THE UNIVERSITY OF MANITOBA

OPTIMIZATION OF NONLINEAR PULSE-FREQUENCY MODULATED CONTROL SYSTEMS

By

© SHADIA A. ELGAZZAR

A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements for the Degree of
Doctor of Philosophy

Department of Electrical Engineering

Winnipeg, Manitoba

June, 1981

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ABSTRACT

Pulse-frequency modulated (PFM) control systems are systems in which the control effort is a sequence of standard pulses. Variations in the control effort are accomplished by changes in the polarity of, and the time between, successive pulses.

The objective of the present study is to find the optimum control function for PFM systems with nonlinear plants.

First, the modified maximum principle (MMP) is proved to be applicable to nonlinear open-loop systems. A method for obtaining the optimum control function for such systems is then developed and illustrated by an example.

Using the MMP, an optimal feedback controller and a modulator are designed for systems with quadratic performance indices and fixed final time, and hence the optimal control sequence is obtained.

The problem of finding an optimal controller for PFM systems containing a modulator of the second kind with fixed final time is also studied, and the optimal control is found. For this case, the two-point boundary-value problem resulting from the split boundary conditions could not be solved using conventional methods. A successful method of solution for the initial values of the costates is presented. This method combines analytical and numerical techniques and is demonstrated for a system containing a first order linear plant and an integral pulse-frequency modulator. A second method, based on computer simulation, is also investigated and its limitations discussed.

ACKNOWLEDGEMENTS

I gratefully acknowledge the help and encouragement I have received from my supervisor, Dr. S. Onyshko, during the course of this study.

Part of this work was completed while working at the National Research Council of Canada. I like to express my thanks to Mr. F.V. Cairns of the Electronics section, Division of Electrical Engineering for allowing me to take the time to complete this work. Special thanks are due to Mr. D. O'Hara for his continuous support and encouragement. I also wish to express my appreciation to the staff and colleagues in the Electronics and the Computer Graphics sections for their support, and to the staff of the drafting office for preparation of the figures.

I am also grateful to the National Research Council of Canada for the financial support received while at the University of Manitoba.

Special gratitude to my husband whose help and encouragement have contributed much to the successful completion of this work, also to my son and daughter for their great sacrifices during the course of the study.

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LIST OF SYMBOLS

$\underline{A}$	Constant matrix
$\underline{B}$	Constant vector
d	Minimum allowable dead time between pulses
$\underline{f}(x,u,t)$	Time derivative of $\underline{x}(t)$
$F(x,u,t)$	Integral of the performance index J
H	Hamiltonian function
I	Time integral of H
I^*	Optimized integral using the MMP
J	Performance index to be optimized
M	Pulse magnitude
N	Number of pulses in the interval $(0,t_f)$
N_u	Maximum allowable number of pulses
$\underline{p}$	Variable costate vector
$\underline{p}_0$	Initial value of the costate vector
$\underline{Q}$	Positive semi-definite matrix
$\underline{r}$	Reference state vector
s	Threshold in Type-II modulators
S	Pontryagin function
t	Time
t_i	Time of initiation of the ith pulse
t_f	Final time
t_0	Initial time

t_{iL}	Lower limit on t
t_{iH}	Upper limit on t
T	Sampling interval
u	Control function
u^*	Optimum control function
$u_s(.)$	Step function
$\underline{x}$	State vector
$\underline{x}_0$	Initial state vector
z	Modulated parameter
α	Cost applied on pulses
η	Modulating function
τ	Pulse width
$\delta(.)$	Dirac delta function

Chapter I

INTRODUCTION

There has been, in recent years, an increasing interest in the use of pulse-frequency modulated systems. Engineers are attracted by their effectiveness and simplicity in engineering realization, mathematicians by the novelty and peculiarity of the problem, and physiologists by the possible analogies with certain processes taking place in the neuronal nets.

There are three basic types of modulation schemes used in nonlinear pulsed controllers. If the width of the pulses, initiated at equal intervals, is varied with the signal, pulse-width modulation (PWM) is obtained. On the other hand, if the distance of the pulses varies from their reference positions in the equally spaced pulse train, one gets the pulse-position modulation (PPM). In the third case, pulse-frequency modulation (PFM) is obtained by representing the modulating signal by a series of identical pulses of variable spacing, and possibly, different polarities.

This dissertation is concerned with the optimization of pulse-frequency modulated control systems. The modified maximum principle is extended to include systems with nonlinear plants. The optimal controller and the modulator for systems containing nonlinear plants with a quadratic performance index are found. Also studied is the problem of finding the optimal feedback controller for systems containing a modulator of the second kind. A method of solution for the resulting two-point boundary value problem (TPBVP) is developed and demonstrated for a system containing a first order linear plant and an integral pulse-frequency modulator.

In this chapter, pulse-frequency modulation and the existing types of pulse-frequency modulators are presented. Also discussed are the configurations of the control systems containing a pulse-frequency modulator and possible uses of PFM control systems.

The previous work involving PFM is presented in Chapter 2 along with a detailed description of the modified maximum principle (MMP) as applied to linear separable plants. In Chapter 3, the admissible controls are defined and the problem considered in this dissertation is formulated.

1.1 PULSE-FREQUENCY MODULATION

PFM is a form of pulse-time modulation where the modulating signal, which in the case of control systems often is the error signal, is represented by a series of identically shaped pulses. The information is contained in the time between the initiation of these pulses and possibly their polarities. In most practical cases, the shape of the pulses is of minor significance. Although rectangular pulses of predetermined magnitude and width are used in this study, it would require minor modification to consider other pulse shapes. The frequency is limited to some maximum value since it is not desired to consider the case where the pulses overlap. Usually, a pulse generator only produces one pulse at a time; a reset time called the dead time is also required before the generation of the next pulse. However, in the case of two consecutive pulses of different polarity, the overlapping of pulses may be considered. This would be the case when different pulse generators are used for the different polarities.

In the three modulation schemes mentioned earlier, the pulse representation of the signal is a nonlinear process. Of the three schemes, PFM is probably the most difficult since the sampling frequency is completely signal dependent and the modulating process is nonlinear. This precludes the application of existing sampled-data theory, and introduces complexity to the analysis.

1.2 TYPES OF MODULATORS

There are basically two types of PFM schemes commonly known as PFM of the first and second kind. Systems using pulse-frequency modulators of the first kind (type-I) are systems in which the sampling interval T_k, that is the interval between the kth and the (k+1)th pulses, depends on the value of the modulating function $\eta(t)$ at the kth time instant t_k, i.e. $T_k = f(\eta(t_k))$. A block diagram which represents this operation is shown in Figure 1.1. The pulse element in Figure 1.1 is the one discussed by Clark [4]. The output pulses will occur starting at each sampling instant. The values of η between the time instants t_k are immaterial for this type of modulator. The instantaneous value $\eta(t_n)$ immediately determines the pulse interval T_n which will follow the time instant t_n. The polarity of the pulse at t_n is usually the same as the sign of $\eta(t_n)$. A number of possible functions $f(\eta)$ are proposed in [4].

Systems using pulse-frequency modulators of the second kind (type-II) are systems in which the modulated parameter $z(t)$ is a function of the modulating function $\eta(t)$. Modulators of this type emit a pulse whenever $z(t)$ reaches a preset threshold s, which may vary with time, then a reset operation is performed on $z(t)$. Systems of this kind have been widely investigated.

Since the optimization of systems containing type-II modulators is one of the topics addressed in this dissertation, a more detailed description of type-II modulators is presented next.

A modulator of the second kind is a sigma pulse-frequency modulator, which is a generalization of the well known integral pulse-frequency modulation (IPFM) scheme. Figure 1.2 presents the block diagram of such modulator. The

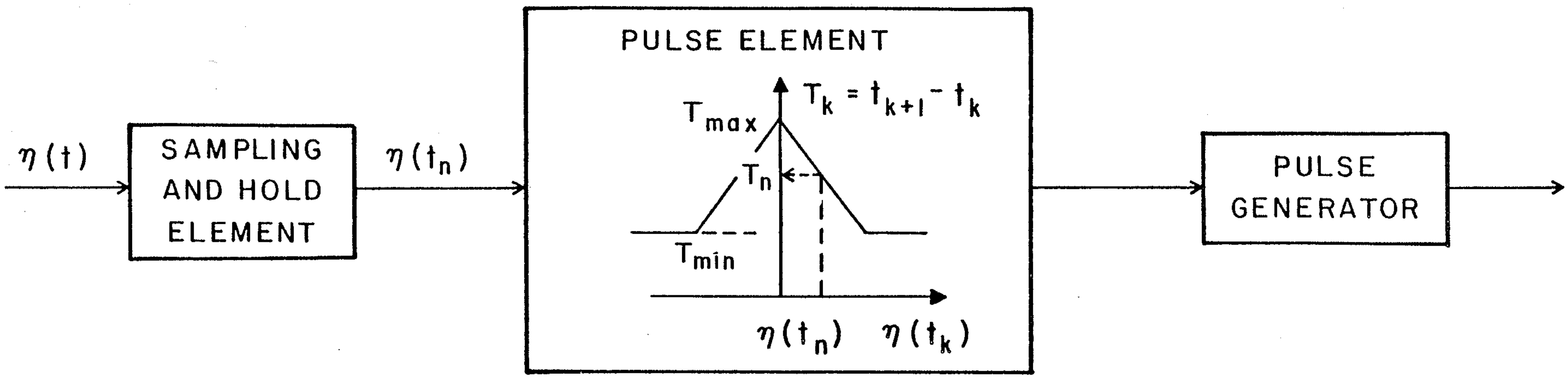


Figure 1.1 Type-I pulse-frequency modulator

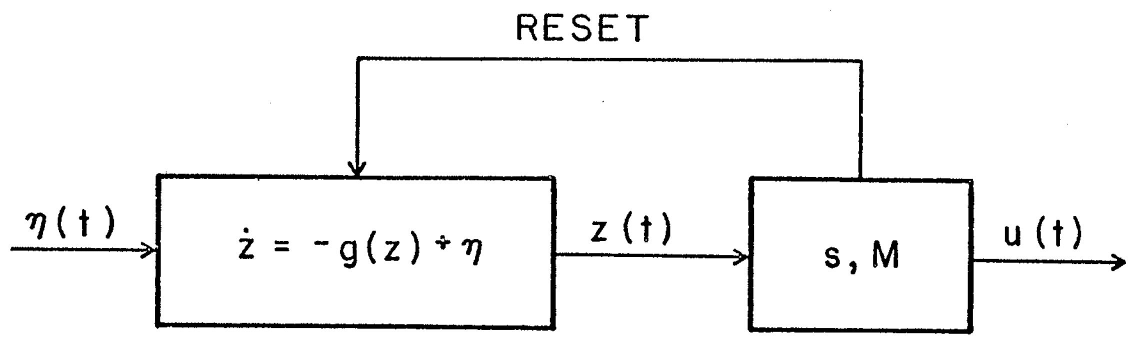


Figure 1.2 Block diagram of a Sigma-PFM

modulating signal $\eta(t)$ is fed to a first order lowpass filter (possibly nonlinear or time varying) that decides on the emission of a pulse when its output reaches a certain level s. The equations describing the ΣPFM are

$$\dot{z} + g(z) = \eta(t) - s\ \mathrm{sgn}(z)\ \delta(|z|-s) \tag{1.1}$$

$$u(t) = W\ \mathrm{sgn}(z)\ \delta(|z|-s) \tag{1.2}$$

where $z(t)$ is the modulated parameter and $u(t)$ is the train of pulses (delta functions) of strength W emitted whenever $|z| = s$. The second term on the right-hand side of Equation (1.1) represents the resetting of $z(t)$ to the zero value [38]. If $g(z)$ is linear in z, Equation (1.1) represents the relaxation pulse-frequency modulator. If, on the other hand, $g(z)$ is set to zero, the IPFM scheme is obtained.

1.3 PULSE-FREQUENCY MODULATED CONTROL SYSTEMS

In this section, an open loop and a closed loop configuration of control systems containing a pulse-frequency modulator are presented.

For the open-loop system, Figure 1.3, the control law is known and does not depend on subsequent changes of the states of the controlled plant. The open-loop structure may result from an optimal control approach and may be feasible for systems that, for all practical purposes, can be

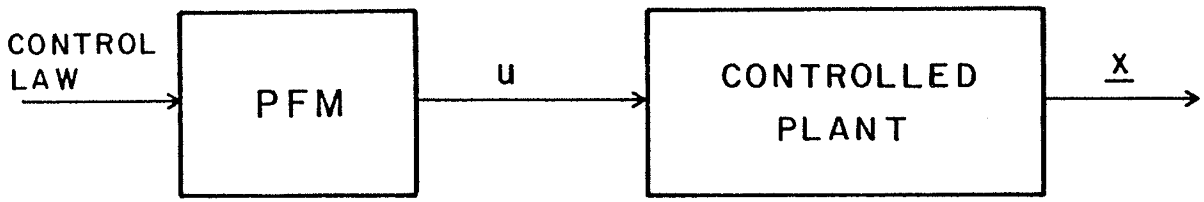


Figure 1.3   An open-loop PFM control system

considered deterministic. The disadvantages of the open-loop control are that external disturbances or changes in the controlled plant cannot be recognized. Hence, in some cases, it isnecessary to have a secondary control system to monitor the output of the plant and readjust the control law from time to time.

For the closed-loop system, Figure 1.4, the control system consists of a comparator, a pulse-frequency modulator, a controlled plant, and a feedback element. The error signal $\eta(t)$ is modulated to give a pulse sequence $u(t)$. This error depends on subsequent changes in the state vector $\underline{x}$ of the controlled plant. That is, the closed-loop system will sense a disturbance and act accordingly.

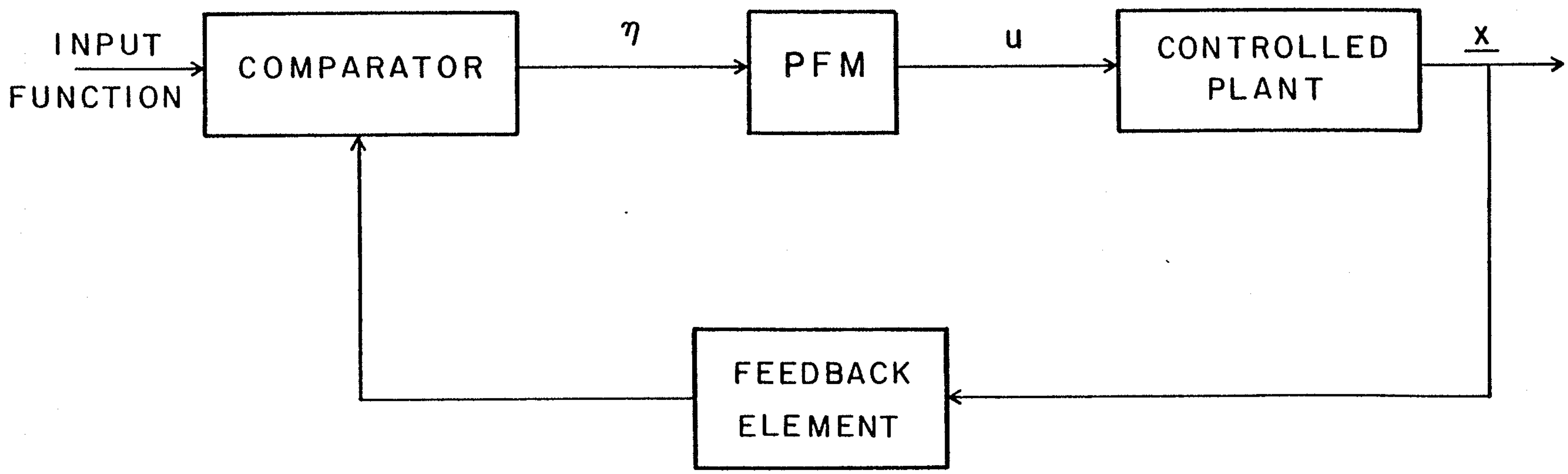


Figure 1.4 A closed-loop PFM control system

1.4 SOME APPLICATIONS OF PULSE-FREQUENCY MODULATION

Pulse-frequency modulation has had many applications as a means of coding information. More recently, this form of modulation has been used to generate pulse sequences for control purposes.

PFM possesses the special features of high noise immunity [15, 16, 24, 50] and uniform pulse size. Because of this immunity PFM can be used in control systems where certain parts are contaminated by strong noise. The uniform pulse size appears to have distinct advantage when precise fuel metering is desired.

Because of its convenience in implementation PFM is used in adaptive control [2, 33] and in attitude control of spacecraft [10]. PFM is also used for conservation of control energy. In remote control systems, pulses may be generated by the system itself and only the triggering is remote. This is advantageous when the application of pulses requires more energy than what can be transmitted economically from the controlling source. Another application is in systems where the fuel consumption of the controlling gear is critical. This type of controlling system, for example in spacecrafts with limited supplies, uses much less fuel than a continuous system. Kolk [23] indicated that using PFM signals to drive actuators in a feedback control system has the advantage of conserving control energy over PWM. Rochelle [41] concluded that for applications of maximum range with the requirement for low weight and low power drain, PFM performs most satisfactorily. PFM has been successfully employed as the encoding technique in a number of small U.S. earth satellites.

PFM has also been applied to the control of thyristors [3, 40] and in optical communication systems including recently fibre optics communication systems [6, 49].

Another motivation for the study of such systems lies in their potential promise in providing a model for a number of functions of the nervous systems. It was noticed that the neural signal in a single nerve fiber is a train of pulses of approximately constant magnitude and uniform shape but its instantaneous frequency varies with the strengh of the stimulation. PFM is thus believed to exist in the neural communication networks of physiological control systems [18, 36, 46]. A number of models for neurons and neural nets have been proposed [1, 36, 45].

PFM has been used to develop a model for the automatic control of the natural pacemaker frequency [17, 32]. It was also used in the realization of a hardware digital device for the assessment of heart-rate variability which resulted in a great increase in speed for processing electrocardiogram signals recorded on tape [5, 42].

Chapter II

BACKGROUND

The design of PFM control systems has not been based on optimizing a given performance index except for the work of few researchers. In this chapter these works will be discussed along with a brief resume of a previous research in the field. In addition, the modified maximum principle (MMP) of Onyshko and Noges [35] will be discussed in length in Section 2.2 of this chapter since in the present dissertation an extension to their work is presented.

2.1 PREVIOUS WORK

Since the appearance of the first work on PFM [43] in 1949, many theoretical and practical aspects of this scheme of modulation have been investigated. Ross [43] was mainly concerned with the reconstruction of sinusoidal signals from a PFM approximation for communication applications.

The next few articles on PFM originated from studies of physiological systems. Jones, Li, Meyer and Pinter [18] studied these systems and found the existence of pulse trains along nerve fibers that possess both logarithmic and direct relationships between frequency and stimulus intensity. Integral pulse-frequency modulation (IPFM), in which a pulse is emitted whenever the integral of the input reaches a preset threshold, was proposed as a likely mechanism and was studied in detail by Li [29], Li and Jones [30], and Meyer [31]. Meyer examined the application of IPFM to control systems in general and emphasized that the IPFM modulator is not intended to directly simulate the physiological type of PFM but is defined , from an engineering standpoint, as a modulator that lends itself to mathematical analysis. In addition to IPFM, Meyer [31] has defined the relaxation pulse-frequency modulator (RPFM) which represents a first order relaxation oscillator. Pavlidis and Jury [38] considered a generalization of IPFM and referred to it as Sigma PFM (ΣPFM) which has some advantages over other schemes, such as IPFM. The most significant advantages are improved stability and simpler physical implementation of the modulator. IPFM application to space vehicle attitude control has been discussed by Farenkoff, Sabroff and Wheeler [10].

Several investigators studied the stability of single modulator pulse-frequency systems (PFS), for example [4, 7, 10, 11, 18, 19, 20, 27, 29, 31, 38, 47, 48], but few results have been presented for multi-modulator systems [12, 13, 14, 26]. The importance of the latter kind of systems arises from the presence of interconnections of pulse-frequency modulators in the nervous system. The methods of solution used in most of these studies were Lyapunov's second method

[4, 10, 20, 27] and the frequency response criteria (Popov's method) [7, 11, 19]. Another approach, namely the direct application of basic functional properties of the system equations, has been discussed in [13]. Besides the stability of pulse-frequency systems, Pavlidis and Jury [38] studied the dynamic response analysis of PFM. The existence of sustained oscillations was also studied by developing a quasi-describing function for the modulation.

Limited work has been done in the area of optimum PFM control systems. Pavlidis [39] presented a solution of the minimum time and the minimum fuel problems for PFM systems . The derivation of the optimal control was done by heuristic arguments. The problem of finding the optimum control function for PFM systems was considered by Onyshko and Noges [35] using two different techniques: the modified maximum principle (MMP), and the dynamic programming. The MMP is based on Pontryagin's maximum principle and is applicable to open loop systems with linear plants and fixed operating time. The performance index is assumed to be a linear combination of the final values of the state variables but not necessarily of the control function. The dynamic programming is applicable to linear and nonlinear open- and closed-loop systems. However, to make this technique applicable, a restriction had to be placed on the control function. This restriction, which is required to make the performance index Markovian, allows the pulses to be initiated only at predetermined instants of time. A computational method to obtain the optimal control in PFM systems was presented by Elgazzar in [8]. This method is used in parts of this thesis work.

Upper and lower bounds on the optimal values of the performance index were established by Vander Stoep and Alexandro [51]. A gradient method for the optimization of PFM systems was discussed in [28] and [34]. Using this method, which is applicable to linear and nonlinear systems, Lermontov [28] obtained the same numerical results as those obtained by the MMP. Nardizzi and El-Hakeem [34] presented an algorithm for the case where the number of pulses is known apriori. They used the discrete PFM (DPFM) technique described in [51] to obtain this information.

Optimization of DPFM systems was investigated by Kuhler and Yeh [25]. DPFM control sequences were obtained under total variation constraint to ease computation. By choosing to remove the freedom of shifting the control pulses on the time axis, the computation was simplified.

Since PFM shares certain similarities with PWM, some literature in this field is of considerable interest. The problem of determining optimal PWM control sequences for both linear and specific nonlinear plants was considered by Kirk [22] and by Yeh and Kuhler [52]. An extended maximum principle [53, 54, 55] was applied to obtain the conditions of optimality for systems whose control signals are restricted to partially specified functions, provided they are piecewise continuous and admit small variations.

2.2 MODIFIED MAXIMUM PRINCIPLE

The modified maximum principle was developed by Onyshko and Noges [35] to overcome the difficulty resulting from the nature of the control function in PFM. The control function is "not controllable" for a period of time following the initiation of each pulse. The optimization problem considered is as follows:

Consider the system described by the state equations,

$$\dot{\underline{x}} = \underline{f}(\underline{x},u,t) = \underline{\underline{A}}\underline{x} + \underline{\phi}(u) \tag{2.1}$$

where

$\underline{x}$ is an n-state vector

$\underline{A}$ is an nxn matrix

$\underline{\phi}(u)$ is an n-vector valued function of u

and u(t) is a scalar control.

Given the initial condition $\underline{x}(0)$, the final time t_f and the the Pontryagin function S, which is assumed to be a linear combination of the final values of the state variables, i.e

$$S = \sum_{i=1}^{n} c_i \, x_i(t_f) , \tag{2.2}$$

it is then required to find the admissible control sequence such that S is maximized (minimized).

Optimizing S does not exclude the problem of optimizing the system with respect to an integral provided that the integrand is linear with respect to the state variables, but not necessarily with respect to the control function. This optimization can be carried out by augmenting the system with a new state variable x_{n+1} representing the integrand, and by choosing the constants c_i in Equation (2.2) as

$$c_1 = \ldots. = c_n = 0 \qquad \text{and} \qquad c_{n+1} = 1 .$$

The function S becomes

$$S = x_{n+1}(t_f) = J . \tag{2.3}$$

Onyshko and Noges [35] have shown that a pulse sequence is optimal for the above problem if the MMP is satisfied. The MMP is stated in the following theorem.

Theorem

For the control function u(t) to be optimum, that is, for it to maximize (minimize) the Pontryagin function S, it is necessary and sufficient that the scalar function

$$I^*(u) = \int_0^{t_f} \sum_{i=1}^{n} p_i(t)\ \phi_i(u)\ dt \tag{2.4}$$

be minimized (maximized) by the optimum control function. The costate functions $p_i(t)$ are defined by

$$\dot{p}_i(t) = -\sum_{j=1}^{n} p_j \frac{\partial f_j}{\partial x_i} , \quad p_i(t_f) = -c_i . \qquad (2.5)$$

The MMP as developed here is useful in optimizing open-loop linear systems.

The MMP has reduced the original optimization problem to the problem of extremizing the function $I^*(u)$ with respect to admissible control sequences. Extremizing that function, which is determined by inspection or trial and error, is not easy for nontrivial problems.

In order to find the optimum control sequence, Elgazzar [8] developed an iterative method to solve Equation (2.4). A detailed description of this method and a listing of the flowcharts and the FORTRAN programs may be found in [8].

Chapter III

PROBLEM DEFINITION

In this chapter, the admissible controls and the control system for this study are defined, and the problem treated in this dissertation is formulated.

3.1 CONTROL PROBLEM

3.1.1 Admissible controls

Since the control function $u(t)$ is restricted to be a series of standard pulses having a minimum dead time between any two consecutive ones, the variation δu of the control function must be of a special form.

This variation shifts pulses by infinitesimal times along the time axis, but does not vary their size or shape. These pulses are assumed to be continuous except possibly at the end points. Consider an arbitrary admissible control function $u(t,\lambda)$, where λ is an arbitrary small number, such that

$$\lim_{\lambda \to 0} u(t,\lambda) \triangleq u(t) \tag{3.1}$$

and

$$u(t,\lambda)-u(t) = \delta u. \tag{3.2}$$

The control $u(t,\lambda)$ is said to be in the λ neighbourhood of $u(t)$ if

$$\int_{t_0}^{t_f} |u(t,\lambda)-u(t)|dt < K|\lambda| \tag{3.3}$$

where K is a positive constant, t_0 is the initial time, and t_f is the final time.

For the case where the control function $u(t)$ is made up of a series of rectangular pulses, the variation $\delta_k u$ in the kth pulse is of the following form

$$\delta_k u = \begin{cases} -M\,\mathrm{sgn}(\delta t_k) & \text{on } (t_k, t_k+\delta t_k] \text{ if } \delta t_k>0 \\ & \quad\;\; (t_k+\delta t_k, t_k] \text{ if } \delta t_k<0 \\ +M\,\mathrm{sgn}(\delta t_k) & \text{on } (t_k+\tau, t_k+\tau+\delta t_k] \text{ if } \delta t_k>0 \\ & \quad\;\; (t_k+\tau+\delta t_k, t_k+\tau] \text{ if } \delta t_k<0 \\ 0 & \text{otherwise.} \end{cases}$$

where M is the pulse magnitude, τ the pulse width, t the time of initiation of the kth pulse, $\delta t_k = \lambda\Delta_k$ the variation in the kth pulse, and Δ_k a finite constant.

3.1.2 Control system

Assume that the dynamics of the system to be controlled optimally are represented by a set of n first-order differential equations

$$\dot{x}_i = f_i(\underline{x},u,t), \quad x_i(t_0) = x_{i0} \tag{3.4}$$

for $t_0 \leq t \leq t_f$ and $i = 1, 2, \ldots, n$, where $\underline{x}$ is the state vector, u is the admissible control and the functions f_i are assumed to be continuous in u. It is desirable to extremize the Pontryagin function

$$S = \sum_{i=1}^{n} c_i \; x_i(t_f) \tag{3.5}$$

with respect to the control function u(t). This Pontryagin function S is general enough to include an integral performance index,

$$J = \int_{t_0}^{t_f} F(\underline{x},u,t)\, dt$$

by augmenting the system with a new state variable, such that

$$\dot{x}_{n+1} = F(\underline{x},u,t), \quad x_{n+1}(t_0) = 0. \tag{3.6}$$

Thus the appropriate Pontryagin function S is given by

$$S = \sum_{i=1}^{n+1} c_i \; x_i(t_f) \tag{3.7}$$

where

$$c_i = 0 \quad \text{for} \quad i = 1, \ldots., n, \qquad c_{n+1} = 1.$$

3.2 PROBLEM STATEMENT

As mentioned in Chapter 2, little work has been done in the optimization of PFM systems and in the implementation of feedback controllers. The objective of the present work is to investigate the problem of finding the optimal control of PFM systems. This includes:

1. the extension of the modified maximum principle (MMP) to nonlinear systems and the development of an off-line method to obtain the optimal control,
2. the determination of optimal PFM control sequences for a quadratic performance index and the implementation of the on-line controller and the modulator, and
3. the determination of the optimal control sequences for a system containing a pulse frequency modulator of the second kind.

In Chapter 4, the applicability of the MMP to nonlinear systems is proved and a method for obtaining the optimal control is developed. The optimal controller for PFM systems with a quadratic performance index is found in Chapter 5. The optimal controller for systems containing a modulator of the second kind is presented in Chapter 6. In Chapter 7, methods for the solution of the resulting TPBVP are developed and an illustrative example presented.

Chapter IV

MODIFIED MAXIMUM PRINCIPLE FOR NONLINEAR PFM CONTROL SYSTEMS

The modified maximum principle for optimally controlling PFM systems, as proposed by Onyshko and Noges [35], is applicable to open-loop systems with linear plants and fixed operating time. In this chapter the MMP is proved to be applicable to systems with nonlinear plants. The admissible controls as described in the previous chapter are included in the proof of the extended MMP which is based on Rozonoer's work [44]. A procedure for finding the optimal control is developed and demonstrated by an illustrative example. Some of the computer programs used in this part of the study were developed and listed in [8]. Listings of the additional programs used here are given in Appendix B.

4.1 MMP FOR NONLINEAR PFM CONTROL SYSTEMS

In the present section , the MMP is extended to include nonlinear open-loop PFM systems with respect to the admissible controls defined in Chapter 3.

An expression for the total variation δS in the Pontryagin function S due to a variation δu in the control function u(t) can be found in [44] as

$$\delta S = - \int_{t_0}^{t_f} (H(\underline{x},\underline{p},u+\delta u,t) - H(\underline{x},\underline{p},u,t))dt - \varepsilon , \qquad (4.1)$$

where H is defined as

$$H = \sum_{i=1}^{n} p_i \ f_i$$

and

$$|\varepsilon| \leqslant A \left(\int_{t_0}^{t_f} |\delta u\,(z)\,|dz\right)^2, \qquad (4.2)$$

where A is a constant. Defining I(u) as

$$I(u) = \int_{t_0}^{t_f} H(\underline{x},\underline{p},u,t)dt, \qquad (4.3)$$

δS can be written as follows

$$\delta S = - (I(u+\delta u) - I(u)) - \varepsilon \ .$$

Consider the following theorem

<u>Theorem</u>

For the control function u(t) to maximize the Pontryagin function S, it is necessary that I(u) be minimized by u(t).

One way to prove this theorem is to show that $|\varepsilon|$ is negligible with respect to the first term on the right-hand side of Equation (4.1) which will be denoted by β. By virtue of the continuity of H in u, and assuming that all the derivatives of H with respect to u exist, Taylor's formula can be applied to the integrand of β, yielding

$$\beta = \int_{t_0}^{t_f} (a_1 \delta u + a_2 \delta u^2 + \ldots. + a_n \delta u^n + \ldots.)dt, \qquad (4.4)$$

where $a_n = \frac{1}{n!} H^{(n)}(\underline{x},\underline{p},u,t)$, δu is an admissible variation and $H^{(n)}$ is the nth derivative of the Hamiltonian.

Generally, δu consists of infinitesimal shifts in a number of pulses and Equation (4.4) sums the effects of these shifts. To show that $|\varepsilon|$ is negligible, it will be shown that this is true for infinitesimal shifts in a single pulse; thus, it is true for shifts in any number of pulses.

Consider the variation δu due to an infinitesimal shift δt in a pulse of general shape. This δu is different from zero for a duration of time $\Delta \equiv \tau + |\delta t|$ where τ is the pulse width. Accordingly, Equation (4.4) can be written in the form

$$\beta = \int_{\Delta} (a_1 \delta u + \ldots.. + a_n \delta u^n + \ldots.)dt. \qquad (4.5)$$

The integral in Equation (4.5) can be expressed in three parts successively over the intervals P, Q, and R (Figure 4.1). The intervals P and R are equal to $|\delta t|$ and Q is equal to $\tau - |\delta t|$. For the first and last intervals, since the integrand is bounded, the integral is of the order of δt. In the middle region, the upper bound on $|\delta u|$ is equal to $|D_{max} \delta t|$, and the lower bound to $|D_{min} \delta t|$, where D_{max}

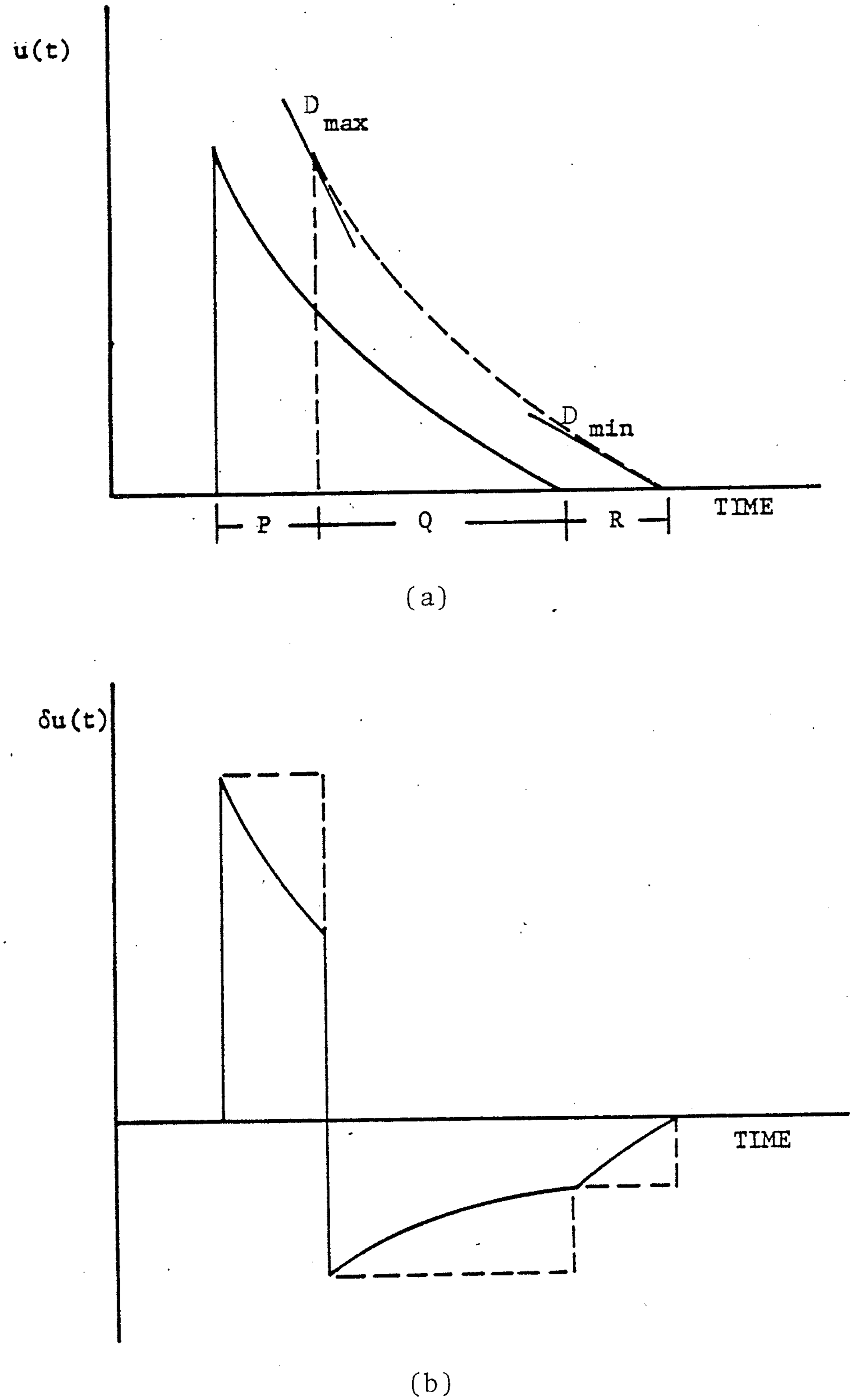


Figure 4.1 (a) Typical pulse, (b) the variation δu for negative δt.

and D_{min} are the maximum and minimum magnitudes of the slopes of the pulse, respectively. Since, the upper and lower bounds are proportional to δt and $|\delta u|$ is bounded by these two values, then the integral over this region is also of the order of δt. Therefore,

$$\beta = O(\delta t). \tag{4.6}$$

Recalling, from Section 3.1, that

$$\int_{t_0}^{t_f} |u(t,\lambda) - u(t)| dt < K |\lambda|$$

and substituting into Equation (4.2), $|\varepsilon|$ can be written as follows

$$|\varepsilon| < K^* (\delta t)^2 \tag{4.7}$$

where K* is a constant. Comparing Equations (4.6) and (4.7), it can be argued that $|\varepsilon|$ is negligible as δt approaches zero. Thus

$$\delta S = -(I(u+\delta u)-I(u)). \tag{4.8}$$

For the case where S is to be maximized, the total variation of S, due to a variation δu from the optimum control $u^*(t)$, must satisfy

$$\delta S \leqslant 0 .$$

From Equation (4.8), it is seen that the necessary condition for the above to be true is that

$$I(u+\delta u)-I(u) \geqslant 0 .$$

Thus, for a Pontryagin function S to be maximized, I(u) should be minimized and vice-versa. This completes the proof of the theorem.

If the state equations of the plant are given in a more specific form, δS can be simplified. A form of such functions, which describes significant classes of nonlinear systems, is

$$\dot{x}_i = g_i(\underline{x})+\phi_i(u)+m_i(\underline{x})\ell_i(u). \tag{4.9}$$

Consider the next lemma:

<u>Lemma</u>

The control u(t) which minimizes the function

$$G(u) = \int_{t_0}^{t_f} \sum_{i=1}^{n} p_i(t)(m_i(\underline{x})\ell_i(u) + \phi_i(u))dt \tag{4.10}$$

also minimizes the function I(u), defined in Equation (4.3).

Proof: By substituting Equation (4.3) into Equation (4.8) and letting

$$\int_{t_0}^{t_f} \sum_{i=1}^{n} p_i(t)(m_i(\underline{x})\ell_i(u) + \phi_i(u))\, dt$$

equal to G(u), the following equation is obtained

$$\delta S = -(G(u+\delta u) - G(u)). \tag{4.11}$$

Comparing Equations (4.8) and (4.11) it is seen that the variations in I(u) and G(u), due to δu, are identical. Therefore, the control u(t) which minimizes G(u) also minimizes I(u). This completes the proof of the lemma.

A special case of Equation (4.9) is obtained when the term $m_i(\underline{x})\ell_i(u)$ does not exist. In this case, G(u) is of the simple form

$$G(u) = \int_{t_0}^{t_f} \sum_{i=1}^{n} p_i(t)\, \phi_i(u)\, dt\, . \tag{4.12}$$

4.2 SOLUTION PROCEDURE

Given the system described by

$$\dot{x}_i = f_i(\underline{x},u,t), \quad x_i(t_0) = x_{i0}, \quad t_0 \leqslant t \leqslant t_f\, ,$$

the costate equations and their boundary conditions can be found using

$$\dot{p}_i = -\sum_{j=1}^{n} p_j \frac{\partial f_j}{\partial x_i}\, , \quad p_i(t_f) = -c_i\, .$$

Thus a set of first-order differential equations is obtained for the state and costate equations which, unfortunately, has split boundary conditions. Since these equations represent a nonlinear two-point boundary-value problem, numerical techniques have to be used to solve them. Another difficulty is that these equations depend on the optimal control $u^*(t)$ which is not known initially.

To circumvent these difficulties the following off-line method to obtain the optimal control is developed. First, the function $I(u)$ for the system under consideration is formed. Then, the control $u_0(t)$ that satisfies the constraints

$$u_0(t) = \pm M, \quad t_0 \leqslant t \leqslant t_f \tag{4.13}$$

and optimizes $I(u)$ is found. M is the mean value of the pulse magnitude and the form of $u_0(t)$ is that of a bang-bang process. We use $u_0(t)$ as an initial guess of $u^*(t)$ for the PFM system. An iterative numerical technique is then used to solve the state and costate equations and thus obtain $\underline{x}^{(j)}(t)$ and $\underline{p}^{(j)}(t)$, where j is a running index indicating the iteration number. These functions, $\underline{x}^{(j)}(t)$ and $\underline{p}^{(j)}(t)$ are used to obtain the control $u^{(j)}(t)$, which is a series of pulses that optimizes $I(u)$. The time of initiation of the pulses as well as their number N are obtained using the method suggested by the author in a previous study [8, 9]. The termination criterion used requires the differences in the time of application of corresponding pulses in two consecutive control functions to be within a prescribed limit.

The procedure is again summarized in the following steps:

1. form $I(u)$ for the given system;
2. find $u_0(t)$ that optimizes $G(u)$ and set the iteration index j to equal one;
3. find corresponding $\underline{x}^{(j)}(t)$ and $\underline{p}^{(j)}(t)$;
4. find $u^{(j)}(t)$ that optimizes $I(u)$;
5. check if the termination criterion is satisfied. If it is, the optimal control $u^*(t)$ is obtained; if not, increase j by one and return to step 3.

An example illustrating this method of solution is given below. The method of variation of extremals [21] was used to solve the nonlinear two-point boundary-value problem. The problem was simulated using the S/360 Continuous System Modeling Program (S/360 CSMP). The example is the one used in a previous study [9] but a nonlinearity is added. This helps to compare the results of the two cases.

Example

Given the second order system described by the differential equation

$$\ddot{x} + \dot{x} + x + ax^2 = u, \quad \dot{x}(0) = x(0) = 0, \quad 0 \leqslant t \leqslant t_f, \tag{4.14}$$

it is required to maximize

$$J = \dot{x}(t_f) - \int_0^{t_f} \alpha |u| dt, \tag{4.15}$$

where α is a weighting constant. That is, it is required to minimize G(u).

Letting $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = J$, the state equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - x_1 - ax_1^{\,2} + u \\ \dot{x}_3 &= -x_2 - x_1 - ax_1^{\,2} + u - \alpha|u| \end{aligned} \tag{4.16}$$

where $x_1(0) = x_2(0) = x_3(0) = 0$. Comparing Equations (4.16) and (4.9), then $\phi_1 = 0$, $\phi_2 = u$ and $\phi_3 = u - \alpha|u|$. The constants in the Pontryagin function are $c_1 = c_2 = 0$ and $c_3 = 1$.

The auxiliary variables p_i must satisfy

$$\begin{aligned} \dot{p}_1 &= (1 + 2\,a\,x_1)p_2 + (1 + 2\,a\,x_1)p_3 \\ \dot{p}_2 &= -p_1 + p_2 + p_3 \\ \dot{p}_3 &= 0 \end{aligned} \tag{4.17}$$

where $p_1(t_f) = p_2(t_f) = 0$, and $p_3(t_f) = -1$. Then G(u) becomes (see Equation (4.12))

$$G(u) = \int_0^{t_f} ((p_2(t) - 1)u + \alpha|u|)\,dt.$$

Thus, $u_0(t)$ that optimizes G(u) is

$$u_0(t) = \begin{cases} -M \operatorname{sgn}(p_2(t)-1) & \text{if} \quad |p_2(t)-1| \geqslant \alpha \\ 0 & \text{if} \quad |p_2(t)-1| < \alpha \end{cases} \tag{4.18}$$

This $u_0(t)$ is used in the computer program as the initial guess for $u^*(t)$. Using rectangular pulses and the following numerical values,

t_f = 10 s final time τ = 0.2 s pulse width

M = 0.1 V pulse magnitude d = 0.3 s dead time

three cases are considered, which are

Case 1. No cost on pulses (that is α = 0) and a = 0.1

Case 2. No cost on pulses and a = 2.

Case 3. A cost is applied on pulses (α = 0.2) and a = 2.

Tables 4.1, 4.2 and 4.3 include, for the three cases, the polarity and time of initiation of each pulse constituting the control u(t). These tables contain the results of each iteration in the process of finding the optimal control $u^*(t)$ which is given in the final iteration. The state variables, the function $P(t) = p_2(t)-1$ and the optimal control $u^*(t)$ for the three cases considered are illustrated in Figures 4.2, 4.3, and 4.4.

In case 1, the parameter 'a' was given a small value to make the system close to the linear case presented in a previous study [9]. It is observed that the control that maximizes the performance index is the same except for the first three pulses where the differences in the time of application are 0.02 s.

As noticed in the figures, $x_1(t)$ is much smoother than $x_2(t)$; this is due to the fact that $x_2(t)$ is affected more directly by the pulses in $u^*(t)$. The variable $x_1(t)$ is the integral of $x_2(t)$; this provides for more low-pass filtering. Also, since the initial conditions are zero, the states shown in Figure 4.4(a), are zeros as long as there is no input.

TABLE 4.1

Time of Pulse Initiation (s) and Pulse Polarities in Parentheses - Case 1.

Pulse	Iteration Number		
Number	1	2	3
1	0.02(-)	0.02(-)	0.02(-)
2	0.52(-)	0.52(-)	0.52(-)
3	1.02(-)	1.02(-)	1.02(-)
4	1.79(+)	1.80(+)	1.80(+)
5	2.29(+)	2.30(+)	2.30(+)
6	2.79(+)	2.80(+)	2.80(+)
7	3.29(+)	3.30(+)	3.30(+)
8	3.79(+)	3.80(+)	3.80(+)
9	4.29(+)	4.30(+)	4.30(+)
10	4.79(+)	4.80(+)	4.80(+)
11	5.43(-)	5.43(-)	5.43(-)
12	5.93(-)	5.93(-)	5.93(-)
13	6.43(-)	6.43(-)	6.43(-)
14	6.93(-)	6.93(-)	6.93(-)
15	7.43(-)	7.43(-)	7.43(-)
16	7.93(-)	7.93(-)	7.93(-)
17	8.43(-)	8.43(-)	8.43(-)
18	9.30(+)	9.30(+)	9.30(+)
19	9.80(+)	9.80(+)	9.80(+)

TABLE 4.2

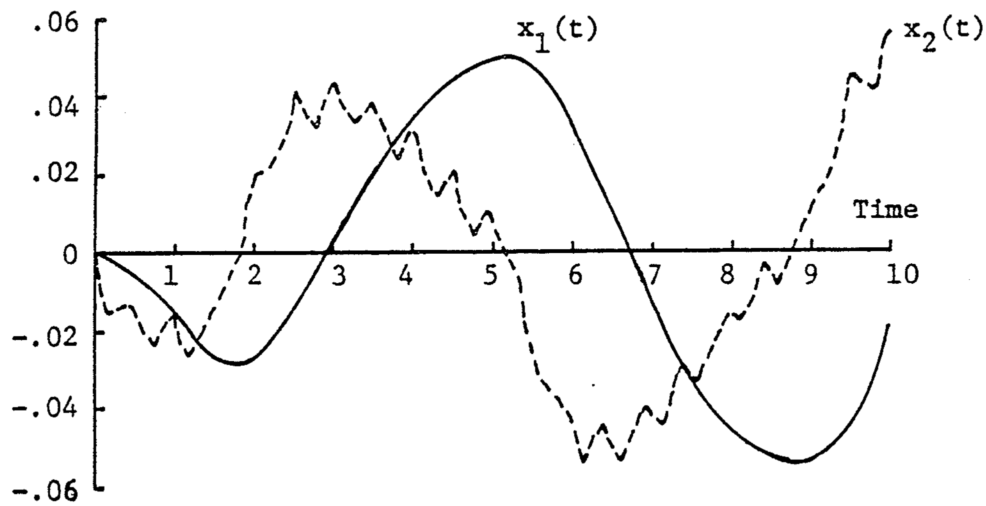
Time of Pulse Initiation (s) and Pulse Polarities in Parentheses - Case 2.

Pulse Number	Iteration Number 1	2	3	4	5
1	0.00(-)	0.00(-)	0.05(-)	0.00(-)	0.00(-)
2	0.50(-)	0.50(-)	0.55(-)	0.50(-)	0.50(-)
3	1.08(+)	1.00(-)	1.05(-)	1.00(-)	1.00(-)
4	1.58(+)	1.80(+)	1.55(+)	1.80(+)	1.80(+)
5	2.08(+)	2.30(+)	2.05(+)	2.30(+)	2.30(+)
6	2.58(+)	2.80(+)	2.55(+)	2.80(+)	2.80(+)
7	3.08(+)	3.30(+)	3.05(+)	3.30(+)	3.30(+)
8	3.58(+)	3.80(+)	3.55(+)	3.80(+)	3.80(+)
9	4.30(-)	4.30(+)	4.05(+)	4.30(+)	4.30(+)
10	4.80(-)	4.80(-)	4.55(+)	4.80(+)	4.80(+)
11	5.30(-)	5.30(-)	5.05(+)	5.30(-)	5.30(-)
12	5.80(-)	5.80(-)	5.80(-)	5.80(-)	5.80(-)
13	6.30(-)	6.30(-)	6.30(-)	6.30(-)	6.30(-)
14	6.80(-)	6.80(-)	6.80(-)	6.80(-)	6.80(-)
15	7.30(-)	7.30(-)	7.30(-)	7.30(-)	7.30(-)
16	7.80(-)	7.80(-)	7.80(-)	7.80(-)	7.80(-)
17	8.30(-)	8.30(-)	8.30(-)	8.30(-)	8.30(-)
18	8.80(+)	8.80(+)	8.80(+)	8.80(+)	8.80(+)
19	9.30(+)	9.30(+)	9.30(+)	9.30(+)	9.30(+)
20	9.80(+)	9.80(+)	9.80(+)	9.80(+)	9.80(+)

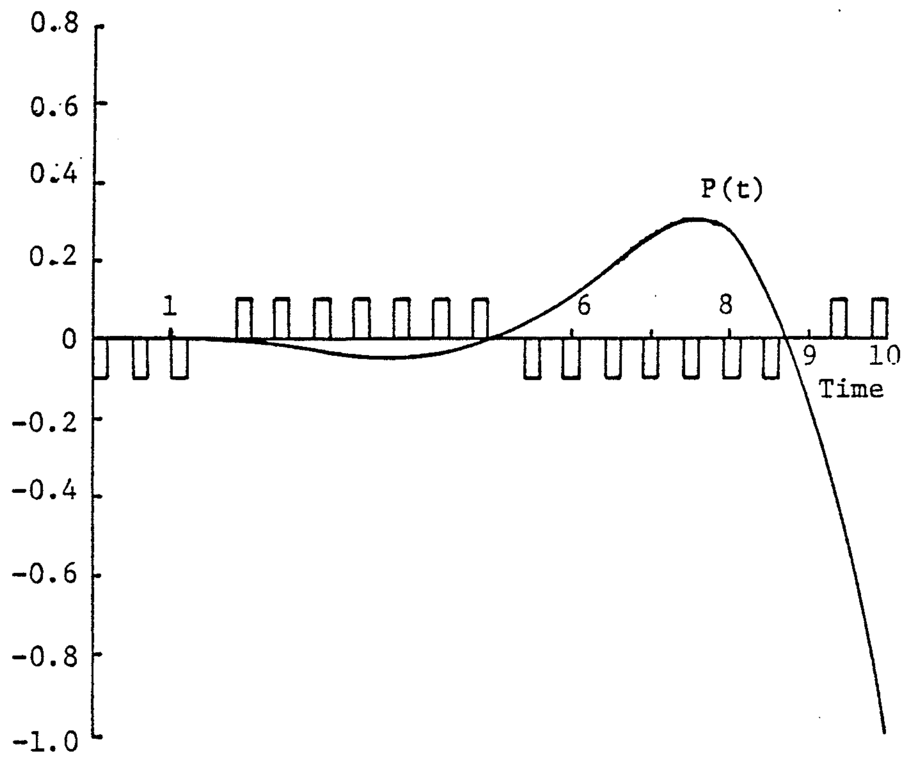
TABLE 4.3

Time of Initiation (s) and Pulse Polarities in Parentheses - Case 3.

Pulse Number	Iteration Number 1	2	3	4	5
1	6.26(-)	6.51(-)	6.83(-)	6.85(-)	6.85(-)
2	6.76(-)	7.01(-)	7.33(-)	7.35(-)	7.35(-)
3	7.26(-)	7.51(-)	7.83(-)	7.85(-)	7.85(-)
4	7.76(-)	8.01(-)	9.30(+)	9.30(+)	9.30(+)
5	9.30(+)	9.30(+)	9.80(+)	9.80(+)	9.80(+)
6	9.80(+)	9.80(+)			

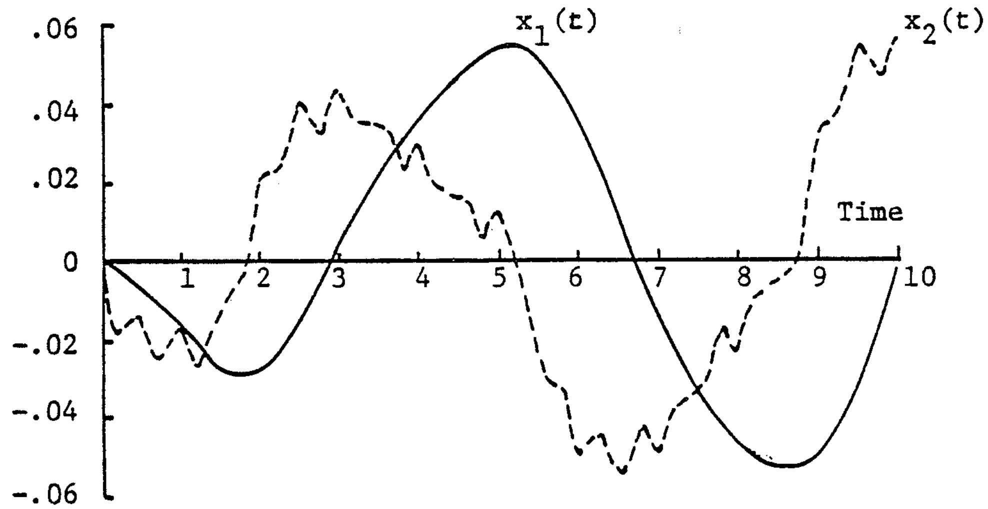


(a)

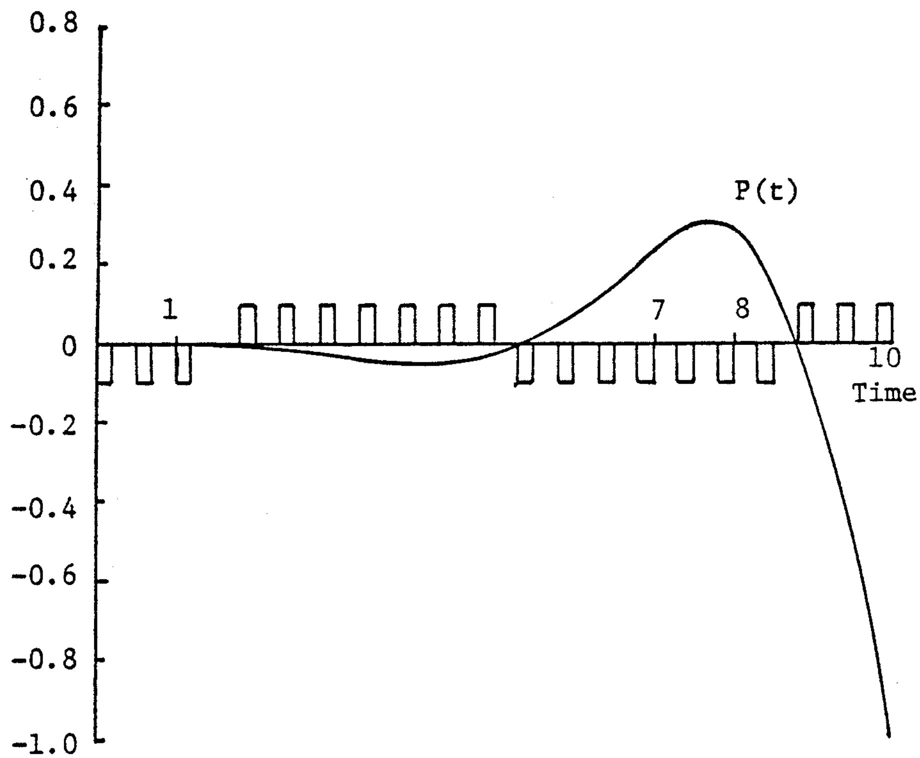


(b)

Figure 4.2 Case 1, a = 0.1 and α = 0.0.
(a) State variables, (b) optimal control u(t) and function P(t).



(a)



(b)

Figure 4.3 Case 2, a = 2.0 and α = 0.0.
(a) State variables, (b) optimal control u(t) and function P(t).

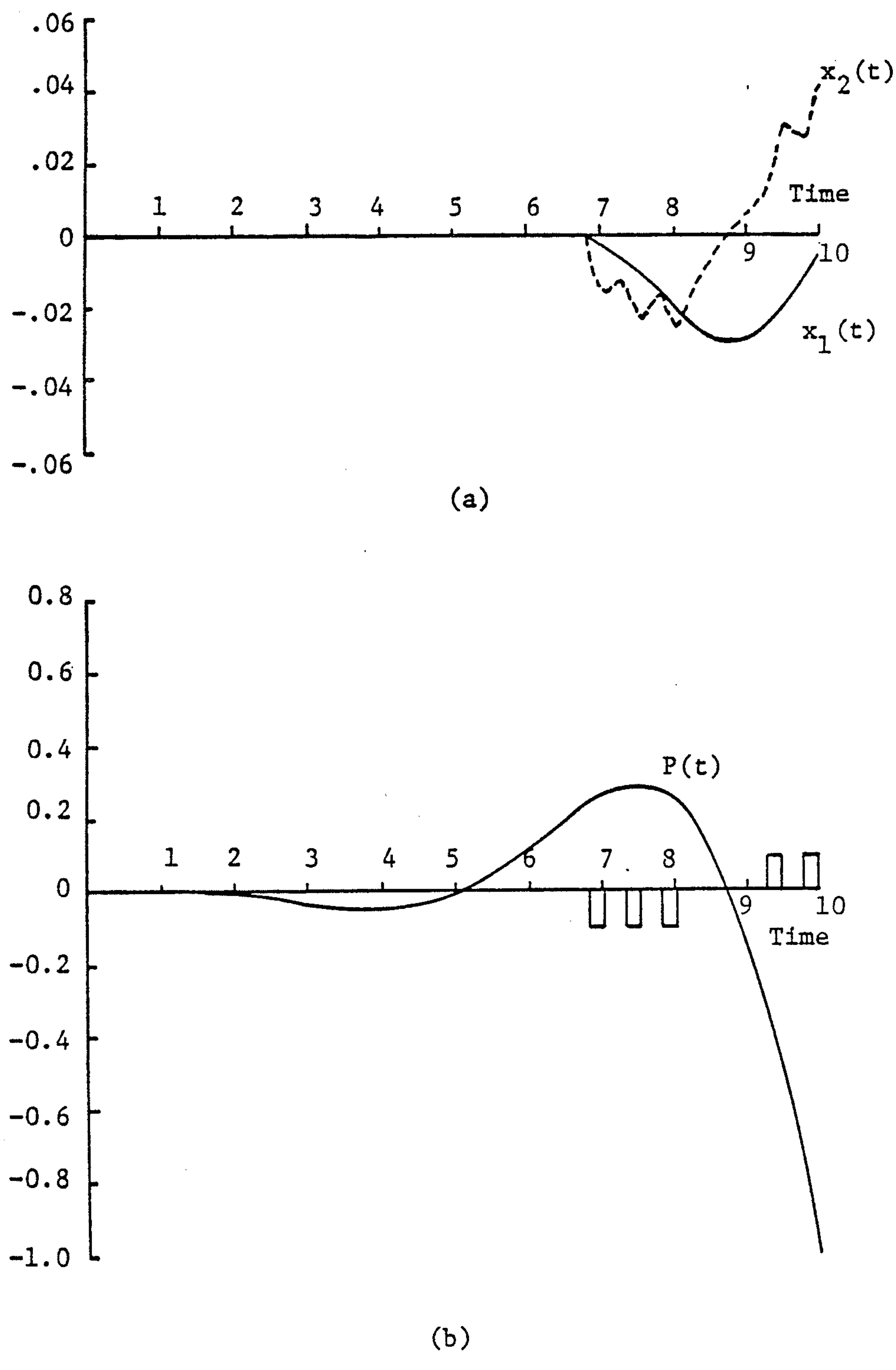


Figure 4.4 Case 3, a = 2.0 and α = 0.2.
(a) State variables, (b) optimal control u(t) and function P(t).

Chapter V

FEEDBACK CONTROLLER FOR SYSTEMS WITH QUADRATIC PERFORMANCE INDEX

In this chapter, the problem of finding the optimum PFM control sequence for minimizing a quadratic performance index is studied. Consider the nth order plant

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},u,t), \quad \underline{x}(0) = \underline{x}_0 . \tag{5.1}$$

The control function u(t) is represented as

$$u(t) = M \sum_k \mathrm{sgn}(y) \ (u_s(t-t_k) - u_s(t-t_k-\tau)),$$

where

- y is the input to the modulator to be synthesized,
- t_k is the time of initiation of the kth pulse,
- τ is the pulse width, and
- u_s is a step function.

The function y is also the output of the controller as shown in Figure 5.1.

The optimal control problem is to synthesize the pulse train u(t) which is the output of the modulator such that the performance index

$$J = \int_{t_0}^{t_f} ((\underline{x}-\underline{r})^T \underline{Q} (\underline{x}-\underline{r}) + h\,u^2)\,dt \tag{5.2}$$

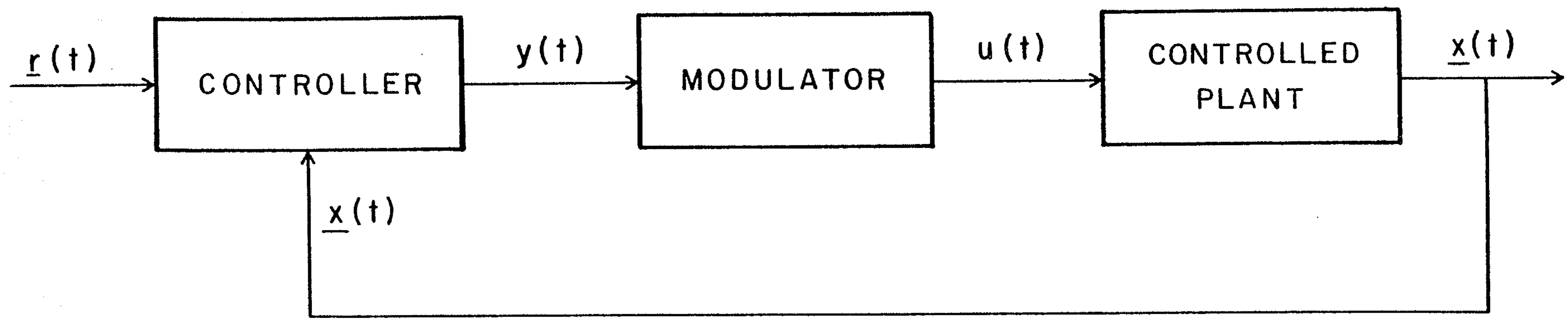


Figure 5.1 General block diagram for a PFM feedback controller

is minimized. The final state vector $\underline{x}(t_f)$ is free, the vector $\underline{r}(t)$ is the reference state vector, $\underline{Q}(t)$ is a positive semidefinite matrix, and h is a positive constant.

The modified maximum principle for nonlinear PFM control systems, derived earlier in Chapter 4, can be applied to obtain the conditions of optimality. To minimize J, the optimal control u(t) that maximizes the function $I(u) = \int H(\underline{x},\underline{p},u,t)\,dt$ should be found from the set of admissible controls. The Hamiltonian of the system is given by

$$H = \underline{p}^T \underline{f}\,(\underline{x},u,t) - (\underline{x}-\underline{r})^T \underline{Q}\,(\underline{x}-\underline{r}) - hu^2 \tag{5.3}$$

and the adjoint n-vector $\underline{p}(t)$ satisfies the differential equation

$$\dot{\underline{p}}(t) = -\frac{\partial H}{\partial \underline{x}} = -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T \underline{p} + (\underline{Q}+\underline{Q}^T)(\underline{x}-\underline{r}),\ \underline{p}(t_f) = 0 \tag{5.4}$$

where $p_{n+1}(t) = -1$ corresponding to the performance index is already included in Equations (5.3) and (5.4). To maximize I(u) the following condition should be satisfied

$$I(u+\delta u) - I(u) \leqslant 0. \tag{5.5}$$

If the nonlinear function $\underline{f}(\underline{x},u,t)$ is given in a more specific form, the control function u(t) may be found more explicitly. A form which describes significant classes of nonlinear systems is given by Equation (4.9) as

$$\dot{x}_i = g_i(\underline{x}) + \phi_i(u) + m_i(\underline{x})\ \ell_i(u)$$

Note that the system defined in the above equation is augmented by an extra state to specify the performance index. Now it is required to find u(t) that maximizes G(u) defined in Equation (4.10) as

$$G(u) = \int_{t_0}^{t_f} \sum_{i=1}^{n+1} p_i(t)(m_i(\underline{x})\ \ell_i(u) + \phi_i(u))\, dt$$

Since $\phi_{n+1}(u) = hu^2$, G(u) can be written as follows

$$G(u) = \int_{t_0}^{t_f} (\sum_{i=1}^{n} p_i(t)(m_i(\underline{x})\ \ell_i(u) + \phi_i(u)) - h\,u^2)\, dt \qquad (5.6)$$

For example if $\underline{\ell}(u) = u$, and $\underline{\phi}(u) = \underline{B}u$, where $\underline{B}$ is an n-constant vector, Equation (5.6) can be expressed in a matrix form as follows

$$G(u) = \int_{t_0}^{t_f} (\underline{p}^T u(m(\underline{x}) + \underline{B}) - h\,u^2)\, dt \qquad (5.7)$$

The state and costate equations are

$$\dot{\underline{x}} = g(\underline{x}) + \underline{B}\,u + \underline{m}(\underline{x})\,u \qquad (5.8)$$

$$\dot{\underline{p}} = -\left[\frac{\partial g(\underline{x})}{\partial \underline{x}} + u\,\frac{\partial \underline{m}(\underline{x})}{\partial \underline{x}}\right]^T \underline{p} + (\underline{Q} + \underline{Q}^T)(\underline{x} - \underline{r}) \qquad (5.9)$$

The optimal control u(t) is then found to be

$$u = \begin{cases} M \,\mathrm{sgn}\,(\underline{p}^T(\underline{m}(\underline{x}) + \underline{B}))[u_s(t - t_k) - u_s(t - t_k - \tau)] & \\ \qquad\qquad\qquad\qquad \text{if } |\underline{p}^T(\underline{m}(\underline{x}) + \underline{B})| > Mh & \\ 0 \qquad\qquad\qquad\qquad \text{if } |\underline{p}^T(\underline{m}(\underline{x}) + \underline{B})| \le Mh & \end{cases} \tag{5.10}$$

The block diagram of such a system is portrayed in Figure 5.2. The $\underline{p}(0)$ are adjusted such that the end conditions are satisfied. When $|\underline{p}^T(\underline{m}(\underline{x})+\underline{B})|$ is greater than Mh, the pulse generator initiates a train of pulses having a minimum dead time between each two consecutive pulses. When $|\underline{p}^T(\underline{m}(\underline{x})+\underline{B})|$ is less than or equal to Mh the generation of pulses stops. However, if a pulse was on at this time it can not be turned off.

Since it is impossible to get the exact positions t_i of the pulses on-line, the control obtained represents a suboptimal control only. If a pulse is on at the final time t_f, the final time should be allowed to change to $(t_f+\tau-\varepsilon)$, where $(t_f-\varepsilon)$ is the time of initiation of the last pulse. If the final time is not allowed to change, the difference between t_f and $(t_i+\tau)$ should always be monitored and no pulse would be generated if that difference is less than $(d+\tau)$.

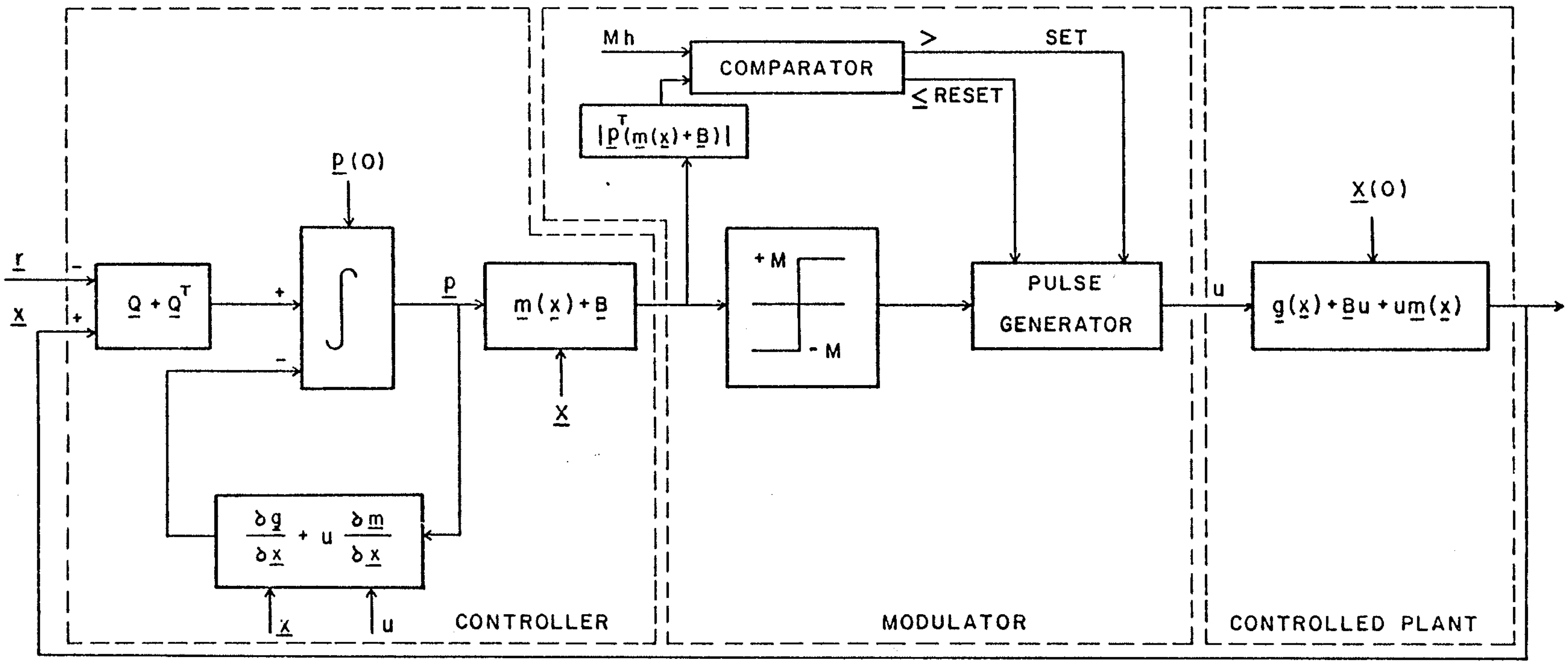


Figure 5.2 Feedback controller for plant Equation (5.8) and a quadratic performance index

In the following, for comparison purposes, a simple first order system is solved using both the on-line method described in this section and the off-line method described in Section 4.2.

Given the first order nonlinear system described by the differential equation

$$\dot{x} + ax - x^2 = u, \quad x(0) = 1, \quad 0 \le t \le t_f ,$$

it is desired to minimize

$$J = \int_0^{t_f} ((x-1)^2 + h\,u^2)\,dt ,$$

where h is a positive constant. This is equivalent to maximizing G(u), where

$$G(u) = \int_0^{t_f} (p_1\,u - h\,u^2)\,dt .$$

Letting $x_1 = x$, and $x_2 = J$, the state equations are

$$\dot{x}_1 = -ax_1 + x_1^2 + u , \quad x_1(0) = 1$$

$$\dot{x}_2 = (x_1 - 1)^2 + h\,u^2 , \quad x_2(0) = 0.$$

The constants in the Pontryagin function are $c_1 = 0$ and $c_2 = 1$. The auxiliary variable p_i must satisfy

$$\dot{p}_1 = ap_1 - 2x_1p_1 + 2x_1 - 2$$

$$\dot{p}_2 = 0$$

where $p_1(t_f) = 0$ and $p_2(t_f) = -1$. G(u) then becomes

$$G(u) = \int_0^{t_f} (p_1 u + p_2 h u^2)\, dt$$

and the optimal control is

$$u^*(t) = \begin{cases} M \operatorname{sgn}(p_1)\ [u_s(t - t_k) - u_s(t - t_k - \tau)] & \text{if } |p_1| > Mh \\ 0 & \text{if } |p_1| \leq Mh. \end{cases}$$

Figure 5.3 shows the block diagram of this system. The following numerical values are assumed

t_f = 3 s, final time	τ = 0.2 s, pulse width
M = 0.2 V, pulse magnitude	d = 0.3 s, dead time

and h = a = 3. The results for both the on-line and the off-line cases are shown in Figure 5.4. The state variable x_1, the costate function p_1 and the optimal control u*(t) are illustrated in this figure.

The initial costate value for the on-line and the off-line cases were found to be 0.496247 and 0.49705 respectively, and the final values for the function G(u) were 5.85603×10^{-3} and 5.95431×10^{-3}, respectively. Table 5.1 includes, for both cases, the polarity and time of initiation of the pulses constituting the control u*(t). Comparing the resulting controls for the two cases, it is noticed that for the off-line case there is no change in the number of pulses N. However the initiation time of the pulses is shifted to the right by 0.087 s resulting in a 1.66% improvement in the value of G(u). The problem was simulated using the Continuous System Modeling Program, CSMP III. The method of variation of extremals [21] was used to solve the nonlinear two-point boundary-value problem. A listing of the computer program used in this part of the study can be found in Appendix B.

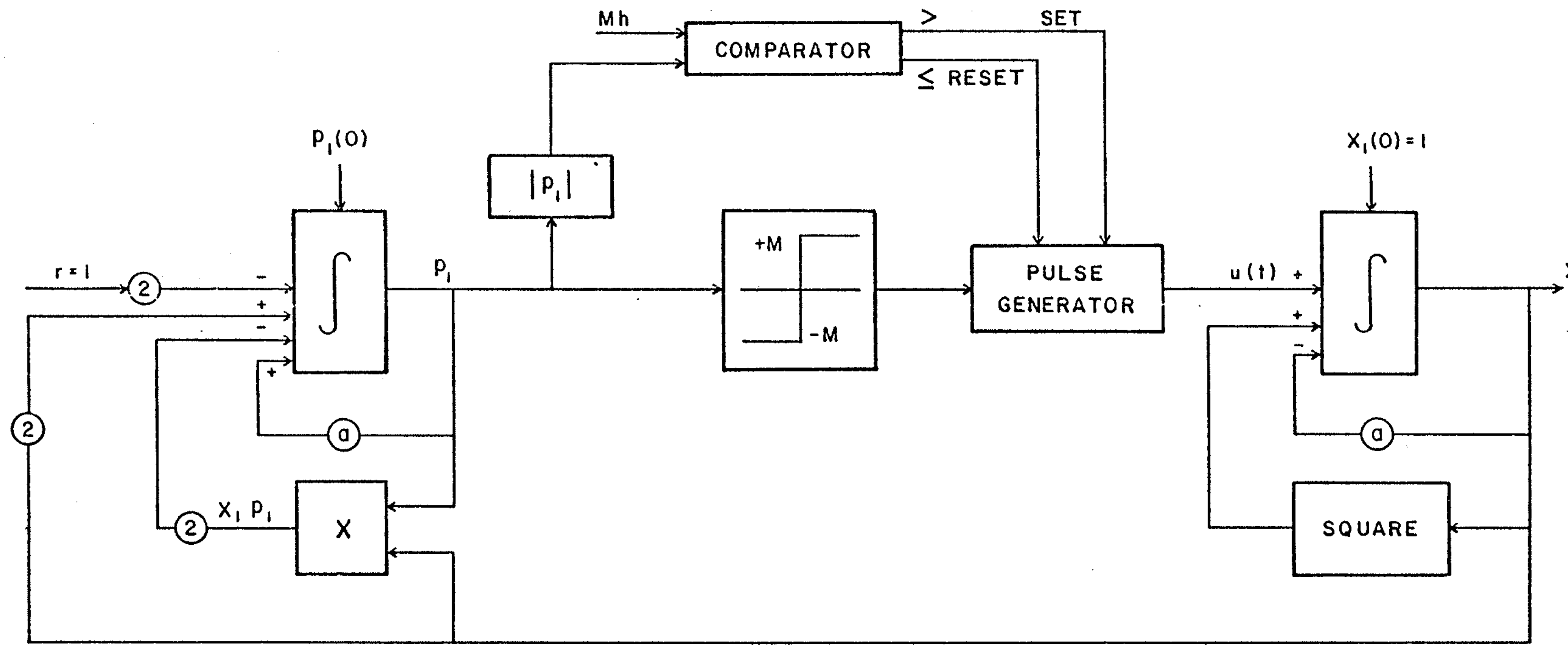


Figure 5.3   Feedback controller for the on-line example

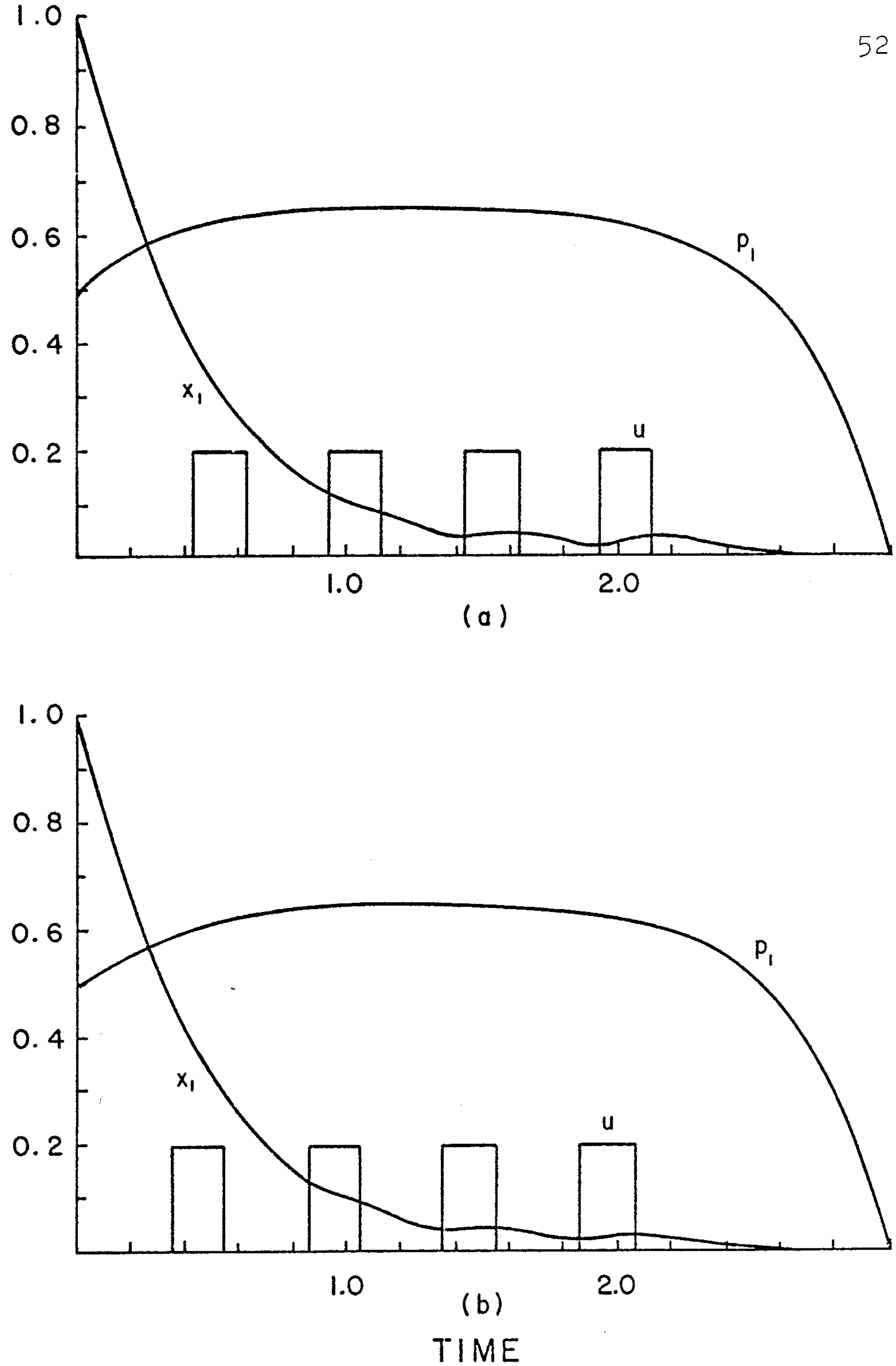


Figure 5.4 $x_1(t)$, $p_1(t)$ and $u(t)$ for (a) the off-line case, (b) the on-line case

Table 5.1

Time of Pulse Initiation (s) and Pulse Polarities in Parentheses for the On-line and Off-line Cases

Pulse Number	On-line Case	Off-line Case
1	0.338 (+)	0.425 (+)
2	0.838 (+)	0.925 (+)
3	1.338 (+)	1.425 (+)
4	1.838 (+)	1.925 (+)

Chapter VI

OPTIMAL CONTROLLER FOR PFM SYSTEMS CONTAINING TYPE-II MODULATORS

The theory developed in Chapter 4 is now applied to a control system containing a pulse-frequency modulator of the second type. The problem is formulated, the assumptions are specified, and the optimal controller is then obtained.

6.1 SYSTEM STATES

6.1.1 Control Problem

The system considered in this chapter contains a type-II modulator and a plant which can be linear or nonlinear. The problem is defined as follows:

For the system

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}\ , u\ , t) \tag{6.1}$$

$$\dot{z}(t) + g(z) = \eta(t) - s\ \mathrm{sgn}(z)\ \delta(|z| - s) \tag{6.2}$$

where

$$u(t) = W\ \mathrm{sgn}(z)\ \delta(|z| - s) \tag{6.3}$$

and $\underline{x}$ is a n-vector, given the performance index J , the initial states, and the admissible controls, it is required to determine the optimal controller, that is the optimal control function u(t) which optimizes the performance index J, where the final time t_f is fixed.

Note that u(t) as defined in Equation (6.3) is made up of impulses. This is the commonly used representation of the output of the modulator. Later in this chapter, Equation (6.3) is redefined as a function of rectangular pulses.

By examining Equations (6.2) and (6.3), it is realized that the control u(t) and the input to the modulator $\eta(t)$ are totally dependent on each other. An impulse in u(t) can not be generated except when $|z|$ is equal to the threshold s and, at the same time, all the constraints imposed on the control u(t) should be satisfied. Hence, it is concluded that solving for an optimum $\eta(t)$ is equivalent to finding the optimal control function u(t). The performance index to be optimized is then of the form

$$J(\eta) = \int_0^{t_f} F(\underline{x}-\underline{r}\ ,\ \eta,\ t)\ dt. \tag{6.4}$$

The block diagram of the system under consideration is shown in Figure 6.1, where $\underline{r}(t)$ is an m-vector input to the controller.

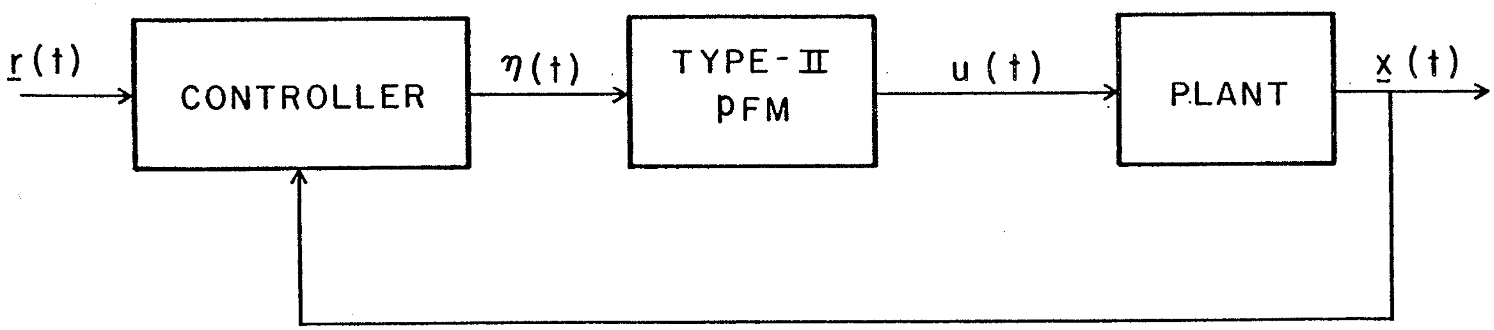


Figure 6.1 Block diagram of the control system considered in Chapter 6

6.1.2 Alternative Form for u(t)

As mentioned earlier, the control u(t) as defined in Equation (6.3) is made up of a series of impulses. However, throughout this study it is assumed that u(t) is a train of rectangular pulses. Equation (6.3) is modified to reflect this as follows:

$$u(t) = M \sum_{i=1}^{N} \operatorname{sgn} (z(t_i^-))[u_s(t - t_i) - u_s(t - t_i - \tau)] , \tag{6.5}$$

where M is the pulse magnitude $(M = W/\tau)$

N is the number of pulses in the interval $(0,t_f)$,

τ is the pulse width,

t_i is the time of initiation of the ith pulse, and

$u_s(.)$ represents a step function.

6.1.3 Resetting of the Pulse-Frequency Modulator

Pavlidis and Jury [38] expressed the resetting of the modulator to the zero value as the second term on the right-hand-side of Equation (6.2). This term is independent of the optimization problem and its presence is confusing when applying the MMP. An equivalent way to express the modulator's equation is

$$\dot{z} + g(z) = \eta \tag{6.6}$$

with the conditions

$$z(0) = z(t_i^{+}) = 0 \ , \quad |z(t_i^{-})| = s \ , \quad i = 1, \ldots, N, \tag{6.7}$$

where s is a preset threshold. Equations (6.6) and (6.7) will replace Equation (6.2) in the remaining part of this study.

A practical method for resetting type-II modulators is developed next. The conditions represented by Equation (6.7) can be fulfilled by computing new thresholds after the initiation of each pulse. A counter, initialized to zero, is used and is incremented (or decremented) whenever a positive (or negative) pulse is initiated. Usually, the absolute value of z is compared to s, but equivalently z can be compared to the threshold $s_1 = +s$ or $s_2 = -s$. With the existence of a counter of a value k_i, two new thresholds s_1 and s_2 can always be computed after the initiation of each pulse according to the relations

$$\begin{aligned} s_1 &= k_i s + s \\ s_2 &= k_i s - s \ , \qquad i = 1, \ldots N, \end{aligned} \tag{6.8}$$

where reaching s_1 means initiating a positive pulse and reaching s_2 means initiating a negative pulse. An example illustrating this operation is shown in Figure 6.2 with the block diagram of such pulse-frequency modulator given in Figure 6.3.

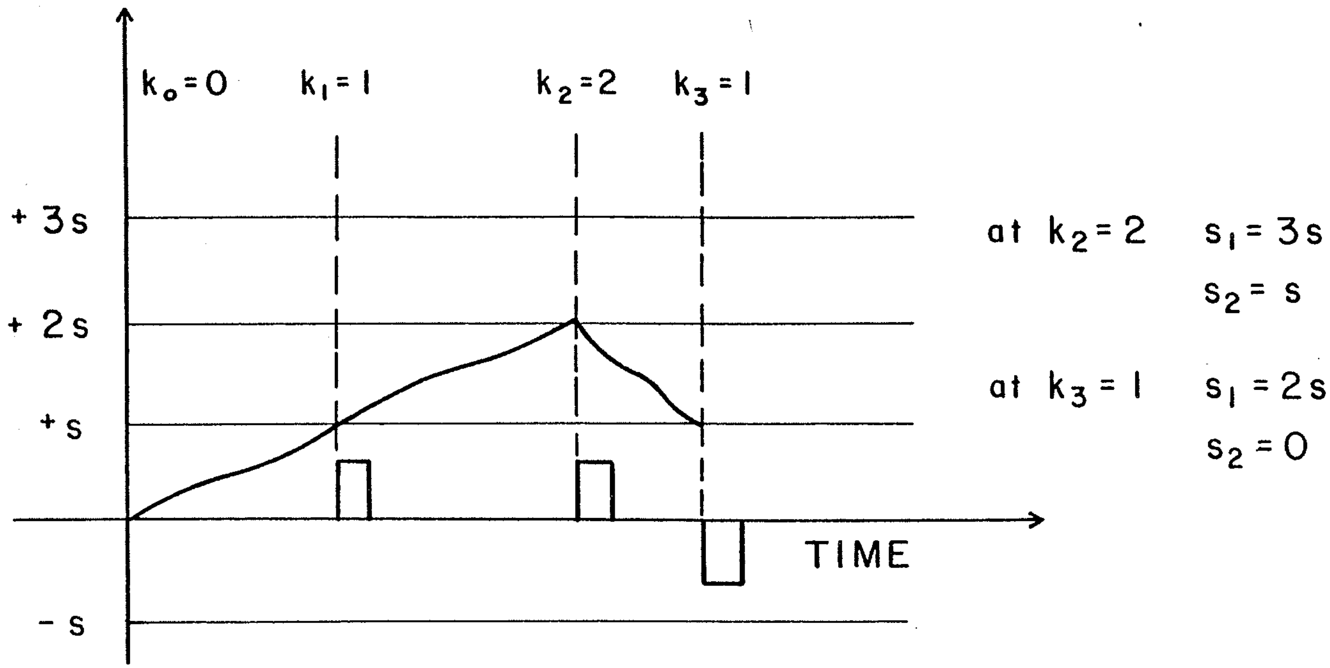


Figure 6.2 Example representative of a particular implementation of the resetting of PFM

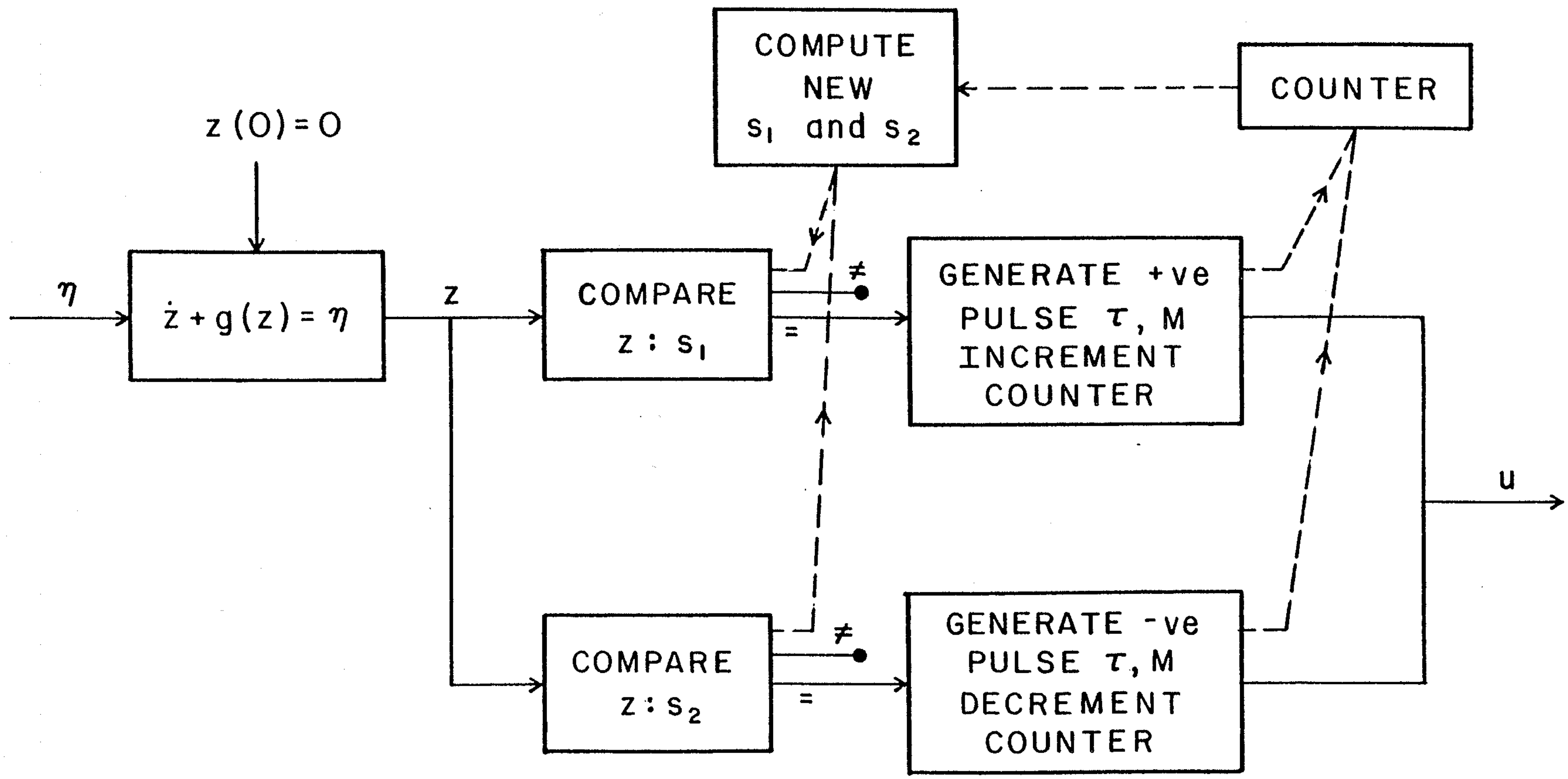


Figure 6.3 Practical implementation of a PFM

6.1.4 State Equations

In order to apply the modified maximum principle, the state equations of the system are augmented to include the performance index as the (n+2)th state variable, and the state variable z is renamed to x_{n+1}. Thus the state equations representing the system are

$$\begin{aligned}
\dot{\underline{x}} &= f(\underline{x},u,t), & \underline{x}(0) &= \underline{x}_0 \\
\dot{x}_{n+1} &= -g(x_{n+1}) + \eta, & x_{n+1}(0) &= x_{n+1}(t_i^+) = 0 \\
& & |x_{n+1}(t_i^-)| &= s \\
\dot{x}_{n+2} &= F(\underline{x},\eta,t) & x_{n+2}(0) &= 0
\end{aligned} \tag{6.9}$$

where

$$\underline{x} = [x_1\, x_2\, \ldots\, x_n]^T$$

$$x_{n+1} = z$$

$$x_{n+2} = J\,.$$

To apply the MMP, specific forms of the functions $\underline{f}$, F, and g are assumed. That is

1. a linear plant of the form

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,u\,, \qquad \underline{x}(0) = \underline{x}_0\,, \tag{6.10}$$

2. a quadratic performance index

$$J(\eta) = \tfrac{1}{2} \int_0^{t_f} ((\underline{x}-\underline{r})^T \, \underline{Q} \, (\underline{x}-\underline{r}) + c\,\eta^2) \, dt, \qquad (6.11)$$

where $\underline{Q}$ is a real symmetric positive semi-definite matrix and c is a positive constant, and

3. an integral pulse-frequency modulator, that is $g(x_{n+1})$ is equal to zero.

Hence, the state equations of the assumed system are

$$\begin{aligned}
&\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,u\,, && \underline{x}(0) = \underline{x}_0 \\
&\dot{x}_{n+1} = \eta\,, && x_{n+1}(0) = x_{n+1}(t_i^{+}) = 0 \\
& && |x_{n+1}(t_i^{-})| = s, \quad i = 1, \ldots, N \qquad (6.12) \\
&\dot{x}_{n+2} = \tfrac{1}{2}(\underline{x}-\underline{r})^T \, \underline{Q} \, (\underline{x}-\underline{r}) + \tfrac{1}{2}\, c\, \eta^2.
\end{aligned}$$

These assumptions do not mean that other forms are not possible. The presence of a modulator in the system increases its complexity and it is preferred here to deal with this complexity without any added nonlinearities. Once a solution method is established, the limitations on any of the functions $\underline{f}$, F, and g will be discussed.

6.2 APPLICATION OF THE MODIFIED MAXIMUM PRINCIPLE

The application of the MMP to the system described in the previous section is now presented. It is required to find the optimal control $\eta(t)$ which minimizes the performance index $J(\eta)$ in Equation (6.11). This is equivalent to finding $\eta(t)$ that maximizes I, where

$$I = \int_0^{t_f} H \, dt$$

and H is the Hamiltonian defined as

$$H = \sum_{i=1}^{n+2} p_i f_i . \tag{6.13}$$

The costate functions $p_i(t)$ of the system are defined as

$$\dot{p}_i(t) = - \sum_{j=1}^{n+2} p_j \frac{\partial f_j}{\partial x_i} , \quad p_i(t_f) = - c_i , \tag{6.14}$$

where $c_i = 0$ for $i = 1, \ldots, n+1$

and $c_{n+2} = 1$.

The costate equations, obtained by substituting Equation (6.12) into Equation (6.14), are

$$\dot{\underline{p}}(t) = -\underline{Q}(\underline{x}-\underline{r}) - \underline{A}^T \underline{p}, \qquad p_i(t_f) = 0, \quad i = 1, \ldots, n$$

$$\dot{p}_{n+1}(t) = -\left(\underline{B}\frac{\partial u}{\partial x_{n+1}}\right)^T \underline{p}, \qquad p_{n+1}(t_f) = 0 \tag{6.15}$$

$$\dot{p}_{n+2}(t) = 0, \qquad p_{n+2}(t_f) = -1$$

and hence, $p_{n+2}(t) = -1$.

The function I is

$$I = \int_0^{t_f} (\underline{p}^T \underline{A}\,\underline{x} + \underline{p}^T \underline{B}\,u + p_{n+1}\,\eta - \tfrac{1}{2}(\underline{x}-r)^T \underline{Q}(\underline{x}-\underline{r}) - \tfrac{1}{2} c\,\eta^2)\, dt$$

and the control $\eta(t)$ that maximizes I is found to be

$$\eta^* = p_{n+1} / c. \tag{6.16}$$

Equations (6.12), (6.15), and (6.16) form the state and costate equations of the optimal control system.

Next, the derivative $\partial u/\partial x_{n+1}$ in Equation (6.15) is evaluated. Recalling that

$$u(t) = M \sum_{i=1}^{N} \operatorname{sgn}(x_{n+1}(t_i^-))(u_s(t-t_i) - u_s(t-t_i-\tau)),$$

and differentiating the ith term of u(t) with respect to x_{n+1} gives

$$\left.\frac{\partial u}{\partial x_{n+1}}\right|_i = \frac{\partial u}{\partial t_i}\frac{\partial t_i}{\partial x_{n+1}} = \frac{\partial u}{\partial (t-t_i)}\frac{\partial (t-t_i)}{\partial t_i}\frac{\partial t_i}{\partial x_{n+1}}$$

but $\dfrac{\partial u}{\partial (t-t_i)} = \delta(t-t_i) - \delta(t-t_i-\tau)$, $\quad \dfrac{\partial (t-t_i)}{\partial t_i} = -1$

and $\dfrac{\partial t_i}{\partial x_{n+1}} = \operatorname{Lim} \dfrac{\Delta t_i}{\Delta x_{n+1}} = -\dfrac{1}{\dot{x}_{n+1}(t_i^-)}$.

It follows that

$$\frac{\partial u}{\partial x_{n+1}} = M \sum_{i=1}^{N} \frac{\operatorname{sgn}(x_{n+1}^-(t_i^-))}{\dot{x}_{n+1}(t_i^-)} (\delta(t-t_i) - \delta(t-t_i-\tau)) . \qquad (6.17)$$

Figure 6.4 demonstrates how the derivative $\partial t_i/\partial x_{n+1}$ is obtained.

It is important to note that x_{n+1} does not change sign at the time instants where the derivatives are taken. The time instants t_i are the times at the intersections of x_{n+1} with the threshold s, that is at either $x_{n+1} = s$ or $x_{n+1} = -s$. Due to the way in which the modulator functions, any change in sign in x_{n+1} occurs at time t_i^+.

To summarize, the optimal control is found by solving the following 2(n+1) differential equations;

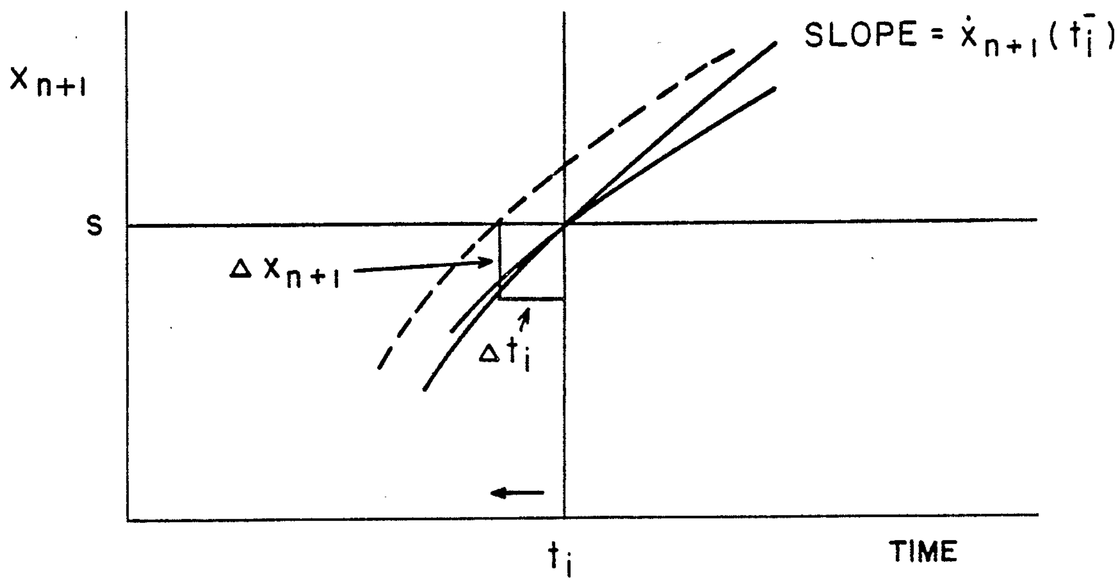


Figure 6.4 The derivative $\partial t_i / \partial x_{n+1}$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,u\,, \qquad \underline{x}(0) = \underline{x}_0\,, \qquad \text{n Equations}$$

$$\dot{x}_{n+1} = p_{n+1}/c\,, \qquad x_{n+1}(0) = x_{n+1}(t_i^{+}) = 0, \quad \text{1 Equation}$$

$$|x_{n+1}(t_i^{-})| = s \tag{6.18}$$

$$\dot{\underline{p}} = -\underline{Q}(\underline{x}-\underline{r}) - \underline{A}^T\underline{p}\,, \qquad \underline{p}(t_f) = \underline{0}\,, \qquad \text{n Equations}$$

$$\dot{p}_{n+1} = -\frac{\partial u}{\partial x_{n+1}}\,\underline{B}^T\underline{p}\,, \qquad p_{n+1}(t_f) = 0\,, \qquad \text{1 Equation}$$

where $\partial u/\partial x_{n+1}$ is given by Equation (6.17).

The block diagram for the optimal control system is shown in Figure 6.5a. A more detailed diagram of the same system is given in Figure 6.5b.

The set of equations (6.18) contains impulses, must fulfill many constraints, and in addition has split boundary conditions. Due to this complexity, no existing method could be applied to obtain a solution directly. The next chapter illustrates how these problems are overcomed.

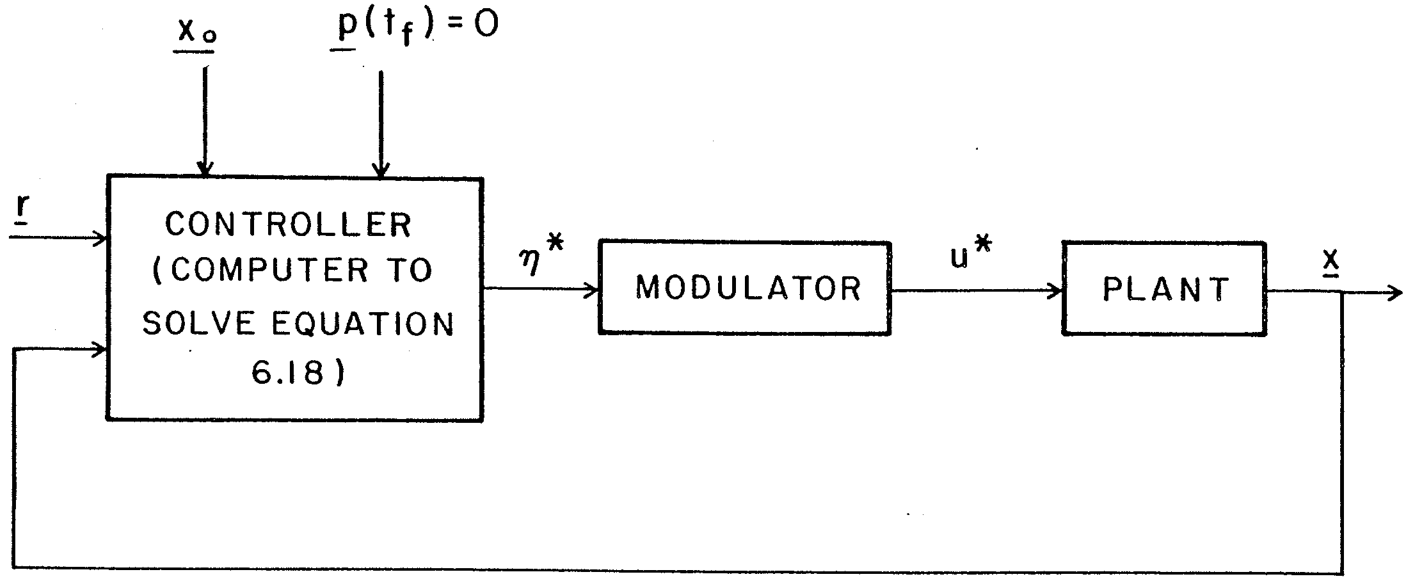


Figure 6.5a Optimal feedback control system

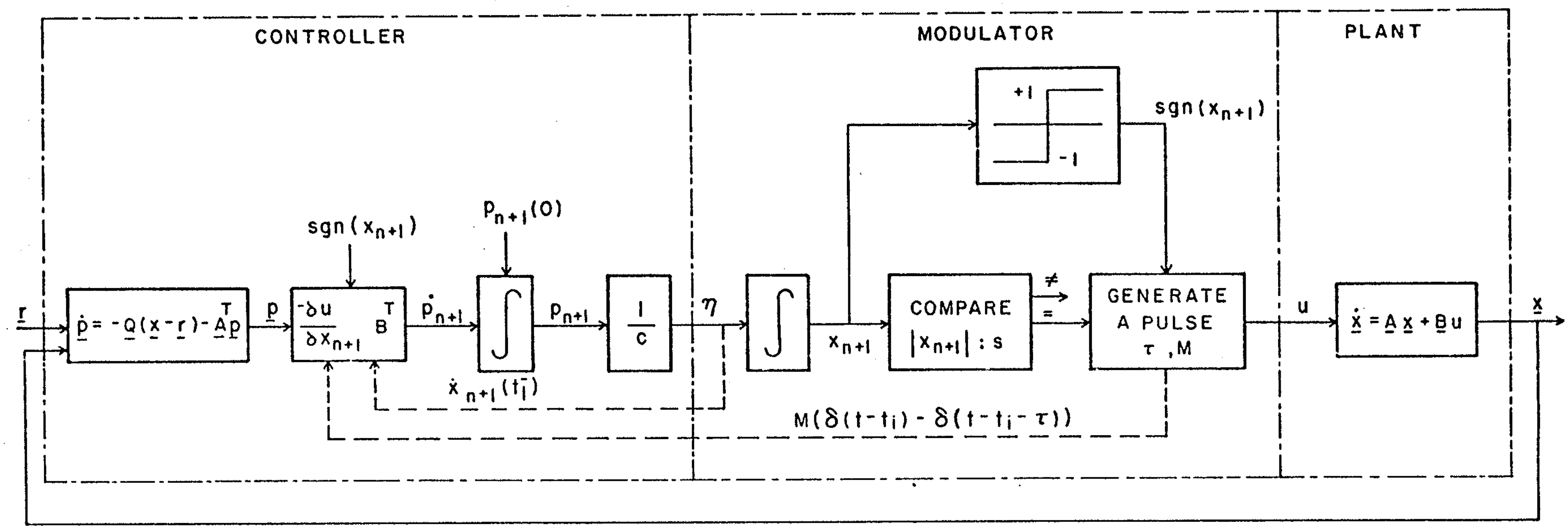


Figure 6.5b Plant, modulator and optimal feedback controller

Chapter VII

SOLUTION METHOD

In this chapter, two methods of solution are presented along with a discussion of their limitations. To make the discussion easier to express and follow, a first order linear plant is chosen. For such a plant, Equation (6.18) can be expressed as

$$
\begin{aligned}
\dot{x}_1 &= ax_1 + b\,u\,, & x_1(0) &= x_0 \\
\dot{x}_2 &= p_2 / c\,, & x_2(0) &= x_2(t_i^+) = 0 \\
& & |x_2(t_i^-)| &= s \qquad\qquad (7.1) \\
\dot{p}_1 &= -\,ap_1 + x_1 - r\,, & p_1(t_f) &= 0 \\
\dot{p}_2 &= -\,bp_1 \frac{\partial u}{\partial x_2}\,, & p_2(t_f) &= 0
\end{aligned}
$$

where the matrix $\underline{Q}$ of the performance index is assumed equal to one and

$$\frac{\partial u}{\partial x_2} = M \sum_{i=1}^{N} \frac{\mathrm{sgn}(x_2(t_i^-))}{\dot{x}_2(t_i^-)} (\delta(t-t_i) - \delta(t-t_i-\tau)). \qquad (7.2)$$

7.1 SOLUTION METHOD I: COMPUTER SIMULATION

When examining Equations (7.1), it appears that a direct way to solve them might be to simulate the system and its constraints using CSMP and then use the method of variational of extremals to solve the existing two-point boundary-value problem. The case here is not so straightforward and some assumptions are to be made in order to satisfy the constraints imposed on the problem. Consider the following arguments concerning the present problem.

1. What is meant by $p_2(t_f) = 0$?

   Due to the nature of $\dot{p}_2$, which is a series of impulses, p_2 is formed of step functions of different sizes. There exists two alternatives;

   a) to assume that a zero is reached if p_2 changes signs at t_f at the trailing edge of a pulse or

   b) p_2 actually reaches zero value at or just before the final time.

   These two alternatives are always considered when solution method I is used.

2. How can the minimum dead time constraint be satisfied for large p_2?

   When $p_2(t)$ is so large that $x_2(t)$ reaches the threshold in a time less than the pulse width τ plus the dead time d, the constraint of minimum dead time between consecutive pulses is bound to be violated. In this case, again two alternatives are considered;

a) when x_2 reaches the threshold in a time less than $\tau+d$, x_2 is held at this threshold level until a time $\tau+d$ has lapsed. A pulse, the polarity of which depends on the sign of x_2, is generated and x_2 is reset to zero.

b) x_2 is computed for the whole interval and its value is checked at the time $t_i+\tau+d$. If x_2 is greater than or equal to the threshold, a pulse is generated at that time; if not proceed with the computation.

The first method has the disadvantage of having a 'do not care' period where the value of p_2 is not accounted for. The second method is more accurate since it takes into account the value of p_2 over the whole interval; this method is the one chosen to satisfy the minimum dead time constraint.

3. Is it easier to solve the system backward in time ?
The answer is no; it is not possible to solve the system backward in time. The system under consideration is a decision making system. The value of $p_2(t_f^-)$ is impossible to predict and the slope $\dot{x}_2(t_i^-)$ can not be known in advance.

A computer program for this method written in CSMP is given in Appendix B. The rectangular method of integration is used since it is considered to be the best method for dealing with discontinuties. The program was run on the

IBM/370 computer at the National Research Council of Canada. The program was successful in simulating the system for given initial conditions. However the final conditions for the costates could not be satisfied. To solve the TPBVP, the method of variation of extremals was tried. All the trials arrived at the trivial solution, that is starting with $p_2(0) = 0$, $\dot{p}_2(t_f)$ remains at zero with no pulses generated.

After a few runs with different $\underline{p}(0)$ it was noticed that any small change in the value of $\underline{p}(0)$ caused a random large change in the final value. This indicated how crucial the initial guess is for the convergence. Making a good initial guess presents a very difficult problem. To the author's knowledge there is no existing numerical TPBVP method that is capable of solving such a complex problem.

We have attempted to solve the problem by trial and error using computer simulation. A grid of possible values of $\underline{p}(0)$ is formed with a very small step size. The program is then run for these thousands of points until a region of good initial guess is found. This method is found to be impractical and time consuming; therefore it is not pursued any further.

The solution method I was tried for the same problem but with different pulse shapes, also with a relaxation pulse-frequency modulator. The solution difficulties remained.

A final comment on the method is on the accuracy of the results. As mentioned earlier, the rectangular method of integration is the one recommended for this case. To achieve the required accuracy with this method, a very small step size has to be used which leads to large round-off errors. Double precision integration is only available in CSMP with the Runge-Kutta method.

7.2 SOLUTION METHOD II : ANALYTIC-NUMERIC

It is concluded from the above that the approach to solve the optimization problem, defined by Equation (7.1), should be different than the method suggested in Section 7.1. A new approach, presented in this section, is summarized in the following points.

1. The differential equations are solved analytically as a function of the n+1 initial costates $\underline{p}(0)$ and the N time instants t_i of the initiation of pulses (N is the number of pulses which is unknown).
2. The intermediate N conditions

$$x_{n+1}(t_i^-) = s \text{ sgn } (x_{n+1}(t_i^-))$$

   and the n+1 final conditions

$$\underline{p}(t_f) = 0$$

   are then satisfied, thus obtaining N+n+1 nonlinear equations in N+n+1 unknowns.
3. To solve for the unknowns, a numerical algorithm is developed since it is quite impossible to solve the equations analytically. An example illustrating this method of solution is given later in this chapter.

7.2.1 Solution of the State-Costate Differential Equations

Equations (7.1) are arranged as follows

$$\dot{x}_1 = ax_1 + b\,u\,, \qquad x_1(0) = x_{10} \tag{7.3}$$

$$\dot{p}_1 = -\,ap_1 + x_1 - r\,, \qquad p_1(t_f) = 0 \tag{7.4}$$

$$\dot{p}_2 = -\,b\,M\,p_1 \sum_{i=1}^{N} \frac{\operatorname{sgn}(x_2\,(t_i^-))}{\dot{x}_2\,(t_i^-)}\,(\delta(t-t_i) - \delta(t-t_i-\tau))\,,$$

$$p_2(t_f) = 0 \tag{7.5}$$

$$\dot{x}_2 = p_2\,/\,c\,, \qquad x_2(0) = x_2\,(t_i^+) = 0$$

$$|x_2\,(t_i^-)| = s. \tag{7.6}$$

To solve these equations, there are three regions to be considered, as shown in Figure 7.1. The first region is the interval $[0,t_1]$ before the initiation of the first pulse, the second is the interval $[t_i,t_i+\tau]$ and the third interval is $[t_i+\tau,\ t_{i+1}]$, for $i = 1,2,\ldots,N$, where as after the final pulse t_{N+1} is equal to t_f.

Equations (7.3) to (7.6) can either be solved directly for the general case of N pulses, or can be solved successively for the cases of one and two pulses and then, by induction, for N pulses. The two methods were used and the

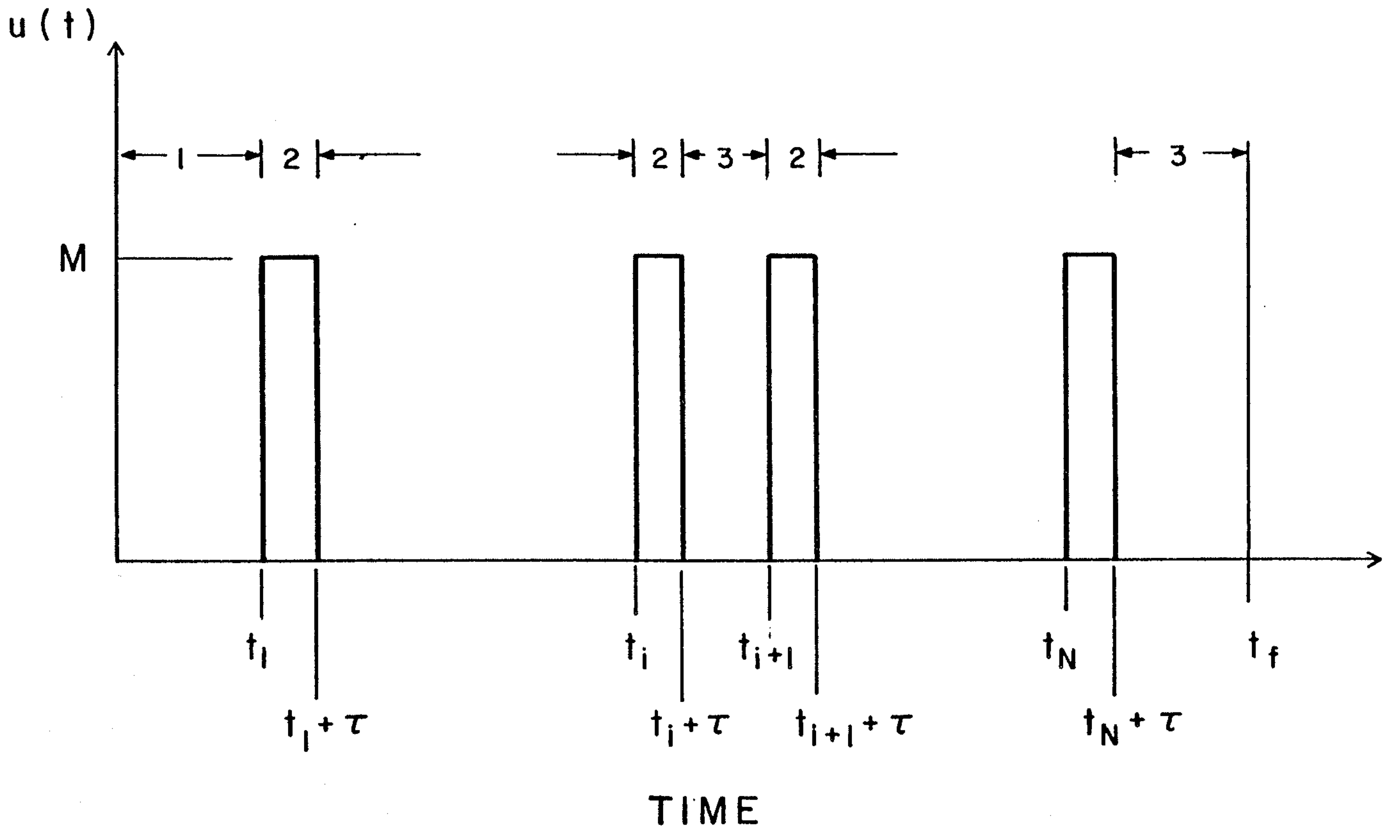


Figure 7.1   Regions of integration

results are in agreement. The solution by induction is given in detail in Appendix A. The results are as follows:

1. The state $x_1(t)$

$0 \leqslant t \leqslant t_1$,

$$x_1(t) = x_{10}\, e^{at} \tag{7.7a}$$

$t_m \leqslant t \leqslant t_m + \tau, \qquad m = 1, 2, \ldots, N$

$$x_1(t) = x_{10}\, e^{at} + \frac{Mb}{a}(1-e^{-a\tau})\left(\sum_{i=1}^{m-1} \operatorname{sgn}(x_2(t_i^-))\, e^{-at_i}\right) e^{at}$$

$$+ \frac{Mb}{a} \operatorname{sgn}(x_2(t_m^-))\left(e^{a(t-t_m)} - 1\right) \tag{7.7b}$$

$t_m + \tau \leqslant t \leqslant t_{m+1}, \qquad m = 1, \ldots, N, \qquad t_{N+1} \equiv t_f$

$$x_1(t) = x_{10}\, e^{at} + \frac{Mb}{a}(1-e^{-a\tau})\left(\sum_{i=1}^{m} \operatorname{sgn}(x_2(t_i^-))e^{-at_i}\right) e^{at}. \tag{7.7c}$$

2. The costate $p_1(t)$

$0 \leqslant t \leqslant t_1$

$$p_1(t) = \left(p_{10} + \frac{r}{a} - \frac{x_{10}}{2a}\right) e^{-at} - \frac{r}{a} + \frac{x_{10}}{2a} \quad e^{at} \tag{7.8a}$$

$t_m \leqslant t \leqslant t_m + \tau, \quad m = 1, \ldots, N$

$$\begin{aligned} p_1(t) = & \left(p_{10} + \frac{r}{a} - \frac{x_{10}}{2a}\right) e^{-at} - \frac{r}{a} + \frac{x_{10}}{2a} \quad e^{at} \\ & + \frac{bM}{2a^2} \sum_{i=1}^{m-1} \operatorname{sgn}(x_2\,(t_i^-)) \\ & \cdot \left[e^{a(t-t_i)} + e^{-a(t-t_i)} - e^{a(t-t_i-\tau)} - e^{-a(t-t_i-\tau)}\right] \\ & + \frac{bM}{2a^2} \operatorname{sgn}(x_2\,(t_m^-)) \left[e^{a(t-t_m)} + e^{-a(t-t_m)} - 2\right] \end{aligned} \tag{7.8b}$$

$t_m + \tau \leqslant t \leqslant t_{m+1}, \quad p = 1, \ldots, N$

$$\begin{aligned} p_1(t) = & \left(p_{10} + \frac{r}{a} - \frac{x_{10}}{2a}\right) e^{-at} - \frac{r}{a} + \frac{x_{10}}{2a} \quad e^{at} \\ & + \frac{bM}{2a^2} \sum_{i=1}^{\bar{m}} \operatorname{sgn}(x_2\,(t_i^-)) \\ & \cdot \left[e^{a(t-t_i)} + e^{-a(t-t_i)} - e^{a(t-t_i-\tau)} - e^{-a(t-t_i-\tau)}\right]. \end{aligned} \tag{7.8c}$$

3. The costate $p_2(t)$

$0 \leqslant t < t_1$

$$p_2(t) = p_{20} \tag{7.9a}$$

$t_m \leqslant t < t_m + \tau, \qquad m = 1, 2, \ldots, N$

$$p_2(t) = p_{20} - \sum_{i=1}^{m-1} M\,b \, \frac{\operatorname{sgn}(x_2(t_i^-))}{\dot{x}_2(t_i^-)} \left[p_1(t_i) - p_1(t_i - \tau)\right]$$

$$- M\,b \, \frac{\operatorname{sgn}(x_2(t_m^-))}{\dot{x}_2(t_m^-)} \, p_1(t_m) \tag{7.9b}$$

$t_m + \tau \leqslant t < t_{m+1}\,, \qquad m = 1, 2, \ldots, N$

$$p_2(t) = p_{20} - \sum_{i=1}^{m} M\,b \, \frac{\operatorname{sgn}(x_2(t_i^-))}{\dot{x}_2(t_i^-)} \left[p_1(t_i) - p_1(t_i + \tau)\right]. \tag{7.9c}$$

4. The state $x_2(t)$

$0 \leqslant t < t_1$

$$x_2(t) = p_{20}\, t / c \tag{7.10a}$$

$t_m \leqslant t < t_m + \tau , \quad m = 1, 2, ..., N$

$$x_2(t) = \left\{ p_{20} - \sum_{i=1}^{m-1} M b \frac{\operatorname{sgn}(x_2 (t_i^-))}{\dot{x}_2 (t_i^-)} [p_1(t_i) - p_1(t_i+\tau)] - M b \frac{\operatorname{sgn}(x_2 (t_m^-))}{\dot{x}_2 (t_m^-)} p_1(t_m) \right\} (t-t_m)/c \tag{7.10b}$$

$t_m + \tau \leqslant t < t_{m+1} , \quad m = 1, 2, ..., N$

$$x_2(t) = - \tau M b \frac{\operatorname{sgn}(x_2 (t_m^-))}{c \dot{x}_2 (t_m^-)} p_1(t_m+\tau) + \left\{ p_{20} - \sum_{i=1}^{m} M b \frac{\operatorname{sgn}(x_2 (t_i^-))}{\dot{x}_2 (t_i^-)} [p_1(t_i) - p_1(t_i-\tau)] \right\} \cdot (t-t_m) / c \tag{7.10c}$$

7.2.2 Substitution with Intermediate and Final Conditions

The intermediate conditions are

$$x_2 (t_m^-) = s \operatorname{sgn}(x_2 (t_m^-)). \tag{7.11}$$

Substituting Equation (7.11) into Equation (7.10c), the first N equations are obtained as

$$c\,s\,\mathrm{sgn}(x_2\,(t_m^-)) = -\tau M b \frac{\mathrm{sgn}(x_2\,(t_{m-1}^-))}{\dot{x}_2\,(t_{m-1}^-)} p_1(t_{m-1}+\tau)$$

$$+ \left\{ p_{20} - \sum_{i=1}^{m-1} M b \frac{\mathrm{sgn}\,(x_2\,(t_i^-))}{\dot{x}_2\,(t_i^-)} [p_1(t_i) - p_1(t_i+\tau)] \right\} (t_m - t_{m-1}),$$

$$m = 1, 2, \ldots, N. \qquad (7.12)$$

From Equation(7.12), the N unknowns t_m are

$$t_m = A / B \qquad (7.13)$$

where

$$A = c\,s\,\mathrm{sgn}(x_2\,(t_m^-)) + p_{20}\,t_{m-1} + \tau M b \frac{\mathrm{sgn}(x_2\,(t_{m-1}^-))}{\dot{x}_2\,(t_{m-1}^-)} p_1(t_{m-1}+\tau)$$

$$- M b \sum_{i=1}^{m-1} \frac{\mathrm{sgn}(x_2\,(t_i^-))}{\dot{x}_2\,(t_i^-)} [p_1(t_i) - p_1(t_i+\tau)]\; t_{m-1}$$

and

$$B = p_{20} - \sum_{i=1}^{m-1} M b \frac{\mathrm{sgn}(x_2\,(t_i^-))}{\dot{x}_2\,(t_i^-)} [p_1(t_i) - p_1(t_i+\tau)].$$

The final conditions are

$$p_1(t_f) = 0 \quad \text{and} \quad p_2(t_f) = 0.$$

Substituting $p_1(t_f) = 0$ in Equation (7.8c), and $p_2(t_f) = 0$ in Equation (7.9c), the initial costates p_{10} and p_{20} are given by

$$p_{10} = -\frac{r}{a} + \frac{x_{10}}{2a} + \frac{r}{a}e^{at_f} - \frac{x_{10}}{2a}e^{2at_f}$$

$$-\frac{bM}{2a^2}e^{at_f}\sum_{i=1}^{N}\mathrm{sgn}(x_2(t_i^-))$$

$$\cdot\left[e^{a(t_f-t_i)} + e^{-a(t_f-t_i)} - e^{a(t_f-t_i-\tau)} - e^{-a(t_f-t_i-\tau)}\right] \tag{7.14}$$

$$p_{20} = \sum_{i=1}^{N} Mb\,\frac{\mathrm{sgn}(x_2(t_i^-))}{\dot{x}_2(t_i^-)}\,[p_1(t_i) - p_1(t_i+\tau)]. \tag{7.15}$$

Equations (7.13), (7.14) and (7.15) form N+2 nonlinear equations that should be solved simultaneously for p_{10}, p_{20}, and t_i ($i=1,2,\ldots,N$). The algorithm developed for the solution of these equations is presented in the following section.

7.2.3 Estimation of the Initial Costates

The set of Equations (7-13), (7-14) and (7-15), which are functions of p_{10}, p_{20}, $t_1,\ldots,t_N$, did not reduce the complexity of the solution or the its sensitivity to the initial guesses of the costates. So, basically the same situation as in method I exists. For these reasons, even a graphical technique for obtaining an approximate initial value for the costates is bound to fail.

It is here required to minimize the performance index (or equivalently maximize the function I). For every assumed value of N the corresponding suboptimal control is found and the performance index evaluated. The optimal control is then the one that minimizes the performance index.

An upper estimate of the value of N, the number of pulses forming the control u(t), is given here before discussing the approach used to solve the problem. The maximum allowable number of pulses is

$$N_u = \left\lfloor \frac{t_f + d}{\tau + d} \right\rfloor \tag{7.16}$$

where t_f is the final time

τ is the pulse width

d is the dead time

and $\lfloor \; \rfloor$ means the nearest lower integer.

Note that the threshold of the modulator affects the number of pulses; the higher the threshold the less the number of pulses. That is, varying the threshold is equivalent to varying the cost on the pulses.

The approach and the algorithm used to find an estimate for the initial costates are described in the following.

1. Approach

Given all the fixed parameters of the problem, assume that the control u(t) is formed of two pulses initiated at times t_1 and t_2. The lower and upper limits on t_1 and t_2 for the interval $[0, t_f]$ are known (t_{1L}, t_{1H} and t_{2L}, t_{2H} respectively). For a certain set (t_1, t_2), Equations (7.14) and (7.15) and another version of Equation (7-12) given below, are used to find the exact p_{10} and p_{20} for the case of two pulses. If N_u of (7-16) is larger than two, the values of p_{10} and p_{20} are used as initial estimates to find the exact initial costates for $N = 3, 4, \ldots,$ up to N_u.

The new version of Equation (7-12) is obtained next. For $i = 1$, Equation (7-12) gives

$$c\ s\ \mathrm{sgn}(x_2(t_1^-)) = p_{20}\ t_1,$$

but
$$\mathrm{sgn}(x_2(t_1^-)) = \mathrm{sgn}(p_{20}),$$

then
$$s = p_{20}\ t_1 / c\ \mathrm{sgn}(p_{20}),$$

or
$$s = |p_{20}|\ t_1 / c. \tag{7.17}$$

For $i = 2$, Equation (7-12) can be written as follows

$$c\ s\ \mathrm{sgn}(x_2(t_2^-)) = p_{20}(t_2 - t_1) - \tau M b \frac{\mathrm{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} p_1(t_1+\tau)$$

$$- M b \frac{\mathrm{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} [p_1(t_1) - p_1(t_1+\tau)](t_2 - t_1), \tag{7.18}$$

but $\dot{x}_2(t_1^-) = p_{20}$

and $$\frac{\operatorname{sgn} x_2(t_1^-)}{\dot{x}_2(t_1^-)} = \frac{\operatorname{sgn} p_{20}}{p_{20}} = \frac{1}{|p_{20}|}.$$

Equation (7-18) can then be written as follows

$$c\, s\, \operatorname{sgn}(x_2(t_2^-))\ |p_{20}| = p_{20}\ |p_{20}|\ (t_2-t_1) - \tau\, M\, b\ p_1(t_1+\tau)$$

$$- M\, b\, [p_1(t_1) - p_1(t_1+\tau)]\ (t_2-t_1). \tag{7.19}$$

Substituting Equation (7-17) and p_{20} $\operatorname{sgn}(p_{20})$ for $|p_{20}|$ in Equation (7-19) and with some manipulation one obtains

$$p_{20}^2 = \frac{-\tau\, M\, b\ p_1(t_1+\tau) - M\, b\, [p_1(t_1) - p_1(t_1+\tau)]\ (t_2-t_1)}{t_1\ \operatorname{sgn} x_2(t_2^-) - (t_2-t_1)\ \operatorname{sgn} p_{20}} \tag{7.20}$$

which is an alternative form for Equation (7.12) to be used in the solution.

2. Algorithm

Equations (7-17) and (7-20) are used along with Equations (7-14) and (7-15) to find p_{10} and p_{20} in two steps:

Step 1: Gross search

In this step the possible regions where a solution exists are located through the following procedure

i) assume the polarity of pulses

ii) change the value of t_2 from t_{2L} to t_{2H} ($t_{2L} = t_{1L}+\tau+d$)

iii) change the value of t_1 from t_{1L} to t_{1H} ($t_{1H} = t_{2H}-d-\tau$)

iv) compute p_{10} from Equation (7.14)

v) compute p_{20}^2 from Equation (7.20). If this value is negative go to (iii), if not continue to next step

vi) compute p_{20}

vii) check if the results satisfy Equation (7.15). If it does not go to (iii); however if it does, continue to next step

viii) compute corresponding threshold s' from Equation (7.17). If s' is not equal to the specified threshold s repeat the above procedure starting at (ii). If s' is found to equal the specified threshold, the values of p_{10}, p_{20}, t_1, and t_2 are successfully obtained.

A flowchart of the above method is given in Figure 7.2. The algorithm is executed once for each set of possible pulse polarities. For this case (two pulses), four possibilities exist. When the wrong polarity is assumed, it is usually found that the value of $p_2^2(0)$ is negative or that the error in the final costate $p_2(t_f)$ is very large. This error value is denoted by F_2 and is obtained by evaluating Equation (7.9c) at $t = t_f$. In this procedure the incremental steps in t_1 and t_2 should be reasonably large.

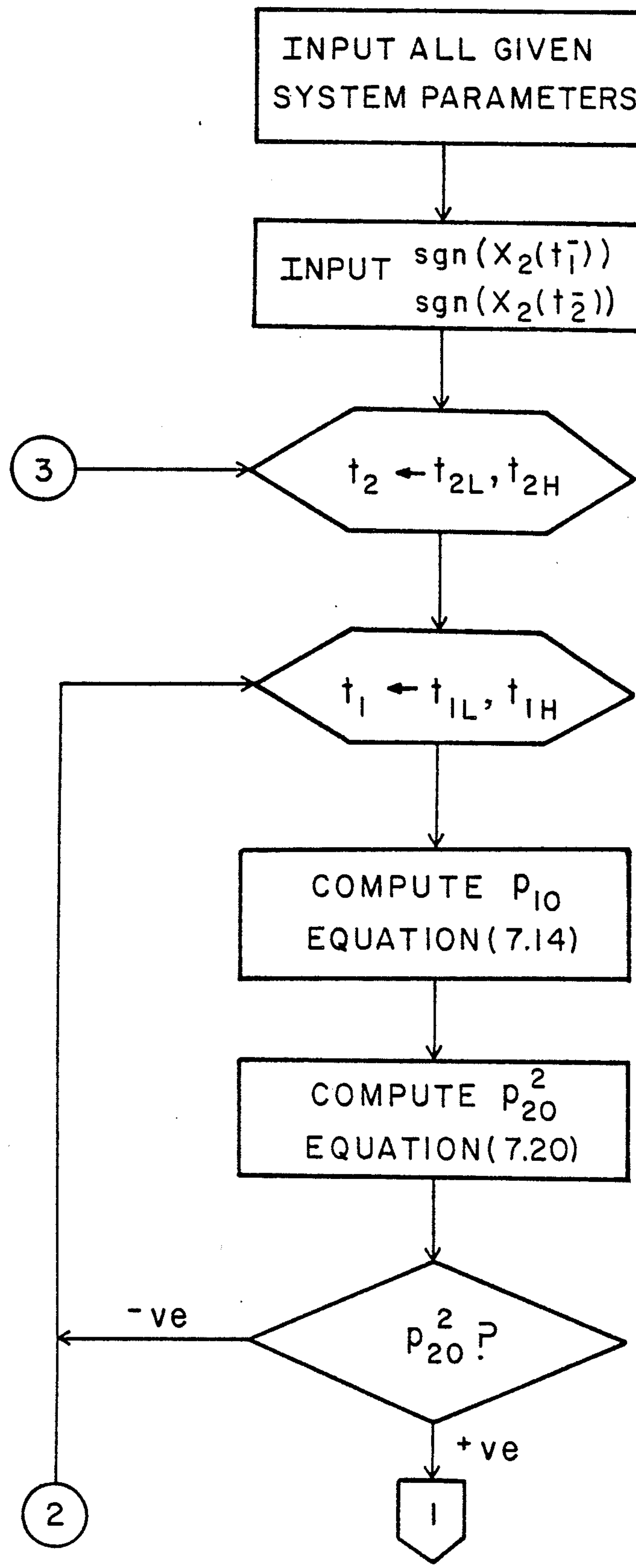


Figure 7.2 Flowchart for proposed algorithm to find initial costates for two pulses

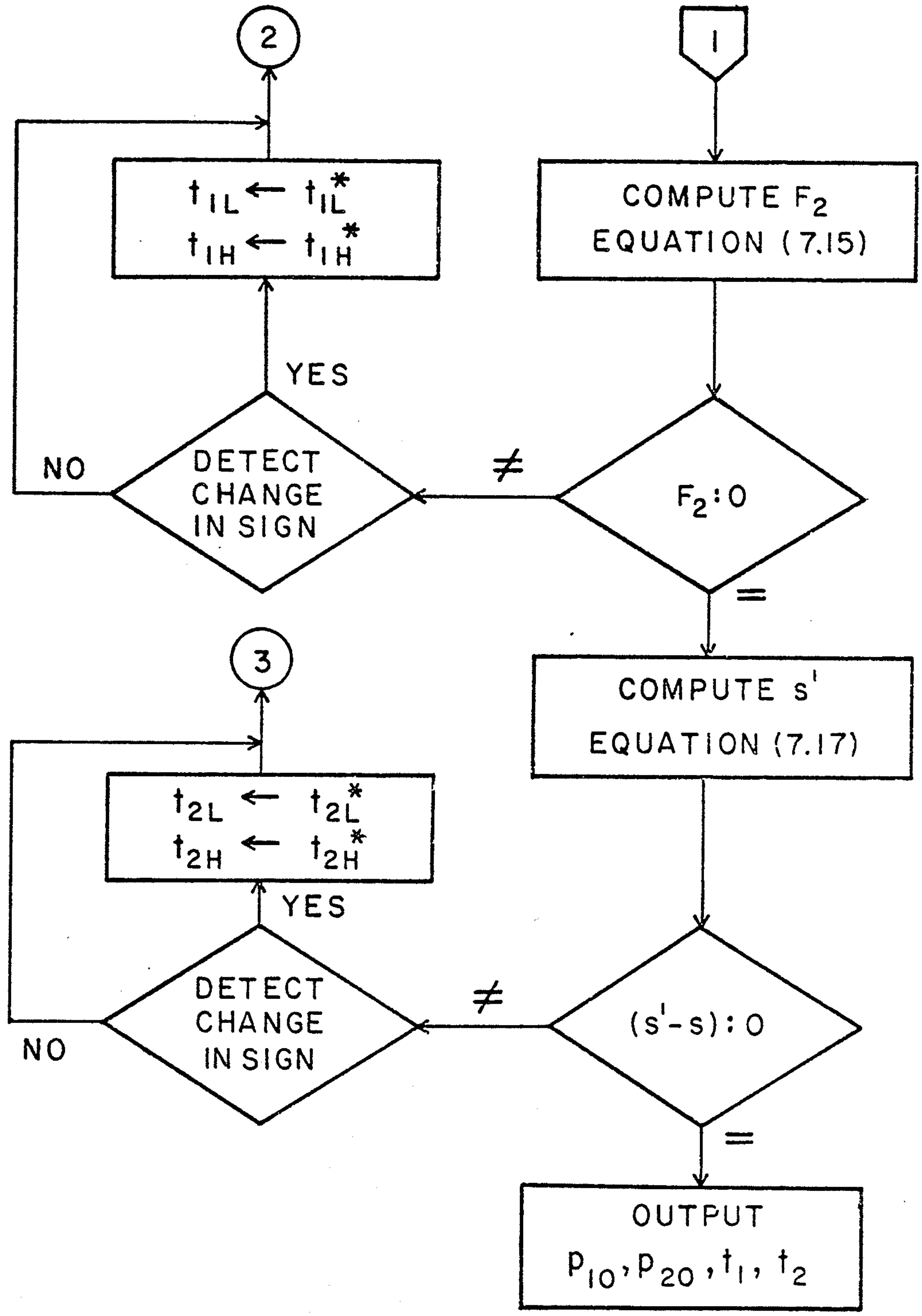


Figure 7.2 Continued

As a result of these preliminary calculations, new limits (t^*_{1L}, t^*_{1H} and t^*_{2L}, t^*_{2H}) on the optimum t_1 and t_2 are found. Within these limits F_2 is changing sign and the computed threshold s' is close to the desired threshold s. The FORTRAN program denoted by "MAG2" which is used to perform the above calculations is listed in Appendix B.

Step 2: Fine search

Once the region of approximate p_{10} and p_{20} is found another calculation procedure is performed to find the exact values of t_1, t_2, p_{10} and p_{20} that satisfy the conditions $F_2 = 0$ and $s' = s$. This calculation is performed by the program "COST" and the subroutine "COSTS". These are essentially the same as "MAG2" mentioned above except that the right polarities of the pulses are supplied. Again, the error function F_2 changes sign in the region $[t^*_{1L}, t^*_{1H}]$ and $(s'-s)$ changes sign in the interval $[t^*_{2L}, t^*_{2H}]$. Accordingly, using an iterative method it is easy to find the exact zeros of the functions F_2 and $(s'-s)$.

The method used here comprises a search over a defined interval until a change in sign is detected. The interval is then narrowed (10 times smaller) and the search restarted.

7.2.4 The Optimal Control

At this point a good estimate of the region of initial guesses for the initial costates is found. The procedure to find the exact initial costate vector for the case of $N_u>2$, is summarized in the following

1. increment the value of N
2. form a grid of values of p_{10} and p_{20} around the initial guess
3. compute the time of initiation t_i of the pulses from Equation (7.13)
4. compute the final costates $F_1 = p_1(t_f)$ and $F_2 = p_2(t_f)$ from Equations (7.8c) and (7.9c), respectively
5. find the approximate initial costate vector graphically by plotting the contours of F_1 and F_2 for different grid values of p_{10} and p_{20}. The desired solution is when $p_1(t_f) = 0$ and $p_2(t_f) = 0$ are satisfied simultaneously.
6. use an iterative technique to locate the point of exact zero final states, and hence the optimal control sequence is obtained.

The flowchart shown in Figure 7.3 illustrates the main steps for the computation of the time instants t_i of the pulse initiation and the final costate vector $\underline{p}(t_f)$ for a given intial costate vector $\underline{p}(0)$.

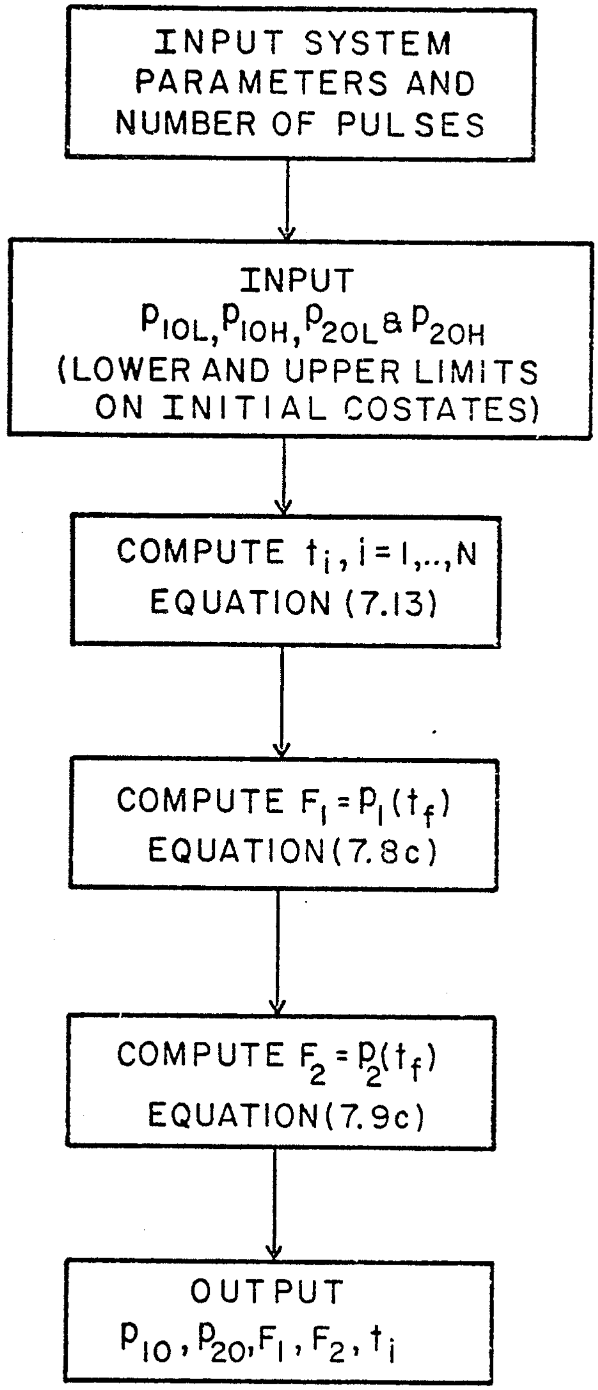


Figure 7.3 Flowchart for method to compute $\underline{p}(t_f)$, t_i for a given $\underline{p}(0)$

The above procedure is repeated for incremented values of N until it reaches the value N_u of (7.16). A solution is not guaranteed to exist for $N = N_u$ or even some smaller N due to the constraints on the problem, mainly the threshold. If N_u is large, N can be incremented by steps greater than one as long as the initial guess still holds. If no convergence occurs at a certain region of the solution, the increment size is reduced until an optimum N is found. This N is the number of pulses forming the optimal control and minimizing the performance index. A check should always be performed to assure that the constraint of the minimum dead time between consecutive pulses is satisfied. If this condition is not met, the previous N then gives the optimum solution.

For practical reasons, it may be preferred to allow $p(t_f)$ to be within a small tolerance v from zero. As a result, the control sequence may contain an extra pulse. In this case, the limiting factors on the solution are that

1. $t_N \leq t_f - \tau$

   where t_N is the time of initiation of the last pulse in the control sequence

2. the minimum dead time should exist between any two consecutive pulses.

The iterative technique used in step 6 above is simple and converges quickly. Starting with the initial guess, the values of F_1 and F_2 (i.e, $p_1(t_f)$, $p_2(t_f)$) are computed. New values of p_{10} and p_{20} are then found using the formulae

$$p_{10N} = p_{10O} + \varepsilon_1 F_1$$
$$p_{20N} = p_{20O} + \varepsilon_2 F_2 \qquad (7.21)$$

where ε_1 and ε_2 are step sizes for adjustment and N and O denote new and old values respectively. To speed up convergence, small values of ε_1 and ε_2 may be used during early iterations. They are then increased as F_1 and F_2 converge to their zero values.

The FORTRAN programs used for this part of the study are

SOLVE : to compute t_i (i=1,...,N), F_1, and F_2 for certain p_{10} and p_{20}

SOLVI : which is the same as SOLVE with the iteration algorithm added to it.

The program listings for "SOLVE" and "SOLVI" are given in Appendix B.

7.2.5 Example

The method described above is applied to the system represented by Equation (7.1) which minimizes the performance index given in Equation (6.11). The following parameter values are used:

$a = -2.0$	$x_{10} = -2.0$
$b = 1.0$	$s = 0.1$
$c = 1.0$	$\tau = 0.2$ s
$r = 0.5$	$d = 0.2$ s
$M = 1.0$	$t_f = 2.0$ s

The method is illustrated in the following steps:

1) The maximum allowable number of pulses is calculated as

$$N_u = \left\lfloor \frac{2.0 + 0.2}{0.2 + 0.2} \right\rfloor = 5 \text{ pulses.}$$

2) Applying the algorithm presented in Section 7.2.3 for two pulses, it is realized that, for this set of parameters, only the possibility of two positive pulses results in a solution. The lower and upper bounds on t_1 and t_2 that guarantee a region where $p_2(t_f)$ is zero and a computed threshold that is close to the desired value is found to be

$$t^*_{1L} = 0.17 \text{ s} \qquad t^*_{1H} = 0.25 \text{ s}$$
$$t^*_{2L} = 1.42 \text{ s} \qquad t^*_{2H} = 1.50 \text{ s.}$$

3) Using the programs COST and COSTS, the optimal solution is found for this case as

$$\begin{aligned}
p_1(0) &= 0.7168508\\
p_2(0) &= 0.4615744\\
p_1(t_f) &= 0.0\\
p_2(t_f) &= -0.0000045\\
t_1 &= 0.2166519\\
t_2 &= 1.4843864\\
s &= 0.1000009\\
J &= 1.2296241.
\end{aligned}$$

4) For N = 3, the above $p_1(0)$ and $p_2(0)$ are chosen as initial guess and $p_1(t_f)$ and $p_2(t_f)$ are computed for a grid of point around the initial guess. The graphs shown in Figure 7.4 are produced using program SOLVE. From these graphs, it was found that $p_1(t_f)$ and $p_2(t_f)$ are approximately zero at $p_1(0) = 0.7082$ and $p_2(0) = 0.575$.

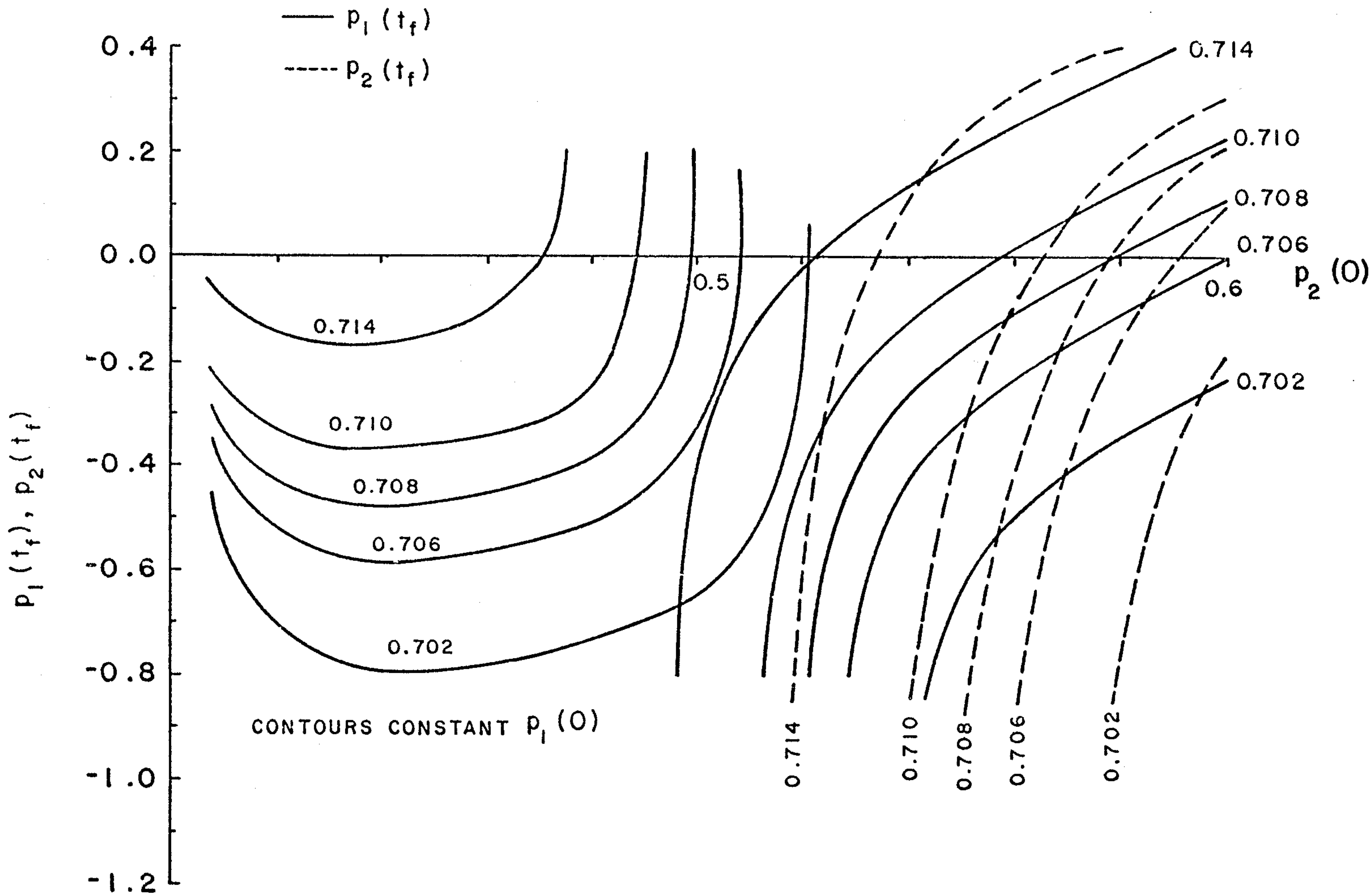


Figure 7.4 Contours of $\underline{p}(0)$ and $\underline{p}(t_f)$ for three pulses

5) Program SOLVI is then used to find the exact values of $p_1(0)$ and $p_2(0)$. The following values are obtained

$$
\begin{aligned}
p_1(0) &= 0.708226 \\
p_2(0) &= 0.576350 \\
p_1(t_f) &= 0.000001 \\
p_2(t_f) &= -0.000001 \\
t_1 &= 0.173506 \\
t_2 &= 0.857185 \\
t_3 &= 1.744180 \\
J &= 1.182445.
\end{aligned}
$$

6) Repeating step 4 for N = 4, Figure 7.5 is obtained and the approximate values for $\underline{p}(0)$ are found to be $p_1(0) = 0.6992$ and $p_2(0) = 0.673$.

7) Performing step 5 for N = 4 and using the results of step 6, the following results are obtained

$$
\begin{aligned}
p_1(0) &= 0.699592 \\
p_2(0) &= 0.669943 \\
p_3(t_f) &= -0.000048 \\
p_4(t_f) &= 0.000034 \\
t_1 &= 0.149266 \\
t_2 &= 0.629537 \\
t_3 &= 1.208169 \\
t_4 &= 1.889115.
\end{aligned}
$$

It should be noticed here that the condition $t_N \leq t_f - \tau$ is violated. It is, therefore, concluded that the optimal control contains a sequence of 3 pulses as given by step 5, where the performance index is minimized.

8) For practical applications, $p_1(t_f)$ and $p_2(t_f)$ may be allowed to be within a small tolerance from zero. For such case, a number of initial values of $\underline{p}(0)$ can be found that satisfy the dead time condition and the final time condi-

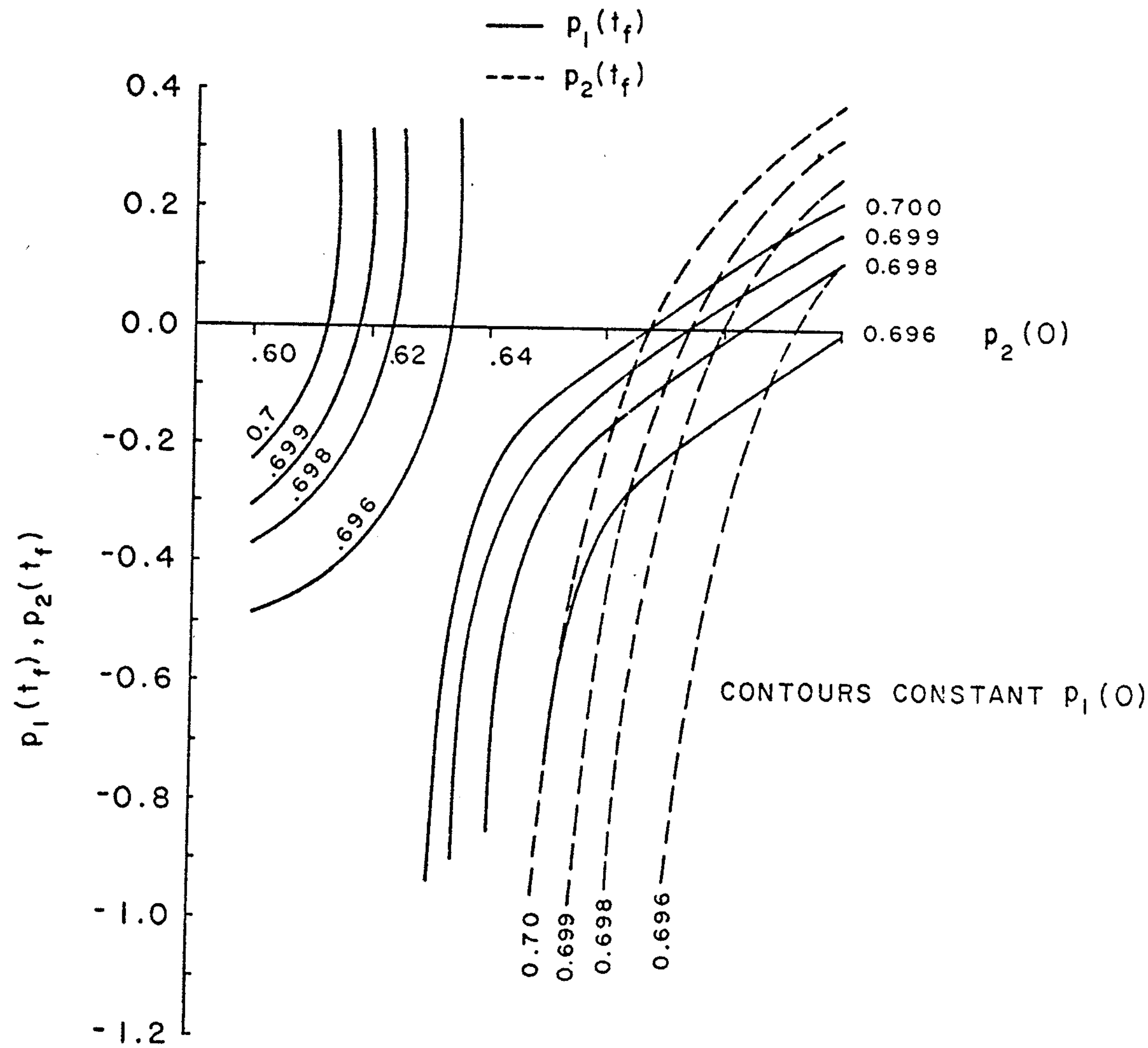


Figure 7.5 Contours of $\underline{p}(0)$ and $\underline{p}(t_f)$ for four pulses

tion. The choice of $\underline{p}(0)$ then depends on the performance index. For this example three points were chosen that satisfy the following final condition (these are given in Table 7.1)

$$p_1(t_f) < 0.05 \quad \text{and} \quad p_2(t_f) < 0.05.$$

From Table 7.1, the solution that minimizes the performance index is the second one; that is

$$\begin{aligned}
p_1(0) &= 0.697300 \\
p_2(0) &= 0.684800 \\
p_3(t_f) &= -0.036530 \\
p_4(t_f) &= -0.000230 \\
t_1 &= 0.146028 \\
t_2 &= 0.603248 \\
t_3 &= 1.152449 \\
t_4 &= 1.799910 \\
J &= 1.138605.
\end{aligned}$$

9) For $N = N_u = 5$, the solution violates the two conditions of final time and dead time and this case is rejected.

The results of this example are summarized in Tables 7.2 and 7.3. The optimal control $u(t)$ contains three pulses. Figure 7.6a illustrates the state variable $x_1(t)$, the costate $p_1(t)$ and the optimal control $u(t)$. Figure 7.6b illustrates the state variable $x_2(t)$ and the costate $p_2(t)$.

Depending on the application, one may choose a suboptimal control that contains four pulses and minimizes the performance index. Figures 7.7a and 7.7b illustrate the state and costate functions and the control $u(t)$ for this case.

Table 7-1

Initial and Final Costates for Three Controls

Containing Four Pulses Each.

P_{10}	P_{20}	$P_1(t_f)$	$P_2(t_f)$	t_4	J
0.6976	0.6841	-0.02342	0.02235	1.8000	1.1388371
0.6973	0.6848	-0.03653	-0.00023	1.7999	1.1386048
0.6979	0.6836	-0.00900	0.04799	1.7985	1.1390718

Table 7-2

Initial and Final Costates and Performance Index for the Optimal Control (*) and Two Suboptimal Controls.

N	p_{10}	p_{20}	$p_1(t_f)$	$p_2(t_f)$	J
2	0.7168502	0.4615744	0.0	0.0000045	1.2296241
3*	0.708226	0.57635	0.000001	-0.000001	1.182445
4	0.6973	0.6845	-0.03653	-0.00023	1.1386048

Table 7-3

Time of Pulse Initiation (s) for the Optimal Control (*) and Two Suboptimal Controls

N	t_1	t_2	t_3	t_4
2	0.2166519	1.4843864	-	-
3*	0.173506	0.857185	1.73418	-
4	0.146028	0.603248	1.152449	1.79991

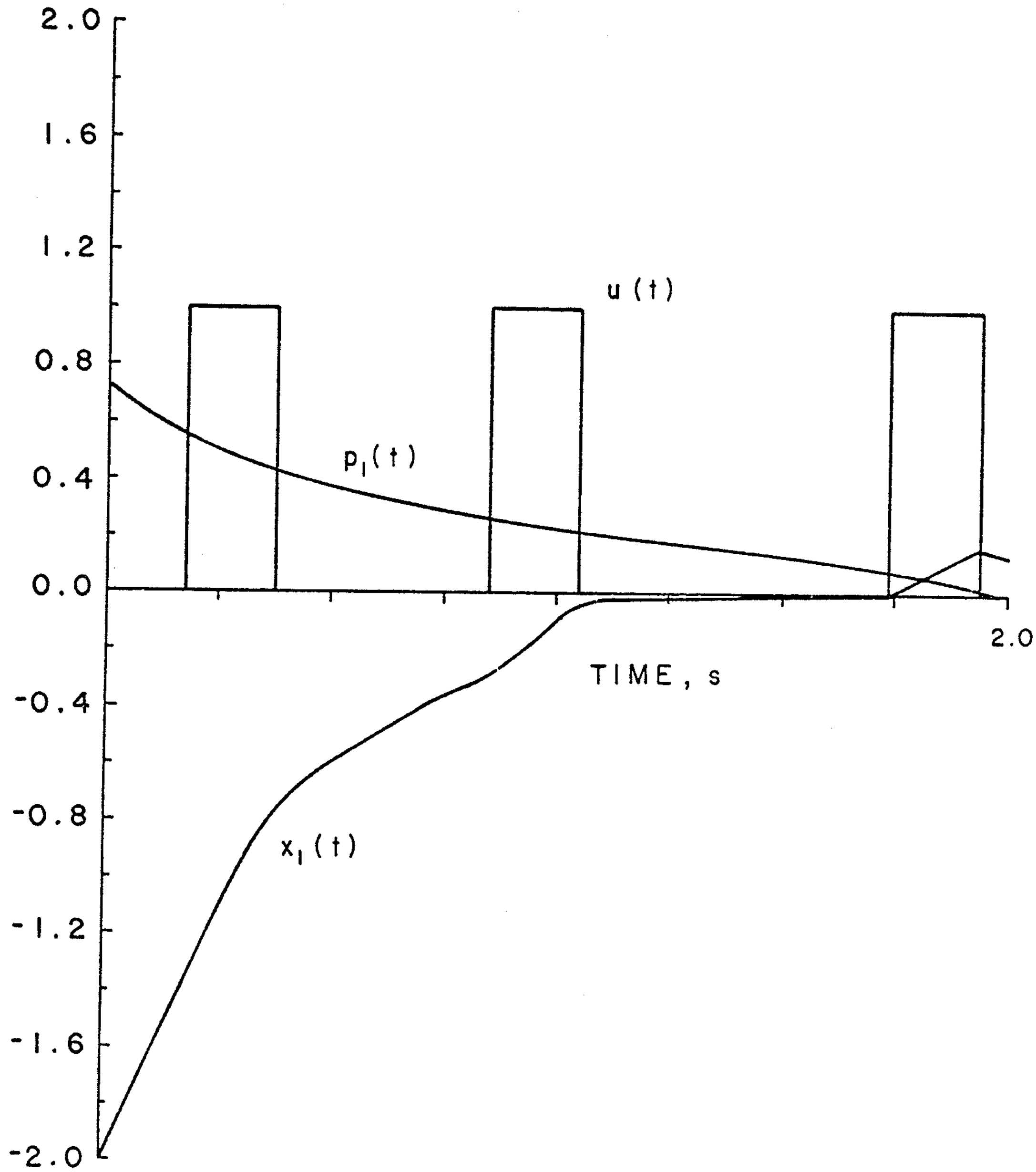


Figure 7.6a State $x_1(t)$, costate $p_1(t)$ and optimal control $u(t)$

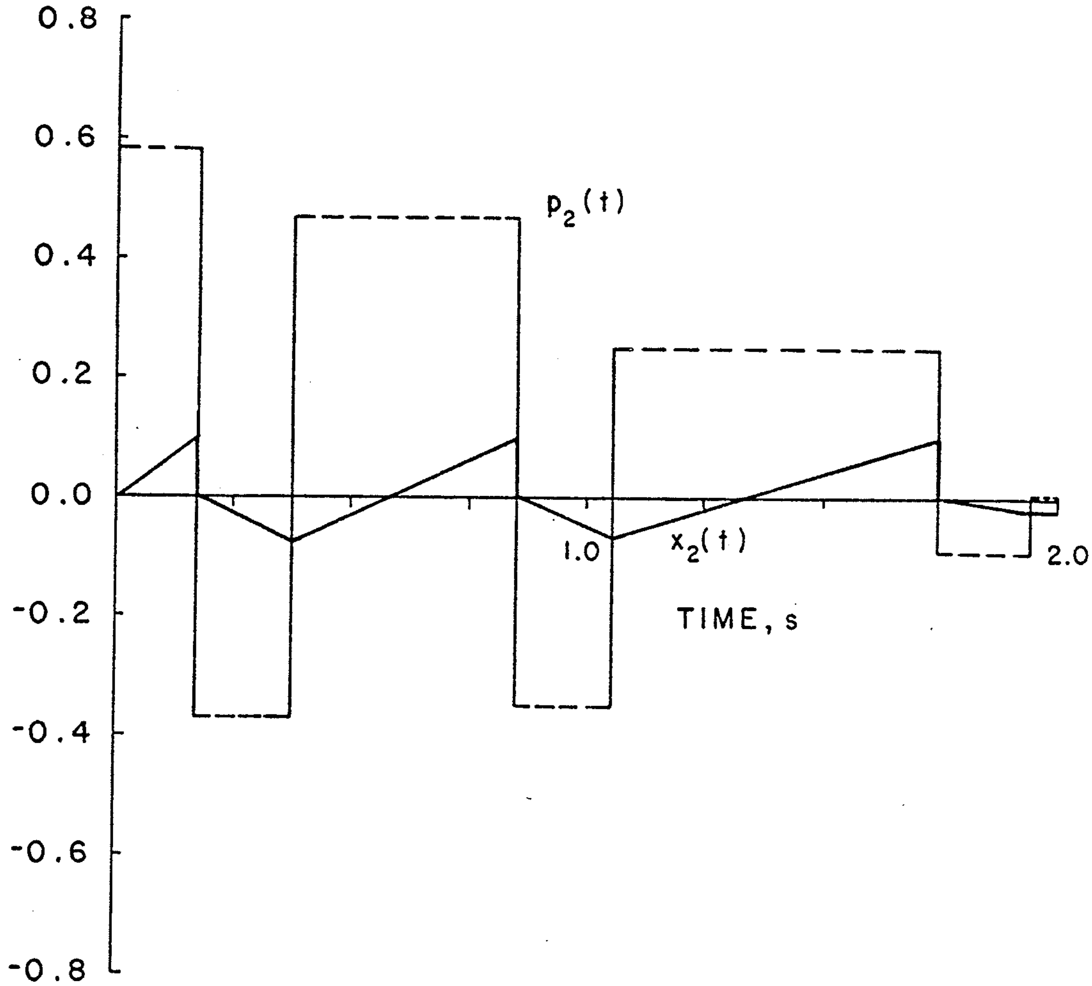


Figure 7.6b State $x_2(t)$ and costate $p_2(t)$

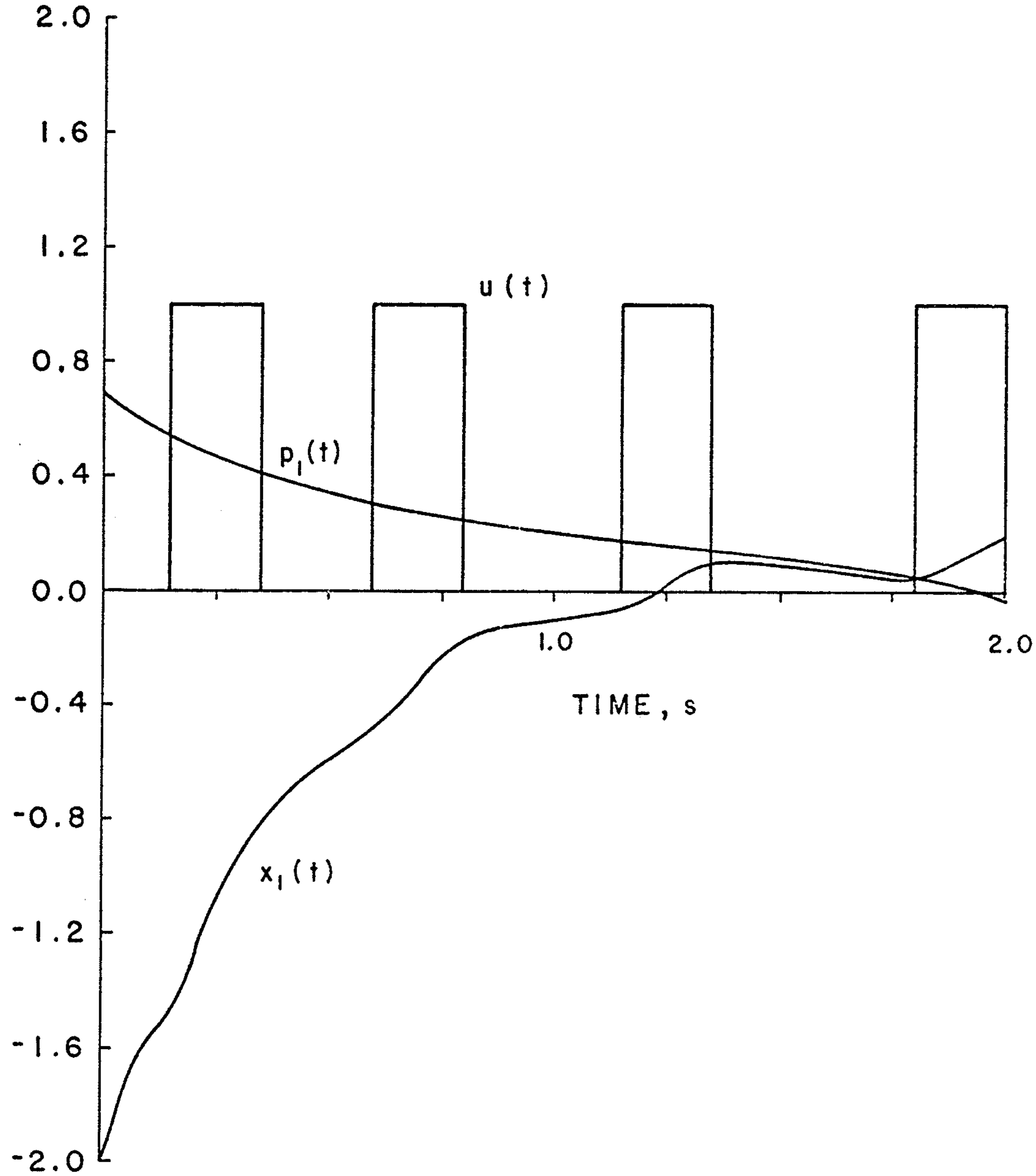


Figure 7.7a State $x_1(t)$, costate $p_1(t)$ and suboptimal control u(t)

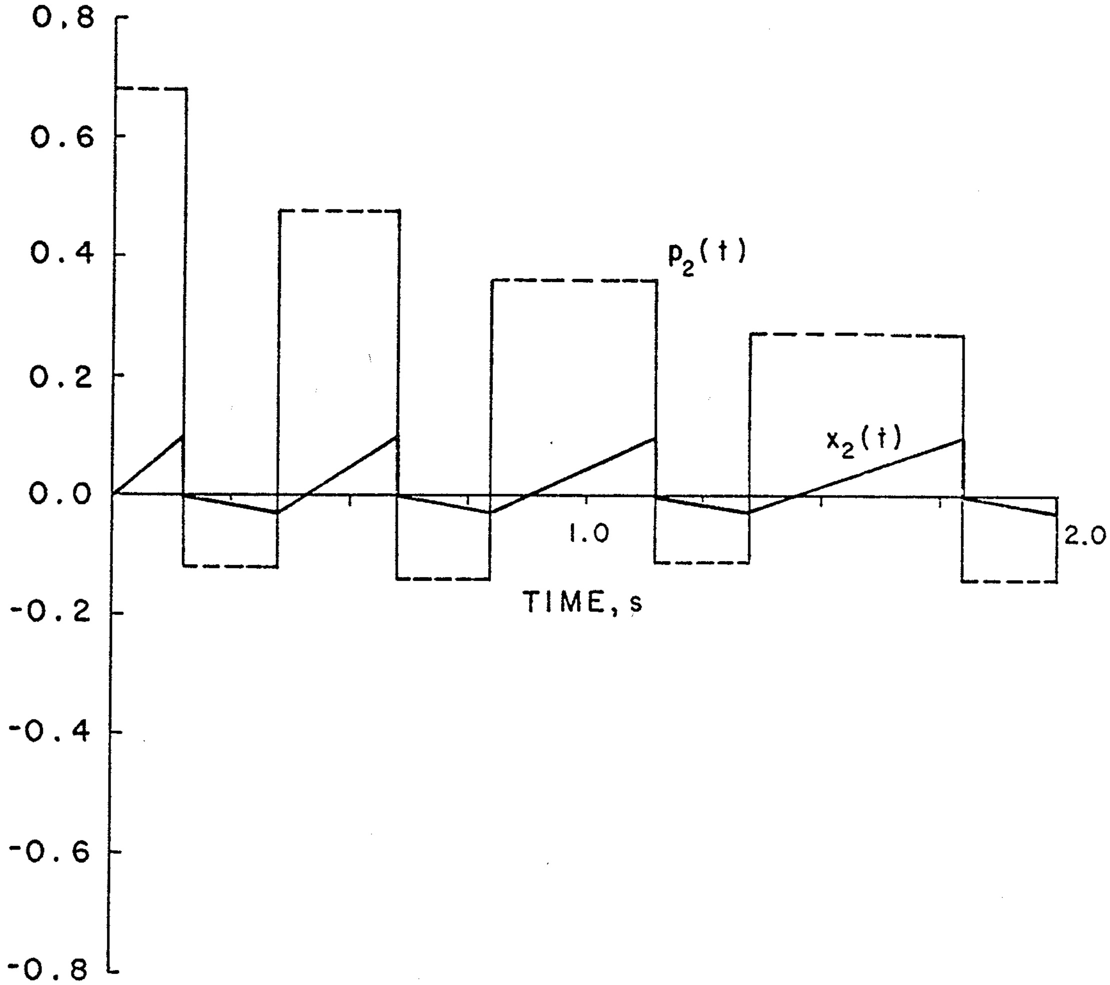


Figure 7.7b State $x_2(t)$ and costate $p_2(t)$ for four pulses

7.3 FURTHER COMMENTS

It is assumed in this chapter that the system contains a first order linear plant and an integral pulse-fequency modulator. Also, the performance index is assumed quadratic. These assumptions are only made to help find the solution and to make the presentation clearer.

In general, any linear or nonlinear plant can be used as long as an analytic solution can be found. Linearization methods can be applied to some nonlinear plants. The real limitation is the complexity introduced by a high order system. A system of order n results in 2n state-costate equations and n final conditions to satisfy.

There is no limitation on the modulator or the performance indices used. Any type-II modulator and commonly used performance index will introduce no difficulties to the solution.

The idea of using CSMP for some of the steps of solution method II is discarded. CSMP is found to be not versatile enough for this problem and its use for finding an initial guess is not possible. The CSMP method was tried for the final iterations and it converged to zero final costates yet with the wrong results. This is due to a very large round-off error in addition to the errors due to numerical integration.

Chapter VIII

SUMMARY AND CONCLUSIONS

In this dissertation the optimal feedback controller and, consequently, the optimal control sequence for pulse-frequency modulated systems with nonlinear plants and which may contain type-II modulators were studied.

The modified maximum principle of Onyshko and Noges was extended to nonlinear open-loop systems. This extended MMP provides only necessary conditions for optimality. Its application to a second order nonlinear plant was then demonstrated.

The optimal controller and the modulator that provide the optimum PFM control sequence for minimizing a quadratic performance index were studied. An off-line as well as an on-line method for obtaining the control sequence was given. The method of variation of extremals was found to be adequate for the solution of the resulting two-point boundary-value problem (TPBVP) and the Continuous System Modeling Program (CSMP) was used for the simulation of the system.

The optimal controller for systems containing a modulator of the second kind were also considered. The presence of the modulator introduced complexity in the solution of the TPBVP. The simulation using CSMP failed to give meaningful results due to a very large round-off error and due to a lack of numerical technique to solve the TPBVP of such a complex system.

An alternative method to solve this problem was applied in two steps

1. The first step is the analytic solution of the state-costate equations to be solved for the variables $\underline{p}(0)$ and the times of pulse initiation.
2. The second is a numerical solution to find an estimate of the initial values $\underline{p}(0)$, then, using graphic techniques and an iterative method, the optimal control is found.

The method is demonstrated for a system containing a first order linear plant and an integral pulse-frequency modulator with a quadratic performance index.

The optimization problem treated in this dissertation is extremely complex. The results, as developed here, are theoretical and not perhaps as practical as was hoped. However, the present study is a preliminary step in the area of optimization of nonlinear PFM systems. It is hoped that it will serve as a stimulus for further research in this area.

More work is still needed to improve the algorithms developed in this dissertation. For specific application, specialized computer program packages can be written.

The optimization problem would much benefit from improvements in the simulation techniques, also from developments of methods for the solution of discrete nonlinear two-point boundary value problems.

Possible extensions of the MMP to closed loop systems should be considered. The control function in closed loop systems is a function of the state variables and of the time rather than just the time alone.

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Appendix A

SOLUTION OF STATE-COSTATE EQUATIONS - CHAPTER 6

SOLUTION BY INDUCTION

1. <u>The state variable $x_1(t)$</u>

$$\dot{x}_1 = ax_1 + bu , \qquad x_1(0) = x_{10} . \tag{A-1}$$

Solving for $x_1(t)$

$$x_1(t) = x_{10}\, e^{at} + b\, e^{at} \int_0^t u\, e^{-at}\, dt . \tag{A-2}$$

For <u>$0 \leqslant t < t_1$</u> , $u = 0$

$$x_1(t) = x_{10}\, e^{at} . \tag{A-3}$$

For <u>$t_1 \leqslant t < t_1 + \tau$</u> , $u = M\, \mathrm{sgn}(x_2\,(t_1^-))$

$$x_1(t) = x_{10}\, e^{at} + b\, e^{at} \int_{t_1}^{t} u\, e^{-at}\, dt$$

$$x_1(t) = x_{10}\, e^{at} + \frac{bM}{a}\, \mathrm{sgn}(x_2\,(t_1^-)) \left(e^{a(t-t_1)} - 1 \right) . \tag{A-4}$$

For <u>$t_1 + \tau \leqslant t < t_2$</u> , $u = 0$

$$x_1(t) = x_{10}\, e^{at} + b\, e^{at} \int_{t_1}^{t_1+\tau} u\, e^{-at}\, dt$$

$$x_1(t) = x_{10}\, e^{at} + \frac{bM}{a}\, \mathrm{sgn}(x_2\,(t_1^-))(1 - e^{-a\tau})\, e^{a(t-t_1)} . \tag{A-5}$$

For $\underline{t_2 \leqslant t < t_2 + \tau}$, $u = M \operatorname{sgn}(x_2(t_2^-))$,

$$x_1(t) = x_{10} e^{at} + b e^{at} \int_{t_1}^{t_1+\tau} u e^{-at} dt + b e^{at} \int_{t_2}^{t} u e^{-at} dt$$

$$x_1(t) = x_{10} e^{at} + \frac{bM}{a} \operatorname{sgn}(x_2(t_1^-)) \left(1 - e^{-a\tau}\right) e^{a(t-t_1)}$$

$$+ \frac{bM}{a} \operatorname{sgn}(x_2(t_2^-)) \left(e^{a(t-t_2)} - 1\right) . \qquad \text{(A-6)}$$

For $\underline{t_2 + \tau \leqslant t < t_3}$, $u = 0$,

$$x_1(t) = x_{10} e^{at} + b e^{at} \int_{t_1}^{t_1+\tau} u e^{-at} dt + b e^{at} \int_{t_2}^{t_2+\tau} u e^{-at} dt$$

$$x_1(t) = x_{10} e^{at} + \frac{bM}{a} e^{at} \left(1 - e^{-a\tau}\right) [\operatorname{sgn}(x_2(t_1^-)) e^{-at_1}$$

$$+ \operatorname{sgn}(x_2(t_2^-)) e^{-at_2}] . \qquad \text{(A-7)}$$

In general, for the m^{th} pulse, $m = 1, 2, \ldots, N$, and by induction

for $t_m \leqslant t < t_m + \tau$, $u = M \operatorname{sgn}(x_2(t_m^-))$

$$x_1(t) = x_{10} e^{at} + \frac{bM}{a} \left(1 - e^{-a\tau}\right) e^{at} \sum_{i=1}^{m-1} \operatorname{sgn}(x_2(t_i^-)) e^{-at_i}$$

$$+ \frac{bM}{a} \operatorname{sgn}(x_2(t_m^-)) \left(e^{+a(t-t_m)} - 1\right) , \qquad \text{(A-8)}$$

and for $t_m + \tau \leqslant t < t_{m+1}$

$$x_1(t) = x_{10}\, e^{at} + \frac{bM}{a}\, e^{at}\left(1 - e^{-a\tau}\right)\sum_{i=1}^{m} \mathrm{sgn}\,(x_2\,(t_i^-))\; e^{-at_i}. \tag{A-9}$$

2. The Costate Variable $p_1(t)$

$$\dot{p}_1 = -\,ap_1 + x_1 - r\,, \qquad p_1(0) = p_{10} \tag{A-10}$$

solving for $p_1(t)$

$$p_1(t) = p_{10}\, e^{-at} + e^{-at}\int_0^t (x_1 - r)\, e^{at}\, dt$$

but since r is assumed constant

$$p_1(t) = \left(p_{10} + \frac{r}{a}\right) e^{-at} - \frac{r}{a} + e^{-at}\int_0^t x_1\, e^{at}\, dt\,. \tag{A-11}$$

For $0 \leqslant t < t_1$, $x_1(t)$ as in (A-3)

$$p_1(t) = \left(p_{10} + \frac{r}{a}\right) e^{-at} - \frac{r}{a} + e^{-at}\int_0^t x_{10}\, e^{2at}\, dt$$

$$p_1(t) = \left(p_{10} + \frac{r}{a} - \frac{x_{10}}{2a}\right) e^{-at} - \frac{r}{a} + \frac{x_{10}}{2a}\, e^{at} \tag{A-12}$$

$$= p_{11}(t)\,.$$

For $\underline{t_1 \leqslant t < t_1 + \tau}$, $x_1(t)$ as in (A-4)

$$p_1(t) = \left(p_{10} + \frac{r}{a}\right) e^{-at} - \frac{r}{a} + e^{-at} \int_0^t x_{10}\, e^{2at}\, dt$$

$$+ e^{-at} \int_{t_1}^{t} x_{11}(t)\, e^{at}\, dt$$

where $x_{11}(t) = \frac{bM}{a} \operatorname{sgn}(x_2(t_1^-)) \left[e^{a(t-t_1)} - 1\right]$.

Then

$$p_1(t) = p_{11}(t) + \frac{bM}{2a^2} \operatorname{sgn}(x_2(t_1^-)) \left[e^{a(t-t_1)} + e^{-a(t-t_1)} - 2\right]. \quad \text{(A-13)}$$

For $\underline{t_1 + \tau \leqslant t < t_2}$, $x_1(t)$ as in (A-5)

$$p_1(t) = p_{11}(t) + e^{-at} \int_{t_1}^{t_1+\tau} x_{11}(t)\, e^{at}\, dt$$

$$+ e^{-at} \int_{t_1+\tau}^{t} x_{12}(t)\, e^{at}\, dt$$

where $x_{12}(t) = \frac{bM}{a} \operatorname{sgn}(x_2(t_1^-)) \left(1 - e^{-a\tau}\right) e^{a(t-t_1)}$.

Then

$$p_1(t) = p_{11}(t) + \frac{bM}{2a^2}\,\mathrm{sgn}(x_2\,(t_1^-))\,\Big[e^{a(t-t_1)} - e^{a(t-t_1-\tau)}$$

$$+ e^{-a(t-t_1)} - e^{-a(t-t_1-\tau)}\Big]. \qquad \text{(A-14)}$$

For $\underline{t_2 \leqslant t < t_2 + \tau}$, $x_1(1)$ as in (A-6)

$$p_1(t) = p_{11}(t) + e^{-at}\int_{t_1}^{t_1+\tau} x_{11}(t)\,e^{at}\,dt$$

$$+ e^{-at}\int_{t_1+\tau}^{t_2} x_{12}(t)\,e^{at}\,dt + e^{-at}\int_{t_2}^{t} x_{13}(t)\,e^{at}\,dt$$

where $x_{13}(t) = \frac{bM}{a}\,\mathrm{sgn}\,(x_2\,(t_1^-))\left(1 - e^{-a\tau}\right)\,e^{a(t-t_1)}$

$$+ \frac{bM}{a}\,\mathrm{sgn}\,(x_2\,(t_2^-))\left(e^{a(t-t_2)} - 1\right).$$

Then

$$p_1(t) = p_{11}(t) + \frac{bM}{2a^2}\,\mathrm{sgn}\,(x_2\,(t_1^-))$$

$$\cdot\,\Big[e^{a(t-t_1)} - e^{a(t-t_1-\tau)} + e^{-a(t-t_1)} - e^{-a(t-t_1-\tau)}\Big]$$

$$+ \frac{bM}{2a^2}\,\mathrm{sgn}(x_2\,(t_2^-))\Big[e^{a(t-t_2)} + e^{-a(t-t_2)} - 2\Big]. \qquad \text{(A-15)}$$

For $\underline{t_2 + \tau \leq t < t_3}$, $x_1(t)$ as in (A-7)

$$p_1(t) = p_{11}(t) + e^{-at} \int_{t_1}^{t_1+\tau} x_{11}(t)\, e^{at}\, dt$$

$$+ e^{-at} \int_{t_1+\tau}^{t_2} x_{12}(t)\, e^{at}\, dt$$

$$+ e^{-at} \int_{t_2}^{t_2+\tau} x_{13}(t)\, e^{at}\, dt$$

$$+ e^{-at} \int_{t_2+\tau}^{t} x_{14}(t)\, e^{at}\, dt$$

where $x_{14}(t) = \frac{Mb}{a}\, e^{at}\left(1 - e^{-a\tau}\right)\left[\operatorname{sgn}\,(x_2\,(t_1^-))\, e^{-at_1}\right.$

$$\left. + \operatorname{sgn}(x_2\,(t_2^-))\, e^{-at_2}\right].$$

Then

$$p_1(t) = p_{11}(t) + \frac{Mb}{2a^2} \operatorname{sgn}\, x_2(t_1^-)$$

$$\cdot \left[e^{a(t-t_1)} - e^{a(t-t_1-\tau)} + e^{-a(t-t_1)} - e^{-a(t-t_1-\tau)}\right]$$

$$+ \frac{Mb}{2a^2} \operatorname{sgn}\, x_2(t_2^-)$$

$$\cdot \left[e^{a(t-t_2)} - e^{a(t-t_2-\tau)} + e^{-a(t-t_2)} - e^{-a(t-t_2-\tau)}\right]. \quad \text{(A-16)}$$

In general, for the m^{th} pulse, $m = 1, 2, \ldots, N$, and by induction

for $\underline{t_m \leqslant t < t_m + \tau}$

$$p(t) = p_{11}(t) + \frac{bM}{2a^2} \sum_{i=1}^{m-1} \operatorname{sgn}(x_2(t_i^-))$$

$$\cdot \left[e^{a(t-t_i)} - e^{a(t-t_i-\tau)} + e^{-a(t-t_i)} - e^{-a(t-t_i-\tau)} \right]$$

$$+ \frac{bM}{2a^2} \operatorname{sgn}(x_2(t_m^-)) \left[e^{a(t-t_m)} + e^{-a(t-t_m)} - 2 \right] \tag{A-17}$$

for $\underline{t_m + \tau \leqslant t < t_{m+1}}$

$$p(t) = p_{11}(t) \frac{bM}{2a^2} \sum_{i=1}^{m} \operatorname{sgn}(x_2(t_i^-))$$

$$\cdot \left[e^{a(t-t_i)} - e^{a(t-t_i-\tau)} + e^{-a(t-t_i)} - e^{-a(t-t_i-\tau)} \right] \tag{A-18}$$

where $p_{11}(t)$ is given in (A-12).

3. <u>The Costate Variable $p_2(t)$</u>

$$\dot{p}_2 = -b\, p_1 \frac{\partial u}{\partial x_2} , \quad p_2(0) = p_{20} \tag{A-19}$$

solving for $p_2(t)$

$$p_2(t) = p_{20} - b \int_0^t p_1 \frac{\partial u}{\partial x_2}\, dt \tag{A-20}$$

where

$$\frac{\partial u}{\partial x_2} = M \sum_{i=1}^{N} \frac{\mathrm{sgn}(x_2(t_i^-))}{\dot{x}_2(t_i^-)} \left[\delta(t-t_i) - \delta(t-t_i-\tau)\right].$$

For $\underline{0 \leq t < t_1}$

$$p_2(t) = p_{20} . \qquad \text{(A-21)}$$

For $\underline{t_1 \leq t < t_1 + \tau}$

$$p_2(t) = p_{20} - b \int_{t_1}^{t} p_1 \frac{\partial u}{\partial x_2} dt$$

$$p_2(t) = p_{20} - Mb \frac{\mathrm{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} p_1(t_1) . \qquad \text{(A-22)}$$

For $\underline{t_1 + \tau \leq t < t_2}$

$$p_2(t) = p_{20} - b \int_{t_1}^{t_1+\tau} p_1 \frac{\partial u}{\partial x_2} dt$$

$$p_2(t) = p_{20} - Mb \frac{\mathrm{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} \left[p_1(t_1) - p_1(t_1+\tau)\right]. \qquad \text{(A-23)}$$

For $\underline{t_2 \leq t < t_2+\tau}$

$$p_2(t) = p_{20} - b \int_{t_1}^{t_1+\tau} p_1 \frac{\partial u}{\partial x_2} dt - b \int_{t_2}^{t} p_1 \frac{\partial u}{\partial x_2} dt ,$$

$$p_2(t) = p_{20} - Mb \frac{\operatorname{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} [p_1(t_1) - p_1(t_1+\tau)]$$

$$- Mb \frac{\operatorname{sgn}(x_2(t_2^-))}{\dot{x}_2(t_2^-)} p_1(t_2). \qquad \text{(A-24)}$$

For $\underline{t_2 + \tau \leqslant t < t_3}$

$$p_2(t) = p_{20} - b \int_{t_1}^{t_1+\tau} p_1 \frac{\partial u}{\partial x_2} dt - b \int_{t_2}^{t_2+\tau} p_1 \frac{\partial u}{\partial x_2} dt$$

$$p_2(t) = p_{20} - Mb \frac{\operatorname{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} [p_1(t_1) - p_1(t_1+\tau)]$$

$$- Mb \frac{\operatorname{sgn}(x_2(t_2^-))}{\dot{x}_2(t_2^-)} [p_1(t_2) - p_1(t_2+\tau)]. \qquad \text{(A-25)}$$

In general, for the m^{th} pulse, $m = 1, \ldots, N$, and by induction

for $\underline{t_m \leqslant t < t_m + \tau}$

$$p_2(t) = p_{20} - Mb \sum_{i=1}^{m-1} \frac{\operatorname{sgn} x_2(t_i^-)}{\dot{x}_2(t_i^-)} [p_1(t_i) - p_1(t_i+\tau)]$$

$$- Mb \frac{\operatorname{sgn} x_2(t_m^-)}{\dot{x}_2(t_m^-)} [p_1(t_m)] \qquad \text{(A-26)}$$

for $\underline{t_m + \tau \leqslant t < t_{m+1}}$

$$p_2(t) = p_{20} - Mb \sum_{i=1}^{m} \frac{\operatorname{sgn} x_2(t_i^-)}{\dot{x}_2(t_i^-)} [p_1(t_i) - p_1(t_i+\tau)]. \qquad \text{(A-27)}$$

4. The State Variable $x_2(t)$

$$\dot{x}_2 = p_2 / c , \qquad x_2(0) = x_2(t_i^+) = 0 \tag{A-28}$$

solving for $x_2(t)$

$$x_2(t) = p_2 \, t / c + k_1 . \tag{A-29}$$

For $0 \leqslant t < t_1$, $p_2(t)$ as in (A-21)

and $x_2(0) = 0$

$$x_2(t) = p_{20} \, t / c . \tag{A-30}$$

For $t_1 \leqslant t < t_1 + \tau$, $p_2(t)$ as in (A-22)

and $x_2(t_1) = 0$

$$x_2(t) = \frac{1}{c} \, p_2(t)\Big|_{(A-22)} \, t + k_1$$

$$x_2(t) = \frac{1}{c} \left[p_{20} - M b \, \frac{\mathrm{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} \, p_1(t_1) \right] (t - t_1). \tag{A-31}$$

For $t_1 + \tau \leqslant t < t_2$, $p_2(t)$ as in (A-23)

$$x_2(t) = \frac{1}{c} \, p_2(t)\Big|_{(A-23)} \, t + k_1 . \tag{A-32}$$

To find the value of k_1 , $x_2(t_1+\tau)$ from (A-31) should be equal to $x_2(t_1+\tau)$ from (A-32).

Hence $k_1 = \frac{1}{c}\left(p_2\Big|_{(A-22)} \tau - p_2\Big|_{(A-23)} t_1 + \tau\right)$

and

$$x_2(t) = -\tau M b \frac{\operatorname{sgn}(x_2(t_1^-))}{c\,\dot{x}_2(t_1^-)} p_1(t_1+\tau)$$

$$+ \left[p_{20} - M b \frac{\operatorname{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} (p_1(t_1) - p_1(t_1+\tau))\right]$$

$$\cdot (t-t_1)/c. \qquad \text{(A-33)}$$

For <u>$t_2 \leqslant t < t_{2+\tau}$</u> , $p_2(t)$ as in (A-24)

and $x_2(t_2^+) = 0$.

Similar to the derivation of (A-31)

$$x_2(t) = \left[p_{20} - M b \frac{\operatorname{sgn}(x_2(t_1^-))}{\dot{x}_2(t_1^-)} (p_1(t_1) - p_1(t_1+\tau))\right.$$

$$\left. - M b \frac{\operatorname{sgn}(x_2(t_2^-))}{\dot{x}_2(t_2^-)} p_1(t_2)\right] (t-t_2)/c. \qquad \text{(A-34)}$$

For <u>$t_2 + \tau \leqslant t < t_3$</u> , $p_2(t)$ as in (A-25)

and similar to the derivation of (A-33), one obtains

$$x_2(t) = -\tau M b \frac{\operatorname{sgn} x_2 (t_2^-)}{c\, \dot{x}_2 (t_2^-)} p_1(t_2+\tau)$$

$$+ \left[p_{20} - M b \frac{\operatorname{sgn}(x_2 (t_1^-))}{\dot{x}_2 (t_1^-)} (p_1(t_1) - p_1(t_1+\tau)) \right.$$

$$\left. - M b \frac{\operatorname{sgn}(x_2 (t_2^-))}{\dot{x}_2 (t_2^-)} (p_1(t_2) - p_1(t_2+\tau)) \right]$$

$$\cdot (t-t_2) / c. \tag{A-35}$$

In general, for the m^{th} pulse, $m = 1, \ldots, N$, and by induction

for $\underline{t_m \leqslant t < t_m+\tau}$

$$x_2(t) = \left[p_{20} - \sum_{i=1}^{m-1} M b \frac{\operatorname{sgn}(x_2 (t_i^-))}{\dot{x}_2 (t_i^-)} (p_1(t_i) - p_1(t_i+\tau)) \right.$$

$$\left. - M b \frac{\operatorname{sgn}(x_2 (t_m^-))}{\dot{x}_2 (t_m^-)} p_1(t_m) \right] (t-t_m) / c \tag{A-36}$$

and for $\underline{t_m+\tau \leqslant t < t_{m+1}}$

$$x_2(t) = -\tau M b \frac{\operatorname{sgn}(x_2 (t_m^-))}{\dot{x}_2 (t_m^-)} p_1(t_m+\tau)$$

$$+ \left[p_{20} - \sum_{i=1}^{m} M b \frac{\operatorname{sgn} x_2 (t_i^-)}{\dot{x}_2 (t_i^-)} \right.$$

$$\left. \cdot (p_1(t_1) - p_1(t_1+\tau)) \right] (t-t_m) / c. \tag{A-37}$$

Appendix B

COMPUTER PROGRAMS

B.1 LIST OF SYMBOLS

A	Plant parameter
B	Plant parameter
D	Minimum dead time between consecutive pulses
E1	Adjustement size for P10
E2	Adjustement size for P20
EF1	Allowable tolerance in F1
EF2	Allowable tolerance in F2
F1	Error in final costate $p_1(t_f)$
F2	Error in final costate $p_2(t_f)$
M	Pulse magnitude
N	Number of pulses
NP1	Polarity of P10
NP2	Polarity of P20
PM	Pulse Magnitude
P10	Initial value of the costate $p_1(t)$
P20	Initial value of the costate $p_2(t)$
R	Input to controller
S	Desired threshold
SP	Computed threshold
SP20	Sgn(P20)

SP2T1w	$\mathrm{Sgn}(x_2(t_2^-))$
SS	Difference between computed and desired thresholds
TF	Final time
TN	Time of initiation of the Nth pulse
TW	Pulse width
T1	Time of initiation of first pulse
T2	Time of initiation of second pulse
T11	Lower limit on T1
T12	Upper limit on T1
T21	Lower limit on T2
T22	Upper limit on T2
W	Pulse width
X10	Initial value of plant state

B.2 LIST OF PROGRAMS

This list of programs is complemented by programs developed in a previous study [8].

```
*
*   PROGRAM USED IN CHAPTER 4
*
*     HERE U EQUAL +1 OR -1
*
LABEL STATE AND COSTATE EQUATIONS
TITLE TWO-POINT BOUNDARY VALUE PROBLEM
INITIAL
      IC1=F1-(Z1+Z2)
      IC2=F2-(Z3+Z4)
 CONSTANT  Z1=0., Z2=0., Z3=0.,Z4=0., F1=0., F2=1., ALPHA=0.
DYNAMIC
      E=2.
 NOSORT
      P=P2-1.
      IF (ALPHA) 4,3,4
    4 IF (ABS(P).LT.ALPHA) GO TO 5
    3 IF (P) 6,7,7
    6 U=0.1
      GO TO 2
    7 U=-0.1
      GO TO 2
    5 U=0.0
    2 CONTINUE
 SORT
      X1DOT=X2
      X2DOT=-X2-X1-E*X1*X1+U
      P1DOT=(1.0+2.*E*X1)*P
      P2DOT=-P1+P2-1.
      P11XD=P21X
      P12XD=P22X
      P21XD=(-1.-2.*E*X1)*P11X-P21X
      P22XD=(-1.-2.*E*X1)*P12X-P22X
      P11PD=(1.+2.*E*X1)*P21P+2.*E*P2*P11X
      P12PD=(1.+2.*E*X1)*P22P+2.*E*P2*P12X
      P21PD=-P11P+P21P
      P22PD=-P12P+P22P
      X1=INTGRL(0.0,X1DOT)
      X2=INTGRL(0.0,X2DOT)
      P1=INTGRL(IC1,P1DOT)
      P2=INTGRL(IC2,P2DOT)
      P11X=INTGRL(0.0,P11XD)
      P12X=INTGRL(0.0,P12XD)
      P21X=INTGRL(0.0,P21XD)
      P22X=INTGRL(0.0,P22XD)
      P11P=INTGRL(1.0,P11PD)
      P12P=INTGRL(0.0,P12PD)
      P21P=INTGRL(0.0,P21PD)
      P22P=INTGRL(1.0,P22PD)
TERMINAL
      IF(ABS)P1).LT.0.00500.AND.ABS(P2).LT.0.00500) GO TO 1
      DET=P11P*P22P-P12P*P21P
      Z1=(P22P/DET)*P1
      Z2=(-P12P/DET)*P2
```

```
      Z3=(-P21P/DET)*P1
      Z4=(P11P/DET)*P2
      F1=IC1
      F2=IC2
      CALL RERUN
    1 CONTINUE
 TIMER DELT=0.0001,FINTIM=10.0,PRDEL=0.01
END
 TIMER OUTDEL=0.2
PRINT P
END
STOP
```

```
*
*   PROGRAM USED IN CHAPTER 4
*
*   HERE U IS FORMED OF PULSES
*
LABEL STATE AND COSTATE EQUATIONS
TITLE TWO-POINT BOUNDARY VALUE PROBLEM
INITIAL
      IC1=F1-(Z1+Z2)
      IC2=F2-(Z3+Z4)
 CONSTANT  Z1=0., Z2=0., Z3=0.,Z4=0., F1=0., F2=1.
DYNAMIC
      E=2.
      P=P2-1.
      U=-0.1*(-STEP(0.0)+STEP(0.2)-STEP(.5)+STEP(.7)-STEP(1.)...
       +STEP(1.2)+STEP(1.8)-STEP(2.0)+STEP(2.3)-STEP(2.5)...
       +STEP(2.8)-STEP(3.0)+STEP(3.3)-STEP(3.5)+STEP(3.8)...
       -STEP(4.)+STEP(4.3)-STEP(4.5)+STEP(4.8)-STEP(5.)...
       -STEP(5.3)+STEP(5.5)-STEP(5.8)+STEP(6.0)-STEP(6.3)...
       +STEP(6.5)-STEP(6.8)+STEP(7.0)-STEP(7.3)+STEP(7.5)...
       -STEP(7.8)+STEP(8.0)-STEP(8.3)+STEP(8.5)+STEP(8.8)...
       -STEP(9.0)+STEP(9.3)-STEP(9.5)+STEP(9.8)-STEP(10.))
      X1DOT=X2
      X2DOT=-X2-X1-E*X1*X1+U
      P1DOT=(1.0+2.*E*X1)*P
      P2DOT=-P1+P2-1.
      P11XD=P21X
      P12XD=P22X
      P21XD=(-1.-2.*E*X1)*P11X-P21X
      P22XD=(-1.-2.*E*X1)*P12X-P22X
      P11PD=(1.+2.*E*X1)*P21P+2.*E*P2*P11X
      P12PD=(1.+2.*E*X1)*P22P+2.*E*P2*P12X
      P21PD=-P11P+P21P
      P22PD=-P12P+P22P
      X1=INTGRL(0.0,X1DOT)
      X2=INTGRL(0.0,X2DOT)
      P1=INTGRL(IC1,P1DOT)
      P2=INTGRL(IC2,P2DOT)
      P11X=INTGRL(0.0,P11XD)
      P12X=INTGRL(0.0,P12XD)
      P21X=INTGRL(0.0,P21XD)
      P22X=INTGRL(0.0,P22XD)
      P11P=INTGRL(1.0,P11PD)
      P12P=INTGRL(0.0,P12PD)
      P21P=INTGRL(0.0,P21PD)
      P22P=INTGRL(1.0,P22PD)
TERMINAL
      IF(ABS)P1).LT.0.00500.AND.ABS(P2).LT.0.00500) GO TO 1
      DET=P11P*P22P-P12P*P21P
      Z1=(P22P/DET)*P1
      Z2=(-P12P/DET)*P2
      Z3=(-P21P/DET)*P1
      Z4=(P11P/DET)*P2
      F1=IC1
```

```
      F2=IC2
      CALL RERUN
    1 CONTINUE
 TIMER DELT=0.0001,FINTIM=10.0,PRDEL=0.01
END
 TIMER OUTDEL=0.2
PRTPLT X1,X2,P1,P2,P,U
END
STOP
```

```
*
*   PROGRAM - ON-LINE CASE
*       CHAPTER 5
*
FIXED N,I
CONSTANT  U=0.0, M=0.2, TW=0.2, D=0.3, H=3.0
 PARAMETER Z1=0.0, P0=0.496247292, A=3.
 INITIAL
      N=1
      MH=M*H
      TWD=TW+D
      FT=FINTIM
 DYNAMIC
      X=INTGRL(1.0,DXDT)
      P=INTGRL(P0,DPDT)
      G=INTGRL(0.0,DGDT)
      DXDT=-A*X+X*X+U
      DPDT=A*P-2.*X*P+2.*X-2.
      DGDT=P*U-H*U*U
      PX=INTGRL(0.0,PXD)
      PP=INTGRL(1.0,PPD)
      PXD=(-A+2.*X)*PX
      PPD=(2.-2.*P)*PX+(A-2.*X)*PP
  NOSORT
      IF (ABS(P).GT.MH) GO TO 2
      IF (ABS(P).LE.MH.AND.UU.GT.0.001) GO TO 6
      U=0.0
      N=1
      GO TO 3
   2  IF (N.EQ.0) GO TO 6
      N=0
      TI=TIME
      IF ((FT-TI).LT.TW) GO TO 3
      SGNP=P/ABS(P)
      T1=1.0
   6  IF (TIME.GE.(TI+0.01)) T1=-1.0
      UU=PULSE(TW,T1)
      U=M*SGNP*UU
      IF (TIME.GE.(TI+TWD)) N=1
   3  CONTINUE
 TERMINAL
      IF(ABS(P).LT.0.0002) GO TO 10
      I=I+1
      IF(I.GT.50) GO TO 4
      Z1=P/PP
      F1=P0
      P0=F1-Z1
      WRITE (10,80) P0
  80  FORMAT (5X,'P0= ',F14.12)
      CALL RERUN
   4  WRITE (10,70) I
  70  FORMAT (5X,'I= ',I3)
  10  CONTINUE
TIMER FINTIM=3.0 ,DELT=0.001 ,PRDEL=0.01
METHOD ADAMS
PRINT X,P,U,G
END
```

```
*
*   PROGRAM - METHOD I
*      CHAPTER 7
*
FIXED I
 MACRO          Y=LIMINT(IC,YDOT,P1,P2)
  PROCEDURE DYDT=LIM(Y,YDOT,P1,P2)
            DYDT=YDOT
        IF (Y.LE.P1) DYDT=AMAX1(0.0,YDOT)
        IF (Y.GE.P2) DYDT=AMIN1(0.0,YDOT)
    ENDPROCEDURE
        Y=INTGRL(IC,DYDT)
  ENDMAC
 CONSTANT A=-2., B=1., S=0.1, X10=-2.0, F1=0.708,Z1=0.0,...
            Z2=0.0, Z3=0.0, Z4=0.0, I=0, T3=-1.,...
          G=1., F2=0.57, D=0.2, TW=0.2, M=1.0
INITIAL
      P10=F1-Z1-Z2
      P20=F2-Z3-Z4
      P2=P20
      U=0.0
      TI=20.
      T11=TW-DELT
      TWD=TW+D
DYNAMIC
       NOSORT
      X2=LIMIT(-S,S,X2)
      P2G=P2/G
      IF((ABS(X2).GE.(S-0.00001)).AND.(TIME.LT.TI)) GO TO 10
      IF (TIME.LT.TI) GO TO 100
      IF (TIME.LT.(TI+TWD)) GO TO 30
      IF (ABS(X2).GE.(S-0.00001)) GO TO 10
      GO TO 100
   10 TI=TIME
      T1=1.0
      SGNX2=X2/ABS(X2)
      X2DOTT=SGNX2/P2G
      TIU=TI+T11
      X2=0.0
   30 IF (TIME.GE.(TI+0.001)) T1=-1.0
      IF (ABS(TIME-TIU).LT.0.0001) T3=1.0
      IF (TIME.GT.(TIU+0.0001)) T3=-1.0
      UU=PULSE(T11,T1)
      DUD=PULSE(0.0001,T1)-PULSE(0.0001,T3)
      U=M*SGNX2*UU
      DUDX2=M*DUD*X2DOTT/DELT
  100 CONTINUE
 SORT
      X1DOT=A*X1+B*U
      P1DOT=-A*P1+X1
      P2DOT=-B*P1*DUDX2
      X2DOT=P2G
      X1=INTGRL(X10,X1DOT)
      P1=INTGRL(P10,P1DOT)
```

```
      P2=INTGRL(P20,P2DOT)
      X2=LIMINT(0.0,X2DOT,-S,S)
      P11XD=A*P11X+B*DUDX2*P21X
      P12XD=A*P12X+B*DUDX2*P22X
      P21XD=P21P/G
      P22XD=P22P/G
      P11PD=P11X-A*P11P
      P12PD=P12X-A*P12P
      P21PD=-B*DUDX2*P11P
      P22PD=-B*DUDX2*P12P
      P11X=INTGRL (0.0,P11XD)
      P12X=INTGRL (0.0,P12XD)
      P21X=INTGRL (0.0,P21XD)
      P22X=INTGRL (0.0,P22XD)
      P11P=INTGRL (1.0,P11PD)
      P12P=INTGRL (0.0,P12PD)
      P21P=INTGRL (0.0,P21PD)
      P22P=INTGRL (1.0,P22PD)
 TERMINAL
      IF (ABS(P1).LT.0.005.AND.ABS(P2).LT.0.005) GO TO 3
      I=I+1
      IF (I.GT.50) GO TO 4
      DET=P11P*P22P-P12P*P21P
      Z1=P22P*P1/DET
      Z2=-P12P*P2/DET
      Z3=-P21P*P1/DET
      Z4=P11P*P2/DET
      F1=P10
      F2=P20
      CALL RERUN
   4  WRITE (10,70) I
   70 FORMAT (5X,'I= ',I3)
   3  CONTINUE
 TIMER  FINTIM=2., DELT=0.001, PRDEL=0.1
 METHOD RECT
 PRINT X1,X2,P1,P2,U
 TITLE PFM & FIRST ORDER LINEAR PLANT
 TITLE       NONLINEAR FEEDBACK SYSTEM
 END
```

```
                    PROGRAM MAG2
C
C     CASE OF OPTIMAL CONTROL CONTAINING TWO PULSES
C     FIND REGION WHERE P(TF) CHANGES SIGNS, AND,
C     COMPUTED THRESHOLD CLOSE TO DESIRED ONE.
C
      REAL M
      INTEGER SP20,SP2T1W
      WRITE(10,19)
   19 FORMAT (5X,'SX2(T1-) & SX2(T2-); 2I2')
      READ(10,20) SP20,SP2T1W
   20 FORMAT (2I2)
      WRITE(10,22)
   22 FORMAT (5X,'LIMITS ON T1 & INCR.(*100000); 2I6,I4')
      READ(10,21) I1L,I1H,I1
      WRITE(10,23)
   23 FORMAT (5X,'LIMITS ON T2 & INCR. (*100000); 2I6,I4')
      READ(10,21) I2L,I2H,I2
   21 FORMAT (2I6,I4)
      WRITE(10,4)
    4 FORMAT(10X,'T1',10X,'T2',10X,'SP',9X,'P10',9X,'P20',10X,'F2')
      DATA A,W,M,TF,R,X10/-2.,0.2,1.,2.,0.5,-2./
C
C     CHANGE T2 BETWEEN ITS LIMITS
C
      DO 1 I=I2L,I2H,I2
      T2=I/100000.
      TT2=A*(TF-T2)
      TW2=TT2-A*W
      AW=A*W
      EAW=EXP(AW)+EXP(-AW)-2.
      XA=X10/(2.*A)
      AM=M/(2*A*A)
      A1=XA*(1.-EXP(2.*A*TF))
      RA=R/A
      C1=RA*(-1.+EXP(A*TF))
C
C     CHANGE T1 BETWEEN ITS LIMITS
C
      DO 1 I=I1L,I1H,I1
      T1=I/100000.
      TT1=A*(TF-T1)
      TW1=TT1-AW
C
C     COMPUTE P10, EQUATION (7-14)
C
      B1=EXP(A*TF)*(SP20*(EXP(-TT1)+EXP(TT1)-EXP(-TW1)-EXP(TW1)
     1)+SP2T1W*(EXP(-TT2)+EXP(TT2)-EXP(-TW2)-EXP(TW2)))
      P10=A1-B1*AM+C1
      PP=P10+RA-XA
      P1T1=PP*EXP(-A*T1)-RA+XA*EXP(A*T1)
      TT=T1+W
      P1T1W=PP*EXP(-A*TT)-RA+XA*EXP(A*TT)+AM*SP20*EAW
      TT=A*(T2-T1)
      TTW=TT-AW
```

```
      P1T2=PP*EXP(-A*T2)-RA+XA*EXP(A*T2)+AM*SP20*(EXP(TT)
     1-EXP(TTW)+EXP(-TT)-EXP(-TTW))
      P1T2W=PP*EXP(-A*(T2+W))-RA+XA*EXP(A*(T2+W))+AM*SP20*(EXP(TT+AW)
     1-EXP(TT)+EXP(-TT-AW)-EXP(-TT))+AM*SP2T1W*EAW
C
C     COMPUTE P20**2, EQUATION (7-20)
C
      AS=T1*SP2T1W-(T2-T1)*SP20
      BS=M*(-W*P1T1W-(T2-T1)*(P1T1-P1T1W))
      P202=BS/AS
      IF (P202.LT.0.0) GO TO 1
      P22=SQRT(P202)
C
C     COMPUTE THRESHOLD SP, EQUATION (7-17)
C
      SP=T1*P22
      P20=SP20*P22
      P2T1W=P20-M*(P1T1-P1T1W)/P22
      AK1=P2T1W/ABS(P2T1W)
      IF (ABS(AK1-SP2T1W).GT.0.01) GO TO 1
C
C     ERROR F2 IN P2(TF), EQUATION (7-15)
C
      F14=P20
      F24=M*(P1T1-P1T1W)/(P20*SP20)+M*(P1T2-P1T2W)/ABS(P2T1W)
      F2=F14-F24
      WRITE (10,3) T1,T2,SP,P10,P20,F2
    3 FORMAT (5X,7F12.7)
    1 CONTINUE
      STOP
      END
```

```
                    PROGRAM COST
C
C     MAIN PROGRAM FOR INITIAL COSTATE
C     USES APPROXIMATE LIMITS OF T1 & T2 (FOUND BY MAG2)
C     TO FIND EXACT P10 & P20 FOR THE CASE OF TWO PULSES.
C
      N=0
      WRITE (10,9)
    9 FORMAT (5X,'T11, T12 &T1C FORMAT = F5.3')
C
C     T1C IS THE DESIRED INITIAL INCREMENT IN T1
C
      READ (10,11) T11,T12,T1C
      WRITE (10,10)
   10 FORMAT (5X,'T21, T22 & T2C FORMAT = F5.3')
C
C     T2C IS THE DESIRED INITIAL INCREMENT IN T2
C
      READ(10,11) T21,T22,T2C
   11 FORMAT (3F5.3)
      JL=IFIX(0.1E07*T21)
      JH=IFIX(0.1E07*T22)
      JC=IFIX(0.1E07*T2C)
    5 SS0=0.0
      T20=0.0
      K=1
C
C     CHANGE T2 BETWEEN LIMITS
C
      DO 1 J=JL,JH,JC
      T2=J*0.1E-05
      CALL COSTS (T2,T11,T12,T1C,SS,P10,P20,T1,F2,SP)
      N=N+1
      IF (N.EQ.100) GO TO 2
      IF(ABS(SS).LE.0.1E-05) GO TO 2
      IF (K.EQ.1) GO TO 4
C
C     DETECT CHANGE OF SIGN IN SS
C
      IF (SS0.GT.0.0.AND.SS.LT.0.0) GO TO 3
      IF (SS0.LT.0.0.AND.SS.GT.0.0) GO TO 3
    4 SS0=SS
      T20=T2
      K=2
    1 CONTINUE
C
C     SET NEW NARROWER LIMITS FOR T2
C
    3 JL=IFIX(0.1E07*T20)
      JH=IFIX(0.1E07*T2)
      CJC=ABS(T20-T2)
      JC=IFIX(0.1E06*CJC)
      GO TO 5
C
C   OPTIMAL CONTROL
```

```
C
    2  WRITE (10,12) T1,T2
       WRITE (10,13) P10,P20
       WRITE (10,14) F2,SP
       WRITE(10,15) N
   12 FORMAT (5X,' T1 =',F10.7,5X,' T2 =',F10.7)
   13 FORMAT (5X,'P10 =',F10.7,5X,'P20 =',F10.7)
   14 FORMAT (5X,' F2 =',F10.7,5X,' SP =',F10.7)
   15 FORMAT (5X,' N =',I3)
      STOP
      END
```

```
                SUBROUTINE COSTS
C
C      EXTERNAL FOR MAIN INITIAL COSTATES
C
      SUBROUTINE COSTS (T2,T11,T12,T1C,SS,P10,P20,T1,F2,SP)
      REAL M
      INTEGER SP20,SP2T1W
      DATA A,W,M,TF,R/-2.,0.2,1.,2.,0.5/
      DATA X10,S,D,SP20,SP2T1W/-2.0,0.1,0.2,1,1/
C
      TT2=A*(TF-T2)
      TW2=TT2-A*W
      AW=A*W
      EAW=EXP(AW)+EXP(-AW)-2.
      XA=X10/(2.*A)
      AM=M/(2*A*A)
      A1=XA*(1.-EXP(2.*A*TF))
      RA=R/A
      C1=RA*(-1.+EXP(A*TF))
      IL=IFIX(0.1E07*T11)
      IH=IFIX(0.1E07*T12)
      IC=IFIX(0.1E07*T1C)
  2   F20=0.0
      T10=0.0
      K=1
      DO 1 I=IL,IH,IC
      T1=I*0.1E-05
      TT1=A*(TF-T1)
      TW1=TT1-AW
      B1=EXP(A*TF)*(SP20*(EXP(-TT1)+EXP(TT1)-EXP(-TW1)-EXP(TW1)
     1)+SP2T1W*(EXP(-TT2)+EXP(TT2)-EXP(-TW2)-EXP(TW2)))
C
C     COMPUTE P10, EQUATION (7-14)
C
      P10=A1-B1*AM+C1
      PP=P10+RA-XA
      P1T1=PP*EXP(-A*T1)-RA+XA*EXP(A*T1)
      TT=T1+W
      P1T1W=PP*EXP(-A*TT)-RA+XA*EXP(A*TT)+AM*SP20*EAW
      TT=A*(T2-T1)
      TTW=TT-AW
      P1T2=PP*EXP(-A*T2)-RA+XA*EXP(A*T2)+AM*SP20*(EXP(TT)
     1-EXP(TTW)+EXP(-TT)-EXP(-TTW))
      P1T2W=PP*EXP(-A*(T2+W))-RA+XA*EXP(A*(T2+W))+AM*SP20*(EXP(TT+AW)
     1-EXP(TT)+EXP(-TT-AW)-EXP(-TT))+AM*SP2T1W*EAW
C
C     COMPUTE P20**2, EQUATION (7-20)
C
      AS=T1*SP2T1W-(T2-T1)*SP20
      BS=M*(-W*P1T1W-(T2-T1)*(P1T1-P1T1W))
      P202=BS/AS
      IF (P202.LT.0.0) GO TO 5
      P22=SQRT(P202)
C
C     COMPUTE THRESHOLD SP, EQUATION (7-17)
```

```
C
      SP=T1*P22
      P20=SP20*P22
      P2T1W=P20-M*(P1T1-P1T1W)/P22
      AK1=P2T1W/ABS(P2T1W)
      IF (ABS(AK1-SP2T1W).GT.0.01) GO TO 1
C
C     COMPUTE ERROR F2 IN P2(TF), EQUATION (7-15)
C
      F14=P20
      F24=M*(P1T1-P1T1W)/(P20*SP20)+M*(P1T2-P1T2W)/ABS(P2T1W)
      F2=F14-F24
      IF (ABS(F2).LE.0.1E-4) GO TO 26
      IF (K.EQ.1) GO TO 28
      IF (F20.GT.0.0.AND.F2.LT.0.0) GO TO 27
      IF (F20.LT.0.0.AND.F2.GT.0.0) GO TO 27
   28 F20=F2
      T10=T1
      K=2
      GO TO 1
    5 WRITE(10,50) P202
   50 FORMAT (5X,'...WARNING...P202 = ',F10.5)
    1 CONTINUE
      WRITE (10,60) T2
   60 FORMAT('NO ZERO FOR T2 = ',F10.7/)
      T2=T2+0.01
      GO TO 2
C
C     SET NEW NARROWER LIMITS FOR T1
C
   27 IL=IFIX(0.1E07*T10)
      IH=IFIX(0.1E07*T1)
      TTT=ABS(T1-T10)
      IC=IFIX(0.1E06*TTT)
      GO TO 2
   26 SS=SP-S
      WRITE(10,22) SS,F2
   22 FORMAT (5X,'SS=',F10.7,5X,'F2=',F10.7)
      RETURN
      END
```

```
                        PROGRAM SOLVE
C
C     CASE OF CONTROL CONTAINING N PULSES
C     FIND INITIAL COSTATES BY PLOTTING CONTOURS
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION T(10),P1(20),P2(20),SP2(10),SX2D(10),X2(10)
      DATA A,B,G,S,X10,PM,W,R,TF/-2.,1.,1.,0.1,-2.,1.,0.2,0.5,2.0/
C
      WRITE(10,54)
   54 FORMAT (5X,'NO. OF PULSES & SIGN OF P10 AND P20; 3I2')
      READ(10,52) N,NP1,NP2
   52 FORMAT (3I2)
      WRITE(10,55)
   55 FORMAT (5X,'LIMITS AND INCR. ON P10 & P20 (*100000.); 4I6,2I5')
      READ(10,53) L1L,L1H,L2L,L2H,IN1,IN2
   53 FORMAT (4I8,2I7)
C
      XA=X10/(2.*A)
      BM=(B*PM)/(2.*A*A)
      AW=A*W
      EAW=DEXP(AW)+DEXP(-AW)-2.
C
C     CHANGE P10 BETWEEN LIMITS
C
      DO 20 L1=L1L,L1H,IN1
      P10=NP1*L1/10000000.
      PP=P10+R/A-XA
C
C     CHANGE P20 BETWEEN LIMITS
C
      DO 20 L2=L2L,L2H,IN2
      P20=NP2*L2/10000000.
      AP2=DABS(P20)
      SX2D(1)=G/AP2
C
C     COMPUTE T1 , EQUATION (7-13)
C
      T(1)=S*SX2D(1)
      SP2(1)=P20/AP2
      TW=T(1)+W
      P1(1)=PP*DEXP(-A*T(1))-R/A+XA*DEXP(A*T(1))
      P1(N+1)=PP*DEXP(-A*TW)-R/A+XA*DEXP(A*TW)+BM*SP2(1)*EAW
      P2(1)=P20-PM*B*SX2D(1)*P1(1)
      P2(N+1)=P20-PM*B*SX2D(1)*(P1(1)-P1(N+1))
      AA=0.0
      IF (N.EQ.1) GO TO 3
      DO 1 J=2,N
      AA=0.0
      BB=0.0
      BBW=0.0
      J1=J-1
      DO 2 I=1,J1
      II=N+I
      AA=AA+(P1(I)-P1(II))*SX2D(I)
```

```
    2  CONTINUE
       N1=N+J-1
       SP2(J)=P2(N1)/DABS(P2(N1))
   11  SX2D(J)=G*SP2(J)/P2(N1)
C
C      COMPUTE TI, I=2,...,N, EQUATION (7-13)
C
       T(J)=(P20*T(J-1)+G*S*SP2(J)-PM*B*T(J-1)*AA+W*PM*B*P1(N1)
      1*SX2D(J-1))/(P20-PM*B*AA)
       X2(J)=(T(J)-T(J-1))*(P20-PM*B*AA)/G-W*PM*B*SX2D(J-1)*P1(N1)/G
       SX2=X2(J)/DABS(X2(J))
       IF ((SX2-SP2(J)).LE.0.005) GO TO 10
       SP2(J)=SX2
       GO TO 11
   10  DO 9 I=1,J1
       TT=T(J)-T(I)
       BB=BB+SP2(I)*(DEXP(A*TT)-DEXP(A*(TT-W))+DEXP(-A*TT)-
      1DEXP(-A*(TT-W)))
       BBW=BBW+SP2(I)*(DEXP(A*(TT+W))-DEXP(A*TT)+DEXP(-A*(TT+W))-
      1DEXP(-A*TT))
    9  CONTINUE
       NJ=N+J
       P1(J)=PP*DEXP(-A*T(J))-R/A+XA*DEXP(A*T(J))+BM*BB
       P1(NJ)=PP*DEXP(-A*(T(J)+W))-R/A+XA*DEXP(A*(T(J)+W))+BM*(BBW+SP2(J)
      1*EAW)
       P2(J)=P20-PM*B*(AA+P1(J)*SX2D(J))
       P2(NJ)=P20-PM*B*(AA+SX2D(J)*(P1(J)-P1(NJ)))
    1  CONTINUE
    3  BBF=0.0
       DO 4 I=1,N
       TT=TF-T(I)
       BBF=SP2(I)*(DEXP(A*TT)-DEXP(A*(TT-W))+DEXP(-A*TT)-DEXP(-A*(TT-W)))
      1+BBF
    4  CONTINUE
C
C      COMPUTE F1, ERROR IN P1(TF), EQUATION (7-8C)
C
       F1=PP*DEXP(-A*TF)-R/A+XA*DEXP(A*TF)+BM*BBF
C
C      COMPUTE F2, ERROR IN P2(TF), EQUATION (7-9C)
C
       F2=-PM*B*(AA+((P1(N)-P1(2*N))*SX2D(N)))+P20
       PS1=SNGL(P10)
       PS2=SNGL(P20)
       FS1=SNGL(F1)
       FS2=SNGL(F2)
       WRITE(10,50) PS1,PS2,FS1,FS2
C      DO 7 I=1,N
C      TI=SNGL(T(I))
C      WRITE(10,51) I,TI
C   7  CONTINUE
   50  FORMAT (4X,'P10=',F10.6,2X,'P20=',F10.6,2X,'F1=',F13.5,2X,'F2='
      1,F13.5)
   51  FORMAT (5X,'T(',I1,')=',F10.6)

   20  CONTINUE
       STOP
       END
```

```
          PROGRAM SOLVI
C
C     TO OBTAIN THE OPTIMAL CONTROL SEQUENCE
C     BY LOCATING THE EXACT ZERO FINAL COSTATES
C     USING AN ITERATIVE TECHNIQUE.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION T(10),P1(20),P2(20),SP2(10),SX2D(10),X2(10)
      DATA A,B,G,S,X10,PM,W,R,TF/-2.,1.,1.,0.1,-2.,1.0,0.2,0.5,2.0/
C
      WRITE(10,54)
   54 FORMAT (5X,'NO. OF PULSES; I2')
      READ(10,52) N
   52 FORMAT (I2)
      WRITE(10,55)
   55 FORMAT (5X,'P10 AND P20; 2F8.6')
      READ(10,53) P10,P20
   53 FORMAT (2F8.6)
      OF1=0.0
      OF2=0.0
      WRITE(10,56)
   56 FORMAT (5X,'STEP SIZES FOR CONVERGENCE; 2F8.6')
      READ(10,53) E1,E2
      WRITE(10,57)
   57 FORMAT (5X,'ALLOWABLE TOLERANCES IN F1 & F2; 2F8.6')
      READ(10,53) EF1,EF2
C
      XA=X10/(2.*A)
      BM=(B*PM)/(2.*A*A)
      AW=A*W
      EAW=DEXP(AW)+DEXP(-AW)-2.
    6 PP=P10+R/A-XA
      AP2=DABS(P20)
      SX2D(1)=G/AP2
C
C     COMPUTE T1, EQUATION (7-13)
C
      T(1)=S*SX2D(1)
      SP2(1)=P20/AP2
      TW=T(1)+W
      P1(1)=PP*DEXP(-A*T(1))-R/A+XA*DEXP(A*T(1))
      P1(N+1)=PP*DEXP(-A*TW)-R/A+XA*DEXP(A*TW)+BM*SP2(1)*EAW
      P2(1)=P20-PM*B*SX2D(1)*P1(1)
      P2(N+1)=P20-PM*B*SX2D(1)*(P1(1)-P1(N+1))
      AA=0.0
      IF (N.EQ.1) GO TO 3
      DO 1 J=2,N
      AA=0.0
      BB=0.0
      BBW=0.0
      J1=J-1
      DO 2 I=1,J1
      II=N+I
      AA=AA+(P1(I)-P1(II))*SX2D(I)
    2 CONTINUE
```

```
      N1=N+J-1
      SP2(J)=P2(N1)/DABS(P2(N1))
   11 SX2D(J)=G*SP2(J)/P2(N1)
C
C     COMPUTE TI, I=2,...,N, EQUATION (7-13)
C
      T(J)=(P20*T(J-1)+G*S*SP2(J)-PM*B*T(J-1)*AA+W*PM*B*P1(N1)
     1*SX2D(J-1))/(P20-PM*B*AA)
      X2(J)=(T(J)-T(J-1))*(P20-PM*B*AA)/G-W*PM*B*SX2D(J-1)*P1(N1)/G
      SX2=X2(J)/DABS(X2(J))
      IF ((SX2-SP2(J)).LE.0.005) GO TO 10
      SP2(J)=SX2
      GO TO 11
   10 DO 9 I=1,J1
      TT=T(J)-T(I)
      BB=BB+SP2(I)*(DEXP(A*TT)-DEXP(A*(TT-W))+DEXP(-A*TT)-
     1DEXP(-A*(TT-W)))
      BBW=BBW+SP2(I)*(DEXP(A*(TT+W))-DEXP(A*TT)+DEXP(-A*(TT+W))-
     1DEXP(-A*TT))
    9 CONTINUE
      NJ=N+J
      P1(J)=PP*DEXP(-A*T(J))-R/A+XA*DEXP(A*T(J))+BM*BB
      P1(NJ)=PP*DEXP(-A*(T(J)+W))-R/A+XA*DEXP(A*(T(J)+W))+BM*(BBW+SP2(J)
     1*EAW)
      P2(J)=P20-PM*B*(AA+P1(J)*SX2D(J))
      P2(NJ)=P20-PM*B*(AA+SX2D(J)*(P1(J)-P1(NJ)))
    1 CONTINUE
    3 BBF=0.0
      DO 4 I=1,N
      TT=TF-T(I)
      BBF=SP2(I)*(DEXP(A*TT)-DEXP(A*(TT-W))+DEXP(-A*TT)-DEXP(-A*(TT-W)))
     1+BBF
    4 CONTINUE
C
C     COMPUTE F1, ERROR IN P1(TF), EQUATION (7-8C)
C
      F1=PP*DEXP(-A*TF)-R/A+XA*DEXP(A*TF)+BM*BBF
C
C     COMPUTE F2, ERROR IN P2(TF), EQUATION (7-9C)
C
      F2=P20-PM*B*(AA+((P1(N)-P1(2*N))*SX2D(N)))
      IF(DABS(F1).LE.EF1.AND.DABS(F2).LE.EF2) GO TO 5
      IF (OF1.EQ.0.0) GO TO 31
      IF ((DABS(OF1)-DABS(F1)).GT.0.0) GO TO 30
      E1=-E1
   30 IF ((DABS(OF2)-DABS(F2)).GE.0.0) GO TO 31
      E2=-E2
   31 OF1=F1
      OF2=F2
    8 WRITE(10,50) P10,P20,F1,F2
C
C     COMPUTE NEW P10 & P20
C
      P10=P10+E1*F1
```

```
      P20=P20+E2*F2
      GO TO 6
C
C     OUTPUT RESULTS
C
    5 WRITE(10,50) P10,P20,F1,F2
      DO 7 I=1,N
      TI=SNGL(T(I))
      WRITE(10,51) I,TI
    7 CONTINUE
   50 FORMAT (4X,'P10=',F10.6,2X,'P20=',F10.6,2X,'F1=',F10.6,2X,
     1'F2=',F10.6)
   51 FORMAT (5X,'T(',I1,')=',F10.6)
   20 CONTINUE
      STOP
      END
```