PROPERTIES OF EXTREMAL CONVEX BODIES

by

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A Thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfilment of the requirements of the degree of

MASTER OF SCIENCE

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Abstract

In 1948 Besicovitch proved that an affine image of a regular hexagon may be inscribed into an arbitrary planar convex body. We prove Besicovitch's result using a variational approach based on special approximation by triangles and generalize the Besicovitch theorem to a certain new class of hexagons. We survey the results on the Banach-Mazur distance between different classes of convex bodies. We hope that our generalization of the Besicovitch theorem may become useful for estimation of the Banach-Mazur distance between planar convex bodies. We examined our special approximation by triangles in some specific cases, and it showed a noticeable improvement in comparison with known general methods. We also consider the Banach-Mazur distance between a simplex and an arbitrary convex body in the three-dimensional case. Using the idea of an inscribed simplex of maximal volume, we obtain a certain related algebraic optimization problem that provides an upper estimate.

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Chapter 1

Introduction

1.1 Overview

The present work is devoted to some properties of convex sets. By definition, a set K in a linear space L is a convex set if for every two points $A, B \in K$, all the points of the form $(1-\lambda)A + \lambda B, \lambda \in (0,1)$ also belong to K. Examples of convex sets are a triangle and square (with interior) on the plane or a ball in $\mathbb{R}^n$.

Surprisingly, such a simple definition leads to a very rich structure and numerous nice properties. However, there are still many natural questions about convex bodies (closed bounded convex sets with non-empty interiors) even in Euclidian spaces R^n that remain open. We will focus on affine invariant properties of convex bodies.

Results that provide approximation (in some geometric sense) of general convex bodies with some specific "nice" convex bodies are convenient tools for investigating convex bodies. Perhaps the most well-known such result is John's theorem.

Theorem (John, 1948)**.** *If K is a centrally symmetric convex compact set with non-empty interior in $\mathbb{R}^n$ and E is an ellipsoid of maximal volume inscribed into K, then*

$$E \subseteq K \subseteq \sqrt{n}E,$$

where $\sqrt{n}E$ is the scaling of E by factor $\sqrt{n}$ with respect to the center of E.

The idea of maximization of a certain geometric characteristic of a convex body (such as volume) found other applications. Another type of structural result for convex bodies can be obtained using some continuity (intermediate value theorem) arguments. An example of such a result is a theorem by Besicovitch which states that an affine image of a regular hexagon may be inscribed into an arbitrary convex body. Besicovitch used his theorem to prove that every convex body $K \subset \mathbb{R}^2$ contains a centrally symmetric convex body with an area at least $\frac{2}{3}$ of the area of K and contained in centrally symmetric body with an area at most $\frac{4}{3}$ of the area of K and these bounds are sharp. We discuss the Besicovitch theorem in Chapter 2 starting with the outline of his original proof (using continuity). Next we develop a variational approach, based on minimization of a certain quantity involving areas, that leads to an alternative proof of the Besicovitch Theorem (see Section 2.2). Then we generalize our approach and obtain some new results of this type in Section 2.3.

A popular measure of comparison of affine classes of convex bodies is the Banach-Mazur distance, which is discussed in Chapter 3. We begin with the still open problem of finding the largest possible Banach-Mazur distance between an arbitrary convex body and a triangle. The best known results are surveyed in Section 3.1, and a possible application of our technique that might improve the estimate is presented in Section 3.2. The three-dimensional analogue of the above problem is considered in Section 3.3. We apply the idea of an inscribed simplex of maximal volume and obtain an optimization problem in purely algebraic terms. The solution to that problem will provide an upper bound for the Banach-Mazur distance between a simplex and an arbitrary convex body. This allows us to reprove the best known upper estimate and to prove that it is not an exact bound.

In Section 3.4 we survey known results on the Banach-Mazur distance between different classes of convex bodies treating both the case of small dimension and the asymptotic case. We also illustrate how the Besicovitch theorem is applied to obtain the current best known estimate on the Banach-Mazur distance between two arbitrary planar convex bodies.

1.2 Notations

Definition 1.1. By $\overline{AB}$ we denote the straight line segment between the points A and B, where $A, B \in \mathbb{R}^n$.

We will use this notation to emphasize the use of the segment instead of the line AB.

Definition 1.2. The length of a segment $\overline{AB}$, where $A, B \in \mathbb{R}^n$, is denoted by $|AB|$.

Definition 1.3. For a Lebesgue measurable set $S \subset \mathbb{R}^2$ denote its measure (area) as $A(S)$.

Definition 1.4. For a Lebesgue measurable set $K \subset \mathbb{R}^3$ denote its measure (volume) as Vol(A).

Definition 1.5. For a vector $v \in \mathbb{R}^n$ denote its Euclidian norm as $\|v\|$.

Definition 1.6. For a point $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ denote $B(x_0, \varepsilon) = \{x \in \mathbb{R}^n : \|x - x_0\| < \varepsilon\}$ - the open ball or radius ε with the centre at x_0.

Definition 1.7. For $K \subset \mathbb{R}^n$ denote by $\partial K = \{x \in \mathbb{R}^n : \forall \varepsilon > 0\ B(x, \varepsilon) \cap K \neq \emptyset, B(x, \varepsilon) \cap (\mathbb{R}^n \setminus K) \neq \emptyset\}$ the *boundary* of K.

Definition 1.8. For $K \subset \mathbb{R}^n$, the *interior* of K is defined as $\operatorname{Int} K = K \setminus \partial K$.

Definition 1.9. By a *convex body* $K \subset \mathbb{R}^n$ we mean a compact convex set with nonempty interior.

Definition 1.10. A *strictly convex body* is a convex body with no line segments in its boundary.

Definition 1.11. For two non-empty sets $X, Y \subset \mathbb{R}^n$ denote the *distance* $d(X, Y) = \inf_{x \in X, y \in Y} \|x - y\|$.

Definition 1.12. For two non-empty sets $X, Y \subset \mathbb{R}^2$ the *Hausdorff distance* $d_H(X, Y)$ is defined as

$$d_H(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\|\}.$$

Consider an example of Hausdorff distance between two simple sets. For the triangle $X = ABC$ and the square $Y = DEFG$, as shown on Fig. 1.1, $\sup_{x\in X} \inf_{y\in Y} \|x - y\|$ denotes the largest distance between a point of the triangle ABC and the closest point of $DEFG$. We may see from the figure that $|AD| = 4$ is the largest distance between a point of X and the square Y for any other point of X there exists a point of Y on the distance less or equal than 4. Therefore, $\sup_{x\in X} \inf_{y\in Y} \|x-y\| = |AD| = 4$.

The case of $\sup_{y\in Y} \inf_{x\in X} \|x - y\|$ is similar. Triangle ACG is obtuse because $|AG|^2 = 49 < 47 = |AC|^2 + |CG|^2$, therefore distance between G and ABC is equal to $|CG| = \sqrt{34}$. For any $y \in Y$ distance between the point C and y is not more than the distance between the points C and G and then for every $y \in Y:\ \inf_{x\in X} \|x - y\| \leq d(C, y) \leq \sqrt{34}$ and $\sup_{y\in Y} \inf_{x\in X} \|x - y\| = \sqrt{34}$. Finally, $d_H(X, Y) = \max(4, \sqrt{34}) = \sqrt{34}$.

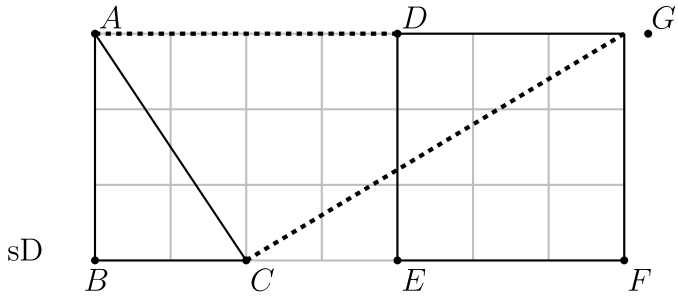

Figure 1.1. Hausdorff distance between a square and a tringle

The Hausdorff distance is a metric on bounded closed sets.

Definition 1.13. We say that a polygon M is *affine-inscribed* into a convex body K if there exists some non-degenerate affine transformation F such that all vertices of $F(M)$ belong to ∂K.

Definition 1.14. For $K \subset \mathbb{R}^n$ and $r \in \mathbb{R}$ denote by $rK = \{rx \mid x \in K\}$ the scaling of K by factor r.

Chapter 2

Besicovitch Theorem and its generalizations

In this chapter we consider convex subsets of $\mathbb{R}^2$ only.

2.1 Besicovitch Theorem on inscribed affine-regular hexagons

In 1948 Besicovitch [3] proved the following theorem:

Theorem 2.1. *For any given convex body Ω there exist points $A, B, C, D, E, F \in \partial\Omega$ such that $\overrightarrow{AB} = \overrightarrow{ED} = \frac{1}{2}\overrightarrow{FC}$ and lines AD, EB and FC intersect at the midpoint of FC.*

Any hexagon satisfying these conditions is an affine image of a regular hexagon. Indeed, denote O the midpoint of FC (see Fig. 2.1) and consider an affine transformation that transforms ABO into a regular hexagon, then all the triangles BCO, CDO, DEO, EFO, FAO will become regular and $ABCDEF$ will become a regular hexagon. We will refer to such hexagons as *affine regular* hexagons.

Besicovitch used his theorem to estimate the area of the largest centrally symmetric convex body inscribed into a given convex body.

We will illustrate the idea of the original proof of Besicovitch for the case of strictly convex bodies.

Proof of Theorem 2.1 (Besicovitch, 1948). For a fixed direction θ choose points on $\partial\Omega$ in the following way: first choose an arbitrary point A, then select B such that AB has direction θ. Then choose ED : $\overrightarrow{ED} = \overrightarrow{AB}$ and finally choose F, C for which FC is equidistant from AB and ED, see Fig. 2.1.

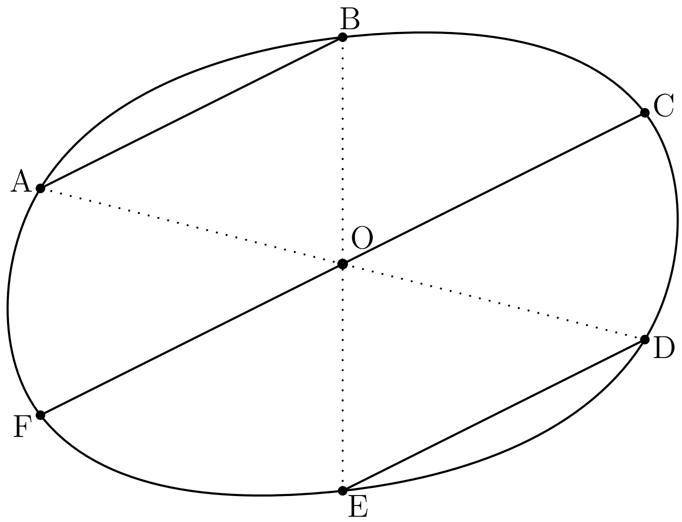

Figure 2.1. Affine-inscribed hexagon

It is possible to choose A for which $|FC| = 2|AB|$. Indeed, at extreme positions of A we have $|AB| = 0$ or $|FC| = |AB|$, and using continuity and the Intermediate Value Theorem, there exists a position of A where the required relation $|FC| = 2|AB|$ holds. Note that FC and AB are unique for each θ since Ω is strictly convex. Let O be the point of intersection of AD, EB, FC. We want to have $\frac{|OF|}{|FC|} = \frac{1}{2}$. Denote $\frac{|OF|}{|FC|} = \alpha(\theta)$. Obviously, $\alpha(\theta) = 1 - \alpha(\theta + \pi)$. Finally, using continuity of $\alpha(\theta)$, there exists θ_0 such that $\alpha(\theta_0) = \frac{1}{2}$ and the corresponding hexagon $ABCDEF$ is affine regular. $\square$

Makeev [11] generalizes the Besicovitch Theorem for the case of centrally symmetric hexagons proving the following theorem:

Theorem 2.2 (Makeev, 1995). *Any centrally symmetric hexagon can be affine-inscribed into any convex body.*

Makeev's proof is entirely different from Besicovitch's proof for the regular hexagon and involves the use of algebraic topology.

2.2 Special approximation by triangles

In this section we will give a different proof of Besicovitch's Theorem using a variational approach. The required affine-regular inscribed hexagon will be obtained from the triangle minimizing a certain objective function.

Theorem 2.3. *Consider an arbitrary strictly convex body Ω. Then there exists a triangle T for which*

$$2A(\Omega\backslash T) + A(T\backslash\Omega) \tag{2.1}$$

is minimal. This triangle intersects $\partial\Omega$ at six points which are vertices of an affine regular hexagon.

The value $2A(\Omega\backslash T)+A(T\backslash\Omega)$ is a partial case of an asymmetric difference metric which was studied before (see [4, p. 467]) in a different context. We could not find any direct applications of this idea to the proof of existence of inscribed affine-regular hexagons.

The proof of Theorem 2.3 requires several lemmas. We will also need the following definitions:

Definition 2.4. For an angle φ denote $e_\varphi = (\cos(\varphi), \sin(\varphi)) \in \mathbb{R}^2$, the unit vector with polar angle φ.

Definition 2.5. For an angle φ and a point x_0 denote $l_{x_0,\varphi} = \{x \in \mathbb{R}^2 : (x-x_0)\cdot e_\varphi = 0\}$ - the line passing through x_0 orthogonal to e_φ.

Definition 2.6. For an angle φ and a point x_0 denote $h_{x_0,\varphi} = \{x \in \mathbb{R}^2 : (x-x_0)\cdot e_\varphi \geq 0\}$- the closed half-plane bounded by $l_{x_0,\varphi}$. Note that the vector e_φ points towards the chosen half-plane.

Definition 2.7. Notation $f(x) \in o(g(x)), x \to a$ means that $\frac{f(x)}{g(x)} \to 0, x \to a$.

For example, $\frac{x+1}{x} = 1 + \frac{1}{x} = 1 + o(x), x \to 0$.

Lemma 2.8. *If Ω is a closed and bounded strictly convex body, $l = l_{x0,\varphi}$ is a fixed line, the intersection of $l_{x0,\varphi}$ and Ω has length s, and $a(h) = A(\Omega \cap h_{x0,\varphi} \cap h_{x0+he_\varphi,\varphi+\pi})$, then $a(h) = sh + o(h),\ h \to 0^+$.*

Proof. The intersection $l \cap \Omega$ is a closed bounded convex subset of a line. So it can be either:

1. a segment AB, or
2. a single point A, or
3. the empty set.

We consider the corresponding cases below:

Case 1: As Ω is strictly convex, there are two points $E, F \in \Omega$ separated by the line AB. If $h > 0$ is small enough, then $l_{x0+he_\varphi,\varphi} \cap \Omega$ is a segment CD. However, as Ω is convex, $C \in GH$ and $D \in IJ$ (see Fig. 2.2). Thus,

$$sh + o(h^2) = A(ABIH) \leq a(h) \leq A(ABDG) = sh + o(h).$$

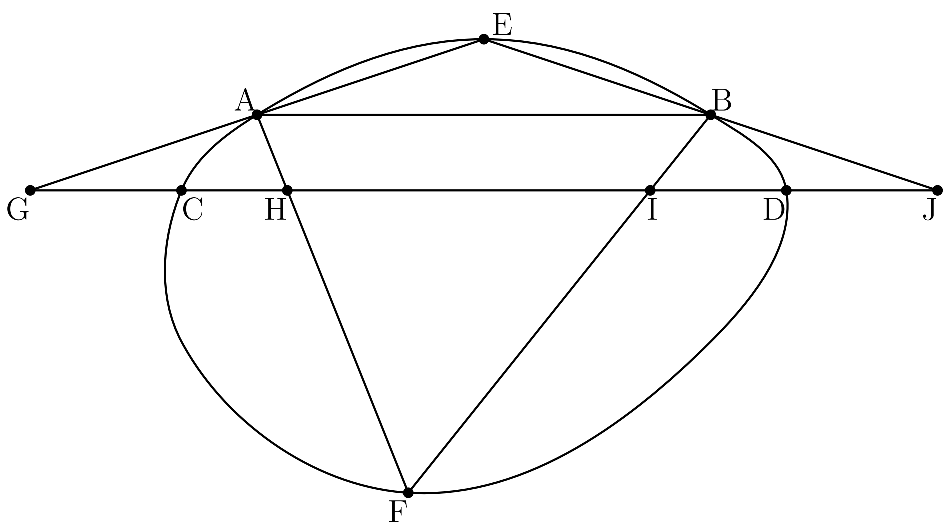

Figure 2.2. Parallel translation of a side

Case 2: Suppose that for small $h > 0$ the intersection $l_{x_0+he_\varphi,\varphi} \cap \Omega$ is a segment A_hB_h. As Ω is closed, all limit points of $\{A_h\}$ and $\{B_h\}$ belong to both l and Ω. Therefore, as $A = l \cap \Omega$ is single point, $A_h \to A,\ h \to 0^+$ and $B_h \to A,\ h \to 0^+$. For every $\varepsilon > 0$ there exist h_ε such that if $h < h_\varepsilon$ then $|A_hB_h| < \varepsilon$ and thus $a(h) \leq h\varepsilon = 0 + o(h)$.

Case 3: As Ω is closed, then obviously $a(h)$ is zero for small enough $h > 0$. $\square$

Lemma 2.9. *Let Ω be a strictly convex body, $l = l_{x,\varphi_0}$ is a fixed line, x is a point outside of Ω. Denote $s(\alpha) = A(\Omega \cap h_{x,\varphi} \cap h_{x,\varphi+\alpha+\pi})$. If $l \cap \Omega$ is a segment AB, then*

$$s(\alpha) = \frac{1}{2}\alpha(|OB|^2 - |OA|^2) + o(\alpha), \quad \alpha \to 0^+.$$

Proof. As Ω is strictly convex, there are points E, F separated by l, that belong to Ω. As we can see on the Fig. 2.3, for small enough α the intersection $l_{x,\varphi_0+\alpha} \cap \Omega$ is a segment CD, where $C \in GH,\ D \in IJ$. We have the bounds

$$A(AHIB) \leq s(\alpha) \leq A(AGJB).$$

Now,

$$\begin{aligned}
A(AHIB) &= \frac{1}{2}\sin(\alpha)(|OB| \cdot |OI| - |OA| \cdot |OH|) \\
&= \frac{1}{2}\sin(\alpha)(|OB| \cdot (|OB| + O(\alpha)) - |OA|(|OA| + O(\alpha))) \\
&= \frac{1}{2}\alpha(|OB|^2 - |OA|^2) + o(\alpha), \quad \alpha \to 0^+,
\end{aligned}$$

and

$$\begin{aligned}
A(AGJB) &= \frac{1}{2}\sin(\alpha)(|OB| \cdot |OJ| - |OA| \cdot |OG|) \\
&= \frac{1}{2}\sin(\alpha)(|OB| \cdot (|OB| + O(\alpha)) - |OA|(|OA| + O(\alpha))) \\
&= \frac{1}{2}\alpha(|OB|^2 - |OA|^2) + o(\alpha), \quad \alpha \to 0^+.
\end{aligned}$$

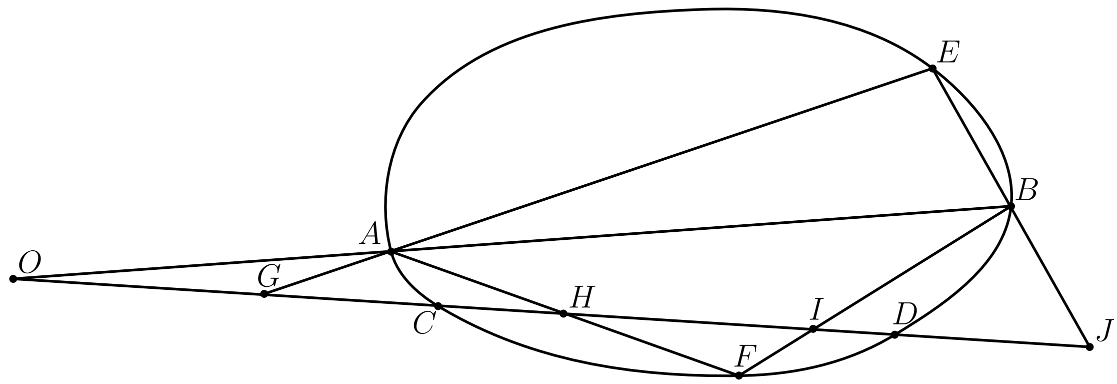

Figure 2.3. Rotation of a side

□

The next lemma shows that the vertices of the triangle minimizing (2.1) cannot be too far away from the body.

Lemma 2.10. *If $\Omega \subset B(0, R)$ is a convex body and T is a triangle, such that*

$$2A(\Omega \setminus T) + A(T \setminus \Omega) > 2A(\Omega),$$

then $T \subset B(0, 13R)$.

Proof. Denote $D = \max_{x \in T}\{\|x\|\}$. Clearly, the triangle has at least one common point with Ω. Hence, one of the sides of T has length at least $D - R$. Let h be the height of the triangle T corresponding to this side. Then

$$2A(\Omega \backslash T) + A(T \backslash \Omega) = 2A(\Omega) + A(T) - 3A(K \cap T) \geq 2A(\Omega) + \frac{(D-R)h}{2} - 6Rh.$$

If $D > 13R$, then the right-hand side is greater than $2A(\Omega)$. □

Proof of Theorem 2.3. We will say that the triangle T is minimal if (2.1) is minimal for this T.

First we prove the existence of a minimal triangle T. Assume that Ω is located inside of the disk $B(0, 1)$. Then by Lemma 2.10 we may assume that every vertex of

T is inside the disk $B = B(0, 13)$. B^3 is compact and (2.1) is a continuous function of the vertices of T, so the minimum is achieved for some T.

Assume that for given Ω the triangle $T = ABC$ minimizes the value of $2A(\Omega\backslash T)+A(T\backslash\Omega)$.

Note that each side of ∂T should intersect $\partial\Omega$ at two points, otherwise, by Lemma 2.8, this side may be moved so that the expression (2.1) is smaller. Assume that AB intersects the boundary of Ω at C_1, C_2. If we move AB parallel to itself by h, then by Lemma 2.8 the expression will change by $h(|AC_1|-2|C_1C_2|+|C_2B|)+o(h)$. By assumption that the expression is maximal at $h = 0$, we have

$$|AC_1| - 2|C_1C_2| + |C_2B| = 0.$$

Now if we consider the rotation of AB around the point A by the angle α, the expression will change by $\frac{1}{2}\sin(\alpha)(|AB|^2-|AC_2|^2-2(|AC_2|^2-|AC_1|^2)+|AC_1|^2)+o(\alpha)$, see Lemma 2.9. Again if $\alpha = 0$ corresponds to the local minimum, we have

$$|AB|^2 - |AC_2|^2 - 2(|AC_2|^2 - |AC_1|^2) + |AC_1|^2 = 0.$$

Denote $|AC_1| = x$, $|C_1C_2| = y$, $|C_2B| = z$. Consider two equations

$$x + z = 2y,$$
$$(x + y + z)^2 - 3(x + y)^2 + x^2 = 0.$$

Solving these equations, we have $x = y = z$. The same arguments may be applied for every side of the triangle T. Therefore, $\partial\Omega$ intersects ∂T at six points, and every side will be divided by two of these points into three equal parts. The points of intersection of ∂T with $\partial\Omega$ are vertices of an affine-regular hexagon as an affine transformation that transforms T into regular triangle will transform points of $\partial T \cap \partial\Omega$ into vertices of regular hexagon. $\square$

Now we will extend Theorem 2.3 to the case of general convex bodies (not necessarily strictly convex).

Lemma 2.11. *For any $\varepsilon > 0$ and any convex body K there exists a strictly convex body $K_\varepsilon \supset K$ such that $d_H(K, K_\varepsilon) < \varepsilon$ (see Definition 1.12).*

Proof. For a given convex body K, we have $K \subset B(0,C)$ for some $C > 0$. Also the function $d(x) = \inf_{y \in K} \|x - y\|$ is convex, and $K = \{x \in \mathbb{R}^2 : d(x) = 0\}$.

For every $\varepsilon > 0$ define $f_\varepsilon(x) = d(x) - \varepsilon(C^2 - \|x\|^2)$, and $\tilde{K}_\varepsilon = \{x : f_\varepsilon(x) \leq 0\}$.

We claim that $\tilde{K}_\varepsilon$ is strictly convex. Indeed, if $x, y \in \partial\tilde{K}_\varepsilon$, then $f_\varepsilon(x) = f_\varepsilon(y) = 0$. We obtain

$$\begin{aligned} f_\varepsilon\Big(\frac{x+y}{2}\Big) &= d\Big(\frac{x+y}{2}\Big) - \varepsilon\Big(C^2 - \Big\|\frac{x+y}{2}\Big\|^2\Big) \\ &< \frac{d(x)+d(y)}{2} - \varepsilon\Big(C^2 - \frac{\|x\|^2 + \|y\|^2}{2}\Big) = 0. \end{aligned}$$

The inequality is strict since $\|x\|^2$ is a strictly convex function.

If $x \in K$, then $x \in \tilde{K}_\varepsilon$, and $\sup_{x \in X} \inf_{y \in \tilde{K}_\varepsilon} d(x,y) = 0$. On the other hand, if $x \in \tilde{K}_\varepsilon$, then $d(x) \leq \varepsilon(C^2 - \|x\|^2) \leq \varepsilon C^2$. Therefore, $d_H(K, \tilde{K}_\varepsilon) \leq \varepsilon C^2$.

So, we can take $K_\varepsilon = \tilde{K}_{\varepsilon C^{-2}}$. Clearly, $K \subset K_\varepsilon$. □

Theorem 2.12. *If K is a convex body, there exists a triangle T such that* (2.1) *is minimal. $\partial K \cap \partial T$ contains six points $E_1, \ldots, E_6$ which are vertices of an affine-regular hexagon.*

Proof. Assume that $K \subset B(0,1)$. By Lemma 2.11, there is a sequence K_n such that $d_H(K, K_n) \leq \frac{1}{n}$. By Theorem 2.3 for each K_n there exists a triangle T_n for which (2.1) is minimal and ∂T_n intersects ∂K_n at the points $E_{1,n}, \ldots, E_{6,n}$. By Lemma 2.10, $T_n \subset B(0,13)$, therefore, using the compactness argument, there is a sequence $\{n_k\}$ such that the vertices of T_{n_k} are convergent to the vertices of some triangle T and E_{i,n_k} are convergent to some points E_i, as $k \to \infty$. Then, using that $d_H(K_n, K) \to 0$, and that the area is continuous in Hausdorff metric, (2.1) is minimal for T, and $E_1, \ldots, E_6 \in \partial K$ are vertices of an affine-regular hexagon. □

2.3 Generalization of The Besicovitch Theorem

In this section we will try to determine which hexagons are affine-inscribed into an arbitrary convex body. Besicovitch's Theorem states that the regular hexagon is an example of such a hexagon. On the other hand, if a hexagon is affine-inscribed into an arbitrary convex body then it is affine-inscribed into a circle.

Proposition 2.13. *A hexagon is affine-inscribed into any convex body only if it is inscribed into an ellipse.*

In fact, this condition is not sufficient.

Example 2.14. *The convex hexagon with vertices* $e_0, e_{\frac{\pi}{10}}, e_{\frac{2\pi}{10}}, e_{\frac{-\pi}{10}}, e_{\frac{-2\pi}{10}}, e_\pi$ *is inscribed into the unit circle, but is not inscribed into any triangle (see Fig. 2.4).*

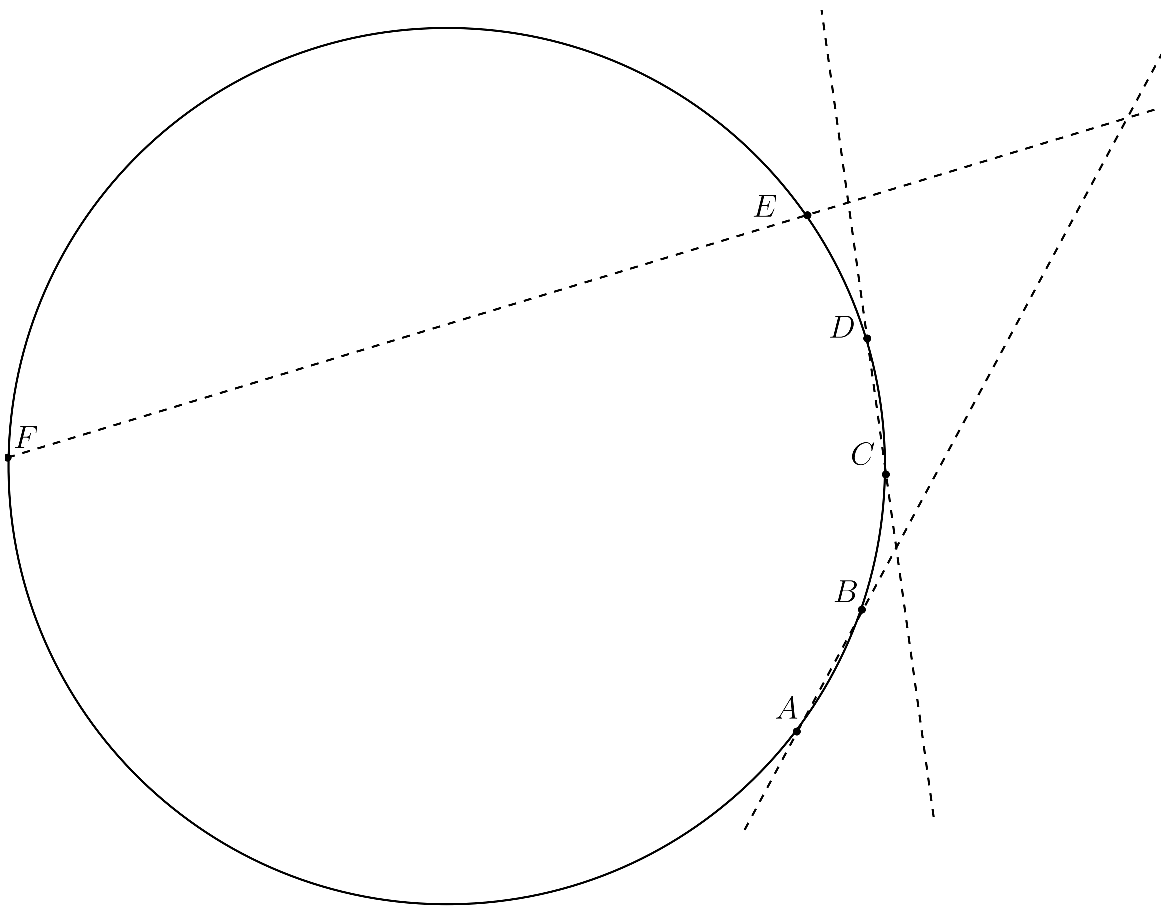

Figure 2.4. Hexagon not inscribed into triangle

Example 2.14 is an example of a hexagon affine-inscribed into a circle which is not affine-inscribed into a triangle. However, in this example, the triangle is not a strictly convex body.

Conjecture 2.15. *Any hexagon affine-inscribed into a circle is affine-inscribed into an arbitrary <u>strictly</u> convex body.*

While strictly convex bodies are dense among all convex bodies, we can not pass to a limit in this problem because the limit inscribed hexagon may be degenerate. Similar difficulties appeared while solving the famous Inscribed Square problem [14]: while a square may be inscribed into any *smooth* Jordan curve, for arbitrary Jordan curves, this is still an open question.

Example 2.14 suggests another necessary condition:

Proposition 2.16. *A hexagon E is affine-inscribed into any convex body only if it is inscribed into a triangle.*

For the general case, we have the following conjecture.

Conjecture 2.17. *If convex hexagon K is inscribed into an ellipse and a triangle, then K is affine-inscribed into an arbitrary convex body.*

Now we will state the *sufficient* condition for a hexagon to be affine inscribed into an arbitrary convex body. We will use a generalization of the special approximation by triangles from Section 2.2.

First consider the case of a strictly convex body. For a strictly convex body K, consider the set B of all ordered triples of distinct points on plane $(t_1, t_2, t_3) \in \mathbb{R}^6$ such that $t_i \notin \mathrm{Int}(K)$ and $\overline{t_i t_j} \cap K \neq \emptyset$. B is the set of all triangles for which each side intersects K and every vertex is outside or on the boundary of K. For given numbers $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ consider the function $f_K : B \to \mathbb{R}$ specified as follows: let $T = T(t_1, t_2, t_3)$ be the triangle with the vertices t_1, t_2, t_3,

$$f_K(t_1, t_2, t_3) = f_K(T) = a_1 S_1 + b_1 M_1 + a_2 S_2 + b_2 M_2 + a_3 S_3 + b_3 M_3, \tag{2.2}$$

where S_1, S_2, S_3 are the areas of the three connected components of $T \setminus K$ corresponding to the vertices t_1, t_2, t_3, and M_1, M_2, M_3 are the areas of the three connected components of $K \setminus T$ corresponding to the sides $\overline{t_2 t_3}, \overline{t_1 t_3}, \overline{t_1 t_2}$ (see fig. 2.5).

Note that B is not bounded and is not closed.

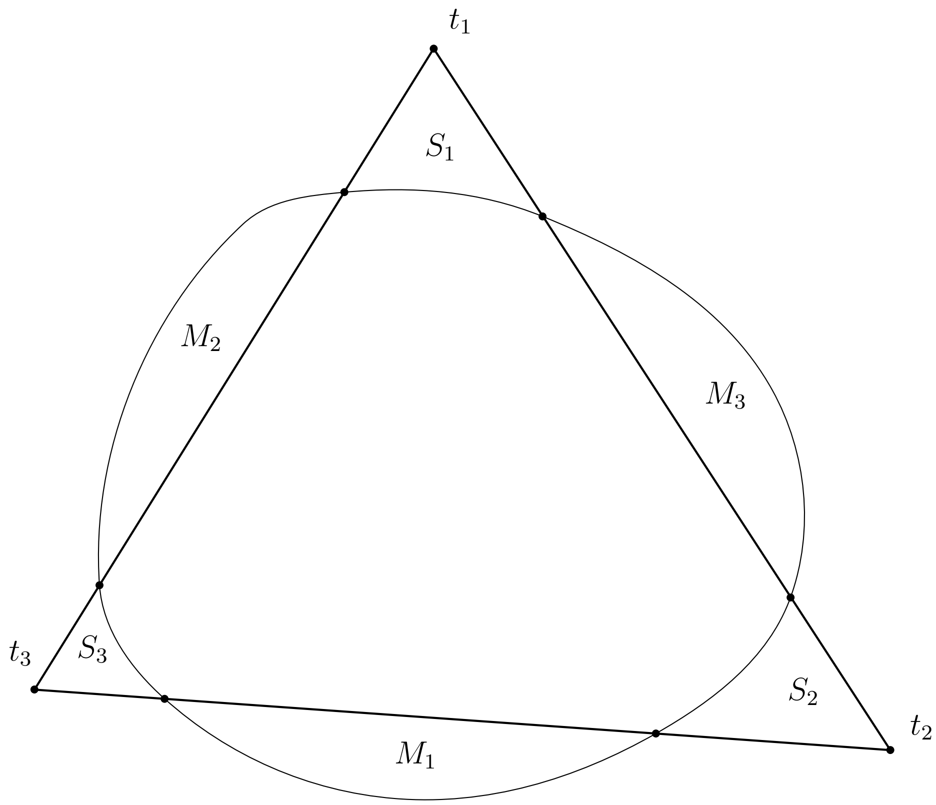

Figure 2.5. Relative position of a triangle and a convex body

Lemma 2.18. *Let $b_1 \le b_2 \le b_3$. If there exists a triangle T' such that $f_K(T') < b_1 A(K)$, then the absolute minimum of f is attained on B.*

Proof. Fix some non-degenerate triangle $T = (t_1, t_2, t_3)$. Consider a half-plane h, bounded by the line passing through t_1, t_2 and not containing t_3. Then $M_3 = A(K \cap h)$ is a continuous function of T, the same is for M_1, M_2. S_1 is continuous as points $\overline{t_1 t_2} \cap \partial K$ and $\overline{t_1 t_3} \cap \partial K$, closest to t_1 depend continuously on T. Therefore f_K is continuous as a linear combination of continuous functions.

Let a_i be the minimum of the numbers a_1, a_2, a_3, and assume that $K \subset B(0,1)$. Now we proceed similarly to the proof of Lemma 2.10. Denote by R the largest distance from the origin to the vertices of T. Then one of the sides of T has length at least $R - 1$. Let h be the height of the triangle T corresponding to this side. We have that

$$\begin{aligned} f_K(T) &\geq a_i A(T \setminus K) + b_1 A(K \setminus T) \\ &= a_i A(T) + b_1 A(K) - (a_i + b_1) A(T \cap K) \\ &\geq a_i (R-1)\frac{h}{2} + b_1 A(K) - (a_i + b_1) 2h \\ &= b_1 A(K) + h(a_i (R-1)/2 - 2a_i - 2b_1). \end{aligned}$$

Therefore, if $R > 5 + 4b_1/a_i$, then $f_K(T) > b_1 A(K) = f_K(T')$, so we can restrict B to the set B' of the triangles with no vertices on distance more than $6 + 4b_1/a_i$ from the origin. B' is bounded and

$$\inf_{T \in B} f_K(T) = \inf_{T \in B'} f_K(T).$$

For any $T \in B'$ we have $f_K(T) \geq b_1(M_1 + M_2 + M_3) \geq b_1(A(K) - A(T))$. Let $m = f_K(T') < b_1 A(K)$ by the condition of the Lemma. Now we can restrict B' to the set B'' of the triangles the with property $A(T) \geq \frac{1}{2}\left(A(K) - \frac{m}{b_1}\right)$. This set B'' is compact and f_K is continuous on B''. Thus there is some triangle $\overline{T}$ such that

$$f_K(\overline{T}) = \min_{T \in B''} f_K(T) = \min_{T \in B} f_K(T).$$

□

Under the conditions of Lemma 2.18, let $\overline{T} = (v_1, v_2, v_3)$ be a triangle minimizing f. Denote by x, y, z the lengths of the three segments in which ∂K divides $\overline{v_1 v_2}$. According to Lemmas 2.8, 2.9

$$b_3 y = a_1 x + a_2 z$$

$$a_1 x^2 + a_2((x+y+z)^2 - (x+y)^2) = b_3((x+y)^2 - x^2).$$

Solving this system, we have

$$\begin{cases} \dfrac{x^2}{z^2} = \dfrac{a_2^2 + a_2 b_3}{a_1^2 + a_1 b_3}, \\ b_3 y = a_1 x + a_2 z, \end{cases} \tag{2.3}$$

this gives the ratios $\frac{x}{z}, \frac{x}{y}$ in terms of the weights $a_1, a_2, a_3, b_1, b_2, b_3$.

Finally, this gives the following theorem:

Theorem 2.19. *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive weights, and let K be a strictly convex body. Denote f_K as in* (2.2)*. If*

$$\text{there exists a triangle } T \text{ such that: } f_K(T) < \min\{b_1, b_2, b_3\}A(K), \tag{2.4}$$

then there exists a triangle $\overline{T}$ such that each side of $\overline{T}$ is divided by the points of $\partial\overline{T} \cap \partial K$ in ratios given by (2.3)*.*

We may apply Lemma 2.11 to obtain a generalization of Theorem 2.19 to the case of arbitrary convex body (not necessarily strictly convex).

Theorem 2.20. *For some positive weights $a_1, a_2, a_3, b_1 \leq b_2 \leq b_3$ and some convex body K denote f_K as in* (2.2)*. If for some triangle T the inequality* (2.4) *holds, then there exists a triangle $\overline{T}$ such that each side of $\overline{T}$ contains points of ∂K that divide it in ratios given by* (2.3)*.*

Proof. Using Lemma 2.11 consider a sequence K_n, $n \in \mathbb{N}$, of strictly convex bodies such that $d_H(K, K_n) < \frac{1}{n}, K \subset K_n$. For each $n \in \mathbb{N}$ choose a triangle T_n in the following way: each vertex of T_n coincides with the corresponding vertex of T if it is outside or on the boundary of K_n. Otherwise, instead of vertex t we choose some point on the boundary of K_n with the distance less than $\frac{1}{n}$ from t. For large enough n, the function f_{K_n} is well-defined at T_n, and since the area is continuous in the Hausdorff metric, $f_{K_n}(T_n) < b_1 A(K_n)$. Therefore, we may apply Theorem 2.19 for K_n and obtain triangle $\overline{T_n} = ABC$ such that $\overline{AB} \cap K_n = \{H_n, I_n\}, \overline{BC} \cap K_n = \{D_n, E_n\}, \overline{CA} \cap K_n = \{F_n, G_n\}$ and

$$\begin{cases} \dfrac{|AH_n|^2}{|I_nB|^2} = \dfrac{a_2^2 + a_2b_3}{a_1^2 + a_1b_3}, \\ b_3|H_nI_n| = a_1|AH_n| + a_2|I_nB|. \end{cases}$$
$$\begin{cases} \dfrac{|BD_n|^2}{|E_nC|^2} = \dfrac{a_2^2 + a_2b_1}{a_3^2 + a_3b_1}, \\ b_1|D_nE_n| = a_2|BD_n| + a_3|E_nC|. \end{cases} \tag{2.5}$$
$$\begin{cases} \dfrac{|CF_n|^2}{|G_nA|^2} = \dfrac{a_3^2 + a_3b_2}{a_1^2 + a_1b_2}, \\ b_2|F_nG_n| = a_3|CF_n| + a_1|G_nA|. \end{cases}$$

Also $A(\overline{T_n}) \geq A(K_n) - \frac{f_K(\overline{T_n})}{b_1}$. Choosing a convergent sequence of $\overline{T_n}$, we may construct the limit triangle $\overline{T}$ and points D, E, F, G, H, I that satisfy (2.5). The obtained triangle $\overline{T}$ is non-degenerate as $A(\overline{T}) \geq A(K) - \frac{f_K(T)}{b_1} > 0$. □

This theorem gives a sufficient condition when a hexagon E is affine-inscribed into any convex body. Assume that $DEFGHI$ is affine-inscribed into circle and into triangle ABC such that

$$D, E \in BC,\ F, G \in AC,\ H, I \in AB.$$

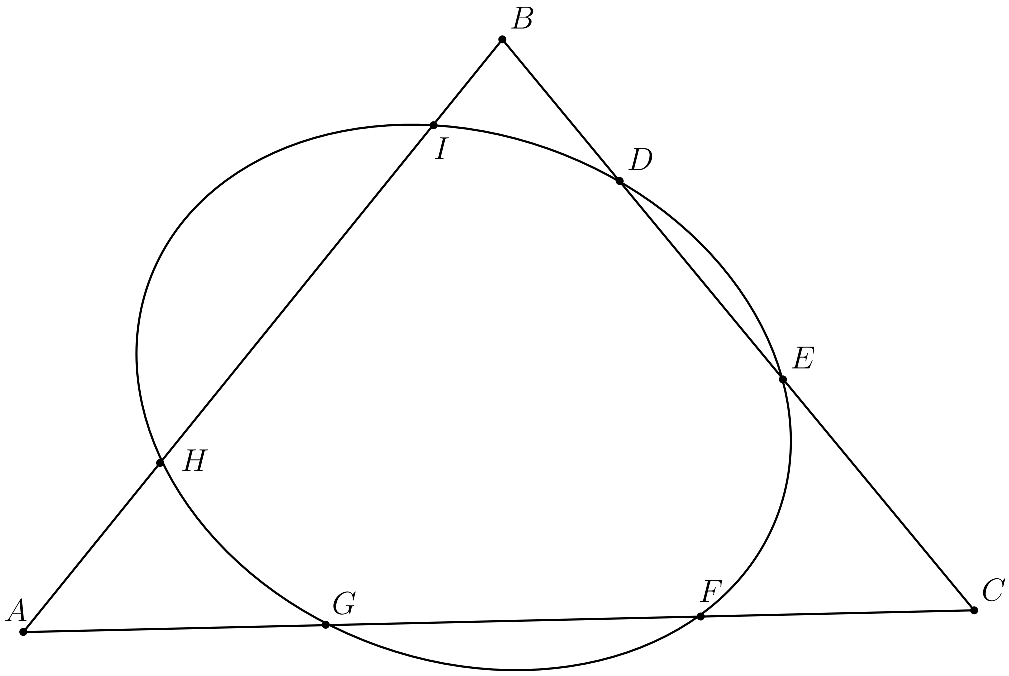

Figure 2.6. Order of points

Lemma 2.21. *If*

$$D, E \in BC,\ F, G \in AC,\ H, I \in AB,$$

and $DEFGHI$ is inscribed into an ellipse, then there exist weights $a_1, a_2, a_3, b_1, b_2, b_3$ such that points D, E, F, G, H, I satisfy (2.5).

Proof. Set $b_3 = 1$. There exists a function $f_1(x)$ such that

$$\frac{|AH|^2}{|IB|^2} = \frac{a_2^2 + a_2}{f_1(a_2)^2 + f_1(a_2)}.$$

Namely, solving a quadratic equation one can find

$$f_1(x) = \frac{-1 + \sqrt{1 + \frac{4|IB|^2}{|AH|^2}(a_2^2 + a_2)}}{2}.$$

Function $f_1(x)$ is monotone and continuous thus one can find a_2 such that

$$|HI| = f_1(a_2)|AH_n| + a_2|I_nB|.$$

Denote $a_1 = f_1(a_2)$. Weights b_3, a_1, a_2 satisfy the system

$$\begin{cases} \dfrac{|AH|^2}{|IB|^2} = \dfrac{a_2^2 + a_2 b_3}{a_1^2 + a_1 b_3}, \\ b_3|HI| = a_1|AH| + a_2|IB|. \end{cases}$$

Set $b_1' = 1$. In the same way we find weights a_2', a_3' satisfying

$$\begin{cases} \dfrac{|BD|^2}{|EC|^2} = \dfrac{a_2'^2 + a_2' b_1'}{a_3'^2 + a_3' b_1'}, \\ b_1'|D_nE_n| = a_2'|BD_n| + a_3'|EC|. \end{cases} \tag{2.6}$$

As the system (2.6) is homogeneous in a_3, a_2, b_1, it is also satisfied by the weights $a_2, a_3 = \frac{a_3' a_2}{a_2'}, b_2 = \frac{a_2}{a_2'}$.

We have found the weights a_1, a_2, a_3, b_3, b_1 satisfying first two systems of (2.5).

We may choose b_2 such that points D, E, F, G', H, I satisfy (2.5) for some point G'. Finally, explicit calculations give that in this case the following equations hold:

$$\frac{BD \cdot BE}{CD \cdot CE} = \frac{a_3^2}{a_2^2},$$
$$\frac{CF \cdot CG'}{AF \cdot AG'} = \frac{a_1^2}{a_3^2},$$
$$\frac{AH \cdot AI}{BH \cdot BI} = \frac{a_2^2}{a_1^2}.$$

Then, points A, B, C, D, E, F, G' belong to a conic (see [1, Problem 15]), and, as five points D, E, F, H, I determine a conic in unique way, $G' = G$. $\square$

Working in definitions of Theorem 2.20 if for some convex body K there exists a triangle T such that $f_K(T) < \min\{b_1, b_2, b_3\}A(K)$, then E is affine-inscribed into K.

Now we will give some easy-to-verify restrictions on weights for which the condition (2.4) holds for every convex body K.

Proposition 2.22. *Consider a convex body K and let f_K be given by the formula* (2.2) *for some weights $a_1, a_2, a_3, b_1, b_2, b_3$. Assume that $b_1 \le b_2 \le b_3$. If $b_3 \le 2b_1$, then* (2.4) *holds.*

Proof. Consider a triangle T of maximal area inscribed into K. It is known [4, p. 467], that $A(T) \ge \lambda A(K)$, where $\lambda = \frac{3\sqrt{3}}{4\pi}$, and then $M_1 + M_2 + M_3 \le (1-\lambda)A(K)$. We may choose the order of the vertices of T so that $M_1 \ge M_2 \ge M_3$. Then

$$\begin{aligned} b_1 A(K) - f_K(T) &\ge \frac{1}{1-\lambda} b_1 (M_1 + M_2 + M_3) - b_1 M_1 - b_2 M_2 - b_3 M_3 \\ &\ge \frac{1}{1-\lambda} b_1 (M_1 + M_2 + M_3) - b_1 M_1 - b_3 (M_2 + M_3) \\ &= \frac{\lambda}{1-\lambda} b_1 M_1 + \Big(\frac{1}{1-\lambda} b_1 - b_3\Big)(M_2 + M_3). \end{aligned}$$

If $\frac{1}{1-\lambda} b_1 - b_3 \ge 0$, then clearly $b_1 A(K) - f(T) > 0$. Otherwise,

$$\begin{aligned} \frac{\lambda}{1-\lambda} b_1 M_1 + \Big(\frac{1}{1-\lambda} b_1 - b_3\Big)(M_2 + M_3) &\ge M_1 \Big(\frac{\lambda}{1-\lambda} b_1 + \frac{2}{1-\lambda} b_1 - 2b_3\Big) \\ &= M_1 \Big(\frac{\lambda+2}{1-\lambda} b_1 - 2b_3\Big). \end{aligned}$$

Finally, if $b_3 < 2.05 < \frac{\lambda+2}{(1-\lambda)2} b_1$, then $f(T) < b_1 A(K)$. ∎

Remark 2.23. *We can replace* 2 *in $b_3 \le 2b_1$ by $\frac{\lambda+2}{(1-\lambda)2} \approx 2.0575$.*

Proposition 2.24. *Consider a convex body K and function f_K given by* (2.2) *for some weights $a_1, a_2, a_3, b_1, b_2, b_3$. If $b_1 = b_2 \le b_3$, then* (2.4) *holds.*

Proof. As K is a compact set, its boundary can contain at most a countable set of linear segments. Then choose a direction θ that does not coincide with direction of any linear segment of ∂K. Let l be a line of direction θ tangent to K. Consider the parallel lines l_h on the distance h from l in the same half-plane with K. For small h we may assume that $l_h \cap \partial K = \{A_h, B_h\}$. As K is convex, for small enough h, $|A_h B_h|$ is increasing and $|A_h B_h| \to 0$, as $h \to 0$. Assume that l_H is tangent to K

at the point X (see Fig. 2.7). Then

$$\begin{aligned} f_K(A_hXB_h) &= b_1M_1 + b_2M_2 + b_3M_3 \\ &= b_1(M_1 + M_2) + b_3M_3 \\ &< b_1(A(K) - A(A_hXB_h)) + b_3M_3 \\ &< b_1A(K) - b_1|A_hB_h|(H-h)/2 + b_3|A_hB_h|h. \end{aligned}$$

If $h\left(\frac{b_3}{b_1} + \frac{1}{2}\right) < \frac{H}{2}$ then $b_1(H-h)/2 > b_3h$, and then $f_K(A_hXB_h) < b_1A(K)$ for a small enough h.

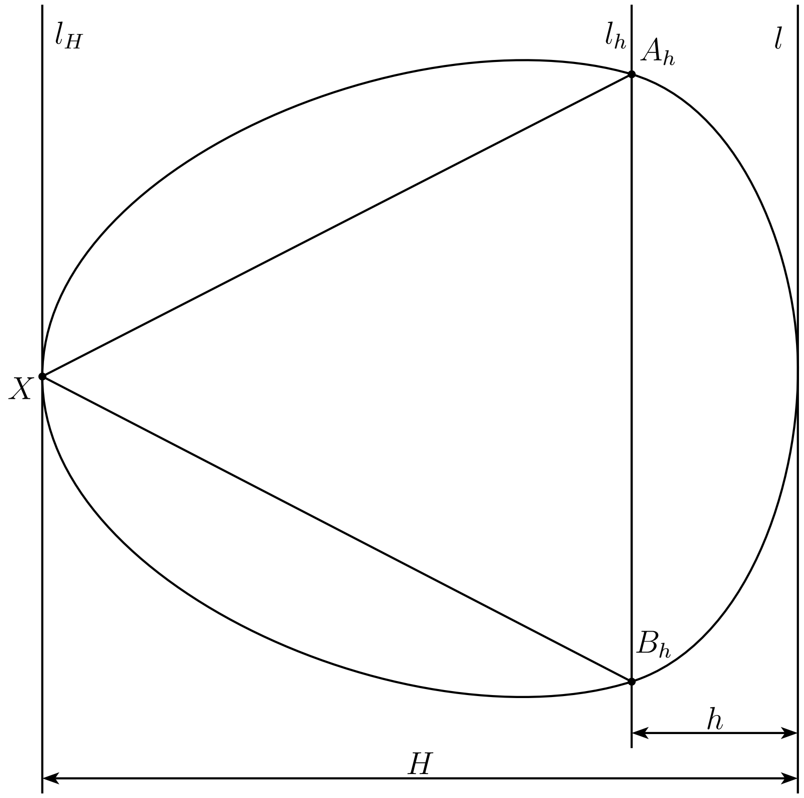

Figure 2.7. Inscribed triangle

□

For certain classes of hexagons, it is easy to find the corresponding weights and to use one of the above propositions. The next example is a new result obtained in our work.

Example 2.25. *For the weights* $a_i = a, b_i = 1$, *the condition* (2.4) *holds for every* *convex body K by Proposition 2.24. Therefore, any hexagon with the angles $\frac{2\pi}{3}$ and sides $1, 2a, 1, 2a, 1, 2a$ (see Fig. 2.8) is affine-inscribed into an arbitrary convex body.*

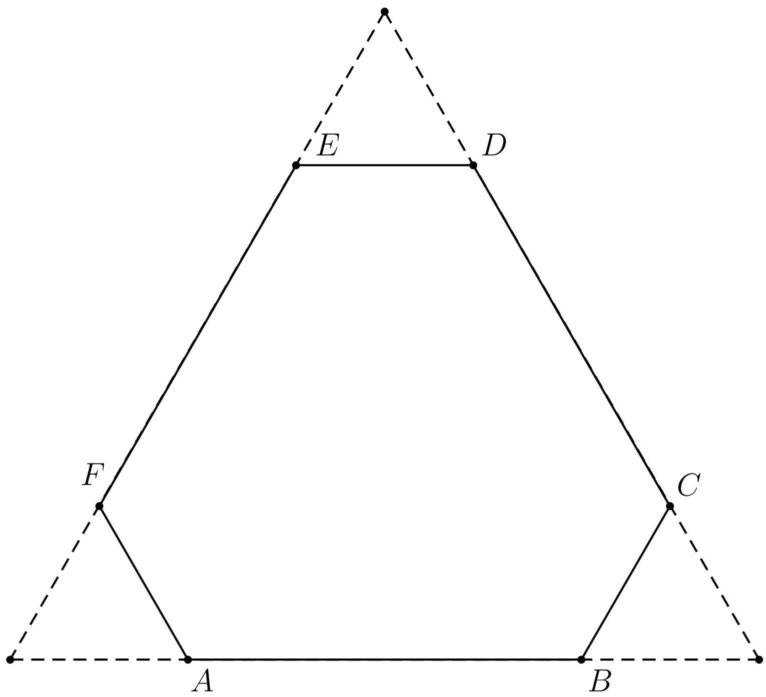

Figure 2.8. Truncated triangle

Remark 2.26. *Makeev* [11] *does not cover this case as the above hexagons are not centrally symmetric (except when $a = \frac{1}{2}$).*

Chapter 3

Banach-Mazur distance

Definition 3.1. For convex bodies P, Q in $\mathbb{R}^n$, we define the *Banach-Mazur distance* $\lambda(P, Q)$ to be the smallest ratio r/s where $r > 0, s > 0$ satisfy $sQ' \subset F(P) \subset rQ''$. Here Q', Q'' are some translates of Q and $F(P)$ is an image of P under a non-degenerate affine transform.

Note that if F is a non-degenerate affine transformation, then the inverse F^{-1} is such a transformation as well. Consider the inclusion $sQ' \subset F(P)$, applying to both sides translation and scaling by a factor that transforms sQ' to rQ'' we obtain inclusion

$$rQ'' \subset r/sF(P)',$$

where $F(P)'$ is some translation of $F(P)$. Applying F^{-1} to the inclusion

$$F(P) \subset rQ'' \subset r/sF(P)',$$

applying a translation and scaling by s we obtain the following relation

$$sP' \subset rsF^{-1}(Q) \subset rP''$$

and hence $\lambda(Q, P) = \lambda(P, Q)$. Banach-Mazur distance between P and Q depends only on affine class of P because its definition contains only affine image $F(P)$. Banach-Mazur distance is symmetric and therefore it depends only on affine class of Q.

Despite the fact that the Banach-Mazur distance is symmetric, it is not a metric on convex sets; however if we consider equivalence classes under non-degenerate affine transforms, then $\ln \lambda$ becomes a metric.

3.1 Approximation by triangles

Consider the following problem: what is the maximal possible value of $\lambda(\Delta, K)$ where Δ is a triangle and K is an arbitrary convex body? Fix some convex body K. We want to find the triangle Δ, inscribed into K, such that $K \subset \lambda\Delta + x$ for some $x \in \mathbb{R}^2$ and λ is minimal. One of the candidates for such Δ is a triangle of maximal area inscribed into K.

The following theorem is proved in [5]:

Theorem 3.2. *If Δ is a triangle of maximal area inscribed into K, then for some $x \in \mathbb{R}^2$ and $\lambda \leq \frac{9}{4}$: $\Delta \subset K \subset \lambda\Delta + x$.*

The idea of the proof is the following: consider a triangle ABC of maximal area, inscribed into K. As the Banach-Mazur distance is affine invariant, assume that ABC is regular. If $A'B'C' = \lambda ABC$ is the minimal triangle containing K with sides parallel to the sides of ABC, then each side of $A'B'C'$ touches K at points A_1, B_1, C_1. It is easy to prove that $\lambda = A(AC_1BA_1CB_1)/A(ABC)$. So if we find an upper estimate for the ratio of the area of a hexagon to the area of the largest inscribed triangle, then we will find the upper estimate for the Banach-Mazur distance between a triangle and an arbitrary convex body. One of the main results of [5] is the following theorem:

Theorem 3.3. *If E is an arbitrary convex hexagon and T is the triangle of maximal area inscribed into E then $\frac{A(E)}{A(T)} \leq \frac{9}{4}$.*

The worst case (the largest ratio of the area of hexagon to the largest inscribed triangle) is obtained for a unique (up to affine transformation) hexagon. However, a triangle of the largest area does not always give the Banach-Mazur distance between two bodies. For instance, if K_5 is a regular pentagon, $\lambda(\Delta, K_5) = 1 + \frac{\sqrt{5}}{2} \approx 2.118$, while a triangle of maximal area gives only $\sqrt{5} \approx 2.24$.

The distance $\lambda(\Delta, K_5)$ is not only the largest known Banach-Mazur distance between triangle and a convex body but also the largest known Banach-Mazur distance between two planar convex bodies. The best known upper estimates are $\frac{9}{4}$ for triangle and a convex body and 3 for two arbitrary convex bodies (see Subsection 3.4.3).

3.2 Possible applications of our technique

Recall that it is conjectured that the maximum possible $\lambda(K, \Delta)$ for arbitrary planar body K and a triangle Δ is achieved when K is a regular pentagon. In this case $\lambda(K, \Delta) = 1 + \frac{\sqrt{5}}{2} \approx 2.118$. The optimal triangle has some of its vertices on the sides of the pentagon. The best known universal upper bound on the Banach-Mazur distance to triangles uses the largest area inscribed triangle, and for the regular pentagon this gives the estimate $\sqrt{5} \approx 2.2361$. In this section we will show that if an approximation triangle T is chosen to minimize $2A(K \setminus T) + A(T \setminus K)$, then there are homotheties of this triangle that will provide an almost optimal estimate, namely $\frac{21}{22}\sqrt{5} - \frac{1}{66} \approx 2.1193$. We conjecture that this approach may improve the bound $\frac{9}{4}$ obtained using the largest area inscribed triangle when approximating a general convex body.

Now we will show the computational details.

Let $K = K_1K_2K_3K_4K_5$ be a regular pentagon and T be a triangle minimizing $2A(K \setminus T) + A(T \setminus K)$. According to Theorem 2.12, $T \cap \partial K$ contains vertices of an affine regular hexagon. There is a unique way (up to numeration of vertices of K) to inscribe the affine-regular hexagon $H = ABCDEF$ into K (see Fig. 3.1). The hexagon H may be inscribed into the isosceles triangle T and homothetic copies T_1, T_2 of T may be inscribed and circumscribed around K.

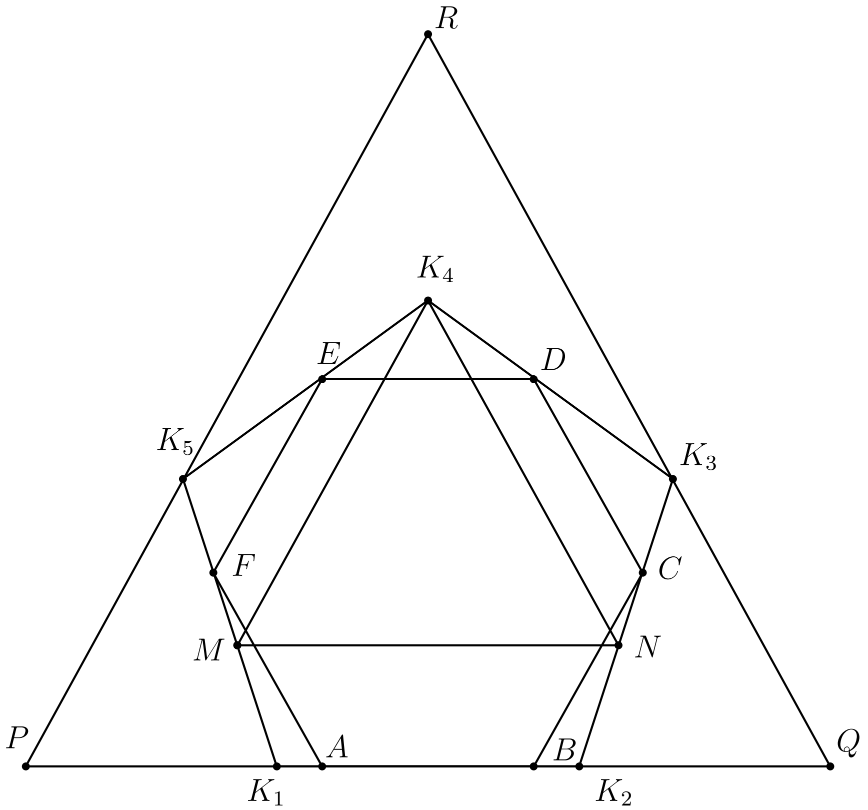

Figure 3.1. Distance between a triangle and a regular pentagon

We may assume that the points A, B belong to the side K_1K_2. Denote $\varphi = \frac{\sqrt{5}+1}{2}$— the golden ratio; we will also use the relation $\varphi^2 = \varphi + 1$. Assume that $|K_1K_2| = 1$, then $K_3K_5 = \varphi$. Stretch K in the direction orthogonal to K_1K_2 so that the distance between K_4 and the line K_3K_5 becomes 1. This is an affine transformation, and thus H will still be affine-regular. If h is the distance between K_4 and ED, then $ED = \varphi h$. Distance between FC and K_1K_2 is $\frac{1+\varphi-h}{2}$. It is easy to see that $|FC| = 1 + \frac{1+\varphi-h}{2} \cdot \frac{\varphi-1}{\varphi}$. Solving the equation $|FC| = 2|ED| = 2\varphi h$, we obtain $h = \frac{3\varphi}{5\varphi+3}$.

Denote $T_1 = K_4MN,\ T_2 = PQR$ and denote by d the distance between MN and AB. Then d satisfies the equation

$$(1+\varphi-d)\frac{2\varphi h}{1+\varphi-h} = 1 + d\frac{\varphi-1}{\varphi}.$$

Solving this equation we obtain

$$d = \left(\frac{(1+\varphi)2\varphi h}{1+\varphi-h} - 1\right)\left(\frac{\varphi-1}{\varphi} + \frac{2\varphi h}{1+\varphi-h}\right)^{-1} = \frac{4\varphi+1}{11},$$

and

$$MN = 1 + d\frac{\varphi - 1}{\varphi} = \frac{3\varphi + 9}{11}.$$

The length $|PQ|$ of the base of T_2 may be found by the formula

$$|PQ| = \varphi + \frac{2\varphi^2 h}{1 + \varphi - h} = \frac{20\varphi - 3}{11}.$$

Calculations show that T_1 and T_2 are homothetic with the coefficient

$$\mu = \frac{|PQ|}{|MN|} = \frac{63\varphi - 32}{33} = \frac{21}{22}\sqrt{5} - \frac{1}{66} \approx 2.1193.$$

3.3 Approximation by simplices in $\mathbb{R}^3$

3.3.1 Reduction to an algebraic optimization problem

Definition 3.4. Denote the plane passing through the points $X, Y, Z \in \mathbb{R}^3$ by (XYZ).

Now we will try to apply the idea of triangle of maximal area to the three dimensional case and find an upper estimate on the Banach-Mazur distance between a simplex and an arbitrary convex body.

Following [5], consider an arbitrary 3-dimensional convex body K and consider a simplex of the largest possible volume contained in it. We may assume that it is a regular simplex $A = A_1A_2A_3A_4$ with the height h. If $C = kA + x$ is homothetic to K with the least possible coefficient k such that

$$A \subset K \subset C,$$

then C touches K at points $C_1, \dots, C_4$. If $h_1, \dots, h_4$ are the distances $d(C_1, (A_2A_3A_4)) = h_1, \dots, d(C_4, (A_1A_2A_3)) = h_4$, then

$$k = \frac{h + h_1 + h_2 + h_3 + h_4}{h} =$$

$$\frac{1}{\mathrm{Vol}(A_1A_2A_3A_4)} \left(\mathrm{Vol}(A_1A_2A_3A_4) + \mathrm{Vol}(C_1A_2A_3A_4) + \mathrm{Vol}(A_1C_2A_3A_4) \right. \tag{3.1}$$

$$\left. + \mathrm{Vol}(A_1A_2C_3A_4) + \mathrm{Vol}(A_1A_2A_3C_4)\right).$$

The analog of this formula is valid in $\mathbb{R}^n$ for every $n \geq 2$.

Proposition 3.5. *If* $\Delta^n \subset \mathbb{R}^n$ *is* n*-dimensional simplex,* K *is arbitrary* n*-dimensional convex body, then* $\lambda(\Delta^n, K) \leq n+2$.

Proof. We may assume that Δ^n is a simplex of maximal volume inscribed into K. Also we may assume that Δ^n is a regular simplex with all altitudes equal to 1. If Δ' is a homothetic image of Δ^n with the least possible coefficient k such that $K \subset \Delta'$, then each face of Δ' touches K at points $A_1, \ldots, A_{n+1}$. Denote by $d_1, \ldots, d_{n+1}$ the distances from A_i to the corresponding faces of Δ^n. As Δ^n is a simplex of maximal volume, inscribed into K, $d_i \leq 1, i = 1, \ldots, n+1$. Finally, using the analog of (3.3.1) in $\mathbb{R}^n$, we obtain

$$\lambda(\Delta^n, K) \leq k = 1 + d_1 + \ldots + d_{n+1} \leq n+2.$$

$\square$

We will prove that under the conditions that K is convex and A has the maximal volume among all tetrahedrons inscribed into K, the following inequality holds: $k < 4\frac{1}{3}$ and thus the simplex of the maximal volume gives the strict upper estimate $4\frac{1}{3}$ for the Banach-Mazur distance between a simplex and an arbitrary convex body in the three-dimensional space.

The non-strict estimate $k \leq 4\frac{1}{3}$ was obtained in [12]. We proved this estimate independently using a similar method. Moreover, a generalization of our method allows us to obtain an estimate in terms of an algebraic optimization problem, and to show that the equality $k = 4\frac{1}{3}$ does not hold.

Theorem 3.6. *Consider all* $(n+1) \times (n+1)$ *real matrices* $B = (b_{ij})$ *with the following properties:*

$$\sum_{j=1}^{n+1} b_{ij} = n, \tag{3.2}$$

$$b_{ij} \in [0, 2], \tag{3.3}$$

$$b_{ii} \in [1, 2], \tag{3.4}$$

$$|\det B'| \leq n \tag{3.5}$$

for every matrix B' obtained from B by replacing some of the rows by the rows containing exactly n ones and one zero.

Denote by M the largest possible value of $(\max \sum_{i=1}^{n+1} b_{ii} - n)$ *over all matrices B satisfying* (3.2)–(3.5). *Then for every convex body K,*

$$\lambda(\Delta^n, K) \leq M.$$

Proof. Consider points in n-dimensional space as points of the hyperplane $x_1 + x_2 + \ldots + x_{n+1} = n$ in $(n+1)$-dimensional space. Assume that T is a simplex with vertices $e_i = (0, 0, \ldots, 0, n, 0, \ldots, 0)$, where n is in the i-th position. Then centres of its faces will form a homothetic simplex T' with coefficient of homothety $-\frac{1}{n}$ with respect to the centre of T. Faces of T' are in the hyperplanes $x_i = 1$ for every $i = 1, \ldots, n+1$, and vertices of T' have coordinates $t_i = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$, where 0 is in the i-th position.

We may assume that T' is the simplex of the largest volume, inscribed into K. Then denote by L the homothetic to T' simplex of the least possible volume, such that $K \subset L$. Denote by $b_i = (b_{i1}, b_{i2}, \ldots, b_{i(n+1)})$, $i = 1, \ldots, n+1$, the points where K touches the corresponding faces of L. Points b_i are inside of K. As T' is the maximal simplex inscribed into K, distances between b_i and planes $x_j = 1$ should be less than or equal to 1; therefore, $|b_{ij} - 1| \leq 1$, or $b_{ij} \in [0, 2]$.

Also $b_{ii} \geq 1$ as the plane $x_i = 1$ should separate the vertices t_i and b_i.

Denote by

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n+1} \end{pmatrix},$$

the matrix consisting of the row-vectors b_i. Also denote by T_i, $i = 1 \ldots n+1$, the simplices obtained from T' when we substitute i-th vertex by b_i. As the distance between b_i and plane $x_i = 1$ is equal to $|b_{ii} - 1|$,

$$\frac{\mathrm{Vol(T_i)}}{\mathrm{Vol(T')}} = |b_{ii} - 1| = b_{ii} - 1.$$

Denote by

$$A = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \ddots & \cdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

the matrix corresponding to T'.

All points $t_i,\ b_i$ are inside of K. Then for every matrix B' consisting of some $n+1$ row-vectors chosen from $\{t_i, b_i\}, i = 1, \ldots, n+1$ the following inequality holds:

$$|\det B'| \leq |\det A| = n,$$

because T' has the maximal volume among all simplices inscribed into K.

The n-dimensional analog of the formula (3.3.1) gives the estimate

$$\lambda(K, \Delta^n) \leq 1 + \sum_{i=1}^{n+1}(b_{ii} - 1) = \sum_{i=1}^{n+1} b_{ii} - n.$$

As the matrix B satisfies the conditions (3.2)–(3.5), the maximal possible value of $\sum_{i=1}^{n+1} b_{ii} - n$ for all matrices satisfying (3.2)–(3.5) is an upper estimate for $\lambda(K, \Delta^n)$.

□

3.3.2 Explicit estimate

Now we will obtain an explicit estimate for the optimization problem from Theorem 3.6 for $n = 3$.

Lemma 3.7. *For any non-negative numbers $s, s_1, \ldots s_n$ consider all matrices $M = (m_{ij})_{i,j=1}^n$ such that*

$$\sum_{j=1}^{n} m_{ij} = s_n \ \ \textit{for all} \ \ i = 1 \ldots n,$$

$$\sum_{j=1}^{n} m_{jj} = s,$$

$$m_{i,j} \geq 0,\ m_{i,i} \geq 1,\ i, j = 1, \ldots, n.$$

Then $\det M$ *is minimal under these restrictions for a matrix which has not more than one nonzero nondiagonal element in each row.*

Proof. First note that restrictions from the statement of the lemma denote a compact set, so the minimum of the determinant is achieved for at least one matrix. Let B be a matrix with minimal determinant which has the least possible number of nonzero elements. If there are two nonzero nondiagonal elements in some row say a_{km}, a_{kl}, denote $B(x)$ the matrix that contains $x(a_{km} + a_{kl})$ on position (k, m) and $(1 - x)(a_{km} + a_{kl})$ on position (k, l). The determinant $\det B(x)$ is a linear function on x and $B = B(\alpha)$ for some $\alpha \in (0, 1)$. Then the determinant of at least one of the matrices $B(0), B(1)$ is less than or equal than $\det B$. Both $B(0)$ and $B(1)$ satisfy the restrictions of the lemma, and thus one of them is a matrix B' satisfying the restrictions of the lemma such that $\det B' \leq \det B$ and B' has less nonzero elements than B. We have a contradiction and B has at most one nonzero element other than the diagonal in each row. □

Lemma 3.8. *Consider the restrictions:*

$$p \in [p_1, p_2], p_1 \geq 0,$$
$$q \in [q_1, q_2], q_1 \geq 0,$$
$$p + q = s.$$

If there is at least one pair of numbers satisfying these conditions, then the product pq *is minimal when* $p = p_i$ *or* $q = q_i,\ i = 1, 2.$

Proof. Restrictions of the lemma determine a compact set, so the minimum is achieved for some numbers p', q'. Assume that $p' \geq q'$. If both of them are inner points of a corresponding interval, there exists $\delta > 0$ such that $p' + \delta \in [p_1, p_2],\ q' - \delta \in [q_1, q_2]$.

$$(p' + \delta)(q' - \delta) = p'q' - \delta(p' - q') - \delta^2 < p'q'.$$

We have a contradiction, so $p' = p_i$ or $q' = q_i,\ i = 1, 2.$ □

Theorem 3.9. *If* K *is a 3-dimensional convex body, then* $\lambda(K, \Delta^3) < 4\frac{1}{3}$.

Proof. According to Theorem 3.6, it is enough to prove that if a matrix $B = (b_{ij})_{i,j=1}^{n+1}$ satisfies the conditions (3.2)–(3.4) and

$$b_{11} + b_{22} + b_{33} + b_{44} \geq 7\frac{1}{3}, \tag{3.6}$$

then $\det B > 3$ or $|\det B'| > 3$ where B' is a matrix obtained from B by substituting some rows by rows containing three ones and one zero.

First we will prove that

$$\text{if } b_{11} + b_{22} + b_{33} + b_{44} \geq 7\frac{1}{3}, \text{ then } \det B \geq 3 \tag{3.7}$$

and we will describe all possible B for which $b_{11} + b_{22} + b_{33} + b_{44} = 7\frac{1}{3}$ and $\det B = 3$.

According to Lemma 3.7 we may consider only matrices with no more than one nonzero nondiagonal element in every row.

If for some k the diagonal element b_{kk} is the only nonzero element in the k-th column, we may set all elements in the k-th row to be 0 as they do not affect $\det B$. So, we may assume that for every k either both k-th row and column have no nonzero elements except b_{kk} or they have exactly one nonzero element except b_{kk}. Let i_k to be k if b_{kk} is the only nonzero element in k-th column (and then all other elements in k-th row are set to 0) and $i_k = l$ if $b_{kl} \neq 0$. We may see that $i_k \neq i_m$ if $k \neq m$ and thus (i_1, i_2, i_3, i_4) is a permutation. As we can rearrange rows and columns of B (in other words, rearrange basic vectors), there are only five possible permutations:

i) (1234),

ii) (1)(234),

iii) (1)(2)(34),

iv) (12)(34),

v) (1)(2)(3)(4).

In each case, the determinant of B may be expressed using the values of diagonal elements, which we will denote by a, b, c, d. We will now prove 3.7 with this notation for each of the cases.

Case i):

$$B = \begin{pmatrix} a & 3-a & 0 & 0 \\ 0 & b & 3-b & 0 \\ 0 & 0 & c & 3-c \\ 3-d & 0 & 0 & d \end{pmatrix},$$

$$detB = abcd - (3-a)(3-b)(3-c)(3-d).$$

Assume that $\det B$ is minimal. Assume that two of the variables are not equal to 2. Without loss of generality, $a, b < 2$. Then

$$\det B = ab(cd - (3-c)(3-d)) + 3(a+b)(3-c)(3-d) - 9(3-c)(3-d) =$$

$$3ab(c+d-3) + 3(a+b)(3-c)(3-d) - 9(3-c)(3-d)$$

Since $a+b > 3,\ a, b \in [1,2]$, we can apply Lemma 3.8 for fixed $a+b$, the expression will be minimal when a or b is equal to 2. So, at least three variables may be assumed to be equal to 2, say $a = b = c = 2$. In this case

$$\det B = 8d - (3-d) = 9d - 3 > 3.$$

Case ii):

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 3-b & 0 \\ 0 & 0 & c & 3-c \\ 0 & 3-b & 0 & d \end{pmatrix},$$

$$\det B = a(bcd + (3-b)(3-c)(3-d)).$$

Similarly, if at least two of b, c, d are not equal to 2 we may assume that $b, c < 2$ and then:

$$\det B = a(bc(d+(3-d))+3(b+c)(3-d)+9(3-d)) = 3a(bc+(b+c)(3-d)+3(3-d));$$

this expression is minimal when $b = 2$ or $c = 2$. Then we may assume that $b = c = 2$ and

$$\det B = a(4d + (3 - d)) = 3a(d + 1) > 3.$$

Case iii):

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 3-c \\ 0 & 0 & 3-d & d \end{pmatrix},$$

$$\det B = ab(cd - (3 - c)(3 - d)) = 3ab(c + d - 3)$$

We may claim that $b = 2$ as the expression is minimal when $a = 2$ or $b = 2$. Assume that:

$$2a(c + d - 3) \leq 1.$$

$a + b + c$ will be maximal if $a = 2, \quad c + d = 3\frac{1}{4}$, and then $a + b + c + d = 7\frac{1}{4} < 7\frac{1}{3}$.

Case iv):

$$B = \begin{pmatrix} a & 3-a & 0 & 0 \\ 3-b & b & 0 & 0 \\ 0 & 0 & c & 3-c \\ 0 & 0 & 3-d & d \end{pmatrix},$$

$$\det B = (ab - (3 - a)(3 - b))(cd - (3 - c)(3 - d)) = 9(a + b - 3)(c + d - 3).$$

Denote $x = a + b - 3; \quad y = c + d - 3$. Then if $x, y \leq 1$, $x + y > 1\frac{1}{3}$: product will be minimal when $x = 1$ but $9xy \geq 9 \times \frac{1}{3} = 3$. This is the only case where $\det B$ can be equal to 3. It happens only if $a = 2, b = 2, c + d = 3\frac{1}{3}$, up to the change in the order of the basic vectors.

Case v)

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

$$\det B = abcd \geq 2 \cdot 2 \cdot 2 \cdot \frac{4}{3} = \frac{32}{3}$$

(here we repeatedly use Lemma 3.8). Thus, in any case $\det B > 3$.

Moreover, we may describe all matrices B satisfying properties $(3.2)-(3.4)$, (3.6)

Let $\tilde{B}$ be such a matrix. We proved that if we use the operations from the proof of the Lemma 3.7 that keep the determinant unchanged and rearrange the order of the basic vectors, then the matrix can be reduced to the case of equality in case iv). Namely, the matrix of the form

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & a & 3-a \\ 0 & 0 & 3-b & b \end{pmatrix}, a+b = 3\frac{1}{3}.$$

In fact, there can be no such operations from Lemma 3.7. Otherwise, consider the last such operation that didn't change the determinant and led to the matrix B_1. Then there is another matrix B_2 (which correspond to $B(0)$ and $B(1)$) such that $\det B_1 = \det B_2 = 3$, and B_1, B_2 differ only by a position of one non-zero non-diagonal element in some row. But then clearly B_2 can't be reduced to case iv, due to the $\det B_2 > 3$, contradiction.

Now consider the matrix

$$B' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & c & 3-c \\ 1 & 1 & 0 & 1 \end{pmatrix};$$

$$\text{then } \det B' = 3c > 3.$$

Therefore, according to Theorem 3.6, $\lambda(K, \Delta^3) < 4\frac{1}{3}$. $\square$

Theorem 3.9 shows that the upper bound $\lambda(K, \Delta^3) \leq 4\frac{1}{3}$ is not exact. It might be improved through finding the solution of the optimization problem described in Theorem 3.6. However, the restrictions in this problem are not linear and there are 12 independent variables, which makes the problem quite difficult.

3.4 Maximal Banach-Mazur distance between convex bodies

A natural question is to determine what is the maximal possible Banach-Mazur distance between two convex bodies in $\mathbb{R}^n$.

3.4.1 Symmetric case

The Banach-Mazur distance between centrally symmetric convex bodies is important for studying the structure of Banach spaces. The Banach-Mazur distance between two Banach spaces is defined as

$$d(X, Y) \equiv \inf\{\|T\| \cdot \|T^{-1}\| : T : X \to Y \text{an isomorphism}\}.$$

If X, Y are finite dimensional spaces denote B_X, B_Y the unit balls in spaces X, Y. Then $d(X, Y) = \lambda(B_X, B_Y)$. As a body B is a unit ball in some finite dimensional Banach space if and only if it is convex and centrally symmetric, studies of Banach-Mazur distance between centrally symmetric convex bodies and between finite dimensional Banach spaces are equivalent problems.

Asplund [2] found the exact upper bound for the Banach-Mazur distance between arbitrary centrally symmetric planar convex bodies. Denote by H and P a regular hexagon and a parallelogram.

Theorem 3.10. *If C is a centrally symmetric planar convex body then $\lambda(C, P) \leq \frac{3}{2}$ and $\lambda(C, P) = \frac{3}{2}$ only if C is an affine regular hexagon.*

Theorem 3.11. *If C is a centrally symmetric planar convex body then $\lambda(C, H) \leq \frac{3}{2}$ and $\lambda(C, H) = \frac{3}{2}$ only if C is a parallelogram.*

These theorems show that any two centrally symmetric convex bodies have Banach-Mazur distance not more than $\frac{3}{2}$, and $\lambda(A, B) = \frac{3}{2}$ only for an affine regular hexagon and parallelogram. In other words, the Banach-Mazur diameter of the set of centrally symmetric planar convex bodies is $\frac{3}{2}$.

Stromquist [15] proved that if B is an arbitrary centrally symmetric planar convex body, and

$$C = \left\{ (x, y) \,\middle|\, |y| \leq 1, \frac{x^2}{2} + y^2 \leq 2, \frac{x^2}{4} + y^2 \leq \frac{4}{3} \right\},$$

then $\lambda(B, C) \leq \sqrt{\frac{3}{2}}$, and therefore he found the Banach-Mazur radius of the set of all centrally symmetric planar convex bodies.

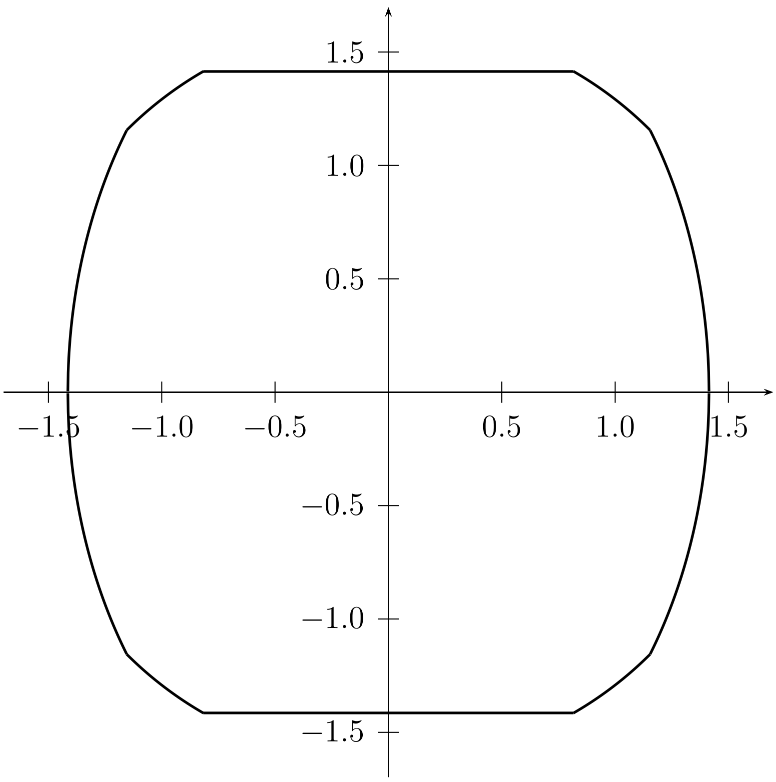

Figure 3.2. Example by Stromquist

Denote by B_∞^n the n-dimensional cube $[-1, 1]^n$. Taschuk [17] proved that for any centrally symmetric convex body K :

$$\lambda(K, B_\infty^n) \leq \sqrt{n^2 - 2n + 2 + \frac{2}{\sqrt{n+1} - 1}}.$$

This result is the best known estimate for the Banach-Mazur distance between the cube and a convex body for small n. There exist estimates that have better asymptotics, the best known is obtained by Giannopoulos [6]: $\lambda(K, B_\infty^n) \leq Cn^{5/6}$ for some absolute constant C and arbitrary centrally symmetric convex body K. Also, the lower estimate with the best asymptotics is found by Szarek [16]: if K is a centrally symmetric convex body then $\lambda(K, B_\infty^n) \geq c\sqrt{n} \log n$ where c is an absolute constant.

3.4.2 General case

In the general case the problem of maximal Banach-Mazur distance between two convex bodies stays unsolved even for $n = 2$. As for asymptotic bounds, according to Rudelson [13], there exist absolute constants C, β such that for arbitrary convex bodies K, L in $\mathbb{R}^n$ we have $\lambda(K, L) \leq Cn^{4/3} \log^\beta n$.

However, for the case when one of the bodies is centrally symmetric and the other body is general, the precise bound was found in paper by Gordon, Litvak, Meyer and Pajor [7].

Theorem 3.12. [7] *If K, L are n-dimensional convex bodies and L is centrally symmetric, then $\lambda(K, L) \leq n$. Moreover, if K' as an affine image of K of maximal volume such that $K' \subset L$, then $K' \subset L \subset nK' + x$.*

This bound is attained when $K = \Delta^n$ is an n-dimensional simplex.

Theorem 3.13. [7] *If K is an n-dimensional centrally symmetric convex body, then $\lambda(\Delta^n, K) \geq n$.*

As the proof of this lower bound is very simple, we will provide it here.

Proof. Suppose $t = \lambda(\Delta^n, K)$ and

$$\Delta^n \subset K \subset t\Delta^n.$$

Let b be the centre of K, then $K = -K + 2b$. We have

$$-K + 2b \subset t\Delta^n,$$

$$K \subset -t\Delta^n + 2b,$$

$$\Delta^n \subset -t\Delta^n + 2b,$$

$$\Delta^n - \frac{2b}{t+1} \subset -t\left(\Delta^n - \frac{2b}{t+1}\right).$$

Therefore, $t \geq n$. □

These two theorems imply $\lambda(\Delta^n, L) = n$ for any centrally-symmetric convex body $L \subset \mathbb{R}^n$. Together with Proposition 3.5, it gives the following bounds for the maximal possible Banach-Mazur distance between a simplex and an arbitrary convex body:

$$n \leq \sup_{K \text{convex}} \lambda(\Delta^n, K) \leq n + 2.$$

3.4.3 Application of The Besicovitch's Theorem

Lassak [10] proves the following theorem, which still remains the best known upper estimate for Banach-Mazur distance between arbitrary planar convex bodies.

Theorem 3.14. *For arbitrary planar convex bodies $P, Q : \lambda(P, Q) \leq 3$.*

Next we will outline the Lassak's proof which uses the Besicovitch's theorem.

Proof. According to Theorem 2.1 there exists an affine regular hexagon $A_1A_2A_3A_4A_5A_6$ inscribed into P. We may assume that this hexagon is regular as Banach-Mazur distance is affine invariant; we may also assume that the origin coincides with the centre of $A_1A_2A_3A_4A_5A_6$. Denote $B_1 = A_6A_1 \cap A_3A_2,\ B_2 = A_1A_2 \cap A_4A_3$ and so on. As P is convex, it is contained in $B_1B_2B_3B_4B_5B_6$. As there exists a regular hexagon, affine-inscribed into Q, there exists a non-degenerate affine transformation F, such that $B_1B_2B_3B_4B_5B_6$ is inscribed into $F(Q)$. Denote $C_1 = B_6B_1 \cap B_3B_2$ and so on. $F(Q) \subset C_1C_2C_3C_4C_5C_6$. Finally, $C_1C_2C_3C_4C_5C_6 = 3A_1A_2A_3A_4A_5A_6$ and thus, $P \subset F(Q) \subset 3P$ (see Fig. 3.3).

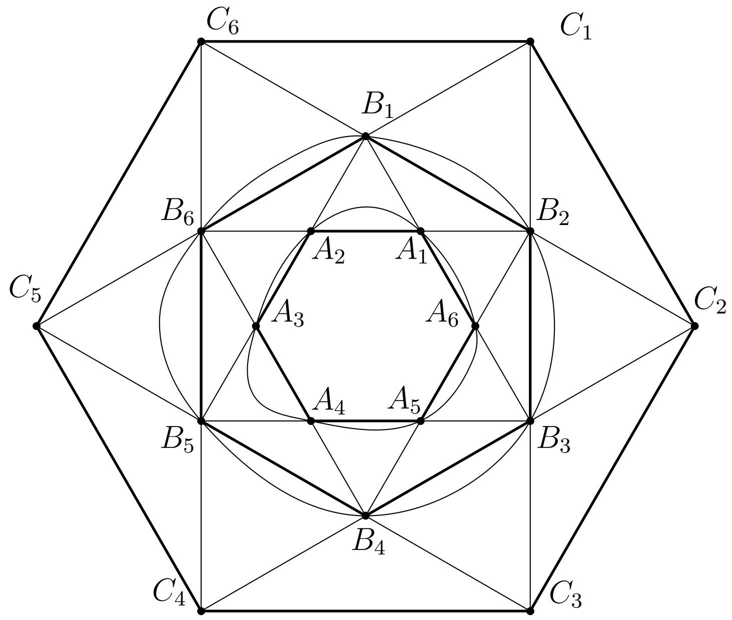

Figure 3.3. Distance between arbitrary planar convex bodies

□

We hope that our generalization of the Besicovitch theorem may become useful in improving the above result.

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