# Seasonal Volatility Models with Applications in Option Pricing

by

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**Abstract**

GARCH models have been widely used in finance to model volatility ever since the introduction of the ARCH model and its extension to the generalized ARCH (GARCH) model. Lately, there has been growing interest in modelling seasonal volatility, most recently with the introduction of the multiplicative seasonal GARCH models.

As an application of the multiplicative seasonal GARCH model with real data, call prices from the major stock market index of India are calculated using estimated parameter values. It is shown that a multiplicative seasonal GARCH option pricing model outperforms the Black-Scholes formula and a GARCH(1,1) option pricing formula. A parametric bootstrap procedure is also employed to obtain an interval approximation of the call price. Narrower confidence intervals are obtained using the multiplicative seasonal GARCH model than the intervals provided by the GARCH(1,1) model for data that exhibits multiplicative seasonal GARCH volatility.

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# Contents

# List of Tables

# List of Figures

# Chapter 1

# Introduction

## 1.1 Motivation

The price (and returns) of a financial asset changes over time. This variability is measured by the asset's volatility. There are many definitions of volatility, but it can be simply defined as the standard deviation of the return series. On a time plot of an asset's returns, we see how the volatility of an asset changes over time. As Mandelbrot [25] noted: "large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes." For example, in Figure 1.1, we see a period of low volatility in the S&P 100 Index from early 2004 to late 2006 and periods of high volatility before 2004 and after 2006. This exhibits the tendency of volatility to cluster. Volatility changes can occur for a variety of reasons. Economic, social, and political crises; macroeconomic news releases; and the amount of information entering a market (for example, information that accumulates on the weekend and enters the market on Mondays) play a large role in causing volatility changes [35], [1]. Time-varying volatility can also be seen in the sample autocorrelation function (ACF) of absolute returns or squared returns.

For example, in Figure 1.2 on page 4, we see a slow decay in the sample ACF of the S&P 100 squared returns.

### 1.1.1 Modelling volatility

There are two general specifications for modelling volatility in the literature. The first viewpoint is that volatility is partially determined by unpredictable events, so it should be modelled as a stochastic process. This has resulted in Stochastic Volatility (SV) models. The second viewpoint is that volatility is a function of past returns such as in ARCH and GARCH models.

Taylor [34] first introduced Stochastic Volatility (SV) models and viewed volatility as a latent random process. For example, unscheduled news entering the market may play a role in determining volatility. It was suggested to model volatility in a discrete-time framework as an approximation of a continuous-time diffusion process. Kawakatsu [23] noted that one of the drawbacks of Taylor's model is its frequent inability to capture infrequent events such as market crashes which fall in the extreme tails of the distribution of asset returns. The probability of such extreme events occurring can be measured by kurtosis, a measure of the thickness of the tails of a distribution, with the normal distribution having a kurtosis of 3. In order to model the leptokurtosis (kurtosis greater than 3, indicating thicker tails than the normal distribution) seen in asset returns, Kawakatsu proposed the quadratic stochastic volatility model.

By modelling volatility by a stochastic process with a random component, estimation of model parameters becomes complicated. Engle [11], on the other hand,

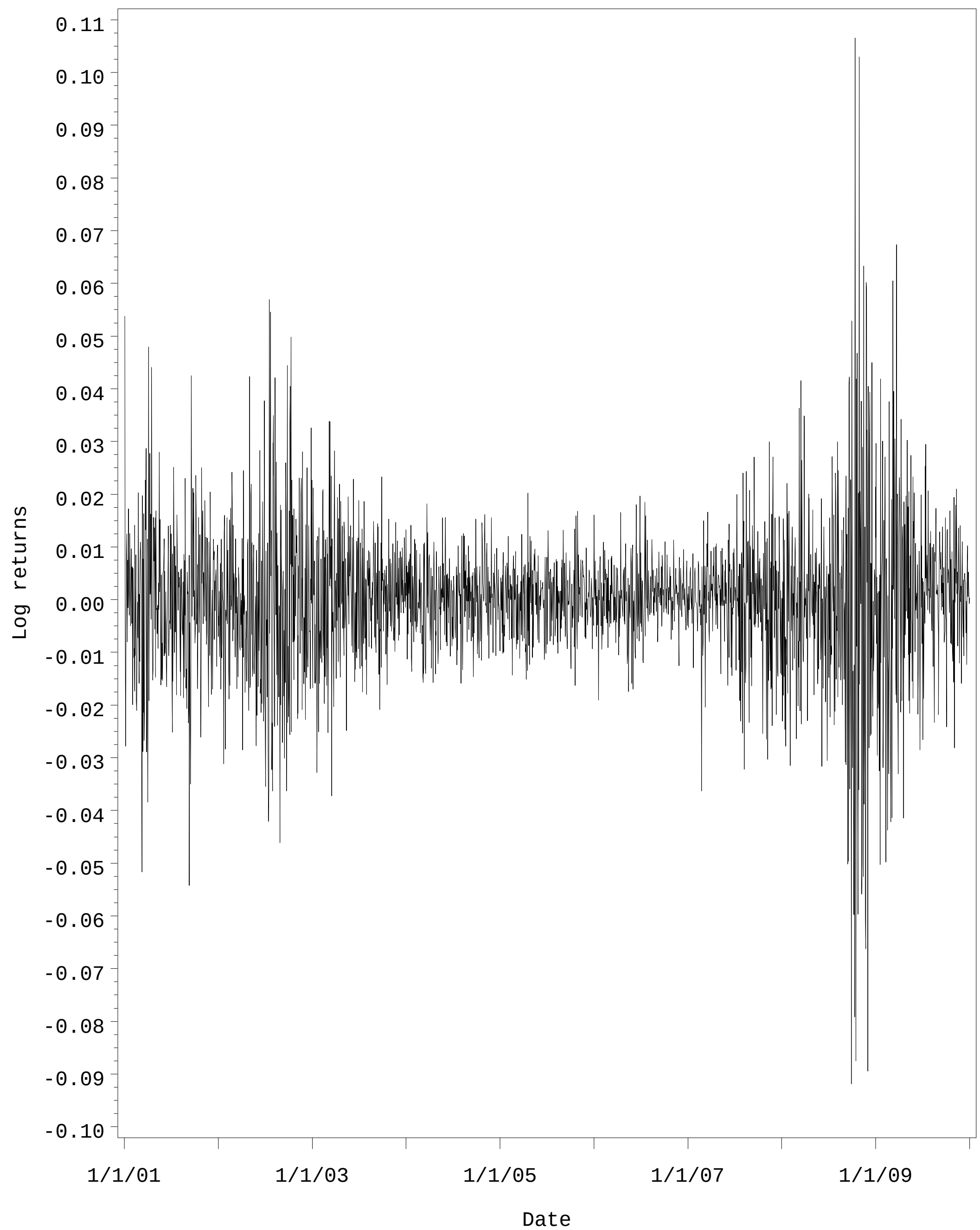


Figure 1.1: Log returns of S&P 100

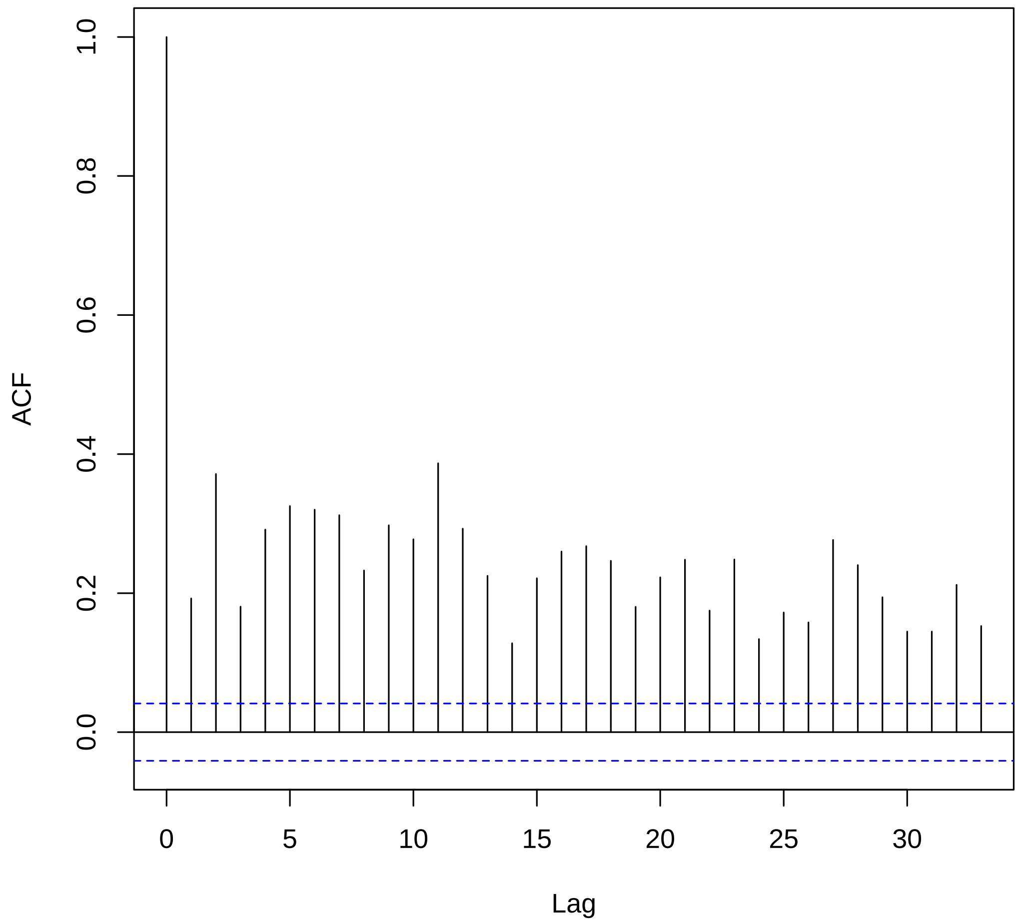


Figure 1.2: Sample ACF of S&P 100 squared returns

used the conditional variance of the return series to model the volatility as a function of past squared residuals in his autoregressive conditional heteroskedastic (ARCH) model. By using squared residuals, the ARCH model in effect models the magnitude of the volatility (Engle et al. [12]), which is of interest due to the tendency of volatility to cluster. In practice, it was found that a good fit using an ARCH model would frequently require a very high order of past squared errors and thus a very large number of parameters to estimate. In order to overcome this difficulty, Bollerslev [7] proposed the generalized ARCH (GARCH($p, q$)) model, where the conditional variance is a function of not only the past $p$ squared errors, but also the past $q$ conditional variances. In practice, a simple GARCH(1,1) model often provides a good fit.

Financial market data typically exhibit various forms of seasonality. In financial markets, seasonal effects such as the opening and closure of the markets and other time-of-the-day effects, weekends and other vacation periods, and day-of-the-week effects have been widely reported. These effects can cause changes in the trading volume and changes in volatility. Therefore, in addition to modelling volatility as a function of past squared errors, several studies have included seasonal factors in the conditional variance equation. In the literature, seasonal volatility of returns have been modelled by using indicator variables in the GARCH volatility equation to account for specific events (e.g., Berument and Kiymaz [5]). Bollerslev and Ghysels [8], Baillie and Bollerslev [3], and Franses and Paap [14] have extended the above indicator-variable GARCH model to more flexible seasonal forms such as the periodic GARCH. Recently, Koopman et al. [24] have proposed a periodic seasonal GARCH model, in which the conditional variance gives equal weight to observations at certain (seasonal) lags.

In the literature, purely seasonal GARCH models have been considered, with autocorrelation only at seasonal lags. However, we should still expect to observe autocorrelation at adjacent (non-seasonal) lags, which a multiplicative seasonal model would account for. In addition, in a multiplicative seasonal model, more weight is given to the most recent occurrences of the same season. In this thesis, a multiplicative seasonal GARCH process is proposed (Doshi et al. [9]) and it is shown to be more appropriate to use in option pricing.

### 1.1.2 Option pricing with volatility models

An option is a financial instrument commonly traded in global financial markets. It is called a derivative because its value is derived from the value of an underlying asset (such as a stock, bond, or currency). There are two kinds of stock options: a call option gives the purchaser the option (not the obligation) to purchase the underlying asset in the future at an agreed upon price (the *strike price*) at an agreed upon date (the *expiration date*); meanwhile, a put option gives the purchaser the option to sell the underlying asset.

The relationship between the price of the underlying asset and the strike price is fundamental to understanding options. When the underlying price exceeds the strike price, the option is said to be in-the-money because if the purchaser of the option chooses to exercise the option by purchasing the asset at the strike price, the purchaser can immediately sell the asset at the underlying price and make a profit. In contrast, when the strike price exceeds the underlying price, the option is said to be out-of-the-money.

The price charged to purchase an option is the subject of option pricing formulae, most famously the Black-Scholes formula (Black and Scholes [6]). The inputs into the formula include the current price of the underlying asset, the strike price, the risk-free interest rate (the interest rate attainable from an investment that carries no risk), the time to expiration, and the volatility of the underlying asset.

The Black-Scholes model assumes that the underlying asset has a constant volatility. To overcome this unrealistic assumption, several continuous-time Black-Scholes models with stochastic volatility and discrete-time GARCH option pricing models have been proposed. Hull and White [21] proposed a Black-Scholes option pricing model in which the volatility follows a continuous-time diffusion process. Their results show that the constant volatility Black-Scholes formula frequently overprices options that are near-the-money (when the underlying asset price is within 10% of the strike price) and underprices options that are deep in- or out-of-the-money. Scott [32] and Wiggins [38] proposed similar Black-Scholes option pricing models with stochastic volatility.

Duan [10] was the first to develop a discrete-time GARCH option pricing model. This type of model includes the Black-Scholes model with constant volatility as a special case. In contrast to the continuous-time Black-Scholes models with stochastic volatility, under the GARCH option pricing model, option values are functions of not only the current underlying asset price, but also lagged asset prices. However, Duan's [10] model requires extensive simulations to obtain option values. Heston and Nandi [20] were the first to develop a GARCH option pricing model with a closed-form solution for option values.

Badescu and Kulperger [2], Barone-Adesi et al. [4], Gong et al. [17], and Thavaneswaran and Singh [37] have computed option prices by modelling the volatility using a GARCH process. Recently, Gong et al. [17] studied option pricing for a Black-Scholes model with GARCH volatility and then developed an option pricing formula which depends on the kurtosis of the observed process and the second moment of the volatility process. By modelling the volatility with a GARCH(1,1) process, they computed call prices for the S&P 100 Index using GARCH parameter estimates and demonstrated the superiority of their method, in terms of percentage error of the observed call price, compared to Duan's [10] option pricing formula and the standard Black-Scholes formula. However, option pricing with seasonal volatility has not been studied in the literature. In this thesis, the superiority of the proposed multiplicative seasonal GARCH model over a GARCH(1,1) model in option pricing is discussed in some detail.

## 1.2 Thesis organization

The thesis is organized as follows. In Chapter 2, first, volatility models such as ARCH and GARCH processes with some of their moment properties, and Stochastic Volatility models are introduced. Then, a new multiplicative seasonal GARCH process which accounts for autocorrelation at seasonal lags and their adjacent non-seasonal lags is introduced and some of its properties are derived. In Chapter 3, the derivation of call prices for a Black-Scholes model with GARCH volatility is discussed in detail. The multiplicative seasonal GARCH model is applied to obtain call prices for a well-known stock market index. It is shown that for this data set, more precise call prices can be obtained than those obtained by the constant volatility Black-Scholes formula or by the GARCH(1,1) option pricing formula of

Gong et al. [17]. In Chapter 4, a bootstrap procedure is used to obtain an interval approximation of the call price. It is shown that for data that exhibits multiplicative seasonal heteroskedasticity, narrower confidence intervals can be obtained using an appropriate multiplicative seasonal GARCH option pricing formula compared to other option pricing formulae that fail to account for the multiplicative seasonal behaviour.

Appendix A contains details of the parametric bootstrap procedure in Chapter 4. First, the proof of the asymptotic correlation between the sample mean and the sample median is given. Second, the theoretical asymptotic correlation for some distributions is derived. The material in Section 2.3 has been published by the author in Doshi et al. [9].

# Chapter 2

# Volatility models

In this chapter, the original GARCH model is extended to model seasonal volatility. In Section 2.1, ARCH and GARCH models and their moment properties are discussed. In Section 2.2, Stochastic Volatility models are given. In Section 2.3, first the multiplicative seasonal GARCH process is introduced and the kurtosis is derived in terms of model parameters. Then, some examples are provided, the expression for the $l$-steps-ahead forecast error is derived, and a simulation study is performed. In Section 2.4, the GARCH models are applied to real data and parameter estimates are obtained.

## 2.1 ARCH and GARCH models

The price process $\{S_t\}$ of an asset is usually non-stationary and is serially correlated. Unlike asset prices $\{S_t\}$, returns $R_t = \frac{S_t - S_{t-1}}{S_{t-1}}$ are weakly stationary and weakly correlated over time, so modelling returns is more desirable (Taylor [35], p. 9). Instead of simple returns, log returns $r_t = \log(1 + R_t) = \log \dfrac{S_t}{S_{t-1}} = \log S_t - \log S_{t-1}$

are frequently used to model asset prices. Log returns can be used when returns are very small because $\log(1+R_t) \approx R_t, R_t \ll 1$. Also, log returns are time additive; that is, the compound log return over $n$ periods is simply the sum of $n$ single-period returns. Finally, if the rate of return $1+R_t$ is assumed to be lognormally distributed, then the log returns are normally distributed. Based on a regression model of the log returns given by

$$r_t = \mu + \epsilon_t,$$

where $\mu = E(r_t)$ is the mean of the returns, in order to model the volatility of the returns, conditional on the past history of returns, Engle [11] introduced the autoregressive conditional heteroskedastic[1] (ARCH) model of order $p$ for the errors $\epsilon_t$ as:

$$\epsilon_t = \sqrt{h_t} Z_t \tag{2.1.1}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2, \tag{2.1.2}$$

where $h_t$ is the conditional variance of the returns, $Z_t$ is a sequence of independent and identically distributed (i.i.d.) normal random variables with zero mean and unit variance, $h_t$ is independent of $Z_t$, $\omega > 0$, and $\alpha_i \geq 0, i = 1, \ldots, p$.

In order to find the mean and variance of the marginal distribution of the log returns $r_t$, we condition on $\mathscr{F}_{t-1}^r$, the $\sigma$-field generated by returns up to and including time $t-1$. We also use the fact that $h_t$ contains information on past returns but $Z_t$ is independent of the past information. Therefore, the unconditional mean and variance of $r_t$ are given by:

$$E(r_t) = \mu + E(\sqrt{h_t} Z_t) = \mu + E[E(\sqrt{h_t} Z_t | \mathscr{F}_{t-1}^r)] = \mu + E[\sqrt{h_t} E(Z_t)] = \mu$$

[1]Heteroskedasticity refers to non-constant variance.

and

$$
\begin{aligned}
Var(r_t) &= Var(\mu + \sqrt{h_t} Z_t) \\
&= E[Var(\sqrt{h_t} Z_t | \mathscr{F}^r_{t-1})] + Var[E(\sqrt{h_t} Z_t | \mathscr{F}^r_{t-1})] \\
&= E[h_t Var(Z_t)] + Var[\sqrt{h_t} E(Z_t)] \\
&= E[h_t] \\
&= \omega + \sum_{i=1}^{p} \alpha_i E(r_{t-i} - \mu)^2 \\
&= \omega + \sum_{i=1}^{p} \alpha_i Var(r_{t-i}).
\end{aligned}
$$

Now, under the assumption that $\{r_t\}$ is weakly stationary,

$$
Var(r_t) = \frac{\omega}{1 - (\alpha_1 + \cdots + \alpha_p)},
$$

which imposes the stationarity restriction $\sum_{i=1}^{p} \alpha_i < 1$. Moreover, the conditional mean of the returns is given by:

$$
E(r_t | \mathscr{F}^r_{t-1}) = \mu + E(\sqrt{h_t} Z_t | \mathscr{F}^r_{t-1}) = \mu + \sqrt{h_t} E(Z_t) = \mu
$$

and the conditional variance is

$$
Var(r_t | \mathscr{F}^r_{t-1}) = Var(\mu + \sqrt{h_t} Z_t | \mathscr{F}^r_{t-1}) = h_t Var(Z_t) = h_t.
$$

Since $r_t$ is conditionally a linear combination of $Z_t$,

$$
r_t | \mathscr{F}^r_{t-1} \overset{i.i.d.}{\sim} N(\mu, h_t). \tag{2.1.3}
$$

For an ARCH($p$) model, the conditional variance $h_t$ depends only on the $p$ most recent squared errors, as seen in (2.1.2). The autoregressive nature of the ARCH($p$) process can be seen in the squared errors $\epsilon_t^2$. To see this, note that from (2.1.3), we can write

$$\epsilon_t | \mathscr{F}_{t-1}^r \sim N(0, h_t)$$

Since the forecast error $v_t$ of the squared errors is defined as

$$\begin{aligned} v_t &= \epsilon_t^2 - \mathbb{E}(\epsilon_t^2 | \mathscr{F}_{t-1}^r) \\ &= \epsilon_t^2 - h_t, \end{aligned}$$

we can rewrite (2.1.2) as:

$$\epsilon_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + v_t. \tag{2.1.4}$$

Next, we show that $\{v_t\}$ in (2.1.4) is a white noise process so that $\epsilon_t^2$ is an AR($p$) process. First, the mean of $v_t$ is $E(v_t) = E(\epsilon_t^2) - E(h_t)$. Since $E(\epsilon_t^2) = E(r_t - \mu)^2 = Var(r_t)$ and we have shown that $Var(r_t) = E(h_t)$, we have $E(v_t) = 0$. Assuming $h_t$ is weakly stationary, the variance of $v_t$ is given by:

$$\begin{aligned} Var(v_t) &= Var(\epsilon_t^2 - h_t) \\ &= Var(\epsilon_t^2) + Var(h_t) - 2Cov(\epsilon_t^2, h_t) \\ &= Var(h_t Z_t^2) + Var(h_t) - 2Cov(h_t Z_t^2, h_t) \\ &= \left\{ E[Var(h_t Z_t^2 | \mathscr{F}_{t-1}^r)] + Var[E(h_t Z_t^2 | \mathscr{F}_{t-1}^r)] \right\} + Var(h_t) \\ &\quad - 2[E(h_t^2 Z_t^2) - E(h_t Z_t^2) E(h_t)] \\ &= E[h_t^2 Var(Z_t^2)] + Var[h_t E(Z_t^2)] + Var(h_t) - 2[E(h_t^2) - E^2(h_t)] \end{aligned}$$

$$= 2Var(h_t) + 2E^2(h_t) \text{ (since } Z_t^2 \sim \chi^2_{(1)})$$

$$= 2E(h_t^2) = \text{constant},$$

and the autocovariances (for all lags $k \in \mathbb{Z}$, the class of all integers) are given by:

$$Cov(v_t, v_{t-k}) = E(v_t v_{t-k}) - E(v_t)E(v_{t-k}) = E[E(v_t v_{t-k}|\mathscr{F}^r_{t-1})]$$

$$= E[v_t E(v_{t-k}|\mathscr{F}^r_{t-1})] = E(v_t)E(v_{t-k}|\mathscr{F}^r_{t-1}) = 0.$$

Therefore, $\{v_t\}$ is a white noise process and $\epsilon_t^2$ in (2.1.4) is an AR($p$) process.

Generalizing the above to the case where the conditional variance $h_t$ depends on the $p$ most recent squared errors $\epsilon^2_{t-i}, i = 1, \ldots, p$ and the $q$ most recent conditional variances $h_{t-j}, j = 1, \ldots, q$, we have the GARCH($p, q$) model of Bollerslev [7]:

$$\epsilon_t = \log \frac{S_t}{S_{t-1}} - \mathbb{E}\left[\log \frac{S_t}{S_{t-1}}\right] = \sqrt{h_t} Z_t, \quad Z_t \overset{i.i.d.}{\sim} N(0,1) \tag{2.1.5}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \epsilon^2_{t-i} + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{2.1.6}$$

where (2.1.6) may be written in ARMA($\max\{p, q\}, q$) form as:

$$\epsilon_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon^2_{t-i} + \sum_{j=1}^{q} \beta_j \epsilon^2_{t-j} + v_t - \sum_{j=1}^{q} \beta_j v_{t-j}. \tag{2.1.7}$$

## 2.2 Stochastic Volatility models

In contrast to GARCH models, in Stochastic Volatility models, the volatility is modelled by a stochastic process. For example, the time-varying volatility may be modelled by a discrete-time autoregressive moving average (ARMA) process. Based on a regression model of the log returns $r_t = \mu + \epsilon_t$, the Stochastic Volatility (SV) model of Taylor [34] is a normal-AR(1) process for the log of the volatility $\sigma_t$ given by:

$$\epsilon_t = \sigma_t Z_t, \qquad Z_t \overset{i.i.d.}{\sim} N(0,1), \tag{2.2.1}$$

$$\log(\sigma_t) = \alpha_0 + \phi(\log(\sigma_{t-1}) - \alpha_0) + \eta_t, \qquad \eta_t \overset{i.i.d.}{\sim} N(0,\sigma_\eta^2), \tag{2.2.2}$$

where $\{\sigma_t\}$ and $\{Z_t\}$ are stochastically independent. The volatility is conditionally lognormally distributed, which guarantees the volatility will be positive, and has conditional variance $\mathrm{Var}(\sigma_t|\mathscr{F}^r_{t-1}) > 0$.

Recently, Kawakatsu [23] proposed Quadratic Stochastic Volatility (QSV) models, in which the quadratic term allows for better modelling of the leptokurtosis seen in asset returns. The models are given by:

$$r_t = \mu + \epsilon_t, \tag{2.2.3}$$

$$\epsilon_t = \sigma_t Z_t, \tag{2.2.4}$$

$$\log(\sigma_t^2) = \alpha_0 + \alpha_1 x_t + \alpha_2 x_t^2, \tag{2.2.5}$$

$$x_{t+1} = \phi x_t + \sqrt{1-\phi^2}\,v_{t+1}, \quad x_0 \sim N(0,1), \quad |\phi| < 1, \tag{2.2.6}$$

$$\begin{bmatrix} Z_t \\ v_{t+1} \end{bmatrix} \sim N\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right). \tag{2.2.7}$$

Gong et al. [18] have studied properties of Taylor's SV model and Kawakatsu's QSV model and applied them in option pricing.

## 2.3 Multiplicative seasonal GARCH models

In financial applications, non-seasonal GARCH(1,1) models are widely used. However, seasonal volatility effects in the underlying process are common in real data. In this Section, the general form of seasonality that accounts for autocorrelation at seasonal lags and at adjacent non-seasonal lags is introduced. Failure to identify this behavior may lead to misleading results and, in turn, to unreliable forecasts.

Consider the class of GARCH$(p, q)$ models given by:

$$\epsilon_t = \sqrt{h_t} Z_t \tag{2.3.1}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{2.3.2}$$

where $Z_t$ is a sequence of i.i.d. random variables with zero mean and unit variance. If we let the martingale difference sequence $\{u_t\}$ be given by $u_t = \epsilon_t^2 - h_t$, the conditional variance equation (2.3.2) becomes:

$$\left(1 - \sum_{i=1}^{p} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j\right) \epsilon_t^2 = \omega + \left(1 - \sum_{j=1}^{q} \beta_j B^j\right) u_t, \tag{2.3.3}$$

where $B$ is the backshift, or lag, operator, as shown in (2.1.7). Since $\{u_t\}$ is a white noise process, $\epsilon_t^2$ in (2.3.3) is an ARMA$(\max\{p, q\}, q)$ process.

In order to develop a multiplicative seasonal GARCH process, which accounts for autocorrelation at seasonal lags and at adjacent non-seasonal lags, we would need a seasonal ARMA$(\max\{p, q\}, q)$x$(P, Q)_s$ representation for $\epsilon_t^2$. This representation is achieved by the following equation:

$$\{1 - [\phi_p(B) - \theta_q(B)]\} \Phi_P(L) \epsilon_t^2 = \omega + \theta_q(B) \Theta_Q(L) u_t, \tag{2.3.4}$$

where $\phi_p(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$, $\theta_q(B) = 1 - \sum_{i=1}^{q} \theta_i B^i$, $\Phi_P(L) = 1 - \sum_{i=1}^{P} \Phi_i L^i$, $\Theta_Q(L) = 1 - \sum_{i=1}^{Q} \Theta_i L^i$, and $L = B^s$ is the seasonal backshift operator, where $s$ represents the seasonal period.[2]

Note that when $P = Q = 0$, (2.3.4) simplifies to an ARMA($\max\{p, q\}, q$) representation for $\epsilon_t^2$, corresponding to the general GARCH($p, q$) model. Using $u_t = \epsilon_t^2 - h_t$ again, we now introduce a class of multiplicative seasonal GARCH$(p, q)$x$(P, Q)_s$ models for the time series $\epsilon_t$ as:

$$\epsilon_t = \sqrt{h_t} Z_t, \tag{2.3.5}$$

$$\theta_q(B)\Theta_Q(L)h_t = \omega + \alpha(B)\epsilon_t^2, \tag{2.3.6}$$

where $Z_t$ is a sequence of i.i.d. random variables with zero mean and unit variance, and $\alpha(B) = \theta_q(B)\Theta_Q(L) - \{1 - [\phi_p(B) - \theta_q(B)]\}\Phi_P(L)$.

We assume that all the zeros of the polynomial $\{1 - [\phi_p(B) - \theta_q(B)]\}\Phi_P(L)$ lie outside the unit circle; thus, $\epsilon_t^2$ as given in (2.3.4) is stationary. The moving average representation is $\epsilon_t^2 = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}$ where $\{\psi_j\}$ is a sequence of constants and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. The $\psi_j$'s are obtained from $\psi(B)\{1 - [\phi_p(B) - \theta_q(B)]\}\Phi_P(L) = \theta_q(B)\Theta_Q(L)$ where $\psi(B) = 1 + \sum_{j=1}^{\infty} \psi_j B^j$.

### 2.3.1 Kurtosis of seasonal GARCH

Previous research performed in financial markets suggests that return series may not be conditionally normally distributed, but rather conditionally leptokurtic (Boller-

[2] Note that the polynomial $\phi_p(B) - \theta_q(B)$ is of the order $\max\{p, q\}$.

slev [7] and Taylor [35]). Therefore, we study the kurtosis of the seasonal GARCH process. The kurtosis for a multiplicative seasonal GARCH process in terms of $\psi$ weights and the variance of the $\epsilon_t^2$ process are given in Theorem 2.3.1. The theorem is a generalization of the non-seasonal GARCH$(p, q)$ in Thavaneswaran et al. [36].

**Theorem 2.3.1.** *Under the stationarity assumptions and finite fourth moment, the kurtosis $K^{(\epsilon)}$ of the process (2.3.5) is given by*

*(a)* $K^{(\epsilon)} = \dfrac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{j=0}^{\infty} \psi_j^2}$,

*(b) The variance of the $\epsilon_t^2$ process is given by* $\gamma_0^{\epsilon^2} = \sum_{j=0}^{\infty} \psi_j^2 \sigma_u^2$, *where*

$\sigma_u^2 = \dfrac{\mu^2 (K^{(\epsilon)} - 1)}{\sum_{j=0}^{\infty} \psi_j^2}$ *is the variance of $u_t$ and the mean of $\epsilon_t^2$,*

$$\mu = E(\epsilon_t^2) = \frac{\omega}{\left(1 - \sum_{i=1}^{p} \phi_i\right)\left(1 - \sum_{i=1}^{P} \Phi_i\right)}.$$

*Proof.* Proof of part (a) is similar to that of Thavaneswaran et al. [36] and is omitted. For the proof of part (b), we have

$$K^{(\epsilon)} - 1 = \frac{E[\epsilon_t^4]}{(E[\epsilon_t^2])^2} - 1 = \frac{E[\epsilon_t^4] - (E[\epsilon_t^2])^2}{\mu^2} = \frac{Var[\epsilon_t^2]}{\mu^2} = \frac{\sigma_u^2 \sum_{j=0}^{\infty} \psi_j^2}{\mu^2}$$

and hence

$$\sigma_u^2 = \frac{\mu^2(K^{(\epsilon)} - 1)}{\sum_{j=0}^{\infty} \psi_j^2}. \tag{2.3.7}$$

□

### 2.3.2 Special cases

In the following examples of multiplicative seasonal GARCH processes, we obtain the kurtosis of the $\epsilon_t$ process and the variance of $u_t$ using Theorem 2.3.1.

**Example 2.3.1.** *(Multiplicative seasonal GARCH* $(0,1)$x$(0,1)$ *process).*

$$(1 - \theta_1 B)\epsilon_t^2 = \omega + (1 - \theta_1 B)(1 - \Theta_1 L)u_t, \tag{2.3.8}$$

where $\theta_1 < 1$ is required for stationarity. The $\psi$ weights are $\psi_s = -\Theta_1$ and $\psi_j = 0$ otherwise. Therefore, $\sum_{j=0}^{\infty} \psi_j^2 = 1 + \Theta_1^2$. Using part (a) of Theorem 2.3.1, the kurtosis is given by

$$K^{(\epsilon)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1](1 + \Theta_1^2)}. \tag{2.3.9}$$

When the $\epsilon_t$ series is assumed to be conditionally normally distributed, the kurtosis of the multiplicative seasonal GARCH process can be written as

$$K^{(\epsilon)} = \frac{3}{3 - 2(1 + \Theta_1^2)} \tag{2.3.10}$$

and the variance of $u_t$ is given by

$$\sigma_u^2 = \frac{(K^{(\epsilon)} - 1)\mu^2}{1 + \Theta_1^2}. \qquad (2.3.11)$$

**Example 2.3.2.** *(Multiplicative seasonal GARCH* $(0,1)$x$(1,0)$ *process).*

$$(1 - \theta_1 B)(1 - \Phi_1 L)\epsilon_t^2 = \omega + (1 - \theta_1 B)u_t, \qquad (2.3.12)$$

where the roots $c$ of the polynomial $(1 - \theta_1 c)(1 - \Phi_1 c^s)$ must lie outside the unit circle for stationarity. The $\psi$ weights are $\psi_{ks} = \Phi_1^k$, where $k = 1, 2, \cdots$. Therefore, $\sum_{j=0}^{\infty} \psi_j^2 = 1 + \Phi_1^2 + \Phi_1^4 + \cdots = \dfrac{1}{1 - \Phi_1^2}$. Using part (a) of Theorem 2.3.1, the kurtosis is given by

$$K^{(\epsilon)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\left(\dfrac{1}{1 - \Phi_1^2}\right)}. \qquad (2.3.13)$$

When the $\epsilon_t$ series is assumed to be conditionally normally distributed, the kurtosis of the multiplicative seasonal GARCH process can be written as

$$K^{(\epsilon)} = \frac{3}{3 - 2\left(\dfrac{1}{1 - \Phi_1^2}\right)} = \frac{3(1 - \Phi_1^2)}{1 - 3\Phi_1^2} \qquad (2.3.14)$$

and the variance of $u_t$ is given by

$$\sigma_u^2 = (K^{(\epsilon)} - 1)\mu^2(1 - \Phi_1^2). \qquad (2.3.15)$$

**Example 2.3.3.** *(Multiplicative seasonal GARCH* $(1,0)$x$(1,0)$ *process).*

$$(1+\phi_1 B)(1-\Phi_1 L)\epsilon_t^2 = \omega + u_t \tag{2.3.16}$$

where the roots $c$ of the polynomial $(1+\phi_1 c)(1-\Phi_1 c^s)$ must lie outside the unit circle for stationarity. The $\psi$ weights are $\psi_1 = -\phi_1$, $\psi_2 = -\phi_1^2$, $\cdots$, $\psi_{s-1} = -\phi_1^{s-1}$, $\psi_s = \phi_1^s + \Phi_1$, and $\psi_j = -\phi_1\psi_{j-1} + \Phi_1\psi_{j-s} + \phi_1\Phi_1\psi_{j-s-1}$ for $j \geq s+1$. Therefore,

$$\sum_{j=0}^{\infty}\psi_j^2 = 1+\phi_1^2+\phi_1^4+\cdots+\phi_1^{2(s-1)}+(\phi_1^s+\Phi_1)^2+(-\phi_1^{s+1}-\phi_1\Phi_1)^2+(\phi_1^{s+2}-\phi_1^2\Phi_1)^2+$$

$$(-\phi_1^{s+3}-\phi_1^3\Phi_1)^2+\cdots$$

$$= \frac{1-\phi_1^{2s}}{1-\phi_1^2} + \frac{\phi_1^{2s}}{1-\phi_1^2} + 2\phi_1^s\Phi_1 + \frac{2\phi_1^{s+2}\Phi_1}{1+\phi_1^2} + \frac{\Phi_1^2}{1-\phi_1^2}$$

$$= \frac{(1+\phi_1^2)(1+\Phi_1^2)+2\phi_1^s\Phi_1(1+\phi_1^2-2\phi_1^4)}{1-\phi_1^4}.$$

Using part (a) of Theorem 2.3.1, the kurtosis is given by

$$K^{(\epsilon)} = \frac{E(Z_t^4)}{E(Z_t^4)-[E(Z_t^4)-1]\left[\dfrac{(1+\phi_1^2)(1+\Phi_1^2)+2\phi_1^s\Phi_1(1+\phi_1^2-2\phi_1^4)}{1-\phi_1^4}\right]}. \tag{2.3.17}$$

When the $\epsilon_t$ series is assumed to be conditionally normally distributed, the kurtosis of the multiplicative seasonal GARCH process can be written as

$$K^{(\epsilon)} = \frac{3}{3-2\left[\dfrac{(1+\phi_1^2)(1+\Phi_1^2)+2\phi_1^s\Phi_1(1+\phi_1^2-2\phi_1^4)}{1-\phi_1^4}\right]} \tag{2.3.18}$$

and the variance of $u_t$ is given by

$$\sigma_u^2 = \frac{(K^{(\epsilon)}-1)\mu^2(1-\phi_1^4)}{(1+\phi_1^2)(1+\Phi_1^2)+2\phi_1^s\Phi_1(1+\phi_1^2-2\phi_1^4)}. \tag{2.3.19}$$

Empirical studies have found evidence of conditional non-normality in financial data. For example, Baillie and Bollerslev [3] find excess kurtosis in the return series for various exchange rates. Bollerslev [7] finds that the conditional distribution for S&P 500 returns is leptokurtic, and suggests that using a $t$-distribution may be more appropriate. For the above three examples the kurtosis of a multiplicative seasonal GARCH process can be easily found when the $\epsilon_t$ series is conditionally $t$-distributed. In (2.3.9), (2.3.13), and (2.3.17), the kurtosis can be obtained by using $E(Z_t^4) = K(Z_t) = 6/(\nu - 4) + 3$, where the degrees of freedom $\nu > 4$.

### 2.3.3 Forecast errors of seasonal GARCH

Following Thavaneswaran et al. [36], let $\epsilon_n^2(l)$ be the $l$-steps-ahead minimum mean square error forecast of $\epsilon_{n+l}^2$ based on $n$ observations $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and let $e_n(l) = \epsilon_{n+l}^2 - \epsilon_n^2(l)$ be the corresponding forecast error.

**Theorem 2.3.2.** *For any multiplicative seasonal GARCH*$(p,q)$x$(P,Q)$ *process, the variance of the* $l$*-steps-ahead forecast error,* $\mathrm{Var}[e_n(l)]$ *is given by*

$$\mathrm{var}[e_n(l)] = \frac{\omega^2}{\left[\sum_{j=0}^{\infty} \psi_j^2\right] \left[1 - \sum_{i=1}^{p} \phi_i\right]^2 \left[1 - \sum_{i=1}^{P} \Phi_i\right]^2} [K^{(\epsilon)} - 1] \left[1 + \sum_{j=1}^{l-1} \psi_j^2\right]. \quad (2.3.20)$$

*Proof.* The variance of the $l$-steps-ahead forecast error for a stationary ARMA process with error variance $\sigma_u^2$ is $Var[e_n(l)] = \sigma_u^2 \sum_{j=0}^{l-1} \psi_j^2$. The proof follows by using the error variance $\sigma_u^2$ in (2.3.7) and part (b) of Theorem 2.3.1. □

The respective variance of the $l$-steps-ahead forecast error for Examples 2.3.1, 2.3.2, and 2.3.3 are given by

$$\text{GARCH}(0,1)\text{x}(0,1): \quad \text{Var}[e_n(l)] = \frac{(K^{(\epsilon)}-1)\mu^2}{1+\Theta_1^2}\sum_{j=0}^{l-1}\psi_j^2, \tag{2.3.21}$$

$$\text{GARCH}(0,1)\text{x}(1,0): \quad \text{Var}[e_n(l)] = (K^{(\epsilon)}-1)\mu^2(1-\Phi_1^2)\sum_{j=0}^{l-1}\psi_j^2, \tag{2.3.22}$$

$$\text{GARCH}(1,0)\text{x}(1,0): \quad \text{Var}[e_n(l)] = \frac{(K^{(\epsilon)}-1)\mu^2(1-\phi_1^4)}{(1+\phi_1^2)(1+\Phi_1^2)+2\phi_1^s\Phi_1(1+\phi_1^2-2\phi_1^4)}\sum_{j=0}^{l-1}\psi_j^2, \tag{2.3.23}$$

where $K^{(\epsilon)}$ is given in (2.3.9)-(2.3.17).

**Example 2.3.4.** *Consider the random coefficient autoregressive (RCA) model of Nicholls and Quinn [28] with seasonal GARCH errors given by:*

$$r_t = (\beta + b_t)r_{t-1} + \epsilon_t \tag{2.3.24}$$

$$\epsilon_t = \sqrt{h_t}Z_t \tag{2.3.25}$$

$$\theta_q(B)\Theta_Q(L)h_t = \omega + \alpha(B)\epsilon_t^2, \tag{2.3.26}$$

*where* $\begin{pmatrix} b_t \\ \epsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_b^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{pmatrix}\right)$, *and* $\beta^2 + \sigma_b^2 < 1$. *Frank et al. [13] have derived expressions for the variance of the l-steps-ahead forecast error of* $r_{n+l}$ *for three RCA-GARCH*$(p,q)\text{x}(P,Q)_s$ *models as follows:*

$$RCA(1) - GARCH(0,1)\text{x}(0,1)_s: \qquad Var[e_n^{(r)}(l)] = \frac{\omega(1-\beta^2)}{1-\beta^2-\sigma_b^2}\sum_{j=0}^{l-1}\beta^{2j}$$

$$RCA(1) - GARCH(0,1)\text{x}(1,0)_s: \quad Var[e_n^{(r)}(l)] = \frac{\omega(1-\beta^2)}{(1-\Phi)(1-\beta^2-\sigma_b^2)} \sum_{j=0}^{l-1} \beta^{2j}$$

$$RCA(1) - GARCH(1,0)\text{x}(1,0)_s: \quad Var[e_n^{(r)}(l)] = \frac{\omega(1-\beta^2)}{(1-\phi)(1-\Phi)(1-\beta^2-\sigma_b^2)} \sum_{j=0}^{l-1} \beta^{2j}.$$

**Comparison of seasonal GARCH with GARCH(1,1)**

We compare the variance of the 1-step-ahead forecast error for a GARCH(1,1) model fit with that of the multiplicative seasonal GARCH process. We generated 50 samples of size 10,000 from each of the three multiplicative seasonal GARCH processes discussed in subsection 2.3.2: GARCH(0,1)x(0,1) with $(\mu, \omega, \theta, \Theta)' = (0.5, 0.0003, 0.03, -0.09)'$, GARCH(0,1)x(1,0) with $(\mu, \omega, \Phi, \theta)' = (0.5, 0.01, 0.04, 0.03)'$, and GARCH(1,0)x(1,0) with $(\mu, \omega, \Phi, \phi)' = (0.5, 0.01, 0.03, 0.02)'$ with $t_5$-distributed errors and with seasonal period $s = 4$. We used $t_5$-distributed errors since it is common to observe leptokurtosis in the return series. For each sample, we obtained maximum likelihood parameter estimates with the corresponding multiplicative seasonal GARCH process and with the GARCH(1,1) process. The parameter estimates obtained from each sample were used to estimate the variance of the 1-step-ahead forecast error using (2.3.21)-(2.3.23). The mean of these estimates is reported in Table 2.1. For all three models the mean of the variances of the 1-step ahead forecast error is marginally larger when the GARCH(1,1) model is assumed.

| *Estimated model* | *Estimate of* $Var[e_n(1)]$ |
|---|---|
| GARCH(0,1)x(0,1) | 7.23x$10^{-7}$ |
| GARCH(1,1) | 8.26x$10^{-7}$ |
| GARCH(0,1)x(1,0) | 8.54x$10^{-4}$ |
| GARCH(1,1) | 9.59x$10^{-4}$ |
| GARCH(1,0)x(1,0) | 8.45x$10^{-4}$ |
| GARCH(1,1) | 8.90x$10^{-4}$ |

Table 2.1: Forecast error variance comparison of seasonal GARCH processes

## 2.4 Application to financial data

The S&P CNX Nifty (or simply known as the Nifty) is one of the leading stock market indices listed on the National Stock Exchange of India. Daily closing prices of the index for a period of 2246 trading days, from January 1, 2001 to December 31, 2009[3] are studied. These prices are plotted in Figure 2.1. The log returns of the daily closing prices, consisting of 2245 observations, are plotted in Figure 2.2.

The log returns $r_t$ are leptokurtic, with an excess kurtosis of 7.98 and there appears to be some volatility clustering, signaling that the return series may be fit by a GARCH model. Figure 2.3 and 2.4 display the autocorrelation function (ACF) and partial autocorrelation function (PACF) of the squared residuals $(r_t - \hat{\mu})^2$, where $\hat{\mu}$ is the sample mean of the log returns. Significant sample autocorrelations further justify the use of a GARCH model. The ACF and PACF feature exponential decay

[3] Agarwal and Ahuja [1] note that the Securities and Exchange Board of India increased the trading hours of the stock markets beginning in January 2010.

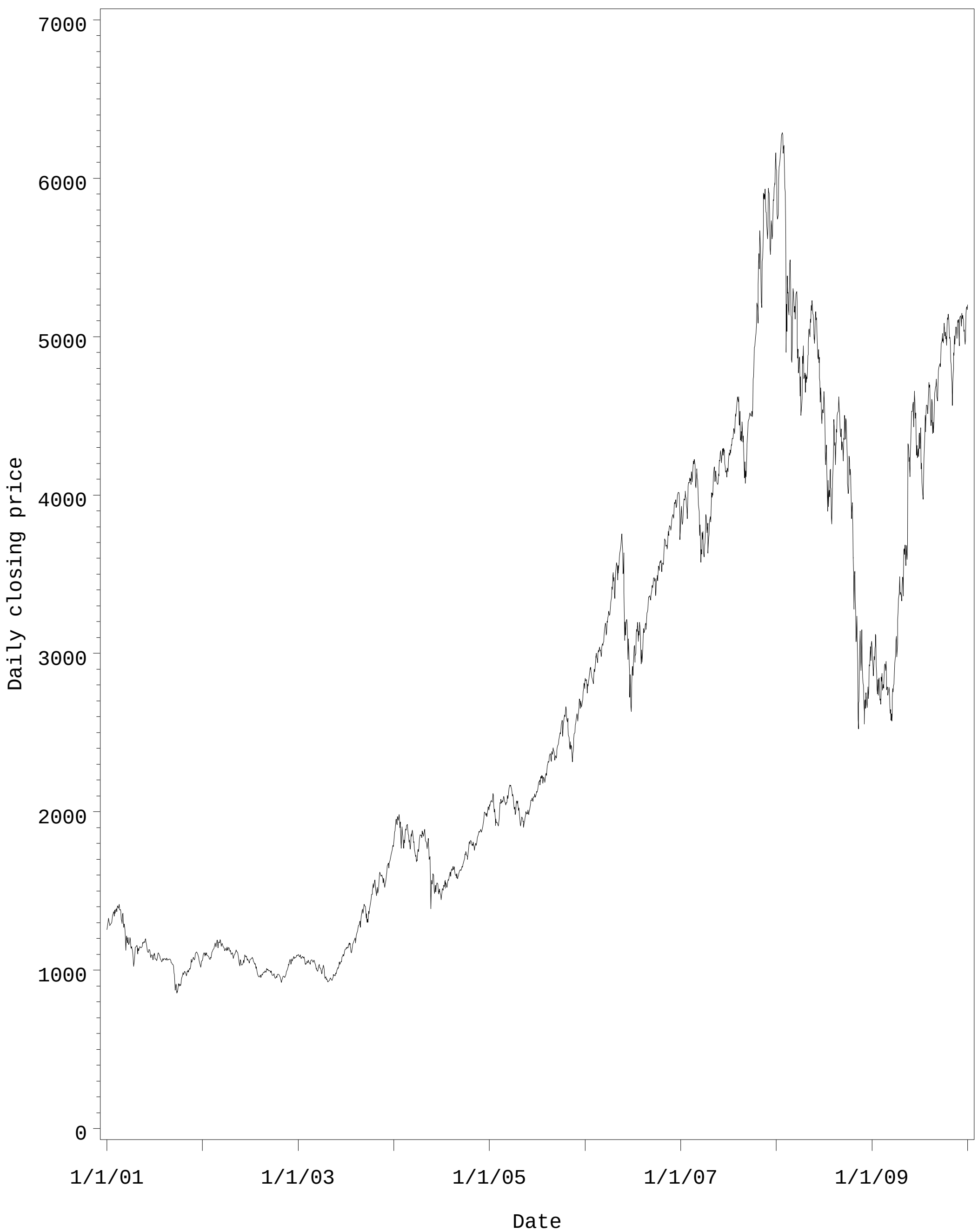


Figure 2.1: Daily closing prices of S&P CNX Nifty

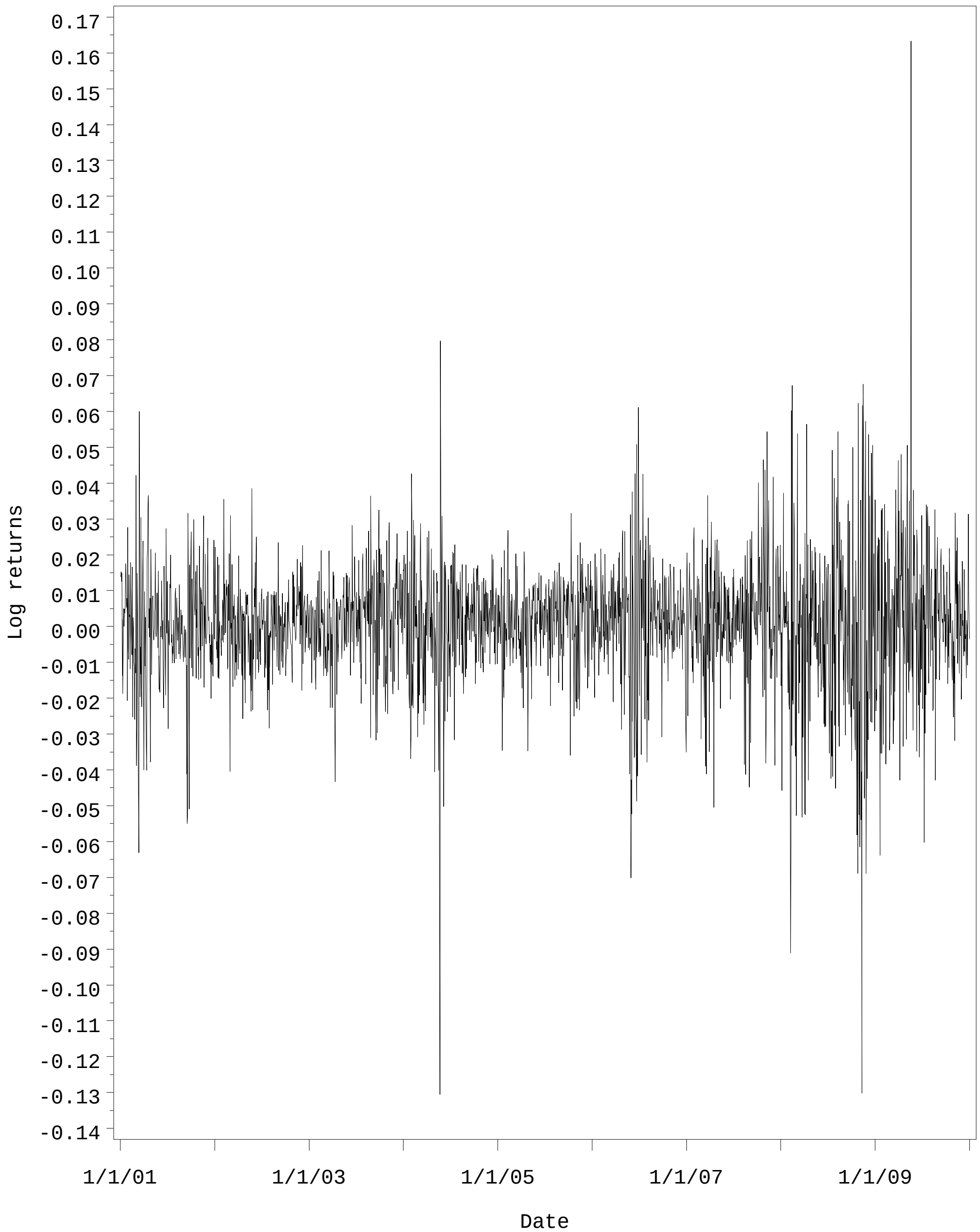


Figure 2.2: Log returns of S&P CNX Nifty

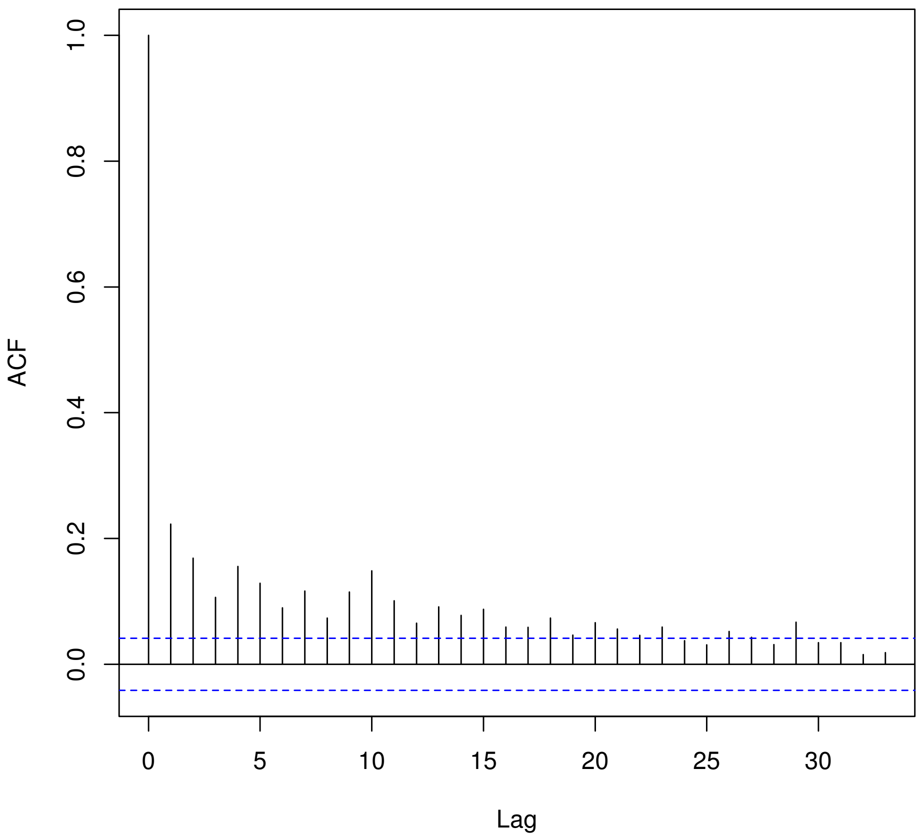


Figure 2.3: Sample ACF of squared residuals of Nifty log returns

with spikes at seasonal lags and at adjacent non-seasonal lags with seasonality index $s = 5$.

We estimated two GARCH models by maximum likelihood for the Nifty data: a GARCH(1,1) and a multiplicative seasonal GARCH(0,1)x$(1,0)_5$. The parameter estimates and standard errors obtained are shown in Tables 2.2 and 2.3. The parameter estimates from each of these GARCH models are used in the next chapter.

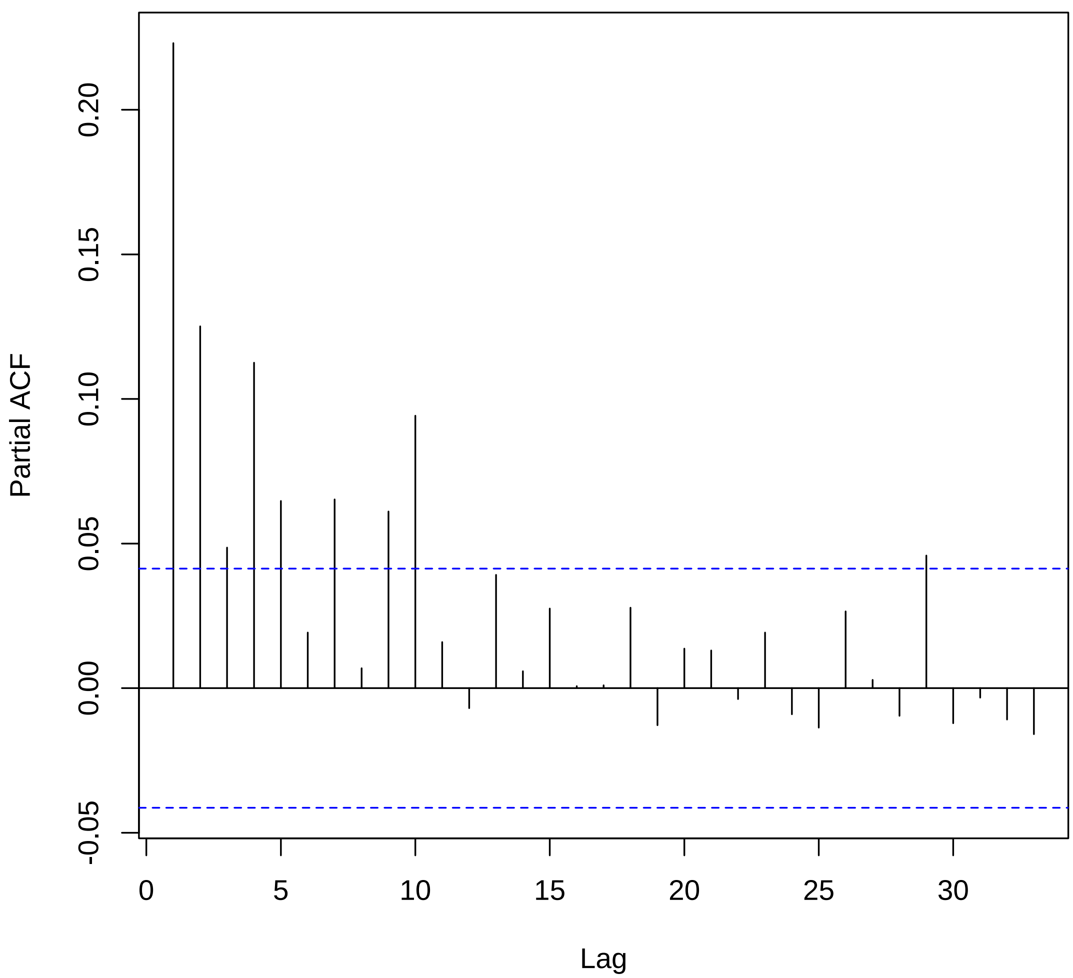


Figure 2.4: Sample PACF of squared residuals of Nifty log returns

| Parameter | Estimate | Standard Error |
|---|---|---|
| $\mu$ | 0.00153 | 0.000259 |
| $\omega$ | 5.36x$10^{-6}$ | 1.36x$10^{-6}$ |
| $\alpha$ | 0.102 | 0.0146 |
| $\beta$ | 0.836 | 0.0214 |

Table 2.2: Estimates and standard errors of GARCH(1,1) model parameters for Nifty data.

| Parameter | Estimate | Standard Error |
|---|---|---|
| $\mu$ | 0.00145 | 0.000275 |
| $\omega$ | 0.000162 | 4.09x$10^{-6}$ |
| $\Phi$ | 0.217 | $< 10^{-10}$ |
| $\theta$ | -0.621 | $< 10^{-10}$ |

Table 2.3: Estimates and standard errors of seasonal GARCH model parameters for Nifty data.

# Chapter 3

# Option pricing

In this chapter, call prices are obtained by using estimated seasonal GARCH parameters from the model derived in Chapter 2 in a Black-Scholes model with seasonal GARCH volatility. In Section 3.1, the foundation of the Black-Scholes model is set up and the popular Black-Scholes formula is derived. In Section 3.2, the assumption of constant volatility is relaxed and an option pricing formula based on the Black-Scholes model that employs GARCH volatility is derived. In Section 3.3, this option pricing formula is applied to price options for the Nifty data by modelling the volatility as a multiplicative seasonal GARCH process.

## 3.1 Black-Scholes

A Wiener process $\{W(t) : t \geq 0\}$, or standard Brownian motion, is a stochastic process with the following properties:

1. $W(t)$ is continuous as a function of $t$.

2. $\{W(t)\}$ has:

(a) independent increments. That is, for all $i = 1, \ldots, n$, the increments $W(t_i) - W(t_{i-1})$ are independent random variables.

(b) stationary increments. That is, the distribution of the increments $W(t+s) - W(t)$ does not depend on $t$.

3. For a fixed $t > 0, W(t) \sim N(0, t)$.

If we let $X(t) = \sigma W(t) + \mu t$, we obtain Brownian motion with drift $\mu$ and infinitesmal variance $\sigma^2$, a stochastic process with the following properties:

1. $X(t)$ is continuous as a function of $t$.

2. $\{X(t)\}$ has independent and stationary increments

3. For a fixed $t > 0, X(t) \sim N(\mu t, \sigma^2 t)$.

If we let $S(t) = S_0 e^{X(t)}$, we obtain geometric Brownian motion (a nonnegative variation of Brownian motion), where $S_0 > 0$ is the initial value. From here, $\log S(t) = \log S_0 + X(t)$. So, for a fixed $t > 0, \log S(t) \sim N(\log S_0 + \mu t, \sigma^2 t)$. Therefore, $S(t) \sim \text{Lognormal}(\log S_0 + \mu t, \sigma^2 t)$. Hence,

$$\mathbb{E}[S(t)] = \mathbb{E}[e^{\log S(t)}] = M_{\log S(t)}(1) = \exp\{\log S_0 + \mu t + \sigma^2 t/2\} = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$$

where $M_Y(s)$ is the moment generating function of the random variable $Y$.

The Black-Scholes model assumes that the price of an underlying asset follows a geometric Brownian motion, which is governed by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{3.1.1}$$

The model gives the value of a European call option when there is no arbitrage opportunity. The nonexistence of an arbitrage opportunity is guaranteed "if and only if there exists a probability measure over the set of outcomes under which all of the wagers have expected return 0" (Ross [31], p. 644). Under this setting, the current price of an asset at time $t = 0$ is the expected value of the asset's price at time $t$ discounted to its present value using the risk-free interest rate $r$. Since the different risk preferences of the market participants is not considered in this setting, we call this a risk-neutral setting. In contrast, in a risk-averse setting, the price of an asset at time $t = 0$ is below the expected value of the asset's price at time $t$ discounted to its present value using the risk-free interest rate $r$. From the continuous compounding interest formula, one dollar invested at time $t = 0$ grows to $e^{rt}$ dollars at time $t$. So, one dollar at time $t$ has a present value of $e^{-rt}$ dollars. Thus, in a risk-neutral universe,

$$S_0 = e^{-rt}\mathbb{E}[S(t)] \tag{3.1.2}$$

$$\Leftrightarrow S_0 = e^{-rt}S_0 e^{\mu t+\frac{1}{2}\sigma^2 t} \tag{3.1.3}$$

$$\Leftrightarrow 0 = -rt + \mu t + \frac{1}{2}\sigma^2 t \tag{3.1.4}$$

$$\Leftrightarrow \mu = r - \frac{1}{2}\sigma^2 \tag{3.1.5}$$

**Remark 3.1.1.** *The discounted price process* $\{e^{-rt}S(t), t \geq 0\}$ *that satisfies* (3.1.2) *in a market with no arbitrage opportunity, where the expectation is taken with respect to the risk-neutral probability measure, is a martingale. A martingale is a stochastic process whose expectation at time* $t$ *conditional on the set of information up to time* $s, s < t$ *is equal to the value of the process at time* $s$*. That is,* $\{X(t), t \geq 0\}$ *is a martingale if* $E(X_t|I_s) = X_s$*, where* $I_s$ *is the information known up to time* $s$ *(Ross*

*[31], pp. 648-649, Taylor [35], p. 35).*

Under the risk-neutral measure, $S(t) = S_0 e^{X(t)} = S_0 e^{rt - \frac{1}{2}\sigma^2 t + \sigma W(t)}$. Now, in a risk-neutral universe, the price of a call option $C(S_0, T)$ with current price $S_0$ and time to expiration $T$ is the expected value of the call option's price at the expiration time discounted to its present value using the risk-free interest rate $r$. If we let $K$ be the strike price, the call option's price at expiration is

$$C(S(T), T) = \begin{cases} S(T) - K, & \text{if } S(T) > K, \text{ i.e., in-the-money} \\ 0, & \text{if } S(T) < K, \text{ i.e., out-of-the-money} \end{cases} \tag{3.1.6}$$

Therefore, the price of the call option is obtained as follows:

$$\begin{aligned} C(S_0, T) &= e^{-rT}\mathbb{E}[\max\{S(T) - K, 0\}] \\ &= e^{-rT}\mathbb{E}[(S(T) - K)\mathbb{I}\{S(T) > K\} + 0 \cdot \mathbb{I}\{S(T) < K\}] \\ &= e^{-rT}\mathbb{E}[(S(T) - K)\mathbb{I}\{S_0 e^{rT - \frac{\sigma^2}{2}T + \sigma W(T)} > K\}] \\ &= e^{-rT}\mathbb{E}\left[(S(T) - K)\mathbb{I}\left\{W(T) > \frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]\right\}\right] \\ &= e^{-rT}\int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} S_0 \exp\left\{rT - \frac{\sigma^2}{2}T + \sigma x\right\} \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}} dx \\ &\quad - Ke^{-rT}\int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}} dx \end{aligned}$$

$$= S_0 \exp\left\{-\frac{\sigma^2}{2}T\right\} \int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} \frac{\exp\left\{-\frac{1}{2T}(x - T\sigma)^2 + \frac{\sigma^2}{2}T\right\}}{\sqrt{2\pi T}} dx$$

$$- Ke^{-rT} \int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}} dx$$

$$= S_0 \int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} \frac{\exp\left\{-\frac{1}{2T}(x - T\sigma)^2\right\}}{\sqrt{2\pi T}} dx$$

$$- Ke^{-rT} \int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}} dx.$$

In the first integral, substitute $y = \dfrac{x - T\sigma}{\sqrt{T}}$, so that $dx = \sqrt{T}dy$ and in the second integral, substitute $y = \dfrac{x}{\sqrt{T}}$ so that $dx = \sqrt{T}dy$. Then,

$$C(S_0, T) = S_0 \int_{\frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{K}{S_0}\right) - rT - \frac{\sigma^2}{2}T\right]}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$- Ke^{-rT} \int_{\frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T\right]}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$= S_0 \left[1 - \Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - rT - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right]$$

$$- Ke^{-rT} \left[1 - \Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right]$$

$$= S_0 \Phi\left(-\frac{\log\left(\frac{K}{S_0}\right) - rT - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(-\frac{\log\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$$

$$= S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + rT - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$$

$$= S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. More generally, if maturity occurs at time $T$, then for any given time $t$, the price of a call option is given by:

$$C(S_0, t) = S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \tag{3.1.7}$$

where $T - t$ is the time to maturity. This is the formula for pricing a call option obtained by Black and Scholes [6].

## 3.2 Black-Scholes with GARCH volatility

Stock prices are commonly assumed to follow a geometric Brownian motion, which is written in differential form as

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.2.1}$$

with the price process $\{S_t\}$ having drift $\mu$ and infinitesmal variance $\sigma$, and $\{W_t\}$ is a standard Brownian motion.[4] If we allow the variance of the stock's log returns $r_t = \log \dfrac{S_t}{S_{t-1}}$ to change with time, i.e. $\sigma = \theta_t$, then given $\theta_t$, the stock's centred log

[4]Nelson [27] has shown that as the time interval decreases and becomes infinitesmal, a discrete-time GARCH model converges to the continuous diffusion process given by (3.2.1).

returns may be written as

$$\epsilon_t = \log \frac{S_t}{S_{t-1}} - \mathbb{E}\left[\log \frac{S_t}{S_{t-1}}\right] = \theta_t Z_t, \quad Z_t \overset{i.i.d.}{\sim} N(0,1) \tag{3.2.2}$$

Under the risk-neutral probability measure, the price of a call option $C(S_0, T)$ with current price $S_0$, and time to expiration $T$ is the expected value of the call option's price at the expiration time discounted to its present value using the risk-free interest rate $r$. If we let $K$ be the strike price, the call price at expiration is

$$C(S(T), T) = \begin{cases} S(T) - K, & \text{if } S(T) > K, \text{ i.e., in-the-money} \\ 0, & \text{if } S(T) < K, \text{ i.e., out-of-the-money} \end{cases} \tag{3.2.3}$$

That is,

$$C(S_0, T) = e^{-rT}\mathbb{E}[\max\{S_T - K, 0\}] \tag{3.2.4}$$

Given $\theta_t$, we know the distribution of $S_T$ since $S_T = S_0 e^{X_T}$, where $X_T$ is the random variable at time $T$ from the Brownian motion process $\{X(t)\}$ having drift $\mu$ and infinitesmal variance $\theta_t$. Since $X_T|\theta_t \sim N(\mu T, \theta_t^2 T), (\log S_T - \log S_0)|\theta_t \sim N(\mu T, \theta_t^2 T)$.

In a risk-neutral universe, we have $(\log S_T - \log S_0)|\theta_t \sim N((r - \frac{1}{2}\theta_t^2)T, \theta_t^2 T)$ or $Y|\theta_t \sim N(-\frac{1}{2}\theta_t^2 T, \theta_t^2 T)$, where $Y = \log S_T - \log S_0 - rT$. Now, the price of the call option is obtained as follows:

$$\begin{aligned} C(S_0, T) &= e^{-rT}\mathbb{E}[\max\{S_T - K, 0\}] \\ &= e^{-rT}\mathbb{E}_{\theta_t}\left[\mathbb{E}_Y\left[\max\{S_0 e^{Y+rT} - K, 0\}\right]\right] \\ &= e^{-rT}\mathbb{E}_{\theta_t}\left[\mathbb{E}_Y\left[(S_0 e^{Y+rT} - K)\mathbb{I}\{S_0 e^{Y+rT} > K\}\right]\right] \end{aligned}$$

$$= e^{-rT}\mathbb{E}_{\theta_t}\left[\mathbb{E}_Y\left[(S_0e^{Y+rT}-K)\mathbb{I}\left\{Y > \log\left(\frac{K}{S_0}\right)-rT\right\}\right]\right]$$

$$= e^{-rT}\mathbb{E}_{\theta_t}\left[\int_{\log\left(\frac{K}{S_0}\right)-rT}^{\infty} S_0e^{y+rT}\frac{\exp\left\{-\frac{1}{2\theta_t^2}(y+\frac{1}{2}\theta_t^2)^2\right\}}{\sqrt{2\pi\theta_t^2}}dy\right]$$

$$- Ke^{-rT}\mathbb{E}_{\theta_t}\left[\int_{\log\left(\frac{K}{S_0}\right)-rT}^{\infty} \frac{\exp\left\{-\frac{1}{2\theta_t^2}(y+\frac{1}{2}\theta_t^2)^2\right\}}{\sqrt{2\pi\theta_t^2}}dy\right]$$

$$= S_0\mathbb{E}_{\theta_t}\left[\int_{\log\left(\frac{K}{S_0}\right)-rT}^{\infty} \frac{\exp\left\{-\frac{1}{2\theta_t^2}[(y+\frac{1}{2}\theta_t^2)^2-2y\theta_t^2]\right\}}{\sqrt{2\pi\theta_t^2}}dy\right]$$

$$- Ke^{-rT}\mathbb{E}_{\theta_t}\left[\int_{\log\left(\frac{K}{S_0}\right)-rT}^{\infty} \frac{\exp\left\{-\frac{1}{2\theta_t^2}(y+\frac{1}{2}\theta_t^2)^2\right\}}{\sqrt{2\pi\theta_t^2}}dy\right]$$

$$= S_0\mathbb{E}_{\theta_t}\left[\int_{\log\left(\frac{K}{S_0}\right)-rT}^{\infty} \frac{\exp\left\{-\frac{1}{2\theta_t^2}(y-\frac{1}{2}\theta_t^2)^2\right\}}{\sqrt{2\pi\theta_t^2}}dy\right]$$

$$- Ke^{-rT}\mathbb{E}_{\theta_t}\left[\int_{\log\left(\frac{K}{S_0}\right)-rT}^{\infty} \frac{\exp\left\{-\frac{1}{2\theta_t^2}(y+\frac{1}{2}\theta_t^2)^2\right\}}{\sqrt{2\pi\theta_t^2}}dy\right].$$

In the first integral, substitute $z = \frac{y-\frac{1}{2}\theta_t^2}{\theta_t}$, so that $dy = \theta_t dz$ and in the second integral, substitute $z = \frac{y+\frac{1}{2}\theta_t^2}{\theta_t}$, so that $dy = \theta_t dz$. Then,

$$C(S_0,T) = S_0\mathbb{E}_{\theta_t}\left[\int_{\frac{1}{\theta_t}\left[\log\left(\frac{K}{S_0}\right)-rT-\frac{1}{2}\theta_t^2\right]}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right]$$

$$- Ke^{-rT}\mathbb{E}_{\theta_t}\left[\int_{\frac{1}{\theta_t}\left[\log\left(\frac{K}{S_0}\right)-rT+\frac{1}{2}\theta_t^2\right]}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right]$$

$$= S_0 \mathbb{E}_{\theta_t}\left[\left(1 - \Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - rT - \frac{1}{2}\theta_t^2}{\theta_t}\right)\right)\right]$$

$$- Ke^{-rT}\mathbb{E}_{\theta_t}\left[\left(1 - \Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - rT + \frac{1}{2}\theta_t^2}{\theta_t}\right)\right)\right]$$

$$= S_0 \mathbb{E}_{\theta_t}\left[\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + rT + \frac{1}{2}\theta_t^2}{\theta_t}\right)\right]$$

$$- Ke^{-rT}\mathbb{E}_{\theta_t}\left[\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + rT - \frac{1}{2}\theta_t^2}{\theta_t}\right)\right]$$

$$= S_0 \mathbb{E}_{\theta_t}[f(\theta_t^2)] - Ke^{-rT}\mathbb{E}_{\theta_t}[g(\theta_t^2)]$$

where $f(\theta_t^2) = \Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + rT + \frac{1}{2}\theta_t^2}{\theta_t}\right)$ and $g(\theta_t^2) = \Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + rT - \frac{1}{2}\theta_t^2}{\theta_t}\right)$.

We can approximate $f(\theta_t^2)$ and $g(\theta_t^2)$ with a second-order Taylor approximation about $E(\theta_t^2)$. That is,

$$f(\theta_t^2) = f(E[\theta_t^2]) + f'(E[\theta_t^2])(\theta_t^2 - E[\theta_t^2]) + \frac{1}{2}f''(E[\theta_t^2])(\theta_t^2 - E[\theta_t^2])^2 + o([\theta_t^2 - E(\theta_t^2)]^2)$$

and

$$g(\theta_t^2) = g(E[\theta_t^2]) + g'(E[\theta_t^2])(\theta_t^2 - E[\theta_t^2]) + \frac{1}{2}g''(E[\theta_t^2])(\theta_t^2 - E[\theta_t^2])^2 + o([\theta_t^2 - E(\theta_t^2)]^2).$$

Taking expectations gives

$$\mathbb{E}_{\theta_t}[f(\theta_t^2)] \approx f(E[\theta_t^2]) + \frac{1}{2}f''(E[\theta_t^2])E[\theta_t^2 - E(\theta_t^2)]^2$$

and

$$\mathbb{E}_{\theta_t}[g(\theta_t^2)] \approx g(E[\theta_t^2]) + \frac{1}{2}g''(E[\theta_t^2])E[\theta_t^2 - E(\theta_t^2)]^2.$$

Note that the first derivative of $f$ is given by

$$\begin{aligned}
f'(\theta_t^2) &= \phi\left(\frac{\log(\frac{S_0}{K}) + rT + \frac{1}{2}\theta_t^2}{\theta_t}\right) \cdot \frac{d}{d(\theta_t^2)}\left(\frac{\log(\frac{S_0}{K}) + rT + \frac{1}{2}\theta_t^2}{\theta_t}\right) \\
&= \phi\left(\frac{\log(\frac{S_0}{K}) + rT + \frac{1}{2}V}{\sqrt{V}}\right) \cdot \frac{d}{dV}\left(\frac{\log(\frac{S_0}{K}) + rT + \frac{1}{2}V}{\sqrt{V}}\right), V = \theta_t^2 \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left[\log(\frac{S_0}{K}) + rT + \frac{1}{2}V\right]^2}{2V}\right\} \cdot \frac{\frac{1}{2}\sqrt{V} - \left[\log(\frac{S_0}{K}) + rT + \frac{1}{2}V\right]\frac{1}{2\sqrt{V}}}{V} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left\{2[\log(\frac{S_0}{K}) + rT + \frac{1}{2}V]\right\}^2}{8V}\right\} \cdot \frac{2V - 2\left[\log(\frac{S_0}{K}) + rT + \frac{1}{2}V\right]}{4V\sqrt{V}} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left\{2[\log(\frac{S_0}{K}) + rT] + V\right\}^2}{8V}\right\} \cdot \frac{V - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4V\sqrt{V}} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left\{2[\log(\frac{S_0}{K}) + rT] + \theta_t^2\right\}^2}{8\theta_t^2}\right\} \cdot \frac{\theta_t^2 - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4\theta_t^2 \cdot \theta_t}
\end{aligned}$$

and the second derivative of $f$ is given by

$$
\begin{aligned}
f''(\theta_t^2) =& f''(V) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\left\{2[\log(\frac{S_0}{K}) + rT] + V\right\}^2}{8V} \right\} \\
& \left[ \frac{4V\sqrt{V} - \left\{V - 2\left[\log\left(\frac{S_0}{K}\right) + rT\right]\right\} 6\sqrt{V}}{16V^3} \right. \\
& - \left( \frac{2\left\{2\left[\log\left(\frac{S_0}{K}\right) + rT\right] + V\right\} \cdot 8V - 8\left\{2\left[\log\left(\frac{S_0}{K}\right) + rT\right] + V\right\}^2}{64V^2} \right) \\
& \left. \left( \frac{V - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4V\sqrt{V}} \right) \right] \\
=& \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\left\{2[\log(\frac{S_0}{K}) + rT] + V\right\}^2}{8V} \right\} \left[ \frac{2V - 3\left\{V - 2\left[\log\left(\frac{S_0}{K}\right) + rT\right]\right\}}{8V^2\sqrt{V}} \right. \\
& - \left( \frac{2V\left\{2\left[\log\left(\frac{S_0}{K}\right) + rT\right] + V\right\} - \left\{2\left[\log\left(\frac{S_0}{K}\right) + rT\right] + V\right\}^2}{8V^2} \right) \\
& \left. \left( \frac{V - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4V\sqrt{V}} \right) \right] \\
=& \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\left\{2[\log(\frac{S_0}{K}) + rT] + V\right\}^2}{8V} \right\} \left[ \frac{6\left[\log\left(\frac{S_0}{K}\right) + rT\right] - V}{8V^2\sqrt{V}} \right. \\
& \left. - \left( \frac{V^2 - 4\left[\log\left(\frac{S_0}{K}\right) + rT\right]^2}{8V^2} \right) \left( \frac{V - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4V\sqrt{V}} \right) \right] \\
=& \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\left\{2[\log(\frac{S_0}{K}) + rT] + \theta_t^2\right\}^2}{8\theta_t^2} \right\} \left[ \frac{6\left[\log\left(\frac{S_0}{K}\right) + rT\right] - \theta_t^2}{8\theta_t^4 \cdot \theta_t} \right. \\
& \left. - \left( \frac{\theta_t^4 - 4\left[\log\left(\frac{S_0}{K}\right) + rT\right]^2}{8\theta_t^4} \right) \left( \frac{\theta_t^2 - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4\theta_t^2 \cdot \theta_t} \right) \right]
\end{aligned}
$$

Similarly, the second derivative of $g$ is found to be

$$g''(\theta_t^2) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left\{2[\log(\frac{S_0}{K}) + rT] - \theta_t^2\right\}^2}{8\theta_t^2}\right\} \left[\frac{6\left[\log\left(\frac{S_0}{K}\right) + rT\right] + \theta_t^2}{8\theta_t^4 \cdot \theta_t}\right.$$

$$\left. + \left(\frac{\theta_t^4 - 4\left[\log\left(\frac{S_0}{K}\right) + rT\right]^2}{8\theta_t^4}\right)\left(\frac{\theta_t^2 + 2\left[\log(\frac{S_0}{K}) + rT\right]}{4\theta_t^2 \cdot \theta_t}\right)\right]$$

Thus, we can approximate the call price by:

$$\begin{aligned}
C(S_0, T) =& S_0 \left\{f(E[\theta_t^2]) + \frac{1}{2} f''(E[\theta_t^2]) E[\theta_t^2 - E(\theta_t^2)]^2\right\} \\
& - Ke^{-rT} \left\{g(E[\theta_t^2]) + \frac{1}{2} g''(E[\theta_t^2]) E[\theta_t^2 - E(\theta_t^2)]^2\right\} \\
=& S_0 \left\{f(E[\theta_t^2]) + \frac{1}{2} f''(E[\theta_t^2])[E(\theta_t^4) - E^2(\theta_t^2)]\right\} \\
& - Ke^{-rT} \left\{g(E[\theta_t^2]) + \frac{1}{2} g''(E[\theta_t^2])[E(\theta_t^4) - E^2(\theta_t^2)]\right\} \\
=& S_0 \left\{f(E[\theta_t^2]) + \frac{1}{2} f''(E[\theta_t^2]) \left[\frac{E(\theta_t^4)}{E^2(\theta_t^2)} - 1\right] [E^2(\theta_t^2)]\right\} \\
& - Ke^{-rT} \left\{g(E[\theta_t^2]) + \frac{1}{2} g''(E[\theta_t^2]) \left[\frac{E(\theta_t^4)}{E^2(\theta_t^2)} - 1\right] [E^2(\theta_t^2)]\right\} \\
=& S_0 \left\{f(E[\theta_t^2]) + \frac{1}{2} f''(E[\theta_t^2]) \left[\frac{E(\theta_t^4 Z_t^4)}{E(Z_t^4)} \frac{E^2(Z_t^2)}{E^2(\theta_t^2 Z_t^2)} - 1\right] [E^2(\theta_t^2)]\right\} \\
& - Ke^{-rT} \left\{g(E[\theta_t^2]) + \frac{1}{2} g''(E[\theta_t^2]) \left[\frac{E(\theta_t^4 Z_t^4)}{E(Z_t^4)} \frac{E^2(Z_t^2)}{E^2(\theta_t^2 Z_t^2)} - 1\right] [E^2(\theta_t^2)]\right\} \\
=& S_0 \left\{f(E[\theta_t^2]) + \frac{1}{2} f''(E[\theta_t^2]) \left[\frac{1}{\kappa^{(Z)}} \frac{E(\epsilon_t^4)}{E^2(\epsilon_t^2)} - 1\right] [E^2(\theta_t^2)]\right\} \\
& - Ke^{-rT} \left\{g(E[\theta_t^2]) + \frac{1}{2} g''(E[\theta_t^2]) \left[\frac{1}{\kappa^{(Z)}} \frac{E(\epsilon_t^4)}{E^2(\epsilon_t^2)} - 1\right] [E^2(\theta_t^2)]\right\}
\end{aligned}$$

$$=S_0\left\{f(E[\theta_t^2])+\frac{1}{2}f''(E[\theta_t^2])\left[\frac{\kappa^{(\epsilon)}}{\kappa^{(Z)}}-1\right][E^2(\theta_t^2)]\right\}$$

$$-Ke^{-rT}\left\{g(E[\theta_t^2])+\frac{1}{2}g''(E[\theta_t^2])\left[\frac{\kappa^{(\epsilon)}}{\kappa^{(Z)}}-1\right][E^2(\theta_t^2)]\right\} \tag{3.2.5}$$

where $\kappa^{(Z)}$ is the kurtosis of $Z_t$, $\kappa^{(\epsilon)}$ is the kurtosis of the centred log returns $\epsilon_t$,

$$f(E[\theta_t^2])=\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+rT+\frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}\right), g(E[\theta_t^2])=\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+rT-\frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}\right),$$

$$f''(E[\theta_t^2])=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left\{2[\log(\frac{S_0}{K})+rT]+E(\theta_t^2)\right\}^2}{8E(\theta_t^2)}\right\}\left[\frac{6\left[\log\left(\frac{S_0}{K}\right)+rT\right]-E(\theta_t^2)}{8E^2(\theta_t^2)\sqrt{E(\theta_t^2)}}\right.$$

$$\left.-\left(\frac{E^2(\theta_t^2)-4\left[\log\left(\frac{S_0}{K}\right)+rT\right]^2}{8E^2(\theta_t^2)}\right)\left(\frac{E(\theta_t^2)-2\left[\log(\frac{S_0}{K})+rT\right]}{4E(\theta_t^2)\sqrt{E(\theta_t^2)}}\right)\right], \text{and}$$

$$g''(E[\theta_t^2])=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left\{2[\log(\frac{S_0}{K})+rT]-E(\theta_t^2)\right\}^2}{8E(\theta_t^2)}\right\}\left[\frac{6\left[\log\left(\frac{S_0}{K}\right)+rT\right]+E(\theta_t^2)}{8E^2(\theta_t^2)\sqrt{E(\theta_t^2)}}\right.$$

$$\left.+\left(\frac{E^2(\theta_t^2)-4\left[\log\left(\frac{S_0}{K}\right)+rT\right]^2}{8E^2(\theta_t^2)}\right)\left(\frac{E(\theta_t^2)+2\left[\log(\frac{S_0}{K})+rT\right]}{4E(\theta_t^2)\sqrt{E(\theta_t^2)}}\right)\right].$$

This turns out to be the formula given in Gong et al. [17].

**Note 3.2.1.** *The expected value of the conditional variance $E(\theta_t^2)$ is*

*(i) (GARCH(p, q) process)*

*For the GARCH(p, q) process given by:*

$$\epsilon_t=\theta_t Z_t, \quad Z_t\sim N(0,1) \tag{3.2.6}$$

$$\theta_t^2=\omega+\sum_{i=1}^{p}\alpha_i\epsilon_{t-i}^2+\sum_{j=1}^{q}\beta_j\theta_{t-j}^2, \tag{3.2.7}$$

*by taking expectations of the conditional variance equation* (3.2.7)*, we obtain:*

$$E(\theta_t^2) = \omega + \sum_{i=1}^{p} \alpha_i E(\theta_{t-i}^2 Z_{t-i}^2) + \sum_{j=1}^{q} \beta_j E(\theta_{t-j}^2). \tag{3.2.8}$$

*Hence, the expected value of the conditional variance is given by:*

$$E(\theta_t^2) = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}. \tag{3.2.9}$$

*(ii) (GARCH($p, q$)x($P, Q$) process)*

*For the multiplicative seasonal GARCH($p, q$)x($P, Q$) process, Frank et al. [13] have shown that the expected value of the conditional variance is given by:*

$$E(\theta_t^2) = \frac{\omega}{\left(1 - \sum_{i=1}^{p} \phi_i\right)\left(1 - \sum_{i=1}^{P} \Phi_i\right)}. \tag{3.2.10}$$

## 3.3 Application to financial data

Call option prices for the S&P CNX Nifty introduced in Section 2.4 were obtained from the website of the National Stock Exchange of India. Closing option prices on October 3, 2005 for contracts expiring on October 27, November 24, and December 29, 2005 (times to maturity of 24, 52, and 87 days, respectively) were used for various strike prices. The closing price of the underlying asset (the S&P CNX Nifty stock index) was 2630.05 on October 3, 2005.

Using a risk-free interest rate $r$ of either 3% or 5%, for each combination of time to maturity $T$ and strike price $K$, estimated call prices were obtained using three option pricing formulae: the Black-Scholes formula under constant volatility (3.1.7), GARCH(1,1) volatility (3.2.5),(3.2.9), and multiplicative seasonal GARCH(0,1)x$(1,0)_5$ volatility (3.2.5),(3.2.10). These estimates were compared to the observed call option prices and an absolute percent error was calculated in each instance. The results are reported in Tables 3.1 and 3.2. In almost every combination of time to maturity, strike price, and risk-free interest rate, the option pricing formula that takes into account seasonal GARCH volatility provided the closest estimates to the true call price.

Although the multiplicative seasonal GARCH model provides estimated call prices closest to the observed call price, in many instances the percent errors are rather large. Ritchey [30], Melick and Thomas [26], and Guo [19] have used a weighted average of two or more normal densities with different volatilities to estimate call prices using the Black-Scholes formula. In Tables 3.3 and 3.4, a weighted average of the estimated call price under constant volatility and the estimated call price under GARCH volatility is computed. This method provides estimated call prices that are close to the observed call price since the Black-Scholes formula with constant volatility consistently over-prices options while the Black-Scholes formulas with GARCH volatility consistently under-price options. For each combination of strike price and time to maturity, weight estimates are obtained by the following convex combination:

Observed call price =

(Weight) x (Estimated call price under GARCH volatility)

+ (1 - Weight) x (Estimated call price under constant volatility).

The weighted average call price is then computed using the mean of the weight estimates as follows:

Weighted average call price =

(Mean weight) x (Estimated call price under GARCH volatility)

+ (1 - Mean weight) x (Estimated call price under constant volatility).

For most combinations of time to maturity, strike price, and risk-free interest rate, the weighted average call price obtained using multiplicative seasonal GARCH volatility provides closer estimates to the true call price than the weighted average call price obtained using GARCH(1,1) volatility.

| T | $S_0$ | K | Observed Call Price | Seasonal GARCH | | GARCH(1,1) | | Black-Scholes | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Estimated Call Price | Percent Error | Estimated Call Price | Percent Error | Estimated Call Price | Percent Error |
| 24 | 2630.05 | 2600 | 73.05 | 45.84 | 37.25% | 41.54 | 43.13% | 107.74 | 47.48% |
| 24 | 2630.05 | 2610 | 64.00 | 38.90 | 39.22% | 33.62 | 47.47% | 102.38 | 59.97% |
| 24 | 2630.05 | 2620 | 61.75 | 32.59 | 47.22% | 26.55 | 57.01% | 97.20 | 57.41% |
| 24 | 2630.05 | 2630 | 59.95 | 26.94 | 55.06% | 20.48 | 65.84% | 92.20 | 53.79% |
| 24 | 2630.05 | 2640 | 49.00 | 21.98 | 55.14% | 15.49 | 68.38% | 87.37 | 78.32% |
| 24 | 2630.05 | 2650 | 45.25 | 17.70 | 60.88% | 11.59 | 74.38% | 82.73 | 82.82% |
| 24 | 2630.05 | 2660 | 42.00 | 14.08 | 66.48% | 8.68 | 79.33% | 78.25 | 86.32% |
| 24 | 2630.05 | 2670 | 36.40 | 11.07 | 69.59% | 6.62 | 81.82% | 73.95 | 103.17% |
| 24 | 2630.05 | 2680 | 29.20 | 8.61 | 70.50% | 5.20 | 82.19% | 69.82 | 139.12% |
| 52 | 2630.05 | 2500 | 146.10 | 141.13 | 3.40% | 141.19 | 3.36% | 210.79 | 44.28% |
| 52 | 2630.05 | 2550 | 120.00 | 93.08 | 22.43% | 93.08 | 22.44% | 180.12 | 50.10% |
| 52 | 2630.05 | 2600 | 91.65 | 50.26 | 45.16% | 46.60 | 49.15% | 152.47 | 66.37% |
| 52 | 2630.05 | 2650 | 68.65 | 20.23 | 70.53% | 13.84 | 79.84% | 127.85 | 86.23% |
| 52 | 2630.05 | 2700 | 47.70 | 6.00 | 87.41% | 3.96 | 91.69% | 106.19 | 122.61% |
| 87 | 2630.05 | 2600 | 105.95 | 56.04 | 47.11% | 53.18 | 49.81% | 194.37 | 83.45% |
| 87 | 2630.05 | 2650 | 85.00 | 23.73 | 72.08% | 17.20 | 79.76% | 170.05 | 100.06% |
| 87 | 2630.05 | 2700 | 69.90 | 7.37 | 89.46% | 4.58 | 93.45% | 148.02 | 111.76% |

Table 3.1: Call prices based on the Nifty data set (r = 0.03/365)

| T | $S_0$ | K | Observed Call Price | Seasonal GARCH Estimated Call Price | Seasonal GARCH Percent Error | GARCH(1,1) Estimated Call Price | GARCH(1,1) Percent Error | Black-Scholes Estimated Call Price | Black-Scholes Percent Error |
|---|---|---|---|---|---|---|---|---|---|
| 24 | 2630.05 | 2600 | 73.05 | 48.34 | 33.82% | 44.41 | 39.21% | 109.61 | 50.05% |
| 24 | 2630.05 | 2610 | 64.00 | 41.21 | 35.60% | 36.25 | 43.36% | 104.20 | 62.81% |
| 24 | 2630.05 | 2620 | 61.75 | 34.69 | 43.82% | 28.87 | 53.24% | 98.96 | 60.27% |
| 24 | 2630.05 | 2630 | 59.95 | 28.82 | 51.93% | 22.45 | 62.55% | 93.91 | 56.64% |
| 24 | 2630.05 | 2640 | 49.00 | 23.63 | 51.78% | 17.10 | 65.11% | 89.03 | 81.69% |
| 24 | 2630.05 | 2650 | 45.25 | 19.12 | 57.75% | 12.83 | 71.64% | 84.33 | 86.35% |
| 24 | 2630.05 | 2660 | 42.00 | 15.27 | 63.63% | 9.60 | 77.15% | 79.80 | 89.99% |
| 24 | 2630.05 | 2670 | 36.40 | 12.06 | 66.87% | 7.26 | 80.06% | 75.44 | 107.26% |
| 24 | 2630.05 | 2680 | 29.20 | 9.42 | 67.74% | 5.64 | 80.70% | 71.26 | 144.04% |
| 52 | 2630.05 | 2500 | 146.10 | 148.11 | 1.38% | 148.15 | 1.40% | 215.39 | 47.43% |
| 52 | 2630.05 | 2550 | 120.00 | 99.87 | 16.78% | 100.00 | 16.67% | 184.39 | 53.66% |
| 52 | 2630.05 | 2600 | 91.65 | 55.98 | 38.92% | 53.12 | 42.04% | 156.37 | 70.62% |
| 52 | 2630.05 | 2650 | 68.65 | 23.70 | 65.48% | 17.17 | 74.99% | 131.37 | 91.36% |
| 52 | 2630.05 | 2700 | 47.70 | 7.35 | 84.59% | 4.57 | 90.42% | 109.33 | 129.20% |
| 87 | 2630.05 | 2600 | 105.95 | 66.14 | 37.57% | 64.48 | 39.14% | 200.74 | 89.47% |
| 87 | 2630.05 | 2650 | 85.00 | 30.38 | 64.26% | 24.13 | 71.61% | 175.96 | 107.01% |
| 87 | 2630.05 | 2700 | 69.90 | 10.21 | 85.40% | 6.09 | 91.29% | 153.46 | 119.54% |

Table 3.2: Call prices based on the Nifty data set (r = 0.05/365)

| T | $S_0$ | K | Observed Call Price | Weighted Average Seasonal GARCH Estimated Call Price | Weighted Average Seasonal GARCH Percent Error | Weighted Average GARCH(1,1) Estimated Call Price | Weighted Average GARCH(1,1) Percent Error |
|---|---|---|---|---|---|---|---|
| 24 | 2630.05 | 2600 | 73.05 | 70.16 | 3.96% | 69.52 | 4.83% |
| 24 | 2630.05 | 2610 | 64.00 | 63.84 | 0.25% | 62.68 | 2.06% |
| 24 | 2630.05 | 2620 | 61.75 | 57.97 | 6.11% | 56.41 | 8.65% |
| 24 | 2630.05 | 2630 | 59.95 | 52.58 | 12.29% | 50.79 | 15.28% |
| 24 | 2630.05 | 2640 | 49.00 | 47.67 | 2.71% | 45.87 | 6.38% |
| 24 | 2630.05 | 2650 | 45.25 | 43.25 | 4.42% | 41.66 | 7.94% |
| 24 | 2630.05 | 2660 | 42.00 | 39.29 | 6.45% | 38.09 | 9.32% |
| 24 | 2630.05 | 2670 | 36.40 | 35.78 | 1.72% | 35.08 | 3.64% |
| 24 | 2630.05 | 2680 | 29.20 | 32.66 | 11.85% | 32.51 | 11.35% |
| 52 | 2630.05 | 2500 | 146.10 | 168.50 | 15.33% | 170.61 | 16.77% |
| 52 | 2630.05 | 2550 | 120.00 | 127.28 | 6.06% | 129.87 | 8.22% |
| 52 | 2630.05 | 2600 | 91.65 | 90.42 | 1.34% | 91.35 | 0.33% |
| 52 | 2630.05 | 2650 | 68.65 | 62.51 | 8.94% | 62.03 | 9.65% |
| 52 | 2630.05 | 2700 | 47.70 | 45.36 | 4.90% | 47.17 | 1.12% |
| 87 | 2630.05 | 2600 | 105.95 | 110.39 | 4.19% | 112.85 | 6.51% |
| 87 | 2630.05 | 2650 | 85.00 | 81.22 | 4.45% | 81.80 | 3.76% |
| 87 | 2630.05 | 2700 | 69.90 | 62.63 | 10.41% | 65.20 | 6.72% |

Table 3.3: Weighted average call prices based on the Nifty data set (r = 0.03/365)

| T | $S_0$ | K | Observed Call Price | Weighted Average Seasonal GARCH | | Weighted Average GARCH(1,1) | |
|---|---|---|---|---|---|---|---|
| | | | | Estimated Call Price | Percent Error | Estimated Call Price | Percent Error |
| 24 | 2630.05 | 2600 | 73.05 | 70.31 | 3.75% | 69.83 | 4.41% |
| 24 | 2630.05 | 2610 | 64.00 | 63.79 | 0.32% | 62.74 | 1.97% |
| 24 | 2630.05 | 2620 | 61.75 | 57.73 | 6.51% | 56.20 | 8.99% |
| 24 | 2630.05 | 2630 | 59.95 | 52.15 | 13.01% | 50.31 | 16.08% |
| 24 | 2630.05 | 2640 | 49.00 | 47.07 | 3.93% | 45.14 | 7.88% |
| 24 | 2630.05 | 2650 | 45.25 | 42.49 | 6.09% | 40.70 | 10.05% |
| 24 | 2630.05 | 2660 | 42.00 | 38.41 | 8.56% | 36.96 | 12.00% |
| 24 | 2630.05 | 2670 | 36.40 | 34.78 | 4.44% | 33.84 | 7.04% |
| 24 | 2630.05 | 2680 | 29.20 | 31.59 | 8.18% | 31.22 | 6.91% |
| 52 | 2630.05 | 2500 | 146.10 | 172.23 | 17.89% | 174.36 | 19.34% |
| 52 | 2630.05 | 2550 | 120.00 | 130.17 | 8.47% | 132.90 | 10.75% |
| 52 | 2630.05 | 2600 | 91.65 | 91.97 | 0.35% | 93.37 | 1.88% |
| 52 | 2630.05 | 2650 | 68.65 | 62.30 | 9.25% | 61.69 | 10.14% |
| 52 | 2630.05 | 2700 | 47.70 | 43.91 | 7.95% | 45.41 | 4.81% |
| 87 | 2630.05 | 2600 | 105.95 | 114.39 | 7.97% | 117.60 | 10.99% |
| 87 | 2630.05 | 2650 | 85.00 | 82.57 | 2.86% | 83.32 | 1.98% |
| 87 | 2630.05 | 2700 | 69.90 | 61.56 | 11.93% | 63.54 | 9.10% |

Table 3.4: Weighted average call prices based on the Nifty data set (r = 0.05/365)

# Chapter 4

# Bootstrap confidence intervals for call prices

In this chapter, bootstrap confidence intervals for the call price are obtained under option pricing formulae with multiplicative seasonal GARCH volatility and GARCH(1,1) volatility. In Section 4.1, a parametric bootstrap procedure is described for obtaining confidence intervals for the call price under a GARCH model. In Section 4.2, the bootstrap procedure is investigated by simulating data from a multiplicative seasonal GARCH process and comparing estimation with a multiplicative seasonal GARCH process and a GARCH(1,1) process. In Section 4.3, the bootstrap procedure is applied to real data that exhibits multiplicative seasonal GARCH volatility and the width of the confidence intervals is compared for a GARCH(1,1) fit and for a multiplicative seasonal GARCH fit.

## 4.1 Methodology

The bootstrap procedure developed by Pascual et al. [29] for a GARCH(1,1) is modified to obtain a confidence interval for the call price based on a seasonal GARCH volatility model. Ghahramani and Thavaneswaran [16] propose a method to identify the error distribution of the GARCH process which may then be used in a parametric bootstrap procedure.

The seasonal GARCH$(p,q)$x$(P,Q)_s$ volatility model is given by:

$$\epsilon_t = r_t - \mu = \sqrt{h_t} Z_t \tag{4.1.1}$$

$$\theta(B)\Theta(L)h_t = \omega + \alpha(B)\epsilon_t^2, \tag{4.1.2}$$

where $r_t = \log(S_t/S_{t-1}), \mu = \mathbb{E}(r_t), \{S_t\}$ is the price process, $\{Z_t\}$ is an uncorrelated sequence of random variables with zero mean and unit variance,

$\alpha(B) = \theta(B)\Theta(L) - [1 - [\phi(B) - \theta(B)]]\Phi(L)$, $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$,

$\theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i$, $\Phi(L) = 1 - \sum_{i=1}^{P} \Phi_i L^i$, $\Theta(L) = 1 - \sum_{i=1}^{Q} \Theta_i L^i$, and $L = B^s$, where $s$ represents the seasonal period.

The parametric bootstrap procedure consists of the following 7 steps.

**Step 1:**

Fit a normal-seasonal GARCH model as an initial model. That is, assume that $Z_t \overset{\text{i.i.d.}}{\sim} N(0,1)$ in (4.1.1). Obtain estimates of the model parameters $\hat{\mu}, \hat{\omega}, \hat{\phi}_i, \hat{\theta}_i, \hat{\Phi}_i$

and $\hat{\Theta}_i$ by maximum likelihood estimation and obtain the following fitted model:

$$\epsilon_t = r_t - \hat{\mu} = \sqrt{\hat{h}_t} Z_t \tag{4.1.3}$$

$$\hat{\theta}(B)\hat{\Theta}(L)\hat{h}_t = \hat{\omega} + \hat{\alpha}(B)\epsilon_t^2, \qquad t = r+1, \ldots, n, \tag{4.1.4}$$

and

$$\hat{h}_1 = \ldots = \hat{h}_r = \frac{\hat{\omega}}{\left(1 - \sum_{i=1}^{p} \hat{\phi}_i\right)\left(1 - \sum_{i=1}^{P} \hat{\Phi}_i\right)}, \tag{4.1.5}$$

where $\hat{\alpha}(B) = \hat{\theta}(B)\hat{\Theta}(L) - [1 - [\hat{\phi}(B) - \hat{\theta}(B)]]\hat{\Phi}(L)$, $\hat{\phi}(B) = 1 - \sum_{i=1}^{p} \hat{\phi}_i B^i$, $\hat{\theta}(B) = 1 - \sum_{i=1}^{q} \hat{\theta}_i B^i$, $\hat{\Phi}(L) = 1 - \sum_{i=1}^{P} \hat{\Phi}_i L^i$, $\hat{\Theta}(L) = 1 - \sum_{i=1}^{Q} \hat{\Theta}_i L^i$, and $r = \max\{q + Qs, \max(p, q) + Ps\}$.

**Step 2:**

Compute the standardized residuals $\hat{Z}_t = \dfrac{\epsilon_t}{\sqrt{\hat{h}_t}}, t = 1, \ldots, n$, where $n$ is the sample size. Following Ghahramani and Thavaneswaran [16], obtain the method of moments estimate of the asymptotic correlation $\rho$ between the sample mean and the sample median of the standardized residuals.[5] The estimator is given by:

$$\hat{\rho} = \frac{\sum_{t=1}^{n} |\hat{Z}_t - \bar{Z}|/n}{s_Z}. \tag{4.1.6}$$

where $\bar{Z}$ and $s_Z$ denote the sample mean and sample standard deviation of the standardized residuals, respectively. The method of moments estimate is compared

[5] Ghahramani and Thavaneswaran [16] note that the asymptotic correlation between the sample mean and the sample median is the same as the finite sample correlation between the least squares and least absolute deviation estimating functions based on one observation from a random sample of size $n$ from a symmetric population.

| Distribution | $\rho$ |
|---|---|
| Normal(0,1) | 0.798 |
| $t_3$ | 0.637 |
| $t_4$ | 0.707 |
| $t_5$ | 0.735 |
| $t_6$ | 0.750 |
| $t_7$ | 0.759 |
| $t_8$ | 0.765 |
| Laplace(0,$b$) | 0.707 |

Table 4.1: Theoretical $E|X-\mu|/\sigma$ values.

to Table 4.1, the theoretical values of $\rho = E|X-\mu|/\sigma$, where $X$ is from a distribution with mean $\mu$ and variance $\sigma^2$, for some continuous symmetric distributions (Johnson and Kotz [22])[6]. We identify the error distribution of the GARCH process as that distribution whose theoretical correlation is closest to the estimate.

**Step 3:**

Estimate the model parameters by maximum likelihood in (4.1.2) assuming the identified error distribution for $Z_t$ in (4.1.1) and obtain a fitted model from which we compute the standardized residuals $\hat{Z}_t = \frac{\epsilon_t}{\sqrt{\hat{h}_t}}$.

[6] Proof of the asymptotic correlation $\rho$ between the sample mean and the sample median and derivations of the values given in Table 4.1 are detailed in the Appendix.

**Step 4:**

Obtain a bootstrap sample $(\epsilon_1^*, \epsilon_2^*, \ldots, \epsilon_n^*)$ from the following:

$$\epsilon_t^* = r_t^* - \hat{\mu} = \sqrt{\hat{h}_t^*} Z_t^* \tag{4.1.7}$$

$$\hat{\theta}(B)\hat{\Theta}(L)\hat{h}_t^* = \hat{\omega} + \hat{\alpha}(B)\epsilon_t^{*2}, \qquad t = 1, 2, \ldots, n, \tag{4.1.8}$$

where $Z_t^*$ are randomly drawn with replacement from the empirical distribution function of the standardized residuals, and $\hat{h}_j^* = \hat{h}_j, j = 1, \ldots, r$.

**Step 5:**

Using the bootstrap sample, estimate the parameters by maximum likelihood of the model given by:

$$\epsilon_t^* = r_t^* - \mu^* = \sqrt{h_t^*} Z_t \tag{4.1.9}$$

$$\theta^*(B)\Theta^*(L)h_t^* = \omega^* + \alpha^*(B)\epsilon_t^{*2}, \qquad t = 1, 2, \ldots, n, \tag{4.1.10}$$

where the innovations $Z_t$ have the distribution identified in Step 2, $\alpha^*(B) = \theta^*(B)\Theta^*(L) - [1 - [\phi^*(B) - \theta^*(B)]]\Phi^*(L)$, $\theta^*(B) = 1 - \sum_{i=1}^{q} \theta_i^* B^i$, $\Phi^*(L) = 1 - \sum_{i=1}^{P} \Phi_i^* L^i$, and $\Theta^*(L) = 1 - \sum_{i=1}^{Q} \Theta_i^* L^i$. The estimation procedure gives parameter estimates $\widehat{\mu^*}, \widehat{\omega^*}, \widehat{\phi_i^*}, \widehat{\theta_i^*}, \widehat{\Phi_i^*}$, and $\widehat{\Theta_i^*}$. From these, compute

$$\hat{E}(h_t^*) = \frac{\widehat{\omega^*}}{\left(1 - \sum_{i=1}^{p} \widehat{\phi_i^*}\right)\left(1 - \sum_{i=1}^{P} \widehat{\Phi_i^*}\right)}. \tag{4.1.11}$$

Now, we can compute an estimated call price which is given by:

$$\hat{C}(S_0, K, r, T) = S_0 \left\{ f(\hat{E}[h_t^*]) + \frac{1}{2} f''(\hat{E}[h_t^*]) \left[ \frac{\hat{\kappa}^{(\epsilon^*)}}{\kappa^{(Z)}} - 1 \right] [\hat{E}^2(h_t^*)] \right\}$$
$$- Ke^{-rT} \left\{ g(\hat{E}[h_t^*]) + \frac{1}{2} g''(\hat{E}[h_t^*]) \left[ \frac{\hat{\kappa}^{(\epsilon^*)}}{\kappa^{(Z)}} - 1 \right] [\hat{E}^2(h_t^*)] \right\} \quad (4.1.12)$$

where $S_0$ is the current price of the underlying stock, $K$ is the strike price of the call option, $r$ is the risk-free interest rate, $T$ is the time to expiration, $\hat{\kappa}^{(\epsilon^*)} = \dfrac{E(\epsilon_t^{*4})}{E^2(\epsilon_t^{*2})}$ is the sample kurtosis of the bootstrap values $\epsilon^*$, $\kappa^{(Z)}$ is the kurtosis of the innovations $Z_t$ with distribution identified in Step 2,

$$f(\hat{E}[h_t^*]) = \Phi \left( \frac{\log\left(\frac{S_0}{K}\right) + rT + \frac{1}{2}\hat{E}(h_t^*)}{\sqrt{\hat{E}(h_t^*)}} \right), g(\hat{E}[h_t^*]) = \Phi \left( \frac{\log\left(\frac{S_0}{K}\right) + rT - \frac{1}{2}\hat{E}(h_t^*)}{\sqrt{\hat{E}(h_t^*)}} \right),$$

$$f''(\hat{E}[h_t^*]) = \frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{\left\{2[\log(\frac{S_0}{K}) + rT] + \hat{E}(h_t^*)\right\}^2}{8\hat{E}(h_t^*)} \right\} \left[ \frac{6\left[\log\left(\frac{S_0}{K}\right) + rT\right] - \hat{E}(h_t^*)}{8\hat{E}^2(h_t^*)\sqrt{\hat{E}(h_t^*)}} \right.$$
$$\left. - \left( \frac{\hat{E}^2(h_t^*) - 4\left[\log\left(\frac{S_0}{K}\right) + rT\right]^2}{8\hat{E}^2(h_t^*)} \right) \left( \frac{\hat{E}(h_t^*) - 2\left[\log(\frac{S_0}{K}) + rT\right]}{4\hat{E}(h_t^*)\sqrt{\hat{E}(h_t^*)}} \right) \right], \text{and}$$

$$g''(\hat{E}[h_t^*]) = \frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{\left\{2[\log(\frac{S_0}{K}) + rT] - \hat{E}(h_t^*)\right\}^2}{8\hat{E}(h_t^*)} \right\} \left[ \frac{6\left[\log\left(\frac{S_0}{K}\right) + rT\right] + \hat{E}(h_t^*)}{8\hat{E}^2(h_t^*)\sqrt{\hat{E}(h_t^*)}} \right.$$
$$\left. + \left( \frac{\hat{E}^2(h_t^*) - 4\left[\log\left(\frac{S_0}{K}\right) + rT\right]^2}{8\hat{E}^2(h_t^*)} \right) \left( \frac{\hat{E}(h_t^*) + 2\left[\log(\frac{S_0}{K}) + rT\right]}{4\hat{E}(h_t^*)\sqrt{\hat{E}(h_t^*)}} \right) \right].$$

**Step 6:**

Repeat Steps 4 and 5 to obtain 100 replicates of call price estimates.

**Step 7:**

Order the call price estimates in the previous step and use the 2.5% and 97.5% quantiles of the distribution of call price estimates to form the 95% confidence interval for the call price.

## 4.2 Simulated data

In this section, data is simulated from a seasonal GARCH process. For the simulated data, we don't have observed call option prices. Therefore, instead of obtaining a bootstrap confidence interval for the call price, estimates of $E(h_t)$, the expected value of the conditional variance, are obtained, which are used to obtain a bootstrap interval for $E(h_t)$. This procedure is performed twice using the parametric bootstrap described in the previous section, by fitting a seasonal GARCH model and by fitting the popular GARCH(1,1) model. The intervals obtained are compared to the true $E(h_t)$.

### 4.2.1 Seasonal GARCH model

The seasonal GARCH$(0,1)$x$(1,0)_5$ model for the errors of the log returns $r_t = \log \dfrac{S_t}{S_{t-1}}$ is given by:

$$\epsilon_t = r_t - \mu = \sqrt{h_t} Z_t \tag{4.2.1}$$

$$h_t = \omega + \Phi \epsilon_{t-5}^2 - \theta \Phi \epsilon_{t-6}^2 + \theta h_{t-1}, \tag{4.2.2}$$

| Parameter | Estimate | Standard Error |
|---|---|---|
| $\mu$ | 0.00124 | 0.000686 |
| $\omega$ | 0.00335 | <0.0001 |
| $\Phi$ | 0.396 | <0.0001 |
| $\theta$ | -0.689 | <0.0001 |

Table 4.2: Estimates and standard errors of seasonal GARCH model parameters for simulated data.

where $\mu = \mathbb{E}(r_t)$. Five thousand data points were simulated from the above model with $Z_t \overset{\text{i.i.d.}}{\sim} N(0,1)$ and $(\mu, \omega, \Phi, \theta)' = (0.00125, 0.00330, 0.416, -0.679)'$. The number of simulated data points corresponds to the size of commonly modelled daily financial data.

The parameter estimates and standard errors obtained from fitting a seasonal GARCH model by maximum likelihood estimation are shown in Table 4.2, from which estimates of the conditional variance $\hat{h}_t$ were obtained using (4.1.3)-(4.1.5). Next, the standardized residuals $\hat{Z}_t = \epsilon_t/\sqrt{\hat{h}_t}$ were computed, from which $\hat{\rho} = 0.79803$ was computed using (4.1.6). Using Table 4.1, the distribution of the residuals of the simulated data was identified to be normal (as expected, since the data was simulated with normally distributed errors).

In the next step, random draws were made with replacement from the empirical distribution function of the standardized residuals $\hat{Z}_t$ to obtain $Z_t^*$. From here, 100 bootstrap samples[7] of size 5000 were obtained with each bootstrap sample $(\epsilon_1^*, \epsilon_2^*, \ldots, \epsilon_{5000}^*)$ obtained from (4.1.7)-(4.1.8) with $\hat{h}_j^* = \hat{h}_j, j = 1, \ldots, 6$. For each

[7] Following Ghahramani [15], 100 bootstrap sample replicates were used. Increasing the number of replicates did not change the results appreciably.

bootstrap sample replicate, the model given by (4.1.9)-(4.1.10) with $Z_t \overset{i.i.d.}{\sim} N(0,1)$ was fit.[8] From each replicate, parameter estimates were obtained by maximum likelihood estimation, of which we are interested in $\widehat{\omega^*}$ and $\widehat{\Phi^*}$, which were used to compute $\hat{E}(h_t^*) = \dfrac{\widehat{\omega^*}}{1-\widehat{\Phi^*}}$. The 100 estimates of the expected value of the conditional variance were used to form a 95% bootstrap confidence interval for $E(h_t)$ of (0.00523, 0.00588), which contains the true parameter $E(h_t) = \omega/(1-\Phi) = 0.00565$. In the next section we will see that by fitting a GARCH(1,1) model instead of a seasonal GARCH model, we would obtain a confidence interval for $E(h_t)$ that is far from the true value.

### 4.2.2 GARCH(1,1) model

The GARCH(1,1) model for the errors of the log returns $r_t = \log \dfrac{S_t}{S_{t-1}}$ is given by:

$$\epsilon_t = r_t - \mu = \sqrt{h_t} Z_t \tag{4.2.3}$$

$$h_t = \omega + \alpha\epsilon_{t-1}^2 + \beta h_{t-1}, \tag{4.2.4}$$

where $\mu = \mathbb{E}(r_t)$. When a GARCH(1,1) model was fit to the simulated seasonal GARCH data in section 4.2.1, the parameter estimates and standard errors shown in Table 4.3 were obtained by maximum likelihood estimation, from which estimates of the conditional variance $\hat{h}_t$, standardized residuals $\hat{Z}_t$, and $\hat{\rho} = 0.77588$ were obtained. The distribution of the residuals of the simulated data was identified to

[8] Out of 100 bootstrap sample replicates, 4 failed to converge when attempting to fit the model. These replicates were discarded and 4 new bootstrap sample replicates were formed and modelled.

| Parameter | Estimate | Standard Error |
|---|---|---|
| $\mu$ | 0.000670 | 0.000804 |
| $\omega$ | 0.00215 | 0.000632 |
| $\alpha$ | -0.0104 | 0.00952 |
| $\beta$ | 0.345 | 0.191 |

Table 4.3: Estimates and standard errors of GARCH(1,1) model parameters for simulated data.

be normal (as expected, since the data was simulated with normally distributed residuals).

One hundred bootstrap samples of size 5000 were obtained and a GARCH(1,1) model with normally distributed innovations was fit for each replicate. The parameter estimates obtained by maximum likelihood estimation were used to compute $\hat{E}(h_t^*) = \frac{\widehat{\omega^*}}{1 - \widehat{\alpha^*} - \widehat{\beta^*}}$, from which a 95% bootstrap confidence interval of (0.00311, 0.00338) was obtained, which does not come close to the true parameter value of 0.00565.

## 4.3 Application to financial data

In this section, bootstrap confidence intervals for the call option price are obtained for the Nifty data set. First, a seasonal GARCH model is fit to the data, and then a GARCH(1,1) model. It is shown that the seasonal GARCH model provides narrower intervals, and is thus preferred over the popular GARCH(1,1).

### 4.3.1 Seasonal GARCH model

A seasonal GARCH$(0,1)$x$(1,0)_5$ model given by (4.2.1)-(4.2.2) with $Z_t \overset{i.i.d.}{\sim} N(0,1)$ was estimated by maximum likelihood estimation for the Nifty data as an initial model and the parameter estimates and standard errors shown in Table 4.4 under the heading *Normal(0,1) errors* were obtained. From the standardized residuals $\hat{Z}_t$, $\hat{\rho} = 0.71420$ was computed and it was concluded that by fitting a seasonal GARCH model to the Nifty data, the residuals are $t$-distributed with 5 degrees of freedom.[9] A seasonal GARCH model was estimated again for the Nifty data set by maximum likelihood, now using $t_5$-distributed innovations. The parameter estimates and standard errors obtained are reported in Table 4.4 under the heading $t_5$ *errors*. These estimates were used to obtain estimates of the conditional variance $\hat{h}_t$ and the standardized residuals $\hat{Z}_t$.

Next, 100 bootstrap samples of size 2245 were obtained and the seasonal GARCH model with $t_5$-distributed residuals was estimated for each replicate by maximum likelihood.[10] The parameter estimates obtained in each replicate were used to compute $\hat{E}(h_t^*)$, and for particular values of $S_0, K, r$, and $T$, a call price estimate was computed for each replicate using (4.1.12) with $\kappa^{(Z)} = 9$. From these estimates, a 95% bootstrap confidence interval for the call price was obtained. The confidence intervals are reported in Tables 4.6 and 4.7 under the *Seasonal GARCH* heading using an annual risk-free interest rate of 3% and 5% respectively, along with the

[9] A $t$-distribution with 4 degrees of freedom has $\rho = 0.707$ and a $t$-distribution with 5 degrees of freedom has $\rho = 0.750$. In order to compute kurtosis-dependent call prices, a $t$-distribution with at least 5 degrees of freedom is needed. A Laplace distribution for the residuals was also attempted, however estimation of the seasonal GARCH model with Laplace-distributed residuals did not converge.

[10] Out of 100 bootstrap sample replicates, 72 converged when attempting to fit the model. An additional 40 replicates were added in order to end up with 100 replicates that converged.

| | Normal(0,1) errors | | $t_5$ errors | |
|---|---|---|---|---|
| Parameter | Estimate | Standard Error | Estimate | Standard Error |
| $\mu$ | 0.00126 | 0.000314 | 0.00145 | 0.000275 |
| $\omega$ | 0.000329 | 6.38x$10^{-6}$ | 0.000162 | 4.09x$10^{-6}$ |
| $\Phi$ | 0.417 | $< 10^{-10}$ | 0.217 | $< 10^{-10}$ |
| $\theta$ | -0.676 | $< 10^{-10}$ | -0.621 | $< 10^{-10}$ |

Table 4.4: Estimates and standard errors of seasonal GARCH model parameters for Nifty data.

estimated call prices that were obtained previously in section 3.3.

For many combinations of time to maturity, strike price, and risk-free interest rate, the observed call price does not come close to lying in the bootstrap interval. In Tables 4.8 and 4.9, a weighted average of the estimated call price under constant volatility and the confidence limits under multiplicative seasonal GARCH volatility is computed using the method in section 3.3. The weighted average bootstrap confidence intervals are reported under the *Weighted Average Seasonal GARCH* heading.

### 4.3.2 GARCH(1,1) model

In order to obtain bootstrap confidence intervals for the call option price for the Nifty data set using a GARCH(1,1) model, a normal-GARCH model was fit as an initial model by maximum likelihood estimation and the parameter estimates and standard errors shown in Table 4.5 under the heading *Normal(0,1) errors* were obtained. From the standardized residuals $\hat{Z}_t$, $\hat{\rho} = 0.76026$ was computed, which is close to $\rho$ for a $t$-distribution with 7 degrees of freedom. When a GARCH(1,1)

| | Normal(0,1) errors | | $t_7$ errors | |
|---|---|---|---|---|
| Parameter | Estimate | Standard Error | Estimate | Standard Error |
| $\mu$ | 0.00132 | 0.000270 | 0.00153 | 0.000259 |
| $\omega$ | 8.01x$10^{-6}$ | 1.66x$10^{-6}$ | 5.36x$10^{-6}$ | 1.36x$10^{-6}$ |
| $\alpha$ | 0.156 | 0.0188 | 0.102 | 0.0146 |
| $\beta$ | 0.824 | 0.0191 | 0.836 | 0.0214 |

Table 4.5: Estimates and standard errors of GARCH(1,1) model parameters for Nifty data.

model was re-estimated for the Nifty data set by maximum likelihood, now using $t_7$-distributed innovations, the parameter estimates and standard errors reported in Table 4.5 under the heading $t_7$ *errors* were obtained. From these parameter estimates, estimates of the conditional variance $\hat{h}_t$ were obtained and the standardized residuals $\hat{Z}_t$ were calculated.

One hundred bootstrap samples of size 2245 were obtained and a GARCH(1,1) model with $t_7$-distributed residuals was obtained for each replicate by maximum likelihood estimation. In each replicate, $\hat{E}(h_t^*)$ was computed using the parameter estimates and for particular values of $S_0, K, r,$ and $T$, a call price estimate was computed using (4.1.12) with $\kappa^{(Z)} = 5$. Finally, these estimates were used to obtain a 95% bootstrap confidence interval for the call price, as reported in Tables 4.6 and 4.7 under the *GARCH(1,1)* heading. By using the popular GARCH(1,1) model for the Nifty data set, much wider confidence intervals were obtained.

For many combinations of time to maturity, strike price, and risk-free interest rate, the observed call price does not come close to lying in the bootstrap interval. In Tables 4.8 and 4.9, a weighted average of the estimated call price under constant volatility and the confidence limits under GARCH(1,1) volatility is computed using the method in section 3.3. The weighted average bootstrap confidence intervals are reported under the *Weighted Average GARCH(1,1)* heading. The GARCH(1,1) model frequently provides wider and less accurate confidence intervals than the multiplicative seasonal GARCH model.

| T | $S_0$ | K | Observed Call Price | Seasonal GARCH Estimated Call Price | Seasonal GARCH Bootstrap Interval | GARCH(1,1) Estimated Call Price | GARCH(1,1) Bootstrap Interval |
|---|---|---|---|---|---|---|---|
| 24 | 2630.05 | 2600 | 73.05 | 45.84 | (42.33, 47.60) | 41.54 | (38.22, 55.32) |
| 24 | 2630.05 | 2610 | 64.00 | 38.90 | (35.00, 40.81) | 33.62 | (29.46, 48.73) |
| 24 | 2630.05 | 2620 | 61.75 | 32.59 | (28.40, 34.60) | 26.55 | (21.61, 42.67) |
| 24 | 2630.05 | 2630 | 59.95 | 26.94 | (22.59, 29.02) | 20.48 | (15.07, 37.17) |
| 24 | 2630.05 | 2640 | 49.00 | 21.98 | (17.62, 24.06) | 15.49 | (10.06, 32.23) |
| 24 | 2630.05 | 2650 | 45.25 | 17.70 | (13.48, 19.73) | 11.59 | (6.59, 27.85) |
| 24 | 2630.05 | 2660 | 42.00 | 14.08 | (10.12, 16.01) | 8.68 | (4.42, 24.02) |
| 24 | 2630.05 | 2670 | 36.40 | 11.07 | (7.47, 12.87) | 6.62 | (3.17, 20.72) |
| 24 | 2630.05 | 2680 | 29.20 | 8.61 | (5.44, 10.24) | 5.20 | (2.44, 17.89) |
| 52 | 2630.05 | 2500 | 146.10 | 141.13 | (140.82, 141.37) | 141.19 | (140.72, 146.35) |
| 52 | 2630.05 | 2550 | 120.00 | 93.08 | (91.88, 93.82) | 93.08 | (91.42, 100.10) |
| 52 | 2630.05 | 2600 | 91.65 | 50.26 | (47.01, 51.92) | 46.60 | (43.72, 59.51) |
| 52 | 2630.05 | 2650 | 68.65 | 20.23 | (15.90, 22.30) | 13.84 | (8.53, 30.45) |
| 52 | 2630.05 | 2700 | 47.70 | 6.00 | (3.44, 7.38) | 3.96 | (1.78, 14.72) |
| 87 | 2630.05 | 2600 | 105.95 | 56.04 | (53.13, 57.55) | 53.18 | (50.72, 64.96) |
| 87 | 2630.05 | 2650 | 85.00 | 23.73 | (19.36, 25.81) | 17.20 | (11.72, 33.98) |
| 87 | 2630.05 | 2700 | 69.90 | 7.37 | (4.46, 8.89) | 4.58 | (2.13, 16.40) |

Table 4.6: Estimated call prices and bootstrap intervals based on the Nifty data set (r = 0.03/365)

| T | $S_0$ | K | Observed Call Price | Seasonal GARCH Estimated Call Price | Seasonal GARCH Bootstrap Interval | GARCH(1,1) Estimated Call Price | GARCH(1,1) Bootstrap Interval |
|---|---|---|---|---|---|---|---|
| 24 | 2630.05 | 2600 | 73.05 | 48.34 | (44.98, 50.04) | 44.41 | (41.34, 57.69) |
| 24 | 2630.05 | 2610 | 64.00 | 41.21 | (37.44, 43.07) | 36.25 | (32.38, 50.93) |
| 24 | 2630.05 | 2620 | 61.75 | 34.69 | (30.58, 36.67) | 28.87 | (24.19, 44.70) |
| 24 | 2630.05 | 2630 | 59.95 | 28.82 | (24.51, 30.88) | 22.45 | (17.16, 39.01) |
| 24 | 2630.05 | 2640 | 49.00 | 23.63 | (19.25, 25.71) | 17.10 | (11.62, 33.88) |
| 24 | 2630.05 | 2650 | 45.25 | 19.12 | (14.83, 21.17) | 12.83 | (7.64, 29.31) |
| 24 | 2630.05 | 2660 | 42.00 | 15.27 | (11.21, 17.25) | 9.60 | (5.06, 25.30) |
| 24 | 2630.05 | 2670 | 36.40 | 12.06 | (8.32, 13.91) | 7.26 | (3.53, 21.82) |
| 24 | 2630.05 | 2680 | 29.20 | 9.42 | (6.09, 11.11) | 5.64 | (2.66, 18.83) |
| 52 | 2630.05 | 2500 | 146.10 | 148.11 | (147.87, 148.32) | 148.15 | (147.80, 153.09) |
| 52 | 2630.05 | 2550 | 120.00 | 99.87 | (98.86, 100.51) | 100.00 | (98.48, 106.60) |
| 52 | 2630.05 | 2600 | 91.65 | 55.98 | (53.07, 57.50) | 53.12 | (50.66, 64.91) |
| 52 | 2630.05 | 2650 | 68.65 | 23.70 | (19.32, 25.78) | 17.17 | (11.69, 33.95) |
| 52 | 2630.05 | 2700 | 47.70 | 7.35 | (4.45, 8.87) | 4.57 | (2.13, 16.39) |
| 87 | 2630.05 | 2600 | 105.95 | 66.14 | (63.80, 67.42) | 64.48 | (62.44, 74.49) |
| 87 | 2630.05 | 2650 | 85.00 | 30.38 | (26.11, 32.42) | 24.13 | (18.97, 40.53) |
| 87 | 2630.05 | 2700 | 69.90 | 10.21 | (6.74, 11.95) | 6.09 | (2.89, 19.74) |

Table 4.7: Estimated call prices and bootstrap intervals based on the Nifty data set (r = 0.05/365)

| T | $S_0$ | K | Observed Call Price | Weighted Average Seasonal GARCH | | Weighted Average GARCH(1,1) | |
|---|---|---|---|---|---|---|---|
| | | | | Estimated Call Price | Bootstrap Interval | Estimated Call Price | Bootstrap Interval |
| 24 | 2630.05 | 2600 | 73.05 | 70.16 | (68.03, 71.23) | 69.52 | (67.60, 77.47) |
| 24 | 2630.05 | 2610 | 64.00 | 63.84 | (61.47, 65.00) | 62.68 | (60.28, 71.40) |
| 24 | 2630.05 | 2620 | 61.75 | 57.97 | (55.43, 59.19) | 56.41 | (53.56, 65.72) |
| 24 | 2630.05 | 2630 | 59.95 | 52.58 | (49.94, 53.84) | 50.79 | (47.67, 60.43) |
| 24 | 2630.05 | 2640 | 49.00 | 47.67 | (45.03, 48.94) | 45.87 | (42.74, 55.54) |
| 24 | 2630.05 | 2650 | 45.25 | 43.25 | (40.69, 44.48) | 41.66 | (38.77, 51.04) |
| 24 | 2630.05 | 2660 | 42.00 | 39.29 | (36.89, 40.46) | 38.09 | (35.63, 46.94) |
| 24 | 2630.05 | 2670 | 36.40 | 35.78 | (33.59, 36.87) | 35.08 | (33.09, 43.22) |
| 24 | 2630.05 | 2680 | 29.20 | 32.66 | (30.74, 33.65) | 32.51 | (30.92, 39.84) |
| 52 | 2630.05 | 2500 | 146.10 | 168.50 | (168.31, 168.64) | 170.61 | (170.33, 173.59) |
| 52 | 2630.05 | 2550 | 120.00 | 127.28 | (126.55, 127.73) | 129.87 | (128.91, 133.92) |
| 52 | 2630.05 | 2600 | 91.65 | 90.42 | (88.44, 91.43) | 91.35 | (89.68, 98.80) |
| 52 | 2630.05 | 2650 | 68.65 | 62.51 | (59.88, 63.77) | 62.03 | (58.96, 71.62) |
| 52 | 2630.05 | 2700 | 47.70 | 45.36 | (43.81, 46.20) | 47.17 | (45.91, 53.38) |
| 87 | 2630.05 | 2600 | 105.95 | 110.39 | (108.62, 111.30) | 112.85 | (111.43, 119.65) |
| 87 | 2630.05 | 2650 | 85.00 | 81.22 | (78.56, 82.48) | 81.80 | (78.64, 91.49) |
| 87 | 2630.05 | 2700 | 69.90 | 62.63 | (60.86, 63.55) | 65.20 | (63.79, 72.03) |

Table 4.8: Weighted average call prices and bootstrap intervals based on the Nifty data set (r = 0.03/365)

| T | $S_0$ | K | Observed Call Price | Weighted Average Seasonal GARCH | | Weighted Average GARCH(1,1) | |
|---|---|---|---|---|---|---|---|
| | | | | Estimated Call Price | Bootstrap Interval | Estimated Call Price | Bootstrap Interval |
| 24 | 2630.05 | 2600 | 73.05 | 70.31 | (68.15, 71.40) | 69.83 | (67.95, 77.93) |
| 24 | 2630.05 | 2610 | 64.00 | 63.79 | (61.37, 64.98) | 62.74 | (60.38, 71.70) |
| 24 | 2630.05 | 2620 | 61.75 | 57.73 | (55.10, 59.00) | 56.20 | (53.34, 65.85) |
| 24 | 2630.05 | 2630 | 59.95 | 52.5 | (49.39, 53.48) | 50.31 | (47.08, 60.41) |
| 24 | 2630.05 | 2640 | 49.00 | 47.07 | (44.27, 48.41) | 45.14 | (41.80, 55.38) |
| 24 | 2630.05 | 2650 | 45.25 | 42.49 | (39.74, 43.81) | 40.70 | (37.53, 50.76) |
| 24 | 2630.05 | 2660 | 42.00 | 38.41 | (35.80, 39.67) | 36.96 | (34.19, 46.54) |
| 24 | 2630.05 | 2670 | 36.40 | 34.78 | (32.38, 35.97) | 33.84 | (31.56, 42.72) |
| 24 | 2630.05 | 2680 | 29.20 | 31.59 | (29.45, 32.67) | 31.22 | (29.40, 39.27) |
| 52 | 2630.05 | 2500 | 146.10 | 172.23 | (172.08, 172.37) | 174.36 | (174.15, 177.38) |
| 52 | 2630.05 | 2550 | 120.00 | 130.17 | (129.52, 130.58) | 132.90 | (131.97, 136.92) |
| 52 | 2630.05 | 2600 | 91.65 | 91.97 | (90.10, 92.95) | 93.37 | (91.87, 100.56) |
| 52 | 2630.05 | 2650 | 68.65 | 62.30 | (59.49, 63.63) | 61.69 | (58.34, 71.93) |
| 52 | 2630.05 | 2700 | 47.70 | 43.91 | (42.05, 44.88) | 45.41 | (43.92, 52.62) |
| 87 | 2630.05 | 2600 | 105.95 | 114.39 | (112.89, 115.22) | 117.60 | (116.35, 123.71) |
| 87 | 2630.05 | 2650 | 85.00 | 82.57 | (79.83, 83.88) | 83.32 | (80.17, 93.32) |
| 87 | 2630.05 | 2700 | 69.90 | 61.56 | (59.34, 62.68) | 63.54 | (61.59, 71.87) |

Table 4.9: Weighted average call prices and bootstrap intervals based on the Nifty data set (r = 0.05/365)

# Chapter 5

# Conclusions

In this thesis, a new multiplicative seasonal GARCH model has been proposed. The higher order moments of the multiplicative seasonal GARCH process and the variance of the squared process have been derived, and some examples have been given. The variance of the $l$-steps-ahead forecast error has been derived and a simulation study has demonstrated that fitting an appropriate multiplicative seasonal GARCH process instead of a GARCH(1,1) process reduces the variance of the forecast error. The model has been applied to estimate option prices using a Black-Scholes option pricing formula with multiplicative seasonal GARCH volatility. Using the Nifty data set, it has been demonstrated that the use of an option pricing formula that models volatility with a multiplicative seasonal GARCH process is superior as it provides more accurate call price estimates than option pricing formulae that assume constant volatility or model volatility with a GARCH(1,1) process. A bootstrap procedure has been investigated first with simulated multiplicative seasonal data and then the method has been applied to the Nifty data to obtain confidence intervals for the call price obtained under a GARCH(1,1) option pricing formula and under an option pricing formula that employs the multiplicative seasonal GARCH model. Narrower confidence intervals have been obtained with the multiplicative seasonal GARCH model, sometimes resulting in a case where the observed call price falls outside the confidence interval. However, if we treat the estimated call prices as a point estimate, and use the percent errors calculated in Chapter 3 as a standard error, then the multiplicative seasonal GARCH model provides closer estimates to the observed call price than the GARCH(1,1) model. The proposed multiplicative seasonal GARCH model of Chapter 2 can be an asset for market participants to develop new trading strategies and to manage their risk.

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# Appendix A

# Details of Chapter 4

## A.1 Proof of Step 2 in Section 4.1

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $E(X_i) = \mu$, $Var(X_i) = \sigma^2 < \infty$, density function $f$, and distribution function $F$. Let $q_{1/2}$ be the median of $F$. Based on a sample of size $n$, we obtain the joint asymptotic distribution of the sample mean $\bar{X}_n$ and the sample median $Q_{1/2,n}$.

The sample median may be represented as a sample mean as follows (Serfling [33]):

$$Q_{1/2,n} = \frac{1}{n}\sum_{i=1}^{n}\left(q_{1/2} + \frac{\frac{1}{2} - I\{X_i \leq q_{1/2}\}}{f(q_{1/2})}\right) + R_n$$

where $R_n = O\left(\left(\frac{\log n}{n}\right)^{3/4}\right)$ a.s. and thus $\sqrt{n}R_n \xrightarrow{P} 0$. We can now write

$$\begin{pmatrix}\bar{X}_n \\ Q_{1/2,n}\end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} X_i \\ q_{1/2} + \dfrac{\frac{1}{2} - I\{X_i \leq q_{1/2}\}}{f(q_{1/2})}\end{pmatrix} + \begin{pmatrix}0 \\ R_n\end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i + \begin{pmatrix}0 \\ R_n\end{pmatrix}$$

where $\mathbf{Y}_i = \begin{pmatrix} Y_{1i} \\ Y_{2i} \end{pmatrix}$ are i.i.d. random vectors with mean

$$E\begin{pmatrix} X_i \\ q_{1/2} + \frac{\frac{1}{2} - I\{X_i \le q_{1/2}\}}{f(q_{1/2})} \end{pmatrix} = \begin{pmatrix} \mu \\ q_{1/2} + \frac{\frac{1}{2} - F(q_{1/2})}{f(q_{1/2})} \end{pmatrix} = \begin{pmatrix} \mu \\ q_{1/2} + \frac{\frac{1}{2} - \frac{1}{2}}{f(q_{1/2})} \end{pmatrix} = \begin{pmatrix} \mu \\ q_{1/2} \end{pmatrix}$$

and variance

$$Var\begin{pmatrix} Y_{1i} \\ Y_{2i} \end{pmatrix} = \begin{pmatrix} Var(Y_{1i}) & Cov(Y_{1i}, Y_{2i}) \\ Cov(Y_{1i}, Y_{2i}) & Var(Y_{2i}) \end{pmatrix}$$

where $Var(Y_{1i}) = Var(X_i) = \sigma^2$,

$$Var(Y_{2i}) = Var\left(q_{1/2} + \frac{\frac{1}{2} - I\{X_i \le q_{1/2}\}}{f(q_{1/2})}\right) = \frac{F(q_{1/2})(1 - F(q_{1/2}))}{f^2(q_{1/2})} = \frac{1}{4f^2(q_{1/2})},$$

and

$$\begin{aligned}
Cov(Y_{1i}, Y_{2i}) &= \frac{1}{f(q_{1/2})} Cov(X_i, -I\{X_i \le q_{1/2}\}) \\
&= \frac{1}{f(q_{1/2})}\left[-E(X_i I\{X_i \le q_{1/2}\}) + \frac{1}{2}E(X_i)\right] \\
&= \frac{1}{f(q_{1/2})}\left[-E(X_i I\{X_i \le q_{1/2}\}) + \frac{1}{2}E(X_i I\{X_i \le q_{1/2}\})\right. \\
&\qquad \left. + \frac{1}{2}E(X_i I\{X_i > q_{1/2}\})\right] \\
&= \frac{1}{2f(q_{1/2})}[E(X_i I\{X_i > q_{1/2}\}) - E(X_i I\{X_i \le q_{1/2}\})] \\
&= \frac{1}{2f(q_{1/2})}\left[E(X_i I\{X_i > q_{1/2}\}) - \frac{1}{2}q_{1/2} + \frac{1}{2}q_{1/2} - E(X_i I\{X_i \le q_{1/2}\})\right]
\end{aligned}$$

$$= \frac{1}{2f(q_{1/2})} \left[ E(X_i I\{X_i > q_{1/2}\}) - q_{1/2} E(I\{X_i > q_{1/2}\}) \right.$$

$$\left. + q_{1/2} E(I\{X_i \le q_{1/2}\}) - E(X_i I\{X_i \le q_{1/2}\}) \right]$$

$$= \frac{1}{2f(q_{1/2})} \left[ E[(X_i - q_{1/2}) I\{X_i > q_{1/2}\}] + E[(q_{1/2} - X_i) I\{X_i \le q_{1/2}\}] \right]$$

$$= \frac{1}{2f(q_{1/2})} E|X_i - q_{1/2}|.$$

By the Central Limit Theorem and Slutsky's theorem,

$$\sqrt{n} \begin{pmatrix} \bar{X}_n \\ Q_{1/2,n} \end{pmatrix} \xrightarrow{d} N_2 \left( \begin{pmatrix} \mu \\ q_{1/2} \end{pmatrix}, \begin{pmatrix} \sigma^2 & \frac{E|X_i - q_{1/2}|}{2f(q_{1/2})} \\ \frac{E|X_i - q_{1/2}|}{2f(q_{1/2})} & \frac{1}{4f^2(q_{1/2})} \end{pmatrix} \right),$$

and the asymptotic correlation between $\bar{X}_n$ and $Q_{1/2,n}$ for symmetric distributions is given by

$$\rho = \frac{ACov(\bar{X}_n, Q_{1/2,n})}{\sqrt{AVar(\bar{X}_n)}\sqrt{AVar(Q_{1/2,n})}} = \frac{\frac{E|X_i - q_{1/2}|}{2f(q_{1/2})}}{\sigma \frac{1}{2f(q_{1/2})}} = \frac{E|X_i - q_{1/2}|}{\sigma} = \frac{E|X_i - \mu|}{\sigma},$$

where $ACov(\cdot)$ and $AVar(\cdot)$ are the asymptotic covariance and asymptotic variance.

In step 2 of subsection 4.1, the standardized residuals $\hat{Z}_t, t = 1, \ldots, n$ are i.i.d. random variables with mean $\mu$ and variance $\sigma^2$. Since the method of moments estimators (MME) of $\mu$ and $\sigma$ are the sample mean $\bar{Z}$ and the sample standard deviation $s_Z$ respectively, the MME of $\rho$ is given by

$$\hat{\rho} = \frac{\sum_{t=1}^{n} |\hat{Z}_t - \bar{Z}|/n}{s_Z}.$$

## A.2 Derivation of $E|X-\mu|/\sigma$ values in Table 4.1

For the standard normal distribution,

$$\rho = \frac{E|X-\mu|}{\sigma} = E|X| = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x \exp\left\{-\frac{x^2}{2}\right\} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \exp\left\{-\frac{x^2}{2}\right\} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{2} \exp\left\{-\frac{u}{2}\right\} du \quad (u = x^2, du = 2xdx)$$

$$= \sqrt{\frac{2}{\pi}} = 0.798.$$

For Student's t-distribution with degrees of freedom $\nu$, $\mu = 0$, so we need to find $E|X|$ and $\sigma = \sqrt{Var(X)}$:

$$E|X| = \int_{-\infty}^{\infty} |x| \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx$$

$$= \frac{2}{\sqrt{\nu\pi}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_{0}^{\infty} x \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx$$

$$= \frac{2}{\sqrt{\nu\pi}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_{1}^{\infty} \frac{\nu}{2} u^{-\frac{1}{2}(\nu+1)} du \quad \left(u = 1 + \frac{x^2}{\nu}, du = \frac{2x}{\nu} dx\right)$$

$$= \sqrt{\frac{\nu}{\pi}} \frac{(\frac{\nu-1}{2})\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{2}{\nu-1} = \sqrt{\frac{\nu}{\pi}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}.$$

The variance of the $t$-distribution is given by:

$$
\begin{aligned}
Var(X) = E(X^2) &= \int_{-\infty}^{\infty} x^2 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx \\
&= \frac{2}{\sqrt{\nu\pi}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} x^2 \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx \\
&= \frac{2}{\sqrt{\nu\pi}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} u\nu(1+u)^{-\frac{1}{2}(\nu+1)} \frac{\nu}{2\sqrt{u\nu}} du \\
&\qquad \left(u = \frac{x^2}{\nu}, du = \frac{2x}{\nu} dx\right) \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \upsilon^{3/2} \int_0^{\infty} u^{1/2}(1+u)^{-\frac{1}{2}(\nu+1)} du \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \nu \int_0^1 \left(\frac{1-y}{y}\right)^{1/2} y^{\frac{1}{2}(\nu+1)} \frac{1}{y^2} dy \\
&\qquad \left(y = \frac{1}{1+u}, u = \frac{1-y}{y}, du = -\frac{1}{y^2} dy\right) \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \nu \int_0^1 y^{\frac{\nu-2}{2}-1}(1-y)^{\frac{3}{2}-1} dy \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \nu \frac{\Gamma\left(\frac{\nu-2}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}, \nu > 2 \\
&= \frac{\nu}{2} \frac{\Gamma\left(\frac{\nu-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}\frac{\nu-2}{2}\Gamma\left(\frac{\nu-2}{2}\right)}, \nu > 2 \\
&= \frac{\nu}{\nu-2}, \nu > 2.
\end{aligned}
$$

Therefore, for Student's $t$-distribution, we have

$$\rho = \frac{E|X-\mu|}{\sigma} = \sqrt{\frac{\nu}{\pi}}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\sqrt{\frac{\nu-2}{\nu}} = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\nu}{2}-1}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}.$$

For a $t$-distribution with $\nu$ degrees of freedom, we obtain:

| $\nu$ | $\rho$ |
|---|---|
| 3 | $\frac{2}{\pi} = 0.637$ |
| 4 | $\frac{1}{\sqrt{2}} = 0.707$ |
| 5 | $\frac{4}{\sqrt{3}\pi} = 0.735$ |
| 6 | $\frac{3}{4} = 0.750$ |
| 7 | $\frac{16}{3\sqrt{5}\pi} = 0.759$ |
| 8 | $\frac{15}{8\sqrt{6}} = 0.765$ |

matching the values given in Table 4.1.

For the Laplace distribution with scale parameter $b$, $\mu = 0$, so we need to find $E|X|$ and $\sigma = \sqrt{Var(X)}$:

$$E|X| = \int_{-\infty}^{\infty} |x| \frac{1}{2b} \exp\{-|x|/b\}dx$$

$$= \int_{0}^{\infty} \frac{x}{b} \exp\{-x/b\}dx = b.$$

Since the variance of the Laplace distribution is given by $2b^2$, we have

$$\rho = \frac{E|X-\mu|}{\sigma} = \frac{1}{\sqrt{2}} = 0.707.$$