TOPOLOGICAL SPACES POSSESSING COMPACTIFICATIONS WITH ZERO-DIMENSIONAL REMAINDERS

by

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A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY
(MATHEMATICS)
at the

UNIVERSITY OF MANITOBA

1982

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ACKNOWLEDGEMENTS

I am deeply indebted to my advisor, Grant Woods, whose excellent teaching first stimulated my interest in topology. I give sincere thanks to him for his support and guidance during my graduate years. I wish to thank Murray Bell, Alan Dow, Ortwin Förster, Jack Porter, and Marlon Rayburn, each of whom contributed either specifically to my work or more generally to my mathematical development. Cliff Berish, in his unique way, has been a source of encouragement and understanding during the preparation of this thesis. Finally, my sincere appreciation goes to my parents for their continuing support and understanding.

I am grateful to the Canada Council of Canada, the National Sciences and Engineering Research Council of Canada, and the University of Manitoba for their financial assistance throughout my graduate studies.

ABSTRACT

Compactifications of completely regular Hausdorff spaces have been studied in several ways; two of these are by embedding topological spaces in products of other spaces, and by supplying topological spaces with additional structure.

In 1930 Tychonoff showed that a topological space is completely regular and Hausdorff if and only if it can be embedded in some product of closed unit intervals. By embedding a completely regular Hausdorff space X in the product $X^{C^*(X,I)}$, one obtains the compactification of X which is the projective maximum among all compactifications of X. This compactification of X is called the Stone-Cech compactification of X; any other compactification of X is some quotient space of the Stone-Cech compactification of X.

In 1948 Samuel developed a relationship between uniformities and compactifications, and in 1952 Smirnov established the correspondence between proximities and compactifications. In this paper we will consider both quotient spaces of Stone-Cech compactifications, and proximities on topological spaces, to study remainders of compactifications.

More specifically, we wish to determine when a completely regular Hausdorff space has a compactification with a 0-dimensional remainder. Such a compactification will be called 0-dimensional at infinity (denoted by 0.I.); a 0-space is any space possessing a 0.I. compactification. In 1942 Freudenthal showed that a rimcompact separable space is a 0-space. Morita using uniformites, and Skylarenko using proximities, showed in 1952 and 1966 respectively that any rimcompact space X has a compactification which has a basis of open sets whose boundaries are contained in X. Skylarenko showed that a 0-space which is Lindelöf at infinity is rimcompact, but mentioned the existence of non-rimcompact 0-spaces. He proved that the maximum 0.I. compactification of a rimcompact space is the minimum perfect compactification of that space, and in 1969 McCartney showed that any 0-space X has a maximum 0.I. compactification which will also be the minimum perfect compactification of X.

In Chapters 2-4 we develop a theory for a class of spaces intermediate between rimcompact spaces and 0-spaces, which we will call "almost rimcompact spaces". Each almost rimcompact space will possess a compactification in which each point of the remainder has a basis (in the compactification) of open sets whose boundaries do not intersect the remainder of the compactification. The approach will be to show that if a space satisfies a condition similar to rimcompactness, then an easily defined quotient space of the Stone-Cech compactification of X is a compactification of X with the property mentioned above; the converse is also true. A proximal characterization of almost rimcompact spaes is also given. To characterize the larger class of 0-spaces, in Chapter 5 we define a relation α on the power set of a space X. We show that α is a proximity compatible with the topology of X if and only if X is a 0-space, in which case the compactification αX of X associated with α is the maximum 0.I. compactification of X.

In Chapter 6 we consider the problem of extending maps of 0-spaces over their maximum 0.I. compactifications. In 1956 Morita showed that if X and Y are locally compact and paracompact, then any closed map from X into Y extends to a map from the maximum 0.I. compactification of X into the maximum 0.I. compactification of Y. Nowinski, in 1972, showed that it is sufficient for X and Y to be locally compact and metacompact. We shall prove that in order for a closed map of X into Y to extend to a map from the maximum 0.I. compactification of X into the maximum 0.I. compactification of Y, it is necessary and sufficient that the map satisfy a condition imposed on preimages of pairs of points. This result is used to show that if i) X is a realcompact or metacompact 0-space and Y is a rimcompact space in which the set of q-points has discrete complement, or if ii) X is a metacompact 0-space or a locally compact realcompact space, and Y is a rimcompact k-space, then any closed map from X into Y extends over the maximum O.I. compactifications of X and Y.

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CHAPTER 1

PRELIMINARY NOTIONS

In the following, all spaces are assumed to be completely regular and Hausdorff.

The symbols R, N, Q, P and I will denote the real numbers, natural numbers, rational numbers, irrational numbers and unit interval respectively. When used as topological spaces, R is given its usual interval topology, and the remaining spaces are given the subspace topology inherited from R. If X and Y are topological spaces, the collection of continuous functions from X to Y is denoted by C(X,Y). The ring of continuous real-valued functions on a space X is denoted by C(X), and its subring of bounded members by C*(X). If $f \in C(X)$, the set $\{x: f(x) = 0\}$ is called the <u>zeroset</u> of f, and is denoted by Z(f). Two subsets A and B of X are said to be <u>completely separated</u> in X if there exists $f \in C(X)$ such that $A \subset f^{\leftarrow}(0)$, and $B \subset f^{\leftarrow}(1)$. A <u>map</u> is a continuous surjection. A function $f: X \to Y$ is <u>closed</u> if whenever F is a closed subset of X, then f[F] is a closed subset of Y. We use without mention the following well known fact. If $f : X \to Y$ is a closed map, and $f^{\leftarrow}[S] \subset U$, where $S \subset Y$, and U is open in X, then there is an open set V of Y such that $f^{\leftarrow}[S] \subset f^{\leftarrow}[V] \subset U$. A closed function $f: X \to Y$ is <u>perfect</u> if for each $y \in Y$, $f^{\leftarrow}(y)$

is compact. If $f \in C(X,Y)$ and $Z \subset X$, we use $f|_A$ (the restriction of f to A) to denote the map of A into Y defined by $(f|_A)(a) = f(a)$ for each $a \in A$. If $A \subset X$, then A is C-embedded (respectively, C*-embedded) in X if for each $f \in C(A)$ (respectively, $C^*(A)$), there exists $g \in C(X)$ (respectively, $C^*(X)$) such that $g|_A = f$.

If X is a topological space, a compactification KX of X is a compact Hausdorff space in which X is densely embedded. For background information on compactifications, the reader is referred to [GJ] or [Ch]. It is well known that a topological space X is completely regular and Hausdorff if and only if X has a compactification. If KX and JX are two compactifications of X, we write $KX \geq JX$ (and say KX is larger than JX) if there is a map $f: KX \to JX$ such that $f(x) = x$ for all $x \in X$. We write $KX = JX$ (and say KX and JX are equivalent compactifications of X) if there is a homeomorphism $h: KX \to JX$ such that $h(x) = x$ for all $x \in X$.

The following is an easy consequence of 3.2.1 of [En].

1.1 Proposition (Taimanov's theorem) : Let KX and KY be compactifications of X and Y respectively, and let f be a map from X into Y. There is a map $f': KX \to KY$ such that $f'|_X = f$ if and only if for $A, B \subset Y$, $\mathrm{Cl}_{KY}A \cap \mathrm{Cl}_{KY}B = \phi$ implies $\mathrm{Cl}_{KX}f^{\leftarrow}(A) \cap \mathrm{Cl}_{KX}f^{\leftarrow}(B) = \phi$.

In particular, if KX and JX are compactifications of X, then $KX \geq JX$ if and only if for $A, B \subset X$, $Cl_{JX}A \cap Cl_{JX}B = \phi$ implies $Cl_{KX}A \cap Cl_{KX}B = \phi$.

If f and f′ are as in 1.1, we say that f <u>extends</u> to $f' \in C(KX,KY)$.

Let $\underline{K}(X)$ denote the family (of equivalence classes) of compactifications of X. The relation $\geq$ is a partial order on $\underline{K}(X)$, and $\underline{K}(X)$ is a complete upper semilattice when partially ordered by $\geq$. The largest element of $\underline{K}(X)$ is the <u>Stone-Cech compactification</u> of X, denoted by βX. The compactification βX is characterized as that compactification of X in which X is C*-embedded. In the sequel, if $KX \in \underline{K}(X)$, the natural map from βX into KX is denoted by Kf. The following is a consequence of 6.12 of [GJ].

1.2 <u>Proposition</u> : Suppose that KX and JX are compactifications of X, and that $KX \geq JX$. Let f denote the natural map from KX into JX. Then $f[KX \setminus X] = JX \setminus X$.

We will often call $KX \setminus X$ the <u>remainder</u> of KX.

A topological property P is <u>hereditary</u> if whenever a space X has property P, and $S \subset X$, then S has property P. A property P is <u>productive</u> if whenever $\{X_i : i \in I\}$ is a set of spaces, each of which has property P, then $\Pi\{X_i : i \in I\}$ has property P.

It is shown in [ES] that any non-empty family $\{K_i X : i \in I\}$ of compactifications of X has a least upper bound. Let $P = \Pi\{K_i X : i \in I\}$. For each $x \in X$, let e(X) be the element of P each of whose coordinates is x. Then $e : X \to P$ is an embedding of X in P, and we can identify e[X] with X. Then $Cl_P X$ is the least upper bound of $\{K_i X : i \in I\}$. The natural map from $Cl_P X$ to $K_i X$ is the restriction to $Cl_P X$ of the projection $p_i : P \to K_i X$. It follows from 1.2 that $Cl_P X \setminus X \subset \Pi\{K_i X \setminus X : i \in I\}$. Thus we have the following result.

1.3 <u>Proposition</u> : Let P be a topological property which is hereditary and productive. If a space X has a compactification whose remainder has property P, then X has a maximum compactification whose remainder has property P.

A <u>proximity</u> on a space X is a relation δ on $\underline{P}(X)$ satisfying,

(P_1) $\phi \not\delta A$ for any $A \subset X$,

(P_2) if $A, B \subset X$, and $A \cap B \neq \phi$, then $A \delta B$,

(P_3) if $A, B \subset X$, and $A \delta B$, then $B \delta A$,

(P_4) for $A, B_1, B_2 \subset X$, $A \delta (B_1 \cup B_2)$ if and only if $A \delta B_1$ or $A \delta B_2$,

and

(P_5) if $A, B \subset X$, and $A \not\delta B$, then there exists $C \subset X$ such that $A \not\delta C$, and $B \not\delta X \setminus C$.

Our standard reference on proximities is [NW]. The pair (X, δ) is called a <u>proximity space</u>, and is <u>separated</u> if for $x, y \in X$, $x \, \delta \, y$ implies $x = y$. (Although strictly speaking we should use the notation $\{x\} \, \delta \, \{y\}$ we shall simply write $x \, \delta \, y$.) We often write $A <_\delta B$ to mean $A \not\delta X \setminus B$.

If (X, δ) is a proximity space and $A \subset X$, define A^δ to be $\{x \in X: x \, \delta \, A\}$. Then $A \to A^\delta$ is a Kuratowski closure operator, so δ induces a topology τ on X, defined by $\tau(\delta) = \{X \setminus A^\delta: A \subset X\}$. If τ is a given topology on X, then δ is called <u>compatible</u> with τ if $\tau(\delta) = \tau$. In the sequel, any proximity considered on a topological space X is assumed to be compatible with the topology of X. It is well known [NW] that a topological space X is completely regular and Hausdorff if and only if there is some separated proximity on X which induces the topology of X.

If δ_1 and δ_2 are two proximites on a space X, we say $\delta_1 \geq \delta_2$ if for $A, B \subset X$, $A \, \delta_1 \, B$ implies that $A \, \delta_2 \, B$. If (X, δ) and (Y, γ) are proximity spaces, then a function $f: X \to Y$ is a <u>proximity map</u> if for $A, B \subset X$, $A \, \delta \, B$ implies $f[A] \, \gamma \, f[B]$, or equivalently, for $C, D \subset Y$, $C \not\gamma D$ implies $f^{\leftarrow}[C] \not\delta f^{\leftarrow}[D]$.

A compact Hausdorff space X admits a unique compatible proximity δ defined by (for $A, B \subset X$) $A \delta B$ if and only if $Cl_X A \cap Cl_X B = \phi$. Hence if KX is a compactification of X, KX induces a proximity δ on X defined by (for $A, B \subset X$), $A \delta B$ if and only if $Cl_{KX}A \cap Cl_{KX}B \neq \phi$.

The converse is also true. Given a proximity δ on X, we can construct a unique compactification δX of X (called the proximal compactification of X associated with δ) satisfying (for $A, B \subset X$) $A \delta B$ if and only if $Cl_{\delta X}A \cap Cl_{\delta X}B \neq \phi$. Then $\delta_1 X \geq \delta_2 X$ if and only if $\delta_1 \geq \delta_2$, so the partially ordered set of proximities on X is order isomorphic to $\underline{K}(X)$. The proximity β on X inducing βX is the largest proximity on X, and is defined by (for $A, B \subset X$), $A \not\beta B$ if and only if A and B are completely separated in X. If (X,δ) and (Y,γ) are proximity spaces, then according to 1.1 and the preceding remarks, if f is a map from X into Y, then f extends to $F \in C(\delta X, \gamma Y)$ if and only if f is a proximity map.

A <u>decomposition</u> $\underline{D}$ of a space X is a collection of disjoint subsets of X whose union is X. Define a function P (called the natural map) of X into $\underline{D}$ by letting $P(x)$, for $x \in X$, be the element of $\underline{D}$ containing x. Then the <u>decomposition space</u> $X/\underline{D}$ is the quotient space whose elements are the elements of $\underline{D}$, and which has the quotient topology induced by the natural map P. An open set V of X is <u>saturated</u> (with respect to $\underline{D}$) if V is a union of elements of $\underline{D}$. Clearly, if V is saturated with respect to $\underline{D}$, then

$P^{\leftarrow}[P[V]] = V$. Since P is a quotient map, if V is clopen in X and saturated with respect to $\underline{D}$, P[V] is clopen in X/$\underline{D}$. The collection $\underline{D}$ is an <u>upper semicontinuous decomposition</u> of X if for each $D \in \underline{D}$, and each open set U of X containing D, there exists a saturated open set V of X such that $D \subset V \subset U$.

Let $\underline{D}$ be a decomposition of $\beta X \setminus X$ into compact sets, and let $\underline{D}'$ denote the decomposition of βX consisting of $\underline{D} \cup \{\{x\}: x \in X\}$. Then $\beta X/\underline{D}'$ is a compactification of X (where X is identified with $\{\{x\}: x \in X\}$) if and only if $\underline{D}'$ is an upper semicontinuous decomposition of βX [Ke].

A space X is <u>connected</u> if whenever $X = U \cup V$, where U and V are nonempty open subsets of X, then $U \cap V \neq \phi$. The <u>connected component</u> C_x of $x \in X$ is the union of all connected subspaces of X containing x. A space X is <u>totally disconnected</u> if $C_x = \{x\}$ for all $x \in X$. The <u>quasi-component</u> Q_x of $x \in X$ is the intersection of all closed-and-open (denoted <u>clopen</u>) subsets of X containing x. A space X is <u>fully disconnected</u> if $Q_x = \{x\}$ for all $x \in X$.

1.4 <u>Definition</u> : The decomposition of βX consisting of $\{\{x\}: x \in X\} \cup \{C_p: C_p$ is the connected component in $\beta X \setminus X$ of $p \in \beta X \setminus X\}$ is denoted by $\underline{C}(\beta X)$. The decomposition of βX consisting of $\{\{x\}: x \in X\} \cup \{Q_p: Q_p$ is the quasi-component in $\beta X \setminus X$ of $p \in \beta X \setminus X\}$ is denoted by $\underline{Q}(\beta X)$.

It is clear that if V is open in βX and $V \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, then V is saturated with respect to both $\underline{C}(\beta X)$ and $\underline{Q}(\beta X)$.

If U is an open subset of X, and $\delta X \in \underline{K}(X)$, then $\mathrm{Ex}_{\delta X}U$ is defined to be $\delta X \setminus \mathrm{Cl}_{\delta X}(X \setminus U)$. The set $\mathrm{Ex}_{\delta X}U$ is often called the <u>extension</u> of U in δX. It is an easy exercise to verify (i), (ii), (iii) and (iv) of the following proposition. Statement (v) is implicit in the proof of Lemma 2 of [Sk], and (vi) follows from (v).

1.5 <u>Proposition</u> : Let $\delta X \in \underline{K}(X)$.

(i) If W is open in δX, then $W \subset \mathrm{Ex}_{\delta X}(W \cap X)$.

(ii) If U and V are open in X, then $\mathrm{Ex}_{\delta X}(U \cap V) = (\mathrm{Ex}_{\delta X}U) \cap (\mathrm{Ex}_{\delta X}V)$.

(iii) If U is open in X, then $(\mathrm{Ex}_{\delta X}U) \cap X = U$, hence $\mathrm{Cl}_{\delta X}U = \mathrm{Cl}_{\delta X}\mathrm{Ex}_{\delta X}U$.

(iv) If F is closed in X, U is open in X, and $F \cap U = \phi$, then $\mathrm{Cl}_{\delta X}F \cap \mathrm{Ex}_{\delta X}U = \phi$.

(v) If U and V are open in X, then $\mathrm{Ex}_{\delta X}(U \cup V) \setminus (\mathrm{Ex}_{\delta X}U \cup \mathrm{Ex}_{\delta X}V) \subset \mathrm{Cl}_{\delta X}U \cap \mathrm{Cl}_{\delta X}V$.

(vi) If U and V are open in X, and $\mathrm{Cl}_{\delta X}U \cap \mathrm{Cl}_{\delta X}V = \phi$, then $\mathrm{Ex}_{\delta X}(U \cup V) = \mathrm{Ex}_{\delta X}U \cup \mathrm{Ex}_{\delta X}V$.

If U is any open subset of X, then it is easy to verify that $\mathrm{Ex}_{\delta X}U$ is the largest open subset of δX whose intersection with X is the set U. The collection $\{Ex_{\delta X}U: U$ is an open subset of $X\}$ of open sets of δX is easily seen to be a basis for the topology of δX.

If $B \subset X$, the <u>boundary</u> of B in X, denoted by $bd_X B$, is defined to be the set $Cl_X B \cap Cl_X(X \setminus B)$. A compactification δX of X is a <u>perfect compactification</u> of X if for each open subset U of X, $Cl_{\delta X}(bd_X U) = bd_{\delta X}(Ex_{\delta X}U)$. According to the corollary to Lemma 1 of [Sk], βX is a perfect compactification of X.

The equivalence of (i), (ii), and (iii) of the following proposition appear in Theorems 1 and 2 of [Sk].

1.6 <u>Proposition</u> : Let $\delta X \in \underline{K}(X)$. The following are equivalent.

(i) δX is a perfect compactification of X.

(ii) If U and V are disjoint open sets of X, then $Ex_{\delta X}(U \cup V) = Ex_{\delta X}U \cup Ex_{\delta X}V$.

(iii) For each $p \in \delta X$, $(\delta f)^{\leftarrow}(p)$ is a connected subset of βX.

Following [Sk], we say a space X is <u>punctiform</u> if every connected compact subset of X consists of one point. The next proposition follows from Theorem 3 (and its proof) of [Sk].

1.7 <u>Proposition</u> : A space X has a minimal perfect compactification KX if and only if X has at least one compactification with punctiform remainder. In this case the minimal perfect compactification KX is unique; $KX \setminus X$ is punctiform, and KX is the largest of all compactifications of X with punctiform remainder. The collection of sets $\{(Kf)^{\leftarrow}(p) : p \in KX \setminus X\}$ are maximal connected compact subsets of $\beta X \setminus X$.

Suppose that X is a space which has a compactification with totally disconnected remainder. Since total disconnectedness is productive and hereditary, it follows from 1.3 that X has a maximum compactification SX having totally disconnected remainder. Since a totally disconnected space is punctiform, it follows from 1.7 that X has a minimum perfect compactification JX, and that $SX \leq JX$. It is easy to verify that if X has a compactification with totally disconnected remainder, then the maximal compact connected subsets of $\beta X \setminus X$ are precisely the connected components of $\beta X \setminus X$, hence by 1.7 $JX = \beta X / \underline{C}(\beta X)$.

A straightforward computation using 2.5 of [Mc] and 1.6 gives us the following.

1.8 <u>Proposition</u> : Let X be a space, and suppose that K_1X is a compactification of X which is not perfect. Then there is a compactification K_2X such that $K_2X \geq K_1X$, and if $f : K_2X \to K_1X$ denotes the natural map, then f is not a homeomorphism, but $|f^{\leftarrow}(p)| \leq 2$ for each $p \in K_1X \setminus X$.

It is easy to verify that if K_1X and K_2X are as in 1.8, and K_1X is totally disconnected, then K_2X is totally disconnected. Thus we have the following.

1.9 <u>Theorem</u> : Let X be a space which has a compactification with totally disconnected remainder. Then X has a maximum compactification SX having totally disconnected remainder. The compactification $SX = \beta X / \underline{C}(\beta X)$, and is the minimum perfect compactification of X.

A space X is <u>zero-dimensional</u> (denoted by 0-dimensional) if X has a basis of clopen sets. A space X is <u>strongly 0-dimensional</u> if any two disjoint zerosets of X are contained in disjoint clopen sets of X. Any 0-dimensional space X has a maximum 0-dimensional compactification denoted by $\beta_0 X$. (Here, and in the following, a maximum *P*-compactification of X means a compactification of X which is the maximum in the class of compactifications of X with property *P*.) The compactification $\beta_0 X$ can be characterized as that compactification of X to which all continuous $\{0,1\}$-valued functions on X can be continuously extended (where $\{0,1\}$ denotes the two-point discrete space). The proximity β_0 on X inducing $\beta_0 X$ is defined by (for $A, B \subset X$) $A \not{\beta}_0 B$ if and only if A and B are contained in disjoint clopen subsets of X. The compactifications $\beta_0 X$ and βX are equivalent if and only if X is strongly 0-dimensional.

1.10 Definitions : A space X is a 0-space if X has a compactification KX such that KX \ X is 0-dimensional. We will say that KX is a 0-dimensional at infinity (denoted by 0.I.) compactification of X.

It follows from 1.7 that if KX is a 0.I. compactification of X and JX is a perfect compactification of X, then $JX \geq KX$. Hence any perfect 0.I. compactification of X is a maximum 0.I. compactification of X. The converse is also true. If X is a 0-space, then X has a maximum 0.I. compactification (which we shall denote by F_0X) which will also be the minimal perfect compactification of X (2.2, 3.3 of [Mc]). Since F_0X is a perfect, 0.I. compactification of X, for each $p \in F_0X \setminus X$, $(F_0f)^{\leftarrow}(p)$ is a connected compact quasi-component of $\beta X \setminus X$. Then $\underline{C}(\beta X) = \underline{Q}(\beta X) = \{(F_0f)^{\leftarrow}(p): p \in F_0X\}$. Also each element of $\underline{C}(\beta X)$ contained in $\beta X \setminus X$ has a basis in βX of open sets of βX whose intersections with $\beta X \setminus X$ are clopen in $\beta X \setminus X$ (again, by the 0-dimensionality of $F_0X \setminus X$). On the other hand, if $\underline{Q}(\beta X) = \underline{C}(\beta X)$, and $\underline{C}(\beta X)$ is an upper semicontinuous decomposition of βX into compact sets, then $\beta X / \underline{C}(\beta X)$ is a perfect compactification of X. Suppose that in addition, elements of $\underline{C}(\beta X)$ contained in $\beta X \setminus X$ have a basis in βX of open sets whose intersections with $\beta X \setminus X$ are clopen in $\beta X \setminus X$. It follows from 1.2, the remark following

Definition 1.4 and the properties of decomposition spaces, that $\beta X\ /\ \underline{\mathcal{C}}(\beta X)$ is a 0.I. compactification of X. Thus $\beta X\ /\ \underline{\mathcal{C}}(\beta X)$ is a perfect 0.I. compactification of X and hence is F_0X.

Following the terminology of [M_1] and [Sk], we say that an open set U of X is <u>π-open</u> in X if bd_XU is compact. The intersection and union of finitely many π-open sets are π-open, as is the complement of a π-open set. A space X is <u>rimcompact</u> if X has a basis of π-open sets. Any rimcompact space is a 0-space [M_1]. The maximum 0.I. compactification of a rimcompact space X is called the <u>Freudenthal compactification</u> of X, and is denoted by FX.

1.11 <u>Definition</u> : If $F_1, F_2 \subset X$, then F_1 and F_2 are <u>π-separated</u> in X if there is a π-open set U of X such that $F_1 \subset U$, and $Cl_XU \cap F_2 = \phi$. We shall often write "$\{x\}$ and F are π-separated" as "x and F are π-separated".

If X is any topological space, define δ to be a relation on $P(X)$ as follows: (for A, B $\subset$ X) $A \not\delta B$ if and only if A and B are π-separated in X. The relation δ is a proximity on X if and only if X is rimcompact, in which case $\delta X = FX$, the maximum 0.I. compactification of X [Sk].

It is easily verified that if X is 0-dimensional and A, B $\subset$ X, then A and B are contained in disjoint clopen subsets of X if and only if A and B are π-separated in X. Thus if X is 0-dimensional, $FX = \beta_0 X$.

For any space X, the <u>residue</u> of X (denoted by R(X)) is the set of points at which X is not locally compact. If KX is any compactification of X, then $Cl_{KX}(KX \setminus X) = R(X) \cup (KX \setminus X)$.

The notions used from set theory are standard. An ordinal is thought of as the set of its predecessors, and a cardinal as an initial ordinal. The symbol ω_α is used to denote the α'th cardinal. For any set X, $|X|$ denotes the cardinality of X.

CHAPTER 2

ALMOST RIMCOMPACT SPACES

In this chapter we develop a theory of 0.I. compactifications for a class of spaces intermediate between the class of rimcompact spaces and the class of 0-spaces (recall Definition 1.10). This class will be characterized and compared to the class of rimcompact spaces by discussing how the remainder of the maximal 0.I. compactification of a member of the class is embedded in the compactification.

We shall be working with π-open subsets of a space X and related open sets of compactifications of X; we begin by listing some straightforward results.

2.1 Definition : Let $KX \in \underline{K}(X)$, and let W be open in KX. If $bd_{KX}W \subset X$, W is said to be a small boundary (denoted by sb) subset of KX.

2.2 Lemma : Let $KX \in \underline{K}(X)$.

(i) The intersection (union) of finitely many sb open subsets of KX is an sb open subset of KX.

If W is an sb open subset of KX, then

(ii) $W \cap X$ is π-open in X.

(iii) $W = Ex_{KX}(W \cap X)$.

(iv) $KX \setminus Cl_{KX}W$ is sb in KX.

(v) If $\delta X \geq KX$, and if $f: \delta X \to KX$ is the natural map, then $f^{\leftarrow}[W]$ is an sb open subset of δX.

Proof : (i) This follows from the fact that if A, B are subsets of X, then $[bd_X(A \cap B)] \cup [bd_X(A \cup B)] \subset [bd_X A \cup bd_X B]$.

(ii) As $bd_X(W \cap X) \subset bd_{KX}W \cap X = bd_{KX}W$, the set $bd_X(W \cap X)$ is a closed subset of a compact subset of X, and hence is compact.

(iii) It is sufficient to show that $Ex_{KX}(W \cap X) \subset W$, since the reverse inclusion is true for any open subset W of KX. Now $Ex_{KX}(W \cap X) \setminus W \subset Cl_{KX}(W \cap X) \setminus W = Cl_{KX}W \setminus W = bd_{KX}W \subset X$, while $Ex_{KX}(W \cap X) \cap X \subset W$, hence $Ex_{KX}(W \cap X) \subset W$.

(iv) This follows from the fact that $bd_X(X \setminus Cl_X A) \subset bd_X A$, for any subset A of a space X.

(v) This is obvious, since $bd_{\delta X}f^{\leftarrow}[W] \subset f^{\leftarrow}[bd_{KX}W] \subset X$. □

2.3 <u>Lemma</u> : Let KX be a perfect compactification of X. Then :

(i) If V is an open subset of βX, and $V \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, then $(Kf)^{\leftarrow}[(Kf)[V]] = V$.

If U is π-open in X, and F is closed in X, then

(ii) $$Cl_{KX}U \cap (KX \setminus X) = Cl_{KX}Ex_{KX}U \cap (KX \setminus X) = (Ex_{KX}U) \cap (KX \setminus X).$$

(iii) $F \subset U$ if and only if $Cl_{KX}F \subset Ex_{KX}U$.

(iv) $F \cap Cl_X U = \phi$ if and only if $Cl_{KX}F \cap Cl_{KX}U = \phi$.

Proof : (i) Recall that by 1.2 $(Kf)[\beta X \setminus X] = KX \setminus X$, and that $(Kf)|_X$ is the identity map on X. Clearly $(Kf)^{\leftarrow}[(Kf)[V \cap X]] = V \cap X$, so it is sufficient to show that $(Kf)^{\leftarrow}[(Kf)[V \cap (\beta X \setminus X)]] = V \cap (\beta X \setminus X)$. Now KX is a perfect compactification of X, so according to 1.6, for each $p \in KX \setminus X$, $(Kf)^{\leftarrow}(p)$ is a connected subset of $\beta X \setminus X$. This implies that either $(Kf)^{\leftarrow}(p) \subset V \cap (\beta X \setminus X)$, or $(Kf)^{\leftarrow}(p) \cap (V \cap (\beta X \setminus X)) = \phi$. If $p \in (Kf)[V \cap (\beta X \setminus X)]$, then $(Kf)^{\leftarrow}(p) \cap (V \cap (\beta X \setminus X)) \neq \phi$, hence $(Kf)^{\leftarrow}(p) \subset V \cap (\beta X \setminus X)$; in other words, $V \cap (\beta X \setminus X) = (Kf)^{\leftarrow}[(Kf)[V \cap (\beta X \setminus X)]]$.

(ii) Since U is π-open in X, and KX is a perfect compactification of X, $Cl_{KX}Ex_{KX}U \setminus Ex_{KX}U = bd_{KX}Ex_{KX}U = Cl_{KX}bd_X U = bd_X U \subset X$.

(iii) Clearly, if $Cl_{KX}F \subset Ex_{KX}U$, then $F \subset U$. On the other hand, if $F \subset U$, then $Cl_{KX}F \setminus Ex_{KX}U = (Cl_{KX}F \setminus Ex_{KX}U) \cap (KX \setminus X) \subset (Cl_{KX}U \setminus Ex_{KX}U) \cap (KX \setminus X) = \phi$, according to (ii).

(iv) Clearly, if $Cl_{KX}F \cap Cl_{KX}U = \phi$, then $F \cap Cl_X U = \phi$. Conversely, if $F \cap Cl_X U = \phi$, then $Cl_{KX}F \cap Cl_{KX}U = Cl_{KX}F \cap Cl_{KX}U \cap (KX \setminus X) = Cl_{KX}F \cap Ex_{KX}U \cap (KX \setminus X)$, by (ii), while the latter set is empty by 1.5 (iv). □

It is an easy exercise to find examples showing that if KX is not a perfect compactification of X, then none of the statements of 2.3 need be true. For example, let ωR be the one point compactification of R. The set $U = (0,\infty)$ is π-open in R, $F = [1,\infty)$ is closed in R, and $F \subset U$, but $\mathrm{Cl}_{\omega R}F \not\subset \mathrm{Ex}_{\omega R}U$.

Recall that if X is rimcompact, then X is a 0-space and FX denotes the maximum 0.I. compactification of X. The space $FX \setminus X$ is a 0-dimensional subspace of FX; we show that it is embedded in FX in a special way. We need the following definition.

2.4 Definition : If $Z \subset Y$, then Z is 0-dimensionally embedded in Y if Y has a basis of open sets whose boundaries are contained in $Y \setminus Z$.

Recall that F_0X is the maximum 0.I. compactification of a 0-space X.

The proof that (i) implies (ii) in the following proposition appears in [Sk]; that (ii) implies (i) is a trivial consequence of 2.2 (ii).

2.5 Proposition : For any space X, the following are equivalent.

(i) X is rimcompact.

(ii) X is a 0-space, and F_0X has 0-dimensionally embedded remainder.

We remark that the Freudenthal compactification FX of a rimcompact space X was constructed in [M_1], where it was shown that $FX \setminus X$ is 0-dimensionally embedded in FX, and that FX is the maximum in the family of compactifications of X having 0-dimensionally embedded remainders. It was not pointed out that FX is the maximum in the family of 0.I. compactifications of X.

According to 2.5, the Freudenthal compactification of a rimcompact space X satisfies two conditions which would not appear to be necessary for a compactification of X to have 0-dimensional remainder. First, points of X have neighbourhood bases (in FX) of open sets whose boundaries lie in X - we might expect the bases to be somewhat more arbitrary, perhaps consisting of open sets merely saturated with respect to $\underline{C}(\beta X)$ (recall Definition 1.4). Secondly, any element of $FX \setminus X$ has a basis of open sets of FX whose boundaries lie in X. It is not true in general that if an open set in a topological space intersects a subspace in a clopen set, then the boundary (in the large space) of that open set does not intersect the subspace. For example, if $X = I \times I$, $Y = \{(1/n, 0)\}_{n \in \mathbb{N}}$, and $U = X \setminus \{(1/2, 0)\}$, then U is open in X, and $U \cap Y$ is a clopen subset of Y, while $bd_X U \cap Y = \{(1/2, 0)\}$. Then we might expect to find a 0-space X where points of the remainder of F_0X have neighbourhood bases (in F_0X) consisting merely of open sets whose intersections with $F_0X \setminus X$ are clopen in $F_0X \setminus X$.

If X is any 0-space, of the two properties possessed by the Freudenthal compactification of a rimcompact space mentioned above, F_0X has the first if and only if X is rimcompact, whereas the second property is possessed by the Freudenthal compactification of a space as a consequence of the space being rimcompact. Hence if X is a 0-space, and F_0X does not have the second property, F_0X cannot have the first; the converse is not true. The following definition weakens the notion of a 0-dimensional embedding.

2.6 Definition : If $Z \subset Y$, Z is relatively 0-dimensionally embedded in Y if each point of Z has a basis (in Y) of open sets whose boundaries are contained in $Y \setminus Z$.

It is immediate that if a space X has a compactification KX with relatively 0-dimensionally embedded remainder, then $KX \setminus X$ is 0-dimensional, hence X is a 0-space. Also, as a consequence of 2.5, if X is rimcompact, then X has a compactification with relatively 0-dimensionally embedded remainder. We shall formulate an internal condition on a space X that in 2.19 will be shown to be equivalent to X having such a compactification; spaces satisfying this condition will be called "almost rimcompact". We shall see that non-rimcompact, almost rimcompact spaces exist (Example 3.18); there are also 0-spaces which are not almost rimcompact (Example 3.22). However, if $bd_X R(X)$ is compact, then X is a 0-space if and only if X is almost rimcompact (4.4).

We have mentioned that the existence of a 0.I. compactification of a space X is equivalent to $\underline{C}(\beta X)$ (or equivalently, $\underline{Q}(\beta X)$ (recall Definition 1.4)) forming an upper semicontinuous decomposition of βX into compact sets, where elements of $\underline{C}(\beta X)$ contained in $\beta X \setminus X$ have neighbourhood bases consisting of open sets of βX whose intersections with $\beta X \setminus X$ are clopen, and that, in fact, this decomposition of βX is $F_0 X$. In the following "saturated" will mean "saturated with respect to $\underline{C}(\beta X)$". Points of $F_0 X \setminus X$ will be regarded as connected components of $\beta X \setminus X$, or as points of $F_0 X \setminus X$, without explicit mention.

Before formally defining the term "almost rimcompact", we investigate the saturated open sets of βX for a rimcompact space X. We make the following definition.

2.7 Definition : If F and U are closed and open subsets of X, respectively, then F is <u>π-contained</u> in U if there is an π-open set V of X such that $F \subset V \subset Cl_X V \subset U$.

Note that F is π-contained in U if and only if F and $X \setminus U$ are π-separated.

Recall that if δ is a proximity on X, and A, B $\subset$ X, then A $<_\delta$ B means A $\not\delta$ X \ B.

2.8 <u>Proposition</u> : Consider statements (i) and (ii) given below. If U is an open subspace of a 0-space X, then (ii) implies (i). If in addition, X is rimcompact, then (i) implies (ii).

(i) $Ex_{\beta X}U$ is saturated.

(ii) If F is a closed subset of X and F $<_\beta$ U, then F is π-contained in U.

Proof : (i) implies (ii). We will show first that if $p \in \beta X \setminus X$, and C_p is the connected component of p in $\beta X \setminus X$, then C_p has a basis in βX of open sets whose boundaries lie in X. Since X is rimcompact, (Ff)(p) has a basis $\underline{W}$ of open sets of FX whose boundaries lie in X. According to 2.2 (v), for each $W \in \underline{W}$, $(Ff)^{\leftarrow}[W]$ is an open set of βX whose boundary lies in X. Since Ff is a closed map, the collection of sets $\{(Ff)^{\leftarrow}[W]: W \in \underline{W}\}$ is a basis in βX for $(Ff)^{\leftarrow}[(Ff)(p)] = C_p$. Also, since $(Ff)^{\leftarrow}[Cl_{FX}W] = Cl_{\beta X}((Ff)^{\leftarrow}[W]$ and $\underline{W}$ is a basis for (Ff)(p), we can assume without loss of generality that for each $W \in \underline{W}$, there is $W_1 \in \underline{W}$ such that $Cl(_{\beta X}(Ff)^{\leftarrow}[W_1]) \subset (Ff)^{\leftarrow}[W]$.

Now suppose that $Ex_{\beta X}U$ is a saturated open set. If F is any closed subset of X such that F is completely separated from X \ U, then $Cl_{\beta X}F \subset Ex_{\beta X}U$. If $p \in Cl_{\beta X}F \cap (\beta X \setminus X)$, then by hypothesis, $C_p \subset Ex_{\beta X}U$. It follows from 2.2 (ii) and (iii), and the remarks in the previous paragraph, that there is a π-open set $U(p)$ of X such that $C_p \subset Ex_{\beta X}U(p) \subset Cl_{\beta X}U(p) \subset Ex_{\beta X}U$. As X is rimcompact, it follows that if $p \in F$, then there is a π-open set $U(p)$ of X such that $p \in U(p) \subset Cl_{\beta X}U(p) \subset Ex_{\beta X}U$. Then $Cl_{\beta X}F \subset \cup\{Ex_{\beta X}U(p): p \in Cl_{\beta X}F\} \subset Ex_{\beta X}U$, so by compactness, there exists a finite set $\{p_1, p_2, \ldots p_n\} \subset Cl_{\beta X}F$ such that $Cl_{\beta X}F \subset \cup\{U(p_i): 1 \leq i \leq n\} \subset \cup\{Cl_{\beta X}U(p_i): 1 \leq i \leq n\} \subset Ex_{\beta X}U$. If $V = \cup\{U(p_i): 1 \leq i \leq n\}$, then V is a π-open subset of X, and $F \subset V \subset Cl_X V \subset U$, hence F is π-contained in U.

(ii) implies (i). Suppose that $p \in Ex_{\beta X}U \cap (\beta X \setminus X)$. It suffices to show that $C_p \subset Ex_{\beta X}U$. Choose F to be a closed subset of X such that $p \in Cl_{\beta X}F \subset Ex_{\beta X}U$. Now F is completely separated from $X \setminus U$, so by hypothesis F is π-contained in U. That is, there is a π-open subset V of X such that $F \subset V \subset Cl_X V \subset U$. Now $Cl_X V \cap (X \setminus U) = \phi$, so by 2.3 (iv), $Cl_{\beta X}V \cap Cl_{\beta X}(X \setminus U) = \phi$, and $p \in Cl_{\beta X}F \subset Cl_{\beta X}V \subset \beta X \setminus Cl_{\beta X}(X \setminus U) = Ex_{\beta X}U$. As $Cl_{\beta X}V \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, and $C_p \cap Cl_{\beta X}V \neq \phi$, $C_p \subset Cl_{\beta X}V$. Thus $C_p \subset Ex_{\beta X}U$, and the statement follows. □

If X is rimcompact, then the collection of π-open subsets of X is a basis of open sets of X, each of which satisfies the condition imposed on U in 2.8. On the other hand, if X has a basis of open sets, each satisfying (ii) of 2.8, then X is easily verified to be rimcompact.

We are now prepared to define "almost rimcompact". The first of the following definitions is a weakening of Definition 2.7.

2.9 <u>Definitions</u> : (i) If F is closed in X, U is open in X, and $F \subset U$, then F is <u>nearly π-contained</u> in U if there is a compact subset K of F so that whenever F' is a closed subset of F, and $F' \cap K = \phi$, F' is π-contained in U.

(ii) A space X is <u>nearly rimcompact</u> if whenever U is open in X, and $x \in U$, there is an open set W of X such that $x \in W$ and $Cl_X W$ is nearly π-contained in U.

(iii) A space X is <u>quasi-rimcompact</u> if for any $x \in X$, there is a compact set K_x of X, so that whenever F is a closed subset of X and $F \cap K_x = \phi$, then x and F are π-separated.

(iv) A space X is <u>almost rimcompact</u> if X is nearly rimcompact and quasi-rimcompact.

Note that X is rimcompact if and only if whenever U is an open subset of X, and $x \in U$, there is an open subset V of X such that $x \in V$ and $Cl_X V$ is π-contained in U. Equivalently, X is rimcompact if and only if whenever U is an open subset of X, and $x \in U$, then $\{x\}$ is π-contained in U. Clearly the latter formulation is the most straightforward. However the former formulation is an analogue of (i) and (ii) of Definition 2.9, and is the motivation for the terminology developed.

Clearly, every rimcompact space is almost rimcompact. We will show that any almost rimcompact space is a 0-space (2.19). Neither near rimcompactness nor quasi-rimcompactness is sufficient to insure that a space is a 0-space (Examples 3.19 and 3.20), hence neither condition implies the other. However, if X is quasi-rimcompact, then X is a 0-space if and only if X is almost rimcompact (2.19) hence if and only if X is nearly rimcompact. Of the three classes of spaces defined in Definition 2.9, the class of almost rimcompact spaces is the most important because of its characterization in terms of spaces possessing compactifications with relatively 0-dimensionally embedded remainders (2.19). Quasi-rimcompactness of a space X will provide a basis of sb open sets of βX for elements of $\underline{\mathcal{C}}(\beta X)$ contained in $\beta X \setminus X$, while in the presence of quasi-rimcompactness, near rimcompactness provides a basis of saturated open sets for each point of X. We need the following easily proved result.

2.10 <u>Lemma</u> : Let X be any space, and let U, V and W be open subsets of X. If $Cl_X U \cap Cl_X V = \phi$, and $W \subset U \cup V$, then $bd_X(W \cap U) \subset bd_X W$.

2.11 <u>Definition</u> : If $KX \in \mathcal{K}(X)$, and $x \in KX$, then $G(KX,x) = \cap\{Cl_{KX}U$: U is a π-open subset of X and $x \in Ex_{KX}U\}$. The set $G(\beta X,x)$ will be denoted by G_x.

2.12 Lemma : Let X be any space. If $KX \in \mathcal{K}(X)$, and $x \in KX$, then $G(KX,x)$ is connected.

Proof : Suppose that for some $x \in KX$, $G(KX,x)$ is not connected. Let $G(KX,x) = G_1 \cup G_2$, where G_1 and G_2 are disjoint non-empty closed subsets of $G(KX,x)$. Since $G(KX,x)$ is compact, G_1 and G_2 are disjoint compact subsets of KX; hence there are open sets U_1 and U_2 of KX such that $G_i \subset U_i$, $(i = 1,2)$ and $Cl_{KX}U_1 \cap Cl_{KX}U_2 = \phi$. Then $G(KX,x) \cap (KX \setminus (U_1 \cup U_2)) = \phi$, so by compactness, there is a finite collection V_i, $i = 1,2,\ldots n$ of π-open subsets of X such that $x \in Ex_{KX}V_i$, for each i, and $\cap\{Cl_{KX}V_i : 1 \leq i \leq n\} \subset U_1 \cup U_2$. If $V = \cap\{V_i : 1 \leq i \leq n\}$, then V is a π-open subset of X, and by 1.5 (ii), $x \in Ex_{KX}V$.

Let $W_i = V \cap U_i$, $(i = 1,2)$. Then $W_1 \cup W_2 = V$. According to 2.10, $bd_X W_i \subset bd_X V$, $(i = 1,2)$. Hence W_1 and W_2 are π-open subsets of X. As $Cl_{KX}W_1 \cap Cl_{KX}W_2 \subset Cl_{KX}U_1 \cap Cl_{KX}U_2 = \phi$, $x \in Ex_{KX}V = Ex_{KX}W_1 \cup Ex_{KX}W_2$, by 1.5 (vi). Assume without loss of generality that $x \in Ex_{KX}W_1$. Then $G(KX,x) \cap U_2 \subset Cl_{KX}W_1 \cap U_2 = \phi$, which is a contradiction to our choice of U_1 and U_2. The theorem is proved. $\square$

2.13 Theorem : If X is quasi-rimcompact, and $p \in \beta X \setminus X$, the G_p is the (compact connected) quasi-component of p in $\beta X \setminus X$. The set G_p has a basis of open sets of βX whose boundaries lie in X.

Proof : We showed in the previous lemma that G_p is compact and connected. It remains to show that G_p is the quasi-component in $\beta X \setminus X$ of $p \in \beta X \setminus X$. We will first show that if X is quasi-rimcompact, and if $p \in \beta X \setminus X$, then G_p is a subset of $\beta X \setminus X$. If $p \in \beta X \setminus X$, and $x \in X$, then there is a closed subset F of X such that $F \cap K_x = \phi$, and $p \in Cl_{\beta X}F$, (where K_x is as in Definition 2.9 (iii)). Then x and F are π-separated, while $p \in Cl_{\beta X}F$. That is, there is a π-open set U of X such that $x \notin Cl_X U$ and $F \subset U$. Since $p \in Cl_{\beta X}F \subset Ex_{\beta X}U$ by 2.3 (iii), $x \notin G_p$. Thus $G_p \subset \beta X \setminus X$.

Let $\underline{G}_p = \{U: U$ is π-open in X and $p \in Ex_{\beta X}U\}$. Then $G_x = G_x \cap (\beta X \setminus X) = \cap\{Cl_{\beta X}U \cap (\beta X \setminus X): U \in \underline{G}_p\}$. For each $U \in \underline{G}_p$, $Cl_{\beta X}U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$ by 2.3 (ii), hence the quasi-component of p in $\beta X \setminus X$ is contained in G_p. On the other hand, G_p is connected by 2.12. Therefore G_p is contained in the quasi-component of p in $\beta X \setminus X$. That is, G_p is the (connected compact) quasi-component of p in $\beta X \setminus X$.

To prove the last statement, we note that the intersection of finitely many members of $\underline{G}_p$ is again a member of $\underline{G}_p$. Then by compactness, if T is a closed subset of βX such that $G_p \cap T = \phi$, there is $U \in \underline{G}_p$ such that $G_p \subset Cl_{\beta X}U \subset \beta X \setminus T$. Since $G_p \subset \beta X \setminus X$, $G_p \subset Cl_{\beta X}U \cap (\beta X \setminus X) = Ex_{\beta X}U \cap (\beta X \setminus X)$. Then the collection of sets $\{Ex_{\beta X}U: U \in \underline{G}_p\}$ is a basis for G_p consisting of open sets of βX whose boundaries are contained in X. □

2.14 <u>Corollary</u> : Suppose that X is quasi-rimcompact and has a compactification with totally disconnected remainder. Then X is a 0-space, and $F_0X \setminus X$ is relatively 0-dimensionally embedded in F_0X.

Proof : Suppose that X has a compactification with totally disconnected remainder. According to 1.9, $\beta X / \underline{C}(\beta X)$ is a compactification of X. Since X is quasi-rimcompact, it follows from 2.13 that $\underline{C}(\beta X) = \underline{Q}(\beta X)$, and that elements of $\underline{C}(\beta X)$ contained in $\beta X \setminus X$ have a basis of open sets of βX whose boundaries are contained in X. Thus X is a 0-space and $\beta X / \underline{C}(\beta X) = F_0X$ has a relatively 0-dimensionally embedded remainder. $\square$

The following will be useful in several arguments.

2.15 <u>Lemma</u> : If $G_x = \{x\}$, for each $x \in X$, then X is rimcompact.

Proof : Suppose that $x \in X$, and that $G_x = \{x\}$. If F is a closed subset of X, and $x \notin F$, then $Cl_{\beta X}F \cap G_x = \phi$. By compactness, there are π-open sets U_i, $(i = 1,2,\ldots n)$ such that $x \in Ex_{\beta X}U_i$, $(i = 1,2,\ldots n)$ and $Cl_{\beta X}F \cap \cap\{Cl_{\beta X}U_i\colon 1 \le i \le n\} = \phi$. Then $x \in \cap\{U_i\colon 1 \le i \le n\}$ which is a π-open subset of X, while $F \cap Cl_X(\cap\{U_i\colon 1 \le i \le n\}) = \phi$. That is, x and F are π-separated in X. Thus X is rimcompact. $\square$

2.16 Corollary : If X is quasi-rimcompact, and $R(X)$ is totally disconnected, then X is rimcompact.

Proof : According to 2.15, it is sufficient to show that $\{x\} = G_x$. The proof of 2.13 involved showing that if $x \in X$ and $p \in \beta X \setminus X$, then there is a π-open subset U of X such that $x \in U$, and $p \notin Ex_{\beta X}U$. Then $p \notin Cl_{\beta X}U$, so $p \notin G_x$; that is, if $x \in X$, then $G_x \subset X$. If $x \notin R(X)$, then $G_x = \{x\}$. If $x \in R(X)$, then $G_x \subset R(X)$, and therefore is a connected subspace of a totally disconnected space, hence consists of one point. Since $x \in G_x$, $G_x = \{x\}$. □

Note that a space with totally disconnected residue need not be rimcompact. If X is the quotient space $R/\{N\}$, then $R(X) = \{N\}$, so $R(X)$ is totally disconnected. However, it is easy to verify that $\{N\}$ does not have a basis of π-open sets, hence X is not rimcompact.

If U is an open subset of X, let $U^s = \cup\{G_p : p \in \beta X \setminus X \text{ and } G_p \subset Ex_{\beta X}U\} \cup U$.

2.17 Theorem : If X is almost rimcompact, and U is an open subset of X, then U^s is a saturated open subset of βX.

Proof : Clearly U^s is saturated. To show that U^s is open in βX, we show that if $p \in U^s$, then there is an open set W of βX such that $p \in W \subset U^s$. First suppose that $p \in (\beta X \setminus X) \cap U^s$. Then $G_p \subset Ex_{\beta X}U$, so by 2.13, and 2.2 (ii), there is a π-open set V of X such that $p \in G_p \subset Ex_{\beta X}V \subset Ex_{\beta X}U$. Clearly $Ex_{\beta X}V \cap X \subset U^s$. If $q \in Ex_{\beta X}V \cap (\beta X \setminus X)$, then $G_q \subset Ex_{\beta X}V$, since $Ex_{\beta X}V \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. In other words, $q \in U^s$. Since q is an arbitrary element of $Ex_{\beta X}V \cap (\beta X \setminus X)$, $Ex_{\beta X}V \subset U^s$. Then $W = Ex_{\beta X}V$ is an open set of βX having the desired properties.

Now suppose that $p \in U^s \cap X = U$. Since X is nearly rimcompact we can choose V to be an open subset of X such that $p \in V$ and $Cl_X V$ is nearly π-contained in U. We show that $Cl_{\beta X}V \subset U^s$. Suppose $r \in Cl_{\beta X}V \setminus X$. Since $r \notin Cl_{\beta X}K = K$ for any compact subset K of X, there is a closed subset F of $Cl_X V$ such that $r \in Cl_{\beta X}F$ and $F \cap K = \phi$, where K is the compact subset of $Cl_X V$ witnessing the fact that $Cl_X V$ is nearly π-contained in U. Then F is π-contained in U; let V_1 be a π-open subset of X such that $F \subset Cl_X V_1 \subset U$. Then $r \in Cl_{\beta X}V_1 \subset Ex_{\beta X}U$. Since $Cl_{\beta X}V_1 \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, it follows by an argument in the preceding paragraph that $Cl_{\beta X}V_1 \subset U^s$. Since $r \in Cl_{\beta X}V$, and r was chosen to be an arbitrary element of $Cl_{\beta X}V$, $Cl_{\beta X}V \subset U^s$. Then $W = Ex_{\beta X}V$ is the desired open set of βX.

We have shown that if $p \in U^s$, then there is an open subset W of βX such that $p \in W \subset U^s$. Thus U^s is a saturated open subset of βX. □

2.18 Corollary : If X is almost rimcompact, then each $x \in X$ has an open basis in βX of saturated open sets of βX.

Proof : Since the collection of open sets $\{Ex_{\beta X}U$: U is open in X, $x \in U\}$ is a basis for x in βX, the collection $\{U^S$: U is open in X, $x \in U\}$ is a basis for x in βX consisting of saturated open sets. □

We can now characterize almost rimcompact spaces as 0-spaces possessing compactifications with relatively 0-dimensionally embedded remainders.

2.19 <u>Theorem</u> : For any space X, the following are equivalent.

(i) X is almost rimcompact.

(ii) X is a 0-space, and F_0X has relatively 0-dimensionally embedded remainder.

(iii) X has a compactification with relatively 0-dimensionally embedded remainder.

(iv) X is quasi-rimcompact, and has a compactification with totally disconnected remainder.

Proof : (i) implies (ii). According to 2.13 and 2.18, if X is almost rimcompact, then $\underline{C}(\beta X) = \underline{Q}(\beta X)$, and is an upper semicontinuous decomposition of βX into compact sets, where elements of $\underline{C}(\beta X)$ contained in $\beta X \setminus X$ have neighbourhood bases in βX of open sets whose boundaries lie in X. Then $F_0X = \beta X \,/\, \underline{C}(\beta X)$ is a 0.I. compactification of X with relatively 0-dimensionally embedded remainder.

(ii) implies (iii). This is obvious.

(iii) implies (i). Suppose that KX is a compactification of X with relatively 0-dimensionally embedded remainder. We first show that X is quasi-rimcompact.

If $x \in X$, and $p \in KX \setminus X$, there is an open set U_p of KX such that $x \notin Cl_{KX}U_p$, $p \in U_p$ and $bd_{KX}U_p \subset X$. Clearly $KX \setminus Cl_{KX}U_p$ is an sb set of KX containing x and not containing p. Let $K_x = \cap\{Cl_{KX}U$: U is an sb open set of KX, $x \in Ex_{KX}U\}$. Clearly K_x is a compact subset of X containing x. Suppose that F is a closed subset of X, and that $F \cap K_x = \phi$. Then $Cl_{KX}F \cap K_X = \phi$. By compactness, there is a finite collection $U_1, U_2, \ldots U_n$ of sb open sets of KX such that $x \in U_i$, $i = 1,2,\ldots n$, and $Cl_{KX}F \cap (\cap\{Cl_{KX}U_i\colon 1 \le i \le n\}) = \phi$. Then $\cap\{U_i \cap X\colon 1 \le i \le n\}$ is a π-open subset of X which witnesses the fact that x and F are π-separated.

A similar argument will show that X is nearly rimcompact. For suppose that U is open in X, and that $x \in U$. Choose V to be an open subset of X such that $x \in V \subset Cl_{KX}V \subset Ex_{KX}U$. Since $KX \setminus X$ is relatively 0-dimensionally embedded in KX, for each $p \in Cl_{KX}V \setminus X$, there is an sb open set U(p) of KX such that $p \in U(p) \subset Cl_{KX}U(p) \subset Ex_{KX}U$. Let $K = Cl_{KX}V \setminus \cup\{U(p)\colon p \in Cl_{KX}V \setminus X\}$. Then K is a compact subset of X. Suppose that F is a closed subset of Cl_XV and that $F \cap K = \phi$. Then $Cl_{KX}F \subset \cup\{U(p)\colon p \in Cl_{KX}V \setminus X\}$. By compactness, there is a finite set $\{p_1, p_2, \ldots p_n\} \subset Cl_{KX}V \setminus X$ such that $Cl_{KX}F \subset \cup\{U(p_i)\colon 1 \le i \le n\} \subset \cup\{Cl_{KX}U(p_i)\colon 1 \le i \le n\}$

$\subset Ex_{KX}U$. It follows from 2.2 (i) and (ii) that $W = (\cup\{U(p_i): 1 \leq i \leq n\}) \cap X$ is a π-open subset of X witnessing the fact that F is π-contained in U. Therefore $Cl_X V$ is nearly π-contained in U and X is nearly rimcompact.

(iv) implies (ii). This is 2.14.

(iii) implies (iv). This is obvious, since (iii) implies that X is almost rimcompact and is a 0-space. □

Theorem 2.19 states that if a space X has a compactification with relatively 0-dimensionally embedded remainder, then $F_0 X$ has relatively 0-dimensionally embedded remainder. The following stronger statement is true.

2.20 Theorem : Let δX be a compactification of X with 0-dimensionally (respectively, relatively 0-dimensionally) embedded remainder. If KX is a 0.I. compactification of X, and $KX \geq \delta X$, then $KX \setminus X$ is 0-dimensionally (respectively, relatively 0-dimensionally) embedded in KX.

Proof : According to 2.2 (v), it is sufficient to prove that KX has relatively 0-dimensionally embedded remainder.

Suppose that T is a closed subset of KX and that $p \in (KX \setminus X) \setminus T$. If $f: KX \to \delta X$ is the natural map, let $T' = f^{\leftarrow}(f(p)) \cap T$. Now $KX \setminus X$ is 0-dimensional, hence $f^{\leftarrow}(f(p))$ is a compact 0-dimensional subspace of KX. Since $p \notin T'$, there are disjoint closed subsets B_1 and B_2 of

$f^{\leftarrow}(f(p))$ such that $p \in B_1$, $T' \subset B_2$, and $B_1 \cup B_2 = f^{\leftarrow}(f(p))$. As B_1 and B_2 are disjoint compact subsets of KX, it follows that there are open sets V_1 and V_2 of KX such that $B_i \subset V_i$, $(i = 1,2)$ and $Cl_{KX}V_1 \cap Cl_{KX}V_2 = \phi$. Since f is a closed map, and $f^{\leftarrow}(f(p)) \subset V_1 \cup V_2$, there is an open set W of δX such that $f^{\leftarrow}(f(p)) \subset f^{\leftarrow}[W] \subset f^{\leftarrow}[Cl_{\delta X}W] \subset V_1 \cup V_2$. Now $\delta X \setminus X$ is relatively 0-dimensionally embedded in δX, so we can assume without loss of generality that $bd_{\delta X}W \subset X$, and hence $bd_{KX}f^{\leftarrow}[W] \subset X$. If $W_i = f^{\leftarrow}[W] \cap V_i$, then by 2.10, $bd_{KX}W_i \subset X$ $(i = 1,2)$. Also, $p \in W_1$, while $T' \subset W_2$.

Let $S = T \setminus W_2$. Then $f^{\leftarrow}(f(p)) \cap S = \phi$. Since f is closed, there is an open set W_3 of δX such that $f^{\leftarrow}(f(p)) \subset f^{\leftarrow}[W_3] \subset KX \setminus S$. Again, without loss of generality we can assume $bd_{\delta X}W_3 \subset X$. Then $f^{\leftarrow}[W_3]$ is an sb subset of KX, therefore by 2.2 (i) $W_1 \cap f^{\leftarrow}[W_3]$ is an sb subset of KX containing p, while $T \cap (W_1 \cap f^{\leftarrow}[W_3])$

$$
\begin{aligned}
&= [(T \cap W_2) \cap W_1 \cap f^{\leftarrow}[W_3]] \cup [(T \setminus W_2) \cap W_1 \cap f^{\leftarrow}[W_3]] \\
&\subset (T' \cap W_1) \cup (S \cap f^{\leftarrow}[W_3]) \\
&= \phi.
\end{aligned}
$$

Thus each point of $KX \setminus X$ has a basis of sb subsets of KX. In other words, KX has relatively 0-dimensionally embedded remainder. □

CHAPTER 3

EXAMPLES OF ALMOST RIMCOMPACT SPACES

In this chapter our main intention is to construct examples of almost rimcompact non-rimcompact spaces. We show in Example 3.18 that if Y is any 0-dimensional space which is not strongly 0-dimensional, then Y can be written as $\beta X \setminus X$ for (i) a rimcompact space X, and (ii) an almost rimcompact non-rimcompact space X. The space X will be a subspace of the product space $KY \times (\omega_1+1)$, where KY is a perfect compactification of Y. A particular example of such a space is discussed in Example VI.13 and Exercise VI.7 of [Is]. We give a general outline of the construction and show that X is rimcompact if and only if $R(X) = KY \setminus Y$ is 0-dimensional.

The results we prove in order to outline the general construction also lead to some interesting observations concerning the conditions under which 0-spaces are rimcompact. We will show that if X is a 0-space in which (i) any two distinct points of X are π-separated in X or (ii) R(X) is locally compact and 0-dimensional, then X is rimcompact (3.7 and 3.11 respectively). We give examples to show that conditions (i) and (ii) are incomparable.

Finally, in 3.22 we present an example of a 0-space which is not almost rimcompact.

In the following we implicitly use the trivial fact that if KX is a compactification of X, and T is a subset of $Cl_{KX}(KX \setminus X)$, then the closure of T in $Cl_{KX}(KX \setminus X)$ equals $Cl_{KX}T$.

The following result and its corollary will be useful in several arguments.

3.1 <u>Lemma</u> : Suppose that KX is a compactification of X, and that W is a compact clopen subset of $Cl_{KX}(KX \setminus X)$. Then

(i) There is $f \in C(KX, [0,1])$ such that

$f[W] = 0$,

$f[(Cl_{KX}(KX \setminus X)) \setminus W] = 1$,

and $Cl_X f^{\leftarrow}[(0,1)]$ is compact.

(ii) There is an sb open set (recall Definition 2.1) U of KX such that $bd_{KX}U \subset X \setminus R(X)$, and

$Ex_{KX}(U \cap X) \cap Cl_{KX}(KX \setminus X) = Cl_{KX}(U \cap X) \cap Cl_{KX}(KX \setminus X) = W$. If $W \cap R(KX \setminus X) = \phi$, then $Cl_X(U \cap X) \cap R(X) = \phi$.

Proof : Suppose that W is a compact clopen subset of $Cl_{KX}(KX \setminus X)$. If $V = [Cl_{KX\setminus X}(KX \setminus X)] \setminus W$, then V and W are disjoint clopen subsets of $Cl_{KX}(KX \setminus X)$ whose union is $Cl_{KX}(KX \setminus X)$. Define a map $g : Cl_{KX}(KX \setminus X) \to \{0,1\}$ as follows :

$g(x) = 0$ if $x \in W$,

$g(x) = 1$ if $x \in V$.

Then g is continuous. Since $Cl_{KX}(KX \setminus X)$ is compact, and hence is C*-embedded in KX, there is a function $h : KX \to [0,1]$ such that $h|\mathrm{Cl}_{KX}(KX \setminus X) = g$.

Define $f : KX \to [0,1]$ as follows.

$f(x) = 0$ if $x \in h^{\leftarrow}[[0,1/3]]$

$f(x) = 3h(x) - 1$ if $x \in h^{\leftarrow}[[1/3,2/3]]$,

$f(x) = 1$ if $x \in h^{\leftarrow}[[2/3,1]]$

Then f is well-defined and continuous on KX. Clearly $f[W] = 0$ and $f[V] = 1$. Also, $\mathrm{Cl}_X f^{\leftarrow}[(0,1)] \subset \mathrm{Cl}_X h^{\leftarrow}[(1/3,2/3)] \subset h^{\leftarrow}[[1/3,2/3]]$, which is a compact subset of X. Thus f has the desired properties.

(ii) Let f be as in (i). Since $\mathrm{Cl}_{KX}(KX \setminus X) \subset f^{\leftarrow}(0) \cup f^{\leftarrow}(1)$, $f^{\leftarrow}[[0,1/3)]$ is an sb open set of KX whose boundary is contained in $X \setminus R(X)$. It follows from 2.2 (iii), that

$$(KX \setminus X) \cap f^{\leftarrow}[[0,1/3)] = (KX \setminus X) \cap \mathrm{Ex}_{KX}[f^{\leftarrow}[0,1/3)] \cap X]$$
$$= (KX \setminus X) \cap \mathrm{Cl}_{KX}[f^{\leftarrow}[[0,1/3)] \cap X].$$

In fact, since $\mathrm{bd}_{KX} f^{\leftarrow}[[0,1/3)] \subset X \setminus R(X)$, it follows that

$$\mathrm{Cl}_{KX}(KX \setminus X) \cap f^{\leftarrow}[[0,1/3)]$$
$$= \mathrm{Cl}_{KX\setminus X}(KX \setminus X) \cap \mathrm{Ex}_{KX}[f^{\leftarrow}[[0,1/3)] \cap X]$$
$$= \mathrm{Cl}_{KX}(KX \setminus X) \cap \mathrm{Cl}_{KX}[f^{\leftarrow}[[0,1/3)] \cap X].$$

Note that $R(KX \setminus X) = \mathrm{Cl}_{KX} R(X) \setminus X$. If we also assume that $W \cap R(KX \setminus X) = \phi$, then $R(X) \subset \mathrm{Cl}_{KX}(KX \setminus X) \setminus W \subset f^{\leftarrow}(1)$. As $\mathrm{Cl}_X[f^{\leftarrow}[[0,1/3)] \cap X] \subset f^{\leftarrow}[[0,1/3]] \cap X$, it follows that $\mathrm{Cl}_X[f^{\leftarrow}[[0,1/3)] \cap X] \cap R(X) = \phi$. Then $U = f^{\leftarrow}[[0,1/3)]$ has the desired properties, and the theorem is proved. □

We make the following easily proved result explicit.

3.2 <u>Lemma</u> : Suppose that S, T are closed subsets of X, and that $S \cap (T \cup R(X)) = \phi$. If S is compact, then there is an open set U of X such that $Cl_X U$ is compact, $S \subset U$, and $T \cap Cl_X U = \phi$.

3.3 <u>Corollary</u> : Let X be a space, and let $KX \in \mathcal{K}(X)$. Suppose that T is a closed subset of KX, that W is a compact clopen subset of $Cl_{KX}(KX \setminus X)$ and that $T \cap W = \phi$. Then there is an sb open set U of KX such that $bd_{KX}U \subset X \setminus R(X)$, $W = U \cap Cl_{KX}(KX \setminus X)$, and $T \cap Cl_{KX}U = \phi$.

Proof : If W is a compact clopen subset of $Cl_{KX}(KX \setminus X)$, then by 3.1 (ii) there is an sb open set U_1 of KX such that $bd_{KX}U_1 \subset X \setminus R(X)$ and $Ex_{KX}(U_1 \cap X) \cap Cl_{KX}(KX \setminus X) = Cl_{KX}(U_1 \cap X) \cap Cl_{KX}(KX \setminus X) = W$. Since $T \cap W = \phi$, $T \cap Cl_{KX}(KX \setminus X) \cap Cl_{KX}(U_1 \cap X) = \phi$, hence $T \cap Cl_{KX}(U_1 \cap X)$ is a compact set contained in $X \setminus R(X)$. According to 3.2, there is an open set V of X such that $Cl_X V$ is a compact subset of $X \setminus R(X)$, and $T \cap Cl_{KX}(U_1 \cap X) \subset V$. Let $U_2 = KX \setminus Cl_X V$. Then U_2 is an sb open set of KX by 2.2 (iv), and $W \subset U_2$. If $U = U_1 \cap U_2$, then U_2 is an sb open set of KX by 2.2 (i), and $Ex_{KX}(U \cap X) = Ex_{KX}(U_1 \cap X) \cap Ex_{KX}(U_2 \cap X)$ by 1.5 (ii). Also $bd_{KX}U \subset bd_{KX}U_1 \cup bd_{KX}U_2 \subset R(X)$. It then follows from 2.2 (iii) that $U \cap Cl_{KX}(KX \setminus X) = Ex_{KX}(U \cap X) \cap Cl_{KX}(KX \setminus X) = W$, while $T \cap Cl_{KX}U \subset T \cap Cl_{KX}U_1 \cap Cl_{KX}U_2 = \phi$. The statement is proved. $\square$

3.4 <u>Lemma</u> : Let X be a 0-space. Suppose that X has a 0.I. compactification KX of X such that $Cl_{KX}(KX \setminus X)$ is 0-dimensional. Then X is rimcompact.

Proof : It suffices to show that if $x \in R(X)$, then x has a basis in X of π-open subsets of X. Suppose that T is a closed subset of X, and that $x \in R(X) \setminus T$. Then $x \notin Cl_{KX}T$. Let $S = Cl_{KX}T \cap Cl_{KX}(KX \setminus X)$. Then S is closed in $Cl_{KX}(KX \setminus X)$ and $x \in [Cl_{KX}(KX \setminus X)] \setminus S$. Since $Cl_{KX}(KX \setminus X)$ is compact and 0-dimensional, there is a compact clopen set W of $Cl_{KX}(KX \setminus X)$ such that $x \in W$, and $S \cap W = \phi$. Then $(Cl_{KX}T) \cap W = \phi$, so by 3.3, there is an sb open set U of KX such that $U \cap Cl_{KX}(KX \setminus X) = W$ and $Cl_{KX}U \cap Cl_{KX}T = \phi$. Then by 2.2 (ii), $U \cap X$ is a π-open subset of X, $x \in U \cap X$, and $T \cap U \cap X = \phi$. Thus x has a basis in X of π-open sets, and X is rimcompact. □

3.5 <u>Theorem</u> : Suppose that T is a locally compact space such that $T = X \cup Y$, where $X \cap Y = \phi$, and X, Y are totally disconnected. If T is not 0-dimensional, then there is a closed connected subset C of T such that $Cl_C(X \cap C) = Cl_C(Y \cap C) = C$.

Proof : If T is a locally compact space which is not 0-dimensional, then there is a closed connected subset C of T such that $|C| > 1$. Since X and Y are totally disconnected, $C \cap X$ and $C \cap Y$ are nonempty. Suppose that

$Cl_C(C \cap Y) \neq C$, and let $z \in C \setminus Cl_C(C \cap Y)$. Since C is a locally compact subspace of T, there is an open set U of C such that $Cl_C U$ is compact, and $z \in U \subset Cl_C U \subset C \setminus Cl_C(C \cap Y)$. Now $Cl_C U$ is a closed subset of X, and hence is a compact 0-dimensional subspace of X, since X is totally disconnected. Hence there is a compact clopen (in $Cl_C U$) subset V of $Cl_C U$ such that $z \in V \subset U$. Then V is a compact clopen subset of C, which contradicts the fact that C is connected. Thus $Cl_C(C \cap Y) = C$. Similarly, $Cl_C(C \cap X) = C$, and the theorem is proved. □

The previous result leads to an interesting sufficient condition for a space to be rimcompact. We need the following definiton.

3.6 <u>Definition</u> : A space X is <u>pointwise rimcompact</u> if whenever x, y are distinct points of X, then x and y are π-separated in X.

There are non-rimcompact, pointwise rimcompact spaces. For example, if X is the quotient space $R/\{N\}$, then X is pointwise rimcompact since $|R(X)| = 1$ but is not rimcompact. Any fully disconnected space is pointwise rimcompact.

3.7 <u>Theorem</u> : Let X be a space. If X has a compactification with totally disconnected remainder, then the following are equivalent.

(i) X is rimcompact.

(ii) X is pointwise rimcompact.

Proof : (i) implies (ii). This is obvious.

(ii) implies (i): Suppose that $x \in X$. Recall that $G_x = \cap\{Cl_{\beta X}U$: U is π-open in X and $x \in Ex_{\beta X}U\}$. According to 2.13, G_x is a connected subset of βX. To show that X is rimcompact, by 2.15 it suffices to show that $G_x = \{x\}$. Since X is pointwise rimcompact, $G_x \cap X = \{x\}$. Suppose that $G_x \cap (\beta X \setminus X) \neq \phi$. Let KX be a compactification of X with totally disconnected remainder. Then $(Kf)[G_x]$ is a connected subset of KX, $(Kf)[G_x] \cap X = \{x\}$, and $[(Kf)[G_x]] \setminus \{x\}$ is totally disconnected. According to 3.5 applied to the sets $\{x\}$, $[(Kf)[G_x]] \setminus \{x\}$, there is a connected subset C of $(Kf)[G_x]$ such that $Cl_C(C \cap \{x\}) = Cl_C[[(Kf)[G_x] \setminus \{x\}] \cap C]$. This is clearly impossible, so it follows that $G_x \cap (\beta X \setminus X) = \phi$. Thus $G_x = \{x\}$ and the theorem is proved. □

If X is fully disconnected, and has a compactification with totally disconnected remainder, then it follows as a special case of 3.7 that X is rimcompact. It is easy to verify that a fully disconnected rimcompact space is 0-dimensional, thus any fully disconnected non-0-dimensional space is a pointwise rimcompact non-rimcompact space. Example 3.10 will illustrate that a totally disconnected rimcompact space need not be 0-dimensional.

We now outline a construction that we will use to produce many of our examples.

A collection of infinite subsets of N is called <u>almost disjoint</u> if the intersection of two distinct members is finite. Zorn's lemma implies that there exists a maximal collection of almost disjoint infinite subsets of N. In the following $\underline{R}$ will denote a maximal such collection. The following topology on $N \cup \underline{R}$ is credited to Isbell in [GJ]. Each point of N is isolated, and $\lambda \in \underline{R}$ has as an open base $\{\{\lambda\} \cup (\lambda \setminus F) : F$ is a finite subset of $N\}$. It is noted in 5I of [GJ] that such spaces $N \cup \underline{R}$ are first countable, locally compact, 0-dimensional and pseudocompact. The following is 2.1 of [Te].

3.8 <u>Proposition</u> : Any compact metric space without isolated points is homeomorphic to the remainder $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ for a suitably chosen maximal almost disjoint collection $\underline{R}$.

In the sequel, when we choose a maximal almost disjoint collection $\underline{R}$ such that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to a compact metric space X having no isolated points, we identify points of $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ with points of X in the obvious manner, and consider $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ to be the space X.

Let P be a topological property. Following the terminology of [HI] we say that X has property P <u>at infinity</u> if $\beta X \setminus X$ has property P.

The following is Theorem 10 of [Sk].

3.9 <u>Proposition</u> : If X is Lindelöf at infinity, then X is a 0-space if and only if X is rimcompact.

Since any countable space is Lindelöf, it follows from 3.9 that if X is a space which has a compactification with countable remainder, then X is rimcompact.

3.10 <u>Example</u> : Choose a maximal collection $\underline{R}$ such that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to the unit interval I. Let $X = N \cup \underline{R} \cup (P \cap I)$, where P denotes the irrationals.

Then $\beta X \setminus X = Q \cap I$, so $|\beta X \setminus X| = \omega_0$. According to the remark following 3.9, X is rimcompact. Note that X is totally disconnected. For if $p \in N \cup \underline{R}$, then the connected component of p in X is $\{p\}$, since $N \cup \underline{R}$ is locally compact and 0-dimensional. If $p \in P \cap I$, then the connected component C_p of p in X is contained in $P \cap I$, which is totally disconnected and thus $C_p = \{p\}$. We claim that X is not fully disconnected and hence is not 0-dimensional. Choose p_1 and p_2 to be distinct points of $P \cap I$. Suppose that there is a clopen subset U of X such that $p_1 \in U$, while $p_2 \notin U$. Then $\mathrm{Cl}_{\beta X}U$ is a clopen subset of βX such that

$p_1 \in Cl_{\beta X}U$, and $p_2 \notin Cl_{\beta X}U$. This impies that I is not connected, which is a contradiction. Thus p_1 and p_2 cannot be contained in disjoint clopen subsets of X, hence X is not 0-dimensional. □

The next result also follows from 3.4 and 3.5.

3.11 Corollary : Let X be a space for which R(X) is locally compact and 0-dimensional. Then the following are equivalent.

(i) X is rimcompact.

(ii) X is a 0-space.

(iii) X has a compactification with totally disconnected remainder.

Proof : Clearly (i) implies (ii) implies (iii).

(iii) implies (i) : Suppose that KX is a compactification of X in which $KX \setminus X$ is totally disconnected. We claim that if R(X) is locally compact and 0-dimensional, then $Cl_{KX}(KX \setminus X)$ is 0-dimensional. It then follows from 3.4 that X is rimcompact, since $KX \setminus X$ is 0-dimensional.

Suppose that $Cl_{KX}(KX \setminus X)$ is not 0-dimensional. Since R(X) and $KX \setminus X$ satisfy the hypotheses of 3.5, there is a compact connected subset C of $Cl_{KX}(KX \setminus X)$ such that $Cl_C(C \cap R(X)) = Cl_C(C \cap (KX \setminus X)) = C$. Then $[Cl_C(C \cap R(X))] \setminus [R(X) \cap C] = C \cap (KX \setminus X)$. However, $R(X) \cap C$ is locally compact, which implies that

$[Cl_C(R(X) \cap C)] \setminus [R(X) \cap C]$ is compact. Then $C \cap (KX \setminus X)$ cannot be dense in C, which is a contradiction. Thus $Cl_{KX}(KX \setminus X)$ is 0-dimensional, and X is rimcompact. □

Suppose that X is a space which has a compactification with totally disconnected remainder. It follows from 3.11 that if R(X) is locally compact and 0-dimensional, then X is pointwise rimcompact. The converse is not true, as the space Q of rational numbers illustrates. The following example shows that the local compactness and 0-dimensionality of R(X) do not generally imply that X is pointwise rimcompact. Note that this shows that the hypotheses of 3.11 do not imply that X is rimcompact.

3.12 <u>Example</u> : Let $N_1 = \{n + 1/2 : n \in N\}$, and let D be the decomposition of R consisting of the sets $\{N, N_1, \{r : r \in R \setminus (N \cup N_1)\}\}$. Let $X = R/D$. It is a straightforward computation to show that if U is an open subset of X containing N and not containing N_1, then the boundary in X of U is not compact. Thus N and N_1 are not π-separated in X, hence X is not pointwise rimcompact. It is easily verified that $R(X) = \{N_1, N\}$, hence R(X) is locally compact and 0-dimensional.

3.13 <u>Theorem</u> : Let X be a space. Suppose that there is a perfect compactification KX of X such that $Cl_{KX}(KX \setminus X)$ is a perfect compactification of $KX \setminus X$. Suppose also that whenever F is a closed subset of $R(X)$, and $x \in R(X) \setminus F$, then x and F are π-separated in X. Then $R(X)$ is 0-dimensional.

Proof : Suppose that F is a closed subset of $R(X)$, and that $x \in R(X) \setminus F$. We show that there is a clopen subset U of $R(X)$ such that $x \in U$, and $U \cap F = \phi$. By hypothesis there exists a π-open subset W of X such that $x \in W$, while $F \cap Cl_X W = \phi$. Let $V = Ex_{KX} W \cap (KX \setminus X)$. It follows from 2.3 (ii) that V is clopen in $KX \setminus X$. As $Cl_{KX}(KX \setminus X)$ is a perfect compactification of $KX \setminus X$, it follows that $Cl_{KX} V$ is a clopen subset of $Cl_{KX}(KX \setminus X)$. Let $U = Cl_{KX} V \cap R(X)$. Then U is a clopen subset of $R(X)$. Since $x \in R(X) \cap W \subset Cl_{KX}(KX \setminus X) \cap Ex_{KX} W$, $x \in Cl_{KX}[Ex_{KX} W \cap Cl_{KX}(KX \setminus X)] = Cl_{KX}[Ex_{KX} W \cap (KX \setminus X)] = Cl_{KX} V$, hence $x \in U$. Also, $Cl_{KX} V \subset Cl_{KX} W$, so $F \cap U \subset F \cap Cl_{KX} V \subset F \cap Cl_{KX} W \cap X = \phi$. Thus U is the desired clopen subset of $R(X)$, and $R(X)$ is 0-dimensional. □

Recall that if X is 0-dimensional, then $\beta_0 X$ is the maximum 0-dimensional compactification of X, and equals FX.

3.14 <u>Corollary</u> : If X is rimcompact, and $Cl_{FX}(FX \setminus X)$ is a perfect compactification of $FX \setminus X$, then $R(X)$ is 0-dimensional. Thus $R(X) \cup (FX \setminus X) = \beta_0(FX \setminus X)$.

Proof : By assumption, X satisfies the hypothesis of 3.13, hence $R(X)$ is 0-dimensional. Then $R(X) \cup (FX \setminus X)$, as a perfect O.I. compactification of $FX \setminus X$, is the maximum O.I. compactification of $FX \setminus X$. That is, $R(X) \cup (FX \setminus X) = F(FX \setminus X)$, which equals $\beta_0(FX \setminus X)$ since $FX \setminus X$ is 0-dimensional. □

3.15 <u>Theorem</u> : Suppose that X is a 0-space such that $Cl_{F_0X}(F_0X \setminus X)$ is a perfect compactification of $F_0X \setminus X$. Then

a) X is almost rimcompact.

b) The following are equivalent.

(i) X is rimcompact.

(ii) $Cl_{F_0X}(F_0X \setminus X) = \beta_0(F_0X \setminus X)$.

(iii) $R(X)$ is totally disconnected.

(iv) $R(X)$ is 0-dimensional.

Proof : a) To prove that X is almost rimcompact, it suffices by 2.19 to show that $F_0X \setminus X$ is relatively 0-dimensionally embedded in F_0X. Suppose that $p \in (F_0X \setminus X) \setminus T$, where T is a closed subset of F_0X. Choose U to be open in F_0X such that $p \notin Cl_{F_0X}U$ and $T \subset U$. Let $S = Cl_{F_0X}U \cap (F_0X \setminus X)$. Then S is closed in $F_0X \setminus X$, and $p \notin S$. Since $F_0X \setminus X$ is 0-dimensional, there is a clopen subset V of $F_0X \setminus X$ such that $p \in V$, and $S \cap V = \phi$. As $Cl_{F_0X}(F_0X \setminus X)$ is a perfect compactification of $F_0X \setminus X$, it follows that $Cl_{F_0X}V$ is a compact clopen subset of $Cl_{F_0X}(F_0X \setminus X)$ such that $p \in Cl_{F_0X}V$ and $Cl_{F_0X}S \cap Cl_{F_0X}V = \phi$. Since

$$T \cap Cl_{F_0X}(F_0X \setminus X) \subset U \cap Cl_{F_0X}(F_0X \setminus X)$$
$$\subset Cl_{F_0X}[U \cap Cl_{F_0X}(F_0X \setminus X)]$$

$$= \mathrm{Cl}_{F_0X}[U \cap (F_0X \setminus X)]$$
$$= \mathrm{Cl}_{F_0X}S,$$

it follows that $T \cap \mathrm{Cl}_{F_0X}(F_0X \setminus X) \cap \mathrm{Cl}_{F_0X}V = \phi$. That is, $\mathrm{Cl}_{F_0X}V$ is a compact clopen set of $\mathrm{Cl}_{F_0X}(F_0X \setminus X)$ disjoint from T which contains p. It follows from 3.3 that there is an sb open set U of F_0X such that $\mathrm{Cl}_{F_0X}V = U \cap \mathrm{Cl}_{F_0X}(F_0X\setminus X)$, and hence $p \in U$, while $U \cap T = \phi$. Thus each point of $F_0X\setminus X$ has a basis in F_0X of open sets whose boundaries lie in X, and $F_0X \setminus X$ is relatively 0-dimensionallly embedded in F_0X. The statement follows.

b) (i) implies (iv). This is 3.14.

(iv) implies (iii). This is obvious.

(iii) implies (ii). It follows from 1.8 that as a perfect compactification of $F_0X \setminus X$ having totally disconnected remainder, $R(X) \cup (F_0X \setminus X)$ is the minumum perfect compactification of $F_0X \setminus X$. Since $F_0X \setminus X$ is rimcompact, the minimum perfect compactification of $F_0X \setminus X$ is $F(F_0X\setminus X)$. Since $F_0X \setminus X$ is 0-dimensional $F(F_0X \setminus X) = \beta_0(F_0X \setminus X)$ and the statement follows.

(iii) implies (i). This is a special case of 3.4. □

In constructing examples we will often use the following two results to show that for $X \subset Y$, $\beta X = Y$. The first combines Theorems 1 and 4 of [G1]; the second is 6.7 of [GJ].

3.16 <u>Proposition</u> : Let $\{X_\alpha : \alpha \in A\}$ be a set of pseudocompact spaces. Then:

(i) If $\Pi\{X_\alpha : \alpha \in A\}$ is pseudocompact, then $\beta[\Pi\{X_\alpha : \alpha \in A\}] = \Pi\{\beta X_\alpha : \alpha \in A\}$.

(ii) If X_α is locally compact for all but one $\alpha \in A$, then $\Pi\{X_\alpha : \alpha \in A\}$ is pseudocompact.

3.17 <u>Proposition</u> : If X is any space, and $X \subset T \subset \beta X$, then $\beta T = \beta X$.

3.18 <u>Example</u> : Let Y be any 0-dimensional non-strongly 0-dimensional space. Then $\beta Y \neq \beta_0 Y$. Let KY be any perfect compactification of Y, and let $X = (KY \times (\omega_1 + 1)) \setminus (Y \times \{\omega_1\})$. It follows from 3.16 and 3.17 that $\beta X \setminus X = Y \times \{\omega_1\}$. Thus X is a 0-space, and $F_0 X = \beta X$. As $Cl_{\beta X}(\beta X \setminus X) = Cl_{\beta X}(Y \times \{\omega_1\}) = KY \times \{\omega_1\}$, it follows that $Cl_{F_0 X}(F_0 X \setminus X)$ is a perfect compactification of $F_0 X \setminus X$. According to 3.15, X is almost rimcompact, and X is rimcompact if and only if $KY = \beta_0 Y$. In particular, if $KY = \beta Y$, X is an almost rimcompact space which is not rimcompact. $\square$

We stated in Chapter 2 that neither the near rimcompactness nor the quasi-rimcompactness of a space X is sufficient to insure that X is a 0-space, although by 2.19 the presence of both properties is sufficient. The following examples validate this statement.

3.19 <u>Example</u> : Let $Y = I \times \{0,1,1/2,1/3,\ldots\}$, $Z = Y \times (\omega_1+1)$, and $X = Z \setminus (I \times \{1,1/2,1/3,\ldots\} \times \{\omega_1\})$. We claim that X is quasi-rimcompact.

Since $R(X) = I \times \{0\} \times \{\omega_1\}$, which is a compact subset of X, it suffices to show that if F is a closed subset of X such that $F \cap R(X) = \phi$, the F and R(X) are π-separated. Suppose that F is closed in X, and that $F \cap R(X) = \phi$. For each $p \in F$, choose U(p) to be an open subset of X such that $p \in U(p)$, $Cl_X U(p)$ is compact, and $(Cl_X U(p)) \cap R(X) = \phi$. If $p \in Cl_{\beta X} F \setminus F$, then $p \in I \times \{1/n\} \times \{\omega_1\}$, for some $n \in N$. Choose V to be a clopen subset of ω_1+1 containing ω_1, and let $U(p) = I \times \{1/n\} \times V$. Then U(p) is clopen in βX, $U(p) \cap R(x) = \phi$, and $p \in U(p)$. Since $Cl_{\beta X} F \subset \cup\{U(p): p \in Cl_{\beta X} F\}$, by compactness there is a finite subset $\{p_1,p_2,\ldots,p_n\} \subset Cl_{\beta X} F$ such that $Cl_{\beta X} F \subset \cup\{U(p_i): 1 \le i \le n\}$. If $U = \cup\{U(p_i): 1 \le i \le n\}$, then $U \cap X$ is π-open in X, $F \subset U \cap X$, and $Cl_X U \cap R(X) = \phi$. Thus F is π-separated from R(X), and X is quasi-rimcompact.

We claim that X is not a 0-space. For if X is a 0-space, then $\underline{Q}(\beta X)$ is an upper semicontinuous decomposition of βX. The elements of $\underline{Q}(\beta X)$ contained in $\beta X \setminus X$ are of the form $I \times \{1/n\} \times \{\omega_1\}$ for $n \in N$. Since a basic neighbourhood in βX of $p \in R(X)$ intersects all but finitely many of these components, if these quasi-components are collapsed to points, then distinct points of $R(X)$ do not have disjoint neighbourhoods. Thus $\underline{\mathcal{Q}}(\beta X)$ is not an upper semicontinuous decomposition of βX and X is not a 0-space.

□

3.20 Example : Choose $\underline{R}$ to be a maximal almost disjoint collection of infinite subsets of N such that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to I. Let $X = [\beta(N \cup \underline{R}) \times (\omega_1+1)] \setminus [(N \cup \underline{R} \cup \{1/2\}) \times \{\omega_1\}]$. Then it follows from 3.16 and 3.17 that $\beta X = \beta(N \cup \underline{R}) \times (\omega_1+1)$. Thus $\beta X \setminus X = (N \cup \underline{R} \cup \{1/2\}) \times \{\omega_1\}$. We claim that X is nearly rimcompact. Define Z to be $X \cup \{(1/2,\omega_1)\}$. Then $\beta Z \setminus Z = \beta X \setminus Z = (N \cup \underline{R}) \times \{\omega_1\}$, which is 0-dimensional. According to 3.15, Z is almost rimcompact. Hence Z is a 0-space and is nearly rimcompact. Note that if U is a π-open subset of Z such that $(1/2,\omega_1) \notin Cl_Z U$, then $U \cap X = U$, and U is a π-open subset of X. Suppose that $x \in V$, where V is open in X. Then V is open in Z, so there is an open set V_1 of Z such that $x \in V_1 \subset Cl_Z V_1 \subset V$, and $Cl_Z V_1 \subset X$. Since Z is almost rimcompact, there is an open subset W of Z such that $x \in W$ and $Cl_Z W$ is nearly π-contained (in Z) in V_1. Since W is open in X, and $Cl_X W = Cl_Z W$, it follows from the previous remark, and the definition of near π-containment that $x \in W$ and $Cl_X W$ is nearly π-contained (in X) in V_1. Thus X is nearly rimcompact.

It is clear that $\beta X \setminus X = (N \cup \underline{R} \cup \{1/2\}) \times \{\omega_1\}$ is totally disconnected. However, $\beta X \setminus X$ is not 0-dimensional.

For suppose F is any non-compact closed subset of $\beta X \setminus X$ such that $(1/2,\omega_1) \notin F$. Suppose there is a clopen subset U of $\beta X \setminus X$ containing $(1/2,\omega_1)$ such that $U \cap F = \phi$. Since $Cl_{\beta X}(\beta X \setminus X) = \beta(\beta X \setminus X)$, $Cl_{\beta X}U$ is a clopen subset of $Cl_{\beta X}(\beta X \setminus X)$. The point $(1/2,\omega_1) \in Cl_{\beta X}U$, while $[Cl_{\beta X}F \setminus (\beta X \setminus X)] \cap Cl_{\beta X}U = \phi$. However, $Cl_{\beta X}F \setminus (\beta X \setminus X) \subset I \times \{\omega_1\}$, which implies that $I \times \{\omega_1\}$ is not connected. Since this is a contradiction, $(1/2,\omega_1)$ and F are not contained in disjoint clopen sets of $\beta X \setminus X$. Thus $\beta X \setminus X$ is not 0-dimensional. Since βX is clearly the maximum compactification of X having totally disconnected remainder, it follows from 1.8 that βX is the minimum perfect compactification of X. Since βX is not a 0.I. compactification of X, X cannot be a 0-space. □

We have seen in 2.16 that if X is a quasi-rimcompact space which has a compactification with totally disconnected remainder, then X is almost rimcompact. The previous example shows that a nearly rimcompact space which has a compactification with totally disconnected remainder need not even be a 0-space.

Since any space with a countable basis is Lindelöf at infinity, it follows from 3.9 that any 0-space having a countable basis is rimcompact. We point out that a space X has a countable basis if and only if X is a separable metric space. A collection $\underline{B}$ of non-empty open sets of X is a

π-basis for X if for any non-empty open set U of X, there is $B \in \underline{B}$ such that $B \subset U$. (Note that the concepts of "π-open" and "π-basis" are unrelated.) The following example shows that a 0-space X having a countable π-basis need not be rimcompact, even if X is almost rimcompact.

3.21 Example : Choose $\underline{R}$ to be a maximal collection of almost disjoint infinite subsets of N such that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to I. Let $X = [\beta(N \cup \underline{R})]^2 \setminus [(N \cup \underline{R}) \times \{1\}]$. Then it follows from 3.16 and 3.17 that $\beta X = [\beta(N \cup \underline{R})]^2$, hence $\beta X \setminus X = (N \cup \underline{R}) \times \{1\}$. Then X is a 0-space and $F_0 X = \beta X$. Since $Cl_{\beta X}(\beta X \setminus X) = \beta(N \cup \underline{R}) \times \{1\}$, $Cl_{\beta X}(\beta X \setminus X)$ is a perfect compactification of $\beta X \setminus X$. It follows from 3.15 (a), that X is almost rimcompact. However X is not rimcompact, by 3.15 (b), since $R(X) = I \times \{1\}$ is not 0-dimensional. Since $N \times N$ is a countable dense set in $[\beta(N \cup \underline{R})]^2$, $[\beta(N \cup \underline{R})]^2$ has a countable π-basis; hence X has a countable π-basis. □

The difference between rimcompact spaces and almost rimcompact spaces lies in the nature of the saturated (with respect to $\underline{C}(\beta X)$) open sets of the Stone-Cech compactification which form a basis for points of the space. If X is rimcompact, and $x \in X$, then the collection $\{Ex_{\beta X}U$: U is π-open in X and $x \in U\}$ is a basis at x of saturated open sets of βX, whereas this is not true if X is not rimcompact. However, in both the case where X is rimcompact or almost rimcompact, non-rimcompact, the 0-dimensionality of $F_0X \setminus X$ is witnessed by a particularly nice collection of open sets of βX, namely open sets whose boundaries are contained in X. In general it is not true that if X is a 0-space, then $F_0X \setminus X$ is relatively 0-dimensionally embedded in F_0X. In the following example, we build on Example VII.26 of [Is] to produce a non-almost rimcompact space X for which $\beta X \setminus X$ is 0-dimensional.

3.22 Example : In Example VII.26 of [Is], a compact space Y is constructed which has the following properties. First, there is a 0-dimensional subspace Z of Y such that $Y \setminus Z$ is dense in Y and Z has only one non-isolated point z. Also, there is a point $p \in Y \setminus Z$ such that if U is any open subset of Y containing z, and $bd_Y U \subset Y \setminus Z$, then $p \in U$.

Let $X = (Y \times (\omega_1+1)) \setminus (Z \times \{\omega_1\})$. Then by 3.16 and 3.17, $\beta X = Y \times (\omega_1+1)$, and so $\beta X \setminus X = Z \times \{\omega_1\}$. Then X is a 0-space and $F_0X = \beta X$. We show that X is not almost rimcompact by showing that (z,ω_1) does not have a basis in βX of sb open sets of βX. Suppose that V is an sb open set of βX such that $(z,\omega_1) \in V$ and $(p,\omega_1) \notin V$. Let $V_1 = V \cap (Y \times \{\omega_1\})$. Then V_1 is an open subset of $Y \times \{\omega_1\}$ whose boundary (in $Y \times \{\omega_1\}$) is contained in $(Y \setminus Z) \times \{\omega_1\}$ such that $(z,\omega_1) \in V_1$. However $(p,\omega_1) \in V_1$, which is a contradiction. Thus (z,ω_1) does not have a basis in βX of sb open subsets of βX; hence X is not almost rimcompact.

Note that since Z has only the non-isolated point z, if U is any open subset of X such that $(z,\omega_1) \notin \mathrm{bd}_{\beta X}\mathrm{Ex}_{\beta X}U$, then $\mathrm{Ex}_{\beta X}U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X = Z \times \{\omega_1\}$. It is then easy to verify that for each $p \in \beta X \setminus X$, there is a collection $\underline{U}(p)$ of open subsets of X such that $\mathrm{Ex}_{\beta X}U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$ for each $U \in \underline{U}(p)$, and $\{\mathrm{Ex}_{\beta X}U\colon U \in \underline{U}(p)\}$ is a basis in βX for p. $\Box$

CHAPTER 4

PROPERTIES OF ALMOST RIMCOMPACT SPACES

In this chapter we develop the properties of almost rimcompact spaces. We begin by showing in 4.4 that if X is a space in which $bd_X R(X)$ is compact, then X is a 0-space if and only if X is almost rimcompact. Such a space X need not be rimcompact. Next we consider invariant properties of almost rimcompact spaces. Neither perfect images nor perfect preimages of rimcompact spaces need be 0-spaces. However, in 4.11 we show that if the perfect preimage of a rimcompact space is a 0-space, then that perfect preimage is almost rimcompact. Example 4.8 shows that an open subspace of an almost rimcompact space is not necessarily a 0-space, while in 4.7 we prove that any closed subspace of an almost rimcompact (respectively, 0-space) is almost rimcompact (respectively, a 0-space). We obtain a partial answer to the question : If S is a closed subset of a 0-space X, what conditions on S imply that $Cl_{F_0 X} S = F_0 S$?

We prove 4.4 by considering separately the cases where X is nowhere locally compact, and where X has compact residue.

4.1 <u>Lemma</u> : Suppose that X is nowhere locally compact, and that KX is a 0.I. compactification of X. Then KX \ X is relatively 0-dimensionally embedded in KX.

Proof : Suppose that $p \in KX \setminus X$, and that $p \in W$, where W is an open subset of KX. Since $KX \setminus X$ is 0-dimensional, there is a clopen subset V of $KX \setminus X$ such that $p \in V \subset W \cap (KX \setminus X)$, and $Cl_{KX}V \subset W$. Let U be any open subset of KX such that $U \cap (KX \setminus X) = V$. Since $KX \setminus X$ is dense in KX, $Cl_{KX}U = Cl_{KX}(U \cap (KX \setminus X)) = Cl_{KX}V$. Then $(Cl_{KX}U) \cap (KX \setminus X) = Cl_{KX}V \cap (KX \setminus X) = Cl_{KX \setminus X}V = V$. It follows that $bd_{KX}U = Cl_{KX}U \setminus U \subset X$, hence U is an sb open subset of KX. Since $Cl_{KX}U \subset W$, p has a basis in KX of sb open sets of KX. Thus $KX \setminus X$ is relatively 0-dimensionally embedded in KX. □

Let X be a space. In the sequel, $L(X)$ denotes the locally compact part of X; that is $L(X) = X \setminus R(X)$. Note that if $KX \in \underline{K}(X)$, then $L(X) = KX \setminus Cl_{KX}(KX \setminus X)$, and that $L(KX \setminus X) = (KX \setminus X) \setminus R(KX \setminus X) = KX \setminus [X \cup Cl_{KX}R(X)]$.

The following is easy to prove.

4.2 <u>Lemma</u> : If X is a space, $KX \in \underline{K}(X)$, and W is a compact clopen subset of either $L(KX \setminus X)$ or $KX \setminus X$, then W is a compact clopen subset of $Cl_{KX}(KX \setminus X)$.

4.3 <u>Lemma</u> : Suppose that X is a space in which $R(X)$ is compact. If KX is a 0.I. compactification of X, then $KX \setminus X$ is relatively 0-dimensionally embedded in KX.

Proof : Suppose that T is closed in KX, and that $p \in (KX \setminus X) \setminus T$. As $R(X)$ is compact, there is an open set U of KX such that $p \in U$, while $[T \cup R(X)] \cap Cl_{KX}U = \phi$.

Since $U \cap (KX \setminus X)$ is open in $KX \setminus X$, and $KX \setminus X$ is locally compact and 0-dimensional, there is a compact clopen set W of $KX \setminus X$ such that $p \in W \subset U$. Then $W \cap T = \phi$, so by 4.2 and 3.3 there is an sb open set V of KX such that $V \cap Cl_{KX}(KX \setminus X) = W$ and $T \cap Cl_{KX}V = \phi$. Then $p \in V$, and $V \cap T = \phi$. Thus each point of $KX \setminus X$ has a basis in KX of open sets whose boundaries lie in X. That is, $KX \setminus X$ is relatively 0-dimensionally embedded in KX. □

4.4 Theorem : If X is a space in which $bd_X R(X)$ is compact, then the following are equivalent.

(i) X is a 0-space.

(ii) X is almost rimcompact.

(iii) X is a 0-space, and $F_0X \setminus X$ is relatively 0-dimensionally embedded in F_0X.

(iv) If KX is any O.I. compactification of X in which $Cl_{KX}(int_X R(X)) \cap Cl_{KX}(X \setminus R(X)) \subset X$, then $KX \setminus X$ is relatively 0-dimensionally embedded in KX.

Proof : It follows from 2.19 that (iii) implies (ii) and (ii) implies (i).

(i) implies (iv). Suppose that KX is a O.I. compactification of X in which $Cl_{KX}(int_X R(X)) \cap Cl_{KX}(X \setminus R(X)) \subset X$. We claim that $KX \setminus X \subset Ex_{KX}(int_X R(X)) \cup Ex_{KX}(X \setminus R(X))$. As $X \setminus [int_X R(X) \cup (X \setminus R(X))] = bd_X R(X)$, which is a compact subset of X, $KX \setminus X \subset Ex_{KX}[int_X R(X) \cup (X \setminus R(X))]$. If U and V are open subsets of X, and $p \in Ex_{KX}(U \cup V) \setminus (Ex_{KX}U \cup$

$Ex_{KX}V$), then by 1.4 (v), $p \in Cl_{KX}U \cap Cl_{KX}V$. As $Cl_{KX}(int_X R(X)) \cap Cl_{KX}(X \setminus R(X)) \subset X$, it follows that $KX \setminus X \subset Ex_{KX}(int_X R(X)) \cup Ex_{KX}(X \setminus R(X))$, and the claim is proved.

Note that $Cl_X int_X R(X)$ is a nowhere locally compact space. For if V is any open subset of $Cl_X int_X R(X)$, there is an open set U of X such that $U \cap Cl_X int_X R(X) = V$. Then $U \cap int_X R(X)$ is a non-empty open subset of X. Since $int_X R(X)$ is nowhere locally compact, $Cl_X(U \cap int_X R(X))$ is not compact. Then $Cl_X V$, which is the closure in $Cl_X int_X R(X)$ of V, is not compact. Thus no point of $Cl_X int_X R(X)$ has a basis (in $Cl_X int_X R(X)$) of compact closed neighbourhoods, and $Cl_X int_X R(X)$ is nowhere locally compact.

As $Cl_{KX} int_X R(X)$ is a 0.I. compactification of $Cl_X int_X R(X)$, it follows from 4.1 that $Cl_{KX} int_X R(X) \setminus Cl_X int_X R(X)$ which by our claim is just $[Ex_{KX} int_X R(X)] \cap [KX \setminus X]$, is relatively 0-dimensionally embedded in $Cl_{KX} int_X R(X)$. Let $p \in [Ex_{KX} int_X R(X)] \cap [KX \setminus X]$. We show that p has a basis in KX of open sets whose boundaries lie in X. Suppose that $p \in KX \setminus T$, where T is a closed subset of KX. Since $p \notin Cl_{KX}(X \setminus R(X))$, there is an open subset U_1 of KX such that $p \in U_1$ and $Cl_{KX}U_1 \cap [Cl_{KX}(X \setminus R(X)) \cup T] = \phi$. Then U_1 is open in $Ex_{KX} int_X R(X)$, and hence in $Cl_{KX} int_X R(X)$. It follows that there is an sb (with respect to $Cl_X int_X R(X)$) open set U_2 of $Cl_{KX} int_X R(X)$ such that $p \in U_2 \subset U_1$. As $U_1 \subset Ex_{KX} int_X R(X)$, it follows that U_2 is open in KX. Since

$Cl_{KX}U_2 \cap Cl_{KX}(X \setminus R(X)) = \phi$, U_2 is an sb open subset of KX which contains p and has empty intersection with T.

The subset $Cl_X(X \setminus R(X))$ of X is a space with compact residue, so by 4.3, $Cl_{KX}(X \setminus X)$ is a O.I. compactification of X with a relatively 0-dimensionally embedded remainder. If $p \in Cl_{KX}(X \setminus R(X)) \setminus Cl_X(X \setminus R(X))$ (which by our earlier claim equals $Ex_{KX}(X \setminus R(X)) \cap (KX \setminus X)$), then $p \notin Cl_{KX}R(X)$. It follows from an argument similar to that in the preceding paragraph that p has a basis in KX of sb open sets of KX. Thus each point of $KX \setminus X$ has a basis of sb open sets of KX, hence $KX \setminus X$ is relatively 0-dimensionally embedded in KX.

(iv) implies (iii). Since F_0X is a perfect compactification of X, and $bd_XR(X)$ is compact, by 2.3 (ii) and 1.5 (ii), $Cl_{F_0X}(int_XR(X)) \cap Cl_{F_0X}(X \setminus R(X)) \cap (F_0X \setminus X) = Ex_{F_0X}int_XR(X) \cap Ex_{F_0X}(X \setminus R(X)) \cap (F_0X \setminus X) = \phi$. Thus F_0X satisfies the condition imposed on KX in (iv) and hence $F_0X \setminus X$ is relatively 0-dimensionally embedded in F_0X. □

The hypothesis of 4.4 do not imply that X is rimcompact. If in Example 3.18, Y is chosen to be a locally compact 0-dimensional space which is not 0-dimensional, and βY is chosen as the perfect compactification of Y, then $X = (\beta Y \times (\omega_1+1)) \setminus (Y \times \{\omega_1\})$ is a almost rimcompact non-rimcompact space in which R(X) is compact.

The next example shows that, as might be expected, it is not true that if X is rimcompact, and $X \subset T \subset \beta X$, then T is necessarily a 0-space.

4.5 Example : Choose $\underline{R}$ so that $\beta(\mathbb{N} \cup \underline{R}) \setminus \mathbb{N} \cup \underline{R} = I$. Let $X = \mathbb{N} \cup \underline{R}$, and $T = \mathbb{N} \cup \underline{R} \cup \{1\}$. Then X is rimcompact. However, the single connected component of $\beta T \setminus T = \beta X \setminus T$ is $[0,1)$, which is not compact. Thus T is not a 0-space. □

It is clear that if X is a 0-space, and $X \subset T \subset F_0X$, then T is a 0-space. Recall that if $X \subset Y \subset \beta X$, then $\beta Y = \beta X$. The following indicates that the expected relationship between F_0X and F_0T holds.

4.6 Theorem : If X is a 0-space, and $X \subset T \subset F_0X$, then T is a 0-space and $F_0X = F_0T$. If X is almost rimcompact (respectively, rimcompact) then T is almost rimcompact (respectively, rimcompact).

Proof : Clearly F_0X is a 0.I. compactification of T. Suppose that KT is a 0.I. compactification of T such that $KT \geq F_0X$. Then KT is a compactification δX of X. Recall that $\delta f : \beta X \to \delta X$ denotes the natural map. Define $g : \delta X \to F_0X$ to be the natural map. Then $g \circ (\delta f) = F_0f$. Suppose that $p \in F_0X \setminus T$. Since F_0X is a perfect compactification of X, by 1.6, $(F_0f)^{\leftarrow}(p) = (g \circ \delta f)^{\leftarrow}(p)$ is a connected subset of βX. Then $(\delta f)[(F_0f)^{\leftarrow}(p)] = g(p)$ is a connected subset of KT contained in $KT \setminus T$. Since $KT \setminus T$ is 0-dimensional, $|g^{\leftarrow}(p)| = 1$. It follows that $KT = F_0X$, and hence $F_0X = F_0T$.

If each point of $F_0X \setminus X$ has a basis of open sets of F_0X whose boundaries are contained in X, then each point of $F_0X \setminus T$ has a basis of open sets of $F_0X = F_0T$ whose boundaries are contained in T. Thus if X is almost rimcompact, T is almost rimcompact. A similar statement holds if X is rimcompact. □

It is tempting to attempt to shorten the proof of the preceeding theorem by immediately claiming that KT as chosen is a 0.I. compactification of X. However, since the union of two 0-dimensional spaces need not be 0-dimensional, it is not immediately clear that $KT \setminus X$ is 0-dimensional, and further argument of the sort provided in the proof is necessary.

We note in passing the following special case for 4.6. If X is a 0-space, and $X \cup Cl_{F_0X}R(X) \subset T \subset F_0X$, then since $X \cup Cl_{F_0X}R(X)$ is almost rimcompact by 4.3, T is almost rimcompact.

We now consider subspaces of 0-spaces. It is an easy exercise to prove that an open or a closed subspace of an rimcompact space is rimcompact. This contrasts with the fact that while a closed subspace of an almost rimcompact space is almost rimcompact, an open subspace of an almost rimcompact space need not even be a 0-space.

4.7 Theorem : If F is a closed subset of a 0-space (respectively, an almost rimcompact space) X, then F is a 0-space (respectively, almost rimcompact).

Proof : If F is closed in a 0-space X, and KX is any 0.I. compactification of X, then $Cl_{KX}F$ is a 0.I. compactification of F. Thus F is a 0-space.

Suppose that $KX \setminus X$ is relatively 0-dimensionally embedded in KX. We show that $Cl_{KX}F \setminus F$ is relatively 0-dimensionally embedded in $Cl_{KX}F$. Suppose that T is a closed subset of $Cl_{KX}F$ and $p \in (Cl_{KX}F \setminus F) \setminus T$. Then T is closed in KX. Since $KX \setminus X$ is relatively 0-dimensionally embedded in KX, there is an sb open set U of KX such that $p \in U$ and $(Cl_{KX}U) \cap T = \phi$. Consider the open set $U \cap Cl_{KX}F$ of $Cl_{KX}F$. The boundary in $Cl_{KX}F$ of $U \cap Cl_{KX}F$ is

$$\begin{aligned} Cl_{KX}(U \cap Cl_{KX}F) \setminus U \cap Cl_{KX}F &\subset [Cl_{KX}(U \cap Cl_{KX}F) \setminus U] \cap Cl_{KX}F \\ &\subset [(Cl_{KX}U) \setminus U] \cap Cl_{KX}F \\ &\subset bd_{KX}U \cap Cl_{KX}F \\ &\subset X \cap Cl_{KX}F \\ &= F. \end{aligned}$$

Then $U \cap Cl_{KX}F$ is an sb open subset of $Cl_{KX}F$ and a neighbourhood (in $Cl_{KX}F$) of p, while $T \cap (Cl_{KX}F) \cap U = \phi$. Thus each point of $Cl_{KX}F \setminus F$ has a basis of sb open sets of $Cl_{KX}F$. Hence $Cl_{KX}F \setminus F$ is relatively 0-dimensionally embedded in $Cl_{KX}F$. It follows from 2.19 that F is almost rimcompact. □

4.8 <u>Example</u> : If X, Z are as in Example 3.20, then X is an open subspace of Z and Z is almost rimcompact, but X is not a 0-space. □

Continuous images and preimages of rimcompact spaces need not be rimcompact, even if the map is perfect. In fact, since any completely regular space is the image of an extremally disconnected space (ie. a space in which disjoint open sets have disjoint closures) under a perfect irreducible map (see [St]), the perfect image of a rimcompact space need not even be a 0-space. The next example shows that the perfect preimage of a rimcompact space need not be a 0-space. However, we will show in 4.11 that if the perfect preimage of a rimcompact space is a 0-space, then the preimage is almost rimcompact. Example 4.12 shows that the preimage need not be rimcompact.

4.9 <u>Example</u> : Let $Y = I \times \{0,1,1/2,1/3,\ldots\}$, and $X = [Y \times (\omega_1+1)] \setminus [I \times \{1,1/2,1/3,\ldots\} \times \{\omega_1\}]$. It is shown in Example 3.19 that X is not a 0-space. Let $f : I \times \{0,1,1/2,1/3,\ldots\} \times (\omega_1+1) \to \{0,1,1/2,1/3,\ldots\} \times (\omega_1+1)$ be the projection map. Then f is closed, since I is compact. Let $S = [\{0,1,1/2,1/3,\ldots\} \times (\omega_1+1)] \setminus [\{1,1/2,1/3,\ldots\} \times \{\omega_1\}]$. Since $f^{\leftarrow}(y) = I \times \{y\}$, for $y \in S$, f is a perfect map from X into S. The space S, being a subspace of $\{0,1,1/2,1/3,\ldots\} \times (\omega_1+1)$, is 0-dimensional (and hence rimcompact).

The following is 1.2 of [HI].

4.10 <u>Lemma</u> : Let $f : X \to Y$ be a perfect map. If S is a compact subset of Y, then $f^{\leftarrow}[S]$ is a compact subset of X.

4.11 <u>Theorem</u> : Let $f : X \to Y$ be a perfect map. If X is a 0-space, and Y is rimcompact, then X is almost rimcompact.

Proof : We show that X is quasi-rimcompact. It then follows from 2.19 that X is almost rimcompact. If $x \in R(X)$, let $K_x = f^{\leftarrow}(f(x))$. Then K_x is a compact subset of X. Suppose that F is a closed subset of X such that $F \cap K_x = \phi$. Since f is a closed map, and $f^{\leftarrow}(f(x)) \subset X \setminus F$, there is a π-open subset W of Y such that $f^{\leftarrow}(f(x)) \subset f^{\leftarrow}[Cl_Y W] \subset X \setminus F$. As f is a perfect map, and $bd_Y W$ is compact, it follows from 4.10 that $f^{\leftarrow}[bd_Y W]$ is compact. Since $bd_X f^{\leftarrow}[W] \subset f^{\leftarrow}[bd_X W]$, it follows that $f^{\leftarrow}[W]$ is a π-open subset of X. Also, $x \in f^{\leftarrow}[W]$, and $F \cap Cl_X f^{\leftarrow}[W] = \phi$. Thus x and F are π-separated. Hence X is quasi-rimcompact, and the theorem follows. □

4.12 <u>Example</u> : Choose $\underline{R}$ to be a family such that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to I. Then $F(N \cup \underline{R}) = \omega(N \cup \underline{R})$, the one-point compactification of $N \cup \underline{R}$. If $X = [\beta(N \cup \underline{R}) \times (\omega_1+1)] \setminus [(N \cup \underline{R}) \times \{\omega_1\}]$, then according to 3.15, X is almost rimcompact but is not rimcompact. Let $f : \beta(N \cup \underline{R}) \times (\omega_1+1) \to \omega(N \cup \underline{R}) \times (\omega_1+1)$ be the natural map, and let $Z = [\omega(N \cup \underline{R}) \times (\omega_1+1)] \setminus [(N \cup \underline{R}) \times \{\omega_1\}]$. If

$z \in Z$, then $f^{\leftarrow}(z) = \{z\}$ or $f^{\leftarrow}(z) = I \times \{p\}$ for some $p \in (\omega_1+1)$. Also $f^{\leftarrow}[Z] = X$, so $f|_X$ is a perfect map from X into Z. The space Z is 0-dimensional (and hence rimcompact).

According to 6.9 (a) of [GJ], if $S \subset X$, then $Cl_{\beta X}S = \beta S$ if and only if S is C*-embedded in X. Also, according to xxx, if X is 0-dimensional, and $S \subset X$, then $Cl_{\beta_0 X}S = \beta_0 S$ if and only if every {0,1}-valued function on S extends continuously to a {0,1}-valued function on X. According to 4.7, if S is a closed subspace of a 0-space X, then S is a 0-space. We address the following question : if S is a closed subspace of X, when is $Cl_{F_0 X}S = F_0 S$?

We can formulate an answer to this question by means of proximities. Let α_X and α_S denote the proximities on X and S inducing $F_0 X$ and $F_0 S$ respectively. The proximity α_X induces a proximity $\alpha_X|_S$ on S defined as follows : if A, B $\subset$ S, then A $(\alpha_X|_S)$ B if and only if A α_X B. Then $Cl_{F_0 X}S = F_0 S$ if and only if $\alpha_X|_S = \alpha_S$. If X is rimcompact, then this formulation of the answer becomes : $Cl_{FX}S = FS$ if and only if whenever S_1 and S_2 are subsets of S which are π-separated in S, then S_1 and S_2 are π-separated in X. This corresponds to the following statement for βX : $Cl_{\beta X}S = \beta S$ if and only if whenever S_1 and S_2 are subsets of S which are contained in disjoint zerosets of S, then S_1 and S_2 are contained in disjoint zerosets of X. If "βX" and "zerosets" are replaced by "$\beta_0 X$" and "clopen sets" respectively, we obtain the corresponding statement for $\beta_0 X$. We attempt to find a condition on a subspace S of a 0-space X involving the extension of certain continuous functions on S which will be equivalent to the condition that $Cl_{F_0 X}S = F_0 S$. Such a condition for subspaces of almost rimcompact spaces with compact residues is presented in 4.23.

The following is a partial external characterization of those closed subspaces S of a 0-space X for which $Cl_{F_0 X}S = F_0 S$.

4.13 <u>Lemma</u> : Consider statements (i) and (ii) given below. If S is a closed subspace of a 0-space X, then (i) implies (ii). If, in addition, S is C*-embedded in the 0-space X, then (ii) implies (i).

(i) $Cl_{F_0 X}S = F_0 S$.

(ii) $Cl_{\beta X}S$ intersects each quasi-component of $\beta X \setminus X$ in a connected set.

Proof : Recall that $\underline{Q}(\beta X) = \{(F_0 f)^{\leftarrow}(p) : p \in F_0 X\}$.

(i) implies (ii). Let $g : \beta S \to Cl_{\beta X}S$ and $h : Cl_{\beta X}S \to Cl_{F_0 X}S$ denote the natural maps. Then $h \circ g : \beta S \to Cl_{F_0 X}S$ is the natural map, and $h = F_0 f | Cl\beta XS$. Suppose that Q is any quasi-component of $\beta X \setminus X$ such that $Cl_{\beta X}S \cap Q \neq \phi$. Then $F_0 f[Q] = h[Q \cap Cl_{\beta X}S] \in Cl_{F_0 X}S$. If $Cl_{F_0 X}S = F_0 S$, then $Cl_{F_0 X}S$ is a perfect compactification of S. Note that

$h[Q \cap Cl_{\beta X}S]$ is a singleton set. It follows from 1.6 that $(h \circ g)^{\leftarrow}[h[Q \cap Cl_{\beta X}S]]$ is a connected subset of βS. Then $g[(h \circ g)^{\leftarrow}[h[Q \cap Cl_{\beta X}S]] = h^{\leftarrow}[h[Q \cap Cl_{\beta X}S]]$ is a connected subset of $Cl_{\beta X}S$. As $h^{\leftarrow}[h[Q \cap Cl_{\beta X}S]] = (F_0 f)^{\leftarrow}[F_0 f[Q]] \cap Cl_{\beta X}S = Q \cap Cl_{\beta X}S$, it follows that $Q \cap Cl_{\beta X}S$ is connected.

(ii) implies (i). Suppose that $Cl_{\beta X}S$ intersects each quasi-component of $\beta S \setminus X$ in a connected set. If $p \in Cl_{F_0 X}S$, then $(F_0 f | Cl_{\beta X}S)^{\leftarrow}(p) = [(F_0 f)^{\leftarrow}(p)] \cap Cl_{\beta X}S$, hence $(F_0 f | Cl_{\beta X}S)^{\leftarrow}(p)$ is connected. If $Cl_{\beta X}S = \beta S$, then by 1.6, $Cl_{F_0 X}S$ is a perfect compactification of S. Since $Cl_{F_0 X}S$ is a 0.I. compactification of S, $Cl_{F_0 X}S = F_0 S$.

Note that if a closed subset F of a space X has compact boundary in X, then F is C*-embedded in X. Then the following, which is 3.13 of [Mc], follows directly from 4.13.

4.14 Corollary : Suppose that U is an open subspace of a 0-space X. If $bd_X U$ is compact, then $Cl_{F_0 X}U = F_0(Cl_X U)$.

If S is not C*-embedded in X in 4.13, then (ii) does not imply (i), even if X is rimcompact.

4.15 Example : Choose $\underline{R}$ to be a family of subsets of N such that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to I. Let X = $(N \cup \underline{R}) \cup (P \cap I)$ (where P denotes the irrationals). Then X is rimcompact and $\beta X = FX = \beta(N \cup \underline{R})$. If $S = I \cap P$, then S is a closed subset of X, and $Cl_{FX}S = I$. The set $Cl_{\beta X}S$ intersects each quasi-component of $\beta X \setminus X$ in a connected set. However $FS = \beta(P \cap I) \neq I$. □

4.16 Theorem : Let S be a closed C*-embedded subspace of a 0-space X. Suppose that whenever a subset F of X is completely separated in X from S, then F is π-contained in $X \setminus S$. Then $Cl_{F_0X}S = F_0(S)$.

Proof : If S satisfies the hypotheses then $X \setminus S$ satisfies the condition imposed on U in 2.8 (ii). It follows from 2.8 that $Ex_{\beta X}(X \setminus S)$ is saturated, and so $Cl_{\beta X}S$ is saturated. That is, any quasi-component of $\beta X \setminus X$ which intersects $Cl_{\beta X}S$ is contained in $Cl_{\beta X}S$, and hence is a connected subset of $Cl_{\beta X}S$. It follows from 4.13 that $Cl_{F_0X}S = F_0S$. □

In view of the fact that the above results include as a hypothesis the C*-embedding of the closed subset S in X, we point out the following. If S is a closed subset of a 0-space X, it is neither necessary nor sufficient for S to be C*-embedded in X in order that $Cl_{F_0X}S = F_0S$ holds.

4.17 Example : Choose $\underline{R}$ so that $\beta(N \cup \underline{R}) \setminus N \cup \underline{R}$ is homeomorphic to I. Let $X = \beta(N \cup \underline{R}) \setminus \{1\}$. Then X is rimcompact, and $FX = \beta X = \beta(N \cup \underline{R})$. The set $[0,1)$ is a closed subset of X, and $Cl_{FX}[0,1) = [0,1] = F[0,1)$. However, $[0,1)$ is not C*-embedded in X.

4.18 Example : Let $X = \{x \in R : x \geq 0\} = R^{+}$. Then X is rimcompact and FX is the one-point compactification of X. The set N is a closed C*-embedded subset of X, while $FN = \beta N \neq Cl_{FX}N$.

As previously mentioned, our main concern is to find a condition on a subspace S of a 0-space X involving the extension of certain continuous functions which will be equivalent to the condition that $Cl_{F_0 X}S = F_0 S$. Results 4.19 to 4.22 inclusive will be useful; our main results are stated in 4.23 to 4.25.

4.19 Lemma : Let X be a 0-space, and let KX be a 0.I. compactification of X. Suppose that S and T are closed subsets of X such that $Cl_{KX}S \cap [Cl_{KX}T \cup Cl_{KX}R(S)] = \phi$. Then :

(i) There is an sb open set U of KX such that $S \subset U$ and $[Cl_{KX}T \cup Cl_{KX}R(S)] \cap Cl_{KX}U = \phi$, hence

(ii) There is $f \in C^*(KX,[0,1])$ such that $f^{\leftarrow}[(0,1)] \subset X$, $Cl_X f^{\leftarrow}[(0,1)]$ is compact, $Cl_{KX}S \setminus S \subset f^{\leftarrow}(1)$ and $[Cl_{KX}T \setminus T] \cup Cl_{KX}R(X) \subset f^{\leftarrow}(0)$.

Proof : (i) Suppose that $p \in Cl_{KX}S \setminus S$. Then $p \in L(KX \setminus X)$, hence there is a compact clopen set W(p) of $L(KX \setminus X)$ such that $p \in W(p)$ and $W(p) \cap [(Cl_{KX}T \cup Cl_{KX}R(X)) \cap (KX \setminus X)] = \phi$. Then $W(p) \cap [Cl_{KX}T \cup Cl_{KX}R(X)] = \phi$. It follows from 3.3 that there is an sb open set U(p) of KX such that W(p) =

$U(p) \cap Cl_{KX}(KX \setminus X)$ and $Cl_{KX}U(p) \cap [Cl_{KX}T \cup Cl_{KX}R(X)] = \phi$. Then $p \in U(p)$ and $Cl_{KX}U(p) \cap [\, Cl_{KX}T \cup Cl_{KX}R(X)] = \phi$. On the other hand, suppose $p \in S$. By hypothesis $S \cap R(X) = \phi$, hence there is an open set $U(p)$ of X such that $p \in U(p)$, $Cl_X U(p)$ is compact, and $Cl_X U(p) \cap [T \cup R(X)] = \phi$. Then $Cl_{KX}S \subset \cup\{U(p): p \in Cl_{KX}S\}$. By compactness there is a finite subset $\{p_1, p_2, \ldots, p_n\}$ of $Cl_{KX}S$ such that $Cl_{KX}S \subset \cup\{U(p_i): i \leq n\}$. If $U = \cup\{U(p_i): i \leq n\}$ then U is an sb open subset of KX, and $Cl_{KX}U \cap [Cl_{KX}T \cup Cl_{KX}R(X)] = \phi$. The statement follows.

(ii) Let U be chosen as in (i). Since U is sb in KX, and $(Cl_{KX}U) \cap R(X) = \phi$, $Cl_{KX}U \setminus U \subset L(X)$. It follows that $U \cap Cl_{KX}(KX \setminus X) = Ex_{KX}(U \cap X) \cap Cl_{KX}(KX \setminus X) = Cl_{KX}(U \cap X) \cap Cl_{KX}(KX \setminus X)$, hence $U \cap Cl_{KX}(KX \setminus X)$ is a compact clopen subset of $Cl_{KX}(KX \setminus X)$. According to 3.1, there is $f \in C^*(KX,[0,1])$ such that

$$f[U \cap Cl_{KX}(KX \setminus X)] = 1$$

$$f[(Cl_{KX}(KX \setminus X)] \setminus U] = 0$$

and $Cl_X f^{\leftarrow}[(0,1)]$ is compact. The function f clearly has the desired properties. □

If "βX" and "βY" are replaced by "KX" and "KY" respectively in the proof of 1.2 of [Iw], we obtain the following.

4.20 <u>Lemma</u> : Suppose that KX, KY are compactifications of X and Y respectively, and that $f : X \to Y$ is a closed map. If

f extends to $g \in C(KX,KY)$, then $Cl_{KX}f^{\leftarrow}(y) = g^{\leftarrow}(y)$ for each $y \in Y$.

The following is a special case of 6.3. In proving 6.3 we do not rely on any results proved in this chapter.

4.21 Theorem : Suppose that X is a 0-space and that $f : X \to [0,1]$ is a closed map. If $f^{\leftarrow}(y)$ is compact for each $y \in (0,1)$, then f extends to $g \in C(F_0 X, [0,1])$.

4.22 Lemma : Suppose that S is a closed subspace of a 0-space X. Suppose $f \in C(S,[0,1])$, $Cl_X f^{\leftarrow}[(0,1)]$ is compact, $S \cap R(X) \subset f^{\leftarrow}(0)$ and $Cl_{F_0X}f^{\leftarrow}(1) \cap Cl_{F_0X}R(X) = \phi$. Then there exists $g \in C(X,[0,1])$ such that $g|_S = f$ and $g^{\leftarrow}(y)$ is compact for each $y \in (0,1)$.

Proof : Suppose $f \in C(S,[0,1])$ satisfies the hypotheses. Since $Cl_X f^{\leftarrow}[(0,1)]$ is compact, f is a closed map from S into $[0,1]$. In particular $f[S]$ is a closed (and hence compact) subset of $[0,1]$. Then $f : S \to f[S]$ is a map from a 0-space into a rimcompact space. By 4.21, f extends to $f_1 : F_0S \to f[S]$. Suppose that $Cl_{F_0X}S = F_0S$. As $f : S \to f[S]$ is closed, it follows from 4.20 that $Cl_{F_0X}f^{\leftarrow}(y) = Cl_{F_0S}f^{\leftarrow}(y) = f_1^{\leftarrow}(y)$ for each $y \in f[S]$. Then $f^{\leftarrow}(y) = f_1^{\leftarrow}(y)$ if $y \in (0,1)$, and $F_0S \setminus S \subset f_1^{\leftarrow}(0) \cup f_1^{\leftarrow}(1)$.

As $f_1^{\leftarrow}(1) \cap [f_1^{\leftarrow}(0) \cup Cl_{F_0X}R(X)] = \phi$, it follows from 4.19 (i) (applied to the subsets $f^{\leftarrow}(1)$ and $f^{\leftarrow}(0)$ of X), that there is an sb open set U of F_0X such that $f^{\leftarrow}(1) \subset U$ and $[f_1^{\leftarrow}(0) \cup Cl_{F_0X}R(X)] \cap Cl_{F_0X}U = \phi$. Since $bd_{F_0X}U \subset L(X)$, it follows from 2.2 (iii) that $Cl_{F_0X}U \cap Cl_{F_0X}(F_0X \setminus X) = Ex_{F_0X}(U \cap X) \cap Cl_{F_0X}(F_0X \setminus X) = U \cap Cl_{F_0X}(F_0X \setminus X)$. Hence $Cl_{F_0X}U \cap Cl_{F_0X}(F_0X \setminus X)$ is a compact clopen subset of $Cl_{F_0X}(F_0X \setminus X)$. Let $W = Cl_{F_0X}U \cap Cl_{F_0X}(F_0X \setminus X)$. Define a map $h : Cl_{F_0X}(F_0X \setminus X) \cup S \to [0,1]$ as follows :

$$\begin{aligned} h(p) &= 0 \text{ if } p \in [Cl_{F_0X}(F_0X \setminus X)] \setminus W, \\ &= 1 \text{ if } p \in W, \\ &= f_1(p) \text{ if } p \in Cl_{F_0X}S. \end{aligned}$$

Since $Cl_{F_0X}(F_0X \setminus X) \cap Cl_{F_0X}S \subset f_1^{\leftarrow}(0) \cup f_1^{\leftarrow}(1)$, while $f_1^{\leftarrow}(0) \cap Cl_{F_0X}(F_0X \setminus X) \subset [Cl_{F_0X}(F_0X \setminus X)] \setminus W$, and $g^{\leftarrow}(1) \cap Cl_{F_0X}(F_0X \setminus X) \subset W$, h is well-defined and continuous. The domain of h is a compact subset of F_0X, so h extends to a function $h_1 \in C^*(F_0X,[0,1])$. If $g = h_1|_X$, then $g|_S = f$ and $g^{\leftarrow}(y)$ is compact for each $y \in (0,1)$. □

4.23 <u>Theorem</u> : Suppose that X is an almost rimcompact space in which $R(X)$ is compact, and that S is a closed subset of X. The following are equivalent.

(i) $Cl_{F_0X}S = F_0S$.

(ii) Suppose $f \in C^*(S,[0,1])$, $Cl_X f^{\leftarrow}[(0,1)]$ is compact, and $S \cap R(x) \subset f^{\leftarrow}(0)$. Then there exists $g \in C^*(X,[0,1])$ such that $g^{\leftarrow}(y)$ is compact for each $y \in (0,1)$.

Proof : (i) implies (ii). Suppose that $f \in C^*(S,[0,1])$ satisfies the hypotheses of (ii). Then $f^{\leftarrow}(1) \cap R(X) = \phi$. Since $R(X)$ is compact, $Cl_{F_0X}f^{\leftarrow}(1) \cap Cl_{F_0X}R(X) = \phi$. Then f

satisfies the hypotheses of 4.22, hence there exists $g \in C(X,[0,1])$ such that $g|_S = f$, and $g^{\leftarrow}(y)$ is compact for each $y \in (0,1)$.

(ii) implies (i). Since $Cl_{F_0X}S$ is a O.I. compactification of X, $Cl_{F_0X}S \leq F_0S$. To show that $F_0S = Cl_{F_0X}S$, it suffices by 1.1 to show that if S_1 and S_2 are closed subsets of S whose closures in F_0S are disjoint, then S_1 and S_2 have disjoint closures in $Cl_{F_0X}S$, or equivalently, in F_0X. Suppose then that S_1, S_2 are closed subsets of S whose closures in F_0S are disjoint, and choose $p \in (F_0X \setminus X) \cap Cl_{F_0X}S_1$. We will show that $p \notin Cl_{F_0X}S_2$. Since $F_0X \setminus X$ is locally compact (since $R(X)$ is assumed to be compact) there is a compact clopen subset W of $F_0X \setminus X$ such that $p \in W$. As $Cl_{F_0X}R(X) = R(X)$, $R(X)$ is a closed subset of F_0X, and $R(X) \cap W = \phi$. It follows from 3.3 that there is an sb open set U of F_0X such that $U \cap Cl_{F_0X}(F_0X \setminus X) = W$ and $(Cl_{F_0X}U) \cap R(X) = \phi$. Evidently $p \in U$. Let $T_1 = S_1 \cap Cl_X(U \cap X)$, and let $T_2 = [S_2 \cap Cl_X(U \cap X)] \cup [R(X) \cap S]$. The sets T_1 and T_2 are closed subsets of S (hence of X), and $p \in (Cl_{F_0X}S_1) \cap U \subset Cl_{F_0X}(S_1 \cap Cl_X(U \cap X)) = Cl_{F_0X}T_1$. To show that $p \notin Cl_{F_0X}S_2$, it suffices to show that $p \notin Cl_{F_0X}T_2$, since $(Cl_{F_0X}S_2) \cap U \subset Cl_{F_0X}(S_2 \cap Cl_X(U \cap X)) \subset Cl_{F_0X}T_2$. Note that T_1 and T_2 have disjoint closures in F_0S, since S_1 and S_2 have disjoint closures in F_0S and $R(X)$ is compact. It is easy to verify that $R(S) \subset R(X) \cap S$. Then $R(S) \subset T_2$, hence the closures in F_0S of T_1 and $R(S)$ are disjoint. It follows from 4.10 (ii) (applied to T_1, T_2 as subsets of S), that there is a function $g \in C^*(F_0 S,[0,1])$ such that $g^{\leftarrow}[(0,1)] \subset S$, $Cl_S g^{\leftarrow}[(0,1)]$ is compact, $Cl_{F_0 S}T_1 \setminus T_1 \subset g^{\leftarrow}(1)$ and $R(S) \cup [Cl_{F_0 S}T_2 \setminus T_2] \subset g^{\leftarrow}(0)$. We will construct a function f satisfying the hypotheses of (ii).

Let $F = S \cap R(X) \cap g^{\leftarrow}[1/4,1]$. Then $F \subset T_2$. Since $Cl_{F_0 S}T_2 \setminus T_2 \subset g^{\leftarrow}(0)$, F is a compact subset of S. Also, since $R(S) \subset g^{\leftarrow}(0)$, $F \subset L(S)$; that is, $F \cap Cl_{F_0 S}(F_0 S \setminus S) = \phi$. As $F \subset R(X)$, $T_1 \subset Cl_X(U \cap X)$ and $R(X) \cap Cl_X(U \cap X) = \phi$, it follows that $F \cap Cl_{F_0 S}T_1 = \phi$. Choose $f_1 \in C^*(F_0 S,[0,1])$ such that

$$f_1[F] = 0,$$

$$f_1[Cl_{F_0 S}(F_0 S \setminus S) \cup Cl_{F_0 S}T_1] = 1.$$

Define $f_2 : F_0 S \to [0,1]$ as follows :

$$f_2(x) = 0 \text{ if } x \in g^{\leftarrow}[[0,1/3]],$$

$$f_2(x) = 1 \text{ if } x \in g^{\leftarrow}[[2/3,1]],$$

$$f_2(x) = 3f_1(x) - 1 \text{ if } x \in g^{\leftarrow}[[1/3,2/3]].$$

The function $f_2(x)$ is well-defined and continuous. Let $f_3 = f_1 \circ f_2$. Then $Cl_{F_0 S}T_1 \setminus T_1 \subset f_1^{\leftarrow}(1) \cap f_2^{\leftarrow}(1) = f_3^{\leftarrow}(1)$, and $Cl_{F_0 S}T_2 \setminus T_2 \subset f^{\leftarrow}(0) \subset f_2^{\leftarrow}(0) \subset f_3^{\leftarrow}(0)$. We claim that $S \cap R(X) \subset f^{\leftarrow}(0)$. If $x \in R(X) \cap S \cap g^{\leftarrow}[0,1/3]$, then $f_2(x) = 0$. If $x \in R(X) \cap X \cap g^{\leftarrow}[1/4,1]$, then $f_1(x) = 0$. The claim follows. Finally, $F_0 S \setminus S \subset g^{\leftarrow}(0) \cup g^{\leftarrow}(1) \subset f_2^{\leftarrow}(0) \cup [f_1^{\leftarrow}(1) \cap f_2^{\leftarrow}(1)] \subset f_3^{\leftarrow}(0) \cup f_3^{\leftarrow}(1)$.

Define $f_4 : F_0 S \to [0,1]$ as follows :

$$f_4(x) = 0 \text{ if } x \in f_3^{\leftarrow}[[0,1/3]],$$

$f_4(x) = 3f_3(x) - 1$ if $x \in f_3^{\leftarrow}[[1/3,2/3]]$,

$f_4(x) = 1$ if $x \in f_3^{\leftarrow}[[2/3,1]]$.

Then f_4 is well-defined and continuous, and has the properties of f_3 listed in the preceding paragraph. In addition, $Cl_X f_4^{\leftarrow}[(0,1)]$ is compact.

Let $f = f_4|_S$. By assumption f extends to a function $h \in C^*(X,[0,1])$ such that $h^{\leftarrow}(y)$ is compact if $y \in (0,1)$. We claim that $Cl_{F_0X}T_2 \cap (F_0X \setminus X) \subset Cl_{F_0X}h^{\leftarrow}[[0,1/3)]$. We write $T_2 = [T_2 \cap f^{\leftarrow}[[0,1/3]]] \cup [T_2 \cap Cl_S f^{\leftarrow}[(1/3,1)]] \cup [T_2 \cap f^{\leftarrow}(1)]$. Since $Cl_S f^{\leftarrow}[(0,1)]$ is compact, $T_2 \cap Cl_S f^{\leftarrow}[(1/3,1)]$ is compact. Also, since $Cl_{F_0S}T_2 \cap (F_0S \setminus S) \subset Cl_{F_0S}f^{\leftarrow}(0)$, $T_2 \cap f^{\leftarrow}(1)$ is compact. Then $Cl_{F_0X}T_2 \cap (F_0X \setminus X) \subset Cl_{F_0X}[T_2 \cap f^{\leftarrow}[0,1/3)] \subset Cl_{F_0X}h^{\leftarrow}[[0,1/3)]$. The claim is proved. Similarly, $Cl_{F_0X}T_1 \cap (F_0X \setminus X) \subset Cl_{F_0X}h^{\leftarrow}[(1/3,1]]$. Since $h^{\leftarrow}[[0,1/3)]$ is a π-open subset of F_0X, $Cl_{F_0X}h^{\leftarrow}[[0,1/3)] \cap Cl_{F_0X}h^{\leftarrow}[(1/3,1]] \subset X$. Since $p \in Cl_{F_0X}T_1 \cap (F_0X \setminus X)$, $p \notin Cl_{F_0X}T_2$, thus $p \notin Cl_{F_0X}S_2$. As p was an arbitrary element of $Cl_{F_0X}S_1 \cap (F_0X \setminus X)$, $Cl_{F_0X}S_1 \cap Cl_{F_0X}S_2 = \phi$, thus $Cl_{F_0X}S = F_0S$. □

4.24 Corollary : Let X be a 0-space. Suppose that S is a closed subset of X which is π-separated from R(X). Then the following are equivalent.

(i) $Cl_{F_0X}S = F_0S$.

(ii) If $f \in C(S,[0,1])$ and $Cl_S f^{\leftarrow}[(0,1)]$ is compact, then there exists $g \in C(X,[0,1])$ such that $g|_S = f$ and $g^{\leftarrow}(y)$

is compact for each $y \in (0,1)$.

Proof : (i) implies (ii). If S is a closed subspace of a 0-space X which is π-separated from $R(X)$, then S is clearly C.E. separated from $R(X)$. Hence $Cl_{F_0X}R(X) \cap Cl_{F_0X}S = \phi$. If f satisfies the hypotheses of (ii), then f clearly satisfies the hypotheses of 4.22. It follows from 4.22 that f extends to $g \in C(X,[0,1])$ having the desired properties.

(ii) implies (i). Let $Y = X \cup Cl_{F_0X}R(X)$. According to 4.6, $F_0Y = F_0X$. Since $R(Y) \subset Cl_{F_0X}R(X)$, $R(Y)$ is compact; hence Y is almost rimcompact. If f as in (ii), $g \in C(X,[0,1])$ is the hypothesized extension of f, then by 4.21, g extends to $h : F_0X \to [0,1]$. According to 4.20, $Cl_{F_0X}(g^{\leftarrow}(y)) = h^{\leftarrow}(y)$ for each $y \in [0,1]$. In particular, $h^{\leftarrow}(y)$ is a compact subset of X for each $y \in (0,1)$. If $h_1 = h|_Y$, then h satisfies the conditions imposed on g in 4.23 (ii). It follows from 4.23 that the closure in F_0Y of S is F_0S. Since $F_0Y = F_0X$, the theorem is proved. □

The next result is a special case of 4.23.

4.25 <u>Corollary</u> : If X is locally compact, and S is a closed subset of X, then the following are equivalent.

(i) $Cl_{FX}S = FS$

(ii) If $f \in C(S,[0,1])$ and $Cl_S f^{\leftarrow}[(0,1)]$ is compact, then there is $g \in C(X,[0,1])$ such that $g|_S = f$ and $g^{\leftarrow}(y)$ is compact if $y \in (0,1)$.

CHAPTER 5

PROXIMITIES AND 0-SPACES

In this chapter we will present a proximal characterization of 0-spaces.

We have seen in 2.5 that X is rimcompact if and only if X has a compactification with 0-dimensionally embedded remainder. Also, according to 2.19 X is almost rimcompact if and only if X has a compactification with relatively 0-dimensionally embedded remainder. A 0-space X was constructed in Example 3.22 in which the remainder of F_0X is not relatively 0-dimensionally embedded in F_0X; this validates the statement that in order for a compactification to have 0-dimensional remainder, it is not necessary that points of the remainder have neighbourhood bases in the compactification consisting of open sets whose boundaries lie in X.

In this chapter we shall characterize internally (i) those open sets U of βX for which $U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$, and in particular, (ii) those open sets U of X for which $(\mathrm{Ex}_{\beta X}U) \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. This will lead to the promised proximal characterization of 0-spaces. We need some tools for studying clopen sets in remainders of compactifications. These are developed in 5.1 - 5.5 inclusive.

5.1 <u>Definitions</u> : (i) Let X be a space. An open set U of $KX \in \underline{K}(X)$ is <u>clopen at infinity in KX</u> (denoted by KX-C.I.) if $U \cap (KX \setminus X)$ is clopen in $KX \setminus X$. The set U is a <u>full KX-C.I.</u> set if U is KX-C.I., and $U = Ex_{KX}(U \cap X)$. Often a βX-C.I. (respectively, full βX-C.I.) set will simply be called a C.I. (respectively, full C.I.) set.

(ii) A 0-space X is a <u>full 0-space</u> if for each $p \in \beta X \setminus X$, the connected component of p in $\beta X \setminus X$ has a basis in βX of full C.I. sets.

(iii) If $\underline{E}$ is a family of open sets of X, and D is open in X, then D is <u>small with respect to $\underline{E}$</u> if for each $E \in \underline{E}$, $Cl_X(D \cap E)$ is compact.

(iv) A family $\underline{E}$ of open sets of X is <u>clopenly extendible</u> (denoted C.E.) if there is a compact subset K of X so that if U is open in X, and $K \subset U$, there is $E \in \underline{E}$, and D small with respect to $\underline{E}$ such that $X = U \cup E \cup D$. A family $\underline{E}$ is a <u>full C.E. family</u> if $\underline{E}$ is C.E., and $Ex_{\beta X}(\cup\{E: E \in \underline{E}\}) = \cup\{Ex_{\beta X}E: E \in \underline{E}\}$.

According to 2.2 (iii), if $KX \in \underline{K}(X)$, and if W is an sb open set of KX, then W is a full KX-C.I. set. The following shows that if W is any KX-C.I. open set, the the sets W and $Ex_{KX}(W \cap X)$ can only differ in the locally compact part of $KX \setminus X$.

5.2 <u>Proposition</u> : If $KX \in \underline{K}(X)$, and if U is a KX-C.I. set, then $Ex_{KX}(U \cap X) \cap Cl_{KX}R(X) = U \cap Cl_{KX}R(X)$.

Proof : Let U be a KX-C.I. open set, and suppose that $p \in [Ex_{KX}(U \cap X) \cap Cl_{KX}R(X)] \setminus U$. As $p \in (KX \setminus X) \setminus U$, which is clopen in $KX \setminus X$, there is an open subset W of KX such that $p \in W \subset Ex_{KX}(U \cap X)$ and $W \cap (KX \setminus X) \cap U = \phi$. As $p \in Cl_{KX}R(X)$, there is $x \in W \cap R(X)$. Now $W \cap R(X) \subset Ex_{KX}(U \cap X) \cap X = U \cap X$, so $x \in W \cap U$, which is an open set of KX. Also, $x \in R(X)$, so $W \cap U \cap (KX \setminus X) \neq \phi$, which is a contradiction to our choice of W. Then $Ex_{KX}(U \cap X) \cap Cl_{KX}R(X) \subset U \cap Cl_{KX}R(X)$. Since the reverse inclusion is always true, the result is proved. □

We need to extend some results concerning open sets and perfect compactifications.

5.3 <u>Lemma</u> : Let $KX \in \underline{K}(X)$. If K is a compact subset of X, and if U is open in X, then $[Ex_{KX}(U \setminus K)] \cap (KX \setminus X) = (Ex_{KX}U) \cap (KX \setminus X)$. Hence if V is open in X, and $Cl_X(U \cap V)$ is compact, then $(Ex_{KX}U) \cap (Cl_{KX}V) \subset X$.

Proof : Since

$$\begin{aligned} Ex_{KX}(U \setminus K) \cap (KX \setminus X) &= Ex_{KX}(U \cap (X \setminus K)) \cap (KX \setminus X) \\ &= Ex_{KX}U \cap Ex_{KX}(X \setminus K) \cap (KX \setminus X) \\ &= Ex_{KX}U \cap (KX \setminus K) \cap (KX \setminus X) \\ &= Ex_{KX}U \cap (KX \setminus X), \end{aligned}$$

the first statement is true.

Suppose that $Cl_X(U \cap V)$ is compact. Since

$[U \setminus Cl_X(U \cap V)] \cap V = \phi$, by 1.5 (iv), $Ex_{KX}(U \setminus Cl_X(U \cap V)) \cap Cl_{KX}V = \phi$. Then $Ex_{KX}(U \cap (KX \setminus X)) \cap Cl_{KX}V =$ $Ex_{KX}(U \setminus Cl_X(U \cap V)) \cap (KX \setminus X) \cap Cl_{KX}V = \phi$. □

If $\underline{E}$ is a family of open subset of X, let $Ex_{KX}\underline{E} = \cup\{Ex_{KX}E: E \in \underline{E}\}$. The following is an immediate consequence of 5.3.

5.4 <u>Corollary</u> : Let $KX \in \underline{K}(X)$. Suppose that $\underline{E}$ is a family of open sets of X, and that D is open in X. If D is small with respect to $\underline{E}$, then

$Cl_{KX}D \cap Ex_{KX}\underline{E} \cap (KX \setminus X) = \phi$, and

$Ex_{KX}D \cap (\cup\{Cl_{KX}E: E \in \underline{E}\}) \cap (KX \setminus X) = \phi$.

The equivalence of (i) and (ii) in the following theorem appears in Theorem 1 of [Sk]; we will need the equivalence of (i) and (iii).

5.5 <u>Theorem</u> : Let $KX \in \underline{K}(X)$, and let U, V be open in X. Then the following are equivalent.

(i) KX is a perfect compactification of X.

(ii) If $U \cap V = \phi$, then $Ex_{KX}(U \cup V) = Ex_{KX}U \cup Ex_{KX}V$.

(iii) If $Cl_X(U \cap V)$ is compact, then $Ex_{KX}(U \cup V) = Ex_{KX}U \cup Ex_{KX}V$.

Proof : (iii) implies (ii). This is obvious.

(ii) implies (iii). Since $[Ex_{KX}(U \cup V)] \cap X = U \cup V =$

$(\mathrm{Ex}_{KX}U \cup \mathrm{Ex}_{KX}V) \cap X$, it is sufficient to show that $\mathrm{Ex}_{KX}(U \cup V) \cap (KX \setminus X) = (\mathrm{Ex}_{KX}U \cup \mathrm{Ex}_{KX}V) \cap (KX \setminus X)$. If $\mathrm{Cl}_X(U \cap V)$ is compact, then according to 5.3,

$(\mathrm{Ex}_{KX}U \cap (KX \setminus X)) \cup (\mathrm{Ex}_{KX}V \cap (KX \setminus X))$

$= [\mathrm{Ex}_{KX}(U\setminus \mathrm{Cl}_X(U\cap V)) \cap (KX\setminus X)] \cup [\mathrm{Ex}_{KX}(V\setminus \mathrm{Cl}_X(U\cap V)) \cap (KX\setminus X)]$

(as $U\setminus \mathrm{Cl}_X(U\cap V)$ and $V\setminus \mathrm{Cl}_X(U\cap V)$ are disjoint open sets of X),

$= \mathrm{Ex}_{KX}[(U \setminus \mathrm{Cl}_X(U \cap V)) \cup (V \setminus \mathrm{Cl}_X(U \cap V))] \cap (KX \setminus X)$

$= \mathrm{Ex}_{KX}[(U \cup V) \setminus \mathrm{Cl}_X(U \cap V)] \cap (KX \setminus X)$

$= \mathrm{Ex}_{KX}(U \cup V) \cap (KX \setminus X)$,

where the last equality follows from 5.3. The theorem follows. □

If $\underline{E} = \{E(\alpha): \alpha \in A\}$ is a collection of sets, then $\underline{E}^F$ will denote the collection of sets $\{\cup\{E(\alpha_i): 1 \le i \le n\} : \{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ is a finite subset of $A\}$. The following series of results will establish a correspondence between C.E. (respectively, full C.E.) families and C.I. (respectively, full C.I.) subsets of compactifications.

5.6 <u>Theorem</u> : Let KX be a perfect compactification of X. If U is a C.I. subset of KX, then there is a C.E. family $\underline{E}$ such that $\mathrm{Ex}_{KX}\underline{E} = U$.

Proof : Since U is an open subset of KX, for each $p \in U$ we can choose an open set E_p of X such that $p \in \mathrm{Ex}_{KX}E_p \subset \mathrm{Cl}_{KX}E_p \subset U$. Let $\underline{E}_1 = \{E_p: p \in U\}$, and $\underline{E} = \underline{E}_1^F$. Note that if $E \in \underline{E}_1$ then $\mathrm{Cl}_{KX}E \subset U$.

Clearly, $Ex_{KX}\underline{E} = U$. In order to show that $\underline{E}$ is a C.E. family, we must construct a compact subset K of X so that if U is open in X, and $K \subset U$, there is $E \in \underline{E}_1$ and D small with respect to $\underline{E}$ such that $X = U \cup D \cup E$. First we construct a second family of open sets of X. Since $U \cap (KX \setminus X)$ is clopen in $KX \setminus X$, for each $p \in (KX \setminus X) \setminus U$, we can choose an open set D_p of X such that $p \in Ex_{KX}D_p$ while $(Cl_{KX}D_p) \cap U \subset X$. Let $\underline{D}_1 = \{D_p: p \in (KX \setminus X) \setminus U\}$, and $\underline{D} = \underline{D}_1^F$. Note that if $D_1 \in \underline{D}_1$ and $E_1 \in \underline{E}_1$, then $Cl_{KX}D_1 \cap Cl_{KX}E_1 \subset X$, hence $Cl_X(E_1 \cap D_1)$ is compact. It follows that if $D \in \underline{D}$ and $E \in \underline{E}$, then $Cl_X(D \cap E)$ is compact (being a finite union of compact sets). In other words, if $D \in \underline{D}$, then D is small with respect to $\underline{E}$.

Let $K = KX \setminus \cup\{Ex_{KX}A: A \in \underline{E} \cup \underline{D}\}$. Then K is a compact subset of X. Suppose that $K \subset V$, where V is open in X. Then the collection of sets $\{Ex_{KX}A: A \in \underline{E} \cup \underline{D}\} \cup \{Ex_{KX}V\}$ is an open cover of KX, so there is a finite subcollection whose union covers KX. Then X is covered by the union of a finite subcollection of $\underline{E} \cup \underline{D} \cup \{V\}$. Since $\underline{E}$ and $\underline{D}$ are closed under finite unions, there are sets $E \in \underline{E}$ and $D \in \underline{D}$ such that $X = V \cup E \cup D$. Since D is small with respect to $\underline{E}$, $\underline{E}$ is a C.E. family. □

It is a straightforward computation to verify that if $KX = \beta X$, and if U is a full C.I. subset of βX, then $\underline{E}$ as defined in the proof of 5.6 is a full C.E. family. We observe that in the proof of 5.6, the only conditions that $\underline{E}$

is required to satisfy are that (i) for each $E \in \underline{E}$, $Cl_{KX}E \subset U$, and (ii) $Ex_{KX}\underline{E} = U$. Therefore, we could have chosen $\underline{E}$ to be $\{V: V \text{ is open in } X \text{ and } Cl_{KX}V \subset U\}$.

5.7 Theorem : Let KX be a perfect compactification of X, and let $\underline{E}$ be a C.E. family of open sets of X. Suppose that $p \in (KX \setminus X) \setminus Ex_{KX}\underline{E}$. Then

(i) There is a set D small with respect to $\underline{E}$ such that $p \in Ex_{KX}D$, hence

(ii) $(Ex_{KX}\underline{E}) \cap (KX \setminus X) = \cup\{Cl_{KX}E: E \in \underline{E}\} \cap (KX \setminus X)$, and

(iii) $Ex_{KX}\underline{E}$ is KX-C.I.

Proof : Let K be a compact subset of X which witnesses the fact that $\underline{E}$ is a C.E. family, and let $p \in (KX \setminus X) \setminus Ex_{KX}\underline{E}$. Since $p \notin Cl_{KX}K = K$, there is an open set U of X such that $K \subset U$, while $p \notin Cl_{KX}U$. Choose D to be small with respect to $\underline{E}$, and choose $E \in \underline{E}$, such that $X = U \cup E \cup D$. Now $X \setminus Cl_XU \subset D \cup E$, so $p \in KX \setminus Cl_{KX}U = Ex_{KX}(X \setminus Cl_XU) \subset Ex_{KX}(E \cup D) = Ex_{KX}E \cup Ex_{KX}D$, where the last equality follows from 5.5. Since $p \notin Ex_{KX}E$, it follows that $p \in Ex_{KX}D$.

(ii) and (iii): Suppose that $p \in (KX \setminus X) \setminus Ex_{KX}\underline{E}$. According to (i) and 5.4, there is an open set D of X such that $p \in Ex_{KX}D$, and $Ex_{KX}D \cap (\cup\{Cl_{KX}E: E \in \underline{E}\}) \subset X$. Then $Ex_{KX}\underline{E} \cap (KX \setminus X) = (\cup\{Cl_{KX}E: \ E \in \underline{E}\}) \cap (KX \setminus X)$. Thus, $p \notin Cl_{KX\setminus X}[Ex_{KX}\underline{E} \cap (KX \setminus X)]$, so $(Ex_{KX}\underline{E}) \cap (KX \setminus X)$ is clopen in $KX \setminus X$. □

It follows easily from the above that if $\underline{E}$ is a full C.E. family, then $Ex_{\beta X}\underline{E}$ is a full C.I. subset of βX.

When we defined a C.E. family $\underline{E}$, we did not specify that $\underline{E}$ is to be closed under finite unions, although the C.E. family $\underline{E}$ constructed in the proof of 5.6 is closed under finite unions. The following result shows that it is not necessary to specify this property in the definition of a C.E. family.

5.8 Theorem : Let KX be a perfect compactification of X, and let $\underline{E}$ be a C.E. family of open sets of X. Then

(i) $\underline{E}^F$ is a C.E. family.

(ii) $Ex_{KX}\underline{E} = Ex_{KX}(\underline{E}^F)$.

(iii) If B is a closed subset of X, then $Cl_{KX}B \subset Ex_{KX}\underline{E}$ if and only if there is $E \in \underline{E}^F$ such that $B \subset E$.

Proof : (i) Note that if D is small with respect to $\underline{E}$, then D is small with respect to $\underline{E}^F$. It is then clear that if $\underline{E}$ is a C.E. family, $\underline{E}^F$ is also.

(ii) If U and V are any open subsets of a space X, and if δX is any compactification of X, then an easy computation shows that $(Ex_{\delta X}U) \cup (Ex_{\delta X}V) \subset Ex_{\delta X}(U \cup V) \subset Cl_{\delta X}(U \cup V) = Cl_{\delta X}U \cup Cl_{\delta X}V$. Then $Ex_{KX}\underline{E} \subset Ex_{KX}\underline{E}^F \subset \cup\{Cl_{KX}E\colon E \in \underline{E}^F\} \subset \cup\{Cl_{KX}E\colon E \in \underline{E}\} = Ex_{KX}\underline{E}$, where the last equality follows from 5.7 (ii), hence $Ex_{KX}\underline{E}^F = Ex_{KX}\underline{E}$.

(iii) Note that $Ex_{\delta X}U \cup Ex_{\delta X}V \subset Ex_{\delta X}(U \cup V)$, for any compactification δX of X, and open sets U, V of X. Hence if $Cl_{KX}B \subset Ex_{KX}\underline{E}$, by compactness there is a set $E \in \underline{E}^F$ such that $Cl_{KX}B \subset Ex_{KX}E$; that is, $B \subset E$. On the other hand, if $B \subset E$, where $E \in \underline{E}^F$, then $Cl_{KX}B \subset Cl_{KX}E \subset Ex_{KX}\underline{E}^F = Ex_{KX}\underline{E}$, where the last inclusion and the equality follow from 5.7 (ii), and (ii) of the present result respectively. $\square$

In the following results, we will assume without loss of generality that any C.E. family is closed under finite unions.

The correspondence between C.I. open sets and C.E. families developed in 5.7 has an interesting form in the special situations discussed below.

5.9 Proposition : Let U be an open subset of X. Then

(i) $\{U\}$ is a C.E. family if and only if $bd_X U$ is compact.

(ii) $Ex_{\beta X}U$ is C.I. in βX if and only if $\{V: Cl_X V <_\beta U\}$ is a C.E. family.

Proof : (i) Suppose that $\{U\}$ is a C.E. family. Then by 5.7 (ii) $Cl_{\beta X}U \cap (\beta X \setminus X) = Ex_{\beta X}U \cap (\beta X \setminus X)$. That is, $bd_{\beta X}Ex_{\beta X}U = Cl_{\beta X}bd_X U \subset X$.

Conversely suppose that $bd_X U$ is compact and let $K = bd_X U$. If $K \subset V$, where V is open in X, then $X = U \cup V \cup (X \setminus Cl_X U)$. Since $Cl_X(U \cap (X \setminus Cl_X U)) = \phi$, $\{U\}$ is a C.E. family.

(ii) Suppose that $\underline{U}' = \{V: Cl_X V <_\beta U\}$ is a C.E. family. Then by 5.7 (iii), $Ex_{\beta X}\underline{U}'$ is a C.I. set of βX which clearly equals $Ex_{\beta X}U$.

On the other hand, suppose that $Ex_{\beta X}U$ is a C.I. subset of βX. According to the remark following 5.6, the family $\{V: Cl_X V <_\beta U\}$ is a C.I. family. □

If X is almost rimcompact, the connected components of $\beta X \setminus X$ have a particularly nice form. According to 2.14, the connected component in $\beta X \setminus X$ of $p \in \beta X \setminus X$ is the set $\cap\{Cl_{\beta X}U: U \text{ is } \pi\text{-open in } X,\ p \in Ex_{\beta X}U\}$. Indentifying the connected components of $\beta X \setminus X$ in this way allows us to show directly that $\underline{C}(\beta X)$ is an upper semicontinuous decomposition of βX with certain special properties. The connected components of $\beta X \setminus X$ are not as easily identified for an arbitrary 0-space X. Rather than working with $\underline{C}(\beta X)$, we will characterize 0-spaces in terms of proximity theory. We would like to motivate this characterization by first considering almost rimcompact spaces from the viewpoint of proximities.

Recall that for a rimcompact space X, the proximity δ associated with FX is defined as follows: for $A, B \subset X$, $A \not\delta B$ if and only if A and B are π-separated in X. If X is any space, define γ to be a relation on $\underline{P}(X)$ as follows: for $A, B \subset X$, $A \not\gamma B$ if and only if there is a compact subset K of $Cl_X A$, so that if A' is a closed subset of $Cl_X A$ and $A' \cap K = \phi$, then A' and B are π-separated. For the rest of this chapter, γ will be defined as above.

If δ is as in the previous paragraph, then δ is clearly symmetric, while it is not clear that γ is symmetric. It is not necessary to build symmetry into the definition of γ. Recall that if $KX \in \underline{K}(X)$, and ρ is the relation on $\underline{P}(X)$ defined by (for $A, B \subset X$) $A \rho B$ if and only if $Cl_{KX}A \cap Cl_{KX}B \neq \phi$, then ρ is a proximity on X. We apply this fact to prove that if X is almost rimcompact, then γ is a proximity on X and therefore is symmetric (and satisfies the remaining defining properties of a proximity).

5.10 Theorem : For any space X, the following are equivalent.

(i) X is almost rimcompact.

(ii) γ is a proximity on X.

If γ is a proximity on X, then $\gamma X = F_0 X$.

Proof : (i) implies (ii). If X is almost rimcompact, then by 2.19, X is a 0-space and $F_0 X \setminus X$ is relatively 0-dimensionally embedded in $F_0 X$. We will show both that γ is a proximity on X and that $\gamma X = F_0 X$ by showing that if F_1, F_2 are subsets of X, then $Cl_{F_0 X}F_1 \cap Cl_{F_0 X}F_2 = \phi$ if and only if $F_1 \not\gamma F_2$.

Suppose that $Cl_{F_0 X}F_1 \cap Cl_{F_0 X}F_2 = \phi$. For each

$p \in Cl_{F_0X}F_1 \setminus Cl_XF_1$, choose an π-open subset $U(p)$ of X such that $p \in Ex_{F_0X}U(p)$, and $Cl_{F_0X}U(p) \cap Cl_{F_0X}F_2 = \phi$. Let $K = Cl_{F_0X}F_1 \setminus \cup\{U(p): p \in Cl_{F_0X}F_1 \setminus X\}$. Then K is a compact subset of Cl_XF_1. Suppose that F_1' is a closed subset of Cl_XF_1 and that $F_1' \cap K = \phi$. Then $Cl_{F_0X}F_1' \subset \cup\{Ex_{F_0X}U(p): p \in Cl_{F_0X}F_1 \setminus X\}$. By compactness there is a finite set $\{p_1, p_2, \ldots, p_n\} \subset Cl_{F_0X}F_1 \setminus X$ such that $Cl_{F_0X}F_1' \subset \cup\{Ex_{F_0X}U(p_i): 1 \le i \le n\}$. Then $F_1' \subset \cup\{U(p_i): 1 \le i \le n\}$, which is a π-open subset of X whose closure has empty intersection with F_2. In other word, F_1' and F_2 are π-separated, so $F_1 \not\pi F_2$.

Conversely, suppose that $F_1 \not\pi F_2$, and let K be a compact subset of Cl_XF_1 witnessing this fact. Let $p \in Cl_{F_0X}F_1 \setminus Cl_XF_1$. There is a closed subset F_p of Cl_XF_1 such that $p \in Cl_{F_0X}F_p$, and $(Cl_{F_0X}F_p) \cap K = \phi$. Thus $p \in Cl_{F_0X}F_p$, and by our choice of K, F_p is π-separated from F_2. Since F_0X is a perfect compactification of X, according to 2.3 (iii) and (iv), $Cl_{F_0X}F_p \cap Cl_{F_0X}F_2 = \phi$. Then $p \notin Cl_{F_0X}F_2$, and as p was arbitrarily chosen in $Cl_{F_0X}F_1$, $Cl_{F_0X}F_1 \cap Cl_{F_0X}F_2 = \phi$.

(ii) implies (i). Suppose that γ is a proximity on X. We will show that the proximal compactification γX associated with γ has relatively 0-dimensionally embedded remainder, and therefore, by 2.19, that X is almost rimcompact.

Note that if U is a π-open subset of X, and if A, B are closed subsets of X contained in U, $X \setminus Cl_X U$ respectively, then A and B are π-separated in X, hence $A \not\approx B$. That is, $Cl_{\gamma X}A \cap Cl_{\gamma X}B = \phi$.

We now claim that if U is a π-open subset of X, then $bd_X U = bd_{\gamma X}Ex_{\gamma X}U$. For suppose that $p \in bd_{\gamma X}Ex_{\gamma X}U \setminus bd_X U$. Then $p \in Cl_{\gamma X}Ex_{\gamma X}U \cap Cl_{\gamma X}(X \setminus U)$. As U is π-open in X, $bd_X U$ is closed in γX. Hence we can choose an open subset W of X such that $p \in Ex_{\gamma X}W$, and $Cl_{\gamma X}W \cap bd_X U = \phi$. Since $p \in Cl_{\gamma X}U \cap Ex_{\gamma X}W$, $p \in Cl_{\gamma X}(W \cap U)$. Similarly, $p \in Cl_{\gamma X}(W \cap (X \setminus U)) = Cl_{\gamma X}(W \cap (X \setminus Cl_X U))$, since $W \cap bd_X U = \phi$. Hence $p \in Cl_{\gamma X}(W \cap U) \cap Cl_{\gamma X}(W \cap (X \setminus Cl_X U))$. However,

$$Cl_X(W \cap U) \subset Cl_X W \cap Cl_X U \subset (Cl_X W) \cap U,$$

while

$$Cl_X(W \cap (X \setminus Cl_X U)) \subset Cl_X W \cap Cl_X(X \setminus Cl_X U) \subset (Cl_X W) \cap (X \setminus U).$$

Then $Cl_X(W \cap U)$ and $Cl_X(W \cap (X \setminus Cl_X U))$ are π-separated in X, hence $Cl_{\gamma X}(W \cap U) \cap Cl_{\gamma X}(W \cap (X \setminus Cl_X U)) = \phi$, which contradicts our choice of p. Therefore $bd_X U = bd_{\gamma X}Ex_{\gamma X}U$ and our claim is verified.

Suppose that T is a closed subset of γX, and that $p \in (\gamma X \setminus X) \setminus T$. Choose open sets U and V of X such that $p \in Ex_{\gamma X}U$, $T \subset Ex_{\gamma X}V$, and $Cl_{\gamma X}U \cap Cl_{\gamma X}V = \phi$. Then $Cl_X U \not\approx Cl_X V$; let K be a compact subset of $Cl_X U$ witnessing this fact. Since $p \notin K$, there is a closed subset F of $Cl_X U$ such that $p \in Cl_{\gamma X}F$, and $F \cap K = \phi$. Then F is π-separated from $Cl_X V$. Choose W to be a π-open subset of X such that

$F \subset W$, and $Cl_X W \cap Cl_X V = \phi$. Then $Ex_{\gamma X} W$ is an sb open set of γX, and $p \in Cl_{\gamma X} F \cap (\gamma X \setminus X) \subset Cl_{\gamma X} W \cap (\gamma X \setminus X) = Ex_{\gamma X} W \cap (\gamma X \setminus X)$, while $T \cap Ex_{\gamma X} W \subset (Cl_{\gamma X} V) \cap (Ex_{\gamma X} W) = \phi$. This shows that $\gamma X \setminus X$ is relatively 0-dimensionally embedded in γX, as required. □

A proximity similar to γ will be defined using C.E. families instead of π-open sets. Just as in the case of almost rimcompact spaces, when considering 0-spaces we are only concerned with what happens "away from compact subsets" of X.

5.11 <u>Definitions</u> : (i) If $A, B \subset X$, A is <u>C.E.-separated from B</u> if there is a C.E. family $\underline{E}$ such that $A \subset E$ for some $E \in \underline{E}$, and $Cl_X(\cup\underline{E}) \cap Cl_X B = \phi$.
(ii) Let X be any space, and define α to be a relation on $\underline{P}(X)$ as follows: for $A, B \subset X$, $A \not\alpha B$ if and only if there is a compact subset K of $Cl_X A$, so that if A' is a closed subset of $Cl_X A$, and $A' \cap K = \phi$, then A' is C.E.-separated from B.

For the rest of this chapter, α will be as defined above. We shall prove that X is a 0-space if and only if α is a proximity on X, in which case $\alpha X = F_0 X$ (5.15). Unless specifically stated, in the following results α is not assumed to be a proximity on X.

5.12 <u>Lemma</u> : Suppose that KX is a perfect compactification of X, and that F_1, F_2 are closed subsets of X such that $F_1 \not\approx F_2$. Then if $p \in Cl_{KX}F_1 \setminus F_1$, there is a KX-C.I. subset U_p such that $p \in U_p$ and $Cl_X(U_p \cap X) \cap F_2 = \phi$, hence $U_p \cap Cl_{KX}F_2 = \phi$.

Proof : Suppose that $F_1 \not\approx F_2$; let K be a compact subset of F_1 witnessing this fact. If $p \in Cl_{KX}F_1 \setminus F_1$, then $p \notin K$, so there is a closed subset F_1' of F_1 such that $p \in Cl_{KX}F_1'$, and $F_1' \cap K = \phi$. Thus $p \in Cl_{KX}F_1'$ and F_1' is C.E.-separated from F_2. Let $\underline{E}$ be a C.E. family such that $(Cl_X(\cup\underline{E})) \cap F_2 = \phi$, and $F_1' \subset E$, for some $E \in \underline{E}$. Since KX is a perfect compactification of X, by 5.7 (iii), $Ex_{KX}\underline{E}$ is C.I. in KX. Also, $p \in Cl_{KX}F_1' \subset Ex_{KX}\underline{E}$, by 5.8 (iii), while $Cl_X(\cup\underline{E}) \cap F_2 = \phi$, hence $Ex_{KX}\underline{E} \cap Cl_{KX}F_2 = \phi$. □

The following is an immediate consequence of 5.12.

5.13 <u>Corollary</u> : Suppose that KX is a perfect compactification of X, and that F_1, F_2 are closed subsets of X. If $F_1 \not\approx F_2$, then $Cl_{KX}F_1 \cap Cl_{KX}F_2 = \phi$.

5.14 <u>Lemma</u> : Suppose that α is a proximity on X, and that αX is a perfect compactification of X. Then $\alpha X \setminus X$ is 0-dimensional, hence X is a 0-space and $\alpha X = F_0 X$.

Proof : Suppose that T is a closed subset of $\alpha X \setminus X$, and that $p \in (\alpha X \setminus X) \setminus T$. We must find a clopen subset U of $\alpha X \setminus X$ such that $p \in U$, while $U \cap T = \phi$. Now $p \notin Cl_{\alpha X}T$, so there exist open sets V, W of X such that $p \in Ex_{\alpha X}U$, $Cl_{\alpha X}T \subset Ex_{\alpha X}W$, and $Cl_{\alpha X}V \cap Cl_{\alpha X}W = \phi$. Hence $Cl_X V \not\delta\, Cl_X W$. If αX is a perfect compactification of X, then according to 5.12 there is an αX-C.I. open set U_p such that $p \in U_p$, while $U_p \cap Cl_{\alpha X}W = \phi$. Then $U_p \cap (\alpha X \setminus X)$ is a clopen subset of $\alpha X \setminus X$ having the desired properties. □

5.15 <u>Theorem</u> : If X is any space, then the following are equivalent.

(i) X is a 0-space.

(ii) α is a proximity on X.

Furthermore, if α is a proximity on X, then $\alpha X = F_0 X$.

Proof : (i) implies (ii). Suppose that X is a 0-space. We will prove that α is a proximity on X, and that $\alpha X = F_0 X$ by showing that if F_1, F_2 are closed subset of X, then $Cl_{F_0 X}F_1 \cap Cl_{F_0 X}F_2 = \phi$ if and only if $F_1 \not\delta\, F_2$.

Suppose that $F_1 \not\delta\, F_2$. Since $F_0 X$ is a perfect compactification, according to 5.13, $Cl_{F_0 X}F_1 \cap Cl_{F_0 X}F_2 = \phi$.

On the other hand, suppose that $Cl_{F_0 X}F_1 \cap Cl_{F_0 X}F_2 = \phi$. Since $F_0 X \setminus X$ is 0-dimensional, for each $p \in (Cl_{F_0 X}F_1) \setminus X$, there is an $F_0 X$-C.I. open set $U(p)$ such that $p \in U(p)$ while $Cl_X(U(p) \cap X) \cap F_2 = \phi$. Let $K = Cl_{F_0 X}F_1 \setminus \cup\{U(p): p \in Cl_{F_0 X}F_1 \setminus X\}$. Then K is a compact subset of F_1. If F_1' is a closed subset of F_1 such that $F_1' \cap K = \phi$, then $Cl_{F_0 X}F_1' \subset \cup\{U(p): p \in Cl_{F_0 X}F_1 \setminus X\}$. By compactness, there is a finite subset $\{p_1, p_2, \ldots, p_n\} \subset Cl_{F_0 X}F_1 \setminus X$ such that $Cl_{F_0 X}F_1' \subset$

$\cup\{U(p_i)\colon 1 \le i \le n\}$. Now $\cup\{U(p_i)\colon 1 \le i \le n\}$ is a C.I. open set of F_0X, so by 5.6, there is a C.E. family $\underline{E}$ of open sets of X such that $Ex_{F_0X}\underline{E} = \cup\{U(p_i)\colon 1 \le i \le n\}$. Now $Cl_{F_0X}F_1' \subset Ex_{F_0X}\underline{E}$, so by 5.8 (iii), there is $E \in \underline{E}$ such that $F_1' \subset E$. Also, since $Cl_X(\cup\{U(p_i)\colon 1 \le i \le n\}) \cap F_2 = \phi$, $Cl_X(\cup\underline{E}) \cap F_2 = \phi$. In other words, F_1' is C.E. separated from F_2; that is, $F_1 \not\delta F_2$.

(ii) implies (i). Suppose that α is a proximity on X. According to 5.14, to show that X is a 0-space it suffices to prove that αX is a perfect compactification of X.

First, suppose that V_1 and V_2 are disjoint C.I. subsets of βX. If $y_i \in V_i \cap (\beta X \setminus X)$ $(i = 1,2)$, we claim that $(\alpha f)(y_1) \ne (\alpha f)(y_2)$. To see this, note that there are closed subsets F_i of X such that $y_i \in Cl_{\beta X}F_i \subset V_i$ $(i = 1,2)$. By 5.6 there exists a C.E. family $\underline{E}$ such that $Ex_{\beta X}\underline{E} = V_1$. Since $Cl_{\beta X}F_1 \subset Ex_{\beta X}\underline{E}$, by 5.8 (iii), $F_1 \subset E$, for some $E \in \underline{E}$. Also, $Cl_X(\cup\underline{E}) \cap F_2 \subset (Cl_{\beta X}V_1) \cap V_2 = \phi$, so F_1 is C.E. separated from F_2; that is $F_1 \not\delta F_2$. Then $Cl_{\alpha X}F_1 \cap Cl_{\alpha X}F_2 = \phi$. Since $(\alpha f)(y_i) \in Cl_{\alpha X}F_i$, $(\alpha f)(y_1) \ne (\alpha f)(y_2)$, and our claim is verified.

Now suppose that αX is not a perfect compactification of X. According to 1.5, there is $p \in \alpha X \setminus X$ such that $(\alpha f)^{\leftarrow}(p)$ is not connected. Write $(\alpha f)^{\leftarrow}(p) = T_1 \cup T_2$, where T_1 and T_2 are disjoint closed subsets of $(\alpha f)^{\leftarrow}(p)$. Since $(\alpha f)^{\leftarrow}(p)$ is compact, T_1 and T_2 are disjoint compact subsets of βX, so there are open sets U_1 and U_2 such that $T_i \subset U_i$ $(i = 1,2)$, and $Cl_{\beta X}U_1 \cap Cl_{\beta X}U_2 = \phi$. Since αf is a closed map, and $(\alpha f)^{\leftarrow}(p) \subset U_1 \cup U_2$, there are open sets W_1 and W_2 of X such that $p \in Ex_{\alpha X}W_1 \subset Cl_{\alpha X}W_2 \subset Ex_{\alpha X}W_2$, and $Cl_{\beta X}W_2 \subset (\alpha f)^{\leftarrow}[Cl_{\alpha X}W_2] \subset U_1 \cup U_2$. Now $Cl_{\alpha X}W_1 \cap Cl_{\alpha X}(X \setminus W_2) = \phi$; that is $Cl_X W_1 \not\alpha (X \setminus W_2)$.

In the following $i = 1,2$. Choose $z_i \in (\alpha f)^{\leftarrow}(p) \cap U_i \cap Cl_{\beta X}W_1$. According to 5.12, there are βX-C.I. sets S_i such that $z_i \in S_i$, and $S_i \cap Cl_{\beta X}(X \setminus W_2) = \phi$. Now $S_i \subset U_1 \cup U_2$. Let $S_i' = S_i \cap U_i$. According to 2.11, applied to the sets $Cl_{\beta X}U_1 \cap (\beta X \setminus X)$, $Cl_{\beta X}U_2 \cap (\beta X \setminus X)$ and $S_i \cap (\beta X \setminus X)$, $S_i' \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. In other words, S_i' is a C.I. subset of βX. Also $z_i \in S_i'$, $i = 1,2$, while $S_1' \cap S_2' \subset U_1 \cap U_2 = \phi$. It follows from our earlier claim that $(\alpha f)(y_1) \neq (\alpha f)(y_2)$, which contradicts the fact that $z_i \in (\alpha f)^{\leftarrow}(p)$. Thus $(\alpha f)^{\leftarrow}(p)$ is connected for each $p \in \alpha X \setminus X$, hence αX is a perfect compactification of X. □

The correspondence between full C.I. sets and full C.E. families that is outlined in the remarks following 5.6 and 5.7 allows us to characterize full 0-spaces.

5.16 <u>Definitions</u> : (i) If $A, B \subset X$, then A is <u>fully C.E. separated</u> from B if there is a full C.E. family $\underline{E}$ such that $Cl_X(\cup\underline{E}) \cap Cl_X B = \phi$, while $A \subset E$ for some $E \in \underline{E}$.
(ii) If X is any space, define α' to be a relation on $\underline{P}(X)$ as follows: for $A, B \subset X$, $A \not\alpha B$ if and only if there is a compact subset K of $Cl_X A$ so that if A' is a closed subset of

$Cl_X A$, and $A' \cap K = \phi$, then A' is fully C.E. separated from B.

Then results 5.12 - 5.15 hold, if in the statements and proofs of the results, "C.E.", "C.I.", "α", and "0-space" are replaced by "full C.E.", "full C.I.", "α'", and "full 0-space" respectively, leaving us with the following characterization of full 0-spaces.

5.17 Theorem : If X is any space, then the following are equivalent.

(i) X is a full 0-space.

(ii) α' is a proximity on X.

If α' is a proximity on X, then $\alpha X = F_0 X$.

Example 1.34 is a full 0-space which is not almost rimcompact. We do not have an example of a 0-space which is not full - this question is left open to the reader.

Recall that a closed subset F of X is regular closed in X if $Cl_X int_X F = F$. The following result is 2.4 of [Wo].

5.19 Lemma : If A is a regular closed subset of X, B is closed in X, and $Cl_{\beta X} A \setminus A \subset Cl_{\beta X} B \setminus B$, then $Cl_X(A \setminus B)$ is pseudocompact.

5.20 Proposition : Let U be open in X. If $Ex_{\beta X} U$ is C.I. in

βX, and $p \in bd_{\beta X}(Ex_{\beta X}U) \cap (\beta X \setminus X)$, then there is a closed pseudocompact subset F of X such that $p \in Cl_{\beta X}F$.

Proof : By assumption $Ex_{\beta X}U \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. Note that $bd_{\beta X}(Ex_{\beta X}U) \setminus X = [Cl_{\beta X}U \setminus Ex_{\beta X}U] \setminus X$. If $p \in (Cl_{\beta X}U \setminus Ex_{\beta X}U) \setminus X$, there exists a regular closed subset V of X such that $p \in Ex_{\beta X}V$, while $(Cl_{\beta X}V) \cap Ex_{\beta X}U \subset X$. Let $B = X \setminus U$. Then $Cl_{\beta X}V \cap (\beta X \setminus X) \subset \beta X \setminus Ex_{\beta X}U = Cl_{\beta X}B$. According to 5.19, $Cl_X(V \setminus B)$ is a pseudocompact subset of X. Now $Cl_X(V \setminus B) = Cl_X(V \cap U)$, and it is easily checked that $p \in Cl_{\beta X}(V \cap U)$. The proposition follows. □

5.21 <u>Corollary</u> : Suppose X is a space in which pseudocompact closed subsets are compact. If X is a full 0-space, then X is almost rimcompact.

Proof : Suppose that pseudocompact closed subsets of X are compact. It follows from 5.20 that if $Ex_{\beta X}U$ is any full C.I. subset of βX, then $bd_{\beta X}Ex_{\beta X}U \subset X$. This implies that any connected component of $\beta X \setminus X$ having a basis in βX of full C.I. sets has a basis of open sets whose boundaries are contained in X. In other words, if X is a full 0-space, then by 2.19, X is almost rimcompact. □

5.22 <u>Corollary</u> : If X is realcompact, or metacompact, then X is a full 0-space if and only if X is almost rimcompact.

By slightly generalizing some of the previous definitions, we can determine, in terms of the proximity δ associated with a compactification δX, whether δX is larger than some O.I. compactification of X.

5.23 Definitions : (i) Let K_1X, $K_2X \in \underline{K}(X)$, with $K_1X \geq K_2X$, and let $f: K_1X \to K_2X$ denote the natural map. Then K_2X is perfect with respect to K_1X if $f^{\leftarrow}(p)$ is connected for each $p \in K_2X$.

(ii) Let δ be a proximity on X. A family $\underline{E}$ of open sets of X is C.E. with respect to δ, (denoted by δ-C.E.) if $\underline{E}$ is C.E. and also satisfies: for $E \in \underline{E}$, there is $E_1 \in \underline{E}$ such that $E_1 \not\delta (X \setminus E_2)$.

(iii) If A, B $\subset$ X, then A is δ-C.E. separated from B if there is a δ-C.E. family $\underline{E}$ such that $Cl_X(\cup\underline{E}) \cap Cl_X B = \phi$, while $A \subset E$ for some $E \in \underline{E}$.

(iv) If X is any space, define α_δ to be a relation on $\underline{P}(X)$ as follows: for A, B $\subset$ X, A α_δ B if and only if there is a compact subset K of $Cl_X A$ such that if A' is a closed subset of $Cl_X A$, and $A' \cap K = \phi$, then A' is δ-C.E. separated from B.

If δX is larger than some O.I. compactification of X, then there is a maximal O.I. compactification smaller than δX, denoted by $F(\delta X)$. According to an argument in [Mc], $F(\delta X)$ is perfect with respect to δX. The proof of 5.5 (when modified slightly) also yields the following.

5.24 Theorem : Suppose that KX, $\delta X \in \underline{K}(X)$, and that U, V are open in X. Then the following are equivalent.

(i) KX is perfect with respect to δX.

(ii) If $U \cap V = \phi$, and $Ex_{\delta X}U \cup Ex_{\delta X}V = Ex_{\delta X}(U \cup V)$, then $Ex_{KX}U \cup Ex_{KX}V = Ex_{KX}(U \cup V)$.

(iii) If $Cl_X(U \cap V)$ is compact, and $Ex_{\delta X}U \cup Ex_{\delta X}V = Ex_{\delta X}(U \cup V)$, then $Ex_{KX}U \cup Ex_{KX}V = Ex_{KX}(U \cup V)$.

The following result is analogous to 5.7, but requires a slightly different method of proof.

5.25 <u>Theorem</u> Suppose that KX, $\delta X \in \underline{K}(X)$, that KX is perfect with respect to δX, and that $\underline{E}$ is a δ-C.E. family. Let $p \in (KX \setminus X) \setminus Ex_{KX}\underline{E}$. Then

(i) there is a set D small with respect to $\underline{E}$ such that $p \in Ex_{KX}D$, hence

(ii) $Ex_{KX}\underline{E} \cap (KX \setminus X) = (\cup\{Cl_{KX}E\colon E \in \underline{E}\}) \cap (KX \setminus X)$ and

(iii) $Ex_{KX}\underline{E}$ is C.I. in KX.

Proof : (i) Let $\underline{E}$ be a δ-C.E. family, and suppose that $p \in (KX \setminus X) \setminus Ex_{KX}\underline{E}$. As in the proof of 5.7, we can find a set D small with respect to $\underline{E}$, and $E \in \underline{E}$ such that $p \in Ex_{KX}(E \cup D)$. It is sufficient to show that if D is small with respect to $\underline{E}$, then $Ex_{\delta X}(E \cup D) = Ex_{\delta X}E \cup Ex_{\delta X}D$. As KX is perfect with respect to δX, 5.24 then implies that $Ex_{KX}(E \cup D) = Ex_{KX}E \cup Ex_{KX}D$. As $p \notin Ex_{KX}E$, $p \in Ex_{KX}D$, and (i) is proved.

Suppose that for some $E \in \underline{E}$, there is a point

$p \in Ex_{\delta X}(E \cup D) \setminus (Ex_{\delta X}E \cup Ex_{\delta X}D)$. By 1.4 (v), $p \in Cl_{\delta X}E \cap Cl_{\delta X}D$. Since $\underline{E}$ is a δ-C.E. family, there exists $E_1 \in \underline{E}$ such that $Cl_{\delta X}E \subset Ex\delta XE_1$. Then $(\delta X \setminus X) \cap Ex_{\delta X}E_1 \cap Cl_{\delta X}D \neq \phi$, contradicting 5.4. Hence if D is small with respect to $\underline{E}$, and if $E \in \underline{E}$, it follows that $Ex_{\delta X}(E \cup D) = Ex_{\delta X}E \cup Ex_{\delta X}D$. As in the proof of 5.7, (ii) and (iii) follow from (i). □

Then 5.6, 5.8 and 5.12 - 5.15 are true if in the statements and proofs, "C.E.", "C.I.", "perfect", and "α" are replaced by "δ-C.E.", "δX-C.I.", "perfect with respect to δX", and "α_δ" respectively. This leads to the following result.

5.26 Theorem : Let $\delta X \in \underline{K}(X)$. The following are equivalent.

(i) δX is larger than some O.I. compactification of X.

(ii) α_δ is a proximity on X.

If α_δ is a proximity on X, then $\alpha_\delta = F(\delta X)$.

CHAPTER 6

CLOSED MAPS ON 0-SPACES

In this chapter we consider the following question : If $f : X \to Y$ is a closed map, and if X and Y are 0-spaces, under what conditions on X, Y and/or f will f extend to $g \in C(F_0X, F_0Y)$?

We begin by summarizing the known results on this question. In Lemma 1 of [Di] it is shown that if $f \in C(X,[0,1])$, and the set $\{y \in [0,1] : f^{\leftarrow}(y)$ contains a compact set K such that $X \setminus K$ can be written as $U \cup V$, where U, V are π-open in X and $U \subset f^{\leftarrow}[0,y]$, while $V \subset f^{\leftarrow}[y,1]\}$ is dense in $[0,1]$, then f extends to $g \in C[F_0X,[0,1])$. An argument in the proof of Theorem 3 of [M_3] shows that if $f : X \to Y$ is closed, X and Y are rimcompact and $bd_X f^{\leftarrow}(y)$ is compact for each $y \in Y$, then f extends to $g \in C(FX,FY)$. This result is used to prove Theorem 5 of [M_2] which states that if $f : X \to Y$ is a closed map, and if X and Y are locally compact and paracompact, then f extends to $g \in C(FX,FY)$. In Theorem 4 of [No] it is shown that the paracompactness of X and Y can be weakened to metacompactness.

In the sequel, if X is a 0-space (respectively, rimcompact) then α_X (respectively δ_X) will denote the proximity on X inducing F_0X (respectively, FX). We show in 6.3 that if X is a 0-space and Y is rimcompact, then a closed map $f : X \to Y$ extends to $g \in C(F_0X, FY)$ if and only if for any distinct pair of points $y, z \in Y$, $f^{\leftarrow}(y) \not\delta_X f^{\leftarrow}(z)$. We apply this result to several particular classes of spaces. In particular we show in 6.11 that if X is a metacompact 0-space, Y is a rimcompact quotient space of a locally compact space, and $f : X \to Y$ is a closed map, then f extends to $g \in C(F_0X, FY)$. Note that since the closed image of a metacompact space is metacompact, Y is necessarily metacompact.

The following, which is Lemma 1 of [SK], will simplify the proofs of several results.

6.1 <u>Lemma</u> : Suppose that the compactification θX associated with the proximity θ on X is a perfect compactification of X, and that A, B are disjoint subsets of X. Then $A \,\theta\, B$ if and only if $bd_X A \,\theta\, bd_X B$.

We use without mention the fact that if θX is a compactification of X associated with the proximity θ on X, then $A \,\theta\, B$ if and only if $Cl_{\theta X} A \cap Cl_{\theta X} B \neq \phi$ (see Chapter 1). In the following, if $f : X \to Y$ is a map, the natural map of βX into βY extending f is denoted by βf.

6.2 <u>Definition</u> : A map $f : X \to Y$ is a <u>WZ-map</u> if $Cl_{\beta X} f^{\leftarrow}(y) =$

$(\beta f)^{\leftarrow}(y)$ for each $y \in Y$.

Theorems 1.1, 1.2 and 1.3 of [Iw] show that a closed map is a WZ-map, and that the converse is true if either X is normal, or $bd_X f^{\leftarrow}(y)$ is compact for each $y \in Y$.

Suppose that $f : X \to Y$ is a map, and that θX, γY are compactifications of X and Y associated with the proximities θ and γ respectively. Recall that f extends to $g \in C(\theta X, \gamma Y)$ if and only if for $C, D \subset Y$, $C \not\gamma D$ implies $f^{\leftarrow}[C] \not\theta f^{\leftarrow}[D]$ (see Chapter 1). Suppose that Y is rimcompact, that θX is a perfect compactification of X and that $f : X \to Y$ is a WZ-map. The following result states that to show that f extends to $g \in C(\theta X, FY)$, it suffices to show that $f^{\leftarrow}[C] \not\theta f^{\leftarrow}[D]$, where C and D are singleton subsets of Y.

6.3 <u>Theorem</u> : Suppose that Y is rimcompact, and that f is a WZ-map from a space X into Y. If θX is a perfect compactification of X, then the following are equivalent.

(i) f extends to $g \in C(\theta X, FY)$.

(ii) For any distinct pair of points $y, z \in Y$, $f^{\leftarrow}(y) \not\theta f^{\leftarrow}(z)$.

(iii) If $f^{\leftarrow}(y)$ is completely separated in X from a subset B of X, then $f^{\leftarrow}(y) \not\theta B$.

Proof : (i) implies (iii). If $B \subset X$, and $f^{\leftarrow}(y)$ is completely separated from B, then $Cl_{\beta X} f^{\leftarrow}(y) \cap Cl_{\beta X} B = \phi$. Since f is a WZ-map, $Cl_{\beta X} f^{\leftarrow}(y) = (\beta f)^{\leftarrow}(y)$. Then y =

$(\beta f)[(\beta f)^{\leftarrow}(y)] \not\subseteq (\beta f)[\mathrm{Cl}_{\beta X}B] = \mathrm{Cl}_{\beta Y}f[B]$, hence $y \not\delta_Y f[B]$. Thus if f extends to $g \in C(\theta X, FY)$, $f^{\leftarrow}(y) \not\delta B$.

(iii) implies (ii). Suppose that y, z are distinct points of Y. Then $f^{\leftarrow}(y)$ and $f^{\leftarrow}(z)$ are completely separated in X, hence $f^{\leftarrow}(y) \not\delta f^{\leftarrow}(z)$.

(ii) implies (i). We wish to show that if $C, D \subset Y$ and $C \not\delta_Y D$, then $f^{\leftarrow}[C] \not\delta f^{\leftarrow}[D]$. It is easy to show that if Y is rimcompact, then $C \not\delta_Y D$ if and only if C and D are contained in π-open sets of Y whose closures in Y are disjoint. It then suffices to show that if C and D are disjoint closed subsets of Y with compact boundaries in Y, then $f^{\leftarrow}[C] \not\delta f^{\leftarrow}[D]$.

We claim that (ii) implies the following statement : if C is a closed subset of Y with compact boundary, and $y \in Y \setminus C$, then $f^{\leftarrow}(y) \not\delta f^{\leftarrow}[C]$. If $y \in Y \setminus C$, then $y \notin \mathrm{bd}_Y C$. Hence if $z \in \mathrm{bd}_y C$, (ii) implies that $f^{\leftarrow}(y) \not\delta f^{\leftarrow}(z)$. Then there is an open set $U(z)$ of X such that $\mathrm{Cl}_{\theta X}f^{\leftarrow}(z) \subset \mathrm{Ex}_{\theta X}U(z)$, while $\mathrm{Cl}_{\theta X}U(z) \cap \mathrm{Cl}_{\theta X}f^{\leftarrow}(y) = \phi$. As $(\theta f)^{\leftarrow}[\mathrm{Ex}_{\theta X}U(z)] \subset \mathrm{Ex}_{\beta X}U(z)$, it follows that $\mathrm{Cl}_{\beta X}f^{\leftarrow}(z) \subset \mathrm{Ex}_{\beta X}U(z)$. Since f is a WZ-map, $\mathrm{Cl}_{\beta X}f^{\leftarrow}(z) = (\beta f)^{\leftarrow}(z)$. The map βf is closed, hence there is an open set $V(z)$ of βY such that $(\beta f)^{\leftarrow}(z) \subset (\beta f)^{\leftarrow}[V(z)] \subset \mathrm{Ex}_{\beta X}U(z)$. Let $W(z) = V(z) \cap Y$. Then $f^{\leftarrow}(z) \subset f^{\leftarrow}[W(z)] \subset U(z)$, and so $f^{\leftarrow}[\mathrm{bd}_Y C] \subset \cup\{f^{\leftarrow}[W(z)] : z \in \mathrm{bd}_Y C\}$. It follows that $\mathrm{bd}_Y C \subset \cup\{W(z) : z \in \mathrm{bd}_Y C\}$. As $\mathrm{bd}_Y C$ is compact, there is a finite subset $\{z_1, z_2, \ldots, z_n\} \subset \mathrm{bd}_Y C$ such that $\mathrm{bd}_Y C \subset$

$\cup\{W(z_i): 1 \leq i \leq n\}$. Then $f^{\leftarrow}[bd_Y C] \subset \cup\{f^{\leftarrow}[W(z_i)]: 1 \leq i \leq n\} \subset \cup\{U(z_i): 1 \leq i \leq n\}$. Since $f^{\leftarrow}(y) \not\subset U(z_i)$, $f^{\leftarrow}(y) \not\subset \cup\{U(z_i): 1 \leq i \leq n\}$. As $bd_X f^{\leftarrow}[C] \subset f^{\leftarrow}[bd_Y C]$, $f^{\leftarrow}(y) \not\subset bd_X f^{\leftarrow}[C]$. It then follows from 6.1 that $f^{\leftarrow}(y) \not\subset f^{\leftarrow}[C]$, and the claim is proved.

Suppose then that C and D are disjoint closed subsets of Y whose boundaries are compact. If $p \in bd_Y D$, then $p \notin C$, hence $f^{\leftarrow}(p) \not\subset f^{\leftarrow}[C]$. Then there is an open set $U_1(p)$ of X such that $Cl_{\theta X} f^{\leftarrow}(p) \subset Ex_{\theta X} U_1(p)$, and $f^{\leftarrow}[C] \cap Cl_{\theta X} U_1(p) = \phi$. From an argument essentially identical to that in the preceding paragraph, where $f^{\leftarrow}(y)$ is replaced by $f^{\leftarrow}[C]$, it follows that $bd_X f^{\leftarrow}[D] \not\subset f^{\leftarrow}[C]$. Thus by 6.1 $f^{\leftarrow}[D] \not\subset f^{\leftarrow}[C]$, and the theorem is proved. □

It is an easy exercise to show that Lemma 1 of $[D_1]$ follows from the fact that 6.3 (ii) implies 6.3 (i).

Althought the conclusions of 6.3 hold whenever the hypothesized function f is a WZ-map, in our applications of it we will assume that f is a closed map. As noted above, a closed map is a WZ-map.

6.4 <u>Theorem</u> : Suppose that X is a 0-space, Y is rimcompact, and $f : X \to Y$ is a WZ-map. Then the following are equivalent.

(i) f is closed, and extends to $g \in C(F_0X, FY)$.

(ii) f is closed, and for any distinct pair of points

$y, z \in Y$, $f^{\leftarrow}(y) \not\delta_X f^{\leftarrow}(z)$.

(iii) If B is a closed subset of X, $y \in Y$ and $f^{\leftarrow}(y) \cap B = \phi$, then $f^{\leftarrow}(y) \not\delta_X B$.

If in addition, X is rimcompact, the previous conditions are equivalent to

(iv) For each $y \in Y$, $f^{\leftarrow}(y)$ has a neighbourhood basis in X of π-open sets of X.

Proof : It follows from 6.3 that (ii) implies (i).

(i) implies (ii). Suppose that $y \in Y$, that B is closed in X, and that $f^{\leftarrow}(y) \cap B = \phi$. Since f is a closed map, there is an open set V of Y such that $f^{\leftarrow}(y) \subset f^{\leftarrow}[V] \subset X \setminus B$. Then $y \not\delta_Y U \setminus V$. Since f extends to $g \in C(F_0X, FY)$, $f^{\leftarrow}(y) \not\delta_X f^{\leftarrow}[Y \setminus V] = X \setminus f^{\leftarrow}[V]$. Thus $f^{\leftarrow}(y) \not\delta_X B$.

(iii) implies (ii). If y and z are distinct points of Y, then $f^{\leftarrow}(y) \cap f^{\leftarrow}(z) = \phi$, hence $f^{\leftarrow}(y) \not\delta_X f^{\leftarrow}(z)$. We show that f is a closed map by showing that if $S \subset Y$, and $f^{\leftarrow}[S] \subset U$, where U is open in X, then there is an open set V of Y such that $f^{\leftarrow}[S] \subset f^{\leftarrow}[V] \subset U$. If $y \in S$, then $f^{\leftarrow}(y) \cap (X \setminus U) = \phi$, hence $f^{\leftarrow}(y) \not\delta_X X \setminus U$. Then $Cl_{\beta X}f^{\leftarrow}(y) \cap Cl_{\beta X}(X \setminus U) = \phi$. As $(\beta f)^{\leftarrow}(y) = Cl_{\beta X}f^{\leftarrow}(y)$, since f is a WZ-map, and as βf is a closed map, it follows that there is an open set $V(y)$ of βY such that $(\beta f)^{\leftarrow}(y) \subset (\beta f)^{\leftarrow}[V(y)] \subset Ex_{\beta X}U$. Then $f^{\leftarrow}(y) \subset f^{\leftarrow}[V(y) \cap Y] \subset Ex_{\beta X}U \cap X = U$. If $V = \cup\{V(y) \cap Y: y \in S\}$, then V is open in Y, and $f^{\leftarrow}[S] = \cup\{f^{\leftarrow}(y): y \in S\} \subset \cup\{f^{\leftarrow}[V(y) \cap Y]: y \in S\} = f^{\leftarrow}[V] \subset U$. Thus f is closed.

If X is rimcompact then (iii) and (iv) are clearly equivalent since if X is rimcompact, then $f^{\leftarrow}(y) \not\delta_X B$ if and only if $f^{\leftarrow}(y)$ and B are π-separated in X. □

Example 4.12 shows that the hypotheses of 6.4 do not imply that X is rimcompact.

6.5 <u>Corollary</u> : Suppose that X, Y are rimcompact, and that $f : X \to Y$ is a closed map. If f extends to $g \in C(FX,FY)$, then $g^{\leftarrow}(y) = \cap\{Cl_{FX}U: U \text{ is } \pi\text{-open in } X \text{ and } f^{\leftarrow}(y) \subset U\}$ for each $y \in Y$.

Proof : Since f is closed, if $y \in Y$ then by 4.20 $g^{\leftarrow}(y) = Cl_{FX}f^{\leftarrow}(y)$. If $p \in FX \setminus Cl_{FX}f^{\leftarrow}(y)$, then there is a closed set B of X such that $p \in Cl_{FX}B$, and $B \cap f^{\leftarrow}(y) = \phi$. It follows from 6.4 that there is a π-open set U of X such that $f^{\leftarrow}(y) \subset U$, and $(Cl_XU) \cap B = \phi$. Then by 2.3 (iii) and (iv), $g^{\leftarrow}(y) = Cl_{FX}f^{\leftarrow}(y) \subset Ex_{FX}U$, while $Cl_{FX}U \cap Cl_{FX}B = \phi$. Thus $p \notin Cl_{FX}U$. It follows that $g^{\leftarrow}(y) \supset \cap\{Cl_{FX}U: U \text{ is } \pi\text{-open in } X \text{ and } f^{\leftarrow}(y) \subset U\}$. Since the reverse inclusion clearly holds, the statement is proved. □

6.6 <u>Definition</u> : Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a collection of open sets of X. A subset F of X is <u>$\mathcal{U}$-compact</u> if there exists a finite subset A' of A such that $F \subset \cup\{U_\alpha : \alpha \in A'\}$.

6.7 <u>Theorem</u> : Let $f : X \to Y$ be a WZ-map, where X is a 0-space and Y is rimcompact. Suppose that for any open cover $\underline{U}$ of X, $Y(\underline{U})$ is a discrete subspace of Y, where

$$Y(\underline{U}) = \{y \in Y\colon bd_X f^{\leftarrow}(y) \text{ is not } \underline{U}\text{-compact}\}.$$

If either f is closed, or X is rimcompact, then f extends to $g \in C(F_0X, FY)$.

Proof : According to 6.3, it suffices to show that if y and z are distinct points of Y, then $f^{\leftarrow}(y) \not\delta_X f^{\leftarrow}(z)$. Choose $y, z \in Y$ such that $y \neq z$. Let V be an open subset of Y such that $y \in V$, while $z \notin Cl_Y V$. Then $f^{\leftarrow}(y) \subset f^{\leftarrow}[V]$, and $f^{\leftarrow}(z) \cap f^{\leftarrow}[Cl_Y V] = \phi$. We define an open cover U of X in the following way. If $x \in f^{\leftarrow}[Cl_Y V]$, then $x \not\delta_X f^{\leftarrow}(z)$, so there is an open set $U(x)$ of X such that $x \in U(x)$, and $f^{\leftarrow}(z) \not\delta_X U(x)$. If $x \in X \setminus f^{\leftarrow}[Cl_Y V]$, then $x \not\delta_X f^{\leftarrow}(y)$, so there is an open set $V(x)$ of X such that $x \in V(x)$ and $f^{\leftarrow}(y) \not\delta_X V(x)$. We define $U = \{U(x)\colon x \in f^{\leftarrow}[Cl_y V]\} \cup \{V(x)\colon x \notin f^{\leftarrow}[Cl_Y V]\}$, which is an open cover of X. Note that $f^{\leftarrow}(y) \cap [\cup\{V(x)\colon x \notin f^{\leftarrow}[Cl_Y V]\}] = \phi = f^{\leftarrow}(z) \cap [\cup\{U(x)\colon x \in f^{\leftarrow}[Cl_Y V]\}]$.

Let $Y(\underline{U}) = \{y \in Y\colon bd_X f^{\leftarrow}(y) \text{ is not } \underline{U}\text{-compact}\}$. If $y \notin Y(\underline{U})$, then $bd_X f^{\leftarrow}(y) \subset \cup\{U(x_i)\colon 1 \leq i \leq n\}$, where $\{x_1, x_2, \ldots, x_n\} \subset f^{\leftarrow}[Cl_Y V]$. Since $f^{\leftarrow}(z) \not\delta_X \cup\{U(x_i)\colon 1 \leq i \leq n\}$, it follows from 6.1 that $f^{\leftarrow}(y) \not\delta_X f^{\leftarrow}(z)$, and the theorem is proved. Now suppose that $y \in Y(\underline{U})$. By assumption $Y(\underline{U})$ is a discrete subset of Y, hence there is a π-open set W of Y such that $y \in W \subset V$, and $Cl_Y W \cap Y(\underline{U}) = \{y\}$. If $p \in bd_Y W$, then $p \notin Y(\underline{U})$ so there is an open set $U'(p)$ of X which is a finite union of elements of $\underline{U}$ such that $bd_X f^{\leftarrow}(p) \subset U'(p)$ and

$U'(p) \not\subset_X f^{\leftarrow}(z)$. It follows from 6.1, the choice of $\underline{U}$ and the fact that F_0X is a perfect compactification of X, that there is an open set W(p) of X such that $f^{\leftarrow}(p) \subset W(p)$ and $W(p) \not\subset_X f^{\leftarrow}(z)$. If X is rimcompact, W(p) can be chosen to be a π-open subset of X.

We claim that there is an open set W′(p) of Y such that $f^{\leftarrow}(p) \subset f^{\leftarrow}[W'(p)] \subset W(p)$. This is obvious if f is a closed map. Suppose that X is rimcompact, and that W(p) is π-open in X. Since f is a WZ-map, it follows from 2.3 (iii) that $(\beta f)^{\leftarrow}(p) = Cl_{\beta X} f^{\leftarrow}(p) \subset Ex_{\beta X} W(p)$. Since βf is a closed map, we can again find the desired open set W′(p) of Y, and the claim is true.

Then $f^{\leftarrow}[bd_Y W] \subset \cup\{f^{\leftarrow}[W'(p)]: p \in bd_Y W\}$, so $bd_Y W \subset \cup\{W'(p): p \in bd_Y W\}$. Since $bd_Y W$ is compact, there is a finite set $\{p_1, p_2, \ldots, p_n\} \subset bd_Y W$ such that $bd_Y W \subset \cup\{W'(p_i): 1 \leq i \leq n\}$. Then $f^{\leftarrow}[bd_Y W] \subset \cup\{f^{\leftarrow}[W'(p_i)]: 1 \leq i \leq n\} \subset \cup\{W(p_i): 1 \leq i \leq n\}$. As $f^{\leftarrow}(z) \not\subset_X W(p_i)$, and $bd_X f^{\leftarrow}[W] \subset f^{\leftarrow}[bd_Y W]$, it follows that $f^{\leftarrow}(z) \not\subset_X bd_X f^{\leftarrow}[W]$. Thus by 6.1 $f^{\leftarrow}(z) \not\subset_X f^{\leftarrow}[W]$. Since $f^{\leftarrow}(y) \subset f^{\leftarrow}[W]$, $f^{\leftarrow}(z) \not\subset_X f^{\leftarrow}(y)$ and the theorem is proved. □

The next result is a special case of 6.7.

6.8 Corollary : Suppose that $f : X \to Y$ is a closed map, where X is a 0-space and Y is rimcompact. Let $Y_0 = \{y \in Y: bd_X f^{\leftarrow}(y) \text{ is not compact}\}$. If Y_0 is a discrete subspace of Y, then f extends to $g \in C(F_0X, FY)$.

As mentioned in our summary of known results, it is shown in [M_3] that if X is rimcompact and the set Y_0 defined in 6.8 is empty, then the conclusions of 6.8 hold.

A space X is a <u>k-space</u> if a subset F of X is closed if and only if $F \cap K$ is compact for each compact subset K of X. It is well known that a space X is a k-space if and only if X is the quotient of a locally compact space, and that any first countable space is a k-space.

The following are 1.3 of [Ar] and 7.2 (d) of [Iw] respectively.

6.9 <u>Proposition</u> : Suppose that Y is a k-space, and that f is a closed map from a space X into Y. If $\mathcal{U}$ is any point-finite open cover of X, and $Y(\mathcal{U}) = \{y \in Y: f^{\leftarrow}(y))$ is not $\mathcal{U}$-compact$\}$, then $Y(\mathcal{U})$ is a closed discrete subspace of Y.

6.10 <u>Proposition</u> : Suppose that X is locally compact and realcompact, and that f is a closed map from X into a space Y. If $Y_0 = \{y \in Y: f^{\leftarrow}(y)$ is not compact$\}$, then Y_0 is a closed discrete subspace of Y.

We point out that although the normality of X is included as a hypothesis in 7.2 of [Iw], it is not required in the proof of 7.2 (d).

The following shows that the requirement that X be locally compact in Theorem 4 of [No] is not necessary.

6.11 <u>Theorem</u> : Suppose that Y is a rimcompact k-space, and that X is either (i) locally compact and realcompact, or (ii) a metacompact 0-space. If $f : X \to Y$ is a closed map, then f extends to $g \in C(F_0X, FY)$.

Proof : In the case where X is realcompact and locally compact, the theorem follows immediately from 6.8 and 6.10.

If X is a metacompact 0-space, and $\mathcal{U}$ is any open cover of X, choose $\mathcal{V}$ to be a point-finite open refinement of $\mathcal{U}$. Clearly $Y(\mathcal{U}) \subset Y(\mathcal{V})$, where $Y(\mathcal{U})$ and $Y(\mathcal{V})$ are as in 6.9. The theorem then follows from 6.8 and 6.9. □

We now consider closed maps into q-spaces. If $x \in X$, then x is a <u>q-point</u> of X if there exists a sequence $\{N_i\}_{i \in \mathbb{N}}$ of neighborhoods of x such that if $x_i \in N_i$, for $i \in \mathbb{N}$, and $i \neq j$ implies that $x_i \neq x_j$, then the set $\{x_i : i \in \mathbb{N}\}$ has an accumulation point in X. A space X is a <u>q-space</u> if every point of X is a q-point of X.

Clearly any first countable or locally countably compact space is a q-space. An example of a countably compact space which is not a k-space is outlined in 1.10 of [Bu]. The following example shows that a k-space need not be a q-space.

6.12 Example : Let X be the quotient space $\mathbb{R}/\{\mathbb{N}\}$. Since X is the quotient of a locally compact space, X is a k-space. We show that $\{\mathbb{N}\}$ is not a q-point of X. Let $\{U_n: n \in \mathbb{N}\}$ be a sequence of open neighbourhoods of N in X. For each $n \in \mathbb{N}$, let V_n be an open interval of the form $(n-r_n, n+r_n)$ which is contained in U_n. If $s_n = (n+r_n)/2$, for each n, then $s_n \in U_n$, and $s_n \neq s_m$ if $n \neq m$, but $\{s_n: n \in \mathbb{N}\}$ has no accumulation point in X. □

A subset F of a space X is relatively pseudocompact in X if for each $f \in C(X)$, f is bounded on F. Following the terminology of [Iw], we say that a subset F of X has property (*) if $\inf\{f(x): x \in F\} > 0$ for each $f \in C(X)$ which is positive on F. It is pointed out in [Iw] that a pseudocompact subset of X has property (*), and that a subset with property (*) is relatively pseudocompact.

6.13 Definition : A subset F of a space X has property (**) if for any point-finite collection $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ of open sets of X covering F, there is a finite subset A' of A such that $F \subset \cup\{Cl_X U_\alpha: \alpha \in A'\}$.

6.14 Lemma : If a subset F of a space X has property (**), then F has property (*).

Proof : Let $f \in C(X)$ such that f is positive on F. If $g = f \wedge 1$, then g is positive on F, and $\inf\{g(x): x \in F\} \leq \inf\{f(x): x \in F\}$. For $n \in N$ let $U(n) = g^{\leftarrow}(1/(n+2),1/n)$. Then $\{U(n): n \in N\}$ is a point-finite collection of open sets of X which covers F. Since F has property (**), there is a finite subset $\{n_1,n_2,\ldots,n_m\}$ of N such that $F \subset \cup\{Cl_X U(n_i): 1 \leq i \leq n\}$. If $m = \max\{n_1,n_2,\ldots,n_m\}$, then $F \subset g^{\leftarrow}[1/(m+2),1]$, hence $\inf\{f(x): x \in F\} \geq \inf\{g(x): x \in F\} > 0$. Thus F has property (*). □

It is shown in 2.1 of [Mi] that if $f : X \to Y$ is a closed map, and $y \in Y$ is a q-point of Y, then $bd_X f^{\leftarrow}(y)$ is relatively pseudocompact. It follows from 6.14, and the remarks preceding 6.13, that the next result generalizes this fact.

6.15 <u>Proposition</u> : Suppose that $f : X \to Y$ is a closed map. If $y \in Y$ is a q-point of Y, then $bd_X f^{\leftarrow}(y)$ has property (**).

Proof : Let $\underline{U} = \{U_\alpha: \alpha \in A\}$ be a point-finite collection of open sets of X covering $bd_X f^{\leftarrow}(y)$. Since f is a closed map, and $f^{\leftarrow}(y) \subset int_X f^{\leftarrow}(y) \cup (\cup\{U_\alpha: \alpha \in A\})$, there is an open set V of Y such that $f^{\leftarrow}(y) \subset f^{\leftarrow}[Cl_X V] \subset int_X f^{\leftarrow}(y) \cup (\cup\{U_\alpha: \alpha \in A\})$. Let $\{N_i\}_{i \in N}$ be a sequence of open neighbourhoods of y in Y witnessing the fact that y is a q-point of Y. If $M_i = N_i \cap V$, then $\{M_i: i \in N\}$ witnesses the fact that y is a q-point of Y. Suppose that for any finite subset A' of A, $bd_X f^{\leftarrow}(y) \not\subseteq \cup\{Cl_X A_\alpha : \alpha \in A'\}$. We will construct inductively a closed discrete set $\{x_i : i \in N\}$ of X such that $x_i \in f^{\leftarrow}[M_i]$ for each i, and $f(x_i) = f(x_j)$ implies that $i = j$. Let $x_1 \in bd_X f^{\leftarrow}(y)$. Suppose we have chosen x_i, for $i < n$, such that $x_i \in f^{\leftarrow}[M_i]$ and $f(x_i) \neq f(x_j)$ if $i \neq j$. Let $A_n = \{\alpha \in A: x_i \in U_\alpha, 1 \leq i \leq n\}$. Since $\underline{U}$ is a point-finite collection of subsets of X, $|A_n| < \omega$. Hence $bd_X f^{\leftarrow}(y) \not\subseteq \cup\{Cl_X A_\alpha : \alpha \in A_n\}$. Let $V_n =$ $[f^{\leftarrow}[M_n] \cap (X \backslash \cup\{Cl_X A_\alpha : \alpha \in A_n\})] \setminus [f^{\leftarrow}[f(x_2), f(x_3), \ldots, f(x_n)]]$. Since $f(x_i) \neq y$ if $i > 1$ by our inductive hypothesis, V_n is a non-empty open subset of X which intersects $bd_X f^{\leftarrow}(y)$. Hence there is a point $x_n \in V_n \setminus f^{\leftarrow}(y)$. Clearly $f(x_n) \neq f(x_i)$ for $i < n$.

We claim that $\{x_i : i \in N\}$ is a closed discrete subspace of X. Since $\{x_i\}_{i \in N} \subset Cl_X f^{\leftarrow}[V] \setminus int_X f^{\leftarrow}(y)$, $Cl_X\{x_i : i \in N\} \subset Cl_X f^{\leftarrow}[V] \setminus int_X f^{\leftarrow}(y) \subset \cup\{U_\alpha : \alpha \in A\}$. If $x \in Cl_X f^{\leftarrow}[V]$, let $V_x = \cup\{U_\alpha : x \in U_\alpha\}$. If $V_x \cap \{x_i : i \in N\} = \phi$, then $x \notin Cl_X\{x_i : i \in N\}$. If $V_x \cap \{x_i : i \in N\} \neq \phi$, choose $x_j \in V \cap \{x_i : i \in N\}$. Then $\{x, x_j\} \subset U_\alpha$ for some $\alpha \in A_{j+1}$. By our inductive hypothesis, $\{x_i : i \in N \text{ and } i > j\} \cap [\cup\{U_\alpha : \alpha \in A_{j+1}\}] = \phi$, hence U_α is a neighbourhood of x in X such that $U_\alpha \cap \{x_i : i \in N\} \subset \{x_1, x_2, \ldots, x_j\}$. Thus $x \notin Cl_X\{x_i : i \in N\}$, and the claim is proved.

Since f is a closed map, every subset of $f[\{x_i : i \in N\}]$ is a closed discrete subset of Y. This contradicts the fact that since $f(x_i) \in N_i$, and $f(x_i) \neq f(x_j)$ if $i \neq j$, $\{f(x_i): i \in \mathbb{N}\}$ has an accumulation point in Y. Thus $bd_X f^{\leftarrow}(y) \subset \cup\{Cl_X U_\alpha: \alpha \in A'\}$ for some finite subset A' of A, and so $bd_X f^{\leftarrow}(y)$ has property (**). □

We now have the following.

6.16 Theorem : Suppose that Y is rimcompact, and that the set Y_0 of non q-points of Y is discrete in Y. If $f : X \to Y$ is closed, where X is (i) a metacompact 0-space or (ii) a realcompact 0-space, then f extends to $g \in C(F_0X, FY)$.

Proof : If $y \notin Y_0$, then by 6.15, $bd_X f^{\leftarrow}(y)$ has property (**). If X is realcompact, then it follows from 6.14 and the remarks preceding 6.13, that $bd_X f^{\leftarrow}(y)$ is compact, since any relatively pseudocompact subset of a realcompact space is compact, by 8E.1 of [GJ].

We show that if X is metacompact, and $bd_X f^{\leftarrow}(y)$ has property (**), then $bd_X f^{\leftarrow}(y)$ is compact. According to 17B.1, 17K.2 and 17K.3 of [Wi], it suffices to show that if $\underline{V} = \{V_\alpha: \alpha \in A\}$ is a collection of open sets of X such that $bd_X f^{\leftarrow}(y) \subset \cup\{V_\alpha: \alpha \in A\}$, then there is a finite sub-collection of $\underline{V}$ whose closures cover $bd_X f^{\leftarrow}(y)$. Let $\underline{V}$ be such a collection. Then $\underline{V}' = \underline{V} \cup \{int_X f^{\leftarrow}(y), X \setminus f^{\leftarrow}(y)\}$ is an open cover of X. Choose $\underline{W}$ to be a point-finite open refinement of $\underline{V}'$. Then $\underline{U} = \{W \in \underline{W}: W \cap bd_X f^{\leftarrow}(y) \neq \phi\}$ is a point-finite refinement of $\underline{V}$ which covers $bd_X f^{\leftarrow}(y)$. Since

$bd_X f^{\leftarrow}(y)$ has property (**), there is a finite subcollection of $\underline{\mathcal{U}}$ whose closures cover $bd_X f^{\leftarrow}(y)$. Since $\underline{\mathcal{U}}$ refines $\underline{\mathcal{V}}$, there is a finite subcollection of $\underline{\mathcal{V}}$ whose closures cover $bd_X f^{\leftarrow}(y)$. Thus $bd_X f^{\leftarrow}(y)$ is compact.

It follows that if $Y_1 = \{y \in Y: bd_X f^{\leftarrow}(y) \text{ is not compact}\}$, then $Y_1 \subset Y_0$, hence Y_1 is a discrete subspace of X. Thus by 6.8, f extends to $g \in C(F_0X, FY)$. □

There are examples of maps of rimcompact spaces which do not extend to maps of the respective Freundenthal compactifications. The following is Example 1 of [No].

6.17 Example : Let $X = \omega_1 \times I$, and let $Y = I$. Then X is locally compact and Y is compact. Let f be the projection map from X onto Y. Then f is an open map. Since ω_1 is countably compact, f is also closed. However FX is the one-point compactification of $\omega_1 \times I$. Clearly f does not extend to $g \in C(FX, I)$.

In fact, if X is any countably compact 0-space such that $\beta X \setminus X$ is not 0-dimensional, there exists $f \in C(X,[0,1])$ such that f does not extend to $g \in C(FX,[0,1])$. For any bounded continuous real-valued function on X is closed. Thus if X is not C*-embedded in F_0X, (ie. if $F_0X \neq \beta X$), there is a closed function from X into I which does not extend over F_0X. This is not true if "countably compact" is weakened to "pseudocompact". In the following $\underline{\mathcal{R}}$ is any maximal almost disjoint collectin of subsets of $\mathbb{N}$. Recall that $\mathbb{N} \cup \underline{R}$ is pseudocompact, for any such collection $\underline{R}$.

6.18 Theorem : Let $f : \mathbb{N} \cup \underline{R} \to Y$ be a closed map, where Y is any space. Then Y is 0-dimensional, and f extends to $g \in C(F(\mathbb{N} \cup \underline{R}), FY)$.

Proof : First note that if $y \in Y \setminus f[\underline{R}]$, then $f^{\leftarrow}(y)$ is a closed subset of $\mathbb{N} \cup \underline{R}$ contained in $\mathbb{N}$, hence $f^{\leftarrow}(y)$ is a finite subset of $\mathbb{N} \cup \underline{R}$ contained in $\mathbb{N}$. Then $f^{\leftarrow}(y)$ is open in $\mathbb{N} \cup \underline{R}$. Since f is a quotient map, y is isolated in Y, hence has a basis of clopen subsets of Y.

Since $\underline{R}$ is a closed discrete subset of $\mathbb{N} \cup \underline{R}$, and f is a closed map, $f[\underline{R}]$ is a closed discrete subset of Y. Suppose that $y \in f[\underline{R}] \setminus T$, where T is closed in Y. Then there is an open subset U of Y such that $y \in U$, $U \cap T = \phi$, and $U \cap f[\underline{R}] = \{y\}$. Choose V to be open in Y such that $y \in V \subset Cl_Y V \subset U$. Then $bd_Y V \subset Y \setminus f[\underline{R}]$. Since each point of $bd_Y V$ is isolated in Y, $bd_Y V$ is open in Y, hence $Cl_Y V$ is open in Y. Thus $Cl_Y V$ is a clopen subset of Y such that $y \in Cl_Y V$ and $(Cl_Y V) \cap T = \phi$. It follows that Y is 0-dimensional. Then f extends to $g \in C(\beta_0(\mathbb{N} \cup \underline{R}), \beta_0 Y)$. Since $\beta_0(\mathbb{N} \cup \underline{R}) = F(\mathbb{N} \cup \underline{R})$ and $\beta_0 Y = FY$, the theorem follows. □

It is well known that if $f : X \to Y$ is a map, where X and Y are 0-dimensional, then f extends to $g \in C(FX, FY) = C(\beta_0 X, \beta_0 Y)$. The following generalizes this fact.

6.19 <u>Theorem</u> : Suppose that X is a space, Y is 0-dimensional and KX is a perfect compactification of X. If $f : X \to Y$ is a map, then f extends to $g \in C(KX, \beta_0 Y)$.

Proof : Subsets C and D of Y have disjoint closures in $\beta_0 Y$ if and only if C and D are contained in disjoint clopen subsets U and $Y \setminus U$ of Y respectively. Since $f^{\leftarrow}[U]$, $f^{\leftarrow}[Y \setminus U]$ are then disjoint clopen subsets of X, and KX is a perfect compactification of X, it follows that $Cl_{KX} f^{\leftarrow}[U] \cap Cl_{KX} f^{\leftarrow}[Y \setminus U] = \phi$. Then $Cl_{KX} f^{\leftarrow}[C] \cap Cl_{KX} f^{\leftarrow}[D] = \phi$; thus by 1.1, f extends to $g \in C(KX, \beta_0 Y)$. □

6.20 <u>Definition</u> : A map $f : X \to Y$ is <u>monotone</u> if $f^{\leftarrow}(y)$ is connected for each $y \in Y$.

The following answers a question communicated verbally to R. G. Woods (Topology Conference, 1980) by D. Bellamy.

6.21 <u>Theorem</u> : Let $f : X \to Y$ be a monotone quotient map, and let KX, KY be perfect compactifications of X and Y respectively. If f extends to $g \in C(KX, KY)$, then g is monotone.

Proof : Suppose that there is $p \in KY$ such that $g^{\leftarrow}(p)$ is not connected. Write $g^{\leftarrow}(p) = G_1 \cup G_2$, where G_1 and G_2 are disjoint closed subsets of $g^{\leftarrow}(p)$. Since $g^{\leftarrow}(p)$ is compact, G_1 and G_2 are disjoint compact subsets of KX; hence there are open sets U_1 and U_2 of X such that $G_i \subset Ex_{KX}U_i$, $(i = 1,2)$ and $Cl_{KX}U_1 \cap Cl_{KX}U_2 = \phi$. Since g is a closed map, there is an open set V of Y such that $g^{\leftarrow}(p) \subset g^{\leftarrow}[V] \subset Ex_{KX}U_1 \cup Ex_{KX}U_2$. Let $W_i = g^{\leftarrow}[V] \cap U_i = f^{\leftarrow}[V \cap Y] \cap U_i$, $(i = 1,2)$. Then W_1 and W_2 are disjoint open subsets of X, and $W_1 \cup W_2 = f^{\leftarrow}[V \cap Y]$. Since $f^{\leftarrow}(y)$ is connected for each $y \in Y$, $W_i = f^{\leftarrow}[V_i]$ for some subset V_i of Y, $(i = 1,2)$. Since f is a quotient map, V_i is open in Y $(i = 1,2)$. Then $V \cap Y = V_1 \cup V_2$, while $V_1 \cap V_2 = \phi$. It follows from 1.5 (i) and (ii), and 1.6 that $p \in Ex_{KY}V = Ex_{KY}V_1 \cup Ex_{KY}V_2$, while $Ex_{KY}V_1 \cap Ex_{KY}V_2 = \phi$. Suppose without loss of generality that $p \in Ex_{KY}V_1$. Since $g^{\leftarrow}[Ex_{KY}V_1]$ is an open subset of KX containing $f^{\leftarrow}[V_1]$, $g^{\leftarrow}(p) \subset g^{\leftarrow}[Ex_{KY}V_1] \subset Ex_{KX}f^{\leftarrow}[V_1] = Ex_{KX}W_1 \subset Ex_{KX}U_1$, which contradicts the fact that $g^{\leftarrow}(p) \cap Ex_{KX}U_2 \neq \phi$. Thus $g^{\leftarrow}(p)$ is connected for each $p \in KY$.

6.22 <u>Corollary</u> : Suppose that X is a 0-space and Y is 0-dimensional. If there is a perfect monotone map from X into Y, then X is almost rimcompact and $F_0X \setminus X$ is homeomorphic to $FY \setminus Y$.

Proof : Let $f : X \to Y$ be a perfect monotone map. Then f extends to $g \in C(F_0X,FY)$ by 6.19. Since f is perfect, $g^{\leftarrow}[FY \setminus Y] = F_0X \setminus X$. As f is monotone, it follows from 6.21 that $g^{\leftarrow}(y)$ is connected for each $y \in FY \setminus Y$. Since $F_0X \setminus X$ is 0-dimensional, and $g^{\leftarrow}(y) \subset F_0X \setminus X$, $|g^{\leftarrow}(y)| = 1$. Thus $g|_{F_0X\setminus X} : F_0X \setminus X \to FY \setminus Y$ is a closed continuous one-to-one map, hence g is a homeomorphism. The fact that X is almost rimcompact follows from 4.11. □

Example 4.9 shows that the perfect monotone preimage X of a 0-dimensional space need not be a 0-space, while Example 4.12 shows that even if X is a 0-space, X need not be rimcompact.

BIBLIOGRAPHY

[Ar] A.V. Archangelskii, Closed mappings, bicompact sets and a problem of P.S. Aleksandrov, Amer. Math. Soc. Trans. 78(1968),41-66.

[Bu] D.K. Burke, On p-spaces and wΔ-spaces, Pac. Jour. Math. 35(1970),285-296.

[Ch] R.E. Chandler, Hausdorff compactifications. Lecture Notes in Pure and Applied Mathematics No.23, Marcel Dekker, Inc., New York and Basel, 1976.

[Di] R.F. Dickman, Jr., Some characterizations of the Freudenthal compactification of a semicompact space, Proc. Amer. Math. Soc. 19(1968),631-633.

[En] R. Engelking, General Topology, Polish Scientific Publishers, Warsaw, 1977.

[ES] R. Engelking and E.G. Sklyarenko, On compactifications allowing extensions of mappings, Fund. Math. 53(1963),65-79.

[Fr] H. Freudenthal, Neuaufbau der Erdentheorie, Ann. of Math. 43(2)(1942),261-279.

[GJ] L. Gillman and M. Jerison, Rings of continuous functions. Van Nostrand, New York, 1960.

[Gl] I. Glicksberg, Stone-Cech compactifications of products, Trans. Amer. Math. Soc. 90(1959),369-382.

[HI] M. Henriksen and J.R. Isbell, Some properties of compactifications, Duke Math. J. 25(1958),83-105.

[Is] J.R. Isbell, Uniform spaces. Amer. Math. Soc. Math. Surveys No. 12 (1962).

[Iw] T. Isiwata, Mappings and spaces, Pac. Jour. Math. 20(1967),455-480.

[Ke] J.L. Kelley, General Topology. Van Nostrand, New York, 1955.

[Mc] J.R. McCartney, Maximum zero-dimensional compactifications, Proc. Camb. Phil. Soc. 68(1970), 653-661.

[Mi] E. Michael, A note on closed maps and compact sets, Israel J. Math. 2 (1964),173-176.

[M_1] K. Morita, On bicompactifications of semibicompact spaces, Sci. Rep Tokyo Bunrika Daigaku Sec. A vol. 4, 94(1952).

[M_2] ---------, On closed mappings, Proc. Japan Acad. 32(1956),539-543.

[M_3] ---------, On images of an open interval under closed continuous mappings, Proc. Japan Acad. 32(1956),15-19.

[NW] J.A. Naimpally and B.D. Warrack. Proximity spaces, Camb. Univ. Press, Cambridge, 1970.

[No] K. Nowinski, Closed mappings and the Freudenthal compactification, Fund. Math. 76(1972),71-83.

[Sa] P. Samuel, Ultrafilters and compactifications of uniform spaces, Trans. Amer. Math. Soc. 64(1948),100-132.

[Sk] E.G. Sklyarenko, Some questions in the theory of bicompactifications, Amer. Math. Soc. Trans. 58(1966),216-244.

[S_1] Y.M. Smirnov, On proximity spaces in the sense of V.A. Efremovic, Amer. Math. Soc. Trans. Ser. 2, 38, 1-4.

[S_2] ------------, On proximity spaces, Amer. Math. Soc. Trans. Ser. 2, 38, 5-35.

[St] D.P. Strauss, Extremally disconnected spaces, Proc. Amer. Math. Soc. 18(1967),305-309.

[Te] J. Terasawa, Spaces $N \cup \underline{R}$ and their dimensions, Top. and its Appl. 11(1980),93-102.

[Ty] A. Tychonoff, Über die topologische Erweitungen von Räumen, Mat. Ann. 102(1930),544-561.

[Wi] S. Willard, General topology, Addison-Wesley, Reading, Mass., 1970.

[Wo] R.G. Woods, Co-absolutes of remainders of Stone-Cech compactifications, Pac. Jour. Math. 37(1971),545-560.