\alpha)\mathrm{Ai}(-\lambda^{2/3}\zeta) + \frac{1}{i\lambda^{1/3}} q_n(\alpha)\mathrm{Ai}'(-\lambda^{2/3}\zeta)] \left(\frac{i}{\lambda}\right)^n$$

as $\lambda \to \infty$, uniformly in α near zero. In § 6, we shall have occasion to use the approximation

$$\begin{aligned} P_1(\lambda; 2/\sqrt{27} - \alpha) = e^{i\lambda(1/12 - \alpha/\sqrt{3} + \eta)} \{ & 2\pi\lambda^{-1/3} p_0(\alpha)\mathrm{Ai}(-\lambda^{2/3}\zeta) \cdot \\ & [1 + O(1/\lambda)] - 2\pi i \lambda^{-2/3} q_0(\alpha)\mathrm{Ai}'(-\lambda^{2/3}\zeta)[1 + O(1/\lambda)]\} \end{aligned} \tag{3.10}$$

for $\lambda \to \infty$ as above.

4. The Coefficients $p_0(\alpha)$ and $q_0(\alpha)$

Although expressions for $p_0(\alpha)$ and $q_0(\alpha)$ can be calculated routinely, because of the labour involved in obtaining limiting forms as $\alpha \to 0$, we reproduce some of the requisite analysis. The reader will appreciate the difficulty in calculating higher coefficients from these two examples.

We begin by remarking that the following analysis remains valid for all (small) complex α, provided we choose the principal branch for $\alpha^{1/2}$. For the purpose of exposition, we shall take $\alpha > 0$.

Differentiate (3.1) twice with respect to u; this gives

$$g_{zz}(dz/du)^2 + g_z(d^2z/du^2) = 2u.$$

Evaluation at $z = z_3$, and use of the fact that z_3 corresponds with $\zeta^{1/2}$, gives

$$(2\sqrt{3}z_3 + 3z_3^2)\left(\frac{dz}{du}\right)^2\bigg|_{u=\zeta^{1/2}} = 2\zeta^{1/2} \quad ,$$

whence

$$\left(\frac{dz}{du}\right)\bigg|_{u=\zeta^{1/2}} = \pm\left\{\frac{2\zeta^{1/2}}{2\sqrt{3}z_3 + 3z_3^2}\right\}^{1/2} .$$

Since z_3, for $\alpha > 0$, is a local minimum of $g(z;\alpha)$, $g_{zz}(z_3;\alpha)$ is positive and so the expression under the square root is positive. Also, as z increases with u,

$$\left(\frac{dz}{du}\right)\bigg|_{u=\zeta^{1/2}} = \left\{\frac{2\zeta^{1/2}}{2\sqrt{3}z_3 + 3z_3^2}\right\}^{1/2} .$$

Similarly,

$$\left(\frac{dz}{du}\right)\bigg|_{u=-\zeta^{1/2}} = \left\{\frac{-2\zeta^{1/2}}{2\sqrt{3}z_2 + 3z_2^2}\right\}^{1/2} .$$

Since $g_0(u;\alpha) = dz/du$, we have, from the first two equations of (3.7) with $n = 0$,

$$(4.1) \qquad p_0(\alpha) = \frac{1}{2}\left(\left\{\frac{2\zeta^{1/2}}{2\sqrt{3}z_3 + 3z_3^2}\right\}^{1/2} + \left\{\frac{-2\zeta^{1/2}}{2\sqrt{3}z_2 + 3z_2^2}\right\}^{1/2}\right)$$

$$(4.2) \qquad q_0(\alpha) = \frac{1}{2\zeta^{1/2}}\left(\left\{\frac{2\zeta^{1/2}}{2\sqrt{3}z_3 + 3z_3^2}\right\}^{1/2} - \left\{\frac{-2\zeta^{1/2}}{2\sqrt{3}z_2 + 3z_2^2}\right\}^{1/2}\right).$$

Use of equation (3.2) allows us to express (4.1) and (4.2) completely in terms of the function g.

To investigate limiting behaviour of p_0, q_0 as $\alpha \to 0$, we will need the behaviour of $z_2(\alpha)$, $z_3(\alpha)$ and $\zeta^{1/2}(\alpha)$ as $\alpha \to 0$. Since $z_2(\alpha) = [2\sin\phi - 1]/\sqrt{3}$ and $\phi = \pi/6 - (1/3)(\sqrt{27\alpha})^{1/2}(1 + \sqrt{3}\alpha/8 + O(\alpha^2))$ for small α, we see upon developing $[2\sin\phi - 1]/\sqrt{3}$ into its Taylor series at $\phi = \pi/6$ that

$$z_2(\alpha) = a_1\alpha^{1/2} + a_2\alpha + a_3\alpha^{3/2} + O(\alpha^2),$$

with $a_1 = -27^{1/4}/3$, $a_2 = -1/6$ (higher terms are not explicitly required for our purposes). Similarly,

$$z_3(\alpha) = -a_1\alpha^{1/2} + a_2\alpha - a_3\alpha^{3/2} + O(\alpha^2).$$

Thus,

$$g_{zz}(z_2;\alpha) = \alpha^{1/2}[2\sqrt{3}a_1 + (2\sqrt{3}a_2 + 3a_1^2)\alpha^{1/2} + (2\sqrt{3}a_3 + 6a_1a_2)\alpha + O(\alpha^{3/2})]$$

$$g_{zz}(z_3;\alpha) = \alpha^{1/2}[-2\sqrt{3}a_1 + (2\sqrt{3}a_2 + 3a_1^2)\alpha^{1/2} - (2\sqrt{3}a_3 + 6a_1a_2)\alpha + O(\alpha^{3/2})].$$

Also, from (3.2), we have

$$\zeta^{1/2}(\alpha) = (3/2)^{1/3}(a_1^3/\sqrt{3} - a_1)^{1/3}\alpha^{1/2}[1 + b\alpha + O(\alpha^{3/2})],$$

where we have put

$$b = [a_1^3 a_2 + \sqrt{3}(a_1 a_2^2 + a_1^2 a_3) - a_3] \,/\, (3(a_1^3/\sqrt{3} - a_1)).$$

Here, use has been made of the binomial formula.

Continuing, we get

$$\left(\frac{-2\zeta^{1/2}}{2\sqrt{3}z_2 + 3z_2^2}\right)^{1/2} = \left[2(3/2)^{1/3}\left(\frac{a_1^3}{\sqrt{3}} - a_1\right)^{1/3}\left(\frac{-1}{2\sqrt{3}a_1}\right)\right]^{1/2}$$
$$\left\{1 - \frac{3a_1^2 + 2\sqrt{3}a_2}{4\sqrt{3}a_1}\alpha^{1/2} + O(\alpha)\right\}$$

and

$$\left(\frac{2\zeta^{1/2}}{2\sqrt{3}z_3 + 3z_3^2}\right)^{1/2} = \left[2(3/2)^{1/3}\left(\frac{a_1^3}{\sqrt{3}} - a_1\right)^{1/3}\left(\frac{-1}{2\sqrt{3}a_1}\right)\right]^{1/2}$$
$$\left\{1 + \frac{3a_1^2 + 2\sqrt{3}a_2}{4\sqrt{3}a_1}\alpha^{1/2} + O(\alpha)\right\}.$$

Note that $a_1^3/\sqrt{3} - a_1 > 0$ and $-1/(2\sqrt{3}a_1) > 0$. Again, the binomial theorem has been employed. Thus,

$$(4.3)\quad p_0(\alpha) = \left[2(3/2)^{1/3}\left(\frac{a_1^3}{\sqrt{3}} - a_1\right)^{1/3}\left(\frac{-1}{2\sqrt{3}a_1}\right)\right]^{1/2} \cdot [1 + O(\alpha^{1/2})]$$

$$= 3^{-1/6}[1 + O(\alpha^{1/2})]$$

and

$$(4.4)\quad q_0(\alpha) = \left\{(-2\sqrt{3}a_1)^{-1/2}\left(\frac{2\sqrt{3}a_2 + 3a_1^2}{2\sqrt{3}a_1}\right)\Big/\left[2(3/2)^{1/3}\left(\frac{a_1^3}{\sqrt{3}} - a_1\right)^{1/3}\right]^{1/2}\right\}$$
$$\cdot [1 + O(\alpha^{1/2})]$$

$$= \frac{-3^{-5/6}}{2}[1 + O(\alpha^{1/2})]$$

as $\alpha \to 0$.

5. Expansion of P_2

Because Γ_2 begins and ends in valleys at ∞, we see that the contour can be deformed into the steepest descent path leading away from the saddle point t_1. The determination of the steepest descent path follows.

We begin by expanding f about the point $t = t_1$:

$$if(t;\mu) = if(t_1;\mu) + i((3t_1^2-1)/2)(t - t_1)^2 + it_1(t - t_1)^3 + i(t - t_1)^4/4.$$

Let $z = t - t_1 = x + iy$. Steepest descent paths are among the level curves $0 = \mathrm{Im}[if(t;\mu) - if(t_1;\mu)]$. Consequently,

$$(1/2)(3t_1^2-1)(x^2 - y^2) + t_1(x^3 - 3xy^2) + (1/4)(x^4 - 6x^2y^2 + y^4) = 0$$

gives steepest paths. Solving for y in terms of x gives

$$y = \pm(1/\sqrt{2})(6x^2 + 12t_1x + 2(3t_1^2 - 1) \pm ((6x^2 + 12t_1x + 2(3t_1^2 - 1))^2$$

$$- 4(x^4 + 4t_1x^3 + 2(3t_1^2 - 1)x^2))^{1/2})^{1/2},$$

wherever the real square roots are defined. We see that the steepest descent curve through t_1 begins at $\infty e^{-7\pi i/8}$ and ends at $\infty e^{5\pi i/8}$. Figure 2.6 displays hills and valleys in the case $\delta = 0$ (or $\mu = 2/\sqrt{27}$). Shaded regions represent valleys, the unshaded regions being hills. Solid

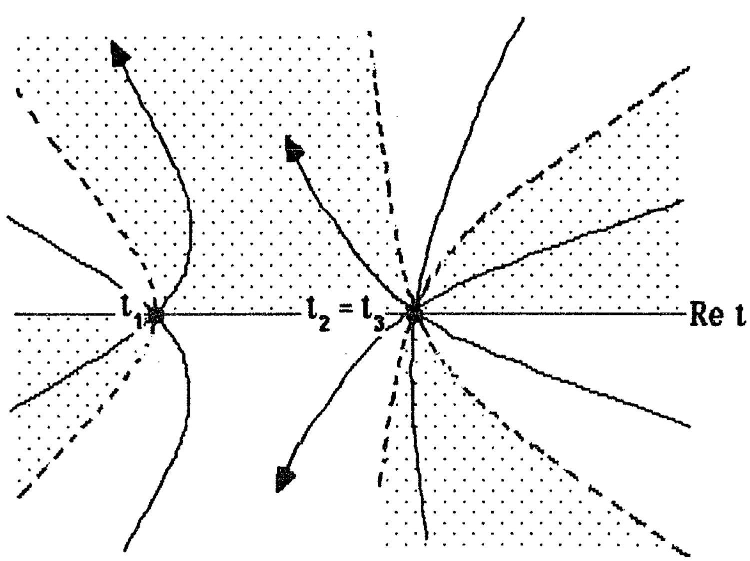


Figure 2.6. Hills and valleys defined by the steepest curves through t_1. Hills and valleys (relative to t_1) through $t_2 = t_3$ are also displayed.

curves (excluding the real axis) represent steepest descent or ascent paths according as they lie in shaded or unshaded regions respectively.

Let Γ_{2+} be the steepest descent curve running from t_1 to $\infty e^{5\pi i/8}$, and let Γ_{2-} be the steepest descent contour running from t_1 to $\infty e^{-7\pi i/8}$. Then we have

$$P_2(\lambda;\mu) = \int_{\Gamma_{2+}} e^{i\lambda f} dt - \int_{\Gamma_{2-}} e^{i\lambda f} dt$$

Let $w = if(t_1;\mu) - if(t;\mu)$ so that on $\Gamma_{2\pm}$, w is positive and increasing as we move away from t_1.

Let t_+ be the solution of $w = if(t_1;\mu) - if(t;\mu)$ on Γ_{2+}, t_- being the corresponding solution on Γ_{2-}. By the Lagrange inversion theorem, since

$$w = -(i/2)(3t_1^2 - 1)(t_\pm - t_1)^2 - it_1(t_\pm - t_1)^3 - (i/4)(t_\pm - t_1)^4$$

on $\Gamma_{2\pm}$ respectively, there are numbers $a_n^\pm$ for which

$$t_\pm = t_1 + \sum_1^\infty \frac{a_n^\pm}{n!} w^{n/2}$$

whence

$$\frac{dt_+}{dw} - \frac{dt_-}{dw} = \sum_1^\infty \frac{a_n^+ - a_n^-}{2(n-1)!} w^{(n/2)-1}$$

Thus

$$P_2(\lambda;\mu) \sim e^{i\lambda f(t_1;\mu)} \sum_1^\infty \frac{a_n^+ - a_n^-}{2(n-1)!} \frac{\Gamma(n/2)}{\lambda^{n/2}}$$

as $\lambda \to \infty$; see arguments in [Cop1, §§ 30 and 33].

The first few terms are easily found to be

$$a_1^+ = e^{\pi i/4}(2/(3t_1^2 - 1))^{1/2}$$
$$a_2^+ = -i(2/(3t_1^2 - 1))(2t_1/(3t_1^2 - 1))$$
$$a_3^+ = e^{3\pi i/4}(2/(3t_1^2 - 1))^{3/2}[-3/2(3t_1^2-1) + 15t_1^2/2(3t_1^2 - 1)^2]$$
$$a_1^- = e^{5\pi i/4}(2/(3t_1^2 - 1))^{1/2}$$
$$a_2^- = a_2^+$$
$$a_3^- = e^{-\pi i/4}(2/(3t_1^2 - 1))^{3/2}[-3/2(3t_1^2 - 1) + 15t_1^2/2(3t_1^2 - 1)^2],$$

whence

$$P_2(\lambda;\mu) = e^{i\lambda f(t_1;\mu)}\left\{\left(\frac{\pi}{3t_1^2 - 1}\right)^{1/2}\frac{1+i}{\lambda^{1/2}} + \frac{(-3 - 6t_1^2)\pi^{1/2}}{4(3t_1^2 - 1)^{7/2}}\frac{1-i}{\lambda^{3/2}} + O(1/\lambda^{5/2})\right\}, \tag{5.1}$$

where the 0-symbol is uniform in μ for μ in a compact interval in $\mathbb{R}_+$ containing $2/\sqrt{27}$. A general expression for the $a_n^\pm$ is available to us via the binomial theorem, but it is too unwieldy to be of intrinsic value. We note that $a_n^+ = a_n^-$ for even n.

6. Uniform Expansion of P at the Caustic

We have obtained expansions of P_1 and P_2. Using (2.3), (3.10) and (5.1), and restoring the large parameter x and the function f, we get the uniform approximation as $x \to \infty$:

$$\begin{aligned} P(-x, (2/\sqrt{27} - \alpha)x^{3/2}) = {} & \exp\{ ix^2[f(t_2;\mu) + f(t_3;\mu)]/2 \}\cdot \\ & \{2\pi x^{-1/6}p_0(\alpha)\mathrm{Ai}(-x^{4/3}\zeta)[1 + O(1/x^2)] \\ & - 2\pi i x^{-5/6}q_0(\alpha)\mathrm{Ai}'(-x^{4/3}\zeta)[1 + O(1/x^2)]\} \\ & + \exp\{ ix^2 f(t_1;2/\sqrt{27} - \alpha) \}(\pi/(3t_1^2 - 1))^{1/2}x^{-1/2}(1 + i)[1 + O(1/x^2)]. \end{aligned} \tag{6.1}$$

Here, $t_1(\mu) = t_1(2/\sqrt{27} - \alpha) = -(2/\sqrt{3})\sin(\pi/3 + \phi)$ where $3\phi = \arcsin(1 - \sqrt{27}\alpha/2)$. The nature of the approximation when ζ is bounded away from 0 is readily available when the asymptotic forms of the Airy function and its derivative are used **[Abr]**; recall that

$$(6.2) \qquad \mathrm{Ai}(z) = \frac{e^{-2z^{3/2}/3}}{2\sqrt{\pi}\, z^{1/4}}(1 + O(z^{-3/2})), \quad |\arg z| < \pi$$

$$(6.3) \qquad \mathrm{Ai}(-z) = \pi^{-1/2} z^{-1/4}[\sin(\pi/4 + 2z^{3/2}/3)(1 + O(z^{-3})) - \cos(\pi/4 + 2z^{3/2}/3)(5z^{-3/2}/48 + O(z^{-9/2}))], \quad |\arg z| < 2\pi/3$$

$$(6.4) \qquad \mathrm{Ai}'(z) = \frac{-z^{1/4} e^{-2z^{3/2}/3}}{2\sqrt{\pi}}(1 + O(z^{-3/2})), \quad |\arg z| < \pi$$

$$(6.5) \qquad \mathrm{Ai}'(-z) = -\pi^{-1/2} z^{1/4}[\cos(\pi/4 + 2z^{3/2}/3)(1 + O(z^{-3})) + \sin(\pi/4 + 2z^{3/2}/3)(-7z^{-3/2}/48 + O(z^{-9/2}))], \quad |\arg z| < 2\pi/3$$

as $z \to \infty$ in the indicated sectors.

If α is positive and bounded away from 0, then so is ζ. Hence

$$x^{-1/6}\mathrm{Ai}(-x^{4/3}\zeta) = \pi^{-1/2}\zeta^{-1/4}x^{-1/2}(\sin(\pi/4 + 2x^2\zeta^{3/2}/3)(1 + O(x^{-4})) - \cos(\pi/4 + 2x^2\zeta^{3/2}/3)(5x^{-2}\zeta^{-3/2}/48 + O(x^{-6})))$$

and

$$x^{-5/6}\mathrm{Ai}'(-x^{4/3}\zeta) = -\pi^{-1/2}\zeta^{1/4}x^{-1/2}(\cos(\pi/4 + 2x^2\zeta^{3/2}/3)(1 + O(x^{-4})) + \sin(\pi/4 + 2x^2\zeta^{3/2}/3)(-7x^{-2}\zeta^{-3/2}/48 + O(x^{-6})))$$

by (6.3) and (6.5). Thus, (6.1) reduces to a sum of three oscillatory terms, each of order $x^{-1/2}$ as $x \to \infty$. This is what we would expect from stationary phase inside the caustic (compare (1.2)).

On the other hand, if α is negative and bounded away from 0, then $\zeta^{3/2}$ lies in a segment of the imaginary axis bounded away from the origin.

Accordingly, $\arg(\zeta) = \pm\pi/3$, so that the asymptotic forms (6.2) and (6.4) show that (6.1) reduces to an oscillatory term of order $x^{-1/2}$, plus two exponentially decaying terms. This is consistent with stationary phase outside the caustic (compare (1.4)).

Finally, we examine (6.1) when $\alpha = 0$. Again from **[Abr]**, we see that $Ai(0) = \Gamma(1/3)/(2\pi 3^{1/6})$ and $Ai'(0) = -3^{1/6}\Gamma(2/3)/(2\pi)$. Since $1 + i = \sqrt{2}\exp(\pi i/4)$, $t_1(2/\sqrt{27}) = -2/\sqrt{3}$, and $\eta = 0$, (6.1) reduces to

$$P(-x, 2x^{3/2}/\sqrt{27}) = e^{-2ix^2/3 + \pi i/4}\sqrt{\frac{2\pi}{3x}} + e^{ix^2/12}\frac{\Gamma(1/3)}{3^{1/3}x^{1/6}}$$

$$- e^{ix^2/12}\frac{i\Gamma(2/3)}{2\cdot 3^{2/3}x^{5/6}} + O(1/x^{2+(1/6)})$$

as x tends to $+\infty$. Thus we recover (1.3). Here use has been made of (4.3) and (4.4).

A full expansion of $P(-x,\mu x^{3/2})$ follows from the expansions of P_1 and P_2, although the coefficients become more complicated as more terms are included. The extension to complex values of x and μ (for the analytic continuation of P) can be accomplished through the expansions for P_1 and P_2 for λ, μ complex, although as applications center on real values, we have not been overly concerned with developing the full range of complex values for which our expansion is valid.

We now turn to the work in **[Sta]**. Here, the authors examine the asymptotic behaviour of the function

$$(6.6) \qquad P^*(X, Y) = \int_{-\infty}^{+\infty} \exp(\, i(t^4 + Xt^2 + Yt)\,)\, dt$$

for large values of the parameters X, Y. P and P* are related by

$$P^*(X, Y) = P(X, Y/\sqrt{2})/\sqrt{2}.$$

In developing the expansions of $P^*(X, Y)$ away from the caustic $27Y^2+8X^3=0$, Stamnes and Spjelkavik first apply the method of stationary phase to integrals of the form

$$(6.7) \qquad \int_{-\infty}^{+\infty} g(t)e^{ikh(t)}\,dt\,, \qquad k \to \infty$$

with $g(t) = 1$, and $h(t) = t^4 + Xt^2 + Yt$. They then set $k = 1$ to yield the desired large X or large Y behaviour of P^*; see **[Sta**, p.1338 § 3.1].

For X and Y near the caustic, the authors formally invoke the method of Chester *et al* **[CFU]**, and, indeed, include a brief outline of the uniform asymptotic theory for integrals of the form (6.7) in an appendix. Let J_U denote the contribution to the integral (6.7) due to the coalescing stationary points (thus, J_U plays the same rôle as our P_1 (cf. (2.4))). The authors claim that

$$(6.8) \qquad J_U \sim 2\pi\exp(\, ik[h(t_2) + h(t_3)]\,) \sum_{m=0}^{2} (p_m F_m + q_m G_m)$$

as $|X|, Y \to \infty$, uniformly valid near the caustic; see **[Sta**, eq'n (3.31)]. Here, $h(t)$ is the phase function of (6.7) with Y replaced by -Y, and t_2 and t_3 are the critical points of $h(t)$ that coalesce as (X, Y) approaches the caustic, (X, Y) remaining bounded away from the origin in the XY-plane. The functions F_m and G_m are given by

$$\begin{aligned} F_0 &= k^{-1/3}\mathrm{Ai}(-\zeta k^{2/3}) & \qquad G_0 &= -ik^{-2/3}\mathrm{Ai}'(-\zeta k^{2/3}) \\ F_1 &= 0 & G_1 &= ik^{-4/3}\mathrm{Ai}(-\zeta k^{2/3}) \\ F_2 &= 2k^{-5/3}\mathrm{Ai}'(-\zeta k^{2/3}) & G_2 &= 2ik^{-4/3}\zeta\mathrm{Ai}(-\zeta k^{2/3}), \end{aligned}$$

where

$$(4/3)\zeta^{3/2} = h(t_3) - h(t_2).$$

The p_m and q_m are determined similarly as in (3.7) of this chapter, and are presented in the appendix of **[Sta**, equations (A18)-(A23)].

However, care must be taken in using expansion (6.8) to obtain the large negative-X behaviour of P^* near the caustic. (There is a typographical error in the term $h(t_2) + h(t_3)$, which should likely be half the stated value.) Throughout the appendix and § 2 of **[Sta]**, k appears as a large positive parameter. Yet, in several places, k is set equal to one prior to examining the large X behaviour of (6.7); see for instance, equations (3.5), (5.3) and the discussion in § 3.2 and note the absence of k in expansion (3.30). This naturally leads one to suspect that the same is being done in (6.8), although no explicit mention of this is made in [Sta]. It should be pointed out that (6.8) is that part of the expansion of

$$P^*(X, Y; k) = \int_{-\infty}^{+\infty} \exp(\, ik(t^4 + Xt^2 + Yt)\,)\, dt$$

due to the coalescing saddles of the phase function $t^4 + Xt^2 + Yt$, and not part of an expansion of $P^{\#}(X, Y)$. However, expansion (6.8) can be used to deduce the large negative-X behaviour of $P^{\#}$ near the caustic via the relation

$$P^{\#}(X,Y) = k^{1/4}P^*(\mathbf{X}, \mathbf{Y}; k),$$

where k is a large parameter, $X = k^{1/2}\mathbf{X}$ and $Y = k^{3/4}\mathbf{Y}$. Note that $27Y^2 + 8X^3 = k^{3/2}(27\mathbf{Y}^2 + 8\mathbf{X}^3)$; thus, if (X, Y) is near the caustic, then so is $(\mathbf{X}, \mathbf{Y})$.

The integral P^* is also related to the Pearcey function $P(x, y)$ given in (1.1) by

$$P^*(X, Y; k) = 2^{-1/2}k^{-1/4}P(k^{1/2}X,\, 2^{-1/2}k^{3/4}Y).$$

By setting $x = -k^{1/2}X$ and $\mu = 2^{-1/2}(-X)^{-3/2}Y$ in our expansion for $P_1(x^2; \mu)$ (cf. equation (3.10)), we obtain the first three terms in (6.8). Thus, modulo misprints, (6.8) appears to be correct.

Finally, we turn to the 'transitional approximation' developed in [Sta]. This was used in the numerical evaluation of $P^{\#}(X, Y)$ for (X, Y) in a band covering the caustic extending 0.05 units in the X-direction on either side of the caustic, with $Y > 0$ (see [Sta, p.1349, second paragraph]). In this calculation, it was assumed that $X^2 + Y^2 > 16$ and $|X|, Y \leq 8$.

The 'transitional approximation' developed is an asymptotic approximation of J_T in the "immediate vicinity of the caustic", where J_T represents the contribution to (6.6) due to coalescing stationary points of

$h(t)$. The authors assert that

$$(6.9) \qquad J_T \sim 2\pi d \cdot \exp(ih_0)[Ai(z) + icAi^{(4)}(z) - (1/2)c^2 Ai^{(8)}(z) - (1/6)ic^3 Ai^{(12)}(z) + \cdots],$$

where

$$h_0 = -5X^2/36 + Y(-X/6)^{1/2} \qquad h_1 = -8[Y/8 - (-X/6)^{3/2}]$$
$$h_3 = -4(-X/6)^{1/2} \qquad h_4 = 1$$
$$d = (3|h_3|)^{-1/3} \qquad \varepsilon = \mathrm{sgn}(h_3)$$
$$z = \varepsilon h_1 d \qquad c = h_4 d^4;$$

see [**Sta**, p.1343, eq'n (3.30)].

There are two points which we wish to make regarding the preceding approximation. First, the derivation is purely formal with no mention being made of the region of validity. Secondly, the authors did not actually use the expansion in the form given in (6.9), but instead used one in which each of the derivatives of the Airy function, $Ai^{(4j)}$, $j = 0,1,2,3$, is replaced by the first term in its Maclaurin expansion; cf. the first three lines on p.1344 of [**Sta**]. That is, the authors replace (6.9) by

$$(6.10) \qquad J_T = 2\pi d e^{ih_0}\left(\sum_{j=0}^{3} c^j Q_j(z) + O(c^4)\right),$$

where each Q_j is an expression of the form α or βz, α and β being constants.

In order for this to be valid, the implied O-terms in (6.10) involving z, resulting from approximating the $Ai^{(4j)}$ by the first terms of their Maclaurin series, must be $O(c^4)$ for large $|X|$. In particular, we must have $z = O(c^4)$. Since $z = -h_1 c^{1/4}$, this is equivalent to

$$h_1 = O(c^{15/4})$$

or, upon restoring X and Y,

$$Y - 8(-X/6)^{3/2} = O((-X)^{-5/2})$$

whence

$$27Y^2 + 8X^3 = O((-X)^{-1}).$$

This displays a condition on how quickly the point (X, Y) must approach the caustic in order for (6.10) to be an asymptotic series.

Before proceeding to the expansion of $Q(x, y, z)$, we note that it is easy to extend the results of this chapter to the problem of developing uniform asymptotic expansions near the caustic of the first order partial derivatives of P, $\partial P/\partial x$ and $\partial P/\partial y$, for a range of complex values of x and y. This is most easily seen by noting that both of the associated integrals P_1 and P_2 are analytic in λ and μ, hence in x and α; cf. (2.4). Since the coefficients p_n and q_n of (3.7) are analytic in α (see [**Urs**, p.52]), we need only differentiate our expansions with respect to λ and μ (or x and α) to obtain uniform expansions of $\partial P/\partial x$ and $\partial P/\partial y$ for large λ (ie., large x). However, this requires no new techniques, and so has been excluded from our discussion.

Chapter Three: The Expansion of $Q(x,y,z)$

1. Introduction

We turn now to the investigation of the asymptotic behaviour of

$$(1.1) \qquad Q(x, y, z) = \int_{-\infty}^{+\infty} \exp\left\{i(t^5/5 + xt^3/3 + yt^2/2 + zt)\right\} dt$$

for large values of the arguments x, y, z. The situation in this case is somewhat more complicated than that of the Pearcey integral, as we now have four saddle points that may coalesce as (x,y,z) varies in $\mathbb{R}^3$. Deform the path of integration into the complex t-plane so that the integration contour begins at $\infty\exp(9\pi i/10)$ and ends at $\infty\exp(\pi i/10)$. Hence, Q can be extended to an entire function in $\mathbb{C}^3$.

In this chapter, we shall restrict our attention to the case where x is non-negative, leaving the case of negative x to a later chapter. Some discussion will shed light on why we elect to treat the case of negative x separately.

The phase function of the integrand of (1.1) has, as its derivative, the polynomial $t^4 + xt^2 + yt + z$, which we call F. If F has zeroes of order two or higher, then the derivative of F, $4t^3 + 2xt + y$, must vanish at points t_0 for

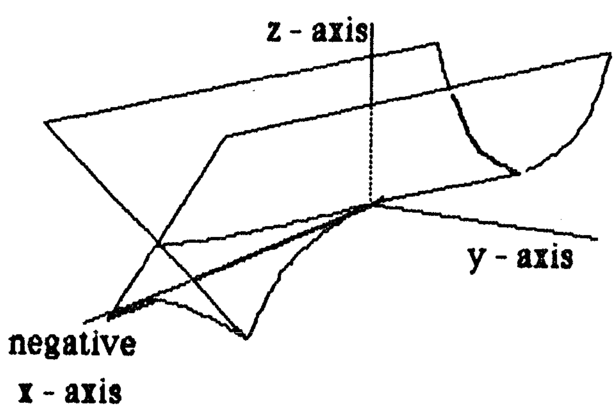


Figure 3.1. Locus of points (x, y, z) for which F and F' have simultaneous real zeroes.

which $F(t_0) = 0$. Zeroes of order three occur at those points t_0 for which $F(t_0) = F'(t_0) = F''(t_0) = 0$, where $F''(t) = 12t^2 + 2x$. Thus we see that (1.1) can have saddles of order three only in the event $x \leq 0$. If x is positive, then the phase function of (1.1) can have saddle points only of order ≤ 2.

Our interest in (1.1) resides in the structure of the real zeroes of F, for if F has a complex zero, the contribution to the asymptotic behaviour of (1.1) due to such a saddle is of exponentially small order; see, for example, (3.7).

If we plot the parameters (x, y, z) for which F and F' have simultaneous real zeroes, then we see that (x, y, z) lies on a self-intersecting surface; see figure 3.1 on the previous page. By taking plane slices x = *constant* for $x > 0$, $x = 0$, and $x < 0$, we produce the graphs in figures 3.2 to 3.4. For $x > 0$, say $x = 1$, we have $F = t^4 + t^2 + yt + z$ and $F' = 4t^3 + 2t + y$ so that if F and F' have simultaneous real zeroes, then we have $(y, z) = (-2t - 4t^3, t^2 + 3t^4)$. Thus, we have displayed the parameters y and z as functions of the coalesced real roots of F. Notice also that replacing t by $-t$ in the parametric forms for y and z on the caustic results only in a change in the sign of y. In view of the

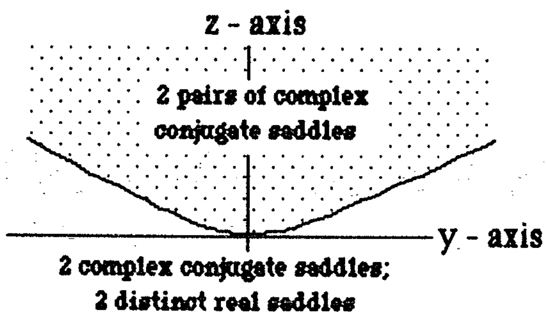


Figure 3.2. The plane slice ($x = 1$) of figure 3.1. The curve separating the two regions is the caustic.

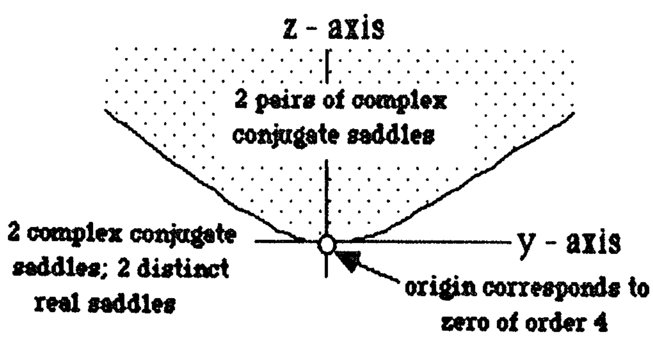


Figure 3.3. The plane slice ($x = 0$) of figure 3.1. The origin is the only point giving a zero of order > 2.

fact that $Q(x, y, z)$ and $Q(x, -y, z)$ are complex conjugates, we need only examine the case $y \geq 0$ when considering the asymptotics of Q, the case of $y < 0$ following by conjugation.

Plotting the $x = 1$ slice of figure 3.1 is accomplished by using the parametric forms of y and z above; see figure 3.2.

If we take the plane slice $x = 0$, we obtain a similarly shaped curve which can, once again, be obtained in parametric form. In this case, we have $F= t^4+yt+z$ and $F'=4t^3+y$ so that at points t for which F and F' have simultaneous real zeroes, we have $y=-4t^3$ and $z=-yt - t^4 = 3t^4$. When we have all of x, y and z equal to zero, F has a zero of order four; see figure 3.3.

If we examine the negative x slice of figure 3.1, we see that the situation is more involved. From $F''(t) =12t^2+2x$, we see that with $x < 0$, there can be zeroes of order 3. For the purpose of illustration, assume $x=-1$. Then there can be zeroes of order 3 when $t=\pm 1/\sqrt{6}$. The parametric equations of the curve in the yz-plane along which F has zeroes of order two or higher become $(y, z) = (2t - 4t^3, - t^2 + 3t^4)$. Plotting the curve(s) defined in this fashion yields the illustration presented as figure 3.4. Here, the two cusps of the curve(s) correspond to values of y and z (for that particular fixed x) for which F has zeroes of order 3. Other points on the curve(s) correspond to zeroes of order 2.

At points off the surface (in figure 3.1), F has at most simple real

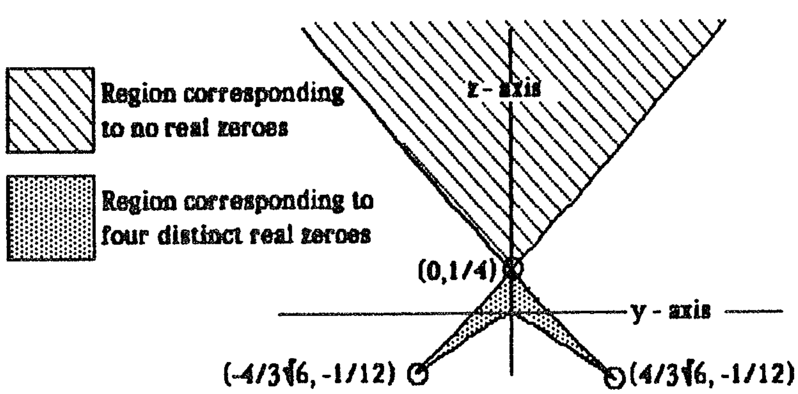


Figure 3.4. The plane slice $(x = -1)$ of figure 3.1. The white region corresponds to those points (y, z) for which F has exactly two distinct real zeroes.

zeroes. The reader interested in a detailed discussion of these plane sections is directed to [Gil, p. 62].

By temporarily restricting $x > 0$, we are in a setting where $Q(x, y, z)$ has at most two coalescing (real) saddle points.

If only one of x, y, or z is large, with the remaining two parameters bounded, then the asymptotic behaviour of (1.1) is readily obtained from the usual stationary phase methods. If two or more parameters are large, the situation is more involved; our analysis will lean heavily on the work of Chester *et al* [CFU] and Ursell [Urs]. It will be apparent, later, that the work required in obtaining the asymptotic behaviour of (1.1) for two parameters tending to infinity with one parameter remaining bounded is as involved as that of obtaining the behaviour of (1.1) with all three parameters growing without bound.

In any case, we will require a detailed description of the zeroes of a quartic polynomial that displays the zeroes as continuous functions of the coefficients. Derivatives of two polynomials in particular will be examined:

(1.2i) $$f_+(t;b,c)=t^5/5+t^3/3+bt^2/2+ct$$

and

(1.2ii) $$h(t;d,e)=t^5/5+dt^3/3+t^2/2+et.$$

These polynomials arise in the following fashion. In (1.1), put $t=x^{1/2}u$ to get

$$Q(x, y, z) = x^{1/2}\int_{\infty e^{9\pi i/10}}^{\infty e^{\pi i/10}} \exp\left\{ix^{5/2}f_+(u; yx^{-3/2}, zx^{-2})\right\} du \ ;$$

on the other hand, if we put $t=y^{1/3}u$, we get

$$Q(x, y, z) = |y|^{1/3}\int_{\infty e^{9\pi i/10}}^{\infty e^{\pi i/10}} \exp\left\{iy^{5/3}h(u; xy^{-2/3}, zy^{-4/3})\right\} du \ ,$$

with f_+ and h as in (1.2).

The asymptotic behaviour of (1.1) in the case where x and one or both of y and z tend to infinity can be deduced from the asymptotic behaviour of

$$(1.3) \qquad I_+(\lambda; b, c) = \int_{\infty e^{9\pi i/10}}^{\infty e^{\pi i/10}} e^{i\lambda f_+(t; b\ c)} dt$$

and the asymptotic behaviour of (1.1) in the case where y and one or both of x and z tend to infinity can be deduced from the asymptotic behaviour of

$$(1.4) \qquad J(\lambda; d, e) = \int_{\infty e^{9\pi i/10}}^{\infty e^{\pi i/10}} e^{i\lambda h(t; d, e)} dt$$

with λ tending to infinity. The integral J must be considered separately in order to examine the asymptotics of $Q(x,y,z)$ with x bounded and at least one of y or z tending to infinity. The integral (1.3) is of little use in the small x and large $|y|+|z|$ analysis of Q, for the parameter λ in (1.3) is no longer large.

To proceed, then, entails an analysis of the zeroes of $f_+'(t;b,c)=t^4+t^2+bt+c$ and $h'(t;d,e)=t^4+dt^2+t+e$.

2. Zeroes of Quartic Polynomials

It is well-known that a solution of quartics by radicals is available and many works in the theory of equations present formulae for this purpose. However, they tend to be unwieldy (due to the number of nested radicals that appear) and cloak the dependence of the zeroes on the coefficients of the polynomials under investigation.

An elegant solution to the problem of extracting zeroes of quartics is provided by the work of A. Greenhill [Gre1, Gre2], published in the late 19th century. We present the results of his work below.

Let

$$(2.1) \qquad U=x^4+6Cx^2+4Dx+E,$$

and let

$$(2.2) \qquad \begin{aligned} g_2 &= E+3C^2 \\ g_3 &= CE-D^2-C^3. \end{aligned}$$

(We note that every quartic can be brought into the form (2.1) by a linear change of variables.) Denote by $\wp(z;g_2,g_3)$ the Weierstrass elliptic function formed with the invariants g_2 and g_3. Let α (in the fundamental period parallelogram) be that number for which

$$(2.3) \qquad \begin{aligned} \wp(2\alpha;g_2,g_3) &= -C \\ \wp'(2\alpha;g_2,g_3) &= -D; \end{aligned}$$

for the existence of such α, see either [Gre1, p. 271 - 272] or [Gre2, p. 152 - 153]. Put

$$(2.4) \qquad S=4s^3-g_2s-g_3$$

and let the discriminant of this cubic be

$$(2.5) \qquad \delta=g_2^3-27g_3^2.$$

Denote the zeroes of S by e_i, $i=1,2,3$, and those of U by x_j, $j=0,1,2,3$. Greenhill found that

$$(2.6) \qquad \begin{aligned} x_0 &= \sqrt{\wp(2\alpha)-e_1} + \sqrt{\wp(2\alpha)-e_2} + \sqrt{\wp(2\alpha)-e_3} \\ x_1 &= \sqrt{\wp(2\alpha)-e_1} - \sqrt{\wp(2\alpha)-e_2} - \sqrt{\wp(2\alpha)-e_3} \\ x_2 &= -\sqrt{\wp(2\alpha)-e_1} + \sqrt{\wp(2\alpha)-e_2} - \sqrt{\wp(2\alpha)-e_3} \\ x_3 &= -\sqrt{\wp(2\alpha)-e_1} - \sqrt{\wp(2\alpha)-e_2} + \sqrt{\wp(2\alpha)-e_3} \end{aligned}$$

We note here that there is no ambiguity in equations (2.6) since the square roots are chosen by

$$(2.7) \qquad \sqrt{(\wp(z)-e_i)} = \exp(-\eta_i z)\,\sigma(z+\omega_i)/[\sigma(z)\sigma(\omega_i)].$$

In (2.7), ω_i is an irreducible half-period, and the numbers η_i are determined by $\eta_i=\zeta(\omega_i)$. The functions ζ and σ are, respectively, the Weierstrass zeta and sigma functions. A number of well-known properties and relations involving the constants in (2.7) can be found in [Cop2]. We mention only a few of importance to our work. For example, it is known that $\wp(\omega_i)=e_i$, and that the square root determined by (2.7) is always $\pm$ the principal branch; see [Cop2, p.367].

By convention, if the e_i are all real, then they are labelled so that $e_1\geq e_2\geq e_3$.

3. Analysis of I_+

Applying the theory of the previous section to f_+' gives

$$(3.1)\qquad \begin{aligned} t_0 &= \sqrt{-1/6-e_1} + \sqrt{-1/6-e_2} + \sqrt{-1/6-e_3} \\ t_1 &= \sqrt{-1/6-e_1} - \sqrt{-1/6-e_2} - \sqrt{-1/6-e_3} \\ t_2 &= -\sqrt{-1/6-e_1} + \sqrt{-1/6-e_2} - \sqrt{-1/6-e_3} \\ t_3 &= -\sqrt{-1/6-e_1} - \sqrt{-1/6-e_2} + \sqrt{-1/6-e_3} \end{aligned}$$

Use of the trigonometric solution for roots of cubic polynomials gives

$$(3.2)\qquad \begin{aligned} e_1 &= \sqrt{c/3+1/36}\,\sin(\pi/3-\psi) \\ e_2 &= \sqrt{c/3+1/36}\,\sin\psi \\ e_3 &= -\sqrt{c/3+1/36}\,\sin(\pi/3+\psi) \end{aligned}$$

where the angle ψ is determined by

$$(3.3)\qquad \sin 3\psi = -(c/6-b^2/16-1/216)/(c/3+1/36)^{3/2}.$$

We note here that e_1 and e_2 coalesce as ψ tends to $\pi/6$, and that the e_i's are real if ψ is real with $\psi\in[-\pi/6,\pi/6]$. We will see that if ψ tends to $\pi/6$ from the left, then the corresponding point (b,c) tends to the caustic from above, and that if ψ tends to $\pi/6$ from the right, then (b,c) tends to the caustic from below (here, the caustic is precisely the caustic of figure

3.2 with y replaced by b, and z replaced by c).

Fortunately, it proves to be unnecessary to use (2.7) to determine the appropriate branches of the square roots appearing in (3.1) [all other square roots are taken with the principal branch unless otherwise stated]. By taking particular values of b and c , and calculating roots for $f_+'=0$, we can compare the formulae (3.1) with various branches (plus or minus the principal branch) of the square roots of $-1/6-e_i$ to determine what choices are appropriate. Once an appropriate choice of branch is made, we know that it holds everywhere in the bc-plane since the square root determined in (2.7) is always ± the principal branch, the square root being a ratio of entire functions.

Proceeding in this fashion, we take b=1 and c=0. We have, to calculator precision, the following as zeroes of $f_+'(t;1,0)$:

$$0, \qquad -0.6823278, \qquad 0.3411639 \pm 1.1615414i.$$

Calculation of the e_i's gives (again, to calculator precision):

$$(-1/6-e_1)^{1/2}=-0.17058195-0.5807707i$$
$$(-1/6-e_2)^{1/2}=-0.17058195+0.5807707i$$
$$(-1/6-e_3)^{1/2}=0.3411639.$$

These roots are taken with their principal values. By taking ± each of these values in the formulae (3.1), we see that only the following choices of branches yield all the zeroes of $f_+'(t;1,0)$:

- *the first two roots are chosen with the principal branch, the remaining root chosen as the negative of the principal branch*
- *the first and last roots are taken with the principal branch, the second root being the negative of the principal branch*
- *the second and third roots are taken with the principal branch, the first root chosen to be the negative of the principal branch*
- *all roots are chosen to be the negative of the principal branch.*

A closer look at these four possibilities reveals that they correspond to a mere relabelling of the t_i - hence, we can choose any of the four possible choices listed. <u>In this work, we will choose the first candidate</u>.

Recall (2.2) and (2.5). With $g_2 = c + 1/12$, $g_3 = c/6 - b^2/16 - 1/216$, we see that the discriminant δ is given by

$$\delta = (c + 1/12)^3 - 27(c/6 - b^2/16 - 1/216)^2;$$

δ is clearly continuous in b and c. At points below the caustic (say, (b, c) = (1, 0)), we see that $\delta < 0$; equivalently, $27g_3^2/g_2^3 > 1$. The ratio $27g_3^2/g_2^3$ is the square of sin 3ψ in (3.3), so that evaluation at (b, c) = (1, 0) implies $\sin 3\psi = 1 + 27/2$. This latter equation further implies that ψ is of the form of $\pi/6 + i\tau$, τ real (this is in fact the case everywhere $\delta < 0$, $b \geq 0$).

Use of this in (3.2) shows that e_1 and e_2 are complex conjugates with only $e_3 = -(c/3 + 1/36)^{1/2}\cosh\tau$ remaining real.

Our choice of square roots for (3.1) then implies that $\sqrt{(-1/6 - e_1)}$ and $\sqrt{(-1/6 - e_2)}$ are always complex conjugates when $\delta < 0$. Thus, only t_0 and t_3 can be real for negative δ. To establish the ordering of t_0 and t_3, we note that evaluation at (b, c) = (1, 0) implies $t_0 = -0.6823278$ and $t_3 = 0$. Thus, $t_0 < t_3$ whenever $\delta < 0$.

This ordering cannot change while $\delta < 0$, for if t_0 were to become greater than t_3 at some point (b, c), then the sign of δ would have to change. Geometrically, t_0 and t_3 vary as (b, c) do, with $t_0 = t_3$ happening only when the point (b, c) "touches" the caustic.

When $\delta = 0$, we have $g_2^3 = 27g_3^2$, or, in view of (3.3), sin $3\psi = \pm 1$. Once again, evaluation at a convenient point, say (b, c) = (0, 0), shows that $\sin 3\psi = +1$ on the caustic whence $\psi = \pi/6$.

If $\psi \to \pi/6^+$, then e_1 and $e_2 \to (1/2)\sqrt{(c/3 + 1/36)}$, with e_1 and e_2 complex conjugates. Thus, by continuity, we see that $\sqrt{(-1/6 - e_1)}$ and $\sqrt{(-1/6 - e_2)}$ tend to purely imaginary numbers which are conjugate. The expressions $\pm[\sqrt{(-1/6 - e_1)} + \sqrt{(-1/6 - e_2)}]$ therefore tend to zero as $\psi \to \pi/6^+$. From (3.1), we see that t_0 and t_3 coincide on the caustic $\delta = 0$ (see (3.5) below).

For $\delta > 0$, the point (b, c) lies above the caustic and all roots of $f_+' = 0$ are members of complex conjugate pairs.

A more extensive treatment of the behaviour of $t_0$and t_3 is presented in § 5.

The non-uniform asymptotic behaviour of I_+ is readily available now. When δ is strictly less than zero, we have for $\lambda \to +\infty$,

$$\text{(3.4)} \qquad I_+(\lambda; b, c) \sim \sqrt{\frac{2\pi}{\lambda}} \Big\{ e^{i\lambda f_+(t_0; b, c) - \pi i/4} |f_+''(t_0)|^{-1/2} + e^{i\lambda f_+(t_3; b, c) + \pi i/4} (f_+''(t_3))^{-1/2} \Big\} .$$

When $\delta = 0$, we find that

$$\text{(3.5)} \qquad t_0 = t_3 = -((c/3 + 1/36)^{1/2} - 1/6)^{1/2}$$

and

$$\text{(3.6)} \qquad I_+ \sim 2^{1/3} 3^{-1/6} \Gamma(1/3) (\lambda |f_+'''(t_0)|)^{-1/3} \exp(i\lambda f_+(t_0))$$

as $\lambda \to +\infty$. For $\delta > 0$, the integral is of exponentially small order. We have, in this case,

$$\text{(3.7)} \qquad I_+ \sim (2\pi)^{1/2} \{\lambda |f_+''(t_0)|\}^{-1/2} \exp(i\lambda f_+(t_0) + i[\pi/2 - \arg f_+''(t_0)]/2)$$

as $\lambda \to +\infty$ [note: $\operatorname{Im} f_+(t_0) > 0$ in (3.7)]. The real part of $if_+(t_0)$ is negative. The saddle point t_3 makes no contribution here as the topography of if_+ does not permit one to join t_0 and t_3 by a path contained in valleys relative to the two saddles; see figure 3.5 [the illustration was constructed using the values $b=0$, $c=1/16$]. Alternatively, one can see that I_+ is of order $O(\lambda^{-n})$ for any positive integer n by repeated partial

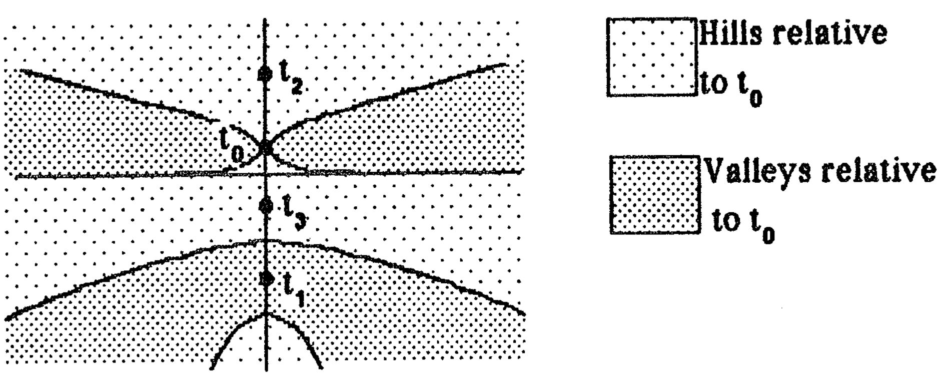


Figure 3.5. The topography of if_+ relative to the saddle t_0.

integration.

Since the two real saddles coalesce as δ tends to 0 from the left, and two complex saddles coalesce as δ tends to 0 from the right, we apply the cubic transformation of Chester *et al* [CFU]:

(3.8) $$f_+(t;b,c)=z^3/3-\zeta z+\eta.$$

We want saddle points of f_+ to correspond to saddle points of the right hand side of (3.8). Thus we want t_0 and t_3 to correspond to $-\zeta^{1/2}$ and $\zeta^{1/2}$ respectively. Therefore,

$$f_+(t_0)=2\zeta^{3/2}/3+\eta, \qquad f_+(t_3)=-2\zeta^{3/2}/3+\eta$$

whence

(3.9) $$\begin{aligned}4\zeta^{3/2}/3&=f_+(t_0)-f_+(t_3)\\ 2\eta&=f_+(t_0)+f_+(t_3).\end{aligned}$$

Note that ζ and η are functions of b and c only.

We consider the mapping (3.8). Use of the trigonometric solution of cubic equations provides three possible candidates for the uniformly analytic 1 - 1 solution to (3.8):

$$\begin{aligned}z_1&=-2\zeta^{1/2}\sin(\pi/3+\phi)\\ z_2&=2\zeta^{1/2}\sin\phi\\ z_3&=2\zeta^{1/2}\sin(\pi/3-\phi)\end{aligned}$$

with

(3.10) $$\sin 3\phi = 3(\eta - f_+)/2\zeta^{3/2};$$

in the event that $\zeta = 0$, ie., $t_0 = t_3$, then (3.8) gives $z = (3(f_+-\eta))^{1/3}$ and again there are 3 branches to consider.

When $t = t_0$, we find

$$\eta - f_+(t_0) = [f_+(t_3) - f_+(t_0)]/2$$

and when $t = t_3$,

$$\eta - f_+(t_3) = [f_+(t_0) - f_+(t_3)]/2.$$

Accordingly, we find

$$\begin{array}{ll} z_1(t_0)=-\zeta^{1/2} & z_1(t_3)=-2\zeta^{1/2} \\ z_2(t_0)=-\zeta^{1/2} & z_2(t_3)=\zeta^{1/2} \\ z_3(t_0)=2\zeta^{1/2} & z_3(t_3)=\zeta^{1/2} \end{array}$$

so that $z_2(t)$ is the desired uniformly analytic 1 - 1 solution we require. Set $z(t)=z_2(t)$.

We can study this mapping much as was done in the case of the Pearcey integral by introducing an intermediate variable T, so that we examine the sequence of mappings

$$\begin{aligned} T &= f_+(t;b,c) \\ \sin 3\phi &= 3(\eta - T)/2\zeta^{3/2} \\ z &= 2\zeta^{1/2}\sin \phi. \end{aligned} \tag{3.11}$$

With the appropriate uniformly analytic 1 - 1 solution of the cubic transformation in hand, we have, from [CFU],

$$I_+(\lambda; b, c) = \int_C e^{i\lambda(z^3/3 - \zeta z + \eta)} g_0(z)\, dz \tag{3.12}$$

where we have set $g_0 = dt/dz$, and C is some contour, which we will determine in the process of analyzing the conformal map $z = z(t)$. At the moment, we need only know that C begins at $\infty\exp(5\pi i/6)$ and ends at $\infty\exp(\pi i/6)$. Along such a contour, the phase function in the integral of (3.12) is exponentially decaying.

Define the function sequences

$$\begin{aligned} p_n(b,c) &= \{ g_n(\zeta^{1/2};b,c) + g_n(-\zeta^{1/2};b,c) \}/2 \\ q_n(b,c) &= \{ g_n(\zeta^{1/2};b,c) - g_n(-\zeta^{1/2};b,c) \}/(2\zeta^{1/2}) \end{aligned} \tag{3.13}$$

$$g_n(z;b,c) = p_n(b,c) + q_n(b,c)z + (z^2 - \zeta)h_n(z;b,c) \tag{3.13 con't}$$
$$g_{n+1}(z;b,c) = h_n'(z;b,c).$$

With judicious partial integration and the use of formulae (3.13), we obtain the uniform asymptotic expansion

$$I_+(\lambda; b, c) \sim e^{i\lambda\eta} \sum_{n=0}^{\infty} \left(\frac{i}{\lambda}\right)^n [p_n(b, c)F(\lambda; \zeta) + q_n(b, c)G(\lambda; \zeta)] \tag{3.14}$$

as $\lambda \to +\infty$. The functions F and G are those of chapter II; cf. equations (2.3.8) and (2.3.9).

Restoring the large parameter x and the function f_+ yields as $x \to +\infty$,

$$Q(x, y, z) \sim \frac{2\pi}{x^{1/3}} \exp\left(ix^{5/2}[f_+(t_0) + f_+(t_3)]/2\right) \bullet$$
$$\sum_{n=0}^{\infty}\left(\frac{i}{x^{5/2}}\right)^n [p_n(yx^{-3/2}, zx^{-2})Ai(-x^{5/3}\zeta) - \frac{i}{x^{5/6}} q_n(yx^{-3/2}, zx^{-2})Ai'(-x^{5/3}\zeta)] \tag{3.15}$$

uniformly valid in a band of fixed positive width containing the caustic $\delta=0$ in the bc-plane.

4. Conformal Mapping

In order to determine the image C of our original integration contour under the mapping (3.8), we shall have to explore the nature of the cubic transformation more closely. A useful device to this end is that exploited in chapter II, that of determining all curves in the t-plane mapped to the real T-axis.

By selecting representative values of (b,c) for each of the cases $\delta < 0$, $\delta=0$, and $\delta > 0$, and setting the imaginary part of $f_+(t;b,c)$ equal to zero, we obtain the curves displayed in figures 3.6 to 3.8 respectively. We note that even though the illustrations provided are for fixed values of b and c, the qualitative behaviour of the level curves displayed will be largely unaffected by variations of the parameters (b,c) unless the sign of δ changes.

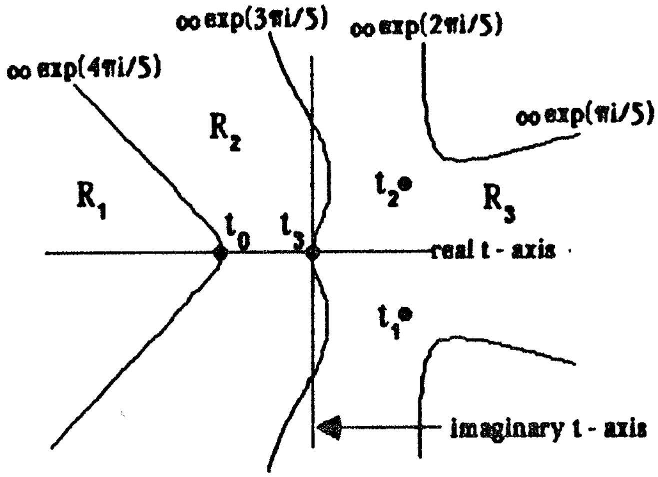


Figure 3.6. t-plane curves that $t \to f_+(t; b, c)$ maps to the real axis ($\delta < 0$, $b = 1$, $c = 0$).

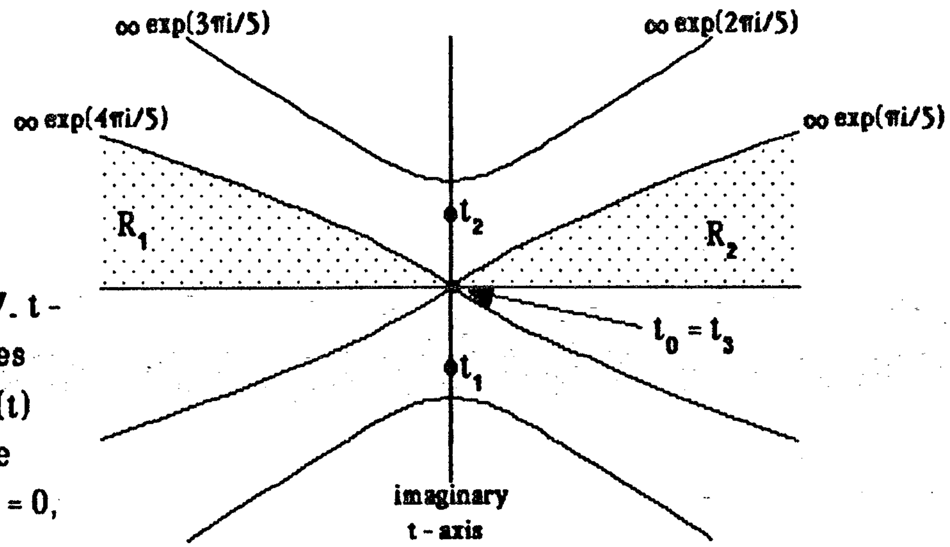
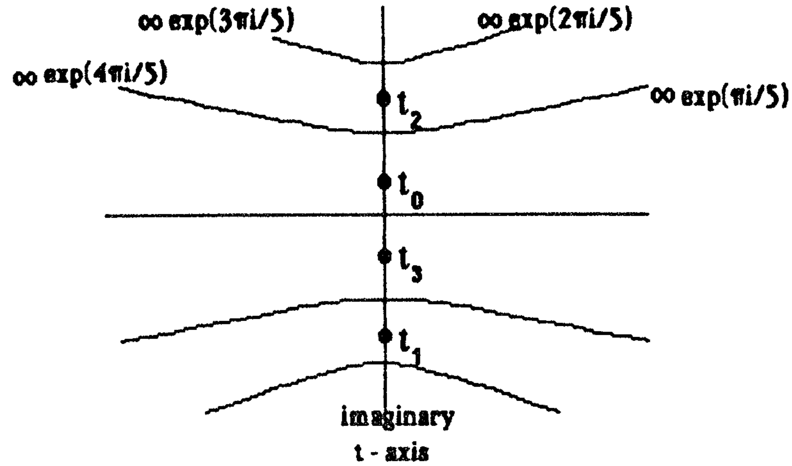


Figure 3.7. t-plane curves that $t \to f_+(t)$ maps to the real axis ($\delta = 0$, $b = c = 0$).

We shall devote considerable attention to the case where δ is negative, and only sketch the (comparably) simpler cases remaining.

Refer to the illustrations in figures 3.9 and 3.10. Note that there are three regions in the upper half of the t-plane touching the real t-axis obtained by partitioning the t-plane by the curves $\mathrm{Im}\, f_+ = 0$. Under the map $T=f_+(t)$, the region R_1 is mapped to the upper half of the T-plane, with the steepest descent curve from t_0 to $\infty\exp(9\pi i/10)$ being sent to the vertical ray in the T-plane beginning at $T(t_0)$ and ending at $T(t_0)+i\infty$. The region labelled R_2 is sent to the lower half of the T-plane, with the steepest ascent curve from t_0 to $\infty\exp(7\pi i/10)$ being sent to the vertical

Figure 3.8. t - plane curves that $t \to f_+(t)$ maps to the real axis ($\delta > 0$, $b = 0$, $c = 1/16$).



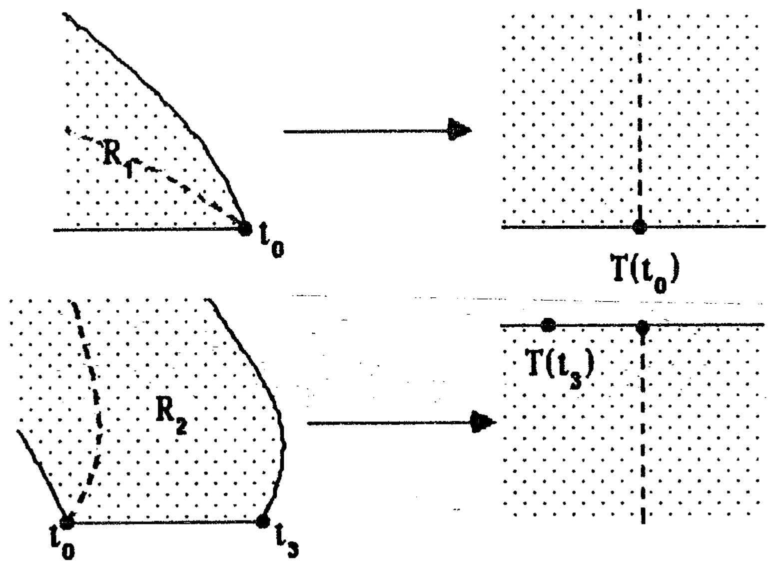


Figure 3.9. Effect of the map $t \to f_+(t)$ between the t- and T- planes. Steepest paths are shown as broken curves.

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ray in the T - plane beginning at $T(t_0)$, and ending at $T(t_0) - i\infty$.

Thus far, the mapping is straightforward, mimicking that of the map studied for the Pearcey integral in chapter II. However, when we encounter the region R_3, we must deal with the presence of a saddle point t_2 in the interior of R_3. As we sweep t from $+\infty$ counterclockwise to the curve $\operatorname{Im} f_+(t) = 0$ at $\infty\exp(\pi i/5)$, then along $\operatorname{Im} f_+(t) = 0$ to $\infty\exp(2\pi i/5)$,

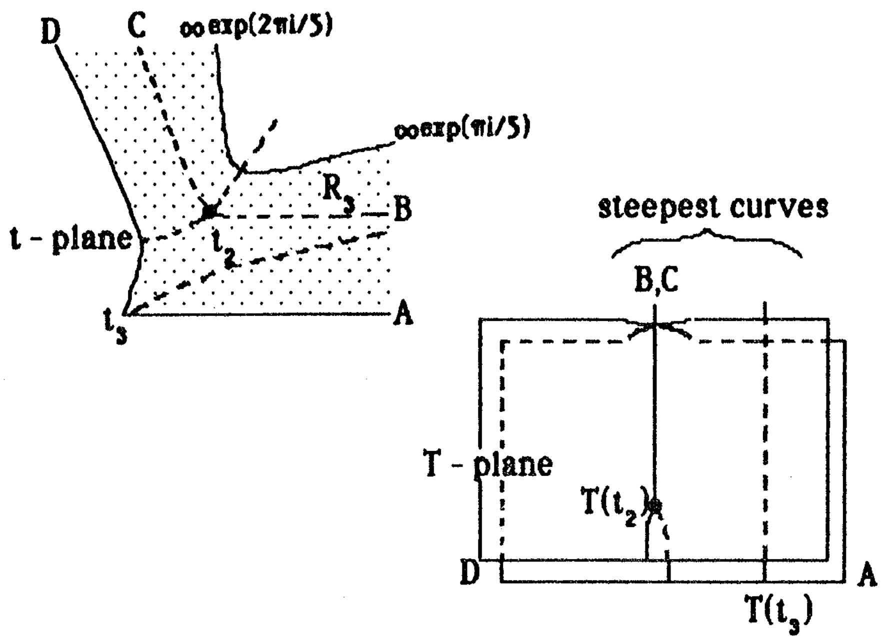


Figure 3.10. The map from the t - plane to the T - plane. Broken curves in R_3 represent steepest paths. Their images in the T - plane are shown as vertical rays.

and finally to the leftmost upper extent of R_3, we see that we have covered the upper half of the T-plane twice under the mapping $T = f_+(t)$. Thus, we shall have to subdivide R_3 further to explore the effects of our change of variables (3.8).

To this end, we consider the steepest ascent and descent curves running through t_2. The steepest descent curve through t_2 begins at $i\infty$, passes through t_2, and then continues on to $\infty\exp(\pi i/10)$. In figure 3.10, this is displayed as a broken curve beginning at C, passing through t_2, and thence on to B. The steepest ascent curve through t_2 is the only other broken path in figure 3.10 coincident with t_2. It begins in R_2 (see figure 3.9) at $\infty\exp(7\pi i/10)$, crosses the boundary shared by R_2 and R_3, continues through t_2, and then passes out of the upper right boundary of R_3 to terminate at $\infty\exp(3\pi i/10)$. The remaining broken curve in the t-plane portion of figure 3.10 represents the steepest descent curve from t_3 to $\infty\exp(\pi i/10)$.

Under the map $T = f_+(t)$, the steepest descent curve from t_3 to $\infty\exp(\pi i/10)$ is mapped to the vertical ray beginning at $T(t_3)$ and ending at $T(t_3) + i\infty$. The steepest descent path from t_2 to $\infty\exp(\pi i/10)$ is mapped to the vertical ray from $T(t_2)$ to $T(t_2) + i\infty$, as is the descent curve from t_2 to $i\infty$.

The steepest ascent curve from t_2 to $\infty\exp(7\pi i/10)$ is sent to the vertical ray running from $T(t_2)$ to $T(t_2) - i\infty$, as is the ascent curve from t_2 to $\infty\exp(3\pi i/10)$. Since only finite portions of these curves lie within R_3, the images of those portions correspond to the line segments running from $T(t_2)$ to the real T-axis (displayed as AD) shown in the T-plane section of figure 3.10.

The image of the region R_3 can be visualized as two copies of the upper half of the T-plane, which we have cut along the ray from $T(t_2)$ to $T(t_2)+i\infty$. The preimage of this cut is the steepest descent curve passing through t_2.

We will restrict our attention to the subregion of R_3 between the steepest descent curve through t_2 and the lower and left-hand boundaries of R_3 (see figure 3.11). Call this subregion R_3^*.

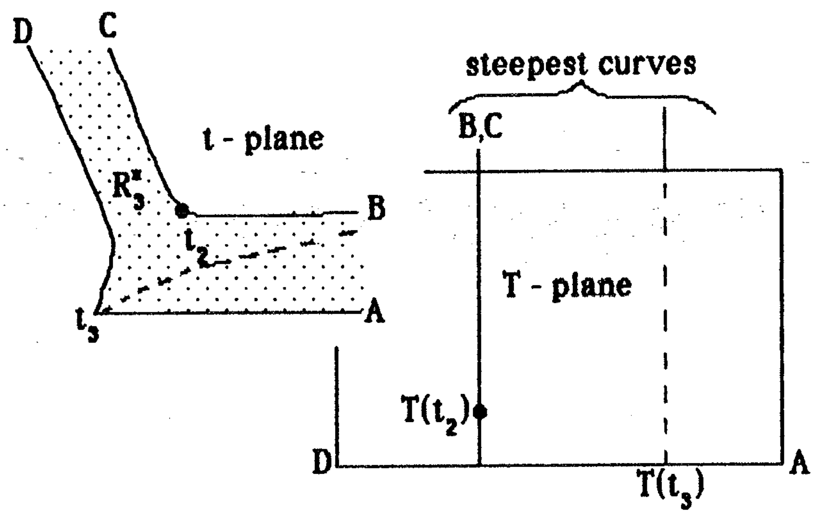


Figure 3.11. Image of R_3^* under the mapping $t \to f_+(t)$.

Examining the image of $T(R_3^*)$ under the mapping $\phi = \phi(T)$ given in equation (3.11) is most easily accomplished by looking at the inverse map $T = T(\phi)$. The point $\phi = \pi/6$ is sent to $T(t_3)$ in view of relation (3.9); the ray $\pi/6 + i\tau$, $\tau > 0$, is sent to the real T-axis to the left of $T(t_3)$. Again, use of (3.9) shows that the point $T(t_0)$ (a point to the right of $T(t_3)$) is sent to $\pi/2$, and the ray $\pi/2 + i\tau$, $\tau > 0$, is sent to the real T-axis to the right of $T(t_0)$. The image of the ray that begins at $T(t_3)$ and goes off to $T(t_3) + i\infty$ under $\phi = \phi(T)$ is determined precisely as in chapter II, p. 18. The image of the steepest descent curve through $T(t_2)$ in this setting is displayed in figure 3.12.

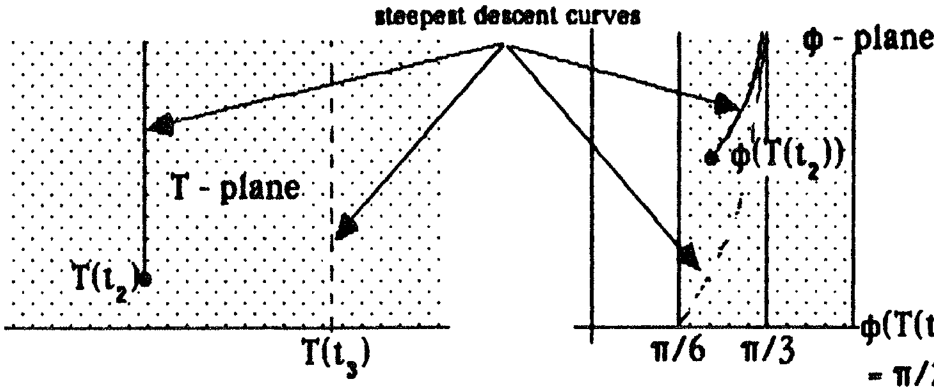


Figure 3.12. Effect of the map from the T- to ϕ- planes.

The final mapping to apply is $z = z(\phi)$ given in equation (3.11). Once again (as in the study of the Pearcey integral), we see that the ray $\pi/6 + i\tau$, $\tau \geq 0$, is sent to the arch of $3x^2 - y^2 = 3\zeta$ in the $z = x + iy$ -plane in the first quadrant; the ray $\pi/2 + i\tau$, $\tau \geq 0$, is sent to the interval $[2\zeta^{1/2}, +\infty[$. The images of the steepest descent curves beginning at $\phi = \pi/6$ and $\phi = \phi(T(t_2))$ are mapped to the steepest descent curves displayed in figure 3.13 (since, under analytic maps, steepest descent curves map to steepest descent curves).

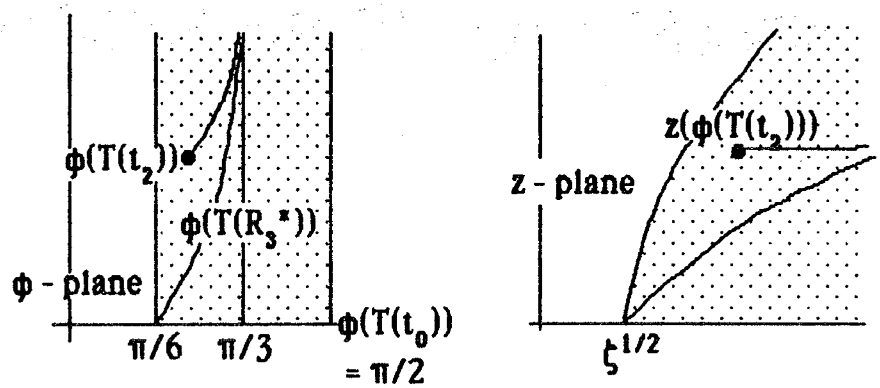


Figure 3.13. Effect of the map $z = z(\phi)$ on $\phi(T(R_3^*))$. Curves within the shaded regions are steepest descent paths.

We treat R_1 and R_2 under the sequence of maps in (3.11) in precisely the same fashion; the resulting analysis for these regions is simpler (there is no saddle point in the interior of either R_1 or R_2), so we only summarize the effects of these maps in figure 3.14.

For $\delta = 0$, the situation becomes less complicated. The curves for which $\operatorname{Im} f_+(t)=0$ are displayed in figure 3.7. Recall that when $\delta = 0$, we are on the caustic - thus, f_+ has a saddle point of order two at $t_0 = t_3$. As there are

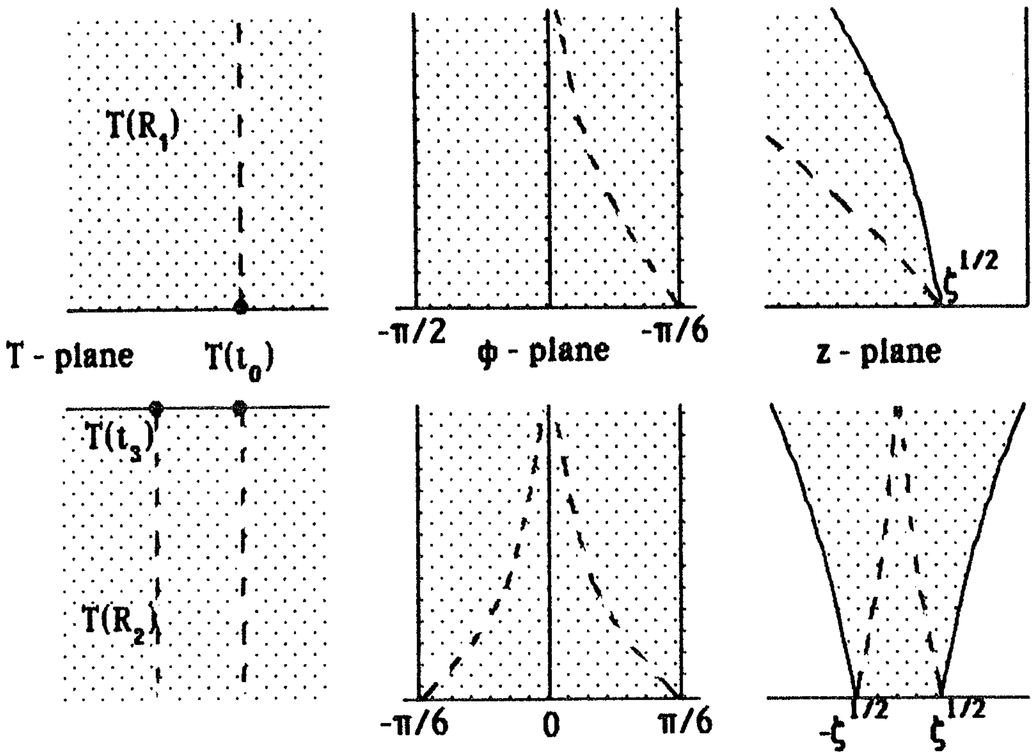


Figure 3.14. Images of R_1 and R_2 under the mappings in (3.11). Steepest curves are shown as broken paths.

only two regions of interest here, we provide a minimal commentary.

The steepest descent curve through the double saddle t_0 begins at $\infty\exp(9\pi i/10)$, passes through t_0, and ends in the valley at $\infty\exp(\pi i/10)$. By examining $z(t) = (\eta - f_+(t))^{1/3}$ and its effect on the regions R_1 and R_2 shaded in figure 3.7, we see that R_1 maps to the region bounded below by the negative real z-axis, and from above by the straight line $y = \sqrt{3}x$ in the second quadrant. The region R_2 maps to the region in the z-plane bounded below by the positive real axis, and above by the straight line segment $y = \sqrt{3}x$ in the first quadrant. The steepest descent curve in the t-plane running through t_0 is sent to the steepest descent curve through the saddle of order two (the origin $z = 0$) of the function $z^3/3 + \eta$, beginning at $\infty\exp(5\pi i/6)$, and ending at $\infty\exp(\pi i/6)$.

When δ is positive, some care must be exercised.

Examine figure 3.8, which shows the curves obtained from the equation $\operatorname{Im} f_+(t)=0$ in the case δ is positive (for the illustrations provided, we are working with $b = 0$, $c = 1/16$). There are six open regions obtained by partitioning the t - plane by these curves, of which only four contain saddle points. Since our integration contour in the t-plane begins at $\infty\exp(9\pi i/10)$ and ends at $\infty\exp(\pi i/10)$, we need only consider the two regions touching the real t-axis. Divide each such region as illustrated in figure 3.15 into two pieces, resulting in the four subregions labelled R_1 to R_4. The image of each R_i in the T-plane is also given in figure 3.15. Rather than examine the intermediate transformation $\phi = \phi(T)$, we shall find it more convenient to study the mapping $T = z^3/3 - \zeta z + \eta$ directly. The curves in the z - plane that map to the real T- axis are presented in figure 3.16. This gives two regions each containing a solitary saddle point. Subdivide the two regions containing saddles into two pieces each by using the imaginary z - axis. This results in four regions R_1' to R_4'. The images of each of these regions under the map $T = T(z)$ is also presented in figure 3.16.

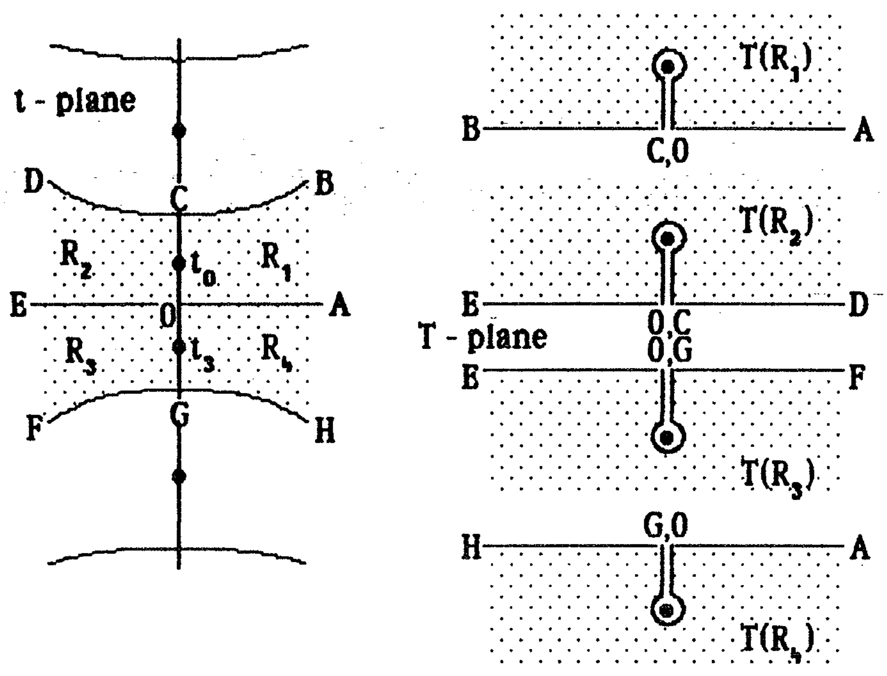
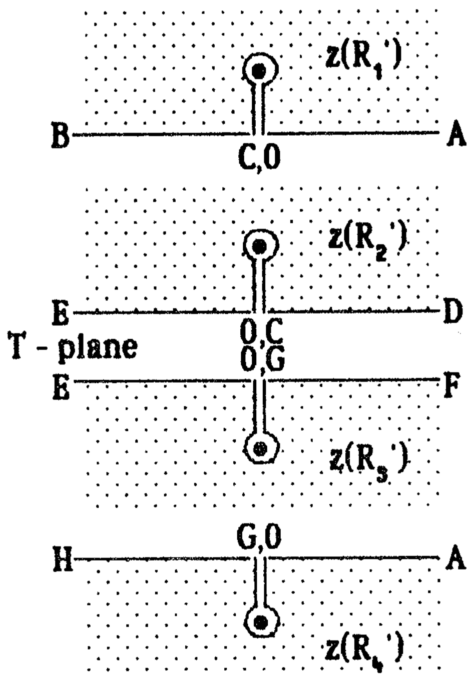


Figure 3.15. T - plane images of the regions R_1 through R_4 in the case $\delta > 0$.

The effect of the uniformly analytic 1 - 1 solution to (3.8) is now apparent: the regions R_i in the T-plane are sent to the regions R_i' in the z-plane. The hyperbola displayed in figure 3.16 is just $y^2 = 3(x^2 - \zeta)$, where $z = x + iy$ (recall that $\zeta < 0$ when $\delta > 0$).

The general behaviour of the solution $z(t)$ of (3.8) (for regions



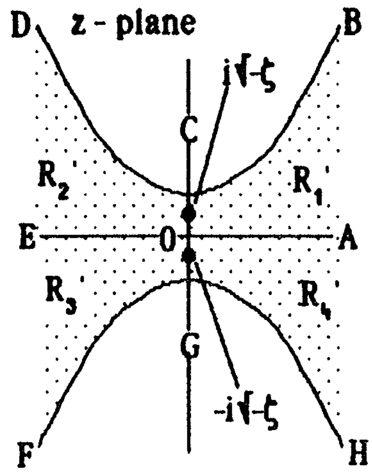


Figure 3.16. T - plane images of the z - plane regions R_1' to R_4' in the case $\delta > 0$.

containing the coalescing saddle points and the integration contour) can now be deduced from our graphical look at the map (3.8). From these considerations, the contour C mentioned in (3.12) can be chosen to be any contour beginning at $\infty\exp(5\pi i/6)$ and ending at $\infty\exp(\pi i/6)$. In particular, if $\delta < 0$, the integration contour for I_+ (cf. equation (1.3)) can be chosen to be the steepest descent curve from $\infty\exp(9\pi i/10)$ to t_0, the straight line segment from t_0 to t_3, and then the steepest descent contour from t_3 to $\exp(\pi i/10)$. The contour C will then be the steepest descent curve from $\infty\exp(5\pi i/6)$ to $-\zeta^{1/2}$, the line segment joining $-\zeta^{1/2}$ and $\zeta^{1/2}$, and then the steepest descent curve from $\zeta^{1/2}$ to $\infty\exp(\pi i/6)$.

In the event that $\delta = 0$, we make the same choice of integration contours, although the line segment joining the saddles has zero length (remember that $\delta=0 \Rightarrow t_0 = t_3$).

For $\delta > 0$, the path of integration for the integral I_+ is chosen to be the steepest descent curve through t_0, and the image of this contour under $z = z(t)$ becomes the steepest descent curve through $(-\zeta)^{1/2}i$ [see, again, figures 3.15 and 3.16].

5. Limiting Forms of the Coefficients

We shall obtain limiting forms for the coefficients p_0 and q_0 appearing in the expansion (3.14) in the limit $\delta \to 0$. Without loss of generality, we can consider δ tending to zero through negative values. The computation is involved, and parallels closely that which was done for the Pearcey integral in the previous chapter. Consequently, we shall be somewhat terser in our presentation, and since the saddles t_0 and t_3 have <u>functions</u> of c as limits (see equation (3.5)), we will not pursue much by way of simplifying the final limiting forms.

We begin by temporarily restricting c_0 to be positive, and we let (b_0, c_0) be a point on the caustic $\delta = 0$ (cf. (2.5); recall that the caustic for I_+ is that of figure 3.2 with y replaced by b, and z replaced by c). Symmetry of the caustic with respect to the c- axis (cf. figure 3.2) allows us to assume, without loss of generality, that $b_0 \geq 0$. With $c_0 > 0$, we may also conclude that $b_0 > 0$. The case where $c_0 = 0$ will be dealt with in the latter part of this section.

Consider $p_0(b_0 + \Delta b, c_0)$ and $q_0(b_0 + \Delta b, c_0)$ for small positive Δb. Then the point $(b_0 + \Delta b, c_0)$ lies in that part of the bc-plane for which f_+ has two real saddles and for which $\delta < 0$. We will obtain the limits of p_0 and q_0 as $(b_0 + \Delta b, c_0)$ tends to (b_0, c_0); ie., as $\Delta b \to 0$.

Recall formulae (3.2). Let

$$C = \{c_0/3 + 1/36\}^{1/2}. \tag{5.1}$$

With the choice of branches made on page 41, we have, from (3.1),

$$t_0 = -\{-1/6 - e_3\}^{1/2} \pm i\{1/6 + e_2\}^{1/2} \mp i\{1/6 + e_1\}^{1/2}. \tag{5.2}$$

Developing $\{1/6 + e_1\}^{1/2}$ and $\{1/6 + e_2\}^{1/2}$ into their Taylor series centred at $\psi=\pi/6$ (note: all square roots in (5.2) are chosen with their principal branches) gives the approximations

$$\{1/6 + e_1\}^{1/2} = E_0-E_1(\psi-\pi/6)-E_2(\psi-\pi/6)^2-E_3(\psi-\pi/6)^3+O((\psi-\pi/6)^4) \tag{5.3}$$
$$\{1/6 + e_2\}^{1/2} =E_0+E_1(\psi-\pi/6)-E_2(\psi-\pi/6)^2+E_3(\psi-\pi/6)^3+O((\psi-\pi/6)^4), \tag{5.4}$$

where

$$E_0 = \{C/2+1/6\}^{1/2}$$

$$E_1 = C\sqrt{3}/(4E_0)$$

(5.5)

$$E_2 = 3C^2/(32E_0^3)+C/(8E_0)$$

$$E_3 = \sqrt{3}(3C^3/(128E_0^5)+C^2/(32E_0^3)-C/(24E_0)).$$

Doing the same for $-\{-1/6-e_3\}^{1/2}$ gives

$$(5.6)\quad -\{-1/6-e_3\}^{1/2} = -\{-1/6+C\}^{1/2}+C/(4[-1/6+C]^{1/2})(\psi-\pi/6)^2+O((\psi-\pi/6)^4).$$

Since $c_0 = 0$ gives $C = 1/6$, we see from (5.6) why we have restricted $c_0 > 0$. Substituting the expressions (5.3), (5.4) and (5.6) into equation (5.2) yields

$$t_0 = -\{-1/6+C\}^{1/2} \pm 2iE_1(\psi-\pi/6)+C/(4[-1/6+C]^{1/2})(\psi-\pi/6)^2 \pm 2iE_3(\psi-\pi/6)^3 + O((\psi-\pi/6)^4);$$

similarly,

$$t_3 = -\{-1/6+C\}^{1/2} \mp 2iE_1(\psi-\pi/6)+C/(4[-1/6+C]^{1/2})(\psi-\pi/6)^2 \mp 2iE_3(\psi-\pi/6)^3 + O((\psi-\pi/6)^4).$$

In order to guarantee that t_0 will be less than t_3 for negative δ, we must take the upper choice of signs in the expressions for t_0 and t_3. This can be seen by first recalling equation (3.3):

$$\sin 3\psi = -(c/6 - b^2/16 - 1/216)/(c/6 + 1/36)^{3/2}.$$

Replacing c by c_0 and b by $b_0 + \Delta b$ yields

$$\sin 3\psi = \frac{-(c_0/6 - b_0^2/16 - 1/216)}{(c_0/3 + 1/36)^{3/2}} + \frac{2b_0\Delta b + (\Delta b)^2}{16(c_0/3 + 1/36)^{3/2}}.$$

Since (b_0, c_0) lies on the caustic $\delta = 0$, we have, from equation (2.5),

$$g_2^3 = 27 g_3^2,$$

and from (2.2) with $E = c_0$, $D = b_0/4$, C(of (2.2)) = 1/6, we have

$$\{c_0 + 1/12\}^3 = 27\{c_0/6 - b_0^2/16 - 1/216\}^2$$

or,

$$\{c_0/3 + 1/36\}^3 = \{c_0/6 - b_0^2/16 - 1/216\}^2.$$

Extracting square roots gives

$$c_0/6 - b_0^2/16 - 1/216 = \pm\{c_0/3 + 1/36\}^{3/2};$$

since we have taken $c_0 > 0$ (and consequently $b_0 > 0$), we must take the "-" choice for the square root (to see this, consider what happens when c_0 and b_0 are very small - then the choice of square root is determined by the reduced equation $-1/216 \approx \pm(1/36)^{3/2}$). Thus, we have

$$c_0/6 - b_0^2/16 - 1/216 = -(c_0/3 + 1/36)^{3/2},$$

so that the previous expression involving $\sin 3\psi$ becomes

$$\sin 3\psi = 1 + ((2b_0\Delta b + \Delta b^2)/[16(c_0/3 + 1/36)^{3/2}]),$$

with the quotient in parentheses being positive. Inverting the sine function gives the result: $\psi = \pi/6 + i\tau$, $\tau > 0$. Substituting this into the Taylor approximations of t_0 and t_3 developed in powers of $(\psi - \pi/6)$ yields the appropriate choice of sign in those expressions.

For notational convenience, we write

$$\begin{aligned} t_0 &= a_0 + a_1(\psi - \pi/6) + a_2(\psi - \pi/6)^2 + a_3(\psi - \pi/6)^3 + O((\psi - \pi/6)^4) \\ t_3 &= a_0 - a_1(\psi - \pi/6) + a_2(\psi - \pi/6)^2 - a_3(\psi - \pi/6)^3 + O((\psi - \pi/6)^4), \end{aligned} \tag{5.7}$$

with

$$
(5.8)\qquad
\begin{aligned}
a_0 &= -(C - 1/6)^{1/2}\\
a_1 &= 2iE_1\\
a_2 &= C/(4[C - 1/6]^{1/2})\\
a_3 &= 2iE_3.
\end{aligned}
$$

To calculate $f_+(t_0) - f_+(t_3)$, needed in the expression for ζ, we expand f_+ about the coalesced real root, say t^*, given in equation (3.5) [notice that $t^* = a_0$]. We have

$$
\begin{aligned}
&f_+(t_0; b_0+\Delta b, c_0) - f_+(t_3; b_0+\Delta b, c_0)\\
&\quad = f_+(t^*) + \Delta b t^*(t_0 - t^*) + (\Delta b/2)(t_0 - t^*)^2 + [(12a_0^2+2)/6](t_0 - t^*)^3\\
&\qquad\qquad + a_0(t_0 - t^*)^4 + (1/5)(t_0 - t^*)^5\\
&\quad -[f_+(t^*) + \Delta b t^*(t_3 - t^*) + (\Delta b/2)(t_3 - t^*)^2 + [(12a_0^2+2)/6](t_3 - t^*)^3\\
&\qquad\qquad + a_0(t_3 - t^*)^4 + (1/5)(t_3 - t^*)^5]
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta b\, a_0(t_0 - t_3) + (\Delta b/2)[(t_0 - t^*)^2 - (t_3 - t^*)^2]\\
&\quad + [(6a_0^2+1)/3][(t_0 - t^*)^3 - (t_3 - t^*)^3]\\
&\quad + a_0[(t_0 - t^*)^4 - (t_3 - t^*)^4] + (1/5)[(t_0 - t^*)^5 - (t_3 - t^*)^5].
\end{aligned}
$$

Upon application of equations (5.7), this becomes

$$
\begin{aligned}
f_+(t_0) - f_+(t_3) &= \Delta b\, a_0[2a_1(\psi - \pi/6) + 2a_3(\psi - \pi/6)^3 + O((\psi - \pi/6)^4))]\\
&\qquad + (\Delta b/2)[4a_1a_2(\psi - \pi/6)^3 + O((\psi - \pi/6)^5)]\\
&+ ((12a_0^2+2)/6)[2a_1^3(\psi - \pi/6)^3 + 2(3a_1a_2^2 + 3a_1^2a_3)(\psi - \pi/6)^5\\
&\qquad + O((\psi - \pi/6)^6)]\\
&+ 8a_0a_1^3a_2(\psi - \pi/6)^5 + (2/5)a_1^5(\psi - \pi/6)^5 + O((\psi - \pi/6)^6).
\end{aligned}
$$

As we are concerned with the limit $\Delta b \to 0$, our next task is to express this in terms of Δb. From the argument preceding (5.7), which gives the form of ψ for $(b_0 + \Delta b, c_0)$ near the caustic, we know that

(5.9) $$\sin 3\psi = 1 + \Delta b(2b_0 + \Delta b)/(16C^3).$$

Use of the approximation

$$\sin^{-1}(1 - z) = \pi/2 - (2z)^{1/2}[1 + z/12 + O(z^2)]$$

for small z coupled with (5.9) gives

$$\psi - \pi/6 = (i/3)[b_0/(4C^3)]^{1/2}(\Delta b)^{1/2}\{1 + [(1/(4b_0)) - (b_0/(96C^3))]\Delta b + O(\Delta b^2)\}$$

valid for small Δb. The choice of the square root is made by recalling that $\psi = \pi/6 + i\tau$ with $\tau > 0$. Let β and β_i be those functions for which

(5.10) $$(\psi - \pi/6)^j = \beta^j(\Delta b)^{j/2}(1 + \beta_j\Delta b + O(\Delta b^2)).$$

In particular, we have

(5.11) $$\beta = (i/3)(b_0/(4C^3))^{1/2}$$

and

(5.12) $$\beta_1 = 1/(4b_0) - b_0/(96C^3).$$

Substituting (5.10) into the above expression for $f_+(t_0) - f_+(t_3)$ gives

$$\begin{aligned} f_+(t_0) - f_+(t_3) = \Delta b^{3/2}\{&[2a_0a_1\beta + ((6a_0^2+1)/3)2a_1^3\beta^3] \\ &+ (2a_0a_1\beta\beta_1 + 2a_0a_3\beta^3 + 2a_1a_2\beta^3 + 8a_0a_1^3a_2\beta^5 + (2/5)a_1^5\beta^5 \\ &+ ((6a_0^2+1)/3)[2a_1^3\beta^3\beta_3 + 2(3a_1a_2^2 + 3a_1^2a_3)\beta^5]\}\,\Delta b \\ &+ O(\Delta b^{3/2})\}, \end{aligned}$$

where we have retained only those terms of order less than Δb^3. Under the assumption that $[2a_0a_1\beta + ((6a_0^2+1)/3)2a_1^3\beta^3]$ is nonzero, we get

$$(5.13) \quad f_+(t_0) - f_+(t_3) = \Delta b^{3/2}[2a_0a_1\beta + ((6a_0^2+1)/3)2a_1^3\beta^3] \\ \{1 + \text{mess}\,\Delta b + O(\Delta b^{3/2})\},$$

where

$$(5.14) \quad \text{mess} = \{2a_0a_1\beta\beta_1 + 2a_0a_3\beta^3 + 2a_1a_2\beta^3 + 8a_0a_1^3a_2\beta^5 + (2/5)a_1^5\beta^5$$

$$+ ((6a_0^2+1)/3)[2a_1^3\beta^3\beta_3 + 2(3a_1a_2^2 + 3a_1^2a_3)\beta^5]\}/[2a_0a_1\beta + ((6a_0^2+1)/3)2a_1^3\beta^3].$$

The denominator of "mess" can never be zero, for, as we shall see in (5.15) below, this would imply that ζ could vanish off the caustic. Such an occurrence violates the uniform nature of the cubic transformation; ie., (b_0, c_0) is on the caustic $\Longleftrightarrow \zeta = 0$.

From (5.13), we find

$$(5.15) \quad \zeta^{1/2} = (3/4)^{1/3}(\Delta b)^{1/2}\{2a_0a_1\beta + ((6a_0^2+1)/3)2a_1^3\beta^3\}^{1/3} \\ \{1 + (\text{mess}/3)\Delta b + O((\Delta b)^{3/2})\}$$

as Δb tends to zero.

Developing f_+'' about t^* gives

$$f_+''(t) = \Delta b + (12a_0^2 + 2)(t - t^*) + 12a_0(t - t^*)^2 + 4(t - t^*)^3,$$

so that evaluating $f_+''(t)$ at t_0, and subsequently replacing t_0 by its Taylor expansion in powers of $(\psi - \pi/6)$ gives

$$f_+''(t_0) = \Delta b + (12a_0^2 + 2)(a_1(\psi - \pi/6) + a_2(\psi - \pi/6)^2 + a_3(\psi - \pi/6)^3 \\ + O((\psi - \pi/6)^4))$$

$$+ 12a_0(a_1^2(\psi - \pi/6)^2 + 2a_1a_2(\psi - \pi/6)^3 + O((\psi - \pi/6)^4))$$

$$+ 4(a_1^3(\psi - \pi/6)^3 + O((\psi - \pi/6)^4)).$$

Use of (5.10) gives

$$(5.16) \qquad f_+''(t_0) = (\Delta b)^{1/2}[(12a_0^2 + 2)a_1\beta + (1 + (12a_0^2 + 2)a_2\beta^2 + 12a_0a_1^2\beta^2)(\Delta b)^{1/2} + O(\Delta b)],$$

where only terms of order less than $(\Delta b)^{3/2}$ have been retained. Similarly, we have

$$(5.17) \qquad f_+''(t_3) = (\Delta b)^{1/2}[-(12a_0^2 + 2)a_1\beta + (1 + (12a_0^2 + 2)a_2\beta^2 + 12a_0a_1^2\beta^2)(\Delta b)^{1/2} + O(\Delta b)]$$

as Δb tends to zero. Hence, (5.15) and (5.16) together yield

$$(5.18) \qquad \begin{aligned} -2\zeta^{1/2}/f_+''(t_0) &= 2(3/4)^{1/3}(2a_0a_1\beta + ((6a_0^2 + 1)/3)2a_1^3\beta^3)^{1/3} \\ &\quad \cdot[-(12a_0^2 + 2)a_1\beta]^{-1} \\ &\quad \cdot\{1 - [1 + (12a_0^2 + 2)a_2\beta^2 + 12a_0a_1^2\beta^2][(12a_0^2 + 2)a_1\beta]^{-1}(\Delta b)^{1/2} + O(\Delta b)\}, \end{aligned}$$

and (5.15) paired with (5.17) gives

$$(5.19) \qquad \begin{aligned} 2\zeta^{1/2}/f_+''(t_3) &= 2(3/4)^{1/3}(2a_0a_1\beta + ((6a_0^2 + 1)/3)2a_1^3\beta^3)^{1/3} \\ &\quad \cdot[-(12a_0^2 + 2)a_1\beta]^{-1} \\ &\quad \cdot\{1 + [1 + (12a_0^2 + 2)a_2\beta^2 + 12a_0a_1^2\beta^2][(12a_0^2 + 2)a_1\beta]^{-1}(\Delta b)^{1/2} + O(\Delta b)\} \end{aligned}$$

as Δb tends to zero. This yields, finally,

$$(5.20) \qquad \begin{aligned} p_0 &= (1/2)([-2\zeta^{1/2}/f_+''(t_0)]^{1/2} + [2\zeta^{1/2}/f_+''(t_3)]^{1/2}) \\ &= 2^{1/2}(3/4)^{1/6}(2a_0a_1\beta + ((6a_0^2 + 1)/3)2a_1^3\beta^3)^{1/6}\ [-(12a_0^2 + 2)a_1\beta]^{-1/2} \\ &\qquad (1 + O(\Delta b)) \end{aligned}$$

as Δb tends to 0 [cf. equation (3.13) with $n = 0$], and

$$q_0 = (1/2\zeta^{1/2})([2\zeta^{1/2}/f_+''(t_3)]^{1/2} - [-2\zeta^{1/2}/f_+''(t_0)]^{1/2})$$

or,

$$q_0 = -2^{1/2}(3/4)^{-1/6}(2a_0a_1\beta + ((6a_0^{\,2} + 1)/3)2a_1^{\,3}\beta^3)^{-1/6}$$

$$\cdot[-(12a_0^{\,2} + 2)a_1\beta]^{-3/2}[1 + (12a_0^{\,2} + 2)a_2\beta^2 + 12a_0a_1^{\,2}\beta^2] \tag{5.21}$$

$$\cdot(1 + O((\Delta b)^{1/2})$$

upon applying (5.18) and (5.19).

For convenience, we recall equations (5.1), (5.5) (5.8) and (5.11) which allow us to express the functions appearing in (5.20) and (5.21) in terms of b_0 and c_0:

$$C = (c_0/3 + 1/36)^{1/2} \tag{5.1}$$

$$\begin{cases} E_0 = (C/2 + 1/6)^{1/2} \\ E_1 = C\sqrt{3/(4E_0)} \\ E_2 = 3C^2/(32E_0^{\,3}) + C/(8E_0) \\ E_3 = \sqrt{3}(3C^3/(128E_0^{\,5}) + C^2/(32E_0^{\,3}) - C/(24E_0)) \end{cases} \tag{5.5}$$

$$\begin{cases} a_0 = -(C - 1/6)^{1/2} \\ a_1 = 2iE_1 \\ a_2 = C/(4[C - 1/6]^{1/2}) \\ a_3 = 2iE_3 \end{cases} \tag{5.8}$$

$$\beta = (i/3)\sqrt{(b_0/(4C^3))}. \tag{5.11}$$

In the event that c_0 equals 0, the form of the analysis is essentially unchanged, except that we have

$$-(-1/6 - e_3)^{1/2} = -\sqrt{(1/6)[\sin(\pi/3 + \psi) - 1]}$$

$$= -\sqrt{(1/3)[-i(\psi - \pi/6) + (i/24)(\psi - \pi/6)^3 + O((\psi - \pi/6)^5)]},$$

$$t_0 = \sqrt{(1/3)}\, i[(\psi - \pi/6) - (1/24)(\psi - \pi/6)^3 + O((\psi - \pi/6)^4)]$$

and $t_3 = 0$ (clearly, 0 is always a root of $f_+'(t; \Delta b, 0) = t^4 + t^2 + \Delta b t$). The remaining analysis is more streamlined now that $f_+(t_3) = f_+'(t_3) = 0$ and $\sin 3\psi = 1 + (27/2)(\Delta b)^2$. We leave this case to the interested reader.

We mentioned at the beginning of this section that we could, without loss of generality, consider only the limiting behaviour of p_0 and q_0 as δ went to zero through negative values (ie., Δb tended to zero on the right). The essential features of the analysis for δ positive are unchanged. The principal differences are that all e_i's are real, and that the angle ψ, figuring in our work so prominently, is real. This comes at the cost of having all of the roots t_i complex. We elected to work with two real roots instead. Of course, the small δ limit of the coefficients p_0 and q_0 are unaffected by the sign of δ as the coefficients are continous functions of the parameters b and c; indeed, they are analytic : see **[Urs**, § 2, p. 52].

6. The Exceptional Integral J

If we apply the theory of § 2 to the polynomial h'(t; d, e) (cf. (1.2ii)), we see that zeroes of h' are given by

(6.1)
$$\begin{aligned}
t_0 &= \sqrt{-d/6 - e_1} + \sqrt{-d/6 - e_2} + \sqrt{-d/6 - e_3}\\
t_1 &= \sqrt{-d/6 - e_1} - \sqrt{-d/6 - e_2} - \sqrt{-d/6 - e_3}\\
t_2 &= -\sqrt{-d/6 - e_1} + \sqrt{-d/6 - e_2} - \sqrt{-d/6 - e_3}\\
t_3 &= -\sqrt{-d/6 - e_1} - \sqrt{-d/6 - e_2} + \sqrt{-d/6 - e_3} \quad ,
\end{aligned}$$

where

$$
(6.2) \qquad
\begin{aligned}
e_1 &= \surd(e/3 + d^2/36)\sin(\pi/3 - \psi) \\
e_2 &= \surd(e/3 + d^2/36)\sin\psi \\
e_3 &= -\surd(e/3 + d^2/36)\sin(\pi/3 + \psi)
\end{aligned}
$$

and the angle ψ is determined in a fashion similar to that employed in section 3:

$$
(6.3) \qquad \sin 3\psi = -(de/6 - 1/16 - d^3/216)/[e/3 + d^2/36]^{3/2}.
$$

As in section three, the square roots can be found by substituting particular values of (d,e) into formulae (6.1) - (6.3) and seeing what choices of square roots yield all roots of $h' = 0$. Continuity can then be used to extend that choice of square roots over the entire region of interest in the de-plane. Indeed, a suitable choice of square roots is that made for § 3, since we may set b=1 in (1.2i), d = 1 in (1.2ii) and c (of (1.2i)) = e (of (1.2ii)) to yield $f_+=h$.

The analysis of the integral in (1.4) now follows with minor modification from that of I_+, except that d will be a very small number.

The caustic associated with the integral J is presented in figure 3.17 below. Of interest is the region near the e - axis (recall § 1 : the integral J arose from consideration of the behaviour of Q(x,y,z) with x bounded and one of y or z large - thus, d is small). Notice that the point Λ in figure 3.17, which corresponds to 2 saddle points of J, one of order 3, is bounded away from the e - axis.

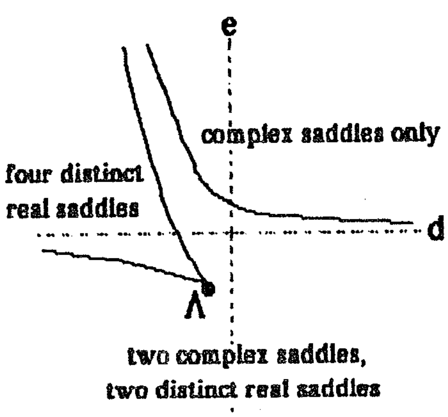


Figure 3.17. The caustic (the solid curves in the illustration) corresponding to the phase function of the integral $J(\lambda; d, e)$.

In the illustration provided, the caustic curves are seen to partition the de-plane into three disjoint open sets: one region contains those values of for which h' has exactly four real, distinct zeroes; one region contains (d,e) for which h' has complex saddles only; the remaining region (lying between the two curves in the plane) contains those values for which h' has one pair of complex conjugate roots, and one pair of distinct real roots.

Because of the similarity in the analysis of the integral J with that of the integral I_+, we only note here that the results of § 3 carry over with little modification to $J(\lambda;\, d,\, e)$, provided we replace the t_i's and the angle ψ of § 3 by those given in equations (6.1) to (6.3) above.

For reasons identical to those presented at the end of chapter II, we note that uniform asymptotic expansions of $\partial Q/\partial x$, $\partial Q/\partial y$ and $\partial Q/\partial z$ can be obtained by termwise differentiation of expansions of Q. For the case of $x \to +\infty$, we differentiate the result given as equation (3.15). When x is bounded and $|y| + |z| \to +\infty$, we would differentiate an analog of (3.15), obtained by performing an analysis of $J(\lambda;\, d,\, e)$ similar to that done for $I_+(\lambda;\, b,\, c)$ in § 3 of the present chapter.

We continue our investigation of $Q(x,\, y,\, z)$ for parameter values near the cusp point.

Chapter Four. The Expansion of Q(x,y,z) [cont'd]

1. Introduction

We continue our investigation into the asymptotic behaviour of the integral given in (3.1.1)., and temporarily restrict our attention to an associated integral similar to that given in (3.1.3); ie., we consider

$$(1.1) \qquad I_-(\lambda; b, c) = \int_{\infty\exp(9\pi i/10)}^{\infty\exp(\pi i/10)} \exp(i\lambda f_-(t; b, c))\, dt\,,$$

where

$$(1.2) \qquad f_-(t; b, c) = t^5/5 - t^3/3 + bt^2/2 + ct.$$

The integral Q(x,y,z) is related to $I_-(\lambda; b, c)$ via the equation

$$(1.3) \qquad Q(-x, y, z) = x^{1/2}\, I_-(x^{5/2}; yx^{-3/2}, zx^{-2}), \qquad x > 0.$$

The case of Q(x, y, z) with $x > 0$ has already been considered in chapter III.

As was done in chapter III, we can examine the saddlepoint structure of I_- by finding those parameter values of (b, c) in (1.2) which yield saddles of various orders. We find, by mimicking the analysis of the function F of § 1, ch. 3, that f_- has saddlepoints of order three when $b = 4/3\sqrt{6}$, $c = -1/12$, and that along three curves in the bc-plane, f_- has saddles of order 2. These curves partition the bc-plane in much the same fashion as the curves in figure 3.4 partition the yz-plane. For convenience, we provide an appropriately modified version of figure 3.4.

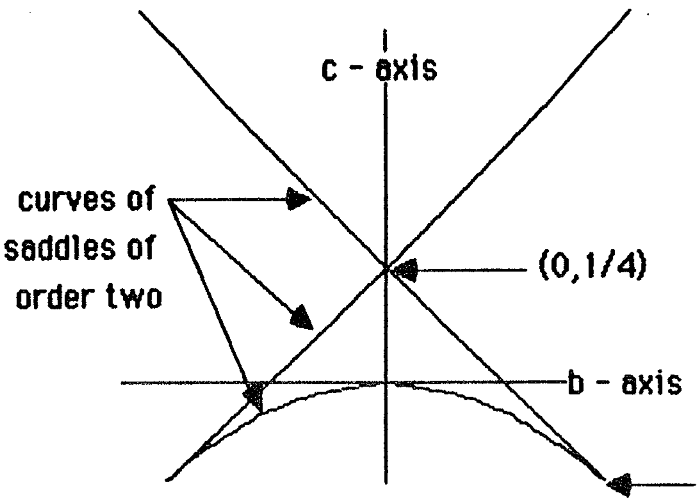


Figure 4.1. Curves in the bc-plane which give rise to saddles of order $\geq$ 2. The region interior to all three curves gives distinct real zeroes of f_-. The saddlepoint behaviour of f_- is similar to that in figure 3.4.

Note the symmetry of the curves in figure 4.1 with respect to the c - axis. For this reason, we may restrict our attention to the case where $b \geq 0$.

We shall devote a considerable amount of study to the problem of finding the asymptotic behaviour of $I_-(\lambda;\, b,\, c)$ for $\lambda \to +\infty$ with (b, c) in various parts of the bc-plane. In particular, we will obtain the large-λ asymptotic behaviour of $I_-(\lambda;b,c)$, uniformly valid in a neighborhood of the cusp point $(4/3\sqrt{6}, -1/12)$. This will be accomplished with the aid of a quartic transformation studied by N. Levinson **[Lev]** in 1961. This, in turn, will yield the large positive-x behaviour of $Q(-x, y, z)$. The large $|y| + |z|$, $|x|$ bounded, behaviour will be examined later in the present chapter.

To proceed further requires an examination of the zeroes of $\partial f_-/\partial t$.

2. The Zeroes of f_-'

With C of (3.2.1) equal to -1/6, D = b/4, and E = c, equations (3.2.2) to (3.2.6) yield

$$
(2.1)\qquad \begin{aligned} g_2 &= c + 1/12 \\ g_3 &= 1/216 - b^2/16 - c/6 \end{aligned}
$$

$$
(2.2)\qquad \begin{aligned} &\wp(2\alpha; g_2, g_3) = 1/6 \\ &\wp'(2\alpha; g_2, g_3) = -b/4, \end{aligned}
$$

so that the zeroes of $\partial f_{-}/\partial t$, subject to an appropriate choice of branches, are given by

$$
(2.3)\qquad \begin{aligned} t_0 &= \sqrt{1/6 - e_1} + \sqrt{1/6 - e_2} + \sqrt{1/6 - e_3} \\ t_1 &= \sqrt{1/6 - e_1} - \sqrt{1/6 - e_2} - \sqrt{1/6 - e_3} \\ t_2 &= -\sqrt{1/6 - e_1} + \sqrt{1/6 - e_2} - \sqrt{1/6 - e_3} \\ t_3 &= -\sqrt{1/6 - e_1} - \sqrt{1/6 - e_2} + \sqrt{1/6 - e_3} \ . \end{aligned}
$$

Here, the e_i are the zeroes of the associated cubic given in (3.2.4). By using the trigonometric solution of cubics, we have

$$
(2.4)\qquad \begin{aligned} e_1 &= (c/3 + 1/36)^{1/2} \sin(\pi/3 - \psi) \\ e_2 &= (c/3 + 1/36)^{1/2} \sin \psi \\ e_3 &= -(c/3 + 1/36)^{1/2} \sin(\pi/3 + \psi), \end{aligned}
$$

where

$$
(2.5)\qquad \sin 3\psi = (c/6 + b^2/16 - 1/216)/[c/3 + 1/36]^{3/2}.
$$

Before proceeding further, we must determine what choices of square roots are appropriate in (2.3) [NB: in the rest of this chapter, a square root is taken to be the principal branch]. As in § 3, Ch. III, this can be determined by calculating zeroes of $\partial f_{-}/\partial t$ ($\equiv f_{-}'$) for some values of (b, c), and comparing the results against the possibilities provided by various choices of branches in formulae (2.3).

Continuing in this fashion, we first take $b = 4/3\sqrt{6}$, $c = -1/12$ so that f_{-}' has a zero of order three. At the triple zero, t^*, we have $f_{-}'(t^*) = f_{-}''(t^*) =$

$f_-'''(t^*) = 0$, so that $f_-'''(t^*) = 12(t^*)^2 - 2 = 0$, or $t^* = \pm 1/\sqrt{6}$. From $f_-'(t) = t^4 - t^2 + (4/3\sqrt{6})t - 1/12$, we see that $t^* = 1/\sqrt{6}$ is the triple zero [since $f_-'(-1/\sqrt{6}) \neq 0$]. With this in hand, it is easy to see that $t = -3/\sqrt{6}$ is the remaining zero of f_-'.

Now, $b = 4/3\sqrt{6}$, $c = -1/12$ gives, from (3.2.4) with g_2 and g_3 as in (2.1), $S = 4s^3$ so that $e_1 = e_2 = e_3 = 0$. Hence (2.3) becomes

$$\begin{aligned}
t_0 &= \pm 1/\sqrt{6} \pm 1/\sqrt{6} \pm 1/\sqrt{6} \\
t_1 &= \pm 1/\sqrt{6} \mp 1/\sqrt{6} \mp 1/\sqrt{6} \\
t_2 &= \mp 1/\sqrt{6} \pm 1/\sqrt{6} \mp 1/\sqrt{6} \\
t_3 &= \mp 1/\sqrt{6} \mp 1/\sqrt{6} \pm 1/\sqrt{6}.
\end{aligned}$$

From this, we see that the only choices of $\pm$ principal branch of each of $(1/6-e_i)^{1/2}$ which can give rise to the roots $1/\sqrt{6}$ and $-3/\sqrt{6}$ are the following:

choice in sign	± principal branch of $\sqrt{(1/6-e_1)}$	$\sqrt{(1/6-e_2)}$	$\sqrt{(1/6-e_3)}$
	−	−	−
	+	+	−
	+	−	+
	−	+	+

We must, in the chart above, choose a row of choices. As was noted for the matter of choosing branches in Ch.III, § 3, these choices are mere permutations of each other. We make the choice + + −, since this is the same choice of branches made in the previous chapter.

With this choice of branches, (2.3) becomes

$$(2.6)\qquad \begin{aligned}
t_0 &= \sqrt{1/6-e_1} + \sqrt{1/6-e_2} - \sqrt{1/6-e_3} \\
t_1 &= \sqrt{1/6-e_1} - \sqrt{1/6-e_2} + \sqrt{1/6-e_3} \\
t_2 &= -\sqrt{1/6-e_1} + \sqrt{1/6-e_2} + \sqrt{1/6-e_3} \\
t_3 &= -\sqrt{1/6-e_1} - \sqrt{1/6-e_2} - \sqrt{1/6-e_3},
\end{aligned}$$

where now all square roots have their principal branches. Before proceeding to order the zeroes t_0, t_1, t_2, t_3, a few observations regarding the e_i should be recorded.

Along the caustic, we have

$$(c/6 + b^2/16 - 1/216)^2 = (c/3 + 1/36)^3$$

since the discriminant δ of the associated cubic must vanish when two or more t_i coincide (cf. equations (3.2.5), (2.1)). Thus, along the caustic (excluding the cusp), (2.5) becomes

$$\sin 3\psi = \pm 1, \qquad \text{or,} \qquad \psi = \pm\pi/6.$$

At the point $(b,c) = (0,0)$, we find $\psi = -\pi/6$, and at the point $(b,c) = (0,1/4)$, we get $\psi = \pi/6$. Thus, along the lower arch of the caustic, $\psi = -\pi/6$, and along the upper arch, $\psi = \pi/6$; see figure 3.1. All level curves $\sin 3\psi = \tau$, $|\tau| \leq 1$, must therefore pass through the cusp. The curve $\psi = -\pi/6$ will prove to be of special interest in later calculations.

We turn now to ordering the t_i. At the point $(b,c) = (0,0)$, we find $t_0 = 0$, $t_1 = 0$, $t_2 = 1$, and $t_3 = -1$ since $\psi = -\pi/6$, $e_1 = 1/6$, $e_2 = e_3 = -1/12$. With $(b,c) = (0,1/4)$, $\psi = \pi/6$, $e_1 = e_2 = 1/6$, $e_3 = -1/3$, we obtain $t_0 = t_3 = -1/\sqrt{2}$ and $t_1 = t_2 = 1/\sqrt{2}$. Finally, at the cusp itself, where $(b,c) = (4/3\sqrt{6}, -1/12)$, we find that all the e_i's vanish since the associated cubic, (3.2.4), reduces to $S = 4s^3$ in view of the fact that g_2 and g_3 vanish; see equation (2.1). In this case, we see that $t_0 = t_1 = t_2 = 1/\sqrt{6}$ and $t_3 = -3/\sqrt{6}$.

Piecing together these values for the t_i's on the caustic leads to the inequality

(2.7) $$t_3 \leq t_0 \leq t_1 \leq t_2 .$$

Since the t_i's are distinct for values of (b,c) inside the caustic, the inequalities in (2.7) must be strict there. Furthermore, we see that the zero t_3 remains isolated from the other zeroes for (b,c) with $b \geq b_0 > 0$ for some b_0.

3. Analysis of I_-

With the description (2.6) of the zeroes of f_- comes the non-uniform asymptotic behaviour of $I_-(\lambda; b, c)$. For (b, c) inside the caustic, we see that f_- has four distinct real saddles, and so stationary phase immediately yields the result

$$(3.1)\quad I_-(\lambda; b, c) \sim \sum_{j=0}^{3} e^{\lambda i f_-(t_j; b, c) + \pi i \,\mathrm{sgn}(f_-''(t_j; b, c))/4} \sqrt{\frac{2\pi}{\lambda |f_-''(t_j; b, c)|}}$$

as λ tends to infinity, with (b, c) fixed. If (b, c) is on the caustic, but not on the cusp, then we have either $t_0 = t_1$ (on the lower arch) or $t_1 = t_2$ (on the upper arch). For the purpose of illustration, assume that $t_0 = t_1$. Then, as λ tends to $+\infty$ with (b, c) fixed, we have

$$(3.2)\quad \begin{aligned} I_-(\lambda; b, c) \sim & \sum_{j=2}^{3} e^{i\lambda f_-(t_j; b, c) + i\pi \,\mathrm{sgn}(f_-''(t_j; b, c))/4} \sqrt{\frac{2\pi}{\lambda |f_-''(t_j; b, c)|}} \\ & + \frac{2^{1/3} 3^{-1/6} \Gamma(1/3) e^{i\lambda f_-(t_0; b, c)}}{(\lambda f_-'''(t_0; b, c))^{1/3}} . \end{aligned}$$

Note that $f_-'''(t_0) \neq 0$, since we have taken (b,c) to be off the cusp.

At the cusp $(b, c) = (4/3\sqrt{6}, -1/12)$, we have the triple zero $t_0 = t_1 = t_2 = 1/\sqrt{6}$, and the simple zero $t_3 = -3/\sqrt{6}$. With this choice of values for (b, c), $I_-(\lambda)$ has the asymptotic behaviour

$$(3.3)\quad \begin{aligned} I_-(\lambda; 4/3\sqrt{6}, -1/12) \sim & \frac{2^{-7/8} 3^{1/8} \Gamma(1/4)}{\lambda^{1/4}} e^{\pi i/8 - i\lambda/45\sqrt{6}} + \\ & 2^{-7/4} 3^{3/4} \sqrt{\frac{\pi}{\lambda}}\, e^{7\lambda i/5\sqrt{6} - \pi i/4} + \frac{2^{3/8} 3^{3/8} \cdot 63 \cdot \Gamma(3/4)}{32 \cdot 25\, \lambda^{3/4}} e^{3\pi i/8 - i\lambda/45\sqrt{6}} \end{aligned}$$

as $\lambda \to +\infty$.

If (b, c) lies outside the caustic, then the asymptotic behaviour of $I_-(\lambda; b, c)$ is given by two terms from (3.1): one term arises from the saddlepoint t_3, the second term being the contribution from the remaining real saddle. The other two saddles form a complex conjugate pair providing at most an exponentially small contribution to the value of $I_-(\lambda)$ due to considerations of the topography of the saddles (cf. the discussion following equation (3.3.7) and figure 3.5).

As we pass from inside the caustic, through the caustic, and then outside, we see that the asymptotic behaviour of I_- is determined by four of the t_i (inside the caustic), two or three of the t_i (on the caustic, possibly at the cusp), or only two t_i. Hence, as two or three of the saddles coalesce as δ tends to zero, we need the quartic transformation used by Ursell **[Urs]**:

$$(3.4) \qquad f_-(t; b, c) = z^4/4 - \zeta z^2/2 + \eta z + \theta.$$

We must first determine the parameters ζ, η, and θ in a fashion which provides for a "local" uniformly analytic 1 - 1 transformation $t \to z$. Thus, the three coalescing saddles of f_- must correspond with the saddles of the right hand side of (3.4). Let $g(z; \zeta, \eta, \theta)$ denote the quartic in (3.4).

From the trigonometric solution of cubic equations, we know that the zeroes of $g'(z; \zeta, \eta, \theta) = z^3 - \zeta z + \eta$ are given by

$$(3.5) \qquad \begin{aligned} z_1 &= 2(\zeta/3)^{1/2} \sin(\pi/3 - \phi) \\ z_2 &= 2(\zeta/3)^{1/2} \sin \phi \\ z_3 &= -2(\zeta/3)^{1/2} \sin(\pi/3 + \phi) \end{aligned}$$

where the angle ϕ is determined by the equation

$$(3.6) \qquad \sin 3\phi = 3^{3/2}\eta/(2\zeta^{3/2}).$$

By examining the functions $\sin(\pi/3 - \phi)$, $\sin\phi$, and $-\sin(\pi/3 + \phi)$ for $\phi \in]-\pi/6, \pi/6[$, we conclude that when the z_j are real, we have $z_3 < z_2 < z_1$.

The quartic g must have $\zeta > 0$ in order to have three real zeroes. Furthermore, the zeroes of g coincide when $(\zeta, \eta) = (0, 0)$.

In view of the inequality (2.7), coupled with the knowledge that t_3 remains isolated from the other t_i, we see that the uniformly analytic 1 -1 solution of (3.4) must preserve the correspondences

$$\begin{aligned} t_2 &\leftrightarrow z_1 \\ t_1 &\leftrightarrow z_2 \\ t_0 &\leftrightarrow z_3. \end{aligned}$$

How shall the parameters ζ, η, and θ be determined? The straightforward approach of substituting the t_i's and their associated z_j's into (3.4) very quickly leads to a mess. A more elegant approach to obtaining our parameters begins with an observation from the classical theory of equations.

If $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is an n^{th} degree polynomial with zeroes $x_1, x_2, \dots, x_n$, then it is well - known that if we put

$$(3.7) \qquad \begin{aligned} &s_1 = \sum_{j=1}^{n} x_j, \qquad s_2 = \sum_{i<j} x_i x_j, \\ &s_3 = \sum_{i<j<k} x_i x_j x_k, \dots, \quad s_n = \prod_{j=1}^{n} x_j \end{aligned}$$

then the s_i's are related to the coefficients $a_0, a_1, \dots, a_{n-1}$ via the formulae

$$(3.8) \qquad \begin{aligned} a_0 &= (-1)^n s_n \\ a_1 &= (-1)^{n+1} s_{n-1} \\ a_2 &= (-1)^{n+2} s_{n-2} \\ &\dots\dots\dots\dots\dots \\ a_{n-1} &= - s_1. \end{aligned}$$

The s_i's are called elementary symmetric functions of the roots of $P(x) = 0$ [symmetric since any permutation of the x_j's leaves the s_i's invariant]; for a discussion of the use of elementary symmetric functions in solving for the zeroes of polynomials, see [**Usp**, p. 61 - 65].

We note that the saddles z_1, z_2, and z_3 are zeroes of the polynomial $z^3 - \zeta z + \eta$. Use of (3.8) yields the relations

$$
\begin{aligned}
&z_1 + z_2 + z_3 = 0 \\
(3.9) \qquad &z_1 z_2 + z_1 z_3 + z_2 z_3 = -\zeta \\
&z_1 z_2 z_3 = -\eta .
\end{aligned}
$$

If we form the functions

$$
\begin{aligned}
&\sigma_1 = f_-(t_0) + f_-(t_1) + f_-(t_2) \\
(3.10) \qquad &\sigma_2 = f_-(t_0)f_-(t_1) + f_-(t_0)f_-(t_2) + f_-(t_1)f_-(t_2) \\
&\sigma_3 = f_-(t_0)f_-(t_1)f_-(t_2),
\end{aligned}
$$

then use of the correspondence between t_i's and z_j's and equation (3.4) will allow us to express the functions (3.10) as functions of z_1, z_2 and z_3. Use of (3.9) will then give (3.10) with the right-hand sides replaced by functions of ζ, η, and θ. With judicious calculation, we will also be able to give computable expressions for ζ, η, and θ in terms of the σ_i $(i = 1, 2, 3)$.

Because of the large number of calculations involved in finding ζ, η, and θ for the mapping (3.4), we first present the method by using a simpler example (the cubic transformation of [**CFU**]), and then display the results for the quartic transformation (3.4). An outline of some of the intermediate calculations for the quartic map is provided in the Appendix.

Let a function $c(t; \alpha)$ be given which has exactly two coalescing saddles as the parameter α tends to zero. The cubic transformation of [**CFU**] gives a change of variables determined by

$$
(3.11) \qquad c(t; \alpha) = z^3/3 - \zeta z + \eta .
$$

If we require that the coalescing saddles t_1 and t_2 of $c(t; \alpha)$ correspond to the saddles of $z^3/3 - \zeta z + \eta$, $\pm \zeta^{1/2}$, then we can obtain expressions for ζ and η by merely solving the pair of equations

$$
\begin{aligned}
c(t_1; \alpha) &= (-\zeta^{1/2})^3/3 - \zeta(-\zeta^{1/2}) + \eta \\
c(t_2; \alpha) &= (\zeta^{1/2})^3/3 - \zeta(\zeta^{1/2}) + \eta
\end{aligned}
$$

for ζ and η [in this case, the correspondence $t_1 \leftrightarrow -\zeta^{1/2}$, $t_2 \leftrightarrow \zeta^{1/2}$ is assumed; if the only other correspondence is desired, we merely interchange t_1 and t_2]. We find the well-known result

$$
\begin{aligned}
\eta &= \{ c(t_1; \alpha) + c(t_2; \alpha) \}/2 \\
\zeta &= (3/4)^{2/3}[c(t_1; \alpha) - c(t_2; \alpha)]^{2/3}
\end{aligned}
\tag{3.12}
$$

by simple arithmetic.

Alternatively, we could proceed by first noting that roots z_1, z_2 of the derivative of the right-hand side of (3.11), $z^2 - \zeta$, satisfy

$$
\begin{aligned}
0 &= z_1 + z_2 \\
-\zeta &= z_1 z_2.
\end{aligned}
$$

If we form the functions

$$
\begin{aligned}
\sigma_1 &= c(t_1; \alpha) + c(t_2; \alpha) \\
\sigma_2 &= c(t_1; \alpha)c(t_2; \alpha)
\end{aligned}
$$

and then use (3.11), we obtain the symmetric functions (of z_1 and z_2)

$$
\begin{aligned}
\sigma_1 &= (z_1^3 + z_2^3)/3 - \zeta(z_1 + z_2) + 2\eta \\
\sigma_2 &= (z_1^3 z_2^3)/9 - (z_1 z_2^3 + z_1^3 z_2)\zeta/3 + (z_1^3 + z_2^3)\eta/3 \\
&\quad + \zeta^2 z_1 z_2 - \eta\zeta(z_1 + z_2) + \eta^2.
\end{aligned}
\tag{3.13}
$$

Use of $z_1 + z_2 = 0$ and $z_1 z_2 = -\zeta$ in (3.13) allows us to write

$$\sigma_1 = (z_1^3 + z_2^3)/3 + 2\eta$$

(3.14)

$$\sigma_2 = -\zeta^3/9 + \zeta^2(z_1^2 + z_2^2)/3 + (z_1^3 + z_2^3)\eta/3 - \zeta^3 + \eta^2$$

$$= -10\zeta^3/9 + \zeta^2(z_1^2 + z_2^2)/3 + (z_1^3 + z_2^3)\eta/3 + \eta^2.$$

To completely express the σ_i's in terms of η and ζ requires the evaluation of the terms $z_1^3 + z_2^3$ and $z_1^2 + z_2^2$. To that end, we square $(z_1 + z_2)$:

$$0 = (z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2 = z_1^2 + z_2^2 - 2\zeta.$$

Hence, $z_1^2 + z_2^2 = 2\zeta$.

For the sum of cubes, we proceed in similar fashion:

$$0 = (z_1 + z_2)^3 = z_1^3 + z_2^3 + 3(z_1^2 z_2 + z_1 z_2^2) = z_1^3 + z_2^3 + 3z_1z_2(z_1 + z_2)$$

$$= z_1^3 + z_2^3 - 3\zeta \cdot 0,$$

or, $z_1^3 + z_2^3 = 0$.

Therefore, the expressions in (3.14) reduce to

$$\sigma_1 = 2\eta$$

(3.15)

$$\sigma_2 = -4\zeta^3/9 + \eta^2.$$

The former equation gives

$$\eta = \sigma_1/2 = [\, c(t_1; \alpha) + c(t_2; \alpha) \,]/2,$$

in complete agreement with the first equation of (3.12). Substitution of $\sigma_1 = 2\eta$ into the latter equation in (3.15) yields

$$\sigma_2 = -4\zeta^3/9 + \sigma_1^2/4,$$

or

$$\zeta = (9/4)^{1/3}[\,\sigma_1^2/4 - \sigma_2\,]^{1/3}.$$

To compare with the second equation in (3.12), we replace the σ_i's by their expressions involving $c(t_i;\alpha)$:

$$\begin{aligned}\zeta &= (9/4)^{1/3}[\,(\,c(t_1;\alpha)^2 + c(t_2;\alpha)^2 + 2\,c(t_1;\alpha)c(t_2;\alpha)\,)/4 \\ &\qquad - c(t_1;\alpha)c(t_2;\alpha)\,]^{1/3} \\ &= (9/4)^{1/3}[\,(\,c(t_1;\alpha)^2 + c(t_2;\alpha)^2 - 2\,c(t_1;\alpha)c(t_2;\alpha)\,)/4]^{1/3}.\end{aligned}$$

Thus,

$$\zeta = (3/4)^{2/3}[\,(\,c(t_1;\alpha) - c(t_2;\alpha)\,)^2\,]^{1/3}.$$

Compare with the second equation in (3.12).

For the problem of determining ζ, η, and θ in (3.4), we begin with equations (3.10) and make use of (3.4). We then reduce the resulting "mess" through repeated application of equations (3.9). As mentioned earlier, we will present the full calculation for σ_1, and state the consequences for σ_2 and σ_3. The interested reader is directed to the Appendix for a summary of some of the intermediate computations.

We have, from (3.4) and (3.10), the expression

$$\sigma_1 = \sum_{i=1}^{3} z_i^4/4 \; - \zeta\sum_{i=1}^{3} z_i^2/2 \; + \; \eta\sum_{i=1}^{3} z_i \; + 3\theta \tag{3.16}$$

which immediately reduces to

$$\sigma_1 = (z_1^4 + z_2^4 + z_3^4)/4 - \zeta(z_1^2 + z_2^2 + z_3^2)/2 + 3\theta \tag{3.17}$$

in view of the first equation of (3.9). To calculate the sum of squares and the sum of fourth powers, we proceed as we did earlier. (3.9) gives

$$\begin{aligned}0 &= (z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_1z_3 + z_2z_3) \\ &= z_1^2 + z_2^2 + z_3^2 - 2\zeta.\end{aligned}$$

Accordingly, $z_1^2 + z_2^2 + z_3^2 = 2\zeta$, so that (3.17) becomes

$$\sigma_1 = (z_1^4 + z_2^4 + z_3^4)/4 - \zeta^2 + 3\theta. \tag{3.18}$$

We square the expression for the sum of squares to get

$$(z_1^2 + z_2^2 + z_3^2)^2 = 4\zeta^2 = z_1^4 + z_2^4 + z_3^4 + 2(z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2).$$

The latter term can be evaluated by squaring the second equation in (3.9):

$$\begin{aligned}(z_1z_2 + z_1z_3 + z_2z_3)^2 = \zeta^2 &= z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 \\ &\quad + 2(z_1^2z_2z_3 + z_1z_2^2z_3 + z_1z_2z_3^2) \\ &= z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 + 2(z_1z_2z_3)(z_1 + z_2 + z_3) \\ &= z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 + 0\end{aligned}$$

since $z_1 + z_2 + z_3 = 0$. Hence, $z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 = \zeta^2$. Use of this in the expression for the sum of fourth powers yields

$$4\zeta^2 = z_1^4 + z_2^4 + z_3^4 + 2\zeta^2$$

so that (3.18) becomes

$$\sigma_1 = -\zeta^2/2 + 3\theta. \tag{3.19}$$

Proceeding in similar fashion using the formulae presented in the appendix gives, additionally,

$$\sigma_2 = \zeta^4/16 - 9\eta^2\zeta/8 - \theta\zeta^2 + 3\theta^2 \tag{3.20}$$

and

$$\sigma_3 = -27\eta^4/64 + \eta^2\zeta^3/32 + \theta\zeta^4/16 - 9\eta^2\theta\zeta/8 - \theta^2\zeta^2/2 + \theta^3. \tag{3.21}$$

The task now before us is that of using equations (3.19) to (3.21) to solve for η, ζ, and θ in terms of σ_1, σ_2, and σ_3.

From (3.20), we see that $\sigma_2\theta - 3\theta^3 + \theta^2\zeta^2/2 = \theta\zeta^4/16 - 9\eta^2\theta\zeta/8 - \theta^2\zeta^2/2$. Using this in (3.21) gives

$$\sigma_3 = -27\eta^4/64 + \eta^2\zeta^3/32 + \sigma_2\theta + \theta^2\zeta^2/2 - 2\theta^3. \tag{3.22}$$

We use (3.19) in (3.20) to isolate η^2: from (3.19), we have

$$\theta = \sigma_1/3 + \zeta^2/6 \tag{3.23}$$

and this in (3.20) gives

$$\eta^2 = -\zeta^3/54 + 8(\sigma_1^2 - 3\sigma_2)/(27\zeta). \tag{3.24}$$

Similarly, (3.23) can be applied to eliminate all occurrences of θ in (3.22). The result of this is:

$$\begin{aligned}\sigma_3 = -27\eta^4/64 + \eta^2\zeta^3/32 &+(\sigma_1\sigma_2 - 2\sigma_1^3/9)/3 \\ &- (\sigma_1^2 - 3\sigma_2)\zeta^2/18 + \zeta^6/216.\end{aligned} \tag{3.25}$$

If we replace every instance of η^2 by the right-hand side of (3.24), we obtain

$$\begin{aligned}\sigma_3 = \zeta^6/256 - (\sigma_1^2 - 3\sigma_2)\zeta^2/24 &+ (\sigma_1\sigma_2 - 2\sigma_1^3/9)/3 \\ &- (\sigma_1^2 - 3\sigma_2)^2/(27\zeta^2)\end{aligned} \tag{3.26}$$

so that multiplying (3.26) by $256\zeta^2$ yields

$$\begin{aligned}\zeta^8 - 32(\sigma_1^2 - 3\sigma_2)\zeta^4/3 &+ 256(9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3)\zeta^2/27 \\ &- 256(\sigma_1^2 - 3\sigma_2)^2/27 = 0.\end{aligned} \tag{3.27}$$

We note here that

$$9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3 = \sigma_1^3 - 3\sigma_1(\sigma_1^2 - 3\sigma_2) - 27\sigma_3. \tag{3.28}$$

(3.28) will be of some value later.

Put $Z = \zeta^2$ in the octic (3.27). The resulting quartic,

$$Z^4 - 32(\sigma_1^2 - 3\sigma_2)Z^2/3 + 256(9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3)Z/27 \tag{3.29}$$
$$- 256(\sigma_1^2 - 3\sigma_2)^2/27 = 0,$$

can then be solved using Greenhill's formulae (cf. chapter 3, § 2). To that end, we calculate the invariants of the required elliptic function.

From (3.2.1) and (3.2.2), we get

$$g_2 = - 256(\sigma_1^2 - 3\sigma_2)^2/27 + 3[-16(\sigma_1^2 - 3\sigma_2)/9]^2 = 0$$

$$g_3 = [-16(\sigma_1^2 - 3\sigma_2)/9][- 256(\sigma_1^2 - 3\sigma_2)^2/27] \tag{3.30}$$
$$- 64^2(9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3)^2/27^2 + 16^3(\sigma_1^2 - 3\sigma_2)^3/9^3$$
$$= 4(16^3)(\sigma_1^2 - 3\sigma_2)^3/9^3 - 64^2(9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3)^2/27^2.$$

The first equation in (3.30) represents a happy state of affairs.

The associated cubic of the quartic in Z is $S = 4s^3 - g_2 s - g_3$, so that $S = 4s^3 - g_3$. Hence, the zeroes of S are given by $\omega^j(g_3/4)^{1/3}$, $j = 0, 1, 2$, where ω is the cube root of unity $\omega = e^{2\pi i/3}$, and $(g_3/4)^{1/3}$ is taken with its principal value. Use of (3.30) permits us to write

$$(g_3/4)^{1/3} = [16(\sigma_1^2 - 3\sigma_2)/9]\{ 1 - (\text{num}/\text{denom}) \}^{1/3}, \tag{3.31}$$

where we have set

$$\text{num} = (9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3)^2 \tag{3.32}$$
$$\text{denom} = 4(\sigma_1^2 - 3\sigma_2)^3.$$

Also, equation (3.2.3) gives

$$\wp(2\alpha) = 16(\sigma_1^2 - 3\sigma_2)/9. \tag{3.33}$$

Applying Greenhill's formulae (3.2.6) with the e_j's replaced by the $\omega^j(g_3/4)^{1/3}$ gives

$$
\begin{aligned}
Z_0 &= \frac{4}{3}\sqrt{\sigma_1^2 - 3\sigma_2}\,\sum_{j=0}^{2}\sqrt{1 - \omega^j\left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}} \\
Z_1 &= \frac{4}{3}\sqrt{\sigma_1^2 - 3\sigma_2}\left\{\sqrt{1 - \left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}} - \sum_{j=1}^{2}\sqrt{1 - \omega^j\left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}}\right\} \\
Z_2 &= \frac{4}{3}\sqrt{\sigma_1^2 - 3\sigma_2}\left\{\sqrt{1 - \omega\left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}} - \sum_{j=0,2}\sqrt{1 - \omega^j\left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}}\right\} \\
Z_3 &= \frac{4}{3}\sqrt{\sigma_1^2 - 3\sigma_2}\left\{\sqrt{1 - \omega^2\left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}} - \sum_{j=0}^{1}\sqrt{1 - \omega^j\left(1 - \frac{\text{num}}{\text{denom}}\right)^{1/3}}\right\}
\end{aligned}
\tag{3.34}
$$

where the first surd, $\sqrt{(\sigma_1^2 - 3\sigma_2)}$, is taken to have its principal value. The other surds, as was the case in chapter 3, § 2 and in § 2 of the present chapter, must have their branches chosen with care. To do this, we examine the Z_i's for values of b and c on the caustic; ie., $\delta = 0$.

$\delta = 0$ implies that

$$(\sigma_1^3 - 3\sigma_1(\sigma_1^2 - 3\sigma_2) - 27\sigma_3)^2 = 4(\sigma_1^2 - 3\sigma_2)^3; \tag{3.35}$$

for $\delta = 0$ implies that at least two t_i's coincide. If, for the sake of argument, we have $t_0 = t_1 \neq t_2$, then $f\_(t_0) = f\_(t_1) \neq f\_(t_2)$. (3.10), together with some arithmetic, provides us with

$$9\sigma_1\sigma_2 - 2\sigma_1^3 - 27\sigma_3 = 2[\,f\_(t_0) - f\_(t_2)\,]^3$$

and

$$\sigma_1^2 - 3\sigma_2 = [\,f\_(t_0) - f\_(t_2)\,]^2.$$

Use of these two equations, together with (3.28), establishes (3.35). Similarly, we obtain (3.35) in the case $t_0 \neq t_1 = t_2$.

(3.35) leads to the conclusion num = denom (cf. equation (3.32)), so that the expressions for the Z_i's reduce to

$$(3.36)\qquad \begin{aligned} Z_0 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}(\pm 1\ \pm 1\ \pm 1) \\ Z_1 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}(\pm 1\ \mp 1\ \mp 1) \\ Z_2 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}(\mp 1\ \pm 1\ \mp 1) \\ Z_3 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}(\mp 1\ \mp 1\ \pm 1), \end{aligned}$$

where in only the first row are signs independently chosen, the choice of signs subsequently being determined according to (3.2.6). With (3.35) and (3.18), the quartic (3.29) reduces to

$$(3.37)\qquad Z^4 - 32(\sigma_1^2 - 3\sigma_2)Z^2/3 + 512(\sigma_1^2 - 3\sigma_2)^{3/2}Z/27 - 256(\sigma_1^2 - 3\sigma_2)^2/27 = 0.$$

The form of the Z_i's in (3.36) strongly suggests putting $Z_i = (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}\varepsilon_i$. In this event, (3.37) yields the reduced equation

$$(3.38)\qquad \varepsilon_i^4/3 - 2\varepsilon_i^2 + 8\varepsilon_i/3 - 1 = 0.$$

Experiment is now easy, and shows, for example, that if we choose the square roots appearing in the summation in the first equation of (3.34) to be all principal branch choices, then $\varepsilon_0 = 3$ does not satisfy (3.38), but if we choose the square roots so that the first two have their principal branch, the last one being the negative of the principal branch, then we have $\varepsilon_0 = 1 = \varepsilon_1 = \varepsilon_2$ and $\varepsilon_3 = -3$. This latter choice provides ε_i's, all of which satisfy (3.38). Hence, we make, as we have done in the past, this latter selection of branches for the surds appearing in (3.34).

All other choices of branches for the surds in (3.34) amount to a reordering of the Z_i's under either of the two choices of branches discussed above.

For the sake of notational ease, we let

$$(3.39)\qquad \chi = [1 - (\mathrm{num}/\mathrm{denom})]^{1/3}.$$

Then, (3.34) with our choice of branches becomes

$$
(3.40)\quad
\begin{aligned}
Z_0 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}[\,(1-\chi)^{1/2} + (1-\omega\chi)^{1/2} - (1-\omega^2\chi)^{1/2}] \\
Z_1 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}[\,(1-\chi)^{1/2} - (1-\omega\chi)^{1/2} + (1-\omega^2\chi)^{1/2}] \\
Z_2 &= (4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}[-(1-\chi)^{1/2} + (1-\omega\chi)^{1/2} + (1-\omega^2\chi)^{1/2}] \\
Z_3 &= -(4/3)(\sigma_1^2 - 3\sigma_2)^{1/2}[\,(1-\chi)^{1/2} + (1-\omega\chi)^{1/2} + (1-\omega^2\chi)^{1/2}],
\end{aligned}
$$

where all fractional powers are taken to have their principal value.

For our change of variables (3.4), we require ζ positive. Recall that $Z=\zeta^2$. Hence, we must determine which of the Z_i's is real and nonnegative. With square roots chosen to be the principal branch, we know that

$$(3.41)\quad \sqrt{\bar{z}} = \overline{\sqrt{z}}$$

whence the observation that ω and ω^2 are complex conjugates implies that $(1-\omega\chi)^{1/2}$ and $(1-\omega^2\chi)^{1/2}$ are conjugates. Thus, of the Z_i's in (3.40), we see that only Z_2 and Z_3 can be real.

To determine which of Z_2 or Z_3 gives rise to real square roots, we examine the limiting behaviour of Z_2 and Z_3 as we approach the caustic. From (3.35), we know, for δ near zero, that the ratio (num/denom) is approximately 1 in which case χ is tending to zero. Applying the binomial expansion yields the approximation

$$(3.42)\quad (1-\omega^j\chi)^{1/2} = 1 - \omega^j\chi/2 + O(\chi^2).$$

Use of this in the expressions for Z_2 and Z_3 gives

$$
\begin{aligned}
&-(1-\chi)^{1/2} + (1-\omega\chi)^{1/2} + (1-\omega^2\chi)^{1/2} \\
&= 1 - (\omega + \omega^2 - 1)\chi/2 + O(\chi^2) \\
&= 1 + \chi + O(\chi^2)
\end{aligned}
$$

and

$$(1-\chi)^{1/2}+(1-\omega\chi)^{1/2}+(1-\omega^2\chi)^{1/2}$$

$$=3-(\omega+\omega^2+1)\chi/2+O(\chi^2)$$

$$=3+O(\chi^2).$$

Thus, Z_2 is positive, and Z_3 is negative so that

$$\zeta=\pm(4/3)^{1/2}(\sigma_1^{\,2}-3\sigma_2)^{1/4}\cdot$$
$$[-(1-\chi)^{1/2}+(1-\omega\chi)^{1/2}+(1-\omega^2\chi)^{1/2}]^{1/2}.$$

Use of the fact that ζ must be positive inside the caustic yields

$$\zeta=(4/3)^{1/2}(\sigma_1^{\,2}-3\sigma_2)^{1/4}\cdot [-(1-\chi)^{1/2}+(1-\omega\chi)^{1/2}+(1-\omega^2\chi)^{1/2}]^{1/2}. \tag{3.43}$$

(3.23) and (3.24) therefore yield computable expressions for θ and η respectively.

Before continuing with the development of the uniform expansion of $I_-(\lambda;b,c)$, we examine the sign of the term $\sigma_1^{\,2}-3\sigma_2$ appearing in all our formulae for ζ, η, and θ. We shall also provide an alternate expression for η, potentially of use in numerical work, which does not involve ratios of terms which vanish as we approach the cusp, as is the case in (3.24).

Straightforward calculation reveals

$$\sum_{i<j}[f_-(t_i)-f_-(t_j)]^2=2[\,f_-(t_0)^2+f_-(t_1)^2+f_-(t_2)^2$$
$$-f_-(t_0)f_-(t_1)-f_-(t_1)f_-(t_2)-f_-(t_0)f_-(t_2)\,]$$
$$=2(\sigma_1^{\,2}-3\sigma_2),$$

whence, inside and on the caustic, $\sigma_1^{\,2}-3\sigma_2\geq 0$. Furthermore, passing outside the caustic (with $b>0$), we see that one of the t_i's, with $0\leq i\leq 2$, is real, the other two being complex conjugates (t_3 is always real for $b>0$). In such case, two of the $f_-(t_i)$ are complex conjugates with the remaining one real. Thus, $\sigma_1^{\,2}-3\sigma_2$ is real outside the caustic as well, although it may be negative.

As to the calculation of η, we observe that (3.20) implies

$$\sigma_2\theta + \theta^2\zeta^2 - 3\theta^3 = \theta\zeta^4/16 - 9\eta^2\theta\zeta/8.$$

From this and (3.21), we have

$$\sigma_3 = -27\eta^4/64 + \eta^2\zeta^3/32 + \sigma_2\theta + \theta^2\zeta^2 - 3\theta^3 - \theta^2\zeta^2/2 + \theta^3$$

$$= -27\eta^4/64 + \eta^2\zeta^3/32 + \sigma_2\theta + \theta^2\zeta^2/2 - 2\theta^3.$$

Using (3.19) gives

$$\sigma_3 = -27\eta^4/64 + \eta^2\zeta^3/32 + \sigma_2\theta + 3\theta^3 - \sigma_1\theta^2 - 2\theta^3$$

$$= -27\eta^4/64 + \eta^2\zeta^3/32 + \sigma_2\theta - \sigma_1\theta^2 + \theta^3,$$

or

$$\eta^4 - 2\eta^2\zeta^3/27 - 64(\theta^3 - \sigma_1\theta^2 + \sigma_2\theta - \sigma_3)/27 = 0.$$

The quadratic formula yields

$$\eta^2 = \zeta^3/27 \pm \sqrt{\zeta^6/27^2 + 64(\theta^3 - \sigma_1\theta^2 + \sigma_2\theta - \sigma_3)/27}\ ,$$

or

$$\eta^2 = \frac{1}{27}\left[\zeta^3 \pm \sqrt{\zeta^6 + 27\cdot 64\prod_{i=0}^{2}(\theta - f_-(t_i))}\right]. \tag{3.44}$$

The appropriate sign in (3.44) can be determined from (3.24)

When taking the square root of η^2 in either (3.24) or (3.44), we choose it so that it is negative on the arc of the caustic joining (0, 0) to $(4/3\sqrt{6}, -1/12)$, and positive on the arc joining $(4/3\sqrt{6}, -1/12)$ to $(0, 1/4)$; see figure 4.1.

It is interesting to note that we have obtained closed form expressions for the parameters η, ζ and θ appearing in (3.4). Ursell, in examining the quartic transformation (3.4), remarks that η, ζ and θ can, in principle, be constructed from convergent power series (a consequence of the use of Levinson's theorem), and that the power series approach is not practical computationally. He further states that the parameters η, ζ and θ can be obtained without explicit reference to the uniformly analytic one-to-one solution to (3.4), but does not go on to provide expressions for the parameters; cf. [**Urs**, p. 64 - 65]. This has been accomplished here.

With our transformation (3.4) completely determined, we can proceed to the uniform expansion of $I_-(\lambda)$.

Since t_3 remains isolated from t_0, t_1, t_2 for $b > 0$, we rewrite $I_-(\lambda)$ as the sum of two path integrals:

$$I_-(\lambda; b, c) = \int_{\Gamma_1} e^{i\lambda f_-(t;\, b,\, c)}\, dt \;+\; \int_{\Gamma_2} e^{i\lambda f_-(t;\, b,\, c)}\, dt \tag{3.45}$$

$$= I_1(\lambda; b, c) + I_2(\lambda; b, c)$$

where $I_j(\lambda; b, c)$ denotes the integral of $\exp(i\lambda f_-(t; b, c))$ over the contour Γ_j, with the Γ_j as depicted in figure 4.2 below.

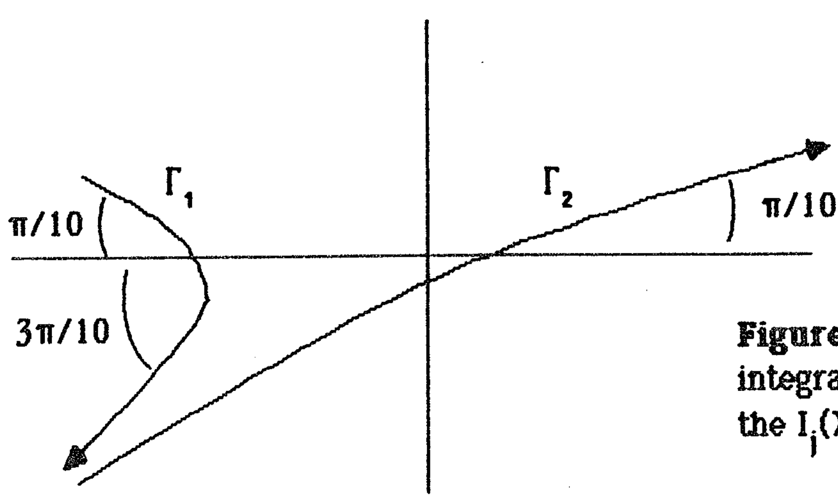


Figure 4.2. t-plane integration paths for the $I_j(\lambda; b, c)$.

Γ_1 may be taken to be the steepest descent curve through t_3 beginning at $\infty e^{9\pi i/10}$ and ending at $\infty e^{13\pi i/10}$. Thus, $I_1(\lambda;\ b,\ c)$ has an asymptotic expansion of the form

$$I_1(\lambda; b, c) \sim e^{i\lambda f_-(t_3; b, c) - \pi i/4} \sum_{j=1}^{\infty} a_j(b, c)\, \lambda^{-j/2} \tag{3.46}$$

as λ tends to $+\infty$. The leading term has the coefficient

$$a_1(b, c) = \sqrt{\frac{2\pi}{-f_-''(t_3; b, c)}} \tag{3.47}$$

which is well-behaved for $b \geq b_0 > 0$. Note that $f_-''(t_3) < 0$ since t_3 is a local maximum of the quintic f_-.

For the integral $I_2(\lambda;\ b,\ c)$, we invoke the quartic transformation (3.4), and introduce the function sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$, $\{g_n\}$ and $\{h_n\}$ defined by

$$\begin{aligned} g_0(z; \zeta, \eta) &= p_0 + q_0 z + r_0 z^2 + (z^3 - \zeta z + \eta) h_0(z; \zeta, \eta) \\ \partial h_k(z; \zeta, \eta)/\partial z &= g_{k+1}(z; \zeta, \eta) \\ &= p_{k+1} + q_{k+1} z + r_{k+1} z^2 + (z^3 - \zeta z + \eta) h_{k+1}(z; \zeta, \eta) \\ &\qquad \text{for } k = 0, 1, 2, \ldots \end{aligned} \tag{3.48}$$

where $g_0(z; \zeta, \eta) = dt/dz$ (cf. equation (3.4)). The coefficients p_n, q_n, and r_n are functions of ζ and η which, in view of our expressions for ζ, η and θ in terms of b and c, in turn can be regarded as functions of b and c.

Through repeated substitution and partial integration, we obtain

$$\begin{aligned} e^{-i\lambda\theta} I_2(\lambda; b, c) &= \int_C e^{i\lambda(z^4/4 - \zeta z^2/2 + \eta z)} g_0(z; \zeta, \eta)\, dz \\ &= p_0 F_0(\lambda; \zeta, \eta) + q_0 F_1(\lambda; \zeta, \eta) + r_0 F_2(\lambda; \zeta, \eta) \\ &\quad + \int_C e^{i\lambda(z^4/4 - \zeta z^2/2 + \eta z)} (z^3 - \zeta z + \eta) h_0(z; \zeta, \eta)\, dz\ , \end{aligned}$$

and

$$e^{-i\lambda\theta}\, I_2(\lambda; b, c) = p_0 F_0(\lambda; \zeta, \eta) + q_0 F_1(\lambda; \zeta, \eta) + r_0 F_2(\lambda; \zeta, \eta)$$

$$+ \frac{i}{\lambda}\int_C e^{i\lambda(z^4/4 - \zeta z^2/2 + \eta z)}\, g_1(z; \zeta, \eta)\, dz$$

$$= \cdots = \sum_{j=0}^{n}\left(\frac{i}{\lambda}\right)^j [p_j\, F_0(\lambda; \zeta, \eta) + q_j\, F_1(\lambda; \zeta, \eta) + r_j\, F_2(\lambda; \zeta, \eta)]$$

$$+ \left(\frac{i}{\lambda}\right)^{n+1}\int_C e^{i\lambda(z^4/4 - \zeta z^2/2 + \eta z)}\, g_{n+1}(z; \zeta, \eta)\, dz$$

which yields

$$(3.49)\quad I_2(\lambda; b, c) \sim e^{i\lambda\theta}\sum_{j=0}^{\infty}\left(\frac{i}{\lambda}\right)^j [p_j\, F_0(\lambda; \zeta, \eta) + q_j\, F_1(\lambda; \zeta, \eta) + r_j\, F_2(\lambda; \zeta, \eta)]$$

as λ tends to $+\infty$, uniformly valid for (ζ, η) "near" the caustic $27\eta^2 - 4\zeta^3 = 0$; ie., uniformly valid near the caustic in the bc-plane with $b \geq b_0 > 0$, and (b, c) within a band of fixed distance from the caustic.

The functions F_k appearing in (3.49) are given by the integrals

$$(3.50)\qquad F_k(\lambda; \zeta, \eta) = \int_C e^{i\lambda(z^4/4 - \zeta z^2/2 + \eta z)}\, z^k\, dz$$

with $k = 0, 1, 2$. C is a contour in the z-plane beginning at $\infty e^{9\pi i/8}$ and ending at $\infty e^{\pi i/8}$. With the change of variables $u = z/\lambda^{1/4}$, we find

$$F_0(\lambda; \zeta, \eta) = \lambda^{-1/4} P(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta)$$

$$(3.51)\qquad F_1(\lambda; \zeta, \eta) = -i\lambda^{-1/2} P_y(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta)$$

$$F_2(\lambda; \zeta, \eta) = -2i\lambda^{-3/4} P_x(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta)$$

where the function $P(x, y)$ is the Pearcey function of chapter 2, and the functions P_x and P_y are the first order partial derivatives of $P(x, y)$ with respect to x and y respectively. Use of (3.51) in (3.49) yields the expansion

$$I_2(\lambda; b, c) \sim e^{i\lambda\theta} \sum_{j=0}^{\infty} \left(\frac{i}{\lambda}\right)^j \Big[p_j \lambda^{-1/4} P(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta)$$

$$- iq_j \lambda^{-1/2} P_y(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta) - 2ir_j \lambda^{-3/4} P_x(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta)\Big] \tag{3.52}$$

as λ tends to $+\infty$, uniformly valid for (b, c) in a band of fixed distance from the caustic, with $b \geq b_0 > 0$ for some fixed positive b_0.

The expansion of $I_-(\lambda; b, c)$ is therefore given by the sum of (3.52) and (3.46). In particular, we obtain the uniform approximation

$$I_-(\lambda; b, c) = e^{i\lambda f_-(t_3; b, c) - \pi i/4} \sqrt{\frac{2\pi}{-\lambda f_-''(t_3; b, c)}} [1 + O(1/\sqrt{\lambda})]$$

$$+ e^{i\lambda\theta}(p_0(b, c)\, \lambda^{-1/4} P(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta)$$

$$- iq_0(b, c)\, \lambda^{-1/2} P_y(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta) \tag{3.53}$$

$$- 2ir_0(b, c)\, \lambda^{-3/4} P_x(-\lambda^{1/2}\zeta, \lambda^{3/4}\eta))[1 + O(1/\lambda)]$$

where the O-symbols are independent of (b, c).

The expansion of $Q(-x, y, z)$ now follows directly from (3.52) through the use of (1.3).

In the next section, we devote some time to analyzing the nature of the mapping (3.4). In particular, we will determine the effect of (3.4) on integration paths, and verify that the change of variables determined by (3.4) is valid for a large region in the complex plane.

4. Conformal Mapping

We begin by supposing that a point (b, c) lies inside the caustic (recall figure 4.1). Then there are four real saddles for (1.1), $t_3 < t_0 < t_1 < t_2$, of which only t_0, t_1 and t_2 are involved with confluence since b is taken to be positive. As was the case in chapters two and three, we determine the curves for which $\operatorname{Im} f_-(t; b, c) = 0$. These are displayed below in figure 4.3.

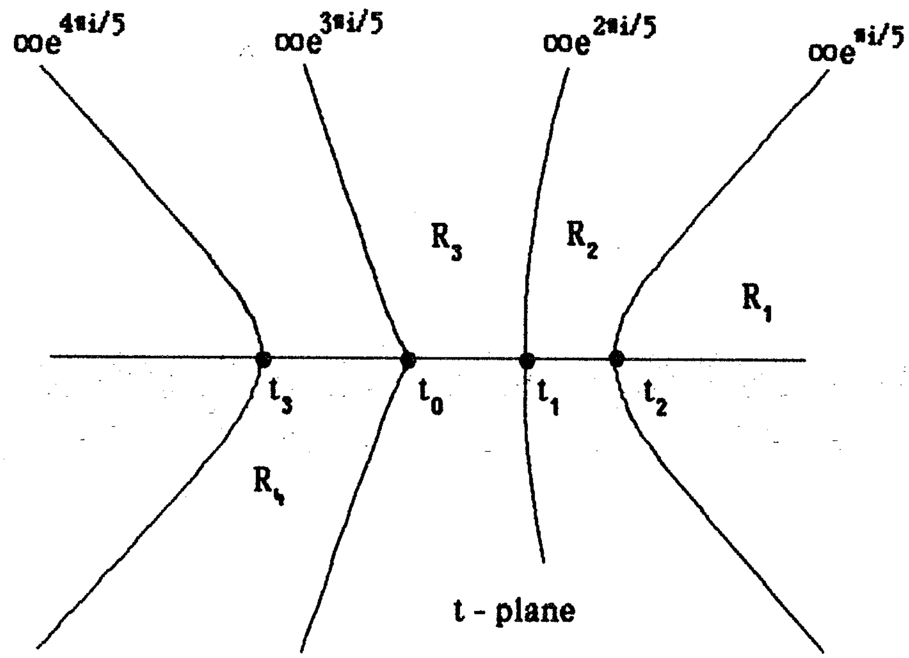


Figure 4.3. Curves in the complex t-plane for which $\operatorname{Im} f_-(t; b, c) = 0$. The real axis is included as such a curve.

These curves, which include the real axis, partition the t-plane into several disjoint regions. We shall consider only those labelled $R_1, \dots, R_4$, as our integration contour for $I_2(\lambda; b, c)$ of equation (3.45), Γ_2, can be chosen to lie entirely within $R_1 \cup R_2 \cup R_3 \cup R_4$, or this union's closure.

We will see that the (local) uniformly analytic one-to-one solution of (3.4) is indeed one-to-one on this union.

Since the saddles of $g(z; \xi, \eta, \theta)$ under the uniformly analytic solution of (3.4) correspond with the t_i's in the fashion $t_2 \leftrightarrow z_1$, $t_1 \leftrightarrow z_2$, $t_0 \leftrightarrow z_3$, we have $z_3 < z_2 < z_1$ for (b, c) inside the caustic (this can also be seen from (3.5)). Hence, the z-plane curves for which $\operatorname{Im} g(z; \xi, \eta, \theta) = 0$ partition the z-plane into the disjoint regions displayed in figure 4.4. Again, the real axis is a curve for which $\operatorname{Im} g(z) = 0$.

We will be concerned with the regions labelled $\Omega_1, \dots, \Omega_4$.

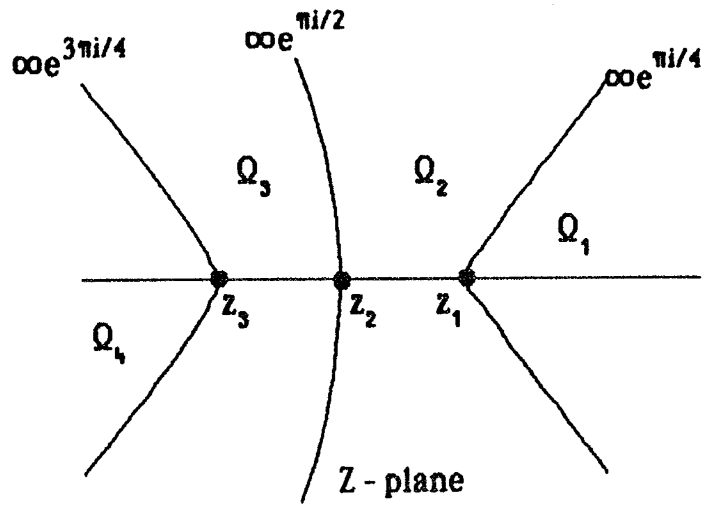


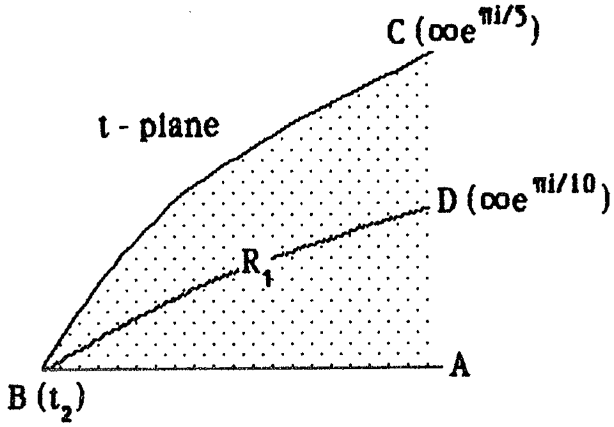
Figure 4.4. Curves in the complex z-plane for which $\operatorname{Im} g(z)=0$. As in the previous figure, the real axis is such a curve.

Introduce an intermediate variable Z so that (3.4) becomes

$$f_-(t; b, c) = Z = g(z; \zeta, \eta, \theta). \tag{4.1}$$

We consider the effects of the maps $t \to Z = f_-(t; b, c)$ and $z \to Z = g(z; \zeta, \eta, \theta)$ on domains R_i and Ω_i respectively.

Consider R_1 in the t-plane as depicted in figure 4.5.



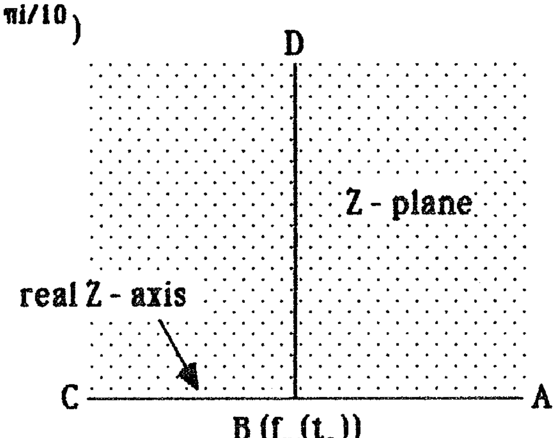


Figure 4.5. Effect of mapping $t \to Z$ on the t-plane region R_1.

The arc BD is the steepest descent curve from t_2 to $\infty e^{\pi i/10}$. The arc BC is the upper extent of the region R_1. Under (4.1), the image of R_1 is the upper half-Z-plane depicted in the right hand illustration in figure 4.5.

The map $z \to Z$ on Ω_1 produces a similar effect. Refer to figure 4.6. In the left-hand illustration, the curve BD is the steepest descent curve of $ig(z)$ beginning at z_1 and ending at $\infty e^{\pi i/8}$. BC is the upper extent of Ω_1. Under (4.1), Ω_1 maps to the illustration at the right of figure 4.6.

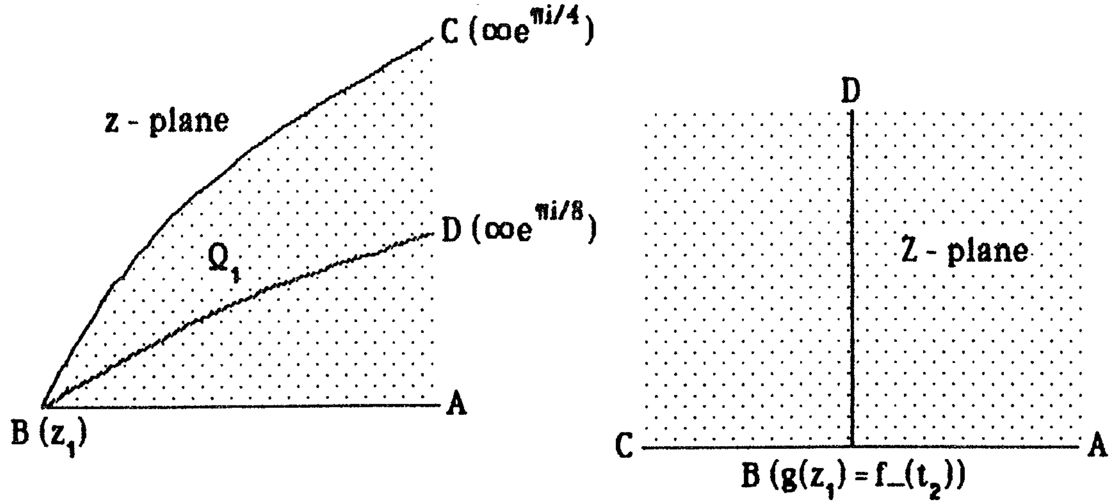


Figure 4.6. Effect of the mapping $z \to Z$ on the z-plane region Ω_1.

Thus, we see that, under (4.1), the region R_1 maps to Ω_1.

We now direct our attention to R_2. In figure 4.7, the arcs CD and BA bound the region R_2 ; these are curves for which $\mathrm{Im}\, f_-(t; b, c) = 0$. The arc BE is the steepest ascent curve from t_2 to $\infty e^{3\pi i/10}$. Under (4.1), the region maps to the Z-plane as illustrated on the right of figure 4.7.

It is easy to see that B, C reverse ordering upon application of the map $t \to Z$ since t_1 is a local maximum of f_-, and t_2 is an adjacent local minimum.

Consider $z \to Z$ acting on Ω_2 ; refer to figure 4.8. CD and BA are curves for which $\mathrm{Im}\, g(z) = 0$; these two arcs bound Ω_2 on two sides. BE is the steepest ascent curve from z_1 to $\infty e^{3\pi i/8}$. Under (4.1), Ω_2 maps to the half-Z-plane given in the right-hand side of figure 4.8.

As was the case for R_1 and Ω_1, we see that $f_-(t) = g(z)$ provides us with a one-to-one map from R_2 onto Ω_2.

Similar analysis applies to the mapping from R_3 to Ω_3 (see figure 4.9).

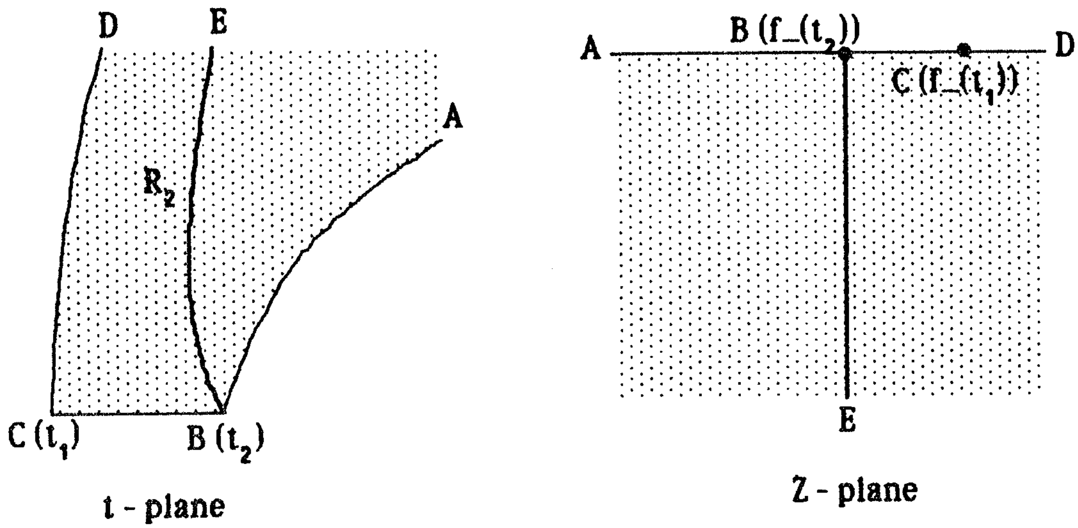


Figure 4.7. Effect of the mapping $t \to Z$ on the t-plane region R_2.

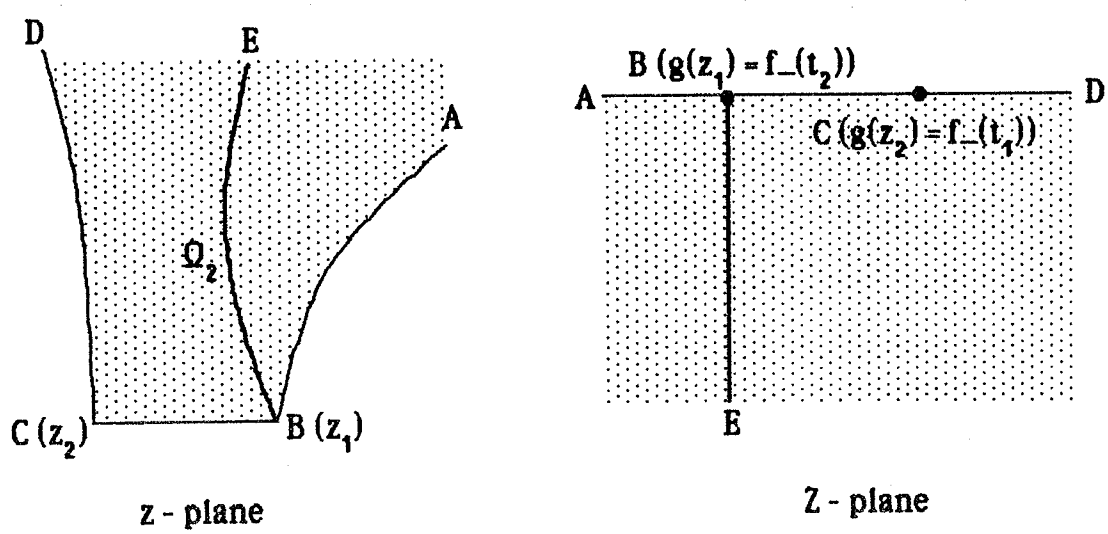


Figure 4.8. Effect of the mapping $z \to Z$ on the z-plane region Ω_2.

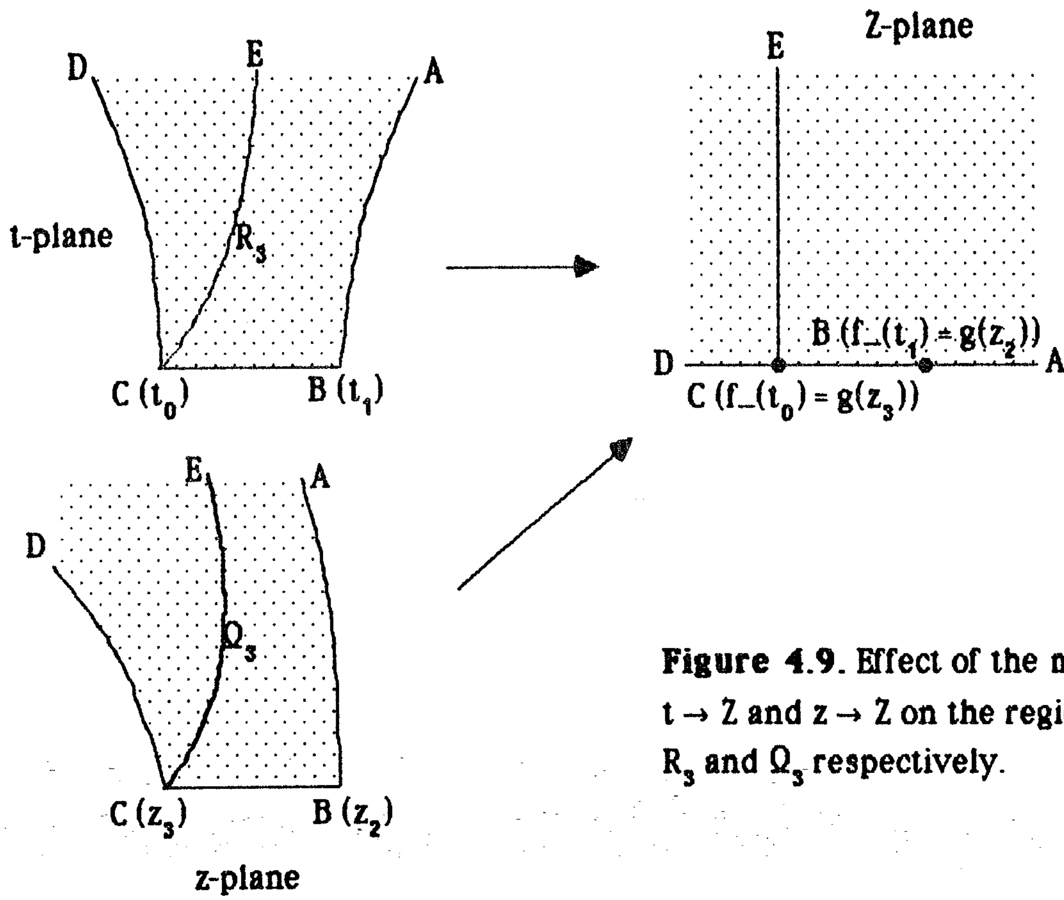


Figure 4.9. Effect of the maps $t \to Z$ and $z \to Z$ on the regions R_3 and Ω_3 respectively.

Finally, we turn to $R_4 \to \Omega_4$. In figure 4.10, in the t-plane, CE is the steepest descent curve from t_0 to $\infty e^{13\pi i/10}$, and the corresponding curve in the z-plane is the steepest descent curve from z_3 to $\infty e^{9\pi i/8}$.

We note here that f_- decreases steadily from $+\infty$ to $-\infty$ as t moves from $A = \infty e^{6\pi i/5}$ to $B = t_3$, then to $C = t_0$, and then to $D = \infty e^{7\pi i/5}$. Similarly, g decreases steadily from $+\infty$ to $-\infty$ as z moves from $A = +\infty$ to $C = z_3$ to $D = \infty e^{5\pi i/4}$. Since $g(z_3) = f_-(t_0)$, this implies the existence of a point B' in the interval $]-\infty, z_3[$ such that $g(B') = f_-(t_3)$. This is indicated in figure 4.10.

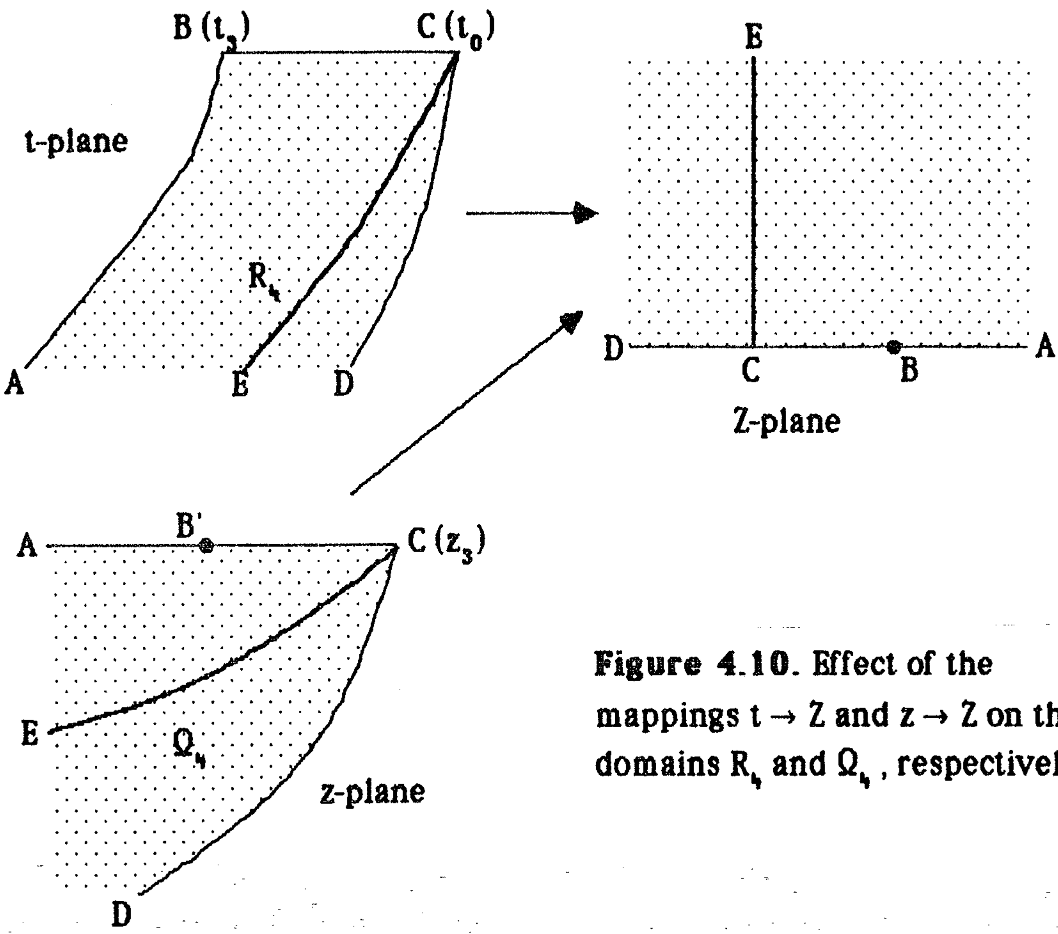


Figure 4.10. Effect of the mappings $t \rightarrow Z$ and $z \rightarrow Z$ on the domains R_4 and Ω_4 , respectively.

Thus, $R_4 \rightarrow \Omega_4$ in a one-to-one, onto fashion. To obtain the mappings of regions conjugate to the R_i to regions conjugate to the Ω_i , we merely "flip" all illustrations about the real t, Z, and z-axes, and replace "ascent" by "descent" and vice versa.

If (b, c) lies on the caustic (but not at the cusp), then two possibilities arise: either $t_0 = t_1 \neq t_2$, or $t_0 \neq t_1 = t_2$. No other possibilities exist, for if we had $t_0=t_2$, then we would have $t_0 = t_1 = t_2$ (in which case, (b, c) is at the cusp). Assume that $t_0 = t_1 \neq t_2$ (it is easy to modify our treatment for the case $t_0 \neq t_1 = t_2$).

Figure 4.11 graphically presents the regions R_i and Ω_i in the case $\delta = 0$ with $t_0=t_1 \neq t_2$ (and, consequently, $z_3 = z_2 \neq z_1$). As it is readily apparent how these regions map, no further commentary will be provided.

The case where all of t_0 , t_1 , and t_2 coincide is displayed in figure 4.12. This case corresponds to that of (b, c) equal to the cusp $(4/3\sqrt{6}, -1/12)$. Again, as the mapping from R_i to Ω_i is immediate, no further discussion

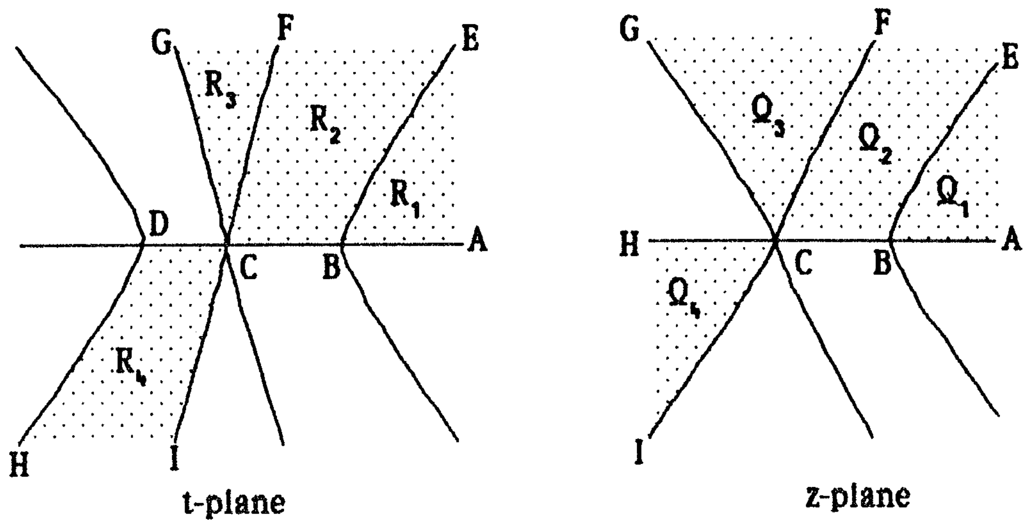


Figure 4.11. The regions $R_1, \ldots, R_4$ and $\Omega_1, \ldots, \Omega_4$ in the case where (b, c) lies on the caustic, not at the cusp ($t_0 = t_1 \neq t_2$)

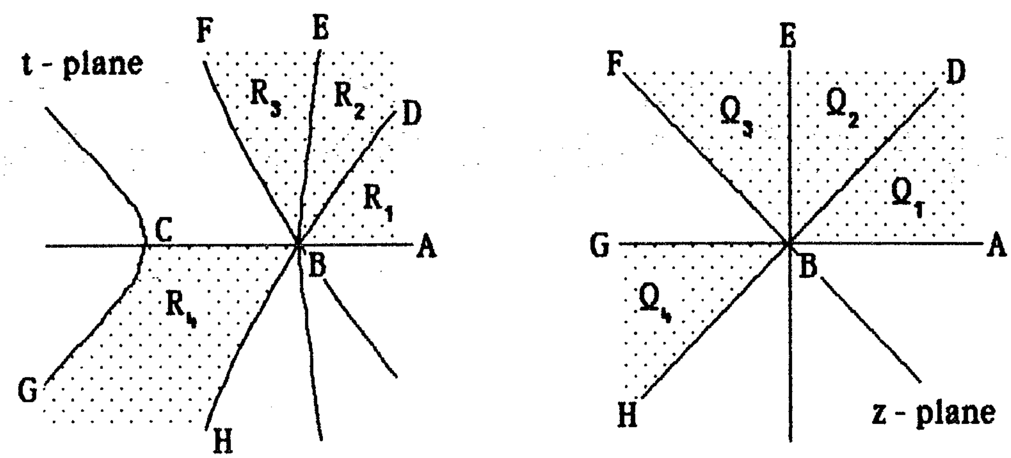


Figure 4.12. The regions R_i and Ω_i in the case where (b, c) = ($4/3\sqrt{6}$, $-1/12$), the cusp of the caustic.

will be given, aside from noting that at the cusp, we have $\zeta = \eta = 0$ and

$$g(z; \zeta, \eta, \theta) = z^4/4 + f_-(1/\sqrt{6}; 4/3\sqrt{6}, -1/12).$$

This concludes our treatment of the mapping (4.1) when the roots t_i, for $i = 0, 1, 2, 3$, are real.

Suppose (b, c) lies below the caustic so that $t_3 < \operatorname{Re} t_0 = \operatorname{Re} t_2 < t_1$ with t_0 and t_2 complex conjugates, and $\operatorname{Im} t_2 > 0$ [this can happen in the situation where (b, c) lies below the caustic on the $\psi = 0$ curve $b^2 = 2/27 - 8c/3$; see (2.5)]. Our treatment of (4.1) proceeds in a fashion similar to that used for the study of the conformal mapping properties of f_+ in the case of complex roots: we do not rely strictly on curves $\operatorname{Im} f_-(t) = 0$, but instead use steepest curves through the complex saddles.

We consider the curves in figure 4.13.

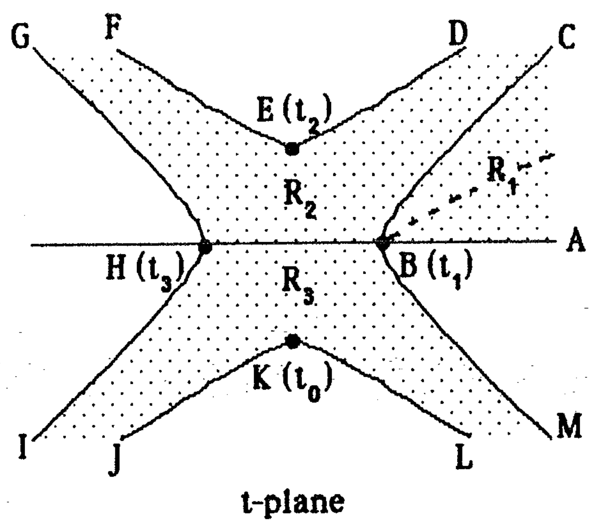


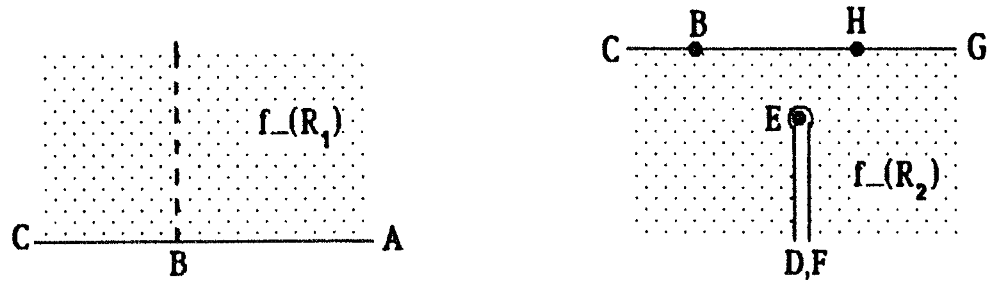
Figure 4.13. Curves partitioning the t-plane for the case of two complex saddles.

Here, CBM is the curve $\operatorname{Im} f_-(t) = 0$ passing through t_1. It begins at $\infty e^{-\pi i/5}$ and ends at $\infty e^{\pi i/5}$. Similarly, GHI is the curve $\operatorname{Im} f_-(t) = 0$ passing through t_3. GHI begins at $\infty e^{4\pi i/5}$ and ends at $\infty e^{6\pi i/5}$.

The broken curve issuing from B is the steepest descent curve to $\infty e^{\pi i/10}$.

The curve DEF is the steepest ascent curve passing through t_2, beginning at $\infty e^{7\pi i/10}$ and ending at $\infty e^{3\pi i/10}$. The curve JKL is the steepest descent curve passing through t_0 beginning at $\infty e^{13\pi i/10}$ and ending at $\infty e^{17\pi i/10}$.

The regions R_1, R_2, and R_3 map to the Z -plane as displayed in figure 4.14.



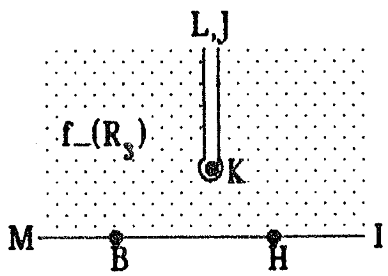


Figure 4.14. Z-plane images of the regions R_1, R_2, and R_3 of figure 13.

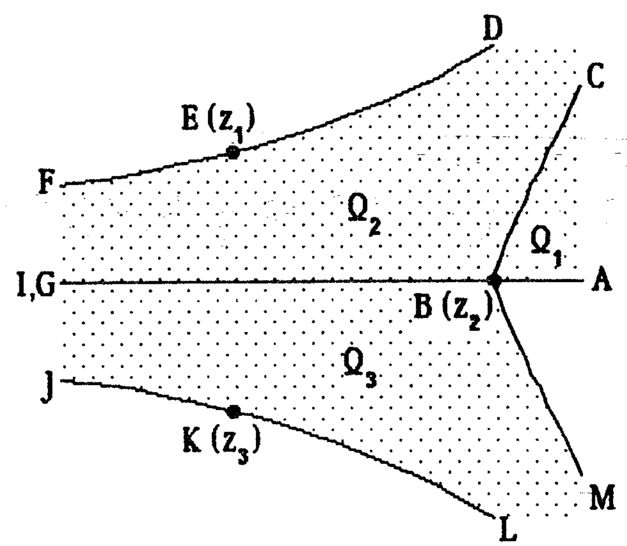


Figure 4.15. Curves partitioning the z-plane for the case of two complex saddles of g(z)

In figure 4.15, we note that CBM is the curve Im $g(z) = 0$ passing through z_2, beginning at $\infty e^{-\pi i/4}$ and ending at $\infty e^{\pi i/4}$. DEF is the steepest ascent curve through z_1 beginning at $\infty e^{7\pi i/8}$ and ending at $\infty e^{3\pi i/8}$. JKL is the steepest descent curve through z_3 starting at $\infty e^{9\pi i/8}$ and ending at $\infty e^{13\pi i/8}$.

The images of Ω_1, Ω_2 and Ω_3 under the mapping $z \to Z = g(z)$ in the Z-plane are given next in figure 4.16.

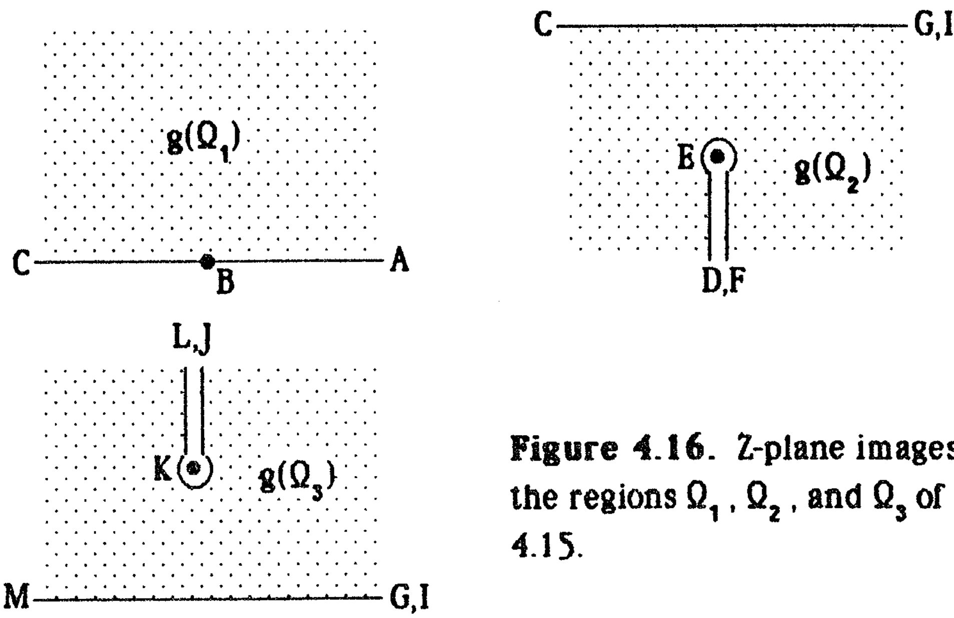


Figure 4.16. Z-plane images of the regions Ω_1, Ω_2, and Ω_3 of figure 4.15.

Thus, the mapping (4.1) or rather, the uniformly analytic one-to-one solution to (3.4) taking $t_0 \to z_3$, $t_1 \to z_2$ and $t_2 \to z_1$, is one-to-one and onto from $R_i \to \Omega_i$ for i = 1, 2, 3.

Accordingly, the integration contour Γ_2 in (3.45) can be chosen so that its image in the z-plane, C (cf. (3.50)), is:

i) for (b, c) inside the caustic, Γ_2 is chosen to be the steepest descent curve from $\infty e^{13\pi i/10}$ through to t_0, followed by the straight line segment $[t_0, t_2]$, followed in turn by the steepest descent curve from t_2 to $\infty e^{\pi i/10}$.

Under (4.1), this Γ_2 maps to the steepest curve beginning at $\infty e^{9\pi i/8}$ passing through to z_3, followed by the line segment $[z_3, z_1]$, followed in turn by the steepest descent curve from z_1 to $\infty e^{\pi i/8}$.

ii) for (b, c) on the caustic (even at the cusp), the same description in (i) applies, except the intervals $[t_0, t_2]$ and $[z_3, z_1]$ may now be points.

iii) for (b, c) off the caustic with, say, t_1 real, Γ_2 may be chosen to be the steepest descent curve beginning at $\infty e^{13\pi i/10}$, passing through to t_0, then along the straight line segment joining t_0 to t_1, and then followed by the steepest descent curve from t_1 to $\infty e^{\pi i/10}$.

The image of Γ_2, C, then is the steepest descent curve from $\infty e^{9\pi i/8}$ to z_3, followed by a curve joining z_3 to z_2 (contained within the region Ω_3; see figure 4.15), followed thereafter by the steepest descent curve from z_2 to $\infty e^{\pi i/8}$.

5. A Limiting Case

As an example of the use of the quartic transformation formulae developed in § 3, and as a check on the validity of our results, we determine the limiting form of the coefficients in the approximation (3.53) when our parameters (b, c) tend to the cusp $(4/3\sqrt{3}, -1/12)$. This calculation showcases the determination of the parameters (ζ, η, θ), and makes use of the explicit formula for ζ presented as equation (3.43).

We begin by taking (b, c) to lie on the caustic. From the discussion on page 70, it is readily seen that the $\psi = -\pi/6$ level curve in the bc - plane contains the lower arch of the caustic joining (0, 0) to $(4/3\sqrt{3}, -1/12)$, and that the $\psi = \pi/6$ level curve contains the upper right arch of the caustic joining $(4/3\sqrt{3}, -1/12)$ to (0, 1/4); see figure 4.1. We will take (b, c) to lie on that part of the caustic joining (0, 0) to $(4/3\sqrt{3}, -1/12)$, so that $\psi = -\pi/6$ with $c \neq -1/12$ ($c = -1/12$ would place us at the cusp). With these choices for ψ and c, it is easy to see that (2.5) becomes

$$c/6 + b^2/16 - 1/216 = -(c/3 + 1/36)^{3/2},$$

a quadratic equation in b. Put $c = \Delta c - 1/12$. Use of this in the previous equation implies

$$b^2 = (8/27)[1 - 9\Delta c - 54(\Delta c/3)^{3/2}],$$

so that if b is positive, we may write

$$(5.1) \qquad b = (4/3\sqrt{6})[1 - 9\Delta c - 54(\Delta c/3)^{3/2}]^{1/2}$$

for a point (b, c) on the arch of the caustic under examination. We will use (5.1) extensively to develop $\Delta c \to 0^+$ limits for a variety of quantities needed in the calculation of p_0, q_0 and r_0 in (3.53).

Before proceeding, we shall determine computable expressions for p_0, q_0 and r_0. First note that on the $\psi = -\pi/6$ portion of the caustic, we have $t_0 = t_1 < t_2$ so that $z_3 = z_2 < z_1$ in view of the correspondence $t_2 \leftrightarrow z_1$, $t_1 \leftrightarrow z_2$, $t_0 \leftrightarrow z_3$. Furthermore, from $\phi = -\pi/6$ we have $\eta = -2(\zeta/3)^{3/2}$(cf. equation (3.6)), $z_1 = 2\sqrt{(\zeta/3)}$, and $z_2 = z_3 = -\sqrt{(\zeta/3)}$.

From the first equation of (3.48), we get

$$(5.2) \qquad g_0(z) = dt/dz = p_0 + q_0 z + r_0 z^2 + (z^3 - \zeta z - 2(\zeta/3)^{3/2})h_0(z),$$

which implies

$$(5.3) \qquad g_0(z_1) = p_0 + 2(\zeta/3)^{1/2}q_0 + (4\zeta/3)r_0$$

and

$$(5.4) \qquad g_0(z_2) = p_0 - (\zeta/3)^{1/2}q_0 + (\zeta/3)r_0\,,$$

in view of the fact that $(z^3 - \zeta z - 2(\zeta/3)^{3/2})h_0(z)$ vanishes at z_1 and z_2. As (5.3) and (5.4) are a pair of linear equations in the three unknowns p_0, q_0, r_0, (5.3) and (5.4) are insufficient for determining p_0, q_0 and r_0 uniquely. A third (linearly) independent equation can be formed by differentiating (5.2) with respect to z:

$$g_0'(z) = q_0 + 2zr_0 + (3z^2 - \zeta)h_0(z) + (z^3 - \zeta z - 2(\zeta/3)^{3/2})h_0'(z).$$

Evaluation of the last equation at z_2 implies

$$(5.5) \qquad g_0'(z_2) = q_0 - 2(\zeta/3)^{1/2}r_0.$$

Thus, (5.3), (5.4) and (5.5) determine p_0, q_0 and r_0.

Subtract (5.4) from (5.3) to get

$$(5.6) \qquad g_0(z_1) - g_0(z_2) = 3(\zeta/3)^{1/2}q_0 + \zeta r_0.$$

Multiply (5.5) by $3(\zeta/3)^{1/2}$ to get

$$(5.7) \qquad 3(\zeta/3)^{1/2}g_0'(z_2) = 3(\zeta/3)^{1/2}q_0 - 2\zeta r_0.$$

Subtracting (5.7) from (5.6) yields

$$(5.8) \qquad g_0(z_1) - g_0(z_2) - 3(\zeta/3)^{1/2}g_0'(z_2) = 3\zeta r_0$$

or, upon solving for r_0,

$$(5.9) \qquad r_0 = [g_0(z_1) - g_0(z_2) - 3(\zeta/3)^{1/2}g_0'(z_2)]/(3\zeta).$$

Use of this in (5.5) then provides us with

$$(5.10) \qquad q_0 = g_0'(z_2)/3 + 2[g_0(z_1) - g_0(z_2)]/\sqrt{(27\zeta)}.$$

Finally, (5.9) and (5.10) together in (5.4) give

$$(5.11) \qquad p_0 = g_0(z_1)/9 + 8g_0(z_2)/9 + (2/3)(\zeta/3)^{1/2}g_0'(z_2).$$

Thus, once we have determined $g_0(z_1)$, $g_0(z_2)$ and $g_0'(z_2)$, equations (5.9) - (5.11) will provide us with values for our coefficients.

To calculate the $g_0(z_i)$ and $g_0'(z_2)$, we return to the mapping (3.4). If we differentiate (3.4) twice with respect to z (bearing in mind that $g_0(z) = dt/dz$), we obtain

$$f_-''(t)g_0^2(z) + f_-'(t)g_0'(z) = 3z^2 - \zeta.$$

Evaluation at $t = t_0 \leftrightarrow z = z_2$ yields no information (since $f_-'(t_0) = f_-''(t_0) = 0$

at the order two saddle t_0), while evaluation at $t = t_2 \leftrightarrow z = z_1$ yields

$$f_-''(t_2)g_0^2(z_1) = 3z_1^2 - \zeta = 3(2(\zeta/3)^{1/2})^2 - \zeta = 3\zeta,$$

or

$$g_0(z_1) = \pm \sqrt{(3\zeta/f_-''(t_2))}.$$

Reasoning as was done for the Pearcey integral (cf. Ch. II, § 4) shows that we must take the positive square root (briefly, $f_-''(t_2) > 0$ since t_2 is a local minimum of f_- so we see that the ratio inside the square root is positive; since z must increase with t, the positive square root must be extracted). Therefore,

$$g_0(z_1) = \sqrt{(3\zeta/f_-''(t_2))}. \tag{5.12}$$

To continue, we differentiate $f_-''(t)g_0^2(z) + f_-'(t)g_0'(z) = 3z^2 - \zeta$ again with respect to z to get

$$f_-'''(t)g_0^3(z) + 3f_-''(t)g_0(z)g_0'(z) + f_-'(t)g_0''(z) = 6z. \tag{5.13}$$

Evaluation of (5.13) at $z = z_2 \leftrightarrow t = t_0$ yields

$$f_-'''(t_0)g_0^3(z_2) = 6z_2 = 6(-(\zeta/3)^{1/2})$$

or

$$g_0(z_2) = [-6(\zeta/3)^{1/2}/f_-'''(t_0)]^{1/3}. \tag{5.14}$$

Differentiate (5.13) with respect to z to get

$$f_-''''(t)g_0^4(z) + 6f_-'''(t)g_0^2(z)g_0'(z)$$

$$+ 3f_-''(t)(g_0'(z))^2 + 4f_-''(t)g_0(z)g_0''(z) + f_-'(t)g_0'''(z) = 6.$$

Evaluating this at $z = z_2 \leftrightarrow t = t_0$ gives us, after some arithmetic,

$$(5.15) \qquad g_0'(z_2) = [6 - f_-''''(t_0)g_0^4(z_2)]/(6f_-'''(t_0)g_0^2(z_2)).$$

It is, at this point, clear that $g_0(z_1)$, $g_0(z_2)$ and $g_0'(z_2)$ can be expressed in terms of f_-, its derivatives, and ζ.

We turn to the calculation of these latter quantities. To begin, we apply the binomial theorem in the form

$$(1 + x)^{1/2} = 1 + x/2 - x^2/8 + x^3/16 + O(x^4)$$

to (5.1) with $x = -9(\Delta c)(1 + 2(\Delta c/3)^{1/2})$ to obtain

$$(5.16) \qquad b = (4/3\sqrt{6})[1 - 9(\Delta c)/2 - 9(\Delta c)^{3/2}/\sqrt{3} - 81(\Delta c)^2/8$$

$$- 81(\Delta c)^{5/2}/(2\sqrt{3}) - 27 \cdot 35(\Delta c)^3/16 + O(\Delta c)^{7/2}].$$

For the limiting forms of the saddles t_0 and t_2, we note that $\psi = -\pi/6$ implies, from (2.4),

$$(5.17) \qquad \begin{aligned} e_1 &= (\Delta c/3)^{1/2} \\ e_2 = e_3 &= -(\Delta c/3)^{1/2}/2; \end{aligned}$$

recall that we have set $c = \Delta c - 1/12$.

Therefore,

$$\begin{aligned} t_0 &= \sqrt{(1/6 - e_1)} + \sqrt{(1/6 - e_2)} - \sqrt{(1/6 - e_3)} \\ &= \sqrt{(1/6 - e_1)} \\ &= (1/\sqrt{6})\sqrt{(1 - 6(\Delta c/3)^{1/2})}. \end{aligned}$$

We apply the binomial formula again, but unlike the case for b, we must use the higher order approximation

$$(1+x)^{1/2} = 1 + x/2 - x^2/8 + x^3/16 - 5x^4/128 + 7x^5/256 - 21x^6/1024 + O(x^7)$$

with $x = -6(\Delta c/3)^{1/2}$ in order to provide an approximation of t_0 to the same order as that for b in (5.16). Doing this gives

$$\begin{aligned}(5.18)\qquad t_0 - 1/\sqrt{6} = (-1/\sqrt{6})[\sqrt{3}(\Delta c)^{1/2} &+ 3(\Delta c)/2 + 9(\Delta c)^{3/2}/(2\sqrt{3}) \\ &+ 45(\Delta c)^2/8 + 7\cdot 27(\Delta c)^{5/2}/(8\sqrt{3}) \\ &+ 21\cdot 27(\Delta c)^3/16 + O(\Delta c)^{7/2}]\end{aligned}$$

as $\Delta c \to 0$.

We proceed similarly for t_2:

$$\begin{aligned}t_2 &= -\sqrt{(1/6 - e_1)} + \sqrt{(1/6 - e_2)} + \sqrt{(1/6 - e_3)} \\ &= -\sqrt{(1/6 - e_1)} + 2\sqrt{(1/6 - e_2)} \\ &= -t_0 + (2/\sqrt{6})\sqrt{(1 + (3\Delta c)^{1/2})}.\end{aligned}$$

Applying the binomial approximation used above, in addition to the approximation (5.18), yields

$$\begin{aligned}(5.19)\qquad t_2 - 1/\sqrt{6} = (1/\sqrt{6})[2\sqrt{3}(\Delta c)^{1/2} &+ 3(\Delta c)/4 + 15\sqrt{3}(\Delta c)^{3/2}/8 \\ &+ 7\cdot 45(\Delta c)^2/64 + 17\cdot 63\sqrt{3}(\Delta c)^{5/2}/128 \\ &+ 21\cdot 27\cdot 31(\Delta c)^3/512 + O(\Delta c)^3].\end{aligned}$$

We are now in a position to calculate $f_-(t_0) - f_-(t_2)$ which we shall see is required in the calculation of ζ. We will use a Taylor expansion of f_- about the point $t = 1/\sqrt{6}$ in this computation:

$$\begin{aligned}(5.20)\qquad f_-(t) = f_-(1/\sqrt{6}) &+ f_-'(1/\sqrt{6})(t - 1/\sqrt{6}) + (1/2)f_-''(1/\sqrt{6})(t - 1/\sqrt{6})^2 \\ &+ (1/\sqrt{6})(t - 1/\sqrt{6})^4 + (1/5)(t - 1/\sqrt{6})^5.\end{aligned}$$

From (5.16), we find

(5.21) $$f_-(1/\sqrt{6}) = -1/45\sqrt{6} + (\Delta c)/2\sqrt{6} - (\Delta c)^{3/2}/3\sqrt{2} - 9(\Delta c)^2/8\sqrt{6} - 3(\Delta c)^{5/2}/2\sqrt{2} - 3\cdot 35(\Delta c)^3/16\sqrt{6} + O(\Delta c)^{7/2},$$

(5.22) $$f_-'(1/\sqrt{6}) = -[2(\Delta c)^{3/2}/\sqrt{3} + 9(\Delta c)^2/4 + 9(\Delta c)^{5/2}/\sqrt{3} + 3\cdot 35(\Delta c)^3/8 + O(\Delta c)^{7/2}]$$

and

(5.23) $$f_-''(1/\sqrt{6}) = -(4/\sqrt{6})[3(\Delta c)/2 + \sqrt{3}(\Delta c)^{3/2} + 27(\Delta c)^2/8 + 27(\Delta c)^{5/2}/2\sqrt{3} + 9\cdot 35(\Delta c)^3/16 + O(\Delta c)^{7/2}].$$

Thus, use of (5.20) - (5.23) with each of (5.18) and (5.19) gives

(5.24) $$f_-(t_0) = -1/45\sqrt{6} + (\Delta c)/2\sqrt{6} - (\Delta c)^{3/2}/3\sqrt{2} - 3(\Delta c)^2/8\sqrt{6} - 3(\Delta c)^{5/2}/10\sqrt{2} - 15(\Delta c)^3/16\sqrt{6} + O(\Delta c)^{7/2}$$

and

(5.25) $$f_-(t_2) = -1/45\sqrt{6} + (\Delta c)/2\sqrt{6} - (\Delta c)^{3/2}/3\sqrt{2} - 57(\Delta c)^2/8\sqrt{6} - 42(\Delta c)^{5/2}/5\sqrt{2} - 15\cdot 83(\Delta c)^3/32\sqrt{6} + O(\Delta c)^{7/2}.$$

Subtracting (5.25) from (5.24) gives

(5.26) $$f_-(t_0) - f_-(t_2) = (27/4\sqrt{6})(\Delta c)^2 [1 + 6\sqrt{3}(\Delta c)^{1/2}/5 + 45(\Delta c)/8 + O(\Delta c)^{3/2}].$$

Since $\sigma_1^2 - 3\sigma_2 = (f_-(t_0) - f_-(t_2))^2$ on this portion of the caustic (see the discussion following (3.35)), we have $(\sigma_1^2 - 3\sigma_2)^{1/2} = f_-(t_0) - f_-(t_2)$.

Observe that (3.35) is equivalent to the expression num = denom, where num and denom are presented in (3.32). This in turn implies that χ, in (3.39), is zero. Hence, (5.26) together with $\chi = 0$ and (3.43) yield

(5.27) $$\zeta = 2^{-1/4}3^{3/4}(\Delta c)[1 + 3\sqrt{3}(\Delta c)^{1/2}/5 + 9\cdot 101(\Delta c)/400 + O(\Delta c)^{3/2}].$$

With ζ in hand, we can proceed to the calculation of $g_0(z_1)$, $g_0(z_2)$ and $g_0'(z_2)$. We obtain $g_0(z_1)$ first. Differentiating (5.20) twice with respect to t and using (5.19) and (5.23) in the result gives

$$(5.28) \qquad f_-''(t_2) = (18/\sqrt{6})\,[(\Delta c) + \sqrt{3}(\Delta c)^{3/2} + 45(\Delta c)^2/16 + 33\sqrt{3}(\Delta c)^{5/2}/8 + O(\Delta c)^3].$$

Dividing (5.28) into 3ζ gives

$$3\zeta/f_-''(t_2) = 2^{-3/4}3^{1/4}\,[1 - 2\sqrt{3}(\Delta c)^{1/2}/5 + 33(\Delta c)/50 + O(\Delta c)^{3/2}]$$

so that, from (5.12),

$$(5.29) \qquad g_0(z_1) = 2^{-3/8}3^{1/8}\,[1 - \sqrt{3}(\Delta c)^{1/2}/5 + 27(\Delta c)/100 + O(\Delta c)^{3/2}].$$

To get $g_0(z_2)$, we first compute the limiting behaviour of $f_-'''(t_0)$. Differentiate (5.20) once more and use (5.18) to get

$$(5.30) \qquad f_-'''(t_0) = -4\sqrt{3}\,(\Delta c)^{1/2} + O(\Delta c)^{7/2}.$$

With (5.14), (5.27), (5.30) and the binomial approximations

$$(1 + x)^{1/2} = 1 + x/2 - x^2/8 + O(x^3)$$

and

$$(1 + x)^{1/3} = 1 + x/3 - x^2/9 + O(x^3),$$

we have

$$(5.31) \qquad g_0(z_2) = 2^{-3/8}3^{1/8}[1 + \sqrt{3}(\Delta c)^{1/2}/10 + 3\cdot 81(\Delta c)/800 + O(\Delta c)^{3/2}].$$

To obtain $g_0'(z_2)$, we differentiate (5.20) one last time and put $t = t_0$, with t_0 as given in (5.18). This gives

$$(5.32) \qquad f_-^{(iv)}(t_0) = (24/\sqrt{6})[1 - \sqrt{3}(\Delta c)^{1/2} - 3(\Delta c)/2 + O(\Delta c)^{3/2}].$$

Using (5.30) to (5.32) in (5.15), we arrive at

$$(5.33) \qquad g_0'(z_2) = -2^{-5/4}3^{3/4}5^{-1}[1 + 21\sqrt{3}(\Delta c)^{1/2}/40 + O(\Delta c)].$$

With $g_0(z_1)$, $g_0(z_2)$ and $g_0'(z_2)$ at our disposal, we are in a position to calculate the coefficients p_0, q_0 and r_0. We do r_0 first.

From (5.29) and (5.31), we have

$$(5.34) \qquad g_0(z_1) - g_0(z_2) = -2^{-3/8}3^{1/8}[3\sqrt{3}(\Delta c)^{1/2}/10 + 27(\Delta c)/800 + O(\Delta c)^{3/2}],$$

and from (5.33) and (5.27),

$$3(\zeta/3)^{1/2} g_0'(z_2) = -2^{-11/8}3^{13/8}5^{-1}(\Delta c)^{1/2}[1 + 33\sqrt{3}(\Delta c)^{1/2}/40 + O(\Delta c)].$$

Taking the difference of these two gives

$$g_0(z_1) - g_0(z_2) - 3(\zeta/3)^{1/2}g_0'(z_2) = (2^{-11/8}3^{13/8}3\cdot 21\cdot\sqrt{3}/400) \cdot(\Delta c)[1 + O(\Delta c)^{1/2}],$$

so that

$$(5.35) \qquad r_0 = (2^{-1/8}3^{3/8}63/(32\cdot 25))[1 + O(\Delta c)^{1/2}]$$

upon division by 3ζ.

From (5.10), (5.27), (5.33) and (5.34), we have

$$(5.36) \qquad q_0 = -2^{-5/4}3^{3/4}5^{-1}[1 + O(\Delta c)^{1/2}].$$

In view of the fact $\zeta \to 0$ as $\Delta c \to 0$, we see, from (5.11), (5.29) and (5.31), that

$$(5.37) \qquad p_0 = 2^{-3/8}3^{1/8}[1 + O(\Delta c)^{1/2}].$$

To compare the limit, as $\Delta c \to 0$, of (3.53) with the classically obtained result (3.3) requires the calculation of $P(0, 0)$, $P_x(0, 0)$ and $P_y(0, 0)$, where P is the Pearcey integral of Chapter II. Standard techniques give

$$P(0,0) = 2^{-1/2}\Gamma(1/4)e^{\pi i/8}$$
$$P_x(0,0) = i2^{-1/2}\Gamma(3/4)e^{3\pi i/8}$$
$$P_y(0,0) = 0,$$

so that use of these values for the Pearcey function and its derivatives, along with (5.35) - (5.37), in (3.53) gives

$$\begin{aligned} &[p_0(4/3\sqrt{6}, -1/12)\lambda^{-1/4}P(0,0) \\ &\quad - i\cdot q_0(4/3\sqrt{6}, -1/12)\lambda^{-1/2}P_y(0,0) \\ &\quad - 2ir_0(4/3\sqrt{6}, -1/12)\lambda^{-3/4}P_x(0,0)]\, e^{i\lambda f_-(1/\sqrt{6})} \\ &= 2^{-7/8}3^{1/8}\lambda^{-1/4}\Gamma(1/4)e^{\pi i/8 - i\lambda/(45\sqrt{6})} \\ &\quad + (2^{3/8}3^{3/8}\lambda^{-3/4}\cdot 63\cdot\Gamma(3/4)e^{3\pi i/8 - i\lambda/(45\sqrt{6})})/(32\cdot 25). \end{aligned} \tag{5.38}$$

(5.38), together with $f_-(-3/\sqrt{6}) = 7/5\sqrt{6}$ and $f_-''(-3/\sqrt{6}) = -32/3\sqrt{6}$ ($t_3 = -3/\sqrt{6}$ at the cusp), shows that (3.53) agrees with the classically obtained approximation given in (3.3).

6. Asymptotics Elsewhere on the Caustic

Recall figure 4.1. Our discussion of the asymptotics of $I_-(\lambda; b, c)$ (and therefore of $Q(-x, y, z)$ for large x; cf. (1.3)) has been concerned with values of (b, c) near the caustic with $b \geq b_0 > 0$, close to the arcs joining (0, 0) to $(4/3\sqrt{6}, -1/12)$ or (0, 1/4) to $(4/3\sqrt{6}, -1/12)$. In view of the fact that $I_-(\lambda; b, c)$ and $I_-(\lambda; -b, c)$ are complex conjugates (recall the discussion in § 1 of chapter III), we see that we can readily obtain the uniform asymptotic behaviour of $I_-(\lambda; b, c)$ near the caustic for $b \leq b_0 < 0$, close to the arcs joining (0, 0) to $(-4/3\sqrt{6}, -1/12)$ or joining $(-4/3\sqrt{6}, -1/12)$ to (0, 1/4).

The restriction $b \geq b_0 > 0$ can be weakened in our discussion of the asymptotics of $I_-(\lambda; b, c)$ to $b \geq 0$, provided we restrict (b, c) to a band around the arc of the caustic joining (0, 0) to $(4/3\sqrt{6}, -1/12)$. To see this, we note that at (b, c) = (0, 0), $f_-'(t) = t^4 - t^2$, which has $t_3 = -1$, $t_0 = t_1 = 0$, and $t_2 = 1$ as zeroes. Thus, our work, based on the ordering of the zeroes t_i, remains valid for $b \geq 0$.

Along the arc of the caustic joining $(4/3\sqrt{6}, -1/12)$ to (0, 1/4), we have ψ (of equation (2.5)) equal to $\pi/6$. Along this portion of the caustic, $t_1 = t_2$ since $e_1 = e_2$ there (cf. (2.4) and (2.6)). As (b, c) approaches (0, 1/4) along this arc, the saddles t_3 and t_0 coalesce to $-1/\sqrt{2}$ with $t_1 = t_2 = 1/\sqrt{2}$. Thus, the asymptotics of $I_-(\lambda; b, c)$ can be dealt with through the use of cubic transformations after the fashion of Chester *et al* **[CFU]**; one cubic transformation would be used with the pair t_3 and t_0, another being applied to the pair t_1 and t_2.

For those portions of the caustic above the line c = 1/4, we are presented with a situation in which exactly two real saddles coalesce, and then split into a complex conjugate pair as we pass through the caustic from below. This setting, as before, is amenable to the method of Chester *et al*.

If it should happen that x is not a large parameter in Q(-x, y, z), and both y and z are large, then we would use the integral J, presented in equation (3.1.4), to examine the uniform asymptotic behaviour of Q (of course, if only one of x, y, or z is large, there is no need for uniform asymptotic methods). The discussion in § 6 of chapter III applies in this setting.

Finally, as we have noted in the previous two chapters, termwise differentiation can be used to obtain uniform asymptotic expansions of the first order partial derivatives of Q for the case where the first argument of Q tends to $-\infty$.

Appendix

From the three identities

(1) $z_1 + z_2 + z_3 = 0$

(2) $z_1 z_2 + z_1 z_3 + z_2 z_3 = -\zeta$

(3) $z_1 z_2 z_3 = -\eta$

(see equations (4.3.9)) we proceed to develop a number of formulae useful in obtaining (4.3.19), (4.3.20) and (4.3.21). Only a brief indication of the calculation involved is given.

(A1) $\sum_i z_i^2 = 2\zeta$

Square (1) and apply (2).

(A2) $\sum_{i<j} (z_i^2 z_j + z_j^2 z_i) = 3\eta$

Form the product $0 = (z_1 z_2 + z_1 z_3 + z_2 z_3)(z_1 + z_2 + z_3)$ and apply (3).

(A3) $\sum_i z_i^3 = -3\eta$

Form the product $0 = (z_1^2 + z_2^2 + z_3^2)(z_1 + z_2 + z_3)$ and apply (A2).

(A4) $\sum_{i<j} z_i^2 z_j^2 = \zeta^2$

Square (2) and apply (1).

(A5) $$\sum_i z_i^4 = 2\zeta^2$$

Square (1) and apply (A4).

(A6) $$\sum_{i<j} (z_i^4 z_j + z_j^4 z_i) = 5\eta\zeta$$

Form the product $(z_1z_2 + z_1z_3 + z_2z_3)(z_1^3 + z_2^3 + z_3^3) = 3\eta\zeta$ and apply (A1) and (3).

(A7) $$\sum_{i<j} z_i^3 z_j^3 = 3\eta^2 - \zeta^3$$

Form the product $(z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2)(z_1z_2 + z_1z_3 + z_2z_3) = -\zeta^3$ and apply (3) and (A2).

(A8) $$\sum_{i<j} (z_i^4 z_j^2 + z_j^4 z_i^2) = 2\zeta^3 - 3\eta^2$$

Square $(z_1^2z_2 + z_1z_2^2 + z_1^2z_3 + z_1z_3^2 + z_2^2z_3 + z_2z_3^2) = 3\eta$ and apply (3), (A7), and (A3).

(A9) $$\sum_{i<j} z_i^4 z_j^4 = \zeta^4 - 4\eta^2\zeta$$

Square $(z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2) = \zeta^2$ and apply (3) and (A1).

(A10) $$\sum_{i<j} (z_i z_j^3 + z_j z_i^3) = -2\zeta^2$$

Form the product $(z_1z_2 + z_1z_3 + z_2z_3)(z_1^2 + z_2^2 + z_3^2) = -2\zeta^2$ and apply (1).

The derivation of $\sigma_1 = -\zeta^2/2 + 3\theta$ has been done in chapter 4, § 3; to obtain

(4.3.20), we begin with

$$\sigma_2 = \sum_{i<j} z_i^4 z_j^4/16 - \zeta \sum_{i<j} (z_i^2 z_j^4 + z_j^2 z_i^4)/8 + \eta \sum_{i<j} (z_i z_j^4 + z_j z_i^4)/4$$

$$+ \theta \sum_{i} z_i^4/2 + \zeta^2 \sum_{i<j} z_i^2 z_j^2/4 - \zeta\eta \sum_{i<j} (z_i z_j^2 + z_j z_i^2)/2$$

$$- \zeta\theta \sum_{i} z_i^2 + \eta^2(z_1 z_2 + z_1 z_3 + z_2 z_3) + 2\eta\theta(z_1 + z_2 + z_3) + 3\theta^2.$$

(4.3.21) is obtained by beginning with

$$\sigma_3 = f_-(t_0) f_-(t_1) f_-(t_2) = g(z_1) g(z_2) g(z_3)$$

$$= z_1^4 z_2^4 z_3^4/64 - (\zeta/32)(z_1^2 z_2^2 z_3^2) \sum_{i<j} z_i^2 z_j^2 + (\eta/16)(z_1 z_2 z_3) \sum_{i<j} z_i^3 z_j^3$$

$$+ (\theta/16) \sum_{i<j} z_i^4 z_j^4 + (\zeta^2/16)(z_1^2 z_2^2 z_3^2) \sum_{i} z_i^2 - (\eta\zeta/8)(z_1 z_2 z_3) \sum_{i<j} (z_i z_j^3 + z_j z_i^3)$$

$$- (\theta\zeta/8) \sum_{i<j} (z_i^2 z_j^4 + z_j^2 z_i^4) + (\eta^2/4)(z_1 z_2 z_3) \sum_{i} z_i^3$$

$$+ (\eta\theta/4) \sum_{i<j} (z_i z_j^4 + z_j z_i^4) + (\theta^2/4) \sum_{i} z_i^4 - (\zeta^3/8)(z_1^2 z_2^2 z_3^2)$$

$$+ (\eta\zeta^2/4)(z_1 z_2 z_3)(z_1 z_2 + z_1 z_3 + z_2 z_3) + (\theta\zeta^2/4) \sum_{i<j} z_i^2 z_j^2$$

$$- (\eta^2\zeta/2)(z_1z_2z_3)(z_1 + z_2 + z_3) - (\eta\zeta\theta/2)\sum_{i<j}(z_iz_j^2 + z_jz_i^2)$$

$$- (\zeta\theta^2/2)\sum_i z_i^2 + \eta^3(z_1z_2z_3) + \theta\eta^2(z_1z_2 + z_1z_3 + z_2z_3)$$

$$+ \eta\theta^2(z_1 + z_2 + z_3) + \theta^3.$$

Equations (4.3.20) and (4.3.21) follow upon application of (1) - (3) and (A1) - (A10).

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