A BASIS FOR A FREE MALCEV ALGEBRA

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Abstract

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When studying algebraic systems one is often faced with the problem of obtaining a basis for the system. This thesis considers an approach to solving this problem for free non-associative algebras, and examines the process in detail for free Malcev algebras. A brief description is given of a computer program which may be used to assist in finding a basis for such algebras.

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1. Introduction

In (3) a method is given for obtaining a basis of a free Lie ring. It is our purpose to examine the procedure and the possibility of extending it to any free non-associative algebra, with particular reference to Malcev algebras.

2. Preliminaries

2.1 - A non-associative algebra A over a commutative ring R with unit element is a left R-module such that for each pair (x,y) $x,y \in A$ there is defined a product $xy \in A$ for which

2.2 $$(x_1 + x_2)y = x_1y + x_2y$$
$$x(y_1 + y_2) = xy_1 + xy_2$$

2.3 $$\lambda(xy) = (\lambda x)y = x(\lambda y) \qquad \lambda \in R$$

2.4 - A non-associative algebra is a Lie algebra if the multiplication satisfies

2.5 $$xx = 0$$

2.6 $$x(yz) + y(zx) + z(xy) = 0$$

Notice that from (2.5) we obtain

$$(x + y)^2 = x^2 + xy + yx + y^2 = xy + yx = 0 \quad \text{or}$$

2.7 $$xy = -yx.$$

Conversely, setting $x = y$ in (2.7) yields $2x^2 = 0$ so that if the characteristic is not 2, then (2.5) and (2.7) are equivalent.

2.8 - A non-associative algebra is Malcev (4) if

2.9 $\qquad xx = 0$

2.10 $\qquad (xz)(yw) + x((yz)w) + y((zw)x)$

$$+ z((wx)y) + w((xy)z) = 0$$

2.11 - Let X be any unstructured set. A free algebra generated by X is a pair (A,i) where A is an algebra and $i:X \to A$ is a mapping such that, given any algebra U and any mapping $k:X \to U$, there exists a unique homomorphism $k':A \to U$ for which $k = ik'$.

i.e.:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "k"'] & A \arrow[d, "k'"] \\ & U \end{tikzcd}$$

commutes.

2.12 - A magma (5) is a set M together with a map $M \times M \longrightarrow M$ denoted $(x,y) \longrightarrow xy$.

2.13 - Let X be any set, and define inductively a family of sets X_n $(n \geq 1)$ as follows

a) $X_1 = X$

b) $X_n = (X_p \times X_q) \qquad p + q = n \geq 2$

Set $M_x = \overset{\infty}{\cup} X_n$, and define $M_x \times M_x \longrightarrow M_x$ by:

$(x_p,x_q) \to (x_p,x_q)$ for $x_p \in X_p$ and $x_q \in X_q$.

Note $(x_p,x_q) \in X_{p+q}$

Then M_x is a magma called the free magma on X.

2.14 - If N is any magma and $f:X \to N$ is any map, then there exists a unique magma homomorphism $F:M_x \longrightarrow N$ which extends f.

Proof - This follows immediately by defining F inductively as

$$F(u,v) = F(u)F(v) \qquad u,v \in X_p \times X_q$$

2.15 - An element w of a free magma M_x is called a non-associative word on X. The length, $l(w)$, of w is the unique n such that $w \in X_n$.

2.16 - Let R be a commutative ring with unit, and A_x the algebra over R of a free magma M_x. A_x is called a free algebra on X.

An element $a \in A_x$ is a finite sum $\sum \lambda_m m$ $m \in M_x$, $\lambda_m \in R$. The multiplication in A_x extends the multiplication in M_x.

2.17 - Let B be an algebra and $f: X \longrightarrow B$ a map. Then there exists a unique algebra homomorphism $F: A_x \longrightarrow B$ which prolongs f.

Proof - By (2.14), f can be extended to a unique magma homomorphism $f': M_x \longrightarrow B$ (where B is viewed as a magma under multiplication). By linearity this map extends to a linear map $F: A_x \longrightarrow B$. Now F is an algebra homomorphism, and since X generates A_x, it is unique.

2.18 - Let I be the two sided ideal of A_x generated by all elements of the form aa and (a,b,c) $a,b,c \in A_x$,

where $(a,b,c) = a(bc) + b(ca) + c(ab)$

The quotient algebra A_x/I is called the free Lie algebra on S.

2.19 - Let J be the two sided ideal of A_x generated by all elements of the form aa and $(ac)(bd) + (a,b,c,d)$

where $(a,b,c,d) = a((bc)d) + b((cd)a) + c((da)b) + d((ab)c)$

The quotient algebra A_x/J is the free Malcev algebra on X.

The remainder of this section is mainly a resume of the treatment of Lie algebras given in (3).

Consider a free Lie algebra L generated by a well ordered set $\sigma = x_1, x_2, \ldots$ of generators x_i over a commutative ring R with unit element.

2.20 - Using the concept of the length of a word (2.15) together with the ordering on the set σ, define an order $>$ between words $w_1, w_2 \in L$ inductively as follows:

a) if $l(w_2) > l(w_1)$, then $w_2 > w_1$

b) if $l(w_2) = l(w_1) = 1$, then $w_1, w_2 \in B$ and are ordered according to the order on σ.

c) if $l(w_2) = l(w_1) > 1$, then $w_2 = w_{21}w_{22}$ and $w_1 = w_{11}w_{12}$ with $l(w_{ij}) > l(w_k)$ for all $i,j,k = 1,2$.

Now if i) $w_{21} > w_{11}$, then $w_2 > w_1$,

or if ii) $w_{21} = w_{11}$ and $w_{22} > w_{12}$, then $w_2 > w_1$.

Let $S(\sigma)$ be the set of all words in L for which either

a) $l(w) = 1$

or b) if $l(w) > 1$, then $w = w_1w_2 \in S(\sigma)$ if $w_1, w_2 \in S(\sigma)$ and $w_2 > w_1$.

The set $S(\sigma)$ is then ordered under the relation $>$.

2.21 - An "a" reduction on L is a procedure

$$S = \sum \lambda_i w_i \longrightarrow S' = \sum \lambda_i' w_i' \qquad \lambda_i, \lambda_i' \in R$$

in which one of the following operations is performed.

a) $\lambda_i \ldots (w_1w_2) \ldots$ is replaced in S by $- \lambda_i \ldots (w_2w_1) \ldots$ to give S' if $w_1, w_2 \in S(\sigma)$ and $w_1 > w_2$.

b) $\lambda_i \ldots (ww) \ldots$ is deleted from S to give S'.

If S' is empty, write $S' = \emptyset$.

c) $\lambda_i w + \lambda_j w$ is replaced in S by $(\lambda_i + \lambda_j)w$ to give S'.

d) Ow is deleted from S to give S'

(where O is the zero element in σ).

2.22 - A "j" reduction is a procedure $S \longrightarrow S'$ in which $\lambda_i \,..\, w_3(w_1w_2) \,..$ is replaced in S by $\lambda_i \,..\, w_2(w_1w_3) + w_1(w_2w_3) \,..$ to give S' if $w_3(w_1w_2) > w_2(w_1w_3)$, $w_1(w_2w_3)$.

The paper shows that a sequence of these reduction procedures, when applied to a linear combination of words in L will, in a finite number of steps, yield a linear combination of words which are in $S(\sigma)$. Further it is shown by an induction based on word length that the resulting linear combination of words is unique up to the order of its summands.

Let K be the ideal of L generated by all elements which reduce to the empty word, and for S, T $\in$ L define $S \overset{\mathcal{L}}{=} T$ if $S \equiv T$ mod K. Then L is a free Lie algebra over R under $\overset{\mathcal{L}}{=}$ with the set of irreducible elements as a basis.

3. The General Case

Let us now examine the possibility of extending the treatment of Lie algebras in (3) to the more general case.

Consider a free non-associative algebra A generated by a well ordered set σ of generators x_i over a commutative ring R with unit element. Let the monomials of A satisfy a (finite) number of relations of the form

$$f(w_1, w_2, \,..) = f(w_i) = \sum \lambda_j w_j = 0 \qquad w_i, v_j \in A, \quad \lambda_j \in R$$

e.g.: In the case of a Lie ring we have (2.5 & 2.6)

$$f_1(w) = w^2 = 0$$

$$f_2(w_1, w_2, w_3) = w_1(w_2w_3) + w_2(w_3w_1) + w_3(w_1w_2) = 0$$

Given two words $w_1, w_2 \in A$ define $w_1 < w_2$ inductively as before (2.20) by

a) $l(w_1) < l(w_2) \Rightarrow w_1 < w_2$

b) $l(w_1) = l(w_2) = 1 \Rightarrow w_1, w_2 \in \sigma$ and are ordered according to the order assigned to the set σ of generators.

c) $l(w_1) = l(w_2) > 1 \Rightarrow w_1 = w_{11}w_{12}, w_2 = w_{21}w_{22}$ with $l(w_{ij}) > l(w_k)$ for all i, j, k = 1, 2

i) $w_{11} < w_{21} \Rightarrow w_1 < w_2$

ii) $w_{11} = w_{21}$ and $w_{12} < w_{22} \Rightarrow w_1 < w_2$

For economy of notation the following convention will be adopted with respect to bracketing:

$$x_1(x_2x_3 \ldots x_n) = x_1x_2x_3 \ldots x_n \qquad n = 3,4,5, \ldots$$

Consider a relation f_α on A and a set of words $w_i \in A$ for which $f_\alpha(w_i) = \sum \lambda_j v_j = 0$. Let v_α be the element of this set maximal with respect to the ordering $<$, and define a relation $\rho_\alpha : A \longrightarrow A$ by

$\rho_\alpha : \lambda_\alpha v_\alpha \rightarrow \lambda_\alpha v_\alpha - f(w_i)$

(Note that $v_j < v_\alpha$ for all v_j in the expression $\lambda_\alpha v_\alpha - f_\alpha(v_i)$

e.g.: for the case of a Lie algebra with words w_1, w_2, w_3 satisfying $w_1 < w_2 < w_3$ we have

$$f_2(w_1, w_2, w_3) = w_1w_2w_3 + w_2w_3w_1 + w_3w_1w_2 = 0$$

Now $v_2 = w_3w_1w_2 > w_1w_2w_3, w_2w_3w_1$

so that $\rho_2 : w_3w_1w_2 \longrightarrow -(w_1w_2w_3 + w_2w_3w_1)$

Let Σ be the set of all words which are irreducible under any of the possible reductions ρ_α. An element is said to be in standard form, or to be a standard element if it is either void or a linear combination of members of Σ. This set of standard words then serves as a set of basis elements.

We now show that any word can be completely reduced in a finite number of steps:

For consider a word w which reduces to $\Sigma \lambda_i v_i$. Each v_i is composed of the same generators as w, reordered and/or rebracketed in some way. This is also true if any reduction is applied to any of the v_i. Thus at any stage of a reduction process beginning with a word w, we have a linear combination of words v_i each of which is composed of the same generators as w. Consequently, since $l(w) < \infty$ there can be only a finite number of permutations and rebraketings of the generators in w. Consequently, any reduction process necessarily terminates in a finite number of steps.

To show that a word w is uniquely reducible is proven by induction over the length of w, in conjunction with the Birkhoff condition (6) i.e.:

Consider a set S on which there is defined a reduction, that is, a binary relation $a \longrightarrow b$ such that

1) no infinite sequence exists of the form

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \cdots$$

2) if there exist two reductions $a \begin{smallmatrix} \nearrow b \\ \searrow c \end{smallmatrix}$ with the same initial element, then there exists an element d and reduction chains of the form

$$\begin{matrix} b \longrightarrow b_2 \longrightarrow \cdots \longrightarrow b_{p-1} \\ c \longrightarrow c_2 \longrightarrow \cdots \longrightarrow c_{q-1} \end{matrix} \begin{matrix} \searrow \\ \nearrow \end{matrix} d$$

Then the Birkhoff condition states that for any two complete reductions

$$\begin{array}{l} a \to a_1 \to \cdots \to a_{r-1} \to z_1 \\ a \to b_1 \to \cdots \to b_{s-1} \to z_2 \end{array} \quad z_1, z_2 \text{ irreducible}$$

we have $z_1 = z_2$.

Proof - Trivially this condition is necessary for unique reduction.

To see that it is sufficient, consider the set S^* of all elements of S which terminate in a unique irreducible element. S^* is non-empty since it contains all irreducible elements. Suppose $a \in S - S^*$. Then there exist sequences

$$a \begin{array}{l} \nearrow a_1 \to \cdots \to c \\ \searrow b_1 \to \cdots \to c^* \neq c \end{array} \quad \text{with } c \text{ and } c^* \text{ irreducible.}$$

By hypothesis there exists a $d \in S$ such that

$$a \begin{array}{l} \nearrow a_1 \to \cdots \to a_{p-1} \searrow \\ \searrow b_1 \to \cdots \to b_{q-1} \nearrow \end{array} d \to d_1 \to \cdots \to d^*$$

d^* irreducible

If $a \in S^*$, then all sequences beginning with a_1 must end in d^*, and similarly for b_1. We then have $c = d^* = c^*$. Consequently, since $c \neq c^*$, at least one of $a_1, b_1 \notin S^*$. Without loss of generality assume $a_1 \notin S^*$. Repeating the preceding argument yields an element $a_2 \notin S^*$ for which $a \to a_1 \to a_2$. Continuing in this way yields an infinite sequence of the form $a \to a_1 \to a_2 \to \ldots$ which contradicts the initial hypothesis.

Hence $S = S^*$ and each element reduces to a unique irreducible element.

Returning to the problem of proving unique reduction, consider a word w. A simple induction on the length $l(w)$ shows that w is uniquely reducible: If $l(w) = 1$ then w is irreducible, and so is certainly uniquely reducible. If $l(w) = n > 1$, then by the induction hypotheses all subwords of w (which have lengths less than n) are uniquely reducible. Consequently if the Birkhoff condition is satisfied, the induction is complete and w is uniquely reducible. Notice that this induction includes the possibility of w being reduced in two (or more) ways by one reduction procedure, and/or the possibility of being reduced by two distinct reduction processes. Having verified the Birkhoff condition we will then have a method of obtaining a basis of a free non-associative algebra generated by a well ordered set of generators over a commutative ring with unit element. To this end let us look at the particular case of a Malcev algebra.

4. Malcev algebras

We first consider an example. Let w_1, w_2, w_3, w_4 be words in a Malcev algebra satisfying

$$w_1 < w_2 < w_3 < w_4 < w_1w_2 < w_1w_4 < w_2w_3.$$

We have by (2.10) that

$$f_2(w_1,w_2,w_3,w_4) = (w_1w_3)w_2w_4 + w_1(w_2w_3)w_4 + w_2(w_3w_4)w_1 + w_3(w_4w_1)w_2 + w_4(w_1w_2)w_3 = 0$$

Since $v_2 = (w_1w_3)w_2w_4 > w_1(w_2w_3)w_4,\ w_2(w_3w_4)w_1,$ $w_3(w_4w_1)w_2,\ w_4(w_1w_2)w_3,$

then $$\rho_2\colon (w_1w_3)w_2w_4 \to -w_1(w_2w_3)w_4 - w_2(w_3w_4)w_1 - w_3(w_4w_1)w_2 - w_4(w_1w_2)w_3$$

From (2.7) we have that ρ_1: $-w_1(w_2w_3)w_4 \to w_1w_4w_2w_3$, $-w_2(w_3w_4)w_1 \to w_2w_1w_3w_4$, $-w_3(w_4w_1)w_2 \to -w_3w_2w_1w_4$, and $-w_4(w_1w_2)w_3 \to w_4w_3w_1w_2$.

Thus $(w_1w_3)w_2w_4 \to w_1w_4w_2w_3 + w_2w_1w_3w_4 - w_3w_2w_1w_4 + w_4w_3w_1w_2$ and no further reduction is possible.

This reduction was obtained using $f_2(w_1,w_2,w_3,w_4)$. Alternately consider $f_2(w_3,w_2,w_1,w_4)$ from which we obtain

ρ_2: $(w_3w_1)w_2w_4 \to -w_3(w_2w_1)w_4 - w_2(w_1w_4)w_3 - w_1(w_4w_3)w_2 - w_4(w_3w_2)w_1$.

Application of ρ_1 where possible yields

$(w_1w_3)w_2w_4 \to -w_3w_4w_1w_2 + w_2w_3w_1w_4 - w_1w_2w_3w_4 - w_4w_1w_2w_3$

and no further reduction is possible.

Comparison of these two reductions for $(w_1w_3)w_2w_4$ now reveals a difficulty - the Birkhoff condition is not satisfied, i.e.: we have two possible reductions for the same element.

In an effort to achieve unique reduction, two more reduction formulas were developed (empirically)

$$f_3(w_1,w_2,w_3,w_4) = (w_1,w_2,w_3w_4) + (w_2,w_3,w_4w_1) + (w_3,w_4,w_1w_2) + (w_4,w_1,w_2w_3) = 0$$

where $(a,b,c) = abd + bca + cab$

and $f_4(w_1,w_2,w_3,w_4) = -(w_1,w_2,w_3,w_4) - (w_4,w_3,w_2,w_1) = 0$

where $(a,b,c,d) = a(bc)d + b(cd)a + c(da)b + d(ab)c$.

With the aid of an IBM 1620 computer, it was shown that the Birkhoff condition is now satisfied using the four reductions which derive from f_4, f_3, together with

$$f_2(w_1,w_2,w_3,w_4) = (w_1w_3)w_2w_4 + w_1(w_2w_3)w_4 + w_2(w_3w_4)w_1 + w_3(w_4w_1)w_2 + w_4(w_1w_2)w_3 = 0$$

and $f_1(w_1,w_2) = w_1w_2 + w_2w_1 = 0$

5. Verifying the Birkhoff condition

Verification that the Birkhoff condition is satisfied is reasonably straightforward, tedious, and somewhat prone to error if done by hand, i.e.: it is ideally suited to being done by a computer. For this reason a program is discussed here which was run on an IBM 1620 to assist in establishing that the Birkhoff condition was satisfied for Malcev algebras.

The purpose of the program is to examine alternative reductions of a word, and to verify that each of these reductions yields the same result.

A word is reduced according to a particular reduction procedure. Each word which is obtained as a result of this reduction is tested for further reduction. When no further reduction is possible, the result is stored and another reduction is tested. The results of each reduction are compared to that first obtained. If the two reductions are not the same, the program is terminated. Otherwise the program continues until all possible reductions have been tested. The program itself is examined in greater detail in appendex A.

Given the results of the program it is now possible to complete the verification of the Birkhoff condition. Consider a word $w = .. u .. v ..$ in which u and v are reducible.

Then two possibilities must be considered:

I. u and v are disjoint -

In this case it is evident that regardless of whether u or v is reduced first, w reduces uniquely.

II. u and v are not disjoint --

In this case either $u = v$, u is a subword of v, or u contains v as a subword. Consider for the moment the case in which $u = v$. There are several subcases:

$$1)\quad \left.\begin{array}{l} u = u_2u_1 \\ v = v_3v_2v_1 \end{array}\right\} \Rightarrow u_2 = v_3,\ u_1 = v_2v_1$$

Here u and v are reducible only by the anti-commutative law. Thus we have for $v_1 < v_2 < v_1v_2 = u_1 < u_2 = v_3$

$$u = u_2u_1 \to -u_1u_2 = -(v_2v_1)u_2$$
$$\to (v_1v_2)u_2$$
$$v = v_3v_2v_1 \to -v_3v_1v_2$$
$$\to (v_1v_2)v_3$$

and the reduction is unique.

$$2)\quad \left.\begin{array}{l} u = u_2u_1 \\ v = v_4v_3v_2v_1 \end{array}\right\} \Rightarrow u_2 = v_4,\ u_1 = v_3v_2v_1$$

Assume $v_1 < v_2 < v_3$. In order that u be reducible we must have $u_1 = v_3v_2v_1 < u_2 = v_4$.

Reducing u first by the anti-commutative law gives

$$u = u_2u_1 \to -u_1u_2 = -(v_3v_2v_1)u_2$$

Reducing v_2v_1 by the anti-commutative law we have

$$-(v_3v_2v_1)u_2 \to (v_3v_1v_2)u_2$$

From the results of the program we find that for $v_1 < v_2 < v_3 < v_3v_1v_2 < v_4$ this reduces to

$$(v_3v_1v_2)v_4 \to -v_1(v_2v_3)v_4 + v_2v_1v_3v_4 + v_2v_3v_1v_4$$
$$+ v_3(v_1v_2)v_4 - v_1v_3v_2v_4 - v_1v_2v_3v_4 + v_2(v_1v_3)v_4 + (v_1v_2)v_3v_4.$$

Alternately, reducing v first by means of f_4 we have

$$v_4v_3v_2v_1 = v_1v_4v_2v_3 + v_2v_1v_3v_4 + v_3v_2v_4v_1 + v_4v_1v_3v_2$$
$$+ v_3v_4v_2v_1 + v_2v_3v_1v_4 + v_1v_2v_4v_3.$$

Applying the anti-commutative law where possible yields

$$v = - v_1(v_2v_3)v_4 + v_2v_1v_3v_4 - v_3v_2v_1v_4 + (v_1v_2v_3)v_4$$
$$+ v_3(v_1v_2)v_4 + v_2v_3v_1v_4 - v_1v_2v_3v_4.$$

Again, under the specified conditions, the results of the program show that $(v_1v_2v_3)v_4$ is reducible to $v_2(v_1v_3)v_4 + v_3v_2v_1v_4 - v_1v_3v_2v_4 + (v_1v_2)v_3v_4$. Consequently (5.4) becomes

$$v = - v_1(v_2v_3)v_4 + v_2v_1v_3v_4 - v_3v_2v_1v_4 + v_2(v_1v_3)v_4$$
$$+ v_3v_2v_1v_4 - v_1v_3v_2v_4 + (v_1v_2)v_3v_4 + (v_3(v_1v_2))v_4 + v_2v_3v_1v_4 - v_1v_2v_3v_4$$
$$= - v_1(v_2v_3)v_4 + v_2v_1v_3v_4 + v_2(v_1v_3)v_4 - v_1v_3v_2v_4$$
$$+ (v_1v_2)v_3v_4 - v_3(v_1v_2)v_4 + v_2v_3v_1v_4 - v_1v_2v_3v_4$$

Thus again we have unique reduction.

Another 35 cases can arise as a consequence of having two reducible words which are not disjoint. To illustrate the procedure when u is a proper subword of v consider: $w = w_7w_6w_5w_4w_3w_2w_1$ with $w_1 < w_2 < w_3 < w_4 < w_1w_2 < w_1w_4 < w_2w_3 < w_4w_3w_1w_2 < w_5 < w_6 < w_7$. Let $u = u_4u_3u_2u_1$ with $u_4 = w_7$, $u_3 = w_6$, $u_2 = w_5$ and $u_1 = w_4w_3w_2w_1$. Then u is reducible by f_4 to $u_1u_4u_2u_3 + u_2u_1u_3u_4 - u_3u_2u_1u_4 - u_4u_1u_2u_3 - u_3u_4u_1u_2 + u_2u_3u_1u_4 - u_1u_2u_3u_4$.

Thus $w \to (w_4w_3w_2w_1)w_7w_5w_6 + w_5(w_4w_3w_2w_1)w_6w_7 - w_6w_5(w_4w_3w_2w_1)w_7 - w_7(w_4w_3w_2w_1)w_5w_6 - w_6w_7(w_4w_3w_2w_1)w_5 - w_5w_6(w_4w_3w_2w_1)w_7 - (w_4w_3w_2w_1)w_5w_6w_7$.

Each term of this reduction contains the subword $w_4w_3w_2w_1$ which is reducible by the anti-commutative law to $- w_4w_3w_2w_1$ and thence (from the results of the program using the case $4 < 12 < 14 < 23$) to

$$w_1w_4w_2w_3 + w_2w_1w_3w_4 - w_3w_2w_1w_4 - w_4w_1w_2w_3 - w_3w_4w_1w_2 + w_2w_3w_1w_4$$

$- w_1w_2w_3w_4$. Consequently

$$
\begin{aligned}
w \rightarrow\ & (w_1w_4w_2w_3)w_7w_5w_6 + w_5(w_1w_4w_2w_3)w_6w_7 - w_6w_5(w_1w_4w_2w_3)w_7 \\
& - w_7(w_1w_4w_2w_3)w_5w_6 + (w_2w_1w_3w_4)w_7w_5w_6 - (w_1w_4w_2w_3)w_5w_6w_7 \\
& + w_5w_6(w_1w_4w_2w_3)w_7 - w_6w_7(w_1w_4w_2w_3)w_5 + w_5(w_2w_1w_3w_4)w_6w_7 \\
& - w_6w_5(w_2w_1w_3w_4)w_7 - w_7(w_2w_1w_3w_4)w_5w_6 - w_6w_7(w_2w_1w_3w_4)w_5 \\
& - w_5(w_3w_2w_1w_4)w_6w_7 - (w_3w_2w_1w_4)w_7w_5w_6 - (w_2w_1w_3w_4)w_5w_6w_7 \\
& + w_5w_6(w_2w_1w_3w_4)w_7 + w_7(w_3w_2w_1w_4)w_5w_6 + w_6w_7(w_3w_2w_1w_4)w_5 \\
& + w_6w_5(w_3w_2w_1w_4)w_7 - w_5w_6(w_3w_2w_1w_4)w_7 + w_6w_5(w_4w_1w_2w_3)w_7 \\
& - w_5(w_4w_1w_2w_3)w_6w_7 - (w_4w_1w_2w_3)w_7w_5w_6 + (w_3w_2w_1w_4)w_5w_6w_7 \\
& + w_7(w_4w_1w_2w_3)w_5w_6 + w_6w_7(w_4w_1w_2w_3)w_5 - w_5w_6(w_4w_1w_2w_3)w_7 \\
& + (w_4w_1w_2w_3)w_5w_6w_7 + w_7(w_3w_4w_1w_2)w_5w_6 + w_6w_5(w_3w_4w_1w_2)w_7 \\
& - w_5(w_3w_4w_1w_2)w_6w_7 - (w_3w_4w_1w_2)w_7w_5w_6 + w_6w_7(w_3w_4w_1w_2)w_5 \\
& - w_5w_6(w_3w_4w_1w_2)w_7 + (w_3w_4w_1w_2)w_5w_6w_7 + (w_2w_3w_1w_4)w_7w_5w_6 \\
& + w_5(w_2w_3w_1w_4)w_6w_7 - w_6w_5(w_2w_3w_1w_4)w_7 - w_7(w_2w_3w_1w_4)w_5w_6 \\
& - w_6w_7(w_2w_3w_1w_4)w_5 - w_5(w_1w_2w_3w_4)w_6w_7 - (w_1w_2w_3w_4)w_7w_5w_6 \\
& - (w_2w_3w_1w_4)w_5w_6w_7 + w_5w_6(w_2w_3w_1w_4)w_7 + w_6w_5(w_1w_2w_3w_4)w_7 \\
& + w_7(w_1w_2w_3w_4)w_5w_6 + w_6w_7(w_1w_2w_3w_4)w_5 - w_5w_6(w_1w_2w_3w_4)w_7 \\
& + (w_1w_2w_3w_4)w_5w_6w_7.
\end{aligned}
$$

Alternately, $v = v_4v_3v_2v_1$ with $v_4 = w_4$, $v_3 = w_3$, $v_2 = w_2$ and $v_1 = w_1$ is (from the program) reducible to $v_1v_4v_2v_3 + v_2v_1v_3v_4 - v_3v_2v_1v_4 - v_4v_1v_2v_3 - v_3v_4v_1v_2 + v_2v_3v_1v_4 - v_1v_2v_3v_4$. Now

$$
\begin{aligned}
w \rightarrow\ & w_7w_6w_5w_1w_4w_2w_3 + w_7w_6w_5w_2w_1w_3w_4 - w_7w_6w_5w_3w_2w_1w_4 \\
& - w_7w_6w_5w_4w_1w_2w_3 - w_7w_6w_5w_3w_4w_1w_2 + w_7w_6w_5w_2w_3w_1w_4 \\
& - w_7w_6w_5w_1w_2w_3w_4.
\end{aligned}
$$

Again from the program it can be seen that each term of this reduction may be further reduced to give

$$
\begin{aligned}
w \rightarrow\ & (w_1w_4w_2w_3)w_7w_5w_6 + (w_2w_1w_3w_4)w_7w_5w_6 - (w_3w_2w_1w_4)w_7w_5w_6 \\
& - (w_4w_1w_2w_3)w_7w_5w_6 - (w_3w_4w_1w_2)w_7w_5w_6 + (w_2w_3w_1w_4)w_7w_5w_6 \\
& - (w_1w_2w_3w_4)w_7w_5w_6 + w_5(w_1w_4w_2w_3)w_6w_7 + w_5(w_2w_1w_3w_4)w_6w_7 \\
& - w_5(w_3w_2w_1w_4)w_6w_7 - w_5(w_4w_1w_2w_3)w_6w_7 - w_5(w_3w_4w_1w_2)w_6w_7 \\
& + w_5(w_2w_3w_1w_4)w_6w_7 - w_5(w_1w_2w_3w_4)w_6w_7 - w_6w_5(w_1w_4w_2w_3)w_7 \\
& - w_6w_5(w_2w_1w_3w_4)w_7 + w_6w_5(w_1w_2w_3w_4)w_7 - w_7(w_1w_4w_2w_3)w_5w_6 \\
& + w_6w_5(w_3w_2w_1w_4)w_7 + w_6w_5(w_4w_1w_2w_3)w_7 + w_6w_5(w_3w_4w_1w_2)w_7 \\
& - w_6w_5(w_2w_3w_1w_4)w_7 - w_7(w_2w_1w_3w_4)w_5w_6 + w_7(w_3w_2w_1w_4)w_7 \\
& + w_7(w_4w_1w_2w_3)w_5w_6 + w_7(w_3w_4w_1w_2)w_5w_6 - w_7(w_2w_3w_1w_4)w_5w_6 \\
& + w_7(w_1w_2w_3w_4)w_5w_6 - w_6w_7(w_1w_4w_2w_3)w_5 - w_6w_7(w_2w_1w_3w_4)w_5 \\
& + w_6w_7(w_1w_2w_3w_4)w_5 + w_6w_7(w_4w_1w_2w_3)w_5 + w_6w_7(w_3w_4w_1w_2)w_5 \\
& - w_6w_7(w_2w_3w_1w_4)w_5 + w_6w_7(w_3w_2w_1w_4)w_5 + w_5w_6(w_1w_4w_2w_3)w_7 \\
& + w_5w_6(w_2w_1w_3w_4)w_7 - w_5w_6(w_3w_2w_1w_4)w_7 - w_5w_6(w_4w_1w_2w_3)w_7 \\
& - w_5w_6(w_3w_4w_1w_2)w_7 + w_5w_6(w_2w_3w_1w_4)w_7 - w_5w_6(w_1w_2w_3w_4)w_7 \\
& - (w_1w_4w_2w_3)w_5w_6w_7 - (w_2w_1w_3w_4)w_5w_6w_7 + (w_3w_2w_1w_4)w_5w_6w_7 \\
& + (w_4w_1w_2w_3)w_5w_6w_7 + (w_3w_4w_1w_2)w_5w_6w_7 - (w_2w_3w_1w_4)w_5w_6w_7 \\
& + (w_1w_2w_3w_4)w_5w_6w_7.
\end{aligned}
$$

Finally, comparison of these two reductions shows that the Birkhoff condition is once more satisfied. The remaining 34 cases can be treated in a similar manner.

6. In conclusion

The direction to go from here would be to determine necessary and sufficient conditions that a reduction process may be derived immediately from the defining equations of a given algebra. In any particular case it is conceivable that a reduction process could be obtained by deriving sufficiently many reduction formulas, provided the Birkhoff condition was satisfied, but this is not too satisfactory.

Bibliography

Hall, M., Jr. "A Basis for Free Lie Rings and Higher Commutators in Free Groups," Proc. Amer. Math. Soc. 1:575-81, 1950.

Jacobson, N. Lie Algebras. New York, Interscience Publishers, 1962.

Noonan, B. "An Explicit Construction for a Basis of a Free Lie Algebra." Unpublished.

Sagle, A. A. "Malcev Algebras," Trans. Amer. Math. Soc. 101:426-58, 1961.

Serre, J. P. "Lie Algebras and Lie Groups." New York, W. A. Benjamin Incorporated, 1965.

Zassenhaus, H. "Lie Rings and Lie Algebras." Summer Seminar, Can. Math. Cong. (1957).

APPENDIX

Appendix A

Program to assist in verifying the Birkhoff condition for Malcev algebras

A word $w_1w_2w_3w_4 = + w_1(w_2(w_3w_4))$ is presented in the form 816263477.

Here + is represented by 8 (- would be represented by 9)

1,2,3,4 are the subscripts of the subwords,

and 6,7 represent open (left) and closed (right) brackets.

Note that all brackets and the algebraic sign are included.

Thus for example we have

$$w_1(w_2w_3)w_4 = + w_1((w_2w_3)w_4) \quad \text{<->} \quad 816623747$$

$$-(w_1w_2)w_3w_4 = -(w_1w_2)(w_3w_4) \quad \text{<->} \quad 961276347 \quad \text{etc.}$$

The initial word is placed in minimal form with respect to the anti-commutative law and then recorded. In order to examine this reduction in detail, consider the particular case of the monomial a(bc)d. Notice that there are three comparisons to be made:

1) a with (bc)d, 2) bc with d, and 3) b with c.

The location of each of the subwords within the monomial is obtained, and also their relative size (with respect to the order $<$). The relevant comparisons are made and, if necesssary, the subwords are rearranged in order to minimize the word. In this particular case, if $a < b < c < d < ab$, we have $a < bc$ and so the monomial would become adbc.

The word is now substituted into one of the reduction formulas and expanded accordingly. Again to examine the process in greater detail, consider the case in which one wants to obtain a reduction for (ab)cd according to the formula

$$(w_1w_2)w_3w_4 + w_1(w_3w_2)w_4 + w_3(w_2w_4)w_1 + w_2(w_4w_1)w_3 + w_4(w_1w_3)w_2.$$

A word of the same form is located in the reduction formula, and the two words are assigned the same subscripts. In the particular case under consideration, the required word in the formula is $(w_1w_2)w_3w_4$. Thus the given monomial becomes $(a_1b_2)c_3d_4$. The relation is then expanded according to the subscripts, which in this case would yield

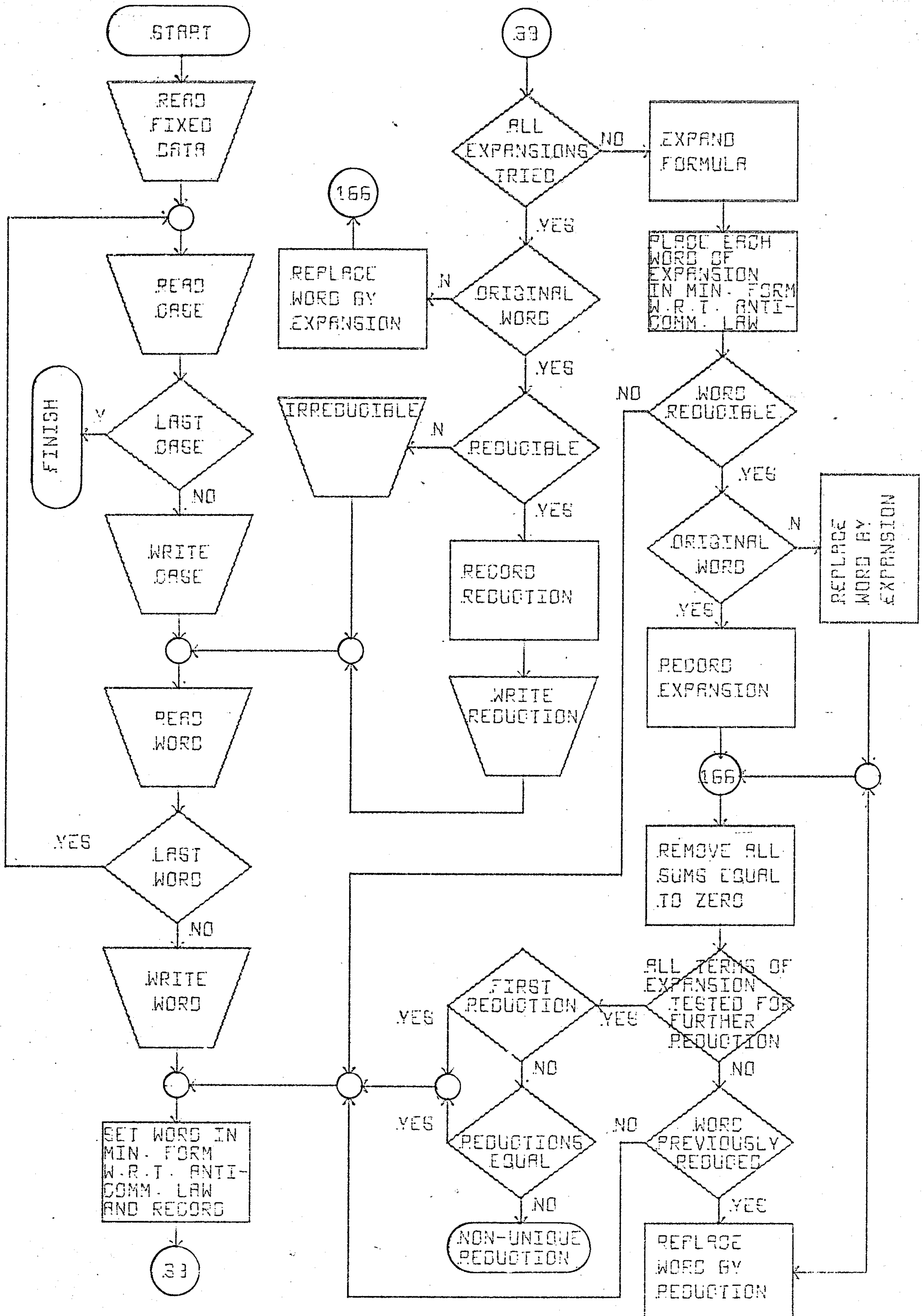
$$(ab)cd + a(cb)d + c(bd)a + b(da)c + d(ac)b.$$

Each monomial in the expansion is placed in minimal form with respect to the anti-commutative law and compared with the original word. If the original word is less than any of the words in the expansion, then the word is irreducible by this method. In this case the reduction under consideration is terminated and the original word is subjected to a second reduction procedure. This process continues until all possible reductions have been applied.

In any case in which the original word is reducible, the expansion is recorded and each word in the expansion is tested for further possible reductions. The entire procedure is repeated until all possible reductions have been applied. Each reduction is compared to the first to determine if they are the same. If they are, the word is uniquely reducible. But if any two reductions are not the same, the word is not uniquely reducible and the program terminates.

The program as given has several limitations. First, it can be used only in connection with Malcev algebras, although the method of approach should be applicable to other non-associative algebras. Second, the program only examines words composed of four subwords. This does not result in any loss of generality for words composed of only two or three subwords since such monomials are reducible only by the anti-commutative law. Words of length five or greater are examined four subwords at a time, so that, given the results of the program as it now stands, it is not difficult (nor excessively time consuming) to obtain the reduction of such words with a not unreasonable amount of hand computation.

Copies of the program may be obtained from the author.


START
READ FIXED DATA
READ CASE
LAST CASE
Y
FINISH
NO
WRITE CASE
READ WORD
LAST WORD
YES
NO
WRITE WORD
SET WORD IN MIN. FORM W.R.T. ANTI-COMM. LAW AND RECORD
33
166
REPLACE WORD BY EXPANSION
N
IRREDUCIBLE
N
33
ALL EXPANSIONS TRIED
NO
EXPAND FORMULA
YES
ORIGINAL WORD
YES
REDUCIBLE
YES
RECORD REDUCTION
WRITE REDUCTION
PLACE EACH WORD OF EXPANSION IN MIN. FORM W.R.T. ANTI-COMM. LAW
NO
WORD REDUCIBLE
YES
ORIGINAL WORD
N
REPLACE WORD BY EXPANSION
YES
RECORD EXPANSION
166
REMOVE ALL SUMS EQUAL TO ZERO
ALL TERMS OF EXPANSION TESTED FOR FURTHER REDUCTION
YES
FIRST REDUCTION
YES
NO
YES
REDUCTIONS EQUAL
NO
NON-UNIQUE REDUCTION
NO
WORD PREVIOUSLY REDUCED
NO
YES
REPLACE WORD BY REDUCTION

```
DIMENSION KLPH(20), MA(5, 6), NRW(4), KRW(3, 8, 9), IKW(2, 25, 9),
XLSW(5, 6), KCW(22, 7), NSW(10, 6), KASE(70), MK(25), JKW(16, 15, 9
X), IR(16)
READ FIXED DATA
READ (1,2) (NRW(NR), NR = 1, 4)
READ (1,2) (((KRW(NR, I, K), K = 1, 9), I = 1, 8), NR = 1, 3)
FORMAT (8(9I1, 1H  ))
READ (1,5) (KLPH(I), I = 1, 10)
READ (1,3) ((MA(I, J), J = 1, 6), I = 1, 5)
READ (1,3) ((LSW(I, J), J = 1, 6), I = 1, 5)
FORMAT (6(1H , I2))
READ CASE
READ (1,5) (KASE(I), I = 1, 35)
FORMAT (35A1)
IF (KASE(1) - KLPH(10)) 6, 177, 175
IS = 1
WRITE (3,7) (KASE(I), I = 1, 35)
FORMAT (1H1, 35A1)
READ WORD
READ (1,8) ((KCW(I, K), K = 1, 7), I = 1, 22)
FORMAT (10(7I1, 1H  ))
READ (1,2) (IKW(2, 1, K), K = 1, 9)
IF (IKW(2, 1, 9) - 9) 10, 4, 175
SET COUNTERS
IA = 1
FORMAT (' '9I2)
IB = IA
IC = 0
NS = 0
IH = 0
IQ = 1
DO 13 K = 1, 9
JKW(1, 1, K) = IKW(2, IA, K)
NR = 0
FORMAT (' '9I3)
MK(IA) = 0
NR = NR + 1
L = 1
NB = NRW(NR) + 2
NA = 1
GO TO 37
RECORD MINIMAL FORM OF WORD
IF (IA - 1) 175, 18, 22
DO 19 K = 1, 9
IKW(2, 1, K) = JKW(1, 1, K)
N = 1
IU = 1
IF (IC) 175, 130, 22
WRITE (3,21) (IKW(1, 1, K), K = 2, 9)
FORMAT (1H0, 8A1)
DO 23 K = 1, 9
JKW(1, 15, K) = JKW(1, 1, K)
NA = 2
GO TO 37
K = 5
GO TO 26
K = 6
IC = MK(IA) + 1
IF (IC - NRW(NR)) 27, 27, 28
MK(IA) = IC
```

```
IF (KRW(NR, IC, K) - 6) 26, 33, 26
IF (JKW(1, 1, 5) - 6) 29, 29, 30
IF (NR - 3) 15, 31, 175
IF (NR - 2) 15, 31, 175
IF (IA - 1) 175, 32, 82
IF (IH) 175, 154, 123
EXPAND FORMULA
DO 34 K = 2, 9
I = KRW(NR, IC, K)
JKW(1, 2, I) = JKW(1, 1, K)
NA = 1
L = 3
JKW(1, L, 1) = KRW(NR, L - 2, 1)
JKW(1, L, 1) = KRW(NR, L - 2, 1)
DO 36 K = 2, 9
I = KRW(NR, L - 2, K)
JKW(1, L, K) = JKW(1, 2, I)
SET WORD IN MINIMAL FORM W.R.T. ANTI - COMMUTATIVE LAW
ALSO - SET WORD IN FORM FOR EXPANDING
IF (JKW(1, L, 2) - 6) 38, 39, 175
IF (JKW(1, L, 4) - 6) 40, 41, 175
IF (JKW(1, L, 6) - 6) 42, 44, 43
I = 1
GO TO (46, 24), NA
I = 2
N = 4
GO TO (46, 52), NA
I = 3
N = 6
GO TO (46, 52), NA
I = 4
N = 6
GO TO (46, 52), NA
GO TO (45, 25), NA
I = 5
N = 1
J = MA(I, N) - 1
JA = LSW(I, N)
K = 0
K = K + 1
IF (K - 22) 49, 49, 175
DO 50 JB = 1, JA
M = J + JB
IF (KCW(K, JB) - JKW(1, L, M)) 48, 50, 48
NSW(L, N) = K
A = N
B = A / 2.
C = N / 2
IF (B - C) 175, 51, 68
IF (NSW(L, N) - NSW(L, N - 1)) 52, 63, 68
JA = MA(I, N - 1)
DO 53 K = 1, JA
JKW(1, L + 1, K) = JKW(1, L, K)
JB = MA(I, N) - 1
DO 54 K = JA, JB
JC = K + LSW(I, N)
JKW(1, L + 1, JC) = JKW(1, L, K)
JA = MA(I, N) + LSW(I, N) - 1
JB = JB + 1
DO 55 K = JB, JA
```

```
JC = K - LSW(I, N - 1)
JKW(1, L + 1, JC) = JKW(1, L, K)
IF (JA - 9) 56, 58, 175
JA = JA + 1
DO 57 K = JA, 9
JKW(1, L + 1, K) = JKW(1, L, K)
IF (JKW(1, L, 1) - 8) 175, 59, 60
JKW(1, L + 1, 1) = 9
GO TO 61
JKW(1, L + 1, 1) = 8
DO 62 K = 1, 9
JKW(1, L, K) = JKW(1, L + 1, K)
GO TO 37
DO 64 K = 1, 9
JKW(1, L, K) = 0
DO 65 I = 1, 6
NSW(L, I) = 0
IF (L - 1) 175, 66, 77
WRITE (3,67)
FORMAT (1H , 15HREDUCES TO ZERO)
GO TO 16
N = N + 1
IF (N .LE. 6) GO TO 47
IF (L - 1) 175, 17, 69
IF (NSW(1, 6) - NSW(1, 5)) 70, 175, 71
I = 6
J = 5
GO TO 72
I = 5
J = 6
IF (NSW(L, 6) - NSW(L, 5)) 73, 175, 74
M = 6
N = 5
GO TO 75
M = 5
N = 6
IF (NSW(1, I) - NSW(L, M)) 16, 76, 77
IF (NSW(1, J) - NSW(L, N)) 16, 77, 77
L = L + 1
IF (L .LE. NB) GO TO 35
SET COUNTERS
IF (IA - 1) 175, 79, 81
NS = NR
JA = 1
JB = 1
IH = 1
IF (IA - 1) 175, 91, 83
JA = 2
JB = 3
GO TO 83
JA = 1
JB = 2
N = IB
JC = 2
DO 85 K = 2, 9
IF (IKW(2, JC, K) - JKW(1, 15, K)) 90, 85, 90
CONTINUE
GO TO (88, 86, 91), JB
DO 87 K = 1, 9
IKW(1, JC, K) = 0
```

```
      GO TO 90
8     DO 89 K = 1, 9
      IKW(1, JC, K) = 0
9     IKW(2, JC, K) = 0
0     JC = JC + 1
      IF (JC .LE. N) GO TO 84
      GO TO (163, 80, 175), JA
      REPLACE WORD BY EXPANSION
1     M = IB + IC
      DO 92 K = 1, 9
      IKW(1, M, K) = JKW(1, IC + 2, K)
2     IKW(2, M, K) = JKW(1, IC + 2, K)
      IA = IB
      DO 100 L = 3, NB
      IA = IA + 1
      IF (M - IA) 93, 100, 93
3     IF (JKW(1, 15, 1) - IKW(1, M, 1)) 97, 94, 97
4     IF (JKW(1, L, 1) - 8) 175, 96, 95
5     IKW(1, IA, 1) = 8
      IKW(2, IA, 1) = 8
      GO TO 98
6     IKW(1, IA, 1) = 9
      IKW(2, IA, 1) = 9
      GO TO 98
7     IKW(1, IA, 1) = JKW(1, L, 1)
      IKW(2, IA, 1) = JKW(1, L, 1)
8     DO 99 K = 2, 9
      IKW(1, IA, K) = JKW(1, L, K)
9     IKW(2, IA, K) = JKW(1, L, K)
00    IB = IA
      REMOVE SUMS EQUAL TO ZERO
01    L = IB - 1
      DO 107 I = 2, L
      J = I + 1
02    IF (IKW(2, I, 1) - IKW(2, J, 1)) 104, 103, 104
03    J = J + 1
      IF (J - IB) 102, 102, 107
04    DO 105 K = 2, 9
      IF (IKW(2, I, K) - IKW(2, J, K)) 103, 105, 103
05    CONTINUE
      DO 106 K = 1, 9
      IKW(1, I, K) = 0
      IKW(2, I, K) = 0
      IKW(1, J, K) = 0
06    IKW(2, J, K) = 0
07    CONTINUE
      GO TO (83, 88, 163), JA
      RECORD REDUCTION
08    IF (IQ - 1) 175, 109, 115
09    IQ = 2
      IS = IS + 1
      J = 0
      DO 112 I = 1, IB
      IF (IKW(2, I, 1)) 175, 112, 110
10    J = J + 1
      DO 111 K = 1, 9
11    JKW(IS, J, K) = IKW(2, I, K)
12    IR(IS) = J
13    NR = NS
      IA = 1
```

```
      IB = IA
      DO 114 K = 1, 9
14    JKW(1, 1, K) = IKW(2, 1, K)
      GO TO 16
      COMPARE REDUCTIONS
15    J = IR(IS)
      DO 119 I = 1, IB
      IF (IKW(2, I, 1)) 175, 119, 116
16    DO 118 L = 1, J
      DO 117 K = 1, 9
      IF (IKW(2, I, K) - JKW(IS, L, K)) 123, 117, 123
17    CONTINUE
      GO TO 119
18    CONTINUE
      GO TO 120
19    CONTINUE
      GO TO 113
20    WRITE (3,121)
21    FORMAT (1H , 20HNON-UNIQUE REDUCTION)
22    FORMAT (8(9I1, 1H  ))
      GO TO 177
      WRITE REDUCTION
23    WRITE (3,124)
24    FORMAT (1H , 18HUNIQUELY REDUCIBLE)
      NA = IS
      L = IR(IS) / 8
      IF (L) 175, 125, 126
25    N = IR(IS)
      M = 0
      GO TO 128
26    N = 8
      J = 1
27    M = J - 1
28    I = 1
29    IU = 8 * M + I
30    DO 143 K = 1, 9
      IF (JKW(NA, IU, K) - 2) 131, 132, 133
31    IKW(1, IU, K) = KLPH(1)
      GO TO 143
32    IKW(1, IU, K) = KLPH(2)
      GO TO 143
33    IF (JKW(NA, IU, K) - 4) 134, 135, 136
34    IKW(1, IU, K) = KLPH(3)
      GO TO 143
35    IKW(1, IU, K) = KLPH(4)
      GO TO 143
36    IF (JKW(NA, IU, K) - 6) 137, 138, 139
37    IKW(1, IU, K) = KLPH(5)
      GO TO 143
38    IKW(1, IU, K) = KLPH(6)
      GO TO 143
39    IF (JKW(NA, IU, K) - 8) 140, 141, 142
40    IKW(1, IU, K) = KLPH(7)
      GO TO 143
41    IKW(1, IU, K) = KLPH(8)
      GO TO 143
42    IKW(1, IU, K) = KLPH(9)
43    CONTINUE
      IF (NA - 1) 175, 20, 144
44    I = I + 1
```

```
      IF (I .LE. N) GO TO 129
45    IF (M) 175, 146, 148
46    WRITE (3,147) ((IKW(1, I, K), K = 1, 9), I = 1, N)
47    FORMAT (1H , A1, 1H , 8A1, 2H #, 7(1H , A1, 1H , 8A1))
      IF (L) 175, 9, 150
48    WRITE (3,149) ((IKW(1, I + 8, K), K = 1, 9), I = 1, N)
49    FORMAT (1H , 12H            , 7(1H , A1, 1H , 8A1))
50    J = J + 1
      IF (J .LE. L) GO TO 127
      IF (L) 175, 9, 151
51    IF (IR(IS) - 8 * L) 9, 9, 152
52    N = IR(IS) - 8 * L
      L = L + 1
      J = L
      GO TO 127
53    GO TO 9
54    WRITE (3,155)
55    FORMAT (1H , 11HIRREDUCIBLE)
      IS = IS + 1
      IR(IS) = 1
      DO 156 K = 1, 9
56    JKW(IS, 1, K) = IKW(2, 1, K)
      GO TO 9
57    DO 159 I = 2, IS
      DO 158 K = 2, 9
      IF (IKW(1, IA, K) - JKW(I, 1, K)) 159, 158, 159
58    CONTINUE
      GO TO 160
59    CONTINUE
      GO TO 12
60    IF (IR(I) - 1) 175, 161, 166
61    DO 162 K = 1, 9
62    IKW(1, IA, K) = 0
63    IA = IB
64    IF (IKW(1, IA, 1)) 175, 165, 157
65    IA = IA - 1
      IF (IA - 1) 175, 108, 164
      REPLACE WORD BY REDUCTION
66    M = IR(I)
      JA = 3
      N = IA
      IA = IB
      DO 173 L = 2, M
      IA = IA + 1
      IF (IKW(1, N, 1) - JKW(I, 1, 1)) 167, 170, 167
67    IF (JKW(I, L, 1) - 8) 173, 168, 169
68    IKW(1, IA, 1) = 9
      IKW(2, IA, 1) = 9
      GO TO 171
69    IKW(1, IA, 1) = 8
      IKW(2, IA, 1) = 8
      GO TO 171
70    IKW(1, IA, 1) = JKW(I, L, 1)
      IKW(2, IA, 1) = JKW(I, L, 1)
71    DO 172 K = 2, 9
      IKW(1, IA, K) = JKW(I, L, K)
72    IKW(2, IA, K) = JKW(I, L, K)
73    IB = IA
      DO 174 K = 1, 9
      IKW(1, N, K) = 0
```

```
74    IKN(2, N, K) = 0
      GO TO 101
75    WRITE (3,176)
76    FORMAT (1H , 5HERROR)
77    CALL EXIT
      END
```

Appendix B

Results from program MALRED

```
12 < 3 < (12)3 < 4

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
- (1(23))4 = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

(2(13))4
UNIQUELY REDUCIBLE
- (2(13))4 = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

((12)3)4
UNIQUELY REDUCIBLE
+ ((12)3)4 = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
              - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```

```
12 < 3 < 2(13) < 4 < (12)3

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
- (1(23))4 = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

(2(13))4
UNIQUELY REDUCIBLE
- (2(13))4 = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
- 4((12)3) = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
              - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```

12 < 3 < 1(23) < 4 < 2(13)

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
− (1(23))4 = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = − 1((23)4) − 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
− 4((12)3) = + 1((23)4) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
− (23)(14) = + 3(1(24)) + 1(3(24)) − 3((12)4) − 2(3(14)) − 2(1(34)) + 1((23)4) − (12)(34)

```
12 < 3 < 23 < 4 < 1(23)

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
- 4((12)3) = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```

12 < 3 < 13 < 4 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) − 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
− 4((12)3) = − 1(4(23)) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
− (23)(14) = − 2(1(34)) − 3((12)4) + 1(3(24)) − 1(4(23)) + 3(1(24)) − 2(3(14)) − (12)(34)

12 < 3 < 13 < 4 < 14 < 23

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) − 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
− 4((12)3) = − 1(4(23)) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) − 1(4(23)) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) − (12)(34)

```
12 < 3 < 4 < 13 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = + 2(4(13)) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) - 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
- 4((12)3) = - 3((12)4) + 1(3(24)) + 1(2(34)) - 2(1(34)) + 2(4(13)) - 1(4(23)) - 2(3(14))
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) - 3((12)4) - 2(3(14)) + 2(4(13)) - 3(2(14)) + 1(3(24)) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = - 2(1(34)) - 3((12)4) + 1(3(24)) - 1(4(23)) + 3(1(24)) - 2(3(14)) - (12)(34)
```

$12 < 3 < 4 < 13 < 14 < 23$

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = + 2(4(13)) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) − 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
− 4((12)3) = − 3((12)4) + 1(3(24)) + 1(2(34)) − 2(1(34)) + 2(4(13)) − 1(4(23)) − 2(3(14))
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) − 3((12)4) − 2(3(14)) + 2(4(13)) − 3(2(14)) + 1(3(24)) − (12)(34)

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) − 3((12)4) + 1(3(24)) − 2(1(34)) − 1(4(23)) − 2(3(14)) − (12)(34)

```
3 < 12 < 3(12) < 4

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
- (1(23))4 = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

(2(13))4
UNIQUELY REDUCIBLE
- (2(13))4 = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

(3(12))4
UNIQUELY REDUCIBLE
- (3(12))4 = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```

3 < 12 < 2(13) < 4 < 3(12)

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
− (1(23))4 = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

(2(13))4
UNIQUELY REDUCIBLE
− (2(13))4 = − 1((23)4) − 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 1((23)4) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
− (23)(14) = + 3(1(24)) + 1(3(24)) − 3((12)4) − 2(3(14)) − 2(1(34)) + 1((23)4) − (12)(34)

```
3 < 12 < 1(23) < 4 < 2(13)

1(2(34))
IRREDUCIBLE

1(3(24))                                  2((13)4)
IRREDUCIBLE                               IRREDUCIBLE

1((23)4)                                  3(1(24))
IRREDUCIBLE                               IRREDUCIBLE

2(1(34))                                  3(2(14))
IRREDUCIBLE                               IRREDUCIBLE

2(3(14))                                  3((12)4)
IRREDUCIBLE                               IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
- (1(23))4 = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```

3 < 12 < 23 < 4 < 1(23)

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = − 1((23)4) − 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 1((23)4) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
− (23)(14) = + 3(1(24)) + 1(3(24)) − 3((12)4) − 2(3(14)) − 2(1(34)) + 1((23)4) − (12)(34)

```
3 < 12 < 13 < 4 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) - 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = - 1(4(23)) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = - 2(1(34)) - 3((12)4) + 1(3(24)) - 1(4(23)) + 3(1(24)) - 2(3(14)) - (12)(34)
```

3 < 12 < 13 < 4 < 14 < 23

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) − 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = − 1(4(23)) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) − 1(4(23)) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) − (12)(34)

3 < 12 < 4 < 13 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = + 2(4(13)) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) − 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = − 3((12)4) + 1(3(24)) + 1(2(34)) − 2(1(34)) + 2(4(13)) − 1(4(23)) − 2(3(14))
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) − 3((12)4) − 2(3(14)) + 2(4(13)) − 3(2(14)) + 1(3(24)) − (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
− (23)(14) = − 2(1(34)) − 3((12)4) + 1(3(24)) − 1(4(23)) + 3(1(24)) − 2(3(14)) − (12)(34)

3 < 12 < 4 < 13 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = + 2(4(13)) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) − 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = − 3((12)4) + 1(3(24)) + 1(2(34)) − 2(1(34)) + 2(4(13)) − 1(4(23)) − 2(3(14))
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) − 3((12)4) − 2(3(14)) + 2(4(13)) − 3(2(14)) + 1(3(24)) − (12)(34)

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) − 3((12)4) + 1(3(24)) − 2(1(34)) − 1(4(23)) − 2(3(14)) − (12)(34)

```
4 < 12 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3(4(12))
IRREDUCIBLE

4(1(23))
IRREDUCIBLE

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 2(4(13)) + 1(4(23)) - 4(1(23)) + 1(3(24)) - 3(1(24)) - 3(2(14)) + 2(3(14))

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 3(4(12)) + 1(2(34)) - 2(1(34)) - 1(4(23)) + 4(1(23)) + 3(2(14)) - 2(3(14))

(12)(34)
UNIQUELY REDUCIBLE
+ (12)(34) = + 1(3(24)) - 4(1(23)) + 2(4(13)) - 3(2(14))

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) + 4(1(23)) + 3(4(12)) - 2(3(14))

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = - 2(1(34)) + 3(4(12)) - 2(4(13)) - 1(4(23)) + 4(1(23)) + 3(1(24)) + 3(2(14))
             - 2(3(14))
```

```
4 < 12 < 14 < 23

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3(4(12))
IRREDUCIBLE

4(1(23))
IRREDUCIBLE

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 2(4(13)) + 1(4(23)) - 4(1(23)) + 1(3(24)) - 3(1(24)) - 3(2(14)) + 2(3(14))

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 3(4(12)) + 1(2(34)) - 2(1(34)) - 1(4(23)) + 4(1(23)) + 3(2(14)) - 2(3(14))

(12)(34)
UNIQUELY REDUCIBLE
+ (12)(34) = + 1(3(24)) - 4(1(23)) + 2(4(13)) - 3(2(14))

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) + 4(1(23)) + 3(4(12)) - 2(3(14))

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) - 2(4(13)) + 3(4(12)) - 2(1(34)) - 1(4(23)) + 4(1(23)) + 3(2(14))
             - 2(3(14))
```

```
4 < 12 < 14 < 23

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3(4(12))
IRREDUCIBLE

4(1(23))
IRREDUCIBLE

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 2(4(13)) + 1(4(23)) - 4(1(23)) + 1(3(24)) - 3(1(24)) - 3(2(14)) + 2(3(14))

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 3(4(12)) + 1(2(34)) - 2(1(34)) - 1(4(23)) + 4(1(23)) + 3(2(14)) - 2(3(14))

(12)(34)
UNIQUELY REDUCIBLE
+ (12)(34) = + 1(3(24)) - 4(1(23)) + 2(4(13)) - 3(2(14))

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) + 4(1(23)) + 3(4(12)) - 2(3(14))

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) - 2(4(13)) + 3(4(12)) - 2(1(34)) - 1(4(23)) + 4(1(23)) + 3(2(14))
             - 2(3(14))
```

```
3 < 12 < 4 < 13 < 23 < 14

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2(4(13))
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = + 2(4(13)) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) - 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = - 3((12)4) + 1(3(24)) + 1(2(34)) - 2(1(34)) + 2(4(13)) - 1(4(23)) - 2(3(14))
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = + 1(2(34)) - 3((12)4) - 2(3(14)) + 2(4(13)) - 3(2(14)) + 1(3(24)) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = - 2(1(34)) - 3((12)4) + 1(3(24)) - 1(4(23)) + 3(1(24)) - 2(3(14)) - (12)(34)
```

3 < 12 < 23 < 4 < 1(23)

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = − 2((13)4) − 3(2(14)) + 1(3(24)) − (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = − 1((23)4) − 3(1(24)) + 2(3(14)) + (12)(34)

4(3(12))
UNIQUELY REDUCIBLE
+ 4(3(12)) = + 1((23)4) − 2(1(34)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4)
− (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = − 3(2(14)) − 2(3(14)) − 3((12)4) + 1(3(24)) + 1(2(34)) − 2((13)4) − (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
− (23)(14) = + 3(1(24)) + 1(3(24)) − 3((12)4) − 2(3(14)) − 2(1(34)) + 1((23)4) − (12)(34)

```
3 < 12 < 3(12) < 4

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
- (1(23))4 = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

(2(13))4
UNIQUELY REDUCIBLE
- (2(13))4 = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

(3(12))4
UNIQUELY REDUCIBLE
- (3(12))4 = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```

```
12 < 3 < 13 < 4 < 14 < 23

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1(4(23))
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

4(1(23))
UNIQUELY REDUCIBLE
+ 4(1(23)) = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = + 1(4(23)) - 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
- 4((12)3) = - 1(4(23)) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
           - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(14)(23)
UNIQUELY REDUCIBLE
+ (14)(23) = + 3(1(24)) - 1(4(23)) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) - (12)(34)
```

```
12 < 3 < 1(23) < 4 < 2(13)

1(2(34))
IRREDUCIBLE

1(3(24))
IRREDUCIBLE

1((23)4)
IRREDUCIBLE

2(1(34))
IRREDUCIBLE

2(3(14))
IRREDUCIBLE

2((13)4)
IRREDUCIBLE

3(1(24))
IRREDUCIBLE

3(2(14))
IRREDUCIBLE

3((12)4)
IRREDUCIBLE

(1(23))4
UNIQUELY REDUCIBLE
- (1(23))4 = - 2((13)4) - 3(2(14)) + 1(3(24)) - (12)(34)

4(2(13))
UNIQUELY REDUCIBLE
+ 4(2(13)) = - 1((23)4) - 3(1(24)) + 2(3(14)) + (12)(34)

4((12)3)
UNIQUELY REDUCIBLE
- 4((12)3) = + 1((23)4) - 2(1(34)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4)
             - (12)(34)

(12)(34)
IRREDUCIBLE

(13)(24)
UNIQUELY REDUCIBLE
+ (13)(24) = - 3(2(14)) - 2(3(14)) - 3((12)4) + 1(3(24)) + 1(2(34)) - 2((13)4) - (12)(34)

(23)(14)
UNIQUELY REDUCIBLE
- (23)(14) = + 3(1(24)) + 1(3(24)) - 3((12)4) - 2(3(14)) - 2(1(34)) + 1((23)4) - (12)(34)
```