Ordered groups and a theorem of Farrell

by

Irshad Ayoob

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University of Manitoba

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Abstract

This thesis provides a review of left/right orderable and circularly orderable groups. A detailed proof of the classical Farrell's theorem for left orderable groups is given, and is generalized to the case of circularly orderable groups through a cohomological argument. This generalizes the classical Farrell's theorem, in the sense that left orderable group may be viewed as the special case of circularly orderable group.

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1

Introduction

In this chapter we will discuss the purpose of this thesis and outline the goals of each subsequent chapter. We know that algebraic topology uses tools from abstract algebra to study topological spaces. The theory of orderable groups provides one instance where we see a remarkable interplay between topology and group theory. For example, the existence of foliations of certain 3-manifolds and non-zero degree of a map between them is associated with the left orderabilty of their fundamental group [21].

In this thesis we will discuss one such application via Farrell's theorem. This theorem provides conditions that guarantee the existence of an embedding of a covering space $p : \widetilde{X} \to X$ through right orderability of the fundamental group of a given triangulable topological space, and conversely. The main purpose of this thesis is to generalize Farrell's theorem to circular orderable groups. Right orderable groups may be viewed as special cases of circularly orderable groups. The brief outline of the thesis is as follows:

In Chapter 2, the definitions of left orderable and bi-orderable groups are given along with their main properties. Archimedean ordered groups are defined and Hölder's theorem is proved which states that the additive group of reals $(\mathbb{R}, +)$ is universal for

Archimedean groups. Finally, “the universal theorem” for countable left-orderable groups is stated, which states that any countable left-orderable group may be embedded into the group of order preserving homeomorphisms of the real numbers.

In Chapter 3, the proof of Farrell’s theorem is discussed in detail. This theorem connects an embedding of the universal covering space with the right-orderability of the fundamental group of a triangulable space. Our main purpose is to generalize Farrell’s theorem to circular orders which is achieved in Chapter 5.

In Chapter 4, circular and secret left orders on a group are introduced and the relationship between them is established through the second cohomology group. It is discussed how a left/right ordering may arise as a special case of a circular ordering.

In Chapter 5, we provide the generalization of Farrell’s theorem to circular orderable groups in one direction, and provide a counterexample for the converse.

2

Orderable Groups

In this chapter, we give an introduction to left orderable and bi-orderable groups. We shall state and prove some main results concerning left orderability of a group. In particular, we shall introduce Archimedean ordered groups and provide a detailed proof of Hölder's theorem, which states that the additive group of reals $(\mathbb{R}, +)$ is universal for Archimedean ordered groups. Finally, we shall state and discuss what is called the dynamic realization of left orderings, which provides an embedding of any countable left orderable group into the group of order preserving homeomorphisms of the real numbers. There are many useful reference books on ordered groups, such as *Right-ordered groups* by Kopytov and Medvedev [16], *Orderable groups* by Mura and Rhemtulla [19], *Fully ordered groups* by Kokorin and Kopytov [15], *Ordered Groups and Topology* by Adam Clay and Dale Rolfsen [7], *Partially ordered groups* [13] by A. M. W. Glass, and *Groups, orders, and dynamics* [9] by Deroin, Navas and Rivas.

2.1 Left and Bi-orderable groups

We start with the definitions:

A relation $<$ is a strict order on a set X if it is:

1 Irreflexive: $x < x$ does not hold for any $x \in X$.

2 Asymmetric: if $x < y$, then $y < x$ does not hold.

3 Transitive: $x < y$ and $y < z$ implies $x < z$.

A strict order is total if for any $x, y \in X$, either $x < y, y < x$ or $x = y$.

Definition 2.1. A group G is left orderable if there is a strict total order $<$ on G which is left invariant, that is for all $f, g, h \in G$, the relation $g < h$ holds if and only if $fg < fh$ holds.

Definition 2.2. A group G is right orderable if there is a strict total order $<$ on G which is right invariant, that is for all $f, g, h \in G$, the relation $g < h$ holds if and only if $gf < hf$ holds.

Definition 2.3. The group G is said to be bi-orderable if it admits a strict total order $<$ which is both left and right invariant, in the sense of Definition 2.1 and 2.2.

The elements of G which are greater than the identity element are called positive, and the subset of such elements of G is called the positive cone of the ordering.
The additive group of reals $(\mathbb{R}, +)$, rationals $(\mathbb{Q}, +)$ and integers $(\mathbb{Z}, +)$ are bi-orderable groups under their usual orderings.
We give some important properties of left orderable and bi-orderable groups:

Proposition 2.4. We have:
(1) In a left orderable group G one has $1 < g$ if and only if $g^{-1} < 1$.
(2) In a left orderable group G, if $1 < g, 1 < h$ then $1 < gh$.
(3) If f and g are elements of a left-orderable group and $f \neq 1$, then g is strictly between fg and $f^{-1}g$, and strictly between gf and gf^{-1}.

Proof. (1) Suppose $1 < g$, since G is left orderable, we get $g^{-1} < g^{-1}g$ or $g^{-1} < 1$. Conversely, suppose that $g^{-1} < 1$, then by left orderability of G, we have $gg^{-1} < g$

or $1 < g$.

(2) Given $1 < h$, this gives by left-invariance, $g < gh$. Then, by $1 < g$ and transitivity in G, it follows that $1 < gh$.

(3) To show $fg < g < f^{-1}g$, whenever $f \neq 1$, it suffices to show that, $g^{-1}fg < 1 \Leftrightarrow 1 < g^{-1}f^{-1}g$ for every $g \in G$. This is true because, $g^{-1}fg = (g^{-1}f^{-1}g)^{-1}$ so that if $g^{-1}fg < 1$, then $g^{-1}f^{-1}g > 1$ or $fg < g$ then $f^{-1}g < g$. This by transitivity gives, $fg < g < f^{-1}g$.

Now, we show $gf < g < gf^{-1}, f \neq 1$. Since $f \neq 1$ and $<$ is a total order, we can well assume that $f < 1$. This by left-invariance gives $gf < g$. Also by Proposition 2.4, $f < 1$ implies $f^{-1} > 1$ which further by left-invariance implies $gf^{-1} > g$. By transitivity, we get $gf < g < gf^{-1}$. If $f > 1$, a similar argument holds. □

A right ordering on the group G is "the same as" a left ordering through the following correspondence.

Proposition 2.5. If $<$ is a left-ordering on G, then $g \prec h \Leftrightarrow h^{-1} < g^{-1}$ defines a right-ordering $\prec$ which has the same positive cone, that is $1 \prec g \Leftrightarrow 1 < g$.

Proof. We show $\prec$ is right ordering on G. Let $g \prec h$ which means $h^{-1} < g^{-1}$. This, by left-invariance gives $f^{-1}h^{-1} < f^{-1}g^{-1}$ or $(hf)^{-1} < (gf)^{-1}$ which by definition of $\prec$ implies that $hf \prec gf$, in other words, $\prec$ is a right-ordering on G.

Also, suppose that $1 < g$ which by Proposition 2.4 implies $g^{-1} < 1$, which further, by definition of $\prec$ implies that $1 \prec g$. Similarly, $1 \prec g$ implies $1 < g$. □

The following result shows that a left orderable group cannot be finite, except the trivial group.

Proposition 2.6. Left orderable groups are torsion free, hence infinite.

Proof. Suppose G is a non-trivial left orderable group and $g \neq 1 \in G$. Suppose $g > 1$. Then by left invariance $g^2 > g$, so that $g^2 > 1$ by transitivity. Repeating the argument, we have that $g^n > 1$ for every $n \in N$ so that $g^n \neq 1$ for any n. Thus G is torsion free and hence infinite. $\square$

For example, the multiplicative group of non-zero reals $(\mathbb{R}, \cdot)$ cannot be left ordered since the element -1 has order two which is not possible by Proposition 2.6. The converse of Proposition 2.6 is not true, that is, there exists a torsion free group which is not left orderable. More specifically, consider the crystallographic group $G = \langle a, b | a^2ba^2 = b, b^2ab^2 = a \rangle$. This group is generated by the rigid motions $a(x, y, z) = (x+1, 1-y, -z), b(x, y, z) = (-x, y+1, 1-z), c(x, y, z) = (1-x, -y, z+1)$ acting on $\mathbb{R}^3$ with coordinates x, y, z. One can check that $a^2ba^2 = b, b^2ab^2 = a$ and $abc = id$. One generator may be eliminated by the last relation. This group is torsion free but not left orderable (see [7] for details).

Left orderable groups behaves nicely with respect to extensions in the following way:

Proposition 2.7. Suppose G is a group with normal subgroup K and quotient group $H \cong \frac{G}{K}$. In other words, suppose there is an exact sequence,

$$1 \longrightarrow K \hookrightarrow G \xrightarrow{p} H \longrightarrow 1.$$

Further suppose $(H, <_H)$ and $(K, <_K)$ are left-ordered groups. Then we can give G a left-ordering defined in a sort of lexicographic way:
declare that $g < g'$ if and only if $p(g) <_H p(g')$ or $p(g) = p(g')$ (so $g^{-1}g' \in K$) and $1 <_K g^{-1}g'$.

Proof. Suppose $g < g'$, we prove left-invariance and transitivity of $<$.
Left-Invariance:
Case 1: If $p(g) <_H p(g')$ then $p(fg) = p(f)p(g) <_H p(f)p(g') = p(fg')$ for every

$f \in G$. This by definition implies that $fg < fg'$ so that $<$ is left-invariant in his case.

Case 2: If $p(g) = p(g')$ with $1 <_K g^{-1}g'$ then $p(fg) = p(fg')$ with $(fg)^{-1}fg' = g^{-1}f^{-1}fg' = g^{-1}g' >_K 1$ which implies that $fg < fg'$ for every $f \in G$. Hence $<$ is left-invariant in both cases.

Transitivity: Suppose $g < g'$ and $g' < f$, then

Case 1: $p(g) <_H p(g')$ and $p(g') <_H p(f)$ implies that $p(g) <_H p(f)$ which means $g < f$.

Case 2: $p(g) <_H p(g')$ and $p(g') = p(f)$ implies $p(g) <_H p(f)$ so that $g < f$.

Case 3: $p(g) = p(g')$ and $p(g') <_H p(f)$ implies that $p(g) <_H p(f)$ which means $g < f$.

Case 4: If $p(g) = p(g')$ with $1 <_K g^{-1}g'$ and $p(g') = p(f)$ with $1 <_K g'^{-1}f$. For $g'^{-1}f >_K 1$, left-invariance implies $g^{-1}g'(g'^{-1}f) >_K g^{-1}g' >_K 1$ or $g^{-1}(g'g'^{-1})f >_K 1$ or $g^{-1}f >_K 1$ so that $g < f$. Hence the transitivity. □

We provide some important properties of bi-orderable groups:

Proposition 2.8. We have:

(1) A left ordering is a bi-ordering if and only if the ordering is invariant under conjugation.

(2) In a bi-orderable group, $g_1 < h_1, g_2 < h_2$ implies $g_1g_2 < h_1h_2$.

(3) Bi-orderable groups have unique roots, that is, if $g^n = h^n$ for some $n > 0$ then $g = h$.

(4) In a bi-orderable group G, g^n commutes with h if and only if g commutes with h.

(5) Bi-orderable groups do not have generalized torsion: any product of conjugates of a non-trivial element must be non-trivial. In particular, $x^{-1}yx = y^{-1}$ implies $y = 1$.

Proof. (1) Suppose the left-ordering $<$ is bi-ordering. Let $g < h$, then left-invariance gives $f^{-1}g < f^{-1}h$ and right-invariance gives $f^{-1}gf < f^{-1}hf$. Hence, $<$ is invariant under conjugation.

Conversely, suppose the left-ordering $<$ is a invariant under conjugation. Let $g < h$. This gives by conjugation $f^{-1}gf < f^{-1}hf$ which further by left-invariance gives $f(f^{-1}gf) < f(f^{-1}hf)$ or $gf < hf$, that is, $<$ is right-ordering as well. Thus, $<$ is a bi-order on G. (2) Let $g_2 < h_2$, then by left-invariance we get $g_1g_2 < g_1h_2$. Also, by right-invariance $g_1 < h_1$ implies $g_1h_2 < h_1h_2$. Thus by transitivity, we get $g_1g_2 < h_1h_2$.

(3) Suppose $g^n = h^n$ for some $n > 0$ but $g \neq h$. Since $<$ is a total order on G, we must have either $g < h$ or $g > h$. Without loss of generality, assume that $g < h$. Applying (2) repetitively, we get $g^2 < h^2, g^3 < h^3, ..., g^n < h^n$ for every $n > 0$, a contradiction. Hence, $g = h$.

(4) Suppose that $gh = hg$, we prove $g^nh = hg^n$ for every integer $n \neq 0$.

We have $gh = hg$ implies $g^2 = h = ghg = hgg = hg^2$, which further implies $g^3h = ghg^2 = hgg^2 = hg^3$. Thus inductively, we get $g^nh = hg^n$.

Conversely, suppose that $g^nh = hg^n$ (equivalently $h^{-1}g^nh = g^n$) for some non zero integer n. Now consider $(h^{-1}gh)^n = h^{-1}g^nh = g^n$, that is, $(h^{-1}gh)^n = g^n$. By the unique root extraction property of bi-orderable groups, that is by (3), we get $h^{-1}gh = g$ or $gh = hg$. More generally, suppose $g^nh^m = h^mg^n$(equivalently $h^{-m}g^nh^m = g^n$) for some non zero integers m and n. Consider, $(h^{-m}gh^m)^n = h^{-m}g^nh^m = g^n$ or $(h^{-m}gh^m)^n = g^n$, which by a unique root extraction property of bi-orderable groups gives $h^{-m}gh^m = g$ or $gh^m = h^mg$, which by (1) implies that $hg = gh$.

(5) Let G be bi-orderable. Suppose G contains a generalized torsion element, say $g \neq 1$, so that any product of conjugates of g is 1, that is,

$$\prod_i (a_i^{-1}ga_i) = 1.$$

Since $g \neq 1$, $g < 1$ or $g > 1$. Assume $g > 1$, by bi-orderability, this implies $a_i^{-1}ga_i > 1$

for every i, so that any product of such positive elements is positive, contradiction. Similarly, we prove the case when $g < 1$.

Also, suppose $y \neq 1$ but $x^{-1}yx = y^{-1}$. Since, y is non-trivial, any conjugate $x^{-1}yx$ of it must be non-trivial, say $x^{-1}yx < 1$. This gives, by right and left multiplication, $yx < x$ or $y < 1$. But, $x^{-1}yx < 1$ also implies $y^{-1} < 1$ or $y > 1$, contradiction.

$\square$

It is clear that both left orderability and bi-orderablity are preserved under taking subgroups and direct products. If $(A, <_A)$ and $(B, <_B)$ are left orderable (bi-orderable) groups, then their direct product $A \times B$ is also left orderable (bi-orderable) by using the lexicographic ordering, that is, $(a, b) < (a', b')$ if and only if $a <_A a'$ or else $a = a'$ and $b <_B b'$.

Recall by Proposition 2.7 that left orderablility behaves nicely with respect to extensions, however the following example shows that the bi-orderability is not well behaved by the extensions:

Example 2.1.1. The Klein bottle group K is left orderable but not bi-orderable. This example shows that there is a group G that fits into a short exact sequence $1 \to M \to G \to N \to 1$ where M and N are bi-orderable but G is not bi-orderable.

Proof. We have $K = \pi_1(Klein \quad Bottle) \cong \langle x, y \mid xyx^{-1} = y^{-1} \rangle$. Let $\langle y \rangle$ be the subgroup of K generated by y. Then $\langle y \rangle$ is the normal subgroup of K isomorphic to $\mathbb{Z}$ and is invariant under conjugation, that is, for every $y \in \langle y \rangle, xyx^{-1} = y^{-1} \in \langle y \rangle$.

We see that the presentation $H = \langle x, y \mid xyx^{-1} = y^{-1}, y = 1 \rangle$ is the group $\frac{K}{\langle\langle y \rangle\rangle}$ where $\langle\langle y \rangle\rangle$ is the normal closure of y, that is, the smallest normal subgroup of K containing y.

Thus $\langle\langle y \rangle\rangle = \langle y \rangle$ so that $\frac{K}{\langle y \rangle} = H = \langle x, y \mid xyx^{-1} = y^{-1}, y = 1 \rangle$ which is clearly seen to be the infinite cyclic group after applying Tietze transformation, that is, remove y from the list of generators and replace it with the identity so we get $\frac{K}{\langle y \rangle} \cong \mathbb{Z}$. Thus we have the following short exact sequence,

$$1 \longrightarrow \langle y \rangle \hookrightarrow K \xrightarrow{p} \frac{K}{\langle y \rangle} \longrightarrow 1$$

Since $\frac{K}{\langle y \rangle} \cong \langle y \rangle \cong \mathbb{Z}$ is left orderable, it follows from Proposition 2.7 that the group K is also left-orderable.

Also suppose that the group K can be given a bi-order. Let $y < 1, \quad y \in K$, then by left-invariance $xy < x$ and by right-invariance $xyx^{-1} < xx^{-1} = 1$ or $y^{-1} < 1$. That is, $y < 1$ implies $y^{-1} < 1$, a contradiction by Proposition 2.4.

$\square$

There is another way to prove that the Klein bottle group is not bi-orderable:

Example 2.1.2. The Klein bottle group K does not have unique roots.

Proof. Let $H = \langle x, y | xyx^{-1} = y \rangle$ be the Klein bottle group and $G = \langle a, b | a^2 = b^2 \rangle$. We shall define an explicit function $h : G \to K$ by assigning $h(a)$ and $h(b)$ expressions as words in x and y and show that the relation $a^2 = b^2$ in the domain implies $xyx^{-1} = y$ in the range, so that h is a homomorphism. Similarly, we shall define a homomorphism in the other direction and verify that it is inverse to h.

Define $h : G \to H$ by

$$h(a) = x, h(b) = y^{-1}x$$

which is a well-defined homomorphism since we have:

$$a^2b^{-2} = x^2(y^{-1}x)^{-2} = x^2((y^{-1}x)^{-1})^2 = x^2(x^{-1}y)^2$$

$$= x^2(x^{-1}y)(x^{-1}y) = xxx^{-1}yx^{-1}y = xyx^{-1}y = 1_H.$$

where 1_H is the identity element of H. Next, define $f : H \to G$ by:

$$f(x) = a, f(y) = ab^{-1}$$

which is well-defined since:

$$xyx^{-1}y = a(ab^{-1})a^{-1}ab^{-1} = a^2b^{-1}b^{-1} = a^2b^{-2} = 1_G.$$

where 1_G is the identity element of G. Finally, to show that h is a bijection, we prove f is a two-sided inverse of h as:

$$f(h(a)) = f(x) = a, \quad f(h(b)) = f(y^{-1}x) = f(y^{-1})f(x) = (ab^{-1})^{-1}a = ba^{-1}a = b$$

and

$$h(f(x)) = h(a) = a, \quad h(f(y)) = h(ab^{-1}) = h(a)h(b^{-1}) = x(y^{-1}x)^{-1} = xx^{-1}y = y$$

Thus G isomorphic to H.

The association $a \to x, b \to y^{-1}x$ is a homomorphism from G to H. Since $x \neq y^{-1}x$ but we still have $a^2 = b^2$, that is $x^2 = (y^{-1}x)^2$. Hence by Proposition 2.7 (3), K is not bi-orderable. □

The following results give the characterization of left orderable and bi-orderable groups through their subsets [7].

Theorem 2.9. A group G is left-orderable if and only if there exists a subset $P \subset G$ such that:

(1) $P \cdot P \subset P$ and

(2) $\forall g \in G$, exactly one of $g = 1, g \in P$ or $g^{-1} \in P$ holds.

Proof. Suppose $(G, <)$ is left-orderable, we prove there exists P such that conditions

(1) and (2).

Set $P = \{g \in G \mid g > 1\}$, the set of positive elements in G.

Condition 1: Let $g_1, g_2 \in P$ so that $g_1 > 1, g_2 > 1$. By Proposition 2.4, $g_1 > 1$ implies $g_2 g_1 > g_2 > 1$. Hence, $P \cdot P \subset P$.

Condition 2: Suppose $g \neq 1$ and $g \in P$, that is $g > 1$, which by implies Proposition 2.4 that $g^{-1} < 1$, which further implies that g^{-1} is not in P.

Conversely, given such a $P = \{g \in G \mid g > 1\}$ satisfying conditions (1) and (2). Then we can define a left-order on G as : given $g, h \in G, g < h$ if and only if $g^{-1}h \in P$, $g, h \in G$

Left-Invariance : Suppose $g < h$ so that $g^{-1}h \in P$. Now, given $f \in G, (fg)^{-1}fh = g^{-1}f^{-1}fh = g^{-1}h \in P$, which implies that $fg < fh$.

Transitivity: Suppose $f < g$ and $g < h$ so that $f^{-1}g \in P, g^{-1}h \in P$.

By condition (1), we have $f^{-1}g \cdot g^{-1}h = f^{-1}h \in P$, which implies that $f < h$. Also, the ordering is total by condition (2). □

Theorem 2.10. The group G is bi-orderable if and only if it admits a subset P satisfying the conditions (1) and (2) of Theorem 2.9, and in addition:

(3) $gPg^{-1} \subset P \; \forall g \in P$.

Proof. Suppose $(G, <)$ is bi-orderable, we prove condition (3) for $P = \{g \in G : g > 1\}$.

Let $a \in P$, that is $a > 1$, then by left-invariance $ga > g$ and by right-invariance $gag^{-1} > gg^{-1} = 1$, so that $gag^{-1} \in P$. Hence, $gPg^{-1} \subset P \; \forall g \in P$.

Conversely, suppose that condition (3) holds. It suffices to prove that $(G, <)$ is right invariant, where $<$ is defined by $g < h$ if and only if $g^{-1}h \in P$.

Let $g < h$ so that $g^{-1}h \in P$. Now $(gf)^{-1}hf = f^{-1}g^{-1}hf = f^{-1}(g^{-1}h)f \in P$ and let $f \in G$ be given. The last expression follows from condition (3) since $g^{-1}h \in P$. Thus $fg < fh$.

Note that the condition (3) is equivalent to $g^{-1}Pg \subset P$. □

2.2 Hölder's Theorem and Dynamic Realization of Left Orderings

In this section, we provide some important dynamical properties related to left orderability of a group.

Definition 2.11. A left ordering $<$ of a group is called Archimedean if for every pair of positive elements $x, y \in G$, there exists a natural number n such that $x < y^n$.

The standard orderings of $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ are Archimedean. The additive group $\mathbb{Z}^2$ can be bi-ordered lexicographically, that is, we define order $\prec$ on $\mathbb{Z}^2$ as $(x, y) < (x', y')$ if and only if $x < x'$ or ($x = x'$ and $y < y'$) where $<$ is the usual order on $\mathbb{Z}$. Another way to bi-order $\mathbb{Z}^2$ is to consider $\mathbb{Z}^2$ as sitting in the plane $\mathbb{R}^2$. Let $\vec{m} = (m_1, m_2), \vec{n} = (n_1, n_2) \in \mathbb{Z}^2$ and $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$ with irrational slope. We define a bi-ordering on $\mathbb{Z}^2$ as following:

$$\vec{m} <_v \vec{n} \Longleftrightarrow m_1 v_1 + m_2 v_2 < n_1 v_1 + n_2 v_2.$$

We have the following result related to above ordering of $\mathbb{Z}^2$:

Example 2.2.1. The orderings of $\mathbb{Z}^2$ constructed in previous paragraph are Archimedean, whenever the vector $\vec{v} \in \mathbb{R}^2$ has irrational slope. On the other hand, the lexicographic ordering is not Archimedean.

Proof. Let $\vec{m}, \vec{n} \in (\mathbb{Z}^2, +)$ be positive elements. The mentioned ordering on $\mathbb{Z}^2$ will be denoted by $<_v$ and the usual ordering on $\mathbb{R}$ by $<$. We prove that $\vec{m} < t\vec{n}$ for some integer $t > 0$.

Since $\mathbb{Z}$ is Archimedean, for all $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, there exists integers $t_1 > 0$ and $t_2 > 0$ such that $\vec{m_1} < t_1 \vec{n_1}$ and $\vec{m_2} < t_2 \vec{n_2}$. These give by invariance in $\mathbb{Z}$,

$\vec{m_1}v_1 < t_1\vec{n_1}v_1, \vec{m_2}v_2 < t_2\vec{n_2}v_2$.

Take $t = \max(t_1, t_2)$, then we get, $\vec{m_1}v_1 + \vec{m_2}v_2 < t\vec{n_1}v_1 + t\vec{n_2}v_2$ or $(m_1, m_2) <_v (tn_1, tn_2)$ or $\vec{m} <_v t\vec{n}$. Hence $\mathbb{Z}^2$ is Archimedean with respect to $<_v$.

Finally, we prove that the lexicographic ordering on $\mathbb{Z}^2$ is not Archimedean. Suppose on the contrary that lexicographic ordering on $\mathbb{Z}^2$ is Archimedean. We note that both $(0, 1)$ and $(1, 0)$ are greater than the origin $(0, 0)$ with respect to the lexicographic ordering on $\mathbb{Z}^2$. By the definition of lexicographic order, we see that $(0, 1) < (n, 0)$ for every natural $n \geq 1$, that is, $(0, 1) < n(1, 0)$. This implies that there doesn't exist any $n \geq 1$ such that $(0, 1) > n(1, 0)$, contradiction to the Archimedean property.

$\square$

For the Archimedean ordered groups, we have the following results.

Proposition 2.12 ([8])**.** Every Archimedean left ordering is a bi-ordering.

Proof. Let P be the positive cone of an Archimedean left orderable group $(G, <)$. We use Theorem 2.9 to show that $<$ is a bi-ordering on G, that is, we show that $x^{-1}yx \in P$ for every $x \in G$ and $y \in P$.

Case 1: Let x be positive element in G and let $y \in P$. By Archimedean property, there exists a $n > 0$ such that $x < y^n$. By left invariance, we have $1 < x^{-1}y^n$. This implies $1 < x^{-1}y^nx = (x^{-1}yx)^n$ since the product of positive elements is positive. Since $(x^{-1}yx)^n$ is positive, this means $x^{-1}yx$ is positive so that $x^{-1}yx \in P$.

Case 2: Let x be negative so that x^{-1} is positive and let $y \in P$ be a positive element in G. By Archimedean property, there exists an $n > 0$ such that $x^{-1} < y^n$. By left invariance, we have $1 < xy^n$. This implies $1 < xy^nx^{-1}$ since the product of positive elements is positive. This further implies that xyx^{-1} is positive so that $xyx^{-1} \in P$.

$\square$

Proposition 2.13 (for proof see [7])**.** Every Archimedean left ordered group is abelian.

Recall the Archimedean left orderable group $(\mathbb{Z}^2, <_v)$ from Example 2.2.1. We can view $(\mathbb{Z}^2, <_v)$ as the subgroup of $(\mathbb{R}, +)$ through the following correspondence:

Example 2.2.2. Define a map $\phi : \mathbb{Z}^2 \longrightarrow (\mathbb{R}, +)$ by

$$\phi(\vec{m}) = \frac{\vec{m} \cdot \vec{v}}{||\vec{v}||};$$

where vector $\vec{v}$ has the irrational slope. Then ϕ is order preserving and injective homomorphism.

Proof. Let $\vec{m}$ and $\vec{n}$ be in $\mathbb{Z}^2$, then

$$\phi(\vec{m}) = \phi(\vec{n})$$

$$\Longrightarrow \frac{\vec{m}.\vec{v}}{||\vec{v}||} = \frac{\vec{n} \cdot \vec{v}}{||\vec{v}||}$$

$$\Longrightarrow \vec{m} \cdot \vec{v} = \vec{n} \cdot \vec{v}$$

$$\Longrightarrow m_1 v_1 + m_2 v_2 = n_1 v_1 + n_2 v_2$$

$$\Longrightarrow m_1 + m_2 \alpha = n_1 + n_2 \alpha$$

$$\Longrightarrow m_1 - n_1 = (n_2 - m_2)\alpha$$

where $\alpha = \frac{v_2}{v_1}$ is the slope of $\vec{v}$, which is given to be irrational. Since α is irrational and $\vec{m}, \vec{n} \in \mathbb{Z}^2$, the above equation necessarily implies that to be true

$$m_1 = n_1, m_2 = n_2$$

that is, $\vec{m} = \vec{n}$, hence ϕ is injective.

Let $\vec{m}, \vec{n} \in \mathbb{Z}^2$ with

$$\vec{m} <_v \vec{n}$$

$$\iff m_1 v_1 + m_2 v_2 < n_1 v_1 + n_2 v_2$$

$$\iff \frac{m_1 v_1 + m_2 v_2}{||\vec{v}||} < \frac{n_1 v_1 + n_2 v_2}{||\vec{v}||}$$

$$\iff \phi(\vec{m}) < \phi(\vec{n})$$

Hence ϕ is order-preserving homomorphism. $\square$

The above example shows that the Archimedean group $(\mathbb{Z}^2, <_v)$ can be embedded into the group of reals $(\mathbb{R}, +)$ in such a way that the order is preserved. In fact, this is true for every Archimedean left ordered group. Hölder's theorem asserts that the group of reals $(\mathbb{R}, +)$ acts as a universal for Archimedean left orderable groups. This is our next result to prove.

Theorem 2.14 (Hölder [14])**.** The additive group of reals $(\mathbb{R}, +)$ is universal for Archimedean left orderable groups G, that is, every Archimedean left ordered group G is isomorphic with a subgroup of the additive reals, by an isomorphism under which the ordering of G corresponds the usual order of $\mathbb{R}$.

Proof. Fix $f \in G, f > 0$.

Given any $g \in G$ and a fixed positive integer n, since G is Archimedean there exists an unique integer a_n (dependent on f, g and n) such that $f^{a_n} \leq g^n < f^{a_n+1}$.

For each $g \in G$ and every n, define a map $\phi : G \longrightarrow \mathbb{R}$ by $\phi(g) = \lim_{n\to\infty} \frac{a_n}{n}$ so that $f^{a_n} \leq g^n < f^{a_n+1}$. Without loss of generality, we can assume that $\phi(f) = 1$, and let $\phi_n(g) = \frac{a_n}{n}$ so that $\phi = \lim_{n\to\infty} \frac{a_n}{n}$. We need to prove that the sequence $\{\phi_n\}_{n=1}^{\infty}$ is convergent so that the map ϕ is well-defined. It suffices prove the convergence of the subsequence $\{\psi_n = \frac{a_n}{2^n}\}_{n=1}^{\infty}$. Letting $\psi_n = \frac{a_n}{2^n}$, we prove that $\{\psi_n\}_{n=1}^{\infty}$ is Cauchy. Since G is Archimedean hence biorderable, by the Proposition 2.8 (2), the inequality $f^{a_n} \leq g^{2^n} < f^{a_n+1}$ implies that $f^{2a_n} \leq g^{2^{n+1}} < f^{2(a_n+1)}$. This further implies that ψ_{n+1} lies between $\frac{2a_n}{2^{n+1}} = \psi_n$ and $\frac{2(a_n+1)}{2^{n+1}} = \psi_n + \frac{1}{2^n}$, that is, $\psi_n \leq \psi_{n+1} < \psi_n + \frac{1}{2^n}$. Letting n goes to infinity, we see that the sequence $\{\psi_n\}_{n=1}^{\infty}$ is Cauchy and hence convergent.

1. ϕ is a homomorphism: Let $g_1, g_2 \in G$, then we have the existence of a_n and b_n with $f^{a_n} \leq g_1^n < f^{a_n+1}$ and $f^{b_n} \leq g_2^n < f^{b_n+1}$. Since G is bi-orderable and abelian, we get

$$f^{a_n+b_n} \leq g_1^n g_2^n = (g_1 g_2)^n < f^{a_n+b_n+2}.$$

This means, $\phi_n(g_1g_2)$ lies between $\frac{a_n+b_n}{n} = \phi_n(g_1) + \phi_n(g_2)$ and $\frac{a_n+b_n+2}{n} = \frac{a_n}{n} + \frac{b_n}{n} + \frac{2}{n} = \phi_n(g_1) + \phi_n(g_2) + \frac{2}{n}$, that is, we have

$$\phi_n(g_1) + \phi_n(g_2) \leq \phi_n(g_1g_2) < \phi_n(g_1) + \phi_n(g_2) + \frac{2}{n}$$

Letting n goes to infinity, we get $\phi(g_1g_2) = \phi(g_1) + \phi(g_2)$, since $\lim_{n\to\infty} \phi_n(g) = \phi(g)$ by definition.

2. ϕ is order preserving: Let $g_1 < g_2$, this gives, $g_1^n < g_2^n$ for all $n > 0$. Let a_n and b_n be the largest possible values with $f^{a_n} \leq g_1^n$ and $f^{b_n} \leq g_2^n$.
Since, $g_1^n < g_2^n$, we must have $a_n \leq b_n$. Thus $\phi_n(g_1) = \frac{a_n}{n} \leq \frac{b_n}{n} = \phi_n(g_2)$. This gives, $\phi(g_1) \leq \phi(g_2)$, so that ϕ is order preserving.

3. ϕ **is injective:** Let $h \in G$ such that $\phi(h) = 0$. We claim h must be an identity. Suppose that $h \neq 0$, then either $h > 0$ or $h < 0$.

Assume first that $h > 0$, then by Archimedean property in G, exists integer $n > 0$ such that $f < h^n$. Since ϕ is an order preserving homomorphism, we get $\phi(f) < n\phi(h)$. This gives, $1 < n \cdot 0$, a contradiction.

If $h < 0$, then $h^{-1} > 0$, thus there exists integer $m > 0$ such that $f < (h^{-1})^n$. Since ϕ is order preserving, we get $\phi(f) < n(\phi(h))^{-1}$. This gives $1 < n \cdot 0$, a contradiction. Thus h must be the identity. Hence ϕ is injective. □

Thus we have seen that Hölder's theorem gives an order preserving embedding from Archimedean group G into $(\mathbb{R}, +)$. Next we discuss the universal theorem for left orderable countable group G. The theorem asserts that a countable group G is left-orderable if and only if there is an embedding from G into $Homeo_+(\mathbb{R})$, where $Homeo_+(\mathbb{R})$ denotes the group of all order preserving homeomorphisms of the real line. The standard way of constructing such an embedding is called the dynamic realization. We prove the one direction in the next example:

Example 2.2.3. $Homeo_+(\mathbb{R})$ is left orderable.

Proof. Let $\{x_1, x_2, \dots\}$ be a countable dense subset of reals. For two functions $f, g \in Homeo_+(\mathbb{R})$, define an order by choosing $m = m(f, g)$ to be the minimum i for which $f(x_i) \neq g(x_i)$ and the declare $f \prec g$ if and only if $f(x_m) < g(x_m)$ in the natural ordering of $\mathbb{R}$. We prove this is indeed a left ordering on $Homeo_+(\mathbb{R})$.

Left Orderability: Let $f \prec g$, this means that $f(x_m) < g(x_m)$ where m is the minimum i such that $f(x_i) \neq g(x_i)$. For $h \in Homeo_+(\mathbb{R})$, this implies $h(f(x_i)) < h(g(x_i))$ or $hf < hg$, equivalently, $hf \prec hg$.

Transitivity: Let $f \prec g$ and $g \prec h$. Further, let $m_1 = m_1(f, g)$ be the minimum i with $f(x_i) \neq g(x_i)$ and $f(x_i) < g(x_i)$, and $m_2 = m_2(g, h)$ be the minimum k with

$g(x_k) \neq h(x_k)$ and $g(x_k) < h(x_k)$.

Take, $t = \min(i,k)$, then $f(x_t) < h(x_t)$ with t being minimum such that $f(x_t) \neq h(x_t)$. Hence, transitivity.

Ordering is Total: Since, $\{x_i\}_{i=1}^{\infty}$ is dense in $\mathbb{R}$ and if $f, g \in Homeo_+(\mathbb{R})$, then $f = g$ if and only if $f(x_i) = g(x_i)$ for every i. $\square$

Finally we state the important result of this section, a variant of of which is to be used in the proof of Farrell's Theorem:

Theorem 2.15. A countable group G is left-orderable if and only if G is isomorphic to a subgroup of $Homeo_+(\mathbb{R})$.

We have proved the one direction in Example 2.2.3. For the other direction of the proof, one may see [20, 7].

3

Proof of Farrell's Theorem

In this chapter, we discuss the proof of Farrell's theorem [11, 7] in detail. This theorem connects an embedding of the universal covering space with the right-orderability of the fundamental group of the base space, and hence provides an example of interplay between topology and and the theory of orderability of groups.

3.1 Embedding Problem

One of the fundamental problems in various branches of mathematics is the embedding problem, which asks for the conditions under which one object can be made to sit inside another object. Farrell's theorem deals with the topological embedding problem for which we review we review the following definitions:

Definition 3.1. An injective continous map $f : X \to Y$ between topological spaces X and Y is a topological embedding if f yields a homeomorphism between X and $f(X)$.

Let M and N be smooth manifolds and $f : M \to N$ be a smooth map. Then f is called an immersion if its derivative is everywhere injective.

Definition 3.2. A smooth embedding is defined to be an injective immersion which is an embedding in the above topological sense.

A topological embedding lets us treat X as a subspace of Y and smooth embedding lets us treat the image of X as a submanifold of Y.
One of the most famous embedding theorem in topology is as follows:

Theorem 3.3 (Whitney Embedding Theorem)**.** Any smooth real manifold M of dimension n can be embedded in $\mathbb{R}^{2n}$.

Here we are concerned about the embedding problem applied to covering spaces. We summarise the basic results from the theory of covering spaces below [1].

3.2 Background from Covering Spaces

We start with the definition of covering space:

Definition 3.4. Let X be a topological space. A covering space $\tilde{X}$ of X is a topological space $\tilde{X}$ together with a continuous surjective map $p : \tilde{X} \longrightarrow X$ satisfying the following properties:
(1) For every $x \in X$, there exists an open neighbourhood U of x, such that $p^{-1}(U)$ is a disjoint union of open subsets $\{U_\alpha\}_{\alpha \in J}$ of $\tilde{X}$.
(2) Each U_α is mapped onto U homeomorphically by p.

The neighbourhood U is called evenly covered neighbourhood of x, the open sets $\{U_\alpha\}_{\alpha \in J}$ are referred to as sheets lying above U and for $x \in X$, the subset $p^{-1}(x)$ of $\tilde{X}$ is called the fiber over x. The covering $\tilde{X}$ is called universal covering space if it is simply connected.

Theorem 3.5 (Path Lifting Lemma [1])**.** If $p : \tilde{X} \longrightarrow X$ is a covering space with $x \in X, r \in \tilde{X}, p(r) = x$ and if α is a path in X from x to y, then there is a unique path $\tilde{\alpha}$ in $\tilde{X}$ starting at r with $p \circ \tilde{\alpha} = p$.

Theorem 3.6 ([1])**.** Suppose $p : \tilde{X} \longrightarrow X$ and $\tilde{\alpha_1} : [0,1] \longrightarrow \tilde{X}$ and $\tilde{\alpha_2} : [0,1] \longrightarrow \tilde{X}$ are two lifts of path $\alpha : [0,1] \longrightarrow X$ such that $\tilde{\alpha_1}(t_0) = \tilde{\alpha_2}(t_0)$ for some $t_0 \in [0,1]$. Then the two lifts agree on the whole interval namely, $\tilde{\alpha_1}(t) = \tilde{\alpha_2}(t)$ for every $t \in [0,1]$.

Definition 3.7. A covering transformation or deck transformation of a covering space $\tilde{X}$ is defined as a homeomorphism $T : \tilde{X} \longrightarrow \tilde{X}$ such that $p \circ T = p$.

The set of all deck transformations of $\tilde{X}$ forms a group under composition of maps, denoted by $Deck(\widetilde{X}, X)$. We recall the standard correspondence between the deck $Deck(\widetilde{X}, X)$ transformation and the fundamental group $\pi_1(X)$ when $\widetilde{X}$ is universal. Choose $\widetilde{x_0} = p^{-1}(x_0) \in \widetilde{X}$. Since $T \in Deck(\widetilde{X}, X)$ is a permutation on $p^{-1}(x_0)$, choose a path α from $\tilde{x_0}$ to $T(\tilde{x_0})$ in $\widetilde{X}$. Now $p \circ \alpha$ is some closed path at x_0, say γ in X. The map M which sends $T \in Deck(\widetilde{X}, X)$ to $[\gamma] \in \pi_1(X, x_0)$ is an isomorphism. Thus we have the following result:

Theorem 3.8 ([1])**.** If $\widetilde{X}$ is the universal covering space of X, then $Deck(\widetilde{X}, X) \simeq \pi^{-1}(X)$.

There is an action of $\pi_1(X, x_0)$ on the fibres $p^{-1}(x_0)$. Take $[\gamma] \in \pi_1(X, x_0)$ and send it to the deck transformation T satisfying $T(\widetilde{x_0}) = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the lift of γ starting at base point $\widetilde{x_0}$. Thus T is defined to be the end point of a lift of the path γ.

For simplicity, we write this action as: $[\gamma](\tilde{x_0}) = \tilde{\gamma}(1)$, thinking of $[\gamma]$ as acting on $p^{-1}(x_0)$ given by above the correspondence.

3.3 Farrell's theorem

Now we specify the exact problem we are going to address in this chapter. Let $p : \widetilde{X} \to X$ be a universal covering space, where the space X has a countable fundamental group, and admits a triangulation. Also let $q : X \times \mathbb{R} \to X$ be projection on first factor, where $\mathbb{R}$ denotes a real line. Then the problem we are going to address is the following:

Question 3.9. Does there exists an embedding $f : \widetilde{X} \to X \times \mathbb{R}$ such that $q \circ f = p$?

Farrell's theorem answers this question in terms of the right-orderability of the fundamental group of the space X. Before proceeding to the proof of Farrell's theorem which deals with the embedding of covering spaces, we prove the following trivial result about the embedding of n-sheeted coverings:

Theorem 3.10. Let $p : Y \to X$ be an n-sheeted covering map, where X is a manifold. If X is compact, then Y can be immersed in $\mathbb{R}^{2k}$ for some natural number k.

Proof. Since X is a manifold, the covering space Y is also a manifold because of the local homeomorphism provided by the covering map. We prove Y is compact.
Let C_1 be any open covering of Y. For each $p \in X$, we choose an open set $p \in U_\alpha \subset X$ such that

(i) Y is trivial over U

(ii) each lift of U is contained in some element of C_1.

Then C_2 is an open covering of X which has a finite subcovering $D_1 = \{U_i\}_i^n$ since X is compact. Then the collection $D_2 = \{p^{-1}(U_i)\}_i^n$ is the finite subcover of C_1 so that Y is compact. Hence by the Whitney embedding theorem, Y can be embedded in $\mathbb{R}^{2k+2}$ and immersed in $\mathbb{R}^{2k}$ for some natural number k. □

The next two propositions prove Farrell's theorem, but before that we have a lemma:

Lemma 3.11. If A is a countable, totally ordered set, then there exists an order preserving injection $\psi : A \to \mathbb{R}$ with discrete image in $\mathbb{R}$ (set of reals).

Proof. Let $\{x_i\}_{i=0}^{\infty}$ be an enumeration of A. We will define $\psi(x_n)$ by induction on n. This can be done by defining a sequence (I_j) of disjoint bounded open intervals.

We start by mapping x_0 to any arbitrary real number and let I_0 be any bounded open interval around $\psi(x_0)$. In particular, we can take I_0 symmetrically so that $\psi(x_0)$ is its mid-point. Suppose $\psi(x_0), \psi(x_1), \ldots, \psi(x_{n-1})$ have been already defined in the above manner. Now we map x_n appropriately so that $\psi(x_n)$ is not contained in $\bigcup_{j<n} I_j$.

Let $S_n = \{x_i \mid i \leq n\}$. Then x_n has a certain position in S_n. Let x_n^- be the largest element in S_n which is smaller than x_n and let x_n^+ be the smallest element in S_n which is larger than x_n, so that we have $x_n^- < x_n < x_n^+$. Also by induction assumption, let $\psi(x_n^-) \in I_j$ and $\psi(x_n^+) \in I_k$ for some $j, k < n$, where I_j and I_k are disjoint intervals. Then we map x_n to something in the interval $(\psi(x_n^-), \psi(x_n^+))$. To be specific, let a be the right end point of I_j and b be the left end point of I_k, then $\psi(x_n^-) < a < b < \psi(x_n^+)$. Let $I_n = (c, d)$ such that $a < c < d < b$ and we choose $\psi(x_n) \in (c, d)$. In particular, we can take $\psi(x_n) = \frac{c+d}{2}$. Note that by construction, we have that $x_n^- < x_n < x_n^+$ implies $\psi(x_n^-) < \frac{c+d}{2} < \psi(x_n^+)$ or $\psi(x_n^-) < \psi(x_n) < \psi(x_n^+)$ so that ψ is ordering preserving. Also note that ψ is injective with discrete image since the collection I_j has been chosen as disjoint and ψ maps the elements of A onto the mid-points of I_j inductively.

$\square$

Proposition 3.12 ([11])**.** Let $p : \widetilde{X} \to X$ be a universal covering space and $b_0 \in X$. If there exists a continuous function $h : \widetilde{X} \to \mathbb{R}$ such that the map $f : \widetilde{X} \to X \times \mathbb{R}$ defined by $f(x) = (p(x), h(x))$ is an injection, then $\pi_1(X, b_0)$ is right-orderable.

Proof. Define an ordering $\prec$ on $\pi_1(X, b_0)$ as follows: $[\alpha] \prec [\beta]$ if and only if

$$h([\alpha]\widetilde{b_0}) < h([\beta]\widetilde{b_0}) \tag{3.1}$$

where $\tilde{b_0}$ is a lift of $b_0 \in X$ through the map p and $<$ is the usual ordering of $\mathbb{R}$. We prove by contradiction that $(\pi_1(X, b_0), \prec)$ is a right-ordered group. Assume that the ordering is not right invariant, so that there exists $[\alpha], [\beta], [\gamma] \in \pi_1(X, b_0)$ with

$$[\alpha] \prec [\beta]$$

but

$$[\alpha][\gamma] \succ [\beta][\gamma]. \tag{3.2}$$

Since γ is a loop based at $b_0 \in X$, and $[\alpha]\tilde{b_0}$ and $[\beta]\tilde{b_0}$ are the points in $\widetilde{X}$, hence by Theorem 3.5, there exists lift $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ of γ such that $\tilde{\gamma_1}(0) = [\alpha]\tilde{b_0}$ and $\tilde{\gamma_2}(0) = [\beta]\tilde{b_0}$. Also $\tilde{\gamma_1}(1) = [\alpha \circ \gamma]\tilde{b_0} = [\alpha][\gamma]\tilde{b_0}$ and $\tilde{\gamma_1}(1) = [\beta \circ \gamma]\tilde{b_0} = [\beta][\gamma]\tilde{b_0}$.

Define a map $r : [0, 1] \to \mathbb{R}$ by $r(t) = h(\tilde{\gamma_1}(t)) - h(\tilde{\gamma_2}(t))$. Then we have that $r(0) = h(\tilde{\gamma_1}(0)) - h(\tilde{\gamma_2}(0)) = h([\alpha]\tilde{b_0}) - h([\beta]\tilde{b_0}) < 0$ by the prescription (3.1). Also $r(1) = h(\tilde{\gamma_1}(1)) - h(\tilde{\gamma_2}(1)) = h([\alpha][\gamma]\tilde{b_0}) - h([\beta][\gamma]\tilde{b_0}) > 0$, since by the assumption (3.2), $[\alpha][\gamma] \succ [\beta][\gamma]$ so that $h([\alpha][\gamma]\tilde{b_0}) > h([\beta][\gamma]\tilde{b_0})$. Therefore we have $r(0) < 0$ and $r(1) > 0$, hence by intermediate value theorem, there exists a real number t_0 such that $r(t_0) = 0$, that is, $h(\tilde{\gamma_1}(t_0)) = h(\tilde{\gamma_2}(t_0))$. This implies $f(\tilde{\gamma_1}(t_0)) = f(\tilde{\gamma_2}(t_0)))$. Since f is injective, we get $\tilde{\gamma_1}(t_0) = \tilde{\gamma_2}(t_0)$. We know that if the two lifts agree at one point agree everywhere. In particular, the lifts γ_1 and γ_2 must also agree at initial points $\tilde{\gamma_1}(0) = [\alpha]\tilde{b_0}$, and $\tilde{\gamma_2}(0) = [\beta]\tilde{b_0}$. This implies $[\alpha]\tilde{b_0} = [\beta]\tilde{b_0}$ forcing $[\alpha] = [\beta]$, a contradiction to the assumption that $[\alpha] \prec [\beta]$. □

Proposition 3.13 ([11])**.** Suppose that X is a space admitting a triangulation with countable fundamental group $\pi_1(X)$ and $x_0 \in X$. If $\pi_1(X, x_0)$ is right-orderable then there exists a map $h : \widetilde{X} \to \mathbb{R}$ such that the map $f : \widetilde{X} \to X \times \mathbb{R}$ given by $f(g) = (p(g), h(g))$ is an embedding, where p is a universal covering map of X.

Proof. Since X is triangulable, we fix a triangulation on X. This induces a corresponding triangulation on $\widetilde{X}$ through the map p, that is, the simplexes on $\widetilde{X}$ are the liftings of simplexes of X. For each vertex $x \in X$, we choose a point $\tilde{x} \in p^{-1}(x)$ in $\widetilde{X}$.

Assume that $(\pi_1(X, x_0), <)$ is right-orderable. Then by Lemma 3.1, there exists an order preserving injection $\psi : \pi_1(X) \to \mathbb{R}$ with discrete image.

Step 1: We construct h on the vertices of $\widetilde{X}$ as following.

For each vertex $x \in X$ and each element $[\gamma] \in \pi_1(X)$, we define $h : \widetilde{X} \to \mathbb{R}$ by

$$h([\gamma]\tilde{x}) = \psi([\gamma]).$$

Note that this definition makes sense through Theorem 2.1, which gives a correspondence between the deck transformations and the fundamental group.

Then we extend h linearly to the rest of $\tilde{X}$ using barycentric coordinates. Let $g \in \widetilde{X}$ be a point lying in a simplex with vertices as $[\gamma_0]\tilde{x_0}, [\gamma_1]\tilde{x_1}, ..., [\gamma_n]\tilde{x_n}$, then we can write g as:

$$g = t_0[\gamma_0]\tilde{x_0} + t_1[\gamma_1]\tilde{x_1} + ... + t_n[\gamma_n]\tilde{x_n},$$

where $t_i \in [0, 1]$ with $\sum_{i=0}^{n} t_i = 1$.

Then define h as:

$$h(g) = t_0 h([\gamma_0]\tilde{x_0}) + t_1 h([\gamma_1]\tilde{x_1}) + ... + t_n h([\gamma_n]\tilde{x_n})$$

$$= t_0\psi([\gamma_0]) + t_1\psi([\gamma_1]) + ... + t_n\psi([\gamma_n])$$

With this choice of h, we prove that $f(g) = (p(g), h(g))$ is an embedding.

Step 2 (f is injective): To prove an injectivity of f, we take $c, d \in \widetilde{X}$ with $c \neq d$ and show that $f(c) \neq f(d)$.

First note that if $c, d \in \tilde{X}$ satisfy $p(c) \neq p(d)$ then $f(c) \neq f(d)$, so we only deal with $p(c) = p(d)$. Suppose $c, d \in \widetilde{X}, c \neq d$ such that $p(c) = p(d)$. This is possible since p is covering map and we can take c and d from the same fibre.

Let Δ_1 and Δ_2 be the simplexes in $\widetilde{X}$ such that $c \in \Delta_1, d \in \Delta_2$, and $p(\Delta_1) = p(\Delta_2) = \Delta$ where Δ is the simplex in X containing $p(c)$. Since each deck transformation acts by permutation on $p^{-1}(\Delta)$ and by the correspondence Theorem 3.8, there exists an element $[\gamma] \in \pi_1(X, x_0)$ such $[\gamma](\Delta_1) = \Delta_2$ and $[\gamma]^{-1}(\Delta_2) = \Delta_1$. Let 1 denote the identity of $\pi_1(X)$, then either $[\gamma] > 1$ or $[\gamma] < 1$, since $\pi_1(X, x_0)$ is right-orderable. Without loss of generality, we may assume that $[\gamma] > 1$. We work with this special $[\gamma]$.

Let $x_0, x_1, \dots, x_n$ be the vertices of Δ, then there exists elements $[\gamma_i]$ in $\pi_1(X, x_0)$ such that the points $[\gamma_i]\tilde{x_i}$ are the vertices of Δ_1. Since $[\gamma](\Delta_1) = \Delta_2$, the vertices of Δ_2 are the points $[\gamma]([\gamma_i]\tilde{x_i})$ where $i = 0, 1, \dots, n$. Thus c can be written in barycentric coordinates as

$$c = t_0[\gamma_0]\tilde{x_0} + t_1[\gamma_1]\tilde{x_1} + \cdots + t_n[\gamma_n]\tilde{x_n}$$

and consequently,

$$d = [\gamma]c = t_0[\gamma][\gamma_0]\tilde{x_0} + t_1[\gamma][\gamma_1]\tilde{x_1} + \cdots + t_n[\gamma][\gamma_n]\tilde{x_n}.$$

Therefore by the definition of h, we have that

$$h(c) = t_0\psi([\gamma_0]) + t_1\psi([\gamma_1]) + \cdots + t_n\psi([\gamma_n])$$

and

$$h(d) = t_0\psi([\gamma][\gamma_0]) + t_1\psi([\gamma][\gamma_1]) + \cdots + t_n\psi([\gamma][\gamma_n]).$$

But $[\gamma] > 1$ implies that $[\gamma][\gamma_i] > [\gamma_i]$ for all i since $\pi_1(X, x_0)$ is right-orderable. This implies $\psi([\gamma][\gamma_i]) > \psi([\gamma_i])$ for each $i = 0, 1, \dots, n$, since ψ is an order preserving map with discrete image (so that inequality is strict). This gives us $h(d) > h(c)$. Though $p(c) = p(d)$ we have $h(c) < h(d)$, hence $f(c) \neq f(d)$ so that f is an injection.

Step 2 (f is continuous): Since the covering map p is continuous, we only need to show that h is continuous. Now recall the construction of h, which was first defined on the vertices of $\widetilde{X}$, then on the simplexes of $\widetilde{X}$ through barycentric coordinates, and then linearly extended to the rest of $\widetilde{X}$. Since simplexes are closed subsets of $\widetilde{X}$, by the Pasting Lemma [18] h is continuous, so that f is also continuous.
Hence f is an embedding. $\square$

Combining the Proposition 3.12 and 3.13, we have proved the following main result of this chapter:

Theorem 3.14 (Farrell's theorem [11])**.** Suppose X is a space admitting triangulation with countable fundamental group $\pi_1(X)$. Then $\pi_1(X)$ is right-orderable if and only if there is an embedding $f : \widetilde{X} \to X \times \mathbb{R}$ so that $q \circ f = p$, where $q : X \times \mathbb{R} \to X$ is a projection on the first factor and $p : \tilde{X} \longrightarrow X$ is the universal cover.

We note that Farrell actually proved a more general result by assuming that X is a Hausdorff, paracompact space with countable fundamental group, and that $\widetilde{X}$ is an arbitrary regular covering space of X. The result can be found be in [11].

4

Circular Orders and the Second Cohomology Group

In this chapter, we introduce circular orders and secret left orders on a group G, and establish a relationship between them through the second cohomology group. We discuss how a left ordering on a group may arise as a special case of circular ordering. The results discussed in this chapter will be used to reformulate the generalized version of Farrell's theorem in Chapter 5.

4.1 Second Cohomology Group

In this section, a brief introduction is given to second cohomology group of a group. One may refer to *Cohomology of Groups* [5] by *K.S Brown* for detailed exposition. We start with the definitions:

Definition 4.1 (Exact Sequence)**.** A sequence

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} G_3 \xrightarrow{f_4} \dots \xrightarrow{f_n} G_n$$

of groups and group homomorphisms is called *exact* if $Image(f_k) = Kernel(f_{k+1})$.

Definition 4.2 (Short Exact Sequence)**.** The sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called *short exact* sequence if f is a monomorphism and g is an epimorphism.

Since image of f is equal to the kernel of g, we can think of A embedded in B through the embedding f. Also C can be thought of as the quotient $B \big/ Image(f)$ since g is onto.

Definition 4.3 (Split)**.** The above short exact sequence is called *(right) split* if there exists a homomorphism $h : C \longrightarrow B$ such that the composition $g \circ h = 1_C$ is the identity map on C.

Definition 4.4 (Group Extension)**.** If A and C are two groups, then G is called an *extension* of A by C if there is a short exact sequence

$$1 \to A \xrightarrow{i} G \xrightarrow{\pi} C \to 1.$$

Definition 4.5 (Equivalent Extensions)**.** The two group extensions

$$1 \to A \xrightarrow{i} G \xrightarrow{\pi} C \to 1$$

and

$$1 \to A \xrightarrow{i'} G' \xrightarrow{\pi'} C \to 1$$

are *equivalent* if there exists a group isomorphism $F : G \longrightarrow G'$ making the following diagram commutative.

$$\begin{tikzcd}
 & & G \arrow[dd, "F"] \arrow[dr] & & \\
1 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 1 \\
 & & G' \arrow[ur] & &
\end{tikzcd}$$

A trivial extension is an extension $1 \to A \to G \to C \to 1$ which is equivalent to the extension $1 \to A \to A \times C \to C \to 1$ where the left and right arrows are the inclusion and the projection maps respectively. The question which group G is an extension of A by C (in the sense of above equivalence) is called an extension problem.

Definition 4.6 (Central Extensions)**.** A *central extension* is an extension $1 \to A \xrightarrow{i} G \xrightarrow{\pi} C \to 1$ such that A is in $Z(G)$, the center of group G.

Let $\mathcal{E}(G)$ denotes the set of all isomorphism classes of central extensions of A by G. There is a one-one correspondence between $\mathcal{E}(G)$ and $H^2(G; A)$, the second cohomology group. With $A = \mathbb{Z}$, we discuss this correspondence.

We look at the extension $1 \to \mathbb{Z} \xrightarrow{i} C \xrightarrow{\pi} G \to 1$, not necessarily split. Consider a set theoretical section $s : G \longrightarrow C$ satisfying $\pi \circ s = 1_G$ which is not necessarily a homomorphism. We may measure to what extent it fails to be a homomorphism in the following sense:

$$s(g)s(h) = i(f(g,h))s(gh) \tag{4.1}$$

where $f : G \times G \longrightarrow \mathbb{Z}$ is a function. The section s is said to be *normalized* if $s(1) = 1$. The condition $s(1) = 1$ is called normalization and it implies

$$f(g,1) = f(1,g) = 0. \tag{4.2}$$

We wish to define a group structure on $C = G \times \mathbb{Z}$ by the following prescription:

$$(a,g) \cdot (b,h) = (a + b + f(g,h), gh) \tag{4.3}$$

However we cannot start with an arbitrary function $f : G \times G \longrightarrow \mathbb{Z}$ since the multiplication fails to be associative in general. Let $(a,g),(b,h),(c,k) \in G \times \mathbb{Z}$, then computation shows that

$$((a,g)\cdot(b,h))\cdot(c,k) = (a,g)\cdot((b,h)\cdot(c,k)) + f(h,k) - f(gh,k) + f(g,hk) - f(g,h)$$

so that for associativity to hold we must have

$$f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0 \tag{4.4}$$

With this restriction on f, the prescription in equation (4.3) makes $G \times \mathbb{Z}$ into a group with identity $(0,1)$, and inverse given by $(a,g)^{-1} = (-a - f(g^{-1},g), g^{-1})$. A function f satisfying condition (4.4) is called an inhomogeneous 2-cocycle. So the extensions $1 \to \mathbb{Z} \to C \ \to G \to 1$, with normalized section $s : G \longrightarrow C, s(1) = 1$ are classified by inhomogeneous 2-cocycles f.

Two or more sections may give rise to the same central extension $G \times \mathbb{Z}$ for given 2-cocycle f. Thus we must identify the 'same' sections. Consider another normalized section $s' : G \longrightarrow C$ of the same extension. Then s and s' must differ by some function $r : G \longrightarrow \mathbb{Z}$, that is

$$s'(g) = i(r(g))s(g). \tag{4.5}$$

Since $s(1) = 1$ and $s'(1) = 1$, we get $r(1) = 0$. This section s' gives rise to another inhomogeneous 2-cocycle $f' : G \times G \longrightarrow \mathbb{Z}$. Using equations (4.1) and (4.5), the computation shows:

$$s'(g)s'(h) = i(r(g) + r(h) + r(g,h) - r(gh))s'(gh). \tag{4.6}$$

This measures the extent to which s differs from s'. Thus for sections s and s' to be equal, we must have $f'(g,h) = r(g) + r(h) + f(g,h) - r(gh)$ or

$$f'(g,h) - f(g,h) = r(g) + r(h) - r(gh). \tag{4.7}$$

The functions of the form $(g,h) \to r(g) + r(h) - r(gh), \quad r(1) = 0$ are called 2-coboundaries. We consider sections s and s' as same if the condition (4.7) is satisfied. Thus the central extensions are classified by the following group called the second cohomology group:

$$H^2(G;\mathbb{Z}) = \frac{\{\text{2-cocycles} \quad f : G \times G \longrightarrow \mathbb{Z}, f(g,1) = f(1,g) = 0\}}{\{\text{2-coboundaries} \quad (g,h) \to r(g) + r(h)) - r(gh), r(1) = 0\}} \tag{4.8}$$

where f also satisfies the condition (4.3). If the functions f and r are bounded, then the above group is called bounded cohomology group and is denoted by $H_b^2(G;\mathbb{Z})$. The above discussion tells us that there is an injection from $H^2(G;\mathbb{Z})$ into the set $\mathcal{E}(G)$, that is, we picked up an element $[f] \in H^2(G;\mathbb{Z})$ and got a central extension $1 \to \mathbb{Z} \to K \ \to G \to 1$, where the group operation on $K = G \times \mathbb{Z}$ is defined through the map f by the prescription (4.3).

Conversely, we can take a central extension $1 \to A \xrightarrow{i} C \ \xrightarrow{\pi} G \to 1$ and choose a set theoretic section $s : G \longrightarrow C, \quad \pi \circ s = 1_G, \quad s(1) = 1$. Define a function $f : G \times G \longrightarrow \mathbb{Z}$ by $i(f(g,h)) = s(g)s(h)s(gh)^{-1}$. Note that $f(g,1) = f(1,g) = 0$ for all $g \in G$. A computation shows that f is an inhomogeneous 2-cocycle and different choices of s gives rise to cohomologous choices for f, that is, such two functions differ in the sense of equation (4.7) for some function $r : G \longrightarrow \mathbb{Z}$ so that $[f] \in H^2(G;\mathbb{Z})$. All this informal discussion leads us to the following theorem which we shall use in the next section:

Theorem 4.7. Let $\mathcal{E}(G)$ denotes the set of all equivalence classes of central extensions of $\mathbb{Z}$ by G. Then $H^2(G;\mathbb{Z})$ is in bijective correspondence to $\mathcal{E}(G)$.

4.2 Circular Orders

We start this section with the definition of circular ordering [3, 6] on a group G.

Definition 4.8. Let G be a group. A left invariant circular ordering is a function $c : G \times G \times G \longrightarrow \{\pm 1, 0\}$ such that

1. c is non-degenerate: $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for some $i \neq j$.

2. c satisfies homogeneous 2-cocycle condition:

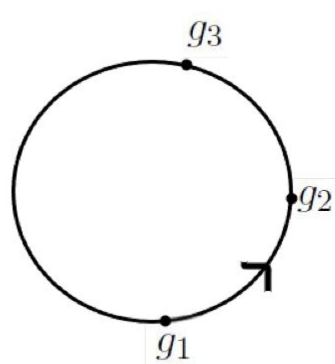
$$c(g_1, g_2, g_3) - c(g_0, g_2, g_3) + c(g_0, g_1, g_3) - c(g_0, g_1, g_2) = 0$$

   for any $g_0, g_1, g_2, g_3 \in G$.

3. c is left invariant: $c(gg_1, gg_2, gg_3) = c(g_1, g_2, g_3)$ for all $g, g_1, g_2, g_3 \in G$.

If the group G is S^1, we may visualize c as a function on S^1. If g_1, g_2 and g_3 are distinct and positively oriented (anticlockwise direction) on the circle, then we take $c(g_1, g_2, g_3) = 1$ and if they are negatively oriented, we take $c(g_1, g_2, g_3) = -1$. If any of the two elements from g_1, g_2, g_3 coincide, we take $c(g_1, g_2, g_3) = 0$, which we call a degenerate case.

Figure 4.1: Circular order on circle.



A right invariant circular ordering is a circular ordering which satisfies $c(g_1g, g_2g, g_3g) = c(g_1, g_2, g_3)$ for all $g, g_1, g_2, g_3 \in G$ instead of condition (3) in the Definition 4.8. The following result shows that left invariant and right invariant circular orderings are essentially same:

Proposition 4.9. Let $c : G \times G \times G \longrightarrow \{\pm 1, 0\}$ be a given left invariant circular order, then the map $d(g_1, g_2, g_3) = c(g_1^{-1}, g_2^{-1}, g_3^{-1})$ defines a right invariant circular on $G \times G \times G$.

Proof. 1. d is non-degenerate:

Since c is a circular order, we have $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for $i \neq j$. This implies $c(g_1^{-1}, g_2^{-1}, g_3^{-1}) = 0$ if and only if $g_i^{-1} = g_j^{-1}$ for $i \neq j$. By the definition of d, this further implies that $d(g_1, g_2, g_3) = 0$ if and only if $g_i^{-1} = g_j^{-1}$ or $g_i = g_j$ for $i \neq j$.

2. d satisfies homogeneous 2-cocycle condition: Let $g_0, g_1, g_2, g_3 \in G$. By the definition of d we have $d(g_1, g_2, g_3) - d(g_0, g_2, g_3) + d(g_0, g_1, g_3) - d(g_0, g_1, g_2) = c(g_1^{-1}, g_2^{-1}, g_3^{-1}) - c(g_0^{-1}, g_2^{-1}, g_3^{-1}) + c(g_0^{-1}, g_1^{-1}, g_3^{-1}) - c(g_0^{-1}, g_1^{-1}, g_2^{-1}) = 0$ since all $g_i^{-1} \in G$ and c is a 2-cocycle.

3. d is right invariant: Let $g, g_1, g_2, g_3 \in G$. Then by the definition of d we have $d(g_1g, g_2g, g_3g) = c((g_1g)^{-1}, (g_2g)^{-1}, (g_3g)^{-1}) = c(g^{-1}g_1^{-1}, g^{-1}g_2^{-1}, g^{-1}g_3^{-1}) = c(gg^{-1}g_1^{-1}, gg^{-1}g_2^{-1}, gg^{-1}g_3^{-1}) = c(g_1^{-1}, g_2^{-1}, g_3^{-1}) = d(g_1, g_2, g_3)$. The third last equation follows by multiplying g on left since c is left invariant. This completes the proof.

□

Due to Proposition 4.9, any result which is true for left invariant circular orders would be true for right invariant circular orders. In particular we shall be considering right invariant circular ordering in Theorem 5.8 instead of left invariant circular ordering. The reason for this is the use of concatenation of paths in the fundamental group of X which is done from left to right.

Let $Homeo_+(S^1)$ be the group of all orientation preserving homeomorphisms of S^1. Recall that a countable group is left orderable if and only if it embeds in $Homeo_+(\mathbb{R})$, the group of orientation preserving homeomorphisms of $\mathbb{R}$. A similar result holds for circularly orderable groups. More specifically, we have the following result [6, 17]:

Theorem 4.10. Let G be a countable group. Then G is circularly orderable if and only if there exists an injective homomorphism from G into $Homeo_+(S^1)$.

Every left orderable group can be considered as circularly ordered in the following way:

Proposition 4.11. Every left orderable group is circular orderable.

Proof. Let $<$ be a left order on a group G. Declare a circular order on G by

$$c(g_1, g_2, g_3) = \begin{cases} +1 & \text{if } g_1 < g_2 < g_3 \quad or \quad g_2 < g_3 < g_1 \quad or \quad g_3 < g_1 < g_2 \\ -1 & \text{if } g_1 < g_3 < g_2 \quad or \quad g_2 < g_1 < g_3 \quad or \quad g_3 < g_2 < g_1 \\ 0 & \text{otherwise} \end{cases}$$

That is, we define $c(g_1, g_2, g_3) = sign(\sigma)$ if and only if $g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$ for all distinct $g_1, g_2, g_3 \in G$, where $\sigma \in S_3$ is a permutation. □

The circular orderings on a group G which arise in this way are called *secret left orderings.*

Definition 4.12. A circular order c on a group G is *secret left order* if there exists a left order $<$ on G such that $c(g_1, g_2, g_3) = sign(\sigma)$ if $g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$ for all distinct $g_1, g_2, g_3 \in G$.

In general, a circularly orderable group may not be left orderable. We wish to look for the conditions under which circularly orderable group may be promoted into a left orderable group. Amazingly, the answer is provided through a cohomological argument.

We note that circular ordering c on a group G is a bounded homogeneous 2-cocycle (in the sense of condition (2) of definition 4.8). The inhomogeneous form of c is defined by $c' : G \times G \longrightarrow \{\pm 1, 0\}, (g,h) \longrightarrow c(1,g,gh)$. We note that $c'(1,g) = c'(g,1) = c(g,g^{-1}) = 0$ for all $g \in G$. We construct a dehomogenized 2-cocycle from c. Following [22], consider the function $f_c : G \times G \longrightarrow \{0,1\}$ given by

$$f_c(a,b) = \begin{cases} 0 & \text{if } a = id \quad or \quad b = id \quad or \quad c(1,a,ab) = 1, \\ 1 & \text{if } ab = id (a \neq id) \quad or \quad c(1,a,ab) = -1. \end{cases} \tag{4.9}$$

The function f_c satisfies equation (4.4), and hence is an inhomogeneous 2-cocycle. By Theorem 4.7, f_c will give rise to a central extension say $\tilde{G}_c$ of G. The group $\tilde{G}_c$ is in fact $\mathbb{Z} \times G$ equipped with group structure $(n,a) \cdot (m,b) = (n + m + f_c(a,b), ab)$. Setting $P = \{(x,a) | n \geq 0\} \setminus \{(0,id)\}$, then P is a positive cone of a left ordering $<_c$ on $\tilde{G}_c$. We summarize these facts as following (for proof see [4, 22])

Proposition 4.13. 1. The function $f_c : G \times G \longrightarrow \{0,1\}$ defined in equation (4.9) is an inhomogeneous 2-cocycle.

2. The central extension $\tilde{G}_c$ of G arising from f_c in the sense of Theorem 4.7 is a left orderable group.

Supposing $[f_c] = id \in H^2(G;\mathbb{Z})$, then by equation (4.1), $s(g)s(h) = s(gh)$, so that the section $s : G \longrightarrow C$ is a homomorphism. This implies that the extension is split and hence by splitting lemma $\tilde{G}_c$ is isomorphic to $\mathbb{Z} \times G$. Since by Proposition 4.13, $\tilde{G}_c$ is left orderable, we have that G is also left orderable. Thus we have [4]:

Proposition 4.14. If G is a circular ordered group and $[f_c] = id \in H^2(G; \mathbb{Z})$, then G is left orderable.

In the above proposition, the circular ordering c doesn't provide us with an explicit left ordering on G. However, if the circular ordering c is a secret left ordering then the left order on G is explicit. When this happens is characterized through the second bounded cohomology group:

Proposition 4.15. Let G be a circularly ordered group with circular ordering c. Then the circular order c is secretly a left order if and only if $[f_c] = id \in H_b^2(G; \mathbb{Z})$.

A detailed proof of Proposition 4.15 is given in [4]. We use Proposition 4.14 and Proposition 4.15 in the next chapter to formulate a generalization of Farrell's theorem.

5

Generalizing Farrell's theorem

In this chapter, we discuss the generalization of Farrell's theorem in two different possible ways. The first possible direction is to replace the real line $\mathbb{R}$ in Farrell's theorem by any other topological space A and look for the existence of an embedding $f : \tilde{X} \longrightarrow X \times A$ in terms of the right orderability of $\pi_1(X)$. The second direction is to look for the existence of an embedding $f : \tilde{X} \longrightarrow X \times A$ in terms of the circular orderability of $\pi_1(X)$. More specifically, we have the following questions. Suppose X is a topological space admitting a triangulation and that the fundamental group $\pi_1(X)$ is countable. Henceforth we shall be dealing only with the topological spaces with these conditions.

Question 5.1. For what topological spaces A is it true that $\pi_1(X)$ is right orderable if and only if there is an embedding $f : \widetilde{X} \to X \times A$ with $q \circ f = p$?

Here $q : X \times A \to X$ denotes a projection on the first factor and $p : \tilde{X} \longrightarrow X$ is the universal cover.

Question 5.2. For what topological space A is it true that $\pi_1(X)$ is circular-orderable if and only if there is an embedding $f : \widetilde{X} \to X \times A$ with $q \circ f = p$?

For Question 5.2, we are specifically interested when A is a circle S^1. We have seen in the previous chapter how every left ordering gives rise to a circular ordering, and investigated the converse through the cohomological argument (Proposition 4.15). We note that in Proposition 3.12 of Farrell's theorem, we are exploiting the group structure and orderability of real line $\mathbb{R}$ to construct the map h. Henceforth, we assume that the topological space A has a group structure, though we do not not need it to be a topological group.

5.1 Answering Question 5.1

Let us take A to be a one dimensional second countable Hausdorff manifold. This case is interesting to investigate because the Farrell's theorem deals with the case $A = \mathbb{R}$ which is a 1-manifold. Thus we wish to explore the Farrell's theorem for other 1-manifolds. We have the following classification theorem for one dimensional manifolds:

Theorem 5.3. Any connected 1-manifold is homeomorphic to one of the following four manifolds [12]:

(1) $\mathbb{R}$

(2) $[0, 1]$

(3) $[0, 1)$

(4) S^1

We wish to generalize Proposition 3.12 (one direction of Farrell's theorem) by replacing $\mathbb{R}$ with any of the above 1-manifolds. Note that $[0, 1]$ and $[0, 1)$ do not have any natural group structures. Since there is a continuous injection between $[0, 1], [0, 1)$ into $\mathbb{R}$, we can transport the left-orderable group structure to them from $\mathbb{R}$ and we get the following result.

Proposition 5.4. Let $p : \widetilde{X} \to X$ be a universal covering space. If there exists a continuous function $h : \widetilde{X} \to A$ where $A = [0,1]$ or $[0,1)$ such that the map $f : \widetilde{X} \to X \times A$ defined by $f(x) = (p(x), h(x))$ is an injection, then $\pi_1(X, b_0)$ is right-orderable, where $b_0 \in X$.

Proof. Note that in the proof of Proposition 3.12, we used the intermediate value theorem but it does not hold for $A = [0,1]$ or $[0,1)$. We discuss the case $A = [0,1]$; the other case is similar. Since the inclusion map $i : [0,1] \longrightarrow \mathbb{R}$ is a continuous injection, we can define a right ordering on $\pi_1(X, b_0)$ by $[\alpha] < [\beta]$ if and only if

$$i \circ (h([\alpha]\widetilde{b_0})) < i \circ (h([\beta]\widetilde{b_0}))$$

where all notations are same as in Proposition 3.1.
The proof is exactly same as in the Proposition 3.1 except that the map $r : [0,1] \to \mathbb{R}$ is defined by $r(t) = i \circ (h(\tilde{\gamma}_1(t))) - i \circ (h(\tilde{\gamma}_2(t)))$. □

The next result shows that S^1 cannot be left ordered.

Lemma 5.5. A circle S^1 cannot be left ordered with its usual group structure of complex multiplication.

Proof. Suppose there exists a linear order on S^1 which is left invariant. Without loss of generality, let $-1 < 1$. By left multiplication this implies $-i < i$, which further implies that $-i^2 < i^2$, that is $1 < -1$, a contradiction. □

In Proposition 5.4, we have exploited the fact that there exists a continuous injection from $[0,1]$ into $\mathbb{R}$. Even if there exists some other group structure on S^1 with respect to which it is a left ordedrable group, the following result shows that the same proof strategy may not be applied to $A = S^1$.

Proposition 5.6. There is no continuous injection from S^1 from $\mathbb{R}$.

Proof. Suppose that $h : S^1 \longrightarrow \mathbb{R}$ is a continuous injection. This implies that the image of f in $\mathbb{R}$ is connected and compact. Thus we must have $f(S^1) = [a,b]$ for some

$a, b \in \mathbb{R}$. Since S^1 is compact and $[a, b]$ is Hausdorff, f must be a homeomorphism which is not possible by Theorem 5.3. □

Lemma 5.5 shows that S^1 is not left orderable with respect to the usual group structure of complex multiplication. But it may be left orderable with respect to some other group structure and the Question 5.1 remains open in this case. Also Proposition 5.6 shows that even if there exists some group structure on S^1, we cannot replace A by S^1 in Question 5.1 by using a similar argument as in Proposition 5.4, which essentially uses intermediate value theorem. We note that for $A = [0, 1]$ and $A = [0, 1)$, we use inclusion map (which is a continuous injection) in Proposition 5.4 to transport the argument from the classical proof of Farrell's theorem. Since no such continuous map exists in case of $A = S^1$ due to Proposition 5.6, we cannot apply the similar argument.

5.2 Farrell's theorem and circular orders

Now we provide a positive answer to Question 5.2 for $A = S^1$ in one direction as a Theorem 5.8, and provide a counterexample for the converse as an Example 5.2.1. We need the following proposition:

Proposition 5.7. Let $\{A, B, C\} \subset S^1 \times \{1\}$ and $\{A', B', C'\} \subset S^1 \times \{0\}$ be distinct points on the top and bottom of the cylinder $S^1 \times [0, 1]$ respectively with the triples $\{A, B, C\}$ and $\{A', B', C'\}$ oppositely oriented (see Figure 5.1). Suppose the points A and B are connected by non-intersecting continuous paths $\alpha : [0, 1] \longrightarrow S^1 \times [0, 1]$ and $\beta : [0, 1] \longrightarrow S^1 \times [0, 1]$, with z coordinate strictly monotonic, to the points A' and B' respectively. Then any continuous path $\gamma : [0, 1] \longrightarrow S^1 \times [0, 1]$ connecting points C and C' must intersect path α or β at some point.

Proof. Without loss of generality, we may assume the following:

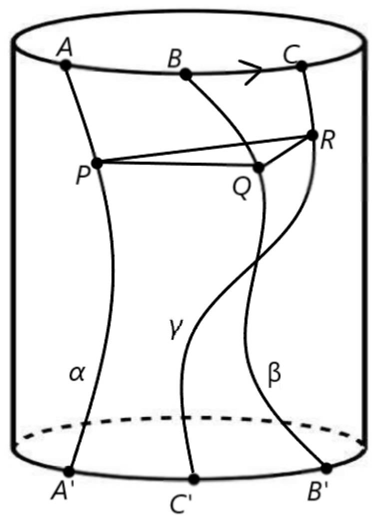
$$\alpha(0) = A', \beta(0) = B', \gamma(0) = C',$$

$$\alpha(1) = A, \beta(1) = B, \gamma(1) = C.$$

and in general, $\alpha(t), \beta(t), \gamma(t) \in S^1 \times \{t\}$ for all $t \in [0, 1]$.

Consider the triangle PQR on the surface of a cylinder (see Figure 5.1) at height t

Figure 5.1: Cylinder depicting Proposition 5.7.



with vertices $P = \alpha(t), Q = \beta(t)$ and $R = \gamma(t), t \in [0, 1]$. Since the z-coordinate of each path α, β and γ is monotone, we may work in the plane $\mathbb{R}^2$ determined by the points P, Q and R, and at the height t where t is fixed but arbitrary. Thus we may regard our paths as the functions from $[0, 1]$ into the plane $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{t\} \subset \mathbb{R}^3$, that is, we can write $\alpha, \beta, \gamma : [0, 1] \longrightarrow \mathbb{R}^2$ with coordinates as $\alpha(t) = (\alpha_1(t), \alpha_2(t)), \beta(t) = (\beta_1(t), \beta_2(t)), \gamma(t) = (\gamma_1(t), \gamma_2(t))$. The reason for this identification is to apply the determinant formula to compute the planar area of triangle PQR which lies at fixed but arbitrary height t. The signed area $Area(P, Q, R, t)$ of the triangle PQR is given by:

$$Area(P, Q, R, t) = \frac{1}{2}\begin{vmatrix} \alpha_1(t) & \alpha_2(t) & 1 \\ \beta_1(t) & \beta_2(t) & 1 \\ \gamma_1(t) & \gamma_2(t) & 1 \end{vmatrix}$$

which is a continuous function of t.

Since the points A, B, C are oriented in an anticlockwise direction, the area is positive, that is $Area(A, B, C, 1) > 0$. Also the points A', B', C' are oriented in a clockwise direction so that $Area(A', B', C', 0) < 0$. By the Intermediate Value Theorem, there exists $t_0 \in [0, 1]$ such that $Area(P, Q, R, t_0) = 0$. Geometrically, this means that there exists some collinear points P, Q and R on the surface of cylinder for some t_0. But three points (not lying along the axis) on the surface of cylinder cannot be collinear unless two of them coincide. By assumption α and β do not intersect so $P \neq Q$ for all t. Thus we must have $P = R$ which is equivalent to $\alpha(t_0) = \gamma(t_0)$ or $Q = R$ which is equivalent to $\beta(t_0) = \gamma(t_0)$ for some $t_0 \in [0, 1]$. Hence the path γ must intersect one of the paths α or β at some point.

□

The above proposition can also be proved on the two boundary circles of an annular region in the plane by using the Jordan curve theorem. Note that Figure 5.1 shows the case where the z-coordinate of each path is assumed strictly monotonic increasing or decreasing. It seems that the theorem is true without the z-coordinate being monotonic as well, but will require only the monotonic case in this thesis.

Finally we prove the main result of this thesis:

Theorem 5.8. Let $p : \widetilde{X} \to X$ be a universal covering space and $b_0 \in X$. If there exists a continuous function $h : \widetilde{X} \to S^1$ such that the map $f : \widetilde{X} \to X \times S^1$ defined by $f(x) = (p(x), h(x))$ is an injection, then $\pi_1(X, b_0)$ is circular orderable.

Proof. Let c be the usual circular ordering of S^1, we define a function $d : \pi_1(X, b_0) \times \pi_1(X, b_0) \times \pi_1(X, b_0) \longrightarrow \{\pm 1, 0\}$ as following:

$$d([\alpha], [\beta], [\gamma]) = c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0}))$$

where α, β, γ are the loops at $b_0 \in X$ with $[\alpha], [\beta], [\gamma] \in \pi_1(X, b_0)$ and $\tilde{b_0}$ is a lift of b_0 through the covering map p (see Figure 5.2). We prove the map d is a circular ordering on $\pi_1(X, b_0)$. We recall that we need to prove three conditions for d to be a circular order: (i) right-invariance (ii) cocycle condition (iii) d is non-degenerate in the sense of Definition (4.8) of a circular order.

To prove the right invariance of d, we need to show that for any $[\alpha], [\beta], [\gamma], [\delta] \in \pi_1(X, b_0)$, we have:

$$d([\alpha], [\beta], [\gamma]) = d([\alpha][\delta], [\beta][\delta], [\gamma][\delta])$$

or

$$c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0})) = c(h([\alpha][\delta]\tilde{b_0}), h([\beta][\delta]\tilde{b_0}), h([\gamma][\delta]\tilde{b_0})). \tag{5.1}$$

Since δ is a loop based at $b_0 \in X$ and $[\alpha]\tilde{b_0}, [\beta]\tilde{b_0}$ and $[\gamma]\tilde{b_0}$ are points in $\widetilde{X}$, by the path lifting lemma (Theorem 3.5), there exists unique lifts $\tilde{\delta_1}, \tilde{\delta_2}$ and $\tilde{\delta_3}$ of δ such that:

$$\tilde{\delta_1}(0) = [\alpha]\tilde{b_0}, \quad \tilde{\delta_2}(0) = [\beta]\tilde{b_0}, \quad \tilde{\delta_3}(0) = [\gamma]\tilde{b_0} \tag{5.2}$$

and

$$\tilde{\delta_1}(1) = [\alpha * \delta]\tilde{b_0} = [\alpha][\delta]\tilde{b_0} \tag{5.3}$$

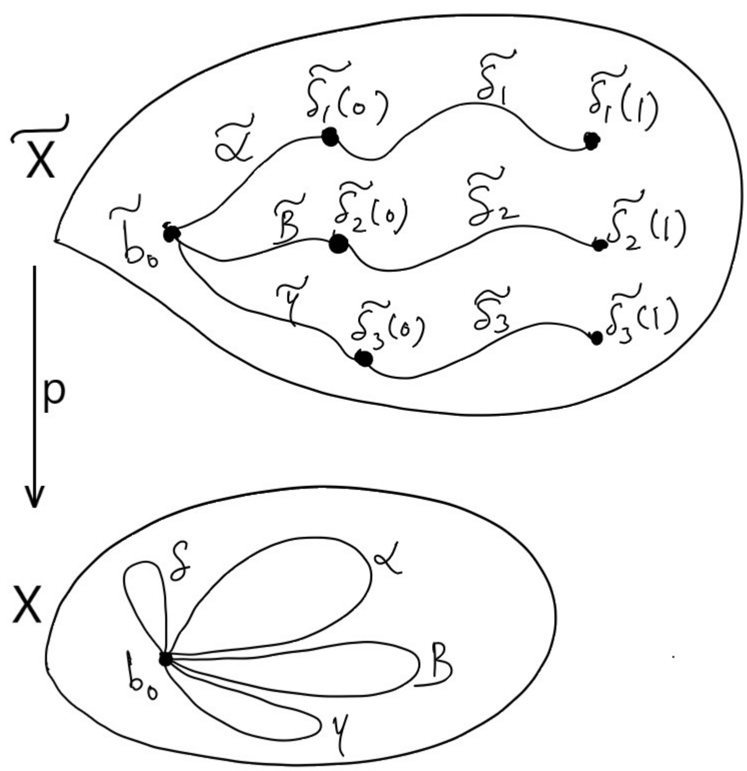
$$\tilde{\delta_2}(1) = [\beta * \delta]\tilde{b_0} = [\beta][\delta]\tilde{b_0} \tag{5.4}$$

$$\tilde{\delta_3}(1) = [\gamma * \delta]\tilde{b_0} = [\gamma][\delta]\tilde{b_0} \tag{5.5}$$

where $*$ denotes the concatenation of paths (see Figure 5.2). Now two cases arise:

Case I: (*Degenerate Case*) Suppose that $\tilde{\delta_1}(t_0) = \tilde{\delta_2}(t_0)$ or $\tilde{\delta_1}(t_0) = \tilde{\delta_3}(t_0)$ or

Figure 5.2: Topological space X and its covering space $\widetilde{X}$ depicting depicting the proof of Theorem 5.8.



$\tilde{\delta_2}(t_0) = \tilde{\delta_3}(t_0)$ for some point $t_0 \in [0, 1]$.

We discuss the case when $\tilde{\delta_1}(t_0) = \tilde{\delta_2}(t_0)$, other cases can be discussed in a similar manner. Since the lifts $\tilde{\delta_1}$ and $\tilde{\delta_2}$ agree at one point t_0, they must agree everywhere. In particular, they must agree at initial points $\tilde{\delta_1}(0) = [\alpha]\tilde{b_0}$ and $\tilde{\delta_2}(0) = [\beta]\tilde{b_0}$. This implies

$$[\alpha]\tilde{b_0} = [\beta]\tilde{b_0} \tag{5.6}$$

Also they must agree at the end points $\tilde{\delta_1}(1) = [\alpha][\delta]\tilde{b_0}$ and $\tilde{\delta_2}(1) = [\beta][\delta]\tilde{b_0}$. This implies

$$[\alpha][\delta]\tilde{b_0} = [\beta][\delta]\tilde{b_0} \tag{5.7}$$

To prove the right invariance in the degenerate case, we substitute (5.6) and (5.7) in the left and right hand sides of equation (5.1) to check if they are same. The left hand side of equation (5.1) gives:

$$c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0})) = c(h([\alpha]\tilde{b_0}), h([\alpha]\tilde{b_0}), h([\gamma]\tilde{b_0})) = 0$$

since two of the entries are the same. Similarly the right hand side of equation (5.1) gives,

$$c(h([\alpha][\delta]\tilde{b_0}), h([\beta][\delta]\tilde{b_0}), h([\gamma][\delta]\tilde{b_0})) = c(h([\alpha][\delta]\tilde{b_0}), h([\alpha][\delta]\tilde{b_0}), h([\gamma][\delta]\tilde{b_0})) = 0.$$

Therefore the map d is right-invariant in this case.

Next we prove the cocycle condition for the degenerate case. To prove the cocycle condition for d, we need to prove that for any $[\alpha], [\beta], [\gamma], [\psi] \in \pi_1(X, b_0)$ where $\alpha, \beta, \gamma, \psi$ are the loops based at $b_0 \in X$ the following holds:

$$d([\alpha], [\beta], [\gamma]) - d([\alpha], [\beta], [\psi]) + d([\alpha], [\gamma], [\psi]) - d([\beta], [\gamma], [\psi]) = 0$$

which by definition of d means

$$c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0})) - c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\psi]\tilde{b_0}))$$

$$+c(h([\alpha]\tilde{b_0}),h([\gamma]\tilde{b_0}),h([\psi]\tilde{b_0}))-c(h([\beta]\tilde{b_0}),h([\gamma]\tilde{b_0}),h([\psi]\tilde{b_0}))=0 \tag{5.8}$$

With $[\alpha]\tilde{b_0}=[\beta]\tilde{b_0}$, the first two terms in the left hand side of equation (5.8) become zero and the third term cancels with the fourth term so that the whole left hand side becomes zero. This proves the cocycle condition for d in this case. The other degenerate cases are all similar.

Case II: (*Non-degenerate Case*) Suppose that $\tilde{\delta_1}(t)\neq\tilde{\delta_2}(t)$ and $\tilde{\delta_1}(t)\neq\tilde{\delta_3}(t)$ and $\tilde{\delta_2}(t)\neq\tilde{\delta_3}(t)$ for all $t\in[0,1]$. We claim that $h(\tilde{\delta_i}(t))\neq h(\tilde{\delta_j}(t))$ for all $t\in[0,1]$ and for all $i\neq j$.

For suppose that $h(\tilde{\delta_i}(t_0))=h(\tilde{\delta_j}(t_0))$ for some $t\in[0,1]$. Then $f(\tilde{\delta_i}(t_0))=f(\tilde{\delta_j}(t_0))$ implies that $\tilde{\delta_i}(t_0)=\tilde{\delta_j}(t_0)$ since f is injective. Then two lifts which agree at one point must agree everywhere, that is $\tilde{\delta_i}(t)=\tilde{\delta_j}(t)$ for all $t\in[0,1]$, a contradiction to the given assumptions $\tilde{\delta_1}(t)\neq\tilde{\delta_2}(t),\tilde{\delta_1}(t)\neq\tilde{\delta_3}(t)$ and $\tilde{\delta_2}(t)\neq\tilde{\delta_3}(t)$ for all $t\in[0,1]$.

In particular, $h(\tilde{\delta_i}(t))\neq h(\tilde{\delta_j}(t))$ implies that $h(\tilde{\delta_i}(t))\neq h(\tilde{\delta_j}(t))$ for $t=[0,1]$ and for any $i\neq j$. That is we have

$$h(\tilde{\delta_1}(0))\neq h(\tilde{\delta_2}(0)),h(\tilde{\delta_1}(0))\neq h(\tilde{\delta_3}(0)),h(\tilde{\delta_2}(0))\neq h(\tilde{\delta_3}(0)) \tag{5.9}$$

and

$$h(\tilde{\delta_1}(1))\neq h(\tilde{\delta_2}(1)),h(\tilde{\delta_1}(1))\neq h(\tilde{\delta_3}(1)),h(\tilde{\delta_2}(1))\neq h(\tilde{\delta_3}(1)).$$

Using equations (5.2), (5.3), (5.4) and (5.5), the above two equations further imply that:

$$h([\alpha]\tilde{b_0})\neq h([\beta]\tilde{b_0}) \quad \textit{and} \quad h([\alpha][\delta]\tilde{b_0})\neq h([\beta][\delta]\tilde{b_0}) \tag{5.10}$$

$$h([\alpha]\tilde{b_0})\neq h([\gamma]\tilde{b_0}) \quad \textit{and} \quad h([\alpha][\delta]\tilde{b_0})\neq h([\gamma][\delta]\tilde{b_0}) \tag{5.11}$$

and

$$h([\beta]\tilde{b_0}) \neq h([\gamma]\tilde{b_0}) \quad and \quad h([\beta][\delta]\tilde{b_0}) \neq h([\gamma][\delta]\tilde{b_0}) \tag{5.12}$$

In the next step, we shall apply equations (5.10), (5.11), (5.12) and Proposition 5.7. To apply Proposition 5.7, we need to construct a certain continuous map say, Φ from $[0,1]$ into the cylinder $S^1 \times [0,1]$ with all the three coordinates as strictly monotonic increasing or decreasing. We construct such a map from the map $h(\tilde{\delta}_i(t)) \subset S^1$. Let $x_i(t)$ and $y_i(t)$ be the two coordinates of $h(\tilde{\delta}_i(t))$ in S^1, that is $h(\tilde{\delta}_i(t)) = (x_i(t), y_i(t)), i = 1, 2$. Define $\Phi(t) : [0,1] \longrightarrow S^1 \times [0,1]$ by $\Phi(t) = (x_i(t), y_i(t), t)$. The map Φ satisfies all the assumptions of Proposition 5.7 as following. Suppose both coordinates $x_i(t)$ and $y_i(t)$ are not strictly monotonic increasing or decreasing, then for all $t_1, t_2 \in [0,1], t_1 < t_2$ we must have $x_i(t_1) = x_i(t_2)$ and $y_i(t_1) = y_i(_2)$. In particular, for $t_1 = 0$ and $t_2 = 1$, we have $x_i(0) = x_i(1)$ and $y_i(0) = y_i(1)$. This implies $(x_i(0), y_i(0)) = (x_i(1), y_i(1))$, that is $h(\tilde{\delta}_i(0)) = h(\tilde{\delta}_j(1))$ which is a contradiction to the fact that $h(\tilde{\delta}_i(t)) \neq h(\tilde{\delta}_j(t))$ for all $t \in [0,1]$. This proves the monotonicity assumption for the map h, and hence for the map Φ in order to apply the Proposition 5.7. With the abuse of notation, we may work with the map h rather than Φ since the third coordinate of Φ is just an identity map t.

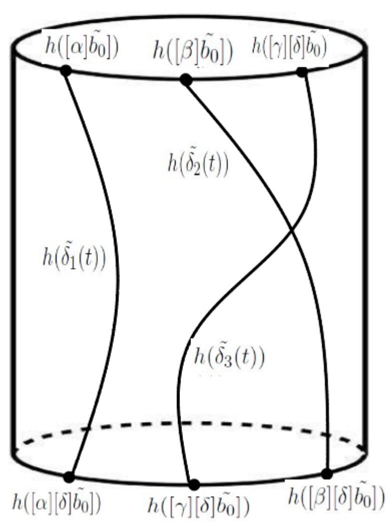
First note that the equations (5.10), (5.11) and (5.12) implies that the left and right hand sides of equation (5.1) cannot be zero so that the case of degeneracy is removed. Suppose that the equation (5.1) is not true and we may well assume without loss of generality that left hand is $+1$ and right hand side is -1, that is $c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0})) = 1$ but $c(h([\alpha][\delta]\tilde{b_0}), h([\beta][\delta]\tilde{b_0}), h([\gamma][\delta]\tilde{b_0})) = -1$. The equation $c(h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0})) = 1$ implies that the points $h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0})$ and $h([\gamma]\tilde{b_0}))$ are oriented in an anticlockwise direction on the circle $S^1 \times \{1\}$ whereas $c(h([\alpha][\delta]\tilde{b_0}), h([\beta][\delta]\tilde{b_0}), h([\gamma][\delta]\tilde{b_0})) = -1$ implies that the orientation is changed after right multiplication by $[\delta]$ so that the points $h([\alpha][\delta]\tilde{b_0}), h([\beta][\delta]\tilde{b_0})$ and $h([\gamma][\delta]\tilde{b_0})$

are oriented in a clockwise direction on the circle $S^1 \times \{0\}$. We claim this cannot happen.

Picture a cylinder $S^1 \times [0,1]$ as in Figure 5.3, and the top of cylinder $S^1 \times \{1\}$ is the circle containing $A = h([\alpha]\tilde{b_0}])$, $B = h([\beta]\tilde{b_0}])$ and $C = h([\gamma]\tilde{b_0}])$ and the bottom $S^1 \times \{0\}$ is the circle containing $A' = h([\alpha][\delta]\tilde{b_0}])$, $C' = h([\beta][\delta]\tilde{b_0}])$ and $B'h([\gamma][\delta]\tilde{b_0}])$, which are circularly ordered in a different sense that than three points on the top of the cylinder. By equations (5.10), (5.11) and (5.12), all of these points A, B, C and A', B', C' are distinct on the top and bottom of the cylinder respectively. Also we know that $h(\tilde{\delta_i}(t)) \neq h(\tilde{\delta_j}(t))$ for any $t \in [0,1]$ and for any $i \neq j$. In particular, $h(\tilde{\delta_1}(t)) \neq h(\tilde{\delta_2}(t))$ which means that the path connecting point $h([\alpha]\tilde{b_0}])$ with $h([\alpha][\delta]\tilde{b_0}])$ does not intersect the path connecting the point $h([\beta]\tilde{b_0}])$ with $h([\beta][\delta]\tilde{_0}])$.

Therefore by Proposition 5.7, we must have that the path connecting C with C' must

Figure 5.3: Cylinder depicting equations (5.10), (5.11) and (5.12).



intersect one of the paths connecting A with A' or B with B'. This further implies that $h(\tilde{\delta_1}(t)) = h(\tilde{\delta_3}(t))$ or $h(\tilde{\delta_2}(t)) = h(\tilde{\delta_3}(t))$ for some $t \in [0,1]$, a contradiction to the fact that $h(\tilde{\delta_i}(t)) \neq h(\tilde{\delta_j}(t))$ for all $t \in [0,1]$ and for all $i \neq j$. Thus the orientation of the points on the bottom circle of the cylinder cannot change after right multiplication by $[\delta]$, that is we must have $c(h([\alpha][\delta]\tilde{b_0}), h([\beta][\delta]\tilde{b_0}), h([\gamma][\delta]\tilde{b_0})) = 1$ as well.

Now we prove the cocyle condition given by equation (5.8) for the non-degenerate case. Let $[\alpha], [\beta], [\gamma], [\psi] \in \pi_1(X, b_0)$ where $\alpha, \beta, \gamma, \psi$ are the loops based at $b_0 \in X$ as considered in the equation (5.8), and the end points of whose lifts are distinct. By equations (5.10), (5.11) and (5.12), we have that $h([\alpha]\tilde{b_0}) \neq h([\beta]\tilde{b_0}), h([\alpha]\tilde{b_0}) \neq h([\gamma]\tilde{b_0})$ and $h([\beta]\tilde{b_0}) \neq h([\gamma]\tilde{b_0})$. The similar argument which we have used to prove the equations (5.10), (5.11) and (5.12) shows that $h([\psi]\tilde{b_0}) \neq h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0}), h([\gamma]\tilde{b_0})$. The only possible values $h([\psi]\tilde{b_0}), h([\alpha]\tilde{b_0}), h([\beta]\tilde{b_0})$ and $h([\gamma]\tilde{b_0}$ are $+1$ or -1. With these values and the fact that c is a circular order, we observe that the left hand side of equation (5.8) becomes zero for all the possible permutations of $\{+1, -1\}$.

Finally, we observe that the map d is non-degenerate in the sense of Definition (4.8) of a circular order, that is triples with repeated entries map to zero and no others map to zero. This follows from the fact that d is defined in terms of a circular order c and from uniqueness of lifts. $\square$

The next two results show how combining the known cohomological relationships (from the Chapter 4) between left orderable and circular orderable groups with this new version of Farrell's theorem directly connects cohomology to the embedding problem. We have the following consequences of Proposition 4.14, Proposition 4.15 and Theorem 5.8:

Theorem 5.9. Let $p : \widetilde{X} \to X$ be a universal covering space and d be the circular ordering on $\pi_1(X, b_0)$ constructed in Theorem 5.8 such that $[f_d] = id \in H^2(G; \mathbb{Z})$ where f_d is the corresponding dehomogenized 2-cocycle given by equation 4.9. If there exists a continuous function $h : \widetilde{X} \to S^1$ such that the map $f : \widetilde{X} \to X \times S^1$ defined by $f(x) = (p(x), h(x))$ is an injection, then $\pi_1(X, b_0)$ is right orderable.

In particular, the above theorem shows that how one direction of classical Farrell's theorem (Proposition 3.12) which holds for right orderable fundamental groups of a topological space, may arise as a special case of a new version of Farrell's theorem (as Theorem 5.8) through the cohomological argument. We note that the Proposition 4.14, and the Theorem 5.9 doesn't provide us an explicit right order arising from the circular order constructed in Theorem 5.8.

Theorem 5.10. Let $p : \widetilde{X} \to X$ be a universal covering space and d be the circular ordering on $\pi_1(X, b_0)$ constructed in Theorem 5.8. If there exists a continuous function $h : \widetilde{X} \to S^1$ such that the map $f : \widetilde{X} \to X \times S^1$ defined by $f(x) = (p(x), h(x))$ is an injection, then the circular order d on $\pi_1(X, b_0)$ is a secret left order if and only if $[f_d] = id \in H^2_b(G; \mathbb{Z})$.

The above theorem shows that if $[f_d] = id \in H^2_b(G; \mathbb{Z})$, then the circular order d constructed in Theorem 5.8 is secretly a right order on $\pi_1(X, b_0)$, that is there is already a left order on $\pi_1(X, b_0)$ giving rise to a circular order in the sense of Definition 4.13. And conversely, if the circular d on $\pi_1(X, b_0)$ is a secret left order, that is, it comes from some left order in the sense of Definition 4.13, then we must have $[f_d] = id \in H^2_b(G; \mathbb{Z})$.

Finally we provide a counterexample to disprove the converse of Theorem 5.8. We recall the famous Borsuk-Ulam theorem, which states that if $f : S^n \to \mathbb{R}^n$ is a continuous map then there exists an $x \in S^n$ such that $f(x) = f(-x)$ [10]. We also recall that S^3 is a covering space for the lens spaces $L(p, q)$ where p and q are coprime integers. More precisely, consider a unit 3-sphere S^3 as a subset in $\mathbb{C}^2$. Consider a homeomorphism $g_{p,q} : S^3 \longrightarrow S^3$ given by $g_{p,q}(z_1, z_2) = (e^{2\pi i/p} \cdot z_1, e^{2\pi iq/p} \cdot z_2)$. It can be seen that p iterations of $g_{p,q}$ is an identity map, so that we get a group action of $\mathbb{Z}_p$ on S^3 with the generator as $g_{p,q}$. Under this group action, the lens space $L(p, q)$ is defined as the quotient of S^3 under the equivalence relation of being in the same orbit. We also have that the spaces $L(p, q)$ and $L(p, q')$ are homeomorphic if and only if $q \equiv q' \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$ [2].

Example 5.2.1. There exists a topological space X such that $\pi_1(X, b_0)$ is circular orderable without the existence of a function f in the sense of Theorem 5.8.

Proof. With the notations in Theorem 8.5, we may take any lens space $X = L(p, q)$ with even p, and $\widetilde{X} = S^3$. Then the fundamental group $\pi_1(X, b_0) \cong \mathbb{Z}_p$ is circular orderable. In particular, we may take $X = L(4, 1)$. Under the group action above, the quotient map $p(x) : S^3 \longrightarrow L(4, 1)$ identifies antipodal points, that is, we have $p(x) = p(-x)$ for all $x \in S^3$. Suppose there is a map $h : S^3 \to S^1 \subset \mathbb{R}^3$ so that $f(x) = (p(x), h(x))$ has the properties guaranteed by Theorem 8.5. Then by Borsuk-Ulam theorem there exists $x \in S^3$ with $h(x) = h(-x)$, however $p(x) = p(-x)$ as well so that f fails to be injective. Therefore the converse of Theorem 8.5 is not true.

$\square$

List of Figures

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