POLYNOMIAL APPROXIMATION IN WEIGHTED SPACES

by

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Abstract

One of the most important problems in Approximation Theory is to connect the rate with which a function can be approximated and the smoothness of this function. The goal is to show direct and inverse estimates in terms of some measure of smoothness. Typically, results are of the following type: "a function can be approximated with a given order if and only if it belongs to a certain smoothness class". We focus on the case of the weighted $\mathbb{L}_p[-1,1]$ spaces with not rapidly changing bounded not vanishing inside interval $(-1,1)$ weights. In order to describe certain smoothness classes we will use moduli of smoothness ω_ϕ^k and $\omega_\phi^{\star k}$ and prove their equivalence. As a final result, we will prove direct theorems for monotone and convex approximation.

Chapter 1

Intoduction

Perhaps the most fundamental theorem in Approximation Theory was proved by Karl Weierstrass in 1885. It states that any continuous function f defined on the real closed interval $[a, b]$ can be estimated by polynomial functions. More precisely, for any positive $\varepsilon > 0$, there exists a polynomial p such that for all x in $[a, b]$, we have $|f(x) - p(x)| < \varepsilon$. However, the original proof was quite complicated, so many famous mathematicians tried to find simpler proofs. A short proof was presented by Sergei Bernstein in 1912. He used what's well known today as Bernstein polynomials:

$$B_n(f, x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f(k/n), \tag{1.1}$$

where $f \in C[0, 1]$, and showed that $B_n(f)$ converge to f uniformly on $[0, 1]$ as $n \to \infty$.

The natural question is "how good is approximation by Bernstein polynomials"? To answer this question we need to define *modulus of smoothness.*

Let

$$\Delta_h^k(f, x, [a, b]) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih), & \text{if } x \pm kh/2 \in [a, b], \\ 0, & \text{otherwise,} \end{cases}$$

be the k-th symmetric difference, and let

$$\overrightarrow{\Delta}_h^k(f, x, [a, b]) = \Delta_h^k(f, x + kh/2, [a, b])$$

and

$$\overleftarrow{\Delta}_h^k (f, x, [a, b]) = \Delta_h^k (f, x - kh/2, [a, b])$$

be the forward and backward k-th differences, respectively.

Definition 1.1. *The modulus of smoothness of order $k \in \mathbb{N}$ (the k-th modulus of smoothness) of a function $f \in C[a, b]$ is defined as follows:*

$$\omega_k(f, \delta; [a, b]) := \sup_{0<h\leq\delta} \sup_{a\leq x\leq b-kh} |\Delta_h^k(f, x, [a, b])|. \tag{1.2}$$

In the trigonometric case for a 2π-periodic function $f \in C(\mathbb{R})$, we will use the following (trigonometric) modulus of smoothness:

$$\omega_k^T(f, \delta) := \sup_{0<h\leq\delta} \sup_{x\in\mathbb{R}} |\Delta_h^k(f, x, \mathbb{R})|. \tag{1.3}$$

Remark 1.2. *The modulus of smoothness of order $k = 1$ is called the modulus of continuity.*

The following estimate in terms of ω_1 was established by T. Popoviciu in 1933 (see [20]):

$$|B_n(f, x) - f(x)| \leq 1.5\omega_1(f, 1/\sqrt{n}), \quad f \in C[0, 1], \quad x \in [0, 1]. \tag{1.4}$$

For a function $f = |x - \frac{1}{2}|$, $\omega_1(f, 1/\sqrt{n}) = 1/\sqrt{n}$. At the same time

$$B_n(f, 1/2) - f(1/2) = 2^{-n-1} \binom{n}{n/2} \sim \frac{1}{\sqrt{2\pi n}},$$

so the estimate (1.4) is accurate. On the other hand, $E_n(f, [0, 1]) \leq n^{-1}$, where

$$E_n(f, [a, b]) := \inf_{p\in\Pi_n} \sup_{a\leq x\leq b} |f(x) - p(x)|$$

is the error of the best polynomial approximation by polynomials Π_n of degree less then n. So Bernstein polynomials are not close to the best approximation.

The investigation of the rate of best approximation is a very important topic in Approximation Theory. Of course, the rate depends on the function, so we want to relate it to some properties of the function being approximated. First result in this area belongs to Dunham Jackson (see [12]):

Theorem 1.3 (Jackson's inequality)**.** *Let f be a 2π-periodic r times differentiable function. Then, for any $n \in \mathbb{N}$,*

$$E_n^T(f) \leq c_r n^{-r} \omega(f^{(r)}, n^{-1}),$$

where c_r is a constant depending on r only, and $E_n^T(f) := \inf_{p \in T_n} \sup_{x \in \mathbb{R}} |f(x) - p(x)|$ is an error of the best trigonometric approximation by trigonometric polynomials T_n of degree less then n.

Jackson's inequality is also valid for approximation by algebraic polynomials.

Theorem 1.4. *Let f be an r times differentiable function on the interval $[a, b]$. Then, for any $n \in \mathbb{N}$,*

$$E_n(f, [a, b]) \leq c_r n^{-r} \omega(f^{(r)}, n^{-1}; [a, b]),$$

where c_r is a constant depending on r only.

The following generalizations of Theorem 1.3 with higher order moduli were proved by Zygmund ($k = 2$) in 1945 (see [26]) and Stečkin ($k > 2$) in 1949 (see [21]):

Theorem 1.5. *Let f be a 2π-periodic r times differentiable function. Then, for any $n \in \mathbb{N}$,*

$$E_n^T(f) \leq c_{k+r} n^{-r} \omega_k^T(f^{(r)}, n^{-1}),$$

where c_{k+r} is a constant depending on $k + r$ only.

We have a similar result in the algebraic case (for example, see [21]).

Theorem 1.6. *Let $f : [0, 1] \mapsto \mathbb{R}$ be an r times differentiable function. Then, for any $n \in \mathbb{N}$,*

$$E_n(f, [0, 1]) \leq c_{k+r} n^{-r} \omega_k(f^{(r)}, n^{-1}),$$

where c_{k+r} is a constant depending on $k + r$ only.

Remark 1.7. *Such theorems are called* Jackson-type inequalities *or* direct theorems*. Reversed inequalities are called* inverse theorems*.*

The first inverse results were proved Bernstein in 1912 (see [2]). Let $C^{k,\alpha}$ denote the Hölder space of all k-times continuously differentiable functions whose kth derivatives satisfy the Hölder condition of order α:

$$\sup_{x \neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} < \infty.$$

For approximation by trigonometric polynomials, we have:

Theorem 1.8. *Let f be a 2π-periodic continuous function and $E_n^T(f) \leq Cn^{-k-\alpha}$, with $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. Then $f \in C^{k,\alpha}$.*

Theorems 1.3 and 1.8 imply the following result.

Corollary 1.9. *For any 2π-periodic continuous function f, any $k \in \mathbb{N}_0$ and $0 < \alpha < 1$, we have*

$$E_n^T(f) = O(n^{-k-\alpha}), \quad n \to \infty \iff \omega(f,\delta) = O(\delta^{k+\alpha}), \quad \delta \to 0+.$$

Unfortunately, a similar result for approximation by algebraic polynomials is not valid. Nikol'skii discovered that algebraic polynomials, retaining on the whole interval the best order of approximation of a function can yield a substantially better approximation at the endpoints of the interval. In fact, it was proved (see, e.g. [10]), that for a function $f \in C[-1,1]$, $\omega_k(f,t) = O(t^\alpha)$, $0 < \alpha < k$, if and only if there exists a sequence of (algebraic) polynomials $\{p_n\}$ of degree n, such that

$$|f(x) - p_n(x)| \leq c(n^{-1}\sqrt{1-x^2} + n^{-2})^\alpha, \qquad x \in [-1,1]. \tag{1.5}$$

To deal with this phenomenon - known as *boundary effect* - Ditzian and Totik suggested to use another modulus of smoothness (see [9]):

Definition 1.10. *For a function $f \in C[-1,1]$, the Ditzian-Totik modulus of smoothness of order k, is the function $\omega_\phi^k(t) = \omega_\phi^k(t; f; [-1,1])$ defined on $(0,\infty)$ by the equality*

$$\omega_\phi^k(f,\delta)_p := \sup_{0<h\le\delta} \|\Delta_{h\phi(\cdot)}^k(f,\cdot,[-1,1])\|_{C[-1,1]}, \tag{1.6}$$

where $\phi(x) = \sqrt{1-x^2}$.

For any continuous on interval $[-1,1]$ function f, any $k \in \mathbb{N}_0$ and $0 < \alpha < k$ we have (see, e.g. [9])

$$E_n(f,[-1,1]) = O(n^{-\alpha}), \quad n \to \infty \iff \omega_\phi^k(f,\delta) = O(\delta^\alpha), \quad \delta \to 0+.$$

As usual, let $\mathbb{L}_p([a,b])$, $p > 0$, be a space of all functions for which the pth power of the absolute value is a Lebesgue integrable. The norm is defined as $\|f\|_{\mathbb{L}_p([a,b])} := \left(\int_a^b |f(x)|^p dx\right)^{1/p}$. For $0 < p < 1$, $\|\cdot\|_{\mathbb{L}_p([a,b])}$ does not satisfy the triangle inequality, so it is only a quasi-norm. For $[a,b] = [-1,1]$ we will use notation $\mathbb{L}_p := \mathbb{L}_p([-1,1])$, and $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p([-1,1])}$.

In the $\mathbb{L}_p$, $p > 0$, spaces, the moduli of smoothness are defined similarly to the continuous case.

Definition 1.11. *The modulus of smoothness of order $k \in \mathbb{N}$ (the k-th modulus of smoothness) of a function $f \in \mathbb{L}_p$ is defined as follows:*

$$\omega_k(f,\delta)_p := \sup_{0<h\le\delta} \|\Delta_h^k(f,\cdot,[-1,1])\|_p. \tag{1.7}$$

In the trigonometric case we will use the trigonometric modulus of smoothness defined by

$$\omega_k^T(f,\delta)_p := \sup_{0<h\le\delta} \|\Delta_h^k(f,\cdot,[-\pi,\pi+hk])\|_p. \tag{1.8}$$

The Ditzian-Totik modulus of smoothness of order k for function $f \in \mathbb{L}_p$, is the function $\omega_\phi^k(t)_p = \omega_\phi^k(t,f,[-1,1])_p$ defined on $(0,\infty)$ by the equality

$$\omega_\phi^k(f,\delta)_p := \sup_{0<h\le\delta} \|\Delta_{h\phi}^k(f,\cdot,[-1,1])\|_p. \tag{1.9}$$

The Direct Theorem is the following (see, e.g. [1]):

Theorem 1.12 (Direct Theorem). *Let $f \in \mathbb{L}_p([-\pi,\pi])$ be a 2π-periodic function, $0 < p < \infty$. Then, for any $n \in \mathbb{N}$,*

$$E_n^T(f,[-\pi,\pi])_p \leq c\omega_k^T(f,n^{-1})_p,$$

where c is a constant depending only on r and p as $p \to 0$, and $E_n^T(f,[-\pi,\pi])_p := \inf\limits_{p \in T_n} \|f-p\|_{\mathbb{L}_p([-\pi,\pi])}$ is an error of the best trigonometric $\mathbb{L}_p$ approximation.

If $f \in \mathbb{L}_p$, $0 < p < \infty$, then, for any $n \in \mathbb{N}$,

$$E_n(f,[-1,1])_p \leq c\omega_\phi^k(f,n^{-1})_p,$$

where c is a constant depending only on k and p as $p \to 0$, and $E_n(f,[-1,1])_p := \inf\limits_{p \in \Pi_n} \|f-p\|_p$ is the error of the best $\mathbb{L}_p$ approximation.

The Inverse Theorem (see, e.g. [11]) in the trigonometric case is the following:

Theorem 1.13 (Inverse Theorem). *Let $f \in \mathbb{L}_p([-\pi,\pi])$ be a 2π-periodic function. Then, for all $k, n \in \mathbb{N}$,*

$$\omega_k^T(f,n^{-1})_p \leq cn^{-k}\sum_{i=1}^n i^{k-1}E_i^T(f)_p, \quad if \quad p \geq 1,$$

and

$$\omega_k^T(f,n^{-1})_p^p \leq cn^{-kp}\sum_{i=1}^n i^{kp-1}E_i^T(f)_p^p, \quad if \;\; 0 < p < 1,$$

where c are constants depending only on k and p as $p \to 0$.

The algebraic versions (see [8]) are the following:

Theorem 1.14. *If $f \in \mathbb{L}_p$, $p \geq 1$, then for all $k, n \in \mathbb{N}$,*

$$\omega_\phi^k(f,n^{-1})_p \leq cn^{-k}\sum_{i=1}^n i^{k-1}E_i(f)_p,$$

where constant c depends on k only.

Theorem 1.15. *If $f \in \mathbb{L}_p$, $0 < p < 1$, then for all $k, n \in \mathbb{N}$,*

$$\omega_\phi^k(f, n^{-1})_p^p \le c n^{-kp} \sum_{i=1}^{n} i^{kp-1} E_i(f)_p^p,$$

where a constant c depends only on k and p.

Another topic that we are interested in is *shape preserving approximation.* The problem of shape preserving approximation is to approximate a given function by polynomials with the same 'shape'. Here by 'shape' of the function f we will understand positivity of its n-th derivative (if f is differentiable). It is known that Bernstein polynomials $B_n(f, x)$ defined in (1.1) have the same shape as initial function $f(x)$. This guaranties the existence of a sequence of polynomials that preserves the shape of a given function f, and converges to it, but does not help to find the rate of shape preserving approximation.

In Chapter 2, we will discuss the weighted shape preserving approximation and introduce main results of this thesis. The norm in the weighted $\mathbb{L}_p([-1, 1])$-space with weight w is defined by $\|f\|_{w,p} := \|wf\|_{\mathbb{L}_p([-1,1])}$. Any non-negative function w could be used as a weight, but we will focus on the weights that do not rapidly change and are not vanishing in the interior of interval $[-1, 1]$, specifically the Jacobi weights $w_{\alpha,\beta}(x) := (1 + x)^\alpha(1 - x)^\beta$.

In Chapter 3, we will prove equivalence of moduli of smoothness ω_ϕ^k and $\omega_\phi^{\star k}$, defined in Chapter 2. Those moduli describe smoothness classes corresponding to approximation with the rate $O(n^{-\alpha})$.

In Chapter 4, we will construct splines to approximate given monotone (convex) functions and polynomials to approximate those splines. This will provide the proof of the main result which is Theorem 2.16.

Chapter 2

Main results

Recall that a weight function on an interval I is a nonnegative function $w : I \mapsto \mathbb{R}$, and the norm in the weighted space $\mathbb{L}_{w,p}$ space is defined by $\|f\|_{\mathbb{L}_p,w} := \|wf\|_{\mathbb{L}_p}$. The error of the best weighted $\mathbb{L}_p$ approximation with weight w on interval $[a,b]$ is defined by

$$E_n(f,[a,b])_{w,p} := \inf_{p\in\Pi_n} \|f-p\|_{\mathbb{L}_p,w}.$$

We will focus on a special class of doubling weights $\mathcal{W}$, that were defined in [14]:

Definition 2.1. *Let $m \in \mathbb{N}$ and $\mathcal{Z} = (z_j)_{j=1}^m$, $-1 \le z_1 < \cdots < z_m \le 1$. We say that doubling weight w belongs to the class $\mathcal{W}(\mathcal{Z})$ if, for any $\epsilon > 0$ and any $x, y \in [-1,1]$ such that $|x-y| \le \epsilon\phi(x) + \epsilon^2$ and dist$([x,y], z_j) \ge \epsilon\phi(z_j) + \epsilon^2$ for all $1 \le j \le m$, the following inequalities are satisfied*

$$cw(y) \le w(x) \le c^{-1}w(y), \tag{2.1}$$

where the constant c depends only on weight w.

It is known that, if the weight $w \neq 1$, the then modulus of smoothness should be modified near zeroes and singularities. We discus only w with zeroes at the ends of interval $I = [-1,1]$, and without singularities. Define now a proper subclass $\mathcal{W} \subset \mathcal{W}(\{-1,1\})$.

Definition 2.2. *We say that the Lebesgue integrable weight function w is in the class $\mathcal{W}$ if it satisfies the following conditions:*

1. $w(x) \geq m \sup\limits_{-1 \leq y \leq 2x+1} w(y), \quad$ *if* $-1 \leq x \leq 0$,
2. $w(x) \geq m \sup\limits_{2x-1 \leq y \leq 1} w(y), \quad$ *if* $0 < x \leq 1$,

where $m > 0$ is a constant depending on w only.

Remark 2.3. *For $w \in \mathcal{W}$, in particular, we have $w(x) \geq mw(x)$, and so $0 < m \leq 1$.*

Lemma 2.4. *$\mathcal{W}$ is a proper subclass $\mathcal{W}(\{-1,1\})$.*

Proof. Firstly, we will show that every $w \in \mathcal{W}$ is a doubling weight.

It is sufficient to check the following condition.

For every interval $[a,b] \subset [-1,1]$

$$\int\limits_a^{\frac{a+b}{2}} |w(x)|dx \sim \int\limits_{\frac{a+b}{2}}^b |w(x)|dx, \tag{2.2}$$

where equivalence constant depends only on w.

Let $c := \frac{a+b}{2} \leq 0$ and $d := \frac{a+c}{2}$. Then for every $x \in [d,c]$ and every $y \in [c,b]$ $w(x) \geq mw(c) \geq m^2 w(y)$. Then

$$\int\limits_a^c |w(x)|dx \geq \int\limits_d^c |w(x)|dx \geq mw(c)(c-d) \geq \frac{1}{2}m^2 \int\limits_c^b |w(x)|dx.$$

Similarly, $\int\limits_c^b |w(x)|dx \geq \frac{1}{2}m^2 \int\limits_a^c |w(x)|dx$. So, w is a doubling weight.

Now we will show that $w \in \mathcal{W}$ satisfies conditions from Definition 2.1 of $\mathcal{W}(\{-1,1\})$.

Suppose that x, y and ϵ satisfy conditions from Definition 2.1. We have to show that $w(x) \geq cw(y)$ and $w(y) \leq cw(x)$.

Consider another weight $w_1(z) := w(-z)$, $z \in [-1,1]$. Then $w_1 \in \mathcal{W}$ with the same constant m. Let also $x_1 := -x$, $y_1 := -y$. Then $\operatorname{dist}([x,y],-1) =$

$\operatorname{dist}([x_1,y_1],1)$, $\operatorname{dist}([x,y],1) = \operatorname{dist}([x_1,y_1],-1)$ and $\phi(x_1) = \phi(x)$. So, x_1, y_1, ϵ also satisfy conditions from Definition 2.1. Then, $cw(y) \le w(x) \le c^{-1}w(y)$ can be written as $cw_1(y_1) \le w_1(x_1) \le c^{-1}w_1(y_1)$. Hence, without loss of generality, we can assume that $x \le 0$.

Note that $\epsilon \le \sqrt{1+x}$, since $1+x \ge \operatorname{dist}([x,y],-1) \ge \epsilon\phi(-1)+\epsilon^2 = \epsilon^2$. Then

$$y \le x + \epsilon\phi(x) + \epsilon^2 \le x + \sqrt{1+x}\sqrt{1-x^2} + 1 + x = 1 + 2x + (1+x)\sqrt{1-x}.$$

If $-1/2 < x \le 0$, then $1+2x > 0$, so $w(x) \ge mw(0) \ge m^2 w(y)$.

If $x \le -1/2$, then since w is a $\mathcal{W}$ weight, we have $w(x) \ge mw(2x+1)$. Also,

$$3 + 4x = 1 + 2x + 2(1+x) \ge 1 + 2x + \sqrt{1-x}(1+x) \ge y,$$

and so $-1 \le y \le 3 + 4x \le 1$.

Since $1 + 2x \le 0$, then

$$w(x) \ge mw(1+2x) \ge m^2 \sup_{-1\le z\le 3+4x} w(z) \ge m^2 w(y).$$

Hence, the first inequality in (2.1) holds with the constant m^2. Since $m \le 1$, it also holds with the constant $c = m^3$.

Let us show the second inequality in (2.1) now. Recall that we assumed that $x \le 0$. First, if $x \le y \le -x$ then $\phi(x) \le \phi(y)$ and $|x-y| \le \epsilon\phi(x) + \epsilon^2 \le \epsilon\phi(y) + \epsilon^2$. In this case, we can repeat the above argument to show that $w(y) \ge m^2 w(x)$.

Now, let $-1 < y < x \le 0$. Then $1 + y = \operatorname{dist}([x,y],-1) \ge \epsilon\phi(-1) + \epsilon^2 = \epsilon^2$. So $\epsilon \le \sqrt{1+y}$.

We will now show that $x \le 7 + 8y$. Indeed, let t satisfy $1 + x = t(1+y)$. Then,

$$x = t - 1 + ty \le y + \sqrt{1+y}\sqrt{1-x^2} + 1 + y,$$

which implies

$$t - 2 + ty - 2y = (t-2)(1+y) \le \sqrt{1+y}\sqrt{1-x^2}.$$

Hence, either $t \le 2$ or

$$(t-2)^2(1+y) \le 1 - x^2 = (1-x)(1+x) < 2(1+x) = 2t(1+y),$$

which yields

$$(t-2)^2 < 2t.$$

Therefore, $t < 3 + \sqrt{5} < 8$, and so $x < 7 + 8y$ as claimed.

If $y \ge -3/4$, then $w(y) \ge m^2 w(0) \ge m^3 w(x)$. If $y < -3/4$, then $w(y) \ge mw(1+2y) \ge m^2 w(3+4y) \ge m^3 w(x)$.

Finally, let $0 \le -x < y < 1$. Then $\operatorname{dist}([x,y],1) = 1 - y \ge \epsilon\phi(1) + \epsilon^2 = \epsilon^2$, and so $\epsilon \le \sqrt{1-y}$. Hence,

$$y \le |x-y| \le \epsilon\phi(x) + \epsilon^2 \le \sqrt{1-y} + 1 - y,$$

which implies that $y \le 3/4$. Then, $w(y) \ge m^2 w(0) \ge m^3 w(x)$.

Therefore, we conclude that w is a $\mathcal{W}(\{-1,1\})$ weight with the constant $c = m^3$. $\square$

Remark 2.5. *The class $\mathcal{W}$ is not a class of $\mathcal{W}(\{-1,1\})$ weights without singularities. More precisely, let $\mathcal{W}_0(\mathcal{Z})$ be the set of bounded weights $w \in \mathcal{W}(\mathcal{Z})$. Then $\mathcal{W} \ne \mathcal{W}_0(\{-1,1\})$.*

Lemma 2.6. *Let $x_n := 2^{-\left(\frac{3}{2}\right)^n}$, $n \in \mathbb{Z}$. Consider functions*

$$f(x) := \begin{cases} x_{2k}^{-1} x^2, & x \in (x_{2k}, x_{2k-1}], k \in \mathbb{Z}; \\ x_{2k+1} x^{-\frac{1}{2}}, & x \in (x_{2k+1}, x_{2k}], k \in \mathbb{Z}; \\ 0, & x = 0 \end{cases}$$

and

$$w(x) = \begin{cases} f(x+1), & x \in [-1,0); \\ 1, & x \in [0,1]. \end{cases}$$

Then $w \in \mathcal{W}_0(\{-1,1\}) \setminus \mathcal{W}$.

Proof. Note that $x_{n+1} = x_n^{\frac{3}{2}}$, $n \in \mathbb{Z}$. Then, for every $k \in \mathbb{Z}$ we have

- $\lim\limits_{x \to (x_{2k}-1)-} w(x) = \lim\limits_{x \to x_{2k}-} f(x) = x_{2k+1} x_{2k}^{-\frac{1}{2}} = x_{2k}^{\frac{3}{2}} x_{2k}^{-\frac{1}{2}} = x_{2k}$;
- $\lim\limits_{x \to (x_{2k}-1)+} w(x) = \lim\limits_{x \to x_{2k}+} f(x) = x_{2k}^{-1} x_{2k}^2 = x_{2k}$;
- $\lim\limits_{x \to (x_{2k+1}-1)-} w(x) = \lim\limits_{x \to x_{2k+1}-} f(x) = x_{2k+2}^{-1} x_{2k+1}^2 = x_{2k}^{\frac{3}{2}} x_{2k}^2 = x_{2k+1}^{\frac{1}{2}}$;
- $\lim\limits_{x \to (x_{2k+1}-1)+} w(x) = \lim\limits_{x \to x_{2k+1}+} f(x) = x_{2k+1} x_{2k+1}^{-\frac{1}{2}} = x_{2k+1}^{\frac{1}{2}}$.

Also, w is continuous on each interval $(x_n - 1, x_{n-1} - 1)$ since f is continuous on (x_n, x_{n-1}).

So, w is continuous on $(-1, 0)$ function and $w(x_{2k} - 1) = x_{2k}$, $w(x_{2k+1} - 1) = \sqrt{x_{2k+1}}$, $k \in \mathbb{Z}$. Then $x + 1 \le w(x) \le \sqrt{x+1}$, $x \in (-1, 0)$. So, $\lim\limits_{x \to -1-} w(x) = 0$ and $\lim\limits_{x \to 0+} w(x) = 1$. Therefore, w is continuous on $[-1, 1]$.

Let us show now that $w \notin \mathcal{W}$.

Assume that $w \in \mathcal{W}$ with constant m. Then for $x = x_{2k} - 1 < 0$ and $y = x_{2k+1} - 1 \in [-1, 2x+1]$ we have $0 < m \le \frac{w(x)}{w(y)}$. However,

$$\frac{w(x)}{w(y)} = \frac{x_{2k}}{\sqrt{x_{2k+1}}} = 2^{-\left(\frac{3}{2}\right)^{2k}} \sqrt{2^{\left(\frac{3}{2}\right)^{2k+1}}} = \left(2^{-\left(\frac{3}{2}\right)^{2k}}\right)^{\frac{1}{4}} \to 0,\ k \to \infty.$$

Therefore, $w \notin \mathcal{W}$.

Let us show now that w is a bounded doubling weight.

Clearly, $w(x) \le 1$, $x \in [-1, 1]$.

We need to check condition (2.2). Let $c := \frac{a+b}{2}$.

Since $w(x) \sim 1$ on any $(-1+\delta, 1]$, $\delta > 0$, it is sufficient to consider only intervals $[a, b]$ with $b < 1$.

We consider 3 cases:

1. $[a, b] \subset [x_{2k} - 1, x_{2k-1} - 1]$, for some $k \in \mathbb{Z}$;
2. $[a, b] \subset [x_{2k+1} - 1, x_{2k} - 1]$, for some $k \in \mathbb{Z}$;

3. $x_n \in (a+1, b+1)$, for some $n \in \mathbb{Z}$.

In each of these cases we have:

1. Let $w(x) = p(1+x)^2$ on $[a,b]$, for some $p > 0$, then

$$\int_a^c w(x)dx = \frac{p}{3}((1+c)^3 - (1+a)^3) \sim p(c-a)(1+c)^2$$

$$\sim p(b-c)(1+b)^2 \sim \frac{p}{3}((1+b)^3 - (1+c)^3) = \int_c^b w(x)dx.$$

2. Let $w(x) = q(1+x)^{-\frac{1}{2}}$ on $[a,b]$, for some $q > 0$, then

$$\int_a^c w(x)dx = 2q((1+c)^{\frac{1}{2}} - (1+a)^{\frac{1}{2}}) \sim q(c-a)(1+c)^{-\frac{1}{2}}$$

$$\sim q(b-c)(1+b)^{-\frac{1}{2}} \sim 2q((1+b)^{\frac{1}{2}} - (1+c)^{\frac{1}{2}}) = \int_c^b w(x)dx.$$

3. Assume that $x_n \in [a,b]$ is the largest. Then $1+c \sim 1+b \sim x_n$, $p(1+x)^2 \leq w(x) \leq q(1+x)^{-\frac{1}{2}}$, $x \in [a, x_n - 1]$ and $q(1+x)^{-\frac{1}{2}} \leq w(x) \leq p(1+x)^2$, $x \in [x_n - 1, b]$, where p, q are chosen so $px_n^2 = w(x_n) = qx_n^{-\frac{1}{2}}$. Let $c \leq x_n - 1$. Then we have

$$\int_a^c p(1+x)^2 dx \leq \int_a^c w(x)dx \leq \int_a^c q(1+x)^{-\frac{1}{2}}dx,$$

but

$$\int_a^c p(1+x)^2 dx \sim p(c-a)(1+c)^2 \sim p(c-a)x_n^2$$

and

$$\int_a^c q(1+x)^{-\frac{1}{2}}dx \sim q(c-a)(1+c)^{\frac{1}{2}} \sim q(c-a)x_n^{\frac{1}{2}} = p(c-a)x_n^2.$$

So

$$\int_a^c w(x)dx \sim p(c-a)x_n^2.$$

Similarly, $\int_c^{x_n} w(x)dx \sim p(x_n - c)x_n^2$ and $\int_{x_n}^b w(x)dx \sim p(b-x_n)x_n^2$. Then,

$$\int_c^b w(x)dx \sim p(b-c)x_n^2 \sim \int_a^c w(x)dx.$$

The case $c > x_n$ is similar.

Therefore w is a doubling weight.

Finally, we need to show that w satisfies conditions from Definition 2.1.

Let x, y satisfy conditions from Definition 2.1 with some ϵ.

If $x, y \in [x_n - 1, x_{n-1} - 1]$, then $w(x) \sim w(y)$ since $(1+x)^2$ and $(1+x)^{-\frac{1}{2}}$ are $\mathcal{W}(\{-1,1\})$ weights. Let $x_n \in [x, y]$ for some $n \in \mathbb{Z}$. Choose p, q such that $px_n^2 = w(x_n) = qx_n^{-\frac{1}{2}}$. Then $p(1+x)^2 \le w(x) \le q(1+x)^{-\frac{1}{2}}$ and $q(1+y)^{-\frac{1}{2}} \le w(y) \le p(1+y)^2$.

Recall that for x, y satisfying conditions from Definition 2.1 (see proof of Lemma 2.4) $\frac{1}{8}(1+y) \le 1+x \le 8(1+y)$. Then $\frac{1}{8}x_n \le 1+x \le 1+y \le 8x_n$. So,

$$p\left(\frac{1}{8}x_n\right)^2 \le p(1+x)^2 \le w(x) \le q(1+x)^{-\frac{1}{2}} \le q\left(\frac{1}{8}x_n\right)^{-\frac{1}{2}} = 8^{\frac{1}{2}}px_n^2$$

and

$$8^{-\frac{1}{2}}px_n^2 = q(8x_n)^{-\frac{1}{2}} \le q(1+y)^{-\frac{1}{2}} \le w(y) \le p(1+y)^2 \le p(8x_n)^2.$$

Therefore $w(x) \sim w(y)$ and $w \in \mathcal{W}_0(\{-1,1\}) \setminus \mathcal{W}$. $\square$

We are now going to define weighted moduli of smoothness as in [9].

The main part weighted modulus of smoothness is defined as

$$\Omega_\phi^k(f, A, \delta)_{w,p} = \sup_{0 \le h \le \delta} \|w(\cdot)\Delta_{h\phi(\cdot)}^k(f, \cdot, [-1+Ah^2, 1-Ah^2])\|_{\mathbb{L}_p[-1+Ah^2, 1-Ah^2]}, \quad (2.3)$$

where A is a positive constant.

Definition 2.7. *The weighted Ditzian-Totik modulus of smoothness of order $k \in \mathbb{N}$ of a function $f \in \mathbb{L}_{w,p}(-1,1)$ is defined as follows:*

$$\omega_\phi^k(f,A,\delta)_{w,p} := \Omega_\phi^k(f,A,\delta)_{w,p} + \overrightarrow{\Omega}_\phi^k(f,A,\delta)_{w,p} + \overleftarrow{\Omega}_\phi^k(f,A,\delta)_{w,p}, \tag{2.4}$$

where

$$\overrightarrow{\Omega}_\phi^k(f,A,\delta)_{w,p} = \sup_{0<h\le 2A\delta^2} \|w\overrightarrow{\Delta}_h^k(f,\cdot,[-1,-1+2A\delta^2])\|_{\mathbb{L}_p[-1,-1+2A\delta^2]},$$

and

$$\overleftarrow{\Omega}_\phi^k(f,A,\delta)_{w,p} := \sup_{0<h\le 2A\delta^2} \|w\overleftarrow{\Delta}_h^k(f,\cdot,[1-2A\delta^2,1])\|_{\mathbb{L}_p[1-2A\delta^2,1]}.$$

Remark 2.8. *In the above definition $\overrightarrow{\Delta}_h^k(f,x,[-1,-1+2A\delta^2]) = 0$, when $x+kh > -1+2A\delta^2$, and $\overleftarrow{\Delta}_h^k(f,x,[1-2A\delta^2,1]) = 0$, when $x-kh > 1-2A\delta^2$.*

Remark 2.9. *We use the main part of modulus Δ_h^k to describe behavior of function in the middle of interval $[-1,1]$; we use $\overrightarrow{\Delta}_h^k$ and $\overleftarrow{\Delta}_h^k$ to describe behavior of function near endpoints -1 and 1 respectively.*

It is difficult to work with modulus $\omega_\phi^k(f,A,\delta)_{w,p}$. To prove the direct result we will use another type of modulus of smoothness with the weight w introduced in [19]:

$$\omega_\phi^{\star k}(f,A,\delta)_{w,p} := \Omega_\phi^k(f,A,\delta)_{w_{\alpha,\beta},p} + E_k(f,[-1,-1+2A\delta^2])_{w,p} + E_k(f,[1-2A\delta^2,1])_{w,p}. \tag{2.5}$$

Definition 2.10. *We say that two quantities A and B are equivalent and write $A \sim B$ if there exists a positive constant C (which we call* "the equivalence constant"*) such that*

$$\frac{1}{C}A \le B \le CA.$$

To show that all results for modulus of smoothes $\omega_\phi^{\star k}$ are also valid for ω_ϕ^k, we will prove their equivalence:

Theorem 2.11. *For $k \in \mathbb{N}$, $A > 0$, $w \in \mathcal{W}$ and $f \in \mathbb{L}_{w,p}$, $p > 0$, there exists a constant $\delta_0 > 0$ such that*

$$\omega_\phi^{\star k}(f, A, \delta)_{w,p} \sim \omega_\phi^k(f, A, \delta)_{w,p},$$

for all $0 < \delta < \delta_0$, where δ_0 depends only on A, and the equivalence constant depend only on k, w and p.

Remark 2.12. *Here and everywhere else "depends on w" means depends on constant m from Definition 2.2.*

Using Theorem 2.11 we can formulate direct and inverse results for both moduli.

Theorem 2.13. *Let $1 \le p \le \infty$, $k \in \mathbb{N}$, $A > 0$, $w \in \mathcal{W}$ and $f \in \mathbb{L}_{w,p}$. Then*

$$E_n(f)_{w,p} \le c\omega_\phi^{\star k}(f, A, n^{-1})_{w,p} \sim c\omega_\phi^k(f, A, n^{-1})_{w,p}$$

and

$$\omega_\phi^k(f, A, n^{-1})_{w,p} \sim \omega_\phi^{\star k}(f, A, n^{-1})_{w,p} \le cn^{-k} \sum_{i=1}^{n} i^{k-1} E_i(f)_{w,p},$$

where constants c depend only on k.

Theorem 2.14. *Let $f \in \mathbb{L}_p$, $w \in \mathcal{W}$, $0 < p < 1$, then for all $k, n \in \mathbb{N}$*

$$E_n(f)_{w,p} \le c\omega_\phi^k(f, A, n^{-1})_{w,p}$$

and

$$\omega_\phi^k(f, \frac{1}{n})_{w,p}^p \le \frac{c}{n^{kp}} \sum_{i=1}^{n} i^{kp-1} E_i(f)_{w,p}^p,$$

where constants c depend only on k and p.

Note that the direct and inverse theorems with $\omega_\phi^{\star k}$ were proved in [14].

Consider now the problem of preserving shape.

Denote by $\Delta^l(a,b)$ the set of all l-monotone functions on (a,b) (i.e., functions with nonnegative l-th order divided difference $[t_1,t_2,\cdots,t_l;f]$ for any choice of distinct points $\{t_1,t_2,\cdots,t_l\}\subset(a,b)$).

Recall that $[t;f]=f(t)$ and, for $l>1$,

$$[t_1,t_2,\cdots,t_l;f]=([t_1,\cdots,t_{l-1};f]-[t_2,\cdots,t_l;f])/(t_l-t_1).$$

In particular, $\Delta^1(a,b)$ and $\Delta^2(a,b)$ are sets of monotone and convex functions on (a,b), respectively. The error of best l-monotone approximation by polynomials of degree less than n is

$$E_n^{(l)}(f,[a,b])_{w,p}:=\inf_{p_n\in\Pi_n\cap\Delta^l(a,b)}\|w(f-p_n)\|_{\mathbb{L}_p([a,b])}.$$

We also denote

$$E_n^{(l)}(f)_{w,p}:=E_n^{(l)}(f,[-1,1])_{w,p}.$$

We denote the Jacobi weights by

$$w_{\alpha,\beta}:=(1+x)^\alpha(1-x)^\beta,\quad \alpha,\beta\in J_p:=\begin{cases}(-1/p,\infty) & \text{if } p<\infty,\\ [0,\infty) & \text{if } p=\infty,\end{cases}$$

Note that, for $\alpha,\beta\geq 0$, $w_{\alpha,\beta}\in\mathcal{W}$ with the constant $m=\max\{2^\alpha,2^\beta\}$.

For Jacobi weights, it is known (see [17]) that the following theorem holds

Theorem 2.15. *Let $l=1$ or $l=2$, $1\leq p\leq\infty$, $A>0$, $\alpha,\beta\in J_p$, and $f\in\mathbb{L}_{w_{\alpha,\beta},p}\cap\Delta^l(-1,1)$. Then*

$$E_n^{(l)}(f)_{w_{\alpha,\beta},p}\leq c\omega_\phi^{\star(l+1)}(f,A,1/n)_{w_{\alpha,\beta},p},\qquad \textit{for all }\ n\geq l+1.$$

We will show the generalization of Theorem 2.15.

The main result of this thesis is the following theorem

Theorem 2.16. *Let $l=1$ or $l=2$, $p>0$, $A>0$, w is a $\mathcal{W}$ weight, and $f\in\mathbb{L}_{w,p}\cap\Delta^l$. Then*

$$E_n^{(l)}(f)_{w,p}\leq c\omega_\phi^{\star(l+1)}(f,A,1/n)_{w,p},\qquad \textit{for all }\ n\geq l+1. \tag{2.6}$$

Theorems 2.11, 2.14 and 2.16 immediately imply the following result.

Corollary 2.17. *Let $l = 1$ or $l = 2$, $p > 0$, $A > 0$, $w \in \mathcal{W}$ and $f \in \mathbb{L}_{w,p}$. Then for $0 < \gamma < l + 1$, we have*

$$E_n^{(l)}(f)_{w,p} = O(n^{-\gamma}) \qquad \Longleftrightarrow \qquad \omega_\phi^{\star(l+1)}(f, A, \delta)_{w,p} = O(\delta^\gamma).$$

Chapter 3

Equivalence of moduli

In this chapter, we prove Theorem 2.11. We start with several auxiliary results.

3.1 Auxiliary Results

We need the following definition of a class of weights defined on $[0,1]$.

Definition 3.1. *We say that a weight function $v : [0,1] \to [0,\infty]$ belongs to the class $\mathcal{V}$ if, for any $x \in [0,1]$ and any $y \in [0, \min\{2x,1\}]$, we have $v(x) \ge mv(y)$, where $m > 0$ depends only on the weight v.*

Recall that

$$\omega_\phi^k(f,A,\delta)_{w,p} := \Omega_\phi^k(f,A,\delta)_{w,p} + \overrightarrow{\Omega}_\phi^k(f,A,\delta)_{w,p} + \overleftarrow{\Omega}_\phi^k(f,A,\delta)_{w,p}$$

and

$$\omega_\phi^{\star k}(f,A,\delta)_{w,p} := \Omega_\phi^k(f,A,\delta)_{w,p} + E_k(f,[-1,-1+2A\delta^2])_{w,p} + E_k(f,[1-2A\delta^2,1])_{w,p}.$$

We assume that $\delta < \delta_0 := \frac{1}{\sqrt{2A}}$, so the intervals $[-1,-1+2A\delta^2]$, $[1-2A\delta^2,1]$ do not intersect.

$\Omega_\phi^k(f,A,\delta)_{w,p}$ is the common part in both moduli, so it is sufficient to show that

1. $\overrightarrow{\Omega}^k_\phi(f,A,\delta)_{w,p} \sim E_k(f,[-1,-1+2A\delta^2])_{w,p}$, and

2. $\overleftarrow{\Omega}^k_\phi(f,A,\delta)_{w,p} \sim E_k(f,[1-2A\delta^2,1])_{w,p}$,

where equivalence constants depend only on k, w and p.

Introducing changes of variables: $x \mapsto \frac{x+1}{2A\delta^2}$ and $x \mapsto \frac{1-x}{2A\delta^2}$, we get

1. $[-1,-1+2A\delta^2] \ni x \mapsto y = \frac{x+1}{2A\delta^2} \in [0,1]$.

   Then with $f_1(y) = f(x) = f(2A\delta^2 y - 1)$, we have

$$\begin{aligned}
E_k(f,[-1,-1+2A\delta^2])_{w,p} &= \inf_{p_k\in\Pi_k} \left(\int\limits_{-1}^{-1+2A\delta^2} |w(x)(f(x)-p_k(x))|^p dx \right)^{1/p} \\
&= \inf_{q_k\in\Pi_k} \left(2A\delta^2 \int\limits_0^1 |v_1(y)(f_1(y)-q_k(y))|^p dy \right)^{1/p} \\
&= \left(2A\delta^2\right)^{1/p} E_k(f_1,[0,1])_{\mathbb{L}_p([0,1]),v_1},
\end{aligned}$$

   where $v_1(y) = w(x) = w(2A\delta^2 y - 1)$ and $q_k(y) = p_k(x) = p_k(2A\delta^2 y - 1)$.

   Also, since $f(2A\delta^2 y - 1 + ih) = f_1(y + ih/(2A\delta^2))$, we have

$$\begin{aligned}
\overrightarrow{\Omega}^k_\phi(f,A,\delta)_{w,p} &= \sup_{0<h\le 2A\delta^2} \|w\overrightarrow{\Delta}^k_h(f,\cdot,[-1,-1+2A\delta^2])\|_{\mathbb{L}_p[-1,-1+2A\delta^2]} \\
&= \sup_{0<h\le 2A\delta^2} \left(\int\limits_{-1}^{-1+2A\delta^2} |w(x)\overrightarrow{\Delta}^k_h(f,x,[-1,-1+2A\delta^2])|^p dx \right)^{1/p} \\
&= \sup_{0<h\le 2A\delta^2} \left(2A\delta^2 \int\limits_0^1 |v_1(y)\overrightarrow{\Delta}^k_{h/(2A\delta^2)}(f_1,y,[0,1])|^p dy \right)^{1/p} \\
&= \left(2A\delta^2\right)^{1/p} \sup_{0<h_1\le 1} \|v_1\overrightarrow{\Delta}^k_{h_1}(f_1)\|_{L_p[0,1]},
\end{aligned}$$

   where $h_1 = h/(2A\delta^2)$.

   For $0 \le x_1 \le 1$ and $0 \le y_1 \le \min\{2x_1, 1\}$, let $x_1 = \frac{x+1}{2A\delta^2}$ and $y_1 = \frac{y+1}{2A\delta^2}$. Then, $x = 2A\delta^2 x_1 - 1 \in [-1,-1+2A\delta^2]$ and $y = 2A\delta^2 y_1 - 1 \in [-1,-1+2A\delta^2]$.

Also, $2x+1 = 4A\delta^2 x_1 - 1 \geq 2A\delta^2 y_1 - 1 = y$, and so $v_1(x_1) = w(x) \geq mw(y) = mv_1(y_1)$. Then, v_1 is a $\mathcal{V}$ weight with the same constant m as weight w.

2. $[1-2A\delta^2, 1] \ni x \mapsto y = \frac{1-x}{2A\delta^2} \in [0,1]$.

Then with $f_2(y) = f(x) = f(1-2A\delta^2 y)$, we have

$$\begin{aligned}
E_k(f, [1-2A\delta^2, 1])_{w,p} &= \inf_{p_k \in \Pi_k} \left(\int\limits_{1-2A\delta^2}^{1} |w(x)(f(x) - p_k(x))|^p dx \right)^{1/p} \\
&= \inf_{q_k \in \Pi_k} \left(2A\delta^2 \int\limits_{0}^{1} |v_2(y)(f_2(y) - q_k(y))|^p dy \right)^{1/p} \\
&= \left(2A\delta^2\right)^{1/p} E_k(f_2, [0,1])_{\mathbb{L}_p([0,1]), v_2},
\end{aligned}$$

where $v_2(y) = w(x) = w(1-2A\delta^2 y)$, $q_k(y) = p_k(x) = p_k(1-2A\delta^2 y)$.

Also,

$$\begin{aligned}
\overleftarrow{\Omega}_\phi^k(f, A, \delta)_{w,p} &= \sup_{0<h\leq 2A\delta^2} \|w \overleftarrow{\Delta}_h^k(f, \cdot, [1-2A\delta^2, 1])\|_{\mathbb{L}_p[-1,-1+2A\delta^2]} \\
&= \sup_{0<h\leq 2A\delta^2} \left(\int\limits_{1-2A\delta^2}^{1} |w(x) \overleftarrow{\Delta}_h^k(f, x, [1-2A\delta^2, 1])|^p dx \right)^{1/p} \\
&= \sup_{0<h\leq 2A\delta^2} \left(-2A\delta^2 \int\limits_{1}^{0} |v_2(y) \overrightarrow{\Delta}_{h/(2A\delta^2)}^k(f_2, y, [0,1])|^p dy \right)^{1/p} \\
&= \left(2A\delta^2\right)^{1/p} \sup_{0<h_2\leq 1} \|v_2 \overrightarrow{\Delta}_{h_2}^k(f_2)\|_{L_p[0,1]},
\end{aligned}$$

where $h_2 = h/(2A\delta^2)$.

For $0 \leq x_2 \leq 1$ and $0 \leq y_2 \leq \min\{2x_2, 1\}$, let $x_2 = \frac{1-x}{2A\delta^2}$ and $y_2 = \frac{1-y}{2A\delta^2}$. Then $x = 1 - 2A\delta^2 x_2 \in [1-2A\delta^2, 1]$ and $y = 1 - 2A\delta^2 y_2 \in [1-2A\delta^2, 1]$. Also, $2x - 1 = 1 - 4A\delta^2 x_2 \leq 1 - 2A\delta^2 y_2 = y$, so $v_2(x_2) = w(x) \geq mw(y) = mv_2(y_2)$. Then, v_2 is a $\mathcal{V}$ weight with the same constant m as w.

In both cases, we have to prove now the following

$$\sup_{0<h<1} \|v_i \overrightarrow{\Delta}_h^k(f_i)\|_{L_p[0,1]} \sim E_k(f_i, [0,1])_{v_i,p}, \qquad i = 1, 2,$$

where $f_i \in \mathbb{L}_{v_i,p}([0,1])$, $v_i \in \mathcal{V}$. The equivalence constants must depend only on k, p and v_i.

Remark 3.2. *Let* $\omega_k(f,\delta)_{v_i,p} = \sup_{0<h<\delta} \|v_i \overrightarrow{\Delta}_h^k(f)\|_{\mathbb{L}_p[0,1]}$ *be the classical weighted modulus of smoothness of order* k*. Then,* $\omega_k(f,1)_{v_i,p} = \sup_{0<h<1} \|v_i \overrightarrow{\Delta}_h^k(f_i)\|_{L_p[0,1]}$*, and so we may look on the equivalence above as on a type of Whitney's inequality.*

Remark 3.3. *For a weight* $w \in \mathcal{V}$*, consider* $v(x) := \sup_{0\le y\le x} w(y)$*,* $x \in [0,1]$*. Then,* $v : [0,1] \to [0,\infty]$ *is a non-decreasing function and* $w(x) \le v(x) \le m^{-1}w(x)$*,* $x \in [0,1]$*. So,* $v \in \mathcal{V}$ *and* $v \sim w$*.*

Everywhere below, let $v : [0,1] \mapsto [0,+\infty]$ be a non-decreasing $\mathcal{V}$ weight.

Therefore, Theorem 2.11 follows by the following lemma.

Lemma 3.4. *Let* $p > 0$*,* $k \in \mathbb{N}$*,* $v \in \mathcal{V}$ *is a non-decreasing weight and* $f \in \mathbb{L}_{v,p}[0,1]$*. Then, we have*

$$\omega_k(f,1)_{v,p} \le CE_k(f)_{v,p} \tag{3.1}$$

and

$$E_k(f)_{v,p} \le C\omega_k(f,1)_{v,p}, \tag{3.2}$$

where constants C *depend on* k *and* p *only.*

For $1 \le p < \infty$, Lemma 3.4 could be proved using the same method as in [5, Proposition 4.2]. We will show this method in Section 3.2. Unfortunately, this method does not work for $0 < p < 1$.

In the case $0 < p < 1$, we introduce a new modulus of smoothness $\tilde{\omega}_k(f,\delta)_{v,p}$ defined by

$$\tilde{\omega}_k(f,\delta)_{v,p} = \sup_{\substack{0<h_i<\delta,\\ 1\le i\le k}} \|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|_{\mathbb{L}_p([0,1]),v}, \tag{3.3}$$

where $\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,x,[0,1])$ is defined recursively by

$$\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,x,[0,1]) = \overrightarrow{\Delta}_{h_1}\left(\overrightarrow{\Delta}_{h_2\ldots h_k}(f,x,[0,1]),x,[0,1]\right). \tag{3.4}$$

Remark 3.5. *When $h_1 = h_2 = \cdots = h_k = h$ we deduce*

$$\omega_k(f,\delta)_{v,p} \le \tilde{\omega}_k(f,\delta)_{v,p}.$$

We will show equivalence of $\tilde{\omega}_k(f,\delta)_{v,p}$ and $\omega_k(f,\delta)_{v,p}$. In fact, we will prove the following generalization of Lemma 3.4.

Lemma 3.6. *Let $0 < p < 1$, $k \in \mathbb{N}$, $v \in \mathcal{V}$ is a non-decreasing weight and $f \in \mathbb{L}_{v,p}[0,1]$. Then, we have*

$$\omega_k(f,1)_{v,p}^p \le CE_k(f)_{v,p}^p, \tag{3.5}$$

$$E_k(f)_{v,p}^p \le C\tilde{\omega}_k(f,1)_{v,p}^p, \tag{3.6}$$

and

$$\tilde{\omega}_k(f,1)_{v,p}^p \le C\omega_k(f,1)_{v,p}^p, \tag{3.7}$$

where constants C depend on k only.

Inequality (3.5) is inequality (3.1) raised to power $1/p$. However, we prove (3.1) only for $1 \le p < \infty$. We will prove (3.5) in Section 3.3.

Inequality (3.6) will be proved in Section 3.3.

Inequality (3.7) follows from the following theorem which we will prove in Section 3.4.

Let $\overrightarrow{\Delta}_h$ be the unrestricted, i.e. defined for all $x \in \mathbb{R}$, difference operator defined by

$$\overrightarrow{\Delta}_h f(x) = f(x+h) - f(x)$$

and let T_t be the translation operator defined by

$$T_t f(x) = f(x+t),$$

where $x, h, t \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$.

Note that $\overrightarrow{\Delta}_h f(x) = \overrightarrow{\Delta}_h(f, x, [a, b])$ for $x \in [a, b-h]$, however, $\overrightarrow{\Delta}_h f(x)$ not necessarily vanishing for $x \notin [a, b-h]$.

Theorem 3.7. *Let $n \in \mathbb{N}$, $h_1, h_2, \ldots, h_n$ be positive numbers. Then there exist $M_n \in \mathbb{N}$, collections of non-negative numbers $\{h_i'\}_{i=1}^{M_n}$, $\{t_i\}_{i=1}^{M_n}$, and a collection of signs $\{s_i\}_{i=1}^{M_n}$, $s_i = \pm 1$ such that*

$$\overrightarrow{\Delta}_{h_1, \cdots, h_n} := \overrightarrow{\Delta}_{h_1} \cdots \overrightarrow{\Delta}_{h_n} = \sum_{i=1}^{M_n} s_i \overrightarrow{\Delta}_{h_i'}^n T_{t_i} \tag{3.8}$$

and

$$n h_i' + t_i \le h_1 + \cdots + h_n, \qquad 1 \le i \le M_n. \tag{3.9}$$

Remark 3.8. *If $x, x+h_1+\cdots+h_n \in [a, b]$, then $x+t_i, x+t_i+nh_i' \in [a, b]$, $1 \le i \le M_n$. So, condition (3.9) guaranties that the following analogue of identity (3.8) holds*

$$\overrightarrow{\Delta}_{h_1, \cdots, h_n}(f, x, [a, b]) = \sum_{i=1}^{M_n} s_i \overrightarrow{\Delta}_{h_i'}^n (f, x+t_i, [a, b]), \quad x \in [a, b-h_1-\cdots-h_n]. \tag{3.10}$$

The following lemma immediately follows from [14, lemma A.1].

Lemma 3.9. *Let $p > 0$, $k \in \mathbb{N}$ and $v \in \mathcal{V}$ be a non-decreasing weight. Let also P_k be the polynomial of near best approximation of $f \in \mathbb{L}_{p,v}$ on interval $I \subset [0, 1]$. Then it is a polynomial of near best approximation on any interval J, $I \subset J \subset [0, 1]$ i.e,*

$$\|f - P_k\|_{\mathbb{L}_p(J), v} \le c E_k(f, J)_{v,p},$$

where the constant c depends only on m, p and $|I|$.

Lemma 3.9 yields the following corollary.

Corollary 3.10. *Let $p > 0$, $k \in \mathbb{N}$ and $v \in \mathcal{V}$ be a non-decreasing weight. Then,*

$$E_k(f, [0, 1])_{v,p} \le c E_k(f, [0, 3/4])_{v,p} + c E_k(f, [1/4, 1])_{v,p},$$

where constant c depends only on v and p.

Proof. Let P_k be the polynomial of near best approximation on interval $I = [1/4, 3/4]$. Then by Lemma 3.9 P_k also is the polynomial of near best approximation on intervals $J_1 = [0, 3/4]$ and $J_2 = [1/4, 1]$. Then,

$$\begin{aligned}
E_k(f, [0,1])_{v,p}^p &\le \|f - P_k\|_{\mathbb{L}_p([0,1]),v}^p \\
&= \int_0^1 |v(x)(f(x) - P_k(x))|^p dx \\
&\le \int_0^{3/4} |v(x)(f(x) - P_k(x))|^p dx + \int_{1/4}^1 |v(x)(f(x) - P_k(x))|^p dx \\
&= \|f - P_k\|_{\mathbb{L}_p([0,3/4]),v}^p + \|f - P_k\|_{\mathbb{L}_p([1/4,1]),v}^p \\
&\le cE_k\left(f, [0, 3/4]\right)_{v,p}^p + cE_k\left(f, [1/4, 1]\right)_{v,p}^p,
\end{aligned}$$

and the proof is complete.

$\square$

3.2 Case $1 \le p < \infty$ in Lemma 3.4

We start with the proof of (3.1). That is, we show that if $1 \le p \le \infty$, $k \in \mathbb{N}$, $v \in \mathcal{V}$ is a non-decreasing weight and $f \in \mathbb{L}_{v,p}[0, 1]$, then

$$\omega_k(f, 1)_{v,p} \le CE_k(f)_{v,p}.$$

Proof of (3.1). Since $\overrightarrow{\Delta}_h^k(P_k) = 0$ for any polynomial P_k of degree $< k$, we can write

$$\begin{aligned}
|v(x)\overrightarrow{\Delta}_h^k(f, x, [0,1])| &= |v(x)||\overrightarrow{\Delta}_h^k(f - P_k, x, [0, 1])| \\
&= v(x)\left|\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}(f - P_k)(x + (k - i)h)\right|.
\end{aligned}$$

Now, using the fact that v is a non-decreasing weight function, we can estimate

$|v(x)\overrightarrow{\Delta}_h^k(f,x,[0,1])|$ as follows

$$\begin{aligned}|v(x)\overrightarrow{\Delta}_h^k(f,x,[0,1])| &\le \sum_{i=0}^{k} v(x+(k-i)h)\times\left|\binom{k}{i}(f-P_k)(x+(k-i)h)\right|\\ &= \sum_{i=0}^{k}\left|\binom{k}{i}\left[v(f-P_k)\right](x+(k-i)h)\right|.\end{aligned}$$

Note that we set $v(y)=f(y)=P_k(y)=0$ for $y\notin[0,1]$.

Then we get

$$\begin{aligned}\|v\overrightarrow{\Delta}_h^k(f,\cdot,[0,1])\|_{\mathbb{L}_p[0,1]} &\le \left(\int_0^1\left(\sum_{i=0}^{k}\left|\binom{k}{i}\left[v(f-P_k)\right](x+(k-i)h)\right|\right)^p dx\right)^{1/p}\\ &\le \sum_{i=0}^{k}\binom{k}{i}\left(\int_0^1\left(\left|\left[v(f-P_k)\right](x+(k-i)h)\right|\right)^p dx\right)^{1/p}\\ &\le \sum_{i=0}^{k}\binom{k}{i}\left(\int_0^1\left|v(x)(f(x)-P_k(x))\right|^p dx\right)^{1/p}\\ &= 2^k\|v(f-P_k)\|_{\mathbb{L}_p[0,1]}.\end{aligned}$$

Taking supremum over h and infimum over P_k we deduce (3.1).

□

We will now prove (3.2) for $1\le p<\infty$. We will adopt the idea from [5]. Note that in [5] modulus of smoothness ω_k is defined with $\overrightarrow{\Delta}_h^k f(x)$ instead of $\overrightarrow{\Delta}_h^k(f,x,[0,1])$, i.e., it has different behavior near the right end of the interval $[0,1]$. Therefore, we have to modify the proof from [5].

Proof of (3.2). By Corollary 3.10 we have

$$E_k(f,[0,1])_{v,p}\le cE_k(f,[0,3/4])_{v,p}+cE_k(f,[1/4,1])_{v,p}.$$

It is sufficient to show that

$$E_k(f,[0,3/4])_{v,p}\le C\omega_k(f,1)_{v,p} \tag{3.11}$$

and

$$E_k(f,[1/4,1])_{v,p} \le C\omega_k(f,1)_{v,p}. \tag{3.12}$$

To prove the converse inequality (3.11) for $1 \le p < \infty$ we define the Steklov mean function with $\tau = \frac{1}{2k}$

$$f_\tau(x) = (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} \left(\sum_{i=1}^k (-1)^{i-1} \binom{k}{i} f(x + i\tau(t_1 + \dots + t_k)) \right) dt_1 \dots dt_k.$$

Steklov function is k times continuously differentiable as k-times integrated continuous function. Note that

$$x < x + i\tau(t_1 + \dots + t_k) \le x + k\tau \left(\frac{1}{2k} + \dots + \frac{1}{2k} \right) = x + \frac{k\tau}{2},$$

so $x + i\tau(t_1 + \dots + t_k) \in [0,1]$ for $0 \le x \le 1 - \frac{k\tau}{2}$, and $f_\tau(x)$ is defined on $[0, \frac{3}{4}]$ for $\tau = \frac{1}{2k}$.

Compute now $f_\tau^{(k)}$ for $0 \le x \le 1 - \frac{k\tau}{2} = 1 - \frac{1}{4} = \frac{3}{4}$.

$$\begin{aligned}
f_\tau^{(k)}(x) &= (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} \left(\sum_{i=1}^k (-1)^{i-1} \binom{k}{i} f^{(k)}(x + i\tau(t_1 + \dots + t_k)) \right) dt_1 \dots dt_k \\
&= \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} f^{(k)}(x + i\tau(t_1 + \dots + t_k)) dt_1 \dots dt_k \\
&= \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} \frac{1}{i\tau} \overrightarrow{\Delta}_{\frac{i\tau}{2k}} f^{(k-1)}(x + i\tau(t_2 + \dots + t_k)) dt_2 \dots dt_k \\
&= \dots = \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} (2k)^k (i\tau)^{-k} \overrightarrow{\Delta}^k_{\frac{i\tau}{2k}} f(x).
\end{aligned}$$

Recall that $0 \le x \le \frac{3}{4}$ and so $x + k\frac{i\tau}{2k} \le 1$. Then $\overrightarrow{\Delta}^k_{\frac{i\tau}{2k}} f(x) = \overrightarrow{\Delta}^k_{\frac{i\tau}{2k}}(f, x, [0,1])$ and

$$f_\tau^{(k)}(x) = \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} (2k)^k (i\tau)^{-k} \overrightarrow{\Delta}^k_{\frac{i\tau}{2k}}(f, x, [0,1]). \tag{3.13}$$

Note that the k-times differentiable function f_τ can be estimated by its Taylor polynomial. We write

$$\begin{aligned} E_k(f,[0,3/4])_{v,p} &\le \|v(f-f_\tau)\|_{\mathbb{L}_p[0,\frac{3}{4}]} + \|v(f_\tau - T_k(f_\tau))\|_{\mathbb{L}_p[0,\frac{3}{4}]} \\ &= I_1 + I_2, \end{aligned}$$

where $T_k(f_\tau)$ is the Taylor polynomial of degree $k-1$ of function f_τ at the point $x_0 = \frac{3}{4}$.

Consider now I_1. Note that

$$(2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} dt_1 \dots dt_k = 1,$$

and so

$$\begin{aligned} I_1 &= \left\| v(x)(2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} \left(f(x) + \sum_{i=1}^{k} (-1)^i \binom{k}{i} f(x + i\tau(t_1 + \dots + t_k)) \right) \right\|_{\mathbb{L}_p[0,\frac{3}{4}]} \\ &= \left\| (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} v(x) \overrightarrow{\Delta}^k_{\tau(t_1+\dots+t_k)}(f,x,[0,1]) dt_1 \dots dt_k \right\|_{\mathbb{L}_p[0,\frac{3}{4}]} \\ &\le (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} \|v \overrightarrow{\Delta}^k_{\tau(t_1+\dots+t_k)}(f,\cdot,[0,1])\|_{\mathbb{L}_p[0,\frac{3}{4}]} dt_1 \dots dt_k \\ &\le (2k)^k \int_0^{\frac{1}{2k}} \dots \int_0^{\frac{1}{2k}} \sup_{0<h\le\frac{\tau}{2}} \|v \overrightarrow{\Delta}^k_h(f,\cdot,[0,1])\|_{\mathbb{L}_p[0,\frac{3}{4}]} dt_1 \dots dt_k \\ &\le \sup_{0<h\le 1} \|v \overrightarrow{\Delta}^k_h(f,\cdot,[0,1])\|_{\mathbb{L}_p[0,\frac{3}{4}]} \\ &\le \omega_k(f,1)_{v,p}. \end{aligned}$$

To estimate I_2, we use the remainder term of the Taylor formula. We obtain

$$\begin{aligned} v(x)|f_\tau(x) - T_k(f_\tau, x)| &\le \frac{v(x)}{(k-1)!} \int\limits_x^{\frac{3}{4}} (y-x)^{k-1} |f_\tau^{(k)}(y)| dy \\ &\le \frac{1}{(k-1)!} \int\limits_0^{\frac{3}{4}} (y-x)_+^{k-1} |f_\tau^{(k)}(y)| v(y) dy, \end{aligned}$$

where $x_+ = \begin{cases} x, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$

Using the above inequality we obtain

$$\begin{aligned} I_2 &= \left(\int\limits_0^{\frac{3}{4}} v(x)^p |f_\tau(x) - T_k(f_\tau, x)|^p dx \right)^{\frac{1}{p}} \\ &\le \frac{1}{(k-1)!} \left(\int\limits_0^{\frac{3}{4}} \left[\int\limits_0^{\frac{3}{4}} (y-x)_+^{k-1} |f_\tau^{(k)}(y)| v(y) dy \right]^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Recall that for measurable function $F : S_1 \times S_2 \to \mathbb{R}$ Minkowski's integral inequality is (see [22]):

$$\left(\int_{S_2} \left| \int_{S_1} F(x,y) dx \right|^p dy \right)^{1/p} \le \int_{S_1} \left(\int_{S_2} |F(x,y)|^p \, dy \right)^{1/p} dx.$$

Then, with $S_1 = S_2 = [0, 3/4]$ and $F(x,y) = (y-x)_+^{k-1}|f_\tau^{(k)}(y)|v(y)$ we obtain

$$\begin{aligned} I_2 &\le \frac{1}{(k-1)!} \int\limits_0^{\frac{3}{4}} |f_\tau^{(k)}(y)|v(y) \left(\int\limits_0^1 (y-x)_+^{(k-1)p} dx \right)^{\frac{1}{p}} dy \\ &= \frac{1}{(k-1)!} \int\limits_0^{\frac{3}{4}} |f_\tau^{(k)}(y)|v(y) \left(\int\limits_0^y (y-x)^{(k-1)p} dx \right)^{\frac{1}{p}} dy \\ &= \frac{1}{(k-1)!} \int\limits_0^{\frac{3}{4}} |f_\tau^{(k)}(y)|v(y) \left(\frac{1}{(k-1)p+1} y^{(k-1)p+1} \right)^{\frac{1}{p}} dy \\ &= \frac{1}{(k-1)!((k-1)p+1)^{\frac{1}{p}}} \int\limits_0^{\frac{3}{4}} |f_\tau^{(k)}(y)|v(y) y^{k-1+\frac{1}{p}} dy. \end{aligned}$$

In the last integral, we use Hölder's inequality for functions $|f_\tau^{(k)}(y)|v(y)$ and $y^{k-1+\frac{1}{p}}$ to obtain

$$\begin{aligned} I_2 &\le \frac{1}{(k-1)!((k-1)p+1)^{\frac{1}{p}}} \| |f_\tau^{(k)}(y)|v(y)\|_{\mathbb{L}_p([0,\frac{3}{4}])} \|y^{k-1+\frac{1}{p}}\|_{\mathbb{L}_q([0,\frac{3}{4}])} \\ &= C \left(\int\limits_0^{\frac{3}{4}} |f_\tau^{(k)}(y)v(y)|^p dy \right)^{\frac{1}{p}}, \end{aligned}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, and $C = \frac{1}{(k-1)!((k-1)p+1)^{\frac{1}{p}}} \|y^{k-1+\frac{1}{p}}\|_{\mathbb{L}_q([0,\frac{3}{4}])}$ depends on k and p only.

Then, taking into account (3.13), we obtain

$$I_2 \le C \sup_{1 \le i \le k} \|\overrightarrow{\Delta}^k_{\frac{i\tau}{2k}}(f, \cdot, [0,1])\|_{\mathbb{L}_p([0,1]),v} \le C\omega_k(f,1)_{v,p}.$$

Note that argument is valid for $v \equiv 1$, so $E_k(f, [0, \frac{3}{4}])_p \le C\omega_k(f,1)_p$.

It remains to prove inequality (3.12).

We have $mv(\frac{1}{2}) \le v(x) \le m^{-1}v(\frac{1}{2})$, for $\frac{1}{4} < x < 1$, and so $v(x) \sim v(\frac{1}{2})$ on $[\frac{1}{4}, 1]$. Then it is sufficient to show that $E_k(f, [\frac{1}{4}, 1])_p \le C\omega_k(f,1)_p$. This follows from the previous case with $v \equiv 1$ after change of variable $x \mapsto y = 1 - x$. $\square$

bjk

3.3 Proof of inequalities (3.5) and (3.6)

In this section, we will prove inequalities (3.5) and (3.6) of Lemma 3.6. As mentioned earlier, inequality (3.7) follows from theorem 3.7 which will be proved in Section 3.4.

Recall inequality (3.5):

$$\omega_k(f,1)_{v,p}^p \le C E_k(f)_{v,p}^p,$$

where $0 < p < 1$, $k \in \mathbb{N}$, $v \in \mathcal{V}$ is a non-decreasing weight and $f \in \mathbb{L}_{v,p}[0,1]$.

Proof. We prove inequality (3.5) using the same method as the one used to prove inequality (3.1) in Section 3.2. Recall that

$$|v(x)\overrightarrow{\Delta}_h^k(f,x,[0,1])| \le \sum_{i=0}^{k} \left| \binom{k}{i} [v(f-P_k)](x+(k-i)h) \right|,$$

where $v(y) = f(y) = P_k(y) = 0$ for $y \notin [0,1]$.

Then

$$\begin{aligned}
\|v\overrightarrow{\Delta}_h^k(f,\cdot,[0,1])\|_{\mathbb{L}_p[0,1]}^p &\le \int_0^1 \left(\sum_{i=0}^{k} \left| \binom{k}{i} [v(f-P_k)](x+(k-i)h) \right| \right)^p dx \\
&\le \sum_{i=0}^{k} \binom{k}{i}^p \int_0^1 \big(|[v(f-P_k)](x+(k-i)h)| \big)^p dx \\
&\le \sum_{i=0}^{k} \binom{k}{i} \int_0^1 \big| v(x)(f(x)-P_k(x)) \big|^p dx \\
&= 2^k \|v(f-P_k)\|_{\mathbb{L}_p[0,1]}^p,
\end{aligned}$$

Taking supremum over h and infimum over P_k finishes the proof. □

Now we are going to prove inequality (3.6), which is

$$E_k(f)_{v,p}^p \le C\tilde{\omega}_k(f,1)_{v,p}^p.$$

Whitney's inequality

$$E_k(f)_p \le C\omega_k(f,1)_p$$

was proved in [24]. This is inequality (3.6) with $v \equiv 1$. Recall that, according to Remark 3.5, $\omega_k(f,1) \le \tilde{\omega}_k(f,1)$. So inequality (3.6) holds for $v \sim 1$.

We will prove inequality (3.6) by induction on k.

The base case ($k = 1$) is similar to [7, Lemma 12.5.2].

Lemma 3.11. *Inequality* (3.6) *holds for* $k = 1$, *i.e.*

$$E_1(f,[0,1])^p_{v,p} \le C\tilde{\omega}_1(f,1)^p_{v,p}.$$

Proof. Note that

$$\tilde{\omega}_1(f,1)^p_{v,p} = \omega_1(f,1)^p_{v,p} = \sup_{0<h\le 1} \|v(\cdot)\overrightarrow{\Delta}_h(f,\cdot,[0,1])\|^p_{\mathbb{L}_p[0,1]}.$$

Then, we need to show that

$$\sup_{0<h\le 1} \|v(\cdot)\overrightarrow{\Delta}_h(f,\cdot,[0,1])\|^p_{\mathbb{L}_p[0,1]} \ge cE_1(f,[0,1])^p_{v,p},$$

or, equivalently

$$\sup_{0<h\le 1} \int_0^{1-h} (v(x)|f(x+h)-f(x)|)^p\,dx \ge c\inf_{a\in\mathbb{R}} \int_0^1 (v(x)|f(x)-a|)^p\,dx.$$

Using the fact that supremum is not less then average we obtain

$$\begin{aligned}\sup_{0<h\le 1} \int_0^{1-h} (v(x)|f(x+h)-f(x)|)^p\,dx \ge \\ \int_0^1 \int_0^{1-h} (v(x)|f(x+h)-f(x)|)^p\,dxdh.\end{aligned} \tag{3.14}$$

After change of variables $y = x+h$ in (3.14) we obtain

$$\begin{aligned}
\tilde{\omega}_1(f)^p_{v,p} &\geq \int_0^1 \int_0^y (v(x)|f(y)-f(x)|)^p \, dx dy \\
&\geq \int_{1/2}^1 \int_0^y (v(x)|f(y)-f(x)|)^p \, dx dy \\
&= \int_{1/2}^1 \int_0^{1/2} (v(x)|f(y)-f(x)|)^p \, dx dy + \int_{1/2}^1 \int_{1/2}^y (v(x)|f(y)-f(x)|)^p \, dx dy \\
&= I_1 + I_2.
\end{aligned} \tag{3.15}$$

We leave the first integral without changes. In the second integral, we change the order of integration and swap variables.

$$\begin{aligned}
I_2 &= \int_{1/2}^1 \int_{1/2}^y (v(x)|f(y)-f(x)|)^p \, dx dy \\
&= \int_{1/2}^1 \int_y^1 (v(y)|f(y)-f(x)|)^p \, dx dy \\
&\geq \int_{1/2}^1 \int_y^1 (mv(1/2)|f(y)-f(x)|)^p dx dy \\
&\geq \int_{1/2}^1 \int_y^1 \left(m^2 v(x)|f(y)-f(x)|\right)^p \, dx dy \\
&= I_2'.
\end{aligned}$$

Now, the second integral can be estimated by

$$\begin{aligned}
I_2 &= \frac{m^{2p}}{m^{2p}+1} I_2 + \frac{1}{m^{2p}+1} I_2 \\
&\geq \frac{m^{2p}}{m^{2p}+1} I_2 + \frac{1}{m^{2p}+1} I_2' \\
&= \frac{m^{2p}}{m^{2p}+1} \int_{1/2}^1 \left(\int_{1/2}^y + \int_y^1 \right) (v(x)|f(y)-f(x)|)^p \, dx dy \\
&= \frac{m^{2p}}{m^{2p}+1} \int_{1/2}^1 \int_{1/2}^1 (v(x)|f(y)-f(x)|)^p \, dx dy.
\end{aligned}$$

Taking into account (3.15), we get

$$\begin{aligned}
\tilde{\omega}_1(f,1)^p_{v,p} &\geq I_1 + I_2 \\
&\geq I_1 + c\int_{1/2}^1 \int_{1/2}^1 (v(x)|f(y)-f(x)|)^p \, dxdy \\
&= c\int_{1/2}^1 \left(\int_0^{1/2} + \int_{1/2}^1 \right) (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\geq c\int_{1/2}^1 \int_0^1 (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\geq c \inf_{1/2\leq y\leq 1} \int_0^1 (v(x)|f(y)-f(x)|)^p \, dx \\
&\geq cE_1(f,I)^p_{v,p},
\end{aligned}$$

and the proof is now complete. □

Note that the polynomial (constant) of 'good enough' approximation is equal to $f(y)$ for some $y \in (1/2, 1)$. A more general result is valid that shows that such y could be chosen from any interval $[a,b] \subset [0,1]$ with constant c depending also on $b-a$.

Lemma 3.12. *For any interval $[a,b] \subset [0,1]$, there exists $y \in [a,b]$ such that*

$$\tilde{\omega}_1(f,1)^p_{v,p} \geq c\|f(\cdot) - f(y)\|^p_{\mathbb{L}_p([0,1]),v}$$

where constant c depends on w, p and $b-a$.

Proof. Assume that $2a \geq b$ (if not we can replace interval $[a,b]$ by $[(a+b)/2, b]$).

Then, (3.15) yields

$$
\begin{aligned}
\tilde{\omega}_1(f,1)^p_{v,p} &\geq \int_0^1 \int_0^y (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\geq \int_a^b \int_0^y (v(x)|f(y)-f(x)|)^p \, dxdy \quad + \int_b^1 \int_0^y (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\geq \int_a^b \int_0^y (v(x)|f(y)-f(x)|)^p \, dxdy \quad + \int_b^1 \int_a^b (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\geq \int_a^b \int_0^a (v(x)|f(y)-f(x)|)^p \, dxdy \quad + \int_a^b \int_a^y (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \int_b^1 \int_a^b (v(x)|f(y)-f(x)|)^p \, dxdy \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Similarly to Lemma 3.11 we get

$$
I_2 \geq c \int_a^b \int_y^b (v(x)|f(y)-f(x)|)^p \, dxdy =: I_2',
$$

and

$$
I_3 \geq c \int_a^b \int_b^1 (v(x)|f(y)-f(x)|)^p \, dxdy =: I_3'.
$$

Then

$$
\begin{aligned}
\tilde{\omega}_1(f,1)^p_{v,p} &\geq I_1 + I_2 + I_3 \geq \frac{1}{2}\left(I_1 + I_2 + I_3 + cI_2' + cI_3'\right) \geq c(I_1 + I_3 + I_2' + I_3') \\
&= c\left(\int_a^b \int_0^a + \int_a^b \int_a^y + \int_a^b \int_y^b + \int_a^b \int_b^1\right) (v(x)|f(y)-f(x)|)^p \, dxdy \\
&= c \int_a^b \int_0^1 (v(x)|f(y)-f(x)|)^p \, dxdy \\
&\geq c(b-a) \int_0^1 (v(x)|f(y)-f(x)|)^p \, dx,
\end{aligned}
$$

for some $y \in [a,b]$. $\square$

Now we are going to prove the inductive step for inequality (3.6). Let $k > 1$ be fixed. Assume that

$$
E_{k-1}(f)^p_{v,p} \leq C \tilde{\omega}_{k-1}(f,1)^p_{v,p} \tag{3.16}
$$

holds for every function $f \in \mathbb{L}_{v,p}[0,1]$ with constant C depending on k only. We need to show that

$$E_k(f)^p_{v,p} \leq C\tilde{\omega}_k(f,1)^p_{v,p}.$$

Our strategy is to estimate $\tilde{\omega}_k(f,1)^p_{v,p}$ by

$$\|\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f,\cdot,[0,1]) - \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f,y,[0,1])\|^p_{\mathbb{L}_p([0,1]),v}$$

and use the inductive assumption (3.16). Note that y may depend on h_i. However, if $f = P_k$ is a polynomial of degree less then k on some interval $[a,b] \subset [0,1]$, then for sufficiently small h_i, $1 \leq i \leq k-1$ (for example, $h_i < \frac{b-a}{2k}$), and for any $y \in [a, \frac{a+b}{2}]$ we have $y + h_1 + \cdots + h_{k-1} \in [a,b]$. Then

$$\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f,y,[0,1]) = \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(P_k,y,[0,1]).$$

Let $P_k(y) = a_{k-1}y^{k-1} + \cdots + a_1 y + a_0$. Then

$$\begin{aligned}\overrightarrow{\Delta}_{h_{k-1}}(P_k,y,[0,1]) &= P_k(y+h_{k-1}) - P_k(y)\\ &= a_{k-1}((y+h_{k-1})^{k-1} - y^{k-1}) + \cdots + a_1((y+h_{k-1}) - y) =\\ &= a_{k-1}(k-1)h_{k-1}y^{k-2} + \ldots\end{aligned}$$

is a polynomial of degree less then $k-1$.

Similarly, $\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(P_k,y,[0,1]) = \overrightarrow{\Delta}_{h_1}\left(\overrightarrow{\Delta}_{h_2\ldots h_{k-1}}(P_k,y,[0,1]),x,[0,1]\right)$ is a polynomial of degree less then $k-(k-1) = 1$, i.e. $\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(P_k,y,[0,1])$ is a constant function. Then for any $y \in [a, \frac{a+b}{2}]$ and sufficiently small h_i, $1 \leq i \leq k-1$ we have

$$\begin{aligned}&\|\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f,\cdot,[0,1]) - \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f,y,[0,1])\|_{\mathbb{L}_p([0,1]),v}\\ &\qquad = \|\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f-P_k,\cdot,[0,1])\|_{\mathbb{L}_p([0,1]),v}. \quad (3.17)\end{aligned}$$

Let now Q_k be the polynomial of degree $< k$ of near best approximation of function f on the interval $[1/2, 2/3]$ with the weight v, i.e.,

$$\|f - Q_k\|_{\mathbb{L}_p(1/2,2/3),v} \leq cE_k(f,[1/2,2/3])_{v,p}.$$

Define

$$f_1(x) = f(x)\chi(0,1/2) + Q_k(x)\chi(1/2,1) = \begin{cases} f(x), & x \in (0,1/2); \\ Q_k(x), & x \in (1/2,1), \end{cases}$$

and

$$f_2(x) = Q_k(x)\chi(0,1/2) + f(x)\chi(1/2,1) = \begin{cases} Q_k(x), & x \in (0,1/2); \\ f(x), & x \in (1/2,1). \end{cases}$$

In order to prove inequality (3.6) we split it into the following chain of inequalities (and prove each if these inequalities separately):

$$\begin{aligned} \tilde{\omega}_k(f,1)^p_{v,p} &\geq c\left(\tilde{\omega}_k(f_1,1)^p_{v,p} + \tilde{\omega}_k(f_2,1)^p_{v,p}\right) && (3.18) \\ &\geq c\left(E_k(f_1,[0,1])_{v,p} + E_k(f_2,[0,1])_{v,p}\right) && (3.19) \\ &\geq cE_k(f,[0,1])_{v,p}. && (3.20) \end{aligned}$$

Recall that the modulus of smoothness $\tilde{\omega}_k(f,1)^p_{v,p}$ is defined by equation (3.3) as

$$\tilde{\omega}_k(f,1)^p_{v,p} = \sup_{\substack{0<h_i\leq 1,\\ 1\leq i\leq k}} \|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|_{\mathbb{L}_p([0,1]),v}.$$

Note that

$$\overrightarrow{\Delta}_{nh}(f,x,[0,1]) = \sum_{i=0}^{n-1} \overrightarrow{\Delta}_h(f,x+ih,[0,1]),$$

for $x, x+nh \in [0,1]$. Therefore

$$\overrightarrow{\Delta}_{nh_1,\cdots,nh_k}(f,x,[0,1]) = \sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} \overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,x+i_1h_1+\cdots+i_kh_k,[0,1]),$$

for $x, x+n(h_1+\cdots+h_k) \in [0,1]$. Then,

$$\sup_{\substack{0<h_i\leq 1,\\ 1\leq i\leq k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|^p_{\mathbb{L}_p([0,1])} \leq n^k \sup_{\substack{0<nh_i\leq 1,\\ 1\leq i\leq k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|^p_{\mathbb{L}_p([0,1])}. \tag{3.21}$$

So, it is sufficient to consider $h_i < \frac{1}{6k}$, $1 \leq i \leq k$.

Proof of inequality (3.18). Let $0 \le x \le 1-h_1-\cdots-h_k$. Consider $\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_1, x, [0,1])$. This is the sum of terms of the form $\pm f_1(x + h_{i_1} + \cdots + h_{i_j})$, $1 \le i_1 < \cdots < i_j \le k$. We write $y_S := x+h_{i_1}+\cdots+h_{i_j}$, where $S = \{i_1, \ldots, i_j\}$. Recall that $f_1(y_S) = f(y_S)$, for $0 \le y_S \le 1/2$, and $f_1(y_S) = Q_k(y_S)$, for $1/2 < y_S \le 1$. For $1/2 < y_S \le 1$ we can write $f_1(y_S) = f(y_S) - (f(y_S) - Q_k(y_S))$. Then,

$$\begin{aligned}\left|v(x)\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_1, x, [0,1])\right|^p &= v^p(x)\left|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f, x, [0,1]) + \sum_S \pm(Q_k - f)(y_S)\right|^p \\ &\le \left|v(x)\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f, x, [0,1])\right|^p + \sum_S |v(x)(Q_k - f)(y_S)|^p,\end{aligned}$$

where sums are taken over all subsets $S = \{i_1, \ldots, i_j\} \subset \{1, \ldots, n\}$, such that $y_S := x + h_{i_1} + \cdots + h_{i_j} > 1/2$. Now we used the fact that $v(x) \le v(y_S)$ since v is a non-decreasing weight to obtain

$$\left|v(x)\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_1, x, [0,1])\right|^p \le \left|v(x)\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f, x, [0,1])\right|^p + \sum_S |v(y_S)(Q_k - f)(y_S)|^p.$$

Then

$$\|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_1, \cdot, [0,1])\|^p_{\mathbb{L}_p([0,1]),v} \le \|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f, \cdot, [0,1])\|^p_{\mathbb{L}_p([0,1]),v} + 2^k\|f - Q_k\|^p_{\mathbb{L}_p([1/2,1]),v}. \tag{3.22}$$

By Lemma 3.9 with $I = [1/2, 2/3]$ we get that Q_k is the polynomial of near best approximation on interval $J = [1/2, 1]$, i.e.,

$$\|f - Q_k\|^p_{\mathbb{L}_p(1/2,2/3),v} \le cE_k(f, [1/2, 1])^p_{v,p}.$$

By taking supremum over $0 < h_i \le 1$, $1 \le i \le k$ we get

$$\tilde{\omega}_k(f_1, 1)^p_{v,p} \le \tilde{\omega}_k(f, 1)^p_{v,p} + c2^k E_k(f, [1/2, 1])^p_{v,p}. \tag{3.23}$$

On the interval $[1/2, 1]$ we have $v \sim v(1/2)$, and so

$$E_k(f, [1/2, 1])_{v,p} \sim v(1/2)E_k(f, [1/2, 1])_p \sim v(1/2) \sup_{0<h\le 1} \|\overrightarrow{\Delta}^k_h(f, \cdot, [1/2, 1])\|_{\mathbb{L}_p([1/2,1])}.$$

In particular,

$$E_k(f,[1/2,1])_{v,p} \le c \sup_{0<h\le 1} \|\overrightarrow{\Delta}_h^k(f,\cdot,[1/2,1])\|_{\mathbb{L}_p([1/2,1]),v}.$$

Using Remark 3.5, we can we get the following inequality

$$E_k(f,[1/2,1])_{v,p} \le c \sup_{\substack{0<h_i\le 1,\\ 1\le i\le k}} \|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[1/2,1])\|_{\mathbb{L}_p([1/2,1])}. \tag{3.24}$$

Combining (3.23) and (3.24) we obtain

$$\sup_{\substack{0<h_i\le 1,\\ 1\le i\le k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_1,\cdot,[0,1])\|_{\mathbb{L}_p([0,1])} \le c \sup_{\substack{0<h_i\le 1,\\ 1\le i\le k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|_{\mathbb{L}_p([0,1])}. \tag{3.25}$$

Let us prove now inequality

$$\sup_{\substack{0<h_i\le \frac{1}{6k},\\ 1\le i\le k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_2,\cdot,[0,1])\|_{\mathbb{L}_p([0,1])} \le c \sup_{\substack{0<h_i\le \frac{1}{6k},\\ 1\le i\le k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|_{\mathbb{L}_p([0,1])}. \tag{3.26}$$

Recall that we consider only $h_i < \frac{1}{6k}$, $1\le i\le k$. Thus, $x+h_1+\cdots+h_k < x+\frac{1}{6} \le \frac{1}{2}$, for $0\le x\le \frac{1}{3}$. Also,

$$\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_2,x,[0,1]) = \overrightarrow{\Delta}_{h_1,\cdots,h_k}(Q_k,x,[0,1]) = 0,$$

for $0\le x\le \frac{1}{3}$. Therefore,

$$\sup_{\substack{0<h_i\le \frac{1}{6k},\\ 1\le i\le k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[0,1])\|_{\mathbb{L}_p([0,1])} = \sup_{\substack{0<h_i\le \frac{1}{6k},\\ 1\le i\le k}} \|v\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[1/3,1])\|_{\mathbb{L}_p([1/3,1])}.$$

Similarly to inequality (3.22), we can prove the following

$$\|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_2,\cdot,[1/3,1])\|^p_{\mathbb{L}_p([1/3,1]),v} \le \|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[1/3,1])\|^p_{\mathbb{L}_p([1/3,1]),v} \tag{3.27}$$

$$+ 2^k\|f-Q_k\|^p_{\mathbb{L}_p([1/3,1/2]),v}. \tag{3.28}$$

Using Lemma 3.9 we get

$$\|f-Q_k\|^p_{\mathbb{L}_p([1/3,1/2]),v} \le \|f-Q_k\|^p_{\mathbb{L}_p(1/3,2/3),v} \le cE_k(f,[1/3,2/3])^p_{v,p}.$$

Again, $v(x) \sim v(1/2)$ on $[1/3, 2/3]$ and

$$\|f - Q_k\|^p_{\mathbb{L}_p([1/3,1/2]),v} \leq c \sup_{\substack{0<h_i\leq 1,\\ 1\leq i\leq k}} \|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f,\cdot,[1/3,1/2])\|_{\mathbb{L}_p([1/3,1/2])},$$

which deduces (3.26).

Adding (3.25) and (3.26) gives inequality (3.18). □

Proof of inequality (3.19). We need to prove inequalities

$$\tilde{\omega}_k(f_i, 1)^p_{v,p} \geq cE_k(f_i, [0,1])^p_{v,p}, \qquad i = 1, 2.$$

Recall that

$$\|\overrightarrow{\Delta}_{h_1,\cdots,h_k}(f_1,\cdot,[0,1])\|_{\mathbb{L}_p([0,1]),v} = \|\overrightarrow{\Delta}_{h_k}\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_1,\cdot,[0,1])\|_{\mathbb{L}_p([0,1]),v}.$$

Then, we apply Lemma 3.12 to the function $g_1(x) = \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_1, x, [0,1])$ and the interval $[1/2, 5/6]$:

$$\tilde{\omega}_1(g_1, 1)^p_{v,p} \geq c\|\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_1,\cdot,[0,1]) - \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_1,y,[0,1])\|^p_{\mathbb{L}_p([0,1]),v},$$

where $y \in [1/2, 5/6]$.

Recall that we can consider only $h_i < \frac{1}{6k}$, $1 \leq i \leq k$. Then $y, y+h_1+\cdots+h_{k-1} \in [1/2, 1]$, and so

$$\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_1,y,[0,1])\|_{\mathbb{L}_p([0,1]),v} = \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(Q_k,y,[0,1])\|_{\mathbb{L}_p([0,1]),v}.$$

Then, by (3.17) and the inductive hypotheses (3.16), we have

$$\begin{aligned}
\tilde{\omega}_k(f_1, 1/6)^p_{v,p} &\geq c\|\overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_1 - Q_k,\cdot,[0,1])\|^p_{\mathbb{L}_p([0,1]),v}\\
&\geq cE_{k-1}(f_1 - Q_k, [0,1])^p_{v,p}\\
&\geq c\|f_1 - Q_k - R_{k-1}\|^p_{v,p}\\
&\geq cE_k(f_1, [0,1])^p_{v,p},
\end{aligned}$$

where R_{k-1} is the polynomial of degree less than $k-1$ of near best approximation for $f_1 - Q_k$.

For f_2 we apply Lemma 3.12 to the function $g_2(x) = \overrightarrow{\Delta}_{h_1,\cdots,h_{k-1}}(f_2, x, [0,1])$ and the interval $[0, 1/3]$. Similarly, we get

$$\tilde{\omega}_k(f_2, 1)^p_{v,p} \geq cE_k(f_2, [0,1])^p_{v,p}.$$

$\square$

Proof of inequality (3.20). Since $f_1 = Q_k$ on $[1/2, 1]$, Q_k is a polynomial of the best approximation of f_1 on $[1/2, 1]$. So it is also a polynomial of near best approximation on $[0, 1]$. Similarly, Q_k is a polynomial of near best approximation of f_2 on $[0, 1]$. Then, by Lemma 3.9,

$$E_k(f_1, [0,1])^p_{v,p} + E_k(f_1, [0,1])^p_{v,p} \geq c\|f_1 - Q_k\|^p_{\mathbb{L}_p([0,1]),v} + c\|f_2 - Q_k\|^p_{\mathbb{L}_p([0,1]),v} =$$

$$c\|f - Q_k\|^p_{\mathbb{L}_p([0,1/2]),v} + c\|f - Q_k\|^p_{\mathbb{L}_p([1/2,1]),v} = c\|f - Q_k\|^p_{\mathbb{L}_p([0,1]),v} \geq cE_k(f, [0,1])^p_{v,p}.$$

$\square$

Combining inequalities (3.18), (3.19) and (3.20) finishes the proof of inequality (3.6).

3.4 Proof of Theorem 3.7

We will prove a more general result.

For $d \in \mathbb{N}$, $h, t \in \mathbb{R}^d$, the unrestricted difference operator $\overrightarrow{\Delta}^{(d)}_h$ and the translation operator $T^{(d)}_t$ may be defined as follows.

Let $x \in \mathbb{R}^d$ and $f : \mathbb{R}^d \mapsto \mathbb{R}$. Then

$$\overrightarrow{\Delta}^{(d)}_h f(x) = f(x+h) - f(x)$$

and

$$T_t f(x) := T^{(d)}_t f(x) := f(x+t).$$

We also define the d-dimensional forward n-th difference operator by

$$\overrightarrow{\Delta}_h^n := \overrightarrow{\Delta}_h^{(d),n} := \underbrace{\Delta_h^{(d)} \dots \overrightarrow{\Delta}_h^{(d)}}_{n \text{ times}}.$$

Clearly, $\overrightarrow{\Delta}_h^{(1)} = \overrightarrow{\Delta}_h$ and $T_t^{(1)} = T_t$.

Theorem 3.13. *Let $d, n \in \mathbb{N}$, $h_1, h_2, \dots, h_n \in \mathbb{R}^d$. Then, there exist $M_n \in \mathbb{N}$, collections of vectors $\{h_i'\}_{i=1}^{M_n}$, $\{t_i\}_{i=1}^{M_n}$, and a collection of signs $\{s_i\}_{i=1}^{M_n}$, $s_i = \pm 1$, such that*

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} := \overrightarrow{\Delta}_{h_1,\cdots,h_n}^{(d)} := \overrightarrow{\Delta}_{h_1}^{(d)} \cdots \overrightarrow{\Delta}_{h_n}^{(d)} = \sum_{i=1}^{M_n} s_i \overrightarrow{\Delta}_{h_i'}^{(d),n} T_{t_i}^{(d)}, \tag{3.29}$$

and all vectors t_i, $nh_i' + t_i$, $i = 1, 2, \dots, M_n$, belong to the convex hull V of all vectors of the form $h_S := \sum_{i \in S} h_i$, $S \subset \{1, 2, \dots, n\}$.

Remark 3.14. *For $S = \emptyset$, the sum in h_S is empty, and we define $h_\emptyset := 0$.*

Remark 3.15. *A simple construction was presented in* [3, Lemma 5.4.11]. *Namely, for $D \subset \{1, \dots, n\}$, let $\tilde{h}_D = -\sum_{i \in D} i^{-1} h_i$, $\tilde{t}_D = \sum_{i \in D} h_i$. Then*

$$\overrightarrow{\Delta}_{h_1,\dots,h_n} = \sum_{D \in \{1,\dots,n\}} (-1)^{|D|} \overrightarrow{\Delta}_{\tilde{h}_D}^n T_{\tilde{t}_D}.$$

However, in this construction, vectors $\tilde{t}_D$, $n\tilde{h}_D' + \tilde{t}_D$, $D \subset \{1, \dots, n\}$, do not necessarily belong to the convex hull V. Indeed, let $n \geq 2$ and $D = \{1\}$. Then

$$\tilde{t}_D + n\tilde{h}_D = h_1 - nh_1 = -(n-1)h_1.$$

If all components of vectors h_i, $1 \leq i \leq n$, are positive then vector $-(n-1)h_1$ is not in V.

We will need the following lemmas.

Lemma 3.16. *Let $n \in \mathbb{N}$, $h_1, h_2, \cdots, h_n$ be vectors in $\mathbb{R}^d$. Then*

$$\overrightarrow{\Delta}_{h_1,h_2,\cdots,h_n} f(x) = \sum_{S \subset \{1,2,\cdots,n\}} (-1)^{n-|S|} f(x+h_S), \tag{3.30}$$

where $h_S := \sum_{i \in S} h_i$ and $|S|$ is a number of elements in the set S.

Proof. We will prove this lemma by induction.

Base case $n = 1$:

$$\overrightarrow{\Delta}_{h_1} f(x) = f(x+h_1) - f(x) = f(x+h_{\{1\}}) - f(x+h_\emptyset) = \sum_{S \subset \{1\}} (-1)^{1-|S|} f(x+h_S).$$

Suppose that identity (3.30) holds for $n = k$, and prove it for $n = k+1$ as follows.

$$\begin{aligned}
\overrightarrow{\Delta}_{h_1,h_2,\cdots,h_{k+1}} f(x) &= \overrightarrow{\Delta}_{h_{k+1}} \overrightarrow{\Delta}_{h_1,h_2,\cdots,h_k} f(x) = \overrightarrow{\Delta}_{h_{k+1}} \sum_{S \subset \{1,2,\cdots,k\}} (-1)^{k-|S|} f(x+h_S) \\
&= \sum_{S \subset \{1,2,\cdots,k\}} (-1)^{k-|S|} f(x+h_{k+1}+h_S) - \sum_{S \subset \{1,2,\cdots,k\}} (-1)^{k-|S|} f(x+h_S) \\
&= \sum_{S \subset \{1,2,\cdots,k\}} (-1)^{k+1-|S\cup\{k+1\}|} f(x+h_{S\cup\{k+1\}}) + \sum_{S \subset \{1,2,\cdots,k\}} (-1)^{k+1-|S|} f(x+h_S) \\
&= \sum_{S \subset \{1,2,\cdots,k+1\}} (-1)^{k+1-|S|} f(x+h_S).
\end{aligned}$$

□

Lemma 3.17. *The convex hull V in the statement of Theorem 3.13 could be expressed as*

$$V = \{v \in \mathbb{R}^d \mid v = \sum_{i=1}^{n} \lambda_i h_i, 0 \le \lambda_i \le 1, 1 \le i \le n\}.$$

Proof. We defined V by

$$V := \{v \in \mathbb{R}^d \mid v = \sum_S \lambda_S h_S, \sum_S \lambda_S = 1, \lambda_S \ge 0, S \subset \{1,\ldots,n\}\}.$$

Let

$$V_1 := \{v \in \mathbb{R}^d \mid v = \sum_{i=1}^{n} \lambda_i h_i, 0 \le \lambda_i \le 1, 1 \le i \le n\}.$$

We need to show $V = V_1$. We will do it in two steps.

Step 1: $V \subset V_1$**.**

Consider $v = \sum\limits_S \lambda_S h_S \in V$.

$$v = \sum_S \lambda_S \sum_{i \in S} h_i = \sum_{i=1}^n \sum_{S \ni i} \lambda_S h_i =: \sum_{i=1}^n \lambda_i h_i.$$

Then $0 \leq \lambda_i = \sum_{S \ni i} \lambda_S \leq \sum_S \lambda_S = 1$. So $V \subset V_1$.

Step 2: $V_1 \subset V$**.**

Consider $v = \sum\limits_{i=1}^n \lambda_i h_i \in V_1$.

With out loss of generality assume $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1$. Then

$$\begin{aligned} v = \sum_{i=1}^n \lambda_i h_i =& \lambda_1(h_1 + h_2 + \cdots + h_n) + (\lambda_2 - \lambda_1)(h_2 + h_3 + \cdots + h_n) + \\ & \cdots + (\lambda_{n-1} - \lambda_{n-2})(h_{n-1} + h_n) + (\lambda_n - \lambda_{n-1})h_n =: \sum_S \lambda_S h_S, \end{aligned}$$

where $\lambda_{\{1,\ldots,n\}} = \lambda_1$, $\lambda_{\{2,\ldots,n\}} = \lambda_2 - \lambda_1, \ldots$, $\lambda_{\{n\}} = \lambda_n - \lambda_{n-1}$, $\lambda_\emptyset = 1 - \lambda_n$. For all other S, we have $\lambda_S = 0$. Since for every S, $0 \leq \lambda_S \leq 1$ and $\sum_S \lambda_S = 1$, $V_1 \subset V$.

Since $V \subset V_1$ and $V_1 \subset V$, we have $V = V_1$. $\square$

Proof of Theorem 3.13. First, we will prove theorem for $n = 1, 2$ and then we consider odd and even n separately.

If $n = 1$, we set $M_1 = 1$, $h_1' = h_1$, $t_1 = 0$ and $s_1 = 1$, and so (3.29) clearly holds.

The idea of the proof for $n \geq 2$ is to consider subsets $S \subset \{1, \ldots, n\}$ with $n \notin S$ (there are 2^{n-1} such subsets) and $S' \subset \{1, \ldots, n\}$ with $n \in S'$ (there are 2^{n-1} such subsets). Then, for each $S \not\ni n$ we create a pair $S' := S'(S) \ni n$ and rewrite the right hand side of identity (3.30) in the form

$$\sum_{S \subset \{1,2,\cdots,n\}} (-1)^{n-|S|} f(x + h_S) = \sum_{S \subset \{1,2,\cdots,n-1\}} (-1)^{n-|S|} (f(x + h_S) - f(x + h_{S'}).$$

For an appropriate pairing S, S' we will have

$$\overrightarrow{\Delta}_{h_1,h_2,\cdots,h_n} f(x) = \sum_{S \subset \{1,2,\cdots,n-1\}} (-1)^{n-|S|} \overrightarrow{\Delta}^n_{\frac{h_{S'} - h_S}{n}} T_{h_S} f(x). \tag{3.31}$$

So, our goal is to find such pairings of S and S'.

Let $n = 2$. Then we pair $\emptyset$ with $\{1,2\}$, and $\{1\}$ with $\{2\}$.

Remark 3.18. *Clearly,* $2 \notin \emptyset$, $2 \notin \{1\}$ *and* $2 \in \{1,2\}$, $2 \in \{2\}$.

Now, we have

$$\begin{aligned}\overrightarrow{\Delta}_{h_1,h_2} f(x) &= f(x) - f(x+h_1) - f(x+h_2) + f(x+h_1+h_2) \\ &= f(x) - 2f(x+\frac{h_1+h_2}{2}) + f(x+h_1+h_2) \\ &- f(x+h_1) + 2f(x+\frac{h_1+h_2}{2}) - f(x+h_2) \\ &= \overrightarrow{\Delta}^2_{\frac{h_1+h_2}{2}} f(x) - \overrightarrow{\Delta}^2_{\frac{h_1-h_2}{2}} T_{h_2} f(x).\end{aligned} \tag{3.32}$$

With $M_2 = 2$, $h_1' = \frac{h_1+h_2}{2}$, $h_2' = \frac{h_1-h_2}{2}$, $t_1 = 0$, $t_2 = h_2$, $s_1 = 1$, $s_2 = -1$, identity (3.32) becomes (3.29).

We now check the second condition of the theorem:

$$t_1 = 0 = h_\emptyset \in V,$$

$$t_2 = h_2 = h_{\{2\}} \in V,$$

$$2h_1' + t_1 = h_1 + h_2 = h_{\{1,2\}} \in V,$$

$$2h_2' + t_2 = h_1 = h_{\{1\}} \in V.$$

Suppose now that $n > 2$.

We were unable to find a pairing that satisfies (3.31) in the general case. Hence, we first complete the proof with an additional assumption (Part I) and then (Part II) show how this assumption can be removed.

Part I: proof with an additional assumption

We assume that additional condition

$$(n-i)h_i = ih_{n-i}, \qquad i = 1, 2, \ldots, n-1, \tag{3.33}$$

holds.

For each set $S \subset \{1, \dots, n-1\}$, we define the corresponding set $S' := S'(S) \subset \{1, \dots, n-1, n\}$ as follows:

$$S' := S'(S) := \{1, \dots, n\} \setminus \{n - i \mid i \in S\}. \tag{3.34}$$

Note that for a given S' we can recover S such that $S' = S'(S)$ by the following rule:

$$S = \{1, \dots, n-1\} \setminus \{n - i \mid i \in S'\}.$$

Recall that $h_{\mathcal{S}} := \sum\limits_{i \in \mathcal{S}} h_i$, $\mathcal{S} \subset \{1, \dots, n\}$, and, in particular, for $\mathcal{S} = \emptyset$, $h_{\mathcal{S}} = 0$. Also, note that

$$h_{\mathcal{S}'} = \sum_{i \in \mathcal{S}'} h_i = \sum_{\substack{n-i \notin \mathcal{S} \\ 1 \le i \le n-1}} h_i + h_n = h_n + \sum_{\substack{i \notin \mathcal{S} \\ 1 \le i \le n-1}} h_{n-i}. \tag{3.35}$$

For $S \subset \{1, \dots, n-1\} \setminus \{i\}$, denote $S_i := S \cup \{i\}$ and note that

$$\begin{aligned} S'_i := (S_i)' &= \{1, \dots, n\} \setminus \{n - j \mid j \in S \cup \{i\}\} \\ &= \{1, \dots, n\} \setminus (\{n - j \mid j \in S\} \cup \{n - i\}) \\ &= (\{1, \dots, n\} \setminus \{n - j \mid j \in S\}) \setminus \{n - i\} \\ &= S' \setminus \{n - i\} \end{aligned}$$

and

$$h_{S'_i} = h_{S'} - h_{n-i}.$$

Consider the sum

$$\Sigma := \Sigma(x, f, h_1, \dots, h_n) := \sum_{S \subset \{1, \dots, n-1\}} (-1)^{|S|} \overrightarrow{\Delta}^n_{\frac{h_{S'} - h_S}{n}} T_{h_S} f(x), \tag{3.36}$$

where S' is defined in (3.34).

Each term in the sum (3.36) will be dealt with separately. Since

$$\begin{aligned} (-1)^{|S|} \overrightarrow{\Delta}^n_{\frac{h_{S'} - h_S}{n}} T_{h_S} f(x) &= \sum_{i=0}^{n} (-1)^{|S|+n-i} \binom{n}{i} f(x + h_S + i \frac{h_{S'} - h_S}{n}) \\ &= \sum_{i=0}^{n} (-1)^{|S|+n-i} \binom{n}{i} f(x + \frac{i h_{S'}}{n} + \frac{(n-i) h_S}{n}), \end{aligned}$$

then

$$\Sigma = \sum_{i=0}^{n} \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|+n-i} \binom{n}{i} f(x + \frac{ih_{S'}}{n} + \frac{(n-i)h_S}{n}) = \sum_{i=0}^{n} \Sigma_i,$$

where

$$\Sigma_i := \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|+n-i} \binom{n}{i} f(x + \frac{ih_{S'}}{n} + \frac{(n-i)h_S}{n}), \qquad i = 0,\ldots,n.$$

Hence, for $i \neq 0, n$, using (3.33) we have

$$\begin{aligned}
\frac{\Sigma_i}{\binom{n}{i}} &= \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|+n-i} f(x + \frac{ih_{S'}}{n} + \frac{(n-i)h_S}{n}) \\
&= \left(\sum_{\substack{S\subset\{1,\ldots,n-1\} \\ S\not\ni i}} + \sum_{\substack{S\subset\{1,\ldots,n-1\} \\ S\ni i}} \right) (-1)^{|S|+n-i} f(x + \frac{ih_{S'} + (n-i)h_S}{n}) \\
&= \sum_{\substack{S\subset\{1,\ldots,n-1\} \\ S\not\ni i}} (-1)^{|S|+n-i} \Bigg(f(x + \frac{ih_{S'} + (n-i)h_S}{n}) - \\
&\qquad\qquad f(x + \frac{ih_{(S\cup\{i\})'} + (n-i)h_{S\cup\{i\}}}{n}) \Bigg) \\
&= \sum_{\substack{S\subset\{1,\ldots,n-1\} \\ S\not\ni i}} (-1)^{|S|+n-i} \Bigg(f(x + \frac{ih_{S'} + (n-i)h_S}{n}) - \\
&\qquad\qquad f(x + \frac{i(h_{S'} - h_{n-i}) + (n-i)(h_S + h_i)}{n}) \Bigg) \\
&= \sum_{\substack{S\subset\{1,\ldots,n-1\} \\ S\not\ni i}} (-1)^{|S|+n-i} \Bigg(f(x + \frac{ih_{S'} + (n-i)h_S}{n}) - \\
&\qquad\qquad f(x + \frac{ih_{S'} + (n-i)h_S + (n-i)h_i - ih_{n-i}}{n}) \Bigg) \\
&= \sum_{\substack{S\subset\{1,\ldots,n-1\} \\ S\not\ni i}} (-1)^{|S|+n-i} \Bigg(f(x + \frac{ih_{S'} + (n-i)h_S}{n}) - \\
&\qquad\qquad f(x + \frac{ih_{S'} + (n-i)h_S}{n}) \Bigg) = 0.
\end{aligned}$$

Since $\Sigma_i = 0$ for $i \neq 0, n$, we conclude that

$$\Sigma = \Sigma_0 + \Sigma_n.$$

Now,

$$\Sigma_0 = \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|+n} f(x+h_S)$$

and

$$\Sigma_n = \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|} f(x+h_{S'}),$$

and so

$$\begin{aligned}\Sigma = \Sigma_0+\Sigma_n &= \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|+n} f(x+h_S) + \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|} f(x+h_{S'})\\ &= \sum_{S\subset\{1,\ldots,n-1\}} \big((-1)^{n-|S|} f(x+h_S) + (-1)^{|S|} f(x+h_{S'})\big).\end{aligned}$$

Recall that $S \mapsto S'$ is a bijection. So, if S runs through all subsets of $\{1,\ldots,n-1\}$, then S' runs through all subsets of $\{1,\ldots,n\}$ that contain n. Note also that (3.34) implies $|S|+|S'| = n$. Then,

$$\begin{aligned}\Sigma &= \sum_{S\subset\{1,\ldots,n-1\}} \big((-1)^{n-|S|} f(x+h_S) + (-1)^{n-|S'|} f(x+h_{S'})\big)\\ &= \sum_{S\subset\{1,\ldots,n\}} (-1)^{n-|S|} f(x+h_S),\end{aligned}$$

and, by Lemma 3.16,

$$\Sigma = \overrightarrow{\Delta}_{h_1,\cdots,h_n} f(x).$$

The identity $\Sigma = \overrightarrow{\Delta}_{h_1,\cdots,h_n} f(x)$ can be rewritten as

$$\begin{aligned}\overrightarrow{\Delta}_{h_1,\cdots,h_n} &= \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|} \overrightarrow{\Delta}^n_{\frac{h_{S'}-h_S}{n}} T_{h_S}\\ &= \sum_{S\subset\{1,\ldots,n-1\}} (-1)^{|S|} \overrightarrow{\Delta}^n_{h'_S} T_{t_S},\end{aligned} \tag{3.37}$$

where, by (3.35),

$$h'_S := \frac{1}{n}\left(h_n + \sum_{\substack{i\notin S\\ 1\le i\le n-1}} h_{n-i} - \sum_{i\in S} h_i\right)$$

and

$$t_S := h_S = \sum_{i\in S} h_i.$$

Now we will check that vectors t_S and h'_S in identity (3.37) satisfy the second condition of the theorem:

$$t_S = h_S \in V,$$

$$nh'_S + t_S = h_n + \sum_{\substack{i\notin\mathcal{S}\\ 1\le i\le n-1}} h_{n-i} - \sum_{i\in S} h_i + \sum_{i\in S} h_i = h_n + \sum_{\substack{i\notin\mathcal{S}\\ 1\le i\le n-1}} h_{n-i} = h_{S'} \in V.$$

Hence, if condition (3.33) is satisfied then our proof is complete.

Part II: proof in the general case

In the general case, our goal is to rewrite $\overrightarrow{\Delta}_{h_1,h_2,\cdots,h_n}$ in the following form.

$$\overrightarrow{\Delta}_{h_1,h_2,\cdots,h_n} = \sum_{i=1}^{m} \overrightarrow{\Delta}_{h_1^{(i)},h_2^{(i)},\cdots,h_n^{(i)}},$$

where vectors $h_1^{(i)}, h_2^{(i)}, \cdots, h_n^{(i)}$ satisfy condition (3.33) for all $i = 1, \ldots, m$. Then we rewrite each $\overrightarrow{\Delta}_{h_1^{(i)},h_2^{(i)},\cdots,h_n^{(i)}}$ using (3.37) to find the needed identity of the form (3.29).

We consider the cases for even and odd n separately.

Case 1: n is odd.

Suppose that $n \ge 3$ is odd, i.e., $n = 2k + 1$.

All difference operators in $\overrightarrow{\Delta}_{h_1} \cdots \overrightarrow{\Delta}_{h_n}$ commute, so we can write them in any order. Group $\overrightarrow{\Delta}_{h_i}$ and $\overrightarrow{\Delta}_{h_{n-i}}$, $1 \le i \le k$ together. Note that we left $\overrightarrow{\Delta}_{h_n}$ unpaired.

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \overrightarrow{\Delta}_{h_n}\left(\overrightarrow{\Delta}_{h_1}\overrightarrow{\Delta}_{h_{n-1}}\right)\left(\overrightarrow{\Delta}_{h_2}\overrightarrow{\Delta}_{h_{n-2}}\right)\cdots\left(\overrightarrow{\Delta}_{h_k}\overrightarrow{\Delta}_{h_{n-k}}\right). \tag{3.38}$$

For each pair $\overrightarrow{\Delta}_{h_i}\overrightarrow{\Delta}_{h_{n-i}}$, $1 \le i \le k$ we use (3.32) to get

$$\overrightarrow{\Delta}_{h_i,h_{n-i}} = \overrightarrow{\Delta}^2_{\frac{h_i+h_{n-i}}{2}} - \overrightarrow{\Delta}^2_{\frac{h_i-h_{n-i}}{2}} T_{h_{n-i}}. \tag{3.39}$$

Then we combine (3.39) and (3.38) to obtain

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \overrightarrow{\Delta}_{h_n}\left(\overrightarrow{\Delta}^2_{\frac{h_1+h_{n-1}}{2}} - \overrightarrow{\Delta}^2_{\frac{h_1-h_{n-1}}{2}}T_{h_{n-1}}\right)\cdots\left(\overrightarrow{\Delta}^2_{\frac{h_k+h_{n-k}}{2}} - \overrightarrow{\Delta}^2_{\frac{h_k-h_{n-k}}{2}}T_{h_{n-k}}\right). \tag{3.40}$$

We expand the product in (3.40). Let

$$e := \pm\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}^2_{\frac{h_i\pm h_{n-i}}{2}}T_{\frac{h_{n-i}\mp h_{n-i}}{2}}\right)$$

be an element in the expanded product, and let $A := A_e$ be the set of indexes j, for which the sign $'-'$ was taken in the j-th bracket. Then

$$\begin{aligned} e &= (-1)^{|A|}\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}^2_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2}}T_{\delta_{i,A}h_{n-i}}\right) \\ &= (-1)^{|A|}\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}^2_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2}}\right)T_{\sum_{j\in A}h_{n-j}}, \end{aligned}$$

where

$$\delta_{i,A} := \begin{cases} 1, & \text{if } i\in A, \\ 0, & \text{if } i\notin A. \end{cases}$$

Taking the sum over all sets A we obtain

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \sum_{A\subset\{1,\ldots,k\}}(-1)^{|A|}\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}^2_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2}}\right)T_{\sum_{j\in A}h_{n-j}}. \tag{3.41}$$

Now, we use the fact that

$$f(x+mh)-f(x) = (f(x+mh)-f(x+(m-1)h))+\cdots+(f(x+h)-f(x)),$$

which implies

$$\overrightarrow{\Delta}_h = \sum_{i=0}^{m-1}\overrightarrow{\Delta}_{\frac{h}{m}}T_{\frac{ih}{m}}.$$

Hence, for any $n\geq 2$ and $1\leq i\leq n-1$, we have

$$\overrightarrow{\Delta}^2_h = \overrightarrow{\Delta}_h\overrightarrow{\Delta}_h = \left(\sum_{j=0}^{n-i-1}\overrightarrow{\Delta}_{\frac{h}{n-i}}T_{\frac{jh}{n-i}}\right)\left(\sum_{l=0}^{i-1}\overrightarrow{\Delta}_{\frac{h}{i}}T_{\frac{lh}{i}}\right). \tag{3.42}$$

Using (3.41) and (3.42) we obtain:

$$\begin{aligned}
\overrightarrow{\Delta}_{h_1,\cdots,h_n} &= \sum_{A\subset\{1,\ldots,k\}} (-1)^{|A|}\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\sum_{j=0}^{n-i-1}\overrightarrow{\Delta}_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2(n-i)}}T_{j\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2(n-i)}}\right)\times \\
&\quad\times\left(\sum_{l=0}^{i-1}\overrightarrow{\Delta}_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2i}}T_{l\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2i}}\right)T_{\sum_{j\in A}h_{n-j}} \\
&= \sum_{A\subset\{1,\ldots,k\}} (-1)^{|A|}\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\sum_{j=0}^{n-i-1}\sum_{l=0}^{i-1}\left(\overrightarrow{\Delta}_{\frac{h_{i,A}}{n-i}}\overrightarrow{\Delta}_{\frac{h_{i,A}}{i}}T_{j\frac{h_{i,A}}{n-i}}T_{l\frac{h_{i,A}}{i}}\right)T_{\sum_{j\in A}h_{n-j}} \\
&= \sum_{A\subset\{1,\ldots,k\}} (-1)^{|A|}\overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}_{\frac{h_{i,A}}{n-i},\frac{h_{i,A}}{i}}\sum_{j=0}^{n-i-1}\sum_{l=0}^{i-1}\left(T_{j\frac{h_{i,A}}{n-i}+l\frac{h_{i,A}}{i}}\right)\right)T_{\sum_{j\in A}h_{n-j}} \\
&= \sum_{A\subset\{1,\ldots,k\}} (-1)^{|A|}D_AT_A,
\end{aligned} \tag{3.43}$$

where

$$h_{i,A} := \frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2} = \begin{cases} \frac{h_i-h_{n-i}}{2}, & \text{if } i\in A; \\ \frac{h_i+h_{n-i}}{2}, & \text{if } i\notin A, \end{cases}$$

$$D_A := \overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}_{\frac{h_{i,A}}{n-i},\frac{h_{i,A}}{i}}\right) \tag{3.44}$$

and

$$T_A := \prod_{i=1}^{k}\sum_{j=0}^{n-i-1}\sum_{l=0}^{i-1}\left(T_{j\frac{h_{i,A}}{n-i}+l\frac{h_{i,A}}{i}}\right)T_{\sum_{j\in A}h_{n-j}}. \tag{3.45}$$

We start with (3.44).

$$\begin{aligned}
D_A &= \overrightarrow{\Delta}_{h_n}\prod_{i=1}^{k}\left(\overrightarrow{\Delta}_{\frac{h_{i,A}}{n-i},\frac{h_{i,A}}{i}}\right) \\
&= \overrightarrow{\Delta}_{\frac{h_{1,A}}{n-1}}\overrightarrow{\Delta}_{\frac{h_{2,A}}{n-2}}\cdots\overrightarrow{\Delta}_{\frac{h_{k,A}}{n-k}}\overrightarrow{\Delta}_{\frac{h_{k,A}}{k}}\cdots\overrightarrow{\Delta}_{\frac{h_{2,A}}{2}}\overrightarrow{\Delta}_{\frac{h_{1,A}}{1}}\overrightarrow{\Delta}_{h_n} \\
&= \overrightarrow{\Delta}_{\tilde{h}_1\tilde{h}_2\ldots\tilde{h}_{n-1}h_n},
\end{aligned} \tag{3.46}$$

where

$$\tilde{h}_i := \begin{cases} \frac{h_{i,A}}{n-i}, & \text{if } 1\le i\le k; \\ \frac{h_{n-i,A}}{n-i}, & \text{if } k+1\le i\le n-1. \end{cases}$$

Then, $(n-i)\tilde{h}_i = i\tilde{h}_{n-i} = h_{i,A}$, $1 \le i \le k$, and so $\tilde{h}_1, \ldots, \tilde{h}_{n-1}, h_n$ satisfy condition (3.33). We can now rewrite (3.46) using (3.37):

$$D_A = \sum_{S \subset \{1,\ldots,n-1\}} (-1)^{|S|} \overrightarrow{\Delta}^n_{h'_{A,S}} T_{t_{A,S}}, \tag{3.47}$$

where

$$\begin{aligned}
h'_{A,S} &:= \frac{1}{n}\left(h_n + \sum_{\substack{i \notin S \\ 1 \le i \le n-1}} \tilde{h}_{n-i} - \sum_{i \in S} \tilde{h}_i\right) \\
&= \frac{1}{n}\left(h_n + \sum_{i=1}^{n-1}(1-\delta_{i,S})\tilde{h}_{n-i} - \sum_{i=1}^{n-1} \delta_{i,S}\tilde{h}_i\right) \\
&= \frac{1}{n}\left(h_n + \sum_{i=1}^{k}\left((1-\delta_{i,S})\frac{h_{i,A}}{i} - \delta_{i,S}\frac{h_{i,A}}{n-i}\right)\right. \\
&\qquad \left. + \sum_{i=k+1}^{n-1}\left((1-\delta_{i,S})\frac{h_{n-i,A}}{i} - \delta_{i,S}\frac{h_{n-i,A}}{n-i}\right)\right) \\
&= \frac{1}{n}\left(h_n + \sum_{i=1}^{k}(1-\delta_{i,S}-\delta_{n-i,S})\frac{h_{i,A}}{i} + \sum_{i=1}^{k}(1-\delta_{i,S}-\delta_{n-i,S})\frac{h_{i,A}}{n-i}\right) \\
&= \frac{h_n}{n} - \sum_{i=1}^{k}(\delta_{i,S}+\delta_{n-i,S}-1)\frac{h_{i,A}}{i(n-i)}
\end{aligned}$$

and

$$t_{A,S} := \sum_{i \in S} \tilde{h}_i = \sum_{i=1}^{n-1} \delta_{i,S}\tilde{h}_i = \sum_{i=1}^{k}\left(\delta_{i,S}\frac{h_{i,A}}{n-i} + \delta_{n-i,S}\frac{h_{i,A}}{i}\right).$$

Now, consider the translation part (3.45) and expand the product.

For the term T_τ in the expanded product and each i, $1 \le i \le k$ we pick indices $j := j_i := j_i(\tau)$ and $l := l_i := l_i(\tau)$. Let $J := J(\tau) := \{j_1, j_2, \ldots, j_k\}$ be the set of picked indices j and $L := L(\tau) := \{l_1, l_2, \ldots, l_k\}$ be the set of picked indices l. Note that $0 \le j_i \le n-i-1$ and $0 \le l_i \le i-1$, $1 \le i \le k$. Then,

$$T_\tau = \prod_{i=1}^{k} T_{j_i \frac{h_{i,A}}{n-i} + l_i \frac{h_{i,A}}{i}} T_{\sum_{j \in A} h_{n-j}}$$

and so

$$\tau = \sum_{i=1}^{k}\left(j_i\frac{h_{i,A}}{n-i} + l_i\frac{h_{i,A}}{i}\right) + \sum_{j \in A} h_{n-j} =: t_{A,J,L}.$$

Taking the sum over all J and L we obtain

$$T_A = \sum_J \sum_L T_{t_{A,J,L}}. \tag{3.48}$$

Here and everywhere below for the odd case, sum $\sum\limits_J \sum\limits_L$ is taken over all sets $J = \{j_1, j_2, \dots, j_k\},\ 0 \le j_i \le n-i-1,\ 1 \le i \le k$ and $L = \{l_1, l_2, \dots, l_k\}$ $0 \le l_i \le i-1,\ 1 \le i \le k$.

Now, (3.47) and (3.48) yield

$$\begin{aligned}
\overrightarrow{\Delta}_{h_1,\cdots,h_n} &= \sum_{A\subset\{1,\dots,k\}} (-1)^{|A|} D_A T_A \\
&= \sum_{A\subset\{1,\dots,k\}} (-1)^{|A|} \sum_{S\subset\{1,\dots,n-1\}} (-1)^{|S|} \sum_J \sum_L \overrightarrow{\Delta}^n_{h'_{A,S}} T_{t_{A,S}} T_{t_{A,J,L}} \\
&= \sum_{A\subset\{1,\dots,k\}} \sum_{S\subset\{1,\dots,n-1\}} \sum_J \sum_L (-1)^{|A|+|S|} \overrightarrow{\Delta}^n_{h'_{A,S}} T_{t_{A,S,J,L}},
\end{aligned} \tag{3.49}$$

where

$$t_{A,S,J,L} := t_{A,S} + t_{A,J,L} = \sum_{i=1}^k \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) h_{i,A} + \delta_{i,A} h_{n-i} \right).$$

Now, we need to check the second condition of the theorem.

Recall that by Lemma 3.17

$$V = \{v \in \mathbb{R}^d | v = \sum_{i=1}^n \lambda_i h_i, 0 \le \lambda_i \le 1\}.$$

We rewrite $t_{A,S,J,L}$ as

$$
\begin{aligned}
t_{A,S,J,L} &= \sum_{i=1}^{k} \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) h_{i,A} + \delta_{i,A} h_{n-i} \right) \\
&= \sum_{i=1}^{k} \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) \frac{h_i + (1-2\delta_{i,A}) h_{n-i}}{2} + \delta_{i,A} h_{n-i} \right) \\
&= \sum_{i=1}^{k} \frac{1}{2} \left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) h_i \\
&\quad + \sum_{i=1}^{k} \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) (1/2 - \delta_{i,A}) + \delta_{i,A} \right) h_{n-i} \\
&=: \sum_{i=1}^{n} \lambda_i h_i .
\end{aligned}
$$

We first estimate coefficients λ_i with $1 \leq i \leq k$:

$$
\lambda_i = \frac{1}{2} \left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) \geq \frac{0+0}{2(n-i)} + \frac{0+0}{2i} = 0
$$

and

$$
\lambda_i = \frac{1}{2} \left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) \leq \frac{1+(n-i-1)}{2(n-i)} + \frac{1+(i-1)}{2i} = 1.
$$

Now, consider coefficients λ_i with $k+1 \leq i \leq n-1$. If $i \in A$, then

$$
\lambda_i = 1 - \frac{1}{2} \left(\frac{\delta_{n-i,S} + j_{n-i}}{i} + \frac{\delta_{i,S} + l_{n-i}}{n-i} \right) \geq 1 - \left(\frac{1+(i-1)}{2i} + \frac{1+(n-i-1)}{2(n-i)} \right) = 0
$$

and

$$
\lambda_i = 1 - \frac{1}{2} \left(\frac{\delta_{n-i,S} + j_{n-i}}{i} + \frac{\delta_{i,S} + l_{n-i}}{n-i} \right) \leq 1 - \frac{0+0}{2i} + \frac{0+0}{2n-i} = 1.
$$

If $i \notin A$, then

$$
\lambda_i = \frac{1}{2} \left(\frac{\delta_{n-i,S} + j_{n-i}}{i} + \frac{\delta_{i,S} + l_{n-i}}{n-i} \right) \geq \frac{0+0}{2i} + \frac{0+0}{2(n-i)} = 0
$$

and

$$
\lambda_i = \frac{1}{2} \left(\frac{\delta_{n-i,S} + j_{n-i}}{i} + \frac{\delta_{i,S} + l_{n-i}}{n-i} \right) \leq \frac{1+(i-1)}{2i} + \frac{1+(n-i-1)}{2(n-i)} = 1.
$$

Since $0 \le \lambda_i \le 1$, $1 \le i \le n-1$ and $\lambda_n = 0$, we conclude that $t_{A,S,J,L} \in V$.

Now,

$$\begin{aligned}
t_{A,S,J,L} + nh'_{A,S} &= \sum_{i=1}^{k} \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) h_{i,A} + \delta_{i,A} h_{n-i} \right) \\
&\quad + h_n - n \sum_{i=1}^{k} (\delta_{i,S} + \delta_{n-i,S} - 1) \frac{h_{i,A}}{i(n-i)} \\
&= \sum_{i=1}^{k} \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) h_{i,A} + \delta_{i,A} h_{n-i} \right) \\
&\quad + h_n - \sum_{i=1}^{k} (\delta_{i,S} + \delta_{n-i,S} - 1) \left(\frac{1}{n-i} + \frac{1}{i} \right) h_{i,A} \\
&= h_n + \sum_{i=1}^{k} \left(\left(\frac{1 + j_i - \delta_{n-i,S}}{n-i} + \frac{1 + l_i - \delta_{i,S}}{i} \right) h_{i,A} + \delta_{i,A} h_{n-i} \right) \\
&= \sum_{i=1}^{k} \left(\left(\frac{1 + j_i - \delta_{n-i,S}}{n-i} + \frac{1 + l_i - \delta_{i,S}}{i} \right) \frac{h_i + (1 - 2\delta_{i,A}) h_{n-i}}{2} + \delta_{i,A} h_{n-i} \right) \\
&\quad + h_n \\
&= h_n + \sum_{i=1}^{k} \frac{1}{2} \left(\frac{1 + j_i - \delta_{n-i,S}}{n-i} + \frac{1 + l_i - \delta_{i,S}}{i} \right) h_i \\
&\quad + \sum_{i=1}^{k} \left(\left(\frac{1 + j_i - \delta_{n-i,S}}{n-i} + \frac{1 + l_i - \delta_{i,S}}{i} \right) (1/2 - \delta_{i,A}) + \delta_{i,A} \right) h_{n-i} \\
&=: \sum_{i=1}^{n} \lambda_i h_i.
\end{aligned}$$

To estimate coefficients λ_i with $1 \le i \le k$ we write

$$\lambda_i = \frac{1}{2} \left(\frac{1 + j_i - \delta_{n-i,S}}{n-i} + \frac{1 + l_i - \delta_{i,S}}{i} \right) \ge \frac{1 + 0 - 1}{2(n-i)} + \frac{1 + 0 - 1}{2i} = 0$$

and

$$\lambda_i = \frac{1}{2} \left(\frac{1 + j_i - \delta_{n-i,S}}{n-i} + \frac{1 + l_i - \delta_{i,S}}{i} \right) \le \frac{1 + (n-i-1) - 0}{2(n-i)} + \frac{1 + (i-1) - 0}{2i} = 1.$$

Now, consider coefficients λ_i with $k+1 \le i \le n-1$. If $i \in A$, then

$$\begin{aligned}\lambda_i &= 1 - \frac{1}{2}\left(\frac{1+j_{n-i}-\delta_{i,S}}{i} + \frac{1+l_{n-i-\delta_{n-i,S}}}{n-i}\right)\\ &\ge 1 - \frac{1+(i-1)-0}{2i} - \frac{1+(n-i-1)-0}{2(n-i)} = 0\end{aligned}$$

and

$$\lambda_i = 1 - \frac{1}{2}\left(\frac{1+j_{n-i}-\delta_{i,S}}{i} + \frac{1+l_{n-i}-\delta_{n-i,S}}{n-i}\right) \le 1 - \frac{1+0-1}{2i} - \frac{1+0-1}{2n-i} = 1.$$

If $i \notin A$, then

$$\lambda_i = \frac{1}{2}\left(\frac{1+j_{n-i}-\delta_{i,S}}{i} + \frac{1+l_{n-i}-\delta_{n-i,S}}{n-i}\right) \ge \frac{1+0-1}{2i} + \frac{1+0-1}{2(n-i)} = 0$$

and

$$\begin{aligned}\lambda_i &= \frac{1}{2}\left(\frac{1+j_{n-i}-\delta_{i,S}}{i} + \frac{1+l_{n-i}-\delta_{n-i,S}}{n-i}\right)\\ &\le \frac{1+(i-1)-0}{2i} + \frac{1+(n-i-1)-0}{2(n-i)} = 1.\end{aligned}$$

Since $0 \le \lambda_i \le 1$, $1 \le i \le n-1$ and $\lambda_n = 1$, we have $nh_{A,S,J,L+}t_{A,S,J,L} \in V$.

Case 2: n is even.

If $n = 2k \ge 4$ is even then $k = n-k$ and $h_k = h_{n-k}$ always satisfies condition (3.33).

Again, we pair $\overrightarrow{\Delta}_{h_i}$ and $\overrightarrow{\Delta}_{h_{n-i}}$, $1 \le i \le k-1$. Note that we do not pair $\overrightarrow{\Delta}_{h_k}$ and $\overrightarrow{\Delta}_{h_n}$. We have

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \overrightarrow{\Delta}_{h_n}\left(\overrightarrow{\Delta}_{h_1}\overrightarrow{\Delta}_{h_{n-1}}\right)\left(\overrightarrow{\Delta}_{h_2}\overrightarrow{\Delta}_{h_{n-2}}\right)\cdots\left(\overrightarrow{\Delta}_{h_{k-1}}\overrightarrow{\Delta}_{h_{n-k+1}}\right)\overrightarrow{\Delta}_{h_k}. \tag{3.50}$$

The only difference with the odd case is the extra term $\overrightarrow{\Delta}_{h_k}$.

Similarly to (3.41), after replacing each $\overrightarrow{\Delta}_{h_i}\overrightarrow{\Delta}_{h_{n-i}}$ in (3.50) with $\overrightarrow{\Delta}^2_{\frac{h_i+(1-2\delta_{i,A}h_{n-i})}{2}}$, $1 \le i \le k-1$, and expanding the product, we have the following identity.

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \sum_{A\subset\{1,\ldots,k-1\}} (-1)^{|A|}\overrightarrow{\Delta}_{h_n}\overrightarrow{\Delta}_{h_k}\prod_{i=1}^{k-1}\overrightarrow{\Delta}^2_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2(n-i)}} T_{\sum_{j\in A} h_{n-j}}. \tag{3.51}$$

Now, we combine (3.51) and (3.42) to obtain the following.

$$\begin{aligned}
\overrightarrow{\Delta}_{h_1,\cdots,h_n} &= \sum_{A\subset\{1,\dots,k-1\}} (-1)^{|A|} \prod_{i=1}^{k-1} \left(\sum_{j=0}^{n-i-1} \overrightarrow{\Delta}_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2(n-i)}} T_{j\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2(n-i)}} \right) \times \\
&\quad \times \left(\sum_{l=0}^{i-1} \overrightarrow{\Delta}_{\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2i}} T_{l\frac{h_i+(1-2\delta_{i,A})h_{n-i}}{2i}} \right) T_{\sum_{j\in A} h_{n-i}} \overrightarrow{\Delta}_{h_n} \overrightarrow{\Delta}_{h_k} \\
&= \sum_{A\subset\{1,\dots,k-1\}} (-1)^{|A|} \prod_{i=1}^{k-1} \left(\overrightarrow{\Delta}_{\frac{h_{i,A}}{i},\frac{h_{i,A}}{n-i}} \sum_{j=0}^{n-i-1} \sum_{l=0}^{i-1} \left(T_{j\frac{h_{i,A}}{n-i}+l\frac{h_{i,A}}{i}} \right) \right) T_{\sum_{j\in A} h_{n-i}} \\
&\qquad \overrightarrow{\Delta}_{h_n} \overrightarrow{\Delta}_{h_k} \\
&= \sum_{A\subset\{1,\dots,k-1\}} (-1)^{|A|} D_A T_A,
\end{aligned}$$

where

$$D_A := \overrightarrow{\Delta}_{h_n} \overrightarrow{\Delta}_{h_k} \prod_{i=1}^{k-1} \left(\overrightarrow{\Delta}_{\frac{h_{i,A}}{n-i},\frac{h_{i,A}}{i}} \right)$$

and

$$T_A := \prod_{i=1}^{k-1} \sum_{j=0}^{n-i-1} \sum_{l=0}^{i-1} \left(T_{j\frac{h_{i,A}}{n-i}+l\frac{h_{i,A}}{i}} \right) T_{\sum_{j\in A} h_{n-j}}.$$

Similarly to (3.47) in the odd case we have

$$D_A = \sum_{S\subset\{1,2,\dots,k-1,k+1,\dots,n-1\}} (-1)^{|S|} \overrightarrow{\Delta}^n_{h'_{A,S}} T_{t_{A,S}},$$

where

$$h'_{A,S} := \frac{h_n + (1-2\delta_{k,S})h_k}{n} - \sum_{i=1}^{k} (\delta_{i,S} + \delta_{n-i,S} - 1) \frac{h_{i,A}}{i(n-i)}$$

and

$$t_{A,S} := \sum_{i=1}^{k-1} \left(\delta_{i,S} \frac{h_{i,A}}{n-i} + \delta_{n-i,S} \frac{h_{i,A}}{i} \right) + \delta_{k,S} h_k.$$

Similarly to (3.48) we get

$$T_A = \sum_J \sum_L T_{t_{A,J,L}},$$

where

$$t_{A,J,L} := \sum_{i=1}^{k-1} \left(j_i \frac{h_{i,A}}{n-i} + l_i \frac{h_{i,A}}{i} \right) + \sum_{j\in A} h_{n-j}.$$

Here and below, sum $\sum_J \sum_L$ is taken over all sets $J = \{j_1, j_2, \dots, j_{k-1}\}$, $0 \le j_i \le n-i-1$, $1 \le i \le k-1$ and $L = \{l_1, l_2, \dots, l_{k-1}\}$ $0 \le l_i \le i-1$, $1 \le i \le k-1$.

Finally, we have the following identity.

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \sum_{A\subset\{1,\dots,k-1\}} \sum_{S\subset\{1,\dots,n-1\}} \sum_J \sum_L (-1)^{|A|+|S|} \overrightarrow{\Delta}^n_{h'_{A,S}} T_{t_{A,S,J,L}}, \tag{3.52}$$

where

$$t_{A,S,J,L} := t_{A,S} + t_{A,J,L} = \sum_{i=1}^{k-1} \left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i} \right) h_{i,A} + \delta_{i,A} h_{n-i} \right) + \delta_{k,S} h_k.$$

Checking the second condition of Theorem 3.13 is similar to the odd case. □

Proof of Theorem 3.7. Let $n \in \mathbb{N}$, $h_1, h_2, \dots, h_n > 0$. Consider identity (3.29) for $d = 1$:

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \sum_{i=1}^{M_n} s_i \overrightarrow{\Delta}^n_{h'_i} T_{t_i}.$$

Let

$$h''_i := \begin{cases} h'_i, & h'_i \ge 0, \\ -h'_i, & h'_i < 0, \end{cases}$$

$$t'_i := \begin{cases} t_i, & h'_i \ge 0, \\ t_i + nh'_i, & h'_i < 0, \end{cases}$$

and

$$s'_i := \begin{cases} s_i, & h'_i \ge 0, \\ (-1)^n s_i & h'_i < 0. \end{cases}$$

Since

$$\overrightarrow{\Delta}^n_{h'_i} T_{t_i} = (-1)^n \overrightarrow{\Delta}^n_{-h'_i} T_{t_i + nh'_i},$$

we have

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n} = \sum_{i=1}^{M_n} s'_i \overrightarrow{\Delta}^n_{h''_i} T_{t'_i}.$$

The convex hull $V = [0, h_1 + \cdots + h_n]$. Since both t_i and $nh'_i + t_i$ belong to V, we have $t'_i \ge 0$.

Note that
$$nh_i'' + t_i' := \begin{cases} nh_i' + t_i, & h_i' \geq 0; \\ -nh_i' + t_i + nh_i' = t_i, & -h_i' < 0. \end{cases}$$
In both cases, $nh_i'' + t_i' \in V$, so $nh_i'' + t_i' \leq h_1 + \cdots + h_n$. Then M_n, h_i'', t_i', s_i', $i = 1, \ldots, M_n$, satisfy conditions of Theorem 3.7. $\square$

Remark 3.19. *For numbers M_n in Theorem 3.13 (and, therefore, in Theorem 3.7) we have the following estimates:*

$$M_n \leq 2^{\frac{3(n-1)}{2}}(n-1)!, \quad \text{if } n \text{ is odd}, \tag{3.53}$$

$$M_n \leq 2^{\frac{3n}{2}-1}\frac{(n-1)!}{n}, \quad \text{if } n \text{ is even}. \tag{3.54}$$

Indeed, consider the last sum in identity (3.49)*. For A we have 2^k possible choices, for S – 2^{n-1}, for j_i – $n-i$ and for l_i – i, $1 \leq i \leq k$. Together J and L gives $(n-1)!$ different choices. Therefore, $M_n \leq 2^k 2^{n-1}(n-1)! = 2^{\frac{3(n-1)}{2}}(n-1)!$, if n is odd.*

Similarly, considering identity (3.52)*, we deduce $M_n \leq 2^{\frac{3n}{2}-1}\frac{(n-1)!}{n}$, if n is even.*

However, estimates (3.53) *and* (3.54) *are not accurate for $n \geq 3$.*

For example, let $n = 2k+1 \geq 3$ be odd and consider set S so that for each i, $1 \leq i \leq k$ exactly one of $i, n-i$ be in S. In other words, this means $\delta_{i,S} + \delta_{n-1,S} = 1$. Choose also $j_i := \begin{cases} n-i-1, & i \in S; \\ 0, & n-i \in S. \end{cases}$ *and* $l_i := \begin{cases} 0, & i \in S; \\ i-1, & n-i \in S. \end{cases}$.

Then
$$h'_{A,S} = \frac{h_n}{n} - \sum_{i=1}^{k}(\delta_{i,S} + \delta_{n-i,S} - 1)\frac{h_{i,A}}{i(n-i)} = \frac{h_n}{n}$$

and

$$\begin{aligned} t_{A,S,J,L} &= \sum_{i=1}^{k}\left(\left(\frac{\delta_{i,S} + j_i}{n-i} + \frac{\delta_{n-i,S} + l_i}{i}\right)h_{i,A} + \delta_{i,A}h_{n-i}\right) \\ &= \sum_{i=1}^{k}(h_{i,A} + \delta_{i,A}h_{n-i}) = \frac{1}{2}\sum_{i=1}^{n-1}h_i. \end{aligned}$$

Note that these $h'_{A,S} =: h$ and $t_{A,S,J,L} =: t$ do not depend on A. Consider sum of terms in the RHS of identity (3.49) *with such S, J, L:*

$$\sum_{A\subset\{1,\dots,k\}}\sum_{S}(-1)^{|A|+|S|}\overrightarrow{\Delta}^n_{h'_{A,S}}T_{t_{A,S,J,L}} = \sum_{A\subset\{1,\dots,k\}}\sum_{S}(-1)^{|A|+k}\overrightarrow{\Delta}^n_h T_t$$
$$= \left(\sum_{A\subset\{1,\dots,k\}}(-1)^{|A|}\right)\left(\sum_{S}(-1)^k\overrightarrow{\Delta}^n_h T_t\right) = 0.$$

So we can write identity (3.49) *without them. This allows us to reduce M_n by 2^{2k}. Now we have*

$$M_n \le 2^{\frac{3(n-1)}{2}}(n-1)! - 2^{n-1}, \quad \textit{if } n \ge 3 \textit{ is odd.} \tag{3.55}$$

Similarly,

$$M_n \le 2^{\frac{3n}{2}-1}\frac{(n-1)!}{n} - 2^{n-1}, \quad \textit{if } n \ge 4 \textit{ is even.} \tag{3.56}$$

Corollary 3.20. *Let v be a nondecreasing weight function. Then, inequality* (3.7) *holds, i.e., for every $p > 0$ and for any function $f \in \mathbb{L}_{v,p}([0,1])$,*

$$\sup_{0<h_1,\dots,h_n<1}\|\overrightarrow{\Delta}_{h_1,\cdots,h_n}(f,\cdot,[0,1])\|^p_{\mathbb{L}_p([0,1]),v} \le c\sup_{0<h<1}\|\overrightarrow{\Delta}^n_h(f,\cdot,[0,1])\|^p_{\mathbb{L}_p([0,1]),v},$$

where constant c depends on n only.

Proof. Using Theorem 3.7 we can write

$$\overrightarrow{\Delta}_{h_1,\cdots,h_n}f(x) = \sum_{j=1}^{M_n} s_i\overrightarrow{\Delta}^n_{h'_j}(f, x+t_j),$$

where $t_j, h'_j \ge 0$, $nh'_j + t_j \le h_1 + \cdots + h_n$, $1 \le j \le M_n$.

Then

$$
\begin{aligned}
\|\overrightarrow{\Delta}_{h_1,\cdots,h_n}(f,x,[0,1])\|^p_{\mathbb{L}_p([0,1]),v} &= \|\sum_{j=1}^{M_n} s_j \overrightarrow{\Delta}^n_{h'_j}(f,x+t_j,[0,1])\|^p_{\mathbb{L}_p([0,1]),v} \\
&\le \sum_{j=1}^{M_n} \|\overrightarrow{\Delta}^n_{h'_j}(f,x+t_j,[0,1])\|^p_{\mathbb{L}_p([0,1]),v} \\
&= \sum_{j=1}^{M_n} \int_0^1 \left|v(x)\overrightarrow{\Delta}^n_{h'_j}(f,x+t_j,[0,1])\right|^p dx \\
&\le \sum_{j=1}^{M_n} \int_0^1 \left|v(x+t_j)\overrightarrow{\Delta}^n_{h'_j}(f,x+t_j,[0,1])\right|^p dx \\
&\le M_n \omega_k(f,1)^p_{v,p},
\end{aligned}
$$

and the proof is complete. $\square$

Chapter 4

Direct Theorem

In this chapter we prove Theorem 2.16.

We will use the same proof as in [17]. The idea is to approximate the function f by a monotone (if $l = 1$) or a convex (if $l = 2$) spline g (see Section 4.2), and then approximate this spline by a polynomial having the same shape (see Section 4.3). Theorem 2.16 will be proven in Section 4.4.

We will use splines with Chebyshev knots $x_j = \cos(j\pi/n)$, and also denote $x_j := 1$, $j < 0$ and $x_j := -1$, $j > n$. Additionally, let $I_j := [x_j, x_{j-1}]$ and $I_j^{(\nu)} := [x_{j+\nu}, x_{j-\nu-1}]$ (note that $I_j^{(0)} = I_j$), and, for an interval $I = [a, b] \subset [-1, 1]$, denote $|I| := b - a$.

Also, denote

$$\psi_j(x) := \frac{|I_j|}{|x - x_j| + |I_j|}, \qquad 1 \le j \le n.$$

The restricted average main part modulus was defined in [14] as follows:

$$\widetilde{\Omega}_\phi^k(f, \delta)_{\mathbb{L}_p(I),w} := \left(\frac{1}{\delta} \int_0^\delta \int_S |w(x)\delta_{h\phi(x)}^k(f, x, S)| dx dh \right)^{1/p},$$

where $S \subset [-1, 1]$.

Note that $\widetilde{\Omega}_\phi^k(f, \delta)_{\mathbb{L}_p(I),w} \le \Omega_\phi^k(f, \delta)_{\mathbb{L}_p(I),w}$, since supremum is greater then the average.

4.1 Auxiliary Results

Recall that the main part of modulus of smoothness $\Omega_\phi^k(f, A, 1/n)_{w,p}^p$ is defined by (2.3) and the modulus of smoothness $\omega_\phi^{\star k}$ is defined by (2.5).

The following lemma follows from [14, Lemma 4.2] with $\theta = 1$.

Lemma 4.1. *Let $p > 0$, $w \in \mathcal{W}$, $f \in \mathbb{L}_{w,p}$, $n, k \in \mathbb{N}$ and $A > 0$. Denote*

$$I^\star := \{1 \le i \le n | I_i \in [-1 + A\delta^2, 1 - A\delta^2]\},$$

and suppose that the interval J_i such that $I_i \subset J_i \subset [-1 + A\delta^2, 1 - A\delta^2]$, and $|J_i| < c_0|I_i|$ is given for each $i \in I^\star$. Then

$$\sum_{i \in I^\star} (w(x_i) E_k(f, J_i)_p)^p \le c\, \Omega_\phi^k(f, A, 1/n)_{w,p}^p,$$

where constant c depends only on k, p, c_0, w and A.

The following corollary is the same as [17, Corollary 2.4].

Corollary 4.2. *Let $p > 0$,$w \in \mathcal{W}$, $f \in \mathbb{L}_{w,p}$, $A > 0$, $k \in \mathbb{N}$ and $\nu \in \mathbb{N}_0$. Then for each $n \in \mathbb{N}$, we have*

$$\sum_{i=1}^{n} E_k(f, I_i^{(\nu)})_{w,p}^p \le c\omega_\phi^{\star k}(f, A, 1/n)_{w,p}^p, \tag{4.1}$$

where constant c depends only on k, p, ν, w and A.

We will need the following lemma.

Lemma 4.3. *Let $n \in \mathbb{N}$, $0 < p < 1$, $w \in \mathcal{W}$, and $\gamma_j \ge 0$, $1 \le j \le n - 1$. Then for*

$$\Sigma_p(x) := \Sigma_p(x, (\gamma_j)_{j=1}^{n-1}) := \sum_{j=1}^{n-1} \gamma_j |I_j|^{-1/p} \psi_j^\mu(x)$$

and sufficiently large μ, we have

$$\|\Sigma_p\|_{\mathbb{L}_p([-1,1]),w}^p \le c \sum_{j=1}^{n-1} w^p(x_j) \gamma_j^p.$$

Proof. First, we write

$$\begin{aligned}
\|\Sigma_p\|^p_{\mathbb{L}_p([-1,1]),w} &= \int\limits_{-1}^{1} w^p(x)\left(\sum_{j=1}^{n-1}\gamma_j|I_j|^{-1/p}\psi_j^{\mu}(x)\right)^p dx\\
&\le \sum_{j=1}^{n-1}\int\limits_{-1}^{1} w^p(x)\left(\frac{\gamma_j^p\psi_j^{\mu}(x)}{|I_j|^{1/p}}\right)^p dx\\
&= \sum_{j=1}^{n-1}\frac{\gamma_j^p}{|I_j|}\int\limits_{-1}^{1} w^p(x)\psi_j^{p\mu}(x)dx.
\end{aligned}$$

Now, consider

$$A_j := \int\limits_{-1}^{1} w^p(x)\psi_j^{p\mu}(x)dx = \int\limits_{-1}^{1} w^p(x)\left(\frac{|I_j|}{|x-x_j|+|I_j|}\right)^{p\mu} dx,$$

and separate it into the following three integrals:

$$A_j := \left(\int\limits_{-1}^{x_{j+1}} + \int\limits_{x_{j+1}}^{x_{j-1}} + \int\limits_{x_{j-1}}^{1}\right) w^p(x)\left(\frac{|I_j|}{|x-x_j|+|I_j|}\right)^{p\mu} dx = A_j^{(+)} + A_j^{(0)} + A_j^{(-)}.$$

Consider now $A_j^{(0)}$. On the interval $[x_{j+1}, x_{j-1}]$, $w(x) \le cw(x_j)$ and $\psi_j(x) \le 1$. So

$$A_j^{(0)} \le \int\limits_{x_{j+1}}^{x_{j-1}} cw^p(x_j)dx = cw^p(x_j)(x_{j-1}-x_{j+1}) \le cw^p(x_j)|I_j|.$$

Assume that $x_j < 0$ (the case for $x_j \ge 0$ is similar).

To estimate $A_j^{(+)}$, we note that $w(x) \le m^{-1}w(x_j)$ and $\psi_j(x) = \frac{|I_j|}{x-x_{j-1}}$, for $x \in [-1, x_{j+1}]$. So,

$$\begin{aligned}
A_j^{(+)} &\le \int\limits_{-1}^{x_{j+1}} m^{-p}w^p(x_j)\frac{|I_j|^{\mu p}}{(x-x_{j-1})^{\mu p}}dx\\
&\le \int\limits_{-\infty}^{x_{j+1}} m^{-p}w^p(x_j)\frac{|I_j|^{\mu p}}{(x-x_{j-1})^{\mu p}}dx\\
&= m^{-p}w^p(x_j)\frac{|I_j|^{\mu p}}{(\mu p-1)(x_{j-1}-x_{j+1})^{\mu p-1}} \le cw^p(x_j)|I_j|,
\end{aligned}$$

for $\mu p > 2$, since $|I_j| \le (x_{j-1} - x_{j+1})$.

We now estimate $A_j^{(-)}$. Let $x \in [x_{j-1}, 1]$ and let $2^k(x_j+1) \le x+1 < 2^{k+1}(x_j+1)$, $k \in \mathbb{N} \cup \{0\}$. Then for weight $w \in \mathcal{W}$ we have

$$\begin{aligned} w(x) &\le m^{-1} w(2^k x_j) \le \cdots \le m^{-k-1} w(x_j) \\ &\le m^{-\log_2 \frac{x+1}{x_j+1} - 1} w(x) = m^{-1} w(x_j) \left(\frac{x_j+1}{x+1} \right)^{\log_2 m}. \end{aligned}$$

Also, $\psi_j(x) \le |I_j|(x - x_j)^{-1}$ and so

$$\begin{aligned} A_j^{(-)} &\le \int_{x_{j-1}}^{1} m^{-p} w^p(x_j) \left(\frac{x_j+1}{x+1} \right)^{p \log_2 m} \frac{|I_j|^{\mu p}}{(x-x_j)^{\mu p}} dx \\ &\le \int_{x_{j-1}}^{\infty} m^{-p} w^p(x_j) \left(\frac{(x_{j-1}+1)(x-x_j)}{|I_j|(x_j+1)} \right)^{-p \log_2 m} \frac{|I_j|^{\mu p}}{(x-x_j)^{\mu p}} dx \\ &\le m^{-p} w^p(x_j) 4^{-p \log_2 m} \int_{x_j}^{\infty} \left(\frac{x-x_j}{|I_j|} \right)^{-\log_2 mp} \frac{|I_j|^{\mu p}}{(x-x_j)^{\mu p}} dx \\ &= m^{-p} w^p(x_j) 4^{-p \log_2 m} \frac{|I_j|^{(\log_2 m + \mu)p}}{\big((\log_2 m + \mu)p - 1 \big) (x_{j-1} - x_j)^{(\log_2 m + \mu)p - 1}} \\ &\le c w^p(x_j) |I_j|, \end{aligned}$$

for $\mu > \frac{2}{p} - \log_2 m$. Note that $\frac{2}{p} - \log_2 m$ does not depends on j.

Then, for sufficiently large μ, we have $A_j \le c w^p(x_j)|I_j|$. So

$$\|\Sigma_p\|_{w,p}^p = \sum_{j=1}^{n-1} \frac{\gamma_j^p}{|I_j|} A_j \le c \sum_{j=1}^{n-1} \frac{\gamma_j^p}{|I_j|} w^p(x_j)|I_j| = c \sum_{j=1}^{n-1} \gamma_j^p w^p(x_j),$$

and the proof is complete. □

4.2 Approximation by Splines

For $n, k \in \mathbb{N}$ and $r \in \mathbb{N}_0$, we denote by $\mathbb{S}_{k,n}^r$ the set of all r-times differentiable splines of degree k with n Chebyshev knots.

The following lemma follows from [6, Theorem 1.2].

Lemma 4.4. *Let $p > 0$, $l = 1, 2$. Then for every function $g \in \Delta^l \cap \mathbb{L}_p$, $n \in \mathbb{N}$, there exists spline $S = S(g) \in \mathbb{S}^{l-1}_{l+1,n} \cap \Delta^l$ and an absolute constant $\eta \in \mathbb{N}$ such that*

$$\|g - S\|_{\mathbb{L}_p(I_j)} \le cE_{l+1}(g, I_j^{(\eta)})_p, \qquad 1 \le j \le n,$$

where c depends only on l and p.

The following theorem can be proved the same way as [17, Theorem 3.2] with $w_{\alpha,\beta}$ replaced by w.

Theorem 4.5. *Let $l = 1, 2$, $\nu \in \mathbb{N}_0$, $r \in \mathbb{N}_0 \cup \{-1\}$ and $p > 0$. Suppose that for every $g \in \Delta^l \cap \mathbb{L}_p$ and $n \in \mathbb{N}$, there exists a spline $\tilde{S} = \tilde{S}(g) \in \mathbb{S}^r_{l+1,n} \cap \Delta^l$ such that*

$$\|g - \tilde{S}\|_{\mathbb{L}_p(I_j)} \le c_0 E_{l+1}(f, I_j^{(\eta)})_p, \qquad 1 \le j \le n.$$

Then for all $w \in \mathcal{W}$, $n \in \mathbb{N}$ and $f \in \mathbb{L}_p \cap \Delta^l$, there exists $\nu \in \mathbb{N}$, depending only on η, and a spline $S \in \mathbb{S}^r_{l+1,n} \cap \Delta^l$ such that

$$\|w(f - S)\|_{\mathbb{L}_p(I_j)} \le cE_{l+1}(f, I_j^{(\nu)})_{w,p}, \qquad 1 \le j \le n, \tag{4.2}$$

where c depends only on w, η and c_0.

Lemma 4.4 and Theorem 4.5 immediately imply the following result.

Corollary 4.6. *Let $l = 1$ or $l = 2$, $p > 1$, $w \in \mathcal{W}$, $\eta \in \mathbb{N}$, and $f \in \mathbb{L}_{w,p} \cap \Delta^l$. Then there exists a spline $S \in \mathbb{S}^{l-1}_{l+1,n} \cap \Delta^l$ and an absolute constant $\nu \in \mathbb{N}$ such that*

$$\|f - S\|_{\mathbb{L}_p(I_j),w} \le cE_{l+1}(f, I_j^{(\nu)})_{w,p}, \qquad 1 \le j \le n,$$

where c depends only on weight w.

4.3 Approximation of Splines by Polynomials

In this section we will show that convex spline $g \in \mathbb{S}_{3,n} \cap \Delta^2$ can be approximated by convex polynomials.

Denote by $L_j(x,g)$ the quadratic polynomial interpolating g at x_j, x_{j-1} and x_{j-2}, i.e.,

$$L_j(x,g) = \sum_{j-2\le i\le j} g(x_j) \prod_{j-2\le i\le j, l\ne i} \frac{x-x_l}{x_i-x_l}.$$

For $n \ge 2$, let S be a continuous piecewise quadratic polynomial with knots at x_j, $1 \le j \le n-1$, such that

$$S(x) = \max\{L_j(x,g), L_{j+1}(x,g)\}, \quad x \in I_j, \quad 2 \le j \le n-1$$

$$S(x) = L_2(x,g), \quad x \in I_1, \quad \text{and} \quad S(x) = L_n(x,g), \quad x \in I_n.$$

Now S is convex since g is convex.

We need the following lemma.

Lemma 4.7. *Let $p > 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $G \in \mathbb{S}_{l,n}$, $l \ge 1$. Then for all j, $1 \le j \le n-1$, we have*

$$w(x_j)E_l(G, I_j^{(k)})_p \le cE_l(G, I_j^{(k)})_{w,p}, \tag{4.3}$$

where c depends only on k and weight w.

Proof. First, we consider the case if $I_j^{(k)} \subset [x_{n-1}, x_1]$.

Then $w(x) \sim w(x_j)$ on interval $I_j^{(k)}$. Let P_l be polynomial of degree $< l$ of best $\mathbb{L}_{w,p}$ approximation for spline G on interval $I_j^{(k)}$. Then

$$E_l(G, I_j^{(k)})_{w,p} = \|w(G-P_n)\|_{\mathbb{L}_p(I_j^{(k)})} \ge cw(x_j)\|G-P_n\|_{\mathbb{L}_p(I_j^{(k)})} \ge cw(x_j)E_l(G, I_j^{(k)})_p.$$

Assume now that $1 \in I_j^{(k)}$ (the case for $-1 \in I_j^{(k)}$ is similar).

Let P be a polynomial of degree $< l$ of best $\mathbb{L}_{w,p}$ approximation on interval $I_j^{(k)}$.

Then

$$\begin{aligned}
w^p(x_j)E_l(G, I_j^{(k)})_p^p &\le w^p(x_j)\|G-P\|^p_{\mathbb{L}_p(I_j^{(k)})} \\
&= w^p(x_j)\left(\sum_{i=2}^{j+k}\|G-P\|^p_{\mathbb{L}_p(I_i)} + \|G-P\|^p_{\mathbb{L}_p(I_1)}\right) \\
&\le w^p(x_j)\left(\sum_{i=2}^{j+k}\|G-P\|^p_{\mathbb{L}_p(I_i)} + c\|G-P\|^p_{\mathbb{L}_p(x_1,(1+x_1)/2)}\right) \\
&\le c\sum_{i=2}^{j+k}\|w(G-P)\|^p_{\mathbb{L}_p(I_i)} + c\|w(G-P)\|^p_{\mathbb{L}_p(x_1,(1+x_1)/2)} \\
&= c\|G-P\|^p_{\mathbb{L}_p(I_j^{(k)}),w} \le cE_l(G, I_j^{(k)})^p_{w,p}.
\end{aligned}$$

□

In particular, it follows that for any j, $1 \le j \le n-1$,

$$w(x_j)E_3(g, I_j^{(1)})_p \le cE_3(g, I_j^{(1)})_{w,p}. \tag{4.4}$$

The following inequalities were proved in [17].

$$\|w(g-S)\|_{\mathbb{L}_p(I_j)} \le cE_3(g, I_j^{(1)})_{\mathbb{L}_p(I_j),w}, \qquad 1 \le j \le n. \tag{4.5}$$

It was shown in [13] that all knots x_j, $1 \le j \le n$, can be separated into classes $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ so that

$$\begin{aligned}
S(x) = q_2(x) &+ \sum_{2\le j\le n-1, x_j\in \mathbf{I}\cup\mathbf{II}} A_j\left[(x_{j-1}-x_j)(x-x_j)_+ - (x-x_j)_+^2\right] \\
&+ \sum_{1\le j\le n-2, x_j\in \mathbf{II}\cup\mathbf{III}} -A_{j+1}\left[(x_j-x_{j+1})(x-x_j)_+ + (x-x_j)_+^2\right],
\end{aligned}$$

for some polynomial q_2 and numbers A_j.

Then $\sigma_j(x), R_j(x)$ and $\overline{R}_j(x)$ could be defined (see [13]) so that the polynomial

$$\begin{aligned}
P_n(x) = P_n(x,g) = q_2(x) &+ \sum_{2\le j\le n-1, x_j\in \mathbf{I}\cup\mathbf{II}} A_j\left[(x_{j-1}-x_j)\sigma_j(x) - R_j(x)\right] \\
&+ \sum_{1\le j\le n-2, x_j\in \mathbf{II}\cup\mathbf{III}} -A_{j+1}\left[(x_j-x_{j+1})\sigma_j(x) + \overline{R}_j(x)\right]
\end{aligned}$$

of degree $< cn$ is convex on $[-1,1]$.

Lemma 4.8. *For the spline S the polynomial P_n defined above the following inequalities hold.*

$$\|w(P_n - g)\|^p_{\mathbb{L}_p[-1,1]} \le c \sum_{j=1}^{n} E_3(g, I_j^{(1)})^p_{w,p} \tag{4.6}$$

and

$$\|w(P_n' - g')\|^p_{\mathbb{L}_p[-1,1]} \le c \sum_{j=1}^{n} E_2(g, I_j^{(1)})^p_{w,p}, \tag{4.7}$$

where constants c depend on p and w only.

Proof. We prove inequality (4.6) first.

Note that

$$\|w(P_n - g)\|^p_{\mathbb{L}_p[-1,1]} \le \|w(P_n - S)\|^p_{\mathbb{L}_p[-1,1]} + \|w(S - g)\|^p_{\mathbb{L}_p[-1,1]}. \tag{4.8}$$

Consider the first term in (4.8).

$$\begin{aligned}
|S(x) - P_n(x)| = \Big| & \sum_{2 \le j \le n-1, x_j \in \mathbf{I} \cup \mathbf{II}} A_j \left[(x_{j-1} - x_j)((x - x_j)_+ - \sigma_j(x)) \right. \\
& \left. -((x - x_j)_+^2 - R_j(x)) \right] \\
+ & \sum_{1 \le j \le n-2, x_j \in \mathbf{II} \cup \mathbf{III}} -A_{j+1} \left[(x_j - x_{j+1})((x - x_j)_+ - \sigma_j(x)) \right. \\
& \left. +((x - x_j)_+^2 - \overline{R}_j(x)) \right] \Big| \\
\le & \sum_{2 \le j \le n-1, x_j \in \mathbf{I} \cup \mathbf{II}} A_j \left[(x_{j-1} - x_j)|(x - x_j)_+ - \sigma_j(x)| \right. \\
& \left. +|(x - x_j)_+^2 - R_j(x)| \right] \\
+ & \sum_{1 \le j \le n-2, x_j \in \mathbf{II} \cup \mathbf{III}} -A_{j+1} \left[(x_j - x_{j+1})|(x - x_j)_+ - \sigma_j(x)| \right. \\
& \left. +|(x - x_j)_+^2 - \overline{R}_j(x)| \right].
\end{aligned} \tag{4.9}$$

The following estimates were shown in [13]:

$$\begin{aligned}
|(x - x_j)_+ - \sigma_j(x)| &\le c|I_j|\psi_j^\mu(x), \\
|(x - x_j)_+^2 - R_j(x)| &\le c|I_j|^2\psi_j^\mu(x), \\
|(x - x_j)_+^2 - \overline{R}_j(x)| &\le c|I_j|^2\psi_j^\mu(x).
\end{aligned}$$

Then, from (4.9) we deduce

$$\begin{aligned}
|S(x) - P_n(x)| \le c &\sum_{2\le j\le n-1, x_j\in \mathbf{I}\cup\mathbf{II}} A_j \left[(x_{j-1}-x_j)|I_j|\psi_j^\mu(x)\right. \\
&\quad \left. +|I_j|^2\psi_j^\mu(x)\right] \\
&+ c \sum_{1\le j\le n-2, x_j\in \mathbf{II}\cup\mathbf{III}} -A_{j+1}\left[(x_j - x_{j+1})|I_j|\psi_j^\mu(x)\right. \\
&\quad \left. +|I_j|^2\psi_j^\mu(x)\right] \\
&\le c\sum_{j=2}^{n-1} |A_j||I_j|^2\psi_j^\mu(x) \\
&\le c\sum_{j=2}^{n-1} |I_j|^{-1/p}\psi_j^\mu(x) E_3(g, I_j^{(1)})_{w,p}^p.
\end{aligned} \tag{4.10}$$

Hence, using Lemma 4.3 and (4.4), we have

$$\|w(S-P_n)\|^p_{\mathbb{L}_p([-1,1])} \le c\sum_{j=1}^n E_3(g, I_j^{(1)})^p_{w,p}. \tag{4.11}$$

Combining (4.8), (4.5) and (4.11) we get (4.6).

The proof or (4.7) is identical. □

4.4 Proof of Theorem 2.16

Let $l = 1$ or $l = 2$, $0 \le p < \infty$, $w \in \mathcal{W}$, $f \in \mathbb{L}_{w,p} \cup \Delta^l$, and $n \in \mathbb{N}$ be sufficiently large. Recall that Corollary 4.6 implies that for some spline $g_l \in \mathbb{S}^{l-1}_{l+1,n} \cup \Delta^l$ and $\nu \in \mathbb{N}$

$$\|f - g_l\|_{\mathbb{L}_p(I_j),w} \le cE_{l+1}(f, I_j^{(\nu)})_{w,p}, \qquad 1 \le j \le n. \tag{4.12}$$

Then

$$\|f - g_l\|^p_{\mathbb{L}_p[-1,1],w} = \sum_{j=1}^n \|f-g_l\|^p_{\mathbb{L}_p(I_j),w} \le c\sum_{j=1}^n E_{l+1}(f, I_j^{(\nu)})^p_{w,p}. \tag{4.13}$$

Let $l = 2$ and $P_n := P_n(g_2)$ be the polynomial associated with convex spline g_2. Then inequality (4.6) from Lemma 4.8 together with (4.13) and Corollary 4.2 gives

$$\begin{aligned}
E_n(f,[-1,1])^p_{w,p} &\le \|w(f-P_n)\|^p_{\mathbb{L}_p([-1,1])} \\
&\le \|w(f-g_2)\|^p_{\mathbb{L}_p([-1,1])} + \|w(g_2-P_n)\|^p_{\mathbb{L}_p([-1,1])} \\
&\le c\sum_{j=1}^{n} E_3(g_2, I_j^{(\nu)})^p_{w,p} \\
&\le c\omega_\phi^{\star 3}(f,A,1/n)^p_{w,p}.
\end{aligned}$$

For a monotone function f and monotone spline $g_1(x)$ let $G(x) := \int_{-1}^{x} g_1(u)du$ be the convex antiderivative of $g_1(x)$. Then $Q_n(x) := P_n'(x,G)$ is the polynomial that approximates $g_1(x) = G'(x)$. Similarly to convex case we have

$$\begin{aligned}
E_n(f,[-1,1])^p_{w,p} &\le \|w(f-Q_n)\|^p_{\mathbb{L}_p([-1,1])} \\
&\le \|w(f-g_1)\|^p_{\mathbb{L}_p([-1,1])} + \|w(g_1-Q_n)\|^p_{\mathbb{L}_p([-1,1])} \\
&\le c\sum_{j=1}^{n} E_2(g_1, I_j^{(\nu)})^p_{w,p} \\
&\le c\omega_\phi^{\star 2}(f,A,1/n)^p_{w,p}.
\end{aligned}$$

Hence, Theorem 2.16 is proven.

Chapter 5

Conclusion

We discussed monotone and convex approximation in weighted $\mathbb{L}_p$ spaces, $0 < p < \infty$.

We considered a special class of doubling weights $\mathcal{W}$ defined in Definition 2.2. Weights in this class are bounded and do not rapidly change. In particular, Jacobi weights are in this class. It was shown in Lemma 2.4 that $\mathcal{W}$ is a proper subclass of $\mathcal{W}(\{-1, 1\})$ defined in [14].

Our main goal was to prove the direct theorem (Theorem 2.16) for monotone and convex approximation (the inverse results were established in [14]). The technique of proof is well known: we estimate a monotone (convex) function by a monotone (convex) spline and this spline by a monotone (convex) polynomial.

To measure smoothness we use moduli of smoothness $\omega_\phi^{\star k}$ and ω_ϕ^k defined by (2.4) and (2.5) respectively. Modulus ω_ϕ^k could be easily evaluated and therefore is more practical. However, it is hard to work with ω_ϕ^k directly. So, we proved the direct theorem with modulus $\omega_\phi^{\star k}$ and showed equivalence of the moduli ω_ϕ^k and $\omega_\phi^{\star k}$ (Theorem 2.11). For the proof in the case $p \geq 1$ we used the same method as the one used in [5, Proposition 4.2]. However, this method requires the Hölder inequality, which does not hold for $0 < p < 1$. To prove equivalence in the case $0 < p < 1$ we proved Lemma 3.6. Our main auxiliary result is Theorem 3.13 which is an improvement of [3, Lemma 5.4.11] and may be useful in different contexts.

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