ON SOME ASPECTS OF FUZZY RANDOM UNCERTAINTY IN ASSET PRICING: METHODOLOGY AND POTENTIAL TESTABILITY

BY

Kamal Smimou

A Thesis

Submitted to the Faculty of Graduate Studies

In Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Department of Business Administration

I. H. Asper School of Business

University of Manitoba

Winnipeg, Manitoba

© Kamal Smimou, June 2004

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

*****

COPYRIGHT PERMISSION PAGE

On Some Aspects of Fuzzy Random Uncertainty in Asset Pricing:

Methodology and Potential Testability

BY

Kamal Smimou

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

of

DOCTOR OF PHILOSOPHY

KAMAL SMIMOU ©2004

Permission has been granted to the Library of The University of Manitoba to lend or sell copies of this thesis/practicum, to the National Library of Canada to microfilm this thesis and to lend or sell copies of the film, and to University Microfilm Inc. to publish an abstract of this thesis/practicum.

The author reserves other publication rights, and neither this thesis/practicum nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

Dedication

*To my mother **Zineb**, who instilled in me the curiosity to begin the Ph.D. program and the determination to complete it.*

*To my wife **Hind**, whose support from the beginning made it possible.*

*To my son **Abdalbarr**, whose energy and joy of life made it bearable.*

*To **ALL** my family members overseas, whose appearance could make it worthwhile.*

*To my late **father**, whose presence and absence will always be felt.*

*To **ALL** people who I love and admire from my deepest heart...*

ABSTRACT

Several research topics have dealt with the applications of the fuzzy theory in a variety of areas including finance. However, the possibility of combining both fuzzy and probability theories in finance has not received much attention. This research contribution tackles the application aspects of fuzzy theory by combining fuzzy theory with probability theory. Existing literature reveals that both theories describe uncertainty. Fuzzy theory and probability theory are two paradigms of modeling uncertainty. This thesis is an attempt to integrate the two theories.

The lack of proven practical applications and empirical implications of the fuzzy theory during its early stage of development was a favorite criticism of its opponents. To address that criticism, this research presents a methodology and shows a potential testability process for three major aspects of the field.

The first aspect is the use of the fuzzy random uncertainty theory to find the portfolio that gives the mean variance (E,V) combinations that were attainable through the combination of statistical techniques and expert judgments. As inspired from Markowitz's statement, the expert judgments in this research have been modeled through the use of fuzzy theory. Various sample sizes have been used to show the location of the efficient frontier and the capital market without short sales and with subjective fuzzy measure. Also, we show that the validity of this derivation is unaffected by the use of returns on the assets with subjective measure (width).

The second aspect is fuzzy modeling by the introduction of fuzzy probabilities in measuring risk. Following Philippatos and Wilson, the fuzzy entropy has been fully

developed and then implemented in an empirical example to measure risk.

The third aspect of the research is to consider the application of a modified approach to the estimation of risk premium of commodity futures. The aim of this study is to estimate systematic risk using commodity futures prices with the existence of price limits. An estimation process has been conducted in two different phases. With the help of the Ordinary Least Squares (OLS) method, the systematic risk has been estimated using the settlement prices of the commodity futures, which are assumed to be sharply defined. The second phase investigates the impact and effectiveness of price limits on estimating the beta risk of commodities return by using an optimization model. Then, to complete the estimating process, a test for a significant regression relationship is presented in the last aspect of the research.

Acknowledgments

I want to thank all my committee members. Specifically, I express my deepest gratitude to my research advisors Professor C. R. Bector and G. Jacoby for their support and assistance throughout the Ph.D. program. Also, I am very thankful to Mr. Lieb (The Winnipeg Commodity Exchange Inc.) for his constructive explanation during the collection of data.

Also, I offer my deepest gratitude to all my family members for their continuous moral support and encouragement during my educational endeavors.

I appreciate the financial support provided by Professor C. R. Bector from his grants and by the Faculty of Graduate Studies in the form of a University of Manitoba Graduate Fellowship. Also, I am very thankful to the financial support by I. H. Asper School of Business in many instances during the progress of my research proposal until the dissertation stage.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In many real situations variability is indicated by two kinds of uncertainty: randomness (stochastic variability) and inexactness (vagueness). Here, inexactness means non-statistical uncertainty that is due, for example, to the imprecision of human knowledge or to the inexactness of measurements rather than to the uncertainty of random events. While the former uncertainty is modeled by the concept of random variables, the second one is modeled by the concept of fuzzy mathematics and statistics.

The fuzzy set theory introduced by Zadeh [169] is, as the name implies, a theory of graded sets. Due to their sharp boundaries, classical sets are usually referred to in fuzzy set literature as crisp sets. As in classical set theory, the degree to which an element x belongs to the fuzzy set A is described by a function called the membership function. In contrast to the characteristic function of a set in the classical sense, which takes the value one only if x is a member of A, and zero otherwise, the membership function can take values between zero and one. The value between zero and one is interpreted as the degree of membership of an element x belonging to the fuzzy set A. This means, for example, that we could assign the degree of 0.7 to the temperature 23^0C as a member of the linguistic value "warm".

Some probabilists have been supportive of fuzzy set theory and other novel uncertainty theories. One of them is J. N. Kapur, a well-known contributor to classical (probability-based) information theory. The following excerpt from a published interview [139] expresses his views regarding fuzzy set theory:

> "In mathematics, earlier, algebra and topology were fighting for the soul of mathematics. Ultimately both are co-existing and are enriching each other. Similarly today there is a struggle between probability theory and fuzzy set theory to capture the soul of uncertainty. I am sure ultimately both will co-exist and enrich each other. Already the debate has led to a deeper understanding of what we mean by uncertainty... I believe that uncertainty is too deep a concept to be captured by probability theory alone. Probability theory has had a long history, while fuzzy set theory is relatively of recent origin. Let it grow to its full strength."

Another probabilist endorsing fuzzy theory is Viertl [159]. Moreover, Zadeh redefined the concept of probability vis-à-vis fuzziness [170], pointing out that probability is a special case of fuzziness, and it has two limitations: firstly, it works with bivalent sets A, $A \cap A^c = \phi$; $A \cup A^c = X$. So, $P(A \cap A^c) = 0$, $P(A \cup A^c) = 1$ for all sets A, and that itself draws hard lines between things and non-things, and we cannot do that in the real world. Secondly, probability measures need small infinities. A probability measure maps the sets in a single-algebra to the unit interval [0,1].

Fuzzy theorists explain why people have been wrong in a variety of aspects for so long. The reason is that rounding off and quantifying simplifies life and often costs little. The probability that $x \in A$, for example, means that element x either is or is not an element of set A. However, fuzziness may still exist (x belongs to fuzzy set A with degree $\mu_A(x)$).

Probabilists might wonder whether probability describes anything real. David Hume [62] stated:

> "Though there be no such thing as chance in the world, our ignorance of the real cause of any event has the same influence on the understanding and begets like species of belief."

In another instance, Kosko [80] states:

> "The only subsets of the universe that are not fuzzy are the constructs of classical mathematics. All other sets of particles, cells, tissues, people, ideas, galaxies in principle contain elements to different degrees. Their membership is partial, graded, inexact, ambiguous, or uncertain."

Kosko [81] claims that probability is not a primitive theory. He points out that we can often eliminate it, in favor of a "fuzzy" or multivalued containment operator. Kosko [80] presented this illustrative example showing the difference between fuzziness and probability:

> "Suppose there is a 50% chance that there is an apple in the refrigerator. That is one state of affairs, perhaps arrived at through frequency calculations or a Bayesian state of knowledge. Now suppose there is a half an apple in the refrigerator. That is another state of affairs. Both states of affairs are superficially equivalent in terms of their numerical uncertainty. Yet physically, ontologically, they are distinct. One is 'random', the other 'fuzzy'."

When discussing the physical universe, every assertion of event ambiguity or non-ambiguity is an empirical hypothesis. This is habitually overlooked when applying probability theory. Years of such oversight are perhaps responsible for the deeply entrenched sentiment that uncertainty is randomness, and randomness alone. When looking at an inexact oval, we cannot say that it is probably a circle or ellipse, because nothing is random about it. The situation is deterministic, as all the facts are known. However, uncertainty remains, due to the simultaneous occurrence of two properties: to some extent an oval is an ellipse, and to some extent it is not an ellipse.

Kosko [80] pointed out that conceptually and theoretically, there are differences between randomness and fuzziness. At the same time, there are many similarities. One of the similarities is that both theories express uncertainty in a numerical fashion in the interval [0,1]. Fuzziness describes event ambiguity. It measures the degree to which an event occurs, not whether it occurs.

Randomness describes the uncertainty of event occurrence. An event occurs or does not, and you can bet on it. So, whether an event occurs is "random"; to what degree it occurs is fuzzy. Whether an ambiguous event occurs when we say there is a 20% chance of light snow tomorrow, involves compound uncertainties, or the probability of a fuzzy event. In practice, we regularly apply probabilities to fuzzy events: small errors, satisfied customers, safe investments. We understand that at least around the edges, some satisfied customers can be somewhat unsatisfied, and some safe investments can be somewhat unsafe investments.

Fuzziness has been presented as an alternative to randomness, to describe uncertainty. We may pose the following question: Do the notions of likelihood and probability exhaust our notions of uncertainty? Some people who have been trained in probability and statistics believe so. For example, Bayesian physicist E. T. Jaynes [65] says that:

> "Our method of inference in which we present degree of plausibility by real numbers, is necessarily either equivalent to Laplace's (probability) or inconsistent."

Lindley [95] issued a challenge by saying:

> "Probability is the only sensible description of uncertainty and is adequate for all problems involving uncertainty. All other methods are inadequate."

In contrast, Zadeh [169] suggested that notions of an event and its probability constitute the most basic concepts of probability theory. An event is a collection of points in the sample space. However, in everyday experience, one frequently encounters situations in which an "event" is a fuzzy rather than a sharply defined collection of points. Zadeh [170] presented in his study the following definitions where fuzzy events have been elaborated.

Definition 1 *Let $(\mathbb{R}^n, A, P)$ be a probability space in which A is the $\sigma-field$ of Borel sets in $\mathbb{R}^n$ and P is a probability measure over $\mathbb{R}^n$. Then, a fuzzy event in $\mathbb{R}^n$ whose membership function is $\mu_A(\mu_A : \mathbb{R} \to [0,1])$, is Borel measurable. The probability of a fuzzy event A is defined by the Lebesgue-Stieltjes integral: $P(A)=\int_{\mathbb{R}^n} \mu_A(x)dP = E[\mu_A]$.So, the probability of a fuzzy event is the expectation of its membership function assuming that μ_A is Borel measurable.*

This definition forms a basis for generalizations within the framework of the fuzzy set theory.

One of many researchers who criticized the use of fuzzy theory to model uncertainty is Cheeseman [24]. He points out that probability can solve the same problems that fuzzy approaches claimed to solve by expanding the concept of probabilities to avoid limitations imposed by the frequency of probability. This view has been persuasively argued; see for example Klir [74] and Zadeh [174].

For further discussion on the long-standing controversy of the use of prior probabilities and their interpretations and to find an explanation of various aspects of uncertainty, including uncertainty in scientific inquiry, one may refer to [73], [74], [81], [80], [81] and [174]. It may be noted here that when we are making decisions with uncertain and incomplete information, it is always necessary to specify the assumptions. The concept of fuzzy random variables established by Kwakernaak [86] can be applied to model uncertainty. Puri and Ralescu [121] had a slightly different notion. They defined fuzzy random variables as a generalization of random closed sets. This generalization also includes random variables and random vectors. Thus, the concept of fuzzy random variables has been found to be convenient in studying linear statistical inference, limit theorems and so on. Indeed, many results can be regarded as a generalization of results of real-valued random variables.

1.1 Fuzzy Uncertainty

Consulting a dictionary for the term "uncertainty", we find that it has a broad semantic meaning. For example, Webster's New Twentieth Collegiate Dictionary defines uncertainty as the quality or state of being uncertain. Synonymously, doubt, dubiety, skepticism, suspicion, and mistrust mean lack of sureness about someone or something. Uncertainty may range from falling short of certainty to almost a complete lack of definite knowledge. Dubiety stresses a wavering between conclusions; skepticism implies unwillingness to believe without conclusive evidence; suspicion stresses lack of faith in the truth, reality, fairness, or reliability of something or someone. Mistrust implies a genuine doubt based upon suspicion. Also, we find that uncertain stands for 1) indefinite, indeterminate, 2) problematical (not certain to occur), 3) untrustworthy (not reliable), 4) a) dubious (not known beyond doubt) b) not having certain knowledge: doubtful, c) not clearly identified or defined, 5) not constant. These various meanings are mentioned here to illustrate the richness of the concept of uncertainty and the large spectrum of possible theoretical tools that can be used in dealing with difficult real-world problems.

When we investigate these various meanings, at least two major types of uncertainty emerge naturally: *vagueness* and *ambiguity*. It is easy to see that the meanings mentioned above relate to the concepts of fuzziness and crispness. Keeping in mind that the concept of uncertainty is closely connected to the concept of information, when our uncertainty in a situation is reduced by an action such as performing an experiment or finding a historical record, the action may be viewed as a source of information relating to the situation.

Note here, that the classical mathematical frameworks for characterizing situations as uncertain have been crisp set theory and probability theory. Yet, the fuzzy set theory, by its capability of conceptualizing the main types of uncertainty is relevant and obvious. Membership degrees that accompany fuzzy theory and fuzzy data (in the empirical sense) indirectly express a pertinent measurement of uncertainties. Moreover, an important feature of fuzzy set theory is its ability to capture the vagueness of linguistic terms in statements of subjective and natural languages. In that case, vagueness is a kind of uncertainty that does not result from information deficiency but rather from imprecise meanings of linguistic terms. Crisp set theory is not capable of expressing the imprecise meanings of vague terms and of being transferred to a modeling quantifiable environment.

The lack of proven practical applications and empirical implications of the new uncertainty during its early stage of development was a favorite criticism of its opponents. At the beginning, they were able to embarrass proponents of the theory by simple questions such as: "Can you show us at least one practical application or one empirical implication of the new theory?", and they asked increasingly demanding questions. Later, when the number of applications became overwhelming, the opponents asked whether the proponents could show them at least one problem that could be solved with the help of fuzzy theory but that could not be solved without it. Although the question is still debatable, a prolific body of work has recently emerged with the help of fuzzy theory, where classical efforts have failed; see for example [175]. Fuzzy theory is offered as the basis of a new paradigm of uncertainty.

Currently, the range of applications of fuzzy uncertainty is quite wide. For instance, the fuzzy linear programming, which was developed to tackle problems encountered in real-world applications, shows that applications are diverse and cross disciplinary. Business assignment problems (network location problems) (see Darzentas [29]), transportation problems (Perincherry and Kikuchi [117]) and transshipment problems (Verdegay [158]) represent only a suggestive list list of applications in the area of management science. In the finance area, the number of applications is limited, for example, capital asset pricing model (Ostermark [114]), profit apportionment in a concern (Ostermark [113]), bank hedging decision (Lai and Hwang [88]) and project investment (Lai and Hwang [88]), and there is a room for future research. In marketing, the media selection problem by Zimmermann [176] and the new product development by Smimou et al. [144] remain a non-exhaustive list of applications in the area.

1.2 Fuzzy Modeling in Finance

During the last fifty years, investment theory has been developed around EMT (Efficient Market Theory), Markowitz's Mean-Variance Model (EV) [103], Sharpe's Capital Asset Pricing Model (CAPM) [141], Lintner [97] and Mossin [108], Ross's Arbitrage Pricing

Theory [130] and Black, Scholes and Merton's Option Pricing Theory [13], [107]. Of these, the first two theories are regarded as the backbone of modern portfolio theory. The major difficulty faced by operations researchers in modeling the problem of portfolio selection, which is regarded as the theory that precedes the derivation of the Capital Asset Pricing Model (CAPM), is that it is based on the perception of risk by an investor, which will varies, as different people have different beliefs about the future performance of various assets. In real-world problems, we are faced with imperfect information (data) and must deal with uncertain, imprecise, and vague data. In modeling and analyzing problems of this type, earlier works in finance tended to equate all aspects of imperfect information with uncertainty (of a random character). Thus, a multitude of probabilistic models were proposed. This was also the case with the use of modeling in finance, for example [50], [167], [25], [83], [134], [120], [38], [63], [116] and [143].

However, no simple and adequate methods for handling imprecise data, which may stem, e.g., from the use of natural language and subjective statements was available until the mid-1960's when Zadeh [169] proposed fuzzy sets theory. And indeed, financial modeling has been one of the areas to which fuzzy sets theory has been applied ([113], [114]).

Recently, there has been an increased interest in fuzzy theory in a large number of applications, some of which will be mentioned in this thesis.

Markowitz [103] in the derivation of the efficient portfolio assumed that the return r_i (the return on the i^{th} security) and R (yield on the portfolio as a whole) are assumed to be random variables, and the probability beliefs concerning these variables are given. Of course, he did not discuss the method of how investors form their probability of beliefs. He says:

> "In general we would expect that the investor could tell us, for any two events (A and B), whether he personally considered A more likely than B, B more likely than A, or both equally likely. If the investor were consistent in his opinions on such matters, he would possess a system of probability beliefs. We cannot expect the investor to be consistent in every detail. We can, however, expect his probability beliefs to be roughly consistent on important matters that have been carefully considered. We should also expect that he will base his actions upon these probability beliefs even though they *be in part subjective*."

Markowitz addressed the subjectivity part of the probability beliefs by stating:

> "The calculation of efficient surfaces might possibly be of practical use. Perhaps there are ways, by combining statistical techniques and the judgment of experts, to form reasonable probability beliefs."

In this context, the fuzzy random uncertainty is a suitable theory to find the portfolio which gives the (E, V) combinations that were attainable and the desired combination by the investor through the combination of statistical techniques and expert judgments. Markowitz's mean-variance model [103] assumes that the investor is risk averse, i.e. the investor's utility function is increasing and concave, and the security returns are jointly normally distributed or the utility is a quadratic function, and the risk associated with it is fully identified by its variance. Following the line of Markowitz, the purpose of this thesis is to provide the aspects of fuzzy uncertainty in asset pricing, which would involve a rederivation of the mean-variance theory followed by a rederivation of the fuzzy CAPM model. Generally speaking, in finance, and specifically in investment problems, because we are confronted with decision making situations, the essence of Bellman and Zadeh's [10] approach to decision making under fuzziness adds a new dimension to the modeling effort in this research.

Next, in Chapter 2, we present the rederivation of the Markowitz efficient frontier. In Chapter 3, another aspect of fuzzy modeling in finance by the introduction of fuzzy probabilities is presented, an empirical design is discussed, and the actual testing of the collected data is completed. Chapter 4 presents a brief review of various fuzzy regression approaches and illustrates an application of a modified approach to the estimation of risk premium of commodity futures.

Chapter 2

Mathematical Background/Preliminaries

Inferences and decisions in statistics are based on information supplied by a random experiment associated with a population and on additional information about the experiment. To achieve a statistical inference in terms of certainty and precision is almost impossible. Since the development of fuzzy set theory, many studies have tackled the combination of both fuzzy set and probability theory. The aim of this chapter is to examine methods for handling statistical problems involving fuzziness in the elements of the random experiment, and serves as a point from which to derive the Markowitz frontier in the presence of fuzzy uncertainty and random uncertainty. Gebhardt et al. [42] presented two illustrative figures showing the elements and stages in a random experiment and involving the observation of random variables and fuzziness in the observed report.

In statistics, we traditionally assume that the experimental performance and the parameter value, or state specification in a Bayesian setting, are accomplished under randomness, whereas the remaining stages in the experiment are handled under certain and well-defined conditions. However, fuzziness can arise in some of these remaining stages, that is, in the assessment of the experimental and/or prior distribution. Chapter 2 tackles this point. Also see Walley [163] and Thomas [154]. In the context of the quantification process of the random variable, Chapter 3 presents a special example. In this case, limitations sometimes appear when assessing exact probabilities, so the available information about probabilities is more properly described in terms of imprecise propositions, stating a set of experimental results as "highly probable" or "unlikely". Also, the quantification process in the random variable can associate an imprecise report of the variable value with each experimental outcome. Fuzziness can be involved in getting the experimental outcome or the parameter value of the experimental distribution. Regarding the assessment of fuzzy probabilities, we can see, for example, Zadeh [173], [172], Dubois and Prade [34], Rappoport et. al [123], and Ralescu [122]. Apart from these, there are still many open questions in connection to this topic.

The notion of a fuzzy random variable (see for example, Kwakernaak [86], Puri and Ralescu [121], Kruse and Meyer [84]) provides a valuable model that is manageable in a probabilistic framework. Also, the concept of fuzzy information presented by Zadeh [172] can formalize either the experimental data or the events involving fuzziness. The concept of a fuzzy random variable [121] was defined as a tool for establishing relationships between the outcomes of a random experiment and inexact data. By inexactness, we mean non-statistical inexactness that is due to subjectivity and to imprecision of human knowledge rather than to the occurrence of random events. Korner [77] pointed out that the variability is given by two kinds of uncertainties: randomness (stochastic variability) and imprecision (vagueness). Randomness models the stochastic variability of all possible outcomes of an experiment. Fuzziness describes the vagueness of the given or realized outcome. Randomness answers the question: What will happen in the future? Whereas fuzziness answers the question: What has happened? or What is meant by the data?

Kwakernaak [86] presented another explanation for the difference between randomness and fuzziness. He pointed out that when we consider an opinion poll in which a number of people are questioned, randomness occurs because it is not known which response may be expected from any given individual. Once the response is available, there still is uncertainty about the precise meaning of the response. The latter uncertainty will be characterized by fuzziness.

2.1 Fuzzy Random Variables

In this case, we deal with two types of uncertainty, namely, randomness and possibility (fuzzy). Randomness refers to the description of a random experiment by a probability space (Ω, A, P), where Ω is the set of all possible outcomes of this experiment, A is $\sigma-$field of subsets of Ω (the set of all possible events), and the set-function P, defined on A, is a probability measure. We assume that all the information that is relevant for further analysis of any outcome of the random experiment can be expressed with the aid of a real number, so that we can specify a mapping $U : \Omega \rightarrow \mathbb{R}$, which assigns to each outcome in Ω its random value in $\mathbb{R}$. U is called a random variable and is expected to be measurable with respect to the $\sigma-$field A and the Borel $\sigma-$ field B of the real line. The possibility of a second kind of uncertainty in our discussion of a random experiment has to be involved whenever we are not in the position to fix the random values $U(w)$ as crisp numbers in $\mathbb{R}$, but only to imperfectly specify these values by a possibility distribution on $\mathbb{R}$. In this case the random variable $U : \Omega \rightarrow \mathbb{R}$ changes to fuzzy random variable $X : \Omega \rightarrow F(\mathbb{R})$ with $F(\mathbb{R}) = \{\widetilde{x}/\mu_x : \mathbb{R} \rightarrow [0,1]\}$ denoting the class of all fuzzy subsets. Fuzzy random variable (f. r.v.) is interpreted as a fuzzy perception of an inaccessible usual random variable, $U : \Omega \rightarrow \mathbb{R}$, which is the original of X. The idea is that the corresponding description of a random experiment $U_0(w)$ is imperfect in the sense that its most specific specification is the possibility distribution X_w=X(w). In this case, for any r$\in \mathbb{R}$ the value $X_w(r)$ quantifies the degree of possibility with which the proposition U_0(w)=r is regarded as being true. $X_w(r) = 0$ implies that there is no supporting evidence for the possibility of the truth of $U_0(w)$=r, whereas $X_w(r) = 1$ implies that there is no evidence against the possibility of the truth of $U_0(w)$=r, so that this proposition is fully possible. $X_w(r) \in [0,1)$ reflects that there is evidence that supports the truth of the proposition as well as evidence that contradicts it. A way proposed by Gebhardt et al. [42] of interpreting a possibility distribution X_w:$\mathbb{R} \rightarrow [0,1]$ is viewing X_w in terms of the context approach.

The concept of a fuzzy random variable is a reasonable extension of the concept of a usual random variable in the many practical applications of random experiments, where the implicit assumption of data precision seems to be an inappropriate simplification rather than an adequate modeling of the real physical conditions. Considering possibility distribution allows us to involve uncertainty (due to the probability of occurrence of competing specification contexts) and imprecision (due to the context-dependent set-valued specifications of $U_0(w)$).

Definition 2 *Let (Ω, A, P) be a probability space. A function $X{:}\Omega \to \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable if and only if:*

$$\begin{aligned} \underline{X}_\alpha &: \Omega \to \mathbb{R}, w \to \inf(X(w)_\alpha) \text{ and} \\ \overline{X}_\alpha &: \Omega \to \mathbb{R}, w \to \sup\ (X(w)_\alpha) \end{aligned}$$

are A-B- measurable for all $\alpha \in [0,1]$, with B being the Borel $\sigma - field$ of $\mathbb{R}$.

The notion of a probabilistic set and fuzzy random variable was introduced by several authors in different ways. Kwakernaak's theory [86] is similar to that presented here. Puri and Ralescu [121] considered fuzzy random variables whose values are fuzzy subsets of R^n, or more generally of Banach space.

Theorem 3 *Let $X : \Omega \to \mathcal{F}(\mathbb{R})$ be a finite fuzzy random variable such that $X(\Omega) = \{\widetilde{x}_1, \widetilde{x}_2, .., \widetilde{x}_n\}$ and $p_i = P[\{w \in \Omega / X_w = \widetilde{x}_i\}]$, $i=1,\ldots,n$.*

Then, $\{[\sum_{i=1}^n p_i \inf(\widetilde{x}_i)_\alpha, \sum_{i=1}^n p_i \sup(\widetilde{x}_i)_\alpha]\}_{\alpha \in (0,1]}$ is an $\alpha - cut$ representation of E(coX), where $CoX{:}\Omega \to \mathcal{F}(\mathbb{R})$ is defined by (coX)(w)=Co(X_w) with Co(X_w) denoting the convex hull of X_w.

2.2 Fuzzy Random Variables and Properties

Kwakernaak [86] defines the concept of fuzzy random variable as follows:

Let $I_i : R \to [0,1]$ be the characteristic function of the set w_i. Also, let S be the space of all piecewise continuous functions $R \to [0,1]$. We then define the perception of the random variable U, as described above, as the mapping $X : \Omega \to S$ given by

$$w \xrightarrow{X} X_w$$

with $X_w = I_i$ if and only if $U(w) \in W_i$. This means that we associate with each $w \in \Omega$, not a real number $U(w)$, as in the case of an ordinary random variable, but a characteristic function X_w , which is an element of S.

The map $X : \Omega \to S$ described above characterizes a special type of fuzzy random variable. The random variable U, of which this fuzzy random variable is a perception, is called an *original* of the fuzzy random variable. Many originals may exist. Kwakernaak [86] introduced the notion of a fuzzy random variable as a function F

$$F : \Omega \to F(\mathbb{R})$$

subject to certain measurability conditions, where (Ω, A, P) is a probability space and $F(\mathbb{R})$ denotes all piecewise continuous functions:

$$u : \mathbb{R} \to [0,1]$$

Feron [40] defined a fuzzy random set as a measurable function:

$$F : \Omega \to F(\varkappa)$$

where $\varkappa$ is a topological space, $F(\varkappa) = \{u : \varkappa \rightarrow [0,1]\}$, and $\{x \in \varkappa : F(w)(x) \geqslant \alpha\}$ are closed subsets of x for each $0 \leq \alpha \leq 1, w \in \Omega$.

Puri and Ralescu [121] defined fuzzy random variable slightly differently from Kwakernaak [86]. In [121], fuzzy random variable is defined as a function $X : \Omega \rightarrow F_0(\mathbb{R}^n)$, where (Ω, A, P) is probability space, and $F_0(\mathbb{R}^n)$ denotes all functions (fuzzy subsets of $\mathbb{R}^n$) $u : \mathbb{R}^n \rightarrow [0,1]$ such that $\{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ is non-empty and compact for each $0 < \alpha \leq 1$

2.3 Fuzzy Variables and Their Expectations

Let (Ω, A, P) be a probability space where P is a probability measure. Let $F_0(\mathbb{R}^n)$ denote the set of fuzzy subsets $\mu : \mathbb{R}^n \rightarrow [0,1]$ with the following properties:

(a) $\{x \in \mathbb{R}^n ; \mu(x) \geq \alpha\}$ is compact for each $\alpha > 0$

(b) $\{x \in \mathbb{R}^n ; \mu(x) = 1\} \neq \phi$

Definition 4 [77]. *A fuzzy random variable (fuzzy variable) is a function*

$X : \Omega \rightarrow F_0(\mathbb{R}^n)$

such that: $\{(w,x) : x \in X_\alpha(w)\} \in A \times B$ *for every* $\alpha \in [0,1]$

Where $X_\alpha : \Omega \rightarrow P(\mathbb{R}^n)$ *is defined by*

$X_\alpha(w) = \{x \in \mathbb{R}^n : X(w)(x) \geq \alpha\}$

Definition 5 [109]. *A fuzzy variable X is called integrably bounded if* X_α *is integrably bounded for all* $\alpha \in [0,1]$, *i.e. for any* $\alpha \in [0,1]$ *there exists* $h_\alpha \in L^1(\Omega)$ *such that* $\|x\| \leq h_\alpha(w)$ *for each x, w with* $x \in X_\alpha(w)$. $L^1(\Omega)$ *denotes all functions* $h : \Omega \rightarrow \mathbb{R}$ *which are integrable with respect to the probability measure P. Then, expected value* $E[X]$ *of a fuzzy variable X is defined as:*

$$X:\Omega\to F_0(\mathbb{R}^n);\ \{x\in\mathbb{R}^n:\ (E[X])(x)\geq\alpha\}=\int X_\alpha \text{ for each } \alpha\in[0,1]$$

Theorem 6 ([121],[77]). *If $X:\Omega\to F_0(\mathbb{R}^n)$ is an integrably bounded fuzzy variable, there exists a unique fuzzy set $v\in F_0(\mathbb{R}^n)$ such that$\{x\in\mathbb{R}^n : v(x)\geq\alpha\}=\int X_\alpha$ for every $\alpha\in[0,1]$. This theorem was used to define expected value of a fuzzy random variable $X:\Omega\to F_0(\mathbb{R}^n)$ which is integrably bounded.*

Definition 7 *The expected value of X, denoted by E[X], is the fuzzy set $v\in F_0(\mathbb{R}^n)$*[1] *such that $\{x\in\mathbb{R}:v(x)\geq\alpha\}=\int X_\alpha$ for every $\alpha\in[0,1]$. Existence and uniqueness of v are established in the following theorem $(E[X])(x)=Sup\{\alpha\in[0,1]:x\in\int X_\alpha\}$ and its level sets are given by : $\{x:(E[X])(x)\geq\alpha\}=\int X_\alpha,\ \alpha\in[0,1]$*

2.3.1 Properties of the expected value

Extension of Lebesgue dominated convergence theorem to fuzzy random variables is done by $F_0(\mathbb{R}^n)$ a metric which generalizes the Hausdorff metric, let $u,v\in F_0(\mathbb{R}^n)$, and set $d(u,v)=Sup_{\alpha>0}(L_\alpha(u),L_\alpha(v))$. d_H is Hausdorff metric, and we denote by $L_\alpha(u)=\{x:u(x)\geq\alpha\}$ and $L_\alpha(v)=\{x:v(x)\geq\alpha\}$

Theorem 8 *If the probability measure P is nonatomic, and if $X:\Omega\to F_0(\mathbb{R}^n)$ is an intergrably bounded fuzzy variable, then E[X] is a fuzzy convex set.*

Computation of E[X] with examples to compute expected value of a fuzzy random variable.

Example 9 *Toss a fair coin, outcomes: tail (T) and head (H). A player loses approximately \$10 if the outcome is T, wins an amount much larger than \$100 but not much larger than*

[1]Sets of fuzzy subsets

\$1000 if the outcome is H.

The fuzzy random variable is:

$X : \{T, H\} \to F_0(\mathbb{R}^n)$ $\quad$ $X(T) =$ *"approximately 10"*

$X(H) =$ *"much larger than 100 but not larger than 1000"*

For a technical reason:

$X(T) = u, \quad X(H) = v \quad u, v : \mathbb{R} \to [0,1]$

$u(x) = \left[1 - \frac{(x+10)^2}{4}\right]^+ \quad v(x) = \left[1 - \frac{(x-630)^2}{380^2}\right]^+$

$f^+ = \max(f, 0)$

Since u and v are continuous with compact support, it is easy to show:

$$E[X](x) = Sup_{y+z=2x} \min\left(\left[1 - \frac{(y+10)^2}{4}\right]^+, \left[1 - \frac{(z-630)^2}{380^2}\right]^+\right)$$

In particular, support of E[X] is included in the interval $[119, 501]$.

Example 10 *Let* $X : \Omega \to F_0(\mathbb{R}^n)$ *be a fuzzy variable such that*

$P[X = u_i] = p_i,\ i = 1, ..r$

where $u_i : \mathbb{R} \to [0,1]$ *are continuous with compact support. Then,* $E[X] = \sum_{i=1}^{r} p_i u_i$

2.4 Variance of Fuzzy Random Variables

Fuzzy random variable introduced by Puri and Ralescu [121] as a generalization of compact random sets, combines both randomness and imprecision. Stochastic variability is described by use of probability theory and the vagueness by use of fuzzy sets introduced by Zadeh [169].

The notion of expectation and the notion of variance are parallel to the notions for a linear statistical inference with fuzzy random data.

Define F_c : set of all normal compact convex fuzzy subsets of $\mathbb{R}^n$ and assume that any $\mu_A : \mathbb{R} \to [0,1]$ satisfies:

(1) A is normal, $A^l = \{x \in \mathbb{R}^n : \mu_A(x) = 1\}$ is non-empty.

(2) $\alpha - cuts$ of A, $A^\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ $0 < \alpha \leq 1$ are convex and compact.

(3) The support of A , $\mathrm{A}^0 = \cup_{\alpha \in [0,1]} A^\alpha$ is compact.

Each fuzzy set A corresponds uniquely to its support function.

$S_A(\alpha, u) = Sup\{\langle u, a\rangle : a \in A^\alpha\}, \qquad u \in S^{n-1}, \alpha \in [0,1]$

$S^{n-1}$is the $(n-1)$ dimensional unit sphere of $\mathbb{R}^n$ and $\langle ., .\rangle$ is the inner product of the Euclidean space $\mathbb{R}^n$.

X : is a Borel measurable mapping $X : \Omega \to F_c$. It follows that for each $\alpha \in [0,1]$ the $\alpha - cuts$, X^α are non-empty compact convex random sets.

2.4.1 Expectation, variance and covariance

Let the following measure defined such that

$$Ed^2(X, A) = \inf_{B \in M} Ed^2(X, B) \tag{2.1}$$

$A \in M$ with $\mathrm{Ed}^2(X, A) < \infty$. The infimum of 2.1 is called variance of the random element X.

The least square property of real-valued random variables x is generalized by this principle: $E(x - Ex)^2 = \inf_{B \in F_c}(X, B)$

and $Var(X) = \inf_{B \in F_c} Ed_2^2(X, B)$

Assumption: we restrict ourselves to square integrable fuzzy random variables for which $\mathrm{E}\|X\|_2^2 < \infty$. This assumption ensures that the expectations as well as the variance always exist. Expectation of a fuzzy random variable is defined by generalized Aumann expectation EX. $(EX)^\alpha = Sup\{E\xi : \xi$ is a selection of X^α; $\mathrm{E}\,\|\xi\|_2 < \infty\}$, $\alpha \in [0,1]$, or by Bochner expectation of the corresponding support function of X . $S_{EX}(\alpha, u) = E_{S_X}(\alpha, u)$, $u \in S^{n-1}$, $\alpha \in [0,1]$

S^{n-1} : is the (n-1) dimensional unit sphere of $\mathbb{R}^n$ and $\langle .,. \rangle$ is the inner product of the Euclidean space $\mathbb{R}^n$. It follows that

$s_A(., u)$ represents a fuzzy set for any fixed $u \in S^{n-1}$ and

$\mathrm{S}_A(\alpha, .)$ is the support function of the convex $\alpha - cut$ for any fixed $\alpha \in [0,1]$

Expectation as defined by Puri and Ralescu [121] is the unique fuzzy set $E\widetilde{X}$ with

$$\left(E\widetilde{X} \right)_\alpha = E\left[\widetilde{X}_\alpha \right] \qquad 0 \leq \alpha \leq 1$$

Further, we can define:

$$\int_A \widetilde{X} dP = E\left(\widetilde{X}_{\chi_A} \right) \qquad \forall \text{ A} \in A,$$

where χ_A denotes the indicator of A$\in A$

Following Korner [77], the variance of frv $\widetilde{X}$ is defined as $Var\widetilde{X} = Ed_2^2(\widetilde{X}, E\widetilde{X})$.

Using $\left(E\widetilde{X}\right)_\alpha = E\widetilde{X}_\alpha$ and $s_{E\widetilde{X}_\alpha} = E_{s_{\widetilde{X}_\alpha}}$, this can be written as

$$Var\widetilde{X} = n \int_0^1 \int_{S^{n-1}} Var \ s_{\widetilde{X}_\alpha}(t) \mu(dt) d\alpha$$

Analogously, the covariance between two frv's $\widetilde{X}$ and $\widetilde{Y}$ is defined as:

$$Cov(X,Y) = n\int_0^1 \int_{S^{n-1}} Cov(s_{\widetilde{X}_\alpha}(t), s_{\widetilde{Y}\alpha}(t))\mu(dt)d\alpha$$ [2]

2.4.2 LR-fuzzy numbers

If $l > 0$ and $r > 0$, then the membership function of an *LR*-fuzzy number $\langle \mu, l, r\rangle_{LR}$, A is

$$m_A(x) = \begin{cases} L(\frac{\mu - x}{l}) & if \;\; x < \mu \\ 1 & if \;\; x = \mu \\ R(\frac{x-\mu}{r}) & if \;\; x > \mu \end{cases}$$

Here $L,\ R:\ \mathbb{R}^+ \to [0,1]$ are fixed left-continuous and non-increasing functions with $L(0) = R(0) = 1$. The functions L and R are called left and right shape functions, μ the modal point and $l\ ,r \geq 0$ are respectively the left and right spreads of the LR-fuzzy number. The most commonly used LR-fuzzy numbers are **triangular fuzzy numbers** $\langle \mu, l, r\rangle_\Delta$ with linear shape functions $L(x) = R(x) = Max\,\{0, 1-x\}$ and, especially, the **symmetric triangular fuzzy numbers** $\langle \mu, l\rangle_\Delta$ with $l = r$.

Another useful class of fuzzy numbers with unbounded support is the bell-kind (or Gaussian) fuzzy number defined by $L(x) = R(x) = \exp(-x^2)$ with $\alpha - level$ sets

$$A_\alpha = \left[\mu_A - l_A\sqrt{-\ln(\alpha)}, \mu_A + r_A\sqrt{-\ln(\alpha)}\right] \qquad \alpha \in [0,1],$$

and $\sigma_{AL} = \sqrt{\pi}/4$, such that $\sigma_A = \mu_A + \frac{\sqrt{\pi}}{4}(r_A - l_A)$

[2] For details see [110]

2.4.3 Random LR fuzzy numbers

Denote $\widetilde{Y} = \langle \mu_Y, l_Y, r_Y \rangle_{LR}$ a random LR-fuzzy number with left/right shape function L/R, with the random central value μ_Y and the positive random left and right spreads l_Y and r_Y. The result for $E\widetilde{Y}$ is known:

$$E\widetilde{Y} = \langle E\mu_Y, El_Y, Er_Y \rangle_{LR}$$

Following ([109],[77]) for random LR-fuzzy numbers $Var\widetilde{X}$ and $Cov(\widetilde{X}, \widetilde{Y})$ is given by:

$$Var\widetilde{X} = Var(\mu_X) + a_{l_2} Var(l_X) + a_{r_2} Var(r_X) - 2a_{l_1} Cov(\mu_X, l_X) + 2a_{r_1} Cov(\mu_X, l_X) \quad (2.2)$$

and

$$\begin{aligned} Cov(\widetilde{X}, \widetilde{Y}) &= Cov(\mu_X, \mu_Y) + a_{l_2} Cov(l_X, l_Y) + a_{r_2} Cov(r_X, r_Y) \quad (2.3) \\ &\quad -2a_{l_1} \left[Cov(\mu_X, l_Y) + Cov(\mu_Y, l_Y) \right] + 2a_{r_1} \left[Cov(\mu_X, r_Y) + Cov(\mu_X, r_X) \right], \end{aligned}$$

where

$$a_{l_1} = \frac{1}{2} \int L^{-1}(\alpha) d\alpha, \qquad a_{l_2} = \frac{1}{2} \int \left(L^{-1}(\alpha) \right)^2 d\alpha$$

$$a_{r_1} = \frac{1}{2} \int R^{-1}(\alpha) d\alpha, \qquad a_{r_2} = \frac{1}{2} \int \left(R^{-1}(\alpha) \right)^2 d\alpha$$

Definition 11 [171]. *Let X denote a universal set (also known as universe of discourse).*

A fuzzy subset A of X is characterized by a membership function:

$$\mu_A = X \to [0,1]$$

which associates with each element x of X a real number $\mu_A(x)$ in the interval $[0,1]$, with $\mu_A(x)$ representing the grade of membership of element x in fuzzy set A.

A is completely determined by a set of doublets.

$$A = \{(x, \mu_A(x)/x \in X\}$$

If $X = \{x_1, x_2, ..., x_n\}$ is a finite set and A is a fuzzy set in X, then we often use the notation

$$A = \mu_1/x_1 + \mu_2/x_2 + ... + \mu_n/x_n$$

where μ_i/x_i, $i = 1, ..., n$ identifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union. However, when x is not finite, a fuzzy set A is defined as:

$$A = \int_X \mu_A(x)/x$$

Example 12 *Suppose we want to define the set of natural numbers "close to 1". This can be expressed by*

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4$$

Definition 13 *Extension principle of Zadeh [171] provides a general method for extending non-fuzzy mathematical concepts in order to deal with fuzzy quantities. Then it allows the extension of a mapping f from points in X to fuzzy subsets of X:*

$$\begin{aligned} f(A) &= f(\mu_1/x_1 + \mu_2/x_2 + ... + \mu_n/x_n) \\ &= \mu_1/f(x_1) + \mu_2/f(x_2) + ... + \mu_n/f(x_n) \end{aligned}$$

Example 14 *Consider the fuzzy set "about 7" with a discrete universe and the mapping f representing the square. Then the application of the extension principle results in*

$$(\text{"about } 7^2\text{"}) = 0.0/5^2 + 0.5/6^2 + 1.0/7^2 + 0.5/8^2 + 0.0/9^2 = 0.0/25 + 1.0/49 + 0.5/64 + 0.0/81$$

2.4.4 Fast computation of a_{l_1}, a_{r_1}, a_{l_2}, a_{r_2}

$\alpha - Cuts$ of $A = (\mu, l, r)_{LR}$ are given by the intervals

$$A_\alpha = \left[\mu - L^{-1}(\alpha)l, \mu + R^{-1}(\alpha)r\right]; \qquad \alpha \in [0,1],$$

An LR-fuzzy number $A = (\mu, l, r)_{LR}$ with $L = R$ and $l = r \overset{def}{=} \Delta$ is called symmetric and abbreviated by:

$$A \overset{def}{=} (\mu, \Delta)_L.$$

For a random symmetric fuzzy number ([109], [77]):

$$Y = (\mu, \Delta)_L$$

$$E(\mu, \Delta)_L = (E\mu, E\Delta)_L,$$

and

$$Var(\mu, \Delta)_L = Var(\mu) + 2a_{l_2} Var(\Delta)$$

In particular, the variance of a random triangular fuzzy number is simply given by

$$Var(X) = Var(\mu) + \frac{1}{6}Var(l) + \frac{1}{6}Var(r),$$

and the variance of random bell-kind fuzzy numbers is:

$$Var(X) = Var(\mu) + Var(l) + Var(r).$$

The covariance of two random LR-fuzzy numbers X, Y is given by equation (2.3). This form is more convenient under additional assumptions:

- If $L = R$ (shape symmetric LR-fuzzy number) then

$$\begin{aligned} Cov(X,Y) &= Cov(\mu_{X,}\mu_Y) + a_{l_2}(Cov(l_X, l_Y) + Cov(r_X, r_Y)) \\ &\quad +2a_{l_1}(Cov(\mu_X, r_Y - l_Y) + Cov(\mu_Y, r_X - l_X)) \end{aligned}$$

- If $L = R$, $l_X = r_X$, $l_Y = r_Y$ (symmetric LR-fuzzy number $A_S := A_L - A_L$). Then,

$$Cov(X,Y) = Cov(\mu_X, \mu_Y) + a_{s_2} Cov(l_x, l_Y)$$

2.5 Expected Utility Maximum

First, mathematicians formulated principles of behavior in chance situations by assuming that the proper objective of the individual was to maximize expected monetary return. However, later on, some researchers found that the expected return maximum is not the proper methodology [135] [54]. Therefore, the expected utility rule was proposed as a substitute for the expected return rule ([3], [33]). Instead of maximizing expected return, the rational investor would maximize the expected value of the utility of return [4].

Markowitz ([105], p. 209) says:

> "Some recent commentators, on the other hand, have agreed that the expected utility maxim is not the essence of rational behavior. They show instances in which human action differs from that dictated by the maxim... At least two well-known economists who first wrote as opponents later became adherents of the expected utility maxim. The writer knows of no equally famous conversion in the other direction..."

Thus, following Markowtiz [105] we use the expected utility maximum approach to rederive the efficient frontier in the presence of fuzzy random returns.

2.5.1 Utility function

Following ([41]) and ([59], p 60-61), an individual's utility function may be expanded as a Taylor series around his expected end of period wealth.

$$U(\widetilde{w}) = U(E[\widetilde{w}]) + U'(E[\widetilde{w}])(\widetilde{w} - E[\widetilde{w}]) + \frac{1}{2}U''(E[w])(\widetilde{w} - E[\widetilde{w}])^2 + R_3,$$

where the remainder is:

$$R_3 = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)} \left(E \left[\widetilde{w} \right] \right) \left(\widetilde{w} - E \left[\widetilde{w} \right] \right)^n ,$$

and where $U^{(n)}$ denotes the nth derivative of U. Assuming that the Taylor series converges and that the expectation and summation operations are interchangeable, the individual's expected utility may be expressed as

$$E \left[U \left(\widetilde{w} \right) \right] = U(E[\widetilde{w}]) + \frac{1}{2!} U"(\widetilde{w}) \sigma^2(\widetilde{w}) + E[R_3],$$

where

$$E[R_3] = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)} \left(E \left[\widetilde{w} \right] \right) m^n(\widetilde{w})$$

$m^n(\widetilde{w})$ denotes the nth central moment of $\widetilde{w}$. Assuming quadratic utility (or jointly normal returns), the third and higher order derivatives are zero and, therefore, $E[R_3] = 0$. Hence, an individual's expected utility is defined over the first two central moments of his end of period wealth, $\widetilde{w}$,

$$E[U(\widetilde{w})] = E[\widetilde{w}] - \frac{b}{2} E[\widetilde{w}^2] = E[w] - \frac{b}{2} \left((E[\widetilde{w}])^2 + \sigma^2(\widetilde{w}) \right).$$

Chapter 3

Mean-Variance Theory with Fuzzy Random Returns

3.1 Introduction

The pioneer work in the mean variance theory has been presented by Markowitz ([103], [105]) and Tobin [155]. Later, Sharpe [141] and Lintner [97] presented the Capital Asset Pricing Model (CAPM) which was built on the foundation of the mean-variance theory. The logic behind this correlation is that the identification of the efficient frontier of risky assets with the risk-free asset is provided by the mean-variance theory. That efficient frontier is singled out by the risk-free asset and the tangency frontier portfolio. In equilibrium, after asserting the assumption that *all* investors have identical probability beliefs (share the same information) the amalgamation of the risk-free asset and the portfolio would hold . Therefore, if the portfolio of all risky assets represents the market, then the

CAPM is developed and it is empirically measured. Obviously, without ignoring the Roll's critique [126] that CAPM's view of the market portfolio as it contains every asset is not always available. For example, data of real state or real asset investments are not available, yet are crucial elements in the market. Thus, the applicability of CAPM in its existing form is questionable, because the use of different proxies for market return will reshape the empirical implications.

In this chapter, we question one important assumption made by Markowitz ([103], [105]), which remains a fundamental "*hidden*" assumption in mean-variance theory literature today: that assets are normally distributed or that random uncertainty is the sole way of modeling uncertainty.

Markowitz ([105], p.193) discussed the reasons behind the use of variance as a measure of dispersion in asset pricing instead of other dispersion measures.

> "Many considerations influence the choice of V or S as the measure of variability in a portfolio analysis. These considerations include cost, convenience, familiarity, and the desirability of the portfolios produced by the analysis."

Following Markowitz's articulation of the importance of using variance as a measure of dispersion, in this chapter, the variance analysis is considered. Knowing that the analyses based on S (semi-variance) tend to produce better portfolios than those based on V, the analyses based on semi-variance can be considered in future research endeavors after experience is gained with simpler measures in our context.

Although Markowitz [103] ignores the experts' judgments in the derivation of the efficient frontier, he [105] emphasizes the merit of such a combination of statistical techniques and the judgment of experts in the portfolio selection process. Yet, Markowitz does not propose a method to tackle that issue, and he does not study the efficient set of portfolios for the investor in the presence of fuzziness or any subjective information. White [166] has presented a viable conceptual framework for the uncertainty theories which will be used in this chapter. White [166] divides the uncertainty into so-called "subjective" and "objective". Subjective measures are derivable from observation of choice, whereas objective measures are derived, once the basic data are given, by specific procedures, independent of the problem faced. He [166] has suggested that measures of uncertainty are either formally derived from specified data, or are imputed by observing choice in a given class of problems. Also, he said:

> "It is perhaps not an unreasonable prerequisite that objective and subjective measures should be correlated to some extent."

The objective of this chapter is to re-examine mean-variance theory in the presence of fuzziness articulated by fuzzy returns (LR type). We rederive the Markowitz efficient set and present the Fuzzy Capital Market Line (FCML) and the FCAPM. By illustrating these ideas with an empirical example, a comparative study is obtainable.

3.2 Analytical Derivation of the Efficient Frontier with Fuzzy Random Returns

In this section, we analytically derive the efficient frontier in the presence of subjective information indicated by LR-fuzzy random returns. Firstly, the efficient frontier has been developed assuming an economy consisting of no riskless assets. Then the derivation of the Fuzzy Capital Market Line assuming an economy with both risky and riskless assets is achieved.

Throughout most of this chapter we will use the following set of maintained assumptions:

(A1) Perfect markets: The markets for all assets are perfect with no taxes or transaction costs. Unlimited borrowing and short sales are not permitted. Each asset is infinitely divisible.

(A2) Competition: All investors act as price takers in all markets.

(A3) Homogenous expectations: All investors have identical probability beliefs.

(A4) State-independent utility: Investors are risk averse and maximize the expectation of a Von Neuman-Morgenstern utility function, which depends solely on wealth.

(A5) Complete markets: Each competitive investor can obtain any pattern of returns through the purchase of marketed assets (subject only to his/her own budget constraint) if the number of marketed assets with linearly independent returns is equal to the number of states. Under assumptions A1 through A4 it is known that the CAPM will obtain if investor's utility function is quadratic over the relevant range of outcomes or if all asset returns are drawn from one of the class of "separating distributions" defined by Ross [131].

Following Markowitz [103] in assuming a one-period economy, we assume that the investor applies a buy-and-hold strategy during the entire period. Of course, it is noticeable that the usual variations which we observe in a continuous framework are ignored here. As they are under a multiperiod setting, the investors are willing to rebalance their portfolios over time and single period investment models are not appropriate to help investors to make the optimal allocation of their wealth. Still, it is plausible that the analysis under the one-period model assists in understanding the mean-variance theory in the presence of subjective measure, articulated by the use of fuzzy random returns.

3.2.1 Investor optimization problem

Let us assume that we have N risky assets, indexed by j, where $j = 1, 2, . . .,$ N. Let the symbol " ~" and "* " designate a random fuzzy variable. Let $\tilde{R}_j$ represent the one-period gross return on asset j, where the "gross" return is equivalent to one plus the rate of return. Let $\tilde{a}_j$ and $\tilde{b}_j$ represent the lower limit and maximum limit return of security j.

For example, when the investor faces a situation in which returns are not sharply defined but rather vague, she/he will establish, based on the experts' judgments, an aspiration interval in which the returns are located . In that context, the membership function which measures his/her degree of precision has a symmetric LR linear form. Thus, when $\widetilde{R}_j^*$ is assumed to be vague, we construct the fuzzy random return in the following fashion

$$\widetilde{R}_j^* = \widetilde{R}j \pm \text{width } (l_j), \text{ Thus, } \widetilde{a}_j = \widetilde{R}_j - \widetilde{l}_j \text{ and } \widetilde{b}j = \widetilde{R}_j + \widetilde{l}_j.$$

The experts' judgments provide the investor with the level of tolerance (width) she/he needs to develop the efficient frontier and $\tilde{a}_j$ and $\tilde{b}_j$ represent left-hand width and right-hand returns respectively. The fuzzy random return can be abbreviated by $\widetilde{R}_j^* = \left\langle \widetilde{R}_j, \widetilde{l}_j \right\rangle$

Let R_f represent the gross risk-free rate of return. Let W represent initial wealth, $\tilde{Y}$ represent terminal wealth, B represent the investment in a riskless asset, and V_j represent the investment in a risky asset j.

Given the above assumptions, the investor selects an optimal portfolio that maximizes the expected utility of the investor's end period wealth. It follows that the investor solves the following optimization problem.

$$\begin{aligned} & Max\ E\left[U\left(\tilde{Y}\right)\right] \\ \text{Subject to} \quad & 1 = \tfrac{B}{W} + \textstyle\sum_{j=1}^{N} \tfrac{V_j}{W} \\ & \tilde{Y} = R_f B + \textstyle\sum_{j=1}^{N} V_j \tilde{R}_j \end{aligned}$$

The first constraint is the investor's budget constraint, both sides of which are divided by the investor's initial wealth w. The second constraint is the wealth accumulation constraint, which incorporates fuzziness. The investor can hold an asset long or short. A short position implies $X_j < 0$. We denote the investment weights as $X_j = \frac{V_j}{W}$ for asset j and $X_f = \frac{B}{W}$ for the riskless asset. Restating the optimization problem:

$$\begin{aligned} & Max\ E\left[U\left(\tilde{Y}\right)\right] \\ \text{Subject to} \quad & 1 = X_f + \textstyle\sum_{j=1}^{N} X_j \\ & \tilde{Y} = R_f W X_f + \textstyle\sum_{j=1}^{N} W X_j \tilde{R}_j, \end{aligned}$$

Using Taylor series expansion, we expand the investor's utility function around the expected end of period wealth.

$$U\left(\tilde{Y}\right) = U\left(E\left[\tilde{Y}\right]\right) + U'\left(E\left[\tilde{Y}\right]\right)\left(\tilde{Y} - E\left[\tilde{Y}\right]\right) + \frac{1}{2}U"\left(E\left[\tilde{Y}\right]\right)\left(\tilde{Y} - E\left[\tilde{Y}\right]\right)^2 + T_3$$

where

$$T_3 = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}\left(E\left[\tilde{Y}\right]\right)\left(\tilde{Y} - E\left[\tilde{Y}\right]\right)^n$$

Assuming that the Taylor series converges, and because the expectation and summation operations are interchangeable, the individual's expected utility can be expressed as

$$E\left[U\left(\tilde{Y}\right)\right] = U\left(E\left[\tilde{Y}\right]\right) + \frac{1}{2}U"\left(E\left[\tilde{Y}\right]\right)\sigma^2\left(\tilde{Y}\right) + E\left[T_3\right]$$

where

$$E[T_3] = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}\left(E\left[\tilde{Y}\right]\right) m^n\left(\tilde{Y}\right)$$

and $m^n(\tilde{Y})$ denotes the nth central moment of $\tilde{Y}$.

To maximize expected utility of wealth, the investor will maximize a function of the moments of the portfolio return, taking into account the assumption A4 that all investors are risk averse.

In addition, we know from the previous chapter that the covariance of random

LR-fuzzy random variable is:

$$
\begin{aligned}
Cov[X,Y] &= Cov[m_x, m_y] + a_{l_2}[Cov(l_X, l_Y) + Cov(r_X, r_Y)] \\
&\quad -2a_{l_1}[Cov(m_X, r_Y - l_Y) + Cov(m_Y, r_X - l_X),
\end{aligned}
$$

under the symmetric assumption of the fuzzy LR-fuzzy variable, we get:

$$Cov(X,Y) = Cov[m_x, m_y]+a_{l_2}[Cov(l_X, l_Y)+Cov(r_X, r_Y)]-2a_{l_1}[Cov(m_X, l_Y)+Cov(m_Y, l_X),$$

assuming further that m, r and l are independent,

$$Var(X) = Var(\mu) + \frac{1}{6}Var(l) + \frac{1}{6}Var(r), \tag{3.1}$$

and

$$Cov(X,Y) = Cov(\mu_X, \mu_Y) + \frac{1}{3}Cov(l_X, l_Y). \tag{3.2}$$

Applying the equation (3.2) in the context of the fuzzy random returns, we get:

$$Var\left[\tilde{R}_p^*\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \left[Cov\left[\tilde{R}_j, \tilde{R}_i\right] + \frac{1}{3}Cov\left[\tilde{l}_j, \tilde{l}_i\right]\right],$$

where $\widetilde{R}_p^*$ is portfolio fuzzy random return, and $\widetilde{R}_j$, $\widetilde{R}_i$ are the individual returns of assets j and i respectively. $\widetilde{l}_j, \widetilde{l}_i$ represent their spreads.

Following Markowitz [103], portfolio p is a mean-variance efficient portfolio if there is no portfolio q such that $E\left[\tilde{R}_q^*\right] \geq E\left[\tilde{R}_p^*\right]$ and $Var\left[\tilde{R}_q^*\right] < Var\left[\tilde{R}_p^*\right]$. Thus, the efficient frontier can be presented as the set of portfolios that satisfy the quadratic minimization problem:

$$\begin{cases} & Min\ Var\left[\tilde{R}_p^*\right] \\ \text{Subject to} & \mu_p^* = X_f R_f + \sum_{j=1}^{N} X_j E\left[\tilde{R}_j^*\right] \\ & X_f + \sum_{j=1}^{N} X_j = 1 \end{cases} \tag{3.3}$$

where, $\mu_p^* = E\left[\tilde{R}_p^*\right]$, is the expected portfolio fuzzy random return. Because of the linearity of the expectation in fuzzy random environment, the $E[\widetilde{R}_j^*]$ implies that the expectation of a random LR-fuzzy number $\widetilde{R}_j^*$ is again an LR-fuzzy number:

$$E[\widetilde{R}_j^*] = \left\langle E[\widetilde{R}_j], E[\widetilde{l}_j] \right\rangle_{LR}$$

Thus, the model (3.3) is equivalent to:

$$\begin{cases} & Min\ Var\left[\tilde{R}_p^*\right] \\ \text{Subject to} & \mu_p^* = X_f R_f + \sum_{j=1}^{N} X_j \left\langle E[\widetilde{R}_j], E[\widetilde{l}_j] \right\rangle \\ & X_f + \sum_{j=1}^{N} X_j = 1 \end{cases},$$

using the following notation:

$$\mu_p^* = \langle \mu_p, l_p \rangle ;\ Cov(\widetilde{R}_j, \widetilde{R}_i) = \sigma_{ij};\ Cov(\widetilde{l}_j, \widetilde{l}_i) = L_{ij},$$

the investment problem with only risky assets under fuzzy random environment is as follows:

$$\begin{cases} & Min\ \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \left[\sigma_{ij} + \frac{1}{3}L_{ij}\right] \\ \text{Subject to} & \begin{array}{c} \mu_p = \sum_{j=1}^{N} X_j E[\widetilde{R}_j] \\ l_p = \sum_{j=1}^{N} X_j E[\widetilde{l}_j] \end{array} \\ & \sum_{j=1}^{N} X_j = 1 \end{cases}.$$

We know from Dubois and Prade [34] that the following multiplication has two different outcomes when λ is negative versus a positive value.

$$\lambda \odot (m,\alpha,\beta)_{LR} = \begin{array}{ll} (\lambda m, \lambda\alpha, \lambda\beta)_{LR} & if\ \lambda > 0 \\ (\lambda m, -\lambda\alpha, -\lambda\beta)_{LR} & if\ \lambda < 0 \end{array}$$

In response to this consideration, we will limit our investigation to the case when the proportions have positive values, which means we will be dealing with an investment problem without short sales. Specifically, many investors do not hold short sales due to either choice or regulation (see e.g., [64], [2]).

We know from the existing literature that empirical derivation of the mean variance efficient set, when short sales are allowed, shows that most, if not all efficient frontiers contain some negative investment proportions. For example, Levy [91] has suggested that there are two reasons for short sales: profit and diversification. In another paper Levy [92] empirically finds that without short sales, many securities do not enter the efficient frontier, and the larger N, the smaller the percentage of the securities that will appear in the efficient set. Thus, the efficient frontier grows slowly with an increased sample size. This finding has been duplicated here under fuzzy information.

Ross [129] suggested that in the absence of short sales, except on a single riskless asset, using a geometric approach CAPM holds, as long as the market portfolio is efficient. That assumption is maintained here; so it is intended that we will be able to generate the CAPM.

Also, a portfolio model under a fuzzy random environment without consideration for non-negativity constraint is difficult to model. In response to these considerations, in this chapter, we tackle the analytical derivation of the efficient frontier with fuzzy random returns, under the assumption that there are no short sales of risky assets. So, the model is a quadratic programming one in which some stocks are held long (positive proportions) while other stocks are omitted (held in zero proportions). Efficient frontier is a combination of assets if there are no other combinations with the same (higher) expected return with lower risk, and if there is no other portfolio with the same (or lower) risk and with higher expected return.

3.2.2 Efficient frontier in an economy with risky assets

In this section we want to solve the following utility minimization problem to find the efficient frontier:

$$Min \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \sigma_{ji} + X_j X_i L_{ji} \tag{3.4}$$

$$s.t.$$

$$\mu_p = \sum_{j=1}^{N} X_j E[\widetilde{R}_j] \tag{3.5}$$

$$l_p = \sum_{j=1}^{N} X_j E[\widetilde{l}_j] \tag{3.6}$$

$$\sum_{j=1}^{N} X_j = 1 \tag{3.7}$$

$$X_j \geq 0 \tag{3.8}$$

To find the optimal solution of this quadratic programming, we first write the Lagrangian form as

$$F(X, \lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \sigma_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i L_{ij}$$
$$+\lambda_1 \left(\mu_p - \sum_{j=1}^{N} X_j E\left[\tilde{R}_j\right]\right) + \lambda_2 \left(l_p - \sum_{j=1}^{N} X_j E\left[\tilde{l}_j\right]\right) + \lambda_3 \left(1 - \sum_{j=1}^{N} X_j\right). \tag{3.9}$$

In what follow X is in $\mathbb{R}^n$ and is $X = (X_1, X_2, ..., X_N)$.

Organizing the previous equation (3.9) we obtain:

$$\begin{aligned} F(X,\lambda_1,\lambda_2,\lambda_3) &= \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \left(\sigma_{ij}+L_{ij}\right) \\ &+\lambda_1\left(\mu_p-\sum_{j=1}^{N} X_j E\left[\tilde{R}_j\right]\right)+\lambda_2\left(l_p-\sum_{j=1}^{N} X_j E[\widetilde{l_j}]\right)+\lambda_3\left(1-\sum_{j=1}^{N} X_j\right). \end{aligned} \tag{3.10}$$

The Kuhn-Tucker conditions of equation (3.10) are

$$0 \leq \sum_{j=1}^{N} X_i \sigma^*_{ij} - \lambda_1 E\left[\tilde{R}_j\right] - \lambda_2 E[\widetilde{l_j}] - \lambda_3, j = 1,..,N \tag{3.11}$$

$$0 = \mu_p - \sum_{j=1}^{N} X_j E\left[\tilde{R}_j\right], \tag{3.12}$$

$$0 = l_p - \sum_{j=1}^{N} X_j E\left[\widetilde{l_j}\right], \tag{3.13}$$

$$0 = 1 - \sum_{j=1}^{N} X_j. \tag{3.14}$$

$$0 = \frac{\partial L}{\partial X_j} X_j,\ j = 1,..,N \tag{3.15}$$

$$X_j \geq 0 \tag{3.16}$$

where

$$\sigma^*_{ij} = Cov\left[\tilde{R}_j, \tilde{R}_i\right] + \frac{1}{3} Cov\left[\widetilde{l_j}, \widetilde{l_i}\right] = \sigma_{ij} + \frac{1}{3} L_{ij}$$

If every variable is positive then inequalities (3.11) are equalities because of the complementarity conditions (3.14). The X_j's that satisfy the first order conditions minimize the variance for every given level of expected return and are unique. Equation (3.11) implies

$$\sum_{j=1}^{N} X_i\sigma_{ij}^{*} - \lambda_1 E\left[\tilde{R}_j\right] - \lambda_2 E[\tilde{l}_j] - \lambda_3 = 0,$$

that implies:

$$X_k = \lambda_1 \sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] + \lambda_2 \sum_{i=1}^{N} E[\tilde{l}_i] M_{ki} + \lambda_3 \sum_{i=1}^{N} M_{ki}, \; k = 1, .., N. \tag{3.17}$$

Define Ω^* : Variance-Covariance of fuzzy returns, Ω^{*-1} : the inverse of the matrix Ω^* where M_{ki} denote the elements of the inverse of the variance-covariance matrix of fuzzy random returns, i.e., $\Omega^{*-1} \equiv [M_{ki}]$. Ω represents the sum of the two variance-covariance matrices, $\Omega = [\sigma_{ij}] + \frac{1}{3}[L_{ij}]$. Multiplying both sides of equation (3.17) by $E\left[\tilde{R}_k\right]$, and summing over $k = 1, .., N$, it follows

$$\begin{aligned} \sum_{k=1}^{N} X_k E\left[\tilde{R}_k\right] &= \lambda_1 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E\left[\tilde{R}_k\right] & (3.18) \\ &+ \lambda_2 \sum_{k=1}^{N}\sum_{k=1}^{N} M_{ki} E[\tilde{R}_k] E[\tilde{l}_i] + \lambda_3 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_k\right]. \end{aligned}$$

Also, multiplying both sides of equation (3.17) by $E\left[\tilde{l}_k\right]$, and summing over $k = 1, .., N$, it follows

$$\begin{aligned} \sum_{k=1}^{N} X_k E\left[\tilde{l}_k\right] &= \lambda_1 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E\left[\tilde{l}_k\right] & (3.19) \\ &+ \lambda_2 \sum_{k=1}^{N}\sum_{k=1}^{N} M_{ki} E[\tilde{l}_k] E[\tilde{l}_i] + \lambda_3 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{l}_k\right]. \end{aligned}$$

Then, summing equation (3.17) over $k = 1, .., N$, it follows

$$\sum_{k=1}^{N} X_k = \lambda_1 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] + \lambda_2 \sum_{k=1}^{N}\sum_{i=1}^{N} E[\widetilde{l}_i] M_{ki} + \lambda_3 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki}. \tag{3.20}$$

Next, we define

$$\begin{aligned}
A &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right], \\
B &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E\left[\tilde{R}_k\right], \\
C &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki}. \\
A_1 &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E[\widetilde{l}_k], \\
B_1 &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_i] E[\widetilde{l}_k] \\
C_1 &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_k]
\end{aligned} \tag{3.21}$$

From equations (3.12), (3.13), (3.14), (3.18), (3.19) and (3.20), it follows:

$$\mu_p = \lambda_1 B + \lambda_2 A_1 + \lambda_3 A \tag{3.22}$$

$$l_p = \lambda_1 A_1 + \lambda_2 B_1 + \lambda_3 C_1 \tag{3.23}$$

$$1 = \lambda_1 A + \lambda_2 C_1 + \lambda_3 C. \tag{3.24}$$

Noting here,

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_k\right], \\
\sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_i] E[\widetilde{R}_k] &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_k] E[\widetilde{R}_i], \\
\sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E[\widetilde{l}_k] &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_i] E[\widetilde{R}_k].
\end{aligned}$$

Solving system of equations (3.22), (3.23) and (3.24) for λ_1, λ_2 and λ_3, and defining $\Delta \equiv B(B_1C - C_1^2) - A_1(A_1C - AC_1) + A(A_1C_1 - AB_1)$, and as $\sum_k \sum_i M_{ki}$ is positive because of the positive definiteness of matrix $M \equiv [M_{ki}]$, we obtain

$$\begin{aligned}
\lambda_1 &= \frac{\mu_p(B_1C - C_1^2) - l_p(A_1C - C_1A) + (A_1C_1 - B_1A)}{\Delta} \\
\lambda_2 &= \frac{-\mu_p(A_1C - AC_1) + l_p(BC - A^2) - (BC_1 - A_1A)}{\Delta} \\
\lambda_3 &= \frac{\mu_p(A_1C_1 - AB_1) - l_p(BC_1 - AA_1) + (BB_1 - A_1^2)}{\Delta}
\end{aligned} \tag{3.25}$$

Next, we substitute for λ_1, λ_2 and λ_3, from equation (3.25) into equation (3.17) to solve for X_k. X_k is the proportion of each risky asset k held in a portfolio on the minimum-variance for a given expected return, which is as follows:

$$\begin{aligned}
X_k &= \frac{\begin{array}{c}\mu_p \sum_{i=1}^{N} M_{ki}\left[(B_1C - C_1^2)\, E(\widetilde{R}_i) - (A_1C - AC_1)E(\widetilde{l}_i) + (A_1C_1 - AB_1)\right] \\ -l_p \sum_i M_{ki}\left[(A_1C - C_1A)E(\widetilde{R}_i) - (BC - A^2)E(\widetilde{l}_i) + (BC_1 - AA_1)\right] \\ +\sum_{i=1}^{N} M_{ki}\left[(A_1C_1 - B_1A)\, E(\widetilde{R}_i) - (BC_1 - A_1A)E(\widetilde{l}_i) + (BB_1 - A_1^2)\right]\end{array}}{\Delta}, \\
k &= 1, .., N.
\end{aligned} \tag{3.26}$$

Using the following notations: $(B_1C - C_1^2) = \alpha$; $(A_1C - AC_1) = \beta$; $(A_1C_1 - AB_1) = \gamma$; $(BC - A^2) = \delta$; $(BC_1 - AA_1) = \varphi$; $(BB_1 - A_1^2) = \psi$, the equation (3.26) is equivalent to:

$$\begin{aligned} X_k &= \frac{\begin{array}{c}\mu_p \sum_{i=1}^N M_{ki}\left[\alpha E(\widetilde{R}_i) - \beta E(\widetilde{l}_i) + \gamma\right] - l_p \sum_i M_{ki}\left[\beta E(\widetilde{R}_i) - \delta E(\widetilde{l}_i) + \varphi\right] \\ + \sum_{i=1}^N M_{ki}\left[\gamma E(\widetilde{R}_i) - \varphi E(\widetilde{l}_i) + \psi\right]\end{array}}{\Delta}, \\ k &= 1, .., N. \end{aligned} \tag{3.27}$$

Because $M \equiv [M_{ki}]$ is positive definite and Δ is zero if and only if $\mu^* = \lambda 1$ such that $\mu^* = [\mu_1^*, ..., \mu_n^*]'$; $\mu_j^* = \left\langle E\left[\widetilde{R}_j\right], E[\widetilde{l}_j]\right\rangle$, otherwise $\Delta > 0$.

Theorem 15 ([161]). *Let $\mu^* = [\mu_1^*, ...\mu_j^*]' \neq 1$ for all λ. In the model there exists an open interval $(\mu_{p_0}^*, \mu_{p_1}^*)$ of μ_p^* in which every variable is positive if and only if :*

$$\left(\sum_i M_{pi}\right)\left(\sum_i M_{qi}\mu_i^*\right) < \left(\sum_i M_{pi}\mu_i^*\right)\left(\sum_i M_{qi}\right)$$

for all $p \in I^-$ and $q \in I^+$ and

$$\left(\sum_k \sum_i M_{ki}\mu_k^*\right)\left(\sum_k M_{ki}\mu_k^*\right) - \left(\sum_k \sum_i M_{ki}\mu_k^*\mu_i^*\right)\left(\sum_i M_{ki}\right) < 0$$

for all $i \in I^0$

such that:

$$I^+ = \left\{k/\left(\sum_k\sum_i M_{ki}\right)\left(\sum_i M_{ki}\mu_i^*\right) - \left(\sum_i\sum_k M_{ki}\mu_i^*\right)\left(\sum_i M_{ki}\right) > 0\right\}$$
$$I^- = \left\{k/\left(\sum_k\sum_i M_{ki}\right)\left(\sum_i M_{ki}\mu_i^*\right) - \left(\sum_i\sum_k M_{ki}\mu_i^*\right)\left(\sum_i M_{ki}\right) < 0\right\}$$
$$I^0 = \left\{k/\left(\sum_k\sum_i M_{ki}\right)\left(\sum_i M_{ki}\mu_i^*\right) - \left(\sum_i\sum_k M_{ki}\mu_i^*\right)\left(\sum_i M_{ki}\right) = 0\right\}$$

Proof. Similar to what Vörös [161] presented in his paper. ■

Because of the positivity of the variables and of Δ it follows that:

$$\left[\begin{array}{c} \mu_p \sum_{i=1}^N M_{ki}\left[\alpha E(\widetilde{R}_i) - \beta E(\widetilde{l}_i) + \gamma\right] - l_p \sum_i M_{ki}\left[\beta E(\widetilde{R}_i) - \delta E(\widetilde{l}_i) + \varphi\right] \\ + \sum_{i=1}^N M_{ki}\left[\gamma E(\widetilde{R}_i) - \varphi E(\widetilde{l}_i) + \psi\right] \end{array}\right] > 0 \quad (3.28)$$

If we define

$$h_k = \sum_i M_{ki} E[\widetilde{R}_i];\ f_k = \sum_i M_{ki} E(\widetilde{l}_i) \text{ and } g_k = \sum_i M_{ki},$$

then, the equation (3.28) is equivalent to:

$$\mu_p(\alpha h_k - \beta f_k + \gamma g_k) - l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k) > 0 \quad k = 1, ..., n$$

If $i \in I^0$ then $l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k) < 0$, indices $q \in I^+ \to (\alpha h_k - \beta f_k + \gamma g_k) > 0$, and for $p \in I^-$ and $p \in I^- \to (\alpha h_k - \beta f_k + \gamma g_k) < 0$, then the following inequality holds:

$$\frac{l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k)}{(\alpha h_k - \beta f_k + \gamma g_k)} < \mu_p < \frac{l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k)}{(\alpha h_k - \beta f_k + \gamma g_k)} \tag{3.29}$$

In line with Vörös [161], from the inequality (3.29), the interval in which every variable is positive is given by:

$$\mu_p^1 = \min_{p \in I^-} \left\{ \frac{l_p(\beta h_p - \delta f_p + \varphi g_p) + (\gamma h_p - \varphi f_p + \psi g_p)}{(\alpha h_p - B f_p + \gamma g_p)} \right\}$$

$$\mu_p^0 = \max_{q \in I^+} \left\{ \frac{l_p(\beta h_q - \delta f_q + \varphi g_q) + (\gamma h_q - \varphi f_q + \psi g_q)}{(\alpha h_q - B f_q + \gamma g_q)} \right\},$$

We next multiply equation (3.11) by X_j and sum over j for $j = 1, \ldots, N$, to derive the following:

$$\sum_{j=1}^{N} \sum_{i=1}^{N} X_j X_i \sigma_{ij}^* = \lambda_1 \sum_{j=1}^{N} E\left[\tilde{R}_j\right] X_j + \lambda_2 \sum_{j} X_j E(\tilde{l}_j) + \lambda_3 \sum_{j=1}^{N} X_j. \tag{3.30}$$

From the definition of $\sigma^2\left(\tilde{R}_p^*\right)$, equations (3.12), and (3.13), equation (3.30) implies

$$Var\left[\tilde{R}_p^*\right] = \lambda_1 \mu_p^* + \lambda_2 l_p + \lambda_3. \tag{3.31}$$

Substituting for λ_1, λ_2 and λ_3 from (3.25) into (3.31), to obtain the equation for the minimum-variance frontier. So, for the interval (μ_{p1}^*, μ_{p0}^*), we obtain the functional form of return-variance:

$$\sigma^2\left(\tilde{R}_p^*\right)=\frac{\left(\mu_p^2\alpha+l_p^2\delta-2\mu_p l_p\beta-2l_p\varphi+\mu_p\gamma+\psi\right)}{\Delta}, \tag{3.32}$$

Once all fuzzy components ($l_p = 0$, l_i and $l_k = 0$) have been discarded in the equation (3.32), we will get the standard functional form of return-variance. Thus, the model is a special case of the Markowitz frontier. Next, for the sake of completeness of the analysis, the minimum-variance portfolio in the presence of fuzzy random uncertainty is presented below. Since, the equation (3.32) is a function of two variables of degree 2, partial derivatives and all other properties of multiple variables are applicable. The differentiability is achieved as follows

$$\begin{aligned}\frac{\partial\sigma^2\left(\tilde{R}_p^*\right)}{\partial\mu_p} &= \frac{2\alpha\mu_p-2l_p\beta+\gamma}{\Delta}=0\Longrightarrow\mu_{p\min}=\frac{2l_p\beta-\gamma}{2\alpha},\text{ and}\\ \frac{\partial^2\sigma^2\left(\tilde{R}_p^*\right)}{\partial^2\mu_p} &= \frac{2\alpha}{\Delta}>0.\end{aligned} \tag{3.33}$$

3.2.3 Efficient frontier in an economy where one asset is risk-free

For all investors to achieve the efficient frontier by lending or borrowing against the risky portfolio, and for the separation theorem to hold, following Ross' [129] analysis, by permitting the investor to short sale the riskless asset, the analytical derivation of the efficient frontier is presented. The risk-free asset offers a riskless return of R_f. With short sales restrictions, all assets will appear in positive amounts in the market portfolio. The investor's utility minimization problem is formulated as follows.

$$\begin{cases} \underset{X_j}{Min} \sum_i \sum_j X_i X_j \sigma^*_{ij} \\ \text{Subject to} \quad \mu^*_p = X_f R_f + \sum_{j=1} X_j E[\widetilde{R}^*_j] \\ X_f + \sum_j X_j = 1 \\ X_j \geq 0 \,, \quad j = 1, ..., N \end{cases} \tag{3.34}$$

The above model is equivalent to:

$$\begin{cases} \underset{X_j}{Min} \sum_i \sum_j X_i X_j \sigma^*_{ij} \\ \text{Subject to} \quad \langle \mu_p, l_p \rangle = X_f \langle R_f, l_f \rangle + \sum_{j=1} X_j \left\langle E[\widetilde{R}_j], E[\widetilde{l}_j] \right\rangle \\ X_f + \sum_j X_j = 1 \\ X_j \geq 0 \,, \quad j = 1, ..., N \end{cases} \tag{3.35}$$

which is equivalent to:

$$\begin{cases} \underset{X_j}{Min} \sum_i \sum_j X_i X_j \sigma^*_{ij} \\ \text{Subject to} \quad \begin{matrix} \mu_p = X_f R_f + \sum_{j=1} X_j E[\widetilde{R}_j] \\ l_p = X_f l_f + \sum_{j=1} X_j E[\widetilde{l}_j] \end{matrix} \,, \\ X_f + \sum_j X_j = 1 \\ X_j \geq 0 \,, \quad j = 1, ..., N \end{cases} \tag{3.36}$$

For simplicity, we assume that R_f is sharply defined, which means that $l_f = 0$. In order to find the optimal solution of this quadratic programming, we write the Lagrangian form:

$$\begin{aligned} \Psi\left(X_j, \lambda_1, \lambda_2\right) &= \sum_i \sum_j X_i X_j \sigma^*_{ij} \\ &+\lambda_1 \left(\mu_p - R_f - \sum_{j=1} X_j \left(E[\widetilde{R}_j] - R_f \right) \right) \\ &+\lambda_1 \left(l_p - \sum_{j=1} X_j \left(E[\widetilde{l}_j] - R_f \right) \right) \end{aligned} \tag{3.37}$$

The Kuhn-Tucker conditions of (3.37) are:

$$\frac{\partial \Psi}{\partial X_j} = \sum_{i=1}^{N} X_i \sigma^*_{ij} - \lambda_1 \left(E\left[\tilde{R}^*_j\right] - R_f \right) - \lambda_2 E\left[\widetilde{l}_j\right] \geq 0 \quad j = 1, ..., N \tag{3.38}$$

$$\frac{\partial \Psi}{\partial \lambda_1} = \mu_p - R_f - \sum_{j=1}^{N} X_j \left(E\left[\tilde{R}_j\right] - R_f \right) = 0 \tag{3.39}$$

$$\frac{\partial \Psi}{\partial \lambda_2} = l_p - \sum_{j=1}^{N} X_j E\left[\widetilde{l}_j\right] = 0 \tag{3.40}$$

$$\frac{\partial \Psi}{\partial X_j} X_j = 0 \tag{3.41}$$

$$X_j \geq 0 \qquad j = 1, ..., N \tag{3.42}$$

If every variable X_j is positive (inequalities (3.42) hold) then inequalities (3.38) are equalities because of the complementarity conditions (3.41). So, the equations (3.38) imply that:

$$X_k = \lambda_1 \sum_{i=1} M_{ki} (E[\widetilde{R}_i] - R_f) + \lambda_2 \sum_i M_{ki} E(\widetilde{l}_i) \qquad k = 1, ..., N \tag{3.43}$$

Multiplying both sides of the equation (3.43) by $\left[E[\widetilde{R}_k]-R_f\right]$ and summing over $k=1,...,N$, it follows:

$$\begin{aligned}\sum X_k\left(E\left[\widetilde{R}_k^*\right]-R_f\right) &= \lambda_1\sum_k\sum_i M_{ki}\left(E\left[\widetilde{R}_i^*\right]-R_f\right)\left(E\left[\widetilde{R}_k^*\right]-R_f\right) \\ &+\lambda_2\sum_k\sum_i M_{ki}\left(E\left[\widetilde{l}_i\right]\right)\left(E[\widetilde{R}_k]-R_f\right)\end{aligned} \tag{3.44}$$

Multiplying both sides of the equation (3.43) by $E[\widetilde{l}_k]$ and summing over $k=1,...,N$, it follows:

$$\begin{aligned}\sum X_k E\left[\widetilde{l}_k\right] &= \lambda_1\sum_k\sum_i M_{ki}\left(E\left[\widetilde{R}_i^*\right]-R_f\right)E\left[\widetilde{l}_k\right] \\ &+\lambda_2\sum_k\sum_i M_{ki}E(\widetilde{l}_i)E\left[\widetilde{l}_k\right]\end{aligned} \tag{3.45}$$

From equation (3.39), we deduce that the equation (3.44) using the implied parameters A, B, C, A_1, B_1 and C_1, it follows that:

$$\mu_p-R_f=\lambda_1\left[B-2R_fA+R_f^2C\right]+\lambda_2\left[A_1-R_fC_1\right] \tag{3.46}$$

Also, from equation (3.40), the equation (3.45) implies that:

$$l_p=\lambda_1\left[A_1-R_fC_1\right]+\lambda_2 B \tag{3.47}$$

So, both equations (3.46) and (3.47) imply:

$$\begin{aligned}\lambda_1 &= \frac{(\mu_p - R_f)B - l_p(A_1 - R_f C_1)}{(B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)} \\ \lambda_2 &= \frac{(B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)}{(B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)}\end{aligned} \tag{3.48}$$

Defining $D = (B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)$ and substituting for $\lambda_1$and λ_2 from previous equation (3.48) into equation (3.43) to solve for X_k:

$$\begin{aligned}X_k &= \frac{1}{D}\left[\begin{array}{c}[(\mu_p - R_f)B - l_p(A_1 - R_f C_1)] \sum_{i=1} M_{ki}(E[\widetilde{R}_i] - R_f) + \\ \left[(B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)\right] \sum_i M_{ki} E(\widetilde{l}_i)\end{array}\right] \\ k &= 1, ..., N\end{aligned} \tag{3.49}$$

Using the notation indicated in the previous section,

$$g_k = \sum_i M_{ki} \ ; \ f_k = \sum_i M_{ki} E[\widetilde{l}_i] \text{ and } \ h_k = \sum_i M_{ki} E[\widetilde{R}_i^*],$$

equation (3.49) is equivalent to:

$$\begin{aligned}X_k &= \frac{1}{D}\left[\begin{array}{c}[(\mu_p - R_f)B - l_p(A_1 - R_f C_1)]\,(h_k - R_f g_k) \ + \\ \left[(B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)\right] f_k\end{array}\right] \\ k &= 1, ..., N\end{aligned} \tag{3.50}$$

Under the positive condition of the previous equation (3.50), and in a fashion similar to the previous section, we derive the equation for the frontier using Vörös's method [161].

Because of the positivity of the variables and of the dominator, multiplying equation (3.38) by X_j, summing from $j = 1, ..., N$ and rearranging, we find that:

$$\sum_j \sum_i X_j X_i \sigma^*_{ij} = \lambda_1 \sum_j \left(E\left[\widetilde{R}_i\right] - R_f \right) X_j + \lambda_2 \sum_j X_j E(\widetilde{l}_j) \tag{3.51}$$

From the definition of $\sigma^2(\widetilde{R}^*_p)$ and equations (3.39), (3.40), equation (3.51) implies that:

$$Var(\widetilde{R}^*_p) = \lambda_1 \left(\mu_p - R_f\right) + \lambda_2 l_p. \tag{3.52}$$

Substituting for $\lambda_1$and λ_2 from (3.48) into (3.52), we obtain the equation for the minimum-variance frontier.

$$\sigma^2\left(\tilde{R}^*_p\right) = \frac{1}{D}\left[\begin{array}{c} \left(\mu_p - R_f\right)\left[(\mu_p - R_f)B - l_p(A_1 - R_f C_1)\right] \\ +l_p\left[(B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)\right] \end{array}\right] \tag{3.53}$$

Arranging the above equation we get:

$$\sigma^2\left(\tilde{R}^*_p\right) = \frac{\left[\left(\mu_p - R_f\right)^2 B + l_p^2\left(B - 2AR_f + R_f^2 C\right) - 2l_p(A_1 - R_f C_1)(\mu_p - R_f)\right]}{(B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)}. \tag{3.54}$$

In the mean-standard deviation space, we get the following equation:

$$\sigma\left(\tilde{R}^*_p\right) = \sqrt{\frac{\left[\left(\mu_p - R_f\right)^2 B + l_p^2\left(B - 2AR_f + R_f^2 C\right) - 2l_p(A_1 - R_f C_1)(\mu_p - R_f)\right]}{(B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)}} \tag{3.55}$$

Thus, the minimum-variance frontier in mean-standard space is *non*linear, and equation (3.55) is the Fuzzy Capital Market Line (FCML). We believe with the absence of fuzziness (in every single return $l_j = 0$ and in the portfolio mean $l_p = 0$) in the model, the equation (3.55) will offer the classical capital market line. An empirical implication of this conclusion is shown in the next section.

3.3 Empirical Implications of the Model

In this section we analyze the relationship between risk and return in the presence of fuzzy information, revealed by the use of fuzzy returns, in NASDAQ stocks in the 1990-2000 period.

3.3.1 The impact of the subjective measure on the location of capital market line

In this subsection we use NASDAQ stock data to show the impact of the introduction of fuzziness on the location of Capital Market Line. In real life, the investor will be faced with more than just 15 assets as presented here. However, we limit our investigation in this section to 15 stocks to compare the location of fuzzy capital market line with respect to the location of the original CML. The model can be solved using any optimization software to construct the market line of 15 risky assets. The randomly selected 15 stocks are traded on the NASDAQ. The data, which covered the monthly rate of returns of these stocks for the 10-year period 1990-2000, were taken from the Center for Research and Security Prices (CRSP) and used to estimate the mean and standard deviation of returns. The following tables (3.1 and 3.2) show the returns and the widths (spreads) of fuzzy returns for 15 stocks over the 10-year period.

Table 3.1: 15 NASDAQ returns randomly selected

Permno	M2	M3	M4	M5	M6	M7	M8	M9	M10
10025	6.38E-07	-0.010696	-0.02174	-0.11641	0.036368	0.068993	-0.022473	0.07654	-0.043018
10078	0.121184	-0.182324	-0.044449	0.018019	-0.071705	0.028368	0.00465	0.146501	0.120673
10200	0.033226	-0.210156	-0.365934	0.088947	0.056863	0.014963	-0.184304	0	-0.241162
10271	0.152579	0	0.050586	-0.009009	-0.018265	-0.018605	-0.053024	0.02927	-0.034233
10290	0.030214	-0.05506	-0.045024	0.006557	-0.013158	0.181217	-0.068598	0.111848	0.147325
10588	0.223143	-0.123059	-0.04879	0.150572	-0.242139	0.140356	-0.011976	0.102948	0.042559
10772	0.179152	0.038916	-0.14781	-0.017857	-0.018183	-0.066375	-0.159065	0.188052	0.150061
11293	0	0.098801	0.064103	-0.006558	0.066797	0.085472	-0.014225	-0.002869	0
11701	-0.174717	0	-0.105361	-0.020409	-0.020834	0.010471	-0.053489	0.094311	-0.094311
11917	0.179693	0.013072	-0.102479	-0.082444	-0.677643	0.129678	-0.176931	0.190354	-0.127833
12063	-0.158057	-0.147325	-0.057158	0.028988	-0.264816	0.051825	-0.051825	-0.161268	0.012423
12068	-0.037272	0.033198	-0.00409	-0.012371	0	0.024591	0.031875	-0.015811	-0.003992
12189	0.133531	0.020618	0	0.185719	-0.107245	0.072759	-0.072759	0	0.055059
19546	-0.11441	-0.06252	0.279585	0.115069	-0.079137	0.13206	0.030459	0.04879	0.258695
23318	-0.106314	-0.0977	-0.165619	0.312223	-0.009132	0.04485	-0.146142	-0.015346	-0.020834

Due to the space limitation, the above table does not contain all the observations over the 10-year period; it is a subset from the complete data set. Permno is a number identifying the issuing company

Because there are an infinite number of ways to characterize fuzziness, there are an infinite number of ways to graphically depict the membership and to generate the data. Normally, experts should be able to offer decision makers or investors information regarding the measure of fuzziness. In this context, fuzziness has been used under the following conditions, that it reflects the experts' judgments and that the returns should be around those values. For example, the company with permn 10078 has a 0.211 return. After getting a subjective recommendation from experts, the return that should be used is 0.211 ± 0.000645. In a fuzzy setting with TR type fuzzy membership, that means that the membership function equals 1 for a return 0.211, and it is linearly decreasing on the right and left. Ross T. [132] pointed out that there are more ways to assign membership function values to fuzzy variables than for random variables. The literature on this topic is rich with references, for example [34]. The assignment can be intuitive or based on algorithms or logical operations. We established the table (3.2) based on a combination of the intuition and inference methods presented by Ross T. [132].

Following an inference approach, we use the bid-ask spread to get the width of the fuzzy returns. the logic behind that technique is that a bid-ask spread creates vagueness and imprecision in the investor's choice. It is the irregularities, which may arise from the lack of imprecision in the data, that are a concern here.

Moreover, market-created uncertainty results from the interaction (directly or indirectly) among participants who form their expectations in an ill-defined market. Consequently, each participant will form his/her expectations based on their subjective prediction of other participants' expectations.

We use the bid-ask spread because it affects the stock returns (see, [78]). There are considerable theoretical justifications to the use of a bid-ask spread and to its effects on returns. Heinkel and Kraus [52] pointed out that a component of the bid-ask spread, which is based on information asymmetries, could be considered part of true returns. Hence, the effect of bid-ask spread is that the observed returns differ from the true returns.

Moreover, as pointed out by Amihud and Mendelson [6] rational investors select their assets to maximize their expected return net. These authors showed a strong effect of the spread on returns.

Because the bid-ask spread is related to the availability of information about the asset, the greater the amount of information about an asset, the narrower the spread, which means the closer the true return is to the observable return (see, [32]). In contrast, the more information about an asset is vague, the greater the distance between the true return and the observed return. In this sense, the width between the observed return and the net return, taking into consideration the bid-ask spread has been identified (see, [106]). Also, Merton [106] pointed out that incomplete information about a stock, which is a major factor, is reflected in its bid-ask spread. This conclusion has been supported by the effect of Amihud and Mendelson's spread [5].

In a statement Merton [106] says:

> "I also believe that financial models based on frictionless markets and complete information are often inadequate to capture the complexity of rationality in action."

That lead to the development of the so-called width (tolerance level), which means that the investor uses the net and observable returns to form his/her fundamental returns, assuming that the fuzzy random return sways between them. We employ a method comparable to Amihud and Mendelson [5] in developing the net return, to allow the investors the ability to compress information into fuzzy notions that they can analyse using fuzzy theory. Under these considerations, the following formulas have been derived to generate the widths data in Table (3.2)

$$P_{mt} = \frac{Askprice_t - Bidprice_t}{2},$$

and

$$R_t = \ln\left(\frac{Pmt}{P_{mt-1}}\right), R_{net} = \left(\frac{1 - Spread_t}{1 + Spread_{t-1}}\right) R_t,$$

such that

$$Spread_t = \frac{Askprice_t - Bidprice_t}{Askprice_t + Bidprice_t},$$

then

$$width = l_t = |R_{net} - R_t|$$

Table 3.2: 15 widths of the 15 NASDAQ stocks 1990-2000

Permno	M2	M3	M4	M5	M6	M7	M8	M9	M10
10025	6.38E-08	0.000771	0.001388	0.008332	0.002098	0.003002	0.000999	0.00392	0.002747
10078	0.000645	0.001587	0.000402	0.00016	0.000508	0.000132	2.15E-05	0.000588	0.000428
10200	0.000647	0.005252	0.016981	0.003711	0.001969	0.000365	0.006733	0	0.014312
10271	0.004359	0	0.001566	0.000241	0.000498	0.000517	0.001818	0.000832	0.000846
10290	0.001234	0.00305	0.00199	0.000211	0.000513	0.004135	0.001642	0.003502	0.003996
10588	0.016129	0.010004	0.004648	0.011011	0.019282	0.011222	0.00124	0.009616	0.003406
10772	0.004632	0.001458	0.004932	0.000641	0.000974	0.003143	0.009105	0.010504	0.00611
11293	0	0.005047	0.003068	0.000336	0.002629	0.002612	0.000521	0.000105	0
11701	0.007638	0	0.005277	0.000812	0.000434	0.000323	0.001711	0.002763	0.003
11917	0.005927	0.000335	0.003676	0.003125	0.057284	0.008544	0.011111	0.012208	0.009176
12063	0.005244	0.00462	0.001405	0.000585	0.00999	0.002554	0.002659	0.007898	0.00045
12068	0.00108	0.000803	8.28E-05	0.000255	0	0.00059	0.000741	0.000374	7.89E-05
12189	0.015748	0.002018	0	0.018062	0.011509	0.007281	0.007801	0	0.004692
19546	0.006743	0.00391	0.013475	0.004907	0.004586	0.005464	0.001495	0.003145	0.009637
23318	0.001278	0.001265	0.002054	0.002871	8.34E-05	0.000392	0.001481	0.000157	0.000218

Due to the space limitation, the above table does not contain all the observations over the 10 year period; it is a subset from the complete data set.

The following graph (3.1), which has been plotted in two dimensions, shows the location of the capital market line with fuzziness (blue line) and without accounting for fuzziness (red line); the value of 0.031 has been used for the portfolio width $l_p = 0.031$ and 0.07 as the risk-free rate $R_f = 0.07$ to be able to show the graph in two dimensions[1]. The y axis represents σ and the x axis represents μ_p. By increasing from $l_p = 0.031$ (blue line) to 0.045 (navy line) and 0.061(brown line), the FCML is moving upward, which means that an increase of fuzziness manifested by the portfolio width l_p will cause the market line to be more dominated by the original market line. On the contrary, a small degree of fuzziness in the model, measured by the portfolio width, shows that fuzzy capital market lines are

[1]Although it appears very high, risk-free rate of 0.07 has been used only for illustration purpose. The average of T-Bill rate over the period 1990-2000 could be more appropriate.

dominated by the standard linear capital market line. Also, as presented in the previous section, it is obvious in the following figure that the FCML is nonlinear. The introduction of fuzzy information, then, shifts the intercept of the line relating μ_p and the slope from R_f to another positive value (value $< R_f$).

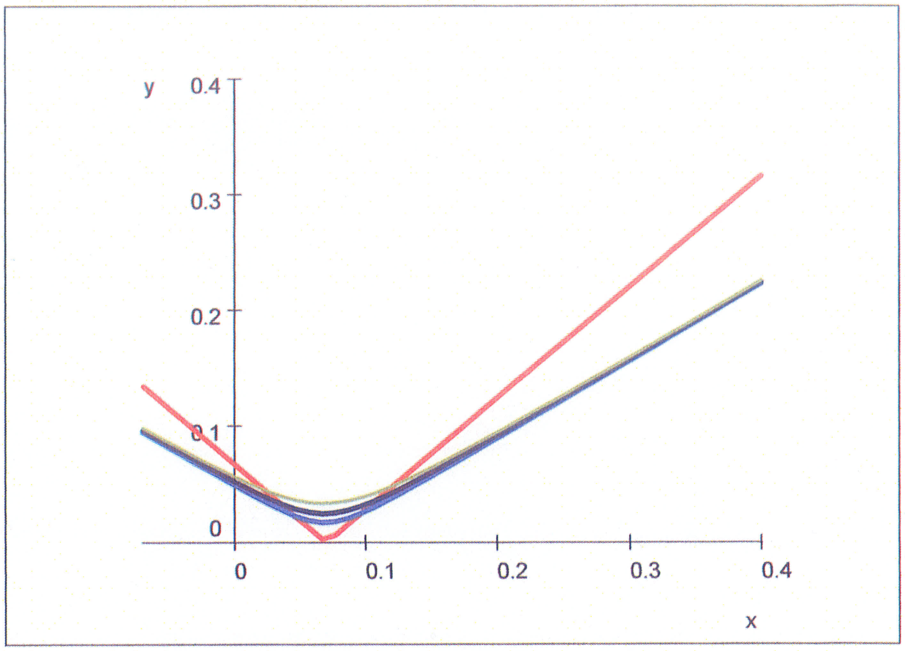


Figure 3.1: Capital Market Line (CML) without accounting for fuzziness (red line) and fuzzy CML (others)

The next graph (3.2) plots the capital market line (blue plane) with the risk free rate ($R_f = 0.07$) and the efficient frontier without the risk free-rate (green plane) in three dimensions; the y-axis represents the portfolio width l_p, x -axis represents the portfolio mean μ_p and z-axis represents the σ_p.

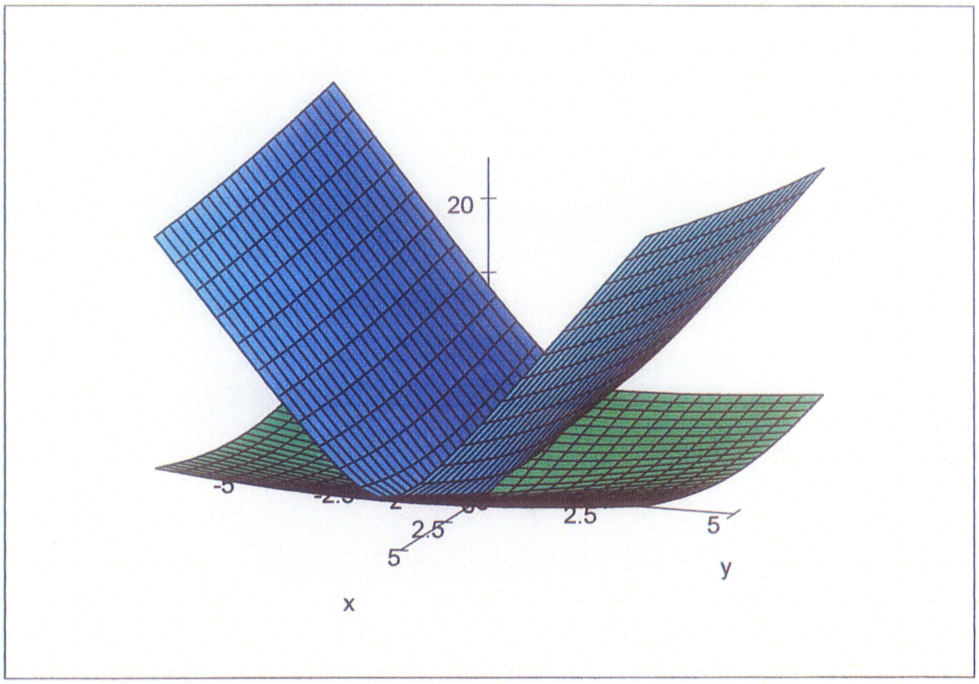


Figure 3.2: 3 D graphical representation of the fuzzy frontier with and without risk-free rate

3.3.2 The impact of fuzziness on the location of efficient frontier

Using another set of data we randomly selected 15, 30 and 50 stocks traded on the NASDAQ. Following the same method discussed in the previous subsection, we generate the widths (spreads) for all the complete data in the form of 15, 30 and 50 widths. Similarly to the case of 15 assets presented previously, the efficient frontiers have been presented for 15, 30 and 50 assets.

The mathematical problem without short sales presented previously (3.4-3.8) has two new constraints (3.6) and (3.8), so it requires programming techniques to handle the problem. With computer capability we are able to achieve the efficient investment strategy for each portfolio return level μ_p with width l_p for all different groups of assets. (15, 30 and

50 assets).

Because the computer program is long and complicated, it has not been included here. The part of the computer code (written in Visual Basic) which generates the N asset mean-variance efficient set with short sales under fuzzy information, is in appendix A.

One aim of this part of the program is to not only do all the necessary computations from stock prices and solve the optimization problem but also to generate a graph of the efficient frontier without short sales. The following figures [(3.3), (3.4), (3.5)] show the efficient frontier without short sales for all three sample sizes (15, 30 and 50 assets). One major element worth elaborating on is that all efficient frontiers are concave arcs, which is consistent with the finding of Szegö. However, the boundary of each sample size turns out not to be a parabola. It is also clearly observed that the arc, which is between minimum and maximum points does not coincide with the original boundary. The minimum (maximum) point represents, as discussed previously and supported by Szegö's finding [147] can be achieved by investing the capital in the investment option with lowest (highest) return.

For comparison, the following graphs represent the case when fuzziness is not included. In accordance with Levy [92], the figure (3.6) plots the efficient frontiers constructed with and without short sales; the efficient frontier without short sales lies inside the efficient frontier with short sales. An investor with short sales will attain a lower utility than an investor with both short and long positions. Also, in all cases (15, 30 and 50 assets), it is clear that the frontier is not a parabola, but an arc of a parabola as suggested by Szegö [147], see figures [(3.7), (3.8)].

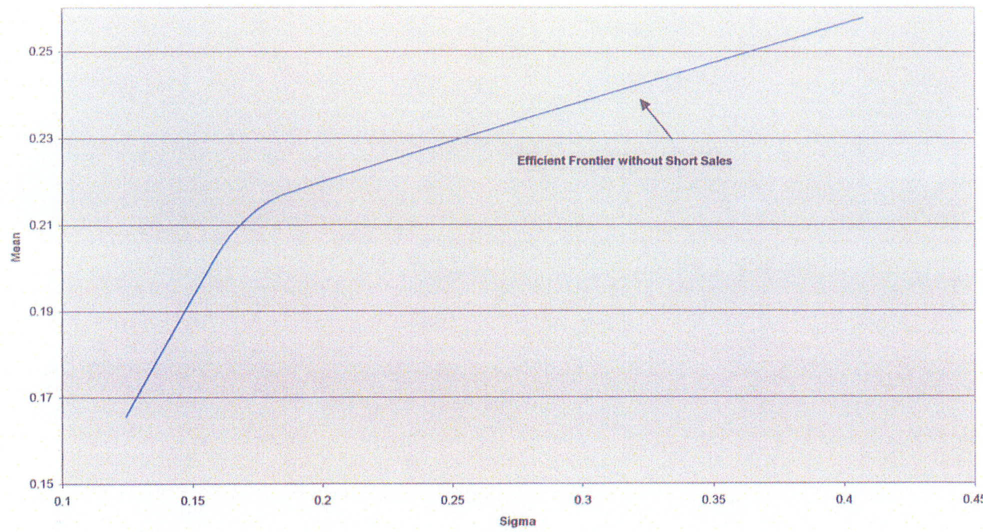


Figure 3.3: Efficient frontier (EF) without short sales for 15 asset prices

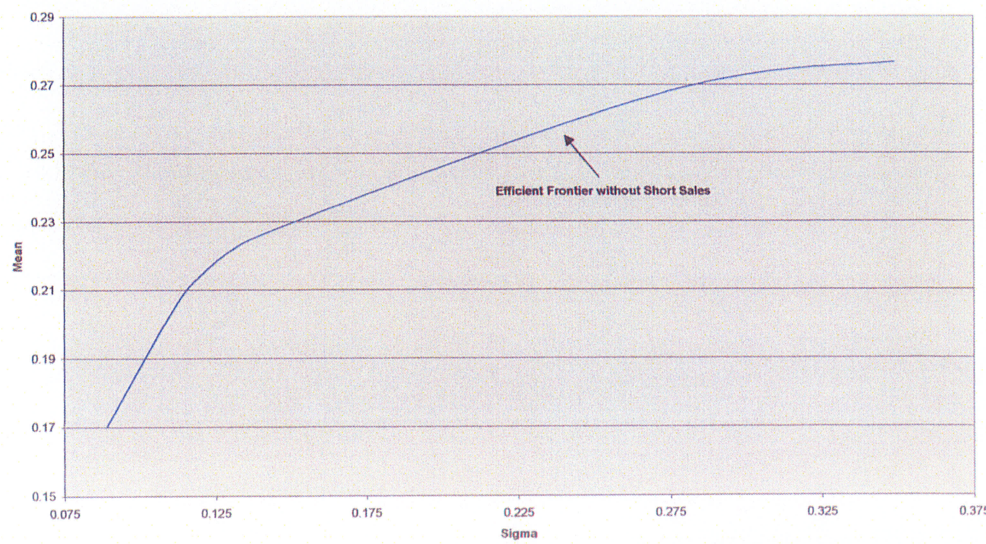


Figure 3.4: Efficient frontier (EF) without short sales for 30 asset prices

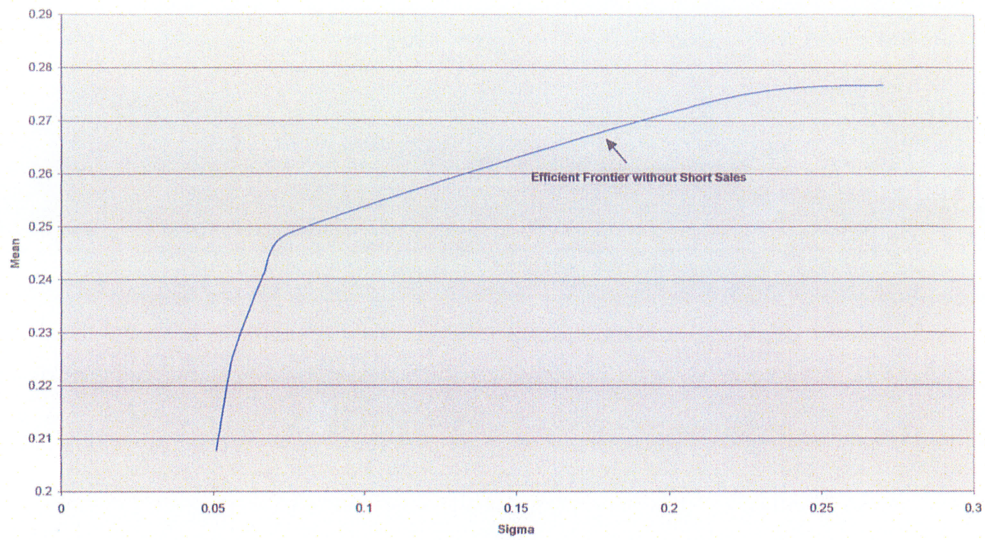


Figure 3.5: Efficient frontier (EF) without short sales for 50 asset prices

Gathering the information together in one graph will generate the following figures (3.6, 3.7 and 3.8). We note here that the efficient frontier without short sales does not coincide with the one with short sales. Yet we may have attached a part of the efficient boundary to the original, so we need to identify the remaining parts of the new efficient frontier. Also, the next three figures reveal that, for all sample sizes, the efficient frontier with short sales dominates the one without short sales. This statement appeared in much of the literature. Because short sales restrictions add a new constraint, it is obvious that the efficient frontier will be dominated. Moreover, various sample sizes show that the efficient frontiers with short sales are parabolas.

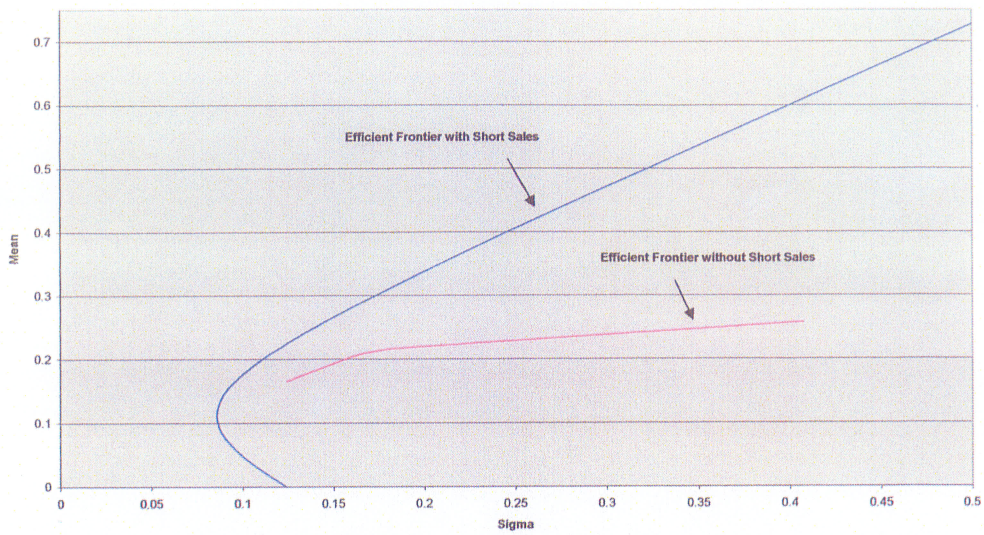


Figure 3.6: Efficient frontier with and without short sales for 15 asset prices

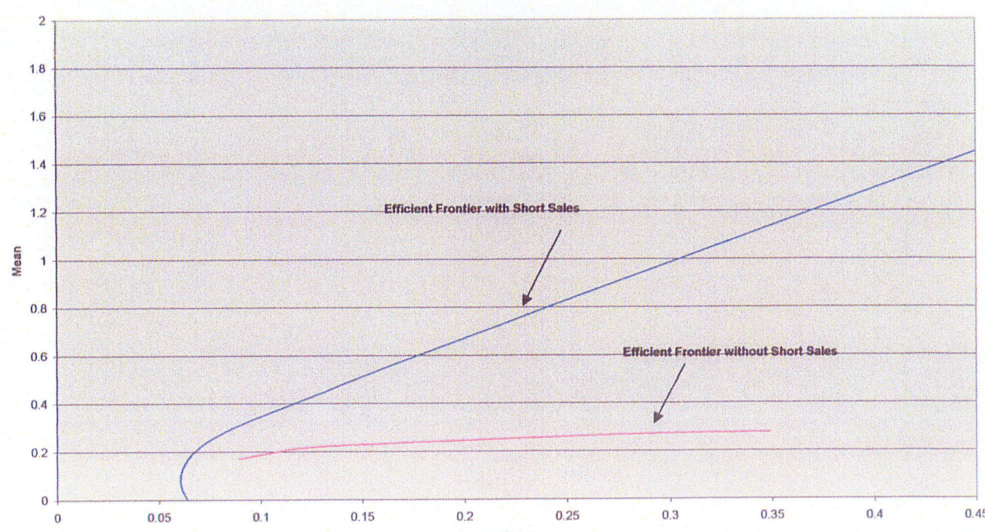


Figure 3.7: Efficient frontier with and without short sales for 30 assets prices

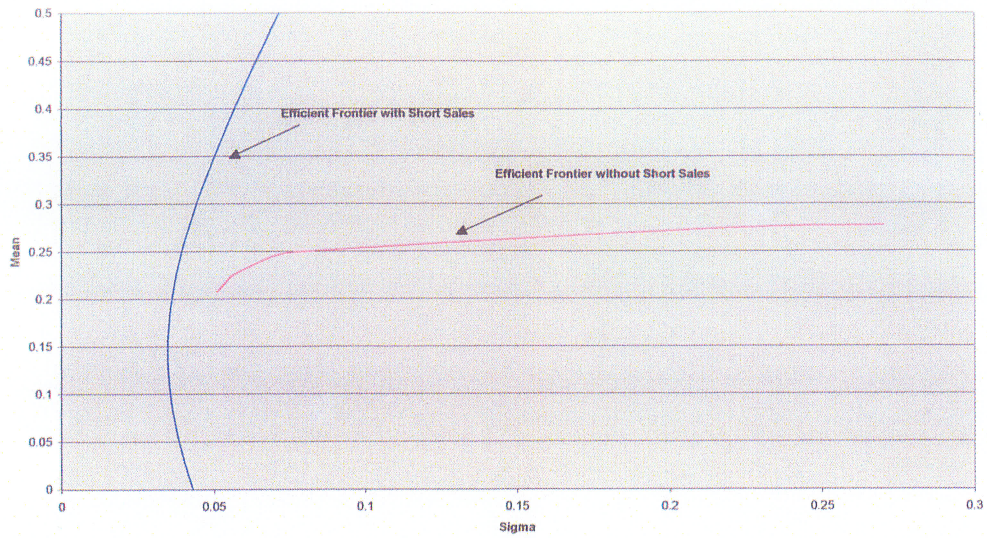


Figure 3.8: Efficient frontier with and without short sales for 50 assets prices

Taking into account various sample sizes, the data suggest a conclusion consistent with Levy's findings [92] that as the sample increases, the efficient frontier with and without shift from the left. It is clear that the distance between is proportional to the data. Levy [92] used a small sample size up to 15 assets; here we expand that finding with a larger sample size. He empirically finds that without short sales, many securities do not enter the efficient portfolios, and the larger N, the smaller the percentage of the securities that appear in the efficient portfolios out of the total number of available securities, N. This finding is supported by the collected data.

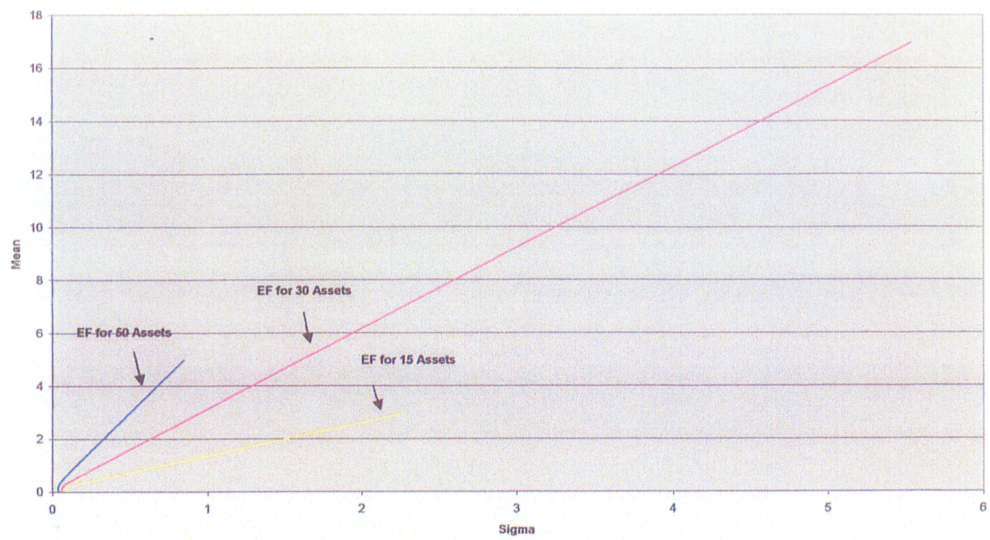


Figure 3.9: Portfolios' efficient frontier(s) (EF)

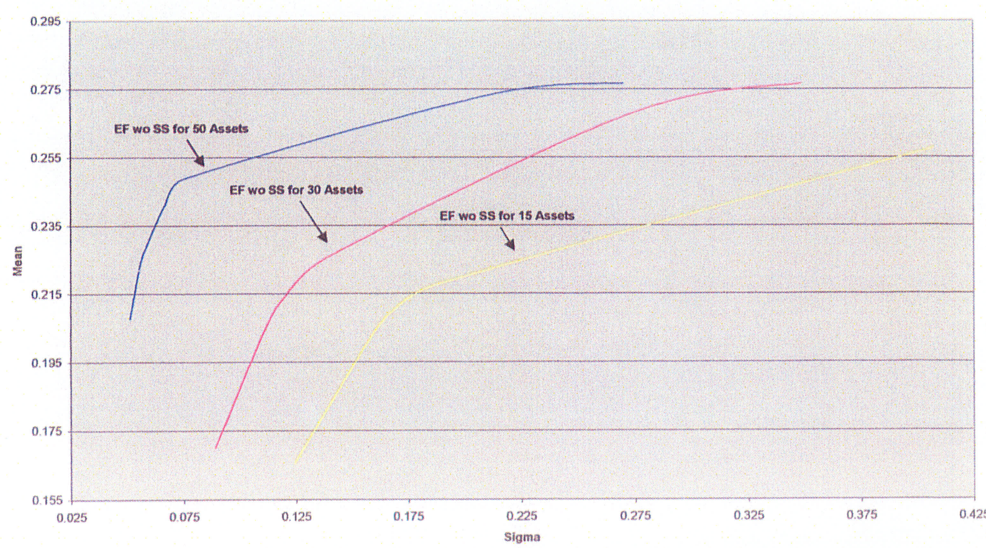


Figure 3.10: Portfolios' efficient frontier(s) without short sales (EF wo SS)

Taking into account fuzzy information, the fuzzy efficient frontiers are represented in XYZ plane as follows for various sample sizes (3.11,3.12 and 3.13):

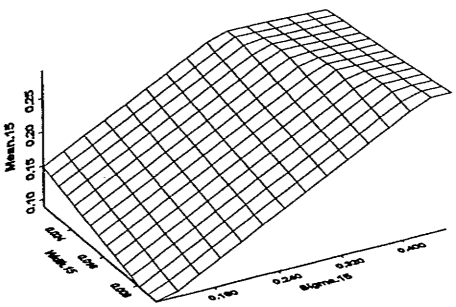


Figure 3.11: Efficient frontier with subjective fuzzy information without short sales (15 assets)

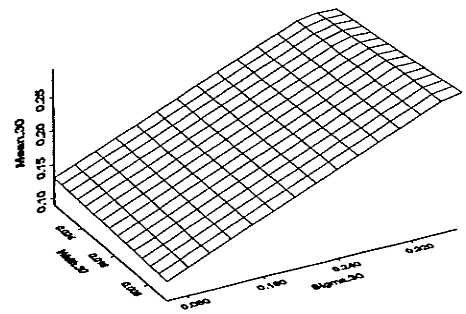


Figure 3.12: Efficient frontier with subjective fuzzy information without short sales (30 assets)

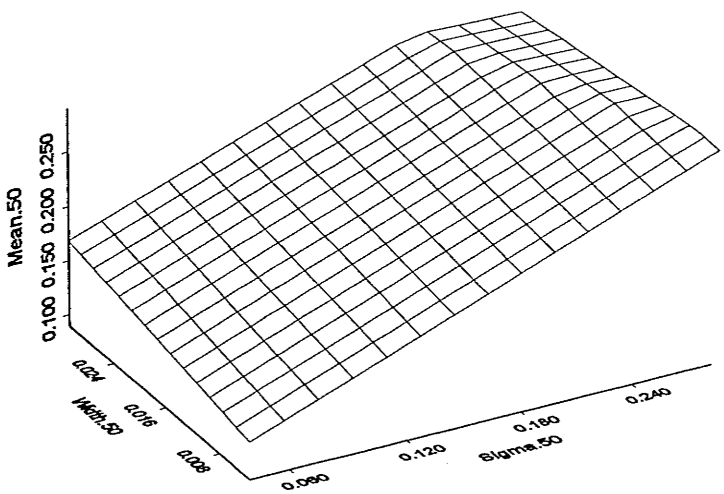


Figure 3.13: Efficient frontier with subjective fuzzy information without short sales (50 assets)

Under a fuzzy information environment, the efficient frontier without short sales has been derived (with the use of a VBA program; part of that program is in appendix A) and plotted for various sample sizes. The portfolio width has been included as a third parameter, and the frontier has been plotted in a three-dimensional graph. In this section, the relationship between risk, return and width, which is used as proxy for the subjective comment of the experts, has been represented by a surface. The efficient frontier portfolios are plotted on a graph with the σ_p in the x-axis, width in the y-axis and the mean in the z-axis. Projecting the graphical representation into a two standard deviation-mean plane figure (3.14) shows an arc, not a parabola, which is consistent with the result reported earlier when the subjective fuzzy measure was discarded from the model. Also, for 15, 30 and 50 asset sample sizes, similar to the case of short sales, we still observe that in the larger sample size, the efficient frontier is shifted to the left; the dominance of the large size sample still holds. In general, the efficient frontier is a combination of assets, if there is no other combination with the same (higher) expected return with lower risk, and if there is no other portfolio with the same (or lower) risk and with higher expected return. In this context, a higher (lower) risk is associated with a higher (lower) return.

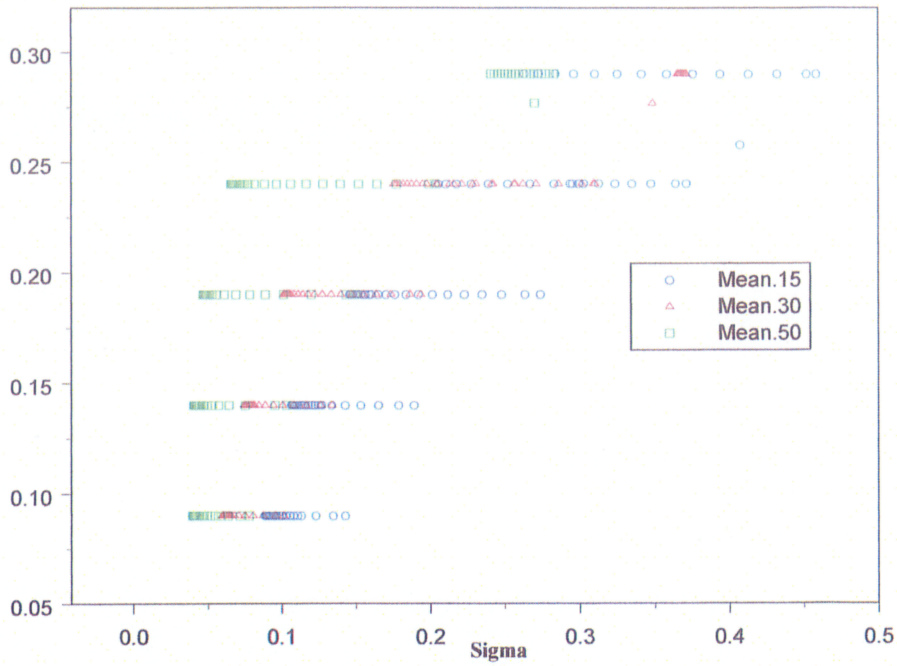


Figure 3.14: Efficient frontiers in a mean-standard deviation plane with subjective fuzzy measure

Also, the following graph (3.15) shows that as the degree of fuzziness increases (flexibility with respect to the portfolio mean improves) there is a slight decrease in the level of risk. Note here that the graph *does not suggest* a strong negative relationship for various sample sizes[2]. Because the widths in our samples are correlated with the returns, we could not see a strong visible (either positive or negative) relationship. Thus, we suggest that as soon as the investor starts getting new subjective information from experts, which is to some extent not primarily correlated with the historical data, we will be able to spot

[2]Also, due to the limited plotted number of observations, we could not see a very strong relationship.

a strong visible relationship between the width size and the risk level. Thus, an investor who is flexible and is acquiring additional subjective information to support the historical data will be flexible to accept a higher risk. Thus, we anticipate a negative relationship. The following graph (3.15) shows a slight negative trend, mainly for a larger size sample. In contrast, an investor with small portfolio width (not flexible with respect to the portfolio mean) tends to accept less risk. For instance, it has been shown in the figures that sometimes there is not a conclusive relationship between portfolio width and risk.

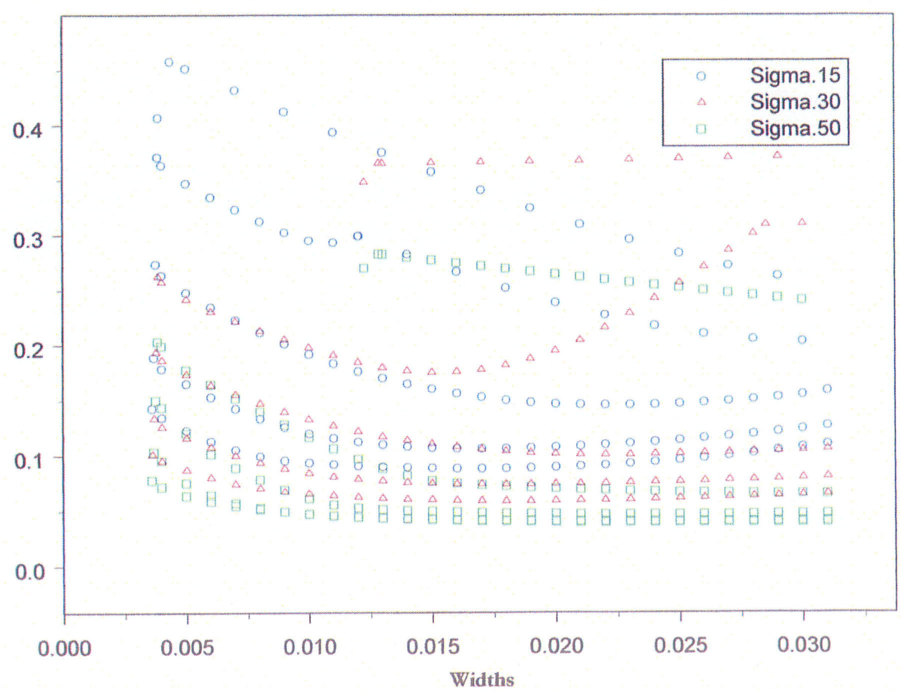


Figure 3.15: Relationship between the widths and sigma for different sample sizes

3.4 Short Sales and the Derivation of the Fuzzy Random CAPM

We derive another fuzzy version of the CAPM that differs from the fuzzy constrained capital asset pricing model attempt derived by Ostermark [114] in which he used fuzzy linear constraints to augment the problem and solved it by parametric methods of linear programming.

One serious deficiency of the method presented by Ostermark is that it allows the introduction of fuzziness only in the linear policy constraints without changing the covariance terms (σ_{ij}). His analysis is amiss, because, if we want to allow some degree of subjective imprecision in the system, we should not ignore that imprecision will influence the covariance terms and the expected return in a similar fashion.

Moreover, limiting the analysis, in the case when the coefficient of policy constraint (returns) and the mean portfolio are fuzzy, is not an appropriate approach to deriving the fuzzy capital asset model with subjective imprecision. This is because, firstly, the returns are used to compute the variance-covariance (Cov-Var) matrix, so if there is imprecision in those coefficients, it must be modeled again in the Cov-Var matrix. Secondly, if we want to capture the managerial imprecision, the theory of CAPM should be extended to deal with the source of fuzziness; for example, when data exclude some observations, when data consist of non-sharply definable observations, or when we want to allow the introduction of a measure of subjectivity in returns, as inspired by Markowitz's statement [105]. Thirdly, because Ostermark assumes that all the proportions have a positive amount (short sales restrictions are imposed) and ignores the riskless asset in the model, the CAPM has been violated. As suggested by Ross ([129], [36] and [37]) the CAPM will not hold if there is a riskless asset, like other risky assets that cannot be shortened, and the CAPM will hold for that subset of assets that can be sold short.

Also, it has been shown in many sources (e.g. [28], [92] and [93]) that capital asset pricing theory has been developed on the basis of mean-variance theory. Thus, we suggest that a better way to handle that derivation task is through an analytical approach.

An approach for deriving the Fuzzy Capital Asset Pricing Model (FCAPM) is presented, using, to some extent, the fuzzy efficient frontier model obtained previously. Let us consider an arbitrarily chosen risky asset J. Let m denote the portfolio lying on the tangency point between the fuzzy random-adjusted CML and the minimum-variance frontier for risky assets. m is a mean-variance efficient portfolio, with $E\left[\tilde{R}^{*}_{m}\right] = \sum_{j=1}^{N}\left\langle E\left[\tilde{R}_j\right], E[\widetilde{l_j}]\right\rangle_{LR} X_j = \left\langle \sum_j X_j E[\widetilde{R}_j], \sum_j X_j E[\widetilde{l_j}]\right\rangle_{LR}$. Consider a portfolio p, consisting of a proportion w invested in security j, an inefficient portfolio, and a proportion $(1-w)$ invested in portfolio m. The optimal portfolio is found through solving the following model:

$$Min\ \frac{1}{2}Var\left(w\left\langle E\left[\tilde{R}_j\right], E[\widetilde{l_j}]\right\rangle + (1-w)\left\langle E\left[\tilde{R}_m\right], E[\widetilde{l_m}]\right\rangle\right) \tag{3.56}$$

Subject to

$$wE\left[\tilde{R}_j\right] + (1-w)\,E\left[\tilde{R}_m\right] = \mu_p$$

$$wE\left[\widetilde{l_j}\right] + (1-w)\,E\left[\widetilde{l_m}\right] = l_p$$

Knowing that such a portfolio will have an expected return equal to

$$\mu_p = E[\widetilde{R}_p] = wE[\widetilde{R}_j] + (1-w)E[\widetilde{R}_m],$$

with spread (or width) equal to

$$l_p = E[\widetilde{R}_p] = wE[\widetilde{l}_j] + (1-w)E[\widetilde{l}_m],$$

we conclude that under fuzzy random setting, the expected value is:

$$\begin{aligned} \mu_p^* &= E[\widetilde{R}_p^*] = wE[\widetilde{R}_j^*] + (1-w)E[\widetilde{R}_m^*] \\ &= w\left\langle E\left[\tilde{R}_j\right], E[\widetilde{l}_j]\right\rangle + (1-w)\left\langle E\left[\tilde{R}_m\right], E[\widetilde{l}_m]\right\rangle, \end{aligned} \tag{3.57}$$

and the standard deviation is equal to

$$\sigma(\widetilde{R}_p^*) = \left[\begin{array}{c} \left(w^2\sigma_j^2 + (1-w)^2\sigma_m^2 + 2w(1-w)\sigma_{jm}\right) \\ +\frac{1}{3}\left(w^2L_j^2 + (1-w)^2 L_m^2 + 2w(1-w)L_{jm}\right) \end{array}\right]^{1/2} \tag{3.58}$$

All such portfolios will lie on a curved line connecting J and m. Of concern is the slope of its curved line with fuzzy random uncertainty. Using equation (3.57), the derivative of E$[\widetilde{R}_p^*]$ with respect to w is taken:

$$\frac{dE[\widetilde{R}_p^*]}{dw} = E[\widetilde{R}_j^*] - E[\widetilde{R}_m^*] \tag{3.59}$$

Second, using equation (3.58), the derivative of σ_P^* with respect to w is taken:

$$\frac{d\sigma_p^*}{dw} = \frac{w\sigma_j^2 - \sigma_m^2 + w\sigma_m^2 + \sigma_{jm} - 2w\sigma_{jm} + \frac{1}{3}\left[wL_j^2 - L_m^2 + wL_m^2 + L_{jm} - 2wL_{jm}\right]}{\left[\left(w^2\sigma_j^2 + (1-w)^2\sigma_m^2 + 2w(1-w)\sigma_{jm}\right) + \frac{1}{3}\left(w^2L_j^2 + (1-w)^2L_m^2 + 2w(1-w)L_{jm}\right)\right]^{1/2}} \tag{3.60}$$

Third, the slope of the curved line Jm can be written:

$$\frac{dE[\widetilde{R}_p^*]}{d\sigma_p^*} = \frac{\frac{dE[\widetilde{R}_p^*]}{dw}}{\frac{d\sigma_p^*}{dw}} \tag{3.61}$$

$$= \frac{\left[E[\widetilde{R}_j^*] - E[\widetilde{R}_m^*]\right] \begin{bmatrix} \left(w^2\sigma_j^2 + (1-w)^2\sigma_m^2 + 2w(1-w)\sigma_{jm}\right) \\ +\frac{1}{3}\left(w^2L_j^2 + (1-w)^2 L_m^2 + 2w(1-w)L_{jm}\right) \end{bmatrix}^{1/2}}{\begin{array}{c}\left[w\sigma_j^2 - \sigma_m^2 + w\sigma_m^2 + \sigma_{jm} - 2w\sigma_{jm}\right] \\ +\frac{1}{3}\left[wL_j^2 - L_m^2 + wL_m^2 + L_{jm} - 2wL_{jm}\right]\end{array}}$$

Since the proportion of w is zero at the endpoint m, the slope of Jm can be calculated by substituting zero for w in equation (3.61). After doing that, many terms drop out, leaving the following:

$$\frac{dE[\widetilde{R}_p^*]}{d\sigma_p^*} = \frac{\left[E[\widetilde{R}_j^*] - E[\widetilde{R}_m^*]\right]\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]^{1/2}}{\left[\sigma_{jm} - \sigma_m^2\right] + \frac{1}{3}\left[L_{jm} - L_m^2\right]} \tag{3.62}$$

At m, the slope of the FCML must coincide with the slope of the curve Jm. Thus, the slope of the curve of Jm at m, as shown on the right-hand side of equation (3.62), is set to be equal to the slope of the FCML assuming that $l_m = 0$:

$$\frac{\left[E[\widetilde{R}_j^*] - E[\widetilde{R}_m^*]\right]\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]^{1/2}}{\left[\sigma_{jm} - \sigma_m^2\right] + \frac{1}{3}\left[L_{jm} - L_m^2\right]} = \frac{E[\widetilde{R}_m^*] - R_f}{\left(\sigma_m^2 + \frac{1}{3}L_m^2\right)^{1/2}} \tag{3.63}$$

$$E[\widetilde{R}_j^*] = \frac{\left[E[\widetilde{R}_m^*] - R_f\right]\left[\left[\sigma_{jm} - \sigma_m^2\right] + \frac{1}{3}\left[L_{jm} - L_m^2\right]\right]}{\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]} + E[\widetilde{R}_m^*] \tag{3.64}$$

$$E[\widetilde{R}_j^*] = \frac{\left[E[\widetilde{R}_m^*] - R_f\right]\left[\left[\sigma_{jm} - \sigma_m^2\right] + \frac{1}{3}\left[L_{jm} - L_m^2\right]\right] + \left[\sigma_m^2 + \frac{1}{3}L_m^2\right]E[\widetilde{R}_m^*]}{\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]} \tag{3.65}$$

$$E[\widetilde{R}_j^*] = \frac{R_f\left[\sigma_m^2 + \frac{1}{3}L_m^2\right] + E[\widetilde{R}_m^*]\left[\sigma_{jm} + \frac{1}{3}L_{jm}\right] - R_f\left[\sigma_{jm} + \frac{1}{3}L_{jm}\right]}{\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]} \tag{3.66}$$

$$E[\widetilde{R}_j^*] = R_f + \frac{\left[E[\widetilde{R}_m^*] - R_f\right]\left[\sigma_{jm} + \frac{1}{3}L_{jm}\right]}{\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]}$$

Thus, we obtain the fuzzy random CAPM (FCAPM) equation as follows:

$$E[\widetilde{R}_j^*] - R_f = \frac{\left[\sigma_{jm} + \frac{1}{3}L_{jm}\right]}{\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]}\left[E[\widetilde{R}_m^*] - R_f\right], or \tag{3.67}$$

$$E[\widetilde{R}_j^*] = R_f + \beta_{jm}\left[E[\widetilde{R}_m^*] - R_f\right]$$

such that

$$\beta_{jm}^* = \frac{\left[\sigma_{jm} + \frac{1}{3}L_{jm}\right]}{\left[\sigma_m^2 + \frac{1}{3}L_m^2\right]} \tag{3.68}$$

Note here that once the fuzzy measure has been ignored in the equation (3.68) by having $L_{jM} = L_M^2 = 0$, we obtain the beta of the original CAPM.

3.5 Summary

This study addressed the implications of relaxing one of the fundamental assumptions associated with mean-variance theory as set down in Markowitz ([103], [105]) and Tobin [155] that asset returns are sharply defined. Theoretical arguments in fuzzy mathematics assume that there are cases in which random uncertainty alone may not serve the purpose and indicate that fuzziness may impact the first two moments of asset return. This suggests that the lack of information associated with market-traded securities challenges the usefulness of standard mean-variance theory for other research and practical portfolio management.

To make the link between existing theory and the subjectivity measure of expert's judgments, we rederived the Markowitz efficient set and dealt with the implications of the rederivation on the Capital Market Line (CML) and the Capital Asset Pricing Model (CAPM). The contribution of this chapter is the presentation of a methodology for the derivation of the attainable efficient frontier in the presence of fuzzy information in the data or when the fuzzy information is imposed in the modeling environment to reflect a subjective measure.

Chapter 4

Fuzzy Probabilities with Applications

4.1 Introduction

Shannon [140], with the intention of measuring the information lost in the process of transmission, pointed out that a measure of information should essentially be a measure of uncertainty. Here, uncertainty usually is associated with a probability distribution P. Also, he showed that $-K\sum p_i Lnp_i$ satisfies the properties for a measure of uncertainty, and he called it a measure of entropy, because a measure of probabilistic uncertainty is equivalent to a measure of entropy. The expected information gained upon complete resolution of uncertainty is a measure of current uncertainty, and K is a positive factor that determines the unit of measurement.

Entropy originated in the field of thermodynamics and statistical mechanics to represent a measure of disorder. Lindley [94] and Good ([43], [44], [45], [46], [47]) presented the relationship between the measure of information and probability and statistics. Both uncertainty and entropy are closely related in describing imperfect knowledge.

The idea behind the principle of entropy, as suggested by Cozzolino and Zahner [27], is that:

> "....the probability distribution desired has maximum uncertainty (minimum information content) subject to representing some explicitly stated *known* information ..."

It is well documented that the scope of entropy applications is not limited; many studies have looked at the use of entropy theory in various subjects. For instance, a comprehensive development and selected empirical bodies of work in business were given by Theil [152], Herinter [53], Abdel-Khalik [1], Philippatos and Wilson ([119],[118]), Saxena [137], Thomas [153], and Nawrocki [111].

Work by Cozzolino and Zahner [27], is considered pioneer work in entropy in financial modeling. They used the principle of maximum entropy to derive a probability distribution of future stock price for an investor having specified expectations. The principle of maximum entropy is used in their study and in others, because it offers a method to generate probability distribution from limited information.

In another effort, Thomas [153] presented a generalized maximum entropy principle to deal with problems involving uncertainty but with initial information about the probability space. Normally, this knowledge is expressed through known moments of a random variable. He suggested adding known bounds on moments to the modeling framework. In fact, Thomas [153] did model system selection in a fuzzy situation, when probabilities might lie between two values. With bounds in event probabilities and moments, his solution to the nonlinear programming problem is achieved by a numerical method (algorithm). In the present study, we relax and expand this restriction somewhat by allowing for information via a random variable with fuzziness in the system.

In another study, Saxena [137] used entropy to select the best alternative investment projects. He presented an algorithm to obtain the probability distribution of variables based on probability ranges, which should be specified at the early stage of a study.

Nawrocki [111] used entropy to measure investment performance (security analysis). He suggested a heuristic algorithm using portfolio analysis with state-value weighting entropy as a measure of investment risk. Philippatos and Gressis [118] provide conditions in which mean-variance, mean entropy and second degree stochastic dominance are equivalent.

In a fuzzy setting, De Luca [99] was the first to define a non-probabilistic entropy with the use of fuzzy theory. He proposed an entropy measure of a quantity of information that is not random in nature. However, as he pointed out in his study, the mathematical efforts were not complete and open for much work.

Even after Philippatos and Wilson [119] argued that entropy is a better statistical measure of risk than variance, because entropy is a non-parametric measure, entropy still did not appear often in published works. As Philippatos and Wilson [119] suggested entropy as a measure of portfolio risk, because it does not make assumptions concerning the probability underlying the returns, we use the same analogy to establish the measure of risk using the proposed fuzzy entropy method.

It is of interest to point out here that neither the Cozzolino and Zahner [27] approach nor the Philippatos and Wilson [119] method include a the situation when there is imprecise information to start from. Consequently, we use the fuzzy theory in conjunction with the entropy theory. This study describes the method of providing a specific distribution by using the fuzzy entropy principle.

Kapur [68] has discussed the views of Jaynes [65], Cheesman [24], and Lindley [96], who are in favor of probabilistic entropy and of Kosko [79] and Klir [73], who are in favor of fuzzy entropy. Kapur [139] emphasized the need for a cooperative effort between probability theory and fuzzy theory to explore the concept of uncertainty for the prosperity of mankind.

To measure uncertainty about facts, events or consequences of actions, we need some kind of probability. However, to measure the indeterminancy that arises from limited knowledge about these matters, we need to use imprecise or fuzzy probabilities. Here, fuzzy probabilities are used as a generic term to cover mathematical models such as upper and lower probabilities. This chapter is concerned with fuzzy probabilistic reasoning, which involves various methods for assessing fuzzy probabilities, taking into account new "fuzzy" information and the derivation of certain results of other probabilities and conclusions.

The focus in the present chapter is on the presentation of a new approach using probability theory, and various other (non-probabilistic) scenarios with their utility in risk modeling. Anyone familiar with the stock market will find that the most challenging decision is to differentiate between the good one stock to buy and the bad stock to sell.

Borch [14] in his book said:

> "However, if we buy the stock in question, there is necessarily a seller who thinks that at the present time and the present price it is right to sell the stock which we consider best to buy. If the seller is just as intelligent and smart as we are, it may be useful to think twice."

In the present study the goal is to draw conclusions about modeling financial risk using fuzzy probabilities. To a lesser extent, we consider decision problems (investment problems), where the goal is to choose an optimal strategy and determine the optimal alternatives. The primary aim of the chapter is to establish the mathematical theory of fuzzy probabilities, based on the measure of entropy.

Fuzzy entropy measures the degree of fuzziness of a set A. The usual entropy measure tells us how equal the probabilities $p_1, p_2, \ldots p_n$ are among themselves or how close the given probability distribution is to the uniform distribution. The measure of fuzzy entropy tells us about the degree of fuzziness of the set A or about how close the given set is to the most fuzzy set.

4.2 A Fuzzy Situation Handled Through Probabilistic Entropy: Discrete Case

Statistical reasoning with imprecise probabilities has been discussed in the literature; for instance, Walley [163] has investigated methods of reasoning and imprecise probabilities. Fuzzy probabilities are not crisp but are imprecise and ambiguous. Specifically, we consider probabilities $p_1, p_2, \ldots, p_n$, satisfying the constraints,

$$a_i \leq p_i \leq b_i, a_i \geq 0, \quad b_i \leq 1; \; i = 1, \ldots, n \tag{4.1}$$

and

$$\sum_{i=1}^{n} p_i = 1 \tag{4.2}$$

There is a fuzziness about these probabilities and we would like to understand it and possibly measure it. As a first step, we attempt to get the most unbiased estimates for the probabilities satisfying constraints (4.1) and (4.2). For this purpose we maximize the probabilistic entropy measure ([67], [69]).

$$-\sum_{i=1}^{n}(b_i - p_i)Ln(b_i - p_i) - \sum_{i=1}^{n}(p_i - a_i)Ln(p_i - a_i) \tag{4.3}$$

This measure is so designed that on its maximization subject to (4.2) , it gives probabilities that automatically satisfy constraint (4.1). Using Lagrange's method to maximize (4.3) subject to (4.2), we get

$$\frac{b_i - p_i}{p_i - a_i} = K \tag{4.4}$$

where K, a Lagrange multiplier associated with (4.2), and is determined by using both (4.2) and (4.4) we get:

$$p_i = \frac{b_i + Ka_i}{1 + K}, \tag{4.5}$$

and using (4.2), we get

$$K = \frac{B-1}{1-A}, \tag{4.6}$$

where A=$\sum_{i=1}^{n} a_i$, and $B = \sum_{i=1}^{n} b_i$.

Using (4.1) and (4.2), we get

$$A \leq 1 \leq B \tag{4.7}$$

and (4.6) and (4.7) yield $K \geq 0$.

Using (4.5) and (4.6) we get

$$p_i = \frac{(b_i - a_i) + Ba_i - Ab_i}{B - A} \tag{4.8}$$

These p_i's yield the most unbiased estimates for the probabilities satisfying the constraints (4.1) and (4.2).

4.2.1 Illustrative Example

Let us assume that we have a set of probabilities p_i that satisfiy the constraints (4.1) and (4.2) such that a_i and b_i are as in the table below. We attempt to get the most unbiased estimates for the probabilities.

i	1	2	3	4
a_i	0.1	0.15	0.20	0.25
b_i	0.2	0.26	0.33	0.35

To evaluate the probabilities p_i such that $(i = 1, \ldots, 4)$ using the method illustrated above we have $A = \sum_i a_i = 0.7, B = \sum_i b_i = 1.14 > 1$. Thus, the most unbiased estimates for the probabilities are

i	1	2	3	4
p_i	0.168	0.225	0.288	0.318

4.2.2 An alternative measure of fuzziness

In this section, we consider the set of probabilities $(p_1, p_2, \ldots, p_n)$ as a fuzzy set with the probabilities $p_1, p_2, ..., p_n$ as values of membership function. A measure of fuzziness of this set [68] is

$$-\sum_{i=1}^{n} p_i Ln(p_i) - \sum_{i=1}^{n} (1 - p_i) Ln\,(1 - p_i) \tag{4.9}$$

An alternative measure of fuzziness as given by (4.3) and (4.4) is

$$\left[-\sum_{i=1}^{n}\left(\frac{(b_i-a_i)+Ba_i-Ab_i}{B-A}\right)Ln\left(\frac{(b_i-a_i)+Ba_i-Ab_i}{B-A}\right)\right]-\left[\sum_{i=1}^{n}\left(1-\frac{(b_i-a_i)+Ba_i-Ab_i}{B-A}\right)Ln\left(1-\frac{(b_i-a_i)+Ba_i-Ab_i}{B-A}\right)\right] \tag{4.10}$$

An alternative measure of fuzziness based on (4.3) and (4.4) is as follows:

$$-\left[\sum_{i=1}^{n}\left(b_i-\frac{b_i+Ka_i}{1+K}\right)Ln\left(b_i-\frac{b_i+Ka_i}{1+K}\right)-\sum_{i=1}^{n}\left(\frac{b_i+Ka_i}{1+K}-a_i\right)Ln\left(\frac{b_i+Ka_i}{1+K}-a_i\right)\right]$$

$$=\left[-\sum_{i=1}^{n}\left(\frac{K(b_i-a_i)}{1+K}\right)Ln\left(\frac{K(b_i-a_i)}{1+K}\right)-\sum_{i=1}^{n}\left(\frac{(b_i-a_i)}{1+K}\right)Ln\left(\frac{(b_i-a_i)}{1+K}\right)\right]$$

$$\begin{aligned} = & -\frac{K}{1+K}\left[\sum_{i=1}^{n}(b_i-a_i)\,Ln\left(\frac{K}{1+K}\right)+\sum_{i=1}^{n}(b_i-a_i)\,Ln\,(b_i-a_i)\right] \\ & -\frac{1}{1+K}\left[\sum_{i=1}^{n}(b_i-a_i)\,Ln\,(b_i-a_i)+\sum_{i=1}^{n}(b_i-a_i)\,Ln(1+K)\right] \end{aligned}$$

$$=-\sum_{i=1}^{n}(b_i-a_i)\,Ln\,(b_i-a_i)-(B-A)\left[\frac{K}{1+K}Ln\frac{K}{1+K}+\frac{1}{1+K}Ln\frac{1}{1+K}\right]$$

$$\begin{aligned} = & (B-A)\left[-\sum_{i=1}^{n}\left(\frac{(b_i-a_i)}{B-A}\right)Ln\left(\frac{(b_i-a_i)}{B-A}\right)-Ln(B-A)\right] \\ & -(B-A)\left[\left(\frac{K}{1+K}\right)Ln(\frac{K}{1+K})+(\frac{1}{1+K})Ln(\frac{1}{1+K})\right] \end{aligned}$$

$$
\begin{aligned}
&= (B-A)\left[-\sum_{i=1}^{n}\frac{(b_i-a_i)}{B-A}Ln\left(\frac{(b_i-a_i)}{B-A}\right)-Ln(B-A)\right] \\
&+(B-A)\left[-\frac{K}{1+K}Ln\frac{K}{1+K}-\frac{1}{1+K}Ln\frac{1}{1+K}\right]
\end{aligned}
\tag{4.11}
$$

The above formula (4.11) consists of three terms:

1. The first term is

$$
(B-A)*\left(\text{entropy of the probability distribution } \frac{(b_1-a_1)}{B-A}, \frac{(b_2-a_2)}{B-A}, \ldots, \frac{(b_n-a_n)}{B-A}\right)
\tag{4.12}
$$

2. The second term is

$$
-(B-A)Ln(B-A) \tag{4.13}
$$

3. The third term is

$$
(B-A)*\left[\text{entropy of the probability distribution } (\frac{K}{1+K}, \frac{1}{1+K})\right] \tag{4.14}
$$

We observe that

(a) Third term in (4.11) is 0 if $K=0$ or $K=\infty$, that is, if $B=1$, or $A=1$, and has maximum value $Ln2$ when $K=1$, that is when $A+B=2$.

(b) If $B-A$ is kept fixed, then the first term in (4.11) is maximum when all (b_i-a_i)'s are equal, the second term is constant and the third term is maximum when $B+A=2$.

Based on the above observations, there are few comments worth mentioning here.

1. Probability theory and fuzzy set theory are considered as giving two different and mutually exclusive approaches for two different types of uncertainties. However, here we have considered both types of uncertainties simultaneously and have tried to discuss fuzziness in terms of the probabilities themselves.

2. We first try to eliminate the fuzziness by using the principle of maximum entropy to get crisp values of probabilities.

3. However, the set of crisp probabilities is itself regarded as a fuzzy set with probabilities giving the values of the membership function, and we find measures of probabilistic uncertainty, which can be measured as the fuzziness of the information that was given to us originally.

4. We started with a fuzzy situation and ended with a probability distribution where the probability distribution, depends upon the fuzziness of the original situation.

5. This measure depends upon all $(b_i - a_i)$'s, $(B - A)$, and the ratio $\dfrac{B-1}{1-A}$.

4.2.3 Generalized Case

Instead of considering probabilities, we now consider any n non-negative numbers, $x_1, x_2, \ldots, x_n$, satisfying the constraints

(i)

$$a_i \leq x_i \leq b_i, \quad a_i \geq 0, \quad b_i \leq 1; \quad i = 1, \ldots, n \tag{4.15}$$

and

$$x_1 + x_2 + \ldots + x_n = C \tag{4.16}$$

so that

$$A \leq C \leq B$$

Proceeding in the same fashion as before, we get

$$x_i = \frac{b_i + Ka_i}{1 + K}$$

$$C = \frac{B + KA}{1 + K}$$

$$K = \frac{B - C}{C - A} \tag{4.17}$$

So that

$$x_i = \frac{C(b_i - a_i) + Ba_i - Ab_i}{B - A} \tag{4.18}$$

which suggests that the fuzziness is still measured by (4.11), but the value of K here is given by (4.17).

4.3 Fuzzy Directed Divergence Measures

In this section we consider two sets of fuzzy probabilities $p_1, p_2, ..., p_n$ and $q_1, q_2, ..., q_n$ such that

(i) First condition

$$a_i \leq p_i \leq b_i, \; e_i \leq q_i \leq f_i, \qquad a_i, e_i \geq 0, \; b_i, f_i \leq 1; \; i = 1, 2, \; . \; . \; ., \; n \tag{4.19}$$

(ii) Second condition

$$\sum_{i=1}^{n} p_i = 1, \quad \text{and} \quad \sum_{i=1}^{n} q_i = 1, \tag{4.20}$$

and, as before, we get the sets of estimates as

$$p_i = \frac{b_i - a_i + Ba_i - Ab_i}{B - A}$$

$$q_i = \frac{f_i - e_i + Fe_i - Ef_i}{F - E}, \tag{4.21}$$

where $F = \sum_{i=1}^{n} f_i$ and $E = \sum_{i=1}^{n} e_i$. We can now use any one of the following measures of directed divergence.

$$\sum_{i=1}^{n} \left[p_i Ln \frac{p_i}{q_i} + (1 - p_i) Ln \frac{1 - p_i}{1 - q_i} \right], \tag{4.22}$$

or

$$\sum_{i=1}^{n} \left[(b_i - p_i)\, Ln \left(\frac{b_i - p_i}{f_i - q_i} \right) + \sum_{i=1}^{n} (p_i - a_i) Ln \left(\frac{p_i - a_i}{q_i - e_i} \right) \right], \tag{4.23}$$

or

$$\sum_{i=1}^{n} \left[(b_i - p_i)\, Ln \left(\frac{b_i - p_i}{f_i - q_i} \right) + \sum_{i=1}^{n} (p_i - a_i) Ln \left(\frac{p_i - a_i}{q_i - e_i} \right) \right]$$
$$+ \sum_{i=1}^{n} \left[(f_i - q_i)\, Ln \left(\frac{f_i - q_i}{b_i - p_i} \right) + \sum_{i=1}^{n} (q_i - e_i) Ln \left(\frac{q_i - e_i}{p_i - a_i} \right) \right]. \tag{4.24}$$

Numerical Example

Let us assume that we have a set of probabilities p_i and q_i that satisfiy the constraints (4.19) and (4.20) such that a_i , b_i , e_i and f_i are specified in the table below. We attempt to get the most unbiased estimates for the probabilities under this scenario.

i	1	2	3	4
a_i	0.1	0.15	0.20	0.25
b_i	0.2	0.26	0.33	0.35
e_i	0.22	0.17	0.4	0.18
f_i	0.26	0.19	0.55	0.2

To evaluate the probabilities p_i and q_i such that $(i = 1, ..., 4)$ using (4.21), $A = \sum_i a_i = 0.7$, $B = \sum_i b_i = 1.14 > 1$, $E = \sum_i e_i = 0.97$, $F = \sum_i f_i = 1.2$. Thus, the most unbiased estimates for the probabilities are

i	1	2	3	4
p_i	0.168	0.225	0.288	0.318
q_i	0.225	0.172	0.419	0.182

4.4 An Application: Measure of Risk in Portfolio Analysis

In usual portfolio analysis, variance of returns is taken as a measure of risk. This requires the investors to know all the variances and covariances for all the n securities. This information may not always be available, or it may exist but not be complete.

An alternative method is to find the minimum and maximum returns for each security. Let the values of these for the i^{th} security be r_i and R_i. Let w_i be the proportionate investment in the i^{th} security, and let

$$a_i = \frac{w_i r_i}{\sum_{i=1}^{n} w_i R_i}, \qquad b_i = \frac{w_i R_i}{\sum_{i=1}^{n} w_i r_i} \tag{4.25}$$

where $a_i \le p_i \le b_i$, i =1, 2, . . ., n, and

$$\sum_{i=1}^{n} p_i = 1 \tag{4.26}$$

Now, proceeding as before, we obtain the following most unbiased estimates $\widehat{p}_i$ for p_i, $i = 1, 2, \ldots, n$

$$\widehat{p}_i = \frac{\dfrac{w_iR_i}{\sum w_ir_i} - \dfrac{w_ir_i}{\sum w_iR_i} + \dfrac{\sum w_iR_i}{\sum w_ir_i}\dfrac{w_ir_i}{\sum w_iR_i} - \dfrac{\sum w_ir_i}{\sum w_iR_i}\dfrac{w_iR_i}{\sum w_ir_i}}{\dfrac{\sum w_iR_i}{\sum w_ir_i} - \dfrac{\sum w_ir_i}{\sum w_iR_i}},$$

or

$$\widehat{p}_i = \frac{\dfrac{w_iR_i}{\sum w_ir_i} - \dfrac{w_ir_i}{\sum w_iR_i} + \dfrac{w_ir_i}{\sum w_ir_i} - \dfrac{w_iR_i}{\sum w_iR_i}}{\dfrac{\sum w_iR_i}{\sum w_ir_i} - \dfrac{\sum w_ir_i}{\sum w_iR_i}},$$

or

$$\widehat{p}_i = \frac{w_i(R_i + r_i)}{\sum w_i(R_i + r_i)} \tag{4.27}$$

Using the notation k_i for $R_i + r_i$ in (4.27) the measure of risk is taken as

$$\ln\ n + \sum \frac{w_ik_i}{\sum w_ik_i} \ln \frac{w_ik_i}{\sum w_ik_i},$$

which is

$$= \ln\ n + \sum \frac{w_ik_i \ln (w_ik_i)}{\sum w_ik_i} - \ln\left(\sum w_ik_i\right) \tag{4.28}$$

We now choose $w_1, w_2, \ldots, w_n$ to maximize

$$\sum w_i\bar{r}_i - \lambda \left[\ln\ n + \frac{\sum k_iw_i \ln w_i}{\sum k_iw_i} + \frac{\sum w_ik_i \ln k_i}{\sum k_iw_i} - \ln \sum k_iw_i\right], \tag{4.29}$$

subject to $\sum w_i = 1$, where $\overline{r}_i$ is the mean return from the i^{th} security, and λ depends upon the attitude of the investor towards risk. This determines points on the mean-entropic risk frontier. Solving these equations by maximizing (4.29) subject to: $\sum w_i = 1$; $i = 1, 2, \ldots, n$

$$\overline{r}_i - \lambda \left[\begin{array}{c} \dfrac{\sum (k_i w_i)\, k_i\,(1 + \ln w_i) - k_i \sum k_i w_i \ln w_i - k_i \ln k_i \sum k_i w_i - k_i \sum w_i k_i \ln k_i}{(\sum k_i w_i)^2} \\ -\dfrac{k_i}{\sum k_i w_i} \end{array} \right] = D \tag{4.30}$$

where D is the Lagrange multiplier corresponding to $\sum w_i = 1$.

Minimization of risk yields that $w_i(R_i + r_i)$, $i = 1, 2, \ldots, n$ should all be equal, so that:

$$w_i = \frac{\dfrac{1}{r_i + R_i}}{\sum \dfrac{1}{r_i + R_i}}, \quad i = 1,\ 2,\ \ldots,\ n \tag{4.31}$$

Thus, the investments in those securities for which the sum of the minimum and the maximum returns are large will be relatively small. A corresponding empirical analysis is presented in the Section 4.12.

An Alternative Approach

In the above discussion we used

$$\frac{r_i}{\sum R_i} \leq b_i \leq \frac{R_i}{\sum r_i} \tag{4.32}$$

that gives rise to the following two possibilities.

(i)

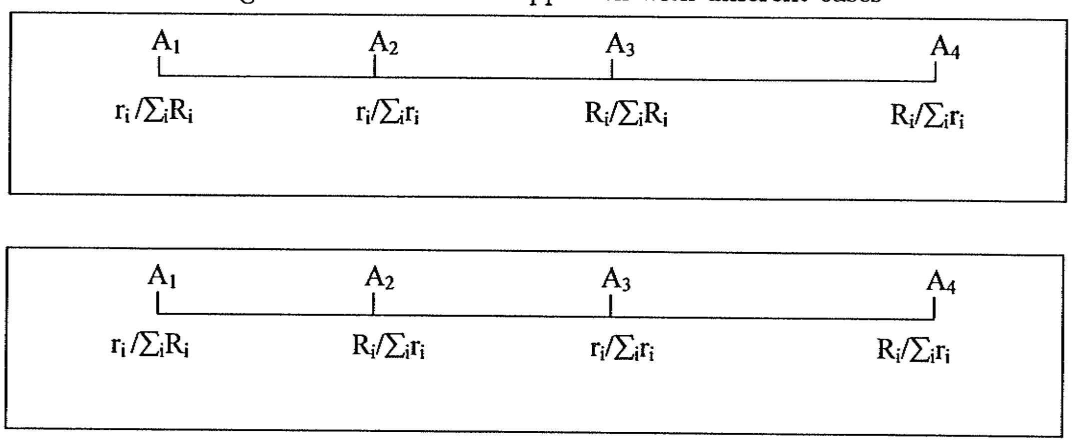
$$\frac{r_i}{\sum r_i} \leq \frac{R_i}{\sum R_i}, \text{ or} \tag{4.33}$$

(ii)

$$\frac{r_i}{\sum r_i} \geq \frac{R_i}{\sum R_i}. \tag{4.34}$$

This in turn yields the following Figure (4.1).

Figure 4.1: Alternative approach with different cases



The procedure of Section 3 gives us probabilities that lie between A_1 and A_2. However, it does not ensure the highly desirable result that these probabilities lie between A_3 and A_4. To overcome this difficulty we define

$$a_i = \min\left(\frac{r_i}{\sum_i r_i}, \frac{R_i}{\sum_i R_i}\right) \qquad \text{for each } i = 1, 2, \ldots, n \tag{4.35}$$

$$b_i = \max\left(\frac{r_i}{\sum_i r_i}, \frac{R_i}{\sum_i R_i}\right) \qquad \text{for each } i = 1, 2, \ldots, n \tag{4.36}$$

and then proceed as before so that each p_i we get will lie between a_i and b_i that is between

$\frac{r_i}{\sum_i r_i}$ and $\frac{R_i}{\sum_i R_i}$ independently of whichever is greater. In other words

$$\min\left(\frac{r_i}{\sum_i r_i}, \frac{R_i}{\sum_i R_i}\right) \leq p_i \leq \max\left(\frac{r_i}{\sum_i r_i}, \frac{R_i}{\sum_i R_i}\right), \qquad i = 1,2, \ldots, n \tag{4.37}$$

Also, it is seen that

$$A = \sum_i a_i \leq 1, \text{ and } B = \sum_i b_i \geq 1 \tag{4.38}$$

Furthermore, p_i divides the line joining a_i and b_i in the ratio $1 - A$ to $B - 1$ as in the following Figure (4.2):

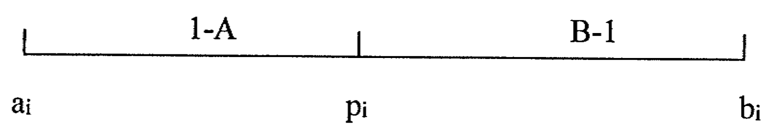


Figure 4.2: Another scenario

Therefore, two cases arise.

Case 1.

$$\frac{r_i}{\sum_i r_i} \leq p_i \leq \frac{R_i}{\sum_i R_i}, \quad \text{or} \quad p_i \sum_i r_i \geq r_i \text{ and } p_i \sum_i R_i \leq R_i \tag{4.39}$$

Case 2.

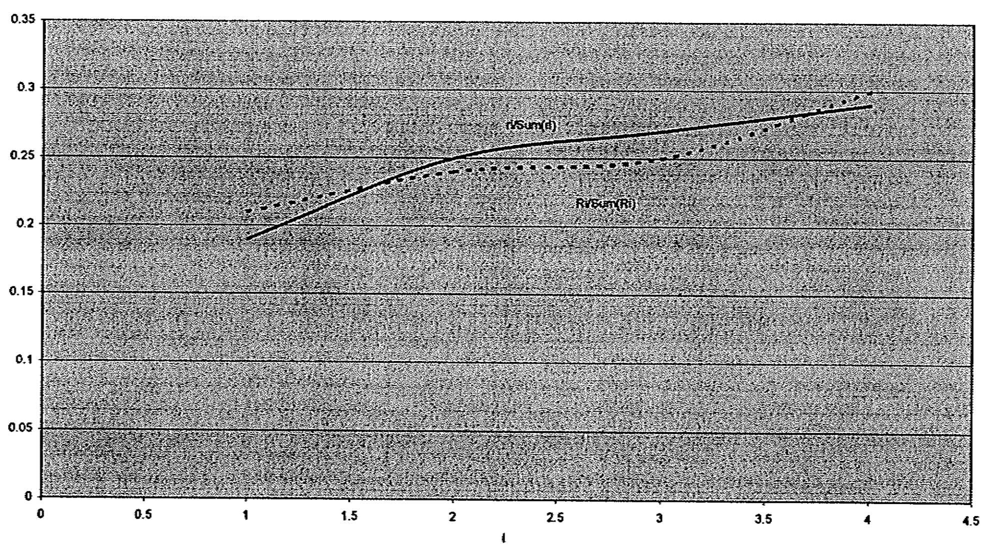
$$\frac{R_i}{\sum_i R_i} \leq p_i \leq \frac{r_i}{\sum_i r_i}, \quad \text{or} \quad p_i \sum_i R_i \geq R_i \text{ and } p_i \sum_i r_i \leq r_i \tag{4.40}$$

It is obvious that neither Case 1 nor Case 2 can arise for all values of i, and if we draw a curve joining points $\left(i, \frac{r_i}{\sum_i r_i}\right)$ and another curve joining points $\left(i, \frac{R_i}{\sum_i R_i}\right)$ for various values of i, the two curves will intersect at some point (or points) (see Figure (4.3))

For example, Let us assume that we have four stocks, and their returns are as follows

i	1	2	3	4
r_i	0.10	0.13	0.14	0.15
R_i	0.125	0.15	0.16	0.19
$\frac{r_i}{\sum r_i}$	0.19	0.25	0.27	0.29
$\frac{R_i}{\sum R_i}$	0.21	0.24	0.25	0.30

Figure 4.3: Curves for a numerical example



4.5 Fuzzy Density Functions: Continuous Case

A recognition of the important role of uncertainty in dealing with problems of organized complexity began another stage that is characterized by the emergence of several new theories (fuzzy set theory, possibility theory, and rough set theory) of uncertainty, different from probability theory, which is capable of capturing only one of several types of uncertainty ([173], [174], [170], [18], [11], [122], [138]).

We now combine fuzzy sets and fuzzy probabilities results in continuous spaces as follows.

Let $f(x)$ and $g(x)$ be two non-negative continuous functions defined over [a,b], and let

$$\int_a^b f(x)dx = F, \text{ and } \int_a^b g(x)dx = G \tag{4.41}$$

So that $\frac{f(x)}{F}$ and $\frac{g(x)}{G}$ represent probability density functions. Furthermore, let

$$a(x) = \min\left(\frac{f(x)}{F}, \frac{g(x)}{G}\right), \quad A = \int_a^b a(x)dx \tag{4.42}$$

and

$$b(x) = \max\left(\frac{f(x)}{F}, \frac{g(x)}{G}\right), \quad B = \int_a^b b(x)dx \tag{4.43}$$

Then, proceeding as before, we get the most unbiased probability density function.

$$p(x) = \frac{[b(x) - a(x)] + Ba(x) - Ab(x)}{B - A} \tag{4.44}$$

4.5.1 Illustrative example

Let

$$f(x) = x, g(x) = x^2, a = 0, b = 1 \tag{4.45}$$

then,

$$F = \int_0^1 xdx = \frac{1}{2}, G = \int_0^1 x^2 dx = \frac{1}{3} \tag{4.46}$$

Set

$$a(x) = min(2x, 3x^2), b(x) = max(2x, 3x^2), \tag{4.47}$$

such that we get the following figure (4.4):

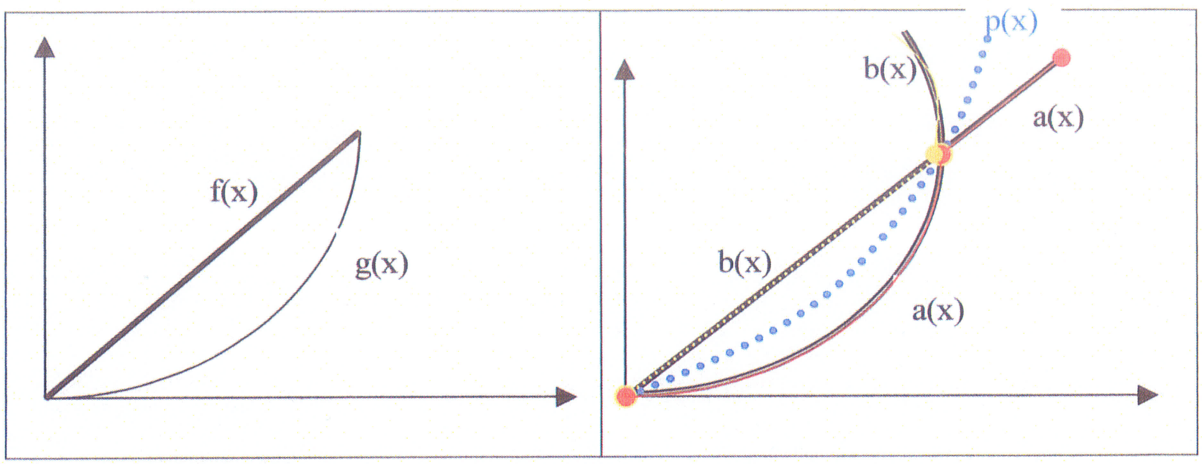


Figure 4.4: Continuous scenario example

where

$$a(x) = \begin{cases} 3x^2 & 0 \leq x \leq \frac{2}{3} \\ 2x & \frac{2}{3} \leq x \leq 1 \end{cases} \tag{4.48}$$

and

$$b(x) = \begin{cases} 2x & 0 \leq x \leq \frac{2}{3} \\ 3x^2 & \frac{2}{3} \leq x \leq 1 \end{cases} \tag{4.49}$$

Obviously both $a(x)$ and $b(x)$ are continuous but are not differentiable. However,

$$a'(x) = \begin{cases} 6x & 0 \leq x \leq \frac{2}{3} \\ 2 & \frac{2}{3} \leq x \leq 1 \end{cases}, \quad b'(x) = \begin{cases} 2 & 0 \leq x \leq \frac{2}{3} \\ 6x & \frac{2}{3} \leq x \leq 1 \end{cases}$$

Thus, at $x = \frac{2}{3}$

$$a'\left(\frac{2}{3}\right) = 4, \quad a'\left(\frac{2}{3}\right) = 2 \tag{4.50}$$

$$b'\left(\frac{2}{3}\right) = 2, \quad b'\left(\frac{2}{3}\right) = 4 \tag{4.51}$$

Now:

$$A = \int_0^{2/3} 3x^2 dx + \int_{2/3}^1 2x dx = \frac{23}{27} \tag{4.52}$$

$$B = \int_0^{2/3} 2x dx + \int_{2/3}^1 3x^2 dx = \frac{31}{27} \tag{4.53}$$

Using (4.48), (4.49), (4.52) and (4.53) into (4.44), we obtain:

$$p(x) = \frac{1}{2}(2x + 3x^2), \tag{4.54}$$

so, that $p(x) = \frac{1}{2}\left(a(x) + b(x)\right)$, when $0 \leq x \leq 2/3$, as well as when $2/3 \leq x \leq 1$

Also, p(x) is continuous and differentiable throughout the interval $[0, 1]$, therefore,

$$\int_0^1 p(x)dx = \int_0^1 \frac{1}{2}\left(2x + 3x^2\right) dx = 1 \tag{4.55}$$

as expected.

4.5.2 General case

In the above example, $p(x) = \frac{1}{2}\left(a(x) + b(x)\right)$. The question may be asked here, whether the same result is true for all non-negative continuous functions $f(x)$ and $g(x)$. A probability density function between $a(x)$ and $b(x)$ is for $0 \leq \lambda \leq 1$

$$P(x) = \lambda\ a(x) + (1-\lambda)\ b(x) \qquad 0 \leq x \leq x_0. \tag{4.56}$$

Alternatively

$$\begin{aligned} P(x) &= \lambda \frac{f(x)}{F} + (1-\lambda)\frac{g(x)}{G} \qquad 0 \leq x \leq x_0 \\ P(x) &= \lambda \frac{g(x)}{G} + (1-\lambda)\frac{f(x)}{F} \qquad x_0 \leq x \leq 1 \end{aligned} \tag{4.57}$$

where x_0 is the point of intersection of

$$y = \frac{f(x)}{F}, \text{ and } y = \frac{g(x)}{G} \tag{4.58}$$

so that the entropy

$$\begin{aligned} &= -\int_0^{x_0} \left(\frac{f(x)}{F} - \lambda\frac{f(x)}{F} - (1-\lambda)\frac{g(x)}{G}\right) Ln\left(\frac{f(x)}{F} - \lambda\frac{f(x)}{F} - (1-\lambda)\frac{g(x)}{G}\right) dx \\ &= -\int_{x_0}^1 \left(\lambda\frac{f(x)}{F} + (1-\lambda)\frac{g(x)}{G} - \frac{g(x)}{G}\right) Ln\left(\lambda\frac{f(x)}{F} + (1-\lambda)\frac{g(x)}{G} - \frac{g(x)}{G}\right) dx \end{aligned} \tag{4.59}$$

$$\begin{aligned} = & -\int_0^1 (1-\lambda)\left|\frac{f(x)}{F} - \frac{g(x)}{G}\right| Ln\left((1-\lambda)\left|\frac{f(x)}{F} - \frac{g(x)}{G}\right|\right) dx \\ & -\int_0^1 \lambda\left|\frac{f(x)}{F} - \frac{g(x)}{G}\right| Ln\left(\left|\frac{f(x)}{F} - \frac{g(x)}{G}\right|\right) dx \end{aligned} \tag{4.60}$$

$$= -(\lambda Ln\lambda + (1-\lambda)Ln(1-\lambda)) - \int_0^1 \left|\frac{f(x)}{F} - \frac{g(x)}{G}\right| Ln\left(\left|\frac{f(x)}{F} - \frac{g(x)}{G}\right|\right) dx$$
$$- \int_0^1 \left|\frac{f(x)}{F} - \frac{g(x)}{G}\right| dx \tag{4.61}$$

This is maximum when $\lambda = \frac{1}{2}$, resulting in the density function

$$p(x) = \frac{1}{2}(a(x) + b(x)) = \frac{1}{2}\left(\frac{f(x)}{F} + \frac{g(x)}{G}\right). \tag{4.62}$$

It may be noted here that we have considered only the set of probability functions of the form

$$\lambda\frac{f(x)}{F} + (1-\lambda)\frac{g(x)}{G}, \tag{4.63}$$

and find that $\lambda = \frac{1}{2}$ gives the maximum entropy density function. In our earlier discussion we considered all other density functions and not density functions of the form (4.63) only.

4.6 Comparison of the Two Solutions

4.6.1 The assumptions and the solutions

In the Section 3 taking $w_i's$ as unity, assuming

$$\frac{r_i}{\sum_i^n R_i} \leq p_i \leq \frac{R_i}{\sum_i^n r_i}, \tag{4.64}$$

and we get the most unbiased probability distribution in (4.27) as

$$\overline{p}_i = \frac{r_i + R_i}{\sum_i (r_i + R_i)}, \qquad i = 1, \ldots, n \tag{4.65}$$

In the second solution (4.37), we assumed a narrower range for p_i

$$\min\left(\frac{r_i}{\sum_i^n r_i}, \frac{R_i}{\sum_i^n R_i}\right) \leq p_i \leq \max\left(\frac{r_i}{\sum_i^n r_i}, \frac{R_i}{\sum_i^n R_i}\right), \qquad i = 1, \ldots, n$$

and we get the most unbiased probability density function as

$$\overline{\overline{p}}_i = \frac{1}{2}\left(\frac{r_i}{\sum_i^n r_i} + \frac{R_i}{\sum_i^n R_i}\right) \tag{4.66}$$

4.6.2 Comparison of the two solutions (discrete case)

We now show that the first solution lies within the range of the second solution.

$$\frac{r_i + R_i}{\sum_i (r_i + R_i)} - \frac{r_i}{\sum_i r_i} = \frac{R_i \sum_i r_i - r_i \sum_i R_i}{(\sum_i r_i)\left[\sum_i (r_i + R_i)\right]}, \tag{4.67}$$

so that

$$\frac{r_i + R_i}{\sum_i (r_i + R_i)} (\leq, \ =, \ \geq) \frac{r_i}{\sum_i r_i},$$

according as

$$\frac{R_i}{\sum_i R_i} (\leq, \ =, \ \geq) \frac{r_i}{\sum_i r_i}$$

Similarly,

$$\frac{r_i + R_i}{\sum_i (r_i + R_i)} - \frac{R_i}{\sum_i R_i} = \frac{r_i \sum_i R_i - R_i \sum_i r_i}{(\sum_i R_i)\left[\sum_i (r_i + R_i)\right]},$$

so that

$$\frac{r_i + R_i}{\sum_i (r_i + R_i)} (\leq, \ =, \ \geq) \frac{R_i}{\sum_i R_i}$$

according as

$$\frac{r_i}{\sum_i r_i}(\leq, \ =, \ \geq)\frac{R_i}{\sum_i R_i}$$

so that if

$$\frac{R_i}{\sum_i R_i} \geq \frac{r_i}{\sum_i r_i}, \qquad \frac{r_i}{\sum_i r_i} \leq \frac{r_i + R_i}{\sum_i (r_i + R_i)} \leq \frac{R_i}{\sum_i R_i}$$

and if

$$\frac{R_i}{\sum_i R_i} \leq \frac{r_i}{\sum_i r_i}, \qquad \frac{R_i}{\sum_i R_i} \leq \frac{r_i + R_i}{\sum_i (r_i + R_i)} \leq \frac{r_i}{\sum_i r_i}$$

In either case $\frac{r_i+R_i}{\sum_i(r_i+R_i)}$ lies between $\frac{r_i}{\sum_i r_i}$ and $\frac{R_i}{\sum_i R_i}$ so that although we have started with possibly a wider range for $\overline{p}_i$ we find that the probabilities for the most unbiased distribution lie in the narrower range.

Also, we show that

$$\frac{r_i + R_i}{\sum_i r_i + R_i} - \frac{1}{2}\left(\frac{r_i}{\sum_i r_i} + \frac{R_i}{\sum_i R_i}\right)(\geq, \ =, \leq,)\, 0, \tag{4.68}$$

equivalent to

$$2\sum_i r_i \sum_i R_i \sum_i (r_i + R_i)\,(\geq, \ =, \ \leq)\ \left(r_i \sum_i R_i + R_i \sum_i r_i\right)\left(\sum_i r_i + \sum_i R_i\right)$$

or

$$r_i \sum_i r_i \sum_i R_i + R_i \sum_i r_i \sum_i R_i \ \ (\geq, \ =, \ \leq)\ r_i \left(\sum_i R_i\right)^2 + R_i \left(\sum_i r_i\right)^2$$

according as

$$r_i \sum_i R_i \left(\sum_i r_i - \sum_i R_i \right) - R_i \sum_i r_i \left(\sum_i r_i - \sum_i R_i \right) (\geq, \ =, \ \leq) \ 0$$

which is equivalent to:

$$R_i \sum_i r_i \, (\geq, \ =, \ \leq) \, r_i \sum_i R_i$$

or

$$\frac{R_i}{\sum_i R_i} (\geq, \ =, \ \leq) \frac{r_i}{\sum_i r_i} \tag{4.69}$$

that is

$$\overline{p}_i \, (\geq, \ =, \ \leq) \, \overline{\overline{p}}_i \tag{4.70}$$

4.6.3 Comparison for the continuous case

1. Proceeding in the same manner as in subsection 5.3 we get the following:

$$\overline{p}(x) \, (\geq, \ =, \ \leq) \, \overline{\overline{p}}(x)$$

corresponding to

$$\frac{g(x)}{G} (\geq, \ =, \ \leq) \frac{f(x)}{F}$$

This is illustrated in the following figure (4.5)

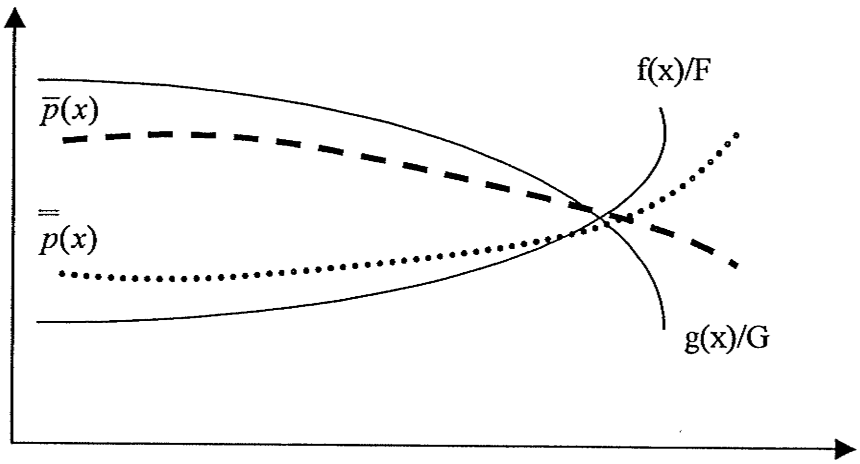


Figure 4.5: Illustrative example

Thus, both probability density functions lie between $\frac{f(x)}{F}$ and $\frac{g(x)}{G}$. The first probability density function is larger for $0 \leq x \leq x_0$, and the second probability density function is larger for $x_0 \leq x \leq 1$. However, the areas under these two different density functions are the same.

4.6.4 New measures of risk

Although we use the same entropy measure in equation (4.28) we get two most unbiased probability density functions according to the range we assumed for them, and we get the following two different measures of risk

$$Ln\ n + \sum_i \left[\frac{w_i\ (R_i + r_i)}{\sum_i w_i\,(R_i + r_i)} \ln \left(\frac{w_i(R_i + r_i)}{\sum_i w_i(R_i + r_i)} \right) \right] \tag{4.71}$$

and

$$Ln\ n + \sum_i \frac{1}{2} \left(\frac{w_i\ r_i}{\sum_i w_i r_i} + \frac{w_i R_i}{\sum_i w_i R_i} \right) \ln \frac{1}{2} \left(\frac{w_i r_i}{\sum_i w_i r_i} + \frac{w_i R_i}{\sum_i w_i R_i} \right) \tag{4.72}$$

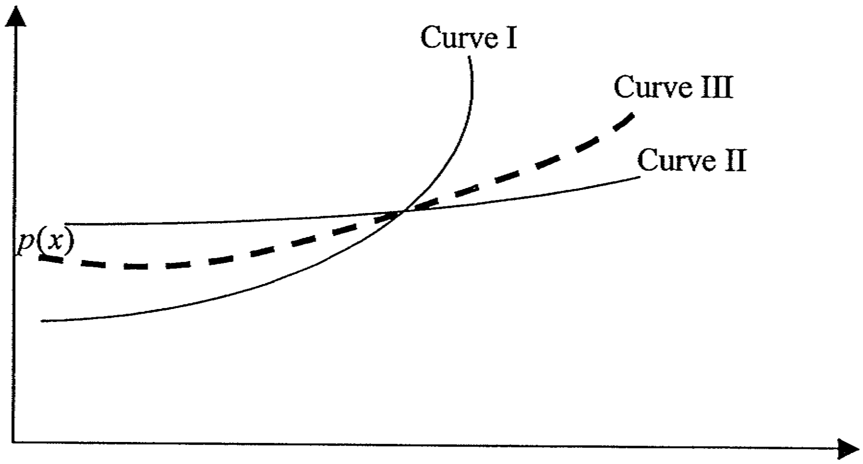


Figure 4.6: Example of two most unbiased probability distributions

We can use either to obtain the mean-entropic risk frontier.

4.7 Heuristic Explanation of the Two Most Unbiased Probability Distributions

(a) In the figure below (4.6), Curve I is for y=$\frac{f(x)}{F}$, and Curve II is for y=$\frac{g(x)}{G}$, and the area under each curve is unity. These two curves must, therefore, intersect in at least one point x_0. We want to find a curve III, $y = p(x)$, such that p(x) lies between $\frac{f(x)}{F}, \frac{g(x)}{G}$ and that

$$\int_a^b p(x)dx = 1 \tag{4.73}$$

There can be an infinity of such curves, such as:

$$y = \lambda \frac{g(x)}{G} + (1-\lambda)\frac{f(x)}{F}, \qquad a \le x \le b, \quad 0 \le \lambda \le 1. \tag{4.74}$$

The area under each curve is $\lambda \int_a^b \frac{g(x)}{G}dx + (1-\lambda)\int_a^b \frac{f(x)}{F}dx = 1$. There are many more such curves, but the question may be asked as to which curve should be chosen. Obviously, the most unbiased choice for λ lying between 0 and 1 is $\lambda = \frac{1}{2}$. This choice of λ yields

$$p(x) = \frac{1}{2}\left[\frac{f(x)}{F} + \frac{g(x)}{G}\right],$$

This is exactly the same result that we got earlier in equation (4.63) by using a relatively more sophisticated argument.

(b) We now consider the Curve IV, $y = \frac{f(X)}{G}$, and Curve V, $y = \frac{g(x)}{F}$. If $f(x) < g(x)$, then the Curve IV will always be below Curve V.

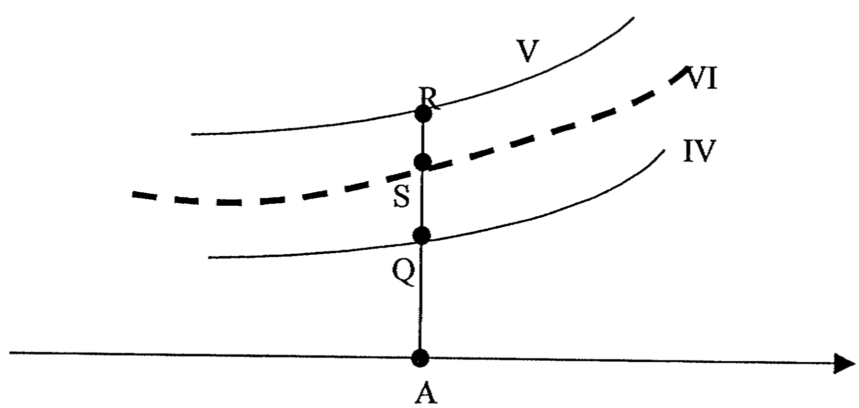


Figure 4.7: Example of three most unbiased probability distributions

The area under the Curve IV is $\frac{F}{G}$ and the area under the Curve V is $\frac{G}{F}$. Now, we want a Curve VI, $y = P(x)$ such that

$$\int_a^b P(x)dx = 1, \tag{4.75}$$

and

$$\frac{f(x)}{G} < P(x) < \frac{g(x)}{F}. \tag{4.76}$$

Using, the Figure (4.7), we get

$$\int_a^b AQdx = \frac{F}{G}, \int_a^b ARdx = \frac{G}{F}$$

such that A, Q and R, represent points in the Figure (4.7) and AQ, AR and AS represent the distance between them.

$$\int_a^b ASdx = 1 \tag{4.77}$$

To be able to have $P(x)$ as presented by the inequality(4.76), the most unbiased choice should be taken as

$$AR : AS : AQ = \frac{g(x)}{F} : 1 : \frac{f(x)}{G}, \tag{4.78}$$

so that,

$$\frac{\frac{g(x)}{F} - \frac{f(x)}{G}}{\frac{g(x)}{F} - P(x)} = \frac{G+F}{G} \tag{4.79}$$

That is the formula of $P(x)$

$$P(x) = \frac{g(x) + f(x)}{G+F}, \tag{4.80}$$

which is exactly the result we got in the first case using relatively more sophisticated arguments.

(c) It may be observed that in our argument in both cases, we have used the principle of maximum entropy by trying to get the most unbiased estimates, although we have not formally stated it.

(d) Suppose in case (b) we take the curves $y = \frac{2g(x)}{F}$, and $y = \frac{f(x)}{2G}$, then proceeding as in the case (b), we get

$$P(x) = \frac{2g(x) + f(x)}{2G + F}.$$

It can be shown that P(x) still lies in the narrower range of $\frac{f(x)}{F}$ to $\frac{g(x)}{G}$. But P(x) does not give an unbiased estimate because we have given different weights to $\frac{f(x)}{G}$ and $\frac{g(x)}{F}$. These different weights result in a biased estimate.

(e) In case (a) we assumed that the curves $y = \frac{f(x)}{F}$ and $y = \frac{g(x)}{G}$ intersect at one point but they can intersect at more than one point (see Figure (4.8)).

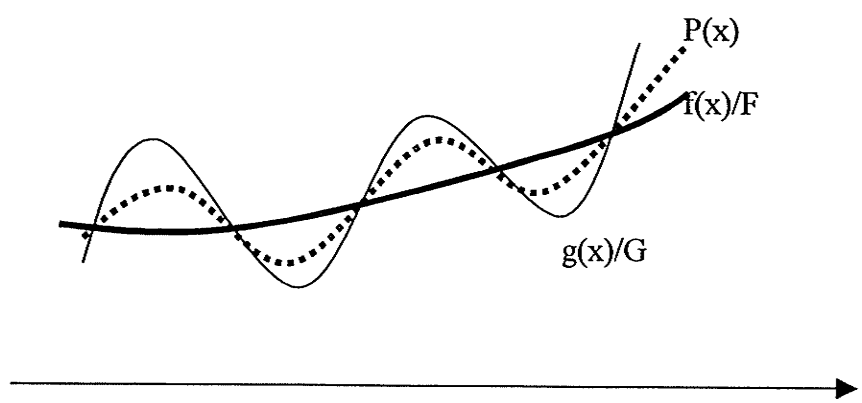


Figure 4.8: Example of another situation

Even in this case, it is easy to show that the most unbiased estimate will still be the same as equation (4.80), and the most unbiased probability density curve will lie exactly mid-way between the two curves $y = \frac{f(x)}{F}$ and $y = \frac{g(x)}{G}$. As an example, let $f(x)$ and $g(x)$ defined over $[a, b]$ such that

$$f(x) = 1 + \sin x, \quad g(x) = 1 + \cos x, \quad a = 0, \; b = 2\pi$$

then

$$F = G = 2\pi$$

and

$$P(x) = \frac{2 + \sin x + \cos x}{4\pi}$$

The curves for $\frac{f(x)}{F}$, $\frac{g(x)}{G}$, $P(x)$ are shown in the figure (4.9) and these intersect in two common points within the interval $[0, 2\pi]$.

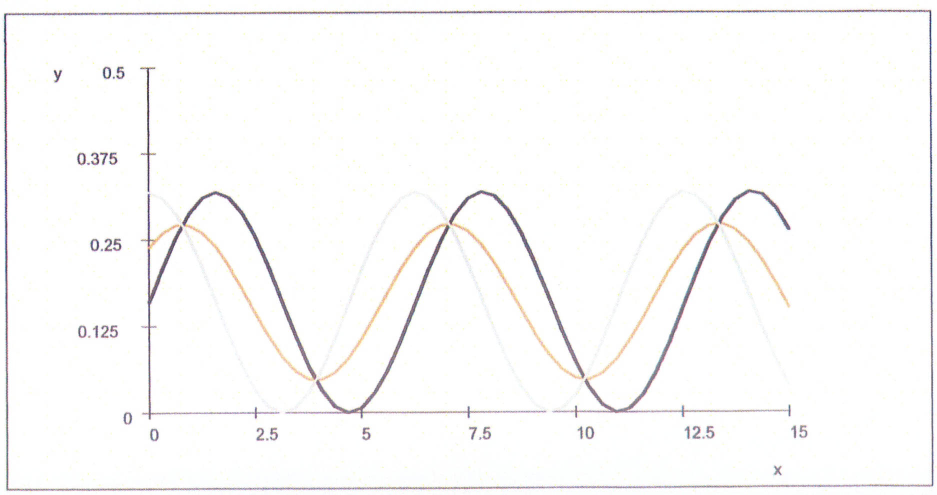


Figure 4.9: Example of three curves

4.8 Weaknesses of Shannon's Measure for the Present Problem

Suppose we are given two probability density functions $f(x)$ and $g(x)$ as distinct functions defined over the closed interval $[a, b]$, and we want to find a probability density function lying between f(x) and g(x) that has the maximum Shannon entropy [140].

Let

$$\phi(x) = \lambda f(x) + (1 - \lambda)g(x), \qquad 0 \leq \lambda \leq 1$$

so that

$$\begin{aligned} \int_a^b \phi(x)dx &= \lambda \int_a^b f(x)dx + (1 - \lambda) \int_a^b g(x)dx, \qquad 0 \leq \lambda \leq 1 \\ &= \lambda + (1 - \lambda) = 1 \end{aligned}$$

Therefore, $\phi(x)$ is a proper density function. Also,

$$\phi(x) - f(x) = (1 - \lambda) [g(x) - f(x)]$$

and

$$g(x) - \phi(x) = \lambda [g(x) - f(x)],$$

so that $\phi(x)$ lies between $f(x)$ and $g(x)$ when $0 \leq \lambda \leq 1$.

Now, Shannon's entropy for the density function $\phi(x)$ is a function of λ and is given by:

$$S(\lambda) = -\int_a^b \phi(x) \ln \phi(x) dx = -\int_a^b \left[\lambda f(x) + (1-\lambda) g(x)\right] \ln \left[\lambda f(x) + (1-\lambda) g(x)\right] dx$$

so that

$$S^{'}(\lambda) = -\int_a^b \left[1 + \ln(\lambda f(x) + (1-\lambda) g(x))\right] \left[f(x) - g(x)\right] dx$$

and

$$S^{''}(\lambda) = -\int_a^b \frac{\left[f(x) - g(x)\right]^2}{(\lambda f(x) + (1-\lambda) g(x))} dx < 0.$$

Therefore, $S(\lambda)$ is a strictly concave function of λ, and thus has a unique global maximum. Additionally, it easily follows that $S^{'}(\lambda)$ is a decreasing function of λ. If at $\lambda = 0$, $S(\lambda) > 0$, and at $\lambda = 1$, $S^{'}(\lambda) < 0$, then $S(\lambda)$ will have a global maximum between $\lambda = 0$ and $\lambda = 1$. In this case Shannon's measure will offer the most unbiased probability density function lying between $f(x)$ and $g(x)$. In all other cases, it will give a probability function lying outside the region contained by $f(x)$ and $g(x)$ as it will correspond to either $\lambda < 0$, or $\lambda > 1$. The above example demonstrates such a situation. If we use the entropy measure given by (4.3), we can always find the most unbiased probability distribution between $f(x)$ and $g(x)$. The reason for this is that the measure given by (4.3) ensures that λ always lies between 0 and 1. For Shannon's measure, it will depend upon the values of $S^{'}(0)$ and $S^{'}(1)$.

4.9 Empirical Illustrative Portfolio Analysis

In this section, following the lines of Philippatos and Wilson [119] and Cozzolino and Zahner [27] in using entropy to measure the portfolio risk, a simple empirical example is presented. One of the suggested measures of risk presented previously in equation (4.28) has been used to compute the optimal strategy. We use some published data in [104], and the returns on the nine securities during the years 1937-54 are presented in the following Table (4.1)

Table 4.1: Returns on nine securities over the period 1937-1954

Year	1 Am.T.	2 A. T. & T.	3 U. S. S.	4 G. M.	5 A. T. & Sfe	6 C. C.	7 Bdn.	8 Frstn.	9 S. S.
1937	-.305	-.173	-.318	-.477	-.457	-.065	-.319	-.400	-.435
1938	.513	.098	.285	.714	.107	.238	.076	.336	.238
1939	.055	.200	-.047	.165	-.424	-.078	.381	-.093	-.295
1940	-.126	.030	.104	-.043	-.189	-.077	-.051	-.090	-.036
1941	-.280	-.183	-.171	-.277	.637	-.187	.087	-.194	-.240
1942	-.003	.067	-.039	.476	.865	.156	.262	1.113	.126
1943	.428	.300	.149	.225	.313	.351	.341	.580	.639
1944	.192	.103	.260	.290	.637	.233	.227	.473	.282
1945	.446	.216	.419	.216	.373	.349	.352	.229	.578
1946	-.088	-.046	-.078	-.272	-.037	-.209	.153	-.126	.289
1947	-.127	-.071	.169	.144	.026	.355	-.099	.009	.184
1948	-.015	.056	-.035	.107	.153	-.231	.038	.000	.114
1949	.305	.038	.133	.321	.067	.246	.273	.223	-.222
1950	-.096	.089	.732	.305	.579	-.248	.091	.650	.327
1951	.016	.090	.021	.195	.040	-.064	.054	-.131	.333
1952	.128	.083	.131	.390	.434	.079	.109	.175	.062
1953	-.010	.035	.006	-.072	-.027	.067	.210	-.084	-.048
1954	.154	.176	.908	.715	.469	.077	.112	.756	.185

The abbreviations represent the ticker symbol of the issuer company of the security, for instance:

Am.T.: American Tobacco; A. T. &T.: American Telephone and Telegraph Comany; U. S. S.: United States Steel; G. M.: General Motors; A. T. & Sfe: Atchison, Topeka & Santa Fee; C. C.: Coca-Cola; Bdn.: Borden; Frstn.: Firestone; S.S.: Sharon Steel.

The data shows the returns, including dividends, for nine securities over an 18-year period. In the previous section in this chapter, in the process to model the portfolio analysis using the entropy method suggested, we started by identifying maximum and minimum returns. Based on the returns table, these minimum and maximum values are as follows (4.2):

Table 4.2: Minimum and maximum returns on nine securities over the period 1937-1954

Items	1 Am.T.	2 A. T. & T.	3 U. S. S.	4 G. M.	5 A. T. & Sfe	6 C. C.	7 Bdn.	8 Frstn.	9 S. S.
r_i	-0.3050	-0.1830	-0.3180	-0.4770	-0.4570	-0.2480	-0.3190	-0.4000	-0.4350
R_i	0.5130	0.3000	0.9080	0.7150	0.8650	0.3550	0.3810	0.7560	0.6390
$r_i + R_i$	0.2080	0.1170	0.5900	0.2380	0.4080	0.1070	0.0620	0.3560	0.2040
W_i	0.0876	0.1557	0.0309	0.0766	0.0447	0.1703	0.2939	0.0512	0.0893

Let us define

$$a_i = \frac{w_i r_i}{\sum w_i R_i}, \qquad b_i = \frac{w_i R_i}{\sum w_i r_i},$$

w_i represents the proportionate investment in security i, $i = 1, \ldots, 9$

Following the formula indicated previously (4.27), we get:

$$\widehat{p}_i = \frac{w_i(R_i + r_i)}{\sum w_i(R_i + r_i)}$$

Maximizing the objective function (4.29) subject to $\sum w_i = 1$, we get

$$w_i = \frac{\frac{1}{r_i+R_i}}{\sum \frac{1}{r_i+R_i}}$$

Using the data illustrated in the Table (4.1), we solve the problem and achieve the optimal proportions given in Table (4.2).

4.10 Summary and Concluding Remarks

In this chapter, we have considered a portfolio for which the minimum and the maximum returns on the i^{th} security are r_i and R_i and the mean return is $\overline{R}_i$. For this purpose, firstly, we find the probabilities $p_1, p_2, ..., p_n$ for which $a_i \leq p_i \leq b_i$, where

$$a_i = \frac{w_i r_i}{\sum_i w_i R_i} \qquad \text{and} \qquad b_i = \frac{w_i R_i}{\sum_i w_i r_i},$$

which maximizes the measure of entropy given by equation (4.3) and offers us the most unbiased probability distribution $\widehat{p}_i$, $i = 1, ..., n$ given by (4.27). We next find the probability distribution, which as in (4.37), lies between values given by (4.35) and (4.36). This gives the most unbiased estimate

$$\widetilde{p}_i = \frac{1}{2}\left[\frac{w_i r_i}{\sum_i w_i r_i} + \frac{w_i R_i}{\sum_i w_i R_i}\right], \qquad i = 1, \ldots, n$$

It can be easily shown that both $\widehat{p}_i$ and $\widetilde{p}_i$ lie between the limits given by (4.35) and (4.36). We also obtained two new entropic measures of risk.

$$\ln n + \sum_i \widehat{p}_i \ln \widehat{p}_i \qquad \text{and} \qquad \ln n + \sum_i \widetilde{p}_i \ln \widetilde{p}_i$$

Furthermore, the results to continuous variate probability density functions are presented. Also, a heuristic explanation of the results without using the principle of Maximum Entropy is provided. Finally, it is shown that the entropy measure (4.3) will always give the most unbiased probability density function, which lies between the two given probability density functions, whereas Shannon's measure may give the most unbiased probability distribution lying outside of the region bounded by the two given density functions.

The aim of the chapter is to present the mathematical theory of approximate probabilities using the measure of entropy. When there is not enough information on which to base our decisions, we cannot expect sharply defined reasoning to reveal the most probable outcome. A substantial amount of research may be needed in this direction.

Chapter 5

Fuzzy Regression with Application

The price limit bounds the daily commodity price to move within the predetermined level above or below the previous day's closing price. Therefore, the equilibrium price is unobserved when it moves outside the limits. Under price limitation, since the observed price is not equal to the equilibrium price, estimating using the observed price may yield biased parameter estimates. Actually, many studies propose econometric analysis to tackle the data distortion caused by price limits. Kodres [76] used the maximum likelihood approach to estimate the parameters of two limit robit models. Roll [127] adopted the proxy variable to substitute the limit move data. The daily commodity price on any trading day cannot be higher (lower) than the previous closing price plus (minus) a limit. The price limits bound the daily commodity price movements and shorten the distribution of equilibrium price changes, allowing for the use of the fuzzy theory developed by Zadeh [169]. Therefore, the equilibrium return may be treated as fuzzy and random. The aim of this study is to estimate systematic risk using commodity futures prices with the existence of price limits. The estimation process has been conducted in two different phases. The systematic risk has been estimated using the settlement price of the commodity futures using the Ordinary Least Squares (OLS) method. Then, an optimization model has been developed to investigate the impact and effectiveness of price limits on estimating the beta risk of commodities return.

In the following section, we present a review of the literature related to price limits and to CAPM when applied to commodity futures. In Section 2, we present the modeling environment of both CAPM and a two-phase fuzzy regression approach, and in Section 3, the data and methodology are demonstrated. Our concluding remarks are offered in Section 4.

5.1 Review of the Literature: Price Limits and CAPM with Futures Markets

Recently, various studies have investigated the modeling of the price limits and their impacts on stock and futures prices, for example ([17], [26], [51], [98], [60], and [115]). A feature of most futures markets is a daily price limit rule. Price limits have been imposed on daily price volatility to stabilize the market. In a market with a daily price limit rule, trading is permitted only at prices within limits determined by the settlement price of the previous day. The settlement price is an average of the transactions' prices in the closing periods of trading or, if trading is halted at the close, it is the relevant price limit. In the stock market, the officers of the exchange have the power to stop when they believe it necessary and desirable. Yet, stops in the stock market are not necessarily related to the size of the price movement. Hopewell and Schwartz [57] found that 92% of the halts on NYSE lasted less than a day. In another example, the Winnipeg Commodity Exchange regulates prices by prohibiting trading during any trading day, in futures of commodities traded at a price that exceeds the settlement price of the previous day's session by a certain amount. Table (5.1) represents the price limits of feed wheat, western barley, canola and flaxseed commodities, excluding the new contract delivery month. Such limits are based upon the Board's lot quotations. In addition, in the case of trading in a contract that is eligible in that month, there shall be no daily limit on price movement on the last day of trading.

Table 5.1: Price limits per commodities futures traded in Winnipeg Commodity Exchange (WCE). Sample period: Jan. 1991 to Dec. 2000

Commodities	Price Limits \$ / Tonne$^{(a)}$	Price Limits \$ / Tonne$^{(b)}$
Western Barley	5.00	7.50
Canola and Flaxseed	10.00	30.00
Feed Wheat	5.00	7.50

(a) daily price limits before October 10, 2000. (b) daily price limits effective October 10, 2000.

Brennan [15] first proposed a theory explaining why a price limit exists in some futures markets. In a market with price limits, when a shock happens, the equilibrium price moves outside the daily maximum allowable increase/decrease interval; it becomes unobserved, and what we observe is merely a limit price. He pointed out:

> "...for agricultural commodities, where the basis risk is typically substantial, we expect to find a role for price limits, at least in the distant contract months."

He found that as the precision of the external signal regarding the equilibrium price increases, the price limits are expected to be either relaxed or ignored. Thus, that precision is not assumed in this chapter.

Roll [127] argued that the price data that is usually used may or may not reflect actual transactions and is determined by members of the exchange at the close of each day's trading. The indicated limits on price movements prevent the price from moving by more than a certain amount from the previous day's settlement price. Roll [127] said that:

> "When a significant event, such as a freeze in Florida, causes the price to move the limit, the settlement price on that day cannot fully reflect all available information. In other words, limit rules cause a type of market information inefficiency (but not a profit opportunity). This might be inconsequential if limit moves occurred rarely; unfortunately, they are rather common."

Hull [61] discussed the concept of the settlement price, which is defined as the average of the prices at which the contract traded immediately before the bell signaled the end of trading for the day. He pointed out that:

> "It is important because it is used for calculating daily gains and losses and margin requirements."

However, in the futures market, some futures contracts are settled in cash. In this case, the settlement price on the last trading day is equal to the closing spot price of the underlying asset, to ensure that the futures prices converges to the spot price.

There are many empirical studies that have dealt with futures prices behavior in practice. Houthakker's [58] study looked at futures prices for wheat, cotton, and corn during 1937-1957, showing that it was possible to earn profits from taking long futures positions. Telser's study [151] constructed the findings of Houthakker [58]. Telser's data covered the period 1926-1950 for cotton and 1927-1954 for wheat and resulted in significant profits for traders taking either long or short positions. Gray's [48] study looked at corn futures prices during 1921-1959 and resulted in findings similar to those of Telser. Dusak's [35] study used data on corn, wheat, and soybeans during 1952-1967. Her study attempted to estimate the systematic risk of an investment in these commodities by calculating the correlation of movements in the commodity prices with movements in the S&P 500. However, the results obtained by Dusak [35] suggest that there is no systematic risk.

Dusak [35] showed that systematic risk and return for wheat, corn, and soybeans futures contracts were near zero. Then, she concluded that these futures contracts are not risky assets when held as part of a large portfolio. Carter, Rausser, and Schmitz [19] (hereafter referred to as CRS) changed Dusak's model by introducing stochastic systematic risk as a function of actual net speculative positions. CRS found that half of the contracts had significantly positive risks. Research done by Chang [22] using the same commodities, supported the existence of a positive systematic risk. Baxter, Conine, and Tamarkin [9] (hereafter referred to as BCT) repeated Dusak's study by using a proxy for the market portfolio consisting of 93.7% of the S&P 500 index and 6.3% of the Dow-Jones cash commodity index. BCT showed insignificant systematic risk for wheat, corn, and soybeans futures. Thus, they could not show a positive systematic risk for the same three futures contracts during 1953-1976. Elam and Vaught [39] (hereafter referred to as EV) investigated the existence of risk and return in cattle and hog futures. They found significant systematic risk for the one hog and four cattle futures contracts. They combined 90% on the S&P 500 index and 10% on the Dow-Jones cash commodity index, as proxy index. Chang, Chen, and Chen (hereafter referred as CCC) [23] used six traded futures for copper, six for silver, and four for platinum. They used the month-end settlement price, which is on the last trading day of the month. They found contrary to other studies, a significant systematic risk in the agricultural and livestock commodity futures.

The estimation in futures markets usually faces the existence of price limit regulations and may call for another approach to estimate systematic risk. Thus, it is important to analyze the behavior of futures prices when the exchange is regulated by price limits. Also, it is well known that the regulation responds to the trading behavior of market members. When traders are confronted with market barriers, they revise their expectations accordingly.

Roll [127] concluded that the use of a possible settlement price implied by limit moves will affect any informational efficiency study. In his empirical study [127], the price on the first day with no limit move was brought back to the day of the first limit move, and all intermediate days were ignored. For example, if a limit occurred on a specific day, he assumed that the settlement price for that day was the price of the following day, which did not have a limit move. Kodres [76] analyzed the impact of price limits on a test of the unbiasedness hypothesis in foreign exchange futures markets. Mao, Rao, and Sears [101] claimed that trading halts mitigate price, enhance informational efficiency, and tend to excessively inflate volatility . Mao et al. ([101], [100]) found that price trends, in general, stabilize or reverse themselves after reaching limits and tend to move back into prelimit price ranges. Other researchers, such as Subrahmanyam [146] argued that limits obstruct informational efficiency. Hall and Kofman [51] proposed a modeling framework to distinguish between observed and theoretical futures prices. Kuserk and Locke's [85] findings contradicted Roll's statement that the limits create information inefficiency, not profit opportunity.[1]

From a probabilistic point of view, Hall and Kofman [51] presented that, in the absence of price limits, observed futures prices should be equal to unobserved fundamental futures prices plus some random noise, to reflect market microstructure effects such as bid-ask bounce. But if there is a fixed and credible upper limit, and the price is close to that limit, the probability of a further increase will be limited, the probability of a decrease will be relatively larger, and the probability distribution of the next price move will become increasingly skewed, the closer it gets to the limit.

Park [115], in his study, investigated price limits in futures markets and pointed out that price limits serve to delay gains and losses that might occur with large price swings. Price limits function similarly to margin accounts by limiting the amount of price exposure risk. According to Park [115]:

> "Unfortunately, there is no generally accepted theory on how price limits influence price behavior."

5.2 The Modeling Environment

5.2.1 Necessary assumptions

As a rule, an observation is subject to different kinds of uncertainty from objective sources (e.g., the coarseness of the computer used to collect and register data) or from subjective ones (e.g., the evaluation of the observer, trader, or investor with respect to the reliability of the observation). Thus, observation is subject to fuzzy structures specification,

[1] Informed traders will try to smooth their trading activities before the price hits a limit, and market makers protect themselves against these movements by increasing the bid-ask spread upward.

which can take place at each daily price. In fuzzy set, it is reflected by "approximately p" and can be enhanced by an evaluation, e.g. "quite surely p". One of many advantages of using a fuzzy approach, here, is that information from different sources or data of different specifications can be utilized. Motivated by these considerations, we assume that futures contracts under price limits are subject to fuzziness. Also, we assume that the trader has some external source of information about the equilibrium future price[2], but that the information is incomplete and not precisely defined (fuzzy).

In fact, Brennan [15] assumes in his study that the price change follows a uniform distribution and that the trader receives a signal equals to the equilibrium price plus a uniformly distributed error term.

What remains unexplained in the literature (e.g. Dusak [35], CRS [19], BCT [9]) that discusses the estimation of systematic risk in futures market, is, first, why a price which is subject to limits is not important in the estimation; and second, why price limits should be ignored in the decision to accept or reject the existence of systematic risk. Furthermore, the findings of various articles explain neither how the price limits could be modeled in the regression analysis, knowing that the maximum allowed change is calculated from the close of the previous day, nor why it is necessary to use the settlement price, which is an average, instead of the equilibrium price.

With a simple treatment, Chou [26] ignores noisy observations, which are obscured by the residual shock. He also treats them as missing data. This approach is questionable, because these discarded observations carry information about the model parameters, and it

[2]Brenan [15] assumes that the trader is able to observe a signal $\widetilde{Y}$ which may be derivable from the spot market for the underlying commodity or asset, from the markets for other futures contracts, or from other sources.

is not appropriate to ignore them in the estimation of the model. In addition, he ignores the residual shocks, which are determined by the price limit regulation and may be carried over from the preceding day and from previous days. The following trading days will show the unrealized excess demand or supply that will accumulate and be carried over to consecutive days [75].

Under a price limit, the settlement price (observed) is not exactly equal to the equilibrium price (unobserved), and estimating without it might imply biased parameter estimates. To preclude the biased parameter estimates introduced by price limits, we treat the futures price as a fuzzy datum and use a two-phase fuzzy approach to estimate the systematic risk.

Following the Brennan paradigm, we assume that there is an external signal suggesting that the equilibrium price is in the boundaries of the observed price and that the equilibrium price is bounded by an upper bound (observed plus half the limit) and lower bound (observed minus half the limit). That is, the equilibrium is partly observable, and the external information suggests that the equilibrium price is between $[p - \frac{l}{2}, p + \frac{l}{2}]$. The membership function, which measures the degree of precision of that equilibrium price is assumed to have a triangular shape function. The reason for this choice is twofold: (i) triangular shapes are easy to construct and manipulate and (ii) most current applications that use fuzzy theory are not significantly affected by their shape.

Let us assume that we have a two-day period, yesterday $(t-1)$ and today (t). If the price hit the up-limit (or down-limit) in $t-1$, we would observe the limit $p_{t-1} = l_u$ (or $p_{t-1} = l_d$); under this scenario, we assume that the equilibrium price is fuzzy random in the interval $\left[p_{t-1} - \frac{l}{2}, p_{t-1} + \frac{l}{2}\right]$. If today ($t$) registered a limit move, the price hits limit up (or down-limit) again, we suggest that the equilibrium price is fuzzy random in the $\left[p_t - \frac{l}{2}, p_t + \frac{l}{2}\right]$. Yet, if the observed price does not hit the limit, we still suggest that the equilibrium price is a fuzzy random in the interval $\left[p_t - \frac{l}{2}, p_t + \frac{l}{2}\right]$ since as observations following a limit move reflect both the associated shocks and the residual shock carried over from previous trading days. Accordingly, prices are correlated, and we will not be able to extract information via the observations to achieve more precise estimates of the model parameters. It is the residual shock that substantially complicates the estimation of the model. To this point, it has been assumed that the equilibrium price that would have been observed in the absence of price limit will be around the settlement price. That is the reason behind the use of fuzzy theory here.

5.2.2 Capital asset pricing model (CAPM)

Sharpe [141], Lintner [97], and Mossin's [108] capital asset pricing models have been investigated for both agricultural and livestock futures during the last decade. Many studies (e.g. [35], [145], [39]) discussed whether futures investors and traders accept any systematic risks and whether there is a reward commensurate with the systematic risk of futures contracts. The systematic risk of commodities is different from other financial assets. For example, holding times for agricultural commodities are relatively short. In addition, they are subject to seasonal production. Because spot and futures prices of a commodity tend to follow each other, it is interesting to look at the relationship between return and systematic risk; the capital asset pricing model, which serves that purpose, has the following standard form [28]:

$$E\left[\widetilde{r}_q\right] = r_f + \beta_{qM}\left[E\left[\widetilde{r}_M\right] - r_f\right] \tag{5.1}$$

for any asset (or portfolio) q where $E[.]$ is the expectation operator, and tildes represent random variables. $\widetilde{r}_q$ is the return on an asset and r_f is the riskless rate of return. $\widetilde{r}_M$ is the return on the market portfolio of all assets. Coefficient β_{qM} is defined by $\frac{Cov(\widetilde{r}_q,\widetilde{r}_M)}{Var(\widetilde{r}_M)}$ which is a measure of the tendency of a security's returns to respond to swings in the broad market.

The model (5.1) assumes that the expected return on a financial asset is composed of a risk premium and the return on the riskless asset. Additionally, the risk premium on a financial asset equals the product of the systematic risk of the asset β_{qM} and the risk premium on the market portfolio $\left[E\left[\widetilde{r}_M\right] - r_f\right]$. In the context of commodity futures, Chang et al. [23] advised that holders of a futures contract can expect a positive risk premium, if changes in contract prices are independent of changes in values of all assets combined. Because CAPM is an ex-ante model, which means that the parameters are unobservable, β_{qM}'s are not observabl and they should be estimated.

Under the assumption of a single-factor return generating process [66], the exposed version of CAPM can be written as:

$$r_q - r_f = e_i + \beta_q(r_M - r_f) \tag{5.2}$$

and the empirical version of CAPM in time series form is as follows:

$$r^e_{it} = \alpha_i + \beta_i r^e_{mt} + \epsilon_{it} \tag{5.3}$$

where $r^e_{it} = r_{it} - r_{ft}$, $r^e_{mt} = r_{mt} - r_{ft}$, ϵ_{it} disturbance term, α_i and $\widehat{\beta}_i$ can be estimated by regressing excess return on asset ex-post returns against excess returns on a proxy for the market portfolio. $\widehat{\beta}_i$ is an estimate of β_i, and α_i measures the mean excess return to the asset i, if the model is well specified.

Elam and Vaught [39] used a slightly modified CAPM to explain returns on futures contracts:

$$E[\widetilde{r}] = \beta [E[\widetilde{r}_m] - r_f] \tag{5.4}$$

without including r_f as an intercept in equation (5.4) because a futures contract represents an agreement to purchase a commodity at some later time. Because the payment for a stock is made up front, the return on a stock should reflect the time value of money (represented by r_f). The returns on a futures contract should not include r_f because no money is put up (or interest can be earned on money put up as margin) to buy a futures contract.

Therefore, we use the following empirical version in this chapter:

$$\mathbf{r}_{it} = \alpha_i + \beta_i (r_{mt} - r_{ft}) + \epsilon_{it} \tag{5.5}$$

The realized return on the contract with a fixed maturity signified by i during period t r_{it} is computed as:

$$r_{it} = Ln\left(\frac{P_{it}}{P_{it-1}}\right) \tag{5.6}$$

where P_{it} represents contract i's settlement price at time t and P_{it-1} represents i's settlement price at time $t-1$. The end of period returns on one-month Treasury Bills are obtained for the same period and serve as a proxy for the monthly risk-free interest rate r_f.

Following Chang et al. [23] we compute Sharpe's [142] performance measure S_i below in (5.7) :

$$S_i = \frac{r_i - r_f}{\sigma_i} \tag{5.7}$$

which is a reward-to-volatility ratio, a ratio of the reward to total volatility trade-off measures. Also, Treynor's [156] measure, provided below, is computed:

$$T_P = \frac{r_i - r_f}{\beta_i} \tag{5.8}$$

which is the ratio of excess return to beta risk (systematic risk).

5.2.3 Fuzzy regression methods

It is impossible to estimate the parameters of the linear model (5.5) CAPM in this case, under these conditions, through the traditional ordinary least squares, because the equilibrium price is not observed. We know that the estimation will be even more complicated when the sample contains consecutive limit moves across more than two days.

Following common practice, we shall assume that the true return is a fuzzy random number, normally distributed so that the proposed fuzzy regression approach is easily implemented.

Tanaka [148] first introduced fuzzy linear regression to determine a linear relationship between a fuzzy dependent and crisp independent variables. Subsequently, many studies (see e.g. [56], [124], [125], [150], [149], [71], and [164]) have been proposed to improve the fuzzy regression method. The literature dealing with fuzzy linear regression and its applications has grown rapidly. For instance, fuzzy regression methods have been widely used in forecasting (see e.g. [55], [164]), engineering (see e.g. [87], [7]), quality control (see e.g. [70]), and health (see e.g. [8]). Several papers have examined fuzzy regression methods and discussed some properties and deficiencies of their methodologies (for example [124], [125]). Tanaka [148] suggested the following model:

$$\begin{cases} \underset{A_i(\alpha_i, c_i)}{Min} \sum c^T |x_i| \\ \text{Subject to} \quad \alpha^T x_i + (1-H)c^T |x_i| \geq y_i + (1-H)e_i \\ -\alpha^T x_i + (1-H)c^T |x_i| \geq -y_i + (1-H)e_i \\ c_j \geq 0, \quad j = 1, ..., p \end{cases}$$

where $A_i = (\alpha_i, c_j)$ are fuzzy coefficients, which are the solution of the fuzzy linear programming problem. Then, $|x| = (1, |x_1|, , ..., |x_p|)^T$ and p is the number of independent variables. $\alpha = (\alpha_0, ..., \alpha_p)^T$ and $c = (c_0, c_1, ..., c_p)^T$.

The fuzzy coefficient can be expressed as "approximate α_i" with center α_i and spread (or width) c_i. $Y_i = (y_i, e_i)$ is the fuzzy output, where y_i is the center, e_i is the fuzzy spread and n is the number of observations.

Savic and Pedrycz [136] introduced another formulation of fuzzy regression method (referring to their approach as fuzzy least squares linear regression) that can resolve the issue of infinite solutions. They formulated the problem as a two-step procedure for building fuzzy regression models.

Phase I: Fit the regression line by using the available information about the center points of the observation, i.e. input data are considered non-fuzzy. Vector α^* is used as one of the input data sets in phase II.

Phase II: Determine the minimal vagueness using the linear constraints presented by the Tanaka method [148] but without α being a vector of decision variables. Savic and Pedrycz proposed some techniques for determining α. They chose to use the least squares method to get α, which is given below:

$$\alpha^* = \left(X^T X\right)^{-1} X^T Y$$

Recently Hojati et al. (hereafter referred to as HBS) [56] presented a new method for fuzzy regression that is simple to use. They developed a model where only the dependent variable is fuzzy and extended it to the case in which both dependent and independent variables are fuzzy. In classical regression setting, we regress the rate of return using the ordinary least squares method (OLS) to get the associated parameters of the model. In a fuzzy regression setting, HBS's method is based on the linear programming approach and minimizes the total absolute deviation to obtain the parameters. The parameters of the model are chosen such that the total deviation of the upper movements of predicted and associated observed intervals and the deviation of the lower movements of these intervals are minimized.

In HBS's model, the objective function is presented by minimizing the total deviations of the upper points of predicted and associated observed intervals, instead of minimizing the total spread of fuzzy parameters. Note that the essence of both papers [56] and [136] has been used here to develop the proposed method.

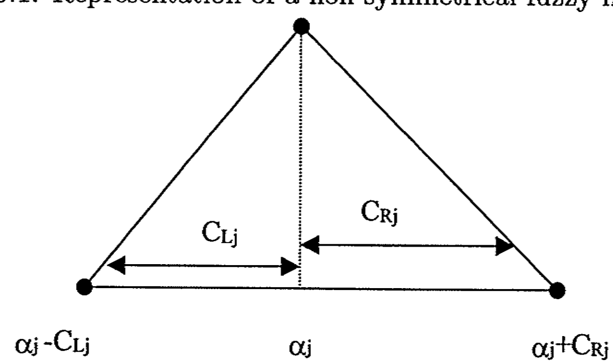
Proposed Method: Modification of Savic and Pedrycz's method

Definition 16 *A non-symmetrical triangular fuzzy number, A_j denoted as $A_j = (C_{Lj}, \alpha_j, C_{Rj})$ is defined as (see figure (5.1)):*

$$\mu_{A_j}(a_j) = \begin{cases} 1 - \frac{(\alpha_j - a_j)}{C_{Lj}} & \alpha_j - C_{Lj} < a_j < \alpha_j \\ 1 + \frac{(\alpha_j - a_j)}{C_{Rj}} & \alpha_j < a_j < \alpha_j + C_{Rj} \end{cases}$$

C_{Rj}, $C_{Lj} > 0$, α_j is a center (interior), C_{Lj} left spread and C_{Rj} right spread. $A_j = (C_{Lj}, \alpha_j, C_{Rj})$.

Figure 5.1: Representation of a non-symmetrical fuzzy number



In Tanaka's model [150] the fuzzy output data are assumed to be a fuzzy number with a symmetric triangular membership function denoted by $Y = (y_i, e_i)$, $i = 1, ..., n$. Extending this model in the case of fuzzy non-symmetric numbers, the linear model is as follows:

$$Y = AX = A_0X_0 + A_1X_1 + ... + A_sX_s = \sum_{j=0}^{s} A_jX_j \tag{5.9}$$

where $A_j = (C_{Lj}, \alpha_j, C_{Rj})$ *and* $A = (C_L, \alpha, C_R)$

Theorem 17 *The fuzzy output* Y *in (5.9) can be represented as:*

Case 1: *If* $X > 0$ *Then* $Y = (C_L X, \alpha X, C_R X)$

Case 2: *If* $X < 0$ *Then* $Y = (|X|\, C_R, \alpha X, |X|\, C_L)$, *where*

$\alpha = (\alpha_1, ..., \alpha_n)$; $C_{Rj} = (C_{R0}, ..., C_{Rs})$; $C_{Lj} = (C_{L0}, ..., C_{Ls})$ *and* $|X| = (|X_1|\,, ..., |X_n|)^t$.

Proof. It is well known from Dubois and Prade [34] that the following equations hold:

$\forall\ \lambda > 0;\ \lambda \odot (C_L, \alpha, C_R) = (\lambda C_L, \lambda\alpha, \lambda C_R)$, and

$\forall\ \lambda < 0;\ \lambda \odot (C_L, \alpha, C_R) = (-\lambda C_R, \lambda\alpha, -\lambda C_L)$. ■

So, the membership functions are written as follows:

In case 1:

$$\mu_Y(y) = \begin{cases} 1 - \frac{\alpha X - y}{X C_L} & If\ \ X C_L < y < \alpha X \\ 1 + \frac{\alpha X - y}{X C_R} & If\ \ \alpha X < y < X C_R \end{cases}$$

In case 2, the membership function of the output is written as:

$$\mu_Y(y) = \begin{cases} 1 - \frac{\alpha X - y}{|X| C_R} & If\ \ |X|\, C_R < y < \alpha X \\ 1 + \frac{\alpha X - y}{|X| C_L} & If\ \ \alpha X < y < |X|\, C_L \end{cases}$$

Linear programming and parameters computation

Following the same procedure presented by Tanaka [150], we provide the linear programming approach to estimate the parameters of the model.

In case 1:

$$\begin{cases} \text{Subject to} & \begin{matrix} Min\ O.F. \\ \mu(x) \geq h \end{matrix} \end{cases} \Leftrightarrow \begin{cases} & \underset{\alpha, C_L, C_R}{Min} \ O.F. \\ \text{Subject to} & 1 - \frac{\alpha X - y}{X C_L} \geq h \\ & 1 + \frac{\alpha X - y}{X C_R} \geq h \end{cases} \Leftrightarrow \begin{cases} \underset{\alpha, C_L, C_R}{Min} \ O.F. \\ -\alpha X + (1-h) X C_L \geq -y \\ \alpha X + (1-h) X C_R \geq y \\ \alpha \in \mathbb{R},\ C_{R,}\ C_L \geq 0 \end{cases}$$

In case 2, the linear programming problem is as follows:

$$\begin{cases} \text{Subject to} & \begin{matrix} Min\ O.F. \\ \mu(x) \geq h \end{matrix} \end{cases} \Leftrightarrow \begin{cases} & \underset{\alpha, C_L, C_R}{Min} \ O.F. \\ \text{Subject to} & 1 - \frac{\alpha X - y}{|X| C_R} \geq h \\ & 1 + \frac{\alpha X - y}{|X| C_L} \geq h \end{cases} \Leftrightarrow \begin{cases} \underset{\alpha, C_L, C_R}{Min} \ O.F. \\ -\alpha X + (1-h)\,|X|\, C_R \geq -y \\ \alpha X + (1-h)\,|X|\, C_L \geq y \\ \alpha \in \mathbb{R},\ C_{R,}\ C_L \geq 0 \end{cases}$$

h: is the degree of the fitting of the fuzzy linear model chosen by the decision maker.

After introducing an aggregate fuzzy linear regression method (aggregate case 1 and case 2), which is designed to minimize the sum of the fuzzy spreads around the y

predictions, the problem is formulated as:

$$\begin{cases} \underset{\alpha,C_L,C_R}{Min} \sum_{j=0}^{1}\sum_{i=1}^{n}(C_{Lj}+C_{Rj})\,|X_{ij}| \\ -\alpha^t X^P + (1-h)\sum_{i}^{p}\sum_{j=0}^{1} X_{ij}C_{Lj} \geq -y_i + (1-h)e_i, \quad for\ i=1,...,p \\ \alpha^t X^P + (1-h)\sum_{i}^{p}\sum_{j=0}^{1} X_{ij}C_{Rj} \geq y_i + (1-h)e_i, \quad for\ i=1,...,p \\ \text{Subject to } -\alpha^t X^N + (1-h)\sum_{i=p+1}^{n}\sum_{j=0}^{1} X_{ij}C_{Rj} \geq -y_i + (1-h)e_i, \quad for\ i=p+1,...,n \\ \alpha^t X^N + (1-h)\sum_{i=p+1}^{n}\sum_{j=0}^{1} X_{ij}C_{Lj} \geq y_i + (1-h)e_i, \quad for\ i=p+1,...,n \\ \alpha\ \in \mathbb{R},\ C_R=(C_{R0},C_{R1}),\ C_L=(C_{L0},C_{L1}) \geq 0 \\ Given\ \ X^P=(X_{1j},..X_{pj}) \geq 0,\ X^N=(X_{p+1j},..X_{nj}) < 0\ \ X_{i0}=1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \underset{\alpha,C_L,C_R}{Min} \sum_{j=0}^{1}\sum_{i=1}^{n}(C_{Lj}+C_{Rj})\,|X_{ij}| \\ -\alpha^t X^P + (1-h)\sum_{i=1}^{p}\sum_{j=0}^{1} X_{ij}C_{Lj} \geq -y_i + (1-h)e_i\ , \quad for\ i=1,...,p \\ \alpha^t X^P + (1-h)\sum_{i=1}^{p}\sum_{j=0}^{1} X_{ij}C_{Rj} \geq y_i + (1-h)e_i, \quad for\ i=1,...,p \\ \text{Subject to } -\alpha^t X^N + (1-h)\sum_{i=p+1}^{n}\sum_{j=0}^{1} X_{ij}C_{Rj} \geq -y_i + (1-h)e_i, \quad for\ i=p+1,...,n \\ \alpha^t X^N + (1-h)\sum_{i=p+1}^{n}\sum_{j=0}^{1} X_{ij}C_{Lj} \geq y_i + (1-h)e_i, \quad for\ i=p+1,...,n \\ \alpha\ \in \mathbb{R},\ C_R=(C_{R0},C_{R1}),\ C_L=(C_{L0},C_{L1}) \geq 0 \\ Given\ \ X^P=(X_{1j},..X_{pj}) \geq 0,\ X^N=(X_{p+1j},..X_{nj}) < 0\ \ X_{i0}=1 \end{cases}$$

e_i : the spread of the output. s: the number of independent variables in the model.

Extending Savic and Pedrycz's method by introducing the non-symmetrical fuzzy number case (center is estimated using the OLS method, thus, α^{*t} is used as one of the input data sets in phase II) and choosing $h = 0$, we get the following optimization problem[3]:

$$\Leftrightarrow \begin{cases} & \underset{C_L, C_R}{Min} \sum_{j=0}^{1} \sum_{i=1}^{n} (C_{Lj} + C_{Rj}) \left| X_{ij} \right| \\ & -\alpha^{*t} X^p + \sum_{i=1}^{p} \sum_{j=0}^{1} X_{ij} C_{Lj} \geq -y_i + e_i, \quad for\ i = 1, ..., p \\ & \alpha^{*t} X^p + \sum_{i=1}^{p} \sum_{j=0}^{1} X_{ij} C_{Rj} \geq y_i + e_i \ , \quad for\ i = 1, ..., p \\ \text{Subject to} & -\alpha^{*t} X^N + \sum_{i=p+1}^{n} \sum_{j=0}^{1} X_{ij} C_{Rj} \geq -y_i + e_i \ , \ for\ i = p+1, ..., n \\ & \alpha^{*t} X^N + \sum_{i=p+1}^{n} \sum_{j=0}^{1} X_{ij} C_{Lj} \geq y_i + e_i, \ for\ i = p+1, ..., n \\ & C_R = (C_{R0}, C_{R1}), \ C_L = (C_{L0}, C_{L1}) \geq 0 \\ & Given\ \ X^P = (X_{1j}, ..X_{pj}) \geq 0, \ X^N = (X_{p+1j}, ..X_{nj}) < 0, \ \ X_{i0} = 1 \end{cases}$$

Because we have non-symmetric fuzzy data, we need to make some modifications to the two-step procedure of Savic and Pedryzc to use it. The proposed method is as follows:

Phase I: α (center) is defined uniquely when X is a full rank matrix. Furthermore, as presented by Savic and Pedrycz, the use of α^* causes the center of the y_i^* fitted values to be closer to the observed values, y_i. That causes higher membership values, because α^* is the optimal vector that minimizes the sum of the squared residuals in ordinary least squares regression analysis ([72], [112])[4].

Phase II: Assuming that $h = 0$ and $c = (c_l, c_r)$ such that c_r is the right spread of the parameter and c_l is the left spread of the parameter, the model is equivalent to:

[3] In section 3, we use the following notation to represent the estimated centers using ordinary least squares (OLS) method $(\alpha^*, \beta^*) = (es\alpha, es\beta)$

[4] Several standard curve-fitting methods may be used in this phase, for example, minimum sum absolute deviations and the Chebyshev minmax criterion [162].

$$\left\{ \begin{array}{ll} & \underset{C_L,C_R}{Min} \sum_{j=0}^{1} \sum_{i=1}^{n} (C_{Lj} + C_{Rj}) \, |X_{ij}| \\ & -\alpha^{*t} X^P + \sum_{i=1}^{p} \sum_{j=0}^{1} X_{ij} C_{Lj} \geq r_{di} \ , \quad for \ i = 1, ..., p \\ & \alpha^{*t} X^P + \sum_{i=1}^{p} \sum_{j=0}^{1} X_{ij} C_{Rj} \geq r_{ui} \ , \quad for \ i = 1, ..., p \\ \text{Subject to} & -\alpha^{*t} X^N + \sum_{i=p+1}^{n} \sum_{j=0}^{1} X_{ij} C_{Rj} \geq r_{di}, \quad for \ i = p+1, ..., n \\ & \alpha^{*t} X^N + \sum_{i=p+1}^{n} \sum_{j=0}^{1} X_{ij} C_{Lj} \geq r_{ui}, \quad for \ i = p+1, ..., n \\ & C_R = (C_{R0}, C_{R1}), \ C_L = (C_{L0}, C_{L1}) \geq 0 \\ & Given \ \ X^P = (X_{1j}, ..X_{pj}) \geq 0, \ X^N = (X_{p+1j}, ..X_{nj}) < 0 \ \ X_{i0} = 1 \end{array} \right. \tag{5.10}$$

where r_d, r_u represent the down and up returns, respectively.

Evidently, it may be noted that the price limits generate a movement interval with a lower bound and an upper bound, which correspond to the lower and upper movements of price, respectively. r_t is the observed commodity return at time t based on the settlement price. It can be seen as pseudo true return with a membership function equal to 1. $\widetilde{r}$ is the fuzzy equilibrium return at time t. Assuming that the given settlement price is not defined sharply, l_t represents the price limits at time t of each commodity futures.

The equation (5.6) under a fuzzy environment generates two returns, up-return and down-return, which represent returns derived from the up movement and the down movement of the commodity prices. We define the up movement (down movement) of the price as the settle price plus (minus) the tolerance level $(l_t/2)$. Note here that l_t represents the limit of the price.

Thus, assuming that the price limit is constant over time, we can reproduce the equation (5.6) as follows:

$$r_{uit} = Ln\left(\frac{P_{it} + l/2}{P_{it-1} + l/2}\right) \simeq \frac{P_{it} - P_{it-1}}{P_{it-1} + l/2} \tag{5.11}$$

and

$$r_{dit} = Ln\left(\frac{P_{it} - l/2}{P_{it-1} - l/2}\right) \simeq \frac{P_{it} - P_{it-1}}{P_{it-1} - l/2} \tag{5.12}$$

The above equations can be rewritten as

$$r_{uit} = \frac{P_{it} - P_{it-1}}{P_{it-1} + l/2} = \frac{P_{it} - P_{it-1}}{P_{it-1}} \frac{P_{it-1}}{P_{it-1} + l/2} = r_{it} * R^{u}_{t-1} \tag{5.13}$$

$$r_{dit} = \frac{P_{it} - P_{it-1}}{P_{it-1} - l/2} = \frac{P_{it} - P_{it-1}}{P_{it-1}} \frac{P_{it-1}}{P_{it-1} - l/2} = r_{it} * R^{d}_{t-1} \tag{5.14}$$

where R^{d}_{t-1}, R^{u}_{t-1} represent the ratios or the multipliers associated with an up and down movement of the price, respectively. Economically, we can conceptualize them as the price limit's movement effect on the returns. Also, it measures the magnitude of the price limit on returns. Similarly, each one is a leftover ratio that represents the unrealized residual shock from trading at time t.

This problem is amenable to comparative static analysis, and the derivation of the equations (5.13) and (5.14)with respect to the price limit can be written into two terms:

$$\frac{dr_{uit}}{dl} = \frac{d}{dl}\left(\frac{P_{it} - P_{it-1}}{P_{it-1}} \frac{P_{it-1}}{P_{it-1} + l/2}\right) = -\frac{1}{2}\frac{P_{it} - P_{it-1}}{P_{it-1}} \frac{P_{it-1}}{(P_{it-1} + l/2)^2} \tag{5.15}$$

and

$$\frac{dr_{dit}}{dl} = \frac{1}{2}\frac{P_{it} - P_{it-1}}{P_{it-1}}\frac{P_{it-1}}{(P_{it-1} + l/2)^2} \tag{5.16}$$

Then

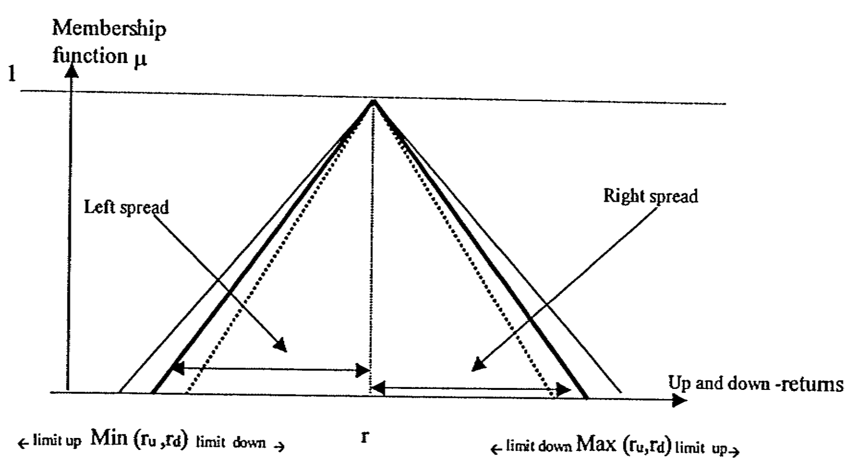
$$If \;\; P_{it} - P_{it-1} \geq 0 \rightarrow \frac{dr_{uit}}{dl} \leq 0 \;\; and \;\; \frac{dr_{dit}}{dl} \geq 0 \tag{5.17}$$

From the equation (5.17), when the price of the commodity is moving upward, assuming that it is moving within the limits, the up-return (down-return) will decrease (increase) with any possible increase in the price limits.

$$If \; P_{it} - P_{it-1} \leq 0 \rightarrow \frac{dr_{uit}}{dl} \geq 0 \;\; and \;\; \frac{dr_{dit}}{dl} \leq 0 \tag{5.18}$$

From the equation (5.18) when the price of the commodity is moving downward, the up-return (down-return) will increase (decrease) with any possible increase in the price limits. The following figure (5.2) illustrates the construction of the up-return and down-return taking into account all possible cases.

Figure 5.2: Representation of fuzzy numbers



It is known that in the Winnipeg Commodity Exchange, before October 10, 2000, the regular daily price limits were \$5.00/tonne for feed wheat and western barley and \$10.00/tonne for canola and flaxseed. These limits could be expanded (increased) in certain situations. If any two of the nearest three contract months closed limit up or limit down for two successive days, the limit was expanded to 1.5 times its normal effective price the following day. If any two of the nearest three contract months closed up or down by the expanded limit for the next two days, the limit was further expanded to twice the normal limit on the following day. When no two of the nearest three contracts closed at their expanded daily limits in the same direction (both up or both down), the daily price limit returned to the normal limit on the following day.

5.3 Data and Methodology

Market return in this chapter has been computed using an approach similar to the one suggested by CRS [19]. However, Marcus [102] suggested a weight of 10% for the commodity index and 60% for the S&P 500 index. The S&P 500 index is a value weighted index of the price of 400 industrial, 40 utility, 40 financial and 20 transportation stocks. The Dow Jones cash commodity futures index[5] is an equal weighted index of five-month-forward futures prices for the 12 commodities: cattle, coffee, copper, corn, cotton, gold, hogs, lumber, silver, soybeans, sugar, and wheat.

Marcus [102] pointed out that when the weight given to commodities in the market portfolio increases, the covariance between r_i and r_m will increase and β, which measures

[5]The index is derived by dividing the price of each commodity on a given date by its price on Dec. 21, 1974, and summing across commodities. The sum is divided by 12 and multiplied by 100 to yield the index.

the systematic risk, will increase. This weighting scheme is based on Marcus's estimate that commodities account for approximately 10% of total wealth. Also, it is approximately the same as the 0.06 weight for commodities used by BCT [9].[6]

Following Marcus [102], Elam and Vaught [39], and Chang et al. [23] we proceed by using the combination of 10% to the monthly log relative return for the TSE 300 index return[7] and 90% of the Dow–Jones cash commodity index return as a proxy for the market portfolio. Then, the above approach is illustrated empirically for four agricultural futures contracts (western barley, canola, flaxseed, and feed wheat) traded at the Winnipeg Commodity Exchange.

The log relative returns for these contracts were divided into six or five groups, based on the time to maturity of the futures contract. Systematic risk was first estimated over the period Januray 1991 to December 2000 for the futures contracts in each of the four groups using ordinary least squares regression for the model (5.5).

5.3.1 Data

Monthly prices for four major traded commodities in the Winnipeg Commodity Exchange (WCE) were obtained. The study period is from January 1991 to December 2000. Table (5.2) summarizes the most important components of the financial data. The risk premium on the portfolio of all assets is estimated by subtracting the risk-free return, proxied by the one-month T-Bill rate, from the rate of return on the market portfolio.[8]

[6]To decide which weighting scheme to use, normally, it is appropriate to look at the porportion of the commodities of the total wealth, which is believed to be 10% or less in the American market. In this paper, we offer 10% to the TSE 300 index.

[7]Both the T-bill rate and TSE 300 Index price were obtained from the Canadian Financial Markets Research Center database (CFRC).

[8] "T-Bill return represents the return on a 91 day T-Bill purchased at the end of last month and sold at the end of this month. If r(t) is the yield (in percent) at the end of month t, then the price last month, P(t-1) of a T-bill with 91 days to

Table 5.2: Means, standard deviations of monthly rates of return and the Sharpe performance and Treynor measures for the four contracts. Sample period Jan. 1991 to Dec. 2000. (Ten percent weight was given to non-commodities in the market portfolio).

Commodity	Mean (%)	SD[a] (%)	S_i[b] (%)	T_i[c] (%)
Western Barley:				
May	0.315	4.608	-3.140	-0.532
August/July	0.359	5.738	-1.755	-0.336
November/October	0.465	5.506	0.096	0.032
November/December	0.454	5.450	-0.105	-0.023
February/March	0.356	5.097	-2.035	-0.487
Canola:				
June/May	0.109	5.745	-6.104	-0.948
June/July	0.138	6.032	-5.318	-0.788
September	0.0357	4.384	-9.672	-1.101
November	0.045	4.487	-9.242	-1.338
January	0.066	5.101	-7.724	-0.980
March	0.069	5.048	-7.742	-2.641
Flaxseed:				
May	0.135	5.450	-5.958	-1.328
July	0.132	5.868	-5.585	1.707
October/September	-0.05160	4.521	-11.309	-4.824
October/November	0.064	4.784	-8.267	-11.667
December/January	0.123	4.778	-7.047	3.432
March	0.127	4.993	-6.663	-2.200
Feed Wheat:				
May	0.465	6.091	0.087	0.013
July	0.415	5.369	-0.825	-0.139
October	0.346	5.994	-1.890	-0.750
December	0.459	5.405	-0.031	-0.006
March	0.422	5.264	-0.718	-0.125

(SD denotes the standard deviation of monthly returns. S_i denotes the Sharpe performance measure which is equal to $\frac{r_i - r_f}{\sigma_i}$, σ_i : denotes the standard deviation of contract return, and $r_i - r_f$: return of bearing risk which is equal to the average contract return minus the average risk-free rate. T_i denotes the Treynor measure which is equal to $\frac{r_i - r_f}{\beta_i}$.)

Sharpe and Treynor measures have been presented in table (5.2). More broadly, the results reveal a visible relationship between maturity and mean and standard deviation of contract returns. Standard deviation for flaxseed and feed wheat declines with the

maturity is $P(t-1) = \frac{100}{[100+r(t)*\frac{91}{365}]}$ and the price today of that same bill with today's yield and only 61 days to maturity is$P(t) = \frac{100}{[100+r(t)*\frac{61}{365}]}$. The return, R(t) is $R(t) = \frac{[P(t)-P(t-1)]}{P(t-1)}$" Excerpt from Canadian Financial Markets Research Center (CFMRC) User's Manual p.11

contract maturity, which is consistent with the assumption of Samuelson [133] concerning the variation of spot and forward prices. Also, it is apparent that there is a noticeable relationship between high volatility and high return. Thus, investors perceive a favorable risk-return trade-off. Over the sample period, feed wheat offered the highest mean return, and canola offered the lowest. However, on average, all futures contracts have approximately an equal volatility. Negative sharp performance measure and Treynor measure indicate that all futures have the lowest return-to-volatility ratio except barley (Nov./Oct. contract) and feed wheat (May contract). Based on that, we can say that these risks have not been well compensated by the market.

5.3.2 Regression methods and results

The results of the estimation method using classical regression and proposed fuzzy regression are reported in tables (5.3), (5.4) and (5.5). Because some of the delivery months are replaced or canceled, we have created two series by combining each canceled month with the closest replaced month to have constant series overtime. For example, the June canola contract has been replaced by two series, May and July, after 1996. So, we have created two series, June data until 1996, then we have continued the data with July data, and we have performed the same process to construct the June/May contract.

A. *Classical regression results: First phase*

Systematic risk is estimated for Jan. 1991 to Dec. 2000 for the futures contracts in each of the six or five groups using ordinary least squares regression. In table (5.3), we report the OLS estimates for regression model (5.5).

Table 5.3: Classical regression parameters for western barley, canola, flaxseed, feed wheat. Sample period Jan. 1991 to Dec. 2000. (Ten percent weight was given to non-commodities in the market portfolio.)

Commodity	α_i	$S(\alpha_i)$	β_i	$S(\beta_i)$	t_β	R^2	DW	SSE
Western Barley:								
May	0.00375	0.00416	0.2722^c	0.13677	1.9905	0.0325	1.673^f	0.2444
August/July	0.00424	0.0052	0.299^b	0.1710	1.751	0.0253	1.558^g	0.3819
November/October	0.00501	0.0050	0.165	0.1654	1.00	0.0084	2.017^e	0.3577
November/December	0.00509	0.0049	0.253	0.1628	1.558	0.020	2.134^e	0.3463
February/March	0.00402	0.0046	0.213	0.1525	1.401	0.0164	1.721^e	0.3041
Average			0.2404			0.0205		
Canola:								
June/May	0.00189	0.0051	0.3702^c	0.1700	2.178	0.0387	2.014^e	0.3776
June/July	0.0022	0.0054	0.407^c	0.1781	2.285	0.0424	1.974^e	0.4146
September	0.00119	0.0038	0.385^a	0.127	3.024	0.0719	2.092^e	0.2123
November	0.00112	0.0043	0.3108^a	0.1323	2.349	0.044	2.155^e	0.2289
January	0.0015	0.0045	0.402^a	0.1494	2.691	0.0578	1.928^e	0.2917
March	0.00101	0.0046	0.148	0.1517	0.977	0.0080	2.102^e	0.3008
Average			0.3371			0.0438		
Flaxseed:								
May	0.00188	0.0049	0.2445	0.1629	1.501	0.0187	1.984^e	0.3468
July	0.00090	0.0053	-0.192	0.1762	-1.094	0.010	1.880^e	0.4057
October/September	-0.0002	0.0041	0.1069	0.1361	0.785	0.0052	2.031^e	0.2420
October/November	0.00071	0.0043	0.0339	0.1443	0.235	0.0005	1.949^e	0.2722
December/January	0.00102	0.0043	-0.0981	0.1439	-0.682	0.0039	1.970^e	0.2706
March	0.00160	0.0045	0.1512	0.150	1.008	0.0085	2.113^e	0.2941
Average			0.0410			0.0078		
Feed Wheat:								
May	0.00553	0.0054	0.4041^c	0.180	2.245	0.041	2.010^e	0.4234
July	0.00484	0.0048	0.319^c	0.1593	2.006	0.0330	1.680^f	0.3318
October	0.00379	0.0054	0.1511	0.1803	0.837	0.0059	1.939^e	0.4250
December	0.00525	0.0048	0.308^b	0.1606	1.919	0.0303	1.747^e	0.3372
March	0.00487	0.0047	0.303^b	0.1564	1.942	0.0310	1.673^e	0.3195
Average			0.2970			0.0282		
Overall Average			0.2288			0.0250		

($S(\widehat{\alpha}_i)$ and $S(\widehat{\beta}_i)$ denote standard errors of the estimated coefficients. a: denotes statistical significance at the 1% level. b: denotes statistical significance at the 5% level. c: denotes statistical significance at the 2.5% level. e and f: denote that Durbin-Watson (DW) statistics do not reject the hypothesis of random residuals of the regressions at the 0.05 and 0.01 levels respectively. g: denotes that the DW test is inconclusive at the 0.01 level.)

The estimated parameters and the corresponding standard errors are reported in table (5.3). In general, the relationship between maturity and estimated beta (esβ) is negative, which means a lower maturity is associated with lower systematic risk. Secondly, the study reveals that a majority of betas are positive. The average betas for western barley, canola, flaxseed and feed wheat are $0.2404, 0.3371, 0.0410$ and 0.2970 respectively. Thirdly, a sufficient number of betas (45% of the estimates) are statistically significant at least at the 5% level. In addition, it is noted that all significant betas are positive. So, all three-commodity futures (barley, canola, and wheat) have a positive systematic risk. Thus, investors will refer to them as risky financial assets, and they will require a risk premium to compensate for the level of risk they are bearing.

Pointing out the magnitude of the parameter α, the table (5.3) shows that all $\alpha's$ are statistically equal to zero. The insignificance of the estimated α (esα) serves as evidence of the non-existence of excess return. This result is consistent with the findings of the previous studies (e.g. [35], [9], [145], [39], and [23]).

The findings of the classical regression, contrary to several previous studies in agricultural and livestock commodity futures, support the existence of a significant systematic risk with an overall average of 0.2288 for the four commodity futures under investigation.

The coefficient of determination (R–square), which provides a measure of goodness fit of the estimated regression equation to the data, is very small (overall average is 0.0250) as previously observed in many studies (e.g. [35], [19], [9], [39], [23]). That simply means that the least square line does not provide a better fit to the data, and the observations are not more closely grouped about the least square line. Note also, that the degree of systematic risk is not constant across contracts for any of the commodity futures shown in table (5.3) and table(5.4). In the case of canola, the highest estimate is 0.407 for the June/July contract and the lowest is 0.148 for the March contract.

Table 5.4: Systematic risk (beta) for barley, canola, flaxseed, feed wheat using different weighting schemes for non-commodities in the market portfolio, Jan. 1991 to Dec. 2000.

Commodity	Weight given to TSE 300 in the Market Portfolio					
	0.00	0.10	0.20	0.30	0.40	0.50
Western Barley:						
May	0.273[c]	0.272[c]	0.256	0.226	0.182	0.134
August/July	0.314	0.299[b]	0.265	0.212	0.148	0.082
November/October	0.157	0.165	0.166	0.158	0.141	0.120
November/December	0.240	0.253	0.256	0.246	0.223	0.191
February/March	0.229	0.213	0.183	0.139	0.087	0.035
Canola:						
June/May	0.374[c]	0.370[c]	0.346[b]	0.300	0.239	0.171
June/July	0.417[c]	0.407[c]	0.372[b]	0.314	0.238	0.157
September	0.380[a]	0.385[a]	0.371[a]	0.336[c]	0.283[b]	0.220
November	0.306[c]	0.310[c]	0.300[b]	0.272	0.229	0.179
January	0.382[a]	0.402[a]	0.404[c]	0.385[c]	0.346[b]	0.293
March	0.153	0.148	0.134	0.111	0.083	0.052
Flaxseed:						
May	0.2446	0.2445	0.231	0.204	0.167	0.124
July	-0.162	-0.192	-0.220	-0.238	-0.245	-0.238
October/September	0.106	0.106	0.101	0.090	0.0738	0.055
October/November	0.0601	0.0339	0.0003	-0.037	-0.073	-0.102
December/January	-0.0711	-0.098	-0.125	-0.150	-0.167	-0.174
March	0.158	0.151	0.134	0.108	0.076	0.043
Feed Wheat:						
May	0.357[b]	0.404[c]	0.440[c]	0.457[c]	0.450[c]	0.421[c]
July	0.307[b]	0.319[c]	0.317	0.298	0.263	0.217
October	0.149	0.151	0.145	0.131	0.110	0.085
December	0.293[b]	0.308[b]	0.310	0.295	0.265	0.224
March	0.284[b]	0.303[b]	0.310	0.302	0.277	0.241

(a denotes statistical significance at the 1% level. b denotes statistical significance at the 5% level. c denotes statistical significance at the 2.5% level.)

To validate whether the weight given to commodities in the market portfolio has an impact on the estimated parameters, different weighting schemes have been used. As Marcus [102] pointed out, as the weight given to commodities in the market portfolio increases, the covariance between return and market return will increase and the estimate of beta will increase. Table (5.4) supports that conjecture for the sample period. It shows that beta decreases as the weight given to commodities decreases, which is consistent with Marcus and

EV arguments. Note here that the American Dow–Jones cash commodity index (DJCCI) has been used in the absence of a Canadian commodity index. The results show that the Canadian commodity market (e.g. WCE) is affected significantly by the American commodity market.

Canola contracts continue to have statistically significant betas although commodities are weighted 40% in the market portfolio. This implies that commodity futures contracts bear a systematic risk that depends to some extent on the proxy index employed. Most notably, based on the table (5.4), all futures contracts except flaxseed were riskier. Results do not show any significance for the futures on flaxseed, which means that the risk premium does not exist statistically.

B. *Fuzzy regression results: Second phase*

In table (5.5), we report the estimates for the fuzzy regression model (5.5) using the phase 2 illustrated in the previous section. Applying the model 5.10, the estimates for our entire sample, covering Jan. 1991 to Dec. 2000 (10-year period) are shown in table (5.5).

Table 5.5: Fuzzy regression parameters for western barley, canola, flaxseed, feed wheat. Sample period: Jan. 1991-Dec. 2000. (Ten percent weight was given to non-commodities in the market portfolio.)

Commodities		Parameters		
	Delivery Months	(c_{0L}, esα, c_{0R})	(c_{1L}, esβ, c_{1R})	O.F.
Western Barley	May	(0.1571,0.00375,0.1403)	(0,0.2722,0)	35.688
	August/July	(0.2714,0.00424,0.235)	(0,0.299,0)	60.768
	November/October	(0.1425,0.00501,0.2324)	(0.5152,0.165,0)	46.41
	November/December	(0.1425,0.00509,0.2315)	(0.6030,0.253,0)	46.549
	February/March	(0.2750,0.00402,0.1308)	(0,0.213,0)	48.696
Average			(0.2236,0.2404,0)	
Canola	June/May	(0.3302,0.00189,0.1813)	(0,0.3702,0)	61.38
	June/July	(0.3308,0.0022,0.1806)	(0,0.407,0)	61.368
	September	(0.1051,0.00119,0.1396)	(0,0.385,0)	29.364
	November	(0.095,0.00112,0.1081)	(0,0.3108,0)	24.372
	January	(0.1302,0.0015,0.1296)	(0,0.402,1.0816)	34.174
	March	(0.1497,0.00101,0.1353)	(0,0.148,0)	34.2
Average			(0,0.3371,0.1802)	
Flaxseed	May	(0.146,0.00188,0.1159)	(0,0.2445,2.1265)	37.327
	July	(0.2594,0.0009,0.0814)	(0,-0.192,1.7455)	45.733
	October/September	(0.1241,-0.0002,0.0832)	(0.8547,0.1069,1.259)	30.735
	October/November	(0.1373,0.00071,0.1146)	(0,0.0339,0.4174)	31.384
	December/January	(0.1323,0.00102,0.1827)	(0,-0.0981,0)	37.8
	March	(0.1089,0.0016,0.0781)	(1.6683,0.1512,1.7483)	31.912
Average			(0.4205,0.0410,1.2161)	
Feed Wheat	May	(0.1907,0.00553,0.316)	(0,0.4041,0)	60.804
	July	(0.1684,0.00484,0.1798)	(0.1742,0.319,0)	42.263
	October	(0.2124,0.00379,0.1717)	(0,0.1511,0)	46.092
	December	(0.1861,0.00525,0.1879)	(0,0.308,0)	44.88
	March	(0.1741,0.00487,0.1744)	(0,0.303,0)	41.82
Average			(0,0.2970,0)	
Overall Average			(0.0348,0.2288,0)	

The data used in the fuzzy regression, with a threshold value for identifying the model, were chosen to be $h = 0$. Following the two-step procedure to estimate the fuzzy parameters (α, β), based on the result provided in table 5, we find that a majority of beta estimates have zero spreads. Hence, the parameters have crisp values. Also, the result indicates the impact of price limits on estimating systematic risk of commodity futures. The result shows, as in the case of canola and flaxseed, that a higher limit corresponds to a higher spread. Additionally, for the four commodity futures, the estimates parameters α's

($es\alpha$) are very small with small spreads. Table 5 demonstrates noticeable patterns between contract maturity and spreads for the four commodities except barley.

The estimated beta with the corresponding objective function -total spreads- are reported in table (5.5). Three observations on the estimated betas merit elaboration. Firstly, the fuzzy systematic risk estimates are all positive with an overall average equal to $(0.0348, 0.2288, 0)$, which means that the fuzzy beta is equal to 0.194 with a membership function equal to 0, and it is equal to 0.2288 with a membership equal to 1. So, it is obvious that beta is always different from zero.

The average betas for canola, feed wheat, barley and flaxseed are $(0, 0.148, 0)$; $(0.0348, 0.2288, 0)$; $(0.2236, 0.2404, 0)$; $(1.66883, 0.1512, 1.7483)$ respectively. Secondly, the majority of the estimates are fuzzily significant (fuzzily acceptable). For example, flaxseed estimates, which have previously been reported to be statistically insignificant, appear to be acceptable under the two-phase fuzzy regression method. Some coefficients have a zero left spread, which means that the estimated fuzzy parameter cannot be below the estimated beta ($es\beta$). Thirdly, the result provides a clear distribution of the spreads in the presence of price limits. Thus, the proposed method offers decision makers or investors the magnitude of the systematic risk in the existence of price limits.

Subsequently, with the help of the two-phase fuzzy approach in the presence of price limits, we have been successful in presenting the movement interval of the systematic risk associated with the membership function, which measures the degree of truth or the degree of precision.

5.4 Conclusion

As Black [12] argued, a major benefit of futures markets is that participants can make production, storage, and process decisions by examining the patterns of futures prices and the risk associated with them. The systematic risk and return in western barley, canola, flaxseed, and feed wheat have been measured after extending the arguments of CRS [19] and Marcus [102]. CRS pointed out that a more appropriate "efficient portfolio" return variable in equation (5.5) would be an index composed of the S&P index of 500 common stocks and the Dow–Jones commodity futures index. They advised that alternative indexes could be proposed. Marcus's suggestion is to construct a reasonable weight for the commodity index in the market index. Consequently, we have constructed a portfolio index composed of 0.90 of DJCCI and 0.10 of TSE 300.

The purpose of the chapter is to estimate the systematic risk of Canadian commodity futures investment. The capital asset pricing model has been the essential component of our analysis. Additionally, CAPM has been structured to be estimated from two complementary phases. Firstly, with the use of classical regression analysis, we have estimated the parameters of the linear model. The result of that regression will serve as the first step of the proposed two-step fuzzy regression method. Secondly, knowing that most futures contracts have a daily price limit specified by the exchange, and the movement of the price is said to have a limit up and limit down, we have constructed fuzzy non-symmetrical data. Then, with the help of the second step, the spreads of the beta estimates have been provided.

Explanation of risk and return for Canadian commodity futures is provided by the use of CAPM. Using a market portfolio based on a weighting of 0.9 for DJCCI and 0.1 for the TSE 300 index, canola, barley, and feed wheat can be considered as low-risk assets, as shown by the results. Thus, a significant portion of the risk associated with holding canola, wheat, and barley cannot be diversified away.

Based on the classical regression, results show that three out of four commodity futures are riskier. Therefore, the usual belief that traders of commodity futures bear above-average risk is supported by the Canadian data. Additionally, a risk premium has been identified for each commodity contract except flaxseed.

The findings of this study, contrary to several previous studies for agricultural and livestock commodity futures, support the existence of a significant systematic risk with an overall average of 0.2288 for the four commodity futures under investigation.

The result of table (5.5) is similar to the result of table (5.3). However, the parameters with their estimated spreads, which were given by the fuzzy regression method, offered a persuasive result with respect to the estimated parameters.

One direction for future research is the examination of the relationship between systematic risk and the size of the firm [30] using a fuzzy regression approach. Further work is needed to examine the fuzzy hypotheses testing by establishing a fuzzy acceptance region (optimal, according to the approach used) [31]. More broadly, we feel that there are more opportunities for fuzzy regression method applications in financial modeling. This method provides an effective way to cope with the uncertainties that are inherent in the financial models.

Chapter 6

Fuzzy Hypothesis and Testing for Significance

6.1 Fuzzy Hypothesis Background

Fuzzy hypothesis has been introduced often in the literature; for example [20], [21], and [128]. Yuan [168] discussed parameter estimation of normal fuzzy parameters in cases when one of the parameters is unknown and when both are unknown. Caslas [20] presented an extension of the problem of testing parameter hypotheses when the information and the hypotheses are fuzzy. By extending the Bayes optimality criterion, Caslas was able to perform that extension. Watanabe and Imaizumi [165] proposed a testing method of a fuzzy hypothesis for random data. Specifically, they examined the case when two population means are *nearly* equal or not. Their method, called the fuzzy statistical test, generates a fuzzy conclusion from the test. Romer and Kandel [128] investigated the impacts of imprecise data on the statistical task of hypothesis testing. They also considered the issue of defuzzification as a way of getting numerical values for the test result. In fact, some researchers in the area are not in favor of this procedure of defuzzifying when the decision maker may be able to understand and make a better judgment based on the result in its original form.

Let $X_1, X_2, ..., X_n$ be a fuzzy sample for a population distribution P_β, where that distribution depends on a parameter β. In the previous chapter the estimation was investigated. But, the actual statistical testing of the validity about the parameter β is presented in the next section to check whether the fuzzy parameter is significant or not. The question here is when to accept or reject a hypothesis about the parameter β in a fuzzy environment (fuzzy data). In test theory, significance testing ($\alpha-$test) is equivalent to making a decision about stochastic quantity ξ that belongs to a class of distribution H_0. So, we suggest that one way to develop such tests is given by the extension principle (e.g. Kruse and Meyer [84], Watanabe and Imaizumi [165] and Viertl [160]).

Definition 18 (Viertl [160]). *A crisp statistic $t(\xi_1, ..., \xi_n)$ is extended to a fuzzy statistic $T(X_1, ..., X_n)$ by $\mu_T(X_1, X_2, ..., X_n)(\beta) = Sup\left\{\min(\mu_{X_1}(\xi_1), ..., \mu_{X_n}(\xi_n)/\ t(X_1, ..., X_n) = \beta\right\}$. Thus, fuzzy tests are obtained. Here, a crisp test for a fuzzy quantity X will be derived. For point estimator $\beta_n = \widehat{\beta}_n(X_1, ..., X_n)$ For the unknown parameter β, we are inclined to reject the hypothesis $H_0 : \beta = \beta_0$ against $H_a : \beta \neq \beta_o$ If the distance $h(\widehat{\beta}_n, \beta_0)$ between the estimator $\widehat{\beta}_n$ and β_0 is too large. The hypothesis H_0 is rejected if $h(\widehat{\beta}_n, \beta_0) > t_{1-\alpha}$, where $t_{1-\alpha}$ is the $(1-\alpha)$ quantile of the distribution of $h(\widehat{\beta}_n, \beta_0)$ and α is the probability of error.*

In our case, where LR-fuzzy parameter is considered, we want to test the following hypothesis:

$$H_0 : \beta = \mu_0 = 0 \text{ against } H_a : \beta \neq \mu_0$$

H_0 is rejected if T_n is larger than the $(1-\alpha)$ quantile of the limit distribution T_n. Usually the distribution of the $\widehat{\beta}_n$ is unknown, but a limit distribution is known when n goes to infinity or very large [see, Krätschmer limit theorem for fuzzy random variables [82]]. Therefore, an asymptotical test be may be obtained. In this chapter, following Nather [109], we use the t-distribution.

Theorem 19 [84]. *Let* $n \in \mathbb{N}$, $\delta \in (0,1)$, *and* $N \in \mathbb{N}$. *Let* $\{\alpha_1, ..., \alpha_N\} \subseteq [0,1)$, $K \in \{1, ..., N\}$ *and* $\mu_0 \in U(\mathbb{R})$. *Let* $[T_n, +\infty)$ *and* $(-\infty, U_n)$ *be two one-sided 100*(*$1-\delta_1$*)% and* $100 * (1-\delta_2)\%$ *(usual) confidence intervals for* y *and* Γ_y *with* $\delta_1 + \delta_2 = \delta\frac{K}{N}$ *and* $T_n \leq U_n$ *Define for* $(\mu_1, ..., \mu_n) \in [F(\mathbb{R})]^n$ *and* $\alpha \in [0,1)$ $A_\alpha[\mu_1, ..., \mu_n]$ *and* $B_\alpha[\mu_1, ..., \mu_n]$.

Define for $(\mu_1, ..., \mu_n) \in [F(\mathbb{R})]^n$.

$$\phi_i(\mu_1, ..., \mu_n) \stackrel{d}{=} \begin{cases} 1 & \text{if } inf(\mu_0)_{\alpha_i} < A_{\alpha_i}[\mu_1, ..., \mu_n] \\ & \quad \text{or } Sup(\mu_0)_{\alpha_i} > B_{\alpha_i}[\mu_1, ..., \mu_n] \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

for $i \in \{1, ..., N\}$, *and*

$$\phi(\mu_1, ..., \mu_n) \stackrel{d}{=} \begin{cases} 1 & \text{if } \sum \phi_i(\mu_1, ..., \mu_n) \geq K \\ & \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

Then $\phi : [F(\mathbb{R})]^n \to \{0,1\}$ *is a test for*

H_0 : *"The convex hull of the fuzzy perception of* y *and* Γ_y *is equal to* μ_0*" against*

H_1 : *"It is not equal to* μ_0*" on the significance level* δ.

Theorem 20 [84]

Let $n \in \mathbb{N}$, $\delta \in (0,1)$, *and* $N \in \mathbb{N}$. *Let* $\{\alpha_1, ..., \alpha_N\} \subseteq [0,1), K \in \{1, ..., N\}$, *and* $\mu_0 \in U(\mathbb{R})$.

(i) *If* $(-\infty, U_n)$ *is a (usual) one-sided 100*(1-*$\delta\frac{K}{N}$*)% confidence interval for* U *and* Γ_y, *let* $B_\alpha[\mu_1, ..., \mu_n]$ *be defined as in [84] (theorem 11.10, p.225) for* $(\mu_1, ..., \mu_n) \in [F(\mathbb{R})]^n$ *and* $\alpha \in [0,1)$.

Define for $(\mu_1, ..., \mu_n) \in [F(\mathbb{R})]^n$. $\phi_i(\mu_1, ..., \mu_n) \stackrel{d}{=} \begin{cases} 1 & \text{if } Sup(\mu_0)_{\alpha_i} > B_{\alpha_i}[\mu_1, ..., \mu_n] \\ 0 & \text{otherwise} \end{cases}$

for $i \in \{1, .., N\}$ *and*

$$\phi(\mu_1, ..., \mu_n) \stackrel{d}{=} \begin{cases} 1 & \text{if } \sum \phi_i(\mu_1, ..., \mu_n) \geq K \\ 0 & \text{otherwise} \end{cases}$$

Then $\phi : [F(\mathbb{R})]^n \to \{0,1\}$ *is a test for*

H_0 : *"The convex hull of the fuzzy perception of* y *and* Γ_y *is greater or equal to* μ_0*"*

against

H_1 : *"It is less than* μ_0*" on the significance level* δ.

(ii) *If* $[T_n, +\infty)$ *is a (usual) one-sided 100*(1-*$\delta\frac{K}{N}$*)% confidence interval for* y *and* Γ_y, *let* $A_\alpha[\mu_1, ..., \mu_n]$ *be defined as in theorem 11.10 for* $(\mu_1, ..., \mu_n) \in [F(\mathbb{R})]^n$:

$$\phi_i(\mu_1, ..., \mu_n) \stackrel{d}{=} \begin{cases} 1 & \text{if } inf(\mu_0)_{\alpha_i} < A_{\alpha_i}[\mu_1, ..., \mu_n] \\ 0 & \text{Otherwise} \end{cases}$$

$$\text{for } i \in \{1,...,N\}, \text{ and } \phi(\mu_1,...,\mu_n) \stackrel{d}{=} \begin{cases} 1 & if \ \sum \phi_i(\mu_1,...,\mu_n) \geq K \\ 0 & Otherwise \end{cases}$$

Then $\phi : [F(\mathbb{R})]^n \to \{0,1\}$ *is a test for*

H_0 : *"The convex hull of the fuzzy perception of* U *and* Γ_y *is less or equal to* μ_0*"*

against

H_1 : *"It is greater than* μ_0*" on the significance level* δ.

6.2 Potential Testing for Significance: Testing a Hypothesis about a Coefficient

6.2.1 Stating the problem

The statistical problem explored most thoroughly is that of hypothesis testing. As the term suggests, one decides whether or not the hypothesis is correct. The choice lies between two decisions: accepting or rejecting the hypothesis. A decision procedure for such a problem is called a test of the hypothesis.

Statistically (also see, [90]) the choice of a level of significance α will usually be somewhat arbitrary, as in most situations, there is no precise limit to the probability of an error of the first kind that can be tolerated. It has become customary to choose for α standard value such as .0005, .01, or .05. Such standardization is convenient, as it reduces certain tables needed for testing. In fact, when choosing a level of significance, one should also weigh the power of the test against various alternatives. If the power is low, one may use much higher values for α than the customary values. The use of α in relation to the power of a test is suggested by Lehman [89]. A low significance level results in the hypothesis being rejected only for a set of values of the observations whose total probability under the hypothesis is small, so that such values would be most unlikely to occur if H were true.

To test for a significant regression relationship, we must conduct a hypothesis test to determine whether the value of β_1 is zero. In classical regression, two tests are commonly used: the t-test and F-test. Both require an estimate of σ^2, the variance of ε in the regression model. To test a hypothesis is to perform an experiment considering this hypothesis; based on the outcome of that experiment we decide whether the hypothesis can be correct. The essential ingredients to establish such fuzzy significance testing are: fuzzy space which is identified by the existence of number of fuzzy parameters (result of the experiment); action space which means whether to accept the null hypothesis (e.g. $\beta = \beta_0 = 0$) or to reject the null of hypothesis and accept the alternative hypothesis ($\beta \neq \beta_0$).

6.2.2 Interval estimation

Regardless of the properties of an estimator, the estimate obtained will vary from sample to sample, and there is some probability that it will be quite erroneous. A point estimate will not provide any information on the likely range of error. The logic behind an interval estimate is that we use the sample data to construct an interval [Lower X, Upper X], such that we can expect this interval to contain the true parameter in some specified proportions of samples, or equivalently, with some desired level of confidence. Clearly, the wider the interval, the more confident we can be that it will, in any given sample, contain the parameter being estimated.

6.2.3 Process of rejecting and accepting

The formal "usual" procedure of hypothesis testing involves a statement of the hypothesis, usually in terms of a "null" or maintained hypothesis and an "alternative", conventionally denoted H_0 and H_1, respectively. The procedure itself is a rule, stated in terms of the data, that dictates whether the null hypothesis should be rejected or not.

For example, the hypothesis might state a parameter is equal to a specified value. But, the decision rule might state that the hypothesis should be rejected if a sample estimate of that parameter is too far from that value (where "far" remains to de defined). The classical, or Neyman-Pearson, methodology involves partitioning the sample space into two regions. If the observed data (i.e., the test statistic) fall in the rejection region (sometimes called the critical region), then the null hypothesis is rejected; if the observed data falls in the acceptance region, then it is not rejected.

6.2.4 Assumptions for testing significance

Assumptions of the classical linear regression model as pointed out by Greene [49] are:

A1. Linear functional forms the relationship $y = \beta X + \varepsilon$

A2. Identifiability of the model parameters X is an $n \times K$ matrix with rank K (identification condition) (the columns of X are linearly independent, and there are at least K observations).

A3. Expected value of the disturbance given observed information.

$$E[\varepsilon/X] = \begin{bmatrix} E[\varepsilon_1/X] \\ \cdot \\ \cdot \\ E[\varepsilon_n/X] \end{bmatrix} = 0$$

A4.

$$Var[\varepsilon_i/X] = \sigma^2$$

A5. Variances and covariances of the disturbances given observed information. $E[\varepsilon\varepsilon'/X] = \sigma^2 I$

A6. Nature of the sample of data on the independent variables. X is a known $n \times k$ of constants (nonstochastic of X_i (regressors)). So, assumptions A3 and A4 can be made unconditional.

A7. Probability distribution of the stochastic part of the model.

A8. So, these assumptions describe the form of the model and relationships among its parts and imply appropriate estimation and inference procedures. It is convenient to assume that the disturbances are normally distributed.

$$\varepsilon/X \sim N(0, \sigma^2)$$

b is linear function of the disturbance vector ε. If we assume that ε has a multivariate normal distribution, we may use the results.[1]

[1]Any linear function of a vector of joint normally distributed variables is also normally distributed. The

6.3 Estimating b and σ^2

To test a hypothesis about β or to form confidence intervals, we will require an estimate of the covariance matrix $Var[b] = \sigma^2(X'X)^{-1}$. The standard error of the regression is s, $s^2 = \frac{e'e}{n-K}$

Est. $Var(b) = s^2(X'X)^{-1}$; standard error of the estimator b_k is $\left[s^2(X'X)^{-1}_{kk}\right]^{1/2}$ (k^{th} diagonal element of $(X'X)^{-1}$).

Assuming normality: $Z_k = \frac{b_k - \beta_k}{\sqrt{\sigma^2 S^{kk}}}$ has a standard normal distribution. So, it is obvious to show that $\frac{b-\beta}{\sigma} = (X'X)^{-1}X'\left(\frac{\varepsilon}{\sigma}\right)$ is independent of $\frac{(n-1)s^2}{\sigma^2}$. If ε is normally distributed, then the least square coefficient estimator b is statistically independent of the residual vector e and therefore, all the functions of e, including s^2, ratio:

$$t_k = \frac{(b_k - \beta_k)/\sqrt{\sigma^2 S^{kk}}}{\sqrt{\left[(n-k)\frac{s^2}{\sigma^2}\right]/(n-k)}} = \frac{b_k - \beta_k}{\sqrt{s^2 S^{kk}}}$$

have a t-distribution with $(n-k)$ degrees of freedom. A common test is whether a parameter β_k is significantly different from zero. The appropriate test statistic:

$$t = \frac{b_k}{S_{b_k}},$$

sum of squared residuals (SSE), is a measure of the variability of the actual observations about the estimated regression line. The mean square error (MSE) provides the estimate

mean vector and covariance matrix of AX, where X is normally distributed, follow the general pattern given earlier. Thus, if X$\sim N(\mu, \sum)$, then AX+b$\sim N\left[A\mu + b, A\sum A'\right]$.

If A does not have a full rank, then $A\sum A'$ is singular and the density does not exist. Nonetheless, the individual element of AX+b will still be normally distributed, and the joint distribution of the full vector is still a multivariate normal.

of σ^2; it is SSE divided by its degrees of freedom.

$$\widehat{y_i} = b_0 + b_1 x$$

SSE can be written as:

$$SSE = \sum_i (y_i - \widehat{y_i})^2 = \sum_i (y_i - b_0 - b_1 x_i)^2$$

Statisticians have shown that SSE has $(n-2)$ degrees of freedom because two parameters (β_0 and β_1) must be estimated to compute SSE. Thus, the mean square is computed by dividing SSE by $(n-2)$. MSE provides an unbiased estimator of σ^2.

$$s^2 = MSE = \sqrt{\frac{SSE}{n-2}}$$

In classical statistics, the least squares estimators are sample statistics that have their own sampling distributions (normal form).

$$\sigma_{b_1} = \frac{\sigma}{\sqrt{\sum_i (x_i - \overline{x})^2}}$$

Because we do not know the values of σ, we develop an estimate of σ_{b_1}, denoted by S_{b_1},

$$S_{b_1} = \frac{s}{\sqrt{\sum_i (x_i - \overline{x})^2}}.$$

Rejection rule is as follows using the t-test: we reject the null hypothesis if one of the following inequalities has been satisfied

$$t = \frac{b_1}{S_{b_1}} > t_{\alpha/2} \qquad \text{or} \quad \frac{b_1}{S_{b_1}} < -t_{\alpha/2}.$$

On the other hand, using the F-test:

$$MSR = \frac{SSR}{\textit{regressors degrees of freedom}} = \frac{SSR}{\#\ \textit{of independent variables}}$$

So,

$$F = \frac{MSR}{MSE}.$$

Thus,

$$F > F_\alpha \Rightarrow \textit{reject } H_0$$

Analogically, we can estimate using the sum squared in a fuzzy environment to get two statistic tests for each estimated coefficients, (C_{0L}, C_{oR}) and (C_{1L}, C_{1R}).

In a fuzzy setting, we suggest

$$S_f = \sqrt{\frac{SSE}{n-4}}$$

and because in our model we did not assume that inputs or independent variables are not fuzzy, we will use the following equation with a slight change, taking into consideration the sum squared method.

$$S_{f_b} = \frac{S_f}{\sqrt{\sum(x_i - \overline{x})^2}}$$

Assuming that t-distribution still holds in this setting, we get the left fuzzy test statistic as follows:

$$t_L = \frac{\widehat{b}_L}{S_{f_b}},$$

and the right fuzzy test statistic

$$t_R = \frac{\widehat{b}_R}{S_{f_b}}.$$

Defining that $\widehat{b}_L = (\widehat{b}_{0_L}, \widehat{b}_{1_R}) = (\widehat{b}_0 - C_{0L}, \widehat{b}_1 - C_{1L})$ and $\widehat{b}_R = (\widehat{b}_{0_R}, \widehat{b}_{1_R}) = (\widehat{b}_0 + C_{0R}, \widehat{b}_1 + C_{1R})$.

Rejection rule, using the t-distribution, limiting the testing to the second coefficient b_1:

$$\begin{aligned} t_L &= \frac{\widehat{b}_L}{S_{f_b}} = \frac{\widehat{b}_1 - C_{1L}}{S_{f_b}} > t_{\alpha/2}, \\ t_R &= \frac{\widehat{b}_R}{S_{f_b}} = \frac{\widehat{b}_1 + C_{1R}}{S_{f_b}} > t_{\alpha/2}. \end{aligned}$$

For the coefficient b_0:

$$\begin{aligned} t_L &= \frac{\widehat{b}_L}{S_{f_b}} = \frac{\widehat{b}_0 - C_{0L}}{S_{f_b}} > t_{\alpha/2}, \\ t_R &= \frac{\widehat{b}_R}{S_{f_b}} = \frac{\widehat{b}_0 + C_{0R}}{S_{f_b}} > t_{\alpha/2}, \end{aligned}$$

which is equivalent to:

$$\frac{\widehat{b}_1}{S_{f_b}} - \frac{C_{1L}}{S_{f_b}} > t_{\alpha/2} \quad \text{or} \quad \frac{\widehat{b}_1}{S_{f_b}} - \frac{C_{1L}}{S_{f_b}} < -t_{\alpha/2} \tag{6.1}$$

and

$$\frac{\widehat{b}_1}{S_{f_b}} + \frac{C_{1R}}{S_{f_b}} > t_{\alpha/2} \quad \text{or} \quad \frac{\widehat{b}_1}{S_{f_b}} + \frac{C_{1R}}{S_{f_b}} < -t_{\alpha/2} \tag{6.2}$$

It appears from the previous inequalities (6.1) that incorporating the fuzziness in the model increases the test statistic value and may become insignificant. With the use of a fuzzy random variable, the testing process becomes more robust and powerful. However, in the right side of the fuzzy parameter, it is noticeable that the test is becoming more significant because we are adding positive value $\frac{C_{1R}}{S_{f_b}}$ which will increase the test statistic. In other words, the fuzzy test generates a more significant relationship. Thus, if the original estimate was not significant, the fuzziness improves the test statistic. In general, the introduction of fuzziness in the model increasingly improves the significance of the test for the right side of the fuzzy parameter and reduces the significance of the test for the left side of the fuzzy parameter. Let us assume that we have an estimated parameter (b) which is statistically insignificant using the ordinary least squares method and t-test statistic. Using the indicated approach by estimating the fuzzy parameter spreads of that parameter, we will get the lower and upper limit of b. It is obvious that an increase of the test statistic value will result in a movement toward significance in the right side of the parameter. The left side test statistics value may remain insignificant even with the use of the fuzziness. We can conclude that the fuzziness introduces partial significance instead of full insignificance. Using the example information above, an increase of the test statistic will result in having a partial significance (partial acceptance of the hypothesis) of that parameter, rather than the complete insignificance of it. Since the procedure illustrated above is simple, testing for significance can be effortlessly applied to any practical situation involving the fuzzy regression.

Our contribution is to establish the impact of fuzzy data on parameter significance tests. The elaborated test procedure serves that purpose. The test result shows whether the data suggests a rejection (belongs to the rejection space) or acceptance of hypothesis. Romer and Kandel [128] suggested that a fuzzy sample might support the rejection and the acceptance of a hypothesis. They warned that the term "acceptance" should be used with care. Indeed, in their paper, they introduced the mathematical background on how to reject the hypothesis to a certain degree, with the use of indices. The acceptance and rejection of indices have been specified by a fuzzy test function. The decision to accept or reject is based on the maximum of both indices from a probability point view; and from the possibilistic (fuzzy) view it is based on the difference to the maximum. However, Romer and Kandel's methods lack practicality use in various situations, like the one under study. Of course, here we are specifically interested in testing the significance of the coefficient that accompanies the fuzzy regression method. That may be a limitation, but further development and extension are necessary, which will be handled in future research.

6.4 Testing for Significance and Graphical Illustration

To test for a significant regression relationship, we must conduct a hypothesis test to determine whether the value of β_1 is zero. In classical regression, two tests are commonly used. Both require an estimate of σ^2, the variance of ε in the regression model. The following figures (6.1) and (6.2) illustrate the movement of the test statistic accounting for the fuzziness in the testing for significance.

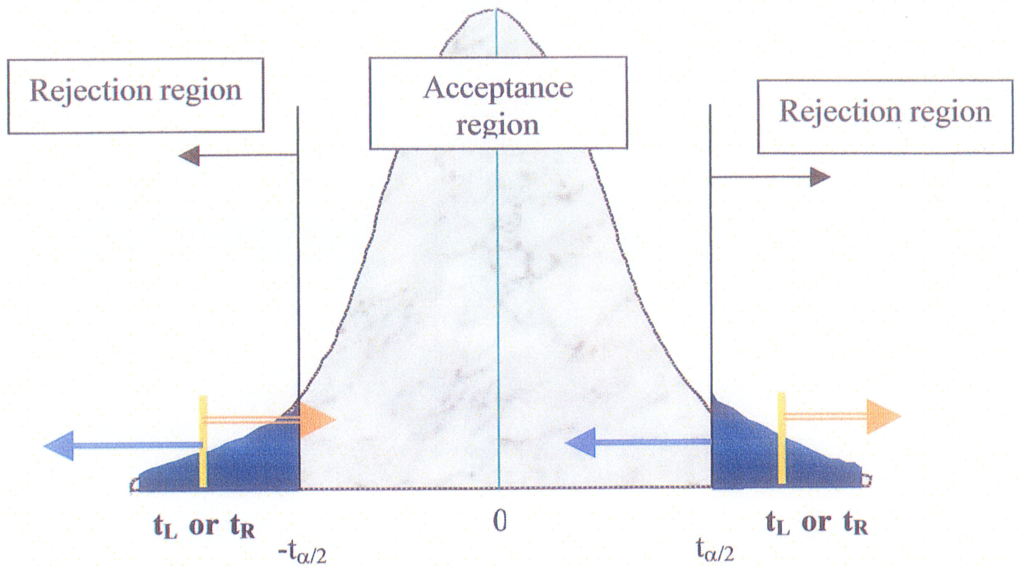


Figure 6.1: T-distribution and fuzzy test statistic

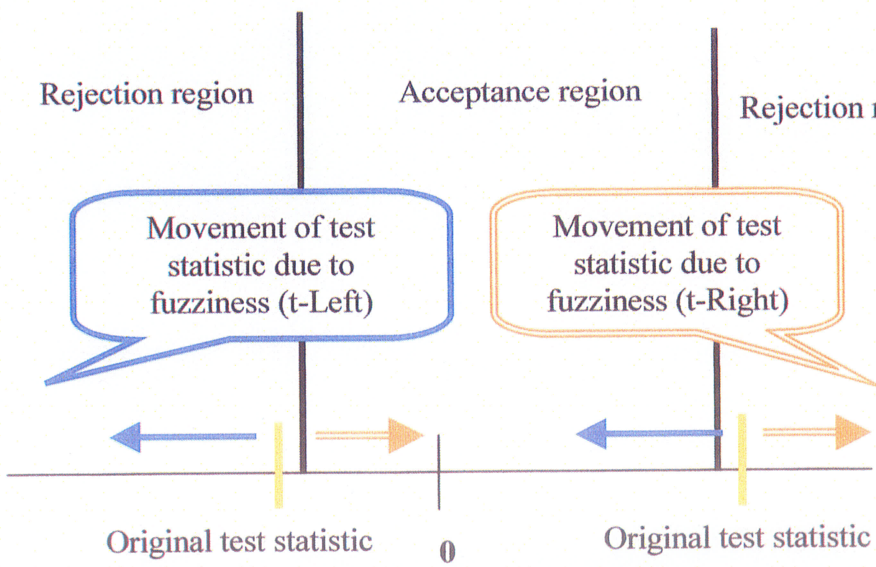


Figure 6.2: Illustration of fuzzy test statistic

6.5 Empirical Example

The results, which have been generated from the previous chapter (Chapter 5) dealing with estimation of systematic risk in the futures commodity market under price limit, have been used to test for the significance of that estimation. The method elaborated on the previous section of this chapter shows the importance of incorporating testing method. The implication of the proposed method to measure the significance of the computed parameters has been the focus of this section. As mentioned in the preceding chapter, the nature of sample data pertaining to futures commodity returns will be used here to test the significance of the result. In favor of a faster computation, LR type of fuzzy parameter has been used in this chapter.

The table of the fuzzy regression parameters for western barley, canola, flaxseed, and feed wheat shows the data objective function values that measure the sum of deviation. The values of sum squared errors have been used to compute the test statistic in a fuzzy setting as examined in the previous section. An important rule employed here is that when both spreads (widths) of the parameter are equal to 0, we need to treat the estimated parameter as crisp. Therefore, the use of the proposed method for significance testing is not required. The following table (6.1) shows the result.

Table 6.1: Results of testing for significance: Statistical versus fuzzy

Commodity		Left spread	esβ	Right spread	t-left	t-right	S-signi	F-signi
Western Barley	May	0	0.2722	0	1.97285	1.972849	Yc	Yb
	Aug./July	0	0.2990	0	1.73366	1.733656	Yb	Yb
	Nov./Oct.	0.5152	0.1650	0	-2.0993	0.98909	N	partial c
	Nov./Dec.	0.603	0.2530	0	-2.1316	1.540818	N	
	Feb./Mar.	0	0.2130	0	1.38485	1.384845	N	N
Canola	June/May	0	0.3702	0	2.15912	2.159121	Yc	Yc
	June/July	0	0.4070	0	2.26577	2.265772	Yc	Yc
	Sep.	0	0.3850	0	2.99389	2.99389	Ya	Ya
	Nov.	0	0.3108	0	2.32919	2.329194	Yc	Ya
	Jan.	0	0.4020	1.0816	2.66786	9.845849	Ya	Ya
	March	0	0.1480	0	0.96731	0.967314	N	N
Flaxseed	May	0	0.2445	2.1265	1.48815	14.43105	N	partial a
	July	0	-0.1920	1.7455	-1.0804	8.741603	N	partial a
	Oct. /Sep.	0.8547	0.1069	1.259	-5.4478	9.950658	N	Ya
	Oct./ Nov.	0	0.0339	0.4174	0.23293	3.100861	N	partial a
	Dec. / Jan.	0	-0.0981	0	-0.6759	-0.675932	N	N
	March	1.6683	0.1512	1.7483	-10.028	12.55561	N	Ya
Wheat	May	0	0.4041	0	2.22589	2.225894	Yc	Yc
	July	0.1742	0.3190	0	0.90125	1.985481	Yc	Partial b
	Oct.	0	0.1511	0	0.83091	0.830913	N	N
	Dec.	0	0.3080	0	1.90148	1.901477	Yb	Yb
	March	0	0.3030	0	1.92083	1.920834	Yb	Yb

Ya denotes statistical or fuzzy significance at the 1% level. Yb denotes statistical or fuzzy significance at the 5% level. Yc denotes statistical or fuzzy significance at the 2.5% level. Partial *a* and partial *b* mean partial significance at the 1% and 5% respectively. N: denotes statistically or fuzzily insignificant

From the Table (6.1) it appears that flaxseed commodity futures, which are proven to be statistically insignificant, have shown partial significance. Also, the result shows that those contracts that have zero spreads are statistically and fuzzily significant.Thus, it is enough to rely on a statistical test, as long as the fuzzy parameter has a zero spread (width), which is equivalent to a crisp value. Another observation worth mentioning is that the March flaxseed contract, which was statistically insignificant, is fully fuzzily significant at 1%. Thus, it is obvious that, as expected, the significance has been improved for some future contracts by the use of price limits.

Chapter 7

Conclusion and contribution

In the present chapter, we discuss the conclusion and the contribution of the thesis. We believe that this research will lead to a number of computational and theoretical investigations. Some directions for further research that uses the methodology and testability strategy have been provided.

7.1 Summary and Conclusion

Lack of proven practical applications and empirical implications of the new uncertainty (fuzzy random uncertainty) during its early stage of development was a favorite criticism of its opponents. Therefore, the present research represents an attempt to present a methodology and a potential testability process for three major aspects.

In chapter 3, we question one important assumption made in Markowitz ([103], [105]), which remains a fundamental "*hidden*" assumption in mean-variance theory literature today, that random uncertainty is the sole means of modeling uncertainty.

Although Markowitz [103] ignores the experts' judgments in the derivation of the efficient frontier, he discusses the value of such a combination of statistical techniques and the judgment of experts, to form reasonable probability beliefs about the portfolio selection process. However, Markowitz does not propose a method to deal with that issue, and he does not examine the efficient set of portfolios for the investor in the presence of fuzziness or any subjective information.

On other hand, White [166] has suggested that measures of uncertainty are either formally derived from specified data, or are imputed by observing choice in a given class of problems. Following along the lines of Markowitz, the purpose of this research is to provide some aspects of fuzzy random uncertainty in asset pricing, which would include a rederivation of the mean-variance theory, followed by the rederivation of the fuzzy CAPM model.

In the present study, we re-examine mean-variance theory in the presence of fuzziness that is articulated by fuzzy returns (LR type). We rederive the Markowitz efficient set and present the Fuzzy Capital Market Line (FCML) and the FCAPM. By illustrating these ideas with an empirical example, a comparative study is obtained.

The boundary of each sample size turns out to not be a parabola. It is also clearly observed that the arc which is between a minimum point and a maximum point does not coincide with the original boundary. The minimum (maximum) point represents, as discussed previously and supported by Szego's finding [147], what can be achieved by investing the capital in the investment option with lowest (highest) return.

The portfolio width has been included as a third parameter, and the frontier has been plotted in three-dimensional graphs. The relationship between risk, return, and width (proxy for the subjective evaluation of the experts) has been represented by a surface. Also, for 15, 30, and 50 asset sample sizes, similarly to the case of short sales, we still observe that the larger the size of the sample the more the efficient frontier is shifted to the left; the dominance of large size sample still holds.

It is discerned in the previous graphs that as the degree of fuzziness increases (flexibility with respect to the portfolio mean), there is a slight decrease in the level of risk. Note here that the graph does not suggest a strong negative relationship for various sample sizes. Because the widths in our samples are correlated with the return, which is derived from historical data, we could not see a strong visible (either positive or negative) relationship. Thus, we suggest that as soon as the investor starts getting new subjective information from experts, which is to some extent not primarily correlated with the historical data, we will be able to spot a strong visible relationship between the width size and the risk level. So, an investor who is very flexible and is acquiring additional subjective information to support the historical data will be liklely to accept a higher risk.

While Philippatos and Wilson [119] argue that entropy is a better statistical measure of risk than variance because entropy is a non-parametric measure, entropy did not appear often in published works. As Philippatos and Wilson [119] suggested entropy as a measure of portfolio risk, because it does not make assumptions concerning the probability underlying the returns, we use the same analogy to establish the measure of risk using the proposed fuzzy entropy method in the second study.

Note here that neither the Cozzolino and Zahner [27] approach nor the Philippatos and Wilson [119] method suggest anything about the situation when there is imprecise information to start from. Consequently, we use the fuzzy theory in conjunction with the entropy theory. This study did extend the method to provide a specific distribution by using the fuzzy entropy principle.

The utilization of variance as a measure of uncertainty is ignored purposely, because for distributions that are non-symmetric or not normally distributed, a new measure of uncertainty is essential. In addition, in a fuzzy environment it is crucial to use a new measure of uncertainty that will differ from the variance, while taking into account the fuzziness existing in the system. In this study, we suggest the use of fuzzy entropy as a measure of uncertainty. Although entropy or expected information has been widely used in many engineering and mathematical subjects, the author deems that the scope of application in finance is limited. The empirical analysis using the Markowitz data has been given to illustrate the use of the method in the construction of the mean-entropy efficient frontier under a fuzzy environment.

This study focused on the presentation of a new approach with the emergence of probability theory and studied its various other (non-probabilistic) manifestations and their utility in risk modeling. Anyone who is familiar with the stock market will find that the most challenging decision is to differentiate between the good stock to buy and the bad stock to sell. To a lesser extent, we considered decision problems (investment problems), where the goal is to choose optimal strategy; some alternatives between actions may be determined. The major aim of the study was to establish the mathematical theory of fuzzy probabilities, based on the measure of entropy.

In the third study, to overcome the biased parameter estimates introduced by price limits in futures markets, we treat the futures price subject to price limits as a fuzzy datum and use a two-phase fuzzy approach to examine the input of price limits and estimate the systematic risk.

Following the Brennan paradigm, we assume that there is an external signal suggesting that the equilibrium price is in the boundaries of the observed price and that the equilibrium price is bounded by an upper bound (observed plus half the limit) and lower bound (observed minus half the limit). It is assumed that the equilibrium price that would have been observed in the absence of a price limit will be around the settlement price.

The study estimates the systematic risk of Canadian commodity futures investment. The capital asset pricing model (CAPM) has been structured to be estimated from two complementary phases. Firstly, with the use of classical regression analysis, we have estimated the parameters of the linear model. The result of that regression will serve as the first step of the proposed two-step fuzzy regression method. Secondly, knowing that most futures contracts have a daily price limit specified by the exchange, and the movement of the price is said to have a limit up and limit down, we have constructed fuzzy non-symmetrical data. Then, with the help of the second step, the spreads of the beta estimates have been provided.

Based on classical regression, results show that three out of four commodity futures are riskier. Therefore, the usual belief that traders of commodity futures bear above-average risk is supported by the Canadian data.

After Phase 2 of the estimation procedure, the parameters with their estimated spreads, which have been given by the fuzzy regression, offered persuasive evidence for the acceptance of some parameters.

The last step was to test if the original estimate was significant or not; we wanted to see if fuzziness improved the test statistic. In general, the introduction of fuzziness in the model increasingly improves the significance of the test for the right side of the fuzzy parameter and decreases the significance of the test for the left side of the fuzzy parameters.

Using an approach that estimates the fuzzy parameter spreads, we get the lower and upper limit of the coefficient. We conclude that the fuzziness introduces partial significance instead of full insignificance. Using the example above, an increase of the test statistic will result in a partial significance (partial acceptance of the hypothesis) of that parameter rather than the complete insignificance of it. Since the procedure illustrated above is simple, the testing for significance can be effortlessly applied to any practical situation involving fuzzy regression.

7.2 Contribution and Further Research

The mean-variance model gained widespread acceptance as a practical instrument for portfolio selection, and it is hoped that the mean-variance frontier will be computed with subjective measures like fuzzy return as part of the portfolio allocation process by many investment advisory firms and pension plans sponsors. The contribution of this research is the presentation of a methodology on how to derive the attainable efficient frontier in the presence of fuzzy information in the data or when the fuzzy information has been imposed in the modeling environment to reflect a subjective measure.

In real-world problems, we are faced with imperfect information (data), and we have to deal with uncertain, imprecise, and vague data. In modeling and analyzing problems of this type, earlier works in finance tended to equate all aspects of imperfect information with uncertainty (of a random character). Thus, a multitude of probabilistic models were proposed. This was also the case with the use of modeling in finance. The suggested method will serve the interest of investors who select their portfolios using a Markowitz-based model with the introduction of fuzziness or any other subjective techniques, like the judgment of experts. An additional important investigation would be to look at the mean-variance, when investors can borrow or lend any amount they want at divergent borrowing and lending rates while we maintain the assumption of fuzzy returns ([16], [157]). In another, we want to see what will happen once we relax the assumption of riskless rate as borrowing and lending rates.

On the other hand, an inevitable consequence of using fuzzy probabilities is that probabilistic reasoning may produce indeterminate conclusions (we may not be able to determine which of two events is more probable), and decision analysis may produce non-decision (we may not be able to choose the best of two actions). When there is not enough information on which to base our conclusions and decisions, we cannot expect sharply defined reasoning to reveal the most probable outcome. We believe that a substantial amount of research should be done in this area.

For instance, one direction for future research is the problem of the two-state variable model of the term structure in which approximate probabilities may be be used. Also, we hope that the suggested method will be useful in solving the problem of valuing the

American put option using the binomial method. The critical-stock-price function, which is the value of the stock price when one is indifferent between exercising and not exercising the put, can be approximated not only to compute accurate put prices but also to provide the boundary of early-excercise.

The third contribution may indicate this direction for future research: the examination of the relationship between systematic risk and the size of the firm, using a fuzzy regression approach. Further work is needed to examine the fuzzy hypothesis testing by establishing a fuzzy acceptance region (optimal according to the approach used) [31]. Generally, we feel that there are more opportunities for fuzzy regression method applications in financial modeling. It provides an effective way to cope with the uncertainties that are inherent in the financial models.

The last contribution was to establish the impact of fuzzy data on parameter significance tests. The elaborated test procedures serve that purpose. The test result shows whether the data suggest a rejection (belongs to the rejection space) or acceptance of the hypothesis. We are specifically interested in testing the significance of the coefficient that accompanies a simple fuzzy regression method. That may be considered a limitation in the presented approach but further development and extension of the method to cover multiple variables is necessary, and will be handled in future research.

Bibliography

[1] Abdel-Khalik, A. R.: 1974, 'The Entropy Law, Accounting Data, and Relevance to Decision-Making'. *Acct. Rev.* **49**, 271–283.

[2] Aitken, M. J., A. Frino, M. S. McCorry, and P. L. Swan: 1998, 'Short Sales Are Almost Instantaneously Bad News: Evidence from the Australian Stock Exchange'. *The Journal of Finance* **53**(6), 2205–2223.

[3] Alchian, A. A.: 1953, 'The Meaning of Utility Measurement'. *American Economic Review* **XLIII**(1), 26–50.

[4] Allais, M.: 1953, 'Le Comportement de L' Homme Rational Devant Le Risque: Critique Des Postulats et Axiomes de L'ecole Americaine'. *Econometrica* **XXI**(4), 503–546.

[5] Amihud, Y. and H. Mendelson: 1986, 'Asset Pricing and the Bid-Ask Spread'. *Journal of Financial Economics* **17**, 223–249.

[6] Amihud, Y. and H. Mendelson: 1989, 'The Effects of Beta, Bid-Ask Spread, Residual Risk, and Size on Stock Returns'. *The Journal of Finance* **44**(2), 479–486.

[7] Bardossy, A., I. Bogardi, and L. Duckstein: 1990, 'Fuzzy Regression in Hydrology'. *Water Resources Research* **26**(7), 1497–1508.

[8] Bardossy, A., I. Bogardi, and L. Duckstein: 1993, 'Fuzzy Non-Linear Regression Analysis of Dose-Response Relationships'. *European Journal of Operational Research* **66**, 36–51.

[9] Baxter, J., T. Conine, and M. Tamarkin: 1985, 'On the Commodity Market Risk Premium: Additional Evidence'. *Journal of Futures Markets* **5**, 121–125.

[10] Bellman, R. E. and L. A. Zadeh: 1970, 'Decision-Making in a Fuzzy Environment'. *Management Science* **17**, 141–164.

[11] Bezdek, J.: 1994, 'Fuzziness Vs. Probability-Again(!?)'. *IEEE Transactions pm Fuzzy Systems* **2**, 1–2.

[12] Black, F.: 1976, 'The Pricing of Commodity Contracts'. *Journal of Financial Economics* **3**, 167–179.

[13] Black, F. and M. S. Scholes: 1973, 'The Pricing of Options and Corporate Liabilities'. *Journal of Political Economy* **8**, 637–654.

[14] Borch, K. H.: 1968, *The Economics of Uncertainty.* Princeton, New Jersey: Princeton University Press.

[15] Brennan, M. J.: 1986, 'A Theory of Price Limits in Futures Markets'. *Journal of Financial Economics* **16**, 213–233.

[16] Brennan, N. J.: 1971, 'Capital Market Equilibrium with Divergent Borrowing and Lending Rates'. *Journal of Financial and Quantitative Analysis* **6**(5), 1197–1205.

[17] Broussard, J. P.: 2001, 'Extreme-Value and Margin Setting with and Without Price Limits'. *Quarterly Review of Economics and Finance* **41**(3), 365–385.

[18] Carlsson, C., R. Fuller, and P. Majlender: 2002, 'A Possibilistic Approach to Selecting Portfolios with Highest Utility Score'. *Fuzzy Sets and Systems* **131**(1), 13–21.

[19] Carter, G. E., G. C. Rausser, and A. Schmitz: 1983, 'Efficient Asset Portfolios and the Theory of Normal Backwardation'. *Journal of Political Economy* **91**, 319–331.

[20] Caslas, M. R.: 1993, 'Bayesian Testing of Fuzzy Parametric Hypotheses from Fuzzy Information'. *Recherche Operationelle* **27**(2), 189–199.

[21] Caslas, M. R. and P. Gil: 1994, 'Bayesian Sequential Test for Fuzzy Parametric Hypotheses from Fuzzy Information'. *Information Sciences* **80**, 283–298.

[22] Chang, E. C.: 1985, 'Returns to Speculators and the Theory of Normal Backwardation'. *Journal of Finance* **40**, 193–208.

[23] Chang, E. C., C. Chen, and S. Chen: 1990, 'Risk and Return in Copper, Platinum and Silver Futures'. *The Journal of Futures Markets* **10**(1), 29–39.

[24] Cheeseman, P.: 1986, 'Probabilistic versus Fuzzy Reasoning'. In: L. N. Kanal and J. F. Lemmer (eds.): *Uncertainty in Artificial Intelligence.* New York: Elsevier Science Publishers B. V., pp. 85–102.

[25] Chen, A. H.: 1977, 'Portfolio Selection with Stochastic Cash Demand'. *The Journal of Financial and Quantitative Analysis* **12**(2), 197–213.

[26] Chou, P. H.: 1999, 'Modeling Daily Price Limits'. *International Review of Financial Analysis* **8**(3), 283–301.

[27] Cozzolino, J. M. and M. J. Zahner: 1973, 'The Maximum-Entropy Distribution of the Future Market Price of a Stock'. *Operations Research* **21**(6), 1200–1211.

[28] Danthine, J. P. and J. B. Donalson: 2002, *Intermediate Financial Theory*. New Jersey: Pearson Education, 1st edition.

[29] Darzentas, J.: 1987, 'On Fuzzy Location Models'. In: J. Kacprzyk and S. A. Orlovski (eds.): *Optimization Models Using Fuzzy Sets and Possibility Theory*. Dordrecht: Reidel.

[30] Daves, P. R., M. C. Ehrhardt, G. A. Kuhlemeyer, and R. A. Kunkel: 2000, 'Increases in the Systematic Risk of Large Firms'. *American Business Review* **18**(2), 62–74.

[31] Delgado, M., J. L. Verdegay, and M. A. Vila: 1985, 'Testing Fuzzy Hypotheses: A Bayesian Approach'. In: M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka (eds.): *Approximate Reasoning in Expert Systems*. Amsterdam: North Holland Publications, pp. 307–316.

[32] Demsetz, H.: 1968, 'The Cost of Transaction'. *Quarterly Journal of Economics* **82**, 33–53.

[33] Dorfman, R., P. A. Samuelson, and R. M. Solow: 1958, *Linear Programming and Economic Analysis*. New York: McGraw-Hill.

[34] Dubois, D. and H. Prade: 1980, *Fuzzy Sets and Systems: Theory and Applications*. New York: Academic Press.

[35] Dusak, K.: 1973, 'Futures Trading and Investor Returns: An Investigation of Commodity Risk Premiums'. *Journal of Political Economy* pp. 1387–1406.

[36] Dybvig, P. H.: 1984, 'Short Sales Restrictions and Kinks on the Mean Variance Frontier'. *Journal of Finance* **39**(1), 239–244.

[37] Dybvig, P. H. and S. A. Ross: 1985, 'The Analytics of Performance Measurement Using Security Market Line'. *Journal of Finance* **40**(2), 401–416.

[38] Easley, D., N. M. Kiefer, M. O'Hara, and J. B. Paperman: 1996, 'Liquidity, Information, and Infrequently Traded Stocks'. *The Journal of Financial and Quantitative Analysis* **51**(4), 1405–1436.

[39] Elam, E. W. and D. Vaught: 1988, 'Risk and Return in Cattle and Hog Futures'. *The Journal of Futures Markets* pp. 79–87.

[40] Feron, R.: 1976, 'Ensembles Aleatoires Flous'. *C. R. Acad. Sci. Paris Ser. A* **282**, 903–906.

[41] Friedman, M. and L. J. Savage: 1984, 'The Utility Analysis of Choices Involving Risk'. *The Journal of Political Economy* **56**, 279–304.

[42] Gebhardt, J., M. A. Gil, and R. Kruse: 1991, 'Fuzzy Set–Theoretic Methods in Statistics'. In: R. Slowinski (ed.): *Fuzzy Sets in Decision Analysis, Operations Research and Statistics.* Boston: Kluwer Academic Publishers.

[43] Good, I. J.: 1952, 'Rational Decisions'. *Journal of the Royal Statistical Society. Series B* **14**(1), 107–114.

[44] Good, I. J.: 1963, 'Maximum Entropy for Hypothesis Formulation, Especially for Multidimensional Contingency Tables'. *The Annals of Mathematical Statistics* **34**(3), 911–934.

[45] Good, I. J.: 1966, 'A Derivation of the Probabilistic Explication of Information'. *Journal of the Royal Statistical Society. Series B* **28**(3), 578–581.

[46] Good, I. J.: 1979, 'Studies in the History of Probability and Statistics. XXXVII A. M. Turing's Statistical Work in World War II'. *Biometrika* **66**(2), 393–396.

[47] Good, I. J.: 1988, 'The Interface Between Statistics and Philosophy of Science'. *Statistical Science* **3**(4), 386–397.

[48] Gray, R. W.: 1961, 'The Search for a Risk Premium'. *Journal of Political Economy* **69**, 250–260.

[49] Greene, W.: 2000, *Econometric Analysis.* New Jersey: Prentice Hall.

[50] Hakansson, N. H.: 1971, 'Capital Growth and the Mean-Variance Approach to Portfolio Selection'. *The Journal of Financial and Quantitative Analysis* **6**(1), 517–557.

[51] Hall, A. D. and P. Kofman: 2001, 'Limits to Linear Price Behavior: Futures Prices Regulated by Limits'. *The Journal of Futures Markets* **21**(5), 463–488.

[52] Heinkel, R. and A. Kraus: 1988, 'Measuring Event Impacts in Thinly Traded Stocks'. *The Journal of Financial and Quantitative Analysis* **23**(1), 71–88.

[53] Herinter, J. D.: 1973, 'An Entropy Model of Brand Purchase Behavior'. *Journal of Marketing Research* **10**, 361–375.

[54] Herstein, I. N. and J. Milnor: 1953, 'An Axiomatix Approach to Measurable Utility'. *Econometrica* XXI(2), 291–297.

[55] Heshmaty, B. and A. Kandel: 1985, 'Fuzzy Linear Regression and its Applications to Forecasting in Uncertain Environment'. *Fuzzy Sets and Systems* **15**, 159–191.

[56] Hojati, M., C. R. Bector, and K. Smimou: 2001, 'A Simple Method for Fuzzy Linear Regression'. In: *Proceedings of the 2001 Annual Conference Management Science Division*, Vol. 22. London, Ontario, pp. 31–42.

[57] Hopewell, M. H. and A. L. Schwartz: 1978, 'Temporary Trading Suspensions in Individual NYSE Securities'. *Journal of Finance* **33**, 1355–1374.

[58] Houthakker, H. S.: 1957, 'Can Speculators Forecast Prices?'. *Review of Economics and Statistics* **39**, 143–151.

[59] Huang, C. and R. H. Litzenberger: 1988, *Foundations for Financial Economics.* New York: North Holland.

[60] Huang, Y., T. W. Fu, and M. C. Ke: 2001, 'Daily Price Limits and Stock Behavior: Evidence from the Taiwan Stock Exchange'. *International Review of Economics and Finance* **10**, 263–288.

[61] Hull, J. C.: 2003, *Options, Futures and Other Derivatives.* New Jersey: Prentice Hall.

[62] Hume, D.: 1995, *An Inquiry Concerning Human Understanding.* New Jersey: Prentice Hall. Originally published in 1748.

[63] Jackwerth, J. C. and M. Rubinstein: 1996, 'Recovering Probability Distributions from Option Prices'. *The Journal of Finance* **51**(5), 1611–1631.

[64] Jarrow, R.: 1980, 'Heterogeneous Expectations, Restrictions on Short Sales, and Equilibrium Asset Prices'. *The Journal of Finance* **35**(5), 1105–1113.

[65] Jaynes, E. T.: 1979, 'Where Do We Stand on Maximum Entropy?'. In: Levine and Tribus (eds.): *The Maximum Entropy Formalism.* Cambridge, Mass: MIT Press.

[66] Jensen, M. C.: 1986, 'The Performance of Mutual Funds in the Period 1945-64'. *Journal of Finance* **23**, 389–416.

[67] Kapur, J. N.: 1989, *Maximum Entropy Models in Science and Engineering.* New Delhi: John Wiley Eastern Ltd.

[68] Kapur, J. N.: 1997, *Measures of Fuzzy Uncertainty.* New Delhi: Mathematical Sciences Trust Society.

[69] Kapur, J. N. and L. Kesvan: 1992, *Entropy Optimization Principles and Their Applications.* Academic Press.

[70] Kim, K. J., H. Moskowitz, A. Dhingra, and G. Evans: 2000, 'Fuzzy Multicriteria Models for Quality Function Deployment'. *European Journal of Operational Research* **121**(3), 504–518.

[71] Kim, K. J., H. Moskowitz, and M. Koksalam: 1996, 'Fuzzy Versus Statistical Linear Regression'. *European Journal of Operational Research* **92**, 417–434.

[72] Kleinbaum, D. G. and L. L. Kupper: 1978, *Applied Regression Analysis and Other Multivariable Methods*. Belmont: Wadsworth.

[73] Klir, G. J.: 1987, 'Where Do We Stand on Measures of Uncertainty, Ambiguity, Fuzziness and the Like?'. *Fuzzy Sets and Systems* **24**, 141–160.

[74] Klir, G. J.: 1999, *Fuzzy Sets: An Overview of Fundamentals, Applications and Personal Views*. Beijing: Beijing Normal University Press.

[75] Kodres, L. E.: 1988, 'Tests of Unbiasedness in Foreign Exchange Futures Markets: The Effects of Price Limits'. *Review of Futures Markets* **7**, 139–175.

[76] Kodres, L. E.: 1993, 'Tests Unbiasedness in Foreign Exchange Futures Markets: An Examination of Price Limits and Conditional Hetroskedasticity'. *Journal of Business* **66**, 463–490.

[77] Korner, R.: 1997, 'On the Variance of Fuzzy Random Variables'. *Fuzzy Sets and Systems* **92**, 83–93.

[78] Koski, J. L.: 1998, 'Measurement Effects and the Variance of Returns After Stock Splits and Stock Dividends'. *The Review of Financial Studies* **11**(1), 143–162.

[79] Kosko, B.: 1986, 'Fuzzy Entropy and Conditioning'. *Information Sciences* **40**, 165–174.

[80] Kosko, B.: 1990, 'Fuzziness Vs. Probability'. *International Journal of General Systems* **17**, 211–240.

[81] Kosko, B.: 1994, 'The Probability Monopoly'. *IEEE Transactions on Fuzzy Systems* **2**(1), 32–33.

[82] Kratschmer, V.: 2002, 'Limit Theorems for Fuzzy-Random Variables'. *Fuzzy Sets and Systems* **126**(2), 253–263.

[83] Kroll, Y. and H. Levy: 1980, 'Sampling Errors and Portfolio Efficient Analysis'. *The Journal of Financial and Quantitative Analysis* **15**(3), 655–688.

[84] Kruse, R. and K. Meyer: 1987, *Statistics with Vague Data.* Holland: D. Reidel Publishing Company.

[85] Kuserk, G. J. and P. R. Locke: 1996, 'Market Making with Price Limits'. *Journal of Futures Markets* **16**, 677–696.

[86] Kwakernaak, H.: 1978, 'Fuzzy Random Variables: Definition and Theorems'. *Inform. Sci.* **15**, 1–29.

[87] Lai, Y. J. and S. I. Chang: 1994, 'A Fuzzy Approach for Multiresponse Optimization: An Off-Line Quality Engineering Problem'. *Fuzzy Sets and Systems* **63**, 117–129.

[88] Lai, Y. J. and C. L. Hwang: 1992, *Fuzzy Mathematical Programming: Methods and Applications.* Heidelberg: Springer-Verlag.

[89] Lehman, E. L.: 1958, 'Significance Level and Power'. *Ann. Math. Stat.* **29**, 1167–1176.

[90] Lehman, E. L.: 1959, *Testing Statistical Hypotheses.* New York: John Wiley.

[91] Levy, H.: 1973, 'The Demand for Assets under Conditions of Risk'. *Journal of Finance* **28**(1).

[92] Levy, H.: 1983, 'The Capital Asset Pricing Model: Theory and Empiricism'. *The Economic Journal* **93**, 145–165.

[93] Levy, H. and M. Sarnat: 1984, *Portfolio and Investment Selection: Theory and Practice.* New Jersey: Prentice Hall.

[94] Lindley, D. V.: 1956, 'On a Measure of the Information Provided by an Experiment'. *The Annals of Mathematical Statistics* **27**(4), 986–1005.

[95] Lindley, D. V.: 1982a, 'The Probability Approach to the Treatment of Uncertainty in Artificial Intelligence and Expert Systems'. *Statistical Science* **2**(3), 3–44.

[96] Lindley, D. V.: 1982b, 'Scoring Rules and the Inevitability of Probability'. *International Statistical Review* **50**, 1–26.

[97] Lintner, J.: 1965, 'The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolio and Capital Budgets'. *Review of Economics and Statistics* **47**, 13–37.

[98] Longin, F. M.: 1999, 'Optimal Margin Level in Futures Markets: Extreme Price Movements'. *The Journal of Futures Markets* **19**(2), 127–152.

[99] Luca, A. D. and S. Termini: 1972, 'A Definition of a Nonprobabilistic Entropy in The Setting of Fuzzy Sets Theory'. *Information and Control* **20**, 301–312.

[100] Mao, C. K., R. P. Rao, and R. S. Sears: 1989a, 'Limit Moves and Price Resolution: The Case of the Treasury Bond Futures Markets'. *Journal of Futures Markets* **9**, 321–335.

[101] Mao, C. K., R. P. Rao, and R. S. Sears: 1989b, 'Volatility, Price Resolution and the Effectiveness of Price Limits'. *Journal of Financial Services Research* **3**, 165–199.

[102] Marcus, A. J.: 1984, 'Efficient Asset Portfolio and the Theory of Normal Backwardation: Comment'. *Journal of Political Economy* **92**, 162–164.

[103] Markowitz, H. M.: 1952, 'Portfolio Selection'. *The Journal of Finance* **7**, 77–91.

[104] Markowitz, H. M.: 2002, *Portfolio Selection: Efficient Diversification of Investments.* Oxford, UK: Blackwell Publishers Ltd.

[105] Markowitz, H. M.: 2003, *Portfolio Selection: Efficient Diversification of Investments.* Cambridge, MA, Oxford, UK: Blackwell.

[106] Merton, R. C.: 1987a, 'A Simple Model of Capital Market Equilibrium with Incomplete Information'. *Journal of Finance* **42**, 483–510.

[107] Merton, R. C.: 1987b, 'Theory of Rational Option Pricing'. *Bell Journal of Economics and Management Science* **4**, 141–183.

[108] Mossin, J.: 1966, 'Equilibrium in a Capital Asset Market'. *Econometrica* **34**(4), 768–783.

[109] Nather, W.: 1997, 'Linear Statistical Inference for Random Fuzzy Data'. *Statistics* **29**, 221–240.

[110] Nather, W.: 2000, 'On Random Fuzzy Variables of Second Order and Their Application to Linear Statistical Inference with Fuzzy Data'. *Metrika* **51**, 201–221.

[111] Nawrocki, D. N. and W. H. Hardling: 1986, 'State-Value Weighted Entropy as a Measure of Investment Risk'. *Applied Economics* **18**, 411–419.

[112] Netter, J., W. Wasserman, and M. H. Kutner: 1985, *Applied Linear Statistical Models.* Irwin: Homewood.

[113] Ostermark, R.: 1988, 'Profit Apportionment in Concerns with Mutual Ownership–an Application of Fuzzy Inequalities'. *Fuzzy Sets and Systems* **26**, 283–297.

[114] Ostermark, R.: 1989, 'Fuzzy Linear Constraints in the Capital Asset Pricing Model'. *Fuzzy Sets and Systems* **30**, 93–102.

[115] Park, C. W.: 2000, 'Examining Futures Price Changes and Volatility on the Trading Day After a Limit-Lock Day'. *The Journal of Futures Markets* **20**(5), 445–466.

[116] Pastor, L.: 2000, 'Portfolio Selection and Asset Pricing Models'. *The Journal of Finance* **55**(1), 179–223.

[117] Perincherry, V. and S. Kikuchi: 1990, 'A Fuzzy Approach to the Transshipment Problem'. *Uncertainty Modeling and Analysis* pp. 330–335.

[118] Philippatos, G. and N. Gressis: 1976, 'Conditions of Equivalence Among EV, SSD and EH Portfolio Selection Criteria: The Case of Uniform, Normal and Lognormal Distributions'. *Management Science* **21**, 617–25.

[119] Philippatos, G. and C. J. Wilson: 1972, 'Entropy, Market Risk and the Selection of Efficient Portfolios'. *Applied Economics* **4**, 209–220.

[120] Porter, D. C.: 1992, 'The Probability of a Trade at the Ask: An Examination of Interday and Intraday Behavior'. *The Journal of Financial and Quantitative Analysis* **27**(2), 209–227.

[121] Puri, M. and D. Ralescu: 1986, 'Fuzzy Random Variables'. *Journal of Mathematical Analysis and Applications* **114**, 409–422.

[122] Ralescu, D. A.: 1995, 'Fuzzy Probabilities and Their Applications to Statistical Inference'. In: M. Bouchon, B. Yager, and L. A. Zadeh (eds.): *Advances in Intelligent Computing: Lectures Notes in Computer Science*, Vol. 945. pp. 217–222.

[123] Rappoport, A., T. S. Wallsten, and J. A. Cox: 1987, 'Direct and Indirect Scaling of Membership Functions of Probability Phrases'. *Mathematical Modelling* **9**, 397–418.

[124] Redden, D. T. and W. H. Woodall: 1994, 'Properties of Certain Fuzzy Linear Regression Methods'. *Fuzzy Sets and Systems* **64**, 361–375.

[125] Redden, D. T. and W. H. Woodall: 1996, 'Further Examination of Fuzzy Linear Regression'. *Fuzzy Sets and Systems* **79**, 203–211.

[126] Roll, R.: 1977, 'A Critique of the Asset Pricing Theory's Tests I: On Past and Potential Testability of the Theory'. *Journal of Financial Economics*.

[127] Roll, R.: 1984, 'Orange Juice and Weather'. *The American Economics Review* **74**(5), 861–880.

[128] Romer, C. and A. Kandel: 1995, 'Statistical Tests for Fuzzy Data'. *Fuzzy Sets and Systems* **72**, 1–26.

[129] Ross, S. A.: 1977, 'The Capital Asset Pricing Model (CAPM), Short-Sale Restrictions and Related Issues'. *Journal of Finance* **32**(1), 177–183.

[130] Ross, S. A.: 1978a, 'The Arbitrage Theory of Capital Asset Pricing Model (CAPM)'. *Journal of Finance* **46**, 605–615.

[131] Ross, S. A.: 1978b, 'Mutual Fund Separation in Financial Theory–The Separating Distributions'. *Journal of Economic Theory* **17**, 254–86.

[132] Ross, T.: 1995, *Fuzzy Logic with Engineering Applications*. New York: McGraw-Hill.

[133] Samuelson, P. A.: 1965, 'Proof That Properly Anticipated Prices Fluctuate Randomly'. *Industrial Management Review* **6**, 41–49.

[134] Samuelson, W. and L. Rosenthal: 1986, 'Price Movements as Indicators of Tender Offer Success'. *The Journal of Finance* **41**(2), 481–499.

[135] Savage, L. J.: 1954, *The Foundations of Statistics*. New York and Chapman and Hall and London: John Wiley and Sons.

[136] Savic, D. A. and W. Pedrycz: 1991, 'Evaluation of Fuzzy Linear Regression Models'. *Fuzzy Sets and Systems* **39**, 51–63.

[137] Saxena, U.: 1983, 'Investment Analysis Under Uncertainty'. *The Engineering Economist* **29**(1), 33–40.

[138] Schmeidler, D.: 1989, 'Subjective Probability and Expected Utility Without Additivity'. *Econometrica* **57**, 517–587. First Version: Subjective expected utility without additivity, Forger Institute Paper (1982).

[139] Seth, A. K.: 1990, 'Prof. J. N. Kapur's Views on Entropy Optimization Principle: Part II'. *Bulletin of Mathematical Assoc. of India* **22**(1), 1–42.

[140] Shannon, C. E.: 1948, 'A Mathematical Theory of Communication'. *The Bell System Technical Journal* **27**(3), 379–423; 623–656.

[141] Sharpe, W. F.: 1964, 'Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk'. *Journal of Finance* **19**, 425–442.

[142] Sharpe, W. F.: 1966, 'Mutual Fund Performance'. *Journal of Business* **39**, 119–138.

[143] Shumway, T.: 2001, 'Forecasting Bankruptcy More Accurately: A Simple Hazard Model'. *The Journal of Business* **74**(1), 101–124.

[144] Smimou, K., S. K. Bhatt, and D. W. Dahl: 2001, 'Fuzzy Analytical Hierarchy Process in New Product Screening'. In: *Proc. of European Society for Fuzzy Logic and Technology (EUSFLAT)*. Leicester, UK, pp. 89–93.

[145] So, J. C.: 1987, 'Commodity Futures Risk Premium and Unstable Systematic Risk'. *Journal of Futures Market* **7**, 311–326.

[146] Subrahmanyam, A.: 1994, 'Circuit Breakers and Market Volatility: A Theoretical Perspective'. *Journal of Finance* **49**, 237–254.

[147] Szego, G. P.: 1980, *Portfolio Theory with Application to Bank Asset Management.* New York: Academic Press.

[148] Tanaka, H.: 1987, 'Fuzzy Data Analysis by Possibilistic Linear Models'. *Fuzzy Sets and Systems* **24**, 363–375.

[149] Tanaka, H. and H. Ishibuchi: 1991, 'Identification of Possibilistic Linear Systems by Quadratic Membership Functions of Fuzzy Parameters'. *Fuzzy Sets and Systems* **41**, 145–160.

[150] Tanaka, H., S. Uejima, and K. Asai: 1982, 'Linear Regression Analysis with Fuzzy Model'. *IEEE Transactions on Systems, Man and Cybernetics* **12**, 903–907.

[151] Telser, L. G.: 1958, 'Futures Trading and the Storage of Cotton and Wheat'. *Journal of Political Economy* **66**, 233–255.

[152] Theil, H.: 1969, 'On the Use of Information Theory Concepts in the Analysis of Financial Statements'. *Management Science* **15**, 459–480.

[153] Thomas, M. U.: 1979, 'A Generalized Maximum Entropy Principle'. *Operations Research* **27**(6), 1188.

[154] Thomas, S. F.: 1995, *Fuzziness and Probability.* Wichita, Kansas: ACG Press.

[155] Tobin, J.: 1985, 'Liquidity Preference as Behavior Towards Risk'. *Review of Economic Studies* **25**, 68–85.

[156] Treynor, J. L.: 1966, 'How to Rate Management Investments Funds'. *Harvard Business Review* **43**, 63–75.

[157] Vasicek, O.: 1971, 'Capital Market Equilibrium with No Riskless Borrowing'. Wells Fargo Bank.

[158] Verdegay, J. L.: 1984, 'Application of Fuzzy Optimization in Operational Research'. *Control and Cybernetics* **13**, 229–239.

[159] Viertl, R.: 1986, 'Is It Necessary to Develop a Fuzzy Bayesian Inference?'. In: R. Viertl (ed.): *Probability and Bayesian Statistics.* New York: Plenum Press.

[160] Viertl, R.: 1996, *Statistical Methods for Non-Precise Data.* Boca Raton, New York, London, Tokyo: CRS Press.

[161] Voros, J.: 1986, 'Portfolio Analysis–An Analytic Derivation of the Efficient Portfolio Frontier'. *European Journal of Operational Research* **23**, 294–300.

[162] Wagner, H. M.: 1975, *Principles of Operations Research.* Englewood Cliffs, New Jersey: Prentice Hall.

[163] Walley, P.: 1991, *Statistical Reasoning with Imprecise Probabilities.* London: Chapman and Hall.

[164] Wang, H. F.: 2000, 'Insight of a Fuzzy Regression Model'. *Fuzzy Sets and Systems* **112**, 355–369.

[165] Watanabe, N. and T. Imaizumi: 1993, 'A Fuzzy Statistical Test of Fuzzy Sets and Systems'. *Fuzzy Sets and Systems* **53**, 167–178.

[166] White, D. J.: 1969, *Decision Theory.* Boston: George Allen Unwin Ltd.

[167] Winkler, R. L. and C. B. Barry: 1975, 'A Bayesian Model for Portfolio Selection and Revision'. *The Journal of Finance* **30**(1), 179–192.

[168] Yuan, C. K.: 1993, 'Parameter Estimations of Normal Fuzzy Variables'. *Fuzzy Sets and Systems* **55**, 179–185.

[169] Zadeh, L. A.: 1965, 'Fuzzy Sets'. *Inform. and Control* **8**, 338–353.

[170] Zadeh, L. A.: 1968, 'Probability Measures of Fuzzy Events'. *Journal of Math. Analysis and Applications* **23**(2), 421–427.

[171] Zadeh, L. A.: 1975, 'The Concept of a Linguistic Variable and its Application to Approximate Reasoning-I'. *Information Sciences* **8**, 199–249.

[172] Zadeh, L. A.: 1978, 'Fuzzy Sets as a Basis for a Theory of Possibility'. *Fuzzy Sets and Systems* **1**, 3–28.

[173] Zadeh, L. A.: 1984, 'Fuzzy Probabilities'. *Information Processing Management* **20**, 363–372.

[174] Zadeh, L. A.: 1986, 'Is Probability Theory Sufficient for Dealing with Uncertainty in AI: A Negative View'. In: L. N. Kanal and J. F. Lemmer (eds.): *Uncertainty in Artificial Intelligence.* North Holland, Amsterdam and New York: Elsevier Science Publishers B. V., pp. 103–116.

[175] Zhang, H., X. Ma, W. Xu, and P. Wang: 1993, 'Design of Fuzzy Controllers for Complex Systems with an Application to 3-Stage Inverted Pendulum'. *Information Sciences* **72**(3), 271–284.

[176] Zimmerman, H. J.: 1997, 'Media Selection and Other Applications of Fuzzy Linear Programming'. In: D. Dubois, H. Prade, and R. Yager (eds.): *Fuzzy Information Engineering. A Guided Tour of Applications.* New York: J. Wiley.

Appendices

Appendix A. Part of the VBA Program code

The following is a part of the used VBA program to do the computation and generate the efficient frontiers.

```
Sub EFwoSSwWgraph()

'graph values without using table

'plots each coordinate

Dim cht As Chart

numSeriesToCreate = Application.Range("rMeans").Rows.Count

Set cht = Charts.Add

cht.SeriesCollection.NewSeries

For i = 1 To numSeriesToCreate

cht.SeriesCollection(i).Name = Application.Range("rMeans").Cells(i,
1)

cht.SeriesCollection(i).Values =
Application.Range("rWidths").Cells(i, 1)
```

```
cht.SeriesCollection(i).XValues =
Application.Range("rSigmas").Cells(i, 1)
If i < numSeriesToCreate Then
cht.SeriesCollection.NewSeries
End If
Next i
cht.Location Where:=xlLocationAsNewSheet, Name:="EFwoSSwW3"
cht.ChartType = xlSurface
With cht
.HasTitle = True
.ChartTitle.Characters.Text = _
"Efficient Frontier with Widths and without Short Sales"
.Axes(xlCategory).HasTitle = True
.Axes(xlCategory).AxisTitle.Characters.Text = "Sigma"
.Axes(xlSeries).HasTitle = True
.Axes(xlSeries).AxisTitle.Characters.Text = "Mean"
.Axes(xlValue).HasTitle = True
.Axes(xlValue).AxisTitle.Characters.Text = "Width"
End With
End Sub
Sub createTable()
'table to create 3d graph
```

```
'table consists of the values to be graphed
Dim ws As Excel.Worksheet
Set ws = Application.Worksheets("Sheet2")
ws.Activate
ws.Range(Cells(1, 2), Cells(1, 226)).Name = "rYValues"
ws.Range("rYValues").FormulaArray = "=TRANSPOSE(rMeans)"
ws.Range(Cells(2, 1), Cells(226, 1)).Name = "rXValues"
ws.Range("rXValues").FormulaArray = "=rSigmas"
ws.Range(Cells(2, 2), Cells(226, 226)).Name = "rZValues"
For i = 1 To 226
ws.Range("rZValues").Cells(i, i) =
Application.Range("rWidths").Cells(i, 1)
Next i
End Sub
Sub graphTable()
'graph values using table
Dim cht As Chart
numSeriesToCreate = Application.Range("rMeans").Rows.Count
Set cht = Charts.Add
cht.SeriesCollection.NewSeries
cht.SeriesCollection(1).XValues = Application.Range("rSigmas")
For i = 1 To numSeriesToCreate
```

```
cht.SeriesCollection(i).Name = Application.Range("rMeans").Cells(i,
1)
cht.SeriesCollection(i).Values =
Application.Range("rZValues").Columns(i)
If i < numSeriesToCreate Then
cht.SeriesCollection.NewSeries
End If
Next i
cht.Location Where:=xlLocationAsNewSheet ', Name:="EFwoSSwW4"
cht.ChartType = xlSurface
With cht
.HasTitle = True
.ChartTitle.Characters.Text = _
"Efficient Frontier with Widths and without Short Sales"
.Axes(xlCategory).HasTitle = True
.Axes(xlCategory).AxisTitle.Characters.Text = "Sigma"
.Axes(xlSeries).HasTitle = True
.Axes(xlSeries).AxisTitle.Characters.Text = "Mean"
.Axes(xlValue).HasTitle = True
.Axes(xlValue).AxisTitle.Characters.Text = "Width"
End With
End Sub
```

Appendix B. Explanation of expansion rules of daily limits in WCE

Before October 10, 2000, the regular daily price limits were \$5.00/tonne for feed wheat and western barley and \$10.00/tonne for canola and flaxseed. These limits could be expanded (increased) in certain situations, as follows:

- Expanded daily limits rule in 1991:

Starting with the March 1991 contracts, there has been a special rule for contracts in delivery. If a contract is in its delivery month and closes at the limit in the same direction (up or down) for two successive days, its daily limit is expanded to 1.5 times normal. If the contract closes at this expanded daily limit for both of the next two days, the limit is increased to two times normal. If the limit has been expanded and the contract does not close at its limit, the limit will return to the normal limit.

- Expanded daily limits changed starting sometime in 1991 or 1992:

If two of the three nearest contract months close limit up or down, the daily limit for that commodity is expanded to 1.5 times normal for the next three days. If two of the nearest three contract months close limit up or down on the third day, the limit will remain at 1.5 times normal for the next three days.

If a contract in its delivery month closes limit up or down, the daily limit for that contract is expanded to 1.5 times normal for the next three days. If that contract closes limit up or down on the third day, the limit will remain at 1.5 times normal for the next three days.

Commodity daily limit expanded to 1.5 times normal expanded to two times normal feed wheat and western barley from $5.00 to $7.50 and to $10.00 canola and flaxseed from $10.00 to $15.00 and to $20.00. There are no expanded daily limits after October 10, 2000.