The Role of Mathematics in Physics Education as Represented in High School and Introductory Level College Physics Textbooks: Using the Law of Universal Gravitation as an Example

by

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Dissertation submitted to the Faculty of Graduate Studies of the

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Abstract

This thesis is concerned with the mode of presentation of the mathematical aspects of the teaching of high school physics. Eight recent high school and introductory level college physics textbooks were analyzed in this study. The topic of Universal Gravitation was chosen as the ideal context for the analysis of the mathematical component of physics in these textbooks. The research generated an instrument for the qualitative analysis of textbooks. The instrument was grounded on a historical inquiry into the relationship between mathematics and physics, and the history of gravity, mainly based on Newton's discovery of the universal law of gravitation. The study paid special attention to the ideas of contemporary learning theories and the requirements of scientific literacy.

It was found that mathematical concepts engaged in the topic of universal gravitation were presented in various modes. However, graphical modes of presentation, which are necessary in visualizing functional relationships, were not used by many of the textbooks. The examined texts demonstrated different ways of establishing connections between mathematical concepts. For example, few of the analyzed textbooks used analogies for the connections between mathematical concepts. Moreover, the textbooks exhibited varying degrees of balance between the qualitative and the quantitative aspects of physics as found in example problems on the law of universal gravitation. The presentation of mathematical concepts through the history and philosophy of science (HPS) in the unit on universal gravitation in these textbooks mostly utilized a descriptive mode rather than both a descriptive and instructional approach.

The findings from this study have several implications for educators and textbook writers. In order to facilitate effective learning, textbooks need to present physics concepts using a variety of modes. The study suggests that numerical data should be presented and used in a more interactive way. It is crucial for graphs to appear in textbooks not as simple illustrations of the narration but as dynamically engaging and interactive vehicles for learning. Pictorial representations provide further rich opportunities for improving students' comprehension. Moreover, the introduction of HPS in physics textbooks can help students understand better the emergence of mathematical relationships used to represent physics concepts, laws, and theories. Textbooks that emphasize the use of mathematical models in science and thought experiments further encourage students' learning. Accordingly, Newton's geometry, a mathematical visual tool which Newton used to conceptualize gravity, should be incorporated into physics textbooks.

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Chapter 1: Introduction

The Research Problem

The mathematical component of physics has been a subject of concern for physics educators (Stinner, 1992; de Berg, 1995; Lavaly, 1990; Rice-Evans, 1992; Jones, 1992; Monk, 1994; Hewitt, 1994). This concern sounded loud, and it seems to be a trend in the 90s. The educators were awaken by the fact established by the research on the conventional physics instruction (McDermott, 1984; Maloney, 1994; Wandersee, Mintzes, & Novak, 1994; Huffman, 1997). This research showed that after conventional (mathematics-based problem-solving) instruction students can not fully explain even the simplest of physics concepts, even though many could work out related problems. The results of the Force Concept Inventory (Hestenes et al., 1992) and the Newton's Laws Test (Heller & Hollabaugh, 1992) both indicated that the explicit strategy did not improve students' conceptual understanding of Newton's laws more than the textbook strategy. In addition, McDermott (1984), who did studies on high school and college students' understanding of mechanics indicates that even after instruction, anywhere from 25% to 50% of the students still fail to answer conceptual questions correctly. As a consequence of such situation in physics education, the drop out rate from physics classes increased, especially physics courses were not popular among girls. Students were finding the mathematical part of physics difficult. It was also discovered that the physical science courses which rely on mathematical formulas and the use of problems that depend mainly on substitution into equations, followed by algorithmic solutions for the unknowns, are perceived by students to be dull and boring. This way of teaching and testing physics has been used since physics textbooks were introduced in the early 19th

century (Stinner, 1992). Generally, students find this approach uninteresting, providing insufficient background to develop a sound understanding of the concepts discussed. The concerns of educators about the algorithmic way of teaching and presenting physics material in textbooks probably motivated Paul Hewitt, the author of *Conceptual Physics*, to write a different physics textbook which considers social perspectives on education. In the introduction to his *Conceptual Physics* (2002) he asserts:

> *The value of teaching physics conceptually is not in minimizing mathematics, but in maximizing the use of students' personal experience in the everyday world and in their everyday language. Students need not see physics as a hodgepodge of mechanistic equations, or only as a classroom or laboratory activity, but should see physics everywhere, as part of everything they experience* (p. 11).

From my teaching experience, I find that students believe that competence in physics is expressed in the knowledge of many formulas. Formulas are "exact" and seem to provide orientation and "scrutiny" in physical contexts. Standard examination tasks, which have often more to do with mathematical rules than with the reflection on physics concepts, strengthen this view, when students mainly have to memorize formulas (Hestenes, 1995). Emphasis on symbolism in physics teaching is a real problem. As De Lozano and Cardenas (2002) note, "in the classroom and in textbooks no attention is given to the importance of providing special attention to the interpretation of the symbolism" (p. 591).

The problem described is significant because what determines the way physics is taught is the way one sees the role mathematics should play in physics education. The criticism of emphasizing mathematical complexity in the absence of meaning applies also to the way the concepts are presented in physical sciences textbooks. Therefore, the studies of textbooks where the treatment of the mathematical component of physics is researched can bring some understanding of how physics textbooks could be improved. Definitely, textbooks are still used as the primary source of information in the science classroom (Yager, 1984; Jeffery & Roach, 1994; Jones, 2000). According to Pratt (1985), Dall'Alba et al. (1993), Chiappetta, Sethna, and Fillman (1991), and Stinner (1998), the textbook is still the principal guide used by teachers to plan a curriculum. For example, Pratt (1985) asserts:

> *In that the textbook may be the single most important indicator of the content of the science curriculum, the manner in which mathematics is treated in the textbook is crucial to understanding how mathematics is used in science courses.* (pp. 394-395)

I would agree with Pratt that the research about what role mathematics plays in science textbooks (and in physics textbooks since physics is an area of science) needs more attention. The way textbooks present the mathematical component often determines how teachers view and use mathematics in their teaching, and how students understand physics when they study from the text (To be able to market their textbooks, textbook publishers have to satisfy the demands of curriculum. Therefore, physics textbooks reflect how curriculum developers view the role of mathematics in physics). Indeed, it is still the fact that textbooks are the primary source of reference both for teachers and students.

Purpose of the Study

As research in science education shows, the mathematical complexity of physics has been identified as one of the major factors that prevent students from studying physics (Lavaly, 1990; Rice-evans, 1992; Jones, 1992; Monk, 1994; Hewitt, 1994). The presentation of the mathematical aspect of physics in teaching and textbooks is criticized for lack of connections among concepts, formal representations, and the real world. The lack of these connections in physics textbooks inhibits students' understanding of physics ideas when they learn from texts. My argument is: **To ensure understanding when students learn from a text, it is crucial that physics textbooks maintain a balance between quantitative and qualitative aspects of physics. To achieve this goal, the mathematics used in physics textbooks must play an appropriate role in placing and finding ways of presentation of physics ideas.**

Therefore, the purpose of this study is to identify, describe and analyze the role mathematics has in high school physics textbooks, to find out if it is used in appropriate sequence in the presentation of physics concepts, and to examine the modes in which mathematics is expressed in the texts. Ultimately, the purpose of this analysis is to understand what role mathematics plays, or is expected to play in the development of concepts and ideas in physics. As physics educators, we want to ensure that mathematics is being utilized for more than a computational tool in solving problems. We also want to make teachers aware that the qualitative and quantitative aspects of physics are clearly identified, in order to achieve the balance between them.

Research Questions

The research problem outlined at the beginning of this chapter determined my motivation to conduct this research. My interest in this area of science education has been sparked by the questions raised by the researchers as well as my own questions that have lingered in my mind throughout my career as a teacher. Based on the research problem, four major questions were formulated that have guided the direction of my research. To address the issues raised by educators about the mathematical complexity of physics, I propose that physics textbooks writers should insure the balance between the qualitative and the quantitative aspects of physics these textbooks present. My expectation is that the answer to the following research question and its sub-questions would validate this proposition:

Research Question 1: What is the rationale for balancing of the qualitative and the quantitative aspects of physics in physics textbooks?

(a) How do contemporary learning theories and the requirements of scientific literacy inform us about the appropriate ways of presenting the qualitative and the quantitative aspects of physics?

(b) What are the pedagogical considerations found in the educational research literature that support the idea of balancing the qualitative and the quantitative aspects of physics?

The mathematician-historian Morris Kline (1959) said that historically, intellectually, and practically, mathematics was primarily man's creation for the investigation of nature. On the other hand, the major mathematical concepts, methods, and theorems have been derived from the study of nature. Mathematics organizes broad classes of natural phenomena into coherent, deductive patterns. Today mathematics is in the heart of our best scientific theories: Newtonian mechanics, the electromagnetic theory of Maxwell, Einsteinian theory of relativity, and quantum theory of Planck and his successors. Indeed, there is a close relationship between mathematics and physics. Do they complement and aid in the development of each other? Does physics, as a science of moving forms of matter, have a mathematical component due to the fundamental connection between physics and mathematics? These kinds of questions lead to the more detailed exploration that can be captured by the following research question.

Research Question 2: What is the historical relationship between mathematics and physics?

Since the main objective of this study is to find how well the balance between the quantitative and the qualitative aspects of physics is maintained in physics textbooks, it is necessary to explore what role mathematics plays in physics education and how it is presented in physics textbooks. The textbook topic of universal gravitation has been chosen because it is conceptually, as well as mathematically rich. Equations, graphs representing functional relationships between variables, and reasoning based on mathematical statements are utilized to construct meanings of the phenomenon of gravity. This topic is also broad enough. Many concepts from other topics are connected to the topic of universal gravitation since they are used for introduction and interpretations of the gravitational phenomena. For example, Newton's inverse square law is used for calculating gravitational force, as it applies locally where **g** is constant. Kepler's three laws of planetary motion are introduced for describing planetary motion governed by gravitational interactions explained by Newton. Uniform circular motion and centripetal acceleration are introduced to calculate the orbital velocity of planets. Projectile motion is also related to gravitational phenomena since the motion of projectiles is governed by the same force as the motion of planets, namely the force of gravity. Equations and their manipulations, conic sections, estimations, calculations represent an extensive mathematical component involved in these subtopics. Thus, the topic of universal gravitation is broad enough to explore the mathematical component of physics in its multifaceted role.

Research consistently shows (Arons, 1990) that the topic of universal gravitation is one of the most difficult for students. Students generally exhibit many misconceptions and are confused about gravity and gravitational effects. According to Arons (1990), these misconceptions "are rarely spontaneously articulated by the students, that frequently pass unnoticed by teachers, and that seriously impede understanding of the material taught" (p. 69). In addition, the mathematical component of the law of gravity is not as simple, as it seems either. As Wigner (1960) points out,

> *...the law...is simple only to the mathematician, not to common sense or to non-mathematically minded freshmen; second, it is a conditional law of very limited scope. It explains nothing about the earth, which attracts Galileo's rocks, or about the circular form of the moon's orbit, or about the planets of the sun. The explanation of these initial conditions is left to the geologist and the astronomer, and they have a hard time with them.* (p. 531)

Since the scope of the major question, as suggested by the title of my thesis, will be within the context of the topic universal gravitation, it is unavoidable that we explore the history of universal gravitation. The history of gravity will have answers on how, for example, Newton and Leibniz invented calculus (Fluxions), which helped Newton to formulate the law of universal gravitation. What background knowledge did Newton have that helped him in the development of his famous law? With whom, if anybody, did he communicate in formulating his law of universal gravitation? What new mathematics did Newton invent for his law? These kinds of questions are impossible to answer if we do not look at the historical development of the concept. All these questions raised lead to the more detailed exploration that can be captured by the following research question and its sub-questions.

Research Question 3: What role did mathematics play in the history of gravity?

(a) What is the history of gravity?

(b) What are the stages of Newton's thinking when he describes universal gravitation?

High school and introductory level college physics textbooks are going to be used in the intended research to explore the role of mathematics in presentation of material on universal gravitation. The presentation of the mathematical component of the physics found in the textbooks can inform us how mathematics is expected to be treated in physics education, and how the balance between the qualitative and the conceptual aspects of physics is insured in these textbooks.

According to contemporary learning theories, there are two mechanisms involved in learning, namely assimilation and accommodation of ideas. Learning, as the result of the process of equilibration between these mechanisms, is the main goal teachers should try to achieve. In this study, I would like to find out to what extent high school physics textbooks present the ideas of learning theories. The pedagogical sequence of introducing and developing concepts in science is crucial in the understanding of the mathematical component of physics if we want to achieve equilibration between assimilation and accommodation. Science textbooks are still the major source of information both for students and teachers, and very often the way physics textbooks present the role of mathematics in the presentation of a topic is how teachers and students are going to perceive it. Therefore, the next research question is:

Research Question 4: How is the rationale for the balancing of the qualitative and quantitative aspects of physics, found from Question 1, reflected in the contemporary high school and introductory level college physics textbooks in the presentation of topic of universal gravitation?

(a) What findings from research Questions 1 and 2 can be used to develop an instrument for the analysis of the mathematical component of physics presented in high school and introductory level college physics textbooks in the topic of universal gravitation?

(b) What are the modes of the mathematical presentation of concepts found in high school and introductory level college physics textbooks in the topic of universal gravitation?

(c) What is the pedagogical sequence of presentation of the mathematical component found in high school and introductory level college physics textbooks in the topic of universal gravitation?

(d) How are the ideas of contemporary learning theories and the requirements of scientific literacy reflected in the presentation of the mathematical component of physics in physics textbooks?

Significance of the Study

To examine the role of mathematics in physics textbooks, an historical inquiry into the qualitative and the quantitative aspects of physics will be undertaken. This will provide teachers, especially inexperienced teachers, with valuable information about the problem of the balancing of these two aspects. This inquiry will help obtain examples of using mathematics as a conceptual tool in physics. The use of mathematics as a conceptual tool in physics textbooks would promote richer treatment of mathematical equations flooding our textbooks. This would enable learners to understand what these equations mean and to go further than using mathematics mainly for performing calculations to give the "right" answer.

The qualitative content analysis used for the intended research of physics textbooks will concentrate on the conceptually and mathematically rich topic of universal gravitation. On the completion of this research, I expect to develop pedagogical suggestions on how to improve significantly the presentation of this topic in physics textbooks and how mathematics can be used to show conceptual richness of the topic Universal Gravitation.

In summary, the significance of this study will be to show that textbooks can inform teachers, administrators, curriculum developers, and textbook writers on the deficiencies and strengths found in the treatment of the mathematical component of physics in the selected high school and introductory level college physics textbooks. Teachers should be aware of how mathematics is used in physics textbooks in order to be able to select textbooks and to plan supplementary material to fill the gaps if mathematical treatment of concepts and relationships between them is not presented in the fullest capacity to insure the balance between the quantitative and the qualitative aspects of physics. This study will not attempt to make recommendations about adopting specific texts because there could be other criteria than the treatment of the mathematical component of physics considered for the selection of a textbook. However, I will provide information to teachers, curriculum developers, and administrators that can assist them in textbook selection.

Limitations of the Study

Using textbooks as an instrument to study the mathematical component of physics has limitations. Education is a multifaceted process that involves many factors. The textbook treatment of the mathematical component of physics is only one factor that can inform us whether the approach and contexts within definitions and concepts are likely to promote a sound understanding of these concepts. Textbooks cannot take on the role of the teacher since learning and conceptual understanding happen in the ecology of a classroom. This study will address only the role of mathematics in physics education as reflected in physics textbooks. From the analysis of physics textbooks, in my study I will not be able to presume that this study will reflect how mathematics is actually used in a physics classroom by students and teachers. Although many teachers use the textbook as their primary source of information, it cannot be assumed that textbooks reflect what is actually taught in the classroom. The full treatment of the mathematical component of physics in a classroom, then, is a question for future studies.

This dissertation will examine only English physics textbooks in North America. However, the textbooks used in England, those in France, and Germany could offer a significantly different treatment of the mathematical component of physics. It is well known, for example, that German textbooks use calculus in their presentation of physics concepts. Is this approach detrimental to the qualitative understanding of physics concepts? Does it promote high-level algorithm memorization at the expense of qualitative understanding? German physics educators seem to think so.

Summary

Algorithmic way of teaching and testing physics has been used since physics textbooks were introduced in the early 19th century (Stinner, 1992). Students' dependence on algorithmic techniques in problem solving seldom generates conceptual understanding. This significant problem is described in this chapter. The way physics is taught determines the perceived way role of mathematics in physics education. On the contrary, the criticism of emphasizing mathematical complexity in the absence of meaning also applies to the way concepts are presented in physical sciences textbooks. The presentation of the mathematical component in textbooks often determines how teachers view and use mathematics in their teaching and how students understand physics when they study from the text. Therefore, the study the treatment of the mathematical component of physics textbooks can bring some understanding of how physics textbooks could be improved. The purpose of this study is to identify, describe and analyze the role mathematics plays in high school physics textbooks, to find out if it is used in an appropriate sequence in the presentation of physics concepts, and to examine the modes in which mathematics is expressed in the texts. Ultimately, the purpose of this analysis is to understand what role mathematics plays, or is expected to play in the development of concepts and ideas in physics.

Organization of the Study

This study critically analyzes physics high school and introductory level college textbooks in order to understand the role that mathematics plays or is expected to play in the development of concepts and ideas in physics. The study begins by identifying prevalence of the research problem in the educational community, formulating the purpose of the study, outlining research questions and their corresponding sub-questions, and presenting the anticipated significance and limitations of the study (Chapter 1).

The literature review that follows (Chapter 2) focuses on textbook research in science education in order to identify gaps in physics textbook research regarding the presentation of the mathematical component of physics. The literature review also includes discussion and criticism of some earlier mathematics-based physics textbooks to make a case for improvement of the presentation of the mathematical component in physics textbooks and initiate search for criteria applied to textbook analysis.

The search for criteria of the analysis of the mathematical component of physics textbooks as well as the design of the study which is guided by the researcher's theoretical assumptions and, consequently, theoretical and methodological framework chosen for this study are described in Chapter 3. This chapter concludes with the description of layers of content analysis of physics textbooks chosen for the study.

The following three chapters inform the construction of the instrument for the analysis of the mathematical component in physics textbooks. Chapter 4 enlightens the reader about the close historical connection between mathematics and physics, discusses examples of the predictive power of mathematics, and specifies the quantitative and the qualitative aspects of physics and mathematics. This chapter concludes with summarizing what educators have learned about the relationship between mathematics and physics from the history and philosophy of science. This summary contributes to the construction of the instrument for textbook analysis.

Chapter 5 presents a historical inquiry into the role of mathematics in the history of gravity. The factors that influenced Newton's reasoning in the development of his law of universal gravitation are critically presented. The described stages of Newton's thinking used in the development of his law of gravity inform a significant part of instrument construction for textbook analysis.

Chapter 6 focuses on the designing of an analytical instrument for textbook research. A conceptual framework for the analysis of textbooks is presented, and themes for the analysis of the mathematical component in physics textbooks are identified. An outline of the instrument for textbook analysis concludes this chapter.

This instrument was further used for textbook research. Chapter 7 reports findings of a qualitative content analysis of physics, the main purpose of which was to evaluate the degree of maintaining balance between the qualitative and the quantitative aspects of physics in the topic universal gravitation in order to establish the role of mathematics in physics textbooks. This chapter concludes with a discussion of the role mathematics plays in high school and introductory level college physics textbooks in the unit on universal gravitation.

The final chapter of this dissertation (Chapter 8) begins with an outline of the knowledge contribution to research in science education. An emphasis is placed specifically on the construction of a comprehensive instrument for the analysis of the mathematical component of physics in textbooks' unit on universal gravitation and on using history and philosophy of science (HPS) as a theoretical and methodological tool in instrument construction. Recommendations for textbook change, teaching and learning, and curriculum development are provided. A discussion of recommendations for textbook selection and implications for future studies concludes this study.

The next chapter will present a review of the literature on textbook research in science education. Gaps in physics textbook research will be identified to plan the forthcoming research on the mathematical component found in physics textbooks. To make a case for improvement of the presentation of the mathematical component in physics textbooks and initiate search for criteria applied to textbook analysis the literature review will include a discussion and criticism of some earlier based physics textbooks.

Chapter 2: Literature Review

Overview

The first part of the literature review (Textbook Research in Science Education) focuses on the interest of the educational community in science textbook research. *Physics Teacher* (1982, 1999) describes areas of textbook research in science education, outlining criteria for physics textbook evaluation, and examples of physics high-school textbook evaluations. Consequently, the gap in physics textbook research is identified, pinpointing limited or lack of analysis of the mathematical component of physics in textbooks as the main concern. However, it is evident that the research of the mathematical component of physics textbooks is evolving and gaining momentum.

Chapter 2 continues with The First Mathematics-Based Physics Textbooks which discusses and criticizes the first mathematics-based textbooks that present the mathematical component of physics, the role of mathematics in these textbooks, and the pedagogical approaches of using mathematics in the presentation of physics material. The discussion develops particularly around the first widely used textbook on elementary physics in the English-speaking world written by William Whewell. The literature review shows the change over time in Whewell's position on the pedagogy of teaching and learning mechanics. The first elementary physics textbooks reflected Whewell's ideas, while some later textbooks differed and veered away from them. Finally, the common limitation of all these texts, the lack of use of mathematics as a rich conceptual tool which goes beyond its computational application, is emphasized.

Textbook Research in Science Education

Textbook research is a current trend in science education research (de Berg, 1989, 1992, 1995; de Berg & Treagust, 1993; Jeffery & Roach, 1994; Good & Wandersee, 1991; Swartz et al., 1999; Vaquero & Santos, 2001; Mamiala, 2002; Rodriguez & Niaz, 2002; Leite, 2002; Pocovi & Finley, 2003). In addition, research on the mathematical component in physics textbooks is gaining momentum (de Berg, 1989, 1992, 1995; de Berg & Treagust, 1993; Swartz et al., 1999). Several areas of textbook research could be identified in the current science educational research. Jeffery and Roach (1994) identified the following aspects of science textbooks evaluation: readability, style, cognitive levels, use of analogies, elaborations, organization, gender bias, and professional productivity of the authors. Physics textbooks have also been evaluated based on similar criteria. In 1982, Lehrman reported the evaluation of high-school physics texts in *The Physics Teacher.* Evaluation of the texts was made on the basis of seven criteria: content, level of difficulty, readability, appearance, science, social problems, and assignments. The last review of high-school physics texts was undertaken in 1999 and reported by Swartz et al. (1999) in *The Physics Teacher.* The review consisted of three parts. In the first part, the texts were evaluated in terms of rigor of treatment and accuracy of presentation, the mathematics and reading level, content distribution, peripherals and special features. The second part of that report listed some of the conceptual and factual mistakes found in textbooks. The third part concerned the use of texts in high schools. The reviewers ask questions about textbooks such as: Do students use them? How are they used? How often are they read or referred to? Should teachers insist that students read them? Are textbooks obsolete? The authors predict that the next review can be expected in 2016, but they are wondering if textbooks as we know them will exist then.

Though the study of textbooks is a current trend in educational research, according to Pratt (1985), analyzing mathematical content of science texts is rare. Pratt made this conclusion more than two decades ago. However, the research of the mathematical component of physics textbooks, as follows from this chapter is still rare. Pratt notes, "the question of the type and quantity of mathematics used in secondary science courses has not been given enough attention" (p. 405). In most of these studies, including the one done by Pratt (1985), the mathematical component of science is mainly analyzed in terms of quantity, kind and the level of difficulty. The role and purpose for which mathematics is used, or how mathematics is used to construct meaning are not analyzed. There are some studies of chemistry textbooks conducted by de Berg (1989, 1992, 1995), and de Berg and Treagust (1993) in which exploring the role of mathematics for constructing meanings is actually the focus of the analysis. However, there are only a few studies of physics textbooks with a similar purpose (Dall' Alba et al., 1993; Bevilacqua 1983). This situation is surprising because according to Pratt's (1985) mathematical category frequency analysis, "*physics* was the most mathematically rich science subject, both in terms of the diversity and frequency of mathematics categories" (p. 402).

If we evaluate a few physics textbooks, we will find that the mathematical component has received limited attention - the factors of evaluation mainly are what kind of mathematics, how much mathematics, and, very rarely, the presentation of mathematics. For example, the following questions about the mathematical component of physics in textbooks have been raised, as quoted from the physics texts review conducted in 1982 (Lehrman et. al., 1982):

> *Does the presentation promote understanding of relationships as opposed to simple memorization?* (p. 513)
>
> *Is the mathematics limited to that which the average 11th-grade student has mastered?* (p. 513)
>
> *Are mathematical statements adequately interpreted in English?* (p. 514)
>
> *Are there enough end-of chapter problems of simple plug-in type?* (p. 514)
>
> *Are there problems requiring student's to combine two or more concepts to obtain a solution?* (p. 514)

As a result of the evaluation of physics textbooks, the authors came to the conclusion that in some textbooks there were insufficient simple, "plug-in problems" for poorer students, while some texts were suitable for superior students only because of the strong demand made on student's ability to follow rigorous, extended algebraic arguments. Some authors felt that only a few books developed concepts, particularly mathematical ones, slowly and carefully, and would reach the poorer students, but may be too simple for those at the top of the class. For those students who are at the top of the class, open-ended questions would be beneficial. However, some authors seems to be concerned with the use of open-ended questions, especially vaguely worded ones, suggesting instead the use of more "plug-in problems", as is obvious from the following quote:

> *Plenty of thought-provoking questions in the text and the ends of the chapters, but many of them are so vaguely worded, and so wide open, that few students will*

know how to attack them. There are...not enough simple, plug-in problems anywhere. (p. 517)

Some evaluators were concerned that the algebraic mode was the predominant way to represent relationships between concepts. The comments were: "*...High vocabulary and too much reliance on the students' ability to comprehend relationships that are expressed solely in mathematical terms*" (p. 518).

It is encouraging to see that in the *Survey of High - School Physics Texts* conducted by Swartz et al. (1999) the analysis of the mathematical component is given a broader scope. In this review the authors looked not only at the type and difficulty level of math but also at how the mathematics was presented, and what mathematical tools can be helpful in learning. Some concerns sound familiar; others are relatively new. The following quotes demonstrate the increased attention to the analysis of the mathematics component in physics textbooks:

> *There are no derivations; occasionally equations support assertions, but are not incorporated into several-step logical processes; fairly rarely students are asked to use such equations to solve one-step problems; very few in-chapter sample problems exist.* (p. 284)
>
> *A few suggestions are included about how to use a graphing calculator, but this kind of activity is not built into the course in a serious way.* (p. 285)
>
> *We believe that more numerical work, in the form of making and interpreting graphs, for example, could have been included without creating insurmountable barriers; the course would be stronger if this had been done.* (p. 285)

It is interesting to see that some of the reviewers give preference to the qualitative representations of concepts to the quantitative ones, at least at the beginning stage, as follows from this comment: "The formal level of mathematics would be accessible to virtually any student in high school. Very few traditional equations are used. Projectile motion is presented without any quantitative calculations at all" (p. 287).
On the other hand, some textbook evaluators are concerned that "the lack of mathematics and equations might make this a difficult textbook for a beginning teacher to use" (p. 287).

Another concern focuses on the "meaning of numbers" and justification of functional relationships between concepts. For example:

> *...Problems are solved to three significant figures (always), but no attention is then paid to the significance of the number obtained...The text is filled with the standard formulas of introductory physics, but practically none of them are derived. They are asserted without any attempt to justify the details or to examine the functional dependencies.* (p. 289)

The First Mathematics-Based Physics Textbooks

The first widely used textbooks in elementary physics in the English-speaking world were written by William Whewell. Though Whewell complimented the continental mathematicians for their analytical skills "in compressing the whole science into a few short formulae" (Stinner, 1992, p. 1), he was deeply concerned about the seductive powers of the finished product of mathematics in teaching physics. As Becher noticed, "for Whewell, mathematics was and remained only a means to an end and not an end in itself" (As cited in Fisch, 1991, p. 2). On the other hand, Becher fails to notice that

"Whewell spoke loud and clear of the decisive superiority of algebraic analysis over geometry for the mathematization of the physics of his day" (As cited in Fisch, 1991, p. 49). Whewell criticised the educational approach to teaching and learning physics in Britain where students learned mechanics from Newton's *Principia* using a geometrical method called the synthetic-geometric approach, and avoiding the analytical method developed by Euler and Lagrange. In 1832, Whewell wrote: "The analytical mode of treating Dynamics is the only method which can now answer the requisites of natural philosophy" (As cited in Fisch, 1991, p. 49).

It is clear that Whewell's position about the pedagogy of teaching and learning mechanics, at least at the beginning of his career as an educator, was quite extreme. Let us assume for a moment that the geometric-synthetic approach could be viewed as analogous to the qualitative approach to teaching and learning physics (The approach to teaching physics where scientifically appropriate meanings are constructed; the focus is on qualitative predictions and explanations, class discussions and debates; in teaching physics the emphasis is placed on concepts and relationships between them, and not on formulas). On the other hand, the analytical approach advocated by Whewell, can then be viewed as a quantitative approach (The approach to teaching physics where concepts and the relationships between them are given in terms of mathematical statements and in the form of formulas and equations; the emphasis is placed on memorization of formulas and definitions, calculations in order to come up with right answer, and not necessarily on the underlying mechanism of the procedures of derivations involved. Students learn formulas and definitions, do "type" problems involving manipulation of symbols and solve mathematical equations for the unknown variable). What I want to show is that there should be a balance between the synthetic-geometric and the algebraic approaches. It is interesting that Whewell later realised that students simply memorised mathematical equations without a sound understanding of the concepts involved. Therefore, he argued later, that there was a need to go back to the geometric-synthetic approach to balance mathematical and conceptual aspects of physical science.

Though the first book in physics by Whewell was extremely mathematical, he later changed his view on the pedagogy of teaching physics concepts. As Fisch noticed, "this early shift of interest from viewing mechanics merely as a sounding-board for mathematical methods, to viewing (and most importantly), to teaching it as the model physical science, marks an important turning point in Whewell's career" (1991, p. 45). Fisch goes on to say that Whewell came to lay the emphasis "in the more elementary textbooks on intuitive geometrical problem-solving technique reminiscent of Newton's *Principia*" (Fisch, 1991, p. 49). He advocated extensive discussion of the origin of, and evidential basis for, mathematical formulations as a pedagogical principle. In *History of the Inductive Sciences* (1858), Whewell made a comparison between the physics of Lagrange and that of Gauss:

> *Lagrange, near the end of his life, expressed his sorrow that the methods of approximation employed in Physical Astronomy rested on arbitrary processes and not on any insight into the results of mechanical action. From the recent biography of Gauss, the greatest physical mathematician of modern times, we learn that he congratulated himself on having escaped this error. He remarked that many of the most celebrated mathematicians…had trusted too much the symbolic calculations of their problems, and would not have been able to give an account of the meaning of*

each successive step in their investigation. He said that he himself, on the other hand, could assert that at each step he took, he always had the aim and purpose of his operations before his eyes without ever turning aside from the way. The same, he remarked, might be said of Newton. (As cited in Fisch, 1991, p. 43)

Whewell embraced the analytical approach of Lagrange and rejected the synthetic approach of Newton. In spite of that, in his textbooks Whewell failed to present mathematical formulations as potentially rich conceptual tools that could lead to understanding of other phenomena.

An attempt to use mathematics as a conceptual tool was made in the textbook *Elements of Natural Philosophy* by E. M. Avery, published in 1878. Unfortunately this approach is not generally used in modern physics textbooks. For example, when discussing free fall using Galileo's experiment with an inclined plane, the author of the textbook provides a table of results from which students can discover patterns, which later help them to understand kinematics equations. The table looks like this (p. 62):

Table 2-1. *Table of Results*

Number of Seconds	Spaces Fallen During Each Second	Velocities at the End of Each Second	Total Number of Spaces Fallen
1	1	2	1
2	3	4	4
3	5	6	9
4	7	8	16
etc.	etc.	etc.	etc.
t	$2t-1$	$2t$	t^2

(This table must be generated from mathematics in the first place. Today we would generate the numbers presented in the table from a stroboscope picture or the inclined plane experiments where the students can do measurements.)

After the patterns have been discovered, formulae for kinematics of falling bodies are given on the next page:

1. $v = gt$ or $\frac{1}{2}$ g x 2t
2. $s = \frac{1}{2} g (2t-1)$ (Note that this equation presented in this textbook appears to be dimensionally inconsistent – it should be $\Delta s/\Delta t = \frac{1}{2} g (2t-1)$.)
3. $s = \frac{1}{2} gt^2$

Then Laws of Falling Bodies are formulated verbally:

1. The velocity of a freely falling body at the end of any second of its descent is equal to 32.16ft. (9.8 m) multiplied by the number of the seconds.
2. The distance traversed by a freely falling body during any second of its descent is equal to 16.08 ft. (4.9 m) multiplied by one less than twice the number of seconds.
3. The distance traversed by a freely falling body during any number of seconds is equal to 16.08 ft. (4.9 m) multiplied by the square of the number of seconds.

This approach (when students start studying motion from experiment, then analyze results, trying to discover patterns, express patterns in mathematical language, and verbalize the rules) leads to better understanding of mathematical equations and the concepts involved.

Unfortunately, not many textbooks in physics were modelled after Avery, and many of Whewell's pedagogical devices to explicate concepts prior to the presentation of the finished product were dropped. In these books (see, e.g., *Natural*

Philosophy, by John Sangster, published in Canada in 1864; *New Practical Physics,* by Henry Black and Harvey Nathaniel Davis, published in 1921), the authors first stated the principles, definitions, and laws and then, sequencing the problems, asked students to work them out as an exercise. In these texts, example problems are worked out to illustrate the applications of formulas only. For example, in *New Practical Physics* textbook, after giving equations (I) $v = at$, II) $s = 1/2at^2$, and (III) $v^2 = 2as$, the authors instruct the student: "It will save time *to memorise* (italics in the original) equations (I), (II), and (III). Notice that there is an equation for each pair of quantities v and t, s and t, and v and s. Always use the equation that gives what is wanted directly from the data" (p. 168). This approach discourages students from using mathematics as a conceptual tool, and makes students believe that substitution of data into formula is what physics is all about. Morris Kline noticed that unfortunately, the relationship of mathematics to the study of nature is not presented in our dry and technique-soaked textbooks (*Mathematics and the Physical World,* 1959). As Stinner (1992) noticed: "It seems that 'post-Whewellian' texts are prototypes for today's physics texts, and they may have set the tone for the format of science texts in general" (p. 1).

However, the relationship of mathematics to the study of physics could be presented differently. The authors of *PSSC Physics* Haber-Schaim et al (1976), fourth edition, a very mathematical physics textbook, see one of the main roles of mathematics in physics education as developing the student's aesthetic sense by learning to appreciate the abstract beauty of a concise mathematical formulation of a natural law.

The textbook *Nuffield Physics,* written by Boulind et al. (1978) is better in the sense of conveying meaning by exploring many good qualitative questions and experimental activities but this book has almost no mathematical component. Moreover, the approach presented in this textbook is the inductivist way of doing physics where knowledge claims are made directly from observation.

Finally, the contemporary physics textbook *Conceptual Physics* by Paul Hewitt (2002) conveys the meaning of mathematical formulations used in the textbook. However, using mathematics as the language of science is very limited. This treatment of the mathematical component of physics is appropriate for beginners but not sufficient for more mature physics students.

In another contemporary physics (college) textbook *Physics matters: an introduction to conceptual physics* by James Trefil and Robert Hazen (2004), the mathematical component is represented in more facets than in the books mentioned above. Though formulae and mathematical derivations play a subsidiary role in treatment of physics concepts, whenever an equation is introduced, it is presented in three steps: first as a verbal statement, then as a word equation, and finally in its traditional symbolic form. In addition, sometimes, graphical representation supports other mathematical forms of expression. In this way, the authors think, students can focus on the meaning rather than on the abstraction of the mathematics. Another valuable feature of this book is that using of simple mathematical calculations in making estimates and determining orders of magnitudes is given proper attention. The limitation of this book, however, is that derivations of formulas showing processes of reasoning are in the large absent.

The uses of mathematics in physics, such as giving definitions, obtaining numerical relations, and formulations of theories, are strongly emphasized in physics textbooks. For example, in the preface to the textbook *Physics*, a text for senior high school by R.W. McKay and D. G Ivey (these are the same people who produced the famous physics education movie for PSSC course "Frames of Reference") published in 1955, the authors assert: "The physicist tries to understand the operation of the physical universe, and wherever possible express his knowledge in mathematical form – because in mathematical form he can state ideas precisely" (pp. 1-2). Unfortunately, the very important role of mathematics as a rich conceptual tool that can lead to formulations of theories is rarely emphasized in school science textbooks. In most physics textbooks mathematics is viewed mostly as a computational tool. The laws of physics expressed mathematically in science can be considered as rich conceptual tools beyond their use as solving problems according to computational algorithms.

Summary

Textbook research is a current trend in science education research. However, analyzing mathematical content of science texts is rare. In most of these studies, the mathematical component of science is mainly analyzed in terms of quantity, kind and level of difficulty. The role and purpose for which mathematics is used or how mathematics is used to construct meaning are not analyzed. The analysis of these aspects of presentation of the mathematical component is essential because different textbooks have employed various methods of presenting the mathematical component of physics.

The first widely used textbooks in elementary physics in the English-speaking world were written by William Whewell. He embraced the analytical approach of Lagrange and rejected the synthetic approach of Newton. In spite of this, Whewell failed to present mathematical formulations in his textbooks as potentially rich conceptual tools that could lead to understanding of other phenomena.

An attempt to use mathematics as a conceptual tool was made in the textbook *Elements of Natural Philosophy* by E. M. Avery, published in 1878. This approach engages students in studying motion from experiment, analyzing results, trying to discover patterns, expressing patterns in mathematical language, and verbalizing the rules. Such an approach leads to better understanding of mathematical equations and the concepts involved.

Unfortunately, few textbooks in physics were modeled after Avery, and many of Whewell's pedagogical devices to explicate concepts prior to the presentation of the finished product were dropped. In these books (see, e.g., *Natural Philosophy,* by John Sangster, published in Canada in 1864; *New Practical Physics,* by Henry Black and Harvey Nathaniel Davis, published in 1921), the authors first stated the principles, definitions, and laws and then sequencing the problems, asked students to work them out as an exercise. In these texts, example problems are worked out to illustrate the applications of formulas only.

In other types of physics textbooks, the relationship of mathematics to the study of physics is presented differently. The authors of *PSSC Physics* Haber-Schaim et al (1976), fourth edition, see one of the main roles of mathematics in physics education as developing the student's aesthetic sense by learning to appreciate the abstract beauty of a concise mathematical formulation of a natural law.

In contrast, other types of physics textbooks had almost no mathematical component. The textbook *Nuffield Physics,* written by Boulind et al. (1978) is effective in conveying meaning by exploring many good qualitative questions and experimental activities but this book had almost no mathematical component. Moreover, the approach presented in this textbook is the inductivist way of doing physics where knowledge claims are made directly from observation.

Finally, the contemporary physics textbook *Conceptual Physics* by Paul Hewitt (2002) conveys the meaning of mathematical formulations used in the textbook. However, using mathematics as the language of science is very limited. In another contemporary physics (college) textbook *Physics matters: an introduction to conceptual physics* by James Trefil and Robert Hazen (2004), the mathematical component is represented in more facets than in the books mentioned above. Though formulae and mathematical derivations play a subsidiary role in treatment of physics concepts, whenever an equation is introduced, it is presented in three steps: first as a verbal statement, then as a word equation, and finally in its traditional symbolic form. In addition, sometimes graphical representation supports other mathematical forms of expression. Using these methods authors postulate that students can focus on the meaning rather than on the abstraction of the mathematics. Another valuable feature of this book is that using simple mathematical calculations in making estimates and determining orders of magnitudes is given proper attention. The limitation of this book, however, is that derivations of formulas showing processes of reasoning are at large absent.

Unfortunately, the very important role of mathematics as a rich conceptual tool that can lead to formulations of theories is rarely emphasized in school science textbooks. In most physics textbooks mathematics is viewed mostly as a computational tool. The laws of physics expressed mathematically in science can be considered as rich conceptual tools beyond their use as solving problems according to computational algorithms.

The next chapter (Chapter 3) includes a description of the researcher's theoretical assumptions that have guided the design of this study, the theoretical and methodological frameworks, and research procedures. The discussion develops around elements such as levels of knowledge representation, studies of experts' and novices' problem solving strategies, application of learning theories in physics education, and procedures for collecting, organizing, analyzing, and synthesizing data in order to construct an instrument for content analysis of physics textbooks.

Chapter 3: Analytical and Methodological Framework

Overview

Chapter 3 commences with a description of the researcher's theoretical assumptions that have guided the design of this study. A presentation of the theoretical frameworks of this study, learning theory and requirements of scientific literacy, are also included. The elements of contemporary learning theories are highlighted. The discussion develops around elements such as levels of knowledge representation, where different types of knowledge are presented, studies of experts' and novices' problem solving strategies, where differences between experts and novices are specified for consideration of potential approaches to be used in presenting physics material in textbooks, and application of learning theories in physics education, where particular examples are described and the importance of considering learning theory in presentation of physics is emphasized.

Additionally, a discussion develops around the requirements of scientific literacy, included as one of the theoretical frameworks utilized in this study. The discussion begins with presenting definitions found in the science education literature followed by the justification of applying requirements of scientific literacy to the analysis of the role of mathematics in physics education. Consequently, the proposition of incorporating history and philosophy of science into the research methodology utilized in this study is introduced and justified.

Chapter 3 presents the central argument in this study: it is crucial that physics textbooks maintain a balance between the qualitative and the quantitative aspects of physics in order to ensure understanding when students learn from textbooks. The explanation of the rationale for balancing the qualitative and quantitative aspects of physics in physics textbooks (Research Question 1) justifies my central argument.

In order to make judgments about the balancing of qualitative and quantitative aspects of physics in physics textbooks, Chapter 3 describes the conceptual framework and the process of its development, where an explanation of which aspects of the role of mathematics in physics will be explored and analysis is provided. In this section the conceptual framework is organized into a table which will later be filled in by the information obtained from further research conducted in developing an instrument for the analysis of physics textbooks in this study.

The chapter concludes with a description of the methodological framework and research procedures. In this section, methods of research for this study are identified and justified. These methods include historical inquiry and qualitative content analysis. The methodology of qualitative content analysis is introduced with a presentation of examples of its application. The appropriateness of methodologies, used by de Berg (1989) and Chiappetta, Sethna, and Fillman (1991), partial application for this study's examples are identified and justified. Finally, procedures for collecting, organizing, analyzing, and synthesizing data are described and presented in an illustration depicting the steps of inductive analysis leading to the construction of an instrument for content analysis of physics textbooks.

Profile of the Researcher

To examine how physics textbooks represent the role of mathematics in expressing ideas of physics and to understand how the authors of these books insure the balance between the qualitative and the quantitative aspects of physics in the presentation of the material to be conceptualized, a researcher has to identify her theoretical assumptions that guide the design of this qualitative study. As Creswell (1998) notes,

> *Qualitative researchers approach their studies with a certain paradigm or worldview, a basic set of beliefs or assumptions that guide their inquiries. These assumptions are related to the nature of reality (the ontology issue), the relationship of the researcher to that being researched (the epistemological issue), the role of values in a study (the axiological issue), and the process of research (the methodological issue).* (p. 74)

Before starting a description of the analytical and theoretical frameworks for this study, following Creswell's advice, I will now present my personal stand on the issue of the study – presentation of the mathematical component in physics textbooks to insure the balance between the qualitative and the quantitative aspects of physics.

As a person who has a passion for physics and mathematics learning since I was introduced to these subjects during my school years, I was always fascinated by the logical aspect of mathematics and the ability of mathematical language to describe the world around us in such a precise and an elegant way. The idea of exploration of how the mathematical component of physics is treated in physics education came during my teaching experience. As a physics teacher, I couldn't help realizing that the mathematical component of physics causes difficulties for students with insufficient background in mathematics. On the other hand, the students who exhibited proficiency in manipulating mathematical equations, also had difficulties, but these were of a different nature – they would often mindlessly use mathematical equations, very often not clearly knowing which one to apply in a particular problem situation. It was evident to me that the students did not have conceptual understanding of the ideas involved, did not understand the assumptions and limitations to be considered for the physical laws studied. From my experience and the reading of physics education literature I realized that mastering the mathematics involved would not make any change in terms of students' understanding of what they learn in physics. With this came the realization that in the presentation of ideas in physics I must use two approaches, namely, a quantitative and a qualitative. In classroom teaching it meant that for successful learning I had to find the balance in presenting the qualitative and the quantitative aspects of physics. Finally, the pedagogical ideas on how to do this came from experience, as well as from reading educational literature and taking courses in my graduate program in science education at the University of Manitoba.

It is a fact that the learning of physics takes place outside the classroom. As was shown in Chapter 1, another source from which students can learn is a textbook itself. It seems to be logical to assume then that students could understand mathematical concepts of ideas presented in physics textbooks provided the presentation of the mathematical component insured the balance between the qualitative and quantitative aspects of physics. Another reason for my interest in physics textbooks research was the opportunity I had in 2003 when I was selected by Manitoba Education, Citizenship and Youth to be on a team for evaluating recent learning resources for Manitoba schools in physics education, primarily in high school physics textbooks. Our task was to determine the correlation between the textbooks submitted for evaluation and the Manitoba curriculum learning outcomes in order to recommend the best identified resources for physics teachers. One of the learning outcomes of the Manitoba Curriculum in Physics requires students to be able to use mathematics in its several modes of presentation in order to express ideas in physics. During the examination of physics textbooks chosen for the analysis I noticed that they differed by the presentation of the mathematical component of physics in terms of levels of difficulty, volume, and modes of expression. However, performing the task of establishing the correlation between curriculum learning outcomes and presentation of material in the analyzed physics textbooks did not provide complete information about the ability of textbooks to present the mathematical component in a way that would be meaningful to students. In other words, I was interested to find out if the presentation of concepts and ideas in physics textbooks insured the balance between the qualitative and the quantitative aspects of physics, which in turn, would promote conceptual understanding. I became convinced that to respond to this problem, research on textbooks had to be undertaken. I also realized that in order to explore what meaning mathematics conveyed in physics textbooks it was not informative to know how much, how often, and what kind of mathematics was presented in textbooks. I suspected that a qualitative analysis of physics textbooks would provide a deeper understanding of the meaning of mathematics in physics textbooks, and how the balance between the qualitative and the quantitative aspects of physics was maintained in these texts. Thus, I selected for my research, examination of mathematical component in physics textbooks as the focus of my study.

After describing the personal position on the question of what to study, a researcher, according to Creswell (1998), should proceed with orienting the theoretical framework so as to inform a reader what will be studied and how it will be studied. In this part of the study, according to Creswell (1998), "topics include the conceptual framework including theory to be used as well as concepts and processes related to the research design" (p. 176). I will follow this advice and continue with describing the theoretical frameworks for my study.

Theoretical Frameworks

The objective of this study is to understand how physics textbooks represent the role and place of mathematics to maintain the balance between the quantitative and the qualitative aspects of physics that lead to an understanding of the ideas presented in physics textbooks. Clearly, to help students understand the basic concepts of physics, textbooks would benefit if they used the findings of learning theories. I would like to find out how physics textbooks reflect the ideas and recommendations made by learning theories in the presentation of the mathematical components of physics to promote conceptual understanding. To insure this understanding, one assumes it is crucial that physics textbooks maintain the balance between the quantitative and the qualitative aspects of physics.

Learning Theories

The task of learning theories is to explain how learning happens and what measures have to be taken to ensure successful learning. The central concept in learning theories, as will be shown in this section, is the concept of learning styles. The term "learning styles" has been defined as "cognitive, affective, and physiological traits that serve as relatively stable indicators of how learners perceive, interact with, and respond to the learning environment" (Keefe, 1982, p. 44). Claxton and Murrell (1987) presented an overview of theory and research in the field of learning styles and discussed significant implications for educational practice. They also stressed the importance of interactions between learning style, developmental stage, disciplinary perspectives, and epistemology. The learning theories presented in this section might offer physics teachers, textbook writers, curriculum developers, and other educational players a better understanding of teaching and learning processes, as well as insights which would enable them to enhance the balanced presentation of the mathematical component in physics. While there are many learning theories where the construct of learning styles includes cognitive, affective, and physiological dimensions, cognitive styles seem to be able to explain new understandings relevant to the process of utilizing knowledge of mathematics in presenting concepts of physics for the purpose of balancing the qualitative and the quantitative aspects of physics.

Contemporary models of learning draw upon research that use Piaget's ideas about levels of reasoning engaged in different stages of epistemological development, earlier cognitive themes models, such as schema structures from the cognitive sciences, and memory structures from the discipline of artificial intelligence.

In Piaget's view, an understanding about the physical world is acquired during specific encounters with objects. This understanding changes during development as thinking progresses through various stages from birth to maturity. Piaget is considered to be a structuralist who tried to identify parts and determine how they are organized into a whole. He proposed that a small set of mental operations underlie a wide variety of thinking episodes. Being concerned with relationships between parts and the whole and between an earlier and a later state, Piaget (1969) and his followers claim that the thinking of younger and older children has similar elements, but these elements are combined in different ways to form the organized whole of thought. Though Piaget's work made substantial impact on the development of cognitive science, many of Piaget's ideas, according to Chandler and Chapman (1991), were modified as a result of subsequent observation and interpretation.

Going beyond Piaget's work, as recognized by contemporary cognitive theories of learning, the mind organizes repeated similar experiences into what psychologists call *schemata* (complex networks of concepts, rules, and strategies). Schemata are described as the fundamental elements upon which all information processing depends (Rumelhart, 1981). Schemata are also employed in the process of interpreting sensory data, in retrieving information from memory, in organizing actions, in determining goals, in allocating resources and generally in guiding the flow of processing in the system. Rumelhart (1981), clearly, assigns schemata a role of providing a data structure for representing generic concepts stored in memory. Thus, schemata represent knowledge about objects, situations, events, sequences of events, actions, and sequences of actions.

According to another view (Marshall, 1989), a schema is a basic storage device where knowledge gets highly organized. Numerous schemata facilitate our understanding of everyday events and are based on previous experience. These schemata are developed by repeatedly doing the same set of actions in a given setting. They have no fixed size and may embed and often overlap. In terms of organization, a schema has a network structure with nodes and links. As Waterworth et al. (2000) describe, information that has something in common is linked in some way, similar to the way a computer's memory is organized, and exists in memory as independent units. These units are connected through links in a hierarchical network. The degree of connectivity among the schema's components determines its strength and accessibility. A schema is a flexible structure, accessible through many channels. These channels comprise different types of knowledge stored in memory. This knowledge about organization of schemata is very important for teachers and textbooks writers. I believe, if we want students to develop conceptual understanding, we have to know how students' minds are organized.

Since schemas are flexible structures, teachers have to develop strategies to help students make strong connections between its elements for better learning. One of these strategies originates from Lev Vygotsky's sociocultural theory and his concept of the *zone of proximal development.* "The zone of proximal development is the distance between what children can do by themselves and the next learning that they can be helped to achieve with competent assistance" (Raymond, 2000, p. 176). This competent assistance teaching strategy is called by Vygotsky scaffolding instruction. He defined scaffolding instruction as the "role of teachers and others in supporting the learner's development and providing support structures to get to that next stage or level" (Ramond. 2000, p. 176). According to Olson and Platt (2000), the activities provided in scaffolding instruction should be beyond the level of what the learner can do alone, so that the scaffolds facilitate a student's ability to build on prior knowledge and internalize new information. These scaffolds, however, are temporary. As the learner's abilities increase, the scaffolding provided by an instructor is progressively withdrawn, so that finally, the learner is able to complete the task or master the concepts independently (Chang, Chen, and Sung, 2002, p. 7).

The scaffolding instruction strategy found application in learning from textbooks in a form of concept mapping. Chang, Chen, and Sung (2002) found that concept mapping (scaffolding) "… may serve as a useful graphic strategy for improving text learning" (p. 21). An important aspect of scaffolding instruction is an appropriate sequence of supports provided to the learner. Therefore, textbooks, I believe, should present the material in such a sequence that it is the most helpful to students to become self-regulated learners.

The degree of connectivity among the schema's components is central to the process of transfer of learning. Transfer of learning is defined as the ability to extend what has been learned in one context to new contexts (Byrnes, 1996). This ability enables students to transfer, for example, learning from one physics problem to another, as well as to place a problem in a broader context. Research has indicated that transfer across contexts happens easier when a subject is taught in multiple contexts rather than in a single context (Bjork and Richardson – Klavhen, 1989). In addition, research has shown that when a subject is taught in multiple contexts, and includes examples that demonstrate many applications of what is being taught, people are more likely to abstract the relevant features of concepts and develop a flexible representation of knowledge (Gick and Holyoak, 1983).

According to Gick and Holyoak (1983), one way to help develop flexibility is to ask learners to solve a specific case and then provide them with an additional, similar case. Relating to similar problem situations, as these researchers believe, would help learners to abstract general principles and lead to more flexible transfer. Through observing similarities and differences across different situations students get many opportunities for knowledge representations. Contrasting different concepts can help students notice new features that previously escaped their attention and learn what features are relevant or irrelevant to a particular concept. The use of well chosen contrasting cases can help students learn the conditions under which new knowledge is applicable.

The Cognition and Technology Group at Vanderbilt (1997) suggests a second way to improve flexibility - to let students learn in a specific context and then engage them in "what – if" problem solving designed to increase the flexibility of their understanding. Bransford et al. (1998) recommend a third way to enhance transfer to novel problems - to generalize the case so that learners are asked to come up with a solution that applies not just to a single problem, but to a whole class of related problems, namely to create mathematical models that characterize a variety of problems.

Contemporary models of learning take into account advances in the developments of psychological theory of acquiring knowledge. The essence of this theory is recognizing the relationship between cognitive learning styles and psychological type theory (Myers and Briggs, 1975). Psychological type theory provides a construct that explains individual favoured natural behaviours and abilities. According to psychological type theory, an attitude associated with awareness and reliance on objects and individuals, external world, represents Extraversion (E). On the other hand, an attitude demonstrating interests centered on an inner world of formulated ideas and concepts in which the individual tends to set his or her own standards with a thoughtful objectivity, reflects Introversion (I). In the view of Myers –Briggs theory, Sensing (S) and Intuition (N) are modes of perceiving the external world, whereas Thinking (T) and Feeling (F) are ways of judging facts and concepts to be perceived. As a mode of Perception (P), Sensing (S) is gathering information directly through observation by way of the five senses. On the other hand, Intuition (N) is associated with Perceiving things indirectly, through hunches, and is accompanied by insight, or imagination. As a mode of Judging (J), Thinking (T) portrays a logical, rational, analytic, orderly, and impersonal, objective approach in contrast to Feeling (F), which also is seeking a rational ordering but is dependent more on drawing conclusions based on personal values and subjective observations. It was established (Myers and McCaulley, 1985) that individuals who prefer the Judgment (J) mode act on facts and concepts in a planned, organized way, whereas those who are inclined to the Perception (P) mode tend to be more curious, flexible, and open to a breadth of experience. Sensing (S) and Intuition (N) serve as functions to implement the Perception (P) orientation, whereas Thinking (T) and Feeling (F) serve as functions to implement the Judgment (J) attitude.

Awareness of psychological type theory is very important for educators. By understanding psychological type preferences of students, we may be able to gain insights into the reasons of choice of their ways of learning in order to identify the cognitive learning styles that are appropriate for particular students in order to help them understand concepts, laws, and theories of physics. For example, S-students focus on things that are practical and observable. I addition, their mind works in a linear fashion. Therefore, the traditional way of presentation of the material – from theory to examples /experimentation would not work for them. It was established that S-students learn best when information is presented in a step-by-step, hands-on manner (Jenson *et al.*, 1998). The example - theory approach would be more effective for these students. On the contrary, intuitive thinkers (N-students) exhibit imagination, acceptance of complexity of abstract concepts and theories, and a tendency to focus on the "big picture". The intuitive thinkers favour a more abstract presentation allowing personal integration of the information into the overall theory (Jenson *et al.*, 1998). Therefore, N-students would benefit from theory – example approach.

We have to realize that this difference in the fundamental approach used by different students poses a difficulty both for the textbook writers and the teachers. I t is a challenge to accommodate the learning needs and preferences of both types of students. Presentation of the mathematical component in physics in both textbooks and teaching is also a challenge given the understanding of mathematics is a cognitive activity dependent on student learning styles. Textbook writers and teachers should be aware of this difficulty. In connection to my research, to choose the effective ways of the representation of the mathematical concepts in physics, educators should have sufficient theoretical knowledge about different levels of knowledge representation.

Levels of Knowledge Representation.

According to Waterworth et al. (2000), three main types of knowledge are stored in memory as analogical, propositional, and distributed representations. They give the following description of these types of knowledge:

Analogical representations are picture-like images, whereas propositional representations are abstract language-like statements that make presuppositions…Distributed representations are a network where the knowledge is in the connections between nodes. Analogical representations and propositional representations are regarded as symbolic representations while distributed representations are considered to be sub-symbolic representations. (p. 9)

In Marshall's (1989) view and in the view of most researchers, schema theory contains three types of knowledge:

- *Declarative* – knowledge that is composed of concepts and facts within a domain, and is static.
- *Procedural* – knowledge that is composed of rules, that consists of skills and techniques, and that determines when a piece of declarative knowledge is applicable and under which circumstances.
- *Schematic* – knowledge that combines procedural and declarative knowledge.

Knowledge types described in the presented learning theories can be represented in different modes. For example, science education researchers identified three different representational levels in chemistry education: macroscopic, symbolic, and submicroscopic (often referred to as microscopic). According to Johnstone (1982), Herron (1996), Hinton and Nakhleh (1999), Nicoll (2003), and Russell et al (1997), the macroscopic level is in the world of the observable phenomena which can be perceived by the senses and can include references to students' everyday experiences. In the macroscopic world, water is a clear liquid and table salt is a white solid. The symbolic level is viewed as the representation of a phenomenon using a variety of media including models, pictures, algebra, and computational forms. The symbolic world is basically the world of formulas and equations. In the symbolic world, water is H_2O and table salt is NaCl. Finally, the submicroscopic level is reality which cannot be observed. Therefore students must develop a mental model of the behaviour, for example, molecules, such as using the particulate theory of matter to describe the movement of chemical particles such as electrons, molecules, and atoms. According to chemistry education researchers, skilled chemists switch easily between these three worlds but new students lack both the basic knowledge and the skills to work with different representations, and therefore fail to make these connections easily (Russell et al, 1997, Seel and Winn, 1997; Kozma, 2000). Herron and Greenbow (1986) found that many students fail to make strong connections between the symbolic signs (chemical formulas) and the physical reality that these signs are representing. Students treat chemical formulas as mathematical puzzles without understanding the chemistry that is underlying these symbols (Kozma, 2000; Marais and Jordan, 2000). Similarly, when we teach physics, we have the same problem.

Studies of Experts' and Novices' Problem-Solving Strategies

Studies on the success of experts, as compared to novices, in physics can provide valuable information about the factors determining success in learning of physics. Insights into experts' plans of action and reasoning employed could help educators understand how experienced problem solvers manage to balance the qualitative and the quantitative approaches when they solve problems in physics.

These studies show that an essential difference between experts and novices involves the structure of domain-specific knowledge. Experts in physics, for example, appear to organize subject-matter knowledge (equations, definitions, and procedures) hierarchically under fundamental concepts such as Newton's second law or the conservation of energy. The knowledge structure of novices in physics tends to be amorphous and based on surface features rather than underlying conceptual frameworks (Larkin et al., 1980; Reif and Heller, 1982). These researchers also found that the difference did not seem to arise from lack of familiarity with the required knowledge. Experts categorize problems according to the fundamental principles and laws required to understand the problem as opposed to novices whose approach to problem solving is based on the superficial features, with little or no activation of fundamental principles. They also tend to use what may be called a "working backward approach" that involves the use of a specific formula or algorithmic procedures with little understanding (Chi et al., 1981; Gabel, Sherwood, and Enochs, 1984; Shoenfeld, 1985). Dhillon (1998) observed that novices used symbols more than experts. Novices had difficulties relating quantities that did not have an obvious relationship, and used symbols to infer similarities and connections between quantities. Experts, in contrast to novices, used the conceptual meaning of the quantities to relate them. This is in agreement with findings by Chi et al. (1981) that novices categorize problems using surface features (for example, rotation, inclined planes, springs) as opposed to experts who categorize using broad physical principles (for example, Newton's laws, conservation of momentum, conservation of energy). Experts use a series of problem representations: the verbal statement of the problem, an illustration of the physical situation described in the problem, conceptual representation (for example, free body diagram), and a set of equations. The final representation, (mathematical equations) is always used by both novices and experts. However, the novice, as opposed to the expert, typically proceeds directly from the problem statement to a mathematical solution using "plug" and "chug" process (Van Heuvelen, 1991). In general, Larkin and Reif (1979) note two main differences between expert and novice problem solver. First, instead of trying to jump directly from a physical situation to quantitative equations, experts seem to interpose an additional step – a qualitative analysis or redescription of the problem, as well as a broad contextualization of the situation described in the problem. Second, experts remember principles in "chunks" or "groups" whereas novices retrieve principles one at a time.

These research findings suggest, according to Dhillon (1998), that knowledge needs to be represented:

- In a descriptive form to help visualize the problem.
- As basic relations, to enable means-end-analysis to be employed.
- In a diagrammatic form, for visual thinkers, to enable transformation of information and to help obtain a total picture.
- With explanations on the applicability of the knowledge.
- Separately, in fundamental blocks, to enable problem decomposition.
- Enabling to make a choice of variable values, allowing the envisioning as well as giving information and producing solution assessment.
- With reference to other similar examples in order to enable the use of analogy.
- With structure, insuring logical sequence.

In my opinion, if these strategies were proven to work successfully in problem solving (which could be an indicator of whether good learning takes place), then they should also work for the presentation of the material in physics textbooks. It is a fact that students still learn mostly from textbooks. If physics textbooks used strategies, proven to be successful for learning and problem solving, then the balance between the qualitative and the quantitative aspects of physics would be maintained. Consequently, mathematical terms and concepts would be meaningful to students.

Application of Learning Theories in Physics Education

As was established earlier in this chapter, there is general agreement among developmental psychologists that the evolution of thinking goes through several cyclic stages. Some of the theories of learning trace the stages of student development from early psychomotor reactions to logical thinking (e.g., Biggs & Collis, 1982; Case, 1985; Fischer, 1980; Stinner, 1998). These theories are reminiscent of Piaget's theory of epistemological development. For example, Osborne (1984) describes mini-theories children use to offer descriptions and provide explanations of ideas about dynamics. He describes three clusters namely "gut dynamics", "lay dynamics" and "physicists' dynamics", which remind us of the stages of epistemological development outlined by Piaget. Osborne (1984) calls "gut dynamics" the dynamics learned through trial and error in the home and based on "direct experience rather than language" (p. 505). He goes on to say that "gut dynamics is about the tangible world and influences motor skills and perception" (p. 505). He then argues that "lay dynamics is the dynamics which is reflected "in the form and content of the language the child grows up to speak and the accounts and images of experiences conveyed by those with whom the child comes in contact, the media, and the authors of the books he or she reads" (p. 506). Finally, he calls "physicists' dynamics" the dynamics learned at school, essentially Newtonian dynamics. He notes that

> *while gut dynamics builds on experience and lay dynamics builds on everyday language, physicists' dynamics has a linguistic and mathematical superstructure of its own...A variable amount of active and self directed experimentation is*

> *possible but the experiences often tend to show the limitations of the idealized theories rather than providing supporting evidence for their usefulness.* (p. 506)

Osborne (1984) raises the concern that these three clusters of learning dynamics do not get integrated. He notes: *" Gut dynamics enables one to play ice hockey, lay dynamics enables one to talk about Star Wars, while physicists' dynamics enables one to do physics assignments"* (p. 506). Osborne (1984) concludes, that *"for many students their gut and lay dynamics are not developed through their physics courses in useful ways, nor are they related to what they are taught in the physics classes"* (p. 506).

Given the discussion of Myers –Briggs' psychological type theory earlier in this chapter, we can make a connection between this theory and Osborne's observation of children learning behaviour. Children, being in the mode of Perception (P), tend to be spontaneous, curious, flexible, and open to the variety of experiences. They behave very similar to an artist who usually learns about the external world through perception of the world. Perception (P), in turn, comprises an artist's external world. In contrast, a physicist internalizes the external world through the constructed worlds and usually operates in an Intuition (N) world, so that the "real" world for a physicist is a mental work. The Perception world for a physicist gets formulated through the constructed worlds and becomes an internal world through Thinking (T) and Judgment (J) in order to act on facts and concepts in a planned, orderly, and organized way.

The insights of the learning theories discussed earlier in this chapter enable us to identify essential elements of the pedagogical sequence necessary to introduce and develop concepts in physics. As Ebenezer and Connor (1999) stress, "it is imperative that students learn to explore their conceptions; identify their assumptions; use critical, logical, and creative thinking; and consider alternative explanations as they continue to study science" (p. 41). For example, Osborne (1984) suggests, as a starting point, the exploration of the child's present ideas. He also argues that the teaching of dynamics must begin at an early age to develop children's gut and lay dynamics in ways that are flexible, unlimited and give appropriate explanations in the real world or are suitable for the subsequent learning of physicists' dynamics. He suggests providing experiences and discussions that would originate "seeds for alternative conceptions upon which the later teaching of physicists' dynamics can be firmly based" (p. 507).

Monk (1994) particularly stresses the importance of practical experience with physical objects in learning physics. He asserts that there is no substitute for direct practical experience if we wish to securely ground knowledge for our students. In other words, "the students must get their hands dirty". Hands-on activities provide students with a good opportunity to construct meanings. Monk warns us that the free play phase of the learning cycle should not be rushed. Students need plenty of time to experience the objects around them before they can shape the way they think about them.

Rowell (1989) addresses the need for the applicability of the epistemology of Piaget to science teaching, stressing the value of two mechanisms for learning introduced by Piaget, namely *assimilation* and *accommodation.* Piaget defines assimilation as the process by which students learn new ideas that match or extend on their existing conceptual knowledge. Accommodation is defined as the process where students learn new ideas that do not fit into their existing conceptual knowledge either because the ideas are new or because the idea conflicts with their present knowledge.

Stinner (1998) discussed Piaget's principle of equilibration (or self-regulation) and noted that the process of equilibration

> *comes into play when we are at a loss to explain a phenomenon or an aspect of a phenomenon using our existing conceptual apparatus. This inability to explain produces a mental discomfort (cognitive disequilibrium) that demands a response. The response consists of conceptual readjustments in a multi-step process involving feedback loops* (p. 42)

Stinner (1998) gives three stages of the equilibration process explaining a progressive sequence of levels as following:

> *In the first stage there is a conservative response to the mismatch, a general resistance to change. In the second stage there is progressive theory change (accommodation), retaining much of the original theory but integrating the disturbance as a new variation. Finally, in the last stage, the reorganization begun in the second stage is completed: the new theory now accommodates the disturbance. The mental discomfort disappears and the new theory is 'symmetrical', that is the initial disturbance is now anticipated and not eliminated. These stages shade into each other and are never clearly delineated.* (p. 42)

According to Redish (1994), students find it much easier to assimilate new ideas because it is easier for the students to learn concepts that fit their view of how things work. Accommodation is much harder because students must change or rethink their existing views. Students often perceive and interpret what they learn in a way that makes sense in terms of their existing beliefs. The tendency to assimilate rather than accommodate is probably one of the reasons that students' conceptual knowledge can be contradictory. According to Posner et al. (1982), in order to change students' existing conceptual understanding, the replacement must have the following characteristics:

- The replacement must be understandable.
- The replacement must be plausible.
- There must be strong conflict with predictions based on the subject's existing conceptual understanding.
- The replacement concept must be seen as useful.

In this view of a mechanism for conceptual change, I intend to see if mathematics used in physics textbooks helps generate the characteristics described above. Does mathematics used in physics textbooks mostly promote memorization? How is mathematics useful in describing the real world? Can it be used to show that one can overcome conflicting ideas by using more plausible and fruitful mathematical treatment of these ideas?

Educational research literature shows that sequencing science content from simple to complex ideas (Hamrick & Harty, 1987; Ausubel, 1968; Novak, 1980; Shavelson, 1972) results in significant gains in science achievement and positive attitudes toward science. There appears to be general agreement among science educators that the teaching of scientific concepts should proceed from the qualitative to the quantitative mode, i.e. simple to complex (Arons, 1984; de Berg, 1993; Hewitt, 1994; Monk, 1994; Mazur, 1996; Stinner, 1994). In the qualitative mode students make verbal statements such as proportionality statements, or conclusions about the results of the lab, for example. When students explain their reasoning, draw a picture of something, describe an observation, or discuss a demonstration or laboratory activity, they are often forced to use concepts in their explanations or descriptions. This pedagogical approach promotes self-awareness. Students cannot change their views significantly unless they are aware of them. Even incorrect predictions can serve the purpose - to demonstrate, for example, that their model has limited applicability, and thus prepare them for further learning.

The next step in the pedagogical sequence, according to Monk (1994) would be using mathematical language to describe ideas, concepts and relationships between them. This stage involves the use of the grammar and language of the signing system developed in the qualitative mode of representation of knowledge to create a new and more powerful knowledge-making capability. In this stage, as Monk (1994) describes, when students transfer from the qualitative mode to the quantitative one, number and scale, and the operations that go with these, can be used to express knowledge that could not be otherwise known.

There is also support for the notion of sequencing quantitative ideas from verbal statements to the algebraic mathematical statements (Arons, 1984; de Berg, 1993; Monk, 1994; Mazur, 1996; Stinner, 1992). The use of the formal language of algebra allows students to make precise predictions. Physics teachers know that students have difficulty with the abstract algebra of physics that is used to model physical systems. Monk (1994) explains why it is so. He says that the ability to use formal operations occurs at the end of a long chain of epistemological justification that takes the learner across realms of reality that are ontologically distinct. It is not a good idea, for example, to introduce Newton's second law of motion with the formula $F = ma$, or start learning about gravity with the formula $F = Gm_1m_2/r^2$. These formulas will not make much sense to students at this time. The students have to go through the stages of epistemological development as outlined by Piaget to understand these laws and concepts involved. Monk (1994) particularly stresses the importance of maintaining a proper epistemological sequence in learning physics concepts. He warns us that if we short circuit this sequence, in a rush to get the students to use algebra, the whole educational enterprise is threatened. He outlines the path for the pedagogical sequence as following:

> *Students may progress to talking about, writing about and reading about their experiences with the phenomenon. They may then move on to thinking about variables and measuring variables. Only lastly should they be helped into dealing with algebraic representations and the prediction of new phenomenological possibilities through the manipulations of the symbols of algebra.* (p. 210)

De Berg (1989) suggested that factors such as *sequence* in terms of physics content sequence, qualitative to quantitative, and the use of ideas of quantification in a verbal to algebraic sequence can be used for a textbook analysis. In this study, the factor of sequence of presentation will be used as one of the themes for the analysis of physics textbooks' topics on universal gravitation.

Another factor for content analysis can be the use of multiple representations generated by educators from contemporary learning theories. As Leonard, Gerace, & Dufresne (1999) suggest, these representations could be linguistic, abstract, verbal, symbolic, experiential, pictorial, physical, or graphical. They go on to say that deep understanding of concepts requires many representations. Hestenes (1992) believes that a single representation is usually insufficient to express the full content and structure of a scientific model. One of the reasons that students experience difficulty in understanding physics concepts is their belief that one representation, mainly algebraic, is sufficient. Consequently, students often don't see interrelationship of these representations, which means that their abstract physics ideas are not well connected to their real world experiences. Using different representations for the same concepts and having students translate between representations (for example from the algebraic to verbal representations) help students connect ideas and to relate them to personal experience. Graphs are especially helpful because they are abstract, like equations, but can be understood qualitatively, like diagrams and pictures. Though mathematics is generally perceived in the symbolic mode, there could be mathematical representations in verbal, pictorial, graphical modes in addition to the commonly known symbolic format. In this view, it would be useful to perform analysis of physics textbooks' contexts to see in what modes mathematics is presented there and how these representations aid conceptual understanding of the topic universal gravitation.

In the process of using multiple representations of concepts and ideas, another factor which is helpful in making connections between different modes of representations is making analogies in the exploration of extended contexts. Moreover, according to Gentner et al. (1997), "analogy is an important mechanism of change of knowledge" (p. 4). Using researchers' studies of transfer of learning, Gentner et al. (1997) concluded that "analogies to prior knowledge can foster insight into new material" (p. 4) and, in the process of learning new material, promote conceptual change. They show that great scientists used analogies very often. For example, Kepler was one of them. Johannes Kepler drew an analogy between planetary motion and clockwork. He is also known for his attempt to relate the speeds of the planets to the musical intervals. Kepler then tried to fit the five regular solids into their orbits. Though these likenesses did not work, they can be seen as the stepping stones of a creative mind (Harrison and Treagust, 1994). Wolfgang Pauli (1955) gave the following description of Kepler's analogies:

> *As living bodies have hair, so does the earth have grass and trees, the cicadas being its dandruff; as living creatures secrete urine in a bladder, so do the mountains make springs; sulphur and volcanic products correspond to excrement, metals and rainwater to blood and sweat; the sea water is the earth's nourishment.* (p. 176)

As for mathematical analogies (which are the interest of the present study), according to Gentner et al. (1997), Kepler believed both that analogy was heuristic, not deductive, and that to be worthwhile analogies ought to preserve interrelationships and structure:

> *I too play with symbols…but I play in such a way that I do not forget that I am playing. For nothing is proved by symbols…unless by sure reasons it can be demonstrated that they are not mere symbolic but are descriptions of the ways in which the two things are connected and of the causes of this connection.* (As cited in Gentner et al., 1997, p. 30)

Gentner et al. (1997) establish that "the open and inclusive character of Kepler's general writing practice offers…encouragement for the belief that the extended analogies used in his text were actually used in his thought processes" (p. 27). They go on to say that "the sheer fecundity of his analogizing suggests that analogy was a natural mode of thought for him" (p.28).

Advocating the use of analogies in instruction, Leonard, Gerace, & Dufresne (1999) warn against superficial use of them. They notice that when simple cases of concepts are introduced, students tend to focus on surface features, and as a result, often generalize incorrectly. For example, when students learn, in an explorative activity about friction, that the force of friction can be found as a product of weight and the coefficient of static or sliding friction (whichever case applies), which is true in the case of horizontal surfaces, they tend to generalize this idea erroneously and extend it to situations with inclined planes. Investigating a broad set of problem situations helps students to refine and to abstract concepts. In this way inappropriate or oversimplified generalizations can be avoided.

When students explore a range of contexts, they are likely to use relevant features and ignore irrelevant ones. Comparing and contrasting of concepts where students explicitly look for distinctions and commonalities between situations also help students in the interrelationship of knowledge. Mathematical language could be used in establishing some of these analogies because mathematics is used to represent relationships between variables. These variables can be different in different contexts but they still could be united in certain mathematical relationships (direct proportion, inverse proportion, power relationship). In addition, the symbolic form of some mathematical relationship is very similar (for example, inverse square law for gravitational force and electrical force). In these terms it would be beneficial to explore the role of mathematics in physics textbooks to see if mathematics in physics textbooks is used as an analogical tool, and if it is, in what capacity.

The main reason that mathematical aspects of physics deserve attention in physics textbooks is the fact that many fundamental theories, laws, principles, and concepts are expressed in mathematical language. These theories, laws, principles, and concepts have a history of development, and represent our knowledge of physics. Knowledge of science and its historical development are part of educating toward scientific literacy.

Scientific Literacy

There are many definitions of scientific literacy. According to Matthews (1994), these definitions vary from a narrow definition where scientific literacy is the ability to recognize formulae and give correct definitions, to a more expansive or liberal definition that includes understanding of concepts and some degree of understanding about the nature of science and its historical and social dimensions. He goes on to say: "There is no one correct definition of science literacy" (p. 31) and suggests the following qualities a scientifically literate person would have:

1) *Understand fundamental concepts, laws, principles and facts in the basic sciences.*
2) *Appreciate the variety of scientific methodologies, attitudes, and dispositions, and appropriately utilize them.*
3) *Connect scientific theory to everyday life and recognize chemical, physical and biological processes in the world around them.*
4) *Recognize the manifold ways that science and its related technology interact with economics, culture and politics of society.*
4) *Understand parts of the history of science, and the ways in which it has shaped, and in turn has been shaped by, cultural, moral and religious forces.* (pp. 32-33)

One of the influential curriculum documents that advocate the achievement of scientific literacy is *Science for All Americans* (1989). This document was published as a result of an extensive national study sponsored by American Association for the Advancement of Science (AAAS). The report *Science for All Americans* (1989) outlines the following characteristics of a scientifically literate person:

> *The scientifically literate person is one who is aware that science, mathematics, and technology are interdependent human enterprises with strengths and limitations; understands key concepts and principles of science; is familiar with the natural world and recognizes both its diversity and unity; and uses scientific knowledge and scientific ways of thinking for individual and social purposes.* (p. 4)

The document above calls for attention to the connections among science, mathematics, the history and philosophy of science. It makes sense then to analyze the role of mathematics in physics education through the lens of scientific literacy. Through examples of history and the philosophy of science one can show how mathematics was used to develop understanding of physical reality, how it helped scientists either to change or to support their conceptions about the physical world. Since the first two questions of the study need inquiry into the history of science, and unavoidably the inquiry processes of scientific investigations, this study will be conducted in the frame of conventionally agreed upon requirements of scientific literacy. Because examples from history of science reflect the nature and the methods of science, and very often their mathematical representation, I have chosen to find out to what extent textbooks use history of science. As one of the factors for textbook analysis in this study, I will try to find out if the history of the universal law of gravitation is presented.

In summary, current learning theory and the requirements of scientific literacy will be the theoretical frameworks for the textbook analysis part of this study.

My central argument in this study is: In order to insure understanding when students learn from textbooks, it is crucial that physics textbooks maintain a balance between the qualitative and the quantitative aspects of physics. My intent to look for the support of this argument was the first step in my research and it was captured by my first research question

– *What is the rationale for the balance of qualitative and quantitative aspects of physics in physics textbooks?*

My expectation is that the findings for the outlined sub-questions listed below -

(a) How do contemporary learning theories and the requirements of scientific literacy inform us about the appropriate ways of presenting the qualitative and the quantitative aspects of physics?

and

(b) What pedagogical considerations are found in educational research literature that support the idea of balancing the qualitative and the quantitative aspects of physics?

– would provide support for the importance of balancing the quantitative and the qualitative aspects of physics in physics textbooks. To obtain these answers, an examination of the contemporary learning theories and the requirements of scientific literacy were carried out earlier in this chapter, so that now I am ready to report the findings.

Answer to Research Question 1. Rationale for Balancing the Qualitative and the Quantitative Aspects of Physics in Physics Textbooks

The following are the reasons why I have argued that physics textbooks should present mathematical concepts in physics in such a way that the qualitative and the quantitative aspects of physics are balanced:

According to epistemological theories of learning described earlier in this chapter, the evolution of students' thinking goes through several cyclic stages. This finding of learning theories means that if the material is presented in the finished form (as a mathematical equation), the epistemological chain of the development of students' thinking would be broken. Consequently, it would be unlikely that the students understand the meaning of concepts involved. Therefore, I would argue that the qualitative expressions (the meaning of which is gained from experience, and then developed verbally) of mathematical concepts in physics cannot be ignored and have to be balanced with the quantitative expressions (symbolic equations, formulas) if the goal is to achieve understanding of physics ideas and concepts.

Cognitive theories of learning inform us that the mind organizes repeated similar experiences in schemata that have a flexible structure, with many channels. These channels comprise different kinds of knowledge (analogical, propositional, and distributional, and are stored in long term memory (Waterworth, 2000). Analogical representations are described as picture-like images, whereas propositional representations are defined as abstract-like statements that make presuppositions. Finally, distributed representation is understood as a network where knowledge is in the connections between nodes of a schema. This information about organization of mind for representation of different kinds of knowledge begs the legitimate question: How do these types of knowledge get invoked in students' mind if the presentation of material in physics utilizes only one of the approaches – qualitative or quantitative? The reasonable answer, in my view, would be to present physics concepts to students in a balanced way when different kinds of knowledge in the operative system of the brain can be connected for students to be able to learn.

According to science educators, these knowledge types can be represented in different levels (macroscopic, symbolic, and microscopic) and in different modes (verbal, numerical, pictorial, graphical, as well as symbolic). Science educators have found that many students fail to make strong connections between the symbolic signs used and physical reality. One of the reasons may be that concepts in science often have a limited representation. The understanding of physics ideas and concepts, in my opinion, is more likely to happen if the mathematical concepts in physics are presented in a balanced way where the qualitative and the quantitative aspects of physics complement each other.

The science educators' goal is to make students understand concepts of science. These concepts represent mental abstractions that consist of "regularities" and "structures". For example, the symbolic statements such as $F = ma$, $W = mg$, represent regularities. On the other hand, gravitational fields, electromagnetic fields, the microscopic world of atoms, represent structures. Structure, according to Hestenes (1992), is one of the most significant general properties of entities in the concrete world. However, structure is an abstraction that does not exist apart from some object. A structure gives any system a certain integrity or wholeness. Hestenes (1992) believes that mathematics supplies conceptual tools and materials for creating models of great clarity, coherence and flexibility. He goes on to say that modeling makes mathematics meaningful. He acknowledges that mathematical symbols have been created to express concepts of order and structure, and mathematics has been often described as "the science of patterns". Thus, concepts in science have to be diversely connected. In order to connect them, I believe, science educators should strive to present regularities and structures in a balanced way where the qualitative and the quantitative aspects of these mental abstractions are properly balanced. In this process of balancing, students would develop better conceptual models.

Studies of experts' and novices' problem solving strategies demonstrated that the reason why experts do better in problem solving than novices is that they can balance properly the qualitative and the quantitative representations of physics concepts by using a series of representations, such as verbal statements, illustrations of physical situations, conceptual representations (models like free-body diagrams), and a set of equations. The sequence experts use in their approach to problem solving also helps them to balance the qualitative and the quantitative aspects of physics – they go from qualitative analysis of the situation (redescription of the problem) to diagramming , and then to thinking about the main principles involved, and finally, to symbolic quantitative equations. Thus, the other reason for the rationale for balancing of the qualitative and the quantitative aspects of physics is to be able to use a proper pedagogical sequence in solving problems and understanding physics ideas during other physics activities.

Educators agree that the learning cycle (exploration, development and application of concepts) should not be rushed if we want students to construct meanings for the concepts they learn. Therefore, the balancing of the qualitative and quantitative aspects of physics in the process of meaningful learning seems to be an effective strategy (given the right sequence of presentation of concepts) to achieve a better understanding.

In the widely used view of a conceptual change model (like Posner's model), balancing the qualitative and the quantitative aspects of physics is also justified, since students would see the replacement of their earlier ideas as plausible and useful when they finally understand the finished product (symbolic equations of physics laws, for example). In this case, when the balancing act is achieved, mathematics will be utilized by students for the purpose of understanding, and not just memorization.

In the view of the requirements of scientific literacy presented earlier in this chapter, a scientifically literate person understands fundamental concepts, laws, principles and facts in the basic sciences, as well as appreciates the variety of scientific methodologies and appropriately utilizes them. Mathematics in all its modes of representation can be considered as a tool to describe physics ideas, and not necessarily to understand physics. Since the goal of the scientifically literate person is to understand, not only to describe concepts, laws, principles, etc. (which are often expressed in the mathematical language), then it is natural to conclude that mathematical representation of concepts alone does not show evidence for conceptual understanding. A balanced way (qualitative and quantitative) of expressing knowledge of physics, in my view, would be a better indicator that a student understands physics ideas and concepts, as well as appreciates the variety of scientific methodologies, and appropriately utilizes them.

The document advocating the achievement of scientific literacy, *Science for All Americans* (1989), calls for attention to the connection among science, mathematics, the history and philosophy of science. This document argues that examples from the history and philosophy of science can show how mathematics was used to describe and then develop understanding of physical reality, and how mathematics helped scientists either to change or support their conceptions about the physical world. From the history of science examples, students can see how scientists themselves struggled to balance their mathematical findings with their qualitative conceptions of the phenomena they studied. For the students, it is some comfort to realize that great scientists also struggled to strike the balance with their experiences or intuitive thinking and mathematical equations obtained at the end of the discovery journey.

One of the objectives of the document *Science for All Americans* is for students to have understanding of the nature and the methods of science. Mathematical representation has its place in the description of laws, principles and methods of science. Students have to learn to distinguish between laws, theories, observations, inferences, and speculations. The appropriate scientific language would not be possible for students to develop if they use only mathematical representations, confusing symbolic statements with qualitative inferences.

Conceptual Framework

In this part of the research I am going to explain in what aspects the role of mathematics in physics will be explored and analyzed in order to make judgments about the balancing of the qualitative and the quantitative aspects of physics in physics textbooks. According to Wang (1998), there is no single perfect approach to framework construction. "Construction of a conceptual framework is closely tied to the nature of the study, and grounded with the purposes of the data needs" (p. 45). The nature of this study is explorative. The role of mathematics in physics textbooks will be explored to establish how the presentation of material in physics textbooks insures the balance between the qualitative and the quantitative aspects of physics and what in turn facilitates understanding of ideas and concepts of physics presented in physics textbooks. Wang (1998) notes that in the past science textbooks studies conducted from 1989 to 1996, basically two approaches were used to establish conceptual frameworks. The first approach was that a framework would be generated based on theoretical support prior to content analysis. The second approach was that the conceptual framework was explored, constructed, and refined during process of content analysis. Since there seems to be no published study about the role of mathematics in balancing of the qualitative and the quantitative aspects of physics, where a conceptual framework has been already developed, a conceptual framework for this study will be drawn from the rationale for balancing of the qualitative and the quantitative aspects of physics that is informed by learning theories and the applications of their findings by science educators, as well as the requirements of scientific literacy. In addition, this informed theoretical framework will be refined after additional themes have been generated from the historical inquiry about the relationship of mathematics in physics and the history of gravity that will be reported in Chapters 4 and 5.

The following conceptual framework was developed in the light of learning theories, science education researchers' findings, and the requirements of scientific literacy. It appears to have three domains: epistemological, cognitive, and contextual (history and philosophy of science, HPS). Every domain has its sub-domains. These sub-domains will be used to categorize the mathematical component found in physics textbooks. This conceptual framework will guide the design of physics textbooks analysis, and is summarized as in the following table:

Table 3-1

The Role of Mathematics in Balancing of the Qualitative and the Quantitative Aspects of Physics Conceptual Framework

1. Epistemological Domain	Concepts in physics textbooks should be presented in different modes: • Numerical • Verbal • Graphical • Pictorial • Symbolic and in appropriate pedagogical sequence • Simple → Complex • Qualitative → Quantitative • Verbal → Algebraic to enrich the presentation of physics knowledge (fundamental concepts, laws, principles and facts); facilitate conceptual understanding; appreciate the variety of scientific methodologies and appropriately utilize them; develop connections between symbols and physical reality.
2. Cognitive Domain	Textbooks could help develop connections of physics concepts to facilitate conceptual understanding, effective problem solving and realization of the unity of different variables in different contexts in certain mathematical relationships (direct proportion, inverse proportion, power relationship) when the presentation of material involves: • Moving between modes of representation • Stating purpose of using a particular mode • Comparing and contrasting concepts by using analogies • Constructing conceptual models • Critical thinking
3. Contextual (HPS) Domain	Textbooks should provide real historical examples of using mathematics by scientists by presenting concepts in physics in historical context to get exposure to the nature and methods of science in order to • Understand conceptual models of scientists. These models would serve as examples for creating students' own conceptual models

- See plausibility and limitations of different historical models and students' own models to facilitate the process of conceptual change
- Appreciate the variety of scientific methodologies

Methodological Framework and Research Procedures

Qualitative researcher Creswell (1998) suggests that in the section of methodology, topics should include "the methods and procedures in preparing to conduct the study, in collecting data, and in organizing, analyzing, and synthesizing the data" (p. 176).

Historical Inquiry

Given the nature of the research questions two and three, *Historical Inquiry* was chosen as the method of research. To explore the historical relationship between mathematics and physics (research question two), I will explore views on the role mathematics played in physics in a historical perspective. Only a few historians (French, 1980; Jenkins, 1979; Gingras, 2001; Garber, 1998) have made comments on the topic of Question Two. One reason must be that not many historians were scientists or mathematicians themselves who could see the importance of the given topic. The other reason might be that even some historians of science or mathematics (for example, Kline, 1959) viewed physics simply as mathematics that is applied to physics problems. Examining views of scientists, science education researchers, and philosophers should shed light on this question. Moreover, some answers can probably be found in the history of the development of mathematics and science and how these are related. Views of the philosophers of science (T. S. Kuhn, K. Popper), the historians of science working on the history of the development of mathematics and science (M. Kline, M. Fisch, B.

Cohen,and M. Hodges), as well as science educators (K. C. de Berg, M. Monk, Tzanakis, 1999, and R. S. Jones) will be explored to get some insight. The historical connection of mathematics and physics will be reported in Chapter 4.

To explore the role that mathematics played in the history of gravity the inquiry into the history of gravity and examination of the stages of Newton's thinking when he describes universal gravitation will be carried out. I will look at available historical accounts (J. D. Bernal, H. Brougham and E. J. Routh, M, Clagett, B. Cohen, A. R. Hall, J. Herival, O. Pedersen, and W. M. Stevens) about Newton's background knowledge that helped him in the development of the law of universal gravitation, and about other people's contributions (if there are any) in formulating his famous law. I will examine the mathematics Newton invented for the formulation of the law of universal gravitation outlined in Newton's *Principia*. The findings will be reported later in Chapter 5. The findings of the historical inquiry for questions two –

What is the historical relationship between mathematics and physics?

and three –

What role did mathematics play in the history of gravity?

will also be used to determine additional themes (sub-domains) for the analytic rubric of the instrument for textbook analysis.

To determine how the rationale for balancing the qualitative and the quantitative aspects of physics, found from Question 1 and reported earlier in this chapter, is reflected in the contemporary high school and introductory level college physics textbooks in the presentation of the topic Universal Gravitation (Research Question 4) the method of *Qualitative Content Analysis* will be applied. To perform this analysis, an instrument has to be developed, and then applied to physics textbooks for the analysis. The findings from research questions one and two will inform the analytic rubric of this instrument for the analysis of the mathematical component of physics presented in high school and introductory level college physics textbooks in the topic universal gravitation (sub-question *(a) - What findings from research questions one and two can be used to develop the instrument for the analysis of the mathematical component of physics presented in high school and introductory level college physics textbooks in the topic universal gravitation?*)

Qualitative Content Analysis

The methodology of the qualitative content analysis to be discussed will be used for the research of textual material to understand what role mathematics plays in physics textbooks.

I will examine five recent high school physics textbooks recommended by the Manitoba Department of Education, Citizenship and Youth, and also used in other provinces of Canada. (Zitzewitz et al., *Glencoe Physics: Principles and Problems*, 2002; Hewitt, P., *Conceptual Physics: The High School Physics Program,* 2002; Nowikow, I., *Physics: Concepts and Connections,* 2002; Edwards, L., *Physics*, 2003; and Giancoli, D., *Physics: Principles with Applications*, 2005*)*. In addition, I will examine three college physics textbooks to see if there is a difference in treatment of the mathematical component of physics (Jones, E. R. and Childers, R. L., *Contemporary College Physics,* 1993; Serway, R. A. and Faughn, J. S, *College Physics,* 1999; Trefil, J. S. and Hazen, R. M., *Physics matters: an introduction to conceptual physics,* 2004). Calculus-based college physics textbooks will not be included in the examination due to extensive use of calculus in the presentation of material. It would be clearly invalid to compare the mathematical component in high school physics textbooks and that of the calculus - based college physics textbooks, even of the introductory level.

Table 3-2

Sample Textbook Overview

Title of Book	**Author(s)**	**Year**	**Publisher**
Glencoe Physics: Principles and Problems	Zitzewitz & Davids	2002	Glencoe/ McGraw-Hill
Conceptual Physics: the High School Physics Program	Hewitt	2002	Prentice-Hall
Physics: Concepts And Connections	Nowicow & Heimbecker	2002	Irwin Publishing Ltd
Physics	Edwards	2003	McGraw-Hill Ryerson
Physics: Principles With Applications	Giancoli	2005, 6th ed.	Pearson/ Prentice Hall
Contemporary College Physics	Jones & Childers	1993, 2nd ed.	Addison-Wesley
College Physics	Serway & Faughn	1999, 5th ed.	Harcourt Brace & Company
Physics Matters: an Introduction to Conceptual Physics	Trefil & Hazen	2004	John Wiley & Sons, Inc.

The selected textbooks will be used to answer the remaining three sub-questions of Research Question 4:

(b) – What are the modes of the mathematical presentation of concepts found in high school and introductory level college physics textbooks in the topic on universal gravitation?,

(c) – What is the pedagogical sequence of presentation of the mathematical component found in high school and introductory level college physics textbooks in the topic on universal gravitation?,

and

(d) – How are the ideas of contemporary learning theories and the requirements of scientific literacy reflected in the presentation of the mathematical component of physics in physics textbooks?

The method of the qualitative content analysis will be used for the examination of the chosen textbooks. A qualitative analysis of written materials can provide valuable insights to science educators (Wandersee, Mintzes, & Arnaudin, 1989). According to Krippendorf (1980) and Wandersee et al. (1989), content analysis is the evaluation of a body of communicated material (textbooks) to determine meaning, in the case of this research, the meaning of the mathematical treatment of physics concepts. They describe an accepted technique of content analysis in which the researcher applies a classification scheme to the material analyzed with respect to the content of interest (in this study – the balancing of the quantitative and the qualitative aspects of physics in physics textbooks, and consequently, the role of mathematics in physics education). The content analysis in this study will involve a search for mathematical terms and concepts to extract significant statements. The meanings formulated from significant statements will then be evaluated to understand what themes they fit in the developed analytic rubric of the instrument for the analysis of the mathematical component of physics textbooks.

For the instrument construction, themes referring to scientific literacy will be taken from the methodology of content analysis of science textbooks suggested by Chiappetta, Sethna, and Fillman (1991). The authors of this methodology quantitatively analyzed a large variety of science textbooks, and, subsequently, came out with a training manual. In this manual four major themes (categories) of scientific literacy (*the knowledge of science, the investigative nature of science, science as a way of thinking, and interaction of science, technology, and society*) and their descriptors were outlined for the content analysis, which had a high rate of recognition. Only certain categories will be selected and modified for this study according to the research questions of this study. Since the purpose of the study is to explore how physics textbooks reflect the role of mathematics in physics in the process of balancing of the quantitative and the qualitative aspects of physics, I am interested in the question of how mathematics is used for constructing meanings, and not in calculating frequencies as the quantitative content analysis methodology requires. Only themes helpful for the qualitative content analysis and those related to the role of mathematics in physics education will be used in this study. Other themes will be taken from the methodology offered by de Berg (1989) in his textbook study of the emergence of quantification in the pressure-volume relationship for gases.

In his study, de Berg described three approaches to learning, namely *static, dynamic,* and *emergent press* approaches. The static approach to learning, according to de Berg (1989), is "the approach in which mathematical formulae appear and are used apparently without any explanation at all. They appear as the "magician's wand" of science in order to get the right answers" (p. 117). The textbooks that use such an approach state mathematical relationships "without recourse to background information, experimental details, how and why the relationship is derived or the usefulness of the relationship" (de Berg, 1992). If a textbook uses mainly a static approach to introduce mathematical formulations, it would be reasonable to conclude that mathematics is used in such textbooks mostly for the memorization of formulas only.

The dynamic approach to introducing mathematical formulations and graphs, as de Berg (1989) defines it, is the approach which "places them in context with an explanatory base for their emergence and use" (p. 117). The emergence profile of mathematical concepts in this case, according to de Berg (1992), "contains background information, explicit experimental details, information as to how and why the mathematical relationship is determined, and a comment on the accuracy of the relationship". In this case, the replacement of existing students' concepts will be understandable and make sense. The dynamic approach is likely to help students use a mathematical component presented in the textbooks in changing their conceptual understanding. There are degrees of explanation in a dynamic approach, and sometimes some meaningful discussion surrounds the emergence of a quantified form; in other cases, there may be justified necessity for the emergence of a quantified form.

For the latter cases, de Berg chose to use the term *emergent press* as an indicator of an expressed need for the emergence of the quantification. The author notes: "The term 'emergent press' refers to intrinsic or extrinsic forces that lead to the expression of a quantified form" (pp. 117-118). If there were such intrinsic or extrinsic forces leading to quantification of ideas, then the replacement of concepts (in the case of quantification, expressing concepts and relationship between them in mathematical language) can be seen to be useful by students. These new conceptions could be attributed partially to the strong cognitive conflict (Posner's conceptual change model) developed in the process of quantifying ideas and concepts. Thus, the other factors related to learning theory, *static-dynamic* description for an *emergence profile* of a quantified form and an associated *emergent press* will also be used in the textbook analysis for this study.

De Berg's methodology is also useful for answering research question 4 (c) of this study (about the pedagogical sequence of presentation of the mathematical component of physics) since his methodology helps to examine the sequence and purpose for introducing mathematical concepts in physics textbooks to make a judgement about the pedagogical appropriateness of the mathematical component presentation sequence.

A similar conceptual model for learning science was offered by Stinner (1992), namely the *LEP conceptual development model.* Stinner says: "In planning successful science teaching we would need to pay attention to all three planes of activity: the logical, the evidential, and the psychological" (p. 6). This conceptual model is much broader than de Berg's model because the *LEP* model is applicable to all participants in learning and teaching science, students, teachers, and the interpretation of textbooks. Since the subject of this study is analysis of physics textbooks for the treatment of the mathematical component, a more specific methodology, oriented mainly to the textbook analysis developed by de Berg would be more suitable.

Procedures for Collecting, Organizing, Analyzing, and Synthesizing Data

After identifying theoretical and methodological frameworks that guide the design of this study it is appropriate to discuss how data will be collected, organized, analyzed, and then synthesized to help make conclusions on the questions of the study.

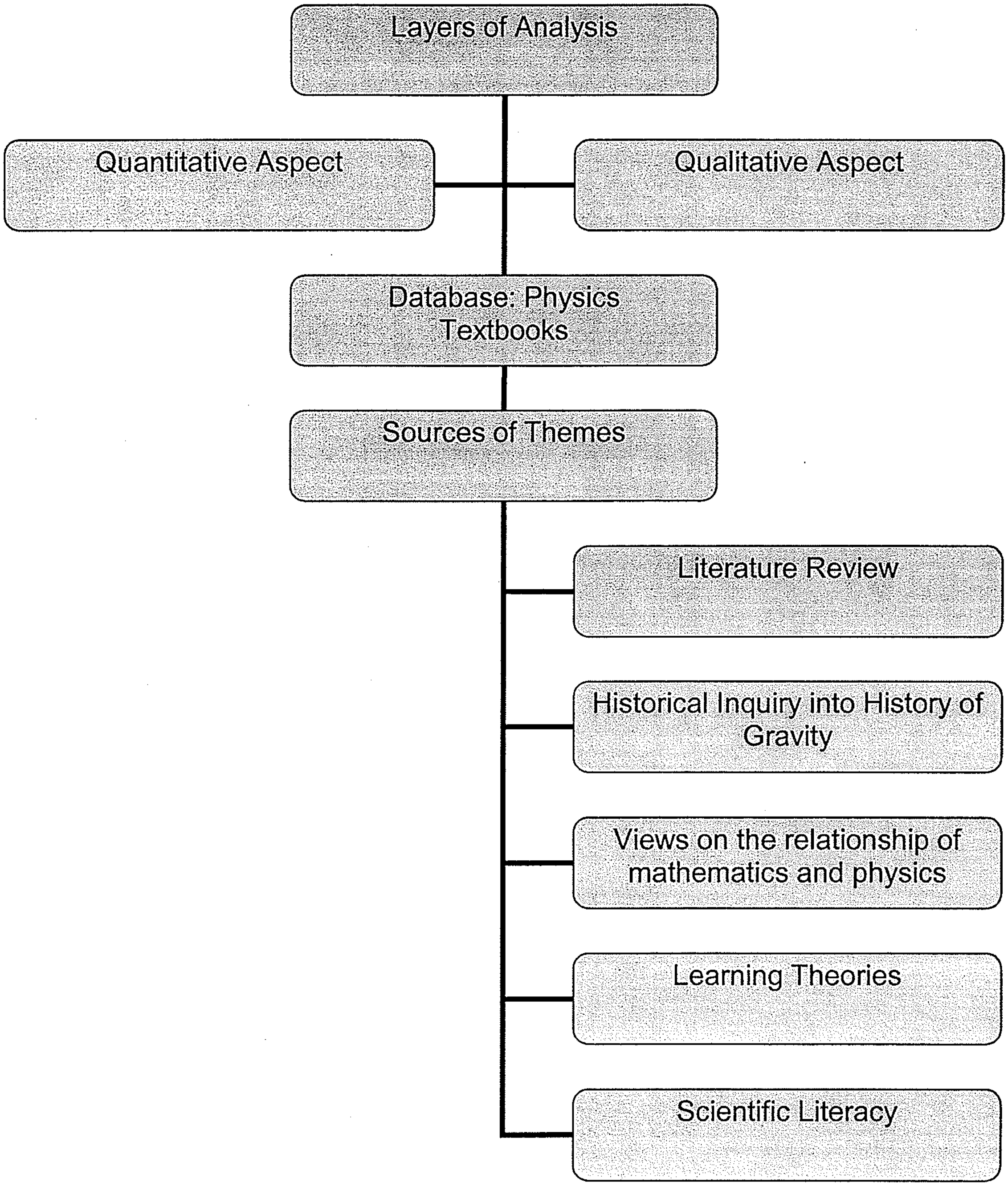
One of the ways to represent data in a qualitative research is a hierarchical tree diagram (Creswell, 1998). According to Creswell (1998), " a hierarchical tree diagram… shows different levels of abstraction, with the boxes in the top of the tree representing the most abstract information and those at the bottom representing the least abstract themes" (p. 145). The generated visual picture would help to conduct the analysis, identify categories, and show how information from the text is grouped. This visual picture also displays the interconnectedness of the categories shown.

The following illustration on the next page shows inductive analysis that begins with sources of information about the role of mathematics in physics. This analysis will later (after examination of identified sources) be broadened to several specific themes to develop the instrument for content analysis of physics textbooks and on to the most general themes represented by the two aspects of the mathematical component of physics – quantitative and qualitative.

In the process of content analysis of the physics textbooks identified, mathematical terms and concepts will be classified, tabulated and evaluated to ascertain meaning for interpreting the mathematical component presented in physics textbooks. The presence or absence of some concepts would also provide valuable information for making inferences. Therefore, a presence matrix will be constructed for data analysis. Pedagogical sequence of presentation of the mathematical component makes a difference for understanding of physics concepts, as has been shown earlier in this chapter, in the section *Learning Theories*. Hence, sequence maps will be constructed to help identify the appropriate pedagogical sequence of presenting the mathematical aspect of physics in textbooks.

The instrument to be developed for the content analysis of physics textbooks will help to establish the meaning of mathematics used in physics textbooks. This would in turn be a determining factor in establishing the extent of balancing between the quantitative and the qualitative approaches. I want to find out if the presentation of the mathematical component in physics textbooks insures the balance between the qualitative and the quantitative aspects of physics, and consequently provides conditions for meaningful learning.

Figure 3-1. Layers of Analysis



Summary

My personal stand on the issue of the study - presentation of the mathematical component in physics textbooks to insure the balance between the qualitative and the quantitative aspects of physics - is that in the presentation of ideas in physics, a quantitative and a qualitative approaches must be used for successful learning.

Current learning theory and the requirements of scientific literacy will be the theoretical frameworks for the textbook analysis part of this study.

The following are the reasons why I have argued that physics textbooks should present mathematical concepts in physics in such a way that the qualitative and the quantitative aspects of physics are balanced:

According to epistemological theories of learning, the evolution of students' thinking goes through several cyclic stages. Therefore, if the material is presented in the finished form (as a mathematical equation), the epistemological chain of the development of students' thinking is broken. Consequently, it would be unlikely that the students understand the meaning of the concepts involved.

Cognitive theories of learning inform us that the mind organizes repeated similar experiences in schemata that have a flexible structure, with many channels. These channels comprise different kinds of knowledge. To invoke these types of knowledge in students' mind, the presentation of material in physics should utilize both of the approaches – qualitative and quantitative – so that different kinds of knowledge in the operative system of the brain could be connected for students to enable learning.

According to science educators, these knowledge types can be represented in different levels (macroscopic, symbolic, and microscopic) and in different modes (verbal, numerical, pictorial, graphical, as well as symbolic). Science educators have found that many students fail to make strong connections between the symbolic signs used and physical reality. One of the reasons may be that concepts in science often have a limited representation.

The science educators' goal is to make students understand concepts of science. These concepts represent mental abstractions that consist of "regularities" and "structures." Mathematical symbols have been created to express concepts of order and structure, and mathematics has been often described as "the science of patterns." Thus, concepts in science have to be diversely connected. In order to connect them, I believe, science educators should strive to present regularities and structures in a balanced way where the qualitative and the quantitative aspects of these mental abstractions are properly balanced. In this process of balancing, students would develop better conceptual models.

Studies of experts' and novices' problem solving strategies demonstrated that the use of a proper pedagogical sequence in problem solving (going from qualitative analysis of the situation to diagramming, and then to thinking about the main principles involved, and finally, to symbolic quantitative equations) helps them to balance the qualitative and the quantitative aspects of physics.

Educators agree that the learning cycle (exploration, development and application of concepts) should not be rushed if we want students to construct meanings for the concepts they learn. Therefore, the balancing of the qualitative and quantitative aspects of physics in the process of meaningful learning seems to be an effective strategy (given the right sequence of presentation of concepts) to achieve better understanding.

In the widely used view of a conceptual change model (like Posner's model), balancing the qualitative and the quantitative aspects of physics is also justified, since students would see the replacement of their earlier ideas as plausible and useful when they finally understand the finished product (symbolic equations of physics laws, for example). In this case, when the balancing act is achieved, mathematics will be utilized by students for the purpose of understanding, and not just memorization.

According to the requirements of scientific literacy, a scientifically literate person understands fundamental concepts, laws, principles and facts in the basic sciences, as well as appreciates the variety of scientific methodologies and appropriately utilizes them. Mathematics in all its modes of representation can be considered as a tool to describe physics ideas, and not necessarily to understand physics. Since the goal of a scientifically literate person is to understand, not only to describe concepts, laws, principles, etc. (which are often expressed in the mathematical language), then it is natural to conclude that mathematical representation of concepts alone does not show evidence for conceptual understanding. A balanced way (qualitative and quantitative) of expressing knowledge of physics, in my view, would be a better indicator that a student understands physics ideas and concepts, as well as appreciates the variety of scientific methodologies and appropriately utilizes them.

The document advocating the achievement of scientific literacy, *Science for All Americans* (1989), calls for attention to the connection among science, mathematics, and the history and philosophy of science. From history of science examples, students can see how scientists themselves struggled to balance their mathematical findings with their qualitative conceptions of the phenomena they studied. Students are provided with some comfort when they realize that great scientists also struggled to strike the balance with their experiences or intuitive thinking and mathematical equations obtained at the end of the discovery journey.

One of the objectives of the document *Science for All Americans* is for students to have understanding of the nature and the methods of science. Mathematical representation has its place in the description of laws, principles and methods of science. Students have to learn to distinguish between laws, theories, observations, inferences, and speculations. The use of appropriate scientific language would not be possible for students to develop if they use only mathematical representations, confusing symbolic statements with qualitative inferences.

The conceptual framework for this study was developed in the light of learning theories, science education researchers' findings, and the requirements of scientific literacy. It appears to have three domains: epistemological, cognitive, and contextual. Every domain has its sub-domains. These sub-domains will be used to categorize the mathematical component found in physics textbooks. This conceptual framework will guide the design of physics textbooks analysis. In addition, this informed conceptual framework will be refined after additional themes have been generated from the historical inquiry on the relationship of mathematics in physics and the history of gravity that will be reported in Chapters 4 and 5.

Historical Inquiry was chosen as the method of research. To explore the historical relationship between mathematics and physics, I will explore views on the role mathematics played in physics from a historical perspective. The historical connection of mathematics and physics will be reported in Chapter 4.

To explore the role that mathematics played in the history of gravity the inquiry into the history of gravity and examination of the stages of Newton's thinking when he describes universal gravitation will be carried out.

The methodology of the qualitative content analysis will be used for the research of textual material to understand what role mathematics plays in physics textbooks.

I will examine five recent high school physics textbooks recommended by the Manitoba Department of Education, Citizenship and Youth, and also used in other provinces of Canada.

For instrument construction, themes referring to scientific literacy will be taken from the methodology of content analysis of science textbooks suggested by Chiappetta, Sethna, and Fillman (1991). Only certain categories will be selected and modified for this study according to the research questions of this study. Only themes helpful for the qualitative content analysis and those related to the role of mathematics in physics education will be used in this study. Other themes will be taken from the methodology offered by de Berg (1989) in his textbook study on the emergence of quantification in the pressure-volume relationship of gases.

The research procedures in this study will commence with an inductive analysis that begins with sources of information about the role of mathematics in physics. This analysis will later (after examination of identified sources) be broadened to several specific themes to develop the instrument for content analysis of physics textbooks.

In the process of content analysis of the physics textbooks, mathematical terms and concepts will be classified, tabulated and evaluated to ascertain meaning for interpreting the mathematical component presented in physics textbooks. The presence or absence of some concepts will also provide valuable information for making inferences. Therefore, a presence matrix will be constructed for data analysis. Sequence maps will be constructed to help identify the appropriate pedagogical sequence of presenting the mathematical aspect of physics in textbooks.

The instrument to be developed for the content analysis of physics textbooks will help to establish the meaning of mathematics in physics textbooks. This would in turn be a determining factor in establishing the extent of balance between the quantitative and the qualitative approaches.

The next chapter (Chapter 4) addresses the question about the historical relationship between mathematics and physics. The close connection between mathematics and physics and the predictive power of mathematics will be described and supported by examples from the history and philosophy of science. The quantitative and the qualitative aspects of physics and mathematics will be discussed, and the value of each will be shown. Finally, it will be revealed what educators have learned about the relationship between mathematics and physics from history and philosophy of science examples.

Chapter 4: Relationship between Mathematics and Physics

Overview

Chapter 4 addresses Research Question 2 – What is the historical relationship between mathematics and physics? Four sections comprise this chapter. The first section of this historical inquiry starts with a description of the close connection between mathematics and physics going as far back as the 3rd century, the time of Euclid and Archimedes. Historical examples from medieval physics show that the connection between mathematics and physics sets the conditions for later development of theoretical physics. The Mean Speed Theorem, Bradwardine's Function, and Buridan's theory of impetus, concepts representing exercises of thought, are examples from the second quarter of the fourteenth century of the existence of a critical connection between mathematics and physics. The discussion continues about the significance of Menaechmus' discovery of conic sections in 375-325 B.C.E, including application of his discovery 1800 years later to the real world when Kepler used conic sections in his laws of planetary motion. Later, Newton used the properties of conic sections and Kepler's laws of planetary motion to discover the law of universal gravitation. The relationship between mathematics and physics was emphasized when Galileo established his conception of scientific method wherein mathematics played a crucial role. The historical inquiry in this chapter reveals how with the invention of calculus by Leibniz and Newton in the seventeenth century, mathematics became the new language of physics, and consequently physics was transformed from a qualitative to a quantitative subject. The first section of Chapter 4 concludes with a discussion about the separation of natural sciences into two parts at the end of the eighteenth century: mechanics (quantitative) and experimental science, or physics (qualitative). The birth of theoretical physics in the middle of the nineteenth century is also highlighted.

The second section of this chapter concentrates on the predictive power of mathematics. Using history of science examples, it is shown how mathematical reasoning enables scientists to predict patterns of motion. Particularly, Galileo's example of studying the motion of a ball on an inclined plane is used to show how nature could be described using the language of mathematics. Philosophers of science held mathematical predictions in high regard, giving these predictions explanatory power. Examples of the predictive power of mathematics, such as calculating trajectories of projectiles and orbits of celestial objects, and examples of the mathematical predictability of physical events happening even on the atomic scale, conclude the second section of this chapter.

The third section of this chapter deals with the quantitative and the qualitative aspects of physics and mathematics. The discussion stresses that both quantitative and qualitative aspects are valuable. The history of science emphasizes this by demonstrating how scientists expressed their ideas in different ways. Scientists' ideas cannot be undervalued, even if they are not expressed mathematically, because the conceptual richness of their ideas remains profound.

The last section of this chapter focuses on what educators learned about the relationship between mathematics and physics from the history and philosophy of science examples. The extent of educators' learning determines to what degree they see mathematics playing a role in physics. The views of educators are presented in this section. The subsequent conclusion formulated is that the history of mathematics has to be treated in connection with the history of physics. The Historic Genetic Approach introduced by Tzanakis (1999) is presented and a suggestion to extend his recommendation for instruction to textbooks' application of his approach is explored.

The History of the Development of Mathematics and Physics

My exploration of the role of mathematics in physics education will be based on the assumption that there is a fundamental connection between physics and mathematics. The first logical step in this exploration would be to establish the relationship between mathematics and physics which is reflected in the history of the development of these two disciplines.

The mathematician, Morris Kline (1959) said that historically, intellectually and practically, mathematics was primarily man's finest creation for the investigation of nature. He claims that the major mathematical concepts, methods and theorems have been derived from the study of nature. He stressed that mathematics is valuable largely because of its contributions to the understanding and mastery of the physical world. This close relationship between mathematics and physics goes as far back as the 3rd century B.C.E., the time of Euclid and Archimedes. In the earliest Greek writings, for example, in describing certain aspects of nature like music, Pythagoreans used whole number ratios. Euclid created geometry that became later the foundation for western mathematics. He used that geometry when writing a book about optics. Archimedes did work on statics and hydrostatics using Euclidean ratios. One of the earliest efforts to use algebraic functions to describe motion was attempted by Aristotle in his pursuit of an explanation of projectile motion. He believed that a body is maintained in motion by the action of a continuous external force (This idea was later developed into so called *Impetus Theory* of motion). This continuous force, as Aristotle thought, was necessary to overcome a resistance offered to the motion. Given this view, in the absence of a proximate force, the body would come to rest immediately (Stinner, 1994). Aristotle wondered what force keeps the projectile in motion after it loses contact with the projector and came up, as Stinner (1994) notes with the only explanation that "the medium somehow provided the necessary force to push the projectile", the air displaced in front of the projectile somehow rushes around it and pushes from behind, thus propelling the projectile along. This explanation is very interesting. Stinner (1994) comments on Aristotle's explanation as following:

> *The paradoxical state of affairs is connected with Aristotle believing that the medium not only sustains the motion but also resists it. Motion in a void was impossible because there was no medium to sustain the motion, and in the absence of resistance the object would eventually move at an infinite speed, clearly an unacceptable solution.* (p. 79)

The essence of the functional relationship Aristotle came up with could be expressed (Stinner, 1994) as the following: "...velocity is directly proportional to the force and inversely proportional to the resistance of the medium, or $V \sim F/R$" (p.78), where V is the object's velocity, F is force acing on the object, and R is resistance of the medium.

Another medieval physicist -Thomas Bradwardine- attempted to show how the dependent variable velocity V was related to the two independent variables force *F* and resistance *R*. He believed that the velocity would vary arithmetically when the proportions of force to resistance are varied geometrically. The function he came up with was incorrect. However, Bradwardine had expressed the Aristotelian law of motion (impetus + resistance accounts for acceleration and deceleration of motion on earth) quantitatively as a function (Bradwardine's Function).

During the Middle Ages Aristotle's ideas of force and motion were first challenged by John Philoponus (sixth century AD). He proposed the much more plausible but still erroneous idea that a projectile moves because of a kinetic force which is impressed on it by the mover (some property of a body, imparted when set in motion) and which exhausts itself during the motion. The functional form of Philoponus's idea differs from Aristotle's one ($V \sim F/R$) as following: $V \sim F - R$ (Stinner, 1994). Philoponus reassessed the role of the medium in the motion of a projectile – he believed that the medium was not responsible for continuation of a projectile's motion. In fact, the medium was an impediment to it. On this basis, Philoponus concluded, against Aristotle, that there was nothing to prevent one from believing that motion could take place through a void (Stinner, 1994).

In the thirteenth and fourteenth centuries, Jean Buridan and Nichole Oresme from France worked out a theory to explain projectile motion. In the 14th century, Jean Buridan named the motion-maintaining property *impetus* and developed impetus theory further. He rejected the view that impetus dissipated spontaneously, arguing that a body would be influenced by the forces of air resistance and gravity which might be opposing its impetus. Buridan saw the necessity of some type of motive force within the projectile. He regarded it as a permanent quality, however, and quantified it simplistically in terms of the primary matter of the projectile and the velocity imparted to it. Although he offered no formal discussion of its mathematical properties, Buridan thought that the impetus of a body increased with the speed at which it was set in motion, and with its quantity of matter. In this respect, Buridan's concept of impetus was similar to Newton's *quantity of motion* (momentum). The following example of Buridan's writing demonstrates how close he came to Newton's concept of force:

> *...after leaving the arm of the thrower, the projectile would be moved by impetus given to it by the thrower and would continue to be moved as long as the impetus remained stronger than the resistance, and would be of infinite duration were it not diminished and corrupted by a contrary force resisting it or by something inclining it to a contrary motion.* (http://encyclopedia.laborlawtalk.com/Inertia)

The example of using mathematics (functional relationships) in describing *impetus theory* is significant in the history of science not only to demonstrate the relationship between mathematics and physics but to show that what was new about the fourteenth-century development was the technical significance given to the concepts in context that is engaged later, in the discussion of gravitational motion. Buridan used his impetus concept to explain the acceleration of falling bodies. He believed that continued acceleration results because the gravity of the body impresses more and more impetus (Dictionary of the History of Ideas, retrieved on April 16, 2005).

Another example of the relationship between mathematics and physics from the history of science is how Nicole Oresme attacked the problem of accelerated motion by graphic constructions.

With Bradwardine and Oresme, the treatment of kinematics problems was posed as imaginary possibilities for theoretical analysis, essentially logical exercises, and without empirical application. Between 1328 and 1350 the work of Thomas Bradwardine, William Heytesbury, Richard Swineshead and John Dumbleton, at Merton College, Oxford, laid the groundwork for further study of space and motion, by clarifying and formalizing key concepts, such as that of instantaneous velocity (Babb, 2003). The Merton School of philosophers at Oxford (13^{th}-14^{th} centuries) and Nicole Oresme created the *Mean Speed Theorem* (Merton Rule) to analyze velocity of a point in time: An object starting from rest and accelerating uniformly over a specified time traverses a distance equal to the distance a second object travels by moving for the same time at a constant speed equal to one-half the maximum speed attained by the first object. According to Clagett (1968), this rule for calculating mean speed was first proposed by Williams Heytesbury in 1335. Commenting on the significance of the *Mean Speed Theorem*, Clagett (1968) said that the invention of the mean speed theorem was one of the true glories of fourteenth century science. According to Clagett (1968), Oresme explained velocity with *longitude* and *latitude* by means of a graph using a geometric approach introduced between 1348 and 1362. In his analysis of motion, Oresme was interested in the forms of qualities, such as velocity. He graphed the extension in time (longitude) of a quality along a horizontal line and the intensity of a quality – velocity (latitude) along a vertical line. Then the graph corresponding to uniform velocity (uniform form of quality) would correspond to a horizontal line, and the graph of motion with constant acceleration (uniformly deformed form of quality) would correspond to a line rising (or falling) at an angle. In each case, distance traveled would be the area under these curves. For constant velocity, distance traveled is equal to the area under the graph which is the area of a rectangle = velocity x time, and for constant acceleration, distance traveled equals the area of a triangle = ½ maximum velocity x time. Since the rectangle and the triangle have equal area, therefore, the mean speed rule is proven (Babb, 2003). Later, Galileo provided a little bit different explanation but he used the same graphs.

The described earlier three achievements of medieval physics in the second quarter of the fourteenth century (Mean Speed Theorem, Bradwardine's Function, and Buridan's theory of *impetus*) were logical exercises which did not require empirical application. The connection of physics and mathematics, in these cases, was an exercise of thought. In other cases, mathematics was used by scientists for the investigation of nature when conducting experiments. According to Kline (1959), mathematics enables the various sciences to draw the implications from their observational and experimental findings because mathematics organizes broad classes of natural phenomena into coherent patterns.

Today mathematics is at the heart of our best scientific theories, including Newtonian mechanics, the electromagnetic theory of Maxwell, Einstein's theory of relativity, and the quantum theory of Planck and his successors. In some of these examples, physics ideas often guided the development of mathematics. For example, Newton had to invent his calculus in order to develop the law of universal gravitation. On the other hand, there were situations in the history of the development of physics and mathematics when discoveries in mathematics found their scientific values hundreds and thousands of years later. One of the examples of this is when a Greek man named Menaechmus discovered the curves of ellipses, parabolas, and hyperbolas (known as conic sections) in 375-325 B.C.E. (The great geometer Appollonius studied them later and wrote eight books regarding his study on conic sections.) During the discovery of these conic sections, Menaechmus was attempting to solve three famous mathematics problems of trisecting the angle, duplicating the cube, and squaring the circle. At the time of his discovery of conic sections, they had no practical uses.

It was not until about 1800 years after conic sections were discovered that people began to see that the curves of conic sections could be applied to the real world. For example, in space, as established by Kepler, a planet orbits around the sun in the curved path of an ellipse. Also, comets travel in the paths of hyperbolas or ellipses, and the path of a projectile can form a parabola. Due to these natural phenomena, conic sections became the main topic in Kepler's laws on planetary motion. Many years after conic sections were discovered, Newton was able to use the properties of conic sections and Kepler's laws of planetary motion to discover the law of universal gravitation. Newton showed that planets orbit the sun because the sun exerts a force of gravitational attraction on them. He was able to show that all Kepler's laws hold true because of his law of universal gravitation. Furthermore, the properties of conic sections helped Newton conclude that the gravitational force between two objects is proportional to the masses of these objects and inversely proportional to the square of the distance between them. The consequences of Newton's use of conic sections go even further: using Kepler's laws and Newton's gravitational law, we can approximate the current positions of the nine known planets at a given time. Also, since we know that asteroids and comets travel in the paths of conic sections, we can use the laws of gravity, laws of motion, and the properties of conic sections to track down current locations of various asteroids and comets in space.

As history of science examples show, the relationship between mathematics and physics became especially strong after Galileo established his concept of scientific method wherein mathematics plays a crucial role. After Galileo, many great mathematicians made contributions to physics: Descartes, Fermat, Leibniz, the Bernoulli brothers in the 17th century, Euler, D'Alembert and Lagrange in the 18th century, Laplace, Cauchy, Gauss and Riemann in the 19th century, Poincare and von Neumann in the 20th century. Newton himself made outstanding contributions to both mathematics and physics. It was Newton's ideas about the "rate of change, or fluxion, of continuously varying quantities, or fluents, such as lengths, areas, volumes, distances and temperatures" (Boyer, 1991, p. 393) that allowed him to formulate some of the mathematical laws that underlie nature. Newton was not the only person to come up with the ideas of the calculus. Leibniz also did so independently at about the same time, but it was Newton who used it in physics. He changed the structure of physics forever. Mathematics became the new language of physics, a form of discourse where physics was transformed from a qualitative to a quantitative subject. This use of mathematics gave physics a new sense of "truth". The use of mathematics as evidence in science, and particularly, physics was largely established on the grounds of the assumption that if the mathematics is correct, the physics must be correct. If a book contains a passage of mathematics that is correct, it is hard to argue against it. No amount of literary prose and argument can outweigh such a powerful discourse. The Scottish historian and philosopher David Hume expressed the power of quantitative thinking and the criteria for a good physics textbook in 1748. In his best-known work entitled *An Enquiry Concerning Human Understanding*, Hume asserts:

> *If we take in our hand any volume - of divinity or school metaphysics for instance - let us ask, Does it contain any abstract reasoning concerning*

> *quantity or number? No. Does it contain any experimental reasoning concerning matter of fact and existence? No. Commit it then to the flames, for it can contain nothing but sophistry and illusion.* (Section xii part 3)

And more than 100 years later Lord Kelvin (1827-1907) said:

> *I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it, but when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of Science, whatever the matter may be.* (As cited in L. Wirth, 1940, p.169)

With Newton's work transforming physics from a qualitative to a quantitative subject, a new period in the development of physics and mathematics commenced, the period of separation of physics from other sciences. After the Newtonian revolution of the 17^{th} century, natural sciences divided into two parts: mechanics which became more mathematical (quantitative) under the influence of mathematicians such as Lagrange, and experimental science which investigated nature in a mostly qualitative way. Towards the end of the 18^{th} century, especially French scientists like Laplace and Poisson used mathematics more widely in physics. By the end of the 19^{th} century, physics had become well established as a separate discipline, in which physicists performed qualitative experiments and explained results with the help of theories framed in the language of mathematics. This development around the middle of the 19^{th} century lead to the emergence of theoretical physics as a separate branch of physics. Consequently, physics split into an experimental and a theoretical part. Speculating about the reasons why this split occurred, historian Elizabeth Garber (1999) in her book *The Language of Physics:*

the calculus and the development of theoretical physics in Europe 1750-1914, contests that much 19th century work, which up to now has been considered theoretical physics, is in fact mathematics. She explains that for a long period, physicists were split personalities. On the one hand, they did experimental work, using mathematics, especially calculus, to quantify their results. On the other hand, they took mathematics, for instance differential equations, as the starting point for new investigations that in turn gave rise to new, purely mathematical results that were given little or no physical interpretation. Garber terms the latter part "mathematical physics", the physics which usually is published in mathematical journals. In Garber's view, this trend persisted until well into the 19th century, when British and German physicists created theoretical physics "by adopting the attitude that mathematics was just a tool to be used within physics ...often to the dismay of mathematicians who watched with horror how physicists ignored proper mathematics to reach their goal" (Kox, 1999, p. 39). One of the striking examples of such use of mathematics was Paul Dirac's introduction of his delta-function, which was widely and successfully used in physics. An illuminating source of information about history of the Dirac delta function is the book by J. Lűtzen *The Prehistory of the Theory of Distributions* (1982).

A characteristic example of the split between mathematics and physics, described by Garber in her book, is the difference of opinion between Albert Einstein and the mathematician David Hilbert on the general theory of relativity. In about 1910, Hilbert had started to work in physics, commenting that "physics was too difficult to leave to the physicists". Among other things, he took up Einstein's early work on general relativity. Hilbert eventually published a theory that was similar to Einstein's final version of general relativity and appeared almost at the same time as Einstein's work. However, whereas Einstein's theory was firmly based on physical principles, Hilbert's was much more an exercise in pure mathematics, based on some dubious assumptions. Although Einstein admired Hilbert's mathematical proficiency, he characterized Hilbert's physics as "infantile".

The consequences of the mathematization of physics have a significant impact on science to the present time. Science with a mathematical basis is often considered "better", having the power to predict events with certainty. Theoretical physics gained an elevated status. This may have led to important qualitative works being overlooked, or looked down upon. One of the historical examples of the experimental work being looked down upon is the original attitude to Hubble's finding about the light from most of the galaxies in the sky being red-shifted. His finding meant that the distant stars must be receding from us as the space between us expands (Doppler shift). According to Arianrhod (2005), "no one had seriously imagined that the universe is expanding before it turned up as an unintended consequence of Einstein's equations for the geometry of four-dimensional spacetime" (p. 186). Mathematics definitely had an elevated status compared to any qualitative work. Arianrhod (2005) goes on to say:

> *With Hubble's monumental discovery, Einstein regretted his lack of mathematical Faith, saying it was the biggest blunder of his life. But the discovery helped Validate the general theory of relativity, revealing it as another example of the Way mathematics can take physicists beyond thought itself* (p. 186).

As for physics education, according to my experience in the classroom, the mathematization of physics lead to the situation when students need much encouragement to include any words at all when they learn physics. They often prefer to answer questions with pages of mathematics thinking that it is easier to understand this way.

One can level the criticism that the mathematical side of physics has gone too far when ideas cannot be easily grasped conceptually or described using language at all (such as the quantum world), but we have to admit that with a new quantitative and incomprehensible nature physics also became powerful. Physics was no longer just about observation, description and possible explanations. It could now use mathematics to explain and predict the natural world (at the limits of observation of the time). Although discoveries like the theory of relativity have shown that Newton's laws hold only in some situations, they are still widely and successfully used in solving many problems in physics. One of the reasons for this success clearly lies in the predictive power of mathematics.

The Predictive Power of Mathematics - Philosophical Reflections

According to Weisheipl (1967) and Kline (1959), there has been a tendency to describe nature in mathematical terms since ancient times. According to these historians, the conviction that nature is mathematical and that every natural process is subject to mathematical law began to take hold in the twelfth century when Europeans first obtained this view from the Arabs, who in turn were quoting the Greeks. The Greeks who most effectively promoted the mathematical investigation of nature were Pythagorous and Plato. For Plato, for example, mathematics constituted a metaphorical bridge that connected the terrestrial to the celestial realm. Plato believed that through mathematics we could understand the world as only an imperfect image of an eternal realm. He did not conceive of mathematics as anything more than a metaphorical or analogical tool to aid understanding and facilitate enlightenment. Roger Bacon, for example, believed that the book of nature is written in the language of geometry. It was a common belief in the thirteenth century that the geometrical laws of optics were the true laws of nature. Leonardo da Vinci did not have a good understanding of mathematics. Nevertheless, he believed in a connection between mathematics and science. He wrote: "No human inquiry can be called true science, unless it proceeds through mathematical demonstrations".

Weisheipl (1967) concludes that the mathematization of motion was accomplished as early as the 14^{th} century at Oxford by Franciscan Thomas Bradwardine and others, and later by Galileo, Descartes and Newton. Weisheipl goes on to stress Bradwardine's significance in establishing the importance of mathematics in understanding and uniting earthly and celestial motions: "It was he who introduced mathematics into scholastic philosophy, initiated the two new sciences of kinematics and dynamics, and made the initial move toward uniting celestial and terrestrial motions under a single mathematics. In a burst of enthusiasm reminiscent of Robert Grosseteste, Roger Bacon and Galileo, Bradwardine declared:

> *It is (mathematics) which reveals every genuine truth, for it knows every hidden secret, and bears the key to every subtlety of letters; whoever, then, has the effrontery to study physics while neglecting mathematics, should know from the start that he will never make his entry through the portals of wisdom.*" (p. 94)

It is evident that this view on the role of mathematics in the studying of physics was shared by Galileo who emphasized that the book of nature was written in the language of mathematics. Drake (1957) in *Discoveries and Opinions of Galileo* translated the following statement from Galileo's *The Assayer* of 1623:

> *Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.* (pp. 237-238)

Thus, Galileo studied and described motion in precise mathematical terms. He discovered and stated the law of falling bodies by first hypothesizing that the speed is proportional to the elapsed time, then showing that the distance must be proportional to the square of the elapsed time. But this was only a hypothesis, and proving it right or wrong turned out to be a difficult task. It was quite difficult to try to measure directly either the velocity or time for a freely falling body. Today we can make measurements easily with high-speed photography, but Galileo had to work with a ruler and a water clock. Confronted with these limitations, he decided to slow down his experiment. He was convinced that the laws of falling bodies would also apply to a ball rolling slowly down an inclined plane. So he set up a gently sloping plank at 7° inclination, about 15 feet long, with a narrow groove cut along the center of it. A polished bronze ball rolled down the groove slowly enough to permit making accurate measurements of time and distance. Galileo's experiments confirmed that the distance traversed by the ball depended upon the square of the elapsed time. Since this relationship was true for any inclination of the plane, he concluded that it would also be true for the limiting angle when the plane was vertical. The ball would merely fall alongside the plane, but its distance at any instant would be proportional to the square of the time of fall. He also proved mathematically that if the distance were proportional to the square of the time, then the velocity at any instant would be proportional to the elapsed time. This way of studying motion is a mathematical description that was based on a conceptualization of a hypothesis and the working out of its measurable consequences.

This example illustrates how mathematical reasoning enables scientists to predict patterns of motion. Some might argue that these laws of falling bodies could be discovered if Galileo performed experiments described above before he came up with his hypothesis. For an argument, I refer to Hanson (1958), who distinguished between 'seeing that' and 'seeing as.' Hanson emphasized that 'seeing as,' the *gestalt* sense of seeing, had been important in the history of science. Kuhn (1962) later developed Hanson's ideas in his famous work *The Structure of Scientific Revolutions.* He believed that the theories that scientists accept significantly affect what scientists observe. Kuhn subscribes to Hanson's idea of the theory-ladenness of observation, according to which what scientists observe depends on the background theory. If one accepts this view, it is unlikely that Galileo could design the experiment described above without having an idea of how the results would look. In this particular case, mathematics, due to its deductive nature, was a necessary tool to derive a functional relationship between distance and time of fall. Granted, Galileo, when referring to mathematics, used Euclidean ratios. However, later physicists used algebraic expressions to make predictions.

Mathematics plays an important role in theory choice because of its predictive power. One of the epistemic values for theory choice according to the scientific community is the ability of a theory to give rise to testable predictions. Evaluating the writings of the philosopher Karl Popper, in particular about the falsifiability criterion of a good theory, Snyder (1998) asserts, "only predictions can count as evidence, because only predictions are 'potential falsifiers' of a theory" (p. 463). Another philosopher McMallin (1998) states, "the goals of predictive accuracy… serve to define the activity of science itself, in part at least" (p. 130). Thomas Kuhn lumped together prediction and explanation, though many philosophers do not agree with him on this matter. However, Kuhn also seemed to think that the most important criterion for theory choice was the ability of the theory to make predictions. He states:

“Ultimately it proves the most nearly decisive of all criteria, partly because it is less equivocal than others but especially because predictive and explanatory powers… are characteristics that scientists are particularly unwilling to give up.” (As cited in Curd and Cover, 1998, p. 104.)

Kuhn (1998) believed that Copernicus was taken seriously because

> *He converted heliocentric astronomy from a global conceptual scheme to mathematical machinery for predicting planetary position. Such predictions were what astronomers valued; in their absence, Copernicus would scarcely have been heard, something which had happened to the idea of a moving earth before.*
>
> (As cited in Curd and Cover, 1998, p. 112)

It is worth noticing, however, that Ptolemy, like many medieval astronomers, did the same with tables of data in terms of mathematical predictions.

According to Jones (1992), beginning with Copernicus' sun-centered model of the solar system, mathematical laws have been treated as *actual descriptions of nature* (pp. 102-103). In a sense, therefore, the emphasis on quantitative descriptions in physics results from the great predictive power of mathematics. As Monk (1994) noted, "the condensed models of the interactions of variables are powerful in the nature of the accurate predictions they can make" (p. 210). For example, Newton related Galileo's equation of free fall to the motion of the moon around the earth. He was the first who understood that the force that causes a rock to fall is the same force that keeps the planets in their orbits. Newton noticed that the parabolic motion of the thrown rock on the earth could change to the elliptical motion of the planets (circular in case of the motion of the moon around the earth). The assumption for this kind of statement would be that for short-ranged projectiles we could neglect earth's curvature, i. e. consider it flat. In this case gravitational field strength vector **g** is the same since there is no change of direction to the center of the earth. If we do consider the earth's curvature, the path of a projectile will change from a parabola to an ellipse provided the projectile is launched with sufficient velocity that the ground does not get in the way. In this case, the projectile will be orbiting earth in an elliptical trajectory.

A standard example in university textbooks (e.g. see R. P. Olenick et al. *The Mechanical Universe*) shows mathematically that Galileo's parabolic trajectory is approximately a small segment of an ellipse with the earth's center at the more distant focus. The following steps can represent a summary of this proof:

The total energy and angular momentum determine the type of orbit.

The total energy E of an object of mass m projected from the surface of the earth at the velocity v_o is the sum of kinetic energy and the gravitational potential energy of the system:

$$E = \frac{1}{2} m v_0^2 - G\frac{Mm}{R},$$

where M is the mass of the earth ($\approx 6 \times 10^{24}$kg) and R is its radius ($\approx 6 \times 10^{6}$ m).

The angular momentum of the object about the center of the earth is

$$L = m v_0 R \sin\theta,$$

where θ is the angle of launching the projectile from the vertical.

The eccentricity e of the orbit of a projectile was derived to be

$$e = \sqrt{1 + \frac{2L^2 E}{D^2 m}},$$

where $D = GMm$. Substituting for E and L and factoring terms, the eccentricity of a projectile can be written as

$$e = \sqrt{1 + \frac{2v_0^2 \sin^2\theta \left(v_0^2/2 - GM/R \right)}{\left(GM/R \right)^2}}.$$

Typical initial velocities of most projectiles are in the range of hundreds of meters per second at the most, so the term $v_o^2/2$ is about 10^4 m^2/s^2. On the other hand, the term $GM/R = 7 \times 10^7$ m^2/s^2. This means that the term $v_o^2/2$ can be ignored compared to the terms GM/R in the numerator of the expression for the eccentricity, so

$$e \approx \sqrt{1 - \frac{2v_0^2 \sin^2\theta}{GM/R}}.$$

This last equation indicates that the eccentricity is less than one, which means that the orbit is part of an ellipse with the earth at the distant focus. In addition, since the velocity is comparatively small, the eccentricity is very nearly 1; that is, the elliptical orbit is extremely elongated. Using the authors' example $v_o = 100$ m/s and $\theta = 45°$, one can find that

$$1 - e \approx 7 \times 10^{-5},$$

so the eccentricity of the orbit differs from that of a parabola by seven parts in 100,000.

Newton established mathematically, using his calculus, that the inverse square law of gravitation must yield a trajectory that is one of the conic sections (parabola, ellipse, or hyperbola). He concluded that the type of trajectory depends on the total energy E. If $E = 0$, the trajectory is a parabola (mathematically, it means that eccentricity $e = 1$). If $E < 0$, the trajectory is an ellipse ($0 \leq e < 1$); and if $E > 0$, the trajectory is a hyperbola ($e > 1$). Thus, Newton showed that if the velocity were high enough, a planet would always be accelerating toward the sun without ever leaving its orbit. This is because an object's motion is the result both of its previous direction of travel and of its speed. Today, with the increasing exploration of space, it is observationally confirmed that with the correct velocity, an object can have an elliptical path; with more velocity, the object can escape the sun's gravity, and the object's path will look like a hyperbola. Only this kind of mathematical analysis could enable Newton in his *Principia* (1687) to connect the three laws of Kepler with his three laws of motion and his law of gravity. Hamming (1980) asserts:

> *Newton used the results of both Kepler and Galileo to deduce the famous Newtonian laws of motion, which together with the law of gravitation are perhaps the most famous examples of the unreasonable effectiveness of mathematics in science. They not only predicted where the known planets would be but successfully predicted the positions of unknown planets, the motion of distant stars, tides, and so forth.* (p. 3)

For example, Newton's method of calculating orbits helped predict the next appearance of Halley's comet. Before Newton, astronomers thought comets paid only a single visit to Earth. Newton showed that comets could travel along closed, elliptical orbits. These orbits have a pronounced elongation (highly eccentric orbits). This is why the comets fly away to great distances from the Sun. Accordingly - they have a long period of revolution. Edmund Halley, applying Newton's method of finding orbits, calculated the moment of return of a famous comet, whose appearance could be traced in the ancient chronicles. The prediction was a striking success: the comet returned periodically every 76 years. Newton's laws have been shown tremendously accurate. Newton's *Principia,* due to its stunning predictive success for over two centuries, continues delivering accurate predictions. Newtonian mechanics is still a good theory because of its ability to predict a big range of physical phenomena. The application of mathematics here has proved to be an indispensable tool for this purpose.

A famous example of predictive power of mathematics in astronomy is the story of the discovery of Neptune (1846). In his book *And There was Light,* Thiel (1957) describes the problem of the deviation of the planet Uranus from its calculated orbit. The French astronomer Leverrier suspected that a strong perturbing factor must be present, a body possessing considerable mass, possibly another planet. Leverrier undertook to calculate the mass and position of the hypothetical planet from the deviation of Uranus. The computations proved to be a subtle and toilsome business, and when he had finished Leverrier himself had no great confidence in his results. According to Thiel (1957), Leverrier wrote to Professor Johann Gottfried Galle in Berlin, where there was an excellent telescope, suggesting that the he look for the computed planet where Leverrier predicted it would be. The new planet Neptune was then soon identified. Thiel summarizes:

> *In this way Neptune was discovered, not by chance, like Uranus, but by the visionary powers of pure intellect, by computation from the universal law of gravitation. This was the very summit of prediction; it was prophecy translated into reality, a dream come true in the fullest sense.* (p. 291)

Of course, somebody can argue that Neptune could have been discovered by observation alone since telescopes were widely available at that time. According to Thiel (1957), a Cambridge student, John Couch Adams, carried out the same computations and came to the same conclusions a year before. His professor though, who had assigned him the task, did not think it was worthwhile to use the excellent telescope at Cambridge Observatory to look for the planet. Adams' results were known at Greenwich but the astronomers' best instruments there were not good enough to detect Neptune. As it turned out, if Neptune had not been predicted by mathematical computations, it might have never been detected (However, other planets have been predicted which were not actually there).

Hamming (1980) provides another example where mathematics was necessary and indispensable. His first experience in the use of mathematics to predict things in the real world was in connection with the design of atomic bombs during the Second World War. He was amazed how the numbers they so patiently computed on the primitive relay computers agreed so well with what happened on the first test shot carried out at Almagordo. Hamming stresses that there were, and could be, no small-scale experiments to check the computations directly. His later experience with guided missiles showed that the use of mathematics as an indispensable tool was not an isolated phenomenon.

One may argue that there are situations where mathematical predictions are not confirmed by experiment. One of these examples could be what philosophers of science call the missing value problem for functional laws. For example, Hooke's law says that the force exerted by a spring is directly proportional to the amount the spring is stretched ($F = kx$). If we stretch the spring several times its normal length, we will not get the magnitude of the force which is predicted by Hooke's law. But the reason for this is not a "faulty" law. Everyone understands that if we try to stretch the spring several times its original length, it is going to break. There are changes in the conditions, which are not accounted for by Hooke's law. It is also assumed that the law holds in the limits of proportionality only. Therefore, the law is fine if we know its limitations. Another example is the ideal gas law ($PV = nRT$) - the pressure times volume of n moles of gas is proportional to the absolute temperature of the gas. It is not hard to understand that only limited number of values of pressure and absolute temperature will be realized since no gas can be practically heated to all possible temperatures. As long as we understand these limitations, there is no problem with missing values. These functional laws, therefore, have great predictive power when used properly. There are, however, problems that are in practice so complicated that they defy formal mathematical analysis (for example, turbulent motion of fluids). Nevertheless, it is believed that this is just a practical limitation, not a fundamental one.

The other argument against predictability of physical events by mathematics could be when one considers events on the atomic scale. One could question the ability of mathematics to make exact predictions of individual events in quantum phenomena. We have to realize that if we have a large population of atomic systems, statistical laws apply, and the behavior of large populations of identical atomic systems is still accurately predictable. It was discovered that the final outcome might be so sensitive to the initial conditions that the long-term situation was in effect unpredictable, that totally different outcomes might be possible. Still, as French (1998) noticed, "statistical predictions of quantum physics are more exquisitely precise than anything in human affairs" (p. 12).

There are many other examples of using mathematics as a predictive mechanism in science. Examples from the history of science show that mathematics was necessary and indispensable in making predictions, often leading to discoveries. Newtonian mechanics and Maxwell's equations for electrodynamics are still valuable in predicting interesting events, even if the observed phenomena were not completely understood.

Quantitative and Qualitative Aspects of Physics and Mathematics

As previously was mentioned, in the early sixteenth century, Galileo emphasized that the book of nature is written in mathematical language. Mechanics had long been the study of the natural laws of moving bodies, but Galileo insisted that the basic concepts must be mathematical. This in turn required that only quantitative, objective characteristics of things - what Galileo referred to as "primary qualities" -could be considered in the science of mechanics. The number of objects, their size and shape, and their position and state of motion in space could all be quantified and become part of a natural law. "Secondary qualities" such as redness, sweetness, noisiness, and foul odor Galileo claimed depend upon the senses and reside only in consciousness (Galileo, *Discoveries and Opinions*, 1957).

At about the same time, Descartes reinforced Galileo's ideas by connecting the knowledge of nature with the knowledge of mathematics. Descartes believed that physics was the science of moving forms of space, just as geometry was the science of resting forms of space. He, too, insisted that an objective understanding of nature is possible only by way of expressing primary qualities in mathematical language, namely size, shape, and quantity. To the Renaissance scientist as to Greeks, mathematics was the key to nature's behaviour. Descartes also formulated the first modern statement of the law of inertia: the natural motion of an isolated body is uniform, along a straight line at constant speed. Departures from uniform inertial motion were attributed to the pushes or pulls exerted by other bodies. Descartes did not succeed in developing a full mathematical science of mechanics, but his ideas were to become the starting point for Newton. His mechanics dealing with bodies accelerated by forces was essentially a grand mathematical design to rationalize all motion that departs from the law of inertia. Newton successfully fulfilled the promise of Galileo and Descartes by constructing a geometric-mathematical theory of planetary and terrestrial motion.

Many historians (M. Kline, M. Hodges, T. Kuhn, K. Popper) give credit to Galileo for leaving very compelling argument, for us to look at our world quite differently than we had in the past. As Miles Hodges (2000) asserts,

> *...the implications of Galileo's thoughts is not that stars do star things, and trees do tree things, and rocks do rock things, and peasants do peasant things, and kings do king things - but that all things operate in a similar manner in accordance to universal principles*

Other historians interpret the view that underlying all life are beautiful mathematical principles, awaiting human discovery, as a simplistic view of the role of mathematics in study of nature. One of them is Karl Popper. He wrote in *The Open Universe: from the Postscripts to the Logic of Scientific Discovery*:

> *...the success... of simple statements, or of mathematical statements... ought not to tempt us to draw the inference that the world is intrinsically simple, or mathematical... All these inferences have in fact been drawn by some philosopher or other; but upon reflection, there is little to recommend them. The world, as we know it is highly complex; and although it may possess structural aspects which are simple in some sense or other, the simplicity of some of our theories - which is of our making - does not entail the intrinsic simplicity of the world* (Popper, 1982).

According to Jones (1992), "the sciences generally, and physics in particular, pride themselves on their quantitative character. The presumption is that quantitative descriptions of things are somehow more meaningful and objective than qualitative ones" (pp. 194 -195). Since Galileo physicists have been using algebraic models to make predictions, and in doing so, physicists used mathematics as a formal description of the actual mechanism behind all natural phenomena. Jones (1992) points out a limiting side of the situation when mathematical laws are treated as *actual descriptions of nature* by saying that "this grand mathematical machine, like the simple mechanical devices it imitated, was mindless and inanimate, acting without meaning or purpose" (pp. 102 - 103).

According to Kline (1959), Galileo, Newton and their successors were criticized (Berkeley, Mach, Popper) for their proposed plan of obtaining quantitative descriptions of scientific phenomena independent of any physical explanations. For example, Galileo's decision to seek the mathematical formulas describing nature's behavior left some critics unimpressed. They saw no real value in these bare mathematical formulas because they explain nothing. They simply describe the motion of the free fall in precise mathematical language. Yet, as Kline (1959) notices, such formulas have proved to be the most valuable knowledge human beings have acquired about nature. He goes on to say that the amazing practical as well as theoretical accomplishments of modern science have been achieved mainly through a quantitative, descriptive knowledge.

The other reason for criticism likely comes from the fact that those successive generations of philosophers and historians of science (from Ernst Mach to Karl Popper and Imre Lakatos) had engaged in a massive research program, the main aim of which was to impose "rational" reconstructions on the history of science. Their goal was to account for the development of science in purely logical terms. According to Naughton (1982), they presented the history of science as a uniform progression from "error" to more and more refined approximation to the "truth" as embodied in modern scientific concepts and theories. One of the consequences of this view was its tendency to undervalue the complexity and sophistication of the thinking of the great scientists of the past.

Kuhn took quite a different approach to the history of science on this matter. The ideas of his thinking are expresses in his book *The Structure of Scientific Revolutions* (1962):

Firstly, when studying the work of past scientists, we should assume that their general mode of cognition was similar to ours. They were no less resourceful, intelligent, reasonable, let alone logical than we are. We should approach their writings with the assumption that they are internally consistent and coherent. Their concepts should be taken seriously, and at the face value accorded them by their authors and their contemporaries.

Secondly, we should assume that the conceptual usage of a scientist is that of the culture in which he worked. We should not reinterpret Carnot's concept of "heat", for example, in the light of the modern concept of entropy.

Thirdly, when seeking explanations of why a particular scientist advocated or believed in certain concepts, we should check that any explanation that is offered is consistent with the specific historical context in which he worked. In the case of Carnot, for example, we should relate his concept of heat to the usage conventionally employed in the texts to which we know he had access.

Clearly, through the application of these principles to the history of science, instead of focusing exclusively on the rationality and perception of the isolated individual researcher and his experiments, Kuhn diverted attention to the complex interaction that took place between a research community, with its received culture, and its intellectual environment. The question is, however, do science textbooks reflect Kuhn's suggestion about the complexity and brilliance of the thinking of the great scientists in the past? For example, is the theory of tides described in the textbooks as only Newton's accomplishment of his theory of gravity? Do the students learn from these books that scientists like Bede, Grosseteste, Descartes and Galileo, each in their time, contributed valuable ideas to the now established theory of tides? Some of these scientists' ideas are not expressed in mathematical terms but for their time their ideas were conceptually advanced and they set the foundation for further development of the theory of tides, let along tremendous practical applications of their ideas. For example, Bede's theory (AD 725) was not expressed mathematically but continues to be used by harbour pilots in all ports of the world today. Robert Grosseteste first recognized that the rise and fall of waters or tidal fluxuations must occur on opposite sides of the earth, simultaneously. Descartes gave a very sophisticated rational for all known tidal phenomena, though it was not mathematical. Galileo used tides to prove that the Copernican schema of planetary circles being centred upon the Sun was correct (however, the proof fails). Finally, Newton explained tides in terms of his new concept of gravity (Cartwright, 1999; Dales, 1973). My argument is that we can not undervalue the work of his predecessors (by not mentioning and discussing contributions of earlier scientists) when we present information to our students if we want them to appreciate science and understand processes of science. If our textbooks present science in a linear fashion as a progression from "error" to the "truth", declaring something as right or wrong, then the students will not understand how scientific theories came about. Consequently, it would be naïve to expect desirable student attitudes towards science, like the appreciation of science, and the educational objective to foster scientific literacy will not be achieved.

The examples from history of science show that scientists expressed their ideas in different ways. Following Kuhn's suggestion about making assumptions that their writings are internally consistent and coherent, and should be taken seriously, I would argue, that the fact that some scientists did not express their ideas in mathematical terms, does not signify that their ideas were less valuable than those which could be expressed in symbols as a functional relationship. The conceptual richness of past scientists' ideas not only helped the development and improvement of theories but also resulted in new ways of expressing knowledge (mathematical as well) by their successors later. Very often scientists conceptualize things by communicating their ideas verbally first. For example, Ohm did not state the law relating electric current, voltage and resistance in the form of a mathematical statement $I = V/R$, as it is generally presented in the textbooks. He only gave a verbal statement from his experiment (1825) about the relationship between the resistance and the length of the wire. Another example of verbalization of ideas is Boyle's law. In textbooks Boyle's law is generally presented as a mathematical equation $P_1V_1 = P_2V_2$. Boyle did not formulate his law in mathematical language. He gave a verbal statement saying that the pressure is inversely proportional to volume. As a matter of fact, the mathematical equation mentioned above appeared about 200 years after Boyle's discovery (K. de Berg, 1995). An historian of science de Berg (1990) discussed the historical profile of the pressure-volume law in detail in his work *The Historical Development of the Pressure-Volume Law for Gases* (1990). He took a Kuhnian approach to the history of science looking at interaction between a research community and its intellectual environment. In fact, up to the time of Descartes (1596 – 1650), and some time after, quantitative relationships were written predominantly in words or in word abbreviations. Flegg (1983) notes that while our modern algebraic notation for quantitative relationships came with Descartes in the seventeenth century, it was not immediately popular with his mathematical contemporaries. By the eighteenth century, however, the new notation had become the norm for mathematicians throughout Western Europe. Flegg (1983), in reflecting on the reasons for the late adoption of abstract symbols in mathematics, makes the following comment:

> *Mathematics ideas were explained in words; mathematical arguments were written in words. To adopt abbreviations of words is therefore a natural step; the change to abstract symbolism demands an intellectual leap of extraordinary magnitude.* (p. 224)

This leap of extraordinary magnitude is witnessed in the forms in which the pressure-volume law has been expressed historically. Algebraic forms of the pressure-volume law for gases appeared as early as in the work of Bernoulli (1738), Lord Kelvin (1849), and Waterston (1892) where new mathematical expressions were generated to provide new information.

In order to conceptualize things, sometimes mathematics can help. Newton, for example, noticed that there was a mathematical pattern in the path of the thrown rock on the earth and the path of the moon. It is very difficult to separate mathematical and conceptual aspects of physics. Sometimes the separation is not clear, sometimes it is clear, sometimes the conceptual and mathematical aspects blend, complementing and enriching each other. It seems then that in physics there is no sharp boundary between the two approaches (qualitative and quantitative), nor do we ever use one entirely to the exclusion of the other.

What did Educators Learn about the Relationship between Mathematics and Physics from the History and Philosophy of Science?

What educators learn about the relationship between mathematics and physics from the history and philosophy of science determines how they see the use of mathematics in physics. For example, Norman Campbell (1963) identified three uses for mathematics in physics:

1. The establishment of systems of derived measurement to give definitions. Density as derived from mass and volume via the equation, $d = m/v$, is quoted as an example.
2. Calculations in the form of combining numerical relations to produce new numerical relations. For example, from Galileo experiments, combining the relations $d_1/d_2 = v_1^2/v_2^2$ and $d_1/d_2 = t_1^2/t_2^2$ gives $v_1/v_2 = t_1/t_2$ as a new relation.
3. Formulation of theories. This formulation may be based on analogies with known laws such as development of the kinetic-molecular theory of gases based on the laws of Newton. On the other hand, the formulations could be also based on a mathematician's sense of form and symmetry. This was the case in Maxwell's treatment of differential equations representing electromagnetic properties of matter.

One of the valuable lessons educators learn from the history and philosophy of science is the fact that mathematical formulations used in physics have a rich historical context behind them. Stinner (1995) is convinced that the relationship between mathematics and physics has foundation in the historical context of the development of mathematical formulations. He strongly believes that educators should delay the presentation of the finished product of the mathematical formulations, for instance, of Newton's laws, such as $F = ma$. Before presenting these formulas, Stinner (1995) recommends that the teacher consider the question "What are the diverse connections that led Newton to his second law?" He reminds us that "historically, there were three empirical connections: the motion of the pendulum, the results of collisions between hardwood balls attached to two pendula, and the motion of the conical pendulum" (p. 284). Stinner attributes to mathematics the role of conceptual tool for "these seemingly disparate phenomena" that were "finally united conceptually by essentially one equation" (p.284).

Since the Law of universal gravitation has been chosen as the context for the study of the role of mathematics in physics, it is unavoidable to explore the history of the mathematical formulations leading to Newton's law of gravity. Exploring the history of gravity and learning about the stages of Newton's thinking in the discovery of universal gravitation would help understand the conceptual richness of mathematical formulations involved in understanding of Newton's law of universal gravitation.

According to a physics educator, Tzanakis (1999), we have learned from the history and philosophy of science that mathematics and physics have always been closely interwoven throughout their historical development, in the sense of a "two-ways process":

- *Mathematics methods* ***are used*** *in Physics. That is, Mathematics is not only the "language" of physics (i.e. the tool for expressing, handling and developing logically physical concepts and theories), but also, it often* ***determines,*** *to a large extent, the* ***content*** *and* ***meaning*** *of physical concepts and theories themselves.*
- *Physical concepts, arguments and modes of thinking are used in Mathematics. That is, Physics is not only a* ***domain of application*** *of Mathematics, providing it with problems "ready-to-be-solved" mathematically by already* ***existing*** *mathematical* ***tools****. It also provides ideas, methods and concepts that are crucial for the creation and development of* ***new*** *mathematical concepts, methods, theories, or even whole mathematical domains.* (p. 103)

Tzanakis warns us that "any treatment of the history of mathematics independent of the history of physics is necessarily incomplete". He asserts: "By accepting the importance of the historical dimension in education, the relation between Mathematics and Physics should not be ignored in teaching these disciplines" (p.103).

Practically, Tzanakis (1999) suggests the pedagogical approach that is called "A Historical-Genetic Approach" and can be used in introducing a historical dimension in teaching. He explains that "such an approach emphasizes less the *way of using* theories, methods and concepts, and more the *reasons* for which these theories, methods and concepts *provide answers* to specific problems and questions, without however disregarding the "technical role of mathematical knowledge" (p. 106).

Tzanakis suggests the following general scheme:

1. *The teacher has a* ***basic*** *knowledge of the historical evolution of the subject, so that he/she is able to identify the* ***crucial steps*** *of this historical evolution and appreciate their significance. These steps consist of key ideas, questions and problems, which opened new research perspectives and enhanced the development of the subject.*
2. *(Some of) these crucial steps, are* ***reconstructed****, by explicitly, or implicitly, integrating historical elements, so that these crucial steps become didactically appropriate.*
3. *Many* ***details*** *of these reconstructions are incorporated into exercises, problems, small research projects and more generally,* ***didactical activities*** *that give the opportunity to the learner to acquire technical skills and a better sense of the concepts and methods used. For instance, one may use sequences of historically motivated problems of an increasing level of difficulty, such that each one presupposes (some) of its predecessors. Their form may vary from simple exercises of a more or less "technical" character, to open questions which presumably should be tackled as parts of a particular study project to be performed by groups of students.*

   (p. 107)

Tzanakis (1999) realizes that presentation of the mathematical component in physics using history could be difficult and too advanced for the learners. We have to understand that

> *the historical evolution of a scientific domain…is almost never straightforward and cumulative. On the contrary, it is rather complicated, involving periods of stagnation and confusion, in which prejudices and misconceptions exist and it is greatly influenced by the more general cultural milieu, in which the evolution*

> *takes place. Moreover, the conceptual framework and the mathematical terminology and notation vary from one period to another. Finally, the didactical, social and cultural conditions of the students today are very different from the corresponding conditions in which mathematicians, who created and developed the subject under consideration, were living. Hence, strictly respecting the historical order makes understanding of the subject more difficult.* (p. 107)

To overcome this difficulty, Tzanakis (1999) suggests creating an historically motivated thinking framework for the student, "in which various aspects of the mathematical subject...can be illustrated". He gives the following guidelines for such a framework:

> *In this respect, the* ***crucial*** *steps of the historical evolution of the subject are didactically important because whether or not a step in the historical evolution is crucial, it is judged* ***a posteriori****. In other words, such* ***a step is crucial*** *exactly* ***because*** *it opened new research paths, it clarified the meaning of new knowledge, it suggested the most convenient and clear formulation of this knowledge and in general* ***it enhanced the development of the subject.*** *Therefore, such a step in the historical evolution is* ***in principle*** *didactically relevant.* (p. 107)

Since students learn from textbooks and teachers use textbooks as a primary source for information and instruction, it is clear that Tzanakis' point of view could be extended beyond recommendations for instruction. Textbooks could also be improved in presentation of mathematical component of physics if they used the similar Historic-Genetic Approach suggested by Tzanakis. After all, his suggestions are drawn upon lessons from the history and philosophy of science about the development of the close interrelation between mathematics and physics.

Summary

The close relationship between mathematics and physics was first recorded as far back as the 3rd century B.C.E., the time of Euclid and Archimedes. As the history of science examples show, the relationship between mathematics and physics became especially strong after Galileo established his conception of scientific method wherein mathematics played a crucial role. Later, Newton changed the structure of physics forever. Mathematics became the new language of physics, a form of discourse where physics was transformed from a qualitative to a quantitative subject.

The consequences of the mathematization of physics have a significant impact on science even today. Theoretical physics has gained an elevated status. As for physics education, the mathematization of physics lead to the situation where students need much encouragement to include any words at all when they learn physics. They often prefer to answer questions with pages of mathematics thinking which is easier to understand.

With a new quantitative and incomprehensible nature, physics also became more powerful. Physics was no longer just about observation, description and possible explanations. It could now use mathematics to explain and predict the natural world. The conviction that nature is mathematical and that every natural process is subject to mathematical law began to take hold in the twelfth century when Europeans first obtained this view from the Arabs, who in turn were quoting the Greeks. Later physicists used algebraic expressions to make predictions. In addition, mathematics plays an important role in theory choice due to its predictive power (one of the epistemic values for theory choice, according to the scientific community, is the ability of a theory to give rise to testable predictions). Examples from the history of science show that mathematics was necessary and indispensable in making predictions, often leading to discoveries. Newtonian mechanics and Maxwell's equations for electrodynamics are still valuable in predicting interesting events, even if the observed phenomena were not completely understood.

In the early sixteenth century, Galileo emphasized that the book of nature is written in mathematical language. The number of objects, their size and shape, and their position and state of motion in space could all be quantified. Galileo referred to these quantitative, objective characteristics of things as "primary qualities". "Secondary qualities" such as redness, sweetness, noisiness, and foul odor Galileo claimed depend upon the senses and reside only in consciousness (Galileo, *Discoveries and Opinions*, 1957).

At about the same time, Descartes reinforced Galileo's ideas by connecting the knowledge of nature with the knowledge of mathematics. Descartes believed that physics was the science of moving forms of space, just as geometry was the science of resting forms of space. He, too, insisted that an objective understanding of nature is possible only by way of expressing primary qualities in mathematical language, namely size, shape, and quantity.

Examples from the history of science show that scientists expressed their ideas in different ways. I would argue that the fact that some scientists did not express their ideas in mathematical terms, does not signify that their ideas were less valuable than those which could be expressed in symbols as a functional relationship. The conceptual richness of past scientists' ideas not only helped the development and improvement of theories but also resulted in new ways of expressing knowledge (mathematical as well) by their successors. It is very difficult to separate mathematical and conceptual aspects of physics. Sometimes the separation is not clear, sometimes it is clear, and sometimes the conceptual and mathematical aspects blend, complementing and enriching each other. It can be inferred that in physics there is no sharp boundary between the two approaches (qualitative and quantitative), nor do we ever use one entirely to the exclusion of the other.

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According to a physics educator, Tzanakis (1999), we have learned from the history and philosophy of science that mathematics and physics have always been closely interwoven throughout their historical development, in the sense of a "two-ways process":

- *Mathematics methods **are used** in Physics. That is, Mathematics is not only the "language" of physics (i.e. the tool for expressing, handling and developing logically physical concepts and theories), but also, it often **determines,** to a large extent, the **content** and **meaning** of physical concepts and theories themselves.*
- *Physical concepts, arguments and modes of thinking are used in Mathematics. That is, Physics is not only a **domain of application** of Mathematics, providing it with problems "ready-to-be-solved" mathematically by already **existing** mathematical **tools**. It also provides ideas, methods and concepts that are crucial for the creation and development of **new** mathematical concepts, methods, theories, or even whole mathematical domains.* (p. 103)

Tzanakis warns us that "any treatment of the history of mathematics independent of the history of physics is necessarily incomplete". Practically, Tzanakis (1999) suggests a pedagogical approach that is called "A Historical-Genetic Approach" and can be used in introducing a historical dimension to teaching.

Chapter 5 addresses the question about the role of mathematics in the history of gravity. The history of gravity with a description of the early Greeks' ideas about gravity followed by medieval scientist's positions and then by Copernicus, Galileo and Kepler's ideas about gravity will be highlighted. In addition, the stages of Newton's thinking in his introduction to universal gravitation will be described and analyzed.

Chapter 5: History of Gravity: From Early Greeks to Newton

Overview

Chapter 5 addresses Research Question 3 with its two sub-questions – What role does mathematics play in the history of gravity?

(a) What is the history of gravity?

(b) What are the stages of Newton's thinking when he describes universal gravitation?

The history of gravity part of the chapter begins with a description of the early Greek ideas about gravity and follows with medieval scientists' positions including Buridan's and Oresme's ideas about gravity. Copernicus, Galileo and Kepler's ideas about gravity conclude the section addressing the history of gravity in sub-question (a).

The next part of this chapter reflects a historical inquiry into which other scientists had an influence on Newton's development of his theory of universal gravitation. A brief description of the history of the development of calculus follows to show its significance in the development of Newton's law of gravity. A thorough analysis of the stages of Newton's development of his law of gravity (sub-question (b)) is conducted. A primary historical source, Newton's *Principia* and a secondary source, Cohen's accounts, are used to describe and analyze the steps of Newton's reasoning in his discovery. The significance of Newton's theory of gravitation is explained next. The chapter concludes with a discussion of the importance of geometry in the presentation of gravity. Examples from science education literature are provided.

Early Greek Ideas about Gravity

The earliest ideas about gravity were probably based on every day experience of observing objects falling if they were not supported, feeling the difficulty of climbing a hill compared to walking on a horizontal surface, and the return of an object after being thrown up. Some objects felt heavy, others were light. Ancient Greeks (Plato, Aristarchos, and, in particular, Aristotle) were the first who began unifying these observations into one idea, the idea of gravity.

Aristotle (384-322 B.C.E.) and other early philosophers thought that all matter was made of four elements: earth, water, air, and fire. They considered earth and water the heaviest. Therefore, an element earth, as the heaviest, was placed in the center of the universe, and all objects would fall towards the earth in straight lines with the exception of planets and stars which were presumed moving in circles around the earth. Air and fire, on the other hand, were considered the lightest, and moved by nature away from the center. The cause of unnatural motion of a heavy object in a direction other than down, according to Aristotle's ideas, was an external force. Alternatively, as the historian of science Wesley Stevens (1998) further describes Aristotle's view, if a light object did not move upward, it was being kept out of its natural place. Aristotle also believed that the earth had a spherical shape because of gravity. As Wesley Stevens (1998) explains, Aristotle reasoned that

> *If every heavy thing moved down, each would strike and move other things that also moved downwards. If they were free to move naturally, all would continue to move naturally, all would continue toward the same center until there was no more space for motion, and each would provide resistance to the motion of*

> *others. This would result in a huge bulk that looked like a ball, because any other shape would allow space for further motion of things to get closer to the center. Thus, the earth is a sphere because of gravity.* (p. 398)

Stevens (1998) goes on to say that this notion of gravity was quite satisfactory for many purposes until the fourteenth century when a new question of gravity arose in a different context.

Medieval Scientists' Ideas about Gravity

The medieval scientists Jean Buridan (1295-1358 C.E.) and later Nicole Oresme (1325-1382 C.E.) were considering *gravitas* from a new point of view. They proposed that there were the element *terra* (earth) and the element *aqua* (water) since the then known continents Asia, Africa, and Europe lay in one-quarter of the orb surrounded by water. Oresme and Buridan could speak of the element *terra* as forming a globe with its own center, and of the element *aqua* as forming the globe with its own center. These centers were not always in the same place (Pedersen, 1993; Stevens, 1998). A scholar like Buridan and Oresme could be wondering, as Stevens (1998) explains, could it be that the element *terra* and the element *aqua* each form a separate natural sphere because of gravity? Going further, Buridan and several scholars after him proposed to describe two spheres of earth and water interacting. As Stevens (1998) clarifies,

> *They argued that the center of the water must move when there is a great flood or a storm at sea changing the shape of the gathered waters; and the center of the element earth must move when large bodies of land shift, as a result of an avalanche or earthquake or volcanic eruption.* (p. 399)

The interesting question one might ask then is about the location of the center of the Kosmos toward which each heavy object moves because of gravity.

As Stevens (1998) notes, the ancient concept of gravity was "severely tested against other cosmological ideas during the fourteenth to sixteenth centuries." Copernicus later rejected the ideas of Oresme and Buridan about terra and aqua spheres, as Stevens explains. Stevens speculates that " it was probably the tension of ideas about spheres of *terra* and *aqua* that Nicolaus Copernicus (1473-1543) had in mind when he said in the preface to *De revolutionibus* (1543) that his new astronomy would once more harmonize the elements" (p. 399).

Copernicus' Ideas about Gravity

In his famous book *De Revolutionibus Orbium Caelestium* (1543), Copernicus claimed that he was only making a mathematical assumption about a heliocentric system, arguing that all celestial spheres revolve about the sun with the sun being a central location in the universe. According to Clagett (1959), it is likely that Oresme influenced Copernicus' idea of heliocentrism. Oresme opposed the theory of a stationary earth 200 years before Copernicus did. He suggested that if gravity, the tendency of earth to move toward the center, were regarded as the attraction of earth to earth, then the earth could revolve around the sun and things would still fall in a straight line. As Clagett (1959) notes, Oresme also reformulated the definitions of "up" and "down." He stated that heavy things go down, toward the center of the earth. However, according to Oresme's reformulations, "up" and "down" no longer refer only to the center and circumference of our world. He argued that all things do not have to orient themselves to our world's center. There could exist a plurality of centers. This assumption denies the idea that all things in the same universe must have a relation to one another. According to this view, two worlds sufficiently removed need not have a relation to one another, but only relations between their own respective parts.

The ancient idea about gravity dealt with heavy things moved by nature. Stevens (1998) gives an example from the history of science of another aspect of gravity when heavy things moved in some cases by attraction rather than by nature or by force, for example, attraction of metals to each other. The work of William Gilbert (1540-1603) on magnetism, he reminds, stimulated many people to experiment with heavy objects. Stevens (1998) concludes that the significance of Gilbert's work was that

> *It also encouraged Johannes Kepler (1571-1630) to speak about the attraction of the Sun toward the planet Mars, as greater when near and less when far away (Astronomia nova, 1609; Harmonices mundi, 1619). This made no sense to Galileo (1564-1642), who had no taste for spiritual forces, and in fact Kepler's "third law" could only describe those motions in geometry but not account for such an attractive force. But it was the formula that Isaak Newton (1642-1727) used to justify his new theory of gravity, though not to prove it.* (p. 399)

Galileo's Views on Gravity

Another significant figure in the development of ideas about gravitation was Galileo (1564-1643). Galileo's work represents a new approach to the study of nature, namely the study of nature by experimentation. This approach was encouraged during the time of the Renaissance. Instead of trying to answer the question "why do objects fall", Galileo explored "how do the objects fall?" According to Hall (1967), Galileo in his work

Discourses disposed of the causes of the acceleration of freely falling bodies as fantasies unworthy of examination:

"At present it is the purpose of our Author merely to investigate and to demonstrate some of the properties of accelerated motion (whatever the cause of this acceleration may be" (as cited in Hall, p. 176). Hall (1967) clarifies, though, that "this does not mean that in replacing the question *why* by the question *how* Galileo has excluded the study of phenomena in terms of cause and effect" (p.176). He goes on to say that Galileo's attitude on this matter was that "the explanation of phenomena at one level is the description of phenomena at a more fundamental level" (p. 177). Hall explains:

> *For Galileo there was no anomaly in recognizing that certain constituents of the physical world had to be accepted as axiomatic; descriptive analysis can only advance gradually from the coarse to the refined, from the lower to the upper levels, each with its appropriate generalizations.* (p. 177)

Hall points out that there could be "constructs which are not reducible to the ultimate physical realities, as was the case for instance with Newtonian mechanics where the law of gravitation had to be taken as descriptively correct, though gravity was not explicable in terms of matter and motion" (p. 177).

Galileo, exploring only the problem of how objects fall, studied the rates at which objects fall. Carrying out experiments, he showed that a body which is not acted upon by any force will continue in constant motion (Contrary to Aristotle's idea of *impetus*). Galileo was especially interested in investigating speed as a function of time. He came up with the following conclusions that are reflected in his well known work *Two New Sciences*:

1. Objects of different weight fall at the same speed
2. As objects fall, their speed increases with time
3. The distance an object falls is proportional to the square of the elapsed time.

The last conclusion was arrived at by the mathematical consequence of Conclusion 2. Galileo checked his conclusions through his experiments with motion of bronze balls along an inclined plane.

Galileo showed not only experimentally that Aristotle's theory of falling objects was incorrect. He also performed a clever thought experiment. Galileo described his thought experiment in *Dialogue Concerning the Two Chief Systems of the World* (1632). The argument goes as follows. Suppose we have two rocks, the first being lighter than the second. If we release the rocks from a height above the surface of the earth, the second rock, being heavier than the first one, falls faster. If they are joined together, argues Galileo, then the combined object should fall at a speed somewhere between that of the light rock and that of the heavy one since the light rock by falling more slowly will retard the speed of the heavier. But if we think of the two rocks tied together as a single object, then it falls more rapidly than the heavy rock. Aristotle poses the question: How do the rocks know if they are one object or two? Clearly, Galileo's stunning thought experiment shows that Aristotle's theory is inconsistent.

Galileo is famous for other innovative ways of thinking. For example, when many people watch a pendulum swinging, they see the bob of the pendulum going back and forth. Galileo, however, saw that the bob went up and down. He perceived the motion of the bob as composed of vertical and horizontal components with gravity being the force in the vertical direction. Galileo established that, although the time taken for the bob to rise and fall depended on the length of the pendulum, it did not depend on the weight of the bob (Stinner and Metz, 2002). Galileo showed again that if we ignored air resistance, the time taken for an object to fall from a given height would not depend on its weight. Described earlier in Chapter 4 Galileo's experiments with rolling balls down inclined planes also allowed him to examine the downward components of the speed of fall.

It is known from the history of the development of ideas about gravity that Galileo was the first to derive the parabolic path of a projectile (Herivel, 1965). His derivation was based on the combination of two independent motions, the one uniform and horizontal, the other vertical and uniformly accelerated by gravity.
Summarizing, Galileo's contribution to the understanding of gravity is that he changed the approach to studying gravity by emphasizing the "how" of gravity. First, he conducted careful experiments to understand the "how" of gravity. Secondly, he gave not only a qualitative account of motion under gravity but also a mathematical quantitative description. Galileo's influence on Newton can be found clearly in the *Principia* (1687), at the beginning of the *Scholium* to the laws of motion. There Newton states:

> *...Galileo discovered that the descent of bodies varied as the square of the time (in duplicata ratione temporis) and that the motion of projectiles was in the curve of a parabola; experience agreeing with both, unless so far as these motions are a little retarded by the resistance of the air.* (p. 21)

Herivel (1965) expresses his opinion about Galileo's influence on Newton this way:

> *Certainly the whole cast of Newton's thought, his humility before Nature, his drive towards exact quantitative results, his delight in experiment, was altogether Galilean, and if he recognized any master in science apart from Archimedes it*

could only have been Galileo. (p. 41)

Kepler's Ideas about Gravity

As Thomas Kuhn (1957) demonstrated, the Copernican Revolution, as we know it, was hardly to be found in Copernicus' own writings, and what has to come to be called the Copernican revolution was, in fact mainly the work of Kepler and Galileo. Newton owed much for preparing the foundation for the development of the idea about universal gravitation to many scientists.

An astronomer Johannes Kepler (1571-1630) who discovered the laws of planetary motion at the beginning of the seventeenth century was a significant figure in the development of ideas about gravitation. Kepler was the first man who realized that planets should move in ellipses and not circles. Kepler's three laws of planetary motion postulate:

- the planets travel in elliptical orbits, one focus of each ellipse being occupied by the sun
- the radius vector connecting sun and planet sweeps over equal areas in equal times
- the squares of the periods of revolution of any two planets are in the same ratio as the cubes of their mean distances from the sun

Kepler's laws were the first accurate mathematical treatment of the universe. They were a turning point in the history of thought. According to an historian of science, Rupert Hall (1967), "it was Kepler who, in the *Cosmographic Mystery,* followed the example of Tycho Brahe in denouncing the traditional belief in material spheres which had been left unchallenged by Copernicus" (p. 127). A journalist, Arthur Koestler (1967), who recognized the contribution of Kepler's laws to our understanding of planetary motion, said so well:

> *They were the first "laws of nature" in the modern sense: precise, verifiable statements, expressed in mathematical terms, about universal relations governing particular phenomena. They put an end to the Aristotelian dogma of uniform motion in perfect circles, which had bedeviled cosmology for two millennia, and substituted for the Ptolemaic universe - a fictitious clockwork of wheels turning on wheels....* (p. 329)

Hall (1967) points out that Kepler's ideas on gravity, on action of forces at distance, certainly "are important factors in the prehistory of the theory of universal gravitation". Though Kepler limited his use of the concept of attraction to heavy bodies cognate with the earth, as Hall points out, "he has stated, for the first time, that the attraction is mutual" (p. 261). He goes on to say that by expressing the motion due to gravitational attraction as $d_1/d_2 = m_2/m_1$, where m_1/m_2 is the ratio of the masses of the two bodies, and d_1/d_2 are the respective distances of these bodies from the earth, Kepler began associating the theory of attraction with a definite dynamical form:

> *Kepler then went on to demonstrate, from the ebbing and flowing of the tides, that this attractive force in the moon does actually extend to the earth, pulling the waters of the seas towards itself; much more likely was it that the far greater attractive force of the earth would reach to the moon, and greater beyond it, so that no kind of earthy matter could escape from it.* (p. 261)

Hall (1967) concludes: clearly, "the genesis of the theory of *universal* gravitation is found in Kepler. He points out that "Newton's hasty calculations of 1666, his later theory of moon, and his theory of tides, are all embryonically sketched in the *Astronomia Nova*" (p.262). Hall, however, specifies what Kepler did not accomplish in his theory of gravitational interactions. He goes on to say:

> *But the attraction was still specific, applicable only to heavy, earthy matter; Kepler himself did not go so far as to suppose that the sun and planets were also mutually attracting masses, or that the dynamic balance he indicated as retaining the earth and moon in their orbits with respect to each other also preserved the stability of the planetary orbits with respect to the sun. He failed, as Copernicus, Gilbert and Galileo failed, to see the full power of gravitational attraction as a cosmological concept.* (p. 262)

There is no doubt that Newton used Kepler's ideas about planetary motion in the development of the theory of universal gravitation. An explicit evidence to that is Propositions I-XI of Book I of *Principia* which deal with all three of Kepler's laws of planetary motion - the law of areas (Propositions I-III), the harmonic law (Corollary 6 to Proposition IV), and the law of elliptical orbits (Proposition XI). As Hall (1967) reminds us, "Kepler's idea that the satellite revolving round a central body is maintained in its path by two forces, one of which is an attraction towards the central body, although applied only to the earth-moon system, holds the key to all that followed and to the *Principia* itself" (p. 262).

Although Galileo and Kepler set the stage for later developments in gravitational theory, their contributions were not thought to be connected until Newton. Kepler's contribution concerned the orbits of the planets round the Sun while Galileo's input concerned motion and the acceleration of falling bodies. It was not suspected until Newton that there was a connection between falling objects and planetary motion.

Newton's Background Knowledge

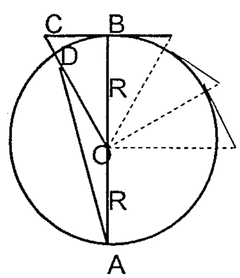
Besides Kepler and Galileo there were other scientists who had influence on Newton's development of his theory of universal gravitation. One of them was Christian Huygens (1629-1695), who, according to Brougham and Routh (1972), fourteen years before the *Principia* was published, first showed how to calculate centrifugal, or center-fleeing forces. Huygens and Descartes had analyzed curved motion in terms of such a centrifugal force. According to Stinner and Metz (2002), Huygens was the first to write the mathematical statement for "centrifugal" acceleration as $a = v^2/r$.

According to Cohen (1981), Descartes had investigated the movement of a ball on the inner surface of a hollow cylinder and the movement of water in a bucket swung in a circle. Since the ball and the water seemed to flee the center of the system, Descartes attributed their motion to the influence of a centrifugal force. We now know that the centrifugal force is just an illusion that comes about when a moving object is viewed from a rotating frame of reference. Giovanni Alphonso Borelli (1608 – 1679), several years earlier than Huygens, had a slightly different idea. As Brougham and Routh (1972) note:

> *Borelli, in treating of the motion of Jupiter's satellites, considers the planets as having a tendency to resign from the sun and the satellites from the planets, but as being "drawn towards and held by those central bodies, and so compelled to follow them in continued revolutions."* (pp. 8-9)

Newton initially used the concept of centrifugal force in the analysis of the circular motion caused by a central force. According to Stinner (1994), "he managed to derive the formula for 'centrifugal' force in a more economical and elegant way" (p. 81). Newton used the results of Galileo's kinematics of free fall and applied them to the dynamics of a revolving object. Stinner (1994) illustrates how it could be done:

Figure 5-1. Derivation of Centripetal Acceleration



Δ ABC $\propto$ Δ DCB for small θ

DB = vt

Let CD ~1/2 $a_c t^2$

Then CD/DB = DB/2R

Substitution gives $a_c = v^2/R$

A similar approach for deriving the formula $a_c = v^2 / R$ is used in many high school physics textbooks today.

Later, Newton realized that the older and misleading notion of a centrifugal force should be replaced by the concept of a centripetal or center-seeking force, which is equal in magnitude to the centrifugal force but has opposite direction. Newton rejected the concept of a centrifugal force because the analysis of the influence of that force seemed to be independent of the properties of the central body. On the other hand, the concept of a centripetal force entirely involves a central body, towards which the revolving object is attracted. Indeed, the fundamental idea in any theory of gravitation is the interaction of the central, attracting body with the revolving, attracted object.

When Newton still spoke about motion in terms of centrifugal force, another scientist, Robert Hooke (1629-1695), joined the discussion about planetary motion. Historians of science consider this period (starting from 1679, when Hooke suggested a philosophical correspondence on scientific topic of mutual interest) as a period of the Newton-Hooke controversy. According to Cohen (1981), Hooke wrote a letter to Newton inviting him to comment on any of Hooke's hypotheses or opinions, particularly on the notion of "compounding the celestiall motions of the planetts [out] of a direct motion by the tangent & an attractive motion towards the central body" (p.167). Cohen goes on to say that this particular sentence was apparently Newton's introduction to the idea of decomposing curved motion into an inertial component and a centripetal one in the *Principia.*

In his letter, Hooke further suggested that the attraction of the sun draws away the planets from moving in straight lines, and proposed that the force of attraction varies inversely with the distance as the square of the separation. It is interesting to learn that Hooke read to the Royal Society a paper explaining the curvilinear motion of the planets by attraction as early as 1666. According to Brougham and Routh (1972), there were other scientists who suspected the inverse square relationship between the force of attraction and the distance of the separation. Brougham and Routh assert:

> *Halley, as well as others, had even hit upon the inverse duplicate ratio, by supposing that the influence from the sun was diffused in a circle, or rather a sphere, and that therefore the areas proportioned to that influence were as the squares of the radii, and that consequently the intensities, being inversely as those areas, were inversely as the squares of the radii or distances.* (p. 9)

It is generally accepted by historians of science that Hooke believed that Newton owed him for the insight about the relationship between the centripetal force and the distance of separation. Newton did not think so because Hooke could not make fruitful conclusions from his idea. It is generally thought that at that point Hooke was stuck. As Cohen (1981) asserts:

> *He could not see the dynamical consequences of his own deep insight and therefore could not make the leap from intuitive hunch and guesswork to exact science. He could go no further because he lacked both the mathematical genius of Newton and an appreciation of Kepler's law of areas, which figured prominently in Newton's subsequent approach to celestial dynamics.* (pp. 167-169)

Clearly, in developing the idea about universal gravitation Newton owes to the insights of many scientists. What differentiated Newton from other scientists who had caught many glimpses of the theory of gravity was, according to Cohen (1981), "Newton's fecund way of thinking about physics, in which mathematics is applied to the external world as it is revealed by experiment and critical observation" (p. 177). This way of thinking Cohen (1981) called the "Newtonian style". As Cohen describes it, the "Newtonian style consists in a repeated give-and-take between a mathematical construct and physical reality" (p. 177). We admire Newton's ability to compare the real world progressively with a simplified mathematical representation of it. One of the tools of Newton's mathematical apparatus was his calculus invented to develop the law of universal gravitation as it applies to the motion of planets.

Newton's Calculus

Newton developed differential calculus, or as he called it, "Theory of Fluxions" in 1666, but the work was not published until 1704 as an appendix to Newton's book *Optics*. It is well known that a German mathematician and philosopher Leibniz discovered the system of differential and integral calculus independently from Newton. It is the signs and symbols of Leibniz which continue to be used today. However, there was a celebrated dispute between Newton and Leibniz over the invention of the calculus, which is known as the Newton-Leibniz controversy.

The significance of the calculus can not be overestimated. Bernal (1954) gave the following evaluation of the importance of the calculus:

> *The calculus, as developed by Newton, could be used and was used by him for the solving of a great variety of mechanical and hydrodynamic problems. It immediately became the mathematical instrument for all understanding of variables and motion, and hence of all mechanical engineering, and remained almost the exclusive one until well into the present century. In a very real sense it was as much an instrument of the new science as the telescope.* (p. 484)

It is interesting to learn from historians of science (Cohen, Hall, Herivel) that, contrary to the common but erroneous supposition that Newton used algebraic analysis and his fluxional calculus to reach his conclusions as they relate to his law of gravity, he employed calculus but transformed it to geometrical arguments. This common supposition is likely based, on the fable Newton himself encouraged towards the end of his life. He claimed: "By the inverse Method of fluxions I found in the year 1677[=1679/80] the demonstration of Kepler's Astronomical Proposition viz. that the

Planets move in Ellipses..."(As cited in Herivel, 1965, p. 34). According to Herivel (1965), Newton's statement, however, runs counter to all documentary evidence, "for with only one proposition in the *Principia* (Book II, Proposition 35, determining the optimum shape for bodies moving through a resisting medium) can any fluxional analysis be connected. In this case he stated his conclusions in the book without proof" (p. 213).

In his *Principia* Newton extensively employed a method of reasoning the essence of which is the elementary conversion of geometrical elements, such as lines and areas, into infinitesimal quantities converging towards finite ratios as they vanish in the limit. Hall (1996) characterizes the calculus Newton used as "an idiosyncratic geometry in which infinitesimal increments of lines and areas perform the functions of first and second order differentials, a geometry intimately integrated with his dynamical principles" (p. 213). As contemporary Italian historian Guicciardini (1999) argues, the reason for Newton giving preference to geometry over the formal methods generated (even by his own calculus) was his philosophical belief that geometry referred to objects, and algebra was merely a heuristic device. Therefore, geometry, but not algebra, was able to provide proofs. Moreover, Newton deeply believed that geometry fitted the context of his *Principia.* By that, he probably meant that there was an intimate intellectual connection between physics and geometry.

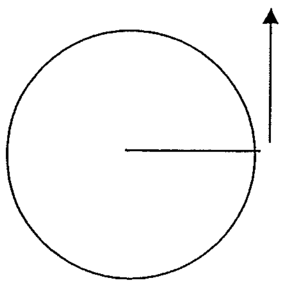
Stages of Newton's Thinking in the Development of his Law of Gravity

Newton's discovery of universal gravitation, according to Cohen (1981), was not "the isolated flush of genius," but the "culmination of a series of exercises in problem solving, ...a product not of induction but of logical deductions and transformations of existing ideas." Cohen is one of the historians of science who extensively analyzed documents related to Newton's work and thinking. These documents enabled Cohen to reconstruct the process that led to Newton's discovery of universal gravitation. Cohen's analysis shows "Newton's fecund way of thinking about physics, in which mathematics is applied to the external world as it is revealed by experiment and critical observation." Cohen, as mentioned earlier, called this kind of thinking the Newtonian style which "consists in a repeated give-and-take between a mathematical construct and physical reality."

According to Cohen, the decisive step on the path of Newton's discovery of universal gravity came in late 1679 and early 1680, and not as Newton claimed, in 1666. In fact, it was initiated by Robert Hooke when he introduced Newton to his new way of analyzing motion along a curve, a motion having two components - an inertial component and a centripetal one. Hooke also suggested that the centripetal attraction is inversely proportional to the square of the distance.

The inverse-square nature of gravitational force for circular orbits was hinted at before by Huygens. In 1673 he published a supplement to a book on the pendulum clock in which he states that for circular motion the centrifugal force is proportional to v^2/R, where v is the velocity of the orbiting body and R is the radius of rotation. According to Cohen, Newton had independently discovered the same relation in the 1666. Since a centrifugal force and a centripetal force differ mathematically only in the direction, the same relation v^2/R holds for a centripetal force. By means of simple algebra, using Kepler's third law of planetary motion and the relation for centripetal force, one can readily show (Halley was the first who did it) that the centripetal force varies inversely with the square of the distance. For circular orbits,

Figure 5-2. Derivation of Kepler's Third Law for Circular Orbits



$$\frac{v^2}{R} = \left(\frac{2\pi R}{T}\right)^2 / R = \left(\frac{4\pi^2 R^2}{T^2}\right)\frac{1}{R} =$$

$$4\pi^2\left(\frac{R^2}{T^2}\right)\frac{1}{R} = 4\pi^2\left(\frac{R^2 R}{T^2 R}\right)\frac{1}{R} =$$

$$4\pi^2\left(\frac{R^3}{T^2}\right)\frac{1}{R^2} = 4\pi^2 K\left(\frac{1}{R^2}\right)$$

In the presented derivation, according to Kepler's third law, R^3/T^2 is a constant K, where R is the radius of the planet's circular orbit and T is the period of the orbit. For a circular orbit, the centripetal force is proportional to v^2/R, where v is the planet's velocity. In time T, the planet makes a complete revolution, covering the circumference of a circle measuring $2\pi R$, the velocity is then $2\pi R/T$.

The question Hooke posed before Newton, according to Cohen, was a reverse one: "If a central attractive force causes an object to fall away from its inertial path and move in a curve, what kind of curve results if the attractive force varies inversely as the square of the distance?"(p.169) As a result of correspondence with Hooke, Newton came up with his work *De Motu (Concerning Motion),* likely finished in 1684, were he showed that an object that has inertial motion and is subject to an inverse-square centripetal force moves in elliptical orbit. According to Cohen, the exact progression of Newton's ideas in the time between his correspondence with Hooke and his completion of *De Motu* is not documented. Nevertheless, Cohen is certain that "it was Hooke's method of analyzing curved motion that set Newton on the right track." Cohen believes that

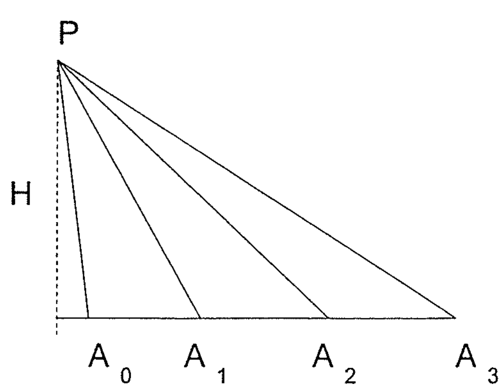
> *the approach Newton takes to terrestrial and celestial dynamics in De Motu, which he further developed...in the first book of the Philosophiae Naturalis Principia Mathematica, represents his thinking on planetary dynamics inspired by his correspondence with Hooke.* (p. 169)

Cohen continues to describe the progression of Newton's thinking in the development of his law on universal gravitation. A series of stages can be identified in this description as follows:

In the *Principia* (and also in the discussion at the beginning of *De Motu*), in the very first proposition, Newton develops the dynamical significance of Kepler's law of areas by proving that the curved motion described by the law is a consequence of centripetal force. This proof has three parts.

As Cohen comments, in the first part of the proof, Newton considers a body moving along a straight line with a constant velocity.

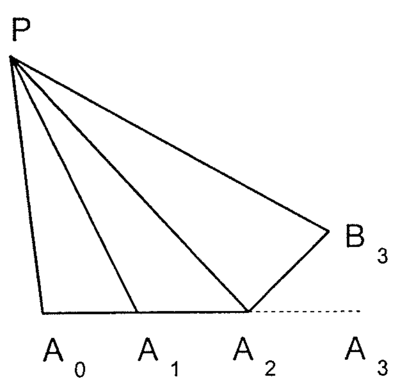
Figure 5-3. The First Step in Newton's Proof of Relation between Inertial Motion and Kepler's Law of Areas



The body starts at A_0 and after successive equal intervals reaches first A_1, then A_2 and so on. A point P is chosen above the line of motion at a distance H. The triangles A_0PA_1, A_1PA_2, A_2PA_3 and so forth all have the same area because they have equal bases and the same altitude. By this analysis, Newton showed the relation between inertial motion and Kepler's law of areas.

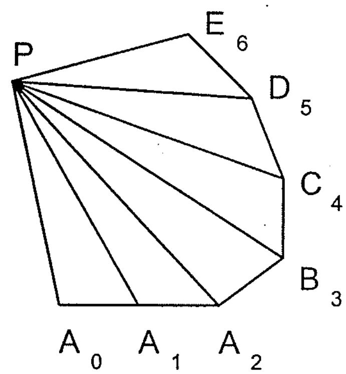
In the second part of the proof, the body begins as before but at A_2 receives an impulsive blow toward P. Now the body moves along a straight line not to A_3 but to B_3. Newton again showed by geometry that the triangles A_1PA_2 and A_2PB_3 have the same area.

Figure 5-4. The Second Step in Newton's Proof of Relation between Inertial Motion and Kepler's Law of Areas



In the third part, the body is given a blow toward P at the end of each interval. Therefore, the body moves in a polygonal path around P. Again, triangles can be formed that have the same area.

Figure 5-5. The Third Step in Newton's Proof of Relation between Inertial Motion and Kepler's Law of Areas



In the limiting case where the time between blows approaches zero the body is subject to a continuous force directed toward P and the polygonal path becomes a smooth curve. Area is still conserved. In this way, Newton proved that a centripetal force generates a curve according to Kepler's law of areas.

In the second proposition of the *Principia* Newton proved the converse: Motion in a curve described by the law of areas implies a centripetal force. As Cohen notes, with these two propositions Newton demonstrated that the law of areas is a necessary and sufficient condition for inertial motion in a central-force field.

The next step for Newton in the development of the concept of universal gravitation was his realization that the planets do not move according to the law of areas in simple Keplerian elliptical orbits with the sun at a focus. Instead, the focus lies at the common center of mass, because as a consequence of Newton's third law of motion (the law of action and reaction), not only does the sun attract each planet but also each planet attracts the sun, and the planets attract one another. Newton realized that the planets move precisely in elliptical orbits and according to Kepler's laws of areas not in the real world but in an artificial situation, a one-body system - a single point mass moving with an initial component of inertial motion in a central-force field. The one-body system reduces the earth to a point mass and the sun to an immobile center of force. In the introduction to the 11th section of the *Principia* (p. 164), Newton explains that he has considered so far a situation where "there is no such thing existent in nature." The situation is artificial

> *For attractions are made towards bodies, and the actions of the bodies attracted and attracting are always reciprocal and equal, by Law III; so that if there are two bodies, neither the attracted nor the attracting body is truly at rest, but both..., being as it were mutually attracted, revolve about a common centre of gravity.* (p. 164)

As Newton continues the discussion in the 11th section of the *Principia*, he takes the next step, from an interactive two-body system to an interactive many-body system. This step is a further consequence of the third law of motion. Since each planet is a center of an attractive force as well as an attracted body, a planet not only attracts and is attracted by the sun but also attracts and is attracted by each of the other planets. Newton asserts:

> *And if there be more bodies, which either are attracted by one body, which either are attracted by one body, which is attracted by them again, or which all attract each other mutually, these bodies will be so moved among themselves, that their common centre of gravity will either be at rest, or move uniformly forwards in a right line.* (p. 164)

Once Newton concluded that all bodies must attract each other, he presented the conclusion and explained why the magnitude of the attractive force is so small in many situations that it is unobservable. He explained that it was possible to observe these forces only in the huge bodies of planets, meaning that gravitational force would depend on the masses of the interacting objects. From his pendulum experiments performed earlier, Newton inferred that, since different masses have identical constant accelerations towards the earth's center, a constant force is acting whose magnitude is proportional to the masses of the bodies concerned (*Principia,* Section VI, pp. 303-326).

This kind of analysis led Newton to conclude that the planets neither move exactly in ellipses nor revolve twice in the same orbit, therefore "there are as many orbits to a planet as it has revolutions." As cited in Cohen (1981), Newton wrote: "To consider simultaneously all these causes of motion and to define these motions by exact laws allowing of convenient calculation exceeds, unless I am mistaken, the force of the entire human intellect" (p. 172).

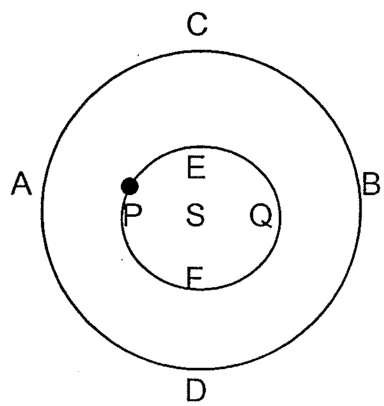
As it turned out, the case of many-body systems still occupies the minds of mathematical physicists trying to find solutions for these kinds of problems up to the present day. One of the revolutionary features of Newtonian celestial dynamics, according to Cohen, is the distinction that Newton draws between the realm of mathematics, in which Kepler's laws are truly laws, and the realm of physics, in which they are only approximations.

Newton realized that a set of point masses circling the central point mass attract one another and perturb one another's orbit when he compared the construct with physical reality. Newton knew that Jupiter and Saturn were the most massive planets, and with the help of John Flamsteed, he found that the orbital motion of Saturn is perturbed when the two planets were closest to each other. Cohen notes, "the process of repeatedly comparing the mathematical construct with reality and then suitably modifying it led eventually to the treatment of the planets as physical bodies with definite shapes and sizes" (p. 178).

The important step to such treatment of the planets was Newton's realization that the earth attracts an object with the same force as if all of the earth's matter were concentrated in one point, at the center of the planet. With the powerful tool of his integral calculus, Newton proved the following theorem: a spherical shell with homogeneously distributed mass attracts a body in the same way as if the entire mass of the shell were concentrated at its center. To prove this theorem Newton goes back to mathematical constructs assuming that the earth can be considered to be a set of concentric spherical shells. He formulates Proposition LXXIII in the *Principia*:

> *If to the several points of a given sphere there tend equal centripetal forces decreasing as the square of the distances from the points, I say, that a corpuscle placed within the sphere is attracted by a force proportional to its distance from the centre.* (p. 196)

Figure 5-6. Proposition LXXIII in the *Principia*



In his proof Newton suggests placing a corpuscle in the sphere ACBD, described about the center S. Then he inscribes an interior sphere PEQF having radius SP. He points out that

> *It is plain...that the concentric spherical surfaces of which the difference AEBF of the spheres is composed, have no effect at all upon the body P, their attractions being destroyed by contrary attractions. There remains, therefore, only the attraction of the interior sphere PEQF.* (p. 196)

After this proposition in his Scholium, Newton specifies his assumptions about orbs being extremely thin, "that is, the evanecent orbs of which the sphere will at last consist, when the number of the orbs is increased, and their thickness diminished without end." The similar assumptions are made about the particles composing lines, surfaces and solids - they "are to be understood equal particles, whose magnitude is perfectly inconsiderable" (p. 196).

Then, in the following Proposition LXXIV, Newton generalizes that "a corpuscle without the sphere is attracted with the force inversely proportional to the square of its distance from the centre (p. 197). Finally, based on this proposition, he concludes the proof of the theorem:

> *For the attraction of every particle is inversely as the square of its distance from the centre of the attracting sphere (by Prop. LXXIV), and is therefore the same as if that whole attracting force issued from one single corpuscle placed in the centre of this sphere.* (p. 197)

Newton presents this conclusion in Proposition LXXV as part of his proof of the theorem concerning the case of two interacting spheres. This theorem says:

> *If to the several points of a given sphere there tend equal centripetal forces decreasing as the square of the distances from the point, I say, that another similar sphere will be attracted by it with a force inversely proportional to the square of the distance of the centres.* (p. 197)

Only after this series of proofs, Newton extends his generalizations for spheres that are dissimilar in terms of density and attractive force in Proposition LXXVI:

> *If spheres be however dissimilar (as to density of matter and attractive force) in the same ratio onwards from the centre to the circumference; but everywhere similar, at every given distance from the centre, on all sides round about; and the attractive force of every point decreases as the square of the distance of the body attracted: I say, that the whole force with which one of these spheres attracts the other will be inversely proportional to the square of the distance of the centres.* (p. 198)

The proof of this theorem comes as a consequence of the previous Proposition LXXV. Newton again employs his method of integral calculus when he represents two interacting spheres as spheres composed of numerous concentric shells having different densities. Then he lets "the number of the concentric spheres be increased *in infinitum*, so that the density of the matter together with the attractive force, in the progress from the circumference to the centre, increase or decrease according to any given law" (p. 199). By this thinking process Newton leads to the idea of interacting objects having definite size and shape: "...so the spheres may acquire any form desired; and the force with which one of these attracts the other will be still, by the former reasoning, in the same inverse ratio of the square of the distance" (p. 199).

In book three of the *Principia,* Newton treats the topic of gravitation essentially the same way, but the treatment is more mathematical. First, Newton conducts what is called the moon test (supposedly done in 1666) where he extends the terrestrial gravity to the moon and demonstrates that the weight force varies inversely with the square of the distance. Newton found that the moon moves as if it were attracted to the earth with the force $1/3600^{th}$ of that with which the earth pulls on objects at its surface. He also knew that the moon is situated at a distance from the earth that is 60 times earth's radius. Therefore, the conclusion that the force of attraction at the distance of the moon's orbit is $1/3600^{th}$ of that on the surface of the earth is consistent with the deduction that the terrestrial gravity extends to the moon and decreases with the square of the distance.

Newton then identifies the gravitational force existing on earth with the force of the sun on the other planets, as well as the force of a planet on its satellites. He now calls all these forces gravity. Referring to the third law of motion, he transforms the concept of a solar force on the planets into the concept of a mutual force between the sun and the planets. In a similar way he transforms the concept of a planetary force on the satellites into the concept of a mutual force between a planet and its satellites. Finally, the resulting transformation is the notion that all bodies interact gravitationally.

The presented stages of Newton's thinking in the development of his law of universal gravitation reflect a new type of thinking. His style of thinking laid the foundation for a new paradigm in exploring science the essence of which is the process of repeatedly comparing the mathematical construct with reality, and if the mathematical construct falls short, modifying it. This way of thinking had an outstanding significance not only for Newton's contemporaries but it is still proves to be indispensable, plausible and fruitful.

Significance of Newton's Theory of Gravitation

The examination of Newton's theory of universal gravitation reveals the high point of the Scientific Revolution, namely the demonstration that terrestrial physics and celestial physics are one and the same. According to Hall (1967), in his theory, Newton gave the proof that the inverse-square law accounts for the elliptical orbit established by Kepler. He provided the general dynamical theory of Kepler's laws of planetary motion. Galileo's law of acceleration, where his curious and unexplained observation that the descent of a free falling object is independent of weight, was accounted for. Newton also proved Huygens' theorems on centrifugal force. The difference between Newton and other scientists who tried to solve problems in celestial mechanics was, as Brougham and Routh (1972) identified, his accomplishments:

> *...the whole subject was at once thoroughly investigated. It was not merely that the general principle hitherto anxiously sought for, and of which others* caught *many glimpses, was now unfolded and established upon appropriate foundations;*

> *but almost every consequence and application of it was either traced, or plainly sketched out.* (pp. 9-10)

The list of these applications or consequences is quite impressive. It includes, for example, Newton's method of determining orbits from observations of position. An approximate solution of the problem (essential to the dynamical theory of the moon's orbit) of determining the motions of three mutually gravitating bodies was the beginning of research area known as the “many-body problem”. The broad theory of the tides was at last established, and the degree of asphericity of the earth caused by its rotation was determined. In all of these and other problems, Newton showed the connection between them and his theory of gravitation. Finally, Newton's extraordinary contribution to physics is that he introduced a new paradigm of thinking about physics termed by Cohen (1981) as "Newtonian style". This style of repeated give-and-take between a mathematical construct and physical reality enabled Newton to lay down the principles of theoretical physics as a mathematical science.

Importance of Geometry in the Presentation of Gravity

The main objective of this study is to explore the role mathematics plays in physics education and to examine how physics textbooks represent it. The examples from the history of gravity highlighted in this chapter show that geometry was used throughout by scientists to get insights. The reason for that could be that geometry assisted in conceptualizing gravity due to its visual quality to represent physical phenomena, and because of this, geometry was very often the only tool ancient scientists used to provide proofs.

For example, in Aristotle's explanation of the spherical shape of the earth (provided in the first section of this chapter) the geometrical concept of the sphere being the shape of minimum surface area helped Aristotle to conclude that "any other shape would allow space for further motion of things to get closer to the center" (As cited in Stevens, 1998, p.398).

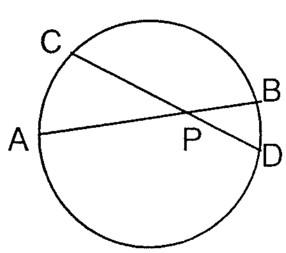
Another example of using geometry is Kepler's description of planetary motions where he applied geometry that Newton used later to test his theory of gravity, namely Kepler's law of areas. Newton proved that the curved motion described by the law of areas was a consequence of centripetal force, and conversely, motion in a curve described by the law of areas implies a centripetal force (as shown in Newton's reasoning stages in the development of his law of universal gravitation). Newton believed that there was an intimate connection between physics and geometry. He showed this connection when he demonstrated that the law of areas was a necessary and sufficient condition for inertial motion in a central-force field.

The example where Newton used (earlier in his developing ideas of gravity) the results of Galileo's kinematics of free fall and applied them to the dynamics of a revolving object to derive the relation $a_c = v^2/R$ (discussed in Stinner, 1994, p. 81) was used in many physics textbooks today of the 1950's and 1960's.

Another impressive example of the significant role geometry plays in understanding gravity was Newton's theorem where he stated that a spherical shell with homogeneously distributed mass attracts a body in the same way as if the entire mass of the shell were concentrated at the center. Here Newton used a geometrical construct to represent the earth as a set of concentric spherical shells (this theorem was discussed earlier in the present chapter). Geometry, in combination with Newton's infinitesimal calculus, provided a logical base in this example for deriving the significant theorem. The derivations in this theorem are not hard to understand for a high school student. The most valuable consequence of Newton's demonstration is that students, having exposure to geometrical proofs, could use the method involved in them as a mode of thinking in solving other physics problems. This is not to say that students should use the geometrical method developed by Newton to study motion, thus reproducing all of Newton's steps from the *Principia.* Clearly, this would not be practical because many of Newton's geometrical proofs are very lengthy, and sometimes beyond the scope of students' knowledge of geometry. In addition, we now know more efficient analytical ways of dealing with problems on motion. The reality is that students master little geometry in high school. Therefore, they should be familiar only with basics of Newton's geometric method in order to use it as a way of thinking when a problem situation warrants it.

For example, a good exercise for students would be to prove Newton's theorem that a test mass within a hollow sphere will feel no push in any direction because of cancellation of forces from opposite sides. This theorem is presented in the *Principia*, Section XII, Proposition LXX, Theorem XXX. Let us examine Newton's proof in detail.

Figure 5-7. Proposition LXX, Theorem XXX in the *Principia*



Let us draw lines AB and CD through a test mass m at point P within this hollow sphere. Since the triangles ACP and BDP are proportional, the arcs AC and BD are also proportional. Therefore, if we construct a faceted hollow sphere, the facets with masses m_1 and m_2 on opposite sides of the sphere will be proportionate. We assume for simplicity that the thickness of the spherical shell is uniform throughout. If this is the case, then the volumes, and consequently masses of the facets on the opposite sides of the sphere would be proportionate to the squares of their distances a and b to test mass m at point P, i.e.:

$$\frac{m_1}{m_2} = \frac{a^2}{b^2}.$$

By the inverse square law of gravitational attraction, the gravitational force acting on the test mass m from the first facet is:

$$F_1 = G\frac{mm_1}{a^2},$$

where G is a gravitational constant.

Similarly, the gravitational force acting on the test mass m from the second facet is:

$$F_2 = G\frac{mm_2}{b^2}.$$

Since

$$\frac{m_1}{m_2} = \frac{a^2}{b^2},$$

then the ratio of the gravitational forces is:

$$\frac{F_1}{F_2} = \frac{m_1 b^2}{m_2 a^2} = 1,$$

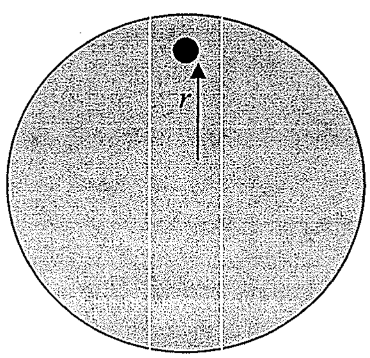
i.e., $F_1 = F_2$.

Similar reasoning for any other pairs of facets would let us conclude that all of them would compensate each other. Thus, the resultant gravitational force from the whole spherical surface would be zero for any point P inside the sphere.

This example, which can be seen as geometric way of introducing the idea of integration, shows students a perfectly sound and an elegant way to approach problems of this kind.

Another example demonstrating the elegance of geometry in combination with Newtonian dynamical principles is a famous, more sophisticated, "tunnel problem" (Eisenkraft and Kirkpatric, 2000).

Figure 5-8. Tunnel Problem



In this problem students are instructed to calculate the period of an object that travels through a hole created along diameter of the Earth. Since on its travel through the tunnel, the object experiences a force due to each part of the Earth, some of the Earth will be pulling inward and some will be pulling outward. A useful observation (proved before by Newton) is that the force at any position is equivalent to that of all of the enclosed mass only, as if that mass were located at the Earth's center. The mass in the external shell has no contribution to the force. At an arbitrary point a distance *r* from the center, the attractive enclosed mass is then

$$M = dv = d\frac{4\pi r^3}{3},$$

where d is density of the Earth, and *v* is volume of the enclosed mass. If the density is constant and there is no friction in the tunnel, we can solve for the force on the object at any point in the tunnel.

$$F = G\frac{Mm}{r^2} = \frac{Gd4\pi r^3 m}{3r^2} = \left(\frac{4\pi Gdm}{3}\right)r = kr.$$

When the r is toward the right, the force is toward the left, and the correct form of this equation is

$$F = -kr.$$

This equation is the same as the equation of a mass on a spring. The motion of the mass in such motion represents simple harmonic motion. The object will oscillate back and forth through the earth. The period of oscillation *T* is given by the equation

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3m}{4\pi Gdm}} = \sqrt{\frac{3\pi}{Gd}}.$$

Assuming that the density of the earth is 5.5×10^3 kg/m^3, we get a period 84 minutes.

It is interesting to compare this result with the period of a satellite orbiting the Earth. Students usually do this problem before considering more sophisticated examples. The idea of a satellite orbiting the Earth first appears famously in Newton's *Principia.*

In low Earth orbit gravitational force according to inverse square law is:

$$F = \frac{GM_E m}{R_E^{\ 2}} = ma_c = \frac{mv^2}{R_E} = \frac{4\pi^2}{T^2}\, mR_E.$$

We assume that the orbit is circular with radius of that of the Earth R_E. Therefore, the satellite is orbiting the Earth at the constant speed $v = 2\pi R_E / T$ under the influence of the only force ma_c according to Newton's second law of motion. This force here is equal to the force represented by inverse square law of gravitational attraction.

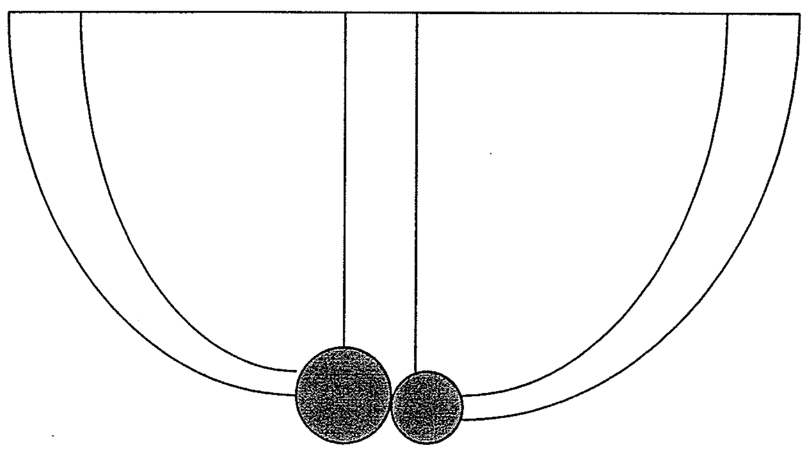
The orbital period can be expressed then by

$$T = 2\pi\sqrt{\frac{R_E^{\ 3}}{GM_E}}.$$

Substitution for radius of the Earth $R_E = 6.37 \times 10^6$ m, mass of the Earth $M_E = 5.98 \times 10^{24}$ kg, and gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 yields a period of 88 minutes. This is very close to the period of an oscillating mass moving through the tunnel of the Earth in the previous example. This example is significant for students' conceptual understanding of a periodic motion because the quantitative coincidence of the periods of a satellite orbiting the Earth and of an object oscillating back and forth through the Earth means that these periodic motions have something in common, mainly they represent simple harmonic motions subjected to the restoring force $F = - kr$ and having period of oscillations $T = 2\pi / k$. This example serves a good opportunity to demonstrate to the students the applicability of the same physical phenomenon to broader contexts (The ability of broader contextualization is an essential characteristic of experts in physics problem solving). For instance, students can discuss the similarity of a simple harmonic motion such as the projection of a steady motion around a circle (circular motion seen edge – on, or motion around a vertical circle shadowed on the ground by vertical sunlight) and that of the to-and-fro motion of a pendulum bob (with small amplitude), or the up-and-down motion of a mass on a spring. In addition, these examples demonstrate the use of mathematics as a rich conceptual tool. Indeed, we could not come to the conclusion of the similarity of motions described in these examples, let alone extend applications of our conclusion to other contexts, if we did not derive mathematical equations describing these examples. On the other hand, our derivations were based on qualitative analysis of the situations represented in these examples. Thus, there is a two-sided connection between qualitative and quantitative approaches in the presentation of physics material.

Finally, another example of using geometry to illuminate the understanding of the third law of motion is Newton's own experiment he describes in his *Principia* in the second Scholium, right after the Laws of Motion and their Corollaries. This experiment plays a vital role in Newton's theory of universal gravitation. It looks like Newton felt that his third law, the law of equality of action and reaction, was a novel statement requiring further justification. In this justification Newton invoked the experiments on elastic impact carried out some years before independently by Christopher Wren, John Wallis, and Christian Huygens (Newton, 1687, p. 22). He collided together the pendulums (about ten feet long) with different masses, to establish that the impacts (forces) experienced by them were equal and opposite as measured by how far they rebounded.

Figure 5-9. Newton's Justification of his Third Law of Motion

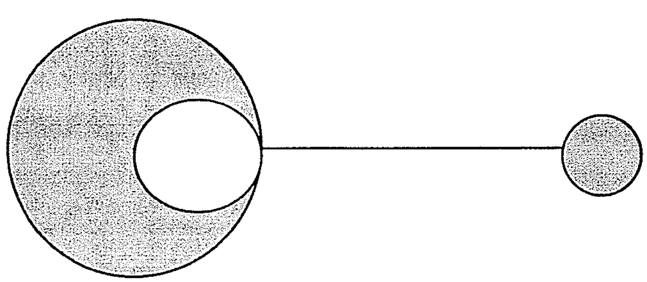


Having performed the analysis of a collision between two bodies of unequal mass in the center of gravity frame of reference, Newton concluded that they had "equal motions" in this frame, both before and after collision. This could only mean the product of mass and velocity, or momentum is conserved (momentums are equal and opposite). He realized and stated that during such a collision the center of mass itself would move at a steady speed. After the description of this experiment in detail, Newton concluded: "And thus the third Law, so far as it regards percussions and reflections, is proved by a theory exactly agreeing with experience" (p. 25). Newton's geometrical justification of the third law of motion in the *Principia* illuminates our understanding that by the third law of motion Newton meant conservation of linear momentum. Unfortunately, Newton's interpretation of the third law of motion is not presented in physics textbooks. The geometrical explanation of the experiment performed by Newton and described in his *Principia* would definitely deepen students' understanding of such important principle of mechanics.

There is no doubt that problems where students model situations using the Newtonian geometric method in combination with his laws of motion, help conceptual understanding of many interesting situations in physics. Once the basics of Newtonian geometric method are understood, students can move on to more advanced problems, often offered at physics olympiads. The following are examples of such problems (Eisenkraft and Kirkpatrick, 2000).

Problem A: A spherical hollow is made in a sphere of radius R such that its surface touches the outside surface of the sphere and passes through the lead sphere's center. The mass of the sphere before hollowing was M. With what force will the lead sphere attract a small sphere of mass m, which lies at a distance d from the center of the lead sphere on the straight line connecting the centers of the spheres and the hollow?

Figure 5-10. Problem A



One way to the solution of this problem could be to imagine that the sphere of radius R was initially filled in with a sphere having the mass $m_{removed}$. If we now remove the filled in part, the gravitational force between the sphere with a hollow and the sphere of mass m would be the difference between two gravitational forces:

$$F = F_1 - F_2 = G\frac{Mm}{d^2} - G\frac{m_{removed}m}{\left(d - \frac{R}{2}\right)^2} =$$

$$\frac{G\rho V_R m}{d^2} - \frac{G\rho V_{R/2} m}{\left(d - \frac{R}{2}\right)^2} = Gm\rho\left(\frac{\frac{4\pi R^3}{3}}{d^2} - 4\frac{\pi\left(\frac{R}{2}\right)^3}{\left(d - \frac{R}{2}\right)^2}\right) =$$

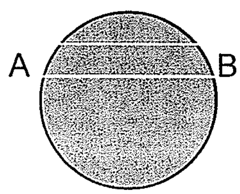
$$Gm\rho\frac{4\pi R^3}{3}\left(\frac{1}{d^2} - \frac{1}{8\left(d - \frac{R}{2}\right)^2}\right) = GMm\left(\frac{1}{d^2} - \frac{1}{8\left(d - \frac{R}{2}\right)^2}\right),$$

Where F_1 is the gravitational force between the sphere with the mass M and the sphere with the mass m, F_2 is the gravitational force between the removed hollow with the mass $m_{removed}$ and the sphere with the mass m, ρ is the density of lead, V_R is the volume of the sphere of the radius R, and $V_{R/2}$ is the volume of the sphere of the radius $R/2$.

Obviously, the solution to this problem includes geometric concepts and some algebraic rearrangements of the equations involved. The final result looks quite complex. It would be useful to ask that students do some sort of error analysis of the final result: If we assume, for example, that $d \gg R$ then the gravitational force $F \approx \frac{7}{8}\frac{GMm}{d^2}$ which is the reasonable answer since the mass of the hollow is less than the mass M of the sphere before hollowing. Students would benefit if the textbooks presented this kind of error analysis in example problems. Unfortunately, the analysis of the answer in textbook example problems is very rare. Students would get a good opportunity to combine their qualitative and quantitative reasoning if textbooks incorporated this step to the process of the problem solutions.

Problem B: A tunnel is drilled along a chord of the Earth connecting points A and B. Calculate the period for an apple to travel from A to B. Comment on the feasibility of such a tunnel for global travel. Does a straight tunnel provide for the fastest journey from A to B?

Figure 5-11. Problem B



The case of "tunnel" through the center of the earth problem was presented earlier in this chapter. The period of oscillations the tunnel turned out to be

$$T = 2\pi\sqrt{\frac{R_E{}^3}{GM_E}}.$$

It is not hard to show that this equation is equivalent to the equation

$T = 2\pi\sqrt{\frac{R}{g}}$, where R is the radius of the earth, and g is acceleration due to gravity (the derivation follows below).

We established earlier that the motion of an object in the earth tunnel is subjected to the force which varies proportionally with the distance x from the center ($F = - kx$). We can find the value of the constant of proportionality k if we set $x = R$ and $F = - mg$ (when the object is on the surface of the earth):

$$-kR = -mg.$$

Therefore, $k = \frac{mg}{R}$, and

$$F = -\frac{mg}{R}x.$$

High school physics textbooks do not introduce differential equations to derive the equation of the period T of oscillations of a harmonic motion. However, the formula

$$T = 2\pi\sqrt{\frac{m}{k}}$$

is introduced qualitatively in many textbooks.

Applying this formula, we will obtain the following equation for the period of oscillations of an object in the tunnel through the center of the earth:

$$T = \sqrt{\frac{m}{\frac{mg}{R}}} = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6370x10^3}{9.8}} \approx 84\,\text{min}.$$

Problem B differs from the case of the "tunnel" through the center of the earth. The difference is that the tunnel does not go through the center of the earth. Let's see if the result will be different.

The difference now is that the apple in this problem situation is oscillating due to the component of the gravitational force in horizontal direction of this tunnel. The gravitational force makes an angle θ with the vertical direction. Therefore, the component of the gravitational force F' in the direction of the tunnel is:

$$F' = F\sin\theta = \frac{mg}{R}x.$$

This equation turns out to be the same equation as the one for the motion of an object in the tunnel through the center of the earth. Therefore, the period of oscillations of an apple in the tunnel AB is the same as in the former case which is $T = 2\pi\sqrt{\frac{R}{g}}$. It would take half of this time to go from point A to point B of the tunnel. Therefore,

$$T = \pi\sqrt{\frac{R}{g}} \approx 42\,\text{min}.$$

As a matter of fact, the formula for the period of oscillations of the apple in the tunnel does not include the distance from A to B. Therefore, it is reasonable to conclude that it would take about 42 min to travel in any straight tunnel in the earth whether the tunnel goes through the center of the earth or not.

The demonstrated examples of applications of the Universal Law of Gravitation show that using elementary geometry is very important in the discussion of gravity. Geometrical proofs can be thought of as valuable conceptual tool for students. Together with Newton's laws of motion geometrical method aids in solving interesting problems, particularly those requiring visualization and constructing of models where students compare mathematical representation with physical reality according to "Newtonian style" of thinking.

Summary

Aristotle and other early philosophers of Greece thought that all matter was made of four elements: earth, water, air, and fire. They considered earth and water the heaviest. Therefore, the element of earth, as the most heaviest, was placed in the center of the universe, and all objects would fall towards the earth in straight lines with the exception of planets and stars which were presumed to be moving in circles around the earth.

The medieval scientists Jean Buridan and later Nicole Oresme considered *gravitas* from a new point of view. They proposed that there were the element *terra* (earth) and the element *aqua* (water). Oresme and Buridan seem to speak of the element *terra* as forming a globe with its own center, and of the element *aqua* as forming the globe with its own center. These centers were not always in the same place, and it was postulated that the element *terra* and the element *aqua* each formed a separate natural sphere because of gravity. Going further, Buridan and several scholars after him proposed to describe the two spheres of earth and water interacting.

Copernicus made a mathematical assumption about the heliocentric system, arguing that all celestial spheres revolve around the sun with the sun being in the center of the universe. It is likely that Oresme influenced Copernicus's idea of heliocentrism. Oresme opposed the theory of a stationary earth 200 years before Copernicus did. He suggested that if gravity, the tendency of earth to move toward the center, were regarded as the attraction of earth to earth, then the earth could revolve around the sun and celestial spheres would still fall in a straight line.

Another significant figure in the development of ideas about gravitation was Galileo. Instead of trying to answer the question "why do objects fall", Galileo explored "how do the objects fall." Galileo's contribution to the understanding of gravity is that he changed the approach to studying gravity by emphasizing the "how" of gravity. First, he conducted careful experiments to understand the "how" of gravity. Secondly, he gave not only a qualitative account of motion under gravity but also a mathematical quantitative description.

The astronomer Johannes Kepler, who discovered the laws of planetary motion at the beginning of the seventeenth century, was a significant figure in the development of ideas about gravitation. Kepler was the first man who realized that planets should move in ellipses and not circles. Kepler's laws were the first accurate mathematical treatment of the universe.

There is no doubt that Newton used Kepler's ideas about planetary motion in the development of the theory of universal gravitation. Although Galileo and Kepler set the stage for later developments in gravitational theory, their contributions were not thought to be connected until Newton brought them to the forefront. Kepler's contribution concerned the orbits of the planets around the sun while Galileo's input concerned motion and the acceleration of falling bodies. It was not suspected until Newton's theories, that there was a connection between falling objects and planetary motion.

Besides Kepler and Galileo there were other scientists who had an influence on Newton's development of his theory of universal gravitation. One of them, Christian Huygens, first showed how to calculate centrifugal, or center-fleeing forces. Huygens and Descartes had analyzed curved motion in terms of such a centrifugal force. Newton initially used the concept of centrifugal force in the analysis of the circular motion caused by a central force. Later, Newton realized that the older and misleading notion of a centrifugal force should be replaced by the concept of a centripetal force, which is equal in magnitude to the centrifugal force but has opposite direction.

When Newton still spoke about motion in terms of centrifugal force, another scientist, Robert Hooke, joined the discussion about planetary motion. Hooke suggested that the attraction of the sun draws away the planets from moving in straight lines, and proposed that the force of attraction varies inversely with distance as the square of the separation.

Clearly, in developing the idea about universal gravitation Newton owes to the insights of many scientists. What differentiated Newton from other scientists who had caught many glimpses of the theory of gravity was Newton's ability to compare the real world progressively with a simplified mathematical representation of it. One of the tools of Newton's mathematical apparatus was his calculus invented to develop the law of universal gravitation as it applies to the motion of planets. The significance of Newton's calculus can not be overestimated. Moreover, contrarily to the common but erroneous supposition that Newton used algebraic analysis and his fluxional calculus to reach his conclusions about his law of gravity, Newton employed calculus and transformed it to geometrical arguments.

Newton's discovery of universal gravitation, according to Cohen (1981), was not an "isolated flush of genius," but the "culmination of a series of exercises in problem solving, …a product not of induction but of logical deductions and transformations of existing ideas." According to Cohen, the decisive step on the path of Newton's discovery of universal gravity came in late 1679 and early 1680. In fact, it was initiated by Robert Hooke when he introduced Newton to his new way of analyzing motion along a curve, a motion having two components - an inertial component and a centripetal one. Hooke also suggested that the centripetal attraction is inversely proportional to the square of the distance. The inverse-square nature of gravitational force for circular orbits was hinted at before by Huygens. By means of simple algebra, using Kepler's third law of planetary motion and the relation for centripetal force, one can readily show (Halley was the first who did it) that the centripetal force varies inversely with the square of the distance.

The question Hooke posed before Newton, according to Cohen, was a reverse one: "If a central attractive force causes an object to fall away from its inertial path and move in a curve, what kind of curve results if the attractive force varies inversely as the square of the distance?"(p.169) As a result of correspondence with Hooke, Newton came up with his work *De Motu (Concerning Motion),* likely finished in 1684, where he showed that an object that has inertial motion and is subject to an inverse-square centripetal force moves in elliptical orbit.

The progression of Newton's thinking in the development of his law on universal gravitation can be described as follows:

In the very first proposition of the *Principia* (and also in the discussion at the beginning of *De Motu*), Newton develops the dynamical significance of Kepler's law of areas by proving that the curved motion described by the law is a consequence of centripetal force. In the second proposition of the *Principia*, Newton proves the converse: Motion in a curve described by the law of areas implies a centripetal force. As Cohen notes, with these two propositions Newton demonstrates that the law of areas is a necessary and sufficient condition for inertial motion in a central-force field.

The next step in the development of the concept of universal gravitation is Newton's realization that the planets do not move according to the law of areas in simple Keplerian elliptical orbits with the sun at a focus. Instead, the focus lies at the common center of mass, because as a consequence of Newton's third law of motion (the law of action and reaction), not only does the sun attract each planet but also each planet attracts the sun, and the planets attract one another. Newton realizes that the planets move precisely in elliptical orbits and according to Kepler's laws of areas not in the real world but in an artificial situation, a one-body system - a single point mass moving with an initial component of inertial motion in a central-force field. The one-body system reduces the earth to a point mass and the sun to an immobile center of force.

Newton then continues the discussion of the next step in the 11th section of the *Principia* focusing on transition from an interactive two-body system to an interactive many-body system. This step is a further consequence of the third law of motion. Since each planet is a center of an attractive force as well as an attracted body, a planet not only attracts and is attracted by the sun but also attracts and is attracted by each of the other planets.

Once Newton concludes that all bodies must attract each other, he presents the conclusion and explains why the magnitude of the attractive force is so small in many situations that it is unobservable. He explains that it is possible to observe these forces only in the huge bodies of planets, meaning that gravitational force would depend on the masses of the interacting objects. From his pendulum experiments performed earlier, Newton inferres that, since different masses have identical constant accelerations towards the earth's center, a constant force is acting whose magnitude is proportional to the masses of the bodies concerned (*Principia,* Section VI, pp. 303-326).

This kind of analysis led Newton to conclude that the planets neither move exactly in ellipses nor revolve twice in the same orbit, therefore "there are as many orbits to a planet as it has revolutions."

One of the revolutionary features of Newtonian celestial dynamics, according to Cohen, is the distinction that Newton draws between the realm of mathematics, in which Kepler's laws are truly laws, and the realm of physics, in which they are only approximations. Newton realized that a set of point masses circling the central point mass attract one another and perturb one another's orbit when he compared the construct with physical reality. This realization caused Newton's treatment of the planets as physical bodies with definite shapes and sizes.

The important step to such treatment of the planets is Newton's realization that the earth attracts an object with the same force as if all of the earth's matter were concentrated in one point, at the center of the planet. With the powerful tool of his integral calculus, Newton proves the following theorem: a spherical shell with homogeneously distributed mass attracts a body in the same way as if the entire mass of the shell were concentrated at its center. Only after this series of proofs, Newton extends his generalizations for spheres that are dissimilar in terms of density and attractive force. Newton again employs his method of integral calculus when he represents two interacting spheres as spheres composed of numerous concentric shells having different densities.

In book three of the *Principia,* Newton treats the topic of gravitation essentially the same way, but the treatment is more mathematical. First, Newton conducts what is called the moon test where he extends the terrestrial gravity to the moon and demonstrates that the weight force varies inversely with the square of the distance.

Newton then identifies the gravitational force existing on earth with the force of the sun on the other planets, as well as the force of a planet on its satellites. He now calls all these forces gravity. Referring to the third law of motion, he transforms the concept of a solar force on the planets into the concept of a mutual force between the sun and the planets. In a similar way he transforms the concept of a planetary force on the satellites into the concept of a mutual force between a planet and its satellites. Finally, the resulting transformation is the notion that all bodies interact gravitationally.

Newton's style of thinking in the development of his law of universal gravitation laid the foundation for a new paradigm in exploring science the essence of which is the process of repeatedly comparing the mathematical construct with reality, and if the mathematical construct falls short, modifying it. This way of thinking had an outstanding significance not only for Newton's contemporaries but it still proves to be indispensable, plausible and fruitful.

The main objective of this study is to explore the role mathematics plays in physics education and to examine how physics textbooks represent it. The examples from the history of gravity highlighted in this chapter show that geometry was used throughout by scientists to gain insight. The reason for this could be that geometry assisted in conceptualizing gravity due to its visual quality to represent physical phenomena, and because of this, geometry was very often the only tool ancient scientists used to provide proofs. There is no doubt that problems where students model situations using the

Newtonian geometric method in combination with his laws of motion, help conceptual understanding of many phenomena in physics.

The next chapter will focus on the preparation stages for instrument construction for the present study. The theoretical and methodological frameworks used in the development of this instrument for the analysis of the mathematical component in physics textbooks (previously described in Chapter 3) will be briefly revisited to identify how themes emerged from the literature review, the inquiry into the historical relationship between mathematics and physics, and how the history of gravity had to be synthesized to develop the instrument for textbook analysis.

Chapter 6: Development of an Instrument for the Present Study

Overview

The discussion within this chapter focuses on the importance of designing an analytical instrument for textbook research. In addition, the preparation stages for instrument construction for the present study are overviewed. Subsequently, a conceptual framework for the analysis of textbooks is presented: Firstly, which conceptual ideas about analyzing the mathematical component in physics textbooks, derived from the literature review presented in Chapter 2, and what further questions about the presentation of mathematics in physics textbooks need to be explored are delineated. Secondly, further themes for the analysis of the mathematical component in physics textbooks are identified based on a historical inquiry on the relationship between mathematics and physics, conducted in Chapter 4. Finally, the findings from the history of gravity, presented in Chapter 5, are used to develop themes for the analysis of the mathematical component of physics in the context of the Law of Universal Gravitation. Chapter 6 briefly revisits the theoretical and methodological frameworks used in the development of this instrument for the analysis of the mathematical component in physics textbooks, previously described in Chapter 3, to identify how methodologies developed by de Berg (1989) and Chiappetta et al. had to be synthesized, and where necessary, modified to fit the purpose of this study. An outline of the instrument for textbook analysis concludes this chapter.

The Task of Designing an Analytical Instrument

Nicholls (2003) argues that "methods for textbook research are fundamentally underdeveloped and in need of further research, … surprisingly little work has been done in terms of setting out clear generic guidelines for analyzing texts." (p. 1). He also notes that designing an analytical instrument is not an easy task but it is a very important stage in textbook research. Accordingly, the American textbook researcher William Fetsko (1992) said, "Time spent in designing the analysis instrument will pay great dividends throughout the process" (p. 133).

The process of instrument development is quite complex. Nicholls (2003) makes the following suggestions on designing an analytical instrument:

> *To "design" the "instrument" researchers must formulate a framework or criteria of categories and questions fine-tuned to the specific aims and objectives of a particular textbook project. The categories and questions are then applied to all the textbooks in the sample from which analysis of the results may proceed.*

(p. 4)

Nicholls (2003) emphasizes that categories and questions arise based on the epistemological orientation of the researcher. There is an intimate relationship between methodology and epistemology. Nicholls asks researchers to consider "whether we construct an analytical instrument based on an idea of what is to be analyzed or on our experience of what is to be analyzed?" (p. 8). I believe designing an analytical instrument incorporates both aspects of this question. In fact, my personal perspective, presented in Chapter 3, has influenced the formulation of this study's categories and questions for textbook research. Instrument development is an integral and complex process, rooted in the researcher's perspective.

Preparation Stages for Instrument Construction

Instrument development involved several stages. My exploration of textbook portrayal of the role of mathematics in physics education began with the identification of relevant information from the literature review (Chapter 2). The gaps in research reflecting the mathematical component in physics textbooks were pointed out and further steps for research were outlined (Chapter 3). A historical inquiry into the relationship between mathematics and physics (Chapter 4) and a historical inquiry into the history of gravity (Chapter 5) were conducted to gain a deeper insight into the role that mathematics has played in physics in the context of universal gravitation.

Based on the literature review and the historical inquiries, I formulated conceptual ideas that have guided instrument construction. These conceptual ideas gave rise to a conceptual framework that was then used to develop an instrument used for the analysis of physics textbooks in this study.

Conceptual Framework for the Analysis of Textbooks

The literature review presented in Chapter 2 provides information on how the role of mathematics has been portrayed in physics textbooks since the first physics books were produced in the English - speaking world. Already in the 19th century Whewell saw that mathematics could play a richer role in physics education than it was usually assigned, namely analytical treatment of mechanics, which in turn became a "sounding-board" for mathematical methods. Despite this, Whewell has failed to present mathematical formulations as potentially rich conceptual tools that could lead to a better understanding of the physical phenomena that the equations were describing.

As the literature review shows, there were other educators (Kline, 1959; Lehrman et al., 1982; Stinner, 1992; Swartz et al., 1999) who saw the problem Whewell attempted to address and who were interested in the improvement of physics textbooks, particularly in the most appropriate presentation of the mathematical component of physics. Recognition of this problem caused educators to start examining the role of the mathematical component in physics textbooks. These educators reiterated Whewell's idea that the mathematical component of physics was not used to its full potential as a conceptual tool. Educators had several concerns regarding the presentation of the mathematical component of physics in textbooks. First, educators were concerned that the presentation of the mathematical component of physics rarely promoted understanding and encouraged only memorization of formulas. Frequently, equations were asserted without justifying details or examining their functional dependencies, often omitting their derivations completely. They found out that the algebraic mode was the predominant way used to represent relationships between concepts, limiting other modes of mathematical representation such as graphs, verbal statements of proportionalities, or numerical order of magnitude analysis to convey the meaning of numbers. The opposite extreme was also observed. Some textbooks conveyed meaning by exploring many good qualitative questions and suggesting good experimental activities while incorporating almost no mathematical components. In addition, the sequencing of mathematical representations did not agree with sound pedagogical practices that emphasize the importance of qualitative representations of concepts rather than quantitative ones, especially at the start of instruction. In summary, to reiterate Morris Kline's (1959) statement, the intrinsic relationship of mathematics to the study of nature is not presented in our dry and technique-based textbooks.

For the purpose of this study I would like to find out if there has been any significant change in the presentation of the mathematical component of physics in recent high school physics textbooks and introductory level college physics textbooks. Specifically, I am looking for answers to the following questions:

1. What are the modes of representation of the mathematics in physics textbooks?
2. Are the equations used in the textbooks justified?
3. What is the sequence of mathematical representations in these texts? Is this sequence in accordance with the ideas of contemporary learning theories discussed in Chapter 3?
4. What function does mathematics have in these physics textbooks?

The historical exploration of the relationship between mathematics and physics described in Chapter 4 represents the views of historians and philosophers of science, scientists, and science education researchers. From this exploration, I gained insight into the role that mathematics plays in physics.

This insight helped me come up with themes for the analysis of physics textbooks. First, my exploration revealed that mathematics organizes broad classes of natural phenomena into coherent patterns. Mathematics was necessary and indispensable in making predictions, often leading to new discoveries in a wide range of physical phenomena. Accordingly, mathematics also plays a vital role in the formulation of theories by using analogies with known laws or using the mathematician's sense of form and symmetry. A good example of using symmetry would be Coulomb's idea to apply Newton's theory of gravity to his theory of electricity. Second, my exploration identified an important function of mathematics as being the language of physics. Moreover, mathematics transformed physics from a qualitative to a quantitative subject. Third, science education researchers learned that very often mathematics is used for calculations in the form of combining numerical relations to produce new numerical relations. It is also used in physics to give definitions through the establishment of derived measurements. Furthermore, educators learned that mathematics can be used beneficially in physics if we consider the assumptions and limitations of its applicability. An equally important lesson for educators was that quantitative relationships in physics can be effectively presented in different modes, including the verbal mode. Finally, educators realized that historical reconstructions of crucial steps in the emergence of a physical phenomenon facilitate the acquisition of mathematical skills which are critical to understand concepts and methods in physics. Therefore, the mathematical component in physics should be presented using historical reconstructions of ideas in a historical context where crucial steps of historical evolution can be identified.

Since the main objective of this dissertation is to examine how physics textbooks demonstrate the role of mathematics in the presentation of the topic of universal gravitation, the history of gravity was explored in Chapter 5. There were many instances in the history of the law of universal gravitation where mathematics played an important role in conceptual understanding of this phenomenon. These instances served as ideas for my conceptual framework from which themes were identified for the qualitative analysis of physics textbooks. Several suggestions about textbook usage of mathematics in the presentation of universal gravitation emerged from an inquiry on the history of gravity. A discussion on the importance of introducing basic geometry in the presentation of gravity at the end of Chapter 5 also contributed to the identification of themes for the analysis of textbooks.

The conceptual framework for the analysis of the mathematical component of physics in the context of the topic on universal gravitation is based on the following propositions: First, mathematics can be used as a conceptual tool in understanding gravity because with the aid of mathematics, as was noted earlier in this chapter, we can generate patterns. This proposition is supported by many examples in history. Galileo, for instance, showed that the descent of bodies varies as the square of the elapsed time ($d{\sim}t^2$). Therefore, it follows conceptually that the curve of projectile motion is a parabola. In another case, the astronomer Halley suggested that areas experiencing influence of the sun (supposing that the influence of the sun was diffused in a sphere) were directly proportional to the squares of the radii ($A{\sim}r^2$). He then reasoned that since the intensities were inversely proportional to these areas ($A{\sim}1/I$), then intensity was inversely proportional to the square of the radius ($I{\sim}1/r^2$). This kind of reasoning is purely conceptual, based on the analysis of proportionalities. Another relevant example of conceptual reasoning based on the analysis of proportionalities is Newton's derivation of the inverse square law which he applied to the centripetal force of gravity for circular orbits ($F{\sim}1/r^2$). This particular case would be very beneficial to present in high school physics textbooks because students have sufficient background in geometry and algebra to handle derivations and understand where they come from. Newton also showed that if motion was described by the inverse square law ($F{\sim}1/r^2$), then the curve of such motion could be a parabola, an ellipse, or a hyperbola. Newton applied Kepler's Law of areas to show that centripetal force generates a curve. Finally, Newton used geometry and calculus to arrive at his law of gravity. What is important for this study is to find out if textbooks mention the idea of the geometry and the calculus Newton used (Newton's mathematical tools) to describe gravity.

Second, mathematics in physics textbooks can be used to demonstrate what the historian B. Cohen (1981) described as the "Newtonian Style", where mathematical modeling is connected and compared with physical reality in an ongoing dialogue. In this regard I would like to find out from the analysis of physics textbooks the answers to the following questions: 1) Do physics textbooks illustrate the use of assumptions in the presentation of the law of universal gravitation? For example, textbooks could reflect Newton's assumptions in the development of his law by mentioning the transition of Newton's ideas from one-body systems to two-body, then to many-body systems, and consequently to the idea about common center of gravity. Finally, these ideas can be applied to real examples, discussing and looking at the problems related to universal gravity. 2) Do physics textbooks describe thought experiments related to the law of gravity? For example, when developing his law of gravity, Newton performed his thought experiment on the motion of satellites. 3) Do physics textbooks illustrate the use of models in the presentation of the material on the law of gravity? For example, textbooks could include Newton's discussion of how the law of universal gravitation applies to objects other than point masses, the ones which have shape and size. 4) Do physics textbooks justify the mathematical relationships presented? Is there any evidence and proof in textbooks' presentation of the material on gravity? In this respect, I would also like to see if there is any mathematical analysis of the relation $F{\sim}m_1m_2/r^2$ found in textbooks, or if textbooks explain why the magnitudes of an attractive force of gravity are so small in many common situations. For example, if the ordinary objects on the surface of the earth are not massive, then the product m_1m_2 is small, and the gravitational force F is small. Textbooks could also give support to the idea of gravity on the surface of the earth being equal to the same force as if all the earth's mass were concentrated in one center point. To demonstrate that mathematics could be used as a rich conceptual tool, textbooks could provide proof of the statement that acceleration due to gravity g is not dependent upon the size of the mass of the test object, as confirmed by observation when comparing mathematical results with physical reality. 5) Do physics textbooks show the enormous applicability of mathematics to solve new problems and make new predictions, thus demonstrating fecundity of mathematics? For example, textbooks could discuss Newton's "Moon Test" where he showed that the gravitational force of the moon is 1/3600 of that of the earth, i.e. F moon = 1/3600 F earth. Textbooks could also show the role of mathematics in making predictions by showing other than earth gravitational phenomena and problems, such as gravity on other planets and different celestial objects, to demonstrate the universality of gravity.

My final proposition is that physics textbooks could show connections between mathematics and physics in the frame of the requirements of scientific literacy, as demonstrated by history of science examples. What is important to find out is whether textbooks show this connection and how textbooks present the theory of universal gravitation. I anticipate that the qualitative content analysis conducted in the presented conceptual framework will shed light not only on the role that mathematics plays in physics textbooks in the unit on universal gravitation but also how the mathematics used in physics textbooks helps conceptualize this phenomenon.

Revisiting Theoretical and Methodological Frameworks for the Development of the Instrument

The analysis of the mathematical component in high school physics textbooks was informed by learning theory and the idea of scientific literacy. Themes representing learning theory were based on the methodology developed by de Berg (1989). He suggested considering the factor of *sequence* for textbook analysis. The *sequence* factor was looked at in terms of physics content sequence, seen as going from qualitative to quantitative, and the ideas of quantification seen as going from verbal to algebraic mode. The other factor was an *emergence profile* of a quantified form where textbooks' treatment of ideas of quantification was analyzed in terms of distinct differentiation between the *static* and the *dynamic* way of presentation.

Themes representing the idea of scientific literacy came from the methodology developed by Chiappetta et al. (1991). As described earlier in Chapter 3, Chiappetta et al. (1991) developed the following themes, referring to scientific literacy: *the knowledge of subject, the investigative nature of science, science as a way of thinking, and interaction of science, technology, and society.*

Methodologies developed by de Berg (1989) and Chiappetta et al. (1991) had to be synthesized and where necessary, modified to fit the research questions of my study. In addition, the emerging methodology should reflect useful ideas of other educators, such as Leonard, et al. (1999) about multiple representations (verbal, symbolic, graphical, etc.), or Gentner's et al. (1997) idea about the role of analogies in representation of concepts.

As a result of the inquiry into the history and philosophy of science (HPS) described in Chapters 4 and 5 and the exploration of the ideas of learning theories (LT) and scientific literacy (SL) in Chapter 3, a number of themes emerged for the content analysis of physics textbooks. These themes (categories) will appear at the end of this chapter. To develop an instrument for this study, these themes had to be revisited and narrowed down to workable categories for the analysis of physics textbooks. The initial categories were then applied to a selected number of textbooks to see if it was possible to use them for my analysis. Some categories were difficult to apply to the analysis. In this case, re-examination was necessary. These categories then had to be modified until the analysis of the textbooks became feasible.

After refining the categories specified for the analysis of the role of mathematics in the presentation of material in physics textbooks, an instrument for the content analysis of physics textbooks was developed and the written material that appeared in physics textbooks in the unit on universal gravitation could finally be properly categorized. The following table represents categories, applicable to the content analysis of physics textbooks, that convey formulated meanings. Outlining descriptors of these categories, analytical tools for data analysis of this textual material, integrated into analytical tools checklists, and inferences that could be drawn from the results of content analysis for interpretation possibilities are parts of the developed instrument for this study. For the convenience to the reader this instrument for the content analysis of physics textbooks is also presented in Appendix A.

Summary

Instrument development involved several stages. My exploration of textbook portrayal of the role of mathematics in physics education began with the identification of relevant information from the literature review (Chapter 2). The gaps in research reflecting the mathematical component in physics textbooks were pointed out and further steps for research were outlined (Chapter 3). A historical inquiry into the relationship between mathematics and physics (Chapter 4) and a historical inquiry into the history of gravity (Chapter 5) were conducted to gain deeper insight into the role that mathematics plays in physics in the context of universal gravitation.

Based on the literature review and the historical inquiries, I formulated conceptual ideas that have guided instrument construction. These conceptual ideas gave rise to a conceptual framework that was then used to develop the instrument used for the analysis of physics textbooks in this study.

The analysis of the mathematical component in high school and college physics textbooks was informed by learning theory and the idea of scientific literacy. Themes representing learning theory were based on the methodology developed by de Berg (1989). Themes representing the idea of scientific literacy came from the methodology developed by Chiappetta et al. (1991). Methodologies developed by de Berg (1989) and Chiappetta et al. (1991) had to be synthesized and where necessary, modified to fit the research questions of my study. In addition, the emerging methodology should reflect useful ideas of other educators, such as Leonard's, et al. (1999) about multiple representations (verbal, symbolic, graphical, etc.), or Gentner's et al. (1997) idea about the role of analogies in representation of concepts.

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In the next chapter the qualitative content analysis of physics textbooks will be carried out and reported. The analysis of the role mathematics in the unit on universal gravitation in terms of balancing qualitative and quantitative aspects of physics will be carried out using the developed instrument for textbook analysis. Eight physics textbooks will be analyzed in terms of modes of representation of mathematical concepts, nature of emergence of a particular mode, purpose for a particular mode, connections of mathematical concepts, sequencing the mathematical content, approaches used in example problems, presentation of mathematical concepts through the history and philosophy of science, and presentation of mathematical concepts based on the view of science as a way of thinking.

Table 6-1

Instrument for Textbooks Analysis

Categories	Descriptors	Inferences	Analytical Tools
Modes of Representation of Mathematical Concepts in the Law of universal gravitation • Numerical • Verbal • Graphical • Pictorial • Symbolic	Mathematical concepts are presented by • Numbers (in tables, charts) • Words • Graphs • Pictures • Algebraic symbols	Multiple representations help develop • Conceptual understanding • Enrich the presentation of physics knowledge • Establish connections between symbols and physical reality Limited representations cause lack of understanding of mathematical concepts.	Constructing a presence/absence of a particular mode matrix, and making judgments about use of different modes according to the following analytic rubric: • Limited use -only one or two modes are present • Moderate use – three or four modes are present • Extensive use - all five modes are present
Nature of Emergence of Mathematical Concepts in the Law of Universal Gravitation	Mathematical formulas, graphs, tables, and verbal formulations appear and are used		Identifying nature of emergence of a particular mode (based on *static/dynamic* descriptors)

Table 6-1. (continued)

• *Static*	• without explanation, or discussion	• Mathematics, in the case of static emergence, is mostly used for memorization of formulas, tables, graphs, pictures • Verbal formulations do not complement each other • Connections between concepts could not be established
• *Dynamic*	• with information about background, experimental details, how mathematical relationship expressed in a	• Presentation of the mathematical relationship in the dynamic way could help change students' conceptions • Connections between concepts could be

Table 6-1. (continued)

	• particular mode is determined, and accuracy of the relationship	• established	
Emergent Press (purpose) for a Particular Mode	Stating purpose of using a particular mode	• The replacement of concepts could be seen by students as useful and plausible • Mindless manipulations of mathematical equations occur in a context where the expressed need for such equations is missing	Identifying purpose of using a particular mode if there is one stated
Connections of Mathematical Concepts	• Moving between modes of representation	Students will likely make connections between concepts.	Constructing maps of tracking movements between modes of representation of concepts and evaluating variability of moves according to the

Table 6-1. (continued)

			following analytic rubric: • Limited –movements are mostly linear, from one mode to another in a single direction with 1-2 back and forth movements between limited kinds of modes • Moderate – movements are mostly not linear, from one mode to another in back and forth directions, and mostly between same kinds of modes • Extensive- movements are not linear, from one mode to another, in back and forth directions between, mostly between different kinds of modes
	• Using analogies	Students would likely use relevant features and	Constructing a presence/absence of

Table 6-1. (continued)

		ignore irrelevant ones when comparing and contrasting concepts what would help in the interrelationship of knowledge.	analogies matrix, and in case of presence, providing examples of analogies
Sequencing Mathematical Content: Simple → complex • Qualitative→quantitative	• From describing qualities (features) of observations, experiences, inferences to describing measurable quantities involved;	This way of presentation is in agreement with learning theory; therefore, meaningful presentation of the mathematical component of physics is likely to happen.	Using maps of tracking movements. The following rubric is applied: • Appropriate – the simple→complex sequence is used
• Verbal→algebraic	• From describing mathematical relationships in words to giving symbolic equations		

Table 6-1. (continued)

Complex→simple • Quantitative→qualitative • Algebraic→verbal	• From presenting measurable quantities to describing features • From describing mathematical relationships in symbols and algebraic equations to describing mathematical relationships in words	This approach is not recommended by educational researchers since cognitive gaps could be formed if such approach is used; therefore, learning of mathematical concepts will be complicated.	• Not appropriate – the complex→simple sequence is used
Balancing Qualitative and Quantitative aspects of Physics in Presentation of Example Problems Problem Solving Approach: • qualitative • quantitative	The approach, where verbal explanations are engaged (conceptual problems) The approach, where calculations, symbolic equations are engaged	• Lack of problems engaging qualitative approach would impede students' learning • Engaging students in solving only	Analyzing the content of example problems and the approach taken to solve them. The following analytic rubric will be used to determine the extent of balancing:

Table 6-1. (continued)

		quantitative problems would signify engagement of primitive levels of thinking which are not suitable for generating conceptual models • Qualitative reasoning combined with quantitative mechanism to communicate thinking strategies would help generate these models and make mathematics meaningful to students	• Limited – mostly quantitative approach is used with almost no qualitative reasoning • Moderate – algebraic equations, calculations are backed up by some qualitative explanations • Extensive – algebraic equations, calculations are backed up by detailed qualitative

Table 6-1. (continued)

			explanations
Presentation of Mathematical Concepts through HPS:			
• Descriptive	Presentation of mathematical concepts referring to HPS with no students' assignments related to the historical context	If HPS is presented descriptively only, students get little exposure to the nature and methods of science and significance of mathematical equations in science could not be understood.	Constructing presence/absence matrix (based on descriptors) featuring descriptive/instructional presentation of mathematical concepts through HPS, and in case of presence, providing illustrative examples
• Instructional	Presentation of mathematical concepts referring to HPS with assignments related to the historical context and requiring from students doing exercises, completing projects, participating in discussions	If HPS is presented in instructional way, students will get exposure to the nature and methods of science by applying presented	

Table 6-1. (continued)

		ideas during construction of conceptual models of their own, gaining experience to evaluate their conceptual models in terms of accuracy, simplicity, plausibility, predictability, and fruitfulness.	
Presentation of Mathematical Concepts Viewing Science as a Way of Thinking	• Illustrating the use of *assumptions*, *models*, and *thought experiments* in the presentation of history of the development of the concept of gravity • Discussing *evidence and proof* • Referring to *Newtonian Style* • Showing fecundity of	• Mathematics would be used as a conceptual tool in learning about gravity, and would facilitate students' construction of their own conceptual models • Mathematical formulations would get	Constructing presence/absence matrix (based on descriptors) featuring presentation of science as a way of thinking, and in case of presence, providing illustrative examples

Table 6-1. (continued)

		mathematics • Referring to *Newton's geometry* and *calculus* for the description of gravity	significance and show their usefulness, fruitfulness, plausibility, and limitations, thus facilitating the process of conceptual change • Higher order thinking would be engaged in understanding NOS and help establish connection between mathematics and physics

Chapter 7: Textbook Analysis for Quantitative – Qualitative Balance

Overview

This chapter addresses parts of Research Question 4 (parts a, b, and c) the main purpose of which was to evaluate the degree of maintaining balance between the qualitative and the quantitative aspects of physics in the topic on universal gravitation. For this purpose, qualitative content analysis of textbooks in regard to the modes of mathematical presentation of concepts, pedagogical sequence of presentation of the mathematical component found in physics textbooks in the topic on universal gravitation, and the reflection of ideas of learning theories and requirements of scientific literacy in the presentation of the mathematical component of physics in physics textbooks was performed and reported in this chapter.

An analysis of role of mathematics in the unit of universal gravitation in terms of balancing qualitative and quantitative aspects of physics was performed using the instrument for textbook analysis developed in Chapter 6 (Table 6-1, or Appendix A). Eight physics textbooks were analyzed in terms of modes of representation of mathematical concepts, nature of emergence of a particular mode, purpose for a particular mode, connections of mathematical concepts, sequencing the mathematical content, approaches used in example problems, presentation of mathematical concepts through the history and philosophy of science, and presentation of mathematical concepts based on the view of science as a way of thinking. The results of this analysis are presented below.

Two sections comprise this chapter. Both of them address parts of the last research question, specifically the reflection of ideas of contemporary learning theories and the requirements of scientific literacy in the presentation of the mathematical component of physics in physics high school and introductory level college physics textbooks in the topic of universal gravitation. The first section focuses on the question of how contemporary learning theories are reflected in the presentation of the mathematical component of physics in textbooks. The report of the findings that answer this question starts with presenting what was discovered about the modes of representation of mathematical concepts in the unit of universal gravitation followed by a discussion of what was uncovered about the nature of emergence and purpose for using a particular mode of representation. Finally, findings about connections between mathematical concepts, sequencing the mathematical content, and balancing the qualitative and the quantitative aspects of physics in textbooks conclude the first section of this chapter.

The second section of Chapter 7 addresses the question of how the requirements of scientific literacy are reflected in the presentation of the mathematical component of physics in textbooks. The findings for this question concentrate around two themes: presentation of mathematical concepts through the history and philosophy of science and presentation of mathematical concepts in physics through the view of science as a way of thinking. The chapter concludes with a discussion of the role mathematics plays in high school and introductory level college physics textbooks in the unit of universal gravitation.

Coding Textbooks

In Chapter 3 of the study the selected textbooks were listed in Table 3-2. The high school textbooks chosen for this research are the recent textbooks which are recommended by the Department of Education in Manitoba and other provinces of Canada. It is a fact that teachers can only purchase a class set of textbooks which are recommended. I found that the timing for my analysis of recent physics textbooks was beneficial because teachers need to know the strengths and deficiencies of the textbooks they are using to fill the gaps in the presentation of the mathematical component in them. The other reason why I chose these books is to give teachers information that they could use when they buy future physics textbooks that are sent to the Department of Education for review. The introductory level college physics textbooks were recommended by a physics professor. These books are in use at the present time at the University of Manitoba. For the convenience of reporting findings from the content analysis of these textbooks letter codes will be used for the identification of the textbooks. The codes of these textbooks appear in the last column of the following table:

Table 7-1

Coded Sample Textbook Overview

Title of Book	Author(s)	Year	Publisher	Code
Glencoe Physics: Principles and Problems	Zitzewitz & Davids	2002	Glencoe/ McGraw-Hill	GP-ZD
Conceptual Physics: the High School Physics Program	Hewitt	2002	Prentice-Hall	CP-H
Physics: Concepts and Connections	Nowicow & Heimbecker	2002	Irwin Publishing Ltd	P-NH
Physics	Edwards	2003	McGraw-Hill Ryerson	P-E
Physics: Principles With Applications	Giancoli	2005, 6th ed.	Pearson/ Prentice Hall	P-G
Contemporary College Physics	Jones & Childers	1993, 2nd ed.	Addison-Wesley	CCP-JC
College Physics	Serway & Faughn	1999, 5th ed.	Harcourt Brace & Company	CP-SF
Physics Matters: An Introduction to Conceptual Physics	Trefil & Hazen	2004	John Wiley & Sons, Inc.	PM-TH

Section 1: Reflection of the Ideas of Learning Theories

Modes of Representation of Mathematical Concepts in the Unit on Universal Gravitation

In the textbooks chosen for the qualitative content analysis a search for different modes of representation (numeric, verbal, graphical, pictorial, and symbolic) of mathematical terms and concepts involved in presentation of Newton's law of universal gravitation was conducted. According to science education researchers Leonard, Gerace, and Dufresne (1999) and Hestenes (1992), multiple representations help students develop conceptual understanding of concepts and facilitate developing connections between symbols and physical reality. The lack of different representations of concepts impedes students' understanding and, consequently, makes learning difficult. The following table where different modes of representation of mathematical concepts were identified and judgment about the degree of variety of them was made according to the rubric described in the instrument in Chapter 6 represents the results of findings:

Table 7-2

Modes of Representation of Mathematical Concepts in the Unit Universal Gravitation

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic	Extent of presence	Reference pages
GP-ZD	√	√	√	√	√	E	175-189
CP-H	√	√	√	√	√	E	169-179
P-NH	√	√	√	√	√	E	159-167, 213-219
P-E	√	√	-	√	√	M	131-133, 577-594, 633
P-G	√	√	-	√	√	M	117-128, 916
CCP-JC	√	√	-	√	√	M	134-135, 142-148
CP-SF	√	√	-	√	√	M	193-198, 201-207
PM-TH	-	√	-	√	√	M	97-110

Note: √ represents the presence of a feature
- represents the absence of a feature
M represents moderate use of modes (3-4 modes)
E represents extensive use of modes (5 modes)

The data in Table 7-2 show that mathematical concepts in the unit of universal gravitation in the selected textbooks are presented in various modes. In three books (GP-

ZD, CP-H, and P-NH) all five modes could be identified. Therefore, it is possible that all these textbooks could help students understand mathematical concepts involved in the unit of universal gravitation and establish connections between symbols and physical reality. However, as the data show, the graphical mode of presentation which is very instrumental in visualizing functional relationships is not engaged by many of the selected physics textbooks. In one of the books (PM-TH), the numerical mode was not used at all. Engaging numbers in the reasoning process also could be a very helpful tool to come up with brilliant ideas, as was illustrated by the history of science examples in Chapter 5 where Newton demonstrated how he deduced the inverse-square relationship between the force of gravity and the distance separating the Earth and the Moon. These limitations (the lack of some modes of representation of mathematical concepts) would probably influence the capacity in which mathematics could be utilized, as well as the degree of balancing of the qualitative and the quantitative aspects of physics what is very important in helping students develop sound understanding of universal gravitation.

Nature of Emergence and Purpose for a Particular Mode of Representation of Mathematical Concepts in the Unit Universal Gravitation

The next category for content analysis was the approach taken in the textbooks in the presentation of the mathematical component in any of the modes of representation used. Some textbooks used a *static* approach that is the approach in which mathematical formulae, graphs, pictures, tables, or verbal statements appear and are used without any explanation at all. Other textbooks used a *dynamic* approach in which mathematical equations, graphs, pictures, tables, and verbal statements were placed in context with an explanation of how they came to be and the purpose of their use. As Reynolds and Baker

(1987) note, "allowing the student to interact with the graph or diagram might draw attention to it, thereby producing better learning" (pp. 161-162).

The purpose of using a particular mode of representation of mathematical concepts in physics textbooks should be obvious to the reader. The significance of clear purpose was convincingly addressed by de Berg (1989):

> *...an important factor which determines if a particular reasoning skill will be used in solving a problem relates to whether the context establishes a clear need for that skill. It would appear, then that the mindless manipulations of mathematical equations ... could easily occur in a context where the expressed need for such equations was missing.* (p. 131)

I would expand this justification of demonstrating clear purpose beyond establishing purpose for using some mathematical equation in physics. Demonstrating purpose for using an equation is not limited to the need of the equation itself. It is not less important to show the purpose of a particular mode engaged in the presentation of material in physics textbooks. Why would students use numerical data, verbal explanations, graphs and pictures in their reasoning process if the need for them is not demonstrated by physics textbooks to be effective, productive and intelligently stimulating?

Thus, exploration of the nature of emergence and the purpose for using of a particular mode of presentation of mathematical concepts was the next step in the qualitative analysis of the chosen physics textbooks. The table representing findings about the nature of emergence and the purpose of using a particular mode of representation of mathematical concepts involved in the law of universal gravitation is presented in the Appendix B of this study (*Nature of Emergence and Purpose for a*

Particular Mode of Presentation of Mathematical Component in the Unit Universal Gravitation)

The data presented in Appendix B show that the physics textbooks chosen had different approaches in using different modes of representation of the mathematical component in the physics of universal gravitation. The difference could be traced not only in the manner of emergence of a particular mode but, as well, the purpose for which a particular mode is used. There is only one textbook (CP-H) where all modes of representation of mathematical component appear dynamically where mathematical formulas, graphs, numbers, pictures, tables, or verbal explanations are placed in context with an explanation of how they came to be and the purpose of their use. For example, the numerical mode in this text is used to make conceptual inferences; the verbal mode is engaged in explanations; the graphical and pictorial modes allow students to interact with the graph or picture by posing questions about them; the symbolic mode is used for making conceptual inferences, as well as for demonstrating usefulness of mathematical equations in obtaining new information and at the same time stressing the esthetic value of the mathematical rule and demonstrating models of reasoning that make the essence of Newtonian Style. Other textbooks used both static and dynamic approaches in the presentation of mathematical components in different modes of representation.

As I expected, most of the textbooks used the dynamic approach when the mathematical components of physics of universal gravitation were presented verbally. This was not surprising because the verbal mode is usually used when explaining, emphasizing or clarifying something. Only in one textbook (CP-SF) was the verbal mode employed using the static approach during the stating of the universal law of gravitation (without any explanation) and listing features of the law (without explaining their origin). Another observation was that most authors of the selected textbooks had difficulty in using the dynamic approach when the mathematical aspect was presented in numerical mode. In one textbook (PM-TH) mathematics was not presented in numerical mode at all. Only in two textbooks (CP-H and CCP-JC) was the dynamic approach used when engaging the numerical mode. The numbers in these texts were used not only for illustrating something but also for making conceptual inferences, interpreting results and explaining discrepancies in results which would likely engage students' thinking. In textbooks P-E and P-G both approaches (dynamic and static) were engaged while presenting the mathematical component in the numerical mode. In these textbooks, numbers were sometimes used just for illustrating, for example, planetary data, other times – as a tool for showing and interpreting relationships, and making conceptual inferences (for example, in drawing conclusions about the inverse square law relationship). In other textbooks (GP-ZD, P-NH, and CP-SF) the numerical mode was utilized in a limited way, using the static approach where numbers were used basically for illustrating data in tables without engaging higher order thinking processes. More interaction between numbers and students would be desirable to produce better learning.

It was disappointing to see that the graphical mode of representation of the mathematical component of physics was used in only three textbooks (GP-ZD, CP-H, and P-NH). None of the examined college physics textbooks utilized graphical mode in the representation of the mathematical concepts. Textbook (GP-ZD) used the graphical mode statically, just for illustrating the law of gravity. However, in two other textbooks (CP-H and P-NH), the dynamic approach was used to present graphs. For example, in text CP-H, questions based on the graph were posed asking students to interact with the graph to find out what would happen to one variable when the other was changed. In text P-NH, explanations of changes were included in discussing graphs. These strategies would make graphs meaningful to students and helpful in their understanding of gravity.

As the collected data in Appendix B show, the pictorial mode of presentation of the mathematical component was found in all textbooks. However, most of the researched textbooks (GP-ZD, P-G, CCP-JC, CP-SF, and PM-TH) used the static approach to present pictures as a mode of representation of mathematical component in physics of gravity. Pictures were primarily used for illustrating, diagramming or giving images of objects. Only two textbooks utilized pictures on a higher level when such processes as posing questions, giving explanations and comparing similar diagrams of different phenomena were engaged for students' interaction with such useful visual tools as pictures. This strategy could definitely set an example for students' use of pictures in their reasoning process while solving problems, working on a model, and many other thinking activities.

I was pleasantly surprised that, in new textbooks, like those chosen for this research, the symbolic mode of representation of mathematical component of the physics of gravity was emerging mostly in a dynamic way. Only in one case, in textbook PM-TH, was the law of gravity stated in symbols without explanations of steps to arrive at it. My observations showed that in a lot of cases, mathematical relationships were derived or their sources explained. The role of mathematics in physics quite clearly emerged. The researched textbooks showed how valuable and useful mathematics could be for establishing connections between different laws, providing new information by obtaining new relationships as a result of calculations and derivations and making conceptual inferences by analyzing proportionalities, thus demonstrating fecundity of mathematics. If students were to see in textbooks processes like deriving, analyzing, calculating and interpreting, they would likely try to use them in the tasks offered for them to do.

Connections of Mathematical Concepts and Sequencing the Mathematical Content in the Presentation of the Law of Universal Gravitation

As was argued in Chapter 3, in the process of using multiple representations of mathematical concepts in science, what really matters is the student ability to see the interrelationship of these representations (Leonard, Gerace, & Dufresne, 1999; Russel et al., 1997; Seel & Winn, 1997; Kozma, 2000; Herron & Greenbow, 1986; Osborne, 1984; de Berg, 1989). The more variability in movements between the different modes of representation of mathematical concepts in physics textbooks, the easier it is for students to make connections between different ideas represented by these concepts. Various movements between numerical (N), verbal (V), graphical (G), pictorial (P) and symbolic (S) modes of representations of the same concepts in physics textbooks can assist students in translating between representations, and consequently, help students connect physics ideas and relate them to personal experience to attain conceptual understanding.

In order to judge how well the qualitative and the quantitative aspects of a particular concept are balanced, and consequently how well students can acquire conceptual understanding of it, it makes sense to focus on a single concept (law of universal gravitation). Therefore, narrowing down the analysis from the whole unit on universal gravitation to that of the law of universal gravitation would be a logical next step to make. To track movements between different modes of representation of mathematical concepts in the law of universal gravitation in eight physics textbooks, maps of movements were constructed and shown below in Table 7-3.

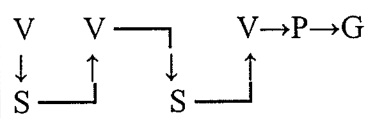
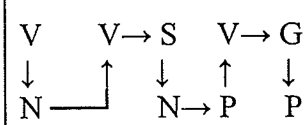
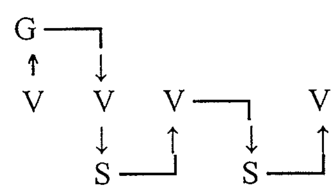
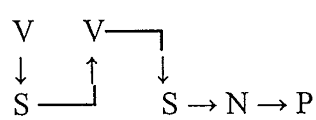
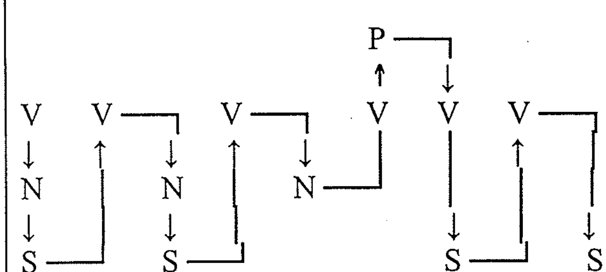
Another helpful factor in the process of making connections between mathematical concepts in physics, as identified earlier in Chapter 3, is using analogies in the presentation of material. In addition to being an important mechanism for conceptual change (Gentner et al., 1997; Waterworth et al., 2000), analogies can foster students' ability to use relevant features of a particular phenomenon and ignore irrelevant ones when comparing and contrasting concepts which in turn would help in the interrelationship of knowledge. A presence/ absence of analogies matrix with supporting examples was constructed to see if analogies were utilized in the eight physics textbooks chosen for the research and presented in Appendix C of this study (*Using Analogies in the Presentation of the Law of Universal Gravitation*).

The judgment about connections of mathematical concepts was made based on the variability of moves between modes of representation (assessed according to the rubric developed in the instrument of the study in Chapter 6) and presence/absence of analogies in the presentation of the law of universal gravitation.

The maps of movements between different modes of representation of mathematical concepts were also used to judge if the sequencing of the mathematical content in the presentation of the law of universal gravitation was appropriate – from simple to complex, i.e. from qualitative to quantitative descriptions or from verbal to algebraic representations of concepts. The following Table 7-3 represents a map of tracking movements between modes of representation of concepts.

Table 7-3

Moving Between Modes of Representation of Mathematical Concepts in the Law of Universal Gravitation

Text	Reference Pages	Movements	Variability of Moves Sequencing
GP-ZD	181-183		M simple→complex (appropriate)
CP-H	169-177		E simple→complex (appropriate)
P-NH	158-161		M simple→complex (appropriate)
P-E	577-578, 632-633		L simple→complex (appropriate)
P-G	117-119		E simple→complex (appropriate)

CCP-JC	88-90, 142	$\begin{tikzcd} & N \arrow[dr] & & \\ V \arrow[d] & V \arrow[u] & V \arrow[d] & V \\ S \arrow[ur] & & S \arrow[ur] & \end{tikzcd}$	M simple→complex (appropriate)
CP-SF	193-195	$\begin{tikzcd} V \arrow[d] & V \arrow[dr] & & V \\ S \arrow[d] & & S \arrow[ur] & \\ P \arrow[uur] & & & \end{tikzcd}$	M simple→complex (appropriate)
PM-TH	96-98	$\begin{tikzcd} & P \arrow[dr] & \\ V \arrow[d] & V \arrow[u] & V \\ S \arrow[ur] & & \end{tikzcd}$	L simple→complex (appropriate)

Note: V represents verbal mode
S represents symbolic mode
N represents numerical mode
G represents graphical mode
P represents pictorial mode
L represents limited variability of movements
M represents moderate variability of movements
E represents extensive variability of movements

Table 7-3 shows that in two (CP-H and P-G) of the eight researched textbooks the variability of movements between different modes of representation of mathematical concepts in the law of universal gravitation was found to be extensive. In these textbooks the movements between verbal, numerical, symbolic, graphical, or pictorial modes appear to be nonlinear, from one mode to another the moves are in back and forth directions, and mostly between different kinds of modes. It is likely that students would establish connections between concepts for better learning, and the balancing between mathematical and conceptual aspects of physics would likely happen. However, only one (CP-H) of these books contain analogies as evident from Appendix C. The textbook (P-G) would benefit in providing students help on establishing connections between concepts if it used analogies in the presentation of material on the law of universal gravitation.

Four of the researched books (GP-ZD, P-NH, CCP- JC, and CP-SF) exhibited moderate variability of movements between different modes of representation of concepts according to the rubric developed in Chapter 6. As the maps of tracking movements (Table 7-3) show, the movements between modes of representation are not linear. However, the movements happen mostly between the same kinds of modes, for example, between verbal and symbolic, in back and forth directions. Three of these books (GP-ZD, P-NH, and CCP- JC) used analogies (Appendix C) which according to research literature (Gentner et al., 1997; Leonard, Gerace, & Dufresne, 1999) were proven to be helpful in establishing connections between concepts in physics. No analogies were found in Textbook CP-SF which limits its use for establishing connections between concepts.

In two of the analyzed textbooks (P-E and PM-TH), according to the constructed maps of movements (Table 7-3), variability of movements between different modes of representation of mathematical concepts is limited. As the maps show, the movements are mostly linear, from one mode to another in a single direction with one or two back and forth movements between limited kinds of modes. The positive finding was that textbook P-E used analogies in presentation of the material (Appendix C). However, no analogies were found in textbook PM-TH which makes this textbook even less instrumental in establishing connections between concepts and less helpful in balancing of mathematical and conceptual aspects of physics in the presentation of the law of universal gravitation.

The constructed maps of tracking movements between modes of representation of concepts (Table 7-3) show that in all textbooks verbal descriptions preceded algebraic ones. Numbers, pictures and graphs were also used to support verbal explanations. This sequence, from qualitative to quantitative, verbal to algebraic is supported by learning theory as educational research (Arons, 1984; de Berg, 1993; Hewitt, 1994; Monk, 1994; Mazur, 1996; Stinner, 1994) reports. This appropriate sequence in the presentation of the law of universal gravitation will likely make students' learning better when conceptual understanding would accompany algebraic symbols involved in the presentation of material. Numbers, symbols, graphs would attain appropriate meaning and serve the tools for balancing the qualitative and the quantitative aspects of physics.

Balancing Qualitative and Quantitative Aspects of Physics in Example Problems on the Law of Universal Gravitation

Chapter 3 described what educators learned from the studies of experts' and novices' problem-solving strategies (Larkin et al., 1980; Reif & Heller, 1982; Chi et al., 1981; Gabel, Sherwood, & Enochs, 1984; Shoenfeld, 1985; Dillon, 1998; Van Heuvelen, 1991). One of the findings from these studies was the experts' use of extensive qualitative reasoning compared to novices' mostly quantitative, often mindless, ways of approaching problems. Indeed, engaging students in solving only quantitative problems enacts primitive levels of thinking which are not suitable for generating conceptual models, and, consequently, impedes students' learning. On the other hand, qualitative reasoning combined with quantitative mechanisms to communicate thinking strategies (mathematics is a language of physics) would help generate these models, and make mathematical concepts in physics meaningful to students. I would like to reiterate the statement I made in Chapter 3. If experts' strategies were proven to be successful in problem solving, then they might work for the presentation of the material in physics textbooks. Indeed, the choice of problems shown in textbooks' examples does determine in what thinking activities students would be engaged, given the fact that students and teachers still use textbooks materials.

To maintain the balance between the qualitative and the quantitative aspects of physics in problems used in textbooks, it is important that students be encouraged to do conceptual analysis of the situations described in these problems, whether the problems require only verbal explanation (conceptual problems), or whether algebraic equations and numbers have to be used to solve the problem. The types of problems textbooks show should engage both the qualitative and the quantitative reasoning to help conceptual understanding. In order to analyze the content of textbooks in terms of their balancing of the qualitative and the quantitative aspects of physics it is important to look at the example problems. The analysis of exercises and end of chapter problems would be less appropriate to do because it is not always obvious what approach in solving them would be taken by students, and what problems would be assigned by a teacher. On the other hand, example problems and the approach used to solve them can give a good idea what types of problems and approaches to the solution are used, and consequently, how well the conceptual and mathematical aspects of physics are balanced in them. Therefore, only example problems on the law of universal gravitation were used for the content analysis of physics textbooks. The table representing illustrative example problems used in eight physics textbooks is given in Appendix D of the study. These examples are categorized based on the type of reasoning demonstrated in solving them: qualitative, quantitative, or both qualitative/quantitative. The extent of balancing of the qualitative and the quantitative aspects of physics in the problems on the law of universal gravitation was judged according to the analytic rubric developed in Chapter 6 in the instrument for textbooks analysis (Appendix A).

I was pleasantly surprised that four examined textbooks (CP-H, P-NH, CCP-JC, and CP-SF) as demonstrated by examples in Appendix D indicated an extensive degree of balancing of the qualitative and the quantitative aspects of physics in the example problems on the law of universal gravitation. In these examples algebraic equations and calculations were backed up by detailed qualitative discussions and explanations. Significant space was devoted to discussing limitations and assumptions which had to be taken into account to develop solutions for these problems.

In three textbooks (P-E, P-G, and PM-TH) the extent of balancing was found moderate because only some qualitative explanations accompanied symbolic equations and calculations used in the solutions of the example problems in these textbooks. The discussions and explanations were very brief and did not cover many related to conceptual understanding issues.

Only one textbook (GP-ZD) was found to give limited attention to qualitative reasoning in the presentation of example problems on the law of universal gravitation. This textbook used mostly the quantitative approach when students are encouraged to select a useful mathematical equation, rearrange it, and solve for the unknown variable. No qualitative discussions or explanations were involved; no limitations or assumptions were considered. It would be hard to expect that this approach would be helpful in developing students' understanding of concepts involved in the law of universal gravitation.

Section 2: Reflecting the Characteristics of Scientific Literacy

Presentation of Mathematical Concepts through the History and Philosophy of Science (HPS)

There is a strong research support (de Berg, 1989, 1992; Chiappetta et al., 1991; Lederman & Niess, 1997; Stinner, 1998; Tzanakis, 1999; Wang, 1998) for the inclusion of history and philosophy of science (HPS) in teaching science. Through historical examples textbooks can show in the presentation of the material how mathematics was used to describe and develop understanding of concepts in science and how mathematics helped scientists either to change or support their conceptions about the physical world. From the history of science examples, students can see how scientists themselves struggled to strike a balance between their experiences or intuitive thinking and mathematical equations obtained at the end of the discovery journey. In the case of the subject of the present study, textbooks could show how the history of gravity developed, what scientists contributed to the development of the theory of universal gravitation and how mathematics could be used to help understand this theory. Research on introducing history and philosophy of science in teaching science is in an evolving state but there are already available suggestions on how to use HPS in the presentation of material in science textbooks. For example, de Berg (1989) suggests introducing HPS not just in a descriptive format when concepts are presented with no student assignments related to the historical context in a certain unit, but more importantly, in an instructional way when presentation of scientific concepts through the history and philosophy of science is given in a more engaging way – offering assignments related to the historical context used in the presentation of the material, requiring that students do complete exercises, complete projects, and participate in discussions. A similar idea was suggested by Tzanakis (1999) in his historical-genetic approach to learning (Chapter 4). Unfortunately, as de Berg (1989) found, from 28 texts he used in his research, "*no texts* use the history of science in an *instructional sense*" (pp. 122-123). This is unfortunate because presenting history and philosophy of science only descriptively provides little exposure for students to the nature and methods of science and, consequently, the significance of mathematical equations in science could not be understood. If history and philosophy of science were presented in textbooks in an instructional way, students would have a better chance to understand and apply scientific ideas and construct their own conceptual models gaining experience to evaluate their models in terms of accuracy, plausibility, predictability, and fruitfulness. It was interesting to see if the situation in presenting HPS in physics textbooks had changed since the time de Berg (1989) conducted his research. The presentation of mathematical concepts through HPS in the unit on universal gravitation is reflected in Appendix E (*Presentation of Mathematical Concepts through HPS*) of the study.

The examples presented in Appendix E show that the presentation of history and philosophy of science in the unit on universal gravitation in eight analyzed textbooks has not changed compared to de Berg's (1989) findings. Most of the examples were found to be used in a descriptive mode, with no tasks offered to students (CP-H, P-NH, P-G, CCP-JC and CP-SF). However, in some textbooks (GP-ZD, P-E, and PM-TH) both approaches – descriptive and instructional were used. Presentation of mathematical concepts through history and philosophy of science in an instructional sense required students to conduct research, to participate in debates, to construct and test arguments. This approach would likely benefit students in understanding the nature and methods of science, and the meaning and significance of mathematical equations in the unit of universal gravitation. In the process of researching, discussing and evaluating scientists' conceptual models students would get motivation for constructing their own conceptual models. The students would also learn how to defend their arguments, to evaluate their models, as well as what steps to take if their models proved to have flaws. In this process of constructing, evaluating and defending arguments students would always have to perform the balancing act when symbols, numbers, graphs, pictures and words interact with each other to yield understanding of powerful ideas in physics.

Presentation of Mathematical Concepts in Physics Viewing Science as a Way of Thinking

Science educators agree that presenting science as a body of knowledge where facts, concepts, principles, laws, hypotheses, theories and models are given as descriptions would not reflect a contemporary, appropriate view of the nature of science (NOS) shared by scientists, educators, and philosophers of science (Chiappetta et al., 1991; Hestenes, 1992; de Berg, 1989, 1992; Tzanakis, 1999). If science were presented in physics textbooks mainly as a body of knowledge students would get a false impression not only of the nature of science but as well, about the role of mathematics in physics. Mathematics utilized in physics given such an approach would be perceived by students as a memorizing mechanism to recall information on tests. Viewing science as a way of thinking reflects the progressive ideas of scientists, educators and philosophers of science concerning the nature of science. Given this view, students might well look at mathematics in a different way if the mathematics used in physics had the possibility of being used as a conceptual tool, in case of this study, a conceptual tool in understanding gravity. This understanding would facilitate students' construction of their own conceptual models. Mathematical formulations would get significance. Students would see them as useful, fruitful, plausible, as well as having limitations. This approach to presenting science as a way of thinking could facilitate the students' process of conceptual change. In the process of engaging in higher order thinking (like modeling) students would be able to understand the NOS and this in turn would help them establish connections between mathematics and physics. According to the descriptors given in the instrument for textbooks analysis presented in Chapter 6 (Appendix A), viewing science as a way of thinking involves illustrating the use of assumptions, models, and thought experiments and discussing evidence and proof. In the context of universal gravitation, viewing science as a way of thinking would involve demonstrating the Newtonian way of approaching nature (frequently comparing a mathematical construct with physical reality, Chapter 5), and showing the mathematical tools he used, such as Newton's geometry or the calculus he invented for the description of gravity. Newton's use of mathematics in describing nature, as shown in Chapter 5, demonstrated its significance in formulating theories, in showing the consequences of the formulated theories, and demonstrating the astonishing fecundity of mathematics. The analysis of textbooks from this point of view was important to understand if the Newtonian Style of thinking about nature was adequately presented in them. In addition, it would reveal if students understand the connection between mathematics and physics and see both aspects of physics – qualitative and quantitative. Table 7-4 presented below shows presence/absence of features illustrating mathematical concepts in the unit on universal gravitation in the view of science as a way of thinking. The supporting illustrative examples for the case of the presence of a feature are shown in Appendix F.

Table 7-4

Presentation of Mathematical Concepts in the Unit on Universal Gravitation Viewing Science as a Way of Thinking

Text	Illustrating the use of assumptions	Describing thought experiments	Illustrating the use of models	Presenting evidence and proof	Demonstrating fecundity of mathematics	Reference to Newton's calculus and geometry
GP-ZD	√	√	√	√	√	√
CP-H	√	√	√	√	√	√
P-NH	√	–	–	√	√	√
P-E	√	√	–	√	√	√
P-G	√	–	√	√	√	√
CCP-J	√	–	√	√	√	√
CP-SF	√	–	–	√	√	–
PM-TH	–	–	–	√	√	√

Note: √ represents the presence of a feature
– represents the absence of a feature

To my pleasant surprise, the data collected in Table 7-4 showed that the mathematical component of physics in the unit of universal gravitation is presented not as a static collection of formulas and facts but as an important component of understanding gravity through the dynamic presentation of science as a way of thinking. As evident from Table 7-4 and supporting examples in Appendix F, illustrating the use of assumptions was the feature which was present in all but one (PM-TH) analyzed textbook. Illustrating assumptions is very important in the sense of students' understanding that any model has limitations, and whatever they come up with as a result of creating the model, they have to be mindful when a particular model is applicable and when it is not.

Unfortunately, thought experiments did not find much reflection in the researched textbooks (No discussions of thought experiments were found in any of the examined college physics textbooks). In only three textbooks (GP-ZD, CP-H, and P-E) were thought experiments used in the presentation of material on gravity. It is unfortunate because the value of thought experiments can not be overestimated. As reported in science education research (Brown, 1986; Helm, 1985a; Helm, 1985b; Kuhn, 1977; Matthews, 1989; Stinner, 1990; Winchester, 1991) thought experiments played an important role in the history of science, and, therefore, should not be neglected in teaching and presenting science.

The textbooks analyzed were not found to contain enough material illustrating the use of models in science and in the presentation of material on universal gravitation. As the data in Tables 7-4 and Appendix F show, four textbooks (GP-ZD, CP-H, P-G, and CCP-J) made some use of models to help students visualize described properties, see similarities and differences between them. These models could be very instrumental in developing conceptual understanding of universal gravitation. Unfortunately, no use of models was found in the other four textbooks (P-NH, P-E, CP-SF, and PM-TH). Indeed, illustrating the use of models is important not only to show how they were used by scientists but also to provide examples which could be helpful in the development of the students' own conceptual models.

It was very encouraging to see (Tables 7-4 and Appendix F) that all analyzed textbooks presented some evidence for stated facts and, where possible, proof for arguments, mathematical statements, or ideas. The proofs were often given both in qualitative and quantitative form, and were grounded in examples from the history of science. Presenting proofs in the form of arguments, discussions, and mathematical derivations engages, I believe, higher order thinking skills, such as critical thinking. Critical thinking enables students to make connections between facts, their own experiences, and mathematical tools used to develop the chain of reasoning. Consequently, learning material would make sense to them.

In the development of proofs, many other mathematical relationships can be derived. This property of mathematics, namely fecundity of mathematics, was demonstrated by all textbooks reflecting the result of approaching nature using the Newtonian Style (described in Chapter 5). Students could see that the law of universal gravitation in combination with other laws of mechanics could provide new information and empower them to calculate quantities they would not be able to obtain otherwise, for example, calculating the mass of planetary objects, orbital speed, period of a satellite, size of different planets, etc.

In Chapter 5 of this study I discussed Newton's mathematical tools (calculus and geometry) used to arrive at his universal law of gravitation. In the textbooks chosen for content analysis I wanted to see if there was any reference made to Newton's calculus and geometry. In all textbooks but one (CP-SF), I found some reference to the mathematical tools used by Newton (Table 7-4) in the presentation of material on universal gravitation. Supporting examples are shown in Appendix F. In Chapter 5 of the study I also discussed the importance of geometry in the presentation of gravity and gave examples of using geometry in teaching this unit. However, no applications of Newton's geometry have been found in the textbooks' example problems, as evident from the content of examples presented in Appendix D. This is unfortunate given the fact that geometry assisted in conceptualization gravity due to its visual quality to represent physical phenomena, and it was used by ancient scientists to provide proofs. It was discussed earlier (Chapter 5) that it is not practical to reproduce all Newton's steps from the *Principia* because many of Newton's geometric proofs are very lengthy, and sometimes, beyond the students' knowledge of geometry. However, if students were exposed to Newton's geometrical proofs (basics of Newton's geometric method) they could use them as a mode of thinking in solving other physics problems.

Conclusion: How do High School and Introductory Level College Physics Textbooks Present the Role that Mathematics Plays in the Unit of Universal Gravitation?

The data show, that mathematical concepts in the unit on universal gravitation in the selected textbooks are presented in various modes. Despite this, the graphical mode of presentation which is very instrumental in visualizing functional relationships is not engaged by many of the selected physics textbooks. In one of the books, the numerical mode was not used at all.

Some textbooks used a *static* approach; other textbooks used a *dynamic* approach. There is only one textbook where all modes of representation of mathematical component appear dynamically. Other textbooks used both static and dynamic approaches in the presentation of the mathematical component in different modes of representation. Most of the textbooks used a dynamic approach when the mathematical component of physics of universal gravitation was presented verbally. Most authors had difficulty in using a dynamic approach when the mathematical aspect was presented in numerical mode. Only in two textbooks was the dynamic approach used when engaging the numerical mode.

The pictorial mode of presentation of the mathematical component was found in all textbooks. However, most of the researched textbooks used the static approach to present pictures as a mode of representation of mathematical component in physics of gravity. The symbolic mode of representation of mathematical component of the physics of gravity was emerging mostly in a dynamic way. Only in one case, was the law of gravity stated in symbols without explanations of steps to arrive at it.

In two of the eight researched textbooks the variability of movements between different modes of representation of mathematical concepts in the law of universal gravitation was extensive. In these textbooks the movement between verbal, numerical, symbolic, graphical, or pictorial modes appear to be nonlinear, from one mode to another the moves are in back and forth directions, and mostly between different kinds of modes. However, only one of these books contains analogies. Four of the researched books exhibited moderate variability of movement between different modes of representation of concepts. The movement between modes of representation are not linear. However, the movement happens mostly between the same kinds of modes, in back and forth directions. Three of these books used analogies. In two of the analyzed textbooks variability of movement between different modes of representation of mathematical concepts is limited. The movements are mostly linear, from one mode to another in a single direction with one or two back and forth movements between limited kinds of modes. One of these books used analogies in presentation of the material. The constructed maps of tracking movement between modes of representation of concepts show that in all textbooks verbal descriptions preceded algebraic ones. Numbers, pictures and graphs were also used to support verbal explanations.

Four examined textbooks indicated an extensive degree of balancing of the qualitative and the quantitative aspects of physics in the example problems on the law of universal gravitation. In these examples algebraic equations and calculations were backed up by detailed qualitative discussions and explanations. Significant space was devoted to discussing limitations and assumptions which had to be taken into account to develop solutions for these problems. In three textbooks the extent of balancing was found moderate because only some qualitative explanations accompanied symbolic equations and calculations used in the solutions of the example problems in these textbooks. The discussions and explanations were very brief and did not cover many related conceptual understanding issues. One textbook gave limited attention to qualitative reasoning in presentation of example problems on the law of universal gravitation. This textbook used mostly the quantitative approach when students are encouraged to select a useful mathematical equation, rearrange it, and solve for the unknown variable. No qualitative discussions or explanations were involved; no limitations or assumptions were considered.

The presentation of HPS in the unit on universal gravitation in most cases was found to be in a descriptive mode, with no tasks offered to students. However, in some textbooks both approaches – descriptive and instructional were used. Presentation of mathematical concepts through HPS in an instructional sense required students to conduct research, participate in debates, construct and test arguments.

The collected data showed that the mathematical component of physics in the unit on universal gravitation is presented not as a static collection of formulas and facts but as an important component of understanding gravity through the dynamic presentation of science as a way of thinking. Illustrating the use of assumptions was the feature which was present in all but one analyzed textbook. Unfortunately, thought experiments did not find much reflection in the researched textbooks. Only in three textbooks, thought experiments were used in the presentation of material on gravity. The textbooks analyzed were not found to contain enough material illustrating the use of models in science and in the presentation of material on universal gravitation. All analyzed textbooks presented some evidence for stated facts and, where possible, proof for arguments, mathematical statements, or ideas. The proofs were often given both in qualitative and quantitative form, and were grounded in examples from the history of science.

In the development of proofs, many other mathematical relationships can be derived. This property of mathematics, namely fecundity of mathematics, was demonstrated by all textbooks reflecting the result of approaching nature using the Newtonian Style (described in Chapter 5). In all textbooks but one I found some reference to the mathematical tools used by Newton in the presentation of material on universal gravitation. However, no applications of Newton's geometry have been found in the textbooks' example problems.

The findings from the qualitative content analysis of the unit on universal gravitation in physics textbooks in this chapter revealed that textbooks present the role of mathematics in many dimensions, serving many purposes. To achieve conceptual understanding in learning physics, mathematics presented in textbooks should be used as a tool for maintaining the balance between the qualitative and quantitative aspects of physics. Mathematics, in the process of balancing the qualitative and quantitative aspects of physics, can be used as a tool for but not limited to the following: establishing connections between mathematical symbols and physical; showing the usefulness of different mathematical representations as being effective, productive, and intellectually stimulating for conceptual understanding of physics; displaying data; aiding in explanations, illustrations, derivations; posing questions; making conceptual inferences, making comparisons between different mathematical relationships; and demonstrating beauty of mathematical relationships.

The usefulness of mathematics in derivations cannot be overestimated. Derivations of mathematical formulas and calculations serve the important purpose of establishing connections between ideas. As a result, new information can be provided by obtaining new relationships what would demonstrate the fecundity of mathematics. Conceptual inferences would also be impossible to make without using ratios of numbers, establishing and analyzing proportionalities, and performing dimensional analysis. In showing the usefulness of mathematics, textbooks demonstrate an important role of mathematics, a calculating tool for problem solving. In problem solving, one cannot underestimate the role of mathematics in generating conceptual models in the process of presenting evidence and proof, or in the process of thought experiments. To succeed in this process, and enable students to learn from textbooks, mathematics has to be used quantitatively and qualitatively in a balanced way.

Summary

The findings from the qualitative content analysis of the unit of universal gravitation in physics textbooks in this chapter revealed that textbooks present the role of mathematics in many dimensions, serving many purposes. To achieve conceptual understanding in learning physics, mathematics presented in textbooks should be used as a tool for maintaining the balance between the qualitative and quantitative aspects of physics. One of the roles of mathematics in the process of balancing the qualitative and quantitative aspects of physics is establishing connections between mathematical symbols and physical reality for the purpose of understanding of the mathematical concepts involved in the material on Universal Gravitation. For this purpose textbooks present mathematical concepts in different modes: numerical, verbal, graphical, pictorial, and symbolic.

The other dimension in the presentation of the role of mathematics by textbooks in learning physics is showing the usefulness of different mathematical representations as being effective, productive, and intellectually stimulating for conceptual understanding of physics. For this purpose many textbooks use the dynamic approach for the presentation of the mathematical component in any of the modes of representation where equations, graphs, pictures, tables, and verbal statements are placed in context with an explanation of how they came to be and the purpose of their use.

As the findings of this study show, the mathematics in physics textbooks could be used as a valuable tool for but not limited to the following: displaying data; aiding in explanations, illustrations, derivations; posing questions; making conceptual inferences, making comparisons between different mathematical relationships; and demonstrating beauty of mathematical relationships. The usefulness of mathematics in derivations cannot be overestimated. Derivations of mathematical formulas and calculations serve the important purpose of establishing connections between ideas. As a result, new information can be provided by obtaining new relationships what would demonstrate the fecundity of mathematics. For example, the information about the mass of planets, stars and galaxies, orbital speeds, period of satellites, acceleration due to gravity on different planets, understanding the discrepancy in some results that could not be obtained without employing mathematical derivations and calculations. Conceptual inferences would also be impossible to make without using ratios of numbers, establishing and analyzing proportionalities, and performing dimensional analysis. In showing the usefulness of mathematics, textbooks demonstrate an important role of mathematics, a calculating tool for problem solving. In problem solving, one cannot underestimate the role of mathematics in generating conceptual models in the process of presenting evidence and proof, or in the process of thought experiments. To succeed in this process and enable students to learn from textbooks, mathematics has to be used quantitatively and qualitatively in a balanced way. Once again, all these listed applications of mathematics in the presentation of material in physics textbooks reflect the major role of mathematics in physics education as an effective aid for balancing the qualitative and quantitative aspects of physics.

The final chapter of this dissertation will outline the knowledge contribution to research in science education. In addition, recommendations for textbook change, teaching and learning, and curriculum development will be made. A discussion of recommendations for textbook selection and implications for future studies will conclude this study.

Chapter 8: Implications and Recommendations

Overview

Chapter 8 begins with an outline of this study's knowledge contribution to research in science education. The emphasis is placed specifically on the construction of a comprehensive instrument for the analysis of the mathematical component of physics in textbooks' unit on universal gravitation and on using HPS as a theoretical and methodological tool in instrument construction. Implications for textbook change and curriculum development are discussed. Recommendations for textbook selection, teaching, curriculum development are provided. A discussion of the directions for future studies is also presented.

Knowledge Contribution to Research in Science Education

The main contribution of this study to research in science education is the construction of the instrument for the qualitative content analysis of the mathematical component of physics in physics textbooks. In most of the studies on evaluation of textbooks the development of the instrument for textbooks analysis was based on existing educational theories, curriculum outcomes, and previous educational research. However, historical inquiry, as a source for the development of themes for content analysis, was largely ignored with the exception of some studies by de Berg. It is unfortunate because, as this study shows, history and philosophy of science can be an invaluable tool as theoretical and methodological approaches to construction of the instrument for content analysis of textbooks. The development of the framework that emerged as a result into inquiry into HPS provided one of the initial stages in the construction of the instrument for content analysis in this study.

The instrument constructed was then applied to the analysis of high school and introductory level physics textbooks to explore the role of mathematics in the presentation of material in the unit on universal gravitation. The exploration of the mathematical component in physics textbooks is still a gap in science education research, as was concluded as a result of conducting the literature review in Chapter 2. In this way, this study is innovative because of its explicit attempt to discover the meanings and purpose of mathematics presented in physics textbooks, the approaches textbooks use to convey them, the ways textbooks contribute to the process of balancing qualitative and quantitative aspects of physics. Until now, no comprehensive instrument for the analysis of the mathematical component of physics in physics textbooks has been developed in identified studies on textbooks in science education. The detailed examination of the mathematical component from multiple perspectives in the physics textbooks used for this study will narrow the gap in the research of mathematical component in science education. In constructing my instrument I immensely appreciated the previous researchers' contributions (discussed in Chapter 3) which gave me a starting point in the development of the instrument for this study. Thus, the contribution of this study is that it provides the basis for future construction of instruments for the analysis of the mathematical component in science education.

The charts found in my dissertation provide useful information for teachers who would like to have a glimpse at the physics books I evaluated without going through many pages of bulky books in order to have idea about strengths and deficiencies of a particular textbook. For example, if they look at the constructed maps of movement between modes of representation of the mathematical component and the sequence of the emergence of these modes, they can make an informative decision on adopting of a particular textbook. In addition, the appendices found in this dissertation could serve as a valuable resource for possible criteria to judge the quality of physics textbooks, for examples of analogies in the presentation of the material, to name a few. The section on the application of geometry in physics problems on gravity (with provided solutions) could be practical for a teacher, an average student, and an advanced student.

Finally, the findings from this dissertation enabled me to give recommendations for textbook writers, teachers, and curriculum developers. I also outlined my recommendations for textbook selection, as well as implications for further studies.

Implications

Implications for Textbook Change

Since textbooks are still used by teachers and students, the findings from this study have implications for textbook improvement. For effective learning and teaching, as learning theories suggest, the concepts of physics have to be presented in different modes. The content analysis of physics textbooks evaluated showed that physics concepts in the unit on universal gravitation are presented in many of them in multiple modes. However, not all modes of representation of mathematical concepts in physics were treated using a dynamic approach when mathematical formulas, graphs, numbers, pictures, tables or verbal explanations are placed in context with an explanation of how they came to be and the purpose of their use. For example, the <u>numerical</u> mode, as the findings of this study showed, was mainly used for illustrating the law of universal gravitation and exhibiting planetary data. Limited use of the dynamic approach of presentation of <u>numerical</u> mode in most of the physics textbooks in this study does not engage higher order thinking skills, such as interpreting results, making conceptual inferences, explaining discrepancy in results and other thinking skills which could help students establish connections in the process of balancing qualitative and quantitative aspects of physics.

Thus, I propose that in physics textbooks (as well as in teaching), **numeric data are presented and used in a more interactive way**. Even if numbers in many physics textbooks are engaged in a static way, teachers can modify the static presentation of numerical information and turn it into dynamic way of their use.

As this research findings showed, the graphical mode of representation of concepts in physics was not used to full capacity and in most textbooks evaluated the graphs are presented in a limited static way. In some researched textbooks there were some questions asked about the presented graphs such as: What would happen to one variable if another variable changes? What is the value of a certain variable when the value of another variable is…? However, **more graphical relationships should be used in physics textbooks and they should not be glossed over in the presentation of material. It is crucial for graphs to appear in textbooks not as simple illustrations of the narration but as dynamically engaging and interactive vehicles for learning**.

Similar limitations were found in terms of using the pictorial mode of representation of mathematical concepts in physics textbooks. Some textbooks used pictures in a dynamic way for posing questions, giving explanations, or comparing similar diagrams of different phenomena. However, most textbooks used the static approach the main purpose of which was illustrating, diagramming or giving images of gravity. Teachers could help fill the gap created by the limited use of the pictorial mode of representation of mathematical concepts in physics textbooks by **providing rich opportunities for using pictures in textbooks.**

As this research showed, in the textbooks researched, the symbolic and verbal modes of mathematical component of the physics of gravity were presented mostly in a dynamic way when mathematical relationships were derived or explained from their origin. It was also established in this research that verbal descriptions preceded algebraic ones which is in agreement with learning theories. However, very often the derivations in physics textbooks were not presented necessarily the way they were obtained historically. The reason for this was probably to reduce the amount of material presented in already bulky textbooks, as well as for pedagogical reasons. Nevertheless, it would be a useful exercise for students to understand the significance and the purpose of the mathematical equations involved if they were asked to research how particular relations historically came to be, who the scientists contributing to the emergence of particular relationships were. Thus, **introducing HPS in physics textbooks to help understand the emergence of mathematical relationships used to represent physics concepts, laws and theories** (not just biographical information about scientists and their achievements) **should be seriously considered by textbook writers**.

In the process of balancing the qualitative and the quantitative aspects of physics in physics textbooks for establishing connections between mathematical concepts used in physics, what is important is not only the variety of modes of representation of these concepts but also the variability of movements between different modes of representation of mathematical concepts in physics textbooks. Textbooks should include an extensive amount of qualitative problems before they move to quantitative problems and mixed qualitative-quantitative problems. Using analogies also help in exploring the interrelationship of knowledge and generation of conceptual models which make mathematical concepts in physics meaningful to students as established by educational research (Gentner et al., 1997; Waterworth et al., 2000). **Extensive use of qualitative problems precedes the quantitative problems** as example problems and as practice problems assigned to students will help balance qualitative and quantitative aspects of the physics students are trying to understand and use in their explanations, problem solving, and other learning activities. Given the results of findings on variability of movement between different modes of representation of mathematical concepts, use of analogies, the degree of balancing qualitative and quantitative example problems in the researched physics textbooks (Chapter 7), my recommendation for textbook improvement would be the following: **Textbooks should present the material in multiple modes where movements between modes of representation of concepts happen in a non-linear way, in back and forth directions and between different kinds of modes.**

The findings on using HPS in the presentation of mathematical concepts in the evaluated physics textbooks showed that most textbook use the descriptive mode in the presentation of the historical component of physics. **Including the instructional mode of presentation of mathematical concepts through HPS** (providing assignments related to the historical context requiring that students do exercises, complete projects, participate in discussions) **in a greater capacity** in addition to using the descriptive mode will expose students to the nature and methods of science and provide a better understanding of mathematical concepts involved in physics material.

The analysis of the chosen physics textbooks in terms of presentation of mathematical concepts in physics through the view of science as a way of thinking showed that the mathematical component of physics in the unit of universal gravitation is presented not as a static collection of formulas and facts but as an important component of understanding about gravity through the dynamic presentation of science as a way of thinking. However, the analyzed textbooks were found not to illustrate well enough the use of models in science in the presentation of material on universal gravitation. Thought experiments did not find much reflection in the researched textbooks as well. Newton's geometry, an important visual mathematical tool which helped Newton to conceptualize gravity did not find any application in example problems of the textbooks examined. Thus, **physics textbooks could be significantly improved if they emphasized the use of mathematical models in science and thought experiments in the presentation of material.** Accordingly, **Newton's geometry,** a mathematical visual tool which Newton used to conceptualize gravity, **should be incorporated in physics textbooks**.

Implications for Curriculum Development

If we assume that one of the textbook's function is to help students meet curriculum outcomes and help teachers present the material in such a way that students could acquire the understanding required for this purpose, then it is reasonable to expect that the objectives of a curriculum should not just specify what topics or concepts have to be covered but also the type of understanding to be developed. As for expectations of the physics curriculum in terms of dealing with mathematical component of physics, **it is necessary to indicate in the curriculum what role is assigned to mathematics**, in the case of this study, in the unit on universal gravitation.

I looked, for example, at the Manitoba curriculum document *Senior 4 Physics (40S): A Foundation for Implementation (2005)* where the unit of universal gravitation was presented in the topic "Fields" in the part called *Exploration of Space*. I could find quite a broad range of use of mathematics reflected in specific learning outcomes of this curriculum such as construction of scale models, description of Kepler's laws, Newton's law of universal gravitation, solving problems, establishing proportions, performing measurements, estimations and dimensional analysis, constructing graphs, developing mathematical models involving linear, power, and inverse relationships among variables, obtaining new relationships using algebraic derivations, using numerical values for qualitative and quantitative reasoning in describing thought experiments, etc. Mathematical concepts of physics were engaged in different modes (numeric, symbolic, verbal, graphical, and pictorial) for particular specific outcomes. However, what I did not find was a statement about the expectation of balancing the qualitative and quantitative aspects of physics (in any kind of expression) in the presentation of the unit of universal gravitation, and what role mathematics was assigned to play in this process of balancing. There is even no indication in any of the curriculum outcomes that students are expected to have qualitative understanding of the key concepts in the course. For example, in one of the specific learning outcomes we read: "Outline Newton's Law of Universal Gravitation and solve problems using $F_g = Gm_1m_2/r^2$" (p. 10 – Topic 2.1 Exploration of Space). In *Notes to the Teacher* on the same page there is no suggestion that the teacher should present the relationship for the law of universal gravitation in a dynamic way, namely going through the important steps of obtaining this law (as presented in Chapter 5 of this study) to show the intriguing, illuminating mathematical processes Newton went through. Reading through the curriculum, I noticed that it is suggested to use history and philosophy of science in introducing the material on gravity and in presentation of some aspects of the law of universal gravitation. However, there is no suggestion of using HPS in presenting the mathematical component of the law. All it says about this mathematical relationship is: "Newton concluded that any two objects in the universe exert a gravitational attraction on each other that is proportional to the product of their masses, and inversely proportional to the square of their separation" (p. 10 Topic 2.1 Exploration of Space). In *Pencil-and-Paper Tasks* suggested on the next page of the curriculum guide it only says: "Students solve problems using Newton's Law of Universal Gravitation" (p. 11-Topic 2.1 Exploration of Space). What kinds of problems (qualitative, quantitative, or both)? In what sequence should the different kind of problems (if there are more than quantitative problems) be solved? It is not clear. According to my observations, the curriculum guide does not place any emphasis on the significance of the mathematical component of physics in learning about gravity, let alone on crucial need to understand the process of balancing the qualitative and the quantitative aspects of physics.

The other limitation of the present Manitoba curriculum is that it does not reflect the Newtonian Style in approaching nature (described in Chapter 5) where mathematics plays an essential role in the description of a phenomenon, making conceptual inferences, and providing ground for a conceptual change. Some aspects of viewing science as a way of thinking found their reflection in learning outcomes in the unit of universal gravitation such as describing some thought experiments, illustrating the use of models, demonstrating fecundity of mathematics. However, insufficient emphasis was placed on illustrating the use of assumptions, presenting evidence and proof, and making reference to Newton's mathematical tools such as calculus and geometry.

Recommendations

Recommendations for Textbook Selection

Given the scope of my study and limited number of textbooks analyzed, I do not intend to make conclusive recommendations for change in physics textbooks, but I am prepared to offer some tentative suggestions. I do not intend to make recommendations about adopting specific texts. My recommendations can not be considered conclusive because I am looking only at the treatment of the mathematical component of physics in the unit of universal gravitation. Obviously, there could be other criteria than the treatment of the mathematical component of physics considered for the selection of a textbook.

Using the results of the analysis of physics textbooks undertaken, I am now ready to make recommendations for the selection of the physics textbook when the criterion for selection is the treatment of the mathematical component in the presentation of material on gravity. Look in a physics textbook for the presence of the following:

- Mathematical concepts of physics are presented in a multimodal way and students are invited to engage interactively with numerical tables, graphs, pictures, symbols, and words.
- Mathematical concepts of physics are presented using the dynamic approach
- The purpose of using a particular mode of representation is clear
- Mathematical concepts are connected by arranging the material in a way that movement between different modes of presentation of mathematical concepts happens in non-linear ways and between different kinds of modes

- Analogies are extensively used in the presentation of material to connect, compare and contrast different concepts
- Mathematical content is sequenced from simple to complex, from qualitative to quantitative approach, and form verbal to algebraic form
- The problems are sequenced from qualitative to quantitative with an extensive use of qualitative problems
- Mathematical concepts are presented using rich contexts through HPS. In using HPS, the instructional approach is extensively applied as opposed to the descriptive approach in the presentation of mathematical concepts of physics
- Presentation of mathematical concepts of the physics of gravity through HPS is enhanced by illustrating the use of assumptions, models, and thought experiments, by discussing evidence and proof, by referring to Newtonian approach to nature, and by showing fecundity of mathematics
- Newtonian geometry and calculus are incorporated in the presentation of the material on gravity and in problem solving

Recommendations for Teaching

The findings of my study showed that the presentation of the mathematical component in physics textbooks has strengths and limitations. It is an obligation of a teacher, I believe, to fill the gaps in the textbook presentation of the material by applying appropriate teaching strategies to help the students develop understanding of mathematical concepts in physics. Therefore, I came up with the following recommendations for the teacher:

Whenever textbooks arrange the presentation so mathematical concepts move from one mode of presentation to another in a linear way, teachers should involve students in discussion of the textbook material in such a way that back and forth directions in moving from one mode to another are engaged, and mostly between different kinds of modes. Students should be given different kinds of assignments where they are required to communicate information either to a teacher or their classmates using not only different modes of representation of mathematical concepts in physics (verbal, symbolic, numerical, graphical, pictorial) but also showing their ability to freely move between these modes. Students also should be encouraged to justify why they chose a particular mode to learn to see the benefits and limitations of a particular mode for a particular task.

Teachers should engage students in discussion during the analysis of example problems in the textbooks. With the help of a teacher, students should fill the gaps of missing explanations and intermediate steps in problem solving, clarify the assumptions, and pose more questions. More advanced students would extend a given problem to a higher level problem, the average students would simplify a given problem to the level suitable for their understanding so they could move easier to the next step of understanding of the problem.

Teachers should encourage students to provide alternative solutions to example problems where the qualitative solutions could complement the quantitative solutions, or the quantitative solutions complement the qualitative solutions presented in textbooks. Students should be encouraged to discuss the benefits and disadvantages of a given approach in a particular problem situation to understand how qualitative and quantitative approaches to problem solving can complement each other to yield conceptual understanding of physics concepts. Teachers should encourage students to find analogies in physics textbooks, as well as create their own analogies with consequent discussion of their benefits and limitations. Students would benefit from assignments that require from them to research different approaches to derive a particular relationship, as well as to participate in the debate on benefits and disadvantages of a particular approach. These kinds of activities not only enrich the presentation of the mathematical component of physics in the symbolic mode but also help students develop their modeling skills which are very essential for problem solving, specifically efficient problem solving.

My recommendation for teachers is to compensate the limitation in the presentation HPS in the instructional mode by giving students assignments where HPS would be presented so as to engage the instructional mode, such as requiring students to conduct research, participate in debates, construct and test arguments, and develop their own conceptual models which subject to peer evaluation. In this way, mathematical equations will gain meaning and value for students, foster students' curiosity for the emergence of many other mathematical equations used to represent physics concepts and laws. I suggest that teachers give students research assignments to find information about models scientists used, about thought experiments scientists performed to come up with their ideas.

My research found that graphs in the researched textbooks are used mainly in a static fashion. Asking the following questions about the graphs presented in textbooks, I believe, would likely increase student engagement in thinking about the information a graph represents:

- From your knowledge of mathematics can you tell what kind of relationship this graph represents?
- What assumptions do we have to make to consider the relationship represented by the graph accurate?
- How would the graph change if we assume…?
- What does the slope of this graph mean (in case of a linear relationship?)
- What does the slope of a tangent to the curve mean (in case of a non-linear relationship)?
- If we find area under the graph, what meaning would it have? Can you do research on this question if you cannot tell now?
- How would you modify the scale of this graph to present better information given in a numerical table you see in the textbook on this topic?

Students, I believe, would benefit if similar kinds of questions were offered and went along with the numerical tables used in the presentation of the material. I suggest that students also get a chance to ask questions once they see the table. They could be asked to think about questions which this table could answer and the ones which require additional numerical information about the objects presented. This approach to using tables would activate students' thinking on a higher level preparing them for solving problems; as well it would foster a students' sense of curiosity and interest in learning physics. To make tables more engaging in learning, I suggest, teachers ask questions like the following:

- How do numerical data for a particular variable (characteristic) compare to each other?

- Which of the objects in the table would have the least (greatest) other characteristic value given numerical values of the present characteristic? How do you know?
- What additional information do you need in order to calculate other characteristics, not presented in the table?
- How would you calculate a numerical value of a certain ratio of variables using data from the table?
- How would you find this ratio by graphing data? What do you think this ratio means?
- How would you use this information to calculate some other characteristic provided you research information about the other characteristic you need to know?

Students also could be encouraged to ask questions about particular pictures, evaluate them critically, as well as to make suggestions on how they would draw a picture describing what they read in the textbook. It could also be a useful learning exercise to ask students to draw a picture on reading material presented on certain pages in the textbook. Particularly, this exercise could be useful to visual learners. Even if a picture is just a simple illustration such as diagramming forces of gravity acting between two objects, the teacher could ask questions which would engage the students in an interactive exploration of the diagram presented. For example,

- How do you interpret this diagram?
- What other similar phenomena can this diagram describe?
- How would this diagram change if you add another object to the system?

- How would this diagram change if you change one of the characteristics presented on it?
- Can you draw your own images associated with gravity? What verbal explanations would you attach to these images?
- Research scientific magazines for images of gravity. What do these images tell you about gravity?
- Make your collection of cartoons about gravity after you research web sites on cartoons about gravity. What do these cartoons demonstrate about gravity?

Recommendations for Curriculum Development

Manitoba curriculum is used as an example to show what recommendations can be made based on the findings of the study. I do not intend to make recommendations for other curricular because the exploration of other curricular was not in the scope of my study, neither was it the focus of my research. Given limitations of the curriculum used in Manitoba, I suggest the following changes for the Manitoba curriculum in physics for the unit on universal gravitation:

- The mathematical component of physics should get special attention in learning outcomes where clear expectations on the role of mathematics in physics and the type of understanding to be developed would be specified.
- The curriculum outcomes should specify how the balance between qualitative and quantitative aspects of physics is expected to be achieved by clearly specifying experiences, activities, resources teachers could use. The curriculum outcomes should state that qualitative understanding of key concepts is expected in the physics course.

- Learning outcomes of the curriculum should include understanding the purpose of using a particular equation. It should be recommended to acquire this understanding through the use of the instructional approach to HPS which could provide significance to the mathematical equations in physics.
- The curriculum should stress the need for the dynamic emergence of a scientific law either during the presentation of the material by the teacher or by research by the student. The mathematics equations only then attain meaning and significance.
- Particular attention should be given to the reflection on the Newtonian approach to nature in curriculum outcomes. There should be a requirement in curriculum outcomes that teachers present all dimensions of Newtonian style and that students are expected to understand and improve on using them in their learning.

Recommendations for Further Study

The instrument developed in this study was designed for the analysis of the mathematical component of physics in high school and introductory level college physics textbooks. The chosen context was universal gravitation. I anticipate future researchers can utilize my instrument, with necessary modifications, in exploration of the mathematical component in any science discipline, for different levels of science textbooks, and in different science contexts. I also see the possibility for a team of researchers to construct a more general instrument for the analysis of the mathematical component of physics where the instrument would contain other categories that would emerge in the process of negotiation between many educators to reflect their multiple perspectives.

The research conducted provides some answers on how the role of mathematics is represented in physics education as reflected in high school and introductory level college physics textbooks in the context of the unit of universal gravitation. As any research, especially, one which is different from most research on textbooks (research of the mathematical component in physics textbooks) leaves many questions unanswered. I see the following possible directions for future research summarized in the following questions:

- How do university level physics textbooks represent the mathematical component of physics?
- How do physics textbooks present the mathematical component of physics in different contexts, other than in the topic of universal gravitation?
- How does the reported presentation of the mathematical component in physics textbooks actually match classroom presentations?
- How can a good textbook, where the presentation of the qualitative and the quantitative aspects of physics are balanced, affect learning outcomes?
- What is the correlation between the curriculum goals in terms of treatment of the mathematical component of physics and physics textbooks' presentation of mathematical concepts?
- What changes have to be made in both textbooks and curriculum guides to insure balance between the qualitative and the quantitative aspects of physics?
- What are the implications of the findings from this study for teacher education?
- How do the teachers view the role of mathematics in physics and in physics education?

- How do the teachers think students learn best about the mathematical component of physics?
- What strategies do the teachers think help insure balancing of the qualitative and the quantitative aspects of physics in the process of learning physics?
- How do students view the role of mathematics in physics?
- What are students' experiences with using mathematics in physics?
- What is the students' opinion on the ways of the best learning about the mathematical component of physics?

In closing, I offer a final suggestion. Writing a science textbook is a large scale endeavor requiring the effort of a team of educators, curriculum developers, scientists, and textbook writers. I see the future physics textbook as an interesting, engaging, and informative resource where units are designed based on the Newtonian Style of approaching nature. All contributors to the writing exhibit a clear vision on the role that mathematics plays in this textbook to insure balancing of the qualitative and the quantitative aspects of physics. This vision is presented and fulfilled by using rich contexts, activities and methods of presentation of the material.

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Appendices

Appendix A. *Instrument for Textbooks Analysis*

Categories	Descriptors	Inferences	Analytical Tools
Modes of Representation of Mathematical Concepts in the Law of Universal Gravitation • Numerical • Verbal • Graphical • Pictorial • Symbolic	Mathematical concepts are presented by • Numbers (in tables, charts) • Words • Graphs • Pictures • Algebraic symbols	Multiple representations help develop • Conceptual understanding • Enrich the presentation of physics knowledge • Establish connections between symbols and physical reality Limited representations cause lack of understanding of mathematical concepts.	Constructing a presence/absence of a particular mode matrix, and making judgments about use of different modes according to the following analytic rubric: • Limited use -only one or two modes are present • Moderate use – three or four modes are present • Extensive use - all five modes are present
Nature of Emergence of Mathematical Concepts in the Law of Universal Gravitation	Mathematical formulas, graphs, tables, and verbal formulations appear and are used		Identifying nature of emergence of a particular mode (based on *static/dynamic* descriptors)

Appendix A. (continued)

• *Static*	• without explanation, or discussion	• Mathematics, in the case of static emergence, is mostly used for memorization of formulas, tables, graphs, pictures • Verbal formulations do not complement each other • Connections between concepts could not be established
• *Dynamic*	• with information about background, experimental details, how mathematical relationship expressed in a	• Presentation of the mathematical relationship in the dynamic way could help change students' conceptions • Connections between concepts could be

Appendix A. (continued)

	• particular mode is determined, and accuracy of the relationship	• established	
Emergent Press (purpose) for a Particular Mode	Stating purpose of using a particular mode	• The replacement of concepts could be seen by students as useful and plausible • Mindless manipulations of mathematical equations occur in a context where the expressed need for such equations is missing	Identifying purpose of using a particular mode if there is one stated
Connections of Mathematical Concepts	• Moving between modes of representation	Students will likely make connections between concepts.	Constructing maps of tracking movements between modes of representation of concepts and evaluating variability of moves according to the

Appendix A. (continued)

			following analytic rubric: • Limited –movements are mostly linear, from one mode to another in a single direction with 1-2 back and forth movements between limited kinds of modes • Moderate – movements are mostly not linear, from one mode to another in back and forth directions, and mostly between same kinds of modes • Extensive- movements are not linear, from one mode to another, in back and forth directions between, mostly between different kinds of modes
	• Using analogies	Students would likely use relevant features and	Constructing a presence/absence of

Appendix A. (continued)

		ignore irrelevant ones when comparing and contrasting concepts what would help in the interrelationship of knowledge.	analogies matrix, and in case of presence, providing examples of analogies
Sequencing Mathematical Content: Simple → complex • Qualitative→quantitative	• From describing qualities (features) of observations, experiences, inferences to describing measurable quantities involved;	This way of presentation is in agreement with learning theory; therefore, meaningful presentation of the mathematical component of physics is likely to happen.	Using maps of tracking movements. The following rubric is applied: • Appropriate – the simple→complex sequence is used
• Verbal→algebraic	• From describing mathematical relationships in words to giving symbolic equations		

Appendix A. (continued)

Complex→simple • Quantitative→qualitative • Algebraic→verbal	• From presenting measurable quantities to describing features • From describing mathematical relationships in symbols and algebraic equations to describing mathematical relationships in words	This approach is not recommended by educational researchers since cognitive gaps could be formed if such approach is used; therefore, learning of mathematical concepts will be complicated.	• Not appropriate – the complex→simple sequence is used
Balancing Qualitative and Quantitative aspects of Physics in Presentation of Example Problems Problem Solving Approach: • qualitative • quantitative	The approach, where verbal explanations are engaged (conceptual problems) The approach, where calculations, symbolic equations are engaged	• Lack of problems engaging qualitative approach would impede students' learning • Engaging students in solving only	Analyzing the content of example problems and the approach taken to solve them. The following analytic rubric will be used to determine the extent of balancing:

Appendix A. (continued)

	quantitative problems would signify engagement of primitive levels of thinking which are not suitable for generating conceptual models • Qualitative reasoning combined with quantitative mechanism to communicate thinking strategies would help generate these models and make mathematics meaningful to students	• Limited – mostly quantitative approach is used with almost no qualitative reasoning • Moderate – algebraic equations, calculations are backed up by some qualitative explanations • Extensive – algebraic equations, calculations are backed up by detailed qualitative explanations

Appendix A. (continued)

Presentation of Mathematical Concepts through HPS:			Constructing presence/absence matrix (based on descriptors) featuring descriptive/instructional presentation of mathematical concepts through HPS, and in case of presence, providing illustrative examples
• Descriptive	Presentation of mathematical concepts referring to HPS with no students' assignments related to the historical context	If HPS is presented descriptively only, students get little exposure to the nature and methods of science and significance of mathematical equations in science could not be understood.	
• Instructional	Presentation of mathematical concepts referring to HPS with assignments related to the historical context and requiring from students doing exercises, completing projects, participating in discussions	If HPS is presented in instructional way, students will get exposure to the nature and methods of science by applying presented	

Appendix A. (continued)

		ideas during construction of conceptual models of their own, gaining experience to evaluate their conceptual models in terms of accuracy, simplicity, plausibility, predictability, and fruitfulness.	
Presentation of Mathematical Concepts Viewing Science as a Way of Thinking	• Illustrating the use of *assumptions*, *models*, and *thought experiments* in the presentation of history of the development of the concept of gravity • Discussing *evidence and proof* • Referring to *Newtonian Style* • Showing fecundity of	• Mathematics would be used as a conceptual tool in learning about gravity, and would facilitate students' construction of their own conceptual models • Mathematical formulations would get	Constructing presence/absence matrix (based on descriptors) featuring presentation of science as a way of thinking, and in case of presence, providing illustrative examples

Appendix A. (continued)

		mathematics • Referring to *Newton's geometry* and *calculus* for the description of gravity	significance and show their usefulness, fruitfulness, plausibility, and limitations, thus facilitating the process of conceptual change • Higher order thinking would be engaged in understanding NOS and help establish connection between mathematics and physics

Appendix B. *Nature of Emergence and Purpose for a Particular Mode of Presentation of Mathematical Component in the Unit Universal Gravitation*

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
GP-ZD	*Static/* **Displaying** planetary data, p. 178	*Dynamic/* **Explaining** Newton's reasoning about necessity for the force of gravity to obey inverse square law, hypothesis about the same nature of force acting between planets and the force causing objects to fall to the Earth, and applicability of Newton's 3rd law to mutually attracting objects, p. 181; Using ratios to explain how to apply the law, p. 182	*Static/* **Illustrating** the change in the force of gravity with distance from Earth, p. 183	*Static/* **Illustrating** - showing variation of the gravitational force when mass/distance changes, p. 182	*Dynamic/* **Usefulness: Establishing connections -deriving** Kepler's 3rd law from the law of universal gravitation; means of providing **new information** by obtaining **new relationships –deriving/ calculating** mass of planets, stars and galaxies, orbital speeds, period of satellites, acceleration on planets, pp. 183-188
CP-H	*Dynamic/* **Conceptual inferences -** using ratios of numbers	*Dynamic/* **Explaining** proportionality in the law, stressing that gravity is universal, p. 170, 172	*Dynamic/* **Posing question** about weight of an apple	*Dynamic/* **Posing questions** when modeling inverse-square law using butter	*Dynamic/* **Conceptual inferences -** using for analysis of proportionality, p. 172; **Usefulness:** means of

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
	representing distances of a falling apple and a falling moon from Earth's center to conclude that F_g dilutes with distance according to inverse square law, pp. 169-170		located at 3d, 4d, and 5d from the surface of the Earth, p. 176	spray example, p.175; **Posing question** about observation and reason using illusion check (different size of hands) example, p. 177	providing new information – **calculating** *G***,** mass of the Earth, pp. 173-174; **Values judgment -** stressing the beauty of the rule which made possible success in science that followed by providing **model of reasoning** (Newtonian Style), p. 179
P-NH	*Static/* **Illustrating** - showing *g* at different locations on the Earth and *g* and *W* on other planets	*Dynamic/* **Explaining** what factors affect $W(F_g)$, assumptions, the process of getting equation of	*Dynamic/* **Explaining** how weight changes with distance from Earth, stressing	*Dynamic/* **Explaining** how *W* depends on gravitational pull showing how it changes on different planets and	*Dynamic/* **Conceptual inferences -** going from proportionality algebraic statements $F_g \sim 1/r^2$ and $F_g \sim m_1m_2/r^2$ to algebraic equation $F_g = Gm_1m_2/r^2$; performing **dimensional analysis** to arrive at the unit of force *N* (canceling

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
	in a table, p. 164	gravitational law from proportionality statement and universality of the law, p. 161	that rapid change in weight is related to the inverse square relationship represented by the shape of the graph, p.160	**diagramming** the situation when the object's *W* is 0, p. 159; **Explaining** why the distance between two gravitating spheres is taken between their centers (reference to Newton's theorem and his calculus), p. 161	factors technique), p. 161; **Usefulness:** means of providing new information - **calculating** F_g, W; **deriving/calculating** orbits' size, orbital speed, period, pp.163-164, 213-216, 219; **obtaining ratios** such as $F_2/F_1=r_1^2/r_2^2$, pp. 165-166; **Establish connections - deriving** Kepler's 3rd law from the law of universal gravitation, p. 218 (margins)
P-E	*Static/* **Illustrating** –showing *g* at different	*Dynamic/* **Explaining** the emergence of the	-	*Dynamic/* **Explaining** how the relative	*Dynamic/* **Conceptual inferences** -performing **unit analysis**, p. 578; **Establishing connections -deriving**

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
	locations on Earth and other planets in a table, p. 132-133; *Dynamic/* **Interpreting** – showing relationship between orbital period and orbital radius for different planets in a table, and asking to make a graph of the data, study the shape of the graph and choose the	relationship in the law of universal gravitation reconstructing historically some of the steps of Newton's reasoning, pp. 577-578		strength of F_g at different distances from Earth's center could be represented with the lengths of arrows, p. 131; **Comparing** the intensity of physical phenomena that obey inverse square law to the spreading out of the surface of a sphere, asking to compare this property of the force of gravity	Kepler's 3rd law from the law of universal gravitation, p. 583; **proving** the equivalence of two ways of calculating g on the Moon showing that centripetal acceleration of the Moon in orbit is exactly equal to the acceleration provided by the force of gravity obeying inverse square law, pp. 632-633; **Usefulness:** means of providing new information by obtaining **new relationships – calculating** W, **deriving/calculating** mass of planets, orbital speeds, orbits' size, pp. 579, 585, 590-592;

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
	appropriate mathematical relationship, p. 573			to the electromagnetic force, p. 633	
P-G	*Static/–* **Illustrating** showing *g* at various locations on Earth, p. 121, and planetary data applied to Kepler's 3rd law in tables, p. 125; *Dynamic/* **Conceptual inferences -** using values for *a* of the	*Dynamic/* **Explaining** Newton's reasoning in arriving at his law of gravity, stating assumptions, referring to Newton's calculus, p. 118-119	-	*Static/* **Illustrating:** downward direction of F_g toward the Earth's center, p. 117; application of Newton's 3rd law in the process of obtaining the law of gravity, p. 118; information given for	*Dynamic/* **Conceptual inferences** -going from proportionality in algebraic algebraic equation $F_g = Gm_1m_2/r^2$, p. 142; **Analyzing** statements $F_g \sim 1/r^2$ and $F_g \sim m_1m_2/r^2$ to algebraic equation $F_g = Gm_1m_2/r^2$; **Usefulness: calculations** – F_g on different planets, pp. 119-120; means of providing new information –**deriving/calculating** mass of planets, stars and galaxies, orbital speeds, period of satellites, acceleration on planets, pp. 121-123, 916;

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
	Moon toward the Earth, *d* of the Moon from the Earth, and *g* of objects on the surface of the Earth, *r* of the Earth to conclude about inverse square law relationship of F_g, p. 118; analysis of the order of magnitude of F_g to explain small F_g between objects on the Earth			example problems, p. 120; orbits of satellites launched at different speeds, and showing how gravity affects the path of a satellite, p. 122	**Establishing connections -deriving** Kepler's 3rd law from the law of universal gravitation, p. 126

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
CCP-JC	*Dynamic/* **Conceptual inferences -** using ratios of values for *a* of the Moon toward the Earth to values of *g* of objects on the surface of the Earth, and ratios of *d* of the Moon from the Earth to *r* of the Earth to conclude about inverse square law relationship of F_g, p. 135; **Explaining discrepancy**	*Dynamic/* **Emphasizing** that gravity prompted Newton to write the *Principia* **Explaining** what factors affect F_g, the process of Getting equation of gravitational law from proportionality statement; **Explaining** Newton's reasoning in arriving at his law of gravity, stating	-	*Static/* **Illustrating:** information given in example problems, p.135, 143, 144, 148; **image** of an object described in example problem, p. 139; **information** about the way of measuring distance for spherical bodies of uniform density when finding F_g between them (reference to	*Dynamic/* **Conceptual inferences –** -going from proportionality in algebraic statements $F_g \sim 1/r^2$ and $F_g \sim m_1m_2/r^2$ to proportionalities, p. 142, 144 (Discussions); **Establishing connections -deriving** Kepler's 3rd law from the law of universal gravitation, p. 144; **Usefulness**: means of providing new information –**deriving/calculating** density of the earth, period of moon's orbit and an artificial earth-orbiting satellite, pp. 145-146, 148

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
	in the results of calculations of the period of the period of the moon's orbit, p. 146	assumptions, referring to Newton's calculus, pp. calculus, pp. 142-143		Newton's theorem and his calculus), p. 143	
CP-SF	*Static/* **Illustrating** – showing in tables g at different altitudes above the surface of the Earth, p. 198, and planetary data applied to Kepler's 3rd law, p. 204	*Static/* **Stating** the universal law of gravitation and **listing** features of the law, p. 194		*Static/* **Illustrating:** the forces of gravity between two objects are attractive and act in pairs, p.194; **image** of a star revolving at a speed v about the center of the galaxy being affected only	*Dynamic/* **Usefulness**: **calculating** – F_g (W), and g on different planets, pp. 197-198; means of providing new information –**deriving/calculating** mass of the Earth, p.197 and mass of the Sun, p. 204, height of a geosynchronous orbit, p.206; orbital speed of a satellite, pp. 206-207; **Establishing connections** - **deriving** Kepler's 3rd law from the law of universal gravitation, p. 203-204;**Conceptual inference** - **analyzing** formula for orbital

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
				by the mass inside its orbit, p. 196; **information given** in the example problems and **free body diagrams,** p. 197, 206	speed v to conclude about its independency on the mass of a satellite, p. 207
PM-TH	-	*Dynamic/* **Explaining** Newton's reasoning about the nature of the force of gravity providing brief history of gravity, p.97, and stressing a	-	*Static/* **Illustrating:** All objects, no matter how far they are from the Earth, experience force of gravity exerted by the Earth, p. 97; Two masses separated by a	*Static/* **Stating** the law of gravity without explanation of steps to arrive at it, p. 98; *Dynamic/* **Usefulness: deriving/calculating** g at the surface of the Earth, p. 101-102, radius of a geosynchronous orbit, p. 108; **deriving** orbit equation $v^2=GM/r$, p. 105; **Conceptual inferences -analyzing**

Appendix B. (continued)

Text	Numerical	Verbal	Graphical	Pictorial	Symbolic
		great triumph of the Newtonian picture of the world, p. 110; **Explaining** proportionality in the law, engaging verbal reasoning by **asking "what if" questions,** stressing that gravity is universal, p. 98		distance with gravitational forces shown between them, p. 98	formula for orbital speed *v* to conclude 1) that for a given distance r, between satellite and its central body, there is only one speed *v*, at which the satellite can move and remain in orbit, and 2) about independency of *v* on the mass of a satellite, p. 105

Note: - represents the absence of a feature

Appendix C. *Using Analogies in the Presentation of the Law of Universal Gravitation*

Text	Presence /Absence	Illustrative Example
GP-ZD	√	"One way to picture how space is affected by mass is to compare space to a large, two-dimensional rubber sheet…The yellow ball on the sheet represents a massive object. It forms an indentation. A marble rolling across the sheet simulates the motion of an object in space. If the marble moves near the sagging region of the sheet, it will be accelerated. In the same way, Earth and the sun are attracted to one another because of the space is distorted by the two bodies" (p. 192).
CP-H	√	"This law applies not only to the spreading butter from a butter gun, and the weakening gravity with distance, but to all cases where the effect from a localized source spreads evenly throughout the surrounding space. More examples are light, radiation, and sound" (p. 175).
P-NH	√	"The equation for Coulomb's law is very similar in form to Newton's universal law of gravitation equation. While the gravitational force depends on the masses of the objects, the electrostatic force depends on their charges. The constant of proportionality for each equation quantifies the difference between each type of force. In both equations, the force is inversely proportional to the square of the distance between the two bodies and directly proportional to the product of the *property* of the object governed by that law (i. e., charge or mass)" (p. 540).
P-E	√	"The force of gravity exerts its influence over very long distances and is the same in all directions, suggesting that the influence extends outward like a spherical surface…How does this property of the force of gravity compare to electromagnetic force" (p. 633)? "…just as Newton was able to develop the mathematics (calculus) that proved that the mass of any spherical object can be considered to be concentrated at a point at the centre of the sphere for all locations outside the sphere, so it might also be proven that if charge is uniformly distributed over the surface of a sphere, then the value of the charge can be considered to be acting at the centre

Appendix C. (continued)

		for all locations outside the sphere" (p. 637).
P-G	–	
CCP-JC	√	"A simple way to visualize the object's motion is to imagine it represented by a bead sliding without friction along a wire bent into the shape of the potential-energy curve. An upward push sends the bead along the wire from r_e to a, where it stops and slides back down to r_e. The bead loses speed as it goes from r_e to a and regains speed as it returns. This analogy can be quite useful, but you must remember that the actual motion of the object is along a straight line directed radially away from the earth" (p. 221). "The idea that a mechanical universe obeyed a single set of laws suggested that all observed behavior could be explained in mechanical terms. Kepler's word *clockwork* is often used to describe this viewpoint" (p. 89).
CP-SF	–	
PM-TH	–	

Note: √ represents the presence of a feature
– represents the absence of a feature

Appendix D. *Problem Solving Approaches in the Example Problems on the Law of Universal Gravitation*

Text	Illustrative Example	Type of Reasoning	Extent of Balancing
GP-ZD	"A satellite orbits Earth 225 km above its surface. What is its speed in orbit and its period?"(p. 187). Strategy: "Determine the radius of the satellite's orbit by adding the height to Earth's radius… Use the velocity equation… Use the definition of velocity to find the orbital period… Rearrange and solve for *T*" (p. 187).	quantitative	**L**
CP-H	"Suppose that an apple at the top of a tree is pulled by Earth's gravity with a force of 1 N. If the tree were twice as tall, would the force of gravity on the apple be only ¼ as strong? Explain your answer" (p. 176). Answer: "No, because the twice-as-tall apple tree is not twice as far from Earth's center. The taller tree would have to have a height equal to the radius of Earth (6370 km) before the weight of the apple would reduce to ¼ N. Before its weight decreases by 1 %, an apple or any object must be raised 32 km – nearly four times the height of Mt. Everest, the tallest mountain in the world. So as a practical matter we disregard the effects of everyday changes in elevation" (p. 176).	qualitative and quantitative	**E**
	"How does g at the surface of Jupiter compare with g at the surface of Earth? Data: Jupiter's mass is about 300 times that of Earth, and its radius is about 10 times greater than the radius of Earth" (p. 184) Answer: For Earth, $g = GM/R^2$. The value of g on Jupiter's surface is $G\ (300M)/(10R)^2 = 300\ GM/(100\ R^2) = 3\ GM/R^2$, or 3 times Earth's g. (More precisely, Jupiter's $g = 2.44$ times Earth's g because its radius is nearly 11 times that of Earth" (p. 184).	quantitative	
P-NH	"A 1000-kg satellite is the payload on a planned shuttle launch. What is its weight 32000 km from Earth's surface?" (p. 165). Strategy: "It is not always necessary to use the universal formula for gravitation to obtain a weight	qualitative and quantitative	**E**

Appendix D. (continued)

	value of an object significantly above a planet's surface…we can solve the problem by finding the ratio between the two forces at the two different locations. The following variables remain the same: G, m_{Earth}, $m_{\text{satellite}}$. The radius r can also be simplified for the second case. r_2, the distance of the satellite from Earth, is 32000 km + 6400 km = 38400 km above the centre of Earth. (Remember, we always use distances from the centre of the object.) r_2 as a ratio of r_1 is 38400 km/6400 km = 6. Therefore, $r_2 = 6r_1$. Similarly, we can take a ratio of the weights, F_1 and F_2. The common factors cancel out… $F_2/F_1 = 1/36$. We can find F_1 easily by using $F = mg$. Therefore, the weight of the satellite on Earth is 1000 kg x 9.8 m/s^2 = 9800 N. From our ratio, $F_2 = F_1/36$ = 272 N… In general, you can do problems like this very quickly by using some fundamental logic. If the object is farther away from the planet, its weight will decrease. If the object moves closer, its weight increases. The factor used to relate the two weights is the square of the ratio of the distances from the centre of the planet" (p. 166).		
P-E	"A 65.0 kg astronaut is walking on the surface of the Moon, which has a mean radius of 1.74 x 10^3 km and a mass of 7.35 x 10^{22}kg. What is the weight of the astronaut?" (p. 579). Solution: **"Frame the Problem** • The weight of the astronaut is the gravitational force on her. • The relationship $F_g = mg$…*cannot* be used in this problem, since the astronaut is not on Earth's surface. • The law of universal gravitation applies to this problem. **Identify the Goal…** **Variables and Constants…** **Strategy** Apply the law of universal gravitation…Substitute the numerical values and solve… The weight of the astronaut is approximately 105 N. Validate Weight on the Moon is known to be much less than	quantitative and some qualitative	**M**

Appendix D. (continued)

	that on Earth. The astronaut's weight on the Moon is about one sixth of her weight on Earth (65.0 kg x 9.81 m/s^2 ≈ 638 N), which is consistent with this common knowledge" (p. 579).		
P-G	"What is the force of gravity acting on a 2000-kg spacecraft when it orbits two Earth radii from the Earth's center (that is, a distance r_E = 6380 km above the Earth's surface…)? The mass of the Earth is M_E = 5.98 x 10^{24} kg. **Approach** We could plug all numbers into Eq…, but there is a simpler approach. The spacecraft is twice as far from the Earth's center as when it is at the surface of the Earth. Therefore, since the force of gravity decreases as the square of the distance (and $1/2^2 = ¼$), the force of gravity on the satellite will be only one-fourth its weight at the Earth's surface. **Solution** At the surface of the Earth, $F_G = mg$. At a distance from the Earth's center of $2r_E$, F_G is ¼ as great: $F_G = ¼\, mg$ = ¼ (2000 kg)(9.80 m/s^2) = 4900 N " (p. 120).	qualitative and quantitative Note: no explanation given why M_E was not needed in the problem	**M**
CCP-JC	"Consider a mass m falling near the earth's surface. Find its acceleration g in terms of the universal gravitational constant G, and draw some conclusions from the form of the answer. **Solution** The gravitational force on the body is $F = GmM_E/r^2$,… We have already noted that the gravitational force on a body at the earth's surface is $F = mg$. Setting the two expressions for the gravitational force on m equal to each other, we get $mg = GmM_E/r^2$, or $g = GM_E/r^2$. Both G and M_E are constants, and r does not change significantly for small variations in height near the surface of the earth. Thus, the right-hand side of this equation does not change appreciably with position on the earth's surface. For this reason we may replace r with the average radius of the earth R_E to get $g = GM_E/R_E{}^2$. **Discussion** The law of gravitation predicts that the acceleration due to gravity of an object at the earth's surface is approximately constant and does	quantitative and qualitative	**E**

Appendix D. (continued)

	not depend on the mass of the object. Experimentally we know that g does not vary appreciably from one place to another. This constancy of g is just what Galileo found. Thus the law of universal gravitation not only describes the forces that hold the planets in their orbits, but also describes the forces on objects close to the earth" (pp. 143-144).		
CP-SF	"A student drops a ball and considers why the ball falls to the ground as opposed to the situation in which the ball stays stationary and the Earth moves up to meet it. The student comes up with the following explanation: "The Earth is much more massive than the ball, so the Earth pulls much harder on the ball than the ball pulls on the Earth. Thus, the ball falls while the Earth remains stationary." What do you think about this explanation? **Explanation** According to Newton's universal law of gravity, the force between the ball and the Earth depends on the product of their masses, so both forces, that of the ball on the Earth, and that of the Earth on the ball, are equal in magnitude. This follows also, of course, from Newton's third law. The ball has large motion compared to the Earth because according to Newton's second law, the force gives a much greater acceleration to the small mass of the ball" (p. 195). "Derive an expression that shows how the acceleration due to gravity varies with distance from the center of the Earth at an exterior point. **Solution** ...assume that the ball is located at some arbitrary distance r from the Earth's center. The first equation in Example..., with r replacing R_E and m_b removed from both sides, becomes $g = GM_E/r^2$ This indicates that the free-fall acceleration at an exterior point decreases as the inverse square of the distance from the center of the Earth. Our assumption... that objects fall with constant acceleration is obviously incorrect in light of the present example. For short falls, however, this change in g is so small that neglecting the variation does not introduce a significant error in the result.	qualitative qualitative and quantitative	**E**

Appendix D. (continued)

	Because the weight of an object is *mg*, we see that a change in the value of *g* produces a change in the weight of the object. For example, if you weigh 800 N at the surface of the Earth, you will weigh only 200 N at a height above the Earth equal to the radius of the Earth. Also, we see that if the distance of an object from the Earth becomes infinitely large, the weight approaches zero…" (p. 198)		
PM-TH	"Suppose that the distance between the Earth and the Sun suddenly doubled. How much would the mass of the Sun have to increase in order to keep the force of gravity between the two the same? The force of gravity depends inversely on the square of the distance between objects, so doubling the distance would cause a decrease in the force of gravity by a factor of four. The force of gravity depends directly on the mass of each object, so the mass of the Sun would have to be four times as big as it is now to compensate for the larger distance" (p. 98).	qualitative	**M**
	"Given that the mass of the Earth is 6×10^{24} kilograms, what is the acceleration due to gravity at the Earth's surface? *Reasoning and solution:* To answer this question, we just have to put the Earth's mass and radius into the expression for *g* given previously. $g = G \times M_E / R_E^{\ 2} \ldots = 9.8$ meters/second^2 This number is the same constant that Galileo and others measured. Notice that because we now understand where *g* comes from, we can predict the appropriate value of gravitational acceleration not only for the Earth, but also for any object in the universe, provided we know its mass and radius" (p. 102).	quantitative with some qualitative discussion	

Note: L represents limited balancing
M represents medium balancing
E represents extensive balancing

Appendix E. *Presentation of Mathematical Concepts through HPS*

Text	Descriptive Approach/Illustrative Examples	Instructional Approach/Illustrative Examples
GP-ZD	"About 400 years ago, Galileo wrote in response to a statement that "gravity" is why stones fall downward, *What I am asking you for is not the name of the thing, but its essence, of which essence you know not a bit more than you know about the essence of whatever moves stars around...we do not really understand what principle or what force it is that moves stones downward*" (p. 175).	"Research and describe the historical development of the concept of gravitational force. Be sure to include Kepler's and Newton's contributions to gravitational physics" (p. 197).
CP-H	"Using geometry, Newton calculated how far the circle of the moon's orbit lies below the straight-line distance the moon otherwise would travel in one second...His value turned out to be about the 1.4-mm distance accepted today. But he was unsure of the distance between Earth and the moon, and whether or not the correct distance to use was the distance between their centers. At this time he hadn't proved mathematically that the gravity of the spherical Earth (and moon) is the same as if all its mass were concentrated at its center. Because of this uncertainty, and also because of criticisms he had experienced in publishing earlier findings in optics, he placed his papers in a drawer, where they remained for nearly 20 years" (p. 170).	–
P-NH	"Besides his fertile imagination, Newton possessed a truly exceptional capacity for mathematics. Starting from Descartes' analytical geometry, Newton invented calculus to be able to solve problems in motion that Galileo had posed...The publication of Newton's *Mathematical Principles of Natural Philosophy (Principia)* in 1687 marked the firm establishment of a physics in which	–

Appendix E. (continued)

	careful experimental measurements provided full support for imaginative mathematical theories" (p. 123).	
P-E	"Halley first met Newton in 1684…He asked Newton what type of path a planet would take if the force attracting it to the Sun decreased with the square of the distance from the Sun. Newton quickly answered, "An elliptical path". When Halley asked him how he knew, Newton replied that he had made that calculation many years ago, but he did not know where his calculations were. Halley urged Newton to repeat the calculations and send them to him" (p. 577).	"Check out the Internet site above to read about – and perhaps even test – Galileo's arguments about logic refuting Aristotle's teachings concerning falling objects. Galileo actually wrote the words! He presented the arguments using two fictitious characters. Salviati voiced the beliefs of Galileo, while Aristotle's ideas were embodied in Simplicio. If you enjoy a good debate, this English translation will captivate you" (p. 133).
P-G	"Galileo's analysis of falling objects made use of his new and creative technique of imagining what would happen in idealized (simplified) cases. For free fall, he postulated that all objects would fall with the *same constant acceleration* in the absence of air or other resistance…To support his claim that falling objects increase in speed as they fall, Galileo made use of a clever argument: a heavy stone dropped from a height of 2 m will drive a stake into the ground much further than will the same stone dropped from a height of only 0.2 m. Clearly, the stone must be moving faster in the former case" (p. 31).	–
CCP-JC	"It was experiments like this, rather than the legendary Tower of Pisa experiment, that led Galileo to state"…I declare that I wish to examine the essentials of motion of a body that leaves from rest and goes with speed always increasing…uniformly with the growth of time…I prove that the spaces passed by such a body to be in the squared ratio of the time…" (p. 46).	–
CP-SF	"He wrote, "I deduced that the forces which keep the planets in their orbs must be reciprocally as squares of their	–

Appendix E. (continued)

	distances from the centers about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with force of gravity at the surface of the Earth; and found them answer pretty nearly" (pp. 193-194).	
PM-TH	"In 1684, Halley visited Newton at Cambridge. Newton told him over dinner that according to his calculations, all bodies subject to a gravitational force would move in orbits shaped like ellipses. Bolstered by this information, Halley analyzed the historical records of some 24 comets…He found that three recorded comets – those that had appeared in 1531, 1607, and 1682 – seemed to be following the same orbit. He realized that the sightings represented not three separate comets, but one comet that was appearing over and over again at intervals of about 75 or 76 years…After some work, Halley predicted that the comet would reappear in 1758. On Christmas day 1758, an amateur astronomer in Germany sighted the comet coming back toward Earth. This so-called recovery of what is now known as Halley's comet marked a great triumph for the Newtonian picture of the world" (p. 110).	"In what sense is the Newtonian universe simpler than Ptolemy's? Suppose observations had shown that the two did equally well at explaining the data. Construct an argument you would make to say that Newton's universe should still be preferred" (p. 116). "Read a biography of Pierre Simon Laplace, who was one of history's most influential scientists. What were his major achievements? What major historical events occurred during his lifetime? How did his research influence his philosophical ideas?" (p. 116).

Note: – represents the absence of a feature

Appendix F. *Illustrative Examples of Presentation of Mathematical Concepts in the Unit on Universal Gravitation Viewing Science as a Way of Thinking*

Text	Feature	Illustrative Example
GP-ZD	Illustrating the use of assumptions	"Newton's law of universal gravitation leads to Kepler's third law. In the derivation of this equation, it is assumed that the orbits of the planets are circles. Newton found the same result for elliptical orbits" (p. 183).
	Describing thought experiments	"Newton used a drawing… to illustrate a thought experiment on the motion of satellites. Imagine a cannon, perched high atop a mountain, firing a cannonball horizontally with a given horizontal speed. The cannonball is a projectile, and its motion has both vertical and horizontal components. Like all projectiles on Earth, it fallows a parabolic trajectory. During its first second of flight, the ball falls 4.9 m. If its horizontal speed were increased, it would travel farther across the surface of Earth, but it would still fall 4.9 m in the first second of flight. Because the surface of Earth is curved, it is possible for a cannonball with just the right horizontal speed to fall 4.9 m at a point where Earth's surface has curved 4.9 m away from the horizontal. This means that, after one second, the cannonball is at the same height above Earth as it was initially. The curvature of the projectile will continue to just match the curvature of Earth, so that the cannonball never gets any closer or farther away from Earth's curved surface. When this happens, the ball is said to be in orbit" (p. 185).
	Illustrating the use of models	"One way to picture how space is affected by mass is to compare space to a large, two-dimensional rubber sheet… The yellow ball on the sheet represents a massive object. It forms an indentation. A marble rolling across the sheet simulates the motion of an object in space. If the marble moves near the sagging region of the sheet, it will be accelerated. In the same way, Earth and the sun are attracted to one another because of the way space is distorted by the two bodies" (p.192).

Appendix F. (continued)

	Presenting evidence and proof	"Newton was able to state his law of universal gravitation in terms that applied to the motion of the planets about the sun. This agreed with Kepler's third law of planetary motion and provided confirmation that Newton's law fit the best observations of the day" (p. 182). Derivation of Keplers's third law follows (pp. 182-183). "Thus, Newton's law of universal gravitation leads to Kepler's third law" (p. 183).
	Demonstrating fecundity of mathematics	Calculating mass of the Earth (pp. 183-184), speed of an object in circular orbit, period for satellite circling Earth (p. 186).
	Reference to Newton's mathematical tools (geometry)	"Newton's drawing shows that Earth curves away from a line tangent to its surface at a rate of 4.9. m for every 8 km" (p. 186).
CP-H	Illustrating the use of assumptions	Our treatment of tides is quite simplified here. We have ignored such complications as interfering land masses, tidal inertia, and friction with the ocean bottom, all of which result in a wide range of tides in different parts of the world" (p. 192).
	Describing thought experiments	"...imagine a hole drilled completely through Earth, say from the North Pole to the South Pole. Forget about impracticalities such as lava and high temperatures, and consider the kind of motion you would undergo if you fell into such a hole. If you started at the North Pole end, you'd fall and gain speed all the way down to the center, and then overshoot and loose speed all the way to the South Pole. You'd gain speed moving toward the center, and lose speed moving away from the canter. Without air drag, the trip would take nearly 45 minutes" (pp. 184-185).
	Illustrating the use of models	"If a ball of taffy is swung on the end of a string, it deforms, with "tidal bulges" on the inner and outer sides. Although the actual Earth-moon interaction differs from this simplified model, the result is similar. Both the taffy and Earth are elongated. Earth elongation is evident in the pair of ocean bulges on opposite sides of Earth" (p. 188). "Using Newtonian physics as a model of reason, Locke

Appendix F. (continued)

	Presenting evidence and proof	and his followers modeled a system of government that found adherents in the 13 British colonies across the Atlantic. These ideas culminated in the Declaration of Independence and the Constitution of the United States of America" (p. 179). "Newton's test was to see if the moon's "fall" beneath its otherwise straight-line path was in correct proportion to the fall of an apple or any object at Earth's surface. He reasoned that the mass of the moon should not affect how it falls, just as mass has no effect on the acceleration of freely falling objects on Earth. How far the moon falls, and how far an apple at Earth's surface falls, should relate only to their respective *distances* from Earth's center. If the distance of fall for the moon and the apple are in correct proportion, then the hypothesis that Earth's gravity reaches to the moon must be taken seriously" (p. 169).
	Demonstrating fecundity of mathematics	"The successes of Newton's ideas ushered in the Age of Reason or Century of Enlightenment. Newton had demonstrated that by observation and reason, people could uncover the workings of the physical universe. How profound it is that all the moons and planets and stars and galaxies have such a beautifully simple rule to govern them, namely, $F = G\, m_1 m_2/d^2$ The formulation of this simple rule is one of the major reasons for the success in science that followed, for it provided hope that other phenomena of the world might also be described by equally simple and universal laws" (p. 179).
	Reference to Newton's mathematical tools (geometry)	"Using geometry, Newton calculated how far the circle of the moon's orbit lies below the straight-line distance the moon otherwise would travel in one second…" (p. 170).
P-NH	Illustrating the use of assumptions	"The constant K in Kepler's equation assumes that the orbiting mass is infinitesimally small. If the mass of the orbiting object is a significant fraction of the larger mass, then the equation $KT^2 = r^3$ is invalid because the objects orbit about the *center of mass* of the two objects, known as the **barycentre**, not about the larger

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		mass" (p. 219).
	Presenting evidence and proof	"Kepler tried various geometric solutions. When he tried elliptical paths, the predictions agreed remarkably well with observations" (p. 218) Derivation of Kepler's third law from Newton's law of gravity (p. 218)
	Demonstrating fecundity of mathematics	Calculating geosynchronous earth orbit (p. 213), obtaining equation for the orbital speed, calculating orbital speed (pp. 215-216).
	Reference to Newton's Mathematical Tools	"Newton possessed a truly exceptional capacity for mathematics. Starting from Descartes' analytical geometry, Newton invented calculus to be able to solve problems in motion that Galileo posed" (p. 123).
P-E	Illustrating the use of assumptions	"At first glance, it would appear to have little relationship to Newton's law of universal gravitation, but a mathematical analysis will yield a relationship. To keep the mathematics simple, you will consider only circular orbits. The final result obtained by considering elliptical orbits is the same, although the math is more complex" (p. 583).
	Describing thought experiments	"Soon after Newton formulated his law of universal gravitation, he began thought experiments about artificial satellites. He reasoned that you could put a cannon at the top of an extremely high mountain and shoot a cannon ball horizontally… The cannon ball would certainly fall toward Earth. If the cannon ball traveled far enough horizontally while it fell, however, the curvature of Earth would be such that Earth's surface would "fall away" as fast as the cannon ball fell" (p. 588).
	Presenting evidence and proof	"…since Kepler published his laws, there has never been a case in which the data for the movement of a satellite, either natural or artificial, did not fit an ellipse" (p. 576). "The values of acceleration due to gravity that were calculated in two completely different ways are in full agreement. The centripetal acceleration of the Moon in orbit is exactly what you would expect it to be if that acceleration was provided by the force of gravity and if

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		the force of gravity obeyed an inverse square law" (p. 633).
	Demonstrating fecundity of mathematics	"The laws of Newton and Kepler, however, have provided scientists and astronomers with a solid foundation on which to explain observations and make predictions about planetary motion, as well as send space probes out to observe all of the planets in our solar system" (p. 594).
	Reference to Newton's mathematical tools (geometry)	"From the size and curvature of Earth, Newton knew that Earth's surface would drop by 4.9 m over a horizontal distance of 8 km" (p. 588)
P-G	Illustrating the use of assumptions	"When extended objects are small compared to the distance between them (as for the Earth-Sun system), little inaccuracy results from considering them as point particles" (p. 119) "Kepler's third law applies only to objects orbiting the same attracting center" (p. 126).
	Illustrating the use of models	"Galileo's analysis of falling objects made use of his new and creative technique of imagining what would happen in idealized (simplified) cases. For free fall, he postulated that all objects would fall with the *same constant acceleration* in the absence of air or other resistance…To support his claim that falling objects increase in speed as they fall, Galileo made use of a clever argument: a heavy stone dropped from a height of 2 m will drive a stake into the ground much further than will the same stone dropped from a height of only 0.2 m. Clearly, the stone must be moving faster in the former case" (p. 31).
	Presenting evidence and proof	"Newton also showed that for any reasonable form for the gravitational force law, only one that depends on the inverse square of the distance is fully consistent with Kepler's laws. He thus used Kepler's laws as evidence in favor of his law of universal gravitation…" (p. 125).
	Demonstrating fecundity of mathematics	Geophysical applications (p. 121), calculating the Sun's mass (p. 127), estimating our Galaxy's mass (p. 916).

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	Reference to Newton's mathematical tools (calculus)	"For an extended object (that is, not a point), we must consider how to measure the distance *r*. This is often best done using integral calculus, which Newton himself invented" (p. 119).
CCP-JC	Illustrating the use of assumptions	"We have derived Kepler's third law only for uniform circular motion of the planet, but the result is true for elliptical orbits if we use the average distance from the sun for *r*" (p. 144). "The result obtained here is slightly greater than the observed period 27.3 days. The discrepancy occurs because we assumed the moon to orbit around a stationary earth. In actuality they both move about a common point that is near, but not at, the center of the earth" (p. 146).
	Illustrating the use of models	"A simple way to visualize the object's motion is to imagine it represented by a bead sliding without friction along a wire bent into the shape of the potential-energy curve. An upward push sends the bead along the wire from r_e to a, where it stops and slides back down to r_e. The bead loses speed as it goes from r_e to a and regains speed as it returns. This analogy can be quite useful, but you must remember that the actual motion of the object is along a straight line directed radially away from the earth" (p. 221).
	Presenting evidence and proof	"Example 5.11 Show that Kepler's third law follows from the law of universal gravitation. [Recall from Chapter 1 that Kepler's third law states that for all planets the ratio $(\text{period})^2/(\text{distance from sun})^3$ is the same.]" (p. 144). Solution follows (p. 144).
	Demonstrating fecundity of mathematics	"Newton's laws are so correct that they have not been modified in the more than 300 years since the *Principia* was first published. Though they do not completely describe the interactions between galaxies or between subatomic particles, these laws cover almost everything in between. The thrust of an airplane's jet engine, the paths of baseballs and comets, how to best hit a tennis ball, and how musical instruments work all can be understood on the basis of Newtonian mechanics" (p. 89).

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	Reference to Newton's mathematical tools (calculus)	Calculating density of the earth (pp. 145-146), period of an artificial earth-orbiting satellite (p. 148), radius of a black hole (p. 223). "Newton's mathematical work …eventually led to his discovery of calculus, with which he proved that the mass of a symmetrical object of uniform density behaves under the law of universal gravitation exactly as if it were concentrated at the point of the object's center of symmetry" (p. 143).
CP-SF	Illustrating the use of assumptions Presenting evidence and proof Demonstrating fecundity of mathematics	"…the free-fall acceleration at an exterior point decreases as the inverse square of the distance from the center of the Earth. Our assumption of Chapter 2 that objects fall with constant acceleration is obviously incorrect in light of the present example. For short falls, however, this change in *g* is so small that neglecting the variation does not introduce a significant error in the result" (p. 198). "Newton later demonstrated that these laws are consequences of a simple force that exists between any two masses. Newton's universal law of gravity, together with his laws of motion, provides the basis for a full mathematical solution to the motions of planets and satellites. More important, Newton's universal law of gravity correctly describes the gravitational attractive force between *any* two masses" (pp. 201-202). Calculating orbital velocity (p. 196), mass of the Earth (p. 197), free-fall acceleration at different altitudes (p. 198)
PM-TH	Presenting evidence and proof Demonstrating fecundity of mathematics	Deriving formula $g = G \, x \, M_E/R_E^2$ and concluding the following: "This result is extremely important. For Galileo, *g* was a number to be measured, but whose value he could not predict. For Newton, on the other hand, *g* was a number that could be calculated purely from the size and mass of the Earth" (p. 101). Deriving the orbit equation (pp. 104-105), calculating geosynchronous orbits (pp. 107-108).

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	Reference to Newton's mathematical tools (calculus)	"The problem of deducing the shape of an orbit under these circumstances is a difficult one, but one that could be dealt with using the new mathematics of calculus, which was invented independently by Isaac Newton and the German mathematician Gottfried Leibniz" (p. 110).