ACTIVE RC SYNTHESIS OF STATE EQUATIONS USING AN OPERATIONAL AMPLIFIER

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ABSTRACT

A necessary condition on the A matrix that it be realizable as an RC network, with one differential-input voltage-controlled operational amplifier, is given. It is shown that if the capacitive sub-network is a star tree, then this condition is also sufficient. A test is given to determine by inspection whether a given second-order A matrix is realizable using this configuration.

Also, state equations obtained from transfer-function matrices by Zadeh and Desoer's method are realized using a current-controlled operational amplifier for the case of a single output.

A current-controlled amplifier is further used to realize a larger class of second-order A matrices than that which is realizable above. It is seen that the second-order A matrix which results from a voltage-controlled amplifier realization is contained in this class. An application to two-port transfer-function synthesis is also given.

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CHAPTER 1

INTRODUCTION

The use of state equations of the form

$$px = Ax + Bu$$

$$y = Cx + Du$$

for the description and analysis of systems is well known [1]. Here x denotes a column matrix of state variables (usually capacitor voltages and inductor currents when dealing with linear electrical networks), y denotes a column matrix of responses, u a column matrix of inputs, A, B, C and D are matrices of constants or time functions (depending upon whether or not the system is time-variant) and p indicates differentiation with respect to time.

As is equally well known, the elimination of inductors from electrical networks is highly desirable since they are often bulky and lossy.

Even though a required realization may be that of a transfer function, it is often simpler to synthesize a network using a state-variable technique [2]. A minimum number of capacitors may then be used. This is a valuable feature since capacitors are the most difficult elements to fabricate in an integrated circuit.

An important problem, then, is the active RC synthesis of state equations.

Solutions have been given using the classical technique of analog computer simulation by Kerwin, Huelsman and Newcomb [2] and Tow [3] using integrators, where n integrators are required for the realization of an n^{th} order set of state equations. A controlled source realization has been given by Martens [4], where $(n+m)^2$ controlled sources may be required for synthesis of an n^{th} order set of state equations with m inputs and responses.

Since it is usually desirable to minimize the number of active elements in a network, this work is concerned with a realization containing only one active element. The operational amplifier is considered as it is readily available in integrated form.

A characterization of the state equations of an RC network containing one differential-input voltage-controlled operational amplifier is given in Chapter 2. Necessary conditions on the second-order A matrix, where the capacitive sub-network is a star tree, are also given in this chapter. It is later shown in Chapter 3 that these conditions are sufficient for such a realization.

In addition, a realization using a current-controlled differential-input operational amplifier is given in Chapter 3 for the diagonal A matrices obtained by Zadeh and Desoer [1] from transfer-function matrices. Single-output transfer-function matrices are also realized here. Finally, a realization is given in this chapter for a class of second-order A matrices along with an application to two-port transfer-function synthesis.

CHAPTER 2

ON STATE EQUATION REALIZATION USING A SINGLE VOLTAGE-CONTROLLED OPERATIONAL AMPLIFIER

2.1 Characterization of State Equations for RC Networks Containing One Differential-Input Voltage-Controlled Operational Amplifier

The use of a voltage-controlled operational amplifier is investigated in this section. In order to determine the usefullness of this device, the state equations for RC networks containing it are characterized.

To facilitate finding a necessary condition on a set of state equations that it be realizable as an RC network with one differential-input voltage-controlled operational amplifier, the following theorem will first be proved:

Theorem I:

For any proper tree* of an RC network with voltage sources, the resistive branch voltages are always expressible as linear combinations of the capacitive and source voltages.

Proof:

Let V_R, V_C and V_S be the column matrices of resistive branch, capacitive and source voltages, respectively. The corresponding current matrices are I_R, I_C and I_S.

Let V_{ch} and I_{ch} be the matrices of chord voltages and currents. (The only chords which exist are resistive.)

* Proper trees only are considered, as excess capacitors are superfluous. Only n capacitors are required for realization of an n^{th} order system.

Then, partitioning the fundamental cut-set matrix of the network yields [5]

$$\begin{pmatrix} Q_{11} & U & 0 & 0 \\ Q_{21} & 0 & U & 0 \\ Q_{31} & 0 & 0 & U \end{pmatrix} \begin{pmatrix} I_{ch} \\ I_S \\ I_R \\ I_C \end{pmatrix} = 0 \tag{1}$$

where U is the unit matrix.

From the second of the above system of equations

$$I_R = -Q_{21} I_{ch} \tag{2}$$

Also, partitioning the fundamental circuit matrix for the network yields [5]

$$\begin{pmatrix} U & B_{12} & B_{13} & B_{14} \end{pmatrix} \begin{pmatrix} V_{ch} \\ V_S \\ V_R \\ V_C \end{pmatrix} = 0 \tag{3}$$

or

$$V_{ch} = -B_{12}V_S - B_{13}V_R - B_{14}V_C \tag{4}$$

Since

$B_f Q_f^T = 0$ [5],

$$\begin{pmatrix} U & B_{12} & B_{13} & B_{14} \end{pmatrix} \begin{pmatrix} Q_{11}^T & Q_{21}^T & Q_{31}^T \\ U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{pmatrix} = 0 \tag{5}$$

From Equation (5):

$$Q_{21}^T = -B_{13} \tag{6}$$

Let G_B and R_C be the (diagonal) matrices of conductive branches and resistive chords.

Then

$$I_R = G_B V_R \tag{7}$$

and

$$V_{ch} = R_c I_{ch} \tag{8}$$

or

$$V_R = R_B I_R \tag{9}$$

and

$$I_{ch} = G_c V_{ch} \tag{10}$$

Equations (6), (4), (10), (2) and (9) may then be combined (where substitutions are made in the order indicated) to yield after premultiplying by G_B:

$$G_B V_R = - Q_{21} G_C (- B_{12} V_S + Q_{21}^T V_R - B_{14} V_C)$$

or

$$(G_B + Q_{21} G_C Q_{21}^T) V_R = Q_{21} G_C (B_{12} \quad B_{14}) \begin{pmatrix} V_S \\ V_C \end{pmatrix} \tag{11}$$

and

$$(G_B + Q_{21} G_C Q_{21}^T) = (U \quad Q_{21}) \begin{pmatrix} G_B & 0 \\ 0 & G_C \end{pmatrix} \begin{pmatrix} U \\ Q_{21}^T \end{pmatrix}$$

The matrix $\begin{pmatrix} G_B & 0 \\ 0 & G_C \end{pmatrix}$ is diagonal with positive entries.

Therefore $(G_B + Q_{21} G_C Q_{21}^T)$ is positive definite (Lemma 6-12 of [5]) and therefore this matrix is non-singular.

Then, from (11):

$$V_R = (G_B + Q_{21} G_C Q_{21}^T)^{-1} Q_{21} G_C (B_{12} \quad B_{14}) \begin{pmatrix} V_S \\ V_C \end{pmatrix} \tag{12}$$

Thus the resistive branch voltages are related to capacitive-branch and source voltages.

The necessary condition will be found by considering the state equations for a general* RC network with one differential-input voltage-controlled operational amplifier.

Kuh and Rohrer [6] have characterized the state equations of an RLC network with sources. They have shown that derivatives of capacitor voltages may be expressed as linear combinations of capacitor voltages and source voltages as follows:

$$pV_C = -\mathcal{C}^{-1}TV_C - \mathcal{C}^{-1} HV_S \tag{13}$$

where T and H are transfer matrices determined by the topology and element values of the resistive sub-network; the diagonal matrix of capacitances is denoted by $\mathcal{C}$.

Partitioning V_S, the voltage source matrix:

$$V_S = \begin{pmatrix} K(v_p - v_q) \\ V_V \end{pmatrix},$$

where $K(v_p - v_q)$ is the dependent voltage source (of the operational amplifier) with gain K and V_V represents the source voltages. The node voltages v_p and v_q are then the controlling voltages for the operational amplifier.

Also partitioning H in (13), the following equation results:

$$pV_C = -\mathcal{C}^{-1}TV_C - \mathcal{C}^{-1}(H_{11} \quad H_{12}) \begin{pmatrix} K(v_p - v_q) \\ V_V \end{pmatrix}$$

* The previous assumption of the existence of a proper tree is still made.

or

$$pV_C = -\mathcal{C}^{-1}TV_C - \mathcal{C}^{-1}H_{11}K(1 \quad -1)\begin{pmatrix} v_p \\ v_q \end{pmatrix} - \mathcal{C}^{-1}H_{12}V_V \tag{14}$$

Since the variables v_p and v_q are neither capacitor voltages (state variables) nor source voltages, they must now be eliminated from the above expression. This may be done by expressing these node voltages as linear combinations of all tree-branch voltages [5]. However, as has been shown in Theorem 1, the resistive branch voltages are themselves linear combinations of capacitive and source voltages. Thus it is possible to express v_p and v_q as linear combinations of capacitive and source voltages. This relationship may be expressed by a partitioned matrix F:

$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \end{pmatrix} \begin{pmatrix} V_C \\ V_V \\ K(v_p - v_q) \end{pmatrix} \tag{15}$$

or

$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} V_C \\ V_V \end{pmatrix} + K\begin{pmatrix} F_{13} \\ F_{23} \end{pmatrix}(1 \quad -1)\begin{pmatrix} v_p \\ v_q \end{pmatrix}$$

and

$$\begin{pmatrix} 1 - KF_{13} & KF_{13} \\ -KF_{23} & 1 + KF_{23} \end{pmatrix} \begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} V_C \\ V_V \end{pmatrix} \tag{16}$$

The coefficient matrix of $\begin{pmatrix} v_p \\ v_q \end{pmatrix}$ in the above equation is non-singular. For if, as Dervisoglu [7] has shown, it is singular for a particular choice of K, elementary row operations will produce at least one row of zeros. This results in at least one equation containing only V_C and V_V, implying that the two are not independent. But, since state variables and source voltages are necessarily independent, the coefficient matrix is always non-singular.

Then, from (16):

$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} 1 - KF_{13} & KF_{13} \\ -KF_{23} & 1 + KF_{23} \end{pmatrix}^{-1} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} V_C \\ V_V \end{pmatrix}$$

$$= \frac{1}{1 + K(F_{23} - F_{13})} \left\{ \begin{pmatrix} F_{11} + KF_{23}F_{11} - KF_{13}F_{21} \\ KF_{23}F_{11} + F_{21} - KF_{13}F_{21} \end{pmatrix} V_C \right.$$

$$\left. \begin{pmatrix} F_{12} + KF_{23}F_{12} - KF_{13}F_{22} \\ KF_{23}F_{12} + F_{22} - KF_{13}F_{22} \end{pmatrix} V_V \right\} \tag{17}$$

Substituting (17) into (14) and simplifying:

$$pV_C = -\mathcal{C}^{-1} \left\{ T + \frac{KH_{11}(F_{11} - F_{21})}{1 + K(F_{23} - F_{13})} \right\} V_C$$

$$-\mathcal{C}^{-1} \left\{ \frac{KH_{11}(F_{12} - F_{22})}{1 + K(F_{23} - F_{13})} + H_{12} \right\} V_V \tag{18}$$

The following may now be stated:

<u>Theorem 2:</u>

A necessary condition on a set of state equations that it be realizable as an RC network with one differential-input voltage-controlled operational amplifier is that it be expressible in the form of equation (18).

If the capacitive sub-network is a star tree, then, each of F_{11} and F_{21} in (15) will contain a 1, and all other entries in these two matrices will be zeros.

Upon defining a new constant term,

$$K_1 = \frac{K}{1 + K(F_{23} - F_{13})},$$

(18) becomes:

$$pV_C = -\mathcal{C}^{-1}\left\{T + K_1 H_{11}(0 \cdots 0\ 1\ 0 \cdots 0\ -1\ 0 \cdots 0)\right\} V_C - \mathcal{C}^{-1}\left\{K_1 H_{11}(F_{12} - F_{22}) + H_{12}\right\} V_V \tag{19}$$

Hence, the following may be stated:

Corollary:

A necessary condition on a set of state equations that it be realizable as an RC network, where the capacitors form a star tree, and one differential-input voltage-controlled operational amplifier is that it be expressible in the form of equation (19).

2.2 Necessary Conditions for Second-Order A Matrices

Let V_1 be a column matrix of capacitor voltages (state variables), V_2 a column matrix of source voltages, V_3 the controlled voltage (amplifier output), and I_1, I_2 and I_3 the currents entering nodes defined by V_1, V_2 and V_3; then upon removal of capacitors from the network, the following node-admittance equations may be written:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = Y \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \tag{20}$$

If the capacitors form a star tree, and all node voltages are taken as positive, then the capacitor currents are actually leaving the nodes, and

$$I_1 = -\mathcal{C}\, pV_1 \tag{21}$$

Substituting (21) into the first equation of (20) yields

$$-\mathcal{C}\, pV_1 = Y_{11}V_1 - Y_{12}V_2 - Y_{13}V_3 \tag{22}$$

If the amplifier inputs are state variables, then the output, V_3, will be:

$$V_3 = K(0 \cdots 0\ 1\ 0 \cdots 0\ -1\ 0 \cdots 0)V_1 \tag{23}$$

where the locations of the non-zero entries in the above row matrix are determined by the capacitor voltages which are amplifier inputs. (Since only one amplifier input is allowed, either the 1 or the -1 may not appear in the above expression.)

Substituting (23) into (22) yields

$$-\mathcal{C}pV_1 = \left\{Y_{11} + KY_{13}(0 \cdots 0\ 1\ 0 \cdots 0\ -1\ 0 \cdots 0)\right\}V_1 + Y_{12}V_2$$

and

$$pV_1 = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(0 \cdots 0\ 1\ 0 \cdots 0\ -1\ 0 \cdots 0)\right\}V_1 - \mathcal{C}^{-1}Y_{12}V_2 \tag{24}$$

The following may then be stated:

Theorem 3:

A sufficient condition on the A matrix that it be realizable as an active RC network using one differential-input voltage-controlled operational amplifier is that it be expressible in the following form *:

$$A = -\mathscr{C}^{-1}\left\{Y_{11} + KY_{13}(0 \cdots 0\ 1\ 0 \cdots 0\ -1\ 0 \cdots 0)\right\} \qquad (25)$$

where Y_{11}, Y_{12} and Y_{13} are submatrices of a realizable node admittance matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{pmatrix} .$$

Thus the necessary condition on the A matrix, given in (19), is also sufficient.

As there are n state variables, but only two amplifier inputs, the connection may be made in several different ways. The first input may be one of n capacitor voltages, leaving (n - 1) voltages. Therefore the second input may be one of (n - 1))capacitor voltages. If both inputs are used, a total of n(n - 1) possibilities exists.

However, either the non-inverting or inverting input may be used alone, yielding another 2n possibilities. The total, then, is n(n + 1) for an n^{th} order system.

The second-order A matrix is here examined in detail. The six second-order cases are:

* As previously mentioned, either the I or the -I may be absent.

1. $A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(1\ \ -1)\right\}$

2. $A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(-1\ \ 1)\right\}$

3. $A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(1\ \ 0)\right\}$

4. $A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(0\ \ 1)\right\}$

5. $A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(-1\ \ 0)\right\}$

6. $A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(0\ \ -1)\right\}$

<u>Case 1:</u>

$$A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(1\ \ -1)\right\}$$

Necessary forms for Y_{11} and Y_{13} are, respectively:

$$\begin{pmatrix} a & -g \\ -g & b \end{pmatrix} \text{ and } \begin{pmatrix} -d \\ -e \end{pmatrix}$$

where

$$g + d \leq a$$

and

$$g + e \leq b,$$

and all variables are non-negative.

Then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}^{-1} \left\{ \begin{pmatrix} a & -g \\ -g & b \end{pmatrix} + K\begin{pmatrix} -d \\ -e \end{pmatrix}(1\ \ -1) \right\}$$

$$= -\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \left\{ \begin{pmatrix} a & -g \\ -g & b \end{pmatrix} + K\begin{pmatrix} -d & d \\ -e & e \end{pmatrix} \right\}$$

$$= \begin{pmatrix} -s_1(a - Kd) & s_1(g - Kd) \\ s_2(g + Ke) & -s_2(b + Ke) \end{pmatrix} \qquad (26)$$

where the a_{ij} are entries of A.

Since all variables are non-negative, the following necessary conditions may immediately be noted (s_1, s_2, and K are, in fact, positive):

$$a_{21} > 0$$

and

$$a_{22} < 0$$

Adding:

$$a_{11} + a_{12} = s_1(g - a)$$

Since $g \leq a$, $s_1(g - a) \leq 0$; therefore:

$$a_{11} + a_{12} \leq 0.$$

Similarly

$$a_{21} + a_{22} \leq 0.$$

Case 2:

$$A = - \mathcal{C}^{-1} \left\{ Y_{11} + KY_{13}(-1 \quad 1) \right\}$$

This differs from Case 1 in that the inputs to the amplifier have been reversed. This is equivalent to considering the two state variables as being interchanged. Then, it is only necessary to interchange the rows and columns of the A matrix in Case 1, and the necessary conditions are:

$$a_{12} > 0,$$

$$a_{11} < 0,$$

$$a_{11} + a_{12} \leq 0,$$

and

$$a_{21} + a_{22} \leq 0.$$

Case 3:

$$A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(1 \quad 0)\right\}$$

Then, from (26):

$$A = \begin{pmatrix} -s_1(a - Kd) & s_1 g \\ s_2(g + Ke) & -s_2 b \end{pmatrix} \tag{27}$$

The necessary conditions then are:

$$a_{12} > 0,$$

$$a_{21} > 0,$$

and

$$a_{22} < 0.$$

Case 4:

$$A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(0 \quad 1)\right\}$$

This differs from Case 3 in that the inputs to the amplifier have been reversed. Once again considering the state variables, and hence rows and columns of the A matrix as being interchanged, the necessary conditions are:

$$a_{21} > 0,$$

$$a_{12} > 0,$$

and

$$a_{11} < 0.$$

Case 5:

$$A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(-1 \quad 0)\right\}$$

This is equivalent to considering K as now being negative in Case 3.

From (27), if K is negative, the following necessary conditions result:

$$a_{11} < 0,$$

$$a_{12} > 0,$$

and

$$a_{22} < 0.$$

Adding:

$$a_{11} + a_{12} = s_1(g - a) + s_1Kd$$

Since $g \leq a$, $s_1(g - a) \leq 0$; and as $K < 0$:

$$a_{11} + a_{12} \leq 0.$$

Similarly

$$a_{21} + a_{22} \leq 0.$$

<u>Case 6:</u>

$$A = -\mathcal{C}^{-1}\left\{Y_{11} + KY_{13}(0 \quad -1)\right\}$$

This differs from Case 5 in that the inputs to the amplifier have been reversed. Once again considering the two state variables, and hence the rows and columns of the A matrix as being interchanged, the resulting necessary conditions are:

$$a_{22} < 0,$$

$$a_{21} > 0,$$

$$a_{11} < 0,$$

$$a_{11} + a_{12} \leq 0$$

and

$$a_{21} + a_{22} \leqslant 0.$$

The necessary conditions on the A matrix for realizability are summarized in Table 1. The sufficiency of these conditions is shown in Section 4 of Chapter 3.

Table 1 Necessary Conditions for Realizability of Second-Order A Matrices.

	Positive	Negative	Non-Positive
Case 1	a_{21}	a_{22}	$a_{11} + a_{12}$ $a_{21} + a_{22}$
Case 2	a_{12}	a_{11}	$a_{11} + a_{12}$ $a_{21} + a_{22}$
Case 3	a_{12} a_{21}	a_{22}	
Case 4	a_{12} a_{21}	a_{11}	
Case 5	a_{12}	a_{11} a_{22}	$a_{11} + a_{12}$ $a_{21} + a_{22}$
Case 6	a_{21}	a_{11} a_{22}	$a_{11} + a_{12}$ $a_{21} + a_{22}$

CHAPTER 3

ON STATE EQUATION REALIZATION USING A SINGLE CURRENT-CONTROLLED OPERATIONAL AMPLIFIER

A procedure for realizing a set of state equations from a transfer-function matrix H(s), where all elements of H(s) are rational and have simple poles, has been given by Zadeh and Desoer [1]. The resultant A matrix is diagonal with negative entries. It is here shown that any such A matrix may be realized with an RC network and one current-controlled operational amplifier.

It is further shown that if H(s) is a single-output multiple-input transfer-function matrix, then it, too, is realizable.

A realization of a larger class of second-order A matrices than that realized above and its application to two-port synthesis is also given in this chapter.

3.1 Network Configuration

A star tree of capacitors is used, and the amplifier output is connected to each capacitor through a resistor. This, however, is equivalent to connecting a current source and resistor in parallel with each capacitor. If the amplifier output is e, and J is the column matrix of current sources, then

$$J = G_L e \qquad (1)$$

where G_L is a column matrix of conductances, and the conductance G_i is connected between the amplifier output and capacitor C_i.

Let the voltage of C_i (with respect to datum) be x_i. The state vector is then X; its dimension is n.

If a conductance g_i^+ is connected between capacitor C_i and the non-inverting input of the amplifier, and a conductance g_i^- is connected between capacitor C_i and the inverting input of the amplifier, then

$$e = K(g_1^+ - g_1^- \quad g_2^+ - g_2^- \quad \cdots\cdots\cdots \quad g_n^+ - g_n^-)X \tag{2}$$

where K is the gain constant of the amplifier.

Substituting (2) into (1):

$$J = G_L K(g_1^+ - g_1^- \quad g_2^+ - g_2^- \quad \cdots\cdots\cdots \quad g_n^+ - g_n^-)X \tag{3}$$

Let V be a column matrix of source voltages; upon removal of capacitors from the network, it may be described by the following node-admittance equations:

$$\begin{pmatrix} I_x + J \\ I_v \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} \begin{pmatrix} X \\ V \end{pmatrix} \tag{4}$$

where I_x is the column matrix of currents entering nodes whose voltages are state variables.

If G_D is a diagonal matrix whose ii element is G_i, G^+ is a diagonal matrix whose ii element is g_i^+, G^- is a diagonal matrix whose ii element is g_i^- and G is a node-admittance matrix containing additional conductances, then

$$Y_{11} = G_D + G^+ + G^- + G \tag{5}$$

Since the node voltages are taken as positive, the capacitor currents are leaving the nodes, and

$$I_x = -\mathcal{C}pX \tag{6}$$

where $\mathcal{C}$ is a diagonal matrix of capacitances.

Substituting (3), (5) and (6) into the first of the two matrix equations in (4):

$$-\mathcal{C}pX = (G_D + G^+ + G^- + G)X$$
$$-G_L K\,(g_1^+ - g_1^- \quad g_2^+ - g_2^- \quad \cdots\cdots \quad g_n^+ - g_n^-)X$$
$$+Y_{12}V$$

or

$$pX = -\mathcal{C}^{-1}(G_D + G^+ + G^- + G - KG_DSG^+ + KG_DSG^-)X - \mathcal{C}^{-1}Y_{12}V \tag{7}$$

where S is an nxn matrix whose entries are all 1.

3.2 Realization of Diagonal A Matrices With Negative Entries

From (7),

$$A = -\mathcal{C}^{-1}(G_D + G^+ + G^- + G - KG_DSG^+ + KG_DSG^-)$$

or

$$K(SG^+ - SG^-) = G_D^{-1}(\mathcal{C}A + G_D + G^+ + G^- + G) \tag{8}$$

The rows of the matrix on the left hand side of the above equation are identical. Since A is a diagonal matrix, $\mathcal{C}A$ is also diagonal, and the only off-diagonal contribution to the right hand side of (8) must come from G. This condition is certainly satisfied if all entries (on both sides of (8)) are identical.

To realize this condition, let

$$g^{+} = g_1^{+} = g_2^{+} = \cdots\cdots = g_n^{+}$$

and

$$g^{-} = g_1^{-} = g_2^{-} \quad \cdots\cdots = g_n^{-}.$$

Furthermore, let each off-diagonal entry of G be $-g$, the ii entry be $g_{ii} + (n - 1)g$ and all entries of G_D be 1.

Since all entries in (8) must be identical,

$$K(g^{+} - g^{-}) = -g \tag{9}$$

and

$$C_i a_i + 1 + g^{+} + g^{-} + g_{ii} + (n - 1)g = -g, \; i = 1, \cdots, n \tag{10}$$

where a_i is the ii entry of A, (and is negative).

From (10),

$$C_i = \frac{g_{ii} + g^{+} + g^{-} + 1 + ng}{-a_i} \tag{11}$$

Substituting (9) into (11);

$$C_i = \frac{g_{ii} + g^{-}(1 + nK) + g^{+}(1 - nK) + 1}{-a_i} \tag{12}$$

Values are now chosen for g_{ii}, g^{-} and g^{+}, and the C_i are determined. It is necessary to choose g^{-} sufficiently larger than g^{+} such that all C_i are positive. (The value chosen for g^{+} may be zero.) Equation (9) then determines g.

3.3 Realization of Transfer-Function Matrices

From (7),

$$B = -\mathcal{G}^{-1}Y_{12}$$

and

$$Y_{12} = -\mathcal{G}B \tag{13}$$

In using the procedure of Zadeh and Desoer for obtaining a set of state equations from a single-output transfer-function matrix, the entries of C may be chosen arbitrarily, but these then determine the entries of B.

If the entries of C are chosen to be identical, then the amplifier output may be taken as the system output:

$$y = e = K(g^{+} - g^{-} \quad g^{+} - g^{-} \quad \cdots\cdots \quad g^{+} - g^{-})X$$
$$+ K(g_{i_1}^{+} - g_{i_1}^{-} \quad \cdots\cdots \quad g_{i_m}^{+} - g_{i_m}^{-})V \tag{14}$$

where D is realized by connecting suitable conductances $g_{i_k}^{+}$ and $g_{i_k}^{-}$, $k = 1, \cdots, m$, between the m sources and amplifier inputs.

From (14), since $g^{-} > g^{+}$, the entries of C are negative. If the entries of the desired C are positive, then the inverted output of the amplifier may be used. (It is assumed that the operational amplifier has an inverted output as well as a non-inverted one; if not, a voltage amplifier with gain -1 may be added.)

Once $\mathcal{G}$ has been found, Y_{12} is determined from (13). For realizability, it is necessary that

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} \left| y_{ij} \right| \leq g_{ii},$$

where m inputs are present and y_{ij} is a typical entry of Y_{12}. That this inequality may always be satisfied is now demonstrated.

If $g^+ = 0$, then all entries of C may be chosen equal to Kg^-. Then, a typical entry of B becomes b_{ij}/Kg^-, where the b_{ij} are the entries of B obtained by the Zadeh and Desoer method provided the entries of C are chosen equal to 1.

Then, from (13)

$$y_{ij} = -\frac{C_i b_{ij}}{Kg^-}$$

Upon substituting (12) into the above equation, then substituting the resulting expression for y_{ij} into the inequality and simplifying:

$$\frac{g^-(1 + nK) + 1}{a_i Kg^-} \sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij} \leq g_{ii}\left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{b_{ij}}{a_i Kg^-}\right) \qquad (15)$$

A sufficiently large g^- will produce a positive right hand side for (15). Clearly, it is then always possible to choose g_{ii} sufficiently large to satisfy (15).

The design procedure is summarized as follows:

1. Choose a sufficiently large g– to produce a positive right hand side for (15).
2. Choose sufficiently large g_{ii} to satisfy (15).
3. Use (12) to determine the C_i.
4. Use (9) to determine g.
5. Use (13) to determine Y_{12}.
6. Realize D by choosing suitable conductances $g^+_{i_k}$ and $g^-_{i_k}$, $k = 1, \ldots, m$, in (14).

The general network configuration is shown in Figure 1.

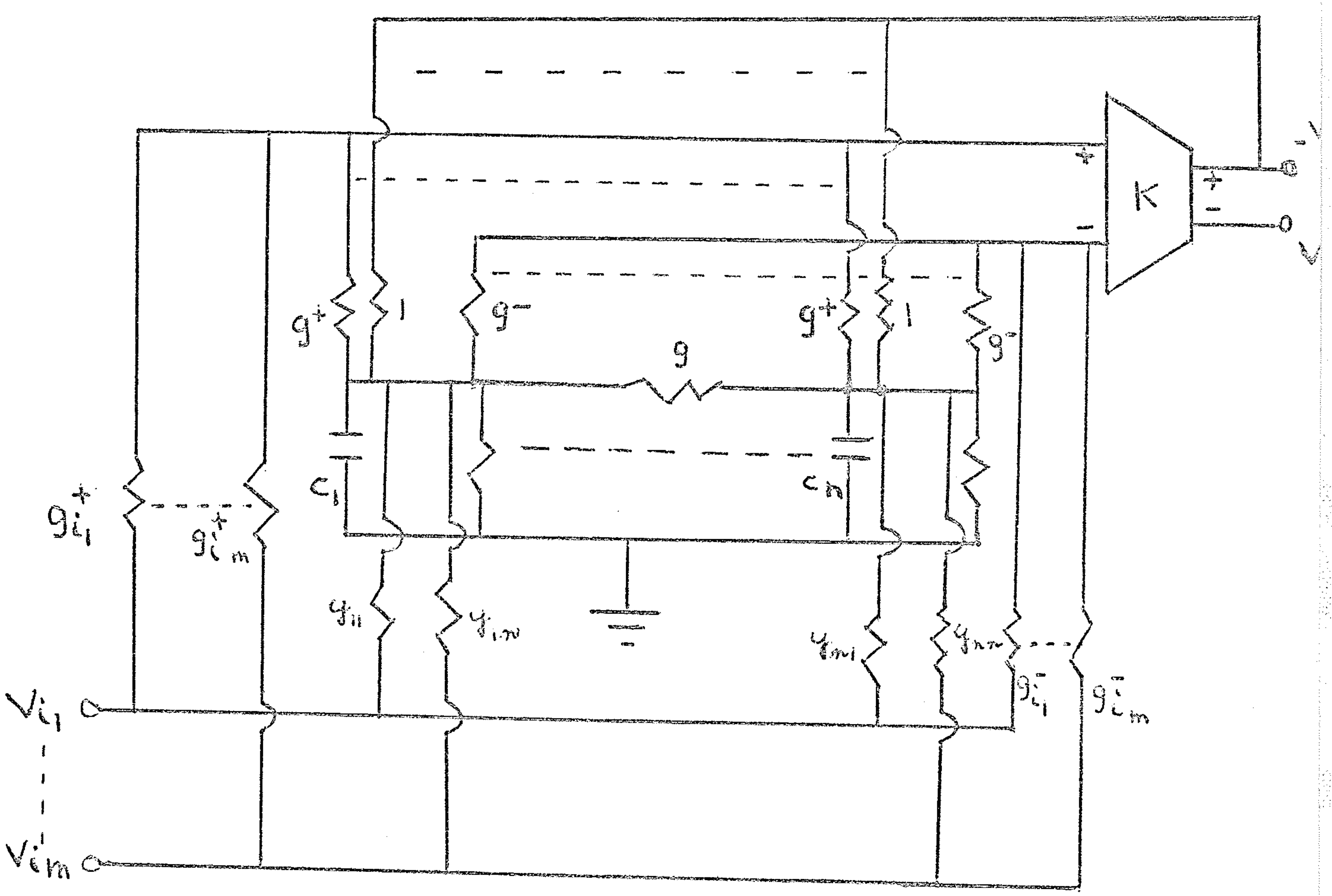

Figure 1. Network for realizing a single-output m-input transfer-function matrix.

An example to illustrate the synthesis procedure is now given.

<u>Example 1:</u>

Given

$$V_o = \begin{pmatrix} \dfrac{2s + 3}{s^2 + 3s + 2} & \dfrac{6s^2 + 25s + 23}{s^3 + 6s^2 + 11s + 6} \end{pmatrix} \begin{pmatrix} V_{i_1} \\ V_{i_2} \end{pmatrix}$$

The resulting state equations are:

$$\begin{pmatrix} px_1 \\ px_2 \\ px_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1/Kg^- & 2/Kg^- \\ 1/Kg^- & 3/Kg^- \\ 0 & 1/Kg^- \end{pmatrix} \begin{pmatrix} V_{i_1} \\ V_{i_2} \end{pmatrix}$$

$$V_o = (Kg^- \quad Kg^- \quad Kg^-) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $g^+ = 0$.

Let $g^- = 0.01$, $g_{11} = g_{22} = g_{33} = 30$ and $K = 1000$.

Then, from (12),

$$C_1 = 61.01, \quad C_2 = \frac{61.01}{2}, \text{ and } C_3 = \frac{61.01}{3}$$

From (9),

$$g = 10$$

From (13),

$$Y_{12} = - \begin{pmatrix} 6.101 & 12.202 \\ 3.0505 & 9.1515 \\ 0 & \dfrac{6.101}{3} \end{pmatrix}$$

The g_{i_k} are zero since $D = 0$. The network shown in Figure 2 then results.

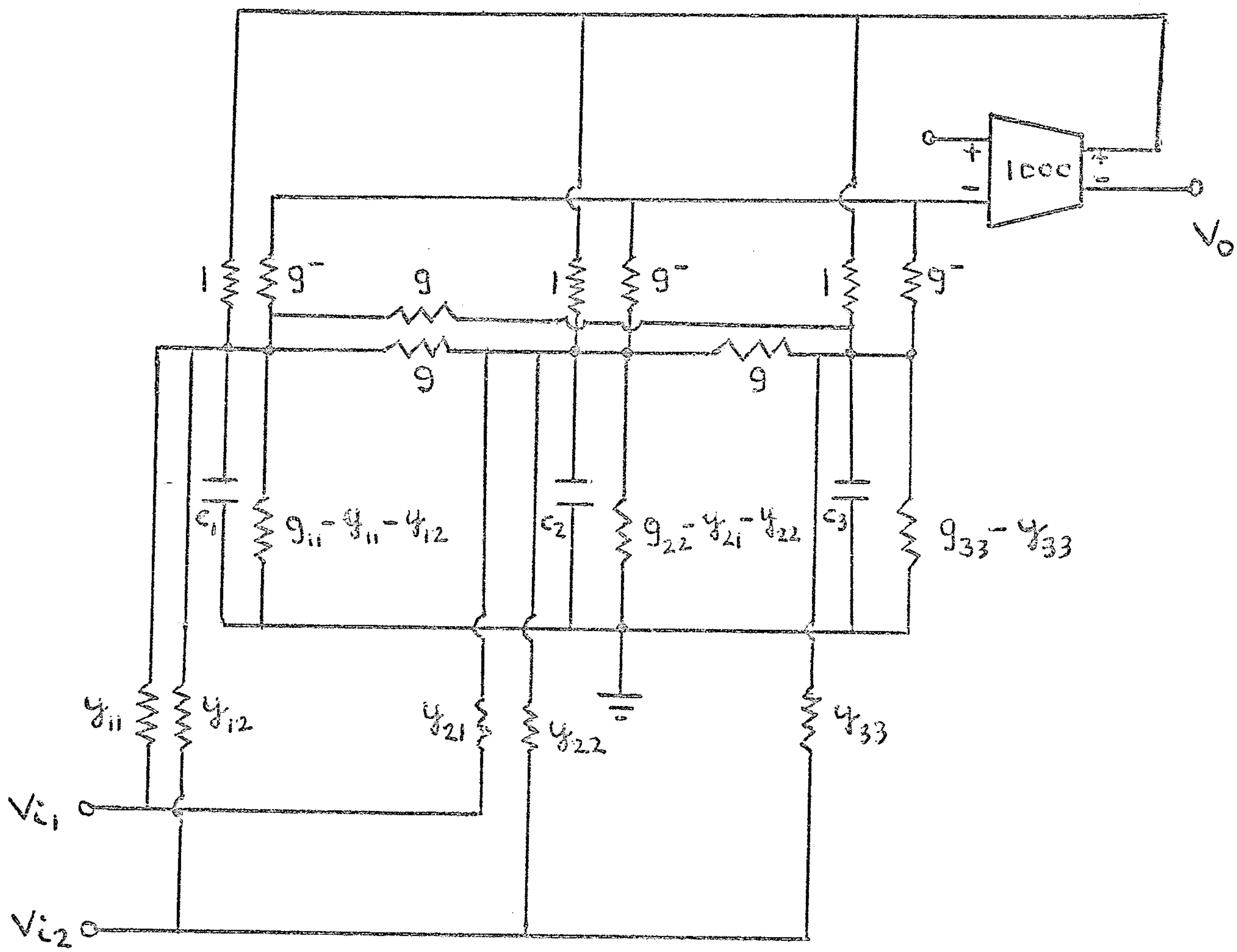

Figure 2. Network for realizing the transfer-function matrix of Example 1.

3.4 Realization of a Class of Second-Order A Matrices

In this section it is shown that, using the configuration described in Section 1 of this chapter, it is possible to realize a much larger class of second-order A matrices than that realizable in Section 2. This class may be defined as follows: If either diagonal entry is positive or zero, then the off-diagonal entry in that column must be positive. (Or, equivalently, the contrapositive of this statement: If either off-diagonal entry is negative or zero, then the diagonal entry in that column must be negative.)

For the second-order case, if all entries of the matrix G are zero, (8) becomes

$$K\begin{pmatrix} g_1^+ - g_1^- & g_2^+ - g_2^- \\ g_1^+ - g_1^- & g_2^+ - g_2^- \end{pmatrix}$$

$$= \begin{pmatrix} 1/G_1 & 0 \\ 0 & 1/G_2 \end{pmatrix}\left\{\begin{pmatrix} c_1a_{11} & c_1a_{12} \\ c_2a_{21} & c_2a_{22} \end{pmatrix} + \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} + \begin{pmatrix} g_1^+ & 0 \\ 0 & g_2^+ \end{pmatrix} + \begin{pmatrix} g_1^- & 0 \\ 0 & g_2^- \end{pmatrix}\right\} \tag{16}$$

Equating entries in the first column:

$$K(g_1^+ - g_1^-) = \frac{C_2a_{21}}{G_2} \tag{17}$$

and

$$K(g_1^+ - g_1^-) = \frac{C_1a_{11} + G_1 + g_1^+ + g_1^-}{G_1} \tag{18}$$

From (18)

$$g_1^+ = \frac{C_1a_{11} + G_1}{KG_1 - 1} + \frac{g_1^-(KG_1 + 1)}{KG_1 - 1} \tag{19}$$

and from (17)

$$g_1^+ = \frac{C_2a_{21} + g_1^-}{KG_2} \tag{20}$$

The expressions on the right hand sides of (19) and (20) may then be equated. Upon simplifying in order to isolate g_1^-:

$$g_1^- = \frac{(KG_1 - 1)C_2a_{21}}{2KG_2} - \frac{(C_1a_{11} + G_1)}{2} \tag{21}$$

Similarly, upon equating entries in the second column of (16), the equation corresponding to (20) is:

$$g_2^+ = \frac{C_1a_{12} + g_2^-}{KG_1} \tag{22}$$

The equation corresponding to (21) is:

$$g_2^- = \frac{(KG_2 - 1)C_1a_{12}}{2KG_1} - \frac{(C_2a_{22} + G_2)}{2} \tag{23}$$

If a_{21} is positive, then a_{11} may be positive, negative or zero. If, however, a_{21} is negative or zero, then a_{11} may only be negative. Similarly, if a_{12} is positive, then a_{22} may be positive, negative or zero. If a_{12} is negative or zero, then a_{22} may only be negative.

If a_{21} is positive, let G_1 be chosen such that $KG_1 > 1$.

If a_{11} is negative, as may be seen in (21), a sufficiently large C_1 will guarantee a positive value for g_1^-. If a_{11} is positive or zero, then a sufficiently large C_2 will guarantee a positive value for g_1^-.

If a_{12} is positive, let G_2 be chosen such that $KG_2 > 1$. If a_{22} is negative, as may be seen in (23), a sufficiently large C_2 will guarantee a positive value for g_2^-. If a_{22} is positive or zero, then a sufficiently large C_1 will produce a positive value for g_2^-.

Once g_1^- and g_2^- are determined, substitution into (20) and (22) yields required values for g_1^+ and g_2^+, respectively. Since a_{21} and a_{12} are positive, positive values for g_1^+ and g_2^+ necessarily result.

If a_{21} is negative or zero, then a_{11} must be negative, and a sufficiently large C_1 will result in a positive value for g_1^-, as seen in (21). Similarly, if a_{12} is negative or zero, then a_{22} must be negative, and a sufficiently large C_2 will result in a positive value for g_2^-, as seen in (23).

Values for g_1^+ and g_2^+ are again determined by substituting into (20) and (22), respectively, g_1^- and g_2^-. Care must now be taken, however, to ensure that g_1^- and g_2^- are made sufficiently large (by choosing C_1 and C_2 sufficiently large) to produce positive values for g_1^+ and g_2^+.

If a_{21} and a_{12} are both negative, G_2 and G_1 should be chosen such that $KG_2 < 1$ and $KG_1 < 1$. Clearly, a solution is then always possible for this case.

3.5 An Application to Two-Port Transfer-Function Synthesis

It is here shown that the A matrix realization of the previous section may be applied to two-port transfer-function synthesis.

Let a set of state equations be obtained from the given transfer function by simulation using integrators where integrator outputs are state variables [8]. The last integrator output is the system output, and the first integrator input is the system input. Hence, for the second-order case:

$$C = (0 \quad 1), \qquad (24)$$

$$D = (0)$$

and

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (25)$$

Substituting (25) into (13):

$$Y_{12} = - \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} C_1 \\ 0 \end{pmatrix}$$

A conductance C_1 must then be connected between the source voltage and capacitor C_1, whose voltage is the first state variable. It is then necessary to add C_1 to the (1,1) entry of the node admittance matrix. The (1,1) entry in the right hand side of (16) then becomes

$$\frac{C_1 a_{11} + G_1 + g_1^{+} + g_1^{-} + C_1}{G_1}$$

or

$$\frac{G_1(a_{11} + 1) + G_1 + g_1^+ + g_1^-}{G_1}$$

Let

$$a_{11}^1 = a_{11} + 1$$

It is then necessary to realize a new A matrix, A^1, where

$$A^1 = \begin{pmatrix} a_{11}^1 & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The following may now be stated:

Theorem:

A sufficient condition on a second-order voltage-transfer function that it be realizable using the configuration described in Section 1 of this chapter is that its A^1 matrix be a member of the class of A matricies realizable in Section 4, where the state equations have been determined using the procedure described in this section.

An example to illustrate the procedure is now given.

Example 2:

Given

$$\frac{V_o}{V_i} = \frac{1}{s^2 + s\sqrt{2} + 1}$$

(a second-order Butterworth low-pass voltage-transfer function)

The resulting state equations are:

$$\begin{pmatrix} px_1 \\ px_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} V_i$$

$$V_o = (0 \quad 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then

$$A^1 = \begin{pmatrix} 1 & -1 \\ 1 & -\sqrt{2} \end{pmatrix}$$

If $K = 1000$, $G_1 = G_2 = 1$, $C_1 = 1$ and $C_2 = 10$, from (20), (22), (19) and (21):

$$g_1^- = 3.995$$

$$g_2^- = 5\sqrt{2} - 0.9995$$

$$g_1^+ = 4.005$$

and

$$g_2^+ = 5\sqrt{2} - 1.0005$$

The network shown in Figure 3 then results.

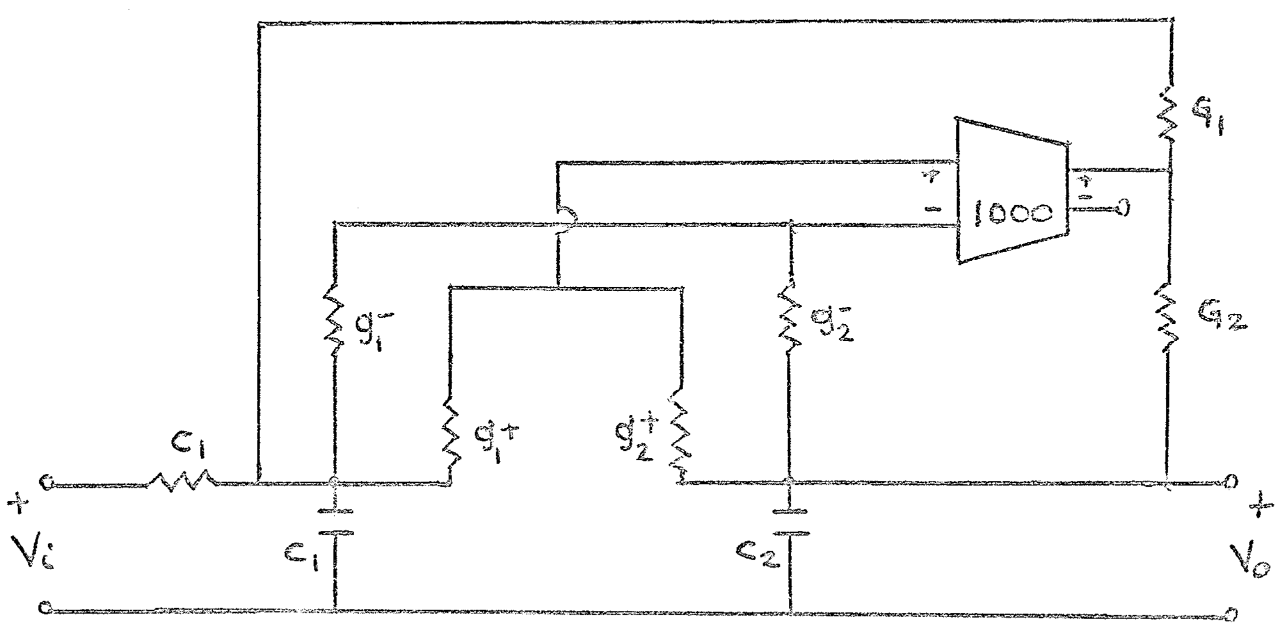

Figure 3. Network for realizing the transfer function of Example 2.

CHAPTER 4

CONCLUSIONS

A necessary condition on the A matrix that it be realizable as an RC network with one differential-input voltage-controlled operational amplifier has been found. It has further been shown that if the capacitive sub-network is a star tree, then this condition is also sufficient.

A test has been given to determine by inspection whether a given second-order A matrix is thus realizable. It was shown that these resulting necessary conditions are also sufficient for realization using a current-controlled operational amplifier.

Also, using a current-controlled operational amplifier, a realization of the diagonal A matrices obtained by Zadeh and Desoer from transfer-function matrices has been found. It was further shown that a larger class of second-order A matrices is realizable using this configuration.

Applications to transfer-function synthesis have been given using a state-equation representation.

Thus it is seen that wide classes of state equations and transfer functions are realizable using only one active element and the minimum number of capacitors, all of which are grounded.

The presence of only one active element and the minimum number of capacitors makes this realization attractive from an economic point of view.

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