# ON FUNCTION CLASSES RELATED TO THE CLASS OF COMPLETELY MONOTONIC FUNCTIONS

BY

SENLIN GUO

A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES
AT THE UNIVERSITY OF MANITOBA
IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA, CANADA
2006

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES
*****
COPYRIGHT PERMISSION

# ON FUNCTION CLASSES RELATED TO THE CLASS OF COMPLETELY MONOTONIC FUNCTIONS

BY

SENLIN GUO

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of

Manitoba in partial fulfillment of the requirement of the degree

OF

DOCTOR OF PHILOSOPHY

# Contents

CONTENTS i

ACKNOWLEDGEMENTS iv

ABSTRACT v

LIST OF NOTATION vi

1 INTRODUCTION 1

2 SOME CLASSES RELATED TO COMPLETELY MONOTONIC FUNCTIONS 24

2.1 Introduction and main results . . . . . . . . . . . . . . . . . . 24

2.2 Proofs of main results . . . . . . . . . . . . . . . . . . . . . . 34

3 THE GAMMA AND RELATED FUNCTIONS 50

4 SOME CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS

INVOLVING THE GAMMA FUNCTION I 55

4.1 Introduction and main results . . . . . . . . . . . . . . . . . . . 57

4.2 Lemmas . . . . . . . . . . . . . . . . . . . . . . . . . . . . 60

4.3 Proofs of main results . . . . . . . . . . . . . . . . . . . . . 62

**5 SOME CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA FUNCTION II 71**

5.1 Introduction and main results . . . . . . . . . . . . . . . . . . . 71

5.2 Proofs of main results . . . . . . . . . . . . . . . . . . . . . 73

**6 SOME CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA FUNCTION III 84**

6.1 Introduction and main results . . . . . . . . . . . . . . . . . . . 84

6.2 Proofs of the main results . . . . . . . . . . . . . . . . . . . . 85

**7 A CLASS OF COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE REMAINDER OF BINET'S FORMULA WITH APPLICATIONS 90**

7.1 Introduction and main results . . . . . . . . . . . . . . . . . . . 90

7.2 Lemmas . . . . . . . . . . . . . . . . . . . . . . . . . . . . 92

7.3 Proofs of the main results . . . . . . . . . . . . . . . . . . . . 94

8 MONOTONICITY AND CONCAVITY PROPERTIES OF SOME FUNCTIONS ASSOCIATED WITH THE GAMMA FUNCTION AND APPLICATIONS 98

8.1 Introduction and main results . . . . . . . . . . . . . . . . . . . 98

8.2 Proof of main results . . . . . . . . . . . . . . . . . . . . . . . 104

A PROOF OF SOME RESULTS RELATED TO COMPLETELY MONOTONIC FUNCTIONS 109

BIBLIOGRAPHY 115

# Acknowledgements

First, I would like to express my great appreciation to my supervisor Professor K. Kopotun for his advice and guidance throughout my PhD program. I also appreciate other Advisory Committee members, Professor J. J. Williams and Professor J. Brewster, for their suggestions, and other professors and office staff in the Mathematics Department for their support.

Secondly, I thank the Mathematics Department at the University of Manitoba for giving me this opportunity to further my study here by providing financial aid and facilities. I also thank the University of Manitoba for giving me the UM Graduate Fellowship and the UMSU Scholarship.

Finally, I am indebted to my parents and my wife who give me their love, always.

# Abstract

In this thesis, we introduce the concept of strongly logarithmically completely monotonic functions, extend the notion of (strongly) complete monotonicity to that of almost (strongly) complete monotonicity, and investigate properties of these and related classes of functions. Several applications involving the Binet's formula, the (di-, poly-)gamma, star-shaped, and super-additive functions are obtained. Some specific classes of logarithmically completely monotonic functions are discussed. Each such function gives rise to an infinitely divisible probability distribution.

# List of notation

| Symbol | Definition |
|---|---|
| $\mathbb{N}$ | The set of all positive integers |
| $\mathbb{N}_0$ | The set of all nonnegative integers |
| $\mathbb{R}^+$ | The set of all positive real numbers |
| $I^+$ | An open interval contained in $\mathbb{R}^+$ |
| $-I$ | $\{-x \mid x \in I\}$ (Here $I \subset \mathbb{R}$) |
| $A^o$ | The interior of set $A$ |
| $C(A)$ | The space of functions continuous on set $A$ |
| $\mathcal{R}(f)$ | The range of function $f$ |
| $\Gamma(z)$ | The gamma function. (See Chapter 3) |
| $\psi(z)$ | The psi (or digamma) function. (See Chapter 3) |

| | |
|---|---|
| $AM$ | $\forall n \in \mathbb{N}_0, f^{(n)}(x) \geq 0$ (See Definition 1.$A$) |
| $CM$ | $\forall n \in \mathbb{N}_0, (-1)^n f^{(n)}(x) \geq 0$ (See Definition 1.$M$) |
| $LCM$ | $\forall n \in \mathbb{N}, (-1)^n [\ln f(x)]^{(n)} \geq 0$ (See Definition 1.Q$'$) |
| $SCM$ | $\forall n \in \mathbb{N}_0, (-1)^n x^{n+1} f^{(n)}(x) \geq 0$ and decreasing (See Definition 2.$A$) |
| $SLCM$ | $\forall n \in \mathbb{N}, (-1)^n x^{n+1} [\ln f(x)]^{(n)} \geq 0$ and decreasing (See Definition 2.1) |
| $ASCM$ | $\forall n \in \mathbb{N}, (-1)^n x^{n+1} f^{(n)}(x) \geq 0$ and decreasing (See Definition 2.2) |
| $ACM$ | $\forall n \in \mathbb{N}, (-1)^n f^{(n)}(x) \geq 0$ (See Definition 2.3) |

# Chapter 1

# Introduction

Let's first introduce the notion of *absolutely monotonic* functions, which is closely associated with that of *completely monotonic* functions.

Bernstein [21] in 1914 first introduced

**Definition 1.A** (AM). *A function $f$ is said to be* absolutely monotonic *on an interval $I$, if $f \in C(I)$, has derivatives of all orders on $I^o$ and for all $n \in \mathbb{N}_0$*

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

We use $AM(I)$ to denote the class of all *absolutely monotonic* functions on $I$.

For the interval $[a, b)$ or $[a, b]$, Bernstein [21] also gave an equivalent definition to Definition 1.A as follows:

**Definition 1.B.** *A function $f$ is* absolutely monotonic *on the interval* $[a, b)$ *if and only if*

$$\Delta_h^n f(x) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh) \geq 0$$

*for all $n \in \mathbb{N}_0$ and for all $x$ and $h$ such that $a \leq x \leq x + nh < b$. If, in addition, $f(b) = f(b-)$, then $f$ is called* absolutely monotonic *on the interval* $[a, b]$.

Although it appears that there is nothing to do with the derivatives $f^{(k)}(x)$ in Definition 1.B, this definition is actually equivalent to Definition 1.A for $I = [a, b)$ or $[a, b]$.

In 1935 Grüss introduced the following

**Definition 1.C.** *A function $f$ is* absolutely monotonic *on the interval* $[0, 1]$ *if $f$ is continuous there and if for all $n \in \mathbb{N}$*

$$\Delta^k f\left(\frac{i}{n}\right) \geq 0, \quad k = 0, 1, \cdots, n; \quad i = 0, 1, \cdots, n-k. \tag{1.1}$$

*Here*

$$\Delta^k f(x) := \Delta^{k-1} f\left(x + \frac{1}{n}\right) - \Delta^{k-1} f(x),$$

*and*

$$\Delta^0 f(x) := f(x).$$

It is easy to modify this definition to apply to the interval $[a, b]$.

Since

$$\Delta^{k-1} f(x) = \Delta^k f\left(x - \frac{1}{n}\right) + \Delta^{k-1} f\left(x - \frac{1}{n}\right),$$

we may replace the condition (1.1) with the following one

$$\Delta^k f(0) \geq 0, \quad k = 0, 1, \cdots, n. \tag{1.2}$$

For the interval $[0, 1]$ (then $[a, b]$), Grüss' definition 1.C is equivalent to Bernstein's definition.

That Bernstein's definition implies that of Grüss is obvious. For the converse part, first we need the following well-known

**Theorem 1.D.** *Suppose that* $f \in C[0, 1]$, *then* $B_n f$ *converges to* $f$ *uniformly on* $[0, 1]$.

Here $B_n f$ is the Bernstein Polynomial of $f$, i.e.

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

Secondly, from this result we can show

**Theorem 1.E.** $f \in C[0, 1]$ *satisfies the condition* (1.2) *if and only if* $f$ *is the uniform limit of a sequence of polynomials with nonnegative coefficients.*

Then by use of Theorem 1.E, we can prove that Grüss' definition 1.C does imply Bernstein's 1.B.

For details of this part and the proof of the equivalence of Bernstein's two definitions, see [92, Chapter IV].

Clearly, if $f, g \in AM(I)$, then $\alpha f + \beta g \in AM(I)$ for $\alpha, \beta \geq 0$, and $fg \in AM(I)$ by use of Leibniz's rule.

From the definition, a convergent series of powers of $(x - a)$ with nonnegative coefficients represents an absolutely monotonic function on $[a, a + \rho)$, where $\rho$ is the radius of convergence of the power series. On the other hand, suppose that $f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n$, $|x - a| < \rho$. If any coefficient of the series is negative, then $f(x)$ can not be absolutely monotonic on any left or right neighborhood of $x = a$. If any coefficient of the series is zero, then $f(x)$ can not be absolutely monotonic on any left neighborhood of $x = a$ unless $f(x)$ is a polynomial. Indeed, if $a_k < 0$, then $f^{(k)}(a) < 0$. Hence $f^{(k)}(x) < 0$ in some left and in some right neighborhood of $a$. If $a_k = 0$, then $f^{(k)}(a) = 0$. If $f(x)$ is absolutely monotonic on a left neighborhood of $a$, then $f^{(k)}(x) \equiv 0$ there for $f^{(k)}(x)$ is increasing. And this means that $f(x)$ is a polynomial.

We also notice

**Theorem 1.F.** *Any function which can be expressed as a series of powers of $(x-a)$ must be the difference of two functions which are absolutely monotonic on a right neighborhood of $a$.*

This result follows from the observation

$$\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} |a_n|\,(x-a)^n - \sum_{n=0}^{\infty}(|a_n| - a_n)(x-a)^n.$$

One of the important properties of absolutely monotonic functions is that it is analytic or holomorphic. More precisely we have

**Theorem 1.G.** *If $f(x)$ is absolutely monotonic on $[a,b)$, then it can be extended analytically into the region of the complex z-plane: $|z-a| < b-a$, where $z = x+iy$.*

From this result, an absolutely monotonic function on $[a,\infty)$ can be extended as an entire function.

Now suppose that $f(x)$ is absolutely monotonic on $(a,b)$ and $f(c) = 0$ for some $c \in (a,b)$. Since $f(x)$ is increasing on $(a,b)$, $f(x) = 0$ for all $x \in (a,c)$. This fact along with Theorem 1.G leads to

**Theorem 1.H.** *Suppose that $f \in AM(I)$. If there exists $x_0 \in I^o$ such that $f(x_0) = 0$, then $f(x) \equiv 0$ on $I$.*

Another concept which is related to completely monotonic functions is the notion of a completely monotonic sequence.

**Definition 1.I** ([92, Chapter III]). *A sequence $\{\mu_n\}_0^\infty$ is called* completely monotonic *if*

$$(-1)^k \Delta^k \mu_n \geq 0, \quad n, k \in \mathbb{N}_0,$$

*where*

$$\Delta^0 \mu_n = \mu_n, \quad \Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n.$$

Such a sequence is called *totally monotone* in [95].

By induction, for $n, k \in \mathbb{N}_0$,

$$\Delta^k \mu_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \mu_{n+k-i} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k+i} \mu_{n+i}.$$

In 1963, Lorch and Moser [62] showed that for a completely monotonic sequence $\{\mu_n\}_0^\infty$, we always have

$$(-1)^k \Delta^k \mu_n > 0, \quad n, k \in \mathbb{N}_0$$

unless $\mu_n = c$, a constant for all $n \in \mathbb{N}$.

Hausdorff [49] in 1921 proved a fundamental result for such sequences

**Theorem 1.J.** *A sequence $\{\mu_n\}_0^\infty$ is completely monotonic if and only if there exists an increasing function $\alpha(t)$ on $[0,1]$ such that*

$$\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0. \tag{1.3}$$

The right side of (1.3) is a Stieltjes integral.

In 1931 Widder [94] gave

**Definition 1.K.** *A sequence* $\{\mu_n\}_0^\infty$ *is called* minimal completely monotonic *if it is completely monotonic and if it will not be completely monotonic when* $\mu_0$ *is replaced by a number less than* $\mu_0$.

For such a class of sequences, Widder proved [94]

**Theorem 1.L.** *A sequence* $\{\mu_n\}_0^\infty$ *is minimal completely monotonic if and only if there exists an increasing function* $\alpha(t)$ *on* $[0,1]$ *with* $\alpha(0) = \alpha(0+)$ *such that*

$$\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0.$$

Apparently not all completely monotonic sequences are minimal. But we can show

**Theorem 1.1.** *For each completely monotonic sequence* $\{\mu_n\}_0^\infty$*, there exists one and only one number* $\mu_0^*$ *such that* $\{\mu_0^*, \mu_1, \mu_2, \cdots\}$ *is minimal completely monotonic.*

In fact, $\mu_0^* = \inf\{\mu | \{\mu, \mu_1, \mu_2, \cdots\}$ is completely monotonic$\}$. For detailed proof of this result see Theorem A.2 in Appendix A.

Now we introduce [92]

**Definition 1.M** (CM). *A function* $f$ *is said to be* completely monotonic *on an interval* $I$*, if* $f \in C(I)$*, has derivatives of all orders on* $I^o$ *and for all*

$n \in \mathbb{N}_0$

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o. \tag{1.4}$$

Some mathematicians use the terminology *completely monotone* instead of *completely monotonic.*

The class of all *completely monotonic* functions on $I$ is denoted by $CM(I)$. If inequality (1.4) is strict, then $f$ is said to be *strictly completely monotonic* on $I$. Such a function class is denoted by $SiCM(I)$.

From Definitions 1.A and 1.M, we have

**Theorem 1.N** ([92]). *A function $f$ is* completely monotonic *on an interval $I$, if and only if the function $f(-x)$ is* absolutely monotonic *on $-I$.*

If $f, g \in CM(I)(SiCM(I))$, and $\alpha, \beta \in \mathbb{R}^+$, then by definition $\alpha f + \beta g \in CM(I)(SiCM(I))$, and by Leibniz's rule $fg \in CM(I)(SiCM(I))$.

For the operation of pointwise convergence, we have (see [92, p. 151])

**Theorem 1.O.** *Suppose that for $n \in \mathbb{N}$, $f_n \in CM(I)$, where $I := (a, b)$ or $(a, b]$. If the limit function $f(x) := \lim_{n\to\infty} f_n(x)$ exists on $I$, then $f \in CM(I)$.*

Clearly this theorem is equivalent to

**Theorem 1.P.** *Suppose that for $n \in \mathbb{N}$, $f_n \in CM(I)$, where $I := (a, b)$ or $(a, b]$. If the function $f(x) := \sum_{n=1}^{\infty} f_n(x)$ exists on $I$, then $f \in CM(I)$.*

We notice that the interval $I$ in Theorem 1.O or 1.P can not be replaced with $[a, b)$ or $[a, b]$. For example, let $f_n(x) := 1/x^n$, and $I := [1, 2)$. Then $f_n \in CM(I)$ and

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1, & x = 1; \\ 0, & x \in (1, 2). \end{cases}$$

Since $f \notin C(I)$, $f \notin CM(I)$.

By Theorem 1.N and Theorem 1.G, or Theorem 1.H, we can obtain the following two results.

**Theorem 1.Q.** *If $f(x)$ is completely monotonic on $(a, b]$, then it can be extended analytically into the region of the complex $z$-plane: $|z - b| < b - a$, where $z = x + iy$.*

**Theorem 1.R.** *Suppose that $f \in CM(I)$. If there exists $x_0 \in I^o$ such that $f(x_0) = 0$, then $f(x) \equiv 0$ on $I$.*

Dubourdieu [32] in 1939 showed

**Theorem 1.S.** *A non-constant completely monotonic function on $(a, \infty)$ is strictly completely monotonic there.*

In 1963 Lorch and Szego [64] gave an alternative proof of this result. In Theorem A.1 of Appendix A, we shall give another alternative proof of

Theorem 1.S. We observe that completely monotonic functions on other intervals may not possess such a property. For example, let $f(x) := x^2, I := (-\infty, 0)$. Then $f \in CM(I)$ and $f^{(k)} \equiv 0$ on $I$ for all $k \geq 3$. Hence $f \notin SiCM(I)$.

In 1987 O'Cinneide [73] proved

**Theorem 1.T.** *If $f \in CM(\mathbb{R}^+)$, then $f(x+\delta)/f(x)$ is strictly increasing in $x$ on $\mathbb{R}^+$ for each $\delta > 0$ unless $f(x) = ce^{-dx}$ for some $c \geq 0$ and some $d \geq 0$.*

In 1939 Feller [38] showed

**Theorem 1.U.** *Suppose that $f, g \in CM(\mathbb{R}^+)$. If there exists a strictly increasing sequence $\{x_k\}_1^\infty \subset \mathbb{R}^+$ with $\sum_{k=1}^\infty (1/x_k)$ diverging such that $f(x_k) = g(x_k), k \in \mathbb{N}$, then $f \equiv g$ on $\mathbb{R}^+$.*

The following result is Theorem 3 of [94].

**Theorem 1.V.** *If $f \in CM(I)$, where $I := (a, \infty)$, then for all $n \in \mathbb{N}_0$*

$$[f(x), f'(x), \cdots, f^{(n)}(x)] \geq 0, \quad x \in I. \tag{1.5}$$

Here, in (1.5),

$$[f(x), f'(x), \cdots, f^{(n)}(x)] := \begin{vmatrix} f(x) & f'(x) & \cdots & f^{(n)}(x) \\ f'(x) & f''(x) & \cdots & f^{(n+1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n)}(x) & f^{(n+1)}(x) & \cdots & f^{(2n)}(x) \end{vmatrix}$$

is a Hankel determinant (see [92]).

From this result, we can show

**Theorem 1.W.** *Suppose that $f \not\equiv 0$ on $I := (a, \infty)$ and that $f \in CM(I)$, then $f$ is log-convex there.*

Indeed, by Theorem 1.R, $f > 0$ on $I$. By Theorem 1.V

$$f(x)f''(x) \geq [f'(x)]^2, \quad x \in I,$$

which means that $[\ln f(x)]'' \geq 0, x \in I$.

We notice that if the interval $(a, \infty)$ is replaced by other kinds of intervals in Theorem 1.V or Theorem 1.W, the conclusion may not be true. For example, let $I := (-\infty, 0)$ and $f(x) := x^2$, then $f \in CM(I)$. But $[f(x), f'(x)] < 0, x \in I$. And consequently $f(x)$ is strictly log-concave on $I$.

There exists a close relationship between completely monotonic functions and completely monotonic sequences.

In 1931 Widder [94] showed

**Theorem 1.X.** *Suppose that $f \in CM[a, \infty)$, then for any $\delta \geq 0$, the sequence $\{f(a + n\delta)\}_{n=0}^{\infty}$ is completely monotonic.*

This result was generalized by Lorch and Newman [63] in 1983 as follows

**Theorem 1.Y.** *Suppose that $f \in CM[a, \infty)$. If the sequence $\{\Delta x_k\}_{k=0}^{\infty}$ is completely monotonic and $x_0 \geq a$, then so is the sequence $\{f(x_k)\}_{k=0}^{\infty}$.*

They [63] also proved the following two results.

**Theorem 1.Z.** *If $f \in AM[0,\infty)$ and if the sequence $\{x_k\}_{k=0}^{\infty}$ is completely monotonic, then so is the sequence $\{f(x_k)\}_{k=0}^{\infty}$.*

**Theorem 1.A′.** *Suppose that $f' \in CM(\mathbb{R}^+)$ and that the sequence $\{\Delta x_k\}_{k=0}^{\infty}$ is completely monotonic. If $x_0$ is in the domain of $f$, then the sequence $\{\Delta f(x_k)\}_{k=0}^{\infty}$ is completely monotonic.*

Now suppose that $f \in CM[0,\infty)$, by Theorem 1.X $\{f(n)\}_0^{\infty}$ is completely monotonic. We may ask if there exists an interpolating function $f \in CM[0,\infty)$ such that $f(n) = \mu_n, n \in \mathbb{N}_0$ for any given completely monotonic sequence $\{\mu_n\}_0^{\infty}$. For this, Widder [94] in 1931 established

**Theorem 1.B′.** *There exists a function $f \in CM[0,\infty)$ such that $f(n) = \mu_n, n \in \mathbb{N}_0$ if and only if the sequence $\{\mu_n\}_0^{\infty}$ is minimal completely monotonic.*

Let $\{\mu_n\}_0^{\infty}$ be any given completely monotonic sequence. By Theorem 1.1 there exists a number $\mu_0^*$ such that $\{\mu_0^*, \mu_1, \mu_2, \cdots\}$ is minimal. Hence by Theorem 1.B′ there is an interpolating function $g \in CM[0,\infty)$ such that $g(0) = \mu_0^*$ and $g(n) = \mu_n, n \in \mathbb{N}$. Then we can replace $g|_{[0,1/2]}$ by the linear function linking the point $(0,\mu_0)$ and the point $(1/2, g(1/2))$, which is completely monotonic on $[0,1/2]$. This discussion leads to

**Theorem 1.2.** *If the sequence* $\{\mu_n\}_0^\infty$ *is completely monotonic, then there exists an interpolating function* $f(x)$ *such that* $f|_{[0,1/2]}$ *and* $f|_{[1/2,\infty)}$ *are both completely monotonic and*

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

For compositions of completely monotonic and related functions, the following are a version of the corresponding Theorems in [92, Chapter IV].

**Theorem 1.C′.** *Suppose that* $f \in AM(I_1), g \in AM(I)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in AM(I)$.

**Theorem 1.D′.** *Suppose that* $f \in AM(I_1), g \in CM(I)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in CM(I)$.

The following example shows that $f \circ g$ may not belong to $CM(I)$ for $f \in CM(I_1), g \in AM(I)$ and $\mathcal{R}(g) \subset I_1$.

Let $f(x) := e^{-x}, g(x) := x^2$, then $f \in CM(\mathbb{R}), g \in AM(\mathbb{R}^+)$. But $f \circ g(x) = e^{-x^2} \notin CM(I)$ since $[f \circ g(x)]'' = 2e^{-x^2}(2x^2 - 1) < 0$ when $x \in (0, \sqrt{2}/2)$.

The next result, which was established in 1983 by Lorch and Newman [63, Theorem 5], is a converse of Theorem 1.D′.

**Theorem 1.E′.** *If for each* $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$*, then* $f \in AM(\mathbb{R}^+)$.

The following result is a generalized form of Theorem 2 of [68].

**Theorem 1.F$'$.** *Suppose that $f \in CM(I_1), g \in C(I), g' \in CM(I^o)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in CM(I)$.*

From this result we obtain that if $f \in CM(I_1)$,where $I_1 := (a, b)$ with $-\infty \leq a < b < \infty$, then $f(b - e^{-x}) \in CM(I)$. Here $I := (-\ln(b - a), \infty)$.

In 1983 Lorch and Newman [63, Theorem 4] gave an interesting result related to Theorem 1.F$'$ as follows.

**Theorem 1.G$'$.** *For each function $f \in CM(I)$, where $I := [0, \infty)$, there exists a function $g$ on $I$ such that $g(0) = 0, f \circ g \in CM(I)$ and $g' \notin CM(\mathbb{R}^+)$.*

This result shows that $g' \in CM(I^o)$ is not a necessary condition in Theorem 1.F$'$.

For the representations of the completely monotonic functions on $\mathbb{R}^+$ or on $[0, \infty)$, the following are the well known Bernstein's Theorems (see [92, Chapter IV, Section 12]).

**Theorem 1.H$'$.** *$f \in CM(\mathbb{R}^+)$ if and only if there exists an increasing function $\alpha(t)$ on $[0, \infty)$ such that*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t). \tag{1.6}$$

**Theorem 1.I′.** *$f \in CM(I)$, where $I := [0, \infty)$, if and only if there exists a bounded, increasing function $\alpha(t)$ on $I$ such that*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t).$$

Here, and throughout the thesis, *increasing* and *decreasing* are understood in a non-strict sense, i.e. *increasing* means *non-decreasing* (and *decreasing* means *non-increasing*). The right side of (1.6) is an improper Stieltjes integral.

In [92] Widder provided three different methods of proof for Theorem 1.I′. In 1944 Pollard [75] also gave a proof for this result.

By use of Theorem 1.N, we can obtain the following two results.

**Theorem 1.J′.** *$f \in AM(I)$, where $I := (-\infty, 0)$, if and only if there exists an increasing function $\alpha(t)$ on $[0, \infty)$ such that*

$$f(x) = \int_0^\infty e^{xt} d\alpha(t).$$

**Theorem 1.K′.** *$f \in AM(I)$, where $I := (-\infty, 0]$, if and only if there exists a bounded, increasing function $\alpha(t)$ on $-I$ such that*

$$f(x) = \int_0^\infty e^{xt} d\alpha(t).$$

In 1928, Bernetein [22] proved Theorem 1.J′. Also Hausdorff [50] in 1921 established a similar result to this theorem.

The completely monotonic functions have remarkable applications in the theory of special functions, probability and statistics theories, physics, etc. For example, in 1972 Muldoon [72] proved

**Theorem 1.L′.** *Suppose that $f > 0$ on $I := (a, \infty)$ with $a \geq 0$ and that $(\ln f)'' \in CM(I)$. If there exists a strictly increasing sequence $\{x_k\}_1^\infty \subset I$ such that $\sum_{k=1}^\infty (1/x_k)$ diverges and $f(x_k) = \Gamma(x_k), k \in \mathbb{N}$, then $f(x) \equiv \Gamma(x)$ on $I$.*

Here $\Gamma$ denotes the gamma function (see Chapter 3). A careful analysis of his proof shows that if $\{x_k\}_1^\infty \subset I$ is strictly decreasing with $x_k \geq c > a$, then the conclusion is also valid (the case in which $\{x_k\}_1^\infty \subset I$ is strictly increasing and bounded is already included in Theorem 1.L′). The next result is also proved in [72].

**Theorem 1.M′.** *Let $f(x)$ be such that $f'(x) > 0$ on $I := (a, \infty)$ with $a \geq \alpha$, where $\alpha := 1.46163\cdots$ is the unique positive zero of $\psi(x)$. If $f/f' \in CM(I)$ and if there exists a strictly increasing sequence $\{x_k\}_1^\infty \subset I$ such that $\sum_{k=1}^\infty (1/x_k)$ diverges and $f(x_k) = \Gamma(x_k), k \in \mathbb{N}$, then $f(x) \equiv \Gamma(x)$ on $I$.*

Here, in Theorem 1.M′, $\psi$ denotes the psi or digamma function (see Chapter 3).

Now we quote the following two results, due to Kimberling [58] in 1974, as examples of applications of completely monotonic functions in probability and statistics.

**Theorem 1.N′.** *Suppose $f$ is a strictly decreasing function from $[0, \infty]$ into $[0, 1]$, that $f(0) = 1$ and $f(\infty) \geq 0$, and that $\{F_n\}$ is a sequence of continuous distribution functions over which $f$ is admissible. Then $f$ is completely monotonic on $[0, \infty)$.*

**Theorem 1.O′.** *Suppose $\{F_n\}$ is a sequence of distribution functions. Suppose $f$ is function from $[0, \infty]$ onto $[0, 1]$ which is completely monotonic on $[0, \infty)$. Then $f$ is admissible over $\{F_n\}$.*

Here, in Theorems 1.N′ and 1.O′, that $f$ is admissible over $\{F_n\}$ means that there exists a probability space $(\Omega, \mathcal{A}, P)$ and a sequence $\{X_n\}$ of random variables defined on that space such that $F_n$ is the distribution function of $\{X_n\}$ for $n \in \mathbb{N}$ and $\{X_n\}$ is exchangeable under $f$. For meanings of other terminologies here see [58]. Also see [59, 74, 84, 86] for more applications in probability and statistics.

For the applications in physics, in 1970, Day [30] established

**Theorem 1.P′.** *In a linear viscoelastic material the relaxation function $G(x)$ has the property that $G(x) - G(\infty)$ is completely monotonic on $[0, \infty)$ if and only if the work done on retraced paths is increased by delay.*

In a non-linear viscoelastic material, the author also proved under certain conditions $G(x)$, the infinitesimal relaxation function, has the property that $G(x) - G(\infty)$ is completely monotonic on $[0, \infty)$ (cf. [30, Theorem 2] for details of this result). For the meanings of terminologies used here see [30]. The applications in physics can also be found, for example, in [39].

In 2004, Qi and Chen [78] first explicitly give the following notion:

**Definition 1.Q′** (LCM). *A function $f$ is said to be* logarithmically completely monotonic *on an interval $I$ if $f > 0, f \in C(I)$, has derivatives of all orders on $I^o$ and for $n \in \mathbb{N}$*

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, \quad x \in I^o. \tag{1.7}$$

The set of all *logarithmically completely monotonic* functions on $I$ is denoted by $LCM(I)$. If inequality (1.7) is strict, then $f$ is said to be *strictly logarithmically completely monotonic.* Such a function class is denoted by $SiLCM(I)$.

By using Dubourdieu's result Theorem 1.S, we can prove

**Theorem 1.R′.** *A logarithmically completely monotonic function which is not identically equal to $c_0 e^{-cx}$ on $(a, \infty)$ for some $c_0 > 0$ and some $c \geq 0$ is* strictly *logarithmically completely monotonic there.*

For the proof of this result, see Theorem A.3 of Appendix A.

From Definition 1.Q$'$, it is apparent that if $f, g \in LCM(I)$, then $fg \in LCM(I)$ and $f^\alpha \in LCM(I)$ for any $\alpha \geq 0$. Notice that if $f \in CM(I)$, we can not guarantee that $f^\alpha \in CM(I)$ for every $\alpha \in \mathbb{R}^+$. For example, let $f(x) := -\ln x, I := (0,1)$. Then $f \in CM(I)$. But for $\alpha = 1/2$, $f^\alpha \notin CM(I)$ since

$$(f^\alpha)''(x) = \frac{-(2\ln x + 1)}{4x^2(-\ln x)^{3/2}} < 0$$

when $x \in (1/\sqrt{e}, 1) \subset I$.

From [61, Result A] by Lorch of 2003, a similar result to O'Cinneide's Theorem 1.T can be easily derived.

**Theorem 1.S$'$.** *If $f \in LCM(I)$, then $f(x+\delta)/f(x)$ is increasing in $x$ for each fixed $\delta > 0$ such that $x + \delta \in I$.*

In [61] Lorch also gave a simple proof for O'Cinneide's Theorem 1.T.

It was proved by Qi and Chen [78] in 2004 that

**Theorem 1.T$'$.**

$$LCM(I) \subset CM(I).$$

The following example shows that

$$LCM(I) \neq CM(I).$$

Let $f(x) := -\ln x, I := (0,1)$. Then $f \in CM(I)$. But $f \notin LCM(I)$ since

$$(\ln f(x))'' = \frac{-(1+\ln x)}{(x \ln x)^2} < 0$$

if $x \in (1/e, 1)$.

It is worth mentioning that by using Lemma 2.18 we can give a simpler and shorter proof of Theorem 1.T$'$ than that in [78] (see Theorem A.4 in Appendix A).

In 1967 Horn [47, Theorem 4.4] established the following result, in terms of logarithmically completely monotonic functions.

**Theorem 1.U$'$.** *Suppose that $f \not\equiv 0$ on $\mathbb{R}^+$. Then $f \in LCM(\mathbb{R}^+)$ if and only if $\sqrt[n]{f} \in CM(\mathbb{R}^+)$ for all $n \in \mathbb{N}$.*

A function such that $\sqrt[n]{f} \in CM(\mathbb{R}^+), n \in \mathbb{N}$ is called infinitely divisible completely monotonic in [47]. We note that, for the interval $I := \mathbb{R}^+$, Theorem 1.T$'$ is a direct consequence of Horn's result Theorem 1.U$'$.

In 2004 Berg [17, Theorem 1.1] reproved Theorem 1.U$'$ following a quite different approach from that of Horn [47].

Before introducing the next result, we need

**Definition 1.V$'$** ([37, Chapter VI, Section 3]). *A distribution $F$ is called infinitely divisible if for each $n \in \mathbb{N}$ there exists a distribution $F_n$ such that $F$ is the n-fold convolution of $F_n$ with itself; i.e.,*

$$F = F_n^{n\star} := \underbrace{F_n \star F_n \star \cdots \star F_n}_{n \text{ times}}.$$

*In other words, $F$ is infinitely divisible if and only if for each $n \in \mathbb{N}$ it can be represented as the distribution of the sum*

$$\boldsymbol{S}_n = \boldsymbol{X}_{1,n} + \cdots + \boldsymbol{X}_{n,n}$$

*of $n$ independent random variables with a common distribution $F_n$.*

For example, all gamma distributions, the Poisson and the compound Poisson distributions are infinitely divisible.

It is known that (see [37, Chapter XIII, Section 7])

**Theorem 1.W$'$.** *A function $\omega$ is the Laplace transform of an infinitely divisible probability distribution if and only if $\omega = e^{-h}$, where $h$ is a function such that $h(0) = h(0+) = 0$ and $h' \in CM(\mathbb{R}^+)$.*

From this result, we can obtain a characterization of logarithmically completely monotonic functions as follows.

**Theorem 1.X$'$.** *A function $f$ with $f(0) > 0$ is in $LCM(I)$, where $I := [0, \infty)$, if and only if the function $f/f(0)$ is the Laplace transform of an infinitely divisible probability distribution.*

For the proof of this result, see Theorem A.5 of Appendix A.

The following form of definition of Stieltjes transform was adopted by Berg and Forst in their book [20, p. 127] published in 1975.

**Definition 1.Y$'$.** *A function $f$ on $\mathbb{R}^+$ is called a Stieltjes transform, if there exist a constant $a \geq 0$ and an increasing function $\mu$ on $[0, \infty)$ such that*

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x+t}, \quad x \in \mathbb{R}^+.$$

For example, let $f(x) := 1/\sqrt{x}$. Since

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{dt}{(x+t)\sqrt{t}}, \quad x \in \mathbb{R}^+,$$

$f(x)$ is a Stieltjes transform.

We note that this definition of Stieltjes transform is a little different from that of Widder [93], [92, p.325]. Under Widder's definition, a Stieltjes transform may be negative.

In 2004 Berg [17, Theorem 1.2] proved

**Theorem 1.Z$'$.** *A non-zero Stieltjes transform belong to $LCM(\mathbb{R}^+)$.*

We now give a brief summary for the rest of this thesis.

In Chapter 2, we discuss some function classes related to the class of completely monotonic functions. Some materials are included in Chapter 3 about the (di-, poly-, incomplete) gamma functions and (Riemann, Hurwitz) zeta functions which will be used in Chapters 4–8. In Chapters 4–6, some classes of logarithmically completely monotonic functions are studied. In Chapter 7, we give a class of completely monotonic functions related to the remainder of Binet's formula. In Chapter 8, we discuss the monotonicity and concavity properties of some functions as well as some applications. In Appendix A, we present short proofs for some interesting results mentioned in this chapter.

# Chapter 2

# Some classes related to completely monotonic functions

## 2.1 Introduction and main results

In 1989, Trimble, et al [87] introduced the following notion:

**Definition 2.A** (SCM). *A function $f$ is said to be* strongly completely monotonic *on $I^+$ if, for all $n \in \mathbb{N}_0, (-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $I^+$.*

The class of such functions is denoted by $SCM(I^+)$. By the definition $\alpha f + \beta g \in SCM(I^+)$ if $f, g \in SCM(I^+)$ and $\alpha, \beta \geq 0$. Using Leibniz's rule we can get $fg \in SCM(I^+)$ if $f, g \in SCM(I^+)$.

Trimble, et al [87] proved

**Theorem 2.B.** *$f \in SCM(\mathbb{R}^+)$ if and only if there exists a nonnegative and increasing function $\phi(t)$ on $[0, \infty)$ such that*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt, \quad x \in \mathbb{R}^+.$$

They also showed that if $f \in SCM(\mathbb{R}^+)$ and if $f \not\equiv c/x$ on $\mathbb{R}^+$ for some $c \geq 0$, then $g := 1/f$ is star-shaped on $\mathbb{R}^+$ (i.e. $g(\alpha x) \leq \alpha g(x)$ for $\alpha \in (0,1)$ and $x \in \mathbb{R}^+$).

Clearly,

$$SCM(I^+) \subset CM(I^+).$$

Let $\alpha \geq 1$, then the functions $f_\alpha(x) = x^{-\alpha} \in LCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+)$. Therefore

$$LCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) \neq \emptyset.$$

In this chapter, we introduce a subclass of functions in $LCM(I^+)$(i.e. $SLCM(I^+)$ , cf. Definition 2.1) which is disjoint from $SCM(I^+)$. We also give a function class(i.e. $ASCM(I^+)$, cf. Definition 2.2) which contains $SLCM(I^+)$ and $SCM(I^+)$, but is different from $CM(I^+)$. Such a function class and $CM(I^+)$ are contained in another function class which shall be introduced in Definition 2.3. Then we study properties of these function classes.

**Definition 2.1** (SLCM). *A positive function $f$ is said to be* strongly logarithmically completely monotonic *on $I^+$ if, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1}[\ln f(x)]^{(n)}$ are nonnegative and decreasing on $I^+$.*

Such a function class is denoted by $SLCM(I^+)$. For example, $e^{1/x} \in SLCM(\mathbb{R}^+)$. But $e^{-x} \in LCM(\mathbb{R}^+) - SLCM(\mathbb{R}^+)$. From the definition, $f^\alpha g^\beta \in SLCM(I^+)$ if $f, g \in SLCM(I^+)$ and $\alpha, \beta \geq 0$; also

$$SLCM(I^+) \subset LCM(I^+).$$

The following example shows that a Stieltjes transform (see Definition 1.Y′) may not belong to $SCM(\mathbb{R}^+) \cup SLCM(\mathbb{R}^+)$: Let $f(x) := 1/\sqrt[3]{x}$. Since

$$f(x) = \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{dt}{(x+t)\sqrt[3]{t}}, \quad x \in \mathbb{R}^+,$$

$f(x)$ is a Stieltjes transform. It is easy to veryfy that $f(x) \notin SCM(\mathbb{R}^+) \cup SLCM(\mathbb{R}^+)$.

**Definition 2.2** (ASCM). *A function $f$ is said to be* almost strongly completely monotonic *on $I^+$ if, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $I^+$.*

The class of almost strongly completely monotonic functions on $I^+$ is denoted by $ASCM(I^+)$. Notice that the difference between Definition 2.A and Definition 2.2 is that $n = 0$ is excluded here.

For example,

$$1 + 1/x^2 \in ASCM(\mathbb{R}^+) - SCM(\mathbb{R}^+).$$

From the definition, $\alpha f + \beta g \in ASCM(I^+)$ if $f, g \in ASCM(I^+)$ and $\alpha, \beta \geq 0$.

By Leibniz's rule for $(fg)^{(n)}$, we have for $n \in \mathbb{N}$

$$(-1)^n x^{n+1} (fg)^{(n)} = \frac{1}{x} \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{k+1} f^{(k)} (-1)^{n-k} x^{n-k+1} g^{(n-k)},$$

hence $fg \in ASCM(I^+)$ if $f, g \in ASCM(I^+)$ and $f, g \geq 0$.

**Definition 2.3** (ACM). *A function $f$ is said to be* almost completely monotonic *on an interval $I$, if $f \in C(I)$, has derivatives of all orders on $I^o$ and for all $n \in \mathbb{N}$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Let's use $ACM(I)$ to denote the set of all such functions on $I$.

For example, the important functions $-\psi(x), -\ln x$ are in $ACM(\mathbb{R}^+)$, but not in $CM(\mathbb{R}^+)$(also, $-\ln x \notin ASCM(\mathbb{R}^+)$).

Directly from the definition, we see $\alpha f + \beta g \in ACM(I)$ if $f, g \in ACM(I)$ and $\alpha, \beta \geq 0$.

It is easy to see that

$$ASCM(I^+) \cup CM(I^+) \subset ACM(I^+). \tag{2.1}$$

The following two results immediately follow from definitions and so the proofs will be omitted.

**Theorem 2.4.**

1. *$-f \in ACM(I)$ if and only if $f \in C(I)$ and $f' \in CM(I^o)$.*

2. *$-f \in ASCM(I^+)$ implies $f' \in SCM(I^+)$.*

We note that the converse of Theorem 2.4(2) is not true. For example, let $f(x) := \ln x$, then $f'(x) = 1/x \in SCM(\mathbb{R}^+)$, while $-f(x) = -\ln x \notin ASCM(\mathbb{R}^+)$ – indeed, $(-1)^1 x^2[-f(x)]' = x$ is strictly increasing on $\mathbb{R}^+$.

**Theorem 2.5.**

1. *$f \in LCM(I)$ if and only if $f > 0$ and $\ln f \in ACM(I)$.*

2. *$f \in SLCM(I^+)$ if and only if $f > 0$ and $\ln f \in ASCM(I^+)$.*

**Theorem 2.6.** *A strongly logarithmically completely monotonic function on $I^+$ must be almost strongly completely monotonic on $I^+$. i. e.*

$$SLCM(I^+) \subset ASCM(I^+).$$

It is easy to see that

$$SLCM(I^+) \cup SCM(I^+) \subset CM(I^+), \tag{2.2}$$

and

$$SCM(I^+) \subset ASCM(I^+). \tag{2.3}$$

From Theorem 2.6 and (2.3),

$$SLCM(I^+) \cup SCM(I^+) \subset ASCM(I^+). \tag{2.4}$$

From (2.1), (2.2) and (2.4), we see that $CM(I^+)$ and $ASCM(I^+)$ all contain $SLCM(I^+) \cup SCM(I^+)$ and are all contained in $ACM(I^+)$. The following examples show that $CM(I^+)$ and $ASCM(I^+)$ do not contain each other.

$$-1 + 1/x^2 \in ASCM(\mathbb{R}^+) - CM(\mathbb{R}^+).$$

$$1/\sqrt{x} \in CM(\mathbb{R}^+) - ASCM(\mathbb{R}^+).$$

**Theorem 2.7.**

$$SLCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) = \emptyset.$$

*In other words, a strongly logarithmically completely monotonic function on $\mathbb{R}^+$ can not be strongly completely monotonic on $\mathbb{R}^+$, or, a strongly completely monotonic function on $\mathbb{R}^+$ can not be strongly logarithmically completely monotonic on $\mathbb{R}^+$.*

**Theorem 2.8.**

1. *Suppose that $f \in C(I), f > 0$ and $f' \in CM(I^o)$, then $1/f \in LCM(I)$.*

2. *Suppose that $f > 0$ and $f' \in SCM(I^+)$. If $xf'(x) \geq f(x), x \in I^+$, then $1/f \in SLCM(I^+)$.*

For example, let $f(x) := \ln x, I^+ := (1, e)$. Then $f'(x) = 1/x \in SCM(I^+)$ and $xf'(x) = 1 \geq f(x) = \ln x, x \in I^+$. By Theorem 2.8(2), $1/f(x) = 1/\ln x \in SLCM(I^+)$.

The condition: $xf'(x) \geq f(x), x \in I^+$ in Theorem 2.8(2) can not be deleted. For example, let $f(x) := \ln x, I^+ := (e, \infty)$. Then $f'(x) = 1/x \in SCM(I^+)$ and the condition $xf'(x) \geq f(x), x \in I^+$ is not satisfied. It is easy to verify that $(-1)^1 x^2 [\ln(1/\ln x)]' = x/\ln x$ is strictly increasing on $I^+$. Therefore $1/f(x) = 1/\ln x \notin SLCM(I^+)$.

**Theorem 2.9.**

1. *Suppose that $f > 0$ and $-f \in ACM(I)$, then $1/f \in LCM(I)$.*

2. *Suppose that $f > 0$ and $-f \in ASCM(I^+)$. Then $1/f \in SLCM(I^+)$.*

**Theorem 2.10.**

1. *Suppose that $f \in AM(I_1), g \in ACM(I)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in CM(I)$.*

2. *Suppose that* $f \in AM(I), g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I$, *then* $f \circ g \in ASCM(I^+)$.

Let $f \in LCM(I)$, then $\ln f \in ACM(I)$. Since $e^x \in AM(\mathbb{R})$, by Theorem 2.10(1), $e^{\ln f} = f \in CM(I)$. Hence Qi and Chen's result Theorem 1.T′ is a simple corollary to Theorem 2.10(1).

**Theorem 2.11.** *Let* $I_1$ *be an open interval and* $f$ *defined on* $I_1$.

1. *If* $f' \in AM(I_1), g \in ACM(I)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in ACM(I)$.

2. *If* $f' \in AM(I_1), g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $-(f \circ g)' \in ASCM(I^+)$.

**Theorem 2.12.**

1. *Suppose that* $f \in ACM(I_1), g \in C(I), g' \in CM(I^o)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in ACM(I)$.

2. *Suppose that* $f \in ASCM(I_1^+), g' \in SCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1^+$. *If* $2xg'(x) \geq g(x), x \in I^+$, *then* $f \circ g \in ASCM(I^+)$.

The condition $2xg'(x) \geq g(x), x \in I^+$ in Theorem 2.12(2) can not be deleted even if $f \in ASCM(I_1^+)$ is replaced by a stronger condition: $f \in SCM(I_1^+)$. For example, let $f(x) := 1/x, g(x) := \ln x$ and $I^+ := (e^2, \infty)$.

Then $f \in SCM(\mathbb{R}^+), g' \in SCM(I^+)$, and the condition $2xg'(x) \geq g(x), x \in I^+$ is not satisfied. It is easy to show that

$$(-1)^1 x^2 [f \circ g(x)]' = \frac{x}{\ln^2 x}$$

is strictly increasing on $I^+$. Hence $f \circ g(x) \notin ASCM(I^+)$.

**Theorem 2.13.**

1. *Suppose that* $f \in ACM(I_1), -g \in ACM(I)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in ACM(I)$.

2. *Suppose that* $f \in ACM(I_1), -g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1$. *Then* $f \circ g \in ASCM(I^+)$.

**Theorem 2.14.**

1. *Suppose that* $f \in LCM(I_1), g \in C(I), g' \in CM(I^o)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in LCM(I)$.

2. *Suppose that* $f \in SLCM(I_1^+), g' \in SCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1^+$. *If* $2xg'(x) \geq g(x), x \in I^+$, *then* $f \circ g \in SLCM(I^+)$.

The condition: $2xg'(x) \geq g(x), x \in I^+$ in Theorem 2.14(2) can not be deleted. For example, let $f(x) := e^{1/x}, g(x) := \ln x$ and $I^+ := (e^2, \infty)$. Then it is easy to verify that $f \in SLCM(\mathbb{R}^+), g' \in SCM(I^+)$, and the condition

$2xg'(x) \geq g(x), x \in I^+$ is not satisfied. We can show that $h(x) := f \circ g(x) = \exp(\frac{1}{\ln x}) \notin SLCM(I^+)$. Indeed

$$(-1)^1 x^2 [\ln h(x)]' = \frac{x}{\ln^2 x} \to \infty$$

as $x \to \infty$. Therefore $x/\ln^2 x$ can not be decreasing on $I^+$.

**Theorem 2.15.**

1. *Suppose that $f \in LCM(I_1), -g \in ACM(I)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in LCM(I)$.*

2. *Suppose that $f \in LCM(I_1), -g \in ASCM(I^+)$ and $\mathcal{R}(g) \subset I_1$. Then $f \circ g \in SLCM(I^+)$.*

**Theorem 2.16.** *Let $I_1$ and $I$ be open interval, and $f$ and $g$ defined on $I_1$ and $I$ respectively.*

1. *If $f' \in CM(I_1), g' \in CM(I)$ and $\mathcal{R}(g) \subset I_1$, then $(f \circ g)' \in CM(I)$.*

2. *If $f' \in LCM(I_1), g' \in LCM(I)$ and $\mathcal{R}(g) \subset I_1$, then $(f \circ g)' \in LCM(I)$.*

**Theorem 2.17.** *Let $f$ and $g$ be defined on $I_1^+$ and $I^+$ respectively. If $0 \leq f' \in ASCM(I_1^+), g' \in SCM(I^+)$, $\mathcal{R}(g) \subset I_1^+$ and $2xg'(x) \geq g(x), x \in I^+$, then $(f \circ g)' \in ASCM(I^+)$.*

The above Theorems are useful tools in dealing with composite functions. For example, let $f(x) := 1/x, g(x) := \ln x$ and $I^+ := (1, e^2)$. Then $f \in ASCM(\mathbb{R}^+), g' \in SCM(I^+)$. It is easy to see that $2xg'(x) \geq g(x), x \in I^+$. Thus by Theorem 2.12(2),

$$f \circ g(x) = \frac{1}{\ln x} \in ASCM(I^+).$$

It is (almost) impossible directly using the definition to show that $1/\ln x \in ASCM(I^+)$ due to the complexity of computation.

## 2.2 Proofs of main results

We need the following lemma to prove the main results of this chapter.

**Lemma 2.18** ([40, p. 21]). *Suppose that the functions $y = y(x), x \in I_1$ and $x = \varphi(t), t \in I$ are $n$ times differentiable, and $\mathcal{R}(\varphi) \subset I_1$. Then, for $t \in I$,*

$$\frac{d^n y}{dt^n} = \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!i_2!\cdots i_n!} \frac{d^m y(\varphi(t))}{dx^m} \left(\frac{\varphi'}{1!}\right)^{i_1} \left(\frac{\varphi''}{2!}\right)^{i_2} \cdots \left(\frac{\varphi^{(n)}}{n!}\right)^{i_n},$$

*where $m = i_1 + i_2 + \cdots + i_n$ and*

$$\Lambda_n := \left\{(i_1, i_2, \cdots, i_n) | i_1, i_2, \cdots, i_n \in \mathbb{N}_0, \sum_{\nu=1}^{n} \nu i_\nu = n\right\}. \tag{2.5}$$

The following is a self-contained proof for Theorem 2.6. By using Theorem 2.5(2) and Theorem 2.10(2), we can give another proof of this result.

*Proof of Theorem 2.6.* Suppose that $f \in SLCM(I^+)$. Let $g(x) = \ln f(x)$. By definition, for $i \in \mathbb{N}$,

$$(-1)^i x^{i+1} g^{(i)}(x) \text{ are nonnegative and decreasing on } I^+. \tag{2.6}$$

By Lemma 2.18, for $n \in \mathbb{N}$,

$$\frac{d^n f}{dx^n} = \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!i_2!\cdots i_n!} e^{g(x)} \left(\frac{g'}{1!}\right)^{i_1} \left(\frac{g''}{2!}\right)^{i_2} \cdots \left(\frac{g^{(n)}}{n!}\right)^{i_n},$$

where $\Lambda_n$ is defined by (2.5).

For any $(i_1, i_2, \cdots, i_n) \in \Lambda_n$, we have

$$(-1)^n x^{n+1} = \frac{(-1)^{i_1+2i_2+\cdots+ni_n} x^{2i_1+3i_2+\cdots+(n+1)i_n}}{x^{i_1+i_2+\cdots+i_n-1}}.$$

Thus

$$(-1)^n x^{n+1} f^{(n)}(x)$$

$$= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!} \frac{e^{g(x)}}{x^{i_1+\cdots+i_n-1}} \left(\frac{(-1)^1 x^2 g'}{1!}\right)^{i_1} \cdots \left(\frac{(-1)^n x^{n+1} g^{(n)}}{n!}\right)^{i_n}. \tag{2.7}$$

Since $(i_1, i_2, \cdots, i_n) \in \Lambda_n$,

$$i_1 + i_2 + \cdots + i_n \geq 1.$$

By taking $i = 1$ in the condition (2.6), we see $g'(x) \leq 0$, that is, $g(x)$ is decreasing on $I^+$. Therefore, for $(i_1, i_2, \cdots, i_n) \in \Lambda_n$, $e^{g(x)} x^{1-(i_1+i_2+\cdots+i_n)}$ are nonnegative and decreasing on $I^+$. Also, by the condition (2.6), each factor in (2.7) following the factor $e^{g(x)}x^{1-(i_1+i_2+\cdots+i_n)}$ is nonnegative and decreasing on $I^+$. So we conclude that, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $I^+$. By definition, $f \in ASCM(I^+)$. The proof is complete. □

*Proof of Theorem 2.7.* Note that it is enough to show that if $f \in SLCM(\mathbb{R}^+)$ then $f \notin SCM(\mathbb{R}^+)$. Now suppose that $f \in SLCM(\mathbb{R}^+)$. Let $g(x) = \ln f(x)$. Then, by definition, $-x^2 g'(x)$ is decreasing on $\mathbb{R}^+$. So

$$\lim_{x\to\infty} (-x^2 g'(x)) \leq -g'(1). \tag{2.8}$$

We claim that $f(x) \notin SCM(\mathbb{R}^+)$.

If $f(x)$ were in $SCM(\mathbb{R}^+)$, then $xf(x)$ would be decreasing on $\mathbb{R}^+$, that is

$$[xf(x)]' \leq 0, \quad x \in \mathbb{R}^+. \tag{2.9}$$

Since

$$\begin{aligned} [xf(x)]' &= f(x) + xf'(x) \\ &= e^{g(x)} + xe^{g(x)}g'(x) \\ &= e^{g(x)}(1 + xg'(x)), \end{aligned}$$

from (2.9),

$$1 + xg'(x) \leq 0, \quad x \in \mathbb{R}^+. \tag{2.10}$$

From (2.10),

$$-x^2 g'(x) \geq x, \quad x \in \mathbb{R}^+.$$

Therefore

$$\lim_{x \to \infty} (-x^2 g'(x)) = \infty.$$

In view of (2.8), we have arrived at a contradiction. And our claim has been proved. □

*Proof of Theorem 2.8(1).* Without loss any generality, we may assume $I$ is open. Let $h(x) := \ln(1/f(x)) = -\ln f(x), x \in I$.

It is easy to verify that for $k \in \mathbb{N}$

$$(\ln x)^{(k)} = \frac{(-1)^{k-1}(k-1)!}{x^k}. \tag{2.11}$$

Since $f' \in CM(I)$,

$$(-1)^i [f'(x)]^{(i)} = (-1)^i f^{(i+1)}(x) \geq 0, x \in I, i \in \mathbb{N}_0. \tag{2.12}$$

By Lemma 2.18 and (2.11), for $n \in \mathbb{N}$,

$$(-1)^n h^{(n)}(x)$$
$$= (-1)^{n+1} \sum_{(i_1, i_2, \cdots, i_n) \in \Lambda_n} \frac{n!}{i_1! i_2! \cdots i_n!} \frac{(-1)^{m-1}(m-1)!}{[f(x)]^m} \left(\frac{f'}{1!}\right)^{i_1} \cdots \left(\frac{f^{(n)}}{n!}\right)^{i_n}$$

$$= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!}(-1)^{2m}\frac{(m-1)!}{[f(x)]^m}\left(\frac{(-1)^0 f'}{1!}\right)^{i_1}\cdots\left(\frac{(-1)^{n-1}f^{(n)}}{n!}\right)^{i_n}, \tag{2.13}$$

where $m = i_1 + i_2 + \cdots + i_n \geq 1$ and $\Lambda_n$ is defined by (2.5).

Combining (2.12) and (2.13) gives

$$(-1)^n h^{(n)}(x) \geq 0, x \in I, n \in \mathbb{N}.$$

Hence $1/f \in LCM(I)$.

The proof of Theorem 2.8(1) is complete.

*Proof of Theorem 2.8(2).*

Let $h(x) := \ln(1/f(x)) = -\ln f(x), x \in I^+$.

Since $f' \in SCM(I^+)$,

$$(-1)^i x^{i+1} f^{(i+1)}(x) \quad \text{are nonnegative and decreasing on } I^+ \text{ for } i \in \mathbb{N}_0. \tag{2.14}$$

By Lemma 2.18 and (2.11), for $n \in \mathbb{N}$,

$$(-1)^n x^{n+1} h^{(n)}(x)$$

$$= (-1)^{n+1} x^{n+1} \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1! i_2! \cdots i_n!}\frac{(-1)^{m-1}(m-1)!}{[f(x)]^m}\left(\frac{f'}{1!}\right)^{i_1}\cdots\left(\frac{f^{(n)}}{n!}\right)^{i_n}$$

$$= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!}(-1)^{2m}\frac{x(m-1)!}{[f(x)]^m}\left(\frac{(-1)^0 x f'}{1!}\right)^{i_1}\cdots\left(\frac{(-1)^{n-1}x^n f^{(n)}}{n!}\right)^{i_n}, \tag{2.15}$$

where $m = i_1 + i_2 + \cdots + i_n \geq 1$ and $\Lambda_n$ is defined by (2.5).

Since $xf'(x) \geq f(x), x \in I^+$,

$$\frac{x}{f(x)} \quad \text{is positive and decreasing on } I^+. \tag{2.16}$$

By setting $i = 0$ in (2.14), we get $f'(x) \geq 0, x \in I^+$, i.e.,

$$f(x) \text{ is increasing on } I^+. \tag{2.17}$$

From (2.16) and (2.17), in view that $\frac{x}{[f(x)]^m} = \frac{1}{[f(x)]^{m-1}} \frac{x}{f(x)}$ and $m \geq 1$, we see

$$\frac{x}{[f(x)]^m} \quad \text{are nonnegative and decreasing on } I^+. \tag{2.18}$$

By (2.14) and (2.18), from (2.15), we conclude that $(-1)^n x^{n+1} h^{(n)}(x)$ are nonnegative and decreasing on $I^+$ for $n \in \mathbb{N}$. Hence $1/f \in SLCM(I^+)$.

The proof of Theorem 2.8(2) is complete. $\square$

*Proof of Theorem 2.9(1).* Since $-f \in ACM(I^o)$ implies $f' \in CM(I^o)$, by Theorem 2.8(1), we obtain that $1/f \in LCM(I)$.

The proof of Theorem 2.9(1) is complete.

*Proof of Theorem 2.9(2).*

Let $h(x) := \ln(1/f(x)) = -\ln f(x), x \in I^+$.

Since $-f \in ASCM(I^+)$,

$$(-1)^{i+1} x^{i+1} f^{(i)}(x) \quad \text{are nonnegative and decreasing on } I^+ \text{ for } i \in \mathbb{N}. \tag{2.19}$$

By Lemma 2.18 and (2.11), for $n \in \mathbb{N}$,

$$
\begin{aligned}
&(-1)^n x^{n+1} h^{(n)}(x) \\
&= (-1)^{n+1} x^{n+1} \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1! i_2! \cdots i_n!} \frac{(-1)^{m-1}(m-1)!}{[f(x)]^m} \left(\frac{f'}{1!}\right)^{i_1} \cdots \left(\frac{f^{(n)}}{n!}\right)^{i_n} \\
&= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1! \cdots i_n!} \frac{(m-1)!}{x^{m-1}[f(x)]^m} \left(\frac{(-1)^2 x^2 f'}{1!}\right)^{i_1} \cdots \left(\frac{(-1)^{n+1} x^{n+1} f^{(n)}}{n!}\right)^{i_n},
\end{aligned}
\tag{2.20}
$$

where $m = i_1 + i_2 + \cdots + i_n \geq 1$ and $\Lambda_n$ is defined by (2.5).

By setting $i = 1$ in (2.19), we get $f'(x) \geq 0, x \in I^+$, i.e.,

$$
f(x) \text{ is increasing on } I^+. \tag{2.21}
$$

Since $m \geq 1$, by (2.19) and (2.21), from (2.20), we conclude that $(-1)^n x^{n+1} h^{(n)}(x)$ are nonnegative and decreasing on $I^+$ for $n \in \mathbb{N}$. Hence $1/f \in SLCM(I^+)$.

The proof of Theorem 2.9(2) is complete. □

*Proof of Theorem 2.10(1).* By Lemma 2.18, for $n \in \mathbb{N}$

$$
\begin{aligned}
&(-1)^n [f \circ g]^{(n)}(x) \\
&= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1! \cdots i_n!} f^{(m)}(g(x)) \left(\frac{(-1)^1 g'}{1!}\right)^{i_1} \cdots \left(\frac{(-1)^n g^{(n)}}{n!}\right)^{i_n}.
\end{aligned}
\tag{2.22}
$$

Since $f \in AM(I_1)$ and $\mathcal{R}(g) \subset I_1$,

$$f(g(x)) \geq 0, \quad x \in I, \tag{2.23}$$

and

$$f^{(i)}(g(x)) \geq 0, \quad x \in I, i \in \mathbb{N}. \tag{2.24}$$

Since $g \in ACM(I)$,

$$(-1)^i g^{(i)}(x) \geq 0, \quad x \in I, i \in \mathbb{N}. \tag{2.25}$$

By (2.24) and (2.25), from (2.22), we have

$$(-1)^n [f \circ g]^{(n)}(x) \geq 0, \quad x \in I, n \in \mathbb{N}. \tag{2.26}$$

(2.26) together with (2.23) means that $f \circ g \in CM(I)$.

The proof of Theorem 2.10(1) is complete.

*Proof of Theorem 2.10(2).*

Since $g \in ASCM(I^+)$, for $i \in \mathbb{N}$,

$$(-1)^i x^{i+1} g^{(i)}(x) \quad \text{are nonnegative and decreasing on } I^+. \tag{2.27}$$

Let $h(x) := f \circ g(x) = f(g(x)), x \in I^+$. By Lemma 2.18, for $n \in \mathbb{N}$,

$$(-1)^n x^{n+1} h^{(n)}(x)$$

$$= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!}\frac{f^{(m)}(g(x))}{x^{m-1}}\left(\frac{(-1)^1x^2g'}{1!}\right)^{i_1}\cdots\left(\frac{(-1)^nx^{n+1}g^{(n)}}{n!}\right)^{i_n}, \tag{2.28}$$

where $m = i_1 + i_2 + \cdots + i_n \geq 1$.

Since $f \in AM(I), g \in ASCM(I^+)$ and $\mathcal{R}(g) \subset I$, $f^{(m)}(x)$ are nonnegative and increasing on $I$, and $g(x)$ is decreasing on $I^+$. Consequently, $f^{(m)}(g(x))$ then

$$\frac{f^{(m)}(g(x))}{x^{m-1}} \quad \text{are nonnegative and decreasing on } I^+. \tag{2.29}$$

By (2.27) and (2.29), from (2.28), we see that for $n \in \mathbb{N}, (-1)^n x^{n+1}h^{(n)}(x)$ are nonnegative and decreasing on $I^+$.

The proof of Theorem 2.10(2) is complete. □

*Proof of Theorem 2.11(1).* Since $g \in ACM(I)$, by Theorem 2.4(1),

$$-g' \in CM(I^o). \tag{2.30}$$

By hypotheses and Theorem 2.10 (1),

$$f'(g(x)) \in CM(I). \tag{2.31}$$

Now

$$-(f\circ g)'(x) = -f'(g(x))g'(x) = f'(g(x))(-g'(x)), \quad x \in I^o. \tag{2.32}$$

By (2.30) and (2.31), from (2.32), we get $-(f \circ g)' \in CM(I^o)$. Then by Theorem 2.4(1), $f \circ g \in ACM(I)$.

The proof of Theorem 2.11(1) is complete.

*Proof of Theorem 2.11(2).*

Since $g \in ASCM(I^+)$, by Theorem 2.4(2),

$$-g' \in SCM(I^+). \tag{2.33}$$

Hence $-g' \geq 0$. By hypotheses and Theorem 2.10(2),

$$0 \leq f'(g(x)) \in ASCM(I^+). \tag{2.34}$$

Now

$$-(f \circ g)'(x) = -f'(g(x))g'(x) = f'(g(x))(-g'(x)), \quad x \in I^+. \tag{2.35}$$

By (2.33) and (2.34), from (2.35), we get $-(f \circ g)' \in ASCM(I^+)$.

The proof of Theorem 2.11(2) is complete. $\square$

*Proof of Theorem 2.12(1).* Without loss generality, we can assume that $I$ is an open interval. Let $h(x) := f \circ g(x) = f(g(x)), x \in I$. By Lemma 2.18, for $n \in \mathbb{N}$,

$$(-1)^n h^{(n)}(x)$$

$$= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!}(-1)^m f^{(m)}(g(x)) \left(\frac{(-1)^0 g'}{1!}\right)^{i_1} \cdots \left(\frac{(-1)^{n-1} g^{(n)}}{n!}\right)^{i_n}, \tag{2.36}$$

where $m = i_1 + i_2 + \cdots + i_n \geq 1$ and $\Lambda_n$ is defined by (2.5).

Since $f \in ACM(I_1)$ and $\mathcal{R}(g) \subset I_1$,

$$(-1)^i f^{(i)}(g(x)) \geq 0, x \in I, i \in \mathbb{N}. \tag{2.37}$$

Since $g' \in CM(I)$,

$$(-1)^j (g'(x))^{(j)} = (-1)^j g^{(j+1)}(x) \geq 0, x \in I, j \in \mathbb{N}_0,$$

or

$$(-1)^{i-1} g^{(i)}(x) \geq 0, x \in I, i \in \mathbb{N}. \tag{2.38}$$

By (2.37) and (2.38), from (2.36), we obtain

$$(-1)^n h^{(n)}(x) \geq 0, x \in I, n \in \mathbb{N}.$$

The proof of Theorem 2.12(1) is complete.

*Proof of Theorem 2.12(2).*

Since $g' \in SCM(I^+)$, $(-1)^j x^{j+1} (g')^{(j)}(x)$ are nonnegative and decreasing on $I^+$ for $j \in \mathbb{N}_0$. That is

$$(-1)^{i-1} x^i g^{(i)}(x) \quad \text{are nonnegative and decreasing on } I^+ \text{ for } i \in \mathbb{N}. \tag{2.39}$$

Let $h(x) := f \circ g(x) = f(g(x)), x \in I^+$.

For $(i_1, i_2, \cdots, i_n) \in \Lambda_n$,

$$i_1 + 2i_2 + \cdots + ni_n = n,$$

$$m = i_1 + i_2 + \cdots + i_n \geq 1$$

and

$$(-1)^n x^{n+1} = x(-1)^{i_1+i_2+\cdots+i_n}(-1)^{i_2+2i_3+\cdots+(n-1)i_n} x^{i_1+2i_2+\cdots+ni_n}.$$

Therefore, by Lemma 2.18, for $n \in \mathbb{N}$

$$(-1)^n x^{n+1} h^{(n)}(x)$$

$$= \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!}(-1)^m x f^{(m)}(g(x)) \left(\frac{(-1)^0 x^1 g'}{1!}\right)^{i_1} \cdots \left(\frac{(-1)^{n-1} x^n g^{(n)}}{n!}\right)^{i_n}. \tag{2.40}$$

By setting $i = 1$ in (2.39), we get $g'(x) \geq 0$, thus

$$g(x) \text{ is increasing on } I^+. \tag{2.41}$$

Since $f \in ASCM(I_1^+)$, for $(i_1, i_2, \cdots, i_n) \in \Lambda_n$,

$$(-1)^m x^{m+1} f^{(m)}(x) \text{ are nonnegative and decreasing on } I_1^+. \tag{2.42}$$

From the results (2.41) and (2.42), we obtain for $(i_1, i_2, \cdots, i_n) \in \Lambda_n$,

$$(-1)^m g^{m+1}(x) f^{(m)}(g(x)) \text{ are nonnegative and decreasing on } I^+. \tag{2.43}$$

From the conditions $\mathcal{R}(g) \subset I_1^+$ and $2xg'(x) \geq g(x), x \in I^+$, it is easy to show that

$$\frac{x}{g^2(x)} \text{ is decreasing on } I^+. \tag{2.44}$$

Since

$$(-1)^m x f^{(m)}(g(x)) = (-1)^m g^{m+1}(x) f^{(m)}(g(x)) \frac{1}{g^{m-1}(x)} \frac{x}{g^2(x)},$$

$m \geq 1$ and $g(x) > 0$, by (2.41), (2.43) and (2.44), we see that

$$(-1)^m x f^{(m)}(g(x)) \text{ are nonnegative and decreasing on } I^+. \tag{2.45}$$

By (2.39) and (2.45), from (2.40), we conclude that $(-1)^n x^{n+1} h^{(n)}(x)$ are nonnegative and decreasing on $I^+$ for $n \in \mathbb{N}$, i.e., $h = f \circ g \in ASCM(I^+)$

The proof of Theorem 2.12(2) is complete. $\square$

*Proof of Theorem 2.13(1).* Since $-g \in ACM(I^o)$ implies $g' \in CM(I^o)$, by Theorem 2.12(1), we obtain $f \circ g \in ACM(I)$

The proof of Theorem 2.13(1) is complete.

*Proof of Theorem 2.13(2).*

Since $-g \in ASCM(I^+)$,

$$(-1)^{i+1} x^{i+1} g^{(i)}(x) \quad \text{are nonnegative and decreasing on } I^+ \text{ for } i \in \mathbb{N}. \tag{2.46}$$

Let $h(x) := f \circ g(x) = f(g(x)), x \in I^+$. By Lemma 2.18, for $n \in \mathbb{N}$,

$$(-1)^n x^{n+1} h^{(n)}(x) =$$

$$\sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!}\frac{(-1)^m f^{(m)}(g(x))}{x^{m-1}}\left(\frac{(-1)^2x^2g'}{1!}\right)^{i_1}\cdots\left(\frac{(-1)^{n+1}x^{n+1}g^{(n)}}{n!}\right)^{i_n}, \tag{2.47}$$

where $m = i_1 + i_2 + \cdots + i_n \geq 1$.

By setting $i = 1$ in (2.46), we get $g'(x) \geq 0$, thus

$$g(x) \text{ is increasing on } I^+. \tag{2.48}$$

Since $f \in ACM(I_1)$, for $(i_1, i_2, \cdots, i_n) \in \Lambda_n$, $(-1)^{m+1}f^{(m+1)}(x) \geq 0$. Hence

$$(-1)^m f^{(m)}(x) \text{ are nonnegative and decreasing on } I_1. \tag{2.49}$$

From the results (2.48) and (2.49), we obtain for $(i_1, i_2, \cdots, i_n) \in \Lambda_n$,

$$(-1)^m f^{(m)}(g(x)) \text{ are nonnegative and decreasing on } I^+. \tag{2.50}$$

By (2.46) and (2.50), from (2.47), we conclude that $(-1)^n x^{n+1}h^{(n)}(x)$ are nonnegative and decreasing on $I^+$ for $n \in \mathbb{N}$, i.e., $h = f \circ g \in ASCM(I^+)$

The proof of Theorem 2.13(2) is complete. □

*Proof of Theorem 2.14(1).* Since $f \in LCM(I_1)$, by Theorem 2.5(1), $\ln f \in ACM(I_1)$. Then by Theorem 2.12(1), we get

$$(\ln f) \circ g \in ACM(I). \tag{2.51}$$

Since $(\ln f) \circ g = \ln(f \circ g)$, from (2.51), by Theorem 2.5(1), we obtain $f \circ g \in LCM(I)$.

The proof of Theorem 2.14(1) is complete.

*Proof of Theorem 2.14(2).*

Since $f \in SLCM(I_1^+)$, by Theorem 2.5(2), we get $\ln f \in ASCM(I_1^+)$. Then by Theorem 2.12(2), $(\ln f) \circ g \in ASCM(I^+)$. Hence

$$\ln(f \circ g) = (\ln f) \circ g \in ASCM(I^+). \tag{2.52}$$

From (2.52), by Theorem 2.5(2), we have $f \circ g \in SLCM(I^+)$.

The proof of Theorem 2.14(2) is complete. □

*Proof of Theorem 2.15.* Since $f \in LCM(I_1)$, by Theorem 2.5(1), we get

$$\ln f \in ACM(I_1). \tag{2.53}$$

Then by Theorem 2.13(1), we have

$$(\ln f) \circ g \in ACM(I). \tag{2.54}$$

Since $(\ln f) \circ g = \ln(f \circ g)$, from (2.54), $\ln(f \circ g) \in ACM(I)$. Therefore $f \circ g \in LCM(I)$ by Theorem 2.5(1).

From (2.53), by Theorem 2.13(2), we get

$$(\ln f) \circ g \in ASCM(I^+).$$

Hence $\ln(f \circ g) \in ASCM(I^+)$. Then by Theorem 2.5(2), we conclude that $f \circ g \in SLCM(I^+)$.

The proof is complete. □

*Proof of Theorem 2.16.*

*I.* By Theorem 1.F′, $f'(g(x)) \in CM(I)$. Hence $(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \in CM(I)$.

*II.* By Theorem 2.14(1), $f'(g(x)) \in LCM(I)$. Thus $(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \in LCM(I)$.

The proof is complete. □

*Proof of Theorem 2.17.* By Theorem 2.12(2),

$$f'(g(x)) \in ASCM(I^+). \tag{2.55}$$

Since $SCM(I^+) \subset ASCM(I^+)$, we get

$$g' \in ASCM(I^+). \tag{2.56}$$

From (2.55) and (2.56), and in view that $f', g' \geq 0$, we have

$$f'(g(x)) \cdot g'(x) = (f \circ g)'(x) \in ASCM(I^+).$$

The proof is complete. □

# Chapter 3

# The gamma and related functions

In this chapter we include definitions of the gamma and related functions as well as some basic properties, which will be used in the following chapters.

It is well-known that the Euler gamma function is defined and denoted for $z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt. \tag{3.1}$$

One of the elementary properties of the gamma function is that $\Gamma(x+1) = x\Gamma(x)$. In particular, $\Gamma(n+1) = n!$. By changing the variable $t$ in (3.1) to $xt$ $(x > 0)$, we can get

$$\frac{1}{x^z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-xt} dt, \quad x, z \in \mathbb{R}^+. \tag{3.2}$$

For integral representations of $\ln\Gamma(z)$, we have

**Theorem 3.A** ([14, Chapter 1]). *For $x \in \mathbb{R}^+$*

*i) (Binet's formula)*

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-xt}}{t}dt.$$

*ii) (Malmstén's formula)*

$$\ln\Gamma(x) = \int_0^\infty \left[(x-1)e^{-t} - \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\right]\frac{dt}{t}.$$

**Definition 3.B** ([65, Chapter 1]).

1. *The psi or digamma function, denoted by $\psi(z)$, is the logarithmic derivative of the Euler gamma function. i.e.*

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}. \tag{3.3}$$

2. *The derivatives of the psi function $\psi^{(k)}, k \in \mathbb{N}$, are called polygamma functions.*

The following are some basic properties of digama and polygamma functions.

**Theorem 3.C** ([40, p. 884],[65, Chapter 1]). *For $n \in \mathbb{N}, x \in \mathbb{R}^+$*

$$i) \quad \psi(x) = \ln x - \frac{1}{2x} - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt} dt. \tag{3.4}$$

$$ii) \quad \psi(x) = \ln x - \frac{1}{2x} - 2\int_0^\infty \frac{tdt}{(t^2 + x^2)(e^{2\pi t} - 1)}. \tag{3.5}$$

$$iii) \quad \psi\left(x + \frac{1}{2}\right) = \ln x + 2\int_0^\infty \frac{tdt}{(t^2 + 4x^2)(e^{\pi t} + 1)}. \tag{3.6}$$

$$iv) \quad \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt. \tag{3.7}$$

$$v) \quad \psi^{(n-1)}(x + 1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n}. \tag{3.8}$$

The function $\psi(x)$ can also be expressed as infinite series as follows.

**Theorem 3.D** ([40, p. 893]).

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^\infty \frac{x}{n(n+x)}, \quad x \in \mathbb{R}^+, \tag{3.9}$$

*where*

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n\right) = 0.5772156649\cdots$$

*is the Euler-Mascheroni constant.*

For large values of $x$, we have

**Theorem 3.E** ([34, p. 47], [65, Chapter 1]). *As $x \to \infty$,*

$$i) \quad \ln\Gamma(x + \beta) = \left(x + \beta - \frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + O\left(\frac{1}{x}\right). \tag{3.10}$$

$$ii) \quad \psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{3.11}$$

$$iii) \quad (-1)^{n+1}\psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + O\left(\frac{1}{x^{n+2}}\right), \quad n \in \mathbb{N}. \tag{3.12}$$

**Definition 3.F** ([35]). *The incomplete gamma functions are defined and denoted for $z > 0$ by*

$$\Gamma(z,x) := \int_x^\infty t^{z-1}e^{-t}dt, \tag{3.13}$$

$$\gamma(z,x) := \int_0^x t^{z-1}e^{-t}dt. \tag{3.14}$$

Other functions related to the gamma function which will be used are

**Definition 3.G** ([14, Chapter 1]).

1. *The Riemann zeta function is defined and denoted for $z > 1$ by*

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

2. *The Hurwitz zeta function is defined by*

$$\zeta(z,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^z}, \qquad -x \notin \mathbb{N}_0.$$

The Riemann and Hurwitz zeta functions have the following representations in terms of definite integrals.

**Theorem 3.H** ([65, Chapter 1]).

$$i) \quad \zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}dt}{e^t - 1}, \qquad z > 1.$$

$$ii) \quad \zeta(z,x) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}e^{-xt}dt}{1 - e^{-t}}, \qquad z > 1, x > 0.$$

We also have the following relation between polygamma functions and Hurwitz zeta functions.

**Theorem 3.I** ([40, p. 894]). *For* $n \in \mathbb{N}, x \in \mathbb{R}^+$

$$\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x).$$

The other important properties of the (di,ploy)gamma functions and zeta functions can be found, for example, respectively, in Chapter I of [34] and [54]. And the important properties of the incomplete gamma functions can be found in Chapter IX of [35].

# Chapter 4

# Some classes of logarithmically completely monotonic functions involving the gamma function I

Partly because of the applications of (logarithmically) completely monotonic functions, a lot of mathematicians are seeking, and have found, functions which are (logarithmically) completely monotonic. Among others, for example, Vogt et al [90] in 2002 proved that the function

$$f_1(x) = 1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$$

is strictly completely monotonic on $(-1, \infty)$ and approaches 1 as $x \to -1$ and approaches 0 as $x \to \infty$.

Muldoon [71] in 1978 proved that the function

$$h_{a,b}(x) = [x^a(\frac{e}{x})^x\Gamma(x)]^b, \quad x \in \mathbb{R}^+ \tag{4.1}$$

is in $CM(\mathbb{R}^+)$ if $a \le \frac{1}{2}$ and $b > 0$. Ismail, et al [51] in 1986 showed that the function $h_{a,b}(x)$, defined by (4.1), is in $CM(\mathbb{R}^+)$ if $a \ge 1$ and $b = -1$.

For the logarithmically complete monotonicity property of functions, Qi et al [80] in 2006 showed that the function

$$f_2(x) = \frac{\Gamma(x+1)^{1/x}}{x}(1+\frac{1}{x})^x$$

belongs to $LCM(\mathbb{R}^+)$.

In 2006 Grinshpan et al [41] proved that for any $a_k \in \mathbb{R}^+(k = 1, \cdots, n)$, the function

$$F_n(x) = \frac{\Gamma(x)\prod_{k=1}^{[n/2]}\prod_{m\in P_{n,2k}}\Gamma(x+\sum_{j=1}^{2k}a_{m_j})}{\prod_{k=1}^{[(n+1)/2]}\prod_{m\in P_{n,2k-1}}\Gamma(x+\sum_{j=1}^{2k-1}a_{m_j})} \tag{4.2}$$

is in $LCM(\mathbb{R}^+)$. Here, in (4.2), $P_{n,k}(k = 1, \cdots, n)$ is the set of all vectors $m = (m_1, \cdots, m_k)$ whose components are natural numbers such that $1 \le m_i < m_j \le n$ for $1 \le i < j \le k$. More functions are proved to be (logarithmically) completely monotonic in more recent works [8, 9, 10, 15, 19, 26, 28, 68, 70, 79, 89].

In this and the next two chapters we present some function classes which are (logarithmically) completely monotonic.

## 4.1 Introduction and main results

Let $\alpha, r$ be real parameters, $\beta$ a non-negative parameter. Define

$$f_{\alpha,\beta,r}(x) := \left[\frac{e^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right]^r, \quad x \in \mathbb{R}^+. \tag{4.3}$$

Qi and Chen [27] proved the following

**Theorem 4.A.** *For $r > 0, \beta \geq 1$, if $\alpha \leq 1/2$, then $f_{\alpha,\beta,r} \in LCM(\mathbb{R}^+)$.*

But for the cases $r > 0, \beta \in (0,1)$ or $r < 0$, the result is unknown.

In this chapter, we mainly consider the case $r > 0, \beta \in (0,1)$(the case $r < 0$ will be considered in the next chapter). In view that $f \in LCM(\mathbb{R}^+)$ implies $(f)^r \in LCM(\mathbb{R}^+)$ for $r > 0$, we only need to consider the functions

$$f_{\alpha,\beta}(x) := \frac{e^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}}, \quad x \in \mathbb{R}^+. \tag{4.4}$$

First we give a necessary condition for $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$, then give some sufficient conditions for $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$ for $0 < \beta < 1$, and a necessary and sufficient condition for $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$ for $1/2 + \sqrt{3}/6 \leq \beta < \infty$. Finally we present the complete monotonicity property of a function class related to the functions $f_{\alpha,\beta}(x)$. The following are the main results of this chapter.

**Theorem 4.1.** *If $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$, then either $\beta > 0$ and $\alpha \leq \min(\beta, 1/2)$ or $\beta = 0$ and $\alpha \leq 1/2$.*

**Theorem 4.2.** *For $\beta \in [1/2, 1]$, if $\alpha \leq \min(3\beta^2 - 3\beta + 1, 1/2)$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

It is easy to verify that $f_{\alpha,0} = f_{\alpha,1}$,

$$3\beta^2 - 3\beta + 1 \leq 1/2 \quad \text{for} \quad \beta \in [1/2, 1/2 + \sqrt{3}/6],$$

and

$$1/2 \leq 3\beta^2 - 3\beta + 1 \quad \text{for} \quad \beta \in [1/2 + \sqrt{3}/6, 1].$$

Hence by Theorem 4.2, we get

**Corollary 4.3.**

1. *If $\beta \in [1/2, 1/2 + \sqrt{3}/6]$, and $\alpha \leq 3\beta^2 - 3\beta + 1$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

2. *If $\beta \in \{0\} \cup [1/2 + \sqrt{3}/6, 1]$, and $\alpha \leq 1/2$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

From Theorems 4.1 and 4.A, and Corollary 4.3, we have

**Corollary 4.4.** *Let $\beta \in \{0\} \cup [1/2 + \sqrt{3}/6, \infty)$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$ if and only if $\alpha \leq 1/2$.*

It is easy to show or see that

$$\beta - 1/3 \leq 3\beta^2 - 3\beta + 1 \quad \text{for} \quad \beta \in [1/2, 3/4] \subset [1/2, 1/2 + \sqrt{3}/6],$$

and that for $\beta \in (3/4, 1]$,

$$\beta - 1/2 \leq 3\beta^2 - 3\beta + 1, \beta - 1/2 \leq 1/2.$$

Then by Corollary 4.3(1) and Theorem 4.2, we obtain the following

**Corollary 4.5.**

1. *For $\beta \in [1/2, 3/4]$, if $\alpha \leq \beta - 1/3$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

2. *For $\beta \in (3/4, 1]$, if $\alpha \leq \beta - 1/2$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

**Theorem 4.6.** *For $\beta \in [0, 1/2)$, if $\alpha \leq \beta - e^{-4}(1-\beta)^2 \exp[2/(1-\beta)]$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

Since $h(1/4) > 0$ and $h(\beta)$ is strictly concave on [0,1/2](see Lemma 4.13), we have, for $\beta \in [1/4, 1/2]$,

$$\begin{aligned}\beta - \frac{1}{4} &= 1 \cdot (\beta - \frac{1}{2}) + \frac{1}{4} \\ &\leq \frac{1/4 - h(1/4)}{1/4}(\beta - \frac{1}{2}) + \frac{1}{4} \\ &\leq \beta - e^{-4}(1-\beta)^2 \exp[2/(1-\beta)].\end{aligned}$$

Then by Theorem 4.6, we obtain

**Corollary 4.7.** *For $\beta \in [1/4, 1/2]$, if $\alpha \leq \beta - 1/4$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

**Theorem 4.8.** *For any $\beta \in [0, \infty)$, if $\alpha \leq 0$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

We remark that for $\beta \in [0, \beta_0)$(see (4.9) for the value of $\beta_0$), Theorem 4.8 is stronger than Theorem 4.6 since $h(\beta) = \beta - e^{-4}(1-\beta)^2 \exp[2/(1-\beta)] < 0$ in $[0, \beta_0)$ (see Lemma 4.13 and remarks right after the proof of Lemma 4.13), that for $\beta \in (\beta_0, 1/2]$, Theorem 4.6 is stronger than Theorem 4.8, and that for $\beta \in [1/2, 1]$, Theorem 4.2 is stronger than Theorem 4.8.

The following theorems deal with the complete monotonicity property of the functions

$$g_{\alpha,\beta}(x) := x + \ln\Gamma(x+\beta) - (x+\beta-\alpha)\ln x, \quad x \in \mathbb{R}^+ \tag{4.5}$$

related to the functions $f_{\alpha,\beta}(x)$. Note that, in (4.5) $\alpha$ and $\beta$ are parameters such that $\alpha \in \mathbb{R}$ and $\beta \geq 0$.

**Theorem 4.9.** *If $g_{\alpha,\beta} \in CM(\mathbb{R}^+)$, then $\alpha = 1/2$.*

**Theorem 4.10.** *If $\alpha = 1/2, \beta \geq 1/2 + \sqrt{3}/6$ or $\beta = 0$, then $g_{\alpha,\beta} \in CM(\mathbb{R}^+)$.*

From Theorems 4.9 and 4.10, we obtain

**Corollary 4.11.** *Let $\beta \in \{0\} \cup [1/2 + \sqrt{3}/6, \infty)$, then $g_{\alpha,\beta} \in CM(\mathbb{R}^+)$ if and only if $\alpha = 1/2$.*

## 4.2 Lemmas

Theorem 3.C, 3.E and the following are needed to prove our main results.

**Lemma 4.12.** *The function $(e^{\theta x} - 1)/x$, where $\theta$ is a real parameter, is increasing on $\mathbb{R}^+$.*

This lemma can be verified by standard arguments. Its proof is omitted.

**Lemma 4.13.** *The function*

$$h(\beta) := \beta - e^{-4}(1-\beta)^2 \exp(\frac{2}{1-\beta}) \tag{4.6}$$

*is strictly increasing and strictly concave from* $[0, 1/2]$ *onto* $[-e^{-2}, 1/4]$.

*Proof.* Direct computations show that

$$h'(\beta) = 1 - 2e^{-4}\beta e^{\frac{2}{1-\beta}},$$
$$h''(\beta) = -2e^{-4}e^{\frac{2}{1-\beta}}\left[1 + \frac{2\beta}{(1-\beta)^2}\right].$$

Hence

$$h''(\beta) < 0, \quad \beta \in [0, 1). \tag{4.7}$$

Since $h'(1/2) = 0$, from (4.7), we get

$$h'(\beta) > 0, \quad \beta \in [0, 1/2). \tag{4.8}$$

By (4.8) and (4.7), $h(\beta)$ is strictly increasing and strictly concave on $[0, 1/2]$. The rest is obvious. The proof is complete. □

By Lemma 4.13, the function $h(\beta)$ has only one zero point in the interval $[0, 1/2]$. This zero point is denoted by $\beta_0$. By using Maple, we get

$$\beta_0 = 0.138537498\cdots. \tag{4.9}$$

## 4.3 Proofs of main results

*Proof of Theorem 4.1.* Suppose that $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$. Then

$$-[\ln f_{\alpha,\beta}(x)]' = \ln x - \psi(x+\beta) + (\beta-\alpha)/x \geq 0, \quad x \in \mathbb{R}^+, \tag{4.10}$$

from which we have

$$\beta - \alpha \geq x[\psi(x+\beta) - \ln x], \quad x \in \mathbb{R}^+. \tag{4.11}$$

If $\beta > 0$, then

$$\beta - \alpha \geq \lim_{x\to 0+} [x\psi(x+\beta) - x\ln x] = 0.$$

That is

$$\alpha \leq \beta. \tag{4.12}$$

By Theorem 3.E, for $\beta > 0$, from (4.11), we obtain

$$\begin{aligned}
\beta - \alpha &\geq \lim_{x\to\infty} x[\ln(x+\beta) - \frac{1}{2(x+\beta)} + O(\frac{1}{x^2}) - \ln x] \\
&= \lim_{x\to\infty} x\ln(1+\beta/x) - \frac{1}{2} \\
&= \beta \lim_{x\to\infty} \frac{x}{\beta}\ln(1+\frac{\beta}{x}) - \frac{1}{2} \\
&= \beta - \frac{1}{2},
\end{aligned}$$

from which we get

$$\alpha \leq \frac{1}{2}. \tag{4.13}$$

Combining (4.12) and (4.13) yields

$$\alpha \le \min(\beta, 1/2) \quad \text{if} \quad \beta > 0. \tag{4.14}$$

If $\beta = 0$, since

$$f_{\alpha,0}(x) = f_{\alpha,1}(x)$$

by (4.14), we have

$$\alpha \le \min(1, 1/2) = 1/2.$$

The proof is complete. □

*Proof of Theorem 4.2 and Theorem 4.6.* It is clear that

$$\ln f_{\alpha,\beta}(x) = x + \ln\Gamma(x+\beta) - (x+\beta-\alpha)\ln x, \quad \text{and}$$

$$[\ln f_{\alpha,\beta}(x)]' = \psi(x+\beta) - \ln x + \frac{\alpha-\beta}{x}.$$

By Theorem 3.C, for $n \ge 2$,

$$(-1)^n[\ln f_{\alpha,\beta}(x)]^{(n)} = (-1)^n\psi^{(n-1)}(x+\beta) - \frac{(n-2)!}{x^{n-1}} + \frac{(\beta-\alpha)(n-1)!}{x^n} \tag{4.15}$$

$$= \int_0^\infty \frac{t^{n-1}}{1-e^{-t}}e^{-(x+\beta)t}dt - \int_0^\infty t^{n-2}e^{-xt}dt + (\beta-\alpha)\int_0^\infty t^{n-1}e^{-xt}dt \tag{4.16}$$

$$= \int_0^\infty \delta(t)\frac{t^{n-2}e^{-xt}}{e^t-1}dt, \tag{4.17}$$

where

$$\delta(t) := (\beta-\alpha)t(e^t-1)+te^{(1-\beta)t}-e^t+1.$$

Direct computations show that

$$\delta'(t) = (\beta-\alpha)te^t+(1-\beta)te^{(1-\beta)t}+e^{(1-\beta)t}+(\beta-\alpha-1)e^t+\alpha-\beta,$$

$$\delta(0) = 0, \tag{4.18}$$

$$\delta'(0) = 0, \tag{4.19}$$

and

$$\delta''(t) = e^t g(t), \tag{4.20}$$

where

$$g(t) := 2\beta-2\alpha-1+(\beta-\alpha)t+2(1-\beta)e^{-\beta t}+(1-\beta)^2te^{-\beta t}. \tag{4.21}$$

Under the conditions of Theorem 4.2 and Theorem 4.6 and by Lemma 4.13,

$$g(0) = 1-2\alpha \geq 0. \tag{4.22}$$

From (4.21) we get

$$g'(t) = \beta-\alpha+(1-\beta)e^{-\beta t}[1-3\beta-\beta(1-\beta)t], \tag{4.23}$$

$$g''(t) = \beta(1-\beta)e^{-\beta t}[2(2\beta-1)+\beta(1-\beta)t]. \tag{4.24}$$

If $\alpha \leq 3\beta^2-3\beta+1$, then

$$g'(0) = 3\beta^2-3\beta+1-\alpha \geq 0. \tag{4.25}$$

If $1/2 \leq \beta \leq 1$, then, from (4.24),

$$g''(t) \geq 0. \tag{4.26}$$

By (4.25) and (4.26), we obtain

$$g'(t) \geq 0, \quad t \geq 0. \tag{4.27}$$

By (4.22) and (4.27), we have

$$g(t) \geq 0, \quad t \geq 0. \tag{4.28}$$

Hence under the conditions of Theorem 4.2, from (4.28) and (4.20),

$$\delta''(t) \geq 0, \quad t \geq 0. \tag{4.29}$$

Combining (4.29) and (4.19) gives

$$\delta'(t) \geq 0, \quad t \geq 0,$$

from which and (4.18), we obtain

$$\delta(t) \geq 0, \quad t \geq 0, \tag{4.30}$$

In view of (4.17), we have proved that under the conditions of Theorem 4.2, for $n \geq 2$,

$$(-1)^n [\ln f_{\alpha,\beta}(x)]^{(n)} \geq 0, \quad x \in \mathbb{R}^+. \tag{4.31}$$

Now if $0 < \beta < 1/2$, Let

$$t_0 := \frac{2(1-2\beta)}{\beta(1-\beta)},$$

from (4.24), it is clear that

$$g''(t) < 0 \quad \text{for} \quad t < t_0$$

and

$$g''(t) > 0 \quad \text{for} \quad t > t_0.$$

Therefore

$$g'(t) \geq g'(t_0), \quad t \geq 0. \tag{4.32}$$

A simple computation shows that

$$g'(t_0) = \beta - \alpha - e^{-4}(1-\beta)^2 \exp[2/(1-\beta)].$$

Hence, if $\alpha \leq \beta - e^{-4}(1-\beta)^2 \exp[2/(1-\beta)]$ and $0 < \beta < 1/2$, then

$$g'(t) \geq 0, \quad t \geq 0. \tag{4.33}$$

If $\beta = 0$, (4.33) is trivial. We have proved that, under the conditions of Theorem 4.6, (4.33) holds. Similar to the arguments from (4.27) to (4.30), by combining (4.33), (4.22), (4.20), (4.19) and (4.18), we obtain

$$\delta(t) \geq 0, \quad t \geq 0,$$

and also using (4.17), we see that, for $n \geq 2$, (4.31) also holds under the conditions of Theorem 4.6.

By using (3.11) of Theorem 3.E,

$$[\ln f_{\alpha,\beta}(x)]' = \ln(1+\frac{\beta}{x}) - \frac{1}{2(x+\beta)} + \frac{\alpha-\beta}{x} + O(\frac{1}{x^2}), \quad \text{as} \quad x \to \infty.$$

Thus for any parameters $\alpha$ and $\beta$

$$\lim_{x\to\infty} [\ln f_{\alpha,\beta}(x)]' = 0. \tag{4.34}$$

By (4.31)

$$[\ln f_{\alpha,\beta}(x)]'' \geq 0, \quad x \in \mathbb{R}^+. \tag{4.35}$$

From (4.34) and (4.35)

$$[\ln f_{\alpha,\beta}(x)]' \leq 0, \quad x \in \mathbb{R}^+.$$

Consequently, for all $n \in \mathbb{N}$, (4.31) holds.

The proof is complete. □

*Proof of Theorem 4.8.* From (4.16), we have, for $n \geq 2$,

$$(-1)^n[\ln f_{\alpha,\beta}(x)]^{(n)} = \int_0^\infty \left[\beta - \alpha + \frac{e^{-\beta t}}{1-e^{-t}} - \frac{1}{t}\right] t^{n-1}e^{-xt}dt \tag{4.36}$$

$$= \int_0^\infty \left[\beta - \alpha + \frac{1}{t}\left(e^{(1-\beta)t}\frac{t}{e^t-1} - 1\right)\right] t^{n-1}e^{-xt}dt. \tag{4.37}$$

It is easy to show that

$$\frac{t}{e^t - 1} > \frac{1}{e^t}, \quad t \in \mathbb{R}^+. \tag{4.38}$$

From (4.37) and (4.38),

$$(-1)^n[\ln f_{\alpha,\beta}(x)]^{(n)} \geq \int_0^\infty \left[\beta - \alpha + \frac{e^{-\beta t} - 1}{t}\right] t^{n-1}e^{-xt}dt.$$

By Lemma 4.12, $(e^{-\beta t} - 1)/t$ is increasing on $\mathbb{R}^+$. Hence, for $n \geq 2$, if $\alpha \leq 0$,

$$(-1)^n[\ln f_{\alpha,\beta}(x)]^{(n)} \geq -\alpha \int_0^\infty t^{n-1}e^{-xt}dt \geq 0. \tag{4.39}$$

From (4.34) and (4.39), we have

$$[\ln f_{\alpha,\beta}(x)]' \leq 0 \quad \text{if} \quad \alpha \leq 0.$$

The proof is complete. □

*Proof of Theorem 4.9.* First we note that $g_{\alpha,\beta}(x) = \ln f_{\alpha,\beta}(x)$.

If $g_{\alpha,\beta} \in CM(\mathbb{R}^+)$, then $f_{\alpha,\beta} \in LCM(\mathbb{R}^+)$. Hence, by Theorem 4.1, we obtain

$$\alpha \leq 1/2. \tag{4.40}$$

On the other hand, by (3.10) of Theorem 3.E,

$$g_{\alpha,\beta}(x) = \ln x(\alpha - 1/2) + \ln\sqrt{2\pi} + O(1/x), \quad \text{as} \quad x \to \infty. \tag{4.41}$$

Since $g_{\alpha,\beta}(x) \geq 0$, from (4.41), we get

$$\alpha - \frac{1}{2} \geq -\frac{\ln\sqrt{2\pi}}{\ln x} - \frac{O(1/x)}{\ln x}.$$

Thus

$$\alpha - \frac{1}{2} \geq \lim_{x\to\infty}\left[-\frac{\ln\sqrt{2\pi}}{\ln x} - \frac{O(1/x)}{\ln x}\right] = 0.$$

That is

$$\alpha \geq \frac{1}{2}. \tag{4.42}$$

By (4.40) and (4.42),

$$\alpha = \frac{1}{2}.$$

The proof is complete. □

*Proof of Theorem 4.10.* Since $g_{\alpha,\beta}(x) = \ln f_{\alpha,\beta}(x)$, by Corollary 4.4, we obtain, for $n \in \mathbb{N}$,

$$(-1)^n g^{(n)}_{\alpha,\beta}(x) \geq 0, \quad x \in \mathbb{R}^+. \tag{4.43}$$

In particular,

$$g'_{\alpha,\beta}(x) \leq 0, \quad x \in \mathbb{R}^+,$$

which means that $g_{\alpha,\beta}(x)$ is decreasing on $\mathbb{R}^+$. By (3.10) of Theorem 3.E, for $\alpha = 1/2$,

$$g_{\alpha,\beta}(x) = \ln\sqrt{2\pi} + O(1/x), \quad \text{as} \quad x \to \infty.$$

Hence

$$g_{\alpha,\beta}(x) \geq \lim_{x\to\infty} g_{\alpha,\beta}(x) = \ln\sqrt{2\pi},$$

from which we see (4.43) is also valid for $n = 0$.

The proof is complete. $\square$

# Chapter 5

# Some classes of logarithmically completely monotonic functions involving the gamma function II

## 5.1 Introduction and main results

In this chapter, we consider the logarithmically complete monotonicity property of the functions $f_{\alpha,\beta,r}(x)$, defined by (4.3), for the case $r < 0$, or equivalently consider the functions

$$g_{\alpha,\beta,r}(x) := \left[\frac{x^{x+\beta-\alpha}}{e^x\Gamma(x+\beta)}\right]^r, \quad x \in \mathbb{R}^+.$$

for $r > 0$.

We only need to consider the functions

$$g_{\alpha,\beta}(x) := \frac{x^{x+\beta-\alpha}}{e^x\Gamma(x+\beta)}, \quad x \in \mathbb{R}^+. \tag{5.1}$$

Here, in (5.1), $\alpha \in \mathbb{R}, \beta \geq 0$ are parameters.

The main results of this chapter are as follows.

**Theorem 5.1.** *If $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$, then either $\beta > 0$ and $\alpha \geq \max(\beta, 1/2)$ or $\beta = 0$ and $\alpha \geq 1$.*

**Theorem 5.2.** *For $\beta \geq 0$, if $\alpha \geq \max(1/2, \beta, 3\beta^2 - 3\beta + 1)$, then $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

**Corollary 5.3.** *$g_{\alpha,0} = g_{\alpha,1} \in LCM(\mathbb{R}^+)$ if and only if $\alpha \geq 1$*

For $\beta \geq 1/2$, we have a simple result.

**Theorem 5.4.** *For $\beta \geq 1/2$, if $\alpha \geq \beta$, then $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.*

For $\beta \in [1/2, 1]$, Theorem 5.2 and Theorem 5.4 are the same since $\beta = \max(1/2, \beta, 3\beta^2 - 3\beta + 1)$. For $\beta > 1$, Theorem 5.4 is stronger than Theorem 5.2 since $\beta < \max(1/2, \beta, 3\beta^2 - 3\beta + 1) = 3\beta^2 - 3\beta + 1$.

**Theorem 5.5.** *let $\beta \in [1/2 - \sqrt{3}/6, \infty)$, then $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$ if and only if $\alpha \geq \max(1/2, \beta)$.*

**Corollary 5.6.**

1. *Let* $\beta \in [1/2-\sqrt{3}/6, 1/2]$, *then* $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$ *if and only if* $\alpha \geq 1/2$.

2. *Let* $\beta \geq 1/2$, *then* $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$ *if and only if* $\alpha \geq \beta$.

The following theorems deal with the complete monotonicity property of the functions

$$h_{a,b,c}(x) := (x+b-a)\ln x - x - \ln\Gamma(x+b) + c, \quad x \in \mathbb{R}^+, \tag{5.2}$$

related to the functions $g_{\alpha,\beta}(x)$. Here in (5.2) $a, b \in \mathbb{R}, b \geq 0$ are parameters.

**Theorem 5.7.** *If* $h_{a,b,c} \in CM(\mathbb{R}^+)$, *then* $a = 1/2, 0 < b \leq 1/2$ *and* $c \geq \ln\sqrt{2\pi}$.

**Theorem 5.8.** *If* $a = 1/2, b \in [1/2-\sqrt{3}/6, 1/2]$ *and* $c \geq \ln\sqrt{2\pi}$, *then* $h_{a,b,c} \in CM(\mathbb{R}^+)$.

From Theorems 5.7 and 5.8, we get

**Corollary 5.9.** *Let* $b \in [1/2-\sqrt{3}/6, 1/2]$, *then* $h_{a,b,c} \in CM(\mathbb{R}^+)$ *if and only if* $a = 1/2$ *and* $c \geq \ln\sqrt{2\pi}$.

## 5.2 Proofs of main results

*Proof of Theorem 5.1.* Suppose that $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$. Then

$$-[\ln g_{\alpha,\beta}(x)]' = \psi(x+\beta) - \ln x - (\beta-\alpha)/x \geq 0, \quad x \in \mathbb{R}^+,$$

from which we have

$$\beta - \alpha \le x[\psi(x+\beta) - \ln x], \quad x \in \mathbb{R}^+. \tag{5.3}$$

If $\beta > 0$, then

$$\beta - \alpha \le \lim_{x \to 0+} [x\psi(x+\beta) - x \ln x] = 0.$$

That is

$$\alpha \ge \beta. \tag{5.4}$$

By Theorem 3.E, for $\beta > 0$, from (5.3), we obtain

$$\begin{aligned}
\beta - \alpha &\le \lim_{x\to\infty} x[\ln(x+\beta) - \frac{1}{2(x+\beta)} + O(\frac{1}{x^2}) - \ln x] \\
&= \lim_{x\to\infty} x \ln(1 + \beta/x) - \frac{1}{2} \\
&= \beta \lim_{x\to\infty} \frac{x}{\beta} \ln(1 + \frac{\beta}{x}) - \frac{1}{2} \\
&= \beta - \frac{1}{2},
\end{aligned}$$

from which we get

$$\alpha \ge \frac{1}{2}. \tag{5.5}$$

Combining (5.4) and (5.5) yields

$$\alpha \ge \max(\beta, 1/2) \quad \text{if} \quad \beta > 0. \tag{5.6}$$

If $\beta = 0$, since

$$g_{\alpha,0}(x) = g_{\alpha,1}(x)$$

by (5.6), we have

$$\alpha \geq \max(1, 1/2) = 1.$$

The proof is complete. □

*Proof of Theorem 5.2.* It is clear that

$$\ln g_{\alpha,\beta}(x) = (x+\beta-\alpha)\ln x - x - \ln\Gamma(x+\beta), \quad \text{and}$$
$$[\ln g_{\alpha,\beta}(x)]' = \ln x - \psi(x+\beta) - \frac{\alpha-\beta}{x}.$$

By Theorem 3.C, for $n \geq 2$,

$$(-1)^n[\ln g_{\alpha,\beta}(x)]^{(n)} = \frac{(n-2)!}{x^{n-1}} - (-1)^n\psi^{(n-1)}(x+\beta) - \frac{(\beta-\alpha)(n-1)!}{x^n} \tag{5.7}$$

$$= \int_0^\infty t^{n-2}e^{-xt}dt - \int_0^\infty \frac{t^{n-1}}{1-e^{-t}}e^{-(x+\beta)t}dt - (\beta-\alpha)\int_0^\infty t^{n-1}e^{-xt}dt \tag{5.8}$$

$$= \int_0^\infty \delta(t)\frac{t^{n-2}e^{-xt}}{e^t-1}dt, \tag{5.9}$$

where

$$\delta(t) := (\alpha-\beta)t(e^t-1) - te^{(1-\beta)t} + e^t - 1.$$

Direct computations show that

$$\delta'(t) = (\alpha-\beta)te^t + (\beta-1)te^{(1-\beta)t} - e^{(1-\beta)t} + (1+\alpha-\beta)e^t - \alpha + \beta,$$

$$\delta(0) = 0, \tag{5.10}$$

$$\delta'(0) = 0, \tag{5.11}$$

and

$$\delta''(t) = e^t g(t), \tag{5.12}$$

where

$$g(t) := 1 + 2\alpha - 2\beta + (\alpha - \beta)t + 2(\beta - 1)e^{-\beta t} - (1 - \beta)^2 te^{-\beta t}. \tag{5.13}$$

If $\beta = 0$ or $\beta = 1$, then

$$g(t) = 2\alpha - 1 + (\alpha - 1)t.$$

Hence under the condition of Theorem 5.2,

$$g(t) \geq 0, \quad t \geq 0 \quad \text{for} \quad \beta = 0 \quad \text{or} \quad \beta = 1. \tag{5.14}$$

Now let $\beta \neq 0$ and $\beta \neq 1$.

From (5.13), we get

$$g(0) = 2\alpha - 1. \tag{5.15}$$

and

$$g'(t) = \alpha - \beta + (\beta - 1)e^{-\beta t}[1 - 3\beta - \beta(1 - \beta)t]. \tag{5.16}$$

Under the condition of Theorem 5.2, from (5.15),

$$g(0) \geq 0. \tag{5.17}$$

From (5.16),

$$g'(0) = \alpha - 3\beta^2 + 3\beta - 1,$$

$$g'(\infty) = \lim_{t\to\infty} g'(t) = \alpha - \beta, \quad \text{and}$$

$$g''(t) = \beta e^{-\beta t}[2(1-2\beta)(1-\beta) - \beta(1-\beta)^2 t]. \tag{5.18}$$

Under the condition of Theorem 5.2,

$$g'(0) \geq 0, \tag{5.19}$$

and

$$g'(\infty) \geq 0. \tag{5.20}$$

*i)* If $(1-2\beta)(1-\beta) \leq 0$, from (5.18), we have

$$g''(t) \leq 0, \quad t \in \mathbb{R}^+. \tag{5.21}$$

By (5.20) and (5.21),

$$g'(t) \geq g'(\infty) \geq 0, \quad t \in \mathbb{R}^+. \tag{5.22}$$

By (5.17) and (5.22),

$$g(t) \geq 0, \quad t \geq 0.$$

*ii)* If $(1-2\beta)(1-\beta) > 0$, from (5.18),

$$g''(t) \begin{cases} > 0 & \text{if} \quad 0 < t < t_0; \\ \leq 0 & \text{if} \quad t \geq t_0, \end{cases} \tag{5.23}$$

where

$$t_0 := \frac{2(1-2\beta)(1-\beta)}{\beta(1-\beta)^2} > 0.$$

By (5.19), (5.20) and (5.23),

$$g'(t) \geq \min(g'(0), g'(\infty)) \geq 0, \quad t \in \mathbb{R}^+. \tag{5.24}$$

From (5.17) and (5.24),

$$g(t) \geq 0, \quad t \geq 0.$$

From the arguments of *i*) and *ii*) above, we have proved that under the condition of Theorem 5.2

$$g(t) \geq 0, \quad t \geq 0 \quad \text{for} \quad 0 < \beta \neq 1. \tag{5.25}$$

Combining (5.14) and (5.25) yields

$$g(t) \geq 0, \quad t \geq 0 \quad \text{for all} \quad \beta \geq 0. \tag{5.26}$$

Hence under the condition of Theorem 5.2, from (5.26) and (5.12),

$$\delta''(t) \geq 0, \quad t \geq 0. \tag{5.27}$$

Combining (5.27) and (5.11) gives

$$\delta'(t) \geq 0, \quad t \geq 0,$$

from which and (5.10), we obtain

$$\delta(t) \geq 0, \quad t \geq 0,$$

In view of (5.9), we have proved that under the condition of Theorem 5.2, for $n \geq 2$,

$$(-1)^n[\ln g_{\alpha,\beta}(x)]^{(n)} \geq 0, \quad x \in \mathbb{R}^+. \tag{5.28}$$

By using (3.11) of Theorem 3.E,

$$[\ln g_{\alpha,\beta}(x)]' = \ln(1+\frac{\beta}{x}) - \frac{1}{2(x+\beta)} + \frac{\alpha-\beta}{x} + O(\frac{1}{x^2}), \quad \text{as} \quad x \to \infty.$$

Thus for any parameters $\alpha$ and $\beta$

$$\lim_{x\to\infty}[\ln g_{\alpha,\beta}(x)]' = 0. \tag{5.29}$$

By (5.28)

$$[\ln g_{\alpha,\beta}(x)]'' \geq 0, \quad x \in \mathbb{R}^+. \tag{5.30}$$

From (5.29) and (5.30)

$$[\ln g_{\alpha,\beta}(x)]' \leq 0, \quad x \in \mathbb{R}^+.$$

Consequently, for all $n \in \mathbb{N}$, (5.28) hold.

The proof is complete. □

*Proof of Theorem 5.4.* From (5.8), we have, for $n \geq 2$,

$$\begin{aligned}(-1)^n[\ln g_{\alpha,\beta}(x)]^{(n)} &= \int_0^\infty \left[\alpha - \beta - \frac{e^{-\beta t}}{1-e^{-t}} + \frac{1}{t}\right] t^{n-1}e^{-xt}dt \\ &= \int_0^\infty \left[\alpha - \beta - \frac{1}{t}\left(e^{(1-\beta)t}\frac{t}{e^t-1} - 1\right)\right] t^{n-1}e^{-xt}dt. \end{aligned} \tag{5.31}$$

It is easy to show that

$$\frac{t}{e^t - 1} < \frac{1}{e^{t/2}}, \quad t \in \mathbb{R}^+. \tag{5.32}$$

From (5.31) and (5.32),

$$(-1)^n [\ln g_{\alpha,\beta}(x)]^{(n)} \geq \int_0^\infty \left[\alpha - \beta - \frac{e^{(1/2-\beta)t} - 1}{t}\right] t^{n-1} e^{-xt} dt.$$

By Lemma 4.12, $(e^{(1/2-\beta)t} - 1)/t$ is increasing on $\mathbb{R}^+$. Hence, for $n \geq 2$, if $\alpha \geq \beta \geq 1/2$,

$$(-1)^n [\ln g_{\alpha,\beta}(x)]^{(n)} \geq (\alpha - \beta) \int_0^\infty t^{n-1} e^{-xt} dt \geq 0. \tag{5.33}$$

From (5.29) and (5.33), we have

$$[\ln g_{\alpha,\beta}(x)]' \leq 0 \quad \text{if} \quad \alpha \geq \beta \geq 1/2.$$

Hence (5.33) is valid for all $n \in \mathbb{N}$.

The proof is complete. □

*Proof of Theorem 5.5.* By Theorem 5.1, the condition is necessary. On the other hand, if $\beta \in [1/2-\sqrt{3}/6, 1/2]$, since $\max(1/2, \beta) = 1/2 = \max(1/2, \beta, 3\beta^2 - 3\beta + 1)$, by Theorem 5.2, we have $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$; if $\beta > 1/2$, since $\max(1/2, \beta) = \beta$, by Theorem 5.4, we get $g_{\alpha,\beta} \in LCM(\mathbb{R}^+)$.

The proof is complete. □

*Proof of Theorem 5.7.* First, we note that

$$h_{a,b,c}(x) = \ln g_{a,b}(x) + c. \tag{5.34}$$

If $h_{a,b,c} \in CM(\mathbb{R}^+)$, then

$$h_{a,b,c}(x) \geq 0, \quad x \in \mathbb{R}^+, \tag{5.35}$$

is decreasing and $g_{a,b} \in LCM(\mathbb{R}^+)$. By (3.10) of Theorem 3.E,

$$h_{a,b,c}(x) = (1/2 - a)\ln x - \ln\sqrt{2\pi} + O(1/x) + c, \quad \text{as} \quad x \to \infty. \tag{5.36}$$

From (5.35) and (5.36), we get

$$\frac{1}{2} - a \geq \frac{\ln\sqrt{2\pi} - c + O(1/x)}{\ln x} \to 0, \quad \text{as} \quad x \to \infty,$$

which means that

$$a \leq \frac{1}{2}. \tag{5.37}$$

On the other hand, since $h_{a,b,c}(x)$ is decreasing on $\mathbb{R}^+$, from (5.36), we have

$$(1/2 - a)\ln x - \ln\sqrt{2\pi} + O(1/x) + c \leq h_{a,b,c}(1)$$

or

$$1/2 - a \leq \frac{\ln\sqrt{2\pi} + O(1/x) - c + h_{a,b,c}(1)}{\ln x} \to 0, \quad \text{as} \quad x \to \infty,$$

from which

$$a \geq \frac{1}{2}. \tag{5.38}$$

Combining (5.37) and (5.38) gives

$$a = \frac{1}{2}. \tag{5.39}$$

From (5.35), (5.36) and (5.39), we obtain

$$c - \ln\sqrt{2\pi} \geq O(1/x) \to 0, \quad \text{as} \quad x \to \infty.$$

That is

$$c \geq \ln\sqrt{2\pi}.$$

Since $g_{a,b} \in LCM(\mathbb{R}^+)$, by Corollary 5.6(2), if $b > 1/2$, then $a \geq b > 1/2$, which is a contradiction to (5.39); by Corollary 5.3, if $b = 0$, then $a \geq 1$; another contradiction to (5.39). So we have proved that

$$0 < b \leq \frac{1}{2}.$$

The proof is complete. $\square$

*Proof of Theorem 5.8.* By Corollary 5.6(1), $g_{a,b} \in LCM(\mathbb{R}^+)$. Then from (5.34), for $n \in \mathbb{N}, c \in \mathbb{R}$,

$$(-1)^n h^{(n)}_{a,b,c}(x) \geq 0, \quad x \in \mathbb{R}^+. \tag{5.40}$$

In particular,

$$h'_{a,b,c}(x) \leq 0, \quad x \in \mathbb{R}^+.$$

Hence $h_{a,b,c}(x)$ is decreasing on $\mathbb{R}^+$. By (3.10) of Theorem 3.E,

$$h_{a,b,c}(x) = (1/2 - a)\ln x + c - \ln\sqrt{2\pi} + O(1/x), \quad \text{as} \quad x \to \infty. \tag{5.41}$$

If $a = 1/2, c \geq \ln\sqrt{2\pi}$, from (5.41),

$$\lim_{x\to\infty} h_{a,b,c}(x) = c - \ln\sqrt{2\pi} \geq 0.$$

Therefore,

$$h_{a,b,c}(x) \geq \lim_{x\to\infty} h_{a,b,c}(x) \geq 0, \quad x \in \mathbb{R}^+, \tag{5.42}$$

which means that (5.40) is also valid for $n = 0$.

The proof is complete. $\square$

# Chapter 6

# Some classes of logarithmically completely monotonic functions involving the gamma function III

## 6.1 Introduction and main results

In this chapter, we give some necessary and sufficient conditions for some classes of functions involving the gamma function to be (logarithmically) completely monotonic.

For simplicity, we denote the interval $(max(0, -\beta), \infty)$ by $I_\beta$ for $\beta \in \mathbb{R}$.

One of the main results of [44] is

**Theorem 6.A.** *Let $\alpha \neq 0, \beta$ be real parameters. Then the function*

$$g_{\alpha,\beta}(x) := \left[\frac{e^x\Gamma(x+1)}{(x+\beta)^{x+\beta}}\right]^\alpha, \quad x \in I_\beta$$

*is in $LCM(I_\beta)$ if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq 1/2$.*

We shall further prove that the condition in Theorem 6.A is also necessary.

**Theorem 6.1.** *Let $\alpha \neq 0, \beta$ be real parameters. Then the function*

$$g_{\alpha,\beta}(x) := \left[\frac{e^x\Gamma(x+1)}{(x+\beta)^{x+\beta}}\right]^\alpha, \quad x \in I_\beta$$

*is in $LCM(I_\beta)$ if and only if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq 1/2$.*

The following result deals with the complete monotonicity property of a function class related to the function $g_{\alpha,\beta}(x)$.

**Theorem 6.2.** *Let $a, b, c \in \mathbb{R}$ be free parameters, then the function*

$$G_{a,b,c}(x) := a[(x+b)\ln(x+b) - x - \ln\Gamma(x+1)] + c, \quad x \in I_b$$

*is in $CM(I_b)$ if and only if $a > 0, b = 1/2$ and $c \geq \frac{a}{2}\ln\frac{2\pi}{e}$.*

## 6.2 Proofs of the main results

**Lemma 6.3.** *The function*

$$\varphi(t) := \frac{1}{t}\ln\frac{e^t - 1}{t} \tag{6.1}$$

*is strictly increasing from $\mathbb{R}^+$ onto $(1/2, 1)$.*

*Proof of Theorem 6.1.*

In view of Theorem 6.A, we only need to prove the necessity.

Suppose that $g_{\alpha,\beta}(x) \in LCM(I_\beta)$. Then $\ln g_{\alpha,\beta}(x)$ is decreasing on $I_\beta$. Hence

$$\lim_{x\to\infty} \ln g_{\alpha,\beta}(x) < \infty. \tag{6.2}$$

By (3.10), we can show that

$$\lim_{x\to\infty} \ln g_{\alpha,\beta}(x) = \infty$$

if either $\alpha < 0$ and $\beta > 1/2$ or $\alpha > 0$ and $\beta < 1/2$. In view of (6.2), we have proved that if $\alpha < 0$, then $\beta \le 1/2$ and that if $\alpha > 0$, then $\beta \ge 1/2$. Now we need to show that if $\alpha > 0$, then $\beta \ge 1$ (although we already know that $\beta \ge 1/2$). When $\alpha > 0$, $I_\beta = \mathbb{R}^+$ since $\beta \ge 1/2$. For $n \ge 2$, by hypothesis,

$$(-1)^n[\ln g_{\alpha,\beta}(x)]^{(n)} \ge 0, \quad x \in \mathbb{R}^+. \tag{6.3}$$

By Theorem 3.I, for $n \ge 2, x > 0$,

$$\begin{aligned}(-1)^n[\ln g_{\alpha,\beta}(x)]^{(n)} &= (-1)^n\alpha\left[\psi^{(n-1)}(x+1) - \frac{(-1)^{n-2}(n-2)!}{(x+\beta)^{n-1}}\right]\\ &= (-1)^n\alpha\left[(-1)^n(n-1)!\zeta(n,x+1) - \frac{(-1)^{n-2}(n-2)!}{(x+\beta)^{n-1}}\right]\end{aligned}$$

$$= \alpha(n-2)!\left[(n-1)\zeta(n,x+1) - \frac{1}{(x+\beta)^{n-1}}\right]. \tag{6.4}$$

Combining (6.3) and (6.4) yields that if $\alpha > 0$, for $n \geq 2$,

$$(n-1)\zeta(n, x+1) - \frac{1}{(x+\beta)^{n-1}} \geq 0, \quad x \in \mathbb{R}^+. \tag{6.5}$$

By taking the limit of both sides of (6.5) as $x \to 0+$, we obtain

$$(n-1)\zeta(n, 1) - \frac{1}{\beta^{n-1}} \geq 0, \quad n \geq 2.$$

That is

$$(n-1)\zeta(n) \geq \frac{1}{\beta^{n-1}}, \quad n \geq 2,$$

from which we have if $\alpha > 0$,

$$\beta \geq \frac{1}{\sqrt[n-1]{n-1}} \frac{1}{\sqrt[n-1]{\zeta(n)}}, \quad n \geq 2, \tag{6.6}$$

Since $\zeta(n)$ is decreasing and greater than 1 for $n \geq 2$,

$$1 \leq \lim_{n \to \infty} \zeta(n) < \infty.$$

Therefore

$$\lim_{n \to \infty} \sqrt[n-1]{\zeta(n)} = 1. \tag{6.7}$$

By using H'Lôpital's rule,

$$\lim_{n \to \infty} \sqrt[n-1]{n-1} = 1. \tag{6.8}$$

From (6.6), (6.7) and (6.8), we conclude that if $\alpha > 0$, then $\beta \geq 1$.

The proof is complete. $\square$

*Proof of Theorem 6.2.* We observe that

$$G_{a,b,c}(x) = \ln g_{-a,b}(x) + c, \quad x \in I_b.$$

Suppose that $a > 0, b = 1/2$ and $c \geq \frac{a}{2} \ln \frac{2\pi}{e}$. Since for $n \in \mathbb{N}$

$$G^{(n)}_{a,b,c}(x) = [\ln g_{-a,b}(x)]^{(n)}, \quad x \in I_b,$$

by Theorem 6.1,

$$(-1)^n G^{(n)}_{a,b,c}(x) \geq 0, \quad n \in \mathbb{N}, x \in I_b, \tag{6.9}$$

from which we see $G_{a,b,c}(x)$ is decreasing on $I_b$. From (6.2), by using (3.10), we have

$$\lim_{x \to \infty} G_{a,b,c}(x) = -a \ln \sqrt{2\pi/e} + c.$$

By hypothesis $c \geq \frac{a}{2} \ln \frac{2\pi}{e}$, thus

$$\lim_{x \to \infty} G_{a,b,c}(x) \geq 0.$$

And then

$$G_{a,b,c}(x) \geq 0, \quad x \in I_b. \tag{6.10}$$

(6.9) together with (6.10) means that $G_{a,b,c}(x)$ is completely monotonic. On the other hand, if $G_{a,b,c}(x)$ is completely monotonic, then $g_{-a,b}(x)$ is logarithmically completely monotonic and $G_{a,b,c}(x)$ is nonnegative and decreasing on $I_b$. Therefore

$$0 \leq \lim_{x \to \infty} G_{a,b,c}(x) < \infty. \tag{6.11}$$

Since $g_{-a,b}(x)$ is logarithmically completely monotonic, by Theorem 6.1, we know that either $a < 0$ and $b \geq 1$ or $a > 0$ and $b \leq 1/2$. By using (3.10), we can show that if either $a < 0$ and $b > 1/2$ or $a > 0$ and $b < 1/2$, then

$$\lim_{x\to\infty} G_{a,b,c}(x) = -\infty,$$

which is impossible compared with the fact of (6.11). So we have proved that $a > 0$ and $b = 1/2$. Furthermore, if $b = 1/2$, then, by using (3.10),

$$\lim_{x\to\infty} G_{a,b,c}(x) = -a \ln \sqrt{2\pi/e} + c. \tag{6.12}$$

From (6.12) and (6.11), we obtain $c \geq \frac{a}{2} \ln \frac{2\pi}{e}$.

The proof is complete. $\square$

# Chapter 7

# A class of completely monotonic functions related to the remainder of Binet's formula with applications

## 7.1 Introduction and main results

The noted Binet's formula (see Theorem 3.A) states that for $x \in \mathbb{R}^+$,

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x), \tag{7.1}$$

where

$$\theta(x) := \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \frac{e^{-xt}}{t} dt \tag{7.2}$$

is called the remainder of Binet's formula (7.1).

Let $p \in \mathbb{R}^+$ and $q, r \in \mathbb{R}$ be parameters, and let

$$f_{p,q,r}(x) := r[\theta(px) - q\theta(x)], \quad x \in \mathbb{R}^+. \tag{7.3}$$

In this chapter, we give a necessary condition and some sufficient conditions for $f_{p,q,r}(x)$ to be completely monotonic, and discuss some applications as well.

**Theorem 7.1.** *Suppose that $r \neq 0$ and $f_{p,q,r}(x)$, defined by (7.3), belong to $CM(\mathbb{R}^+)$, then either $r > 0$ and $(p, q) \in D_1$ or $r < 0$ and $(p, q) \in D_2$, where $D_1 := (0, 1] \times (-\infty, 1] \cup (1, \infty) \times (-\infty, 1)$ and $D_2 := (0, 1) \times (1, \infty) \cup [1, \infty) \times [1, \infty)$*

**Theorem 7.2.** *If $r > 0, q \leq min(1, 1/p)$ or $r < 0, q \geq max(1, 1/p)$, then $f_{p,q,r}(x) \in CM(\mathbb{R}^+)$.*

A function $f$ is said to be star-shaped (see [24]) on $\mathbb{R}^+$ if for all $0 < \alpha < 1$

$$f(\alpha x) \leq \alpha f(x), \quad x \in \mathbb{R}^+.$$

Also recall that a function $f$ is superadditive on $\mathbb{R}^+$ if for all $x, y > 0$

$$f(x + y) \geq f(x) + f(y).$$

If $-f$ is superadditive, then $f$ is said to be subadditive. It is known that a star-shaped function is superadditive(see [24],[66, p. 453]). One of the applications of Theorem 7.2 is the following result.

**Corollary 7.3.**

1. *The function $-\theta(x)$ is star-shaped on $\mathbb{R}^+$.*

2. *The function $\theta(x)$ is subadditive on $\mathbb{R}^+$.*

## 7.2 Lemmas

The following lemmas are needed to prove our results.

**Lemma 7.4.**

1. *The function*

$$\delta(t) := \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}, \quad t \in \mathbb{R}^+ \tag{7.4}$$

*is strictly increasing onto (0,1/2).*

2. *The derivative of $\delta(t)$*

$$\delta'(t) = \frac{1}{t^2} - \frac{1}{(e^t - 1)^2}, \quad t \in \mathbb{R}^+$$

*is strictly increasing onto (0,1/12).*

**Lemma 7.5.** *Let $\delta(t)$ be the function defined by* (7.4), *then*

*i) for* $0 < \alpha < 1$,

$$\alpha \cdot \delta(t) < \delta(\alpha t) < 1 \cdot \delta(t), \quad t \in \mathbb{R}^+. \tag{7.5}$$

*The constants* $\alpha$ *and* 1 *in* (7.5) *are best possible.*

*ii) for* $\alpha > 1$,

$$1 \cdot \delta(t) < \delta(\alpha t) < \alpha \cdot \delta(t), \quad t \in \mathbb{R}^+. \tag{7.6}$$

*The constants* 1 *and* $\alpha$ *in* (7.6) *are best possible.*

*Proof.* Suppose that $0 < \alpha < 1$. By lemma 7.4(1), we have

$$\delta(\alpha t) < \delta(t), \quad t \in \mathbb{R}^+. \tag{7.7}$$

Now let

$$g(t) := \delta(\alpha t) - \alpha\delta(t), \quad t \in \mathbb{R}^+.$$

By Lemma 7.4(1),

$$\lim_{t \to 0+} g(t) = 0. \tag{7.8}$$

Since

$$g'(t) = \alpha[\delta'(\alpha t) - \delta'(t)], \quad t \in \mathbb{R}^+,$$

by Lemma 7.4(2), we obtain

$$g'(t) > 0, \quad t \in \mathbb{R}^+,$$

which means that $g(t)$ is strictly increasing. Then from (7.8), we have

$$g(t) = \delta(\alpha t) - \alpha\delta(t) > 0, \quad t \in \mathbb{R}^+. \tag{7.9}$$

Combining (7.7) and (7.9) yields (7.5). It is easy to see that

$$\lim_{t\to\infty}\frac{\delta(\alpha t)}{\delta(t)}=1. \tag{7.10}$$

From Lemma 7.4(1) and Lemma 7.4(2), by using L'Hôpital's rule, we obtain

$$\lim_{t\to 0+}\frac{\delta(\alpha t)}{\delta(t)}=\lim_{t\to 0+}\frac{\alpha\delta'(\alpha t)}{\delta'(t)}=\alpha. \tag{7.11}$$

From (7.10) and (7.11), we known that the constants $\alpha$ and 1 in (7.5) are best possible.

The proof of part *ii*) is similar to that of part *i*).

The proof is complete. □

## 7.3 Proofs of the main results

*Proof of Theorem 7.1.* From (7.1) and (7.3), after some algebra, we obtain

$$-f'_{p,q,r}(x)=r[q\psi(x)-p\psi(px)+\frac{q-1}{2x}+(p-q)\ln x+p\ln p].$$

By using Theorem 3.D, we have, for $x\in\mathbb{R}^+$,

$$-f'_{p,q,r}(x)=r[p\ln p+(p-q)\gamma+\frac{1-q}{2x}+(p-q)\ln x+\omega(x)], \tag{7.12}$$

where

$$\omega(x):=qx\sum_{n=1}^{\infty}\frac{1}{n(n+x)}-p^2x\sum_{n=1}^{\infty}\frac{1}{n(n+px)}.$$

Suppose that $f_{p,q,r}(x) \in CM(\mathbb{R}^+)$, then $-f'_{p,q,r}(x)$ is nonnegative and decreasing on $\mathbb{R}^+$. Hence

$$0 \le \lim_{x\to 0+}[-f'_{p,q,r}(x)] \le \infty. \tag{7.13}$$

Since

$$\lim_{x\to 0+}\omega(x) = 0,$$

from (7.12),

$$\lim_{x\to 0+}[-f'_{p,q,r}(x)] = r[p\ln p + (p-q)\gamma] + \lim_{x\to 0+}\frac{1}{x}\left[\frac{r(1-q)}{2} + r(p-q)x\ln x\right]. \tag{7.14}$$

From (7.13) and (7.14), we conclude that either $r(1-q) > 0$ or $r(1-q) = 0$ and $r(p-q) \le 0$. From this result, we can get the conclusion of Theorem 7.1.

The proof is complete. $\square$

*Proof of Theorem 7.2.* Let $\delta(t)$ be the function defined by (7.4). By Lemma 7.4(1), we have

$$\delta(t) > 0, \quad t \in \mathbb{R}^+. \tag{7.15}$$

By formula (7.2), for $x \in \mathbb{R}^+$,

$$\begin{aligned} f_{p,q,r}(x) &= r\left(\int_0^\infty \frac{\delta(u)}{u} e^{-pxu} du - q\int_0^\infty \frac{\delta(t)}{t} e^{-xt} dt\right) \\ &= r\left(\int_0^\infty \frac{\delta(t/p)}{t} e^{-xt} dt - q\int_0^\infty \frac{\delta(t)}{t} e^{-xt} dt\right) \\ &= \int_0^\infty k(p,q,r;t)\frac{e^{-xt}}{t} dt, \end{aligned}$$

where

$$k(p,q,r;t) := r[\delta(t/p) - q\delta(t)], \quad t \in \mathbb{R}^+. \tag{7.16}$$

By (7.15) and Lemma 7.5, we conclude that

$$k(p,q,r;t) > 0, \quad t \in \mathbb{R}^+ \tag{7.17}$$

if $p, q$ and $r$ satisfy one of the following conditions:

*i)* $r > 0, 0 < p < 1$ and $q \leq 1$;

*ii)* $r > 0, p > 1$ and $q \leq 1/p$;

*iii)* $r < 0, 0 < p < 1$ and $q \geq 1/p$;

*iv)* $r < 0, p > 1$ and $q \geq 1$.

If $p = 1$, then, from (7.16),

$$k(p,q,r;t) = r(1-q)\delta(t).$$

Since $\delta(t) > 0$, we get

$$k(p,q,r;t) \geq 0, \quad t \in \mathbb{R}^+ \tag{7.18}$$

if $r > 0, p = 1, q \leq 1$ or $r < 0, p = 1, q \geq 1$. Combining results (7.17) and (7.18), we see that if

$$r > 0, q \leq min(1, 1/p)$$

or

$$r < 0, q \geq max(1, 1/p),$$

then

$$k(p, q, r; t) \geq 0, \quad t \in \mathbb{R}^+,$$

which implies that $f_{p,q,r}(x) \in CM(\mathbb{R}^+)$.

The proof is complete. $\square$

*Proof of Corollary 7.3.* Let $r = 1$ and $0 < p = q < 1$, from Theorem 7.2,

$$f_{p,q,r}(x) = \theta(px) - p\theta x \geq 0.$$

Hence $-\theta(x)$ is star-shaped on $\mathbb{R}^+$, which implies that $-\theta(x)$ is supperadditive. Consequently, $\theta(x)$ is subadditive.

The proof is complete. $\square$

# Chapter 8

# Monotonicity and concavity properties of some functions associated with the gamma function and applications [1]

## 8.1 Introduction and main results

We say $a_n \simeq b_n$ $(n \geq n_0)$ if there exist two absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 b_n \leq a_n \leq c_2 b_n \tag{8.1}$$

[1] A version of this chapter has been published in *J. Inequal. Pure Appl. Math.* **7** (2006), Art. 45.

hold for all $n \geq n_0$. The fixed numbers $c_1$ and $c_2$ in (8.1) are called equivalence constants.

In 2002, it was proved by Qi [77] that the functions

$$f(s,r) := \left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}, \quad f(s,r,x) := \left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$$

and

$$g(s,r,x) := \left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$$

are increasing with respect to $r > 0$, $s > 0$, or $x > 0$.

Ramanujan in 1916 in [83] presented the following statement

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{1/6}, \quad x \in \mathbb{R}^+ \tag{8.2}$$

with $3/10 < \theta_x < 1$.

Inspired by (8.2), Ponnusamy and Vuorinen in 1997 in [76] made a conjecture that the function

$$f_1(x) := [g(x)]^6 - (8x^3 + 4x^2 + x),$$

where

$$g(x) := \left(\frac{e}{x}\right)^x \frac{\Gamma(1+x)}{\sqrt{\pi}},$$

is increasing from $(1, \infty)$ into $(1/100, 1/30)$.

In 2001, Karatsuba [53] proved, a little bit more than the above conjecture, that the function $f_1(x)$ is strictly increasing from $[1, \infty)$ onto $[f_1(1), f_1(\infty))$

with

$$f_1(1) = \frac{e^6}{\pi^3} - 13 = 0.0111976\cdots \quad \text{and} \quad f_1(\infty) = \frac{1}{30}.$$

In 2003, Alzer [3] proved that

$$\alpha \le f_1(x) < \frac{1}{30}, \quad x \in \mathbb{R}^+,$$

where

$$\alpha := \min_{x>0} f_1(x) = 0.0100450\cdots = f_1(x_0)$$

for some $x_0 \in [0.6, 0.7]$. Since $f_1(x_0) < f_1(1)$ and

$$f_1(x_0) < \lim_{x\to 0+} f_1(x) = \frac{1}{\sqrt{\pi}},$$

his result shows that $f_1(x)$ is not still monotonic on $(0, 1]$.

In 1997, it was showed by Anderson and Qiu [13] that the function

$$f_2(x) := \frac{\ln \Gamma(x+1)}{x \ln x}$$

is strictly increasing from $(1, \infty)$ onto $(1-\gamma, 1)$, where $\gamma$ is the Euler-Mascheroni constant. In 1998, Alzer [6] proved that $f_2(x)$,with

$$f_2(1) := \lim_{x\to 1} f_2(x) = 1 - \gamma,$$

is strictly increasing on $\mathbb{R}^+$. Also note that the function $f_2(x)$ was proved to be concave on $(1, \infty)$ in 2000 by Elbert and Laforgia [33], and the derivative

$f_2'(x)$ was proved to be strictly completely monotonic on $(1, \infty)$ in 2001 by Berg and Pedersen [19].

In [29, 43, 57, 71, 82, 90], monotonicity property of other functions related to the (di)gamma function was obtained.

For the approximation to $n!$, a well-known result is the following Stirling's formula:

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n},\ n \to \infty,$$

which is a very useful tool in analytical probability theory and statistical physics.

Mermin [67] in 1984 gave an exact identity for $n!$ as follows:

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \prod_{j=n}^{\infty} \frac{(1 + 1/j)^{j+1/2}}{e}.$$

Hsu [48] in 1997 also constructed an identity for $n!$:

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \exp \sum_{k=n}^{\infty} \sum_{j=2}^{\infty} \frac{(j-1)(-k)^j}{2j(j+1)}.$$

In this chapter, we shall give some monotonicity and concavity properties of some functions involving the gamma function and, as application, deduce some equivalence sequences to the sequence $n!$ with best equivalence constants.

**Theorem 8.1.** *The functions*

$$f(x) := \frac{x^{x+\frac{1}{2}}}{e^x \Gamma(x+1)} \tag{8.3}$$

*and*

$$F(x) := \frac{e^x \Gamma(x+1)}{x^x} \tag{8.4}$$

*are strictly log-concave and strictly increasing from* $\mathbb{R}^+$ *onto* $(0, 1/\sqrt{2\pi})$ *and onto* $(1, \infty)$, *respectively.*

**Theorem 8.2.** *The function*

$$g(x) := \frac{e^x \Gamma(x+1)}{(x+1/2)^{x+1/2}} \tag{8.5}$$

*is strictly log-concave and strictly increasing from* $(-1/2, \infty)$ *onto* $\left(\sqrt{\pi/e}, \sqrt{2\pi/e}\right)$.

**Theorem 8.3.** *The function*

$$h(x) := \frac{e^x \Gamma(x+1)\sqrt{x-1}}{x^{x+1}} \tag{8.6}$$

*is strictly log-concave and strictly increasing from* $(1, \infty)$ *onto* $\left(0, \sqrt{2\pi}\right)$.

As an application of these theorems, we have the following corollaries.

**Corollary 8.4.**

$$n! \simeq e^{-n} n^{n+1/2} \quad (n \geq 1).$$

*Moreover, for all* $n \in \mathbb{N}$,

$$\sqrt{2\pi} \cdot e^{-n} n^{n+1/2} < n! \leq e \cdot e^{-n} n^{n+1/2}. \tag{8.7}$$

*The equivalence constants* $\sqrt{2\pi}$ *and* $e$ *in* (8.7) *are best possible.*

**Corollary 8.5.**

$$n! \simeq e^{-n}(n+1/2)^{n+1/2} \quad (n \geq 0).$$

*Moreover, for all* $n \in \mathbb{N}_0$*,*

$$\sqrt{2}\, e^{-n}\left(n+\frac{1}{2}\right)^{n+1/2} \leq n! < \sqrt{\frac{2\pi}{e}}\, e^{-n}\left(n+\frac{1}{2}\right)^{n+1/2}. \tag{8.8}$$

*The equivalence constants* $\sqrt{2}$ *and* $\sqrt{2\pi/e}$ *in* (8.8) *are best possible.*

**Corollary 8.6.**

$$n! \simeq \sqrt{\frac{n}{n-1}}\, e^{-n} n^{n+1/2} \quad (n \geq 2).$$

*Furthermore, for all* $n \geq 2$*,*

$$\left(\frac{e}{2}\right)^2 \sqrt{\frac{n}{n-1}}\, e^{-n} n^{n+1/2} \leq n! < \sqrt{2\pi}\sqrt{\frac{n}{n-1}}\, e^{-n} n^{n+1/2}. \tag{8.9}$$

*The equivalence constants* $(e/2)^2$ *and* $\sqrt{2\pi}$ *in* (8.9) *are best possible.*

In 2000, Sándor and Debnath [85, Theorem 5] proved, with a long proof (three pages), that for $n \geq 2$,

$$\sqrt{2\pi}\, e^{-n} n^{n+1/2} < n! < \left(\frac{n}{n-1}\right)^{1/2} \sqrt{2\pi}\, e^{-n} n^{n+1/2}. \tag{8.10}$$

The famous Wallis' formula was used in their proof. In fact, (8.10) immediately follows from (8.7) and (8.9).

## 8.2 Proof of main results

**Lemma 8.7.** *The function*

$$\varphi(x) := \ln \frac{x+1}{x+\frac{1}{2}} - \frac{1}{2x} \tag{8.11}$$

*is strictly increasing from* $\mathbb{R}^+$ *onto* $(-\infty, 0)$.

*Proof of Theorem 8.1.* Taking the logarithm of $f(x)$ defined by (8.3) and then differentiating yield

$$\ln f(x) = \left(x - \frac{1}{2}\right) \ln x - x - \ln \Gamma(x), \tag{8.12}$$

$$[\ln f(x)]' = \ln x - \frac{1}{2x} - \psi(x). \tag{8.13}$$

Then by formula (3.5) of Theorem 3.C,

$$[\ln f(x)]' = 2 \int_0^\infty \frac{tdt}{(t^2+x^2)(e^{2\pi t}-1)}, \quad x > 0. \tag{8.14}$$

Hence, $[\ln f(x)]' > 0$ for $x \in \mathbb{R}^+$, which means that $\ln f(x)$, and then $f(x)$, is strictly increasing on $\mathbb{R}^+$.

It is easy to see that $\lim_{x\to 0+} f(x) = 0$. By (8.12) and (3.10) of Theorem 3.E, we have

$$\ln f(x) = -\ln \sqrt{2\pi} + O\left(\frac{1}{x}\right) \to \ln \frac{1}{\sqrt{2\pi}}, \quad x \to \infty,$$

which implies $\lim_{x\to\infty} f(x) = 1/\sqrt{2\pi}$.

From (8.4) we have

$$\ln F(x) = x + \ln\Gamma(x+1) - x\ln x, \quad \text{so} \tag{8.15}$$

$$[\ln F(x)]' = \psi(x+1) - \ln x. \tag{8.16}$$

Then by (3.6) of Theorem 3.C, for all $x > 0$,

$$[\ln F(x)]' = \ln\left(1 + \frac{1}{2x}\right) + 2\int_0^\infty \frac{tdt}{[t^2 + 4(x+1/2)^2](e^{\pi t}+1)} > 0. \tag{8.17}$$

Hence, $\ln F(x)$, and then $F(x)$, are strictly increasing on $\mathbb{R}^+$.

It is easy to see that $\lim_{x\to 0+} F(x) = 1$. By using (3.10) of Theorem 3.E, from (8.15),

$$\ln F(x) = \frac{1}{2}\ln x + \ln\sqrt{2\pi} + O\Big(\frac{1}{x}\Big), \quad x \to \infty.$$

Therefore, $\ln F(x)$, and then $F(x)$ tend to $\infty$ as $x \to \infty$.

Formulas (8.14) and (8.17) tell us that $[\ln f(x)]'$ and $[\ln F(x)]'$ are both strictly decreasing. Therefore, $\ln f(x)$ and $\ln F(x)$ are strictly concave, that is, the functions $f(x)$ and $F(x)$ are both log-concave. $\square$

*Proof of Theorem 8.2.* Taking the logarithm of $g(x)$ defined by (8.5) and then differentiating give us

$$\ln g(x) = x + \ln\Gamma(x+1) - \left(x + \frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right), \tag{8.18}$$

$$[\ln g(x)]' = \psi(x+1) - \ln\left(x + \frac{1}{2}\right). \tag{8.19}$$

Then, by formula (3.6) of Theorem 3.C, we have

$$[\ln g(x)]' = 2\int_0^{\infty} \frac{tdt}{[t^2+(2x+1)^2](e^{\pi t}+1)}, \quad x > -\frac{1}{2}. \tag{8.20}$$

So

$$[\ln g(x)]' > 0, \quad x \in \Big(-\frac{1}{2}, \infty\Big), \tag{8.21}$$

which means that $\ln g(x)$, then $g(x)$, are strictly increasing on $(-1/2, \infty)$.

Since $\Gamma(1/2) = \sqrt{\pi}$, it is easy to verify that $\lim_{x\to -1/2} g(x) = \sqrt{\pi/e}$.

From (8.18) and (3.10) of Theorem 3.E, we obtain

$$\ln g(x) = \Big(x+\frac{1}{2}\Big)\ln\frac{x+1}{x+1/2} + \ln\sqrt{2\pi} - 1 + O\Big(\frac{1}{x}\Big), \quad x \to \infty.$$

Hence $\ln g(x) \to \ln\sqrt{2\pi/e}$ as $x \to \infty$, and then $\lim_{x\to\infty} g(x) = \sqrt{2\pi/e}$.

Formula (8.20) shows that $[\ln g(x)]'$ is strictly decreasing. Therefore, $\ln g(x)$ is strictly concave, that is, the function $g(x)$ is log-concave. □

*Proof of Theorem 8.3.* From (8.6), we get

$$\ln h(x) = \ln\Gamma(x) + x + \frac{1}{2}\ln(x-1) - x\ln x, \quad \text{so} \tag{8.22}$$

$$[\ln h(x)]' = \psi(x) + \frac{1}{2(x-1)} - \ln x. \tag{8.23}$$

By setting $x = u+1$ with $u > 0$, we have

$$[\ln h(x)]' = \psi(u+1) + \frac{1}{2u} - \ln(u+1) = [\ln g(u)]' - \varphi(u),$$

where $g(u)$ and $\varphi(u)$ are respectively defined by (8.5) and (8.11). From (8.21) and Lemma 8.7, it is deduced that $[\ln h(x)]' > 0$ for $x > 1$. Therefore, $\ln h(x)$, and then $h(x)$, are strictly increasing on $(1, \infty)$.

It is obvious that $\lim_{x\to 1+} h(x) = 0$. From (8.22) and (3.10) of Theorem 3.E, we see

$$\ln h(x) = \frac{1}{2}\ln\frac{x-1}{x} + \ln\sqrt{2\pi} + O\Big(\frac{1}{x}\Big) \to \ln\sqrt{2\pi}\,, \quad x \to \infty.$$

So $\lim_{x\to\infty} h(x) = \sqrt{2\pi}$.

Since $g(x)$ is strictly log-concave and $\varphi(x)$ is strictly increasing, $[\ln h(x)]'$ is strictly decreasing. Therefore, $\ln h(x)$ is strictly concave, that is, the function $h(x)$ is log-concave. $\square$

*Proof of Corollary 8.4.* By Theorem 8.1, we know that the function $f(x)$ is strictly increasing from $\mathbb{R}^+$ onto $\big(0, 1/\sqrt{2\pi}\,\big)$, hence

$$\frac{1}{e} = f(1) \le f(n) = \frac{n^{n+1/2}}{e^n n!} < \frac{1}{\sqrt{2\pi}} \tag{8.24}$$

for $n \in \mathbb{N}$, and

$$\lim_{n\to\infty}\frac{n^{n+1/2}}{e^n n!} = \frac{1}{\sqrt{2\pi}}\,. \tag{8.25}$$

From (8.24) and (8.25), we see that Corollary 8.4 is true. $\square$

*Proof of Corollary 8.5.* By Theorem 8.2, we see that the function $g(x)$ is strictly increasing from $(-1/2, \infty)$ onto $\left(\sqrt{\pi/e}, \sqrt{2\pi/e}\right)$. So

$$\sqrt{2} = g(0) \le g(n) = \frac{e^n n!}{(n+1/2)^{n+1/2}} < \sqrt{\frac{2\pi}{e}}, \quad n \in \mathbb{N}_0 \tag{8.26}$$

and

$$\lim_{n\to\infty} \frac{e^n n!}{(n+1/2)^{n+1/2}} = \sqrt{\frac{2\pi}{e}}.$$

Inequality (8.26) is equivalent to (8.8). Since the constants $\sqrt{2}$ and $\sqrt{2\pi/e}$ are best possible in (8.26), they are also best possible in (8.8). □

*Proof of Corollary 8.6.* The monotonicity property of $h(x)$ in Theorem 8.3 implies

$$\left(\frac{e}{2}\right)^2 = h(2) \le h(n) = \frac{e^n n! \sqrt{n-1}}{n^{n+1}} < \sqrt{2\pi}, \quad n \ge 2 \tag{8.27}$$

and

$$\lim_{n\to\infty} \frac{e^n n! \sqrt{n-1}}{n^{n+1}} = \sqrt{2\pi}. \tag{8.28}$$

From (8.27) and (8.28), we see that Corollary 8.6 is valid. □

# Appendix A

# Proof of some results related to completely monotonic functions

In this appendix, we present proofs of some interesting results mentioned in Chapter 1.

The following result is due to Dubourdieu [32] in 1939.

**Theorem A.1.** *For any $a \geq -\infty$, a non-constant completely monotonic function on $I := (a, \infty)$ must be strictly completely monotonic there.*

*Proof.* We only need to show that if $(-1)^k f^{(k)}(x_0) = 0$ for some $k \in \mathbb{N}_0$ and for some $x_0 \in I$, then $f(x)$ is a constant on $I$.

Now suppose $(-1)^k f^{(k)}(x_0) = 0$ for some $k \in \mathbb{N}_0$ and for some $x_0 \in I$.

Since $(-1)^k f^{(k)} \in CM(I)$, by Theorem 1.R, $f^{(k)}(x) \equiv 0, x \in I$. And so

$$f(x) = c_0 + c_1 x + \cdots + c_{k-1} x^{k-1}, \quad x \in I. \tag{A.1}$$

We claim that $c_1 = \cdots = c_{k-1} = 0$ in (A.1). Otherwise, $\lim_{x\to\infty} f(x) = \infty$, which contradicts the fact that $f(x)$ is decreasing on $I$; or $\lim_{x\to\infty} f(x) = -\infty$, which contradicts the fact that $f(x) \geq 0$ on $I$. Consequently, $f(x) \equiv c_0, x \in I$. □

**Theorem A.2.** *For each completely monotonic sequence* $\{\mu_n\}_0^\infty$, *there exists one and only one number* $\mu_0^*$ *such that* $\{\mu_0^*, \mu_1, \mu_2, \cdots\}$ *is minimal completely monotonic.*

*Proof.* Let

$$A := \{\mu | \{\mu, \mu_1, \mu_2, \cdots\} \text{ is completely monotonic}\}. \tag{A.2}$$

Since $\{\mu_n\}_0^\infty$ is completely monotonic, $A \neq \emptyset$. Also $A$ is lower bounded since $\mu \geq \mu_1$. Hence $\mu_0^* := \inf A$ is finite.

In order to prove $\{\mu_0^*, \mu_1, \mu_2, \cdots\}$ is completely monotonic, it is suffices to show

$$(-1)^k \Delta^k \mu_0^* \geq 0, \quad k \in \mathbb{N}_0. \tag{A.3}$$

By the definition of $\mu_0^*$, there exist $\mu^{(j)} \in A, j \in \mathbb{N}$ such that

$$\mu^{(j)} \to \mu_0^* \quad (j \to \infty). \tag{A.4}$$

Now for $j \in \mathbb{N}$

$$(-1)^k \Delta^k \mu^{(j)} = \mu^{(j)} + \sum_{m=1}^{k} \binom{k}{m} (-1)^m \mu_m, \quad k \in \mathbb{N}_0. \tag{A.5}$$

Thus for $k \in \mathbb{N}_0$

$$\lim_{j\to\infty} (-1)^k \Delta^k \mu^{(j)} = \lim_{j\to\infty} \mu^{(j)} + \sum_{m=1}^{k} \binom{k}{m} (-1)^m \mu_m \tag{A.6}$$

$$= \mu_0^* + \sum_{m=1}^{k} \binom{k}{m} (-1)^m \mu_m \tag{A.7}$$

$$= (-1)^k \Delta^k \mu_0^* \tag{A.8}$$

Since $\mu^{(j)} \in A$,

$$(-1)^k \Delta^k \mu^{(j)} \geq 0, \quad j \in \mathbb{N}, k \in \mathbb{N}_0. \tag{A.9}$$

Combining (A.8) and (A.9) gives

$$(-1)^k \Delta^k \mu_0^* \geq 0, \quad k \in \mathbb{N}_0.$$

Now suppose that there are two distinct numbers $v_0$ and $w_0$ such that $\{v_0, \mu_1, \mu_2, \cdots\}$ and $\{w_0, \mu_1, \mu_2, \cdots\}$ are both minimal completely monotonic. Say $v_0 < w_0$, since $\{w_0, \mu_1, \mu_2, \cdots\}$ is minimal completely monotonic, by definition $\{v_0, \mu_1, \mu_2, \cdots\}$ is not completely monotonic–a contradiction occurs.

The proof is complete. $\square$

**Theorem A.3.** *A logarithmically completely monotonic function which is not identically equal to $c_0 e^{-cx}$ on $I := (a, \infty)$ for some $c_0 > 0$ and some $c \geq 0$ is strictly logarithmically completely monotonic there.*

*Proof.* We only need to show that if $f \in LCM(I)$ and for some $k \in \mathbb{N}_0$ and for some $x_0 \in I$,

$$(-1)^k \ln^{(k)} f(x_0) = 0, \tag{A.10}$$

then $f(x) = c_0 e^{-cx}$ for some $c_0 > 0$ and some $c \geq 0$ on $I$.

Since $f \in LCM(I)$,

$$g(x) := -[\ln f(x)]' \in CM(I). \tag{A.11}$$

(A.10) means that

$$(-1)^m g^{(m)}(x_0) = 0, \tag{A.12}$$

where $m = k - 1 \in \mathbb{N}_0$. From (A.11) and (A.12), by Theorem A.1, we get

$$g(x) = -[\ln f(x)]' = c \geq 0, \quad x \in I. \tag{A.13}$$

From (A.13), we have

$$f(x) = c_0 e^{-cx}, \quad x \in I,$$

where $c_0$ is a positive constant. □

The following result is due to Qi and Chen [78] in 2004.

**Theorem A.4.** *$LCM(I) \subset CM(I)$.*

*Proof.* Suppose $f \in LCM(I)$. We only need to show that $(-1)^n f^{(n)}(x) \geq 0, x \in I^0$ for $n \in \mathbb{N}$. Let $g(x) := \ln f(x), x \in I$, then $f(x) = e^{g(x)}, x \in I$.

By Lemma 2.18, for $n \in \mathbb{N}$,

$$\frac{d^n f}{dx^n} = \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!i_2!\cdots i_n!} e^{g(x)} \left(\frac{g'}{1!}\right)^{i_1} \left(\frac{g''}{2!}\right)^{i_2} \cdots \left(\frac{g^{(n)}}{n!}\right)^{i_n},$$

where $\Lambda_n$ is defined by (2.5).

Thus

$$(-1)^n f^{(n)}(x) = \sum_{(i_1,i_2,\cdots,i_n)\in\Lambda_n} \frac{n!}{i_1!\cdots i_n!} e^{g(x)} \left(\frac{(-1)^1 g'}{1!}\right)^{i_1} \cdots \left(\frac{(-1)^n g^{(n)}}{n!}\right)^{i_n}. \tag{A.14}$$

Since $(-1)^i g^{(i)}(x) = (-1)^i \ln^{(i)} f(x) \geq 0, x \in I^0, i \in \mathbb{N}$, from (A.14), we have $(-1)^n f^{(n)}(x) \geq 0, x \in I^0$ for $n \in \mathbb{N}$. □

**Theorem A.5.** *A function $f$ with $f(0) > 0$ is in $LCM(I)$, where $I := [0, \infty)$, if and only if the function $f/f(0)$ is the Laplace transform of an infinitely divisible probability distribution.*

*Proof.* Necessity:

Let $g(x) := f(x)/f(0), x \in I$, then $g \in LCM(I)$. Let $h(x) := -\ln g(x), x \in I$, then

$$g(x) = e^{-h(x)}, \quad x \in I \tag{A.15}$$

and

$$h(0) = -\ln g(0) = 0. \tag{A.16}$$

For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}^+$

$$(-1)^n [h'(x)]^{(n)} = (-1)^n h^{(n+1)}(x) = (-1)^{n+1} [\ln g(x)]^{(n+1)} \geq 0$$

for $g \in LCM(I)$. Hence

$$h' \in CM(\mathbb{R}^+). \tag{A.17}$$

By (A.15), (A.16), and (A.17), from Theorem 1.W′, we see that $g = f/f(0)$ is the Laplace transform of an infinitely divisible probability distribution.

Sufficiency:

By Theorem 1.W′

$$\frac{f(x)}{f(0)} = e^{-h(x)}, \quad x \in I \tag{A.18}$$

with

$$h(0) = h(0+) \tag{A.19}$$

and

$$h' \in CM(\mathbb{R}^+). \tag{A.20}$$

From (A.18)

$$\ln f(x) = \ln f(0) - h(x), \quad x \in I.$$

Therefore, for $n \in \mathbb{N}$ and $x \in \mathbb{R}^+$

$$(-1)^n[\ln f(x)]^{(n)} = (-1)^n[-h(x)]^{(n)}$$

$$= (-1)^{n-1}[h'(x)]^{(n-1)} \geq 0 \quad \text{by (A.20)}.$$

That is, $f \in LCM(\mathbb{R}^+)$. Also by (A.19), $f(0) = f(0+)$. Hence $f \in LCM(I)$.

The proof is complete. $\square$

# Bibliography


[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, Dover Publications, Inc., New York, 1966.

[2] H. Alzer, *Sharp inequalities for the digamma and polygamma functions*, Forum Math. **16** (2004), 181–121.

[3] H. Alzer, *On Ramanujan's double inequality for the gamma function*, Bull. London Math. Soc. **35** (2003), 601–607.

[4] H. Alzer, *On a gamma function inequality of Gautschi*, Proc. Edinburgh Math. Soc. **45** (2002), 589–600.

[5] H. Alzer, *Inequalities for the gamma function*, Proc. Amer. Math. Soc. **128** (2000), 141–147.

[6] H. Alzer, *Inequalities for gamma and polygamma functions*, Abh. Math. Sem. Hamburg **68** (1998), 363–372.

[7] H. Alzer, *On some inequalities for the incomplete gamma functions*, Math. Comp. **66** (1997), 771–778.

[8] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373–389.

[9] H. Alzer, *Some gamma function inequalities*, Math. Comp. **60** (1993), 337–346.

[10] H. Alzer and C. Berg, *Some classes of completely monotonic functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 445–460.

[11] H. Alzer and J. Wells, *Inequalities for the polygamma functions*, SIAM J. Math. Anal. **29** (1998), 1459–1466.

[12] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy and M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1713–1723.

[13] G. D. Anderson and S.-L. Qiu, *A monotoneity property of the gamma function*, Proc. Amer. Math. Soc. **125** (1997), 3355–3362.

[14] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[15] V. V. Anh and R. McVinish, *Completely monotone property of fractional Green functions*, Fract. Calc. Appl. Anal. **6** (2003), 157-173.

[16] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), 21-23.

[17] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterranean J. Math. **1** (2004), no. 4, 433–439.

[18] C. Berg, *A pick function related to an inequality for the entropy function*, J. Inequal. Pure Appl. Math. **2** (2001), no. 2, Art. 26.

[19] C. Berg and H. L. Pedersen, *A completely monotone function related to the gamma function*, J. Comp. Appl. Math. **133** (2001), 219–230.

[20] C. Berg and G. Forst, *Potential theory on locally compact Abelian groups*, Springer-Verlag, New York, 1975.

[21] S. Bernstein, *Sur la définition et les propriétés des fonctions analytiques d'une variable réelle*, Math. Ann. **75** (1914), 449–468.

[22] S. Bernstein, *Sur les fonctions absolument monotones*, Acta Math. **51** (1928), 1–66.

[23] S. Bochner, *Completely monotone functions in partially ordered spaces*, Duke Math. J. **9**(1942), 519–526.

[24] A. M. Bruckner and E. Ostrow, *Some function classes related to the class of convex functions*, Pacific J. Math. **12** (1962), 1203–1215.

[25] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, Reidel, Dordrecht, 1988.

[26] J. Bustoz and M. E. H. Ismail, *On gamma function inequalities*, Math. Comp. **47** (1986), no. 2, 659–667.

[27] Ch.-P. Chen and F. Qi, *Logarithmically completely monotonic functions relating to the gamma function*, J. Math. Anal. Appl. to appear (doi:10.1016/j.jmaa.2005.08.056 ).

[28] Ch.-P. Chen and F. Qi, *Logarithmically complete monotonicity properties for the gamma functions*, Aust. J. Math. Anal. Appl. **2** (2005), no. 2, Art.8, 9pp. (electronic).

[29] Ch.-P. Chen and F. Qi, *Monotonicity results for the gamma function*, J. Inequal. Pure Appl. Math. **4** (2003), no. 2, Art. 44.

[30] W. A. Day, *On monotonicity of the relaxation functions of viscoelastic materials*, Proc. Cambridge Philos. Soc. **67** (1970), 503–508.

[31] G. Del Piero and L. Deseri, *Monotonic, completely monotonic, and exponential relaxation functions in linear viscoelasticity*, Quart. Appl. Math. **53** (1995), 273–300.

[32] J. Dubourdieu, *Sur un théorème de M. S. Bernstein relatif à la transformation de laplace-Stieltjes*, Compositio Math. **7** (1939), 96–111.

[33] A. Elbert and A. Laforgia, *On some properties of the gamma function*, Proc. Amer. Math. Soc. **128** (2000), 2667–2673.

[34] A. Erdélyi(ed), *Higher transcendental functions*, Vol 1, McGraw-Hill, New York, 1953.

[35] A. Erdélyi(ed), *Higher transcendental functions*, Vol 2, McGraw-Hill, New York, 1953.

[36] W. Feller, *An introduction to probability theory and its applications*, Vol I, John Wiley & Sons, Inc., New York, 1966.

[37] W. Feller, *An introduction to probability theory and its applications*, Vol II, John Wiley & Sons, Inc., New York, 1966.

[38] W. Feller, *Completely monotone functions and sequences*, Duke Math. J. **5** (1939), 661–674.

[39] T. Franosch and Th. Voigtmann, *Completely monotone solutions of the mode-coupling theory for mixtures*, J. Statist. Phys. **109** (2002), 237–259.

[40] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 6th Edition, Academic Press, New York, 2000.

[41] A. Z. Grinshpan and M. E. H. Ismail, *Completely monotonic functions involving the gamma and q-gamma functions*, Proc. Amer. Math. Soc. **134** (2006), 1153–1160.

[42] G. Grüss, *Bemerkungen zur theorie der voll-bzw. mehrfachmonotonen funktionen*, Math. Z. **39** (1935), 732–741.

[43] B.-N. Guo and F. Qi, *Inequalities and monotonicity for the ratio of gamma functions*, Taiwanese J. Math. **7** (2003), 239–247.

[44] S. Guo, *Some classes of completely monotonic functions involving the gamma function*, Int. J. Pure Appl. Math., accepted.

[45] S. Guo, *Some function classes connected to the class of completely monotonic functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 2, Art. 8 (preprint).

[46] S. Guo, *On the composition of completely monotonic functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 2, Art. 13 (preprint).

[47] R. A. Horn, *On infinitely divisible matrices, kernels, and functions*, Z. Wahrscheinlichkeitstheorie Verw. Geb. **8** (1967), 219–230.

[48] L.C. Hsu, *A new constructive proof of the Stirling formula*, J. Math. Res. Exposition **17** (1997), 5–7.

[49] F. Hausdorff, *Summationsmethoden und momentfolgen I*, Math. Z. **9** (1921), 74–109.

[50] F. Hausdorff, *Summationsmethoden und momentfolgen II*, Math. Z. **9** (1921), 280–299.

[51] M. Ismail, L. Lorch and M. Muldoon, *Completely monotonic functions associated with the gamma function and q-analogues*, J. Math. Anal. Appl. **116** (1986), 1–9.

[52] D. W. Kammler, *Differential approximation of completely monotonic functions*, SIAM J. Numer. Anal. **18** (1981), 900–918.

[53] E. A. Karatsuba, *On asymptotic representation of the Euler gamma function by Ramanujan*, J. Comp. Appl. Math. **135** (2001), 225–240.

[54] A. A. Karatsuba and S. M. Voronin, *The Riemann-Zeta function*, Walter de Gruyter & Co., Berlin, 1992.

[55] J. D. Kečkić and P. M. Vasić, *Some inequalities for the gamma function*, Publ. Inst. Math.(Beograd) (N. S.) **11** (1971), 107–114.

[56] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.

[57] D. Kershaw and A. Laforgia, *Monotonicity results for the gamma function*, Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **119** (1985), 127–133.

[58] C. H. Kimberling, *A probabilistic interpretation of complete monotonicity*, Aequationes Math. **10** (1974), 152–164.

[59] A. Y. C. Kuk, *A litter-based approach to risk assessment in developmental toxicity studies via a power family of completely monotone functions*, J. Roy. Statist. Soc. Ser. C **53** (2004), 369–386.

[60] A. Laforgia, *Further inequalities for the gamma function*, Math. Comp. **42** (1984), 597–600.

[61] L. Lorch, *A property of completely [absolutely] monotonic and other lagorithmically convex [concave] functions*, Indian J. Math. **45** (2003), 207–210.

[62] L. Lorch and L. Moser, *A remark on completely monotonic sequences, with an application to summability*, Canad. Math. Bull. **6** (1963), 171–173.

[63] L. Lorch and D. J. Newman, *On the composition of completely monotonic functions, completely monotonic sequences and related questions*, J. London Math. Soc.(2) **28** (1983), 31–45.

[64] L. Lorch and P. Szego, *Higher monotonicity properties of certain Sturm-Liouville functions*, Acta. Math. **109** (1963), 55–73.

[65] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer, Berlin-New York, 1966.

[66] A. W. Marshall and I. Olkin, *Inequalities: theory of majorization and its applications*, Academic Press, New York, 1979.

[67] N. D. Mermin, *Stirling's formula* , Am. J. Phys. **52** (1984), 362–365.

[68] K. S. Miller and S. G. Samko, *Completely monotonic functions* , Integral Transform. Spec. Funct. **12** (2001), 389–402.

[69] D. S. Mitrinović, *Analytic inequalities*, Springer-verlag, Berlin, 1970.

[70] D. S. Moak, *Completely monotonic functions of the form* $s^{-b}(s^2+1)^{-a}$, Rocky Mountain J. Math. **17** (1987), 719–725.

[71] M. E. Muldoon, *Some monotonicity properties and characterization of the gamma function*, Aequationes Math. **18** (1978), 54–63.

[72] M. E. Muldoon, *Some characterizations of the gamma function involving the notion of complete monotonicity*, Aequationes Math. **8** (1972), 212–215.

[73] C. O'Cinneide, *A property of completely monotonic functions*, J. Austral. Math. Soc. Ser. A **42** (1987), 143–146.

[74] P. M. Peruničić, *Stochastic structure of some completely monotone functions*, Publ. Inst. Math. (Beograd) (N.S.) **46** (1989), 50–54.

[75] H. Pollard, *The Bernstein-Widder theorem on completely monotonic functions*, Duck Math. J. **11** (1944) 427–430.

[76] S. Ponnusamy and M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997) 278–301.

[77] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67.

[78] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[79] F. Qi, R.-Q. Cui, Ch.-P. Chen and B.-N. Guo, *Some completely monotonic functions involving polygamma functions and an application*, J. Math. Anal. Appl. **310**(2005), 303-308.

[80] F. Qi, B.-N. Guo and Ch.-P. Chen, Some completely monotonic functions involving the gamma and polygamma functions, *J. Austral. Math. Soc.* **80** (2006), 81–88.

[81] F. Qi and S. Guo, *Inequalities for the incomplete gamma and related functions*, Math. Inequal. Appl. **2** (1999), 47–53.

[82] S. L. Qiu and M. Vuorinen, *Some properties of the gamma and psi functions, with applications*, Math. Comp. **74** (2005), 723–742.

[83] S. Ramanujan, *Question 754*, J. Indian Math. Soc. **8** (1916), 80.

[84] E. Sandhya and S. Satheesh, *On distribution functions with completely monotone derivative*, Statist. Probab. Lett. **27** (1996), 127–129.

[85] J. Sándor and L. Debnath, *On certain inequalities involving the constant e and their applications*, J Math. Anal. Appl. **249** (2000), 569–592.

[86] S. Satheesh and E. Sandhya, *Distributions with completely monotone probability sequences*, Far East J. Theor. Stat. **1** (1997), 69–75.

[87] S. Y. Trimble, J. Wells and F. T. Wright, *Superadditive functions and a statistical application*, SIAM J. Math. Anal. **20** (1989), 1255–1259.

[88] H. van Haeringen, *Inequalities for real powers of completely monotonic functions*, J. Math. Anal. Appl. **210** (1997), 102–113.

[89] H. van Haeringen, *Completely monotonic and related functions*, J. Math. Anal. Appl. **204** (1996), 389–408.

[90] H. Vogt and J. Voigt, *A Monotonicity property of the* $\Gamma$*-function*, J. Inequal. Pure Appl. Math. **3** (2002), no. 5, Art. 73.

[91] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.

[92] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, New Jersey, 1941.

[93] D. V. Widder, *The Stieltjes transform*, Trans. Amer. Math. Soc. **43** (1938), 7–60.

[94] D. V. Widder, *Necessary and sufficient conditions for the representation of a function as a Laplace integral*, Trans. Amer. Math. Soc. **33** (1931), 851–892.

[95] J. Wimp, *Sequence transformations and their applications*, Academic Press, New York, 1981.