NOTES ON FOREGGER'S CONJECTURE

by

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Abstract

This thesis is devoted to investigation of some properties of the permanent function over the set Ω_n of $n \times n$ doubly stochastic matrices. It contains some basic properties as well as some partial progress on Foregger's conjecture.

Conjecture (Foregger)**.** *For every $n \in \mathbb{N}$, there exists $k = k(n) > 1$ such that, for every matrix $A \in \Omega_n$,*

$$per(A^k) \leq per(A).$$

In this thesis the author proves the following result.

Theorem. *For every $c > 0$, $n \in \mathbb{N}$, for all sufficiently large $k = k(n, c) \in \mathbb{N}$, for all $A \in \Omega_n$ which minimum nonzero entry exceeds c,*

$$\mathrm{per}(A^k) \leq \mathrm{per}(A).$$

This theorem implies that for every $A \in \Omega_n$, there exists $k = k(n, A) > 1$ such that

$$\mathrm{per}(A^k) \leq \mathrm{per}(A).$$

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Contents

Chapter 1

Notation

The following list of notations is used throughout this thesis.

$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{H}$	quaternion algebra
$\overline{n}$	$(1, 2, ..., n)$
π	permutation, *i.e.*, bijection from $\{1, 2, ..., n\}$ to itself
$\mathbf{S}_n$	set of all permutations
$N(\pi)$	number of pairs of elements x, y of $\{1, 2, ..., n\}$ such that $x < y$ and $\pi(x) > \pi(y)$
$\operatorname{sgn}(\pi)$	$(-1)^{N(\pi)}$

$\mathbf{M}_n$	set of all $n \times n$ matrices
A^*	conjugate transpose matrix of A
$\lvert(a_{ij})\rvert$	$(\lvert a_{ij}\rvert)$
$A(i\|j)$	matrix A without ith row and jth column
$O_{n,m}$	$n \times m$ zero matrix
$\mathrm{I_n}$	$n \times n$ identity matrix
J_n	$\left(\frac{1}{n}\right)_{n\times n}$
$\mathbf{P}_n$	set of all $n \times n$ permutation matrices
$\det((a_{ij})_{n\times n})$	$\sum_{\pi\in\mathbf{S}_n} \operatorname{sgn}(\pi)\prod_{i=1}^{n} a_{i\pi(i)}$
$\operatorname{per}((a_{ij})_{n\times n})$	$\sum_{\pi\in\mathbf{S}_n} \prod_{i=1}^{n} a_{i\pi(i)}$
$\sigma_k(A)$	the sum of permanents of all $k \times k$ submatrices of A
$A \oplus B$	if $A \in \mathbf{M}_n$ and $B \in \mathbf{M}_m$, then $A \oplus B = \begin{pmatrix} A & O_{n,m} \\ O_{m,n} & B \end{pmatrix}$
$d(A)$	minimum entry of matrix A
$D(A)$	maximum entry of matrix A
$d^*(A)$	minimum non-zero entry of A
Ω_n	set of all doubly stochastic matrices, *i.e.*, $\Omega_n := \left\{ A = (a_{ij}) \in \mathbf{M}_n : a_{ij} \geq 0, \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1 \right\}$
ρ	metric on $\mathbf{M}_n$, $\rho(A,B)$ is the biggest value among entries of $\lvert A - B\rvert$
$B(A,r)$	open ball in metric ρ with center at A and radius r

Chapter 2

Definitions and basic properties

Definitions that are used throughout this thesis are introduced in this chapter. Since the behaviour of the permanent function on doubly stochastic matrices is the main interest for this thesis, we concentrate on definitions of classes of matrices and some characteristics of matrices (*e.g.*, index of imprimitivity). Also, we provide necessary definitions from probability theory and Clifford algebras which will be used in Section 3.1.3.

Throughout this thesis, we consider several classes of matrices. Positive (nonnegative) matrices are all matrices whose entries are positive (nonnegative). $n \times n$ matrix A is called positive semidefinite if, for every $x \in \mathbb{R}^n$, $x^T Ax \geq 0$. A matrix is called doubly stochastic if it is a nonnegative matrix with row and column sums equal 1. The set of doubly stochastic matrices is denoted by Ω_n. A matrix is called diagonal if all its nonzero entries are on the main diagonal. A $(0,1)$ matrix is a matrix whose entries are either 0 or 1. We denote by Λ_n^k the subset of $(0,1)$ matrices consisting of those matrices that have exactly k 1's in each column and in each row. In other words,

$$\Lambda_n^k := \left\{ A = (a_{ij}) \in \mathbf{M}_n : \, a_{ij} \in \{0,1\}, \, \sum_{i=1}^{n} a_{i\nu} = \sum_{j=1}^{n} a_{\nu j} = k \text{ for all } 1 \leq \nu \leq n \right\}.$$

Definition 2.1.

- *The permanent function of* $A = (a_{ij})_{n \times n}$ *is defined as*

$$\operatorname{per}(A) := \sum_{\pi \in \mathbf{S}_n} \prod_{i=1}^{n} a_{i\pi(i)},$$

  where $\mathbf{S}_n$ *is the set of all permutations of* $\{1, 2, ..., n\}$.

- σ_k *is the sum of permanents of all* $k \times k$ *submatrices. If* $k = n$, *then* $\sigma_n = \operatorname{per}$, *and if* $k = 1$, *then* σ_1 *is the sum of all entries of* A.

There are some immediate consequences of the definition of the permanent function:

- For every $A \in \mathbf{M}_n$, $\operatorname{per}(A) = \operatorname{per}(A^T)$.
- For every $P \in \mathbf{P}_n$, $\operatorname{per}(P) = 1$.
- For every $P, Q \in \mathbf{P}_n$, $\operatorname{per}(PAQ) = \operatorname{per}(A)$.
- If D is a diagonal matrix, then $\operatorname{per}(DA) = \operatorname{per}(AD) = \operatorname{per}(A)\operatorname{per}(D)$.
- Let A and C be square matrices. If

$$X = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

  then $\operatorname{per}(X) = \operatorname{per}(A)\operatorname{per}(C)$.

Note that the permanent of a matrix A does not change if one multiplies A by a permutation matrix, and so we recall some basic properties of permutation matrices.

Lemma 2.2. $P^T = P^{-1}$ *for every permutation matrix* P.

This statement immediately follows from the fact that $PP^T = \mathrm{I}$.

The following definition is useful here in the investigation of Foregger's conjecture (Conjecture 4.1).

Definition 2.3 ([Min88, p. 5])**.** *Matrix* A *is said to be cogredient to a matrix* $\widetilde{A}$*, if there is a permutation matrix* P *such that* $A = P\widetilde{A}P^T$.

Let π be the permutation that corresponds to a permutation matrix P, in other words, $(i, \pi(i))$th entry of P is 1, $1 \leq i \leq n$. Note that PA can be obtained by permutation of rows of A correspondingly to permutation π, while AP^T can be obtained by permutation of columns of A by permutation π. Therefore, A is cogredient to $\widetilde{A}$ if and only if A can be obtained from $\widetilde{A}$ by simultaneous permutation of rows and columns of $\widetilde{A}$.

Cogredient matrices have the following properties:

1. Let P be a permutation matrix. If $A = P\widetilde{A}P^T$ and $B = P\widetilde{B}P^T$, then $AB = P\widetilde{A}\widetilde{B}P^T$, so if A and B are cogredient to $\tilde{A}$ and $\tilde{B}$, respectively, then AB is cogredient to $\widetilde{A}\widetilde{B}$.

2. If A is cogredient to $\widetilde{A}$, then A^n is cogredient to $\widetilde{A}^n$.

Also, in Chapter 4, we discuss the properties of irreducible matrices in superdiagonal block form, and so the following definition is needed.

Definition 2.4 ([Min88, p. 53])**.** *A matrix A in the form*

$$A = \begin{pmatrix} 0 & A_1 & 0 & \dots & 0 \\ 0 & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & A_{m-1} \\ A_m & 0 & 0 & \dots & 0 \end{pmatrix},$$

where the block A_i is an $n_i \times n_{i+1}$ matrix, $i = 1, 2, ..., m-1$, and A_m is an $n_m \times n_1$ matrix, is said to be in the superdiagonal block form, or more specifically, in the superdiagonal $(n_1, n_2, ..., n_m)$-block form.

In this thesis, we consider only superdiagonal block matrices where all n_i are equal.

Definition 2.5 ([Min88, p. 5])**.** *A nonnegative $n \times n$ matrix A, $n \geq 2$, is called reducible if it is cogredient to a matrix of the form*

$$\begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

where B and D are square submatrices. Otherwise, A is irreducible.

Reducible/irreducible matrices are sometimes called decomposable/indecomposable.

Lemma 2.6. *A matrix $A \in \Omega_n$ is reducible if and only if there exists $P \in \mathbf{P}_n$ and square matrices B and C so that*

$$A = P(B \oplus C)P^T,$$

Proof. If $A = P(B \oplus C)P^T$, then, obviously, A is reducible. Conversely, if A is reducible, then there exist $P \in \mathbf{P}_n$ and $m \in \mathbb{N}$ such that $A = P \begin{pmatrix} B & C \\ O & D \end{pmatrix} P^T$, where $B \in \mathbf{M}_m$ and $D \in \mathbf{M}_{n-m}$. Since $A \in \Omega_n$ (column sums equal 1), the sum of all entries of B equals m. $A \in \Omega_n$ (row sums equal 1) implies that the sum of all entries of B plus the sum of all entries of C is m. Therefore, the sum of entries of C equals 0. Since A is a nonnegative matrix, $C = O_{m,n-m}$. □

Note that every matrix $A \in \Omega_n$ can be expressed in the form

$$A = P(A_1 \oplus A_2 \oplus ... \oplus A_l)P^T,$$

where $P \in \mathbf{P}_n$ and A_i is irreducible, $1 \leq i \leq n$.

Definition 2.7 ([Min88, p. 47])**.** *Let A be an irreducible $n \times n$ matrix with maximum eigenvalue r, and suppose that A has exactly m eigenvalues of modulus r. The number m is called the index of imprimitivity of A, or simply the index of A. If $m = 1$, then matrix A is said to be primitive; otherwise, it is imprimitive.*

We denote the index of imprimitivity of A by $\mathrm{ind}(A)$.

The connection between the index of the matrix and its superdiagonal form is shown in Theorem 4.16.

Since Foregger's conjecture (Conjecture 4.1) deals with powers of matrices, we need to determine the form of powers of a matrix in the superdiagonal form.

Lemma 2.8. *Let $A \in \mathbf{M}_{m\mu}$ be in the superdiagonal $\underbrace{(\mu, \mu, .., \mu)}_{m}$-block form,* i.e.,

$$A = \begin{pmatrix} 0 & A_1 & 0 & \dots & 0 \\ 0 & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & A_{m-1} \\ A_m & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $A_i \in \mathbf{M}_\mu$, $1 \le i \le m$. Then, for every $k \in \mathbb{N}_0$ and $0 \le r < m$, there is $P \in \mathbf{P}_n$ such that

$$A^{km+r} = \left(\bigoplus_{i=1}^{m} B_i \right) P$$

where

$$B_i = \left(\prod_{j=i}^{i+m-1} A_j \right)^k A_i A_{i+1} ... A_{i+r-1}.$$

In the case $r = 0$, P is identity permutation.

In the last formula, the index in the product is modulo m, so j should be understood as $j (mod\, m)$.

Proof. Note that in order to find the power of a block matrix, we need to perform all operations as if blocks A_i were numbers. Then the statement is straightforward. $\square$

We also need the following definition.

Definition 2.9.

- *A graph $G = (V, E)$ is bipartite if there is a partition $V = XUY$, $X \cap Y =$, such that $E \subset X \times Y$.*
- *A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V (called partite sets)such that every edge connects a vertex in U to a vertex in V.*
- *A graph is called k-regular if each vertex has degree k.*
- *A perfect matching is a 1-regular subgraph that uses all vertices.*

Let G be a bipartite graph with partite sets $U = \{u_1, u_2, ..., u_n\}$ and $V = \{v_1, v_2, ..., v_n\}$. If $(a_{ij})_{n\times n}$ is the biadjacency matrix for G, then

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge that connects } u_i \text{ and } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Observation 2.10. *The number of perfect matchings in a bipartite graph G coincides with the permanent of the biadjacency matrix for G.*

Proof. Let G be a bipartite graph and A be a biadjacency matrix for G. The perfect matching in G corresponds to the diagonal in A without zeros. Since diagonal products of A can be only 0 and 1, the sum of diagonal products of A over all diagonals is the number of perfect matchings in G. By the definition, this sum is the permanent of A. This finishes proof. □

Therefore, we can restate the problem of finding the permanent of a (0, 1) matrix in terms of the number of perfect matchings.

Also, we define a near-perfect matching as a matching (subgraph in which degree of every vertex is at most 1) that leaves exactly one pair of vertices unmatched. Note that the number of near-perfect matchings corresponds to σ_{n-1} (see, *e.g.*, [Wan99]).

In Section 3.1.3, we discuss methods of approximation of the permanent function. To discuss randomized algorithms, we need some basic definitions from probability theory.

Definition 2.11. *Let 2^X be the set of all subsets of some set X. A set $\mathfrak{F} \subseteq 2^X$ is called a σ-algebra if it satisfies the following condition:*

- *$\mathfrak{F}$ is non-empty: there is at least one $A \subseteq X$ in $\mathfrak{F}$.*
- *$\mathfrak{F}$ is closed under complement: if A is in $\mathfrak{F}$, then so is its complement $X \backslash A$.*
- *$\mathfrak{F}$ is closed under countable unions: if for each $n \in \mathbb{N}$, $A_n \in \mathfrak{F}$, then so is $A = \bigcup_{n \geq 1} A_n$.*

Note that $X \in \mathfrak{F}$.

Definition 2.12. *A probability space is a triple $(X, \mathfrak{F}, \mathbb{P})$ of a sample space X, set $\mathfrak{F}$ and probability $\mathbb{P}$ that satisfies the following condition:*

- *$\mathfrak{F} \subseteq 2^X$ is a σ-algebra. Elements of $\mathfrak{F}$ are called events.*
- *$\mathbb{P} : \mathfrak{F} \to [0, 1]$ is a probability measure,* i.e., *$\mathbb{P}$ satisfies the following conditions.*
    1. *$\mathbb{P}$ is a nonnegative function, $P(\emptyset) = 0$.*

2. *If* $\{A_n\}_{n\geq 1}$ *is a collection of pairwise disjoint subsets of* $\mathfrak{F}$, *then*

$$\mathbb{P}\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mathbb{P}(A_n).$$

3. *The measure of the entire space is 1,* i.e., $\mathbb{P}(X) = 1$.

Definition 2.13.

- $\xi : X \to \mathbb{R}$ *is said to be a real valued random variable, if it is Borel-measurable function.*

- $F : \mathbb{R} \to [0,1]$ *defined by*

$$F(x) := \mathbb{P}\left(\{\omega : \, \xi(\omega) \leq x\}\right)$$

  is called a distribution of real valued random variable ξ.

- *Mathematical expectation of a random variable* ξ *with distribution* F *is defined by*

$$\mathbb{E}\xi := \int_{\mathbb{R}} x dF(x)$$

  if the integral exists.

- *Variance of* ξ *is the value* $D\xi := \mathbb{E}\left[(\xi - \mathbb{E}\xi)^2\right]$.

Let ξ be a random variable with finitely many values. Then we say that ξ is uniformly distributed if it achieves each value with the same probability.

The concept of an independent random variable is basic for all estimators and is used in Section 3.1.3.

Definition 2.14. *Random variables ξ_i, $i = 1, 2, ..., n$, are independent, if $A_i \in \mathfrak{F}$, $i = 1, 2, ..., n$, implies*

$$\mathbb{P}\left(\{\omega : \xi_i(\omega) \in A_i,\ i = 1, 2, ..., n\}\right) = \prod_{i=1}^{n} \mathbb{P}\left(\{\omega : \xi_i(\omega) \in A_i\}\right).$$

In Section 3.1.3 we discuss the approximation algorithm that uses quaternions and Clifford algebras. The quaternions $\mathbb{H}$ is a division algebra, generated by $\{1, i, j, k\}$ with multiplicative rules $i^2 = j^2 = k^2 = ijk = -1$. Note that quaternions is a non-commutative algebra.

Definition 2.15 ([CRS03]). *Let V be an algebra with basis elements $v_{i_1 i_2 \ldots i_k}$, where $1 \leq i_1 < i_2 < \ldots < i_k = m$, together with $v_0 = 1$ and with multiplication rules:*

- *if $i \neq j$, $v_i v_j = v_{ij} = -v_j v_i$,*
- *$v_i^2 = -1$.*

An mth Clifford algebra CL_m is a subalgebra of V with basis $v_{i_1 i_2 \ldots i_k}$, where the number of indeces is even.

Note that CL_1 coincides with the algebra of real numbers $\mathbb{R}$, CL_2 is the algebra of complex numbers $\mathbb{C}$, CL_3 is the quaternion algebra $\mathbb{H}$.

Chapter 3

Basic facts

It seems (see, for example, H. Minc [Min78]) that the permanent function first appeared in 1812 in Binet's [Bin13] and Cauchy's [Cau25] articles.

Definition 3.1. *If* $(a_{ij}) \in \mathbf{M}_n$*, then the permanent function is defined as*

$$\operatorname{per}((a_{ij})) := \sum_{\pi \in \mathbf{S}_n} \prod_{i=1}^{n} a_{i\pi(i)}.$$

Despite the fact that definitions of permanent and determinant look similar, their properties are significantly different. For example, while for every pair of matrices $A, B \in \mathbf{M}_n$,

$$\det(AB) = \det(A)\det(B),$$

we generally cannot state any relation between $\operatorname{per}(AB)$ and $\operatorname{per}(A)\operatorname{per}(B)$.

3.1 Computation

There are several ways how to evaluate the permanent function. If we use the definition

$$\operatorname{per}((a_{ij})_{n\times n}) = \sum_{\pi\in\mathbf{S_n}} \prod_{i=1}^{n} a_{ij},$$

the computation requires $O(n!n)$ arithmetic operations. This is not too efficient!

3.1.1 Pólya's permanent problem

One can see the similarity between the definitions of permanent and determinant and also observe the fact that determinant can be evaluated in polynomial time using Gaussian elimination. Therefore, it would be nice if we could transform any matrix A and get some matrix B such that $\operatorname{per}(A) = \det(B)$. G. Pólya (see, *e.g.*, [McC04, p. 11]) asked if it possible to achieve this by changing some signs of entries of A. The answer is negative in the sense that, if $n \geq 3$, then there is no common pattern of changing signs for all $n \times n$ matrices. For $n = 2$ the pattern of the sign change is trivial:

$$\begin{pmatrix} + & - \\ + & + \end{pmatrix}.$$

For $n \geq 3$, the answer remains negative even for a weaker statement (see, *e.g.*, [BR91]): there is no linear transformation $T : \mathbf{M}_n \to \mathbf{M}_n$ such that $\operatorname{per}(A) = \det(T(A))$, $A \in \mathbf{M}_n$. However, the answer changes for some subclasses of matrices.

If $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$, then we denote $A * B := (a_{ij}b_{ij})_{n\times n}$.

Theorem 3.2 (see, *e.g.*, [BR91, p.237-238]). *Let E be a matrix with entries -1, 0, 1 satisfying* $\mathrm{per}(|E|) = \det(E)$. *Then, for every matrix A that has zeros exactly in the same positions as E,*

$$\mathrm{per}(A) = \det(E * A).$$

For example, let

$$E = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}.$$

Then $\det(E) = \mathrm{per}(|E|) = 6$. Therefore, for every matrix A with the same positions of zeros as in E, $\mathrm{per}(A) = \det(A * E)$.

3.1.2 The fastest algorithms

In 1979, L. Valiant [Val79] proved that the computation of permanent function of (0, 1) matrix is #P-complete. Therefore, under standard assumptions (P$\neq$NP) of theory of complexity, the permanent function cannot be found in polynomial time. The fastest known algorithm (according to [RW08]) for general matrices is due to H. Ryser [Rys63] and it is exponential in time:

$$\mathrm{per}((a_{ij})_{n\times n}) = \sum_{S\subseteq\{1,\ldots,n\}} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j\in S} a_{ij}. \tag{3.1}$$

Using Ryser's formula (3.1), permanent can be evaluated using $O(2^n n^2)$ arithmetic operations.

In 2010, D. Glynn [Gly10] proposed a different algorithm with the same running time.

Theorem 3.3 ([Gly10]). *Let $A = (a_{ij})_{n\times n}$ be a matrix over a field F of a characteristic not two. Then*

$$\mathrm{per}(A) = 2^{1-n}\left[\sum_{\bar{\delta}}\left(\prod_{k=1}^{n}\delta_k\right)\prod_{j=1}^{n}\sum_{i=1}^{n}\delta_i a_{ij}\right],$$

where the outer sum is over all 2^{n-1} vectors $\bar{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in \{\pm 1\}^n$, with $\delta_1 = 1$.

Note that F can be the field of real numbers $\mathbb{R}$.

3.1.3 Approximation of permanent

As discussed above, the fastest known algorithms to find the permanent have exponential running time. Therefore, we are interested in algorithms that allow to approximate the permanent function with an arbitrarily small error and having polynomial running time.

In 2000, L. Linial, A. Samorodnitsky and A. Widgerson [LSW00] showed that there is a polynomial time algorithm of running time $O(n^5 \log n)$ such that, for every nonnegative matrix $A \in \mathbf{M}_n$,

$$e^{-n}\widehat{\mathrm{per}(A)} \leq \mathrm{per}(A) \leq \widehat{\mathrm{per}(A)}, \tag{3.2}$$

where $\widehat{\mathrm{per}(A)}$ is the output of the algorithm. This result is based on the following idea. If we can scale the input matrix A to make it sufficiently close to any $B \in \Omega_n$, then, using van der Waerden-Falikman-Egorychev theorem (see Theorem 3.7 for further discussion), we get bounds on $\mathrm{per}(B)$, namely $e^{-n} < n!/n^n \leq \mathrm{per}(B) \leq 1$, and, therefore, we bound $\mathrm{per}(A)$. This algorithm is fast, however, the interval for $\mathrm{per}(A)$ is huge if n is large.

In 2010, D. Gamarnik and D. Katz [GK10] proposed an algorithm that has better accuracy, but it is in exponential time. If $G = (V_1 \cup V_2, E)$ is a bipartite graph, $|V_1| = |V_2| = n$, then, for every $A \subset V_1 \cup V_2$, we denote the set of neighbors of A by $N(A)$. Let

$$\alpha(G) := \min_{A} \frac{N(A)}{|A|} - 1,$$

where minimization is over all sets A such that $A \subset V_i$, $i = 1, 2$ and $|A| \leq n/2$.

Theorem 3.4 ([GK10])**.** *Let α, $\triangle$ and $\varepsilon > 0$ be fixed, and let G be an n-by-n bipartite graph such that $\alpha(G) \geq \alpha$ and $\triangle$ is a maximum degree of vertices in G. If A is an biadjacency matrix of G, then there is a deterministic polynomial time (in n) approximation algorithm with the output $\widehat{\mathrm{per}(A)}$ satisfying*

$$(1-\varepsilon)^n \widehat{\mathrm{per}(A)} \leq \mathrm{per}(A) \leq (1+\varepsilon)^n \widehat{\mathrm{per}(A)}.$$

The algorithm is not of polynomial time in ε, because the dependence of the running time on ε is in the form $O(n^{\varepsilon^{-\log^{-1}(1+\alpha)\log(\triangle)}})$. However, the accuracy of this algorithm is significantly better than the accuracy in (3.2).

To discuss randomized algorithms, first we state the remarkable result of M. Jerrum, A. Sinclair and E. Vigoda [JSV01] that was a breakthrough in the theory of approximation of permanents.

We need the following definitions. An algorithm for computing permanent has relative error ε means that the output $\widehat{\mathrm{per}(A)}$ satisfies

$$(1-\varepsilon)\widehat{\mathrm{per}(A)} \leq \mathrm{per}(A) \leq (1+\varepsilon)\widehat{\mathrm{per}(A)}$$

with high probability. The randomized algorithm with relative error ε is said to be a fully polynomial randomized approximation scheme (FPRAS) if its running time for the input of size n is bounded by a polynomial in n and ε^{-1}. (see, *e.g.*, [Kar99] for more details).

Theorem 3.5 ([JSV01])**.** *There exists FPRAS for the permanent of a nonnegative matrix.*

The running time of the algorithm, provided in the paper for $\varepsilon^{-1} = cn^5 \log n$ is $O(n^{26}(\log n)^3)$. The power of n is fairly large, but the same authors simplified the algorithm [JSV04] and got the running time $O(\varepsilon^{-2}n^{10}(\log n)^2)$.

Another approach is to use determinants for computing the permanent. The idea is based on the following formula. Let $A = (a_{ij})$. For independent random variables ξ_{ij} with mean 0 and variance 1, let B be a matrix such that its entries are $\sqrt{a_{ij}}\xi_{ij}$. Then

$$\mathbb{E}(|\det(B)|^2) = \mathrm{per}(A).$$

This idea appeared first in the paper of C. Godsil and I. Gutman [GG81], where ξ_{ij} are independent Bernoulli random variables ($\mathbf{P}(\xi = \pm 1) = 1/2$). This estimator is called Godsil-Gutman estimator. However, $|\det(B)|^2$ can have big variance, therefore, it is not commonly used as an estimator of $\mathrm{per}(A)$.

N. Karmarkar, R. Karp, R. Lipton, L. Lovász and M. Luby [KKL+93] proved the following result. If ξ_{ij} are independent Bernoulli random variables, then the (arithmetic) mean of $|\det(B)|^2$ for $3^{n/2}$ (independent) sampling of $\{\xi_{ij}\}$ is FPRAS of running time $p(n)3^{n/2}\frac{1}{\varepsilon^2}$, where p is a polynomial. If ξ_{ij} are independent uniformly distributed over $\{1, -1/2+\sqrt{3}/2i, -1/2-\sqrt{3}/2\}$ random variables, then the (arithmetic) mean of $|\det(B)|^2$ for $2^{n/2}$ (independent) sampling of $\{\xi_{ij}\}$ is FPRAS of running time $p(n)3^{n/2}\frac{1}{\varepsilon^2}$, where p is a polynomial.

In the paper [CRS03], S. Chien, L. Rasmussen and A. Sinclair modified the Godsil-Gutman estimator. Namely, they proved following.

Theorem 3.6 ([CRS03])**.** *Let $A = (a_{ij}) \in \mathbf{M}_n$ be a $(0,1)$ matrix, and let u_{ij} be independent uniformly distributed random variables whose support is*

$$\{\pm e_1, \pm e_2, ..., \pm e_{2^{m-1}}\},$$

where e_i is a basis element of Clifford algebra $CL_m = CL_{4\lceil 1/2\log_2 n\rceil+2}$. If $B = (a_{ij}u_{ij})$, then

$$\mathbb{E}(|\det(B)|^2) = \mathrm{per}(A),$$

and the number of samples needed to approximate the permanent with multiplicative *accuracy $1 \pm \varepsilon$ is bounded by a constant.*

This result is promising, since we need to work in the algebra whose dimension is $2^{4\lceil 1/2 \log_2 n \rceil + 2 - 1} \approx 2n^2$ which is polynomial in n. However, for $m \geq 3$, CL_m is not a communicative algebra, so there is no common definition of the determinant. In the formula above, the definition $\det((b_{ij})_{n \times n}) = \sum_{\pi \in \mathbf{S}_n} (-1)^{\mathrm{sgn}(\pi)} \prod_{i=1}^{n} b_{i\pi(i)}$ (Cayley's determinant) was used. For this definition of the determinant function, there is no known fast method of evaluation determinant for high dimensions of CL_m.

In the same paper [CRS03], the authors modify the estimation for $CL_3 = \mathbb{H}$ (quaternion algebra) in the following way. Let B be a matrix over $\mathbb{H}$. We denote a determinant which is computed using Gaussian elimination only by $\mathrm{Gauss}(B)$ (Dieudonné determinant). Note that $\mathrm{Gauss}(B)$ and $\det(B)$ can be significantly different. $\mathrm{Gauss}(B)$ depends on the way we find it, but $|\mathrm{Gauss}(B)|^2$ does not. Since we use Gaussian elimination, the algorithm for $\mathrm{Gauss}(B)$ in polynomial in time, but the number of trials to get high probability (with factor $1 \pm \varepsilon$) is $(3/2)^n$. The running time of this algorithm is less than for the algorithm in [KKL$^+$93].

3.2 Inequalities

In this section, we list some of the inequalities that represent basic properties of the permanent function.

Van der Waerden's conjecture (now van der Waerden-Falikman-Egorychev Theorem, see Theorem 3.7) is one of the most famous conjectures in the theory of the permanent function. It was posed in 1926 and the full solution was found only in 1981. The conjecture had a strong influence on the permanental theory, and numerous papers containing partial results on this conjecture (see, *e.g.*, [MN59]) were published over the years. Finally, it was resolved independently by D. Falikman [Fal81] and G. Egorychev [Ego81].

Theorem 3.7 (van der Waerden-Falikman-Egorychev, [vdW26])**.** *If $A \in \Omega_n$, then*

$$\operatorname{per}(A) \geq \operatorname{per}(J_n), \tag{3.3}$$

and the equality is obtained only if $A = J_n$.

Note that minimum of the determinant over Ω_n is easy to find using geometrical representation of determinant (the absolute value of determinant is volume of a parallelepiped).

An elegant idea of Falikman's proof of van der Waerden-Falikman-Egorychev theorem is to consider function

$$F_\varepsilon((a_{ij})) := \operatorname{per}((a_{ij})) - \frac{\varepsilon}{\prod_{i=1}^n \prod_{j=1}^n a_{ij}}$$

on the set of positive doubly stochastic matrices and show that, for $\varepsilon > 0$, the minimum of F_ε is attained at J_n which implies that the minimum of $\operatorname{per}(F_0)$ is attained at J_n. However, this proof does not imply that there are no other matrices in Ω_n for which (3.3) can become an equality.

Egorychev's proof is based on London's result (see, *e.g.*, [Min78, p. 85]) that if A is a matrix which provides an equality in the van der Waerden-Falikman-Egorychev theorem, then $\operatorname{per}(A(i|j)) \geq \operatorname{per}(A)$ for all i, j, and the fact that if for $A \in \Omega_n$, $\operatorname{per}(A(i|j)) \geq \operatorname{per}(A)$, then $\operatorname{per}(A(i|j)) = \operatorname{per}(A)$. Recall that $A(i|j)$ denotes $(n-1) \times (n-1)$ matrix obtained from $A \in \mathbf{M}_n$ by deleting ith row and jth column.

Also, we can consider the generalization of van der Waerden conjecture for σ_k. This conjecture was posed by H. Tverberg [Tve63] in 1963 and solved by S. Friedland in 1982.

Theorem 3.8 ([Fri82]). *For each $n \in \mathbb{N}$ and for each $k \in \{2, 3, ..., n\}$, minimum of σ_k over Ω_n is attained uniquely at J_n.*

E. Wang [Wan79] showed that permanent function is convex on Ω_2, but is not convex on Ω_n for $n \geq 3$. However, R. Brualdi and M. Newman [BN65] proved that, for $\lambda \in [0, 1]$ and for every $A \in \Omega_n$,

$$\operatorname{per}(\lambda I_n + (1-\lambda)A) \leq \lambda + (1-\lambda)\operatorname{per}(A).$$

Clearly, we can not state general inequality between determinant and permanent. However, for some classes of matrices we can compare determinant and permanent. Clearly, for all nonnegative matrices A, $\operatorname{per}(A) \geq \det(A)$. The following result provide the comparison for another class of matrices.

Theorem 3.9 ([MN62]). *If A is positive semidefinite hermitian, then*

$$\operatorname{per}(A) \geq \det(A) \geq 0.$$

Equality holds if and only if A has a zero row or A is a diagonal matrix.

Some inequalities for the determinant function can be modified for the permanent function. M. Marcus [Mar66] proved an analog of the equality for determinants:

$$\det(A \otimes B) = (\det(A))^m (\det(B))^n,$$

where $\otimes$ is the tensor or direct product of A and B. This equality can be rewritten in the form

$$|\det(A \otimes B)|^2 = (\det(AA^*))^m (\det(B * B))^n.$$

Theorem 3.10 ([Mar66])**.** *If $A \in \mathbf{M}_n$ and $B \in \mathbf{M}_m$, then*

$$|\operatorname{per}(A \otimes B)|^2 \leq (\operatorname{per}(AA^*))^m (\operatorname{per}(B^*B))^n.$$

Equality holds if and only if either

1. *A has a zero row or B has a zero column, or*

2. *each of A and B is a product of a diagonal matrix and a permutation matrix.*

The other example of the inequality for the determinant that can be modified for the permanent function is Hadamard inequality.

Let A be $n \times n$ complex matrix whose jth column is a_j. $|a_j|$ is the Euclidian norm of a_j. Hadamard inequality for the determinant function is

$$|\det(A)| \leq \prod_{j=1}^{n} |a_j|.$$

Theorem 3.11 ([CLL06]). *For any vectors a_i, $i = 1, 2, ..., n$ in $\mathbb{C}^n$ we have inequality*

$$|\operatorname{per}(A)| \leq \frac{n!}{n^{n/2}} \prod_{j=1}^{n} |a_j|.$$

For $N > 2$, the equality holds if and only if either

1. *at least one of vectors a_j is zero, or*

2. *rank of F is 1 and, for every j, $j = 1, 2, .., n$, all entries of a_j have the same absolute value.*

The matrix is said to be substochastic if it is nonnegative and its row sums does not exceed 1.

Theorem 3.12 ([Gib68]). *If $A \in \mathbb{M}_n$ is a substochastic matrix, then*

$$\operatorname{per}(\mathrm{I}_n - A) \geq \det(\mathrm{I}_n - A) \geq 0.$$

The following inequality is an analog of Cauchy-Schwarz inequality and is useful for proving other inequalities. In particular, the proof of Theorem 4.5 is based on the inequality.

Theorem 3.13 ([MN62]). *If A and B are in $\mathbf{M}_n$, then*

$$|\operatorname{per}(AB)|^2 \leq \operatorname{per}(AA^*) \operatorname{per}(B^*B).$$

If the above inequality holds, then one of the following eventualities must occur:

1. *a row of A or a column of B is zero;*

2. *no row of A and no column of B is zero, and there exist a diagonal matrix D and a permutation matrix P, both in $\mathbf{M}_n$, such that*

$$A^* = BDP.$$

Later D. Đokovic generalized this result for submatrices. Recall that $A[\overline{\alpha}|\overline{\beta}]$ is a matrix on the intersection of $\overline{\alpha}$ rows and $\overline{\beta}$ columns.

Theorem 3.14 ([Đok67]). *Let A, B be complex $n \times n$ square matrices, then*

$$|per(AB[\overline{\alpha}|\overline{\beta}])|^2 \leq per(AA^*[\overline{\alpha}|\overline{\alpha}])\, per((B^*B)[\overline{\beta}|\overline{\beta}]).$$

The equality holds if and only if one of following conditions is satisfied:

- *a row of A or a column of B is zero;*
- *no row of A and no column of B is zero, and there exist a diagonal matrix D and a permutation matrix P, both in $\mathbf{M}_n$, such that $A^*[\overline{n}|\overline{\alpha}] = B[\overline{n}|\overline{\beta}]DP$.*

The following three theorems allow to compare the permanent of some special matrices.

Theorem 3.15 (see,*e.g.*, [Min78, p. 9]). *Suppose $\overline{c} = (c_1, c_2, ..., c_n)$ has positive entries, and let the entries of $\overline{a} = (a_1, a_2, ..., a_n)$ and $\overline{b} = (b_1, b_2, ..., b_n)$ be nonnegative integers. Let A and B be $n \times n$ matrices whose (i, j) entries are $c_i^{a_j}$ and $c_i^{b_j}$, respectively. A necessary and sufficient condition that*

$$\mathrm{per}(A) \leq \mathrm{per}(B)$$

is that there exists $C \in \Omega_n$ such that $a = Cb$.

Recall that the elementwise product of two matrices $A, B \in \mathbf{M}_n$ is denoted by $A * B$.

Theorem 3.16 ([WW07]). *Let $A = (a_{ij}) \in \mathbf{M}_n$ and $B = (b_{ij}) \in \mathbf{M}_n$ are positive matrices such that*

$$\frac{a_{i1}}{a_{i+1\,1}} \leq \frac{a_{i2}}{a_{i+1\,2}} \leq ... \leq \frac{a_{in}}{a_{i+1\,n}}, \quad i = 1, 2, ..., n-1, \tag{3.4}$$

and

$$\frac{b_{i1}}{b_{i+1\,1}} \leq \frac{b_{i2}}{b_{i+1\,2}} \leq ... \leq \frac{b_{in}}{b_{i+1\,n}}, \quad i = 1, 2, ..., n-1. \tag{3.5}$$

Then

$$\frac{\operatorname{per}(A * B)}{n!} \geq \frac{\operatorname{per}(A)}{n!}\frac{\operatorname{per}(B)}{n!}. \tag{3.6}$$

The inequality in (3.6) is reversed for (3.4) and reversed (3.5). The equality in (3.6) holds if and only if $\operatorname{rank}(A) = 1$ *or* $\operatorname{rank}(B) = 1$.

Theorem 3.17 ([Fal97]). *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & A_{12} \\ A_{21} & B_{22} \end{pmatrix}$$

be such that A_{ii}, B_{ii} are hermitian $n_i \times n_i$ matrices such that $A_{ii} - B_{ii}$ is positive semidefinite matrix, $i = 1, 2$, and $A_{12} = A_{21}^*$. Then

$$\operatorname{per}(A) \geq \operatorname{per}(B) \geq 0.$$

Another interesting problem is to find asymptotical bounds on permanent. T. Tao and V. Vu [TV09] established the following relation.

Recall that ξ is called a random variable with Bernoulli distribution if $\mathbb{P}\{\xi = 1\} = \mathbb{P}\{\xi = -1\} = 1/2$. A matrix is said to be a random Bernoulli matrix if its entries are independent random variables with Bernoulli distribution.

Theorem 3.18 ([TV09]). *Let M_n be a random Bernoulli matrix of size n.*

1. *Asymptotically almost surely, $|\operatorname{per}(M_n)| = n^{(1/2+o(1))n}$ as $n \to \infty$.*

2. *There is $c > 0$ such that, for every $\varepsilon > 0$ and $n \geq N(\varepsilon)$,*

$$\mathbb{P}\{|\operatorname{per}(M_n)| \geq n^{(1/2-\varepsilon)n}\} \geq 1 - n^{-c}.$$

3.3 Generalization

In this section, we discuss several generalizations of the permanent and determinant functions. To emphasize similarity of the generalization and the permanent, some results are provided. The idea of all generalizations mentioned below are based on similarity of definitions of the permanent and determinant functions.

Definition 3.19 ([CW05]). *Let $A = (a_{ij}) \in \mathbf{M}_n$ and let H be a subgroup of $\mathbf{S}_n$. Let $\chi : H \to \mathbb{C}$ be a function on H. We define function*

$$d_\chi^H(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If χ is a nontrivial homomorphism, then d_χ^H is called a Schur function. If $H = \mathbf{S}_n$, then such Schur function is called immanent.

In particular, if $H = \mathbf{S}_n$ and $\chi \equiv \text{sgn}$, then d^H_χ is the determinant function. If $H = \mathbf{S}_n$ and $\chi \equiv 1$, then d^H_χ is the permanent function. In this sense, permanent is a partial case of generalized matrix functions.

The following theorem is a generalization of Theorem 3.13.

Theorem 3.20 ([MM65]). *Let $A, B \in \mathbf{M}_n$. Then*

$$|d^H_\chi(AB)| \leq d^H_\chi(AA^*)d^H_\chi(B^*B).$$

In case $\chi \equiv 1$, the equality holds if one of the following conditions is satisfied.

- *A has a zero row;*
- *B has a zero column;*
- *$A = DPB^*$, where D is a diagonal matrix, P is a permutation matrix.*

Also, we are interested when H is a proper subgroup of $\mathbf{S}_n$. For example, if $H \subset \mathbb{S}_n$ is all even permutations and $\chi \equiv 1$, then d^H is called the "even permanent" and is denoted by per_{ev}.

The following equality is a straightforward property of the even permanent. For $A \in \mathbf{M}_n$,

$$\text{per}(A) + \det(A) = 2\,\text{per}_{ev}(A).$$

Similar to van der Waerden - Falikman - Egorychev theorem (see Theorem 3.7), R. Brualdi and B. Liu conjectured that minimum of even permanent over doubly stochastic matrices is $1/2n!/n^n$. However, I. Wanless [Wan08] found counterexamples for $n = 4, 5$. The validity of the conjecture for large n is still unknown.

For the following two generalizations, -1 in the definition of determinant is replaced by a parameter. However, the formula for the determinant can be written in several ways, therefore such generalization is not unique. In this thesis, we discuss two such generalization, namely, the "α-permanent" and the "q-permanent" (defined below).

Definition 3.21 ([Brä12]). *The α-weighted permanent of a square matrix $A = (a_{ij})$ of order n is defined by*

$$\operatorname{per}_\alpha(A) = \sum_{\pi \in \mathbf{S}_n} \alpha^{c(\pi)} \prod_{i=1}^{n} a_{i\pi(i)},$$

where $c(\pi)$ is the number of disjoint cycles in π. The α-weighted determinant is defined by

$$det_\alpha(A) = \alpha^n \operatorname{per}_{1/\alpha}(A).$$

If $\alpha = 1$, then $\operatorname{per}_\alpha$ is the permanent function, and if $\alpha = -1$, then $\operatorname{per}_\alpha$ is the determinant function. Therefore, the α-permanent is a generalization of the permanent and determinant function.

Note that Theorem 3.9 implies that for a positive semidefinite Hermitian matrix A, $\operatorname{per}(A) \geq 0$. The following theorem states for which α the generalization of this statement holds.

Theorem 3.22 ([Brä12]). *Let $D_{\mathbb{R}}$ be the set of all α such that* $\det_\alpha(A) \geq 0$ *for all* *symmetric positive semidefinite $A \in \mathbf{M}_n$ and let $D_{\mathbb{C}}$ be the set of all α such that $\det_\alpha(A) \geq 0$ for all (complex) hermitian positive semidefinite matrices A.*

$$D_{\mathbb{R}} = \left\{-\frac{1}{m+1} : m \in \mathbb{N}\right\} \cup \left\{\frac{2}{m+1} : m \in \mathbb{N}\right\} \cup \{0\},$$

$$D_{\mathbb{C}} = \left\{\pm\frac{1}{m+1} : m \in \mathbb{N}\right\} \cup \{0\}.$$

Also, according to [KM09], it seems that α-permanent is #P-complete for $\alpha \neq 1$, however, it is not proved yet.

Definition 3.23 ([Lal98]). *Let $A = (a_{ij})_{n\times n}$. q-permanent is defined by*

$$\operatorname{per}_q(A) := \sum_{\pi\in\mathbf{S}_n} q^{N(\pi)} \prod_{i=1}^{n} a_{i\pi(i)},$$

where $N(\pi)$ is the number of transpositions in the permutation π.

In this definition, $\operatorname{per}_1 = \operatorname{per}$ and $\operatorname{per}_{-1} = \det$.

Note that the q-permanent sometimes is called μ-permanent (see, *e.g.*, [dF10]).

Recall that for Hermitian positive semidefinite matrix A, $\operatorname{per}(A) \geq 0$ (it is a corollary from Theorem 3.9). For q-permanent the analog of this statement holds for all $q \in [-1, 1]$.

Theorem 3.24 ([Bap92]). *For a Hermitian positive semidefinite matrix A,*

$$\operatorname{per}_q(A) \geq 0, \quad q \in [-1, 1].$$

Moreover, in the case of tridiagonal positive semidefinite matrices, the statement of Theorem 3.9 can be generalized.

Theorem 3.25 ([DF05]). *Let G be a tree, which biadjacency matrix A is a tridiagonal positive semidefinite matrix. Then* $\mathrm{per}_q(A)$ *is increasing function on q in $[-1,1]$. In other words,*

$$\det(A) \le \mathrm{per}_q(A) \le \mathrm{per}(A), \quad -1 \le q \le 1.$$

3.4 Doubly stochastic matrices

Recall that matrix is doubly stochastic if it is nonnegative and its row and column sums equal 1. The set of doubly stochastic matrices is denoted by Ω_n. Note that matrix is doubly stochastic implies that the matrix is square.

R. Brualdi [BR91, p.14-22, p.379-380] provides two following motivations for concentrating on doubly stochastic matrices.

We consider the set

$$\mathcal{F} := \{f : \{1,2,...,n\} \to \{1,2,...,n\} : \ f \text{ is a bijection}\}.$$

Let $g : \Omega \to \mathcal{F}$ be a probability distribution on the functions of $\mathcal{F}$. We associate with it function $g' : \Omega \times \{1,2,...,n\} \to \{1,2,...,n\}$ such that $g'(i) = g'(\omega, i) = (g(\omega))(i)$. Let $a_{ij} := \mathbb{P}(g'(i) = j) = a_{ij})$. We show that $A = (a_{ij})_{n\times n}$ is doubly stochastic. Clearly, $a_{ij} \ge 0$. Note that

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \mathbb{P}(g'(i) = j) = \mathbb{P}(g'(i) \in \{1,2,...,n\}) = 1$$

and

$$\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} \mathbb{P}\{\omega : (g(\omega))(i) = j) = \sum_{i=1}^{n} \mathbb{P}\{\omega : (g(\omega))^{-1}(j) = i\} =$$

$$= \mathbb{P}\{\omega : (g(\omega))^{-1}(j) \in \{1, 2, .., n\} = 1.$$

Therefore, by R. Brualdi [BR91, p.379-380], A is a probabilistic analog of a bijection from $\{1, 2, ..., n\}$ onto itself.

Another motivation for concentration on doubly stochastic matrices is a majorization (see Theorem 3.27).

Definition 3.26 ([BR91, p.15-16]). *Let $\overline{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $\overline{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ be nonincreasing vectors. Then $\overline{x}$ is majorized by $\overline{y}$, denoted $\overline{x} \preceq \overline{y}$, provided their partial sums satisfy*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, ..., n,$$

with equality for $k = n$.

Theorem 3.27 ([HLP34]). *Let $\overline{x} \in \mathbb{R}^n$ and $\overline{y} \in \mathbb{R}^n$ be nonincreasing vectors. Then $\overline{x} \preceq \overline{y}$ if and only if there is a doubly stochastic matrix A such that $X = YA$.*

Famous representatives of doubly stochastic matrices are J_n and all permutation matrices, *i.e.*, $\mathbf{P}_n \subset \Omega_n$.

There is a criterion to determine if a matrix is doubly stochastic.

Theorem 3.28. *Let $u = (1, 1, ..., 1)^T \in \mathbb{R}^n$. The following statements are equivalent:*

1. *$A \in \Omega_n$*

2. *A is nonnegative matrix, $Au = u$ and $u^T A = u^T$*

3. *A is nonnegative matrix $AJ_n = J_n A = J_n$*

4. *A is nonnegative matrix and has eigenpair* $(1, u)$*, and* A^T *has an eigenpair of* $(1, u)$

Note that this criterion (the equivalency of the first and the third statements) implies that set of doubly stochastic matrices is closed under multiplication (see Property 1)

Theorem 3.29. *The spectral radius of doubly stochastic matrix is 1.*

Proof. Let $A = (a_{ij}) \in \Omega_n$. Theorem 3.28 implies that spectral radius $\rho(A)$ is at least 1. Let (λ, v) be an eigenpair of A, *i.e.*, $Av = \lambda v$. Let v_j be a greatest entry of v by absolute value. Then, $\sum_{k=1}^{n} a_{jk} v_k = \lambda v_j$.

$$|\lambda||v_j| = \left| \sum_{k=1}^{n} a_{jk} v_k \right| \leq \sum_{k=1}^{n} a_{jk} |v_k| \leq |v_j| \sum_{k=1}^{n} a_{jk} = |v_j|.$$

Hence, $|\lambda| \leq 1$ and so $\rho(A) \leq 1$. □

The set of doubly stochastic matrices is closed under following operations.

Properties.

1. $A \in \Omega_n$ and $B \in \Omega_n$ implies $AB \in \Omega_n$.

2. If $A, B \in \Omega_n$ and $\lambda \in [0, 1]$, then $\lambda A + (1 - \lambda B) \in \Omega_n$, *i.e.*, Ω_n is a convex set

3. If $A \in \Omega_n$ and $B \in \Omega_m$, then $A \oplus B \in \Omega_{n+m}$.

4. If $A \in \Omega_n$ and $B \in \Omega_m$, then $A \otimes B \in \Omega_{nm}$.

Proof. All above-mentioned properties except the first one can be easily verified by the definition of doubly stochastic matrix.

Let $A, B \in \Omega_n$.

$$(AB)J_n = A(BJ_n) = AJ_n = J_n$$

and

$$J_n(AB) = (J_nA)B = J_nB = J_n.$$

Then, by Theorem 3.28, $AB \in \Omega_n$. $\square$

To discuss inverse matrices of doubly stochastic matrices, we need the following notation.

Definition 3.30. *A matrix is said to be doubly quasi-stochastic, if its row and column sums are 1.*

Note that a nonnegative doubly quasi-stochastic matrix is doubly stochastic.

Theorem 3.31 (see, *e.g.*, [Min88, p. 123])**.** *The inverse of non-singular doubly quasi-stochastic matrix is doubly quasi-stochastic.*

The proof follows from the following observation. A matrix A is doubly quasi-stochastic if and only if $AJ_n = J_nA = J_n$.

The following theorem gives a nice property of doubly stochastic matrices.

Theorem 3.32 ([Kön16])**.** *For every doubly stochastic matrix, there is at least one diagonal which does not contain zero entries.*

Also, this theorem follows from van der Waerden-Falikman-Egorychev theorem (Theorem 3.7).

Since $\mathbf{P}_n \subset \Omega_n$ and Ω_n is convex, the convex hull of $\mathbb{P}_\ltimes$ is a subset of Ω_n. G. Birghoff proved that those two sets coincide.

Theorem 3.33 (Birkhoff, see,*e.g.*,[Min88, p. 117])**.** *Ω_n is a convex hull of $\mathbb{P}_n$.*

Note that Theorem 3.32 is a corollary of this Theorem 3.33.

Note that $\dim\Omega_n = (n-1)^2$. By Carathéodory's theorem, every doubly stochastic matrix can be represented as a convex combination of $\dim \Omega_n + 1 = n^2 + 2n + 2$ permutation matrices.

Chapter 4

Notes on Foregger's Conjecture

4.1 History of Foregger's conjecture

The conjecture of Foregger first appeared in the book of H. Minc "Permanents" [Min78] in 1978. This book contains the most important theorems in the theory of the permanent function and also lists some of the most important open conjectures of that time, such as van der Waerden's conjecture (Conjecture 1)(see Theorem 3.7). Foregger's conjecture appeared as Conjecture 17 there.

To clarify the date when the conjecture was first posed, the author of this thesis contacted T. Foregger who kindly replied that on 17 December 1977, he wrote a letter to H. Minc where this conjecture was stated.

Conjecture 4.1 (Foregger, see [Min78]). *There exists $k = k(n) > 1$ such that, for* *every* $A \in \Omega_n$,

$$per(A^k) \leq per(A). \tag{4.1}$$

We emphasize that k is independent of A in the statement of this conjecture.

This conjecture can be easily validated for $n = 2$ (see the next proposition), is proven by D. Chang in 1990 for $n = 3$ (see below for more details), and remains open for $n \geq 4$.

For $n = 2$, Foregger's conjecture follows from the following proposition.

Proposition 4.2. *If* $A, B \in \Omega_2$, *then* $\operatorname{per}(AB) \leq \operatorname{per}(A)$.

Proof. Let $A = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ and $B = \begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix}$. Then

$$AB = \begin{pmatrix} a+b-2ab & 1-a-b+2ab \\ 1-a-b+2ab & a+b-2ab \end{pmatrix},$$

and so

$$\begin{aligned} \operatorname{per}(AB) - \operatorname{per}(A) =& (a+b-2ab)^2 + (1-a-b+2ab)^2 - a^2 - (1-a)^2 \\ =& -2b(1-b)(1-2a)^2. \end{aligned}$$

Since $b \in [0,1]$, the last expression doesn't exceed zero, so $\operatorname{per}(AB) \leq \operatorname{per}(A)$.

□

So using Proposition 4.2 with $B = A^{k-1}$ shows that any $k > 1$ satisfies Foregger's conjecture for $n = 2$.

The full proof of Foregger's conjecture for $n = 3$ was obtained by D. Chang in 1990.

Theorem 4.3 ([Cha90]). *For $A \in \Omega_3$,*

$$\mathrm{per}(A^8) \leq \mathrm{per}(A).$$

The following proposition is an important part of Chang's proof of Theorem 4.3 and provides a partial proof of Foregger's conjecture for $n \times n$ matrices.

Proposition 4.4 ([Cha90]). *If $A \in \Omega_n$ and $\frac{1}{2} < per(A) < 1$, then for every integer $m \geq 2$*

$$per(A^m) < per(A).$$

Despite the fact that Foregger's conjecture is still unsolved for $n \geq 4$, there are some results verifying it for some subclasses of doubly stochastic matrices.

Theorem 4.5 ([MN62]). *If A is a symmetric positive-definite doubly stochastic matrix and $B \in \Omega_n$ commutes with A, then $\mathrm{per}(AB) \leq \mathrm{per}(A)$. If A is non-singular, then equality holds if and only if B is a permutation matrix.*

Corollary 4.6. *If $A \in \Omega_n$ is a symmetric positive-definite matrix, then, for every $k \in \mathbb{N}$, $\mathrm{per}(A^k) \leq \mathrm{per}(A)$.*

Proof. If we put $B = A^{k-1}$, then all conditions of Theorem 4.5 hold for every $k > 1$. □

Theorem 4.7 ([Cha83]). *Let $W(a)$ be a doubly stochastic matrix with all elements except for the main diagonal entries equal to a. Then, for every $A \in \Omega_n$,*

$$per(AW(a)) \leq per(W(a)).$$

Corollary 4.8. *For every $k \in \mathbb{N}$,*

$$\mathrm{per}(W^k(a)) \leq \mathrm{per}(W(a)).$$

In 1983, D. Chang proved the following theorem.

Theorem 4.9 ([Cha83]). *For any positive integer n and $0 < c < \frac{1}{n}$, there exists an integer $N = N(n, c) > 1$ such that, if $A = (a_{ij}) \in \Omega_n$ with all entries greater than c, then $per(A^{2^k}) \leq per(A)$ for all $k \geq N$.*

This theorem implies that, for every positive doubly stochastic matrix A, there is $k > 1$ such that $\mathrm{per}(A^k) \leq \mathrm{per}(A)$. In this thesis, we generalize this result.

Recall that $d^*(A)$ is the minimum of positive entries of A, and let

$$\mathcal{C}(c) := \{A \in \Omega_n : d^*(A) > c\}.$$

Our main result in this thesis is the following theorem.

Theorem 4.10. *For every $c > 0$, $n \in \mathbb{N}$, for all sufficiently large $k = k(n, c) \in \mathbb{N}$, $A \in \mathcal{C}(c)$ implies*

$$\mathrm{per}(A^k) \leq \mathrm{per}(A).$$

Throughout this thesis, "for sufficiently large k" should be understood as "there exists $N = N(n, c)$ such that for all $k \geq N$".

Corollary 4.11. *For every $A \in \Omega_n$, there exists $k = k(n, A) > 1$ such that*

$$\operatorname{per}(A^k) \leq \operatorname{per}(A).$$

Proof. Note that $d^*(A)$ is the minimum of positive entries of A, therefore, $d^*(A) > 0$. Using Theorem 4.10, for sufficiently large k and for every $B \in \mathcal{C}(d^*(A)/2)$, $\operatorname{per}(B^k) \leq \operatorname{per}(B)$. Since $A \in \mathcal{C}(d^*(A)/2)$, $\operatorname{per}(A^k) \leq \operatorname{per}(A)$. □

Recall that Λ_n^m is the set of $(0,1)$ matrices whose row and column sums equal m. Note that $A \in \Lambda_n^m$ implies $1/mA \in \Omega_n$.

Corollary 4.12. *There exists $k = k(n, m) > 1$ such that, for every $A \in \Lambda_n^m$,*

$$\operatorname{per}(A^k) \leq m^{(k-1)n} \operatorname{per}(A). \tag{4.2}$$

Proof. Every entry of $A \in \Lambda_n^m$ is either 0 or 1. Then, every entry of $1/mA$ is 0 or $1/m$, and so $d^*(1/mA) \geq 1/m$. Since $1/m\Lambda_n^m \subset \Omega_n$, using Theorem 4.10, for sufficiently large k and for every $A \in 1/m\Lambda_n^m$,

$$\operatorname{per}((1/mA)^k) \leq \operatorname{per}(1/mA).$$

Using $\operatorname{per}(cA) = c^n \operatorname{per}(A)$, we get (4.2). □

4.2 Auxiliary lemmas

Lemma 4.13. *If, for some k, inequality (4.1) holds for both matrices $A \in \Omega_n$ and $B \in \Omega_m$, then it also holds for $P(A \oplus B)P^T$, where P is an arbitrary permutation matrix from $\mathbf{P}_{n+m}$.*

Proof. Suppose (4.1) holds for $A \in \Omega_n$ and $B \in \Omega_m$ and for the same k. Then,

$$\operatorname{per}(P(A \oplus B)P^T) = \operatorname{per}(A \oplus B) = \operatorname{per}(A)\operatorname{per}(B),$$

and, by Property 2 of cogredient matrices,

$$\begin{aligned}\operatorname{per}\left(\left(P(A \oplus B)P^T\right)^k\right) &= \operatorname{per}(P(A^k \oplus B^k)P^T) = \operatorname{per}(A^k \oplus B^k) \\ &= \operatorname{per}(A^k)\operatorname{per}(B^k) \leq \operatorname{per}(A)\operatorname{per}(B),\end{aligned}$$

and the proof is complete. □

Corollary 4.14. *If, for some k, inequality (4.1) holds for all $A_i \in \Omega_{n_i}$, $i = 1, 2, .., m$, it also holds for $P(A_1 \oplus A_2 \oplus ... \oplus A_m)P^T$, where P is an arbitrary permutation matrix from $\mathbf{P}_{\sum_i n_i}$.*

The following theorem can be proved using Corollary 4.14. Recall that

$$\mathcal{C}(c) := \{A \in \Omega_n : d^*(A) > c\}.$$

Theorem 4.15. *Let $c \in (0, 1/n)$. Suppose that for every $n \in \mathbb{N}$, there exists $N'(n)$ such that for all $k \geq N'(n)$, and all for every irreducible $A \in \mathcal{C}(c)$,*

$$\operatorname{per}(A^k) \leq \operatorname{per}(A).$$

Then, for every $n \in \mathbf{N}$, for sufficiently large k, $A \in \mathcal{C}(c)$ implies $\operatorname{per}(A^k) \leq \operatorname{per}(A)$.

Proof. Fix $n \in \mathbb{N}$ and let $N(n) := \max_{i=1,2,...,n} N'(i)$. Every $A \in \mathcal{C}(c)$ is cogredient to a direct sum of irreducible matrices, *i.e.*,

$$A = P(A_1 \oplus A_2 \oplus ... \oplus A_m)P^T,$$

where $P \in \mathbf{P}_n$ and A_j is an irreducible matrix, $j = 1, 2, ..., m$. Then, $\text{per}(A) = \prod_{j=1}^{m} \text{per}(A_j)$.

Note that since A is doubly stochastic, so are A_j, $j = 1, 2, ..., m$, and $d^*(A) > c$ implies $d^*(A_j) > c$. Also, note that

$$A^k = P(A_1^k \oplus A_2^k \oplus ... \oplus A_m^k)P^T.$$

Note that $k \geq N(n)$ implies $k \geq N'(i)$, $i = 1, 2, ..., n$, then, for $k \geq N(n)$,

$$\text{per}(A^k) = \prod_{j=1}^{m} \text{per}(A_j^k) \leq \prod_{j=1}^{m} \text{per}(A_j) = \text{per}(A).$$

□

The following theorem shows a connection between the form of a matrix and its index of imprimitivity.

Theorem 4.16 ([Min88], p. 109)**.** *Let $A \in \Omega_n$ be an irreducible matrix with index of imprimitivity* $\text{ind}(A) = m$. *Then m divides n (*i.e., $n = m\mu$*), and the matrix A is cogredient to a matrix in the superdiagonal block form*

$$\begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix}, \tag{4.3}$$

where all the blocks are μ-square.

Theorem 4.17 ([BR91, p. 74]). *Let $A \in \mathbf{M}_n$ be an irreducible nonnegative matrix with index of imprimitivity equal to m. Let k be a positive integer. Then A^k is irreducible if and only if k and m are relatively prime. In general, there is a permutation matrix P of order n (independent of k) such that $A^k = P\left(\bigoplus_{i=1}^r B_i\right)P^T$, where r is the greatest common divisor of k and m. The matrices B_i are irreducible matrices and* $\mathrm{ind}(B_i) = m/r$, $i = 1, 2, ..., r$.

Corollary 4.18. *Suppose that $A \in \Omega_n$ is irreducible and* $\mathrm{ind}(A) = m$, $\mu = n/m$. *For every $k \in \mathbb{N}$, A^{km} is cogredient to $\oplus_{i=1}^m B_i$, where B_i are irreducible matrices with* $\mathrm{ind}(B_i) = 1$, $i = 1, 2, ..., m$.

Using Theorem 4.16, A is cogredient to a matrix (4.3). Then, by Lemma 2.8, A^{km} is cogredient to $\bigoplus_{i=1}^m B_i$. Corollary 4.18 implies that B_i is irreducible, $i = 1, 2, ..., m$.

Note that if A is cogredient to a matrix in the form (4.3), then Lemma 2.8 implies that $B_i = \left(\prod_{j=i}^{i+m-1} A_j\right)^k$.

Theorem 4.19 ([HJ85, p. 520]). *If $A \in \mathbf{M}_n$ is a nonnegative matrix, then A is primitive if and only if A^{n^2-2n+2} is positive.*

Corollary 4.20. *Let $A \in \Omega_n$ be an irreducible matrix with index of imprimitivity m ($\mu = n/m$ is necessarily an integer). Then A is cogredient to the matrix in the form (4.3); $A^{m(n^2-2n+2)!}$ is cogredient to a direct sum of positive $\mu \times \mu$ matrices.*

Proof. Let $A \in \Omega_n$ be an irreducible matrix with index of imprimitivity m, $n = m\mu$. Using Theorem 4.16, we get that A is cogredient to the matrix in the form (4.3).

Lemma 2.8 implies that $A^m = P(B_1 \oplus B_2 \oplus ... \oplus B_m)P^T$, where $P \in \mathbf{P}_n$ and $B_i \in \Omega_\mu$. Using Corollary 4.18 to B_i, we get that B_i is primitive matrix, $i = 1, 2, ..., m$. Note that $\mu^2 - 2\mu + 2$ divides $(n^2 - 2n + 2)!$. Therefore, using Theorem 4.19, we have that $B_i^{(n^2-2n+2)!}$ is positive, $i = 1, 2, ..., m$. Lemma 2.8 implies that

$$A^{m(n^2-2n+2)!} = P\left(B_1^{(n^2-2n+2)!} \oplus B_2^{(n^2-2n+2)!} \oplus ... \oplus B_m^{(n^2-2n+2)!}\right)P^T.$$

Therefore, $A^{m(n^2-2n+2)!}$ is cogredient to a direct sum of positive $\mu \times \mu$ matrices. $\square$

In the following lemmas, the behavior of minimum entries of powers of a matrix is investigated.

Recall that for $A = (a_{ij})_{n\times n}$, $d(A) := \min\{a_{ij} | 1 \leq i, j \leq n\}$ and $D(A) := \max\{a_{ij} | 1 \leq i, j \leq n\}$. We now state several properties of these functions.

Lemma 4.21. *Let $A \in \Omega_n$. Then $D(A) \geq 1 - (n-1)d(A)$.*

Proof. Let $A = (a_{ij})$ and suppose $a_{i^*j^*} = D(A)$. Since $A \in \Omega_n$,

$$a_{i^*j^*} = 1 - \sum_{i \neq i^*} a_{ij^*} \geq 1 - (n-1)d(A).$$

$\square$

Lemma 4.22. *For every $A, B \in \Omega_n$, $d(AB) \geq \max\{d(A), d(B)\}$. In particular, for $A \in \Omega_n$ and $k \in \mathbb{N}$, $d(A^k) \geq d(A)$.*

Proof. Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $AB = (c_{ij})_{n\times n}$. Suppose $d(AB) = c_{i^*j^*} = \arg\min\{c_{ij} | 1 \leq i, j \leq n\}$. Then

$$c_{i^*j^*} = \sum_{r=1}^{n} a_{i^*r} b_{rj^*}.$$

Since all entries of A and B are nonnegative,

$$d(AB) = c_{i^*j^*} = \sum_{r=1}^{n} a_{i^*r} b_{rj^*} \geq \min_r d(B) \sum_{r=1}^{n} a_{i^*r} = d(B)$$

and

$$d(AB) = c_{i^*j^*} = \sum_{r=1}^{n} a_{i^*r} b_{rj^*} \geq \min_r d(A) \sum_{r=1}^{n} b_{rj^*} = d(A).$$

$\square$

Lemma 4.23. *Suppose $A, B \in \Omega_n$. If $d(B) \geq d$ and $d(A) \geq d$, then $d(AB) \geq d(2 - nd)$.*

Proof. Every entry in AB is an inner product of a row in A and a column in B. Let $\overline{a} = (a_1+d, a_2+d, ..., a_n+d)$ be a row in A and $\overline{b} = (b_1+d, b_2+d, ..., b_n+d)$ be a column in B. Note that a_i and b_i are nonnegative, $i = 1, 2, ..., n$, and $\sum_i a_i = \sum_i b_i = 1-nd$. Then

$$\overline{a} \cdot \overline{b} = \sum_{i=1}^{n} (a_i + d)(b_i + d) = 2d - nd^2 + \sum_{i=1}^{n} a_i b_i \geq 2d - nd^2,$$

and hence $d(AB) \geq 2d - nd^2$. $\square$

Lemma 4.24. *If A, B are nonnegative matrices such that $d^*(A) > d_1$ and $d^*(B) > d_2$, then $d^*(AB) > d_1 d_2$. In particular, if $A \in \Omega_n$ is such that $d^*(A) > c$, then $d^*(A^k) > c^k$ for every $k \in \mathbb{N}$.*

Proof. Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $AB = (c_{ij})_{n\times n}$. Suppose that c_{ij} is a positive entry of AB. As A and B are nonnegative matrices and

$$0 < c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

there is m such that $a_{im}b_{mj} > 0$. Hence a_{im} and b_{mj} are non-zero. Thus,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \geq a_{im}b_{mj} \geq d^*(A)d^*(B) > d_1 d_2,$$

and so $d^*(AB) \geq d_1 d_2$. □

Recall that $c_k \uparrow$ means that $\{c_k\}$ is a non-decreasing sequence.

Theorem 4.25 ([Cha84b, Lemma 1])**.** *Let $0 < c_0 \leq d_0 < 1/n$, $\{c_k\}_{k\geq 0}$ be the sequence defined recursively: $c_{k+1} = 2c_k - nc_k^2$, and $\{d_k\}_{d\geq 0}$ be a sequence such that $1/n \geq d_{k+1} \geq 2d_k - nd_k^2$, $k \geq 0$. Then, for $k \geq 0$,*

1. *$c_k \leq d_k \leq 1/n$*
2. *$c_k \uparrow$, $d_k \uparrow$*
3. *$\lim_{k\to\infty} c_k = \lim_{k\to\infty} d_k = 1/n$.*

Corollary 4.26. *Let $n \in \mathbb{N}$ and $c \in (0, 1/n)$. Let $\{c_k\}$ be defined by $c_{k+1} = 2c_k - nc_k^2$, $c_0 = c$. Then for every $A \in \Omega_n$ such that $d(A) \geq c$ and for every $k \geq N$, $d(A^{2^k}) \geq c_k$.*

Proof. Let $A \in \Omega_n$ be such that $d(A) \geq c$. Let $d_k = d(A^{2^k})$. Clearly, $d_k \leq 1/n$. Using Lemma 4.23, we have $d_{k+1} \geq 2d_k - nd_k^2$. Theorem 4.25 completes the proof. □

Lemma 4.27. *If $n \in \mathbb{N}$ and $c \in (0, 1/n)$, then*

$$\lim_{k\to\infty} \inf\{d(A^k) : A \in \Omega_n,\, d(A) \geq c\} = 1/n.$$

Proof. Corollary 4.26 implies that

$$\inf\{d(A^{2^k}) : A \in \Omega_n,\, d(A) \geq c\} \geq c_k.$$

Since $\lim_{k\to\infty} c_k = 1/n$ by Theorem 4.25,

$$\lim_{k\to\infty} \inf\{d(A^{2^k}) : A \in \Omega_n,\, d(A) \geq c\} = 1/n.$$

By Lemma 4.22, we have that $\{d(A^k)\}_{k\geq 1} \uparrow$ for every $A \in \Omega_n$, and so $\inf\{d(A^k) : A \in \Omega_n,\, d(A) \geq c\} \uparrow$.

Since the subsequence $\inf\{d(A^{2^k}) : A \in \Omega_n,\, d(A) \geq c\}$ of a monotonic sequence $\inf\{d(A^k) : A \in \Omega_n,\, d(A) \geq c\}$ converges to $1/n$, so does the sequence $\inf\{d(A^k) : A \in \Omega_n,\, d(A) \geq c\}$. □

In the following lemmas, we show that the inequality $\mathrm{per}(A) \geq \mathrm{per}(A^k)$ holds in a neighborhood of $J_n = (1/n)_{n\times n}$ for every $k \in \mathbb{N}$.

Lemma 4.28. *Let $A, B \in \Omega_n$ be such that $d(B) \geq d(A) \geq \frac{n-2}{(n-1)^2}$. Then* $\mathrm{per}(AB) \leq \mathrm{per}(A)$.

Note that our proof of Lemma 4.28 is similar to the proof of Lemma 2 in [Cha84b]:

Theorem 4.29 ([Cha84b, Lemma 2])**.** *Let $A = (a_{ij}) \in \Omega_n$ be such that*

$$\frac{n-2}{(n-1)^2} \leq d(A) \leq \frac{1}{n},$$

then $\mathrm{per}(A^2) \leq \mathrm{per}(A)$.

To prove Lemma 4.28, we need the following notations. Let $W_n(a) = (w_{ij})$ be an $n \times n$ matrix such that $w_{ii} = a$ and $w_{ij} = (1-a)/(n-1)$, $1 \leq i \neq j \leq n$. Let $T_n(a) = (t_{ij})$ be an $n \times n$ matrix such that $t_{11} = a$, $t_{1j} = t_{i1} = (1-a)/(n-1)$ and $t_{ij} = (n+a-2)/(n-1)^2$ for $2 \leq i, j \leq n$. Note that, if $a \in [0,1]$, then $W_n(a) \in \Omega_n$ and $T_n(a) \in \Omega_n$.

The proof of Lemma 4.28 is based on the following theorems.

Theorem 4.30 ([Cha83, Corollary 2])**.** *Let $b \in [0,1]$ and $a = (1-b)/(n-1)$. Then for every $A = (a_{ij}) \in \Omega_n$ such that $a_{ij} \in [\min(a,b), \max(a,b)]$,*

$$\mathrm{per}(A) \leq \mathrm{per}(W_n(b)).$$

We need this theorem for $b = (1-d)/(n-1)$, where $d \in [(n-2)/(n-1)^2, 1/n]$.

Corollary 4.31. *Let $d \in [0, 1/n]$. Then for every $A = (a_{ij}) \in \Omega_n$ such that*

$$\frac{1-(1-d)/(n-1)}{n-1} \leq a_{ij} \leq \frac{1-d}{n-1},$$

$1 \leq i, j \leq n$, the following inequality holds:

$$\mathrm{per}(A) \leq \mathrm{per}\left(W_n\left(\frac{1-d}{n-1}\right)\right).$$

Proof. Note that

$$\frac{1-\frac{1-d}{n-1}}{n-1} \leq \frac{1-d}{n-1}$$

is equivalent to $d \leq 1/n$, and the proof of the Corollary is complete. □

Theorem 4.32 ([Cha83, Lemma 5]). *Let $0 \leq d \leq 1/n$. Then for $n \geq 2$,*

$$\operatorname{per}\left(W_n\left(\frac{1-d}{n-1}\right)\right) \leq \operatorname{per}(T_n(d)).$$

Theorem 4.33 ([Cha84a, Theorem 1]). *Let $d \in [0,1]$ be such that $\operatorname{per}(T_n(0)) > \operatorname{per}(T_n(d))$. Then for every $A = (a_{ij}) \in \Omega_n$ with $a_{11} = d$,*

$$\operatorname{per}(T_n(d)) \leq \operatorname{per}(A).$$

Theorem 4.34 ([VAP80]). *$\operatorname{per}(T_n(d))$ is decreasing as a function of d on $[0, 1/n]$.*

Corollary 4.35. *For every $d \in [0, 1/n]$ and for every $A \in \Omega_n$ such that $d(A) = d$,*

$$\operatorname{per}(T_n(d)) \leq \operatorname{per}(A).$$

Proof. Let $d \in [0, 1/n]$ and let $A = (a_{ij}) \in \Omega_n$ be such that $d(A) = a_{i^*j^*} = d$. Let P be a permutation such that its $(1, i^*)$th entry is 1 and let Q be a permutation matrix such that its $(j^*, 1)$th entry is 1. Then $(1,1)$th entry of PAQ is d. Theorems 4.33 and 4.34 imply that

$$\operatorname{per}(T_n(d)) \leq \operatorname{per}(PAQ) = \operatorname{per}(A).$$

□

Proof of Lemma 4.28. Since the inequality is obvious if $A = J_n$ or $B = J_n$, we can assume that $A, B \neq J_n$, and so $d(A) < 1/n$ and $d(B) < 1/n$.

Let $d := d(A)$. By Lemma 4.23, $d(AB) \geq d(2 - nd)$. Therefore,

$$1/n \geq d(AB) \geq d(2 - nd) = d + d(1 - nd)$$

$$\geq \frac{n-2}{(n-1)^2} + d\left(1 - n\frac{n-2}{(n-1)^2}\right) = \frac{1-(1-d)/(n-1)}{n-1}.$$

By Lemma 4.21,

$$D(AB) \leq 1-(n-1)d(AB) \leq 1-(n-1)\frac{1-(1-d)/(n-1)}{n-1} \leq \frac{1-d}{n-1}.$$

Using Corollary 4.31, Theorem 4.32 and Corollary 4.35,

$$\operatorname{per}(AB) \leq \operatorname{per}\left(W_n\left(\frac{1-d}{n-1}\right)\right) \leq \operatorname{per}(T_n(d)) \leq \operatorname{per}(A).$$

$\square$

Corollary 4.36. *If $A \in \Omega_n$ is such that $d(A) \geq \frac{n-2}{(n-1)^2}$, then $\operatorname{per}(A^k) \leq \operatorname{per}(A)$ for every k.*

Proof. If $k = 1$, the statement is trivial. If $k > 1$, by Lemma 4.22, $d(A^{k-1}) \geq d(A)$. Lemma 4.28 implies that $\operatorname{per}(A^k) \leq \operatorname{per}(A)$. $\square$

4.3 Proof of the main result

Recall that

$$\mathcal{C}(c) := \{A \in \Omega_n : d^*(A) \geq c\},$$

and denote

$$\mathcal{C}_0(c) := \{A \in \Omega_n : d^*(A) \geq c,\ A \text{ is irreducible}\}.$$

For convenience we introduce the following statement. Let $c > 0$ and let $\mathcal{M}(c)$ be some class of matrices.

Statement 4.37. *There is $N = N(n,c)$ such that, for every $k \geq N$ and every $A \in \mathcal{M}(c)$, the following inequality holds:*

$$\mathrm{per}(A^k) \leq \mathrm{per}(A).$$

If $\mathcal{M}(c) = \mathcal{C}(c)$, then Statement 4.37 coincides with Theorem 4.10. Note that it is enough to prove Theorem 4.10 only for irreducible matrices, *i.e.*, it is enough to prove Statement 4.37 for $\mathcal{M}(c) = \mathcal{C}_0(c)$ (see Theorem 4.15). We split $\mathcal{C}_0(c)$ into finitely many classes (namely, $\mathcal{A}(n,m,P,c)$ and $\mathcal{B}(n,m)$, see definitions below) and we prove Statement 4.37 for each of them.

Recall that $\rho(A,B)$ is the maximum entry of $|A-B|$, and that $B(A,\delta)$ is an open ball in the metric ρ with center at A and radius δ.

Definition 4.38. *Let $n, m \in \mathbb{N}$, $P \in \mathbf{P}_n$, $\mu = n/m \in \mathbb{N}$ and $c \in (0, (n-2)/(n-1)^2)$. We denote by $\mathcal{A} = \mathcal{A}(n,m,c,P)$ the set of all $A \in \mathcal{C}_0(c)$ which satisfy the following conditions:*

1. $\mathrm{ind}(A) = m$;

2. $$A = P \begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix} P^T;$$

3. there is an $i \in \{1, 2, ..., m\}$ *such that* $\rho(A_i, J_\mu) > \frac{1}{\mu(\mu-1)^2}$.

Let

$$\mathcal{J} := \begin{pmatrix} O_{\mu,\mu} & J_\mu & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & J_\mu & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & J_\mu \\ J_\mu & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix}.$$

Remark 4.39. *Since* $\rho(A, P\mathcal{J}P^T) = \max\{\rho(A_i, J_\mu) : 1 \le i \le m\}$, *the condition 3 in Definition 4.38 is equivalent to* $\rho(A, P\mathcal{J}P^T) > \frac{1}{\mu(\mu-1)^2}$.

Definition 4.40. *Let* $m \in \mathbb{N}$. *We denote by* $\mathcal{B} = \mathcal{B}(n, m)$ *the set of all* $A \in \mathcal{C}_0(c)$ *that satisfy the following conditions:*

1. $\operatorname{ind}(A) = m$;

2. A *is cogredient to*

$$\begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix};$$

3. for every $i \in \{1, 2, ..., m\}$, $\rho\left(A_i, J_\mu\right) \le \frac{1}{\mu(\mu-1)^2}$.

We will prove that Statement 4.37 holds for $\mathcal{M}(c) = \mathcal{A}(n, m, c, P)$ and sufficiently large $k = k(n, m, c, P)$. Our goal is to show that

$$\lim_{k\to\infty} \sup_{A\in\mathcal{A}} \operatorname{per}(A^k) = \operatorname{per}(\mathcal{J}) < \inf_{A\in\mathcal{A}} \operatorname{per}(A).$$

If we prove it, then for sufficiently large k, $\inf_{A\in\mathcal{A}} \operatorname{per}(A) > \sup_{A\in\mathcal{A}} \operatorname{per}(A^k)$ which implies Statement 4.37 for $\mathcal{A}$.

Let $A \in \mathcal{A}$. Then A is cogredient to

$$\begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix}$$

and, using Lemma 2.8 and Property 2 of cogredient matrices, A^k is cogredient to $\left(\bigoplus_{i=1}^m B_i\right) Q$, where $Q \in \mathbf{P}_n$ and $B_i \in \Omega_\mu$. Note that Q and B_i, $1 \le i \le m$, depend on k.

Lemma 4.41. *Let $m, n \in \mathbb{N}$, $\mu = n/m \in \mathbb{N}$. For every $c \in (0, 1/n)$ and every $\delta > 0$, there exits $k = k(n, m, c, \delta) \in \mathbb{N}$ with the following property. If $A \in \mathcal{A}(n, m, c, P)$, then*

$$A^k = P(B_1 \oplus B_2 \oplus ... \oplus B_m)P^T,$$

where $B_i \in \Omega_\mu$ and $d(B_i) \ge 1/\mu - \delta/(\mu - 1)$, $i = 1, 2, ..., m$.

Proof. Let $A \in \mathcal{A}(n, m, c, P)$. Then

$$A = P \begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix} P^T,$$

and Corollary 4.20 implies that $A^{m(n^2-2n+2)!}$ is cogredient to a direct sum of positive matrices C_i, $1 \leq i \leq m$. In other words,

$$A^{m(n^2-2n+2)!} = P(C_1 \oplus C_2 \oplus ... \oplus C_m)P^T.$$

Note that positivity of C_i implies $d(C_i) = d^*(C_i)$. Using Lemma 2.8, C_i is a product of $m(n^2 - 2n + 2)!$ A_i's. Note that, for every $i \in \{1, 2, ..., m\}$, $d^*(A_i) \geq d^*(A) \geq c$. Then Lemma 4.24 implies

$$d(C_i) = d^*(C_i) \geq c^{m(n^2-2n+2)!}.$$

By Lemma 4.27, there is $k' \in \mathbf{N}$ such that

$$d\left((C_i)^{k'}\right) \geq 1/\mu - \delta/(\mu - 1).$$

Let $k = m(n^2 - 2n + 2)!k'$. Then,

$$A^k = A^{m(n^2-2n+2)!k'} = \left(A^{m(n^2-2n+2)!}\right)^{k'} = P(C_1^{k'} \oplus C_2^{k'} \oplus ... \oplus C_m^{k'})P^T$$

that finishes the proof. $\square$

Lemma 4.42. *Let $n, m \in \mathbb{N}$, $\mu = n/m \in \mathbb{N}$. For every $c \in (0, 1/n)$ and every $\delta > 0$, there exits $N = N(m, c, \delta)$ with the following property. If $k \geq N$ and $A \in \mathcal{A}(n, m, c, P)$ there exists $Q = Q(n, m, k, P) \in \mathbf{P}_n$ such that*

$$\rho\left(A^k, P\left(\underbrace{J_\mu \oplus J_\mu \oplus ... \oplus J_\mu}_{m}\right)Q^T\right) < \delta. \tag{4.4}$$

Proof. $A \in \mathcal{A}(n, m, c, P)$ implies

$$A = P\begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix} P^T.$$

Lemma 4.41 implies that there exists $\tilde{k} = \tilde{k}(n, m, c, \delta)$ such that

$$A^{\tilde{k}} = P(B_1 \oplus B_2 \oplus ... \oplus B_m)P^T$$

and $d(B_i) > 1/\mu - \delta/(\mu - 1)$.

Let $k \geq \tilde{k}$. Lemma 2.8 implies that there exists $Q \in \mathbf{P}_n$ such that

$$A^k = P\left(\bigoplus_{i=1}^{m} C_i\right)Q^T.$$

Note that B_i and C_i, $1 \leq i \leq m$, defined in Theorem 2.8, have the following property: C_i equals B_i multiplied by some doubly stochastic matrix. Hence, by Lemma 4.22,

$$d(C_i) \geq d(B_i) > 1/\mu - \delta/(\mu - 1).$$

By Lemma 4.21, the last inequality implies $D(C_i) < 1 - (\mu - 1)d(C_i) < 1/\mu + \delta$ and thus $\rho(C_i, J_\mu) < \delta$. Therefore, (4.4) holds. $\square$

Recall that

$$\mathcal{J} := \begin{pmatrix} O_{\mu,\mu} & J_\mu & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & J_\mu & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & J_\mu \\ J_\mu & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix}.$$

Lemma 4.43. *If $\mathcal{A}(n, m, c, P)$ is non-empty,*

$$\inf\{\mathrm{per}(A) : A \in \mathcal{A}\} > \mathrm{per}\left(\mathcal{J}\right).$$

Proof. Since $A \in \mathcal{A}(n, m, c, P)$,

$$A = P \begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix} P^T.$$

Using properties of the permanent function and the van der Waerden-Falikman-Egorychev theorem (see Theorem 3.7),

$$\mathrm{per}(A) = \prod_{i=1}^{m} \mathrm{per}(A_i) \geq \prod_{i=1}^{m} \mathrm{per}(J_\mu) = \mathrm{per}(\mathcal{J}). \tag{4.5}$$

Therefore

$$\inf\{\operatorname{per}(A) : A \in \mathcal{A}\} \geq \operatorname{per}(\mathcal{J}).$$

-Falikman-Egorychev theorem (uniqueness) completes the proof. $\square$

Theorem 4.44. *If $m \in \mathbb{N}$ and $c \in (0, 1/n)$, there exists $N = N(n, m, c)$ such that $per(A^k) \leq per(A)$ for every $A \in \mathcal{A}(n, m, c, P)$ and $k \geq N$.*

Proof. Let $M := \inf\{\operatorname{per}(A) :\ A \in \mathcal{A}\}$. By Lemma 4.43,

$$M > \operatorname{per}(\mathcal{J}) = \operatorname{per}(\underbrace{J_\mu \oplus J_\mu \oplus ... \oplus J_\mu}_{m}).$$

Now we want to show that, for sufficiently large k, $\sup_{A \in \mathcal{A}} \operatorname{per}(A^k) \leq M$.

Using continuity of the permanent function on matrix entries, there is $\delta > 0$ such that, for an open ball $U = U(n, m) = B(\underbrace{J_\mu \oplus J_\mu \oplus ... \oplus J_\mu}_{m}, \delta)$, $C \in U$ implies $\operatorname{per}(C) < M$. Let $V = \cup_{Q \in \mathbf{P}_n} PUQ^T$. Note that $C \in V$ implies that there exists $Q \in \mathbf{P}_n$ such that $P^{-1}CQ \in U$, then $\operatorname{per}(C) = \operatorname{per}(P^{-1}CQ) < M$. We want to show that for sufficiently large k, for every $A \in \mathcal{A}(n, m, c, P)$, $A^k \in V$.

Let $N = N(n, m, c)$ be defined as in Lemma 4.42. Let $A \in \mathcal{A}(n, m, c, P)$ and $k \geq N$. Then there exists $Q \in \mathbf{P}_n$ such that

$$\rho\left(A^k, P\left(\underbrace{J_\mu \oplus J_\mu \oplus ... \oplus J_\mu}_{m}\right)Q^T\right) < \delta.$$

Hence, $A^k \in V$ and the proof is complete. $\square$

We now prove Statement 4.37 for $\mathcal{M} = \mathcal{B}(n, m)$, $1 \leq m \leq n$.

Theorem 4.45. *Let $n, m, k \in \mathbb{N}$, $\mu = n/m \in \mathbb{N}$. Then, for every $A \in \mathcal{B}(n,m)$,*

$$\operatorname{per}(A^k) \leq \operatorname{per}(A).$$

Proof. Since $A \in \mathcal{B}(n,m)$, A is cogredient to a matrix

$$\widetilde{A} = \begin{pmatrix} O_{\mu,\mu} & A_1 & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ O_{\mu,\mu} & O_{\mu,\mu} & A_2 & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \\ \vdots & & & \ddots & & \vdots \\ O_{\mu,\mu} & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & A_{m-1} \\ A_m & O_{\mu,\mu} & O_{\mu,\mu} & \cdots & O_{\mu,\mu} & O_{\mu,\mu} \end{pmatrix}.$$

Since permanents of cogredient matrices are equal, $\operatorname{per}(A^k) \leq \operatorname{per}(A)$ is equivalent to $\operatorname{per}(\widetilde{A}^k) \leq \operatorname{per}(\widetilde{A})$. By the definition of $\mathcal{B}$ (Definition 4.40), $\rho\left(A_i, J_\mu\right) \leq \frac{1}{\mu(\mu-1)^2}$, $1 \leq i \leq m$. Then, by the definition of the metric ρ,

$$d(A_i) \geq \frac{1}{\mu} - \frac{1}{\mu(\mu-1)^2} = \frac{\mu^2 - 2\mu}{\mu(\mu-1)^2} = \frac{\mu-2}{(\mu-1)^2}.$$

Let $k = sm + r$, $0 \leq r < m$. By Lemma 2.8,

$$\widetilde{A}^{sm+r} = \left(\bigoplus_{i=1}^{m} B_i\right) Q,$$

where $Q \in \mathbf{P}_n$ and

$$B_i = \left(\prod_{j=i}^{i+m-1} A_j\right)^s A_i A_{i+1} \ldots A_{i+r-1}.$$

Note that $B_i = A_i C_i$, where $C_i \in \Omega_\mu$ is a product of A_j's. Therefore, by Lemma 4.22,

$$d(C_i) \geq d(A_i) \geq \frac{\mu-2}{(\mu-1)^2}.$$

Lemma 4.28 implies $\mathrm{per}(A_i) \geq \mathrm{per}(A_i C_i) = \mathrm{per}(B_i)$. Therefore,

$$\mathrm{per}(A) = \prod_{i=1}^{m} \mathrm{per}(A_i) \geq \prod_{i=1}^{m} \mathrm{per}(B_i) = \mathrm{per}(\widetilde{A}^k) = \mathrm{per}(A^k).$$

$\square$

Proof of Theorem 4.10. For every $n \in \mathbb{N}$ and $c \in (0, 1/n)$, using Theorem 4.20

$$\bigcup_{m:1\leq m\leq n} \bigcup_{P\in\mathbf{P}_n} (\mathcal{A}(m,c,P) \cup \mathcal{B}(n,m))$$

$$= \{A \in \Omega_n |\, d^*(A) \geq c,\ A \text{ is irreducible}\} = \mathcal{C}_0(c).$$

For each of $\mathcal{A}$ and $\mathcal{B}$, Statement 4.37 holds. Since the union is over a finite set, Statement 4.37 holds for the union, *i.e*, for $\mathcal{M}(c) = \mathcal{C}_0(c)$.

Theorem 4.15 completes the proof of Theorem 4.10. $\square$

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