THREE DIMENSIONAL VECTOR GEOMETRY

by
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ABSTRACT

The central problem discussed is the coordinatisation of three-dimensional affine geometry. Our study begins with the presentation and development of an axiom system for affine geometry. Vectors are introduced into the geometry as equivalence classes of ordered pairs of points. Scalars are defined as functions of vectors onto vectors which in essence preserve parallelism and fix the zero vector.

It is proved that the scalars, under suitably defined operations, form a division ring and the vectors form an abelian group under addition. We next show that the vectors form a vector space of dimension three over the scalars. We further outline a proof that, given a vector space of dimension three over a division ring, under suitable definitions of point, line, plane, incidence, and parallelism, the geometry of the vector space is three-dimensional affine, in the sense of our axiom system. The central problem is thus completely solved.

Axioms of order are then introduced into the system and the notion of direction is discussed with regard to classes of parallel lines and to vectors. Real affine geometry is characterized as an ordered affine geometry where the order relation is analagous to that of the real numbers. Lastly, a metric is introduced into real affine geometry and the usual characterization of vectors in terms of length and direction is obtained.

In developing affine geometry in preparation for a study of vectors, we consider various Desarguesian configurations and prove that the corresponding configuration theorems are valid. In particular, the affine little Desargues Theorem is used to prove that the relation defining vectors is an equivalence relation. The central affine Desargues Theorem is used in the main existence theorem for scalars.

CHAPTER I - INTRODUCTION

In the present chapter, we shall outline the scope of this work and compare its approach with that of the literature on the subject.

Chapter II is devoted entirely to the study of a set of nine axioms for three-dimensional affine geometry. In order to obtain a convenient notation for the point of intersection of two lines, we have chosen to regard lines as nothing more than distinguished subsets of points. The affine theory developed is entirely equivalent to that obtained by regarding lines as entities associated with points by an incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$, for the element-set inclusion symbol '$\in$' plays essentially the same role in our presentation as 'I' does in the more common one.

In line with the above comments, a plane is defined to be a point set of a special type, and it is shown that any three noncollinear points determine a unique plane. Parallelism of lines is defined as an equivalence relation satisfying Playfair's equivalent of Euclid's Parallel postulate. From this, it is proved as Cor. 2.4.6 that two lines are parallel if and only if they are coplanar and do not intersect. Chapter II concludes with a study of finite affine spaces and several counting arguments to compute, for example, the total number of points in a plane or in an affine space.

On referring to D. Hilbert's Grundlagen der Geometrie, we see that Chapter II essentially covers the so-called "Axioms of Connection" and "axiom of parallel". We note that in our system, Euclid's defining characteristic of a plane, namely that it contain all points of a line containing two points of the plane, is obtained as Theorem 2.3. Furthermore, parallelism of planes is defined and proved to be an equivalence relation satisfying an analogue of Euclid's parallel postulate. As indicated in E. Moise, Elementary Geometry from an Advanced Standpoint, this result is usually proved in a metric geometry with the aid of congruence postulates.

Chapter III is concerned with the definition of vectors in a three dimensional affine space. Although similar studies have been made in two dimensions, the specific details for the three-dimensional case do not appear to have been worked out elsewhere. Our approach owes much to G. Papy, Géométrie Affine Plane et Nombres Réels. One important difference between Papy's work and the present study is that, unlike Papy, we do not suppose initially that we are dealing with an ordered infinite geometry. Another is that three-dimensionality allows us to prove, via Desarques' Theorem, that vectors are well-defined. Restricted to two dimensions, Papy postulates well-definedness of vectors. Specifically, we are indebted to Papy for the definition of the relation '$\square$'.

Similar definitions of vectors are to be found in K. Reidemeister, Grundlagen der Geometrie and also in notes from a seminar given by Prof. G. Pickert at Justus Liebig University in Giessen in 1963.

In Chapter IV, we adopt Dr. Pickert's functional definition of multiplication of a vector by a scalar. Theorem 4.3 states that any non zero scalar is a one-to-one mapping of the vectors onto themselves. Corollary 4.8 states that the vectors form a vector space of dimension three over the division ring of scalars, thus indicating the connection of our definition of vectors with the common algebraic definition of a vector space.

Definitions similar to, but not identical with, Prof. Pickert's definition of scalars are found in the literature. In Geometric Algebra, E. Artin studies dilatations, defined as mappings of points into points such that if $P \neq Q$ and ℓ is the unique line through f (P) and parallel to PQ, then f (Q) lies on ℓ. Artin proves that dilatations are determined uniquely by the images of two distinct points and that all nonconstant dilatations are one to one onto functions. In Grundlagen der Elementarmathematik, H. Lenz studies dehnungen, mappings f of points onto points such that either f (A) = f (B) or f (A) f (B) $\parallel$ AB. Such a mapping is called a vektor if it is either the identity mapping or fixed point free. Lenz proves that any dehnung which maps two distinct points into the same point is a constant function. Moreover, if O is a fixed point of the dehnung f and P $\neq$ O, then f (P) lies on OP. Lenz proves the existence theorem that given O, A $\neq$ O, B on OA, there is a unique dehnung f such that f (O) = O, f (A) = B; moreover, given any points A, B, there is exactly one vektor g such that g (A) = B.

In terms of the definitions and notation adopted in the present work, if the vector space is represented at any point P, then any multiplication by a scalar f may be associated with the mapping f^* defined as follows. For any point Q, let $\underline{v} \in \mathcal{V}$ be such that (P, Q) $\in \underline{v}$ and let R $\in \mathcal{P}$ be such that (P, R) $\in$ f ($\underline{v}$). Define f^* (Q) = R. It is readily shown from the results of this thesis that f^* is a function. Moreover, f^* is a dehnung, but clearly not a vektor. Similarly, given a representation of the vector space at a point P, any non-constant dehnung which is not a vektor, may be interpreted as a scalar, with the aid of Desarques' Theorems. The proof consists in representing the vector space at the unique fixed point of the dehnung and arguing as in Theorems 4.1, 4.2.

With regard to dilatations, we note that if (A, B) $\sqsupset$ (C, D) or if (A, B) $\uparrow$ (C, D), then there exists a unique dilatation f such that f (A) = C and f (B) = D, provided A $\neq$ B.

Chapter IV concludes with a comprehensive study of non-degenerate Desarquesian configurations. The author is not aware of any similar study elsewhere in the literature.

In Chapter V, axioms of order are introduced into an affine geometry and direction is defined in terms of equivalence classes of certain types of orders on lines of a parallel class under parallel projection. Positiveness of scalars is defined, and from this an order relation on the scalars is induced so that the scalars form an ordered division ring. Theorems 5. 12 and 5. 13 are the so-called "plane-separation axiom" and "space-separation axiom" respectively. In the book by E. Moise referred to above, these results are each taken as axioms. As a corollary to Th. 5. 12, we prove the classical axiom of Pasch. Adopting Papy's vector definition of mid-point, we prove the existence of unique mid-points and generalize and solve the analagous problem of n-division points.

The proof of Theorem 5. 2 is a simplification of that found in H. Lenz, Grundlagen der Elementarmathematik. Lenz uses axioms of order equivalent to those in Chapter V.

The reader interested in seeing other approaches to order defined in terms of postulates of betweenness and the axiom of Pasch is referred to the works of D. Hilbert and E. Moise mentioned above. An alternative approach to order in which axioms for order of a one dimensional subgroup of vectors are used is to be found in Papy's book.

Chapter VI serves as a transition from ordered affine geometry to metric geometry. By means of Corollary 4. 8 and Theorems 5. 12, 5. 13, we define an orientation of our vector space as an equivalence class of bases, where the relevant equivalence relation is defined with the aid of a ternary "box product" function from the vector space into the scalars. The classical scalar product is used to define the length of a vector. The traditional geometric definition of a vector as a quantity uniquely determined by its length and direction is then obtained as Theorem 6.3. Finally, with the aid of well-known results from linear algebra, the class of metric-preserving linear transformations of the vector space onto itself whose matrix representation has positive determinant is mapped into the real numbers from 0 to π.

The numbering system employed in this work is illustrated in the following table:

Axiom 2.1	First axiom introduced in Chapter II.
Definition 2.1	First definition in Chapter II.
Proposition 3.1	First proposition in Chapter III.
Theorem 2.1	First theorem in Chapter II.
Lemma 2.4.1	First lemma leading to the proof of Th. 2.4.
Corollary 2.1.3	Third corollary to Th. 2.1.

II Axiomatic Study of Affine Spaces

An affine space $\mathcal{Q}$ is a quadruple $(\mathcal{P}, \mathcal{L}, \|, \in)$ where $\mathcal{P}$ is a set of elements called "points", $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$ called "lines", and $\in$ and $\|$ are undefined relations. The statement "$\ell \| m$" may be read "The line ℓ is parallel to the line m." The statement "$P \in \ell$" may be read "The point P lies on the line ℓ" or "The line ℓ passes through the point P." Its negation will be abbreviated as "$P \notin \ell$". We shall understand all statements of the form "$P \notin \ell$" or $P \in \ell$" to imply that P is a point of $\mathcal{Q}$ and ℓ is a line of $\mathcal{Q}$. Lastly, "$P_1, P_2 \in \ell_1, \ell_2$" will be taken to mean that $P_i \in \ell_j$ $(i, j = 1, 2)$.

Axiom 2.1 $P_1, P_2 \in \ell_1, \ell_2 \Rightarrow P_1 = P_2$ or $\ell_1 = \ell_2$ (or both).

Notice that this is simply a concise way of saying that any two distinct points (resp. lines) determine at most one line (resp. point).

Axiom 2.2 $P_1 \neq P_2 \Rightarrow$ there exists ℓ such that $P_1, P_2 \in \ell$.

In view of Ax. 2.1 - 2, it is clear that any two distinct points P_1 and P_2 determine exactly one line which we may denote indifferently by either "P_1P_2" or "P_2P_1". The expression "P_1P_2" will be used only in the event $P_1 \neq P_2$.

Axiom 2.3 Each line passes through at least two distinct points.

Axiom 2.4 Parallelism among lines is an equivalence relation.

We denote the parallelism of any lines ℓ and m by "ℓ // m" or "m // ℓ".

Axiom 2.5 $P \notin \ell \Rightarrow$ there exists exactly one line m such that $P \in m$ and ℓ // m.

The reader will recognize Ax. 2.5 as Playfair's equivalent of the famous parallel postulate of Euclid.

Theorem 2.1 $\ell \neq m, \ell // m \Rightarrow \ell \cap m = \emptyset$ (the null set)

Proof Suppose $Q \in \ell$, m. By Ax. 2.1, 3, there exist points R and S such that $R \in \ell$, $S \in m$, $R \notin m$, $S \notin \ell$, and $R \neq S$. R and S determine the unique line RS. If $Q \in RS$, then we have Q, $R \in \ell$, RS. As $Q \in m$ and $R \notin m$, we may infer $Q \neq R$, whence Ax. 2.1 implies $\ell = RS$. Interchanging the roles of R and S gives $m = RS$ and so $\ell = m$, a contradiction. Thus $Q \notin RS$.

Consequently, there is a unique line $n \parallel RS$ such that $Q \in n$, by Ax. 2.5. Now, Ax. 2.3 gives a point $T \in n$ such that $T \neq Q$. If $T \in m, \ell$, then since $T \neq Q$, Ax. 2.1 implies $\ell = m$, a contradiction. Assuming for definiteness that $T \notin m$, we have a unique line λ through T such that $\lambda \parallel m$.

Suppose $Q \in \lambda$. Since $Q \neq T$ and $Q, T \in \lambda$, we may conclude that $m \parallel \lambda = QT = n \parallel RS$ by Ax. 2.1, 2. We now have $\ell \parallel m \parallel n \parallel RS$ by Ax. 2.4. Further $\ell \neq m$, $RS \neq \ell$, and $RS \neq m$ as $Q \notin RS$. Through Q, m and ℓ are distinct lines parallel to RS, a contradiction to Ax. 2.5. Thus $Q \notin \lambda$. However, $Q \notin \lambda$, $Q \in \ell \cap m$, and $\ell \parallel m \parallel \lambda \Rightarrow \ell = m$ by Ax. 2.5. As this contradicts the hypotheses, no Q satisfies $Q \in \ell, m$. Thus, $\ell \cap m = \emptyset$, as required.

<u>Alternate proof of Th. 2.1:</u>[1]

Let ℓ, m, Q be as in the preceding proof. By Ax. 2.4, 5, since $\ell \neq m$, all lines parallel to ℓ pass through Q.

Now let n be any line through Q other than ℓ or m and let T be any point of n other than Q. By Ax. 2.1, 2, $T \notin \ell$ since $\ell \neq n$. It follows from Ax. 2.5 that there exists a unique line p through T and parallel to ℓ. By our earlier comments $Q \in p$. As $T, Q \in n, p$ and $T \neq Q$, Ax. 2.1 $\Rightarrow$ $n = p$. Hence all lines through Q are parallel to ℓ.

As $Q \notin RS$, there is a **unique** line q through Q and parallel to RS. Since $Q \in q$, it follows that $q \parallel \ell$. Ax. 2.4 implies that $\ell \parallel RS$, whence $Q \in RS$, a contradiction. Thus $\ell \cap m = \emptyset$.

<u>Corollary 2.1.1</u> For any point P and line ℓ, there exists exactly one line m such that $P \in m$ and $\ell \parallel m$.

The proof is immediate from Ax. 2.5 and Th. 2.1.

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[1] Proof due to Dr. W. J. Jonsson.

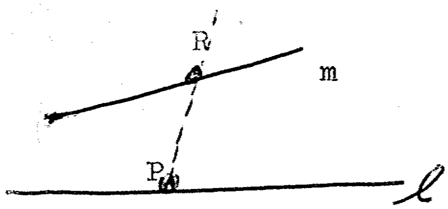
Corollary 2.1.2 All points are contained in the same number of lines. Moreover, this number is the number of parallel classes in $\mathcal{L}$.

The proof follows from the preceding corollary.

Corollary 2.1.3 If two distinct lines exist, then not all lines are parallel.

Proof: Suppose that ℓ and m are two distinct lines. By Ax. 2.1, 3 there exist distinct points R and P such that $P \in \ell$, $P \notin m$, $R \in m$, and $R \notin \ell$. If $RP \parallel \ell$, then Cor. 2.1.1 $\Rightarrow R \in \ell$, a contradiction. Thus, $RP \nparallel \ell$.

Axiom 2.6 If P, Q, and R are distinct non-collinear points such that $P \in \ell \parallel QR$ and $R \in m \parallel PQ$, then there exists a point $S \in \ell \cap m$.



It is easy to show that this familiar process of "completing the parallelogram" leads to a unique S. By Ax. 2.1, it suffices to show $\ell \neq m$. If $\ell = m$, then Ax. 2.1, 2 $\Rightarrow \ell = m = PR \parallel PQ$. However, Th. 2.1 $\Rightarrow PR = PQ$, a contradiction to the non-collinearity of P, Q, and R. Consequently, S is unique.

It is also clear that Ax. 2.3, 4, 6 and Th. 2.1 $\Rightarrow$ parallel lines have the same number of points.

We are now in a position to state the following definitions.

Definition 2.1 If ℓ_1 and ℓ_2 are distinct lines intersecting in a point P, then the plane Π_{ℓ_1, ℓ_2} is defined as $\{x \mid x \in (m_1 \cap m_2),\ m_i \parallel \ell_i \ (i - 1, 2),\ m_i \cap \ell_j \neq \emptyset \text{ for } i \neq j\}$.

Definition 2.2 ℓ is a line of the plane Π_{ℓ_1, ℓ_2} if and only if there exist distinct points P and Q such that $P, Q \in (\ell \cap \Pi_{\ell_1, \ell_2})$.

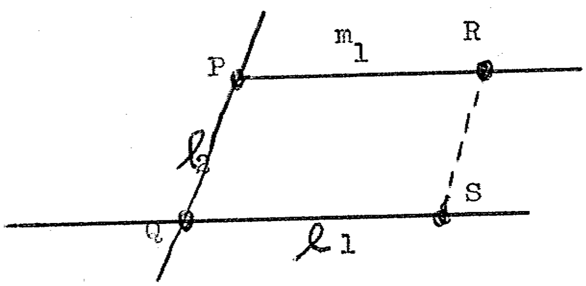
We remark that $P \in \Pi_{\ell_1, \ell_2}$ as one need only let $m_i = \ell_i$ $(i = 1, 2)$. Also any other point of $\ell_1 \cup \ell_2$ is in Π_{ℓ_1, ℓ_2} by Cor. 2.1.1. Ax. 2.3 then guarantees that ℓ_1 and ℓ_2 are each lines of Π_{ℓ_1, ℓ_2}. The next theorem extends these comments.

<u>Theorem 2.2</u> (i) With the notation of Def. 2.1, $m_1 \cap m_2$ contains exactly one point. (ii) $\ell_2 \cap m_1 \neq \emptyset$, $\ell_1 \| m_1 \Rightarrow m_1$ is a line of $\pi\ \ell_1, \ell_2$.

<u>Proof</u> (i) Unless $m_1 = \ell_1$ or $m_2 = \ell_2$, Ax. 2.6 and the comments immediately following it $\Rightarrow m_1 \cap m_2 = \{S\}$ where S is a unique point.

If $m_i = \ell_i$ ($i = 1$ or 2), then $m_1 \cap m_2$ is either $m_1 \cap \ell_2$, $\ell_1 \cap m_2$, or $\ell_1 \cap \ell_2$. Using Ax. 2.1, 4 and $\ell_1 \nparallel \ell_2$, we may readily show that these sets each contain exactly one point.

(ii) Suppose $\ell_1 \cap \ell_2 = \{Q\}$, $m_1 \| \ell_1$ and $m_1 \cap \ell_2 = \{P\}$. By Ax. 2.3, there exists a point $R \neq P$ such that $R \in m_1$. Now by our comments after Def. 2.2, we may assume $\ell_1 \neq m_1$. Hence, by Th. 2.1, $m_1 \cap \ell_1 = \emptyset$ and So P, Q, and R are non-collinear. Then Ax. 2.6 allows us to infer that the line through R parallel to ℓ_2 intersects ℓ_1 in a unique point S. By Def. 2.1, we may conclude that $R \in \pi\ \ell_1, \ell_2$. Since $P \in \pi\ \ell_1, \ell_2$ by our comments after Def. 2.2, Def. 2.2 $\Rightarrow m_1$ is a line $\pi\ \ell_1, \ell_2$.



<u>Axiom 2.7</u> If ℓ and m are any lines of any plane such that $\ell \nparallel m$, then there exists a point P such that $P \in (\ell \cap m)$.

As $\ell \nparallel m \Rightarrow \ell \neq m$, Ax. 2.1 $\Rightarrow$ P is unique.

<u>Theorem 2.3</u> If ℓ is a line of the plane $\pi\ \ell_1, \ell_2$, then for any point P, $P \in \ell \Rightarrow P \in \pi\ \ell_1, \ell_2$.

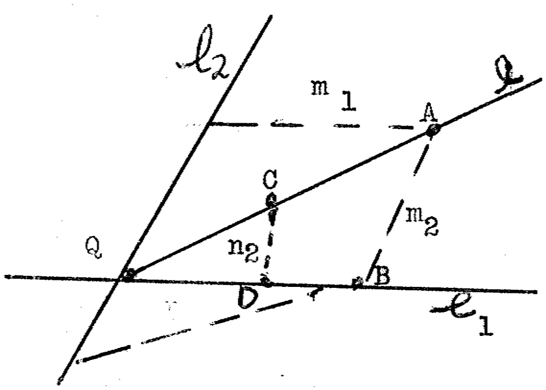
<u>Proof</u>: If $\ell \| \ell_1$, then Ax. 2.4 and Th. 2.1 imply $\ell \nparallel \ell_2$. As ℓ_2 is a line of $\pi\ \ell_1, \ell_2$, we may infer that $\ell \cap \ell_2 \neq \emptyset$. If $\ell \cap \ell_2 = \{T\}$, then $T \in \pi$, as $T \in \ell_2$. For any point R of ℓ such that $R \neq T$, we may apply the

proof of Th. 2.2 (ii) to show that $R \in \pi_{\ell_1, \ell_2}$.

Since $\pi_{\ell_1, \ell_2} = \pi_{\ell_2, \ell_1}$, the above argument takes care of the case $\ell \parallel \ell_2$ as well.

We may now assume $\ell \nparallel \ell_i$ (i = 1, 2). By Ax. 2.7, there exist points Q and R such that $\ell \cap \ell_1 = \{Q\}$, and $\ell \cap \ell_2 = \{R\}$. Depending on whether $Q = R$, we have two cases to consider.

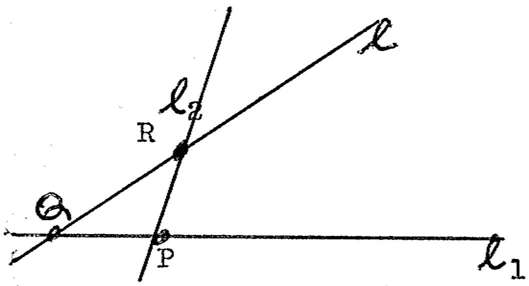
If Q = R, then $\ell \cap \ell_2 = \{Q\}$, by Ax. 2.1. As ℓ is a line of π_{ℓ_1, ℓ_2}, there exists a point $A \in (\ell \cap \pi)$ such that $A \neq Q$. Let m_1, m_2 be as in Def. 2.1 as applied to A.



If $B \in (\ell_1 \cap m_2)$, then Ax. 2.6 as applied to Q, A, and B allows us to conclude that the line through B parallel to AQ intersects ℓ_2. Hence, $B \in \pi_{\ell_2, \ell}$. Since $Q \in \pi_{\ell_2, \ell}$ and $A \neq Q \Rightarrow A \notin \ell_2 \Rightarrow \ell_2 \cap m_2 = \emptyset \Rightarrow B \notin \ell_2 \Rightarrow Q \neq B$, it follows that ℓ_1 is a line of $\pi_{\ell_2, \ell}$.

Now $Q \in \pi_{\ell_1, \ell_2}$. For any point $C \neq Q$ of ℓ, the line through C parallel to ℓ_2 is a line of $\pi_{\ell_2, \ell}$ by Th. 2.2 (ii). This line (n_2 in the figure) is not parallel to ℓ_1, and so Ax. 2.7 $\Rightarrow (n_2 \cap \ell_1)$ contains a single point D. As $Q \notin n_2$, the points Q, C, and D are non-collinear, whence Ax. 2.6 $\Rightarrow$ the line through C and parallel to ℓ_1 intersects ℓ_2. By Def. 2.1, it follows that $C \in \pi_{\ell_1, \ell_2}$ and so the theorem is proved if Q = R.

However, if $Q \neq R$, then suppose $\ell_1 \cap \ell_2 = \{P\}$. It is clear that $Q, P \in \pi_{\ell_2, \ell}$ and $Q \neq P$. Thus, ℓ_1 is a line of $\pi_{\ell_2, \ell}$ and the proof may now be completed as in the previous case with the use of Ax. 2.6.



We note that an immediate consequence of Th. 2.3 is that the point P of Ax. 2.7 is in any plane of which ℓ and m are each lines.

<u>Axiom 2.8</u> If π_1 and π_2 are any planes and P any point such that $P \in (\pi_1 \cap \pi_2)$, then there is at least one line ℓ such that ℓ is a line of both π_1 and π_2.

We shall see that Ax. 2.8 is the principal reason that $\mathbb{Q}$ is three-dimensional.

<u>Axiom 2.9</u> There exist (at least) four distinct non-collinear, non-coplanar points.

From this particular existence axiom, many interesting results follow. In particular, Cor. 2.1.3 $\Rightarrow$ not all lines are parallel. Th. 2.3 implies not all lines lie in the same plane. Ax. 2.9 clearly implies that every point lies on at least three distinct lines. Consequently, non-collinear points must be distinct. Furthermore, non-coplanar lines must be distinct, as any line PQ lies in a plane determined by it and PQ and PR, where R is a point $\notin$ PQ, the existance of R being guaranteed by Ax. 2.9.

<u>Lemma 2.4.1</u> If ℓ_3 is a line of π_{ℓ_1, ℓ_2} such that $\ell_2 \parallel \ell_3$, then $\pi_{\ell_1, \ell_3} = \pi_{\ell_1, \ell_2}$.

<u>Proof</u> Since $\ell_1 \nparallel \ell_2$, Ax. 2.4 $\Rightarrow \ell_1 \nparallel \ell_3$ and Ax. 2.7 $\Rightarrow \ell_1$ intersects ℓ_3 in a unique point, P. Thus π_{ℓ_1, ℓ_3} is defined.

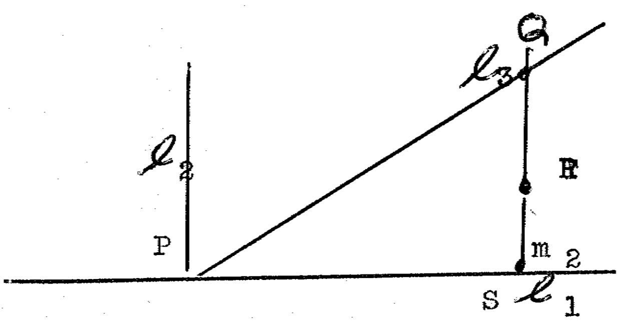
Without loss of generality, $\ell_2 \nparallel \ell_3$. Let R be any point of $\overline{\pi}\,\ell_1, \ell_2$ such that $R \notin (\ell_1 \cup \ell_3)$. Suppose that the line m_2 through R parallel to ℓ_2 intersects ℓ_1 in the point T. (T exists since $R \in \overline{\pi}\,\ell_1, \ell_2$). As $P \notin m_2$ by Th. 2.1, it follows from Ax. 2.6 as applied to R, T, and P that the line through R parallel to ℓ_1 intersects ℓ_3. Thus $R \in \overline{\pi}\,\ell_1, \ell_3$. It is clear that all points of $\overline{\pi}\,\ell_1, \ell_2$ that are in $(\ell_1 \cup \ell_3)$ are also in $\overline{\pi}\,\ell_1, \ell_3$. Consequently, $\overline{\pi}\,\ell_1, \ell_2 \subseteq \overline{\pi}\,\ell_1, \ell_3$.

Now, Th. 2.2 (ii) $\Rightarrow \ell_2$ is a line of $\overline{\pi}\,\ell_1, \ell_3$. Repeating the above argument, we have $\overline{\pi}\,\ell_1, \ell_3 \subseteq \overline{\pi}\,\ell_1, \ell_2$. Hence, $\overline{\pi}\,\ell_1, \ell_2 = \overline{\pi}\,\ell_1, \ell_3$.

<u>Lemma 2.4.2</u> If ℓ_3 is a line of $\overline{\pi}\,\ell_1, \ell_2$ such that $\ell_3 \cap \ell_1 = \ell_3 \cap \ell_2 = \ell_1 \cap \ell_2$, then $\overline{\pi}\,\ell_2, \ell_3 = \overline{\pi}\,\ell_1, \ell_2$.

<u>Proof</u> Since $\ell_1 \cap \ell_3$ contains a single point, $\pi\,\ell_1, \ell_3$ is defined.

Let R be any point of $\overline{\pi}\,\ell_1, \ell_2$ such that $R \notin (\ell_2 \cup \ell_3)$. The line through R parallel to ℓ_2 (denoted m_2 in the figure) intersects ℓ_1 since $R \in \overline{\pi}\,\ell_1, \ell_2$. Suppose $m_2 \cap \ell_1 = \{S\}$. If R = S, then m_2 is a line of $\overline{\pi}\,\ell_1, \ell_2$ by Th. 2.2(ii). If $R \neq S$, then m_2 is a line of $\overline{\pi}\,\ell_1, \ell_2$ by Def. 2.2. Since $\ell_2 \cap \ell_3$ contains a single point, Ax. 2.3, 4 and Th. 2.1 $\Rightarrow \ell_2 \nparallel \ell_3 \Rightarrow \ell_1 \nparallel \ell_3$. Consequently, Ax. 2.7 $\Rightarrow \ell_1 \cap \ell_3 \neq \emptyset$. If $\ell_2 \cap \ell_3 = \{P\}$ and $m_2 \cap \ell_3 = \{Q\}$, then Ax. 2.1 $\Rightarrow Q \notin \ell_2$ since $\ell_2 \nparallel \ell_3$. **Hence**, Ax. 2.6 applied to P, Q, and R im-

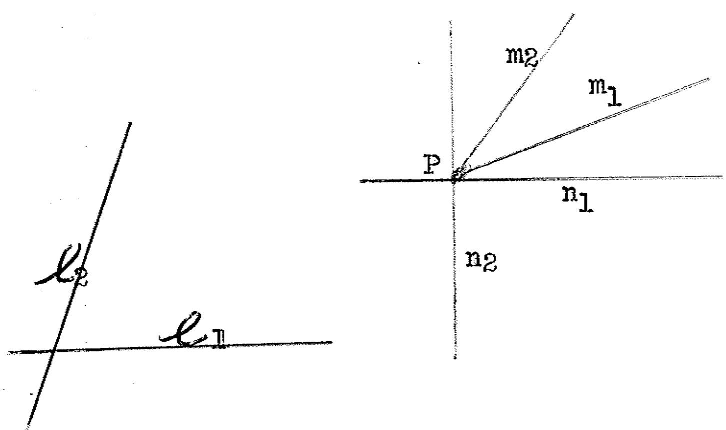


plies that the line through R parallel to ℓ_3 intersects ℓ_2. According to Def. 2.1, we have shown $R \in \overline{\Pi}\, \ell_2, \ell_3$. As in the previous lemma, $\overline{\Pi}\, \ell_1, \ell_2 \subseteq \overline{\Pi}\, \ell_2, \ell_3$.

By the reasoning of the previous lemma, the proof will be complete if we show that ℓ_1 is a line of $\overline{\Pi}\, \ell_2, \ell_3$. With the above notation, it clearly suffices to show that $S \in \overline{\Pi}\, \ell_2, \ell_3$. We already have $S \in m_2 \parallel \ell_2$ such that $m_2 \cap \ell_3 = \{Q\} \neq \emptyset$. It only remains to prove that the line through S parallel to ℓ_3 intersects ℓ_2. As above, this follows by Ax. 2.6.

<u>Theorem 2.4</u> If m_1 and m_2 are any two distinct intersecting lines of $\overline{\Pi}\, \ell_1, \ell_2$, then $\overline{\Pi}\, \ell_1, \ell_2 = \overline{\Pi}\, m_1, m_2$.

<u>Proof</u> Suppose that $m_1 \cap m_2 = \{P\}$ and $P \in n_i \parallel \ell_i$ $(i = 1, 2)$. By Th. 2.3, $P \in \overline{\Pi}\, \ell_1, \ell_2$ and so $n_2 \cap \ell_1 \neq \emptyset$, $n_1 \cap \ell_2 \neq \emptyset$. Then either by Th. 2.2 (ii) or Def. 2.2, n_1 and n_2 are lines of $\overline{\Pi}\, \ell_1, \ell_2$. Lemma 2.4.1 implies $\overline{\Pi}\, \ell_1, \ell_2 = \overline{\Pi}\, \ell_1, n_2 = \overline{\Pi}\, n_2, \ell_1 = \overline{\Pi}\, n_2, n_1$. From the hypotheses, it follows that m_1 and m_2 are each lines of $\overline{\Pi}\, n_1, n_2$. Without loss of generality, $n_i \not\parallel m_j$ $(i, j = 1, 2)$. Then Lemma 2.4.2 implies $\overline{\Pi}\, n_2, n_1 = \overline{\Pi}\, m_1, n_2 = \overline{\Pi}\, m_2, m_1$. Thus $\overline{\Pi}\, \ell_1, \ell_2 = \overline{\Pi}\, m_1, m_2$ and the proof is complete.



<u>Corollary 2.4.1</u> If P_1, P_2 and P_3 are non-collinear points, then there exists a unique plane $\overline{\Pi}$ such that $P_i \in \overline{\Pi}$ $(i = 1, 2, 3)$.

Proof (i) There exists at least one such Π, namely Π P_1P_2, P_2P_3. To verify this one need only guarantee $P_i \neq P_j$ whenever $i \neq j$. This follows from our comments preceding Lemma 2.4.1.

(ii) We must now show that there is at most one such Π. This follows from Def. 2.2 and Th. 2.4 as any such Π equals $\Pi P_1P_2, P_2P_3$.

Corollary 2.4.2 For any point P_1 and line ℓ such that $P_1 \notin \ell$, there exists exactly one plane Π such that $P_1 \in \Pi$ and ℓ is a line of Π.

Proof (i) By Ax. 2.3, there exist distinct points $P_i \in \ell$ $(i = 2, 3)$. Since $P_1 \notin \ell$, we have P_1P_2 and P_3 non-collinear. The preceding corollary implies that there exists a plane Π^* such that $P_i \in \Pi^*$ $(i = 1, 2, 3)$. Since $P_2 \neq P_3$, Def. 2.1 $\Rightarrow \ell$ is a line of Π^*. Hence Π^* is a satisfactory Π.

(ii) We now show that there is at most one such Π, for any satisfactory Π must contain P_i $(i = 1, 2, 3)$ by Th. 2.3 and so by Cor. 2.4.1, Π is unique.

Corollary 2.4.3 Any two distinct planes intersect in either the null set or a line.

Proof Suppose two distinct planes have a non-empty intersection. By Ax. 2.8, that intersection contains a line ℓ. However, no point not on ℓ can be in both planes, for such a point, together with ℓ, determines a unique plane, by Cor. 2.4.2. Of course, the intersection of two distinct planes may be empty.

Corollary 2.4.4 For any two planes Π_1 and Π_2, $\Pi_1 = \Pi_2 \Leftrightarrow$ (for any line ℓ, ℓ is a line of $\Pi_1 \Leftrightarrow \ell$ is a line of Π_2).

Proof (i) If $\Pi_1 = \Pi_2$, then Π_1 and Π_2 contain the same points which by Ax. 2.1, 2 determine the same lines satisfying Def. 2.2.

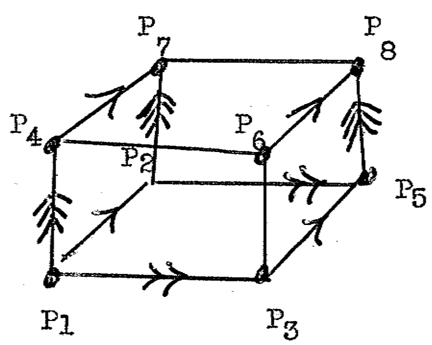
(ii) If $\Pi_1 \neq \Pi_2$, then there are two cases to consider.

If $\Pi_1 \cap \Pi_2 = \emptyset$, then if $\Pi_1 \equiv \Pi \ell_1, \ell_2$, it is clear that ℓ_1 is a line of Π_1, but not of Π_2.

Otherwise, $\Pi_1 \cap \Pi_2 \neq \emptyset \Rightarrow \Pi_1 \cap \Pi_2 = m$ for some line m, by the preceding corollary. As any plane contains three non-collinear points, we may suppose that P_1, P_2, and P_3 are points such that $P_i \in m$ (i.= 1, 2), $P_3 \notin m$, and $P_i \in \Pi_1$ (i = 1, 2, 3). Now, Cor 2.4.1 implies that P_i (i = 1, 2, 3) uniquely determine a plane, which must be Π_1. Since $\Pi_1 \neq \Pi_2$ by assumption and $P_i \in \Pi_2$ (i = 1, 2), it follows that $P_3 \notin \Pi_2$, whence P_1P_3 (or indeed P_2P_3) is a line of Π_1, but not of Π_2.

<u>Corollary 2.4.5</u> There exist at least eight distinct points.[1]

<u>Proof</u> By Ax. 2.9 and Cor. 2.4.2, there exist four distinct non-coplanar points P_i (i = 1, 2, 3, 4), no three of which are collinear. Ax. 2.6 allows us to complete the parallelogram of P_1, P_2, and P_3 with the point P_5. By the reasoning of Th. 2.2 (ii), P_5 lies in the plane determined by P_i(i = 1, 2, 3). Furthermore, P_5 is distinct from P_1, P_2, P_3, and P_4, since $P_3P_5 \cap P_1P_2 = P_2P_5 \cap P_1P_3 = \emptyset$ and P_i (i = 1, 2, 3, 4) are non-coplanar.



By precisely the same argument, there exists a point P_6 in the plane of P_i (i = 1, 3, 4) such that P_1, P_2, P_3, P_4, P_5, and P_6 are all distinct. Similarly, P_i (i = 1, 2, 4) lead to P_7 and P_i (i = 3, 5, 6) lead to P_8 as indicated in the figure.

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[1] Essentially the same treatment of Cor. 2.4.5 and Th. 2.5 is to be found in [8], which was published after the final draft of this thesis.