The Impedance Matching and

Reduction - of - Variation Capabilities

of Cascaded Identical Twoports

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ABSTRACT

The input impedance to a cascade of identical twoports approaches, as the cascade lengthens, an impedance fixed by the parameters of the twoports no matter what termination is used except for a few special cases of no practical interest. For a variable load impedance, the input impedance at the n^{th} twoport lies within a given range of this fixed impedance. With the use of circles which enclose the variations in load and input impedances, equations are given which relate these circles to the ABCD parameters of the network. For certain values of the Thevenin equivalent impedance connected to the input of the network, the ABCD parameters are obtained for one twoport. The insertion loss is shown to be dependent upon the parameters of the network for the above fixed values of the Thevenin equivalent impedance.

A secondary result is the expressing of the Chebychev polynomial of the second kind as a ratio, rather than as a sum, of terms. A discussion on the use of the technique presented in this thesis as a method of designing a cascaded network for matching a variable load impedance to a fixed impedance is presented. Examples illustrating the design of a network of one section, for a given load condition, are also presented.

CONTENTS

LIST OF DIAGRAMS

CHAPTER I

INTRODUCTION

Purpose and Outline

The purpose of this thesis is to investigate the impedance matching capabilities at a fixed frequency of a network of cascaded identical twoports and to determine the parameters required for these twoports to match a variable load condition. To help in the evaluation of this method of impedance matching, the ensuing insertion loss is to be considered.

It is shown that variations in load impedance can be reduced by the use of the cascade of twoports. It is also shown that the insertion loss is dependent only upon the parameters of the twoports if certain conditions on one of the parameters are satisfied.

For the specific case of one twoport, the parameters are obtained for any variation in load impedance. Some examples illustrating the use of the equations obtained are presented.

A discussion of the results of this thesis is presented with some remarks on possible extensions and expansions of the work.

Background Material

Most of the following introductory material can be found in Johnson[1] and Eldring and Johnson[2]. Because this thesis follows a different line of reasoning than that presented in the above papers, the emphasis on the material presented in this section will be different.

Consider a cascade of identical twoports with a variable terminating impedance, Z_L (see Fig. 1). Each section is to be described by its ABCD, or chain, parameters, so that for the $(k + 1)^{th}$ section the following applies:

$$\begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{1}$$

$$= \underline{\underline{M}} \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{2}$$

and,

$$Z_{k+1} = \frac{A Z_k + B}{C Z_k + D} \tag{3}$$

where the symbols are those shown in Fig. 1.

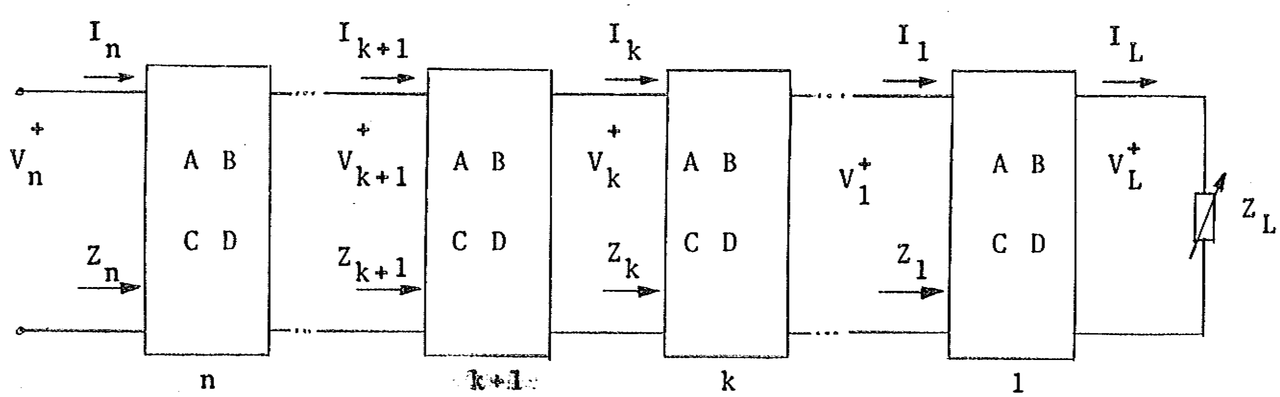

Fig. 1. Cascade of Identical Twoports.

The theorem presented by Johnson[1] makes use of the sequence of input impedances $\{Z_k\}$ and the fixed points of equation (3). It can be shown that the fixed points of (3) (i.e. for $Z_{k+1} = Z_k$) are given by:

$$Z_{s,u} = \frac{(A - D) \pm \sqrt{(A - D)^2 + 4BC}}{2C} \tag{4}$$

The value that is associated with Z_s is the value which, where possible, satisfies the condition:

$$\left| \frac{d\, Z_{k+1}}{d\, Z_k} \right|_{Z_k = Z_s} < 1 \tag{5}$$

The other fixed point, which will yield a magnitude of the above derivative greater than unity, is labelled Z_u.

The theorem states that the sequence of input impedances $\{Z_k\}$ will approach Z_s if the derivative condition can be satisfied. For this reason Z_s is referred to as the "stable" iterative impedance, while Z_u is referred to as the "unstable" iterative impedance. The case for which the derivative condition cannot be satisfied is for the magnitude of the derivative equal to unity. In this case the sequence $\{Z_k\}$ does not approach a definite limit but rather it oscillates about either Z_s or Z_u (see Eldring and Johnson[2]).

A parameter that is used in Ford[3] and Eldring and Johnson[2] to describe the locus of the sequence of input impedances $\{Z_k\}$ is given by equation (6) and is introduced here to simplify further calculations.

$$K = \frac{A - CZ_s}{A - CZ_u} \tag{6}$$

The parameters A and C are two of the ABCD parameters of the twoport and Z_s and Z_u are as defined by (4) and the derivative conditions. The magnitude of K can be shown to be less than unity for all ABCD (see Appendix I).

With a little manipulation, the following useful relations can also be shown:

$$K + \frac{1}{K} = \frac{(A + D)^2}{AD - BC} - 2 \tag{7}$$

and,

$$\frac{(A + D)^2}{AD - BC} = \frac{(K + 1)^2}{K} \tag{8}$$

The parameter K will be used extensively in the following chapters.

It is desirable that the input impedance to the n^{th} section, Z_n, be related directly to the terminating (load) impedance, Z_L. This is accomplished by expressing, in closed form, the n^{th} power of the ABCD matrix, $\underline{M}$. The following work has been presented in Johnson[1] and is presented here to define some symbols to be used later, and to give the full matrix equations that are derived.

It is convenient that the matrix to be taken to the n^{th} power have a unity determinant (the reciprocal case). However, (AD - BC) is not to be assumed equal to unity. Therefore, by normalizing the matrix $\underline{M}$ by dividing by $(AD - BC)^{1/2}$ the non-reciprocal case may be included. That is:

$$\underline{M}' = \begin{bmatrix} \dfrac{A}{(AD - BC)^{1/2}} & \dfrac{B}{(AD - BC)^{1/2}} \\ \\ \dfrac{C}{(AD - BC)^{1/2}} & \dfrac{D}{(AD - BC)^{1/2}} \end{bmatrix}$$

$$= \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \tag{9}$$

for which the characteristic equation is:

$$\lambda^2 - (A' + D')\lambda + 1 = 0 \tag{10}$$

or,

$$\lambda^2 - 2T'\lambda + 1 = 0 \tag{11}$$

where,

$$T' = 1/2(A' + D') = \frac{1/2(A + D)}{(AD - BC)^{1/2}} \tag{12}$$

is the half trace of the matrix $\underline{M}$.

The normalizing procedure must be extended to the voltage-current matrix so that the matrix equation for the

$(k + 1)^{th}$ twoport becomes

$$\frac{1}{(AD - BC)^{1/2}} \begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} = \underline{M}' \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{13}$$

The Cayley - Hamilton theorem states that, if the matrix $\underline{M}'$ be substituted for λ in its characteristic equation, a valid matrix equation is obtained. It is true, therefore, that

$$\underline{M}'^2 = 2T'\underline{M}' - \underline{U} \tag{14}$$

where $\underline{U}$ is the unit matrix of order 2. This leads directly to (see Pease[4])

$$\underline{M}'^n = U_n(T') \; \underline{M}' - U_{n-1}(T) \; \underline{U} \tag{15}$$

where $U_n(T')$ are the Chebyshev polynomials of the second kind given by

$$U_n(T') = \frac{\sinh\,(n \cosh^{-1} T')}{\sqrt{T'^2 - 1}} \tag{16}$$

(This form for $U_n(T')$ is used because T' will probably be complex. Equation (16) can be reduced to the equation given in Johnson[1] (note 11, page 36) for $|U_n(T')| < 1$.)

Now $\underline{M}'^n$ is the transfer matrix for the n - cascaded sections and so is given by

$$\underline{M}'^n = U_n(T') \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} - U_{n-1}(T') \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{M}'^{n} = \begin{bmatrix} U_n A' - U_{n-1} & U_n B' \\ U_n C' & U_n D' - U_{n-1} \end{bmatrix}$$

$$\equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{17}$$

where $U_n(T')$ has been replaced by U_n and $U_{n-1}(T')$ by U_{n-1}.

The normalization also affects the input voltage - current matrix as shown by (13). For the case of n sections instead of one section, (13) can be extended to:

$$\frac{1}{(AD - BC)^{n/2}} \begin{bmatrix} V_n \\ I_n \end{bmatrix} = \underline{M}'^{n} \begin{bmatrix} V_L \\ I_L \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} V_L \\ I_L \end{bmatrix} \tag{18}$$

where V_L and I_L are the load voltage and current and V_n and I_n are the input voltage and current to the n^{th} section, as shown in Fig. 1.

From (18) the input impedance to the n - cascaded sections terminated in Z_L is given by:

$$Z_n = \frac{aZ_L + b}{cZ_L + d} \tag{19}$$

where a, b, c, and d are defined by (17).

The cascade of n identical sections may now be replaced by one section with the input - output voltage - current relations given by (18) and (17). The input impedance has been found to depend upon the load impedance through the bilinear transformation of (19). While it seems that little has been gained by putting the equations describing the cascaded sections in this form, the next two chapters show how helpful equations (16) to (19) are.

CHAPTER II

INSERTION LOSS

The insertion loss is now to be considered because, in the evaluation of it, some assumptions are made which are used in later chapters and because the usefulness of the parameter K is illustrated.

The insertion loss is defined as "the decibel loss in power delivered to the load with the network inserted between the generator and load as compared with that when the generator and load are connected directly"[5].

The network to be considered is the n - cascade of identical sections used in Chapter I. By reference to Figures 2 and 3, the insertion loss is now to be calculated in terms of the ABCD parameters of each section. V_g and Z_g represent the Thevenin equivalent voltage source and impedance, respectively, of the network connected to the input terminals of the cascaded twoports.

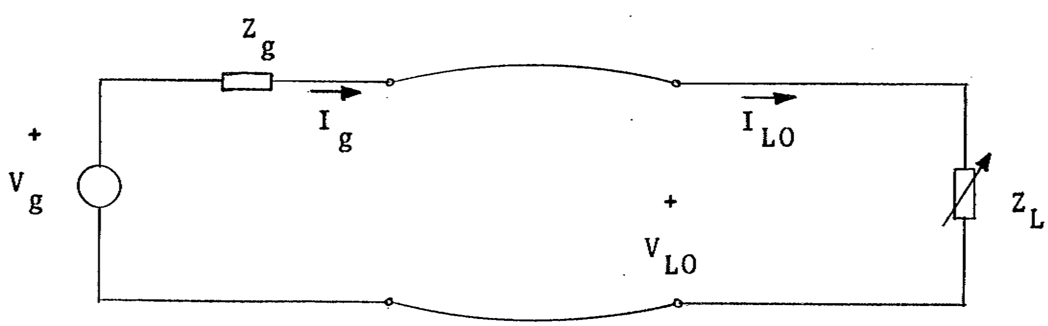

Fig. 2 - Load connected directly to generator

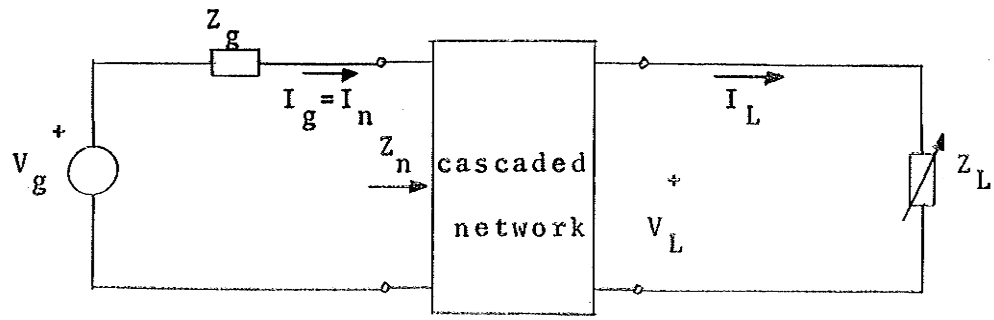

Fig. 3 - Network inserted between load and generator.

The insertion loss, I, can now be given as

$$I = 10 \log_{10} \frac{|V_{LO}|^2 / \; |Z_L'|}{|V_L|^2 / \; |Z_L|}$$

$$= 10 \log \left| \frac{V_{LO}}{V_L} \right|^2$$

$$= 10 \log \; |I_v|^2 \qquad (20)$$

where I_v is defined as the insertion voltage ratio. I_v is now to be determined in terms of K, Z_s, and the ABCD parameters.

For the network in Fig. 3, equation (18) relates the terminal conditions and is rewritten in (21).

$$\begin{bmatrix} V_n \\ I_n \end{bmatrix} = (AD - BC)^{n/2} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} V_L \\ I_L \end{bmatrix} \qquad (21)$$

To solve for the output voltage, (21) can be written as

$$\begin{bmatrix} V_L \\ I_L \end{bmatrix} = \frac{1}{(ad - bc)(AD - BC)^{n/2}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} V_n \\ I_n \end{bmatrix} \qquad (22)$$

or, particularly

$$V_L = \frac{dV_n - bI_n}{(ad - bc)(AD - BC)^{n/2}}$$

$$= \frac{(dZ_n - b)I_n}{(ad - bc)(AD - BC)^{n/2}} \qquad (23)$$

where

$$Z_n \equiv \frac{V_n}{I_n} .$$

But, from Fig. 3, it is evident that

$$I_n = \frac{V_g}{Z_g + Z_n} \qquad (24)$$

and so, there results:

$$V_L = \frac{(dZ_n - b)}{(ad - bc)(AD - BC)^{n/2}} \cdot \frac{V_g}{(Z_g + Z_n)} . \qquad (25)$$

From Fig. 2, it is evident that the voltage across the load is given by (26).

$$V_{LO} = \frac{V_g Z_L}{Z_g + Z_L} \tag{26}$$

However, with the use of (19), Z_L may be put in terms of Z_n to yield:

$$V_{LO} = \frac{V_g (dZ_n - b)}{-cZ_n Z_g + dZ_n + aZ_g - b} \tag{27}$$

Division of (27) by (25) yields:

$$I_v = \frac{V_{LO}}{V_L} = \frac{(ad - bc)(AD - BC)^{n/2}(Z_g + Z_n)}{-cZ_n Z_g + dZ_n + aZ_g - b} \tag{28}$$

Division of both numerator and denominator of (28) by $(Z_n + Z_g)$ and the substitututjon of the values for a, b, c, and d as given in (17), yields (29).

$$I_v = \frac{(ad - bc)(AD - BC)^{n/2}}{\left(- C'Z_g + D' + \frac{C'Z_g^{\,2} + (A' - D')Z_g - B'}{Z_n + Z_g}\right) U_n - U_{n-1}} \tag{29}$$

It can be shown that the roots of the expression

$$C'Z_g^{\,2} + (A' - D')Z_g - B' = 0$$

are

$$Z_g = Z_s - \frac{A - D}{C} = - Z_u \tag{30-a}$$

and

$$Z_g = - Z_s \tag{30-b}$$

by making use of (4) and the facts that

$$Z_s = Z_s' ,$$

(where the prime indicates that the ABCDs are primed) and

$$\frac{A - D}{C} = \frac{A' - D'}{C'} .$$

Therefore, if Z_g is equal to either $-Z_s$ or $-Z_u$, the insertion voltage ratio, and hence the insertion loss, becomes independent of the input impedance Z_n and has the value

$$I_v = \frac{(ad - bc)(AD - BC)^{n/2}}{(-C'Z_g + D')U_n - U_{n-1}} , \quad Z_g = -Z_s \text{ or } -Z_u \tag{31}$$

For the two values of Z_g given by (30), it is possible to obtain I_v in terms of K and n. Before proceeding, however, it is necessary that the Chebyshev polynomial U_n be evaluated in terms of K and n.

Recall that U_n is defined as

$$U_n = \frac{\sinh (n \cosh^{-1} T')}{\sqrt{T'^2 - 1}} \tag{16}$$

where

$$T' = 1/2(A' + D') = \frac{K + 1}{2K^{1/2}} \tag{32}$$

It is known that

$$\cosh^{-1}T' = \log_e(T' + \sqrt{T'^2 - 1})$$

and when T' is replaced by $\frac{(K + 1)}{2K^{1/2}}$ equation (33) results.

$$\cosh^{-1}T' = \ln K^{1/2} \tag{33}$$

By use of (33), the expression

$$\sinh(n \cosh^{-1}T') = 1/2(e^{n \cosh^{-1}T'} - e^{-n \cosh^{-1}T'})$$

becomes

$$\sinh(n \cosh^{-1}T') = \frac{K^n - 1}{2K^{n/2}} . \tag{34}$$

Therefore,

$$U_n = \frac{K^n - 1}{2K^{n/2}\sqrt{\frac{(K + 1)^2}{4K} - 1}}$$

$$= \frac{K^n - 1}{(K - 1)K^{(n-1)/2}} \tag{35}$$

From equation (4) it is known that

$$Z_s + Z_u = \frac{A - D}{C}$$

and hence (6) becomes

$$K = \frac{A - CZ_s}{D + CZ_s} = \frac{A' - C'Z_s}{D' + C'Z_s} \qquad (36)$$

Now, by substituting (32) into (36), it may be seen that

$$D' + C'Z_s = \frac{1}{K^{1/2}} \qquad (37)$$

and

$$A' - C'Z_s = K^{1/2} \qquad (38)$$

For (ad - bc) equal to unity (see Appendix II) and for Z_g equal to $-Z_s$, I_v is found to be given by (39) when (35) and (37) are substituted into (31) and some cancellations are carried out.

$$I_v = K^{n/2}(AD - BC)^{n/2} \qquad (39)$$

Also, for Z_g equal to $-Z_u$, the use of (35) and (38) produces

$$I_v = \frac{(AD - BC)^{n/2}}{K^{n/2}} \qquad (40)$$

It now is possible to write the insertion loss in terms of K, n and the determinant of the chain matrix for two specific values of the generator impedance. That is,

$$I = 10 \log |K(AD - BC)|^n , \quad Z_g = -Z_s \tag{41}$$

and

$$I = 10 \log \left|\frac{AD - BC}{K}\right|^n , \qquad Z_g = -Z_u = Z_s - \frac{A - D}{C} \tag{42}$$

As (AD - BC) can be adjusted to any desired value, the insertion loss is essentially dependent only upon K. For equations (41) and (42) to be equal, K must be equal to unity, thus yielding the situation where Z_s and Z_u are the same impedance. While this condition is theoretically possible (see Ford[3] and Eldring and Johnson[2]) it is unstable and any slight change in parameters would cause a difference between Z_s and Z_u and hence a reduction in the magnitude of K.

Now, since K is always less than unity (for this thesis, at any rate), then, for a given (AD - BC), equation (41) will yield a smaller insertion loss than (42), for all n. However, (41) requires that either Z_g or Z_s must have a negative real part (except where both are lossless). Since Z_s is approximately the input impedance to the cascaded network, the network must have active sources if Z_s is to have a negative real part. Equation (42) suggests that Z_g and Z_s may both be positive real functions and hence the network may be passive. The use of active sources would seem to be indicative of a smaller insertion loss than for a passive network with the same magnitude of (AD - BC). The condition presented by (42) does not rule out active sources in the network, but it does indicate that the network may be composed of passive elements.

The problem of reducing the variations in load impedance by means of the cascade of identical twoports still remains. The equations involved in showing that there is a reduction in load impedance variations are presented in the next chapter.

CHAPTER III

REDUCTION IN IMPEDANCE VARIATIONS

This chapter presents equations which show that the cascade of identical twoports reduces variations in load impedance and changes the impedance about which these variations can be considered to revolve. This is accomplished by employing circles which enclose the load impedance variations and the input impedance variations of the network.

It has been stated in the Introduction that for a cascade of n - identical twoports, the sequence of input impedances $\{Z_k\}$ will approach a fixed impedance Z_s, which is independent of the load impedance Z_L (see Fig. 1 for diagram). If the input impedance to the n - cascaded twoports, Z_n, is required to be within a certain percentage of Z_s, then it is possible to obtain the ABCD parameters of the sections for a given variation in Z_L. By restricting the value of Z_n in this way, the matching of the input impedance and the generator impedance is made possible.

Suppose that Z_n is to be within a value ε of Z_s. That is,

$$|Z_n - Z_s| \leq \varepsilon \tag{43}$$

where

$$\varepsilon = \varepsilon' |Z_s|$$

and ε' is less than unity. Thus, Z_n lies within a circle of radius ε and center Z_s.

Because the variation in load impedance is arbitrary and therefore probably difficult to describe mathematically in the impedance plane, a circle which encloses the complete locus of Z_L may be used to advantage. If the bilinear transformation given by (19) is applied to this circle as well as to the load impedance, it can be shown that Z_n will also remain within the transformed circle. Therefore, wherever this transformed circle falls within the circle drawn about Z_s, the locus of Z_n will fall within ε of Z_s.

The evaluation of the radii and centers of each of the circles in terms of the parameters of the other circle, the ABCD parameters, K, and n is now considered. To aid in this evaluation, however, it is convenient to consider the circle about Z_s of radius ε to be the load impedance circle after n transformations by (3) (or by one transformation (19)).

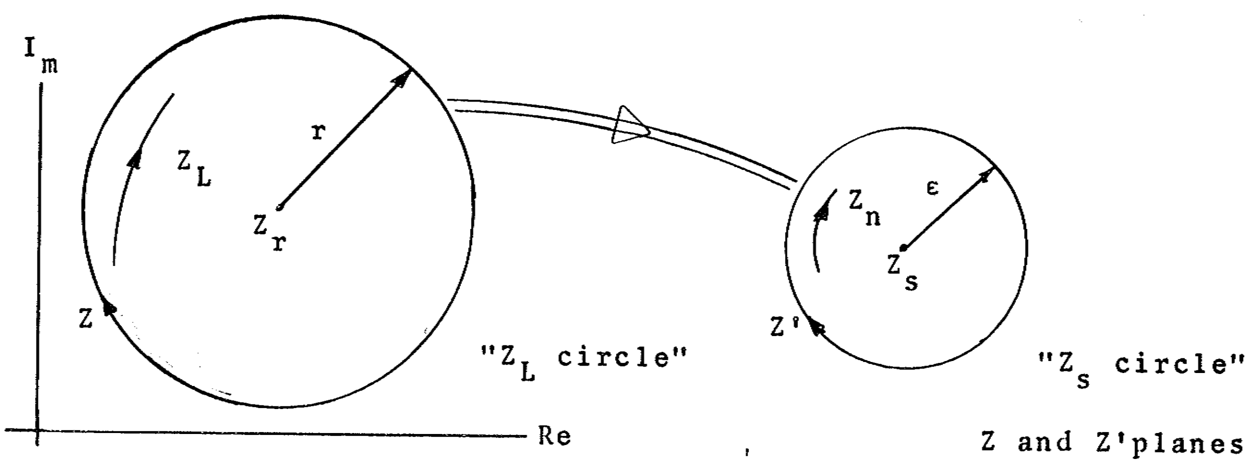

Fig. 4 - Load impedance and input impedance circles, (see below for definition of symbols).

Fig. 4 indicates the transformation to be carried out as well as the relation of the circles to each other. The Z_n and Z_L are arbitrarily drawn and are used only as an illustration of their relative positions in the impedance planes. The equation presented in Fig. 4 is (19) with Z_L replaced by Z and Z_n replaced by Z'. While each circle and the corresponding impedance variation should be in different planes, it is convenient to consider them on the same plane and denote their difference by a change in the plane variable (i.e. Z becomes Z' when Z_L has been transformed to Z_n). It is shown later in this chapter that Z_s must lie within the load impedance circle, but to simplify the diagram this restriction has been overlooked.

The problem now to be solved is that of expressing the center and radius of the load impedance circle (the "Z_L circle") in terms of the center and radius of the input impedance circle (the "Z_s circle"), K, and any of the ABCD parameters that cannot be eliminated. In the next chapter the equations presented here are used to solve for the ABCD parameters for various conditions on the network and the generator impedance.

Using standard notation (see Ford[3]) for the coefficients, the "Z_L circle" is given, in the Z plane, by

$$A_L Z\overline{Z} + B_L Z + \overline{B}_L \overline{Z} + C_L = 0 \qquad (44)$$

where the coefficients A_L and C_L are real and the bar over a symbol represents the complex conjugate of that symbol. Similarly, the "Z_s circle" is given, in the Z' plane, by

$$A_s Z'\overline{Z}' + B_s Z' + \overline{B}_s \overline{Z}' + C_s = 0 \tag{45}$$

where the coefficients A_s and C_s are real.

The transformation taking the "Z_L circle" to the "Z_s circle" is given by (46).

$$Z' = \frac{aZ + b}{cZ + d} \tag{46}$$

where a, b, c, and d are as defined by (17).

The coefficients of the "Z_L circle" are written in terms of those of the "Z_s circle" by substituting (46) into (45) and putting the resulting equations in the form of (44) so that by comparison with (44) the following results:

$$\left.\begin{aligned} A_L &= a\overline{a}A_s + a\overline{c}B_s + c\overline{a}\overline{B}_s + c\overline{c}C_s \\ B_L &= a\overline{b}A_s + a\overline{d}B_s + c\overline{b}\overline{B}_s + c\overline{d}C_s \\ C_L &= b\overline{b}A_s + b\overline{d}B_s + d\overline{b}\overline{B}_s + d\overline{d}C_s \end{aligned}\right\} \tag{47}$$

The center, Z_r, and radius, r, of the "Z_L circle" are given in terms of the coefficients of (44) by (48) and (49).

$$Z_r = \frac{-\overline{B}_L}{A_L} \tag{48}$$

$$r = \frac{(B_L\overline{B}_L - A_LC_L)^{1/2}}{A_L} \tag{49}$$

To obtain Z_r and r in terms of ε, Z_s, K and the A'B'C'D' parameters it is necessary that the coefficients of the "Z_s circle" be expressed in terms of its center and radius. The "Z_s circle" is given, in the Z' plane, by

$$|Z' - Z_s| = \varepsilon$$

or

$$|Z' - Z_s|^2 = (Z' - Z_s)(\overline{Z}' - \overline{Z}_s) = \varepsilon^2$$

and

$$Z'\overline{Z}' - \overline{Z}_sZ' - Z_s\overline{Z}' + (Z_s\overline{Z}_s - \varepsilon^2) = 0 \tag{50}$$

Comparison of (50) with (45) produces:

$$\left.\begin{aligned} A_s &= 1 \\ B_s &= -\overline{Z}_s \\ C_s &= Z_s\overline{Z}_s - \varepsilon^2 \end{aligned}\right\} \tag{51}$$

When the above values of A_s, B_s, and C_s are substituted into (47), A_L becomes

$$A_L = |a - cZ_s|^2 - |c|^2\varepsilon^2$$

With the use of the values of a and c as given by (17)

$$A_L = |U_n(A' - C'Z_s) - U_{n-1}|^2 - \varepsilon^2|C'|^2|U_n|^2$$

For U_n given by (35), $(A' - C'Z_s)$ equal to $K^{1/2}$ (equation (38)), and some manipulation and cancellation, equation (52) is obtained.

$$A_L = |K|^n - \frac{\varepsilon^2 |C'|^2 |K^n - 1|^2}{|K|^{n-1} \quad |K-1|^2} \tag{52}$$

$\overline{B}_L$ and thence Z_r are obtained in a manner similar to that used for obtaining A_L. The work is presented in Appendix III with the final value of Z_L given by (53).

$$Z_r = \frac{1}{C'} \left[-D' + \frac{1}{K^{1/2}} + \frac{\varepsilon^2 |C'|^2 \, (\overline{K}^n - 1)(K-1)}{K^{1/2}(|K|^{2n-1}|K-1|^2 - \varepsilon^2 |C'|^2 |K^n - 1|^2)} \right] \tag{53}$$

Since it is possible to prove equation (54),

$$(B_L \overline{B}_L - A_L C_L)^{1/2} = \varepsilon \tag{54}$$

(See Appendix III - B for details) r is given by (55) when the value of A_L in (52) is used in (49).

$$r = \frac{\varepsilon |K|^{n-1} |K - 1|^2}{|K|^{2n-1}|K - 1|^2 - \varepsilon^2 |C'|^2 |K^n - 1|^2} \tag{55}$$

The center Z_r and the radius r of the "Z_L circle" have now been found in terms of C', K, n, ε, and Z_s (D' is a function of C', K and Z_s, - see (37)). Therefore, for any given n, the load impedance circle is known if K, C' and the "Z_s circle" are given. The converse problem, that of finding K, Z_s, and C' when the load impedance circle is known, is solved in the next chapter, but for one section only.

By use of equations (53) and (55) it is possible to relate the difference between Z_r and Z_s to r. Substituting for D' as given in (37) into (53) and using (55) to simplify the last term of (53), Z_r is given by (56).

$$Z_r = Z_s + \frac{r\varepsilon \overline{C'}\ (\overline{K}^n - 1)}{K^{1/2}|K|^{n-1}\ (\overline{K} - 1)} \tag{56}$$

and

$$|Z_r - Z_s|^2 = \frac{r^2\varepsilon^2|C'|^2}{|K|^{2n-1}} \left|\frac{(\overline{K}^n - 1)}{(K - 1)}\right|^2 \tag{57}$$

But from (55), ε may be obtained in terms of r as

$$\varepsilon = \frac{|K|^{n-1}}{2r|C'|^2} \left|\frac{K - 1}{K^n - 1}\right|^2 \left[- 1 \pm \sqrt{1 + 4r^2|C'|^2|K|\left|\frac{K^n - 1}{K - 1}\right|^2}\ \right] \tag{58}$$

For ε to be greater than zero, the plus sign before the radical of (58) must be used. By substitution of the value for ε given by (58) into (57) and reduction of the resulting equation to its simplest form, equation (59) results:

$$|Z_r - Z_s|^2 = r^2 - \frac{\varepsilon r}{|K|^n} \tag{59}$$

That is, the stable iterative impedance Z_s must lie within the load impedance circle for all n. This definitely is a restriction upon the circle that may be drawn about the load impedance, or, conversely, it is a restriction upon Z_s. Since Z_s may have to be set at a specific value determined either by the insertion loss or the matching problem, there may be some difficulty in describing a load impedance circle which encloses the load impedance as well as Z_s.

It is seen from (59) that the distance between Z_r and Z_s is dependent upon the magnitude of K and the number of sections n. For $|K|^n$ close to unity, which produces minimum insertion loss in the passive network case, Z_s and Z_r will have the maximum separation that is possible. As $|K|^n$ decreases, Z_s moves towards Z_r, and for $|K|^n$ equal to ε/r, Z_s equals Z_r.

While it has been assumed that ε is known, it is, in reality, unknown because it depends upon $|Z_s|$ which is usually unknown for a given problem. However, (59) indicates that the magnitude of Z_s is within r of the magnitude of Z_r and, in most practical cases, probably much less. Therefore, Z_s may be approximated by Z_r so that ε may be given as

$$\varepsilon \approx \varepsilon' |Z_r| . \qquad (60)$$

This means that ε may be assumed known at all times and if ε, as calculated from Z_s, is much different from ε found by using Z_r, then the calculations can be repeated for the new ε.

Up to this point it has been assumed that ε is less than r. While this assumption might not be evident from the work presented, equation (59) shows that it must be true. That is, for (59) to hold,

$$|K|^n \geq \frac{\varepsilon}{r} . \qquad (61)$$

Because $|K|$ is less than unity at all times, ε must be less than r at all times. Therefore, there definitely is a reduction in impedance variation from load to input. Equation (61) also indicates that for increasing n, with K and r constant, ε may be decreased thus reducing the load variation even more.

The problem of solving for some of the parameters in terms of the remaining ones for the specific case of n equal to one, is to be considered in the next chapter.

CHAPTER IV

EVALUATION OF PARAMETERS FOR ONE SECTION ($n = 1$), AND EXAMPLES

The equations needed to solve for some of the parameters in terms of other parameters for one section, or twoport, are now considered. There are many combinations of known and unknown parameters that may be used but only a few of these combinations are presented. In all cases, the final result is the determination of the ABCD parameters.

It is assumed, in all cases, that a load variation is known and must either be matched to a generator (or fixed) impedance, or must have its variation reduced. A network of one twoport is found that satisfies either or both of these conditions. The equations to be used viz: (53), (56), and (55) for $n = 1$ are, respectively:

$$Z_r = \frac{1}{C'}\left(-D' + \frac{\bar{K}^{1/2}}{|K| - \varepsilon^2 |C'|^2}\right) \tag{62}$$

$$= Z_s + \frac{r\varepsilon \bar{C}'}{K^{1/2}} \tag{63}$$

and

$$r = \frac{\varepsilon}{|K| - \varepsilon^2 |C'|^2} \tag{64}$$

The two values of the generator impedance, Z_g, obtained in Chapter II are used extensively in the remainder of this chapter.

Both values are functions of Z_s, K and C', although, at first glance

$$Z_g = -Z_u = Z_s - \frac{A' - D'}{C'} \qquad (30\text{-}a)$$

is not. However, when (37) and (38) are used for A' and D', (30-a) becomes

$$Z_g = -Z_s + \frac{1-K}{C'K^{1/2}} \qquad (65)$$

<u>Case 1</u>

Consider a circle drawn about the load variation thereby fixing Z_r and r. Also assume that ε is known. The problem is to solve for C', K, and Z_s and thence A', B', and D'. To do this, however, it is not enough that Z_r, r and ε are known. Therefore, Z_g is introduced as a known parameter and must have the value of either $-Z_s$ or $-Z_u$, thus ensuring a calculable insertion loss.

<u>(a)</u> $Z_g = -Z_u = Z_s - \dfrac{A' - D'}{C'}$

Substituting for A' from (38) into the above equation, it can be shown that

$$D' = K^{1/2} + C'Z_g \qquad (66)$$

By the use of (66), equation (62) can be put in the following form:

$$C'(Z_r + Z_g) = -K^{1/2} + \frac{\overline{K}^{1/2}}{|K| - \varepsilon^2 \; |C'|^2} \qquad (67)$$

With

$$\frac{r}{\varepsilon} = \frac{1}{|K| - \varepsilon^2 |C'|^2}$$

from (64), (67) becomes

$$\varepsilon C'(Z_r + Z_g) = -\varepsilon K^{1/2} + r\overline{K}^{1/2} \qquad (68)$$

K may be found from (68) by using:

$$C'(Z_r + Z_g) \equiv a_1 + ja_2 = |C'| \; |Z_r + Z_g| e^{j\phi} \qquad (69)$$

and

$$K \equiv |K| e^{j\theta} = |K| \; (\cos\theta + j\sin\theta) \; . \qquad (70)$$

Substituting (69) and (70) into (68) and equating real and imaginary terms yields:

$$|K|^{1/2} \cos\theta/2 = \frac{\varepsilon \; a_1}{r - \varepsilon}$$

and

$$|K|^{1/2} \sin\theta/2 = - \frac{\varepsilon \; a_2}{r + \varepsilon}$$

Therefore, K and $|K|$ are given by (71) and (72) respectively:

$$K = (K^{1/2})^2 = \left(\frac{\varepsilon \; a_1}{r - \varepsilon} - j \; \frac{\varepsilon \; a_2}{r + \varepsilon} \right)^2$$

$$= \varepsilon^2 |C'|^2 |Z_r + Z_g|^2 \left(\frac{\cos\phi}{r - \varepsilon} - j \frac{\sin\phi}{r + \varepsilon} \right)^2 \qquad (71)$$

$$|K| = \varepsilon^2 |C'|^2 \; |Z_r + Z_g|^2 \left(\frac{\cos^2\phi}{(r-\varepsilon)^2} + \frac{\sin^2\phi}{(r+\varepsilon)^2} \right) \qquad (72)$$

The dependence of K on the magnitude of C' may be removed by solving for $|C'|^2$ from (72) and (64) and substituting this value into (71) yielding:

$$|C'|^2 = \frac{1}{\varepsilon r\left[|Z_r + Z_g|^2\left(\frac{\cos^2\phi}{(r-\varepsilon)^2} + \frac{\sin^2\phi}{(r+\varepsilon)^2}\right) - 1\right]} \tag{73}$$

and

$$K = \frac{\varepsilon|Z_r + Z_g|^2\left(\frac{\cos\phi}{r-\varepsilon} - j\frac{\sin\phi}{r+\varepsilon}\right)^2}{r\left[|Z_r + Z_g|^2\left(\frac{\cos^2\phi}{(r-\varepsilon)^2} + \frac{\sin^2\phi}{(r+\varepsilon)^2}\right) - 1\right]} \tag{74}$$

where

$$\phi = \arg(Z_r + Z_g) + \arg C' \tag{75}$$

While K and $|C'|^2$ still depend upon the argument of C', it is possib e to choose this angle such that equations (73) and (74) are simplified. If this is done, C' is then completely determined.

For ϕ either 0° or 180°, $|C'|^2$ and K are given by (76) and (77):

$$|C'|^2 = \frac{1}{\varepsilon r\left[\frac{|Z_r + Z_g|^2}{(r-\varepsilon)^2} - 1\right]} \tag{76}$$

$$K = \frac{\varepsilon}{r\left[1 - \frac{(r-\varepsilon)^2}{|Z_r + Z_g|^2}\right]} \tag{77}$$

For ϕ either 90° or 270°, $|C'|^2$ and K are given by (78) and (79);

$$|C'|^2 = \frac{1}{\varepsilon r\left[\dfrac{|Z_r + Z_g|^2}{(r+\varepsilon)^2} - 1\right]} \tag{78}$$

$$K = \frac{-\varepsilon}{r\left[1 - \dfrac{(r+\varepsilon)^2}{|Z_r + Z_g|^2}\right]} \tag{79}$$

The condition for the existence of $|C'|$ and therefore, K is

$$|Z_r + Z_g| > (r + \varepsilon)$$

It is noticed that K as given by (77) is real and positive while K as given by (79) has an angle of 180°. It is also seen that for $|Z_r + Z_g|$ much greater than $(r + \varepsilon)$:

$$|K| \approx \frac{\varepsilon}{r}$$

no matter what value ϕ has.

The stable iterative impedance Z_s is found by substituting the value for K given in (71) into (63) and solving for Z_s:

$$Z_s = Z_r - \frac{r\,\underline{/-\arg C'}}{|Z_r + Z_g|\left(\dfrac{\cos\phi}{r-\varepsilon} - j\,\dfrac{\sin\phi}{r+\varepsilon}\right)} \tag{80}$$

$$Z_s = Z_r - \frac{r \angle \tan^{-1}\left(\frac{r - \varepsilon}{r + \varepsilon} \tan\phi\right) - \arg C'}{|Z_r + Z_g|\left(\frac{\cos^2 \phi}{(r - \varepsilon)^2} + \frac{\sin^2 \phi}{(r + \varepsilon)^2}\right)^{1/2}} \tag{81}$$

<u>(b)</u> $Z_g = -Z_s$

K is obtained from (63) to yield:

$$K = \frac{(\varepsilon r)^2 \overline{C}'^2}{(Z_r - Z_s)^2} \tag{82}$$

or

$$K = \frac{(\varepsilon r)^2 |C'|^2}{|Z_r + Z_g|^2} \angle -2(\arg C' + \arg(Z_r + Z_g)) \tag{83}$$

To remove the dependence of K on C', take the magnitude of (83) and with (64) solve for $|C'|^2$. Then,

$$|C'|^2 = \frac{|Z_r + Z_g|^2}{\varepsilon r \, (r^2 - |Z_r + Z_g|^2)} \tag{84}$$

Substitution for $|C'|^2$ as given by (84) into (83) produces equation (85).

$$K = \frac{\varepsilon r \angle -2\phi}{r^2 - |Z_r + Z_g|^2} \tag{85}$$

where

$$\phi = \arg C' + \arg(Z_r + Z_g)$$

Therefore, K and C' have been obtained, although as with case 1(a), the argument of C' must be specified. In the case above, arg C' need not be made a multiple of 90° or 180°, although it may be desirable that C' be made such that K becomes real.

For both parts of this case, A', D', and B' are found in terms of K, C' and Z_g as follows. It should be remembered that Z_g is usually known whereas Z_s is known only approximately. For this reason Z_g is used in the equations below rather than Z_s. However, if Z_s is fixed, then Z_g becomes unknown and the equations derived above cannot be solved as they depend upon Z_g being known. Only for case 1(b) may Z_g and Z_s both be known. For this case the following equations still apply although it is now possible to replace Z_g by $-Z_s$. Once K and C' are evaluated, A' is obtained by substituting for D', as given by (37), into (30-a) to yield:

$$A' = \frac{1}{K^{1/2}} - C'Z_g \tag{86}$$

D' has been obtained in the same manner as A' (see (66)) and is given by

$$D' = K^{1/2} + C'Z_g \tag{87}$$

Since (A'D' - B'C') is equal to unity, B' may be found from:

$$B' = \frac{A'D' - 1}{C'} \tag{88}$$

A' and D' may also be obtained by using Z_s (instead of Z_g) and equations (37) and (38). It is better to use Z_g rather than Z_s, however, since Z_g is known and will introduce less error into A', D' and B' than will the use of Z_s.

Since C' is calculated from $|C'|^2$ it can have two different values differing only by 180° (i.e., $C' = \pm\sqrt{|C'|^2}$). Whichever sign before the radical is chosen, the same sign must be chosen for $K^{1/2}$. This is to ensure that Z_s lies within the "Z_L circle".

The ABCD parameters are obtained from the A'B'C'D' parameters by multiplying the normalized parameters by any number desired. This number is the square root of the determinant of the ABCD matrix and will probably be chosen such that one of the parameters is unity or such that the insertion loss is a given value.

Case 2

For K, Z_g, ε, Z_r, and r known, Z_s and the ABCD parameters are obtained for the two values of Z_g.

(a) $Z_g = Z_s - \dfrac{A' - D'}{C'}$

Using the equations of case 1(a) (i.e., (73), (76) or (78)) for $|C'|^2$, the parameter C' is calculated. Z_s is then given by (65) which is rewritten below:

$$Z_s = -Z_g + \frac{1 - K}{C' K^{1/2}} \tag{89}$$

where the sign of C' and $K^{1/2}$ are taken such that Z_s lies within the "Z_L circle" (i.e., $|Z_r - Z_s| < r$).

(b) $Z_g = -Z_s$

This case is somewhat trivial as only C' must be calculated. It is obtained from (63) and its value is given in (90).

$$C' = \frac{(\overline{Z}_r - \overline{Z}_s)\ \overline{K}^{1/2}}{\varepsilon r} \tag{90}$$

The parameters A', B' and D' are calculated as in case 1 from equations (86), (87) and (88). A,B,C, and D are then obtained in the same manner for case 1.

It might be desirable that the generator impedance, Z_g, not be considered in the calculations. This might be the case when a desired K or Z_s is required, independent of the network connected to the input terminals of the twoport. However, if Z_g is to be disregarded, then either Z_s or K must be known before the ABCD parameters may be calculated. Cases 3 and 4 consider the above problem.

Case 3

For this case, K, Z_r, r, and ε are assumed known. The magnitude of C' may be obtained from (64) as

$$|C'|^2 = \frac{r|K| - \varepsilon}{\varepsilon^2 r} \tag{91}$$

for

$$|K| > \frac{\varepsilon}{r} .$$

Z_s may now be found from (63):

$$Z_s = Z_r - \frac{r\,\varepsilon\,\overline{C'}}{K^{1/2}}$$

where arg C' must be assumed and the sign of $K^{1/2}$ chosen so that Z_s lies within r of Z_r.

The remaining ABCD parameters may be obtained by using (86), (87), and (88).

Case 4

The stable iterative impedance Z_s, Z_r, r, and ε are known. $|C'|^2$ and K are calculated from (84) and (85) respectively, with the argument C' still having to be assumed.

It is noticed that in each of the preceding cases, the argument of C' cannot be determined but must be assumed. In most cases, arg C' can be chosen to be either the complement or supplement of arg $(Z_r + Z_g)$ and thus simplify the calculations. For a known Z_s it might be wise to make arg C' the negative of arg Z_s to simplify the calculations of A', B' and D'. It is also possible to make C' a real number as an aid in realizing the network and in simplifying some of the calculations.

Example 1

A network of one section is required that will reduce the load variation given in Fig. 5 to one-twentieth of the given value. The generator impedance is to be 50 ohms.

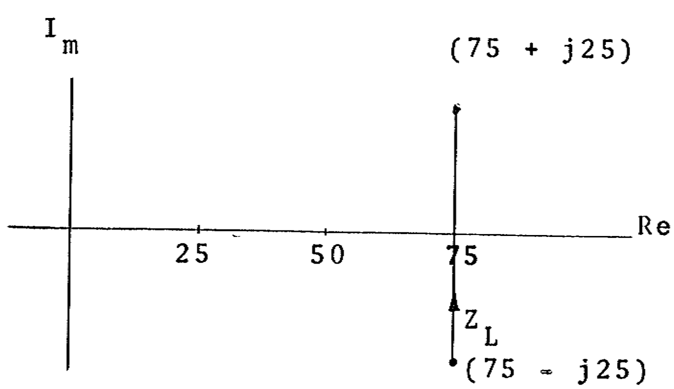

Fig. 5 - Example 1 - Impedance plane.

The equations derived in case 1(a) are used to obtain K, Z_s and then the ABCD parameters of the network. The network is synthesized by a T-network and the values of the impedances are calculated.

The minimum "Z_L circle" that can be drawn about the load variation has a radius of 25 and a center at (75 + j0.). Therefore,

$$Z_r = 75 + j0$$

and

$$r = 25$$

For a reduction in load variation of 1/20,

$$\varepsilon = \frac{r}{20} = 1.25$$

Since Z_r and Z_g are real numbers, ϕ may be made zero by letting

$$\arg C' = 0° .$$

Therefore, from (77), (and using five place logarithms),

$$K = \frac{\varepsilon}{r\left[1 - \dfrac{(r-\varepsilon)^2}{|Z_r + Z_g|^2}\right]}$$

$$K = \frac{1.25}{25\left[1 - \left(\dfrac{23.75}{125}\right)^2\right]}$$

$$= .051872 = \frac{1}{19.378}$$

and from (81),

$$Z_s = Z_r - \frac{r(r - \varepsilon)}{|Z_r + Z_g|}$$

$$= 75 - \frac{25(23.75)}{125}$$

$$= 70.25 \text{ ohms}$$

To calculate C' use (76) to yield:

$$|C'|^2 = \frac{1}{\varepsilon\, r \left[\frac{|Z_r + Z_g|^2}{(r - \varepsilon)^2} - 1 \right]}$$

$$= \frac{1}{(1.25)(25)\left[\left(\frac{125}{23.75}\right)^2 - 1\right]}$$

and

$$C' = .034618 \text{ mhos}$$

For A', D' and B', use equations (86), (87), and (88), respectively, with the plus sign being used for $K^{1/2}$ to produce:

$$A' = \frac{1}{K^{1/2}} - C'Z_g = 2.6598$$

$$D' = K^{1/2} + C'Z_g = 1.9587$$

$$B' = \frac{A'D' - 1}{C'} = 121.60 \text{ ohms}$$

For AD - BC = 1, A = A', B = B', etc. For a network given by these parameters, the insertion loss is given by

$$I = 10 \log_e \frac{1}{|K|} = 10 \log 19.378 = 12.851 \text{ db}$$

The network given by the above parameters may be synthesized by the reciprocal T - network illustrated in Fig. 6. if

$$Z_1 = \frac{A' - 1}{C'} = 47.946 \text{ ohms}$$

$$Z_2 = \frac{D' - 1}{C'} = 27.694 \text{ ohms}$$

$$Z_3 = \frac{1}{C'} = 28.887 \text{ ohms}$$

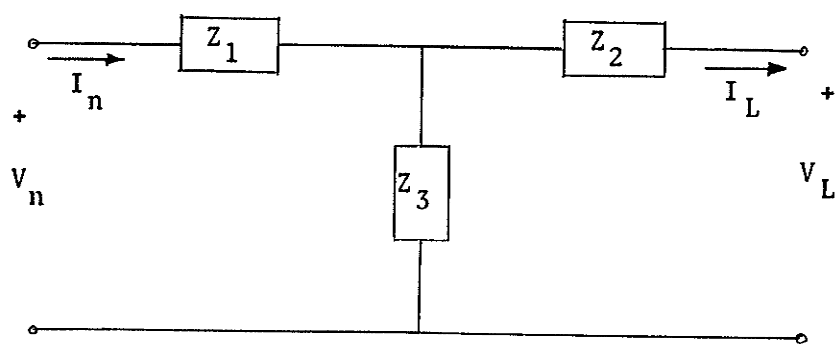

Fig. 6 - Example 1 - T- network.

Example 2

The load variation and the generator impedance are the same as for Example 1. It is now required that Z_s be made approximately equal to Z_g and that ε remain at 1.25 ohms.

The "Z_L circle" must be changed for the minimum case as is evident in the following calculation. The resulting change in the insertion loss and the A'B'C'D' parameters as compared with Example 1 are noted.

Let Z_r be 60 ohms. Hence, for the circle to enclose the load variation,

$$r \geq \sqrt{(75 - Z_r)^2 + (25)^2}$$

$$\geq 29.15 \text{ ohms.}$$

Therefore, let r be 30 ohms. Using (81), Z_s is found to be

$$Z_s = Z_r - \frac{r(r - \varepsilon)}{|Z_r + Z_g|}$$

$$= 60 - \frac{30(28.75)}{110}$$

$$= 52.25 \text{ ohms}$$

which is close enough to Z_g for this example. From (77), for arg $C' = 0°$,

$$K = \frac{\varepsilon}{r\left[1 - \dfrac{(r - \varepsilon)^2}{|Z_r + Z_g|^2}\right]}$$

$$= \frac{1.25}{30\left[1 - \left(\dfrac{28.75}{110}\right)^2\right]}$$

$$= .043324 = \frac{1}{23.081}$$

Using (76), C' is calculated to be

$$C' = .03257 \text{ mhos}$$

when arg C' = 0°. A', D' and B' as calculated from (86), (87) and (88) are as follows:

$$A' = 2.1759$$

$$D' = 1.8369$$

$$B' = 92.0 \quad \text{ohms}$$

The insertion loss is

$$I = 10 \log \frac{1}{|K|} = 13.633 \text{ db}$$

A comparison of the results of Examples 1 and 2 show that by increasing r, decreasing the magnitude of the sum of Z_r and Z_g, and keeping ε constant, the A'B'C'D' parameters are decreased while the insertion loss is increased. Therefore, the advantage gained by matching Z_s to Z_g in Example 2 might be overcome by the increase in insertion loss.

CHAPTER V

DISCUSSION

It is possible that any two different impedances may be matched using a single twoport, or, more likely, a number of twoports. It is difficult, however, to determine the ABCD parameters of the twoport required to match these impedances. This thesis presents equations which not only make it possible to determine the ABCD parameters of a twoport but also allows one of the impedances to vary. The variation in this impedance (the load impedance) is reduced by the network so that the input impedance to the network lies within an arbitrary range of an impedance (Z_s) fixed by the network. This impedance is itself arbitrary and may be made any value desired. It is this impedance which is then matched to a fixed (or generator) impedance. In this way a variable load impedance is simultaneously reduced in its variation and matched to a desired impedance.

By considering circles enclosing the load impedance and the input impedance to the network some observations are made. First of all, the reduction in impedance variation from load to input is given by the parameter K for certain conditions on the load impedance circle and the generator impedance. However, K is directly related to the insertion loss for the two values of the generator impedance mentioned. Therefore, by specifying either the impedance reduction (in terms of the radii of the circles), or the insertion loss, the other value is at least partially specified.

The use of circles also means that, for a large impedance variation and a large reduction in impedance variation, the stable iterative impedance, Z_s, is approximately equal to the center of the load impedance circle. Since Z_s is the impedance which is matched to the generator impedance Z_g, the load circle may be drawn to satisfy these conditions.

Using the above facts that K and Z_s are approximately known, it is a simple matter to obtain the ABCD parameters for one section. The center and radius of the load circle depend upon the relative positions of the load variation and the approximate value of Z_s. The radius of the "Z_s circle", ε, is found either from

$$K \approx \frac{\varepsilon}{r}$$

when r has been determined and K is known approximately, or from a requirement of the problem that there is a given reduction in impedance variation. With ε, r, Z_r, and Z_g now known, K and Z_s are determined exactly from the equations presented in Chapter IV. The ABCD parameters are then obtained in the manner stated in Chapter IV.

It should be possible to repeat the work presented in Chapter IV for more than one section. However, it is quite likely that the solution of K and Z_s will prove to be very difficult, for the equations become increasingly complex for increasing n.

Perhaps the greatest disadvantage of this method of impedance reduction and matching is the fact that the stable iterative impedance Z_s is within the load impedance circle for all n.

While this is an advantage in helping to fix the "Z_L circle", it is also a restriction on the size of this circle. For small impedance variations and a small impedance reduction, it might prove difficult to enclose Z_s within the "Z_L circle" without increasing r drastically. If the "Z_L circle" must be changed from its minimum value (i.e., the smallest circle about Z_L) to enclose the approximate value of Z_s, then there is a decrease in K for a constant ε value and hence a larger insertion loss in the "passive" ($Z_g = Z_s - \frac{A-D}{C}$) case.

The determination of ε is another disadvantage of this method. ε may be determined from K or the impedance variation reduction required and should be obtained this way whenever possible. However, ε was originally defined as a percent of the magnitude of Z_s. It may be assumed that the smaller this percentage, the closer the matching and hence the better the power transfer. Therefore, it should be possible to obtain a relation between ε and either the power input or the maximum available power. Consideration of the scattering parameters might be necessary if this thesis is to be extended to deal with the power aspects of the cascade of identical twoports.

Although this thesis makes little use of ε' (where $\varepsilon = \varepsilon'|Z_s|$) it is possible to relate ε' to the input reflection coefficient of the netowrk. This work is presented here as an indication of the relation between the size of the "Z_s circle" and the input power. Consider the input reflection coefficient as defined by (92).

$$\rho = \frac{\text{voltage of reflected wave at input}}{\text{voltage of incident wave at input}} \tag{92}$$

If Z_n is the input impedance to the network and Z_s is defined as the characteristic impedance of the network, then the magnitude of the reflection coefficient is given by (93).

$$|\rho| = \left| \frac{Z_n - Z_s}{Z_n + Z_s} \right| . \tag{93}$$

But,

$$|Z_n + Z_s| = |Z_n - Z_s + 2Z_s| \leq |Z_n - Z_s| + 2|Z_s|$$

Therefore,

$$|\rho| \geq \frac{|Z_n - Z_s|}{|Z_n - Z_s| + 2|Z_s|}$$

and

$$|Z_n - Z_s| \geq \frac{2|\rho|\ |Z_s|}{1 - |\rho|}$$

But

$$|Z_n - Z_s| \leq \varepsilon' |Z_s|$$

Hence

$$\varepsilon' \geq \frac{2\ |\rho|}{1 - |\rho|} \tag{94}$$

Therefore, for a given reflection coefficient, ε' has a minimum value given by the equality of (94). The reflection coefficient may also be used as a measure of power absorbed by the network although the relation between voltage and power is not linear. That is, it is not possible to replace the word "voltage" in (92) by "power" and satisfy the equation. With a little work, however, it might be possible to relate ε' to the power absorbed by the network using the above results.

Another way has now been found for obtaining ε. However, this method requires that Z_s be known from the beginning. In most cases Z_s is known only approximately. If ε' is chosen correctly, it is likely that (94) can still be satisfied when Z_s has been determined exactly.

Some other observations that should be made deal with the argument of C'. It has been stated in Chapter IV that, for one section, arg C' cannot be calculated and therefore must be assumed. By making arg C' certain values, equations are simplified and calculations are reduced. It might be possible to use the arbitrariness of arg C' to minimize the power loss or in conjunction with ε to determine certain power relations.

A secondary result of this thesis is the expressing of the Chebychev polynomial of the second kind in closed form. Equation (35) gives the Chebychev polynomial in terms of the parameter K.

The results obtained for one section indicate that this method of obtaining a network that will reduce impedance variation and match impedances might be extended to more than one section but not without some difficulty. The main problem would be in solving for K from the equations in Chapter III. It is unlikely that the equations and work presented may be extended to non-identical sections but it should be possible to include the case of a variable frequency.

While this thesis is based on a fixed frequency, the results and equations presented may be used for a variable frequency if the network is synthesized by elements which are not frequency dependent over the range considered. In this case it would then be possible to use the results for such practical purposes as impedance matching on telephone lines or antenna systems where the load impedance is usually frequency dependent.

APPENDICES

APPENDIX I

Proof that the magnitude of K is less than unity for all ABCD for which the derivative condition is fulfilled.

It has been stated in Chapter I that

$$\left| \frac{d\ Z_{k+1}}{d\ Z_k} \right|_{Z_k = Z_s} < 1 \tag{5}$$

where

$$Z_{k+1} = \frac{AZ_k + B}{CZ_k + D} \tag{3}$$

and

$$Z_{s,u} = \frac{(A - D) \pm \sqrt{(A - D)^2 + 4\,B\,C}}{2\,C} \tag{4}$$

with the sign of the radical chosen such that the magnitude condition (5) is satisfied.

Now,

$$\frac{d\ Z_{k+1}}{d\ Z_k} = \frac{(CZ_k + D)A - (AZ_k + B)C}{(CZ_k + D)^2}$$

$$= \frac{AD - BC}{(CZ_k + D)^2} \tag{A-1}$$

Therefore, from (5),

$$\left| \frac{AD - BC}{(CZ_s + D)^2} \right| < 1$$

and

$$|D + CZ_s| > |AD - BC|^{1/2} \tag{A-2}$$

It is also known (see Johnson [1] for proof) that

$$\left| \frac{d\, Z_{k+1}}{d\, Z_k} \right|_{Z_k = Z_u} > 1 \tag{A-3}$$

Therefore, from (A-1),

$$|D + CZ_u| < |AD - BC|^{1/2} \tag{A-4}$$

From (A-2) and (A-4), it is evident that

$$\left| \frac{D + CZ_u}{D + CZ_s} \right| < 1 \tag{A-5}$$

From (4) it can be shown that

$$Z_s + Z_u = \frac{A - D}{C} \tag{A-6}$$

Therefore,

$$D + CZ_u = D + C \left(-Z_s + \frac{A - D}{C} \right)$$

$$= A - CZ_s$$

and

$$D + CZ_s = D + C \left(-Z_u + \frac{A - D}{C} \right)$$

$$= A - CZ_u$$

Inequality (A-5) therefore becomes,

$$\left| \frac{A - CZ_s}{A - CZ_u} \right| < 1 \tag{A-7}$$

But K has been defined as

$$K = \frac{A - CZ_s}{A - CZ_u} \tag{6}$$

Therefore,

$$|K| < 1 \tag{A-8}$$

for all ABCD for which the derivative condition is fulfilled. Q.E.D.

APPENDIX II

Proof that (ad - bc) equals unity for all ABCD.

From equation (17),

$$a = U_n A' - U_{n-1}$$

$$b = U_n B'$$

$$c = U_n C'$$

$$d = U_n D' - U_{n-1}$$

where

$$U_n = \frac{K^n - 1}{(K - 1)\ K^{(n-1)/2}}\ , \tag{35}$$

$$A'D' - B'C' = 1,$$

and

$$A' + D' = \frac{K + 1}{K^{1/2}}\ . \tag{32}$$

Therefore,

$$ad - bc = (U_n A' - U_{n-1})(U_n D' - U_{n-1}) - U_n B' U_n C'$$

$$= U_n^{\ 2}(A'D' - B'C') - U_n U_{n-1}(A'+D') + U_{n-1}^{\ \ 2}$$

$$= \left(\frac{K^n - 1}{(K-1)K^{(n-1)/2}}\right)^2 - \left(\frac{K^n - 1}{(K-1)K^{(n-1)/2}}\right)\left(\frac{K^{n-1} - 1}{(K-1)K^{(n-2)/2}}\right)\left(\frac{K + 1}{K^{1/2}}\right)$$

$$+ \left(\frac{K^{n-1} - 1}{(K-1)K^{(n-2)/2}}\right)^2$$

$$= \frac{\left[(K^n - 1) - K(K^{n-1} - 1)\right]\left[(K^n - 1) - (K^{n-1} - 1)\right]}{(K-1)^2\ K^{n-1}}$$

= 1, for all ABCD. Q.E.D.

This result may be explained in a different manner as follows: Recall that (ad - bc) is the determinant of the matrix for n section, after normalization. That is,

$$
\begin{aligned}
ad - bc &= \det. \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \det. \underline{M}'^{n} \qquad \text{(see (17))} \\
&= \det. \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}^{n} \\
&= \left(\det. \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \right)^{n} \\
&= (1)^{n} = 1
\end{aligned}
$$

APPENDIX III - A

Evaluation of Z_r

From Chapter III, the following are known:

$$Z_r = \frac{-\overline{B}_L}{A_L}, \tag{48}$$

$$\overline{B}_L = b\overline{a}A_s + b\overline{c}B_s + d\overline{a}\overline{B}_s + d\overline{c}C_s, \tag{47}$$

$$A_s = 1, \quad B_s = -\overline{Z}_s, \quad C_s = Z_s\overline{Z}_s - \varepsilon^2 \tag{51}$$

and from (52),

$$A_L = \frac{|K|^{2n-1}\ |K-1|^2 - \varepsilon^2\ |C'|^2\ |K^n - 1|^2}{|K - 1|^2|K|^{n-1}} \tag{A-9}$$

Also it is known that

$$a = U_n A' - U_{n-1}$$

$$b = U_n B'$$

$$c = U_n C'$$

$$d = U_n D' - U_{n-1}$$

where

$$U_n = \frac{K^n - 1}{(K - 1)\ K^{(n-1)/2}} \tag{35}$$

Therefore,

$$\overline{B}_L = b\overline{a} - b\overline{c}\overline{Z}_s - d\overline{a}Z_s + d\overline{c}(Z_s\overline{Z}_s - \varepsilon^2)$$

$$= (b - dZ_s)\ (\overline{a} - \overline{c}\overline{Z}_s) - d\overline{c}\varepsilon^2$$

$$= \left[U_n(B' - D'Z_s) + U_{n-1}Z_s\right]\left[\overline{U}_n(\overline{A}' - \overline{C}'\overline{Z}_s) - \overline{U}_{n-1}\right] - \varepsilon^2\overline{U}_n\overline{C}'(U_nD' - U_{n-1})$$

Now,

$$B' = \frac{A'D' - 1}{C'}$$

Therefore,

$$B' - D'Z_s = \frac{A'D' - 1}{C'} - D'Z_s = \frac{D'(A' - C'Z_s) - 1}{C'}$$

and

$$\overline{B}_L = (1/C')\left[U_n\left(D'(A' - C'Z_s) - 1\right) + U_{n-1}C'Z_s\right]\left[\overline{U}_n\left(\overline{A}' - \overline{C}'\overline{Z}_s\right) - \overline{U}_{n-1}\right]$$

$$- \varepsilon^2\overline{U}_n\overline{C}'(U_nD' - U_{n-1})$$

To remove the $(A' - C'Z_s)$ and $C'Z_s$ terms from the above equation recall that

$$A' - C'Z_s = K^{1/2} \tag{38}$$

and

$$C'Z_s = \frac{1 - D'K^{1/2}}{K^{1/2}} \tag{37}$$

Therefore, $\overline{B}_L$ may now be obtained in terms of K, n, ε, C' and D', as

$$\overline{B}_L = \frac{1}{C'|K-1|^2|K|^{n-1}}\Big[\left((K^n - 1)(D'K^{1/2} - 1) + (K^{n-1} - 1)(1 - D'K^{1/2})\right)$$

$$\left((\overline{K}^n - 1)(\overline{K}^{1/2}) - (\overline{K}^{n-1} - 1)\overline{K}^{1/2}\right) - \varepsilon^2|C'|^2(\overline{K}^n - 1)\left((K^n - 1)D' - K^{n-1} - 1\right)K^{1/2}\Big]$$

To remove the denominator, let

$$N = \overline{B}_L C' \; |K-1|^2|K|^{n-1}$$

Therefore,

$$N = (D'K^{1/2}-1)\overline{K}^{1/2}|(K^n-1)-(K^{n-1}-1)|^2-\varepsilon^2|C'|^2(\overline{K}^n-1)\left[D'(K^n-1)-(K^{n-1}-1)K^{1/2}\right]$$

$$= (D'K^{1/2}-1)\overline{K}^{1/2}|K|^{2(n-1)}|K-1|^2 -\varepsilon^2|C'|^2(\overline{K}^n-1)\left[D'(K^n-1)-(K^{n-1}-1)K^{1/2}\right]$$

The above equation may be simplified by removing the factors in the numerator of (A-9). That is,

$$N = (|K|^{2n-1}|K-1|^2-\varepsilon^2|C'|^2|K^n-1|^2)(D' - \frac{1}{K^{1/2}}) - \frac{\varepsilon^2|C'|^2(\overline{K}^n-1)(K-1)}{K^{1/2}}$$

Solving for Z_r,

$$Z_r = \frac{-\overline{B}_L}{A_L} = \frac{-N}{A_L C'|K-1|^2|K|^{n-1}}$$

$$= \frac{1}{C'}\left[-D' + \frac{1}{K^{1/2}} + \frac{\varepsilon^2|C'|^2(\overline{K}^n-1)(K-1)}{K^{1/2}(|K|^{2n-1}|K-1|^2-\varepsilon^2|C'|^2|K^n-1|^2}\right] \quad \text{(A-11)}$$

APPENDIX III - B

Evaluation of $(B_L\overline{B}_L - A_LC_L)$.

It has been shown previously that

$$A_L = |a-cZ_s|^2 - |c|^2\varepsilon^2 \qquad \text{(see page 16)}$$

and

$$\overline{B}_L = (b-dZ_s)(\overline{a}-\overline{cZ}_s) - d\overline{c}\ \varepsilon^2 \qquad \text{(A-10)}$$

C_L is obtained by substituting equations (51) into (47) to yield:

$$C_L = b\overline{b} - b\overline{dZ}_s - d\overline{b}Z_s + d\overline{d}(Z_s\overline{Z}_s - \varepsilon^2)$$

$$= |b - dZ_s|^2 - |d|^2\varepsilon^2$$

Therefore,

$$B_L\overline{B}_L-A_LC_L = |\overline{B}_L|^2 - A_LC_L$$

$$= |(b-dZ_s)(\overline{a}-\overline{cZ}_s)-d\overline{c}\ \varepsilon^2|^2-(|a-cZ_s|^2-|c|^2\varepsilon^2)(|b-dZ_s|^2-|d|^2\varepsilon^2)$$

$$= |(b-dZ_s)(a-dZ_s)|^2+|dc|^2\varepsilon^4-d\overline{c}(b-dZ_s)(\overline{a}-\overline{cZ}_s)\varepsilon^2$$

$$- c\overline{d}(\overline{b}-\overline{dZ}_s)(a-cZ_s)\varepsilon^2-|(a-cZ_s)(b-dZ_s)|^2-|dc|^2\varepsilon^4$$

$$+ |c|^2|b-dZ_s|^2\varepsilon^2 +|d|^2|a-cZ_s|^2\varepsilon^2$$

$$= \varepsilon^2\left(-d\overline{c}(b-dZ_s)(\overline{a}-\overline{cZ}_s)-c\overline{d}(\overline{b}-\overline{dZ}_s)(a-cZ_s)+|c|^2|b-dZ_s|^2\right.$$

$$\left.+ |d|^2|a-cZ_s|^2\right)$$

$$= \varepsilon^2|c(b-dZ_s) - d(a-cZ_s)|^2$$

$$= \varepsilon^2|bc-ad|^2$$

But

$$ad - bc = 1 \quad \text{(Appendix II)}$$

Therefore,

$$(B_L \bar{B}_L - A_L C_L)^{1/2} = \varepsilon \qquad \text{(A-12)}$$

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