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ANALYSIS OF CURVED FOLDED PLATES OF REVOLUTION USING STRUCTURAL THEORIES

by

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A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

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TO MY WIFE PADMA

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ABSTRACT

The work presented in this thesis deals with the analysis of a curved folded plate of revolution, utilizing curved beam and arch theories. A curved folded plate of revolution is obtained by rotating a folded cross-section about a fixed line. Each unit is independently analysed, while compatibility between units is ensured by applying correction forces at the common joints.

Based on the above analysis a computer programme "CURVFOLD" has been written. This programme can analyse a curved folded plate of fixed and pinned end boundary conditions. Results of analysis of a two unit curved folded plate with different compatible points are presented. Some of the results are compared with those obtained from known finite element analysis and are found to be in reasonable agreement.

CONTENTS

CHAPTER I

INTRODUCTION

1.1 General

Folded plates - also known as hipped plates, prismatic shells or "Faltwerke" - are an example of stressed-skin construction which effectively exploits the monolithic properties of concrete for enclosing large, column-free areas required for a variety of structures such as factory buildings, godowns, gymnasia, assembly halls and auditoria. Their structural behaviour resembles that of shells. In fact, a cylindrical shell can be thought of as a folded plate in the limit. During recent years folded plate construction has found increasing application for roofs of industrial buildings and hangars as well as for the sides and bottoms of elevated bunkers. Such construction is particularly well suited to fairly long spans possessing some of the attributes of thin shell construction, with perhaps the added advantage of somewhat simpler fabrication or forming.

The structural action of a folded plate consists of transverse "slab action" by which the loads are transferred to the joints and longitudinal "plate action" by which they are finally transmitted to the traverses. Because of its great depth and small thickness, each plate offers considerable resistance to bending in its own plane. This "plate action" explains the remarkable rigidity of folded plate construction. A few commonly used shapes of folded plates are shown in Figure 1.1.

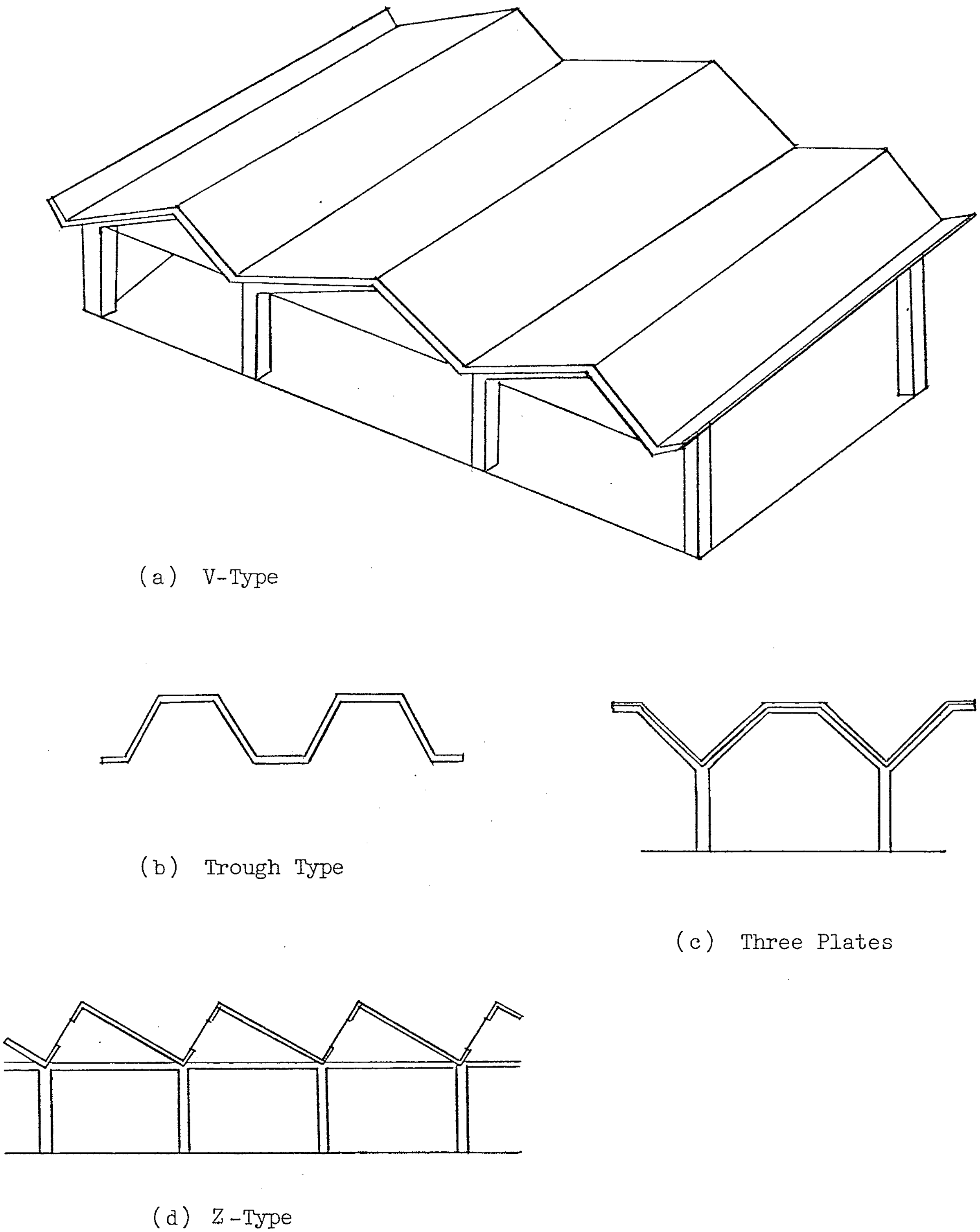

FIGURE 1.1 - Folded Plate Roofs in Common Use

1.2 Historical Development

The first structure of this kind was large coal bunkers designed and erected by G. Ehlers of Germany in 1924-25 and the first paper on the subject was published in 1930 by G. Ehlers [1]. His analysis assumed the longitudinal joints to be hinged, neglecting the transverse moments at the junction of the plates. The displacements of the joints were also ignored. This theory was improved upon in 1932 by E. Gruber [2,3] who included the effects of transverse continuity and joint displacements. Assuming the joints to be hinged as a first approximation, he developed a solution in the form of simultaneous differential equations of the fourth order which were solved by the use of a rapidly converging series. This approach involved (7n + 2) unknowns for (n + 1) plates; thus a roof of 5 plates would involve 30 unknowns. Although Gruber's solution was very laborious, his conclusion that the assumption of hinged or rigid joints would considerably affect the final results was significant. This work was followed by that of Craemer [4] who published a paper in 1953. He proposed rough limits in terms of the length-to-width ratio of individual plates for their classification as "long" and "short". The paper by Winter and Pei [5] published in 1947 is a landmark in the theory on the subject as they, for the first time, reduced the algebraic solution into a stress distribution procedure analogous to the well-known moment distribution method. But they neglected the displacement of the joints. For short folded plate their approach offers a very simple design procedure. However, for long plates the joint deflections cannot be ignored. Girkmann [6] in his book published in 1948, takes into account joint displacements. Treating transverse moments at the joints as the unknowns, he formulated conditions for the compatibility of longitudinal stresses and displacements at joints. The method leads to as many simultaneous equations as the unknown transverse moments. The paper presented by Whitney [7] at the joint ASCE-IABSE meeting in New York is a presentation in English of the Girkmann Method with some modification. Gaafar [8] in 1953 published a modification of the Winter and Pei method extended to include the effect of joint displacements. H. Simpson [9] in 1958 published a new method which involves a number of moment and stress distributions.

Utilization of the two-dimensional theory of elasticity for the determination of membrane stresses, and two-way slab theory for determination of bending and twisting of the slab was introduced by Goldberg and Leve [10]. A comprehensive report on the analyses of folded plate structures was given by an ASCE Task Committee [11] in 1963. In this report, the analyses of folded plates are classified into the following basic categories:

(a) beam method,

(b) folded plate theory neglecting relative joint displacements,

(c) folded plate theory considering relative joint displacements, and

(d) elasticity method.

The use of the finite element method made it possible to analyse many structures subjected to arbitrary loading conditions. However, for this type of analysis a large computer programme and a fair-sized computer were necessary. The finite strip method which closely resembles the finite element method was first used by Cheung [12] to analyse straight folded plate structures. The programme size was considerably reduced by the use of this method which proved very suitable for the analysis of plate structures.

In the next phase the direct stiffness method in conjunction with finite element method was used to analyse folded plate structures. The stresses and displacements in each plate element are computed by classical thin plate bending theory and two dimensional plane-stress elasticity theory. The direct stiffness method facilitated programming of the analysis. Much work was done by Scordelis in this context. The programme 'MUPDl 3' is an example of such work by Scordelis and Lin [18]. Meyaer and Scordelis [14] utilized the same technique to analyse folded plate structures curved in plan and developed a programme called "CURSTR" which mainly deals with shells of revolution.

1.3 Object of Study

Among the available methods, the Winter and Pei procedure is the simplest. But it is applicable only to short folded plates for which the joint displacements can be ignored without appreciable error. Of the methods that are applicable to folded plates of all proportions, those due to Whitney and Simpson are more suitable for use in the design office. Whitney on the basis of Craemer's work, has shown that when the span of the structure is large compared to its depth, analysis can be carried out on the beam theory assuming a straight line distribution of longitudinal stress. This simplified analysis can be used for folded plates with a span to depth ratio of more than 10. But due to the longitudinal bending action the longitudinal tensile stress increases, requiring a large amount of tensile reinforcement. This limits the unsupported clear spans in the case of reinforced concrete folded plates.

There are two ways to overcome this difficulty. One method is to apply prestressing [15] forces to the folded plate structure to account for the tensile stresses. Another method is by arching the folded plates in the longitudinal direction whereby loads are effectively transmitted to the end supports by arch action rather than bending action.

A curved folded plate of revolution is formed by rotating a folded cross section about a fixed line. The structure has the strength characteristic of a double curvature shell and yet it is simple in forming due to straight edges in one direction thus reducing the form work problems. They have a striking appearance which often appeals to the architect. The typical elements of a curved folded plate structure are shown in Figure 1.2.

R.C.P. Hui [16] developed theories for analysing a arch folded plate of fixed end conditions. He demonstrated his theory numerically by establishing compatibilities between two units of folded plates.

In the following studies an attempt to refine the work of Hui, particularly in formulating the flexibility matrix, is made. Further, theories to analyse curved folded plate of pinned-end conditions are derived. And finally a computer programme 'CURVFOLD' has been written, which can analyse both fixed and pinned-end folded plates of symmetric cross section. Several results of analysis are presented. Some results of known finite element analysis are also presented. A listing of the programme 'CURVFOLD' along with computer output is given in Appendix A.

1.4 Assumptions and Limitations

In order to apply one-dimensional structural theories to curved folded plates and to simplify the analysis the following assumptions have

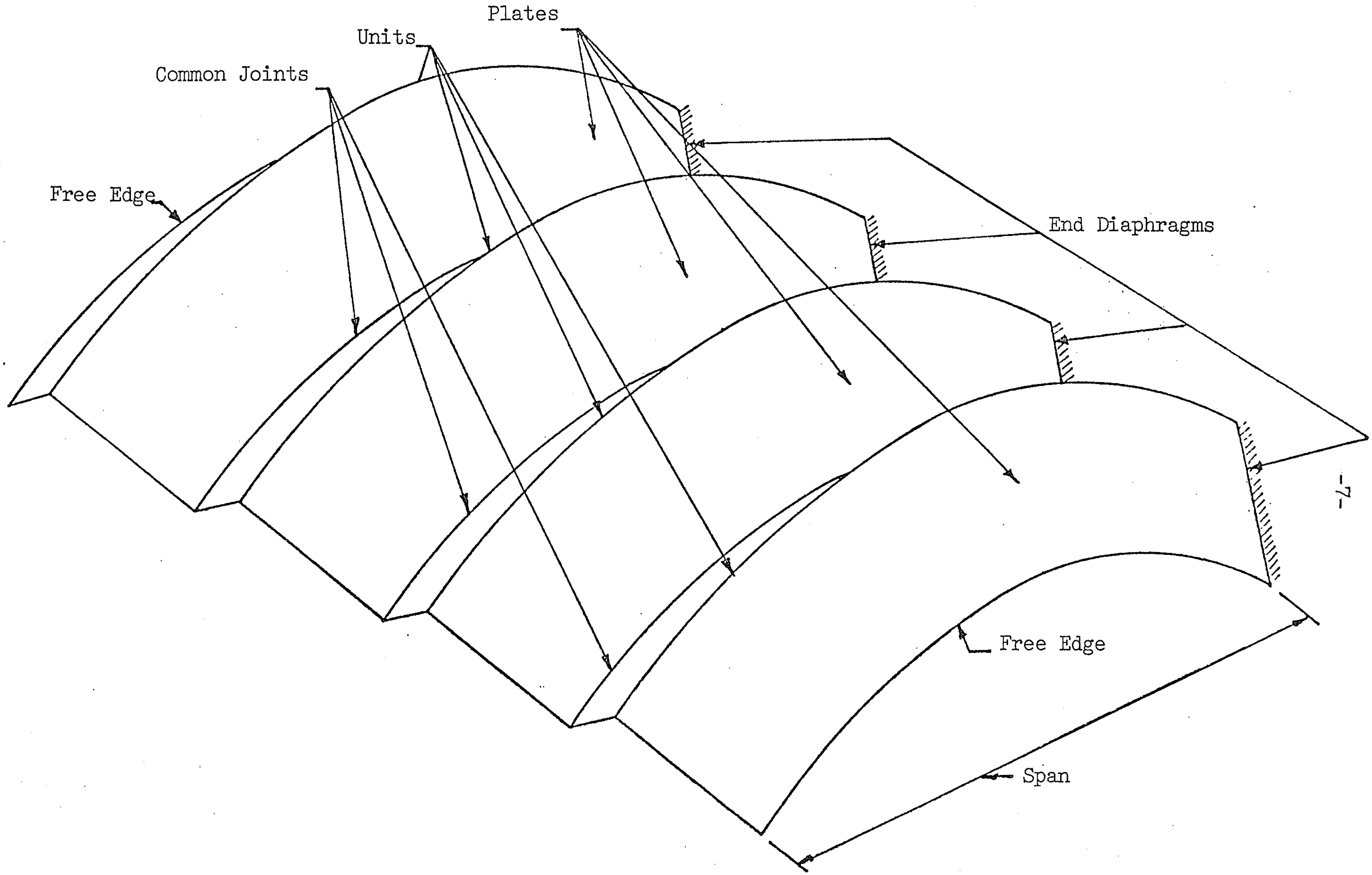

FIGURE 1.2 - Curved Folded Plate Structure

been made:

(i) The material is linearly elastic, isotropic and homogenous.

(ii) The longitudinal distribution of all loads on all units must be symmetrical. (The anti-symmetrical load case will be a subject for future studies).

(iii) The plates carry loads transversely only by bending normal to their planes (transverse continuous one-way slab action).

(iv) The plates carry loads longitudinally by bending and axial force within their planes (longitudinal arch and curved beam actions).

(v) Longitudinal stress varies linearly across the cross section of each unit.

(vi) The cross section of each unit is constant throughout its span length.

(vii) The curved folded plate is symmetrical in the longitudinal direction.

(viii) Each unit can be subjected to in-plane and out-of-plane loadings which can be forces or moments.

(ix) Individual units possess torsional resistance. Torsional stresses due to twisting moments are ignored. Rotational deflection of the cross section due to twisting is considered.

(x) Shear stress in each cross section has negligible effect on the deflection of the cross section.

(xi) The actual deflections are small compared to the overall configuration of the structure. Thus, the principle of superposition holds.

(xii) The two outer longitudinal edges of the structure are assumed to be free.

1.5 Outline of Method of Analysis

The procedure considers arch and curved beam theories for the longitudinal action of each unit and a continuous one-way slab for the transverse action.

Broadly, the analysis may be divided into four parts:

(i) Continuous Transverse Slab Action

The transverse section of the folded plate of unit width is treated as a continuous beam on rigid supports. The joint reactions from the common joints are applied as loads of opposite sign at the common joints. The joint loads thus obtained are resolved into components in the planes of the plates.

(ii) Arch and Curved Beam Longitudinal Action

Now it is temporarily assumed that each unit acts independently. The vertical joint loads and plate loads of each unit are transferred as in-plane and out-of-plane loads to the shear center of each unit. These loads are transmitted to the end supports by arch and curved beam actions. In-plane loads are carried by the arch action and out-of-plane loads are carried by curved beam action. Using the two separate analyses, longitudinal stresses and displacements are calculated for each unit at the common joints. These edge stresses and displacements will be different between the units. But compatibility demands that these should be the same at common edges of adjacent units.

(iii) Correction Forces

Self-equalized correction forces are applied at each common joint. The procedure explained under part (ii) is repeated. The longitudinal stresses and displacements are calculated. The magnitude of these correcting forces are determined by establishing an overall compatibility by formulating a set of simultaneous equations and solving them for the unknowns.

(iv) Superposition

The results of analysis in part (i) combined with those of the correction forces at the common joints yield the final results.

CHAPTER II

CURVED BEAM AND ARCH THEORIES

A. EQUATIONS OF CURVED BEAM

2.1 Introduction

Beams with curved axes under loads normal to the plane of their curvature are termed as curved beams (Figure 2.1). In the following sections the equilibrium equations and general solutions with fixed end conditions for a curved beam are derived, and forces, moments, and displacements for symmetric loading are presented. These are based on Saint-Venant's Equations [17] as derived by Volterra [18] in his paper.

2.2 Equation of Equilibrium

Consider the cantilever curved beam having the constant cross-sectional properties shown in Figure 2.2. The beam has a local cartesian Co-ordinate system x, y, and z with the origin 0 at the centroid of the cross-section of the beam. The shear center S is assumed to coincide with the centroid. The cross-section must also be symmetrical in order to eliminate I_{xy}, the product of inertia. The x and y axes are in the direction of the principal axes of inertia of the cross-section, while the z-axis coincides with the tangent to the elastic line at O. The xz-plane also coincides with the plane of initial curvature of the beam. Other variables, which appear in Figure 2.2 are defined as follows:

M_x = bending moment acting on the cross-section at O about the x-axis,

M_z = twisting moment at O about the z-axis,

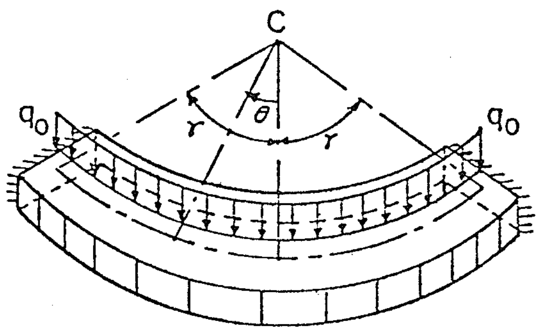

FIGURE 2.1 - Curved Beam with Uniformly Distributed Load

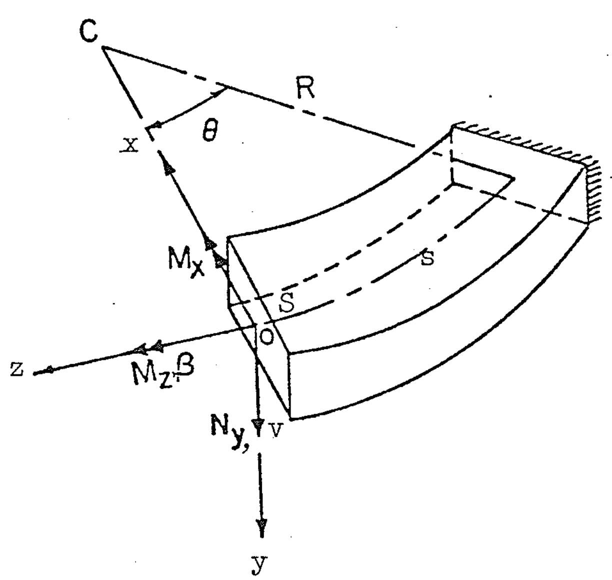

FIGURE 2.2 - Internal Forces and Moments of a Curved Beam

N_y = shear force in the y-direction,

v = centroid displacement in the direction of the y-axis,

β = angle of twist of the cross-section about the z-axis, counter clockwise rotation being positive.

s = arc length of the center line measured from the fixed end,

EI_{xx} = flexural rigidity,

R = radius of curvature of the center line,

J = torsional constant,

G_c = shear modulus,

T = JG_c = torsional rigidity

Saint-Venant's equations for curved beam can be derived as

$$M_x = EI_{xx}\left(\frac{\beta}{R} - \frac{d^2v}{ds^2}\right) \tag{2.1a}$$

$$M_z = T\left(\frac{d\beta}{ds} + \frac{1}{R}\frac{dv}{ds}\right) \tag{2.1b}$$

The displacement equations (2.1) can be written as,

$$M_x = a\left(\beta - \frac{d^2Y}{d\theta^2}\right) \tag{2.2a}$$

$$M_z = \mu a\left(\frac{d\beta}{d\theta} + \frac{dY}{d\theta}\right) \tag{2.2b}$$

where,

$$Y = \frac{v}{R}$$

$$a = \frac{EI_{xx}}{R}$$

$$\mu = \frac{T}{EI_{xx}}$$

and the angle θ is measured from the bisector of the angle 2γ between the two points of support (Figure 2.1).

Figure 2.3 shows an element of length ds of a curved beam with the stress resultants acting on the faces and the distributed loads applied along the elastic line. The positive signs are defined by the directions of all forces shown. By summation of forces in the y-direction and moments about the x and z axes respectively, the following set of equilibrium equations can be obtained.

$$\frac{dM_z}{d\theta} + \frac{d^2M_x}{d\theta^2} = R\frac{dm_x}{d\theta} + R^2 q_y \tag{2.3a}$$

$$\frac{dM_z}{d\theta} - M_x = -Rm_z \tag{2.3b}$$

$$N_y = \frac{1}{R}\left[\frac{dM_x}{d\theta} + M_z\right] - m_x \tag{2.3c}$$

Equations (2.2) and (2.3) are not applicable for curved beams of open cross-sections where shear centers generally do not coincide with centroids. However, according to V.Z. Vlasov [19], equations (2.2) and (2.3) are still valid provided that beams have small initial curvature with the ratio of the largest sectional dimension to the radius of curvature of the order of $\frac{1}{10}$ or less.

In Vlasov's theory, the quantity of $\frac{a_x}{R}$ is neglected when its value is small compared to unity, where a_x is the co-ordinate of the shear center with respect to centroid, and R is the radius of curvature of the centroidal axis. The stress resultant M_x is referred to the centroid, while the torsional moment M_z and shear N_y are referred to the shear center. All internal deflections and external applied loads are referred to the shear center. In deriving equations (2.3), all forces

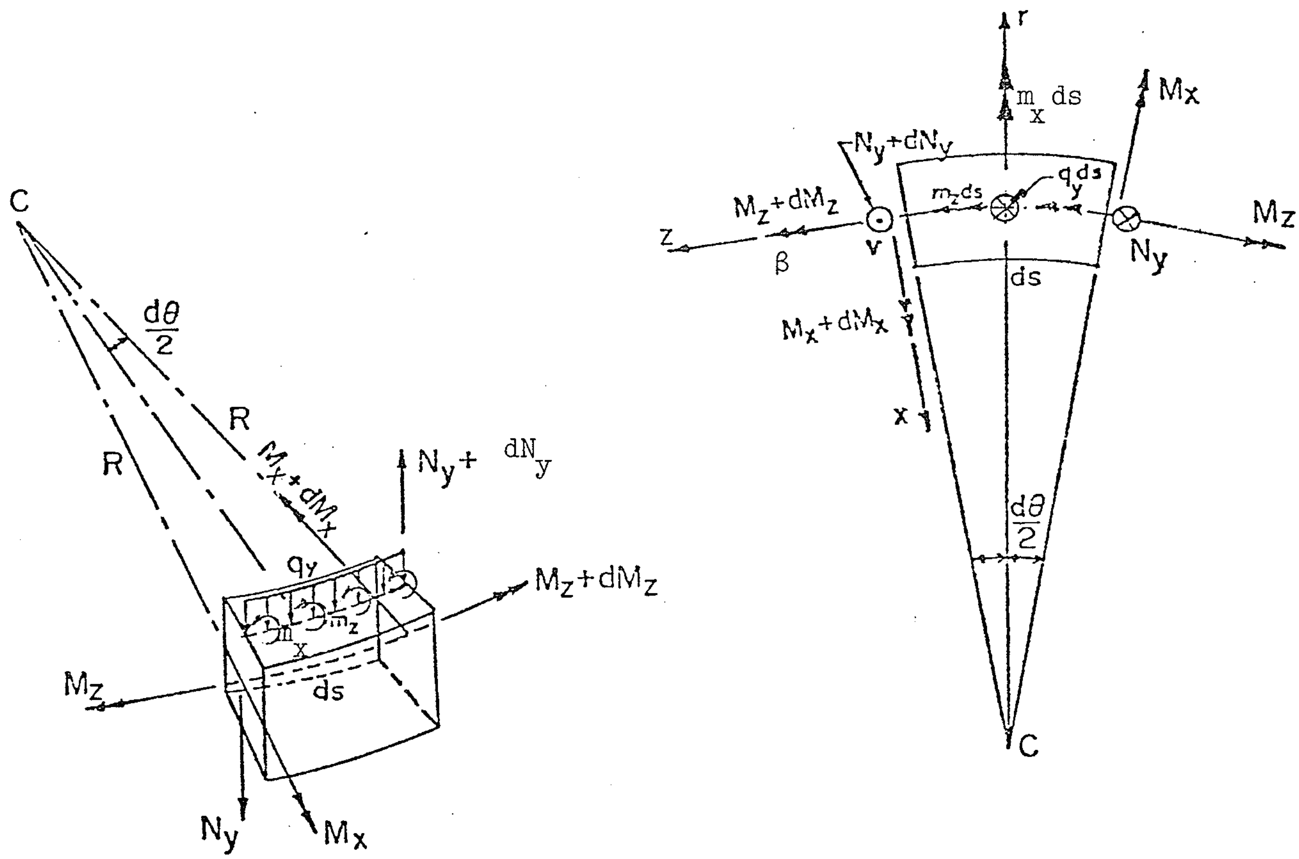

FIGURE 2.3 - Forces and Moments on a Curved Beam Element

are referred to the centrodal axis. This introduces some approximation.

2.3 Solutions of Curved Beam with Fixed-End Condition

The boundary conditions, regardless of the kind of loading, for a fixed ended curved beam are

$$\beta(\gamma) = \beta(-\gamma) = 0$$

$$Y(\gamma) = Y(-\gamma) = 0$$

$$\frac{dY}{d\theta}(\gamma) = \frac{dY}{d\theta}(-\gamma) = 0$$

Where 2γ is the angle subtended between the two end supports.

It is proposed to express the loading in the form of a cosine series as the curved beam is symmetric about the central bisector of the angle 2γ. Hence the loading can be expanded as

$$q_y = \Sigma H_m \cos \frac{m\pi}{2\gamma}\theta$$

$$m_x = \Sigma X_m \sin \frac{m\pi}{2\gamma}\theta$$

$$m_z = \Sigma Z_m \cos \frac{m\pi}{2\gamma}\theta$$

Where $m = 0,1,2,3,....$ and H_m, X_m, and Z_m are the amplitudes of each m^{th} harmonic value.

Assume $n = \frac{m\pi}{2\gamma}$ and q_y, m_x, and m_z are in the positive direction.

The right hand side of equations (2.3) becomes

$$R^2 q_y = R^2 \Sigma H_m \cos n\theta$$

$$R\frac{dm_x}{d\theta} = R\,\Sigma n X_m \cos n\theta$$

$$-R\,m_z = -R\,\Sigma Z_m \cos n\theta$$

By using the following notation

$$A_m \cos n\theta = R^2 H_m \cos n\theta \text{ or } nRX_m \cos n\theta \text{ (with only}$$

one type of loading at a time)

$$B_m \cos n\theta = -RZ_m \cos n\theta$$

equations (2.3) can be written as

$$\frac{dM_z}{d\theta} + \frac{d^2 M_x}{d\theta^2} = A_m \cos n\theta \tag{2.4a}$$

$$\frac{dM_z}{d\theta} - M_x = B_m \cos n\theta \tag{2.4b}$$

$$N_y = \frac{1}{R}\left(\frac{dM_x}{d\theta} + M_z\right) - X_m \sin n\theta \tag{2.4c}$$

The complete solution of equations (2.2) and (2.4) using the boundary conditions are as follows:

$$M_x = C_1 \cos\theta + \left(\frac{A_m - B_m}{1 - n^2}\right) \cos n\theta \tag{2.5a}$$

$$M_z = C_1 \sin\theta + \frac{A_m - n^2 B_m}{1 - n^2} \frac{\sin n\theta}{n} \tag{2.5b}$$

$$N_y = \frac{A_m}{R} \frac{\sin n\theta}{n} - X_m \sin n\theta \tag{2.5c}$$

$$Y = C_2 + C_3 \cos\theta - \frac{C_1(1 + \mu)}{2a\mu} \theta \sin\theta$$

$$\frac{B_m n^2(1 + \mu) - A_m(1 + n^2\mu)}{a\mu\,(n^2 - 1)^2} \left[\frac{\cos n\theta - 1}{n^2}\right] \tag{2.5d}$$

$$\beta = -\left(\frac{C_1}{a\mu} + C_3\right)\cos\theta + \frac{C_1(1+\mu)}{2a\mu}\,\theta\sin\theta + \frac{A_m(1+\mu) - B_m(n^2+\mu)}{a\mu(n^2-1)^2}\cos n\theta \qquad (2.5e)$$

where

$$C_1 = 2\left\{\frac{B_m n^2 - \dfrac{A_m(1+n^2\mu)}{1+\mu}}{(n^2-1)^2}\left(\sin\gamma\cos n\gamma - \cos\gamma\frac{\sin n\gamma}{n}\right) + \frac{(A_m - B_m)\,\mu}{(n^2-1)(1+\mu)}\sin\gamma\cos n\gamma\right\} \div \left[\frac{\mu-1}{\mu+1}\cos\gamma\sin\gamma + \gamma\right] \qquad (2.6a)$$

$$C_2 = \frac{1+\mu}{2a\mu}\left\{(\cos\gamma + \gamma\sin\gamma + \gamma\cos\gamma\cot\gamma)\,C_1 + 2\,\frac{B_m n^2 - \dfrac{A_m(1+n^2\mu)}{1+\mu}}{(n^2-1)^2}\left(\cot\gamma\frac{\sin n\gamma}{n} + \frac{1-\cos n\gamma}{n^2}\right)\right\} \qquad (2.6b)$$

$$C_3 = -\frac{1+\mu}{2a\mu}\left\{(1+\gamma\cot\gamma)\,C_1 + 2\,\frac{B_m n^2 - \dfrac{A_m(1+n^2\mu)}{1+\mu}}{\sin\gamma\,(n^2-1)^2}\,\frac{\sin n\gamma}{n}\right\} \qquad (2.6c)$$

For $n = 0$

$$\frac{\sin n\gamma}{n} = \gamma$$

$$\frac{\sin n\theta}{n} = \theta$$

$$\frac{1-\cos n\gamma}{n^2} = \frac{\gamma^2}{2}$$

$$\frac{\cos n\theta - 1}{n^2} = -\frac{\theta^2}{2}$$

B. EQUATIONS OF THE ARCH

2.4 Introduction

The arch is a well known structural unit. An arch can resist external loads more effectively than a straight beam because its resistance to shear and bending is enhanced by the presence of axial compression. Only the case of the symmetrical circular arch is considered here. The linear differential equations of an arch element along with the force and displacement relationship are presented. The exact solutions of these equations for fixed and pinned-end boundary conditions are also derived.

2.5 Displacement and Equilibrium Equations

The plane of curvature of the arch is assumed to be the plane of symmetry of the cross-section. External loads and any deformations will be only in this plane. The principle of superposition is applicable since only small deformations are assumed. The problem is treated as one of two-dimensional plane strain.

The displacement functions are derived from the Euler-Bernoulli Beam Theory. Stress-strain relationships are established under the plane strain condition. The moment and force equations are obtained by integrating over the entire cross-section.

Figure 2.4 shows a differential element of a straight beam before and after deformation. Figure 2.5 shows a small element of an elastic arch. The displacement functions for a point C within the beam element as given by Euler-Bernoulli can be extended to the arch element and can be expressed as

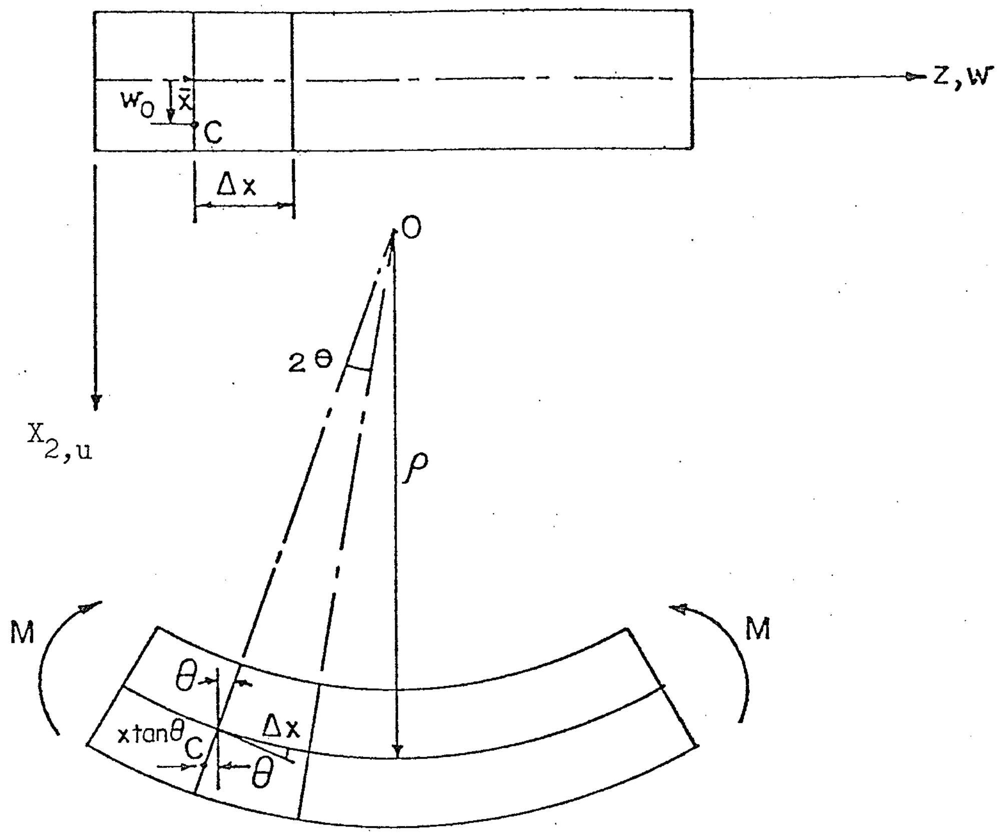

FIGURE 2.4 - Beam Element

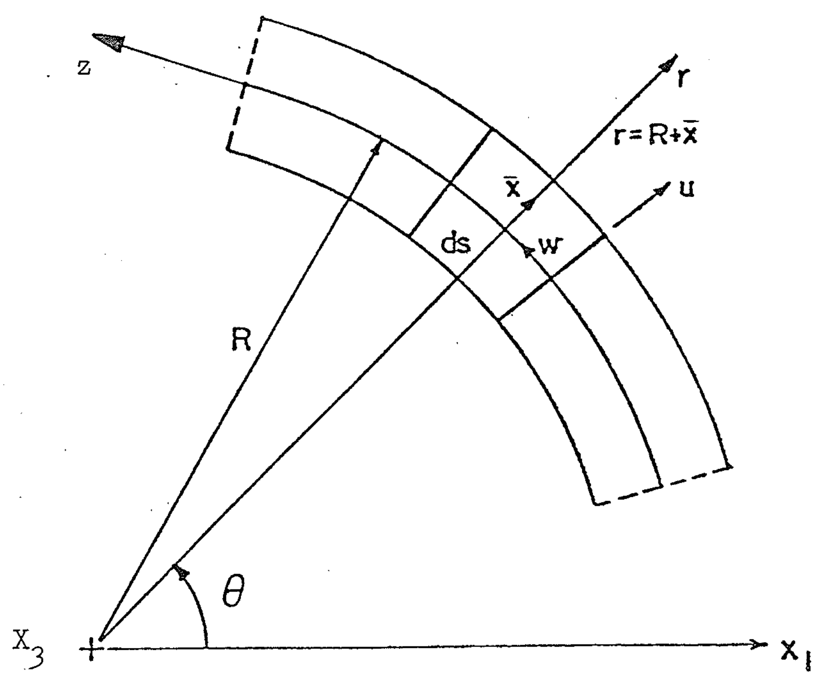

FIGURE 2.5 - Circular Arch Element

$$u = u(\theta)$$
$$w = w_o(\theta) + \bar{x}w_1(\theta) \qquad (2.7)$$
$$v = 0$$

Where u, w, and v, are the displacements along r, θ, and X_3 directions respectively (Figure 2.5). Differentiating equation (2.7) and substituting the values in the strain-displacement relationship expressed in polar co-ordinates and assuming normal and shear strains ε_r, and $\gamma_{r\theta}$ to be zero for plane strain problems of elasticity we get,

$$\sigma_\theta = E.\ \varepsilon_\theta = \frac{E}{R}\left[\left(u + \frac{dw}{d\theta}\right) + \frac{\bar{x}}{R}\left(\frac{dw}{d\theta} - \frac{d^2u}{d\theta^2}\right)\right] \qquad (2.8)$$

The stress resultant N_z and stress couple M_y, defined as the normal force and bending moment acting on the cross-section, are the integrals of stress over the area

$$N_z = \int_A \sigma_\theta \ dA \qquad (2.9a)$$

$$M_y = \int_A \bar{x}\ \sigma_\theta \ dA \qquad (2.9b)$$

Substituting equation (2.8) in equation (2.9) the force-displacement relations are obtained as

$$N_z = \frac{EA}{R}\left(u + \frac{dw}{d\theta}\right) \qquad (2.10a)$$

$$M_y = \frac{EI_{yy}}{R^2}\left(\frac{dw}{d\theta} - \frac{d^2u}{d\theta^2}\right) \qquad (2.10b)$$

Introducing the notation,

$$K = \frac{EA}{R}$$
$$\xi = \frac{I_{yy}}{AR^2}$$

the displacement equations 2.10 can be written as

$$N_z = K(u + \frac{dw}{d\theta}) \quad (2.11a)$$

$$\frac{M_y}{R} = K\xi(-\frac{d^2u}{d\theta^2} + \frac{dw}{d\theta}) \quad (2.11b)$$

Where the angle θ is measured from the bisector of the angle 2γ between the two points of support (Figure 2.6).

Figure 2.7 shows an arch element with stress resultants acting on the faces. The positive directions of all forces are defined as indicated. By summation of forces in the r and z-directions respectively and the moments about the y-axis, the following set of equilibrium equations can be obtained.

$$N_z - \frac{1}{R} \cdot \frac{d^2my}{d\theta^2} = -\frac{dmy}{d\theta} + Rq_x \quad (2.12a)$$

$$\frac{dN_z}{d\theta} + \frac{1}{R}\frac{dM_y}{d\theta} = -R.q_z + m_y \quad (2.12b)$$

$$N_x = -\frac{1}{R}\frac{dM_y}{d\theta} + m_y \quad (2.12c)$$

As in curved beam theory, equations (2.11) and (2.12) are not strictly valid for open cross-section elastic arches. The approximate theory, proposed by V.Z. Vlasov [15] will again be used. All the related assumptions and recommendations are followed as before.

2.6 Loadings on Arch

The various loadings are expressed in the form of Fourier series.

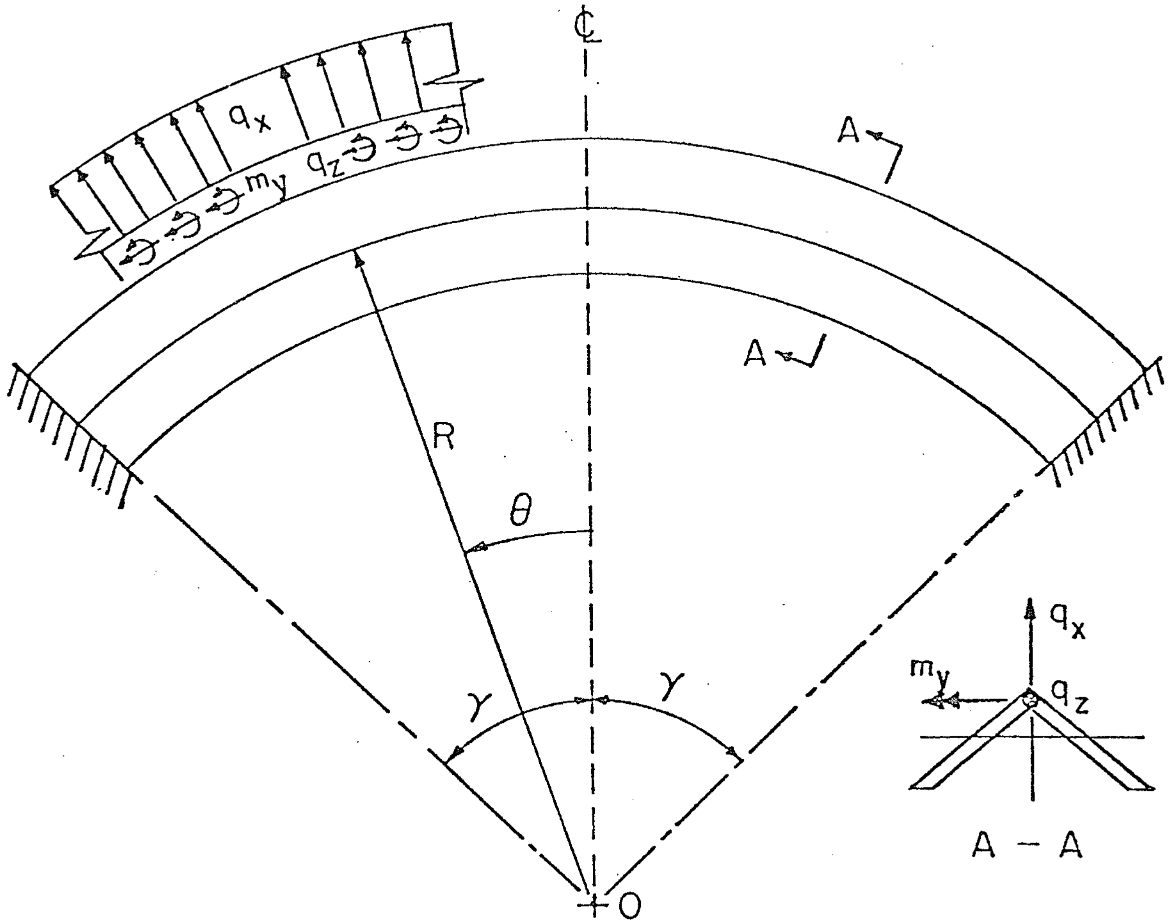

FIGURE 2.6 - Circular Arch

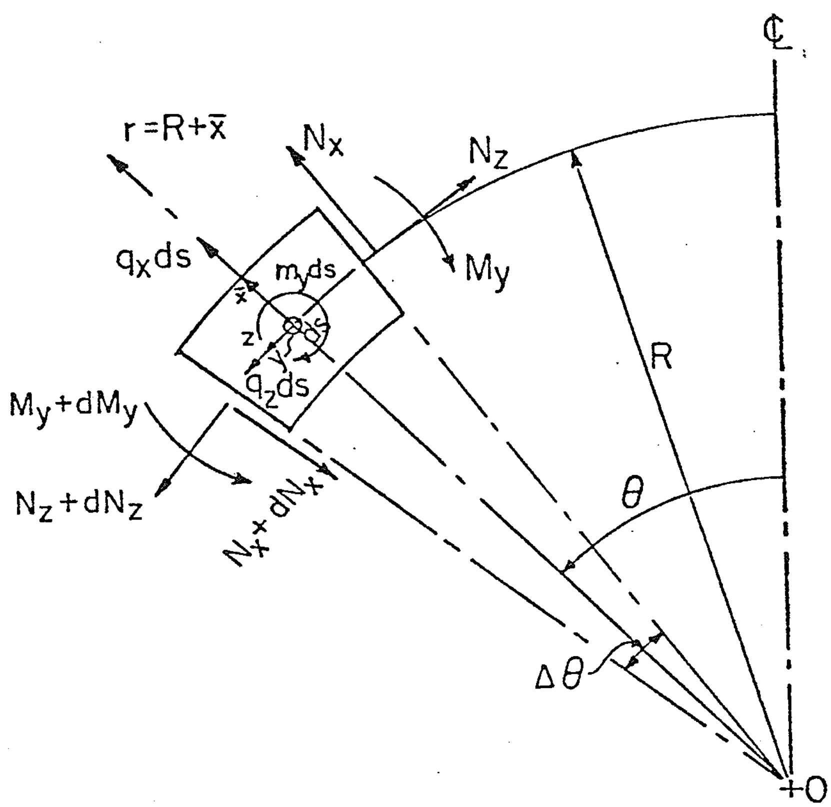

FIGURE 2.7 - Forces and Moments on an Arch Element

$$q_x = \Sigma V_m \cos \frac{m\pi}{2\gamma} \theta$$

$$q_z = \Sigma T_m \sin \frac{m\pi}{2\gamma} \theta$$

$$m_y = \Sigma U_m \sin \frac{m\pi}{2\gamma} \theta$$

where m = 0,1,2,3,.... and V_m, T_m and U_m are the amplitudes of each m^{th} harmonic value.

Assuming $n = \frac{m\pi}{2\gamma}$ and q_x, q_z and m_y are in the positive directions, the right hand sides of equations (2.12) become,

$$R\, q_x = R\, \Sigma V_m \cos n\theta$$

$$\frac{dmy}{d\theta} = -\Sigma n U_m \cos n\theta$$

$$R\, q_z = R\, \Sigma T_m \sin n\theta$$

By using the following notation

$$E_m \cos n\theta = R\, V_m \cos n\theta \text{ or } -nU_m \cos n\theta$$

(one type of loading taken at a time)

$$F_m \sin n\theta = (-RT_m + U_m) \sin n\theta$$

equations (2.12) become

$$N_z - \frac{1}{R} \frac{d^2 M_y}{d\theta^2} = E_m \cos n\theta \qquad (2.13a)$$

$$\frac{dN_z}{d\theta} + \frac{1}{R} \frac{dM_y}{d\theta} = F_m \sin n\theta \qquad (2.13b)$$

$$N_x = -\frac{1}{R} \frac{dM_y}{d\theta} + U_m \sin n\theta \qquad (2.13c)$$

2.7 Solution of Arch with Fixed-End Condition

The boundary conditions for a fixed end arch are:

$$u(\gamma) = u(-\gamma) = 0$$

$$w(\gamma) = w(-\gamma) = 0$$

$$\frac{du}{d\theta}(\gamma) = \frac{du}{d\theta}(-\gamma) = 0$$

Where 2γ is the angle subtended between the two end supports.

The complete solution of equations (2.11) and (2.13) using the above boundary conditions are as follows:

$$N_z = A_1 \cos\theta + \frac{(E_m + nF_m)}{1-n^2} \cos n\theta \qquad (2.14a)$$

$$\frac{M_y}{R} = A_3 - A_1 \cos\theta - \left(\frac{E_m + nF_m}{1-n^2} + \frac{F_m}{n}\right) \cos n\theta \qquad (2.14b)$$

$$N_x = U_m \sin n\theta - A_1 \sin\theta - \frac{(nE_m + F_m)}{1-n^2} \sin n\theta \qquad (2.14c)$$

$$u = -\frac{A_3}{K\xi} + A_2 \cos\theta + \frac{A_1(1+\xi)}{2K\xi} \theta \sin\theta$$

$$+ \frac{1+\xi}{K\xi} \left(\frac{E_m}{(1-n^2)^2} + \frac{F_m(1+n^2\xi)}{n(1+\xi)(1-n^2)^2}\right) \cos n\theta \qquad (2.15d)$$

$$w = \frac{A_3}{K\xi}\theta + \left[\frac{A_1(\xi-1)}{2K\xi} - A_2\right]\sin\theta + \frac{A_1(1+\xi)}{2K\xi}\theta\cos\theta$$

$$+ \frac{1+\xi}{K\xi}\left[\frac{(E_m + nF_m)\xi}{(1+\xi)(1-n^2)^2} - \frac{E_m}{(1-n^2)^2} - \frac{F_m(1+n^2\xi)}{n(1+\xi)(1-n^2)^2}\right]\frac{\sin n\theta}{n} \tag{2.15e}$$

where,

$$A_1 = 2\ \Big\{\left[\frac{E_m}{(1-n^2)^2} + \frac{F_m(1+n^2\xi)}{n(1+\xi)(1-n^2)^2}\right]\left[\frac{\sin n\gamma}{n} - n\sin n\gamma - \gamma\cos n\gamma + n\gamma\cot\gamma\sin n\gamma\right]$$

$$- \frac{(E_m + nF_m)\xi}{(1+\xi)(1-n^2)}\frac{\sin n\gamma}{n}\Big\} \div \left(\gamma\cos\gamma^2\sin\gamma + \gamma^2\cos\gamma\cot\gamma - \frac{2\sin\gamma}{1+\xi}\right) \tag{2.16a}$$

$$A_2 = \frac{1+\xi}{2K\xi}\left\{(1 + \gamma\cot\gamma)A_1 - 2\,\frac{\dfrac{nE_m + F_m(1+n^2\xi)}{(1+\xi)}}{(1-n^2)^2}\,\frac{\sin n\gamma}{\sin\gamma}\right\} \tag{2.16b}$$

$$A_3 = \frac{1+\xi}{2}\ \Big\{(\cos\gamma + \gamma\sin\gamma + \gamma\cos\gamma\cot\gamma)\,A_1$$

$$+ 2\left[\frac{E_m}{(1-n^2)^2} + \frac{F_m(1+n^2\xi)}{n(1+\xi)(1-n^2)^2}\right]\cos n\gamma$$

$$- 2\,\frac{nE_m + F_m\dfrac{1+n^2\xi}{1+\xi}}{(1-n^2)^2}\cot\gamma\sin n\gamma\Big\} \tag{2.16c}$$

For

$$n = 0$$

$$F_m = 0 \, , \quad \frac{F_m}{n} = 0$$

$$\frac{\sin n\gamma}{n} = \gamma$$

$$\frac{\sin n\theta}{n} = \theta$$

2.8 Solutions of Arch with Pinned-End Conditions

The boundary conditions for a pinned end arch are

$$u(\gamma) = u(-\gamma) = 0$$

$$w(\gamma) = w(-\gamma) = 0$$

$$M_y(\gamma) = M_y(-\gamma) = 0$$

where 2γ is the angle subtended between the two end supports.

The complete solution of equations (2.11) and (2.13) using the above boundary conditions are as follows:

$$N_z = D_1 \cos\theta + \frac{E_m + nF_m}{1-n^2} \cos n\theta \tag{2.17a}$$

$$\frac{M_y}{R} = D_2 - D_1 \cos\theta - \left(\frac{E_m + nF_m}{1-n^2} + \frac{E_m}{n}\right) \cos n\theta \tag{2.17b}$$

$$N_x = U_m \operatorname{Sin} n\theta - D_1 \sin\theta - \frac{(nE_m + F_m)}{1-n^2} \sin n\theta \tag{2.17c}$$

$$u = D_3 \cos\theta + \frac{1+\xi}{2K\xi} D_1 \theta \sin\theta - \frac{D_2}{K\xi}$$

$$+ \left[\frac{(1+\xi)E_m}{K\xi(1-n^2)^2} + \frac{(1+n^2\xi)}{K\xi(1-n^2)^2} \frac{F_m}{n}\right] \cos n\theta \qquad (2.17d)$$

$$w = \frac{D_1}{K} \sin\theta - D_3 \sin\theta - \frac{1+\xi}{2K\xi} D_1 (\sin\theta - \theta\cos\theta)$$

$$+ \frac{D_2}{K\xi}\theta - \left[\frac{E_m(1+n^2\xi)}{K\xi(1-n^2)^2} + \frac{F_m(1+n^4\xi)}{nK\xi(1-n^2)^2}\right] \frac{\sin n\theta}{n} \qquad (2.17e)$$

where,

$$D_1 = \frac{2}{1-n^2}\left\{\left[\frac{E_m(1+n^2\xi)}{1-n^2} + \frac{F_m(1+n^4\xi)}{n(1-n^2)}\right]\frac{\sin n\gamma}{n\sin\gamma}\right.$$

$$\left. - \left(E_m + \frac{F_m}{n}\right)\gamma\frac{\cos n\gamma}{\sin\gamma} - \left[\frac{E_m(n^2+\xi)}{1-n^2} + \frac{nF_m(1+\xi)}{1-n^2}\right]\frac{\cos n\gamma}{\cos\gamma}\right\}$$

$$\div\left(\xi - 3 + 2\gamma\cot\gamma + \frac{\gamma(1+\xi)}{\sin\gamma\cos\gamma}\right) \qquad (2.18a)$$

$$D_2 = D_1\cos\gamma + \left[\frac{E_m}{1-n^2} + \frac{F_m}{n(1-n^2)}\right]\cos n\gamma \qquad (2.18b)$$

$$D_3 = \frac{1}{K\xi}\left\{\left(1 - \frac{1+\xi}{2}\gamma\tan\gamma\right)D_1\right.$$

$$\left. - \left[\frac{E_m(n^2+\xi)}{(1-n^2)^2} + \frac{nF_m(1+\xi)}{(1-n^2)^2}\right]\frac{\cos n\gamma}{\cos\gamma}\right\} \qquad (2.18c)$$