Linear and Non-linear Boundary Crossing Probabilities for Brownian Motion and Related Processes

by

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Abstract

We propose a simple and general method to obtain the boundary crossing probability for Brownian motion. This method can be easily extended to higher dimensional of Brownian motion. It also covers certain classes of stochastic processes associated with Brownian motion. The basic idea of the method is based on being able to construct a finite Markov chain such that the boundary crossing probability of Brownian motion is obtained as the limiting probability of the finite Markov chain entering a set of absorbing states induced by the boundary. Numerical results are given to illustrate our method.

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To my family

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 One-dimensional processes

Brownian motion. The Brownian motion is one of the most important continuous time stochastic processes and the boundary crossing probabilities (BCP) or first passage times of one-dimensional Brownian motion processes have tremendous applications in many fields, including nonparametric statistics (Durbin [27], Sen [76], Siegmund [77]), sequential analysis (Anderson [2], Sen [76], Siegmund [77]), mathematical finance (Roberts and Shortland [70]), biology (Ricciardi and Sacerdote [68], Ricciardi *et al.* [67]), medicine (Madec and Japhet [56]), change-point problems (Siegmund [77]), and many engineering problems.

Let $\{W(t) : t \in [0, \infty)\}$ be a stochastic process defined on the real line $\mathbb{R}$. A boundary crossing probability for $W(t)$ is defined by

$$P(a(t) \geq W(t) \text{ or } W(t) \geq b(t), \text{ for some } t \in [0, T]), \tag{1.1}$$

where T is fixed and $a(t)$ and $b(t)$ are continuous functions on $[0, T]$. There are various definitions of Brownian motion (see, e.g., Borodin and Salminen [10]). For our approach, the boundary crossing probability is cast as the limiting probability of a finite Markov chain entering a set of absorbing states, and for this reason we prefer the classical definition of Brownian motion (Karlin and Taylor [42]) given as follows.

Definition 1.1.1. *Brownian motion is a stochastic process* $\{W(t) : t \in [0, \infty)\}$ *which satisfies the following conditions:*

(i) *For* $0 \leq s < t$*, the increment* $W(t) - W(s)$ *is normally distributed with mean 0 and variance* $\sigma^2(t-s)$*;* σ^2 *is a constant.*

(ii) *For any pair of disjoint intervals* $[t_1, t_2]$ *and* $[t_3, t_4]$ *with* $t_1 < t_2 < t_3 < t_4$*, the increments* $W(t_2) - W(t_1)$ *and* $W(t_4) - W(t_3)$ *are independent.*

(iii) $W(0) = 0$ *and* $W(t)$ *is continuous a.s. at* $t = 0$*.*

Given t, Brownian motion has density function

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left\{\frac{x^2}{2t}\right\}, \tag{1.2}$$

which satisfies the differential equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x,t). \tag{1.3}$$

Four known properties of Brownian motion are given in the following.

1. (scaling) For every $c > 0$, $(1/\sqrt{c})W(ct)$ is a Brownian motion.

2. (shifting) For each $s > 0$, $W(s+t) - W(s)$ is a Brownian motion.

3. (time reversal) Consider the time interval $[0,1]$, $W(1-t)-W(1)$ is a Brownian motion, i.e.

$$\{W(1-t)-W(1), t \in [0,1]\} \stackrel{\mathscr{D}}{=} \{W(t), t \in [0,1]\},$$

where $X \stackrel{\mathscr{D}}{=} Y$ represents X and Y have the same distribution.

4. (inversion) $tW(1/t)$ is a Brownian motion starting from 0.

The study of Brownian motion has been over a century since Robert Brown who discovered Brownian motion as the movement of pollen particles in water, Thorvald Thiele who described the mathematics behind Brownian motion and Albert Einstein who gave an explanation of Brownian motion using the kinetic theory of matter. There are many historical and recent results for obtaining the boundary crossing probability for Brownian motion in the literature. We briefly introduce some important ones in the following. For constant one-sided boundary, given $c > 0$, let τ_c be the first time that Brownian motion crosses or hits c and we have

$$\begin{aligned} P(\tau_c \leq T) &= P\left(\sup_{0\leq t\leq T} W(t) \geq c\right) \\ &= P(W(T) > c) + P\left(\sup_{0\leq t\leq T} W(t) \geq c, W(T) \leq c\right). \end{aligned}$$

It follows from the *reflection principle* that

$$\begin{aligned} P\left(\sup_{0\leq t\leq T} W(t) \geq c, W(T) \leq c\right) &= P\left(\sup_{0\leq t\leq T} W(t) \geq c, W(T) \geq c\right) \\ &= P(W(T) \geq c). \end{aligned}$$

Hence, it immediately follows that the boundary crossing probability to constant boundary in the time interval $[0, T]$ is

$$P\left(\sup_{0\le t\le T} W(t) \ge c\right) = 2P\left(W(T) > c\right) = \frac{2}{\sqrt{2\pi T}} \int_c^\infty \exp\left\{\frac{-x^2}{2T}\right\} dx. \qquad (1.4)$$

For a one-sided boundary $\xi(t)$, a well-known *tangent approximation* was first introduced by Strassen [80] and independently established by Daniels [23] using the method of images. The two-sided linear boundary crossing probability for $t \in [0, T]$ was given in an infinite series form in Anderson [2]. Let $b(t) = c_1 + d_1 t$ and $a(t) = c_2 + d_2 t$ be the upper and lower boundaries, respectively. Denote $P_1(T)$ by the probability that the process touches the upper boundary before touching the lower boundary and $P_2(T)$ by the probability that the process touches the lower boundary first. Then,

$$P(T) = 1 - P_1(T) - P_2(T)$$

is the probability that the process always stays between boundaries $a(t)$ and $b(t)$.

If $c_1 > 0$, $c_2 < 0$, $c_1 + d_1T \geq c_2 + d_2T$, then

$$
\begin{aligned}
P_1(T) = 1 - \Phi\left(\frac{d_1T + c_1}{\sqrt{T}}\right)& \\
+ \sum_{r=1}^{\infty} \Bigg\{ & e^{-2[rc_1-(r-1)c_2][rd_1-(r-1)d_2]}\Phi\left(\frac{d_1T + 2(r-1)c_2 - (2r-1)c_1}{\sqrt{T}}\right) \\
& - e^{-2[r^2(c_1d_1+c_2d_2)-r(r-1)c_1d_2-r(r+1)c_2d_1]}\Phi\left(\frac{d_1T + 2rc_2 - (2r-1)c_1}{\sqrt{T}}\right) \\
& - e^{-2[(r-1)c_1-rc_2][(r-1)d_1-rd_2]}\left[1 - \Phi\left(\frac{d_1T - 2rc_2 + (2r-1)c_1}{\sqrt{T}}\right)\right] \\
& + e^{-2[r^2(c_1d_1+c_2d_2)-r(r-1)c_2d_1-r(r+1)c_1d_2]}\left[1 - \Phi\left(\frac{d_1T + (2r+1)c_1 - 2rc_2}{\sqrt{T}}\right)\right]\Bigg\},
\end{aligned}
$$

where $\Phi(\cdot)$ stands for the cumulative distribution of standard normal distribution. The probability $P_2(T)$ can be obtained simply by replacing (c_1, d_1) by $(-c_2, -d_2)$. For one-sided linear boundary $a + bt$, $a > 0$, Robbins and Siegmund [69] showed, using martingale theory,

$$
P\left(\sup_{0\leq t\leq T}(W(t) - bt) \geq a\right) = 1 - \Phi\left(\frac{a}{\sqrt{T}} + b\sqrt{T}\right) + \exp(-2ab)\Phi\left(b\sqrt{T} - \frac{a}{\sqrt{T}}\right).
$$

The first passage time density function for one-sided linear boundary is also known as inverse Gaussian distribution and the density functions is given by

$$
f(t) = \frac{a}{\sqrt{2\pi}t^{3/2}}\exp\left(-\frac{(a-bt)^2}{2t}\right).
$$

Scheike [75] extended Robbins and Siegmund's [69] result to a piecewise linear boundary. Also a number of papers obtained the boundary crossing probability as an integral equation or in integral form. Durbin [27, 28, 29] computed the boundary crossing probabilities using the numerical solution of integral equations. Let $0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a partition of interval [0,T]. Wang and Pötzelberger [84] and Pötzelberger and Wang [65] extended the results of Robbins and Siegmund [69] and Scheike [75] to derive boundary crossing probabilities for one-sided and two-sided piecewise linear functions $a(t)$ and $b(t)$:

$$P\left(a(t) < W(t) < b(t), \text{ for all } t \in [0,T]\right) = Eg(W(t_1), W(t_2), \ldots, W(t_n)),$$

with

$$g(x_1, \ldots, x_n) = \prod_{i=1}^{n} I(\alpha_i < x_i < \beta_i)\left[1 - \sum_{j=1}^{\infty} q(i,j)\right],$$

and

$$\begin{aligned}
q(i,j) = \exp&\left\{-\frac{2}{\Delta t_i}[jd_{i-1} + (\alpha_{i-1} - x_{i-1})][jd_i + (\alpha_i - x_i)]\right\} \\
&- \exp\left\{-\frac{2j}{\Delta t_i}[jd_{i-1}d_i + d_{i-1}(\alpha_i - x_i) - d_i(\alpha_{i-1} - x_{i-1})]\right\} \\
&+ \exp\left\{-\frac{2}{\Delta t_i}[jd_{i-1} - (\beta_{i-1} - x_{i-1})][jd_i - (\beta_i - x_i)]\right\} \\
&- \exp\left\{-\frac{2j}{\Delta t_i}[jd_{i-1}d_i - d_{i-1}(\beta_i - x_i) + d_i(\beta_{i-1} - x_{i-1})]\right\},
\end{aligned}$$

where $t_0 = 0$, $\Delta t_i = t_i - t_{i-1}$, $\beta_i = b(t_i)$, $\alpha_i = a(t_i)$ and $d_i = \beta_i - \alpha_i$. It follows that if it is possible to approximate a general function from above and below by a piecewise linear function then we can easily obtain the upper and lower bounds for the boundary crossing probability using the above formula. Several general non-linear boundaries have been considered and numerical computations were done using Monte Carlo simulation method. Novikov *et al.* [64] also obtained bounds by piecewise approximation for two-sided boundary crossing probabilities. More approximations and computational algorithms for boundary crossing probabilities can be found in Sacerdote and Tomassetti [72] and Di Nardo *et al.* [63].

Diffusion processes An Itô diffusion process $X(t)$ is the solution of the following stochastic differential equation:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \tag{1.5}$$

where $b(t, X(t))$ and $\sigma(t, X(t))$ are called drift and diffusion coefficients, respectively. A diffusion process is said to be time homogeneous if the drift and diffusion coefficients are given by $b(X(t))$ and $\sigma(X(t))$, respectively. The boundary crossing probability for a diffusion process is similarly defined as in Eq. (1.1). The following conditions guarantee the existence of a unique solution of Eq. (1.5): for some constants C_1 and C_2, and $x \in \mathbb{R}$, $t \in [0, T]$,

$$|b(t, x)| + |\sigma(t, x)| \leq C_1(1 + |x|),$$

and

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C_2(|x - y|).$$

There are various methods to obtain boundary crossing probabilities for diffusion processes. The mainstream is to solve the partial differential equation for the transition density function and approximate the first passage time density function based on a system of integral equations. For example, Buonocore et al. [15] determined the first passage time distributions for time homogeneous diffusion processes, satisfying a system of second-kind Volterra integral equations with continuous kernels. An numerical procedure for the solution is also provided. Lehmann [51] proved the existence of a continuous first passage time density of a strong Markov process with continuous sample paths. He also gave a new Volterra integral equation of the second kind for the density.

Another method for finding the boundary crossing probabilities for diffusion processes is to express diffusion processes as functions of Brownian motion, and then the boundary crossing probabilities for diffusion processes can be obtained via the boundary crossing probabilities for Brownian motion with transformed time interval and boundaries. It is known that any time-homogeneous diffusion process can be transformed into a Brownian motion by using random time change and change of variable (see Klebaner [43], page 208). Choi and Nam [19] established the first passage time densities of Ornstein-Uhlenbeck process to exponential boundaries and Brownian bridge to two linear shrinking boundaries by using deterministic time change to transform diffusion processes into functions of a Brownian motion. The first passage time densities of Ornstein-Uhlenbeck process,

$$dX(t) = -\mu X(t)dt + dW(t),$$

to the exponential boundaries $b(t) = ce^{-\mu t}$ is given by

$$f_1(t) = \frac{2c\mu^{3/2}}{\sqrt{\pi}(e^{2\mu t}-1)^{3/2}} \exp\left\{2\mu t - \frac{\mu c^2}{e^{2\mu t}-1}\right\},\ t > 0,$$

and for $b(t) = ce^{\mu t}$,

$$f_2(t) = \begin{cases} f_1(t)\exp\{-\mu c^2(e^{2\mu t}+1)\} & \text{if } 0 < t < \infty, \\ (1-e^{-4\mu c^2})\delta_\infty(t) & \text{if } t = \infty, \end{cases}$$

where $\delta_\infty(t)$ is the Dirac-delta function at $t = \infty$, i.e.

$$\delta_\infty(t) = \begin{cases} \infty & \text{if } t = \infty, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta_\infty(t) dt = 1.$$

Wang and Pötzelberger [85] established a class of diffusion processes which can be transformed into functions of a Brownian motion by using Itô formula and time change. They also developed the existence of such transformation by satisfying certain conditions for the drift coefficient $b(t, X(t))$ and diffusion coefficient $\sigma(t, X(t))$. Many commonly seen diffusion processes are covered in their results, for example Ornstein-Uhlenbeck processes and Growth processes. Downes and Borovkov [25] established the existence of the first passage time density and gave an error bound by replacing the original boundary with one for which the crossing probability can be evaluated.

The boundary crossing probability plays an important role in finance and biology (see, e.g., Allen [1]) as many of the problems can be approximately formulated by the boundary crossing probabilities of diffusion processes. For example, in credit analysis and risk management a firm's asset value is assumed to follow a geometric Brownian motion in first-passage-time models. Given a lower boundary, if the asset value is below the boundary then the firm is declared to be bankrupt and the bankruptcy probability is obtained by finding the boundary crossing probability of a geometric Brownian motion. Black and Cox [6] suggested that the lower boundary for a firm to declare bankruptcy is of the form $Ce^{-\gamma(T-t)}$ where C, γ and T are constants. It also relies on the boundary crossing probabilities to pricing various options, for example barrier options and exotic options (Lin [54]). An Ornstein-Uhlenbeck process is used to model the CD4-cell counts of HIV infected patients by Madec and Japhet [57], and Sæbø, Almøy and Aastveit [73] models the development of mastitis of cow as a first passage time problem of a diffusion process. The probability of outbreak of a disease is equivalent to the boundary crossing probability of the diffusion process.

Random walk to Brownian motion It is well-known that Brownian motion can be approximated by a simple random walk (see, e.g., Kac [41] and Knight [44]); i.e.

$$S_n = \sum_{i=1}^{\lfloor nt \rfloor} X_i \xrightarrow{\mathscr{D}} W(t),$$

with $P(X_i = \pm\Delta x) = 1/2$, $i = 1, \ldots, n$ and $\Delta x^2 = \Delta t = 1/n$, where $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution. The above result can be extended to general independent and identically distributed (i.i.d.) random variables with finite variance. After suitably normalizing, a sequence of partial sums of i.i.d. random variables converges to Brownian motion in distribution, which is the so-called Donsker's invariance principle (see, e.g., Billingsley [5]), i.e.

$$S_n = \frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{\lfloor nt \rfloor} X_i \xrightarrow{\mathscr{D}} W(t),$$

where X_i, $i = 1, \ldots, n$, are i.i.d. random variables with mean 0 and variance σ^2.

In this thesis, a new approach inspired by the above idea is introduced to obtain the boundary crossing probabilities for Brownian motion to nonlinear boundaries by using the strong Markov property of Brownian motion and the idea of absorbing state based on the finite Markov chain imbedding technique (see Fu and Lou [37]). It follows that the boundary crossing probability can be expressed in terms of the product of transition matrices of a discrete Markov chain; i.e.

$$\begin{aligned}
&P\left(W(t) \leq a(t) \text{ or } W(t) \geq b(t),\ \text{for some } t \in [0,T]\right) \\
&= 1 - P\left(a(t) < W(t) < b(t),\ \text{for all } t \in [0,T]\right) \\
&= 1 - \lim_{n\to\infty} \xi_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i\right) \mathbf{1}^{'}, \qquad (1.6)
\end{aligned}$$

where $\mathbf{1}^{'}$ is the transpose of a row vector $\mathbf{1} = (1, \ldots, 1)$, and $\boldsymbol{N}_i$, $i = 1, \ldots, n$, are referred to as fundamental transition probability matrices of discrete Markov chain $\{Y_n\}$ having absorbing states induced by boundaries $a(t)$ and $b(t)$.

To close this section we introduce a recent application to group sequential testing in clinical trials. The boundary crossing probability for Brownian motion was first applied by Lan and DeMets [48] to group sequential tests. In a typical group sequential test, one needs to fix the number of decision times K in advance and usually the lengths of the time intervals between successive looks are set to be equal. They introduced the idea of using *alpha-spending functions* (or boundaries) to maintain the type-I error and relax the conditions of fix number K of interim analyses and of equal length intervals between successive interim analyses. In the paper of Lan and DeMets [48], they studied three different alpha-spending functions:

(i) $\alpha_1^*(0) = 0, t \neq 1$ and $\alpha_1^*(t) = 2 - 2\Phi(z_{\frac{1}{2}\alpha}/\sqrt{t}), 0 < t \leq 1;$

(ii) $\alpha_2^*(t) = \alpha \log(1 + (e-1)t)$;

(iii) $\alpha_3^*(t) = \alpha t$.

Note that the first function $\alpha_1^*(t)$ is corresponding to an one-sided horizontal boundary. All three error spending functions gives prescribed type-I error α at the end of a trial, $t = 1$. If $\alpha_1^*(t)$ is used, a clinical trial is not likely to stop early and hence suitable for a trail that long-term treatment effect is the main concern. The function $\alpha_2^*(t)$ usually results in early stop of a trial, but it will suffer a reduction in power. The function $\alpha_3^*(t)$ would be a compromise between (i) and (ii). Recently, the estimation following group sequential tests has gained popularity in sequential estimation (see e.g. Emerson and Fleming [30], Ferebee [34], Li and DeMets [53], Whitehead [88] and Liu et al. [55]).

1.2 Two or higher-dimensional processes

Let $\{\boldsymbol{X}(t), t \geq 0\}$ be a two-dimensional correlated Brownian motion with drift $t(\mu_1, \mu_2)'$ and covariance matrix

$$t\boldsymbol{\Sigma} = t \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

where μ_1, μ_2, σ_1, σ_2 and ρ are constants. Let $B(t) \subseteq \mathbb{R}^2$ be a convex set with non-empty interior $B^o(t) \notin \emptyset$ for all $t \in [0, T]$. The boundary crossing probability for two-dimensional Brownian motion is defined by

$$P(\boldsymbol{X}(t) \in \partial B(t), \text{ for some } t \in [0, T]),$$

where $\boldsymbol{X}(0) \in B^o(0)$ and $\partial B(t)$ is the boundary of $B(t) \subseteq \mathbb{R}^2$ (can be a function of t). We assume that the boundary crossing probability is well-defined throughout this thesis (see, e.g., Lefebvre and Labib [50]).

Many applications rest on the boundary crossing probabilities for two or higher-dimensional Brownian motion and related stochastic processes in many areas. One of such examples in physics is the study of escape rate from metastable states utilizing the determination of the first passage time (Talkner [81] and Chen [17]), and as an application in the field of chemistry that the time it takes for a particle to pass through a potential barrier for determining reaction-velocities is considered in Kramers [47]. Another typical example of a simple neuron model is described in Crescenzo et al. [21] and Iyengar [40] where either the membrane potential and firing threshold are treated as a two-dimensional random process; or two-dimensional Brownian motion is used to characterize the electrical state and the firing time. Then, the time until the firing threshold is reached is equivalent to the first passage time of two-dimensional Brownian motion. Various examples such as population genetics can also be found in Mason [58] and Soong [78].

Zhou [91] discussed default correlations and multiple defaults using a first-passage-time model. Since nowadays many individual companies are linked together due to the general economic environment or firm-specific conditions, the default correlation is very important in credit analysis such as asset pricing and risk management. A simple example given in Zhou [91] explains the effect of default correlation on multiple defaults. Suppose that firm 1 has a 5% probability of default and firm 2 has 1% probability of default, and the joint default probability of both firms is $5\% \times 1\% = 0.05\%$ provided that they are independent. If their default correlation is 0.2, then in fact the joint default probability becomes 0.48% which is much higher than 0.05%. We quote from Zhou [91]: " *Due to the rapid growth in the credit derivatives market and the increasing importance of measuring and controlling default risks in portfolios of loans, derivative, and other securities, the importance of default correlation analysis has been widely recognized by the financial industry in recent years*". Hence it is evident that considering marginal default probabilities is not sufficient to assess the credit risk in such a complex economic market. In summary, Zhou [91] developed a first-passage-time model of default correlations and multiple defaults and provided an analytical formula for computing the joint default probabilities. A generalization of Zhou's approach is given in Valužis [82].

There are little known analytic results of boundary crossing probability for higher dimensional Brownian motion, except for very special cases such as level boundaries. Several techniques are used to obtain the first passage time distribution for two-dimensional Brownian motion or related processes. One of the methods is to solve the transition probability density function associated with diffusion equations subject to the condition of regarding the boundary as an absorbing barrier. Thus, the first passage time distribution is obtained by integrating the transition probability density over the domain. We list a few papers as follows. Crescenzo et al. [21] obtained the first passage time density function $g(\partial B(t), \tau | \boldsymbol{x}_0, t_0)$, starting from $\boldsymbol{x}_0 = (x_0^1, x_0^2) \notin \partial B(t_0)$ at time t_0, through the linear time-dependent boundary $\partial B(t) = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 = cx_2 + at + b\}$, with some constants a and c, given by

$$g(\partial B(t), \tau | \boldsymbol{x}_0, t_0)$$

$$= \frac{|x_0^1 - h(x_0^2, t_0)|}{\sqrt{2\pi R(\tau - t_0)^3}} \exp\left\{ -\frac{[h(x_0^2, t_0) - x_0^1 - (\mu_1 - c\mu_2 - a)(\tau - t_0)]^2}{2R(\tau - t_0)} \right\}, \quad (1.7)$$

where $h(x_0^2, t) = cx_0^2 + at + b$ and $R = \sigma_2^2 c^2 - 2\sigma_{12} c + \sigma_1^2$. Iyengar [40] and Metzler [59] derived closed forms for the first passage time distribution and joint distribution of the first passage time and location $(\tau, \boldsymbol{X}(\tau))$, etc., by transforming a correlated Brownian motion to a standard one and using the method of images and polar coordinates. Define the first passage time

$$\tau = \min(\tau_1, \tau_2),$$

where

$$\tau_i = \inf\{t \geq 0 : X_i(t) = 0\}, \quad \boldsymbol{X}(0) = \boldsymbol{x}_0.$$

They first transformed a correlated Brownian motion to an independent one via $\boldsymbol{Z}(t) = \boldsymbol{\sigma}^{-1}\boldsymbol{X}(t)$ by writing the covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}'$, where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1\sqrt{1-\rho^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{bmatrix}.$$

Let $\boldsymbol{Z}(0) = \boldsymbol{z}_0$ whose polar coordinates are given by

$$r_0 = \sqrt{\frac{a_1^2 + a_2^2 - 2\rho a_1 a_2}{1-\rho^2}},$$

$$\theta_0 = \begin{cases} \pi + \tan^{-1}\left(\frac{a_2\sqrt{1-\rho^2}}{a_1 - \rho a_2}\right) & \text{if } a_1 < \rho a_2, \\ \frac{\pi}{2} & \text{if } a_1 = \rho a_2, \\ \tan^{-1}\left(\frac{a_2\sqrt{1-\rho^2}}{a_1 - \rho a_2}\right) & \text{if } a_1 > \rho a_2, \end{cases}$$

where $a_i = x_0^i/\sigma_i$. Then, the distribution of $\tau = \min(\tau_1, \tau_2)$ is reduced to the first exit time distribution of $\boldsymbol{Z}(t)$ from the wedge

$$C_\alpha = \{(r\cos\theta, r\sin\theta) : r > 0, 0 < \theta < \alpha\} \subset \mathbb{R}^2,$$

and given by

$$P(\tau > t) = \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{n:\, odd} \frac{1}{n} \sin\frac{n\pi\theta_0}{\alpha} [I_{(\nu_n-1)/2}(r_0^2/4t) + I_{(\nu_n+1)/2}(r_0^2/4t)],$$

where $\nu_n = n\pi/\alpha$ and I_ν is the modified Bessel function of the first kind of order ν. Buckholtz and Wasan [14] derived the first passage time distribution for two-dimensional Brownian motion without drift in a series form by solving a diffusion equation through a scale changing transformation and polar coordinates transformation. Dominé and Pieper [24] used the similar method due to Buckholtz and Wasan [14] to generalize the result for the case with drift. Lefebvre [49] computed the moment generating function of the first passage time density for two-dimensional diffusion processes for a straight line and a circle boundary using methods of separation of variables and of similarity solutions. The joint distribution of hitting time and hitting place is studied by several authors, for example, see Wendel [87] and Yin [90]. More results and properties concerning the hitting time for two or higher-dimensional Brownian motion for nice boundaries, such as straight lines or circles, can be found in Spitzer [79], Bañuelos and Smits [3], Burkholder [16] and Li [52].

Noticeably, many of the papers we mentioned dealt with boundary crossing probabilities with level boundaries, hence the boundary crossing probabilities for high dimensional Brownian motion for general boundaries are still an open challenge as mentioned in Crescenzo et al. [21] that boundary crossing problem in two dimensional is far from being trivial. Since such a wide range of applications are relevant to boundary crossing probabilities for Brownian motion in different areas, a simple method of finding boundary crossing probabilities is of great interest and importance. In this thesis, a unified method is developed for the boundary crossing probabilities for one or higher-dimensional Brownian motion or related diffusion processes by using the finite Markov chain imbedding technique as shown in equation (1.6). The basic idea of our approach is based on being able to construct a finite Markov chain and the boundary crossing probability for Brownian motion is cast as the limiting probability of the finite Markov chain entering a set of absorbing states induced by the boundaries. The method is somewhat simple, but the construction of a finite Markov chain and the sequence of state spaces with absorbing states plays an important role in obtaining the boundary crossing probability. With minor modification, this method can be easily extended to the boundary crossing problems for high-dimensional Brownian motion or general Markov processes.

1.3 Error bound

Let $X_1, \ldots, X_n$ be independent random variables with common distribution F such that $E(X_1) = 0$, $E(X_1^2) = 1$ and $E(|X_1^3|) = c_3$, and $S_n = \sum_{i=1}^n X_i$. The rate of convergence of the distribution of sum of i.i.d. random variables to a normal distribution has been extensively studied in the literature. It is well-known that the Berry-Esseen theorem (see [33], page 542) gives an upper bound for the error as follows:

$$|F_n(x) - \Phi(x)| \leq \frac{3c_3}{\sqrt{n}},$$

where $F_n(x)$ stands for the distribution of the normalized sum

$$(X_1 + \cdots + X_n)/\sqrt{n},$$

and $\Phi(x)$ stands for the distribution of standard normal. This remarkable result is that it depends only on the third moment c_3.

It is often called the invariance principle for partial sums of i.i.d. random variables converging to a Brownian motion, since it does not depend on the underlying distribution F. The rate of convergence for the invariance principle is a general term which represents the convergence rate for the functionals of Brownian motion. In this thesis, the study of convergence rate for the invariance principle means the study of the error given by

$$\text{Error } = \left| P\left(\sup_{1 \leq k \leq n} S_n < \sqrt{n}x) - P(\sup_{0 \leq t \leq 1} W(t) < x \right) \right|.$$

From the Berry-Esseen theorem, we would expect that the upper bound for the above error is also of order $O(1/\sqrt{n})$, which is established in Section 3.1.2.

There are three major methods used for establishing the rate of convergence: (1) Prokhorov distance ; (2) Skorokhod embedding; and (3) inversion formula (characteristic function). The first method was introduced by Prokhorov [66] and he obtained the estimate of the rate $O(\log^2 n/n^{1/8})$. Borovkov [12] improved the convergence rate and showed that the estimate for Prokhorov distance can not be better than $O(n^{-1/4} \log^\beta n)$ by using the method of common probability space, where β depends on the distribution of X_1. For more detail about the Donsker-Prokhorov's invariance principle, readers are referred to the papers by Borovkov [11] and Dudley [26] which contain surveys about recent results. Skorokhod's embedding is another useful approach to study the rate of convergence for invariance principle. The idea is to embed i.i.d. random variables into a Brownian motion with certain stoping times, i.e. there exists a sequence of nonnegative, mutually independent random variables $\tau_1, \tau_2, \ldots, \tau_n$ such that the joint distribution of $W(\tau_1), W(\tau_1+\tau_2)-W(\tau_1), \ldots, W(\tau_1+\cdots+\tau_{n-1})-W(\tau_1+\cdots+\tau_n)$ are identical to the joint distribution of i.i.d. random variables $X_1, \ldots, X_n$. Thus, the distribution of S_i is the same as that of $W(\tau_1+\cdots+\tau_i)$. One early result utilizing Skorokhod's embedding technique is given by Rosenkrantz [71] who obtained the convergence rate as $O((\log n)^{1/2}n^{-\mu})$, where $\mu = \frac{1}{2}a/(a+3)$ and $0 < a \leq 2$. Heyde [39] improved Rosenkrantz's result by releasing the condition $a \leq 2$ and obtained the convergence rate as $O\left((\log n)^{\lambda} n^{-\mu}\right)$ where $\lambda = (1+a/2)/(a+3)(< 1/2)$, $\mu = \min(a, 1+a/2)/(2(a+3))$. There are many other papers improving the convergence rates using the Skorokhod's embedding technique, see for example, Fraser [36] and Sawyer [74]. The anticipated Berry-Esseen bound is achieved by Nagaev [61, 62] using the method of characteristic function and inversion formula and given by

$$\begin{aligned} &|P(a(k/n) < S_k < b(k/n), k = 1, \ldots, n) - \\ &\qquad P(a(t) < W(t) < b(t), t \in [0,1])| \leq L \frac{c_3^2(K+1)}{\sqrt{n}}, \end{aligned} \tag{1.8}$$

where L is an absolute value and $a(t)$ and $b(t)$ are boundaries satisfying the Lipschitz condition

$$|a(t+h) - a(t)| < Kh,$$

$$|b(t+h) - b(t)| < Kh, \quad h > 0,$$

for some positive constant K. Up to now, Eq. (1.8) is the best known result of the rate of convergence for invariance principle, although the constant part can still be improved.

In some sense, the third method using characteristic function is more straightforward to deal with the error problem, hence it may produce better results, since it simply derives the difference of two probabilities. We will establish our error bound with rate $O(1/\sqrt{n})$ by combining the results of Nagaev [61, 62] and Borovkov and Novikov [13] in Section 3.1.2.

To the best of our knowledge, the known error bound for two or high-dimensional Brownian motion by discrete approximation is of order $O(n^{-\frac{1}{8}})$ to date (see, e.g., Fraser [35]). Hence, it remains an open problem to find a better rate of convergence for two or high-dimensional Brownian motion.

1.4 Summary

In Chapter 2, we provide some preliminary results for the FMCI technique and absorption probability of a discrete Markov chain with finite state space. The limiting absorption probability will later be served as the boundary crossing probability.

In Chapter 3, we give the detail for our main results for calculating the boundary crossing probabilities for one-dimensional Brownian motion. Based on the FMCI technique, a finite Markov chain with absorbing states associated with boundaries is constructed and we prove the absorption probability of the finite Markov chain converges to the boundary crossing probability for Brownian motion. In addition, we introduce a class of irregular boundaries called Y-channel boundaries which have not been studied yet in the literature, and show how to calculate boundary crossing probabilities for these. We also derive the error bounds of our approximations. To improve the speed of computation, an efficient algorithm for computing the boundary crossing probabilities for time homogeneous boundaries is given by using eigenvalues and eigenvectors of transition probability matrices.

In Chapter 4, we extend our method to certain classes of diffusion processes. First we introduce a class of diffusion processes which can be transformed to functions of a Brownian motion and, of course, the boundary crossing probabilities for this class of diffusion processes can be obtained through the boundary crossing probabilities for Brownian motion with changed time interval and boundary. Next we further extend the method to jump diffusion processes.

In Chapter 5, the boundary crossing probabilities for two or higher-dimensional Brownian motion are considered. Our results can be directly extended to a standard two-dimensional Brownian motion. For a two-dimensional correlated Brownian motion, it can be transformed into an independent one. Therefore, the boundary crossing probabilities for two-dimensional Brownian motion can be obtained by our method with no additional difficulty.

In Chapter 6, examples and numerical results are given for illustration. We give examples to show the detail of how to implement the FMCI procedure. From the numerical results, it shows our method performs well. We also give an application to pricing the corporate debt showing the potential applications in areas like mathematical finance.

Summary and discussion are given in Chapter 7. We expect that the method can be eventually extended to general higher-dimensional Markov processes.

Chapter 2

Preliminary

2.1 Finite Markov Chain Imbedding

Definition 2.1.1 (Fu and Lou (2003))**.** *An integer-valued random variable X_n is finite Markov chain imbeddable if there exists a finite Markov chain $\{Y_n\}$ defined on a finite state space $\Omega = \{a_1, \ldots, a_m\}$ with initial probability vector $\boldsymbol{\xi}$ such that for every x, we have*

$$P(X_n = x) = P(Y_n \in C_x | \boldsymbol{\xi}), \tag{2.1}$$

where C_x is a subset of Ω corresponding to x, and $\{C_x\}$ forms a partition on the state space Ω.

Suppose $\boldsymbol{M}$ is the transition matrix of the imbedded Markov chain $\{Y_n\}$ defined on the state space $\Omega = \{a_1, \ldots, a_m\}$ with initial distribution $\boldsymbol{\xi} = (P(Y_0 = a_1), \ldots, P(Y_0 = a_m))$. Then, the distribution of a finite Markov chain imbeddable random variable X_n can be obtained via the following theorem.

Theorem 2.1.1 (Fu and Lou (2003))**.** *If X_n is finite Markov chain imbeddable, then*

$$P(X_n = x) = P(Y_n \in C_x | \boldsymbol{\xi}) = \boldsymbol{\xi} \boldsymbol{M}^n \boldsymbol{U}^{'}(C_x), \tag{2.2}$$

where $\boldsymbol{U}(C_x) = \sum_{r : a_r \in C_x} \boldsymbol{e}_r$, $\boldsymbol{e}_r$ is a $1 \times m$ unit row vector corresponding the state a_r. If the Markov chain is non-homogeneous, then Eq. (2.2) becomes the product of matrices $\boldsymbol{M}_t$, $t = 1, \ldots, n$.

Proof. The proof is followed directly by the Chapman-Kolmogorov equation. □

As shown in Fu and Lou [37], many random variables or statistics related to runs and patterns are finite Markov chain imbeddable and the exact distributions can be easily obtained by the FMCI technique, while the results would be tedious or even intractable by using combinatorial methods. A particularly useful characteristic of a Markov chain is the absorption probability which can be considered as a special case of the above theorem. We give more details in the following section.

2.2 Absorption probability of a finite Markov chain

To facilitate our approach, we need a simple result for computing the absorption probability of a finite non-homogeneous Markov chain. Given $n \in J^+ = \{1, 2, \ldots\}$, let us define a sequence of state spaces

$$\Omega_i = \{c_1, c_2, \ldots, c_{m_i}\} \cup \{\alpha_i\}, \text{ for } i = 0, 1, \ldots, n,$$

where α_i stands for an absorbing state (we allow the absorbing state α to be a function of i) and $(m_i + 1)$ is the size of the state space Ω_i (the m_i do not have to be the same). Denote $\boldsymbol{\xi}$ as the initial distribution, and that a finite Markov chain $\{Y_i\}_{i=0}^n$ is defined on the sequence of state spaces $\{\Omega_i\}_{i=0}^n$ with transition probabilities: for $i = 1, 2, \ldots, n$,

$$P(Y_i = k | Y_{i-1} = j) = \begin{cases} p_i(k|j) & \text{if } j \in \Omega_{i-1} \backslash \alpha_{i-1},\ k \in \Omega_i \backslash \alpha_i, \\ p_i(\alpha_i|j) & \text{if } j \in \Omega_{i-1} \backslash \alpha_{i-1}, k = \alpha_i \\ 0 & \text{if } j = \alpha_{i-1},\ k \in \Omega_i \backslash \alpha_i, \\ 1 & \text{if } j = \alpha_{i-1},\ k = \alpha_i. \end{cases} \tag{2.3}$$

The transition probability matrices associated with the finite Markov chain $\{Y_i\}_{i=0}^n$ will have the form:

$$\boldsymbol{M}_i = \left[\begin{array}{c|c} \boldsymbol{N}_i & \boldsymbol{C}_i \\ \hline \mathbf{0} & 1 \end{array} \right], \tag{2.4}$$

where $\boldsymbol{N}_i = (p_i(k|j))$, $j \in \Omega_{i-1} \backslash \alpha_{i-1}$ and $k \in \Omega_i \backslash \alpha_i$, is a $m_{i-1} \times m_i$ rectangular matrix, often referred as the fundamental matrix, $\boldsymbol{C}_i = (p_i(\alpha_i|j))$ is a $m_{i-1} \times 1$ column vector and $\mathbf{0} = (0, \ldots, 0)$ $1 \times m_i$ row vector. It follows from the Chapman-Kolmogorov equation and the structure of the matrices $\boldsymbol{M}_i$ that the probability that the Markov chain $\{Y_i\}_{i=0}^n$ never touches the absorbing states, $P(Y_1 \neq \alpha_1, \ldots, Y_n \neq \alpha_n | \boldsymbol{\xi})$ can be obtained via the following lemma.

Lemma 2.2.1. *Given the state spaces $\{\Omega_i\}_{i=0}^n$ and the finite Markov chain $\{Y_i\}_{i=0}^n$ defined on the state spaces $\{\Omega_i\}_{i=0}^n$ with corresponding transition probability matrices defined by the Eq. (2.4). Then*

$$P(Y_1 \neq \alpha_1, \ldots, Y_n \neq \alpha_n | \boldsymbol{\xi}) = \boldsymbol{\xi}_0 \left(\prod_{i=1}^n \boldsymbol{N}_i \right) \mathbf{1}',$$

where $\boldsymbol{\xi} = (\boldsymbol{\xi}_0, 0)$ and $\boldsymbol{N}_i$ are given by Eqs. (2.3) and (2.4).

Proof. It follows from Theorem 2.1.1 that

$$P(Y_1 \neq \alpha_1, \ldots, Y_n \neq \alpha_n | \boldsymbol{\xi}) = 1 - (\boldsymbol{\xi}_0, 0) \left(\prod_{i=1}^{n} \boldsymbol{M}_i \right) (0, \ldots, 0, 1)^{'}.$$

From Eq. (2.4), we have

$$1 - (\boldsymbol{\xi}_0, 0) \left(\prod_{i=1}^{n} \boldsymbol{M}_i \right) (0, \ldots, 0, 1)^{'} = \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i \right) \mathbf{1}^{'}.$$

Hence, the proof is completed. □

Define the first passage time random variable for the absorbing state α_i as

$$\tau = \inf\{i : Y_i = \alpha_i\}.$$

It follows immediately from the definition of τ that

$$P(\tau > n) = \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i \right) \mathbf{1}'. \tag{2.5}$$

Note that if the finite Markov chain $\{Y_t\}$ is homogeneous, then the above probability becomes

$$P(\tau > n) = \boldsymbol{\xi}_0 \boldsymbol{N}^n \mathbf{1}'.$$

The above result allows us to calculate the absorption probability of a finite Markov chain. In the following, a large deviation approximation is given based on the eigenvalues and eigenvectors decomposition of transition probability matrix of a finite Markov chain. If $\mathbf{1}^{'}$ can be written as a linear combination of the eigenvectors $\boldsymbol{\eta}_i^{'}$, i.e.

$$\mathbf{1}^{'} = \sum_{i=1}^{\omega} a_i \boldsymbol{\eta}_i^{'}, \tag{2.6}$$

where ω is the size of the matrix $\boldsymbol{N}$, then we have the following lemmas.

Lemma 2.2.2. *Let $1 > \lambda_{[1]} \geq |\lambda_{[2]}| \geq \cdots \geq |\lambda_{[\omega]}|$ be the ordered eigenvalues of $\boldsymbol{N}$ of size ω. From Fu and Lou [37], we have*

$$\boldsymbol{\xi}_0 \boldsymbol{N}^n \mathbf{1}' = \sum_{i=1}^{\omega} a_i \boldsymbol{\xi}_0 \boldsymbol{\eta}'_{[i]} \lambda^n_{[i]}, \tag{2.7}$$

where $\boldsymbol{\eta}_{[i]}$'s are the eigenvectors associated with $\lambda_{[i]}$.

Proof. The result of Lemma 2.2.2 is the direct consequence of eigenvalues and eigenvectors decomposition of $\boldsymbol{N}$ (see Fu and Lou [37] for details). □

By Perron-Frobenius theorem for nonnegative matrix, the eigenvalues of $\boldsymbol{N}$ ordered such that $1 > \lambda_{[1]} \geq |\lambda_{[2]}| \geq \cdots \geq |\lambda_{[\omega]}|$ is possible. $\mathbf{1}'$ may not always be able to express as linear combination of the eigenvectors of $\boldsymbol{N}$. Fu and Johnson [38] discussed the eigenvalues and eigenvectors approximation in detail and provided several treatments for the situations where Eq. (2.6) is not possible and λ_1 has algebraic multiplicity greater than 1.

While the above construction is somewhat simple, the construction of the sequence of state spaces and the Markov chain $\{Y_i\}_{i=0}^n$ and absorbing states will play an indispensable role for our method computing the boundary crossing probabilities.

2.3 Simple random walk

We give some boundary crossing results for simple random walk and take this opportunity to show some detail of the FMCI procedure. Let $X_1, \ldots, X_n$ be i.i.d.

random variables taking values on ± 1 and $P(X_i = \pm 1) = 1/2,\ i = 1, \ldots, n$. A simple random walk is the partial sum

$$W_n = X_1 + \cdots + X_n.$$

Since $W_n = W_{n-1} + X_n$, clearly $\{W_n\}$ is a Markov chain taking values on the state space $\Omega = \{\ldots, -1, 0, 1, \ldots\}$ and having transition probabilities

$$P(W_n = k | W_{n-1} = j) = \frac{1}{2}, \text{ if } k - j = \pm 1.$$

Consider the first passage time $\tau_c = \min\{i : W_i = \pm c\}$, and we can define an imbedded Markov chain $\{Y_n\}$ on the state space $\Omega = \{-c+1, \ldots, -1, 0, 1, \ldots, c-1, \alpha\}$, where α is an absorbing state. The transition probability matrix of the imbedded Markov chain $\{Y_n\}$ is given by

$$\boldsymbol{M}_c = \begin{array}{c} -c+1 \\ -c+2 \\ \\ \vdots \\ \\ c-1 \\ \alpha \end{array} \left[\begin{array}{cccccc|c} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ & & \ddots & & \ddots & & \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{N}_c & \boldsymbol{C}_c \\ \hline \boldsymbol{0} & 1 \end{array}\right]. \tag{2.8}$$

It follows from Eq. (2.5) that

$$P(\tau_c > n) = P(Y_n \neq \alpha) = \boldsymbol{\xi}_0 \boldsymbol{N}_c^n \boldsymbol{1}',$$

where $\boldsymbol{\xi}_0 = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the initial distribution.

We can see that the transition matrix $\boldsymbol{N}_c$ is actually a symmetric Toeplitz matrix which has great advantage on computation. Let λ_i and $\boldsymbol{\eta}_i$, $i = 1, \ldots, 2c-$

1, be the eigenvalues and corresponding eigenvectors of the Toeplitz matrix $\boldsymbol{N}_c$, respectively. Let $m = 2c - 1$, it can be shown that $\lambda_i = \cos(i\pi/(m+1))$ and $\boldsymbol{\eta}_i^{'} = (\sin(1i\pi/(m+1), \sin(2i\pi/(m+1)), \ldots, \sin(mi\pi/(m+1)))/\sqrt{m/2}$ (see, e.g., Meyer [60]) after normalization.

It follows from Lemma 2.2.2 that

$$P(\tau_c > n) = \sum_{i=1}^{m} a_i \boldsymbol{\xi}_0 \boldsymbol{\eta}_{[i]}^{'} \lambda_{[i]}^{n},$$

where $a_i = \boldsymbol{\eta}_{[i]} \mathbf{1}^{'} = \sum_{j=1}^{m} \sin(ji\pi/(m+1))/\sqrt{m/2}$ and $\boldsymbol{\xi}_0 \boldsymbol{\eta}_{[i]}^{'} = \sin(ci\pi/(m+1))/\sqrt{m/2}$, and

$$\begin{aligned}
P(\tau_c > n) &= \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \sin\left(\frac{ji\pi}{m+1}\right) \right) \sin\left(\frac{ci\pi}{m+1}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \sin\left(\frac{ji\pi}{m+1}\right) \right) \sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m} \\
&= \sum_{i=1}^{m} \left(\frac{\sin\left(\frac{i\pi}{2}\right) \sin\left(\frac{mi\pi}{2(m+1)}\right)}{\sin\left(\frac{i\pi}{2(m+1)}\right)} \right) \sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m} \\
&= \sum_{i=1}^{m} \left(\frac{\cos\left(\frac{i\pi}{2(m+1)}\right) - \cos\left(\frac{(2m+1)i\pi}{2(m+1)}\right)}{2\sin\left(\frac{i\pi}{2(m+1)}\right)} \right) \sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m}.
\end{aligned}$$

Also, we have

$$\cos\left(\frac{(2m+1)i\pi}{2(m+1)}\right) = \cos\left(\frac{-i\pi}{2(m+1)} + i\pi\right) = (-1)^i \cos\left(\frac{i\pi}{2(m+1)}\right).$$

Hence,

$$
\begin{aligned}
P(\tau_c > n) &= \sum_{i=1}^{m} \left(\frac{\cos\left(\frac{i\pi}{2(m+1)}\right) - (-1)^i \cos\left(\frac{i\pi}{2(m+1)}\right)}{2\sin\left(\frac{i\pi}{2(m+1)}\right)} \right) \sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m} \\
&= \sum_{i=1,odd}^{m} \left(\frac{2\cos\left(\frac{i\pi}{2(m+1)}\right)}{2\sin\left(\frac{i\pi}{2(m+1)}\right)} \right) \sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m} \\
&= \sum_{i=1,odd}^{m} \cot\left(\frac{i\pi}{2(m+1)}\right) \sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right) \frac{2}{m} \\
&= \sum_{i=1,odd}^{m} \frac{\sin\left(\frac{i\pi}{2}\right) \cos^n\left(\frac{i\pi}{m+1}\right)}{\tan\left(\frac{i\pi}{2(m+1)}\right)} \frac{2}{m} \\
&= \sum_{i=1,odd}^{m} \frac{(-1)^{\frac{i-1}{2}} \cos^n\left(\frac{i\pi}{m+1}\right)}{\frac{m}{2}\tan\left(\frac{i\pi}{2(m+1)}\right)} \\
&= \sum_{j=1}^{c} \frac{(-1)^{j-1} \cos^n\left(\frac{(2j-1)\pi}{m+1}\right)}{\frac{m}{2}\tan\left(\frac{(2j-1)\pi}{2(m+1)}\right)}. \qquad (2.9)
\end{aligned}
$$

It is another advantage of the FMCI technique that not only we can calculate the exact probability via transition probability matrices, but also we can derive other identities, for example the explicit form for $P(\tau_c > n)$ given in Eq. (2.9). In a similar fashion, a recursive formula can be derived for the probability $P(\tau_c > n)$ as follows.

Let $\alpha_n(x) = P(Y_n = x)$, $x = -c+1, \ldots, c-1$, and $\alpha_n(c) = P(Y_n = \alpha)$. In view of the transition probability matrix in Eq. (2.8), from backward matrix multiplication it can be shown that the following recursive equations hold: $n \geq 1$,

$$\alpha_0(0) = 1,$$

$$\alpha_n(x) = \frac{1}{2}\alpha_{n-1}(x-1) + \frac{1}{2}\alpha_{n-1}(x+1),\ x = -c+2, \ldots, c-2,$$

$$\alpha_n(c-1) = \frac{1}{2}\alpha_{n-1}(c-2),$$

$$\alpha_n(-c+1) = \frac{1}{2}\alpha_{n-1}(-c+2), \text{ and}$$

$$\alpha_n(c) = \frac{1}{2}\alpha_{n-1}(c-1) + \frac{1}{2}\alpha_{n-1}(-c+1) + \alpha_{n-1}(c).$$

Thus,

$$\begin{aligned} P(\tau_c > n) &= 1 - \alpha_n(c) \\ &= \sum_{i=-c+1}^{c-1} \alpha_n(i). \end{aligned}$$

Note that the above recursive equations still hold for asysmetric simple random walk with $P(X_i = 1) = p$ and $P(X_i = -1) = q$ by simply replacing $1/2$ by p or q.

Chapter 3

Boundary Crossing Probability for One-dimensional Brownian Motion

3.1 Boundary crossing probability

3.1.1 Main results

For given Δt, the Brownian motion $W(t)$ has transition probability density function

$$f(x, y|\Delta t) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left\{-\frac{1}{2\Delta t}(y-x)^2\right\},$$

for all $x, y \in \mathbb{R}$. Throughout this thesis, let $a(t)$ and $b(t)$ denote the lower and upper boundaries, respectively, and satisfy the following conditions:

(A) $a(t) < b(t)$ are continuous for $t \in [0, T]$, and

(B) $a(0) < 0 < b(0)$.

Let $h = \max\left(\sup_{0\le t\le T}|a(t)|, \sup_{0\le t\le T}|b(t)|\right)$. Since $a(t)$ and $b(t)$ are continuous functions defined on the compact set $[0,T]$, we have $0 < h < \infty$. Given a large positive integer m, we define $\Delta x = h/m$ and discretize the real line $\mathbb{R}$ as $\mathbb{R}_m = \{k\Delta x : k = 0, \pm 1, \ldots\}$. We also evenly discretize the time interval $[0,T]$ into $n = m^2T/h^2$ sub-intervals. Without loss of generality, we may assume $T = 1$ and $n = m^2/h^2$ is always an integer for, if not, we may take $n = \lfloor m^2/h^2 \rfloor$, the integer part of m^2/h^2. For given $t \in [0,1]$, we then construct a partial sum

$$\hat{W}_n(t) = \sum_{\jmath=1}^{\lfloor nt \rfloor} \hat{X}_\jmath, \tag{3.1}$$

where $P(\hat{W}_n(0) = 0) = 1$ and $\hat{X}_\jmath{}'s$ are discrete i.i.d. random variables induced by discretizing the $\mathbb{R}$ and the time interval $[0,1]$, having distribution defined by, for $\jmath = 1, \ldots, \lfloor nt \rfloor$,

$$P(\hat{X}_\jmath = k\Delta x) = \begin{cases} \frac{C^{-1}}{\sqrt{2\pi}} \exp\left(-\frac{k^2}{2}\right) & \text{if } k \neq 0, \\ \frac{C^{-1}}{\sqrt{2\pi}} \sum_{\ell\neq 0}(\ell^2 - 1)\exp\left(-\frac{\ell^2}{2}\right) & \text{if } k = 0, \end{cases} \tag{3.2}$$

where $\sum_{\ell\neq 0}$ stands for $\sum_{\ell=-\infty}^{-1} + \sum_{\ell=1}^{\infty}$ and $C = \frac{1}{\sqrt{2\pi}}\sum_{\ell\neq 0}\ell^2\exp(-\ell^2/2)$ is the normalizing constant. Obviously, $\{\hat{W}_n(t)\}$ is a homogeneous Markov chain having transition probabilities

$$\begin{aligned} p(k|j) &= P(\hat{W}_n(t+\Delta t) = k\Delta x | \hat{W}_n(t) = j\Delta x) \\ &= \begin{cases} \frac{C^{-1}}{\sqrt{2\pi}} \exp\left(-\frac{(k-j)^2}{2}\right) & \text{if } k - j \neq 0, \\ \frac{C^{-1}}{\sqrt{2\pi}} \sum_{\ell\neq 0}(\ell^2 - 1)\exp\left(-\frac{\ell^2}{2}\right) & \text{if } k - j = 0. \end{cases} \end{aligned} \tag{3.3}$$

Note that the discrete probability functions defined by Eq. (3.2) have two important characters: (i) they preserve the basic relationship that the variance of $\hat{X}_j$ equals to $\Delta t = \Delta x^2$, and (ii) the transition probabilities depend only on the difference $k-j$. Next we establish that $\hat{W}_n(t)$ converges to a Brownian motion $W(t)$ in distribution.

Theorem 3.1.1. *Given $t \in [0,1]$ and $\Delta t = \Delta x^2$ $(n = m^2/h^2)$, we have*

$$\hat{W}_n(t) \xrightarrow{\mathscr{D}} W(t), \qquad \textit{as } m \to \infty.$$

Proof. We denote $\varphi_X(s)$ as the characteristic function of a random variable X. According to the continuity theorem,

$$\begin{aligned}
&E[e^{is\hat{X}_1}] \\
&= \sum_{k\neq 0} \frac{C^{-1}}{\sqrt{2\pi}}(k^2-1)e^{-\frac{k^2}{2}} + \sum_{k\neq 0} e^{isk\Delta x} \cdot \frac{C^{-1}}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \\
&= \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k\neq 0}(k^2-1)e^{-\frac{k^2}{2}} + \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k=1}^{\infty} e^{-\frac{k^2}{2}} \left(e^{isk\Delta x} + e^{-isk\Delta x}\right) \\
&= \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k\neq 0}(k^2-1)e^{-\frac{k^2}{2}} + \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k=1}^{\infty} e^{-\frac{k^2}{2}} \cdot 2\cos(sk\Delta x) \\
&= \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k\neq 0}(k^2-1)e^{-\frac{k^2}{2}} + \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k\neq 0} e^{-\frac{k^2}{2}} \left(1 - \frac{(sk\Delta x)^2}{2} + \mathcal{O}\left(\Delta x^4\right)\right) \\
&= 1 - \frac{s^2\Delta x^2}{2} \frac{C^{-1}}{\sqrt{2\pi}} \sum_{k\neq 0} k^2 e^{-\frac{k^2}{2}} + \mathcal{O}\left(\Delta x^4\right) \\
&= 1 - \frac{s^2h^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^4}\right).
\end{aligned}$$

Hence, we have

$$\varphi_{\hat{W}_n(t)}(s) = \left(1 - \frac{s^2h^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^4}\right)\right)^{m^2t/h^2} \to \exp\left(-\frac{ts^2}{2}\right), \text{ as } m \to \infty.$$

Let $S(t)$ denote the random variable with distribution function to which $\hat{W}_n(t)$ converges in distribution. For $\epsilon > 0$, let the event $E_n = \{\omega : |S(1/n)| > \epsilon\}$, and we have

$$P(E_n) = P(|S(1/n)| > \epsilon) = 2P(S(1/n) > \epsilon),$$

where $S(1/n)$ follows $N(0, 1/n)$. It follows from Feller [32] (Page 175),

$$P(E_n) < \frac{2}{\epsilon\sqrt{n2\pi}} e^{-\frac{\epsilon^2 n}{2}}.$$

By the integral test for convergence,

$$\sum_{n=1}^{\infty} P(E_n) < \infty \quad \text{if} \quad \int_1^{\infty} \frac{2}{\epsilon\sqrt{2\pi x}} e^{-\frac{\epsilon^2 x}{2}} dx < \infty,$$

and

$$\begin{aligned}\int_1^{\infty} \frac{2}{\epsilon\sqrt{2\pi x}} e^{-\frac{\epsilon^2 x}{2}} dx &\le \int_1^{\infty} \frac{2}{\epsilon\sqrt{2\pi}} e^{-\frac{\epsilon^2 x}{2}} dx \\ &= \frac{2}{\epsilon\sqrt{2\pi}} \left(-\frac{2}{\epsilon^2} e^{-\frac{\epsilon^2 x}{2}} \Big|_1^{\infty} \right) \\ &= \frac{2}{\epsilon\sqrt{2\pi}} \left(\frac{2}{\epsilon^2} e^{-\frac{\epsilon^2}{2}} \right) < \infty.\end{aligned}$$

Hence, we have

$$\sum_{n=1}^{\infty} P(E_n) < \infty.$$

By Borel-Cantelli Lemma, $S(t)$ is continuous at $t = 0$ almost surely. Obviously, $S(t)$ satisfies the independent increment property, therefore, the process $\{S(t), t \in [0, 1]\}$ is a Brownian motion owing to its definition. $\square$

Remark 3.1.1. *It is worth mentioning that the construction of the probability function in Eq. (3.2) which preserve the variance is not unique, for example, we may define the discrete distribution by*

$$P(\hat{X}_i(t) = k\Delta x) = \begin{cases} \frac{C^{-1}}{k^p\sqrt{2\pi}} \exp\left(-\frac{k^2}{2}\right) & \text{if } k \neq 0, \\ \frac{C^{-1}}{\sqrt{2\pi}} \sum_{\ell\neq 0}(\frac{1}{\ell^{p-2}} - \frac{1}{\ell^p}) \exp\left(-\frac{\ell^2}{2}\right) & \text{if } k = 0. \end{cases} \tag{3.4}$$

where $C = (1/\sqrt{2\pi}) \sum_{\ell\neq 0} \frac{1}{\ell^{p-2}} \exp(-\ell^2/2)$ *and* p *is even. As* $p \to \infty$*, the Markov chain* $\{\hat{W}_n(t : p)\}$ *induced by Eq. (3.4) reduces to a simple random walk moving one step in either the right or left direction with equal probability. For* $p = 0$*, it reduces to Eq. (3.2) and* $\{\hat{W}_n(t)\} = \{\hat{W}_n(t : 0)\}$*.*

In the sequel, we define a non-homogeneous Markov chain with absorbing states induced by the homogeneous Markov chain $\{\hat{W}_n(t)\}$ and boundaries $a(t)$ and $b(t)$. Let $t_i = i\Delta t$ and define $a_i = \lfloor a(t_i)/\Delta x\rfloor$ and $b_i = \lfloor b(t_i)/\Delta x\rfloor$. Then the induced boundaries for $\hat{W}_n(t_i)$ are $a^*(i/n) = a_i\Delta x$ and $b^*(i/n) = b_i\Delta x$, for $i = 1, 2, \ldots, n$. We define an imbedded Markov chain $\{Y_n(i)\}_{i=0}^n$ on the state spaces

$$\Omega_i = \{j : a_i < j < b_i\} \cup \{\alpha_i\}, \quad i = 1, 2, \ldots, n, \tag{3.5}$$

by collapsing the values of $\hat{W}_n(t_i)$ greater than $(b_i - 1)\Delta x$ or smaller than $(a_i + 1)\Delta x$ into an absorbing α_i; i.e.

$$Y_n(i) = \begin{cases} \hat{W}_n(t_i)/\Delta x & \text{if } a_i\Delta x < \hat{W}_n(t_i) < b_i\Delta x, \\ \alpha_i & \text{otherwise .} \end{cases}$$

Then $\{Y_n(i)\}_{i=0}^n$ is a non-homogeneous Markov chain with absorbing states $\{\alpha_i\}$ and has transition probabilities given by

$$P(Y_n(i)=k|Y_n(i-1)=j)=\begin{cases} p(k|j) & \text{if } j\in\Omega_{i-1}\backslash\alpha_{i-1},\ k\in\Omega_i\backslash\alpha_i, \\ p_i(\alpha_i|j) & \text{if } j\in\Omega_{i-1}\backslash\alpha_{i-1},\ k=\alpha_i, \\ 1 & \text{if } j=\alpha_{i-1},\ k=\alpha_i, \\ 0 & \text{if } j=\alpha_{i-1},\ k\in\Omega_i\backslash\alpha_i, \end{cases} \tag{3.6}$$

where $p(k|j)$ is given by Eq. (3.3) and

$$p_i(\alpha_i|j)=1-\sum_{k=a_i+1}^{b_i-1}p(k|j),$$

for all $j\in\Omega_{i-1}\backslash\alpha_{i-1}$, and $\Omega_0=\{0\}$ and $P(Y_n(0)=0)\equiv 1$. All of the transition probability matrices of the Markov chain $\{Y_n(i)\}_{i=0}^n$ have the form

$$\boldsymbol{M}_i=\left(\begin{array}{c|c} p(k|j) & p_i(\alpha_i|j) \\ \hline \boldsymbol{0} & 1 \end{array}\right)=\left(\begin{array}{c|c} \boldsymbol{N}_i & \boldsymbol{C}_i \\ \hline \boldsymbol{0} & 1 \end{array}\right),\ i=1,2,\ldots,n, \tag{3.7}$$

where the fundamental matrix $\boldsymbol{N}_i$ are rectangular of size $(b_{i-1}-a_{i-1}-1)\times(b_i-a_i-1)$. It follows from Lemma 2.2.1 that the probability that the Markov chain $\{Y_n(i)\}_{i=0}^n$ never enters the absorbing states α_i is given by the following lemma (see Fu and Lou [37]).

Lemma 3.1.1. *Given m and $n=\lfloor m^2/h^2\rfloor$, we have*

$$P(Y_n(1)\neq\alpha_1,\ldots,Y_n(n)\neq\alpha_n|Y_n(0)=0)=\boldsymbol{\xi}_0\left(\prod_{i=1}^n\boldsymbol{N}_i\right)\boldsymbol{1}',$$

where the $\boldsymbol{N}_i$, $i=1,\ldots,n$ are defined by Eqs. (3.6) and (3.7), and $\boldsymbol{1}=(1,\ldots,1)$ is a row vector of size (b_n-a_n-1).

In view of our constructions and Lemma 3.1.1, we have the following result.

Theorem 3.1.2. *Let $a(t)$ and $b(t)$ be two continuous functions satisfying conditions (A) and (B), and $W(t)$ be a standard Brownian motion. Then*

$$P(W(t) \leq a(t) \text{ or } W(t) \geq b(t), \text{ for some } t \in [0,1])$$

$$= 1 - \lim_{m\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i \right) \mathbf{1}'. \tag{3.8}$$

Proof. We can rewrite the boundary crossing probability as

$$P(a(t) < W(t) < b(t),\, 0 \leq t \leq 1)$$

$$= P(0 < \inf_{0\leq t\leq 1}(W(t) - a(t)) \text{ and } \sup_{0\leq t\leq 1}(W(t) - b(t)) < 0).$$

As the time interval $[0,1]$ is divided into n equal sub-intervals, the boundaries, $a(t)$ and $b(t)$, are also divided into n segments. Then the induced sequences of step functions $a_n(t) = a^*(\lfloor nt \rfloor/n)$ and $b_n(t) = b^*(\lfloor nt \rfloor/n)$ uniformly converge to $a(t)$ and $b(t)$, respectively, on the compact set $[0,1]$. Since $\{a_n(t)\} \to a(t)$, $\{b_n(t)\} \to b(t)$, by Slutsky's theorem, $\hat{W}_n(t) - a_n(t)$ and $\hat{W}_n(t) - b_n(t)$ converge in distribution to $W(t) - a(t)$ and $W(t) - b(t)$, respectively. Also $h_1(x) = (\sup_t x(t), \inf_t x(t))$ is a continuous function. As $m \to \infty$ (or $n \to \infty$), the following holds (see Billingsley [5], page 77),

$$\left(\min_{0\leq i\leq n} \left(\hat{W}_n(t_i) - a^*\left(\frac{i}{n}\right)\right), \max_{0\leq i\leq n} \left(\hat{W}_n(t_i) - b^*\left(\frac{i}{n}\right)\right)\right)$$

$$\xrightarrow{\mathscr{D}} \left(\inf_{0\leq t\leq 1}(W(t) - a(t)), \sup_{0\leq t\leq 1}(W(t) - b(t))\right).$$

Hence, the boundary crossing probability for Brownian motion can be approximated via the constructed Markov chain and calculated using the FMCI technique as follows:

$$
\begin{aligned}
&P\left(a(t) < W(t) < b(t),\, 0 \le t \le 1\right) \\
&\quad = P\left(\inf_{0\le t\le 1}(W(t)-a(t)) > 0 \text{ and } \sup_{0\le t\le 1}(W(t)-b(t)) < 0\right) \\
&\quad = \lim_{m\to\infty} P\left(\min_{0\le i\le n}\left(\hat{W}_n(t_i) - a^*\left(\frac{i}{n}\right)\right) > 0 \text{ and } \max_{0\le i\le n}\left(\hat{W}_n(t_i) - b^*\left(\frac{i}{n}\right)\right) < 0\right) \\
&\quad = \lim_{m\to\infty} P\left(a^*\left(\frac{i}{n}\right) < \hat{W}_n(t_i) < b^*\left(\frac{i}{n}\right),\ \text{ for all } 0\le i\le n\right) \\
&\quad = \lim_{m\to\infty} P\left(Y_n(1) \ne \alpha_1, \ldots, Y_n(n) \ne \alpha_n\right) \\
&\quad = \lim_{m\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2/h^2\rfloor} \boldsymbol{N}_i\right)\mathbf{1}'. \qquad (3.9)
\end{aligned}
$$

This completes the proof. $\square$

Remark 3.1.2. *For the one-sided boundary crossing probability, we simply let $a(t) = -H$ and $H \to \infty$ (or $b(t) = H$ and $H \to \infty$) in our computation. In this case, we use $h = \max(H, \sup_{0\le t\le 1} b(t))$ (or $h = \max(H, \sup_{0\le t\le 1}|a(t)|)$).*

If without assuming the relationship $\Delta t = \Delta x^2$ in advance, then we would have the induced discrete random variables $\hat{X}_\jmath$, $\jmath = 1, \ldots, n$, defined as follows:

$$
P(\hat{X}_\jmath = k\Delta x) = \begin{cases} \frac{C^{-1}\Delta x}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{k^2\Delta x^2}{2\Delta t}\right) & \text{if } k \ne 0, \\ C^{-1}p_0 & \text{if } k = 0, \end{cases} \qquad (3.10)
$$

where C is the normalizing constant and p_0 is adjusted to preserve the variance. We refer Eq. (3.10) as Markov discretization. Let us agree that the symbol $\mathcal{O}$ used below means the corresponding constant is absolute.

Corollary 3.1.1. *Under the Markov discretization, the discrete Markov chain $\{\hat{W}_n(t)\}$ converges in distribution to a Brownian motion if and only if $\Delta t = \mathcal{O}(\Delta x^2)$ provided that the $\hat{X}_j$'s are not degenerate.*

Proof. " $\Rightarrow$ "

Suppose that $\Delta t \neq \mathcal{O}(\Delta x^2)$, and we assume that $\Delta t = \mathcal{O}(\Delta x^{2+a})$, $a \neq 0$, which leads to $\Delta t = c\Delta x^{2+a}$ for some constant $c > 0$. Then the distribution of $\hat{X}_j$ is

$$P(\hat{X}_j = k\Delta x) = \begin{cases} \frac{C^{-1}}{\sqrt{2\pi c\Delta x^a}} \exp\left(-\frac{k^2}{2c\Delta x^a}\right) & \text{if } k \neq 0, \\ C^{-1} p_0 & \text{if } k = 0, \end{cases} \tag{3.11}$$

where $C = \frac{1}{\sqrt{2\pi c\Delta x^a}} \sum_{\ell \neq 0} \exp(-\ell^2/2c\Delta x^a) + p_0$ is the normalizing constant.

Case 1. If $a > 0$, then $\hat{X}_j$ degenerates to 0, as $n \to \infty (\Delta x \to 0)$.

Case 2. If $a < 0$, for $k \neq 0$,

$$\begin{aligned} P(\hat{X}_j = k\Delta x) &= \frac{C^{-1}}{\sqrt{2\pi c\Delta x^a}} \exp\left(-\frac{k^2}{2c\Delta x^a}\right) \\ &\leq \frac{C^{-1}}{\sqrt{2\pi c\Delta x^a}} \to 0, \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\hat{X}_j$ degenerates to 0, as $n \to \infty$.

" $\Leftarrow$ "

If $\Delta t = \mathcal{O}(\Delta x^2)$, then $\Delta t = c\Delta x^2$ for some positive constant c. Then we have the distribution function of $\hat{X}_j$ given by

$$P(\hat{X}_j = k\Delta x) = \begin{cases} \frac{C^{-1}}{\sqrt{2\pi c}} \exp\left(-\frac{k^2}{2c}\right) & \text{if } k \neq 0, \\ C^{-1} p_0 & \text{if } k = 0, \end{cases}$$

where $C = \frac{1}{\sqrt{2\pi c}} \sum_{\ell \neq 0} \exp(-\ell^2/2c) + p_0$ is the normalizing constant. In order to preserve the variance, p_o must satisfy

$$\mathrm{Var}(\hat{X}_j) = \Delta t = \frac{C^{-1}}{\sqrt{2\pi c}} \sum_{k \neq 0} k^2 \Delta x^2 \exp\left(-\frac{k^2}{2c}\right)$$

$$\Rightarrow \quad c\Delta x^2 = \frac{\frac{1}{\sqrt{2\pi c}} \sum_{k \neq 0} k^2 \Delta x^2 \exp\left(-\frac{k^2}{2c}\right)}{\frac{1}{\sqrt{2\pi c}} \sum_{k \neq 0} \exp\left(-\frac{k^2}{2c}\right) + p_0}$$

$$\Rightarrow \quad c\left(\frac{1}{\sqrt{2\pi c}} \sum_{k \neq 0} \exp\left(-\frac{k^2}{2c}\right) + p_0\right) = \frac{1}{\sqrt{2\pi c}} \sum_{k \neq 0} k^2 \exp\left(-\frac{k^2}{2c}\right).$$

Then,

$$p_0 = \frac{1}{\sqrt{2\pi c}} \sum_{k \neq 0} \left(\frac{k^2}{c} - 1\right) \exp\left(-\frac{k^2}{2c}\right).$$

We need to show $p_0 \geq 0$ for all $c > 0$. Let $\epsilon = \sum_{k=1}^{\infty} (k^2/c - 1) \exp(-k^2/2c)$ and $g(k) = (k^2/c - 1) \exp(-k^2/2c)$. We consider two cases when (i) $c < 1/3$ and (ii) $c \geq 1/3$. Then, $g(k)$ is decreasing in k in case (i) and decreasing in $k > \sqrt{3c}$ in case (ii). It follows that

Case (i)

$$\begin{aligned} \epsilon &\geq \int_1^{\infty} \left(\frac{k^2}{c} - 1\right) \exp\left(-\frac{k^2}{2c}\right) dk \\ &= \int_1^{\infty} \frac{k^2}{c} \exp\left(-\frac{k^2}{2c}\right) dk - \int_1^{\infty} \exp\left(-\frac{k^2}{2c}\right) dk. \end{aligned}$$

Set $y = k^2/2c$, $\sqrt{y} = k/\sqrt{2c}$ and $dy = (k/c)dk$, then

$$\begin{aligned}
\epsilon &\geq \int_{\frac{1}{2c}}^{\infty} \frac{k^2}{c} e^{-y} \frac{c}{k} dy - \int_{\frac{1}{2c}}^{\infty} e^{-y} \frac{c}{k} dy \\
&= \int_{\frac{1}{2c}}^{\infty} \sqrt{2cy} e^{-y} dy - \int_{\frac{1}{2c}}^{\infty} \frac{c}{\sqrt{2cy}} e^{-y} dy \\
&= \sqrt{2c} \cdot \Gamma\left(\frac{3}{2}, \frac{1}{2c}\right) - \frac{c}{\sqrt{2c}} \Gamma\left(\frac{1}{2}, \frac{1}{2c}\right),
\end{aligned}$$

where Γ is the upper incomplete gamma function. Since

$$\Gamma\left(\frac{3}{2}, \frac{1}{2c}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}, \frac{1}{2c}\right) + \left(\frac{1}{2c}\right)^{\frac{1}{2}} e^{-\frac{1}{2c}},$$

then

$$\begin{aligned}
\epsilon &\geq \frac{\sqrt{c}}{\sqrt{2}} \Gamma\left(\frac{1}{2}, \frac{1}{2c}\right) + e^{-\frac{1}{2c}} - \frac{\sqrt{c}}{\sqrt{2}} \Gamma\left(\frac{1}{2}, \frac{1}{2c}\right) \\
&= e^{-\frac{1}{2c}}.
\end{aligned}$$

Case (ii)

$$\begin{aligned}
\epsilon &\geq \int_{\lceil \sqrt{3c} \rceil}^{\infty} \left(\frac{k^2}{c} - 1\right) \exp\left(-\frac{k^2}{2c}\right) dk \\
&= \int_{\lceil \sqrt{3c} \rceil}^{\infty} \frac{k^2}{c} \exp\left(-\frac{k^2}{2c}\right) dk - \int_{\lceil \sqrt{3c} \rceil}^{\infty} \exp\left(-\frac{k^2}{2c}\right) dk,
\end{aligned}$$

where $\lceil a \rceil$ is the smallest integer greater than or equal to a. Set $y = k^2/2c$,

$\sqrt{y} = k/\sqrt{2c}$ and $dy = (k/c)dk$, then

$$\begin{aligned}
\epsilon &\geq \int_{\frac{3}{2}}^{\infty} \frac{k^2}{c} e^{-y} \frac{c}{k} dy - \int_{\frac{3}{2}}^{\infty} e^{-y} \frac{c}{k} dy \\
&= \int_{\frac{3}{2}}^{\infty} \sqrt{2cy} e^{-y} dy - \int_{\frac{3}{2}}^{\infty} \frac{c}{\sqrt{2cy}} e^{-y} dy \\
&= \sqrt{2c} \cdot \Gamma\left(\frac{3}{2}, \frac{3}{2}\right) - \frac{c}{\sqrt{2c}} \Gamma\left(\frac{1}{2}, \frac{3}{2}\right).
\end{aligned}$$

Since

$$\Gamma\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}, \frac{3}{2}\right) + \left(\frac{3}{2}\right)^{\frac{1}{2}} e^{-\frac{3}{2}},$$

then

$$\begin{aligned}
\epsilon &\geq \frac{\sqrt{c}}{\sqrt{2}} \Gamma\left(\frac{1}{2}, \frac{3}{2}\right) + \sqrt{3c} e^{-\frac{1}{2c}} - \frac{\sqrt{c}}{\sqrt{2}} \Gamma\left(\frac{1}{2}, \frac{3}{2}\right) \\
&= \sqrt{3c} e^{-\frac{1}{2c}}.
\end{aligned}$$

We have shown $p_0 \geq 0$ for any given $c > 0$, then following exactly the same proof of Theorem 3.1.1, $\hat{W}_n(t)$ converges in distribution to a standard Brownian motion $W(t)$.

□

3.1.2 Error Bound

In this section, we provide the error bound of our approximation for BCP for one-dimensional Brownian motion. The error bound is established based on the results of Nagaev [61] and Borokov and Novikov [13].

Let $\mathscr{F} = \{f_p : \text{the family of distributions given by Eq. (3.4) for } p = 0, 2, 4, \ldots\}$. For the proposed approximation $\hat{W}_n(t : p)$ induced by Eq. (3.4) with $f_p \in \mathscr{F}$, it is important to know its error bound and numerical performance. Let

$$\hat{\mathscr{P}}(a(k/n), b(k/n)) := P(a(k/n) < \hat{W}_n(t_k : p) < b(k/n), k = 1, \ldots, n),$$

$$\mathscr{P}(a(t), b(t)) := P(a(t) < W(t) < b(t), t \in [0, 1]).$$

We present the following theorem.

Theorem 3.1.3. *Given $f_p \in \mathscr{F}$,*

(i) $\hat{W}_n(t : p) \xrightarrow{\mathscr{D}} W(t), \quad$ *as $n \to \infty$, and*

(ii) *for the boundaries $a(t)$ and $b(t)$ satisfying the Lipschitz condition: there exists a constant K such that $|a(t+h) - a(t)| < Kh$ and $|b(t+h) - b(t)| < Kh$, $h > 0$, then the error bound is*

$$|\mathscr{P}(a(t), b(t)) - \hat{\mathscr{P}}(a^*(k/n), b^*(k/n))| = \mathcal{O}(1/m), \quad \text{as } n \to \infty,$$

where $n = \lfloor m^2/h^2 \rfloor$, $a^*(k/n)$ and $b^*(k/n)$ are the boundaries for $\hat{W}_n(t_k : p)$.

The proof of the error bound depends on the results of Nagaev [61] and Borokov and Novikov [13]. We list their results as lemmas, but provide no proofs.

Lemma 3.1.2 (Nagaev)**.** *Assuming that the boundaries $a(t)$ and $b(t)$ satisfy the Lipschitz condition, then there exists a constant c_1 such that, for any $p = 0, 2, \ldots, \infty$,*

$$\left| \mathscr{P}(a(t), b(t)) - \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right), b\left(\frac{k}{n}\right)\right) \right| < \frac{c_1}{\sqrt{n}},$$

where c_1 is a constant which may depend on p.

Lemma 3.1.3 (Borovkov and Novikov)**.** *Given small $\delta > 0$, there exists a constant c_2 such that*

$$|\mathscr{P}(a(t)-\delta, b(t)+\delta) - \mathscr{P}(a(t)+\delta, b(t)-\delta)| < c_2\delta.$$

Lemma 3.1.4. *Given small $\delta > 0$,*

$$\begin{aligned}&\left|\hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right) - \hat{\mathscr{P}}\left(a^*\left(\frac{k}{n}\right), b^*\left(\frac{k}{n}\right)\right)\right| \\ &\qquad \leq \left|\hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right) - \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)+\delta, b\left(\frac{k}{n}\right)-\delta\right)\right|.\end{aligned}$$

Proof. From the definitions of $a^*(k/n)$ and $b^*(k/n)$, the following two inequalities hold for $k = 1, 2, \ldots, n$:

$$a\left(\frac{k}{n}\right) - \delta < a^*\left(\frac{k}{n}\right) < a\left(\frac{k}{n}\right) + \delta \quad \text{and} \quad b\left(\frac{k}{n}\right) - \delta < b^*\left(\frac{k}{n}\right) < b\left(\frac{k}{n}\right) + \delta.$$

The lemma follows immediately from the above two inequalities. □

Proof of Theorem 3.1.3. The proof of part (i) follows along the same lines as the proof of Theorem 3.1.1 and is thus omitted. Given small $\delta > 0$, it follows from triangle inequality that

$$\begin{aligned}&\left|\mathscr{P}(a(t), b(t)) - \hat{\mathscr{P}}\left(a^*\left(\frac{k}{n}\right), b^*\left(\frac{k}{n}\right)\right)\right| \\ &\quad \leq \left|\mathscr{P}(a(t), b(t)) - \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right)\right| \\ &\qquad + \left|\hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right) - \hat{\mathscr{P}}\left(a^*\left(\frac{k}{n}\right), b^*\left(\frac{k}{n}\right)\right)\right| \\ &\quad = A_n + B_n.\end{aligned}$$

Furthermore, using the triangle inequality, we have

$$\begin{aligned}
A_n &\leq |\mathscr{P}(a(t), b(t)) - \mathscr{P}(a(t)-\delta, b(t)+\delta)| \\
&\quad + \left| \mathscr{P}(a(t)-\delta, b(t)+\delta) - \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right) \right| \\
&= C_n + D_n.
\end{aligned}$$

By Lemma 3.1.4, we have

$$\begin{aligned}
B_n &\leq \left| \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right) - \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)+\delta, b\left(\frac{k}{n}\right)-\delta\right) \right| \\
&\leq \left| \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)-\delta, b\left(\frac{k}{n}\right)+\delta\right) - \mathscr{P}(a(t)-\delta, b(t)+\delta) \right| \\
&\quad + |\mathscr{P}(a(t)-\delta, b(t)+\delta) - \mathscr{P}(a(t)+\delta, b(t)-\delta)| \\
&\quad + \left| \mathscr{P}(a(t)+\delta, b(t)-\delta) - \hat{\mathscr{P}}\left(a\left(\frac{k}{n}\right)+\delta, b\left(\frac{k}{n}\right)-\delta\right) \right| \\
&= E_n + F_n + G_n.
\end{aligned}$$

It follows from Lemma 3.1.2 the D_n, E_n and G_n terms tend to zero with order $\mathcal{O}(1/\sqrt{n})$. By Lemma 3.1.3, the C_n and F_n terms tend to zero with order $\mathcal{O}(\delta)$. Part (ii) is an immediate consequence of taking $\delta = h/m$ and $n = \lfloor m^2/h^2 \rfloor$. This completes the proof of part (ii). $\square$

For $p = \infty$, it is a simple random walk (SRW). The result $\hat{W}_n(t:\infty) \xrightarrow{\mathscr{D}} W(t)$ was given in Kac [41] and we expect the rate of convergence of $\hat{W}_n(t:\infty)$ to be slower than that of $\hat{W}_n(t:0)$. Table 3.1 provides a small numerical study to illustrate the errors and rates of convergence of the boundary crossing probabilities of $\hat{W}_n(t:p)$ for various p.

Table 3.1: Error (|exact - approximation|) for (a) one-sided Daniels' boundary $\frac{1}{2} - t\log(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}}))$ and (b) two-sided boundary $\pm(1+t)$ under various p.

(a)

m	$p=0$	$p=2$	$p=6$	$p=\infty$ (SRW)
2000	6.605×10^{-4}	8.780×10^{-4}	9.725×10^{-4}	9.799×10^{-4}
4000	3.220×10^{-4}	4.306×10^{-4}	4.778×10^{-4}	4.809×10^{-4}
8000	1.581×10^{-4}	2.124×10^{-4}	2.360×10^{-4}	2.376×10^{-4}
12000	1.046×10^{-4}	1.408×10^{-4}	1.565×10^{-4}	1.576×10^{-4}

(b)

m	$p=0$	$p=2$	$p=6$	$p=\infty$ (SRW)
2000	1.651×10^{-4}	2.220×10^{-4}	2.467×10^{-4}	2.485×10^{-4}
4000	8.158×10^{-5}	1.101×10^{-4}	1.224×10^{-4}	1.233×10^{-4}
8000	4.035×10^{-5}	5.461×10^{-5}	6.013×10^{-5}	6.124×10^{-5}
12000	2.615×10^{-5}	3.623×10^{-5}	3.762×10^{-5}	4.065×10^{-5}

Several things can be seen from the Table 3.1: (i) the error bound tending to 0 is with order c_p/m with unknown constant c_p. (ii) The constant c_p not only depends on p but also on the boundaries. For example, in Table 3.1 (a), for $p=0$ and $m=8000$ the error is 1.581×10^{-4}, but, for $p=\infty$ (SRW), it requires $m=12000$ to have the same error. This phenomenon can also be found in Table 3.1 (b). (iii) The $\hat{W}_n(t:0)$ has the fastest rate of convergence to a Brownian motion $W(t)$ among all $f_p \in \mathscr{F}$. Technically speaking, we have not yet determined if f_0 has the best rate among all possible discretizations.

3.1.3 Y-channel boundary

We consider a special class of boundaries called *Y-channel boundaries*. The Y-channel boundary has the shape as the letter Y. Two examples are given in Figure 3.1. Let us consider a standard Y-channel boundary (solid lines) as part (a) of Figure 3.1. The boundary splits into two channels at certain time $t_Y \in [0, 1]$ which is called a *split point*. As long as the Brownian motion enters one channel after time t_Y, then it can not enter another channel without crossing the boundary. As a variation of Y-channel boundary, we call part (b) of Figure 3.1 *land boundary* (solid lines and curves). For our method developed in Section 3.1.1, it creates no additional difficulty for computing the boundary crossing probabilities for Y-channel boundaries. An imbedded Markov chain can be constructed as usual before time t_Y, but the transition probability matrices after time t_Y will require some minor modification. The details are given below.

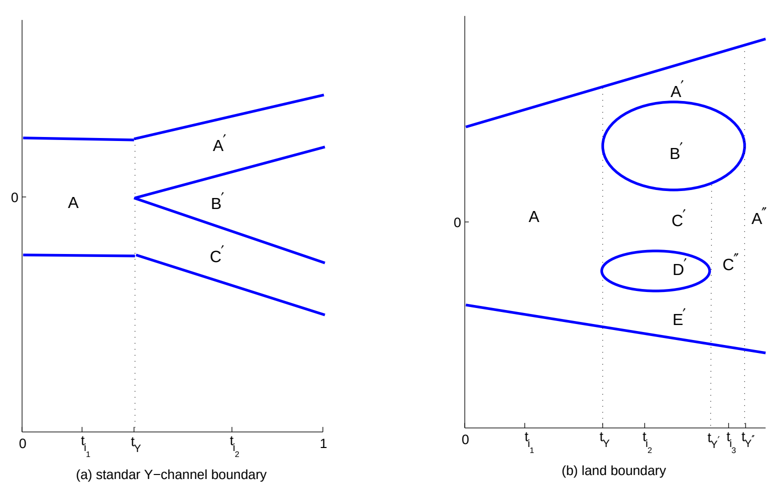


Figure 3.1: Y-channel boundaries

Standard Y-channel boundary For a usual band boundary (with only upper and lower boundaries), at each time t, the set of values within boundaries can be viewed as an interval. We consider a Y-channel boundary in part (a) of Figure 3.1. At each time t, the interval between the outer boundaries is partitioned into several sub-intervals. For example, there is only one interval at time $t_{i_1} < t_Y$, denoted by A_{i_1}, and there are three intervals at time $t_{i_2} \geq t_Y$, denoted by A'_{i_2}, B'_{i_2} and C'_{i_2}. Note that once the Brownian motion enters the area A' after time t_Y, it can never enter area C' without crossing the boundary.

As the states associated with the fundamental matrix indicate the situation that the process stays inside the boundaries, the above idea can then be used to partition the fundamental matrices of the imbedded Markov chain to accommodate the irregular boundaries. At each time t_i, the fundamental matrix $\boldsymbol{N}_i$ is partitioned into certain number of blocks according to the number of intervals induced by the boundaries as we illustrated before. For example, in part (a) of Figure 3.1, the fundamental matrix $\boldsymbol{N}_{i_2}$ at time t_{i_2} is of the form

$$\boldsymbol{N}_{i_2} = \begin{pmatrix} \boldsymbol{N}_{A'_{i_2-1}A'_{i_2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{N}_{C'_{i_2-1}C'_{i_2}} \end{pmatrix}, \tag{3.12}$$

where $\boldsymbol{N}_{A'_{i_2-1}A'_{i_2}}$ and $\boldsymbol{N}_{C'_{i_2-1}C'_{i_2}}$ are rectangular blocks composed of the transition probabilities from states in A'_{i_2-1} and C'_{i_2-1}, at time t_{i_2-1}, to states in A'_{i_2} and C'_{i_2}, at time t_{i_2}, respectively. Also note that transition probability matrices will change the form after the split point. For simplicity, we omit the subscripts of the intervals, for example A_i is written as A. If $t_{j-1} < t_Y \leq t_j$ for some j, then, for all $i < j$, the fundamental matrix $\boldsymbol{N}_i$ has the form

$$\boldsymbol{N}_i = (\boldsymbol{N}_{AA}).$$

After the split point t_Y, the fundamental matrix $\boldsymbol{N}_j$, is then changed to the form

$$\boldsymbol{N}_j = \begin{pmatrix} \boldsymbol{N}_{AA'} & \mathbf{0} & \boldsymbol{N}_{AC'} \end{pmatrix}, \tag{3.13}$$

and for each $i' > j$, the fundamental matrix $\boldsymbol{N}_{i'}$ is of the form in Eq. (3.12).

The detailed construction of the imbedded Markov chain is given as follows. For each i such that $t_i < t_Y$, the fundamental matrix $\boldsymbol{N}_i$ is constructed in the same way in Section 3.1.1. For each i such that $t_i \geq t_Y$, let $a_i^U = \lfloor A_i'^U/\Delta x \rfloor$ and $a_i^L = \lfloor A_i'^L/\Delta x \rfloor$ be the discrete boundaries for the upper channel where $A_i'^U$ and $A_i'^L$ are the upper and lower limits of the interval A_i', respectively. The discrete boundaries c_i^U and c_i^L for the lower channel can be defined analogously. Thus, at each time $t_i \geq t_Y$, the state space is

$$\Omega_i = \Omega_i^{A'} \cup \Omega_i^{C'} \cup \{\alpha_i\},$$

where $\Omega_i^{A'} = \{j : a_i^L < j < a_i^U\}$ and $\Omega_i^{C'} = \{j : c_i^L < j < c_i^U\}$. The transition probabilities for the fundamental matrix $\boldsymbol{N}_{A'_{i-1}A'_i}(\boldsymbol{N}_{C'_{i-1}C'_i})$ are given by, for $j \in \Omega_{i-1}^{A'}(\Omega_{i-1}^{C'})$ and $k \in \Omega_i^{A'}(\Omega_i^{C'})$,

$$\begin{aligned}
(\boldsymbol{N}_{A'_{i-1}A'_i})_{jk} &= P(Y_n(i) = k | Y_n(i-1) = j) \\
&= \begin{cases} \frac{C^{-1}}{\sqrt{2\pi}} \exp\left(-\frac{(k-j)^2}{2}\right) & \text{if } k - j \neq 0, \\ \frac{C^{-1}}{\sqrt{2\pi}} \sum_{\ell \neq 0} (\ell^2 - 1) \exp\left(-\frac{\ell^2}{2}\right) & \text{if } k - j = 0. \end{cases}
\end{aligned}$$

Then the boundary crossing probability for the standard Y-channel boundary can be calculated by our unified formula

$$1 - \lim_{m\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i \right) \mathbf{1}',$$

where $\boldsymbol{N}_i$ can be obtained by the above construction. In a similar fashion, the above construction of fundamental matrices can be generalized to boundaries with finite many split points. As a variation of standard Y-channel boundary, the example given in part (b) of Figure 3.1 which has three split points is in another class of Y-channel boundaries, called land boundaries.

Land boundary The fundamental matrices play an important role in our method. Here, we only show how to form the fundamental matrices, as the construction of the imbedded Markov chain and its state spaces and transition probabilities can be carried out in a similar fashion in the previous section.

At each time $t_i, i = 1, \ldots, n$, the fundamental matrix is of the form given below. For each $i < j$ such that $t_{j-1} < t_Y \le t_j$, the fundamental matrix has the form

$$\boldsymbol{N}_i = (\boldsymbol{N}_{AA}),$$

for $t = t_j$ after the first split point t_Y, the form of the fundamental matrix $\boldsymbol{N}_j$ is changed to

$$\boldsymbol{N}_j = \begin{pmatrix} \boldsymbol{N}_{AA'} & \mathbf{0} & \boldsymbol{N}_{AC'} & \mathbf{0} & \boldsymbol{N}_{AE'} \end{pmatrix},$$

for each $j < i < j^{'}$ such that $t_{j^{'}-1} < t_{Y^{'}} \leq t_{j^{'}}$, the fundamental matrix has the form

$$\boldsymbol{N}_i = \begin{pmatrix} \boldsymbol{N}_{A^{'}A^{'}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{N}_{C^{'}C^{'}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{N}_{E^{'}E^{'}} \end{pmatrix},$$

for $t = t_{j^{'}}$ after the second split point $t_{Y^{'}}$, the form of the fundamental matrix is changed to

$$\boldsymbol{N}_{j^{'}} = \begin{pmatrix} \boldsymbol{N}_{A^{'}A^{'}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{N}_{C^{'}C^{''}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{N}_{E^{'}C^{''}} \end{pmatrix},$$

for each $j^{'} < i < j^{''}$ such that $t_{j^{''}-1} < t_{Y^{''}} \leq t_{j^{''}}$, the fundamental matrix has the form

$$\boldsymbol{N}_i = \begin{pmatrix} \boldsymbol{N}_{A^{'}A^{'}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{N}_{C^{''}C^{''}} \end{pmatrix},$$

for $t = t_{j^{''}}$ after the third split point $t_{Y^{''}}$, the form of the fundamental matrix is changed to

$$\boldsymbol{N}_{j^{''}} = \begin{pmatrix} \boldsymbol{N}_{A^{'}A^{''}} \\ \mathbf{0} \\ \boldsymbol{N}_{C^{''}A^{''}} \end{pmatrix},$$

and for all $i > j^{''}$, the fundamental matrices are again back to whole matrices

$$\boldsymbol{N}_i = \left(\boldsymbol{N}_{A^{''}A^{''}}\right).$$

Thus, our unified formula

$$1 - \lim_{m\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i \right) \mathbf{1}',$$

still applies for land boundaries with only minor modification in the fundamental matrices of the imbedded Markov chain. The construction of fundamental matrices adapted to the combination of finite many standard Y-channel and land boundaries is a straightforward extension.

3.1.4 Comparison under different underlying distributions

In the preceding section, we construct finite Markov chains based on the family of distributions given in Eq. (3.4) with $p = 0$ induced by the distribution of $W(\Delta t)$. In this section, we conduct a numerical comparison study on the convergence rates of approximations based on different underlying distributions for boundary crossing probabilities for Brownian motion. We discuss the performances of approximations based on different discrete distributions induced by: (a) standard normal distribution, (b) Student's $t(\nu)$ distributions with ν degrees of freedom and (c) uniform distributions.

Finite Markov chain under $t(\nu)$ distributions We want to construct finite Markov chains under distributions induced by the $t(\nu)$ distribution, converging to a standard Brownian motion. For a given degrees of freedom ν, the mean and variance of $t(\nu)$ distribution are 0 and $\nu/(\nu-2)$, respectively, hence, $t(\nu)\sqrt{\nu - 2/\nu n}$

has mean 0 and variance $1/n$. Thus, we have the induced discrete i.i.d. random variables with distribution defined by

$$P(\hat{X}_\jmath = k\Delta x) = \begin{cases} C^{-1} k t_\nu(k)\sqrt{\frac{\nu-2}{\nu}} & \text{if } k \neq 0, \\ C^{-1}\sum_{k\neq 0}(k^2-1)k t_\nu(k)\sqrt{\frac{\nu-2}{\nu}} & \text{if } k = 0, \end{cases}$$

where $t_\nu(k)$ stands for the density function of t-distribution with ν degrees of freedom and $C = \sum_{k\neq 0} k^3 t_\nu(k)\sqrt{(\nu-2)/\nu}$ is the normalizing constant. The construction of partial sums or Markov chains and the corresponding transition probability matrices can be carried out in a similar fashion in Section 3.1.1. Tables 3.2 and 3.3 show the boundary crossing probabilities for one-sided boundaries $1+t$ and $\frac{1}{2} - t\log\left(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}})\right)$ under various degrees of freedom ν.

Table 3.2: BCP with boundary $1+t$ under $t(\nu)$ distribution with various degrees of freedom.

ν	$m = 1000$	2000	4000	8000
3	0.08969239	0.09006336	0.09022228	0.09031081
4	0.09054673	0.09047365	0.09044276	0.09042948
5	0.09068311	0.09054484	0.09047957	0.09044826
10	0.09079789	0.09060366	0.09050937	0.09046326
50	0.09084067	0.09062534	0.09052028	0.09046873
100	0.09084462	0.09062734	0.09052128	0.09046924
500	0.09084762	0.09062886	0.09052204	0.09046962

Finite Markov chain under Uniform distributions A uniform distribution is used to construct the Markov chains which converge to a Brownian motion. The discrete versions of uniform distributions are given by

$$P(\hat{X}_\jmath = k\Delta x) = \frac{1}{2\ell+1},\ k = 0, \pm 1, \ldots, \pm \ell.$$

Table 3.3: BCP with boundary $\frac{1}{2} - t\log\left(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}})\right)$ under $t(\nu)$ distribution with various degrees of freedom.

ν	$m = 1000$	2000	4000	8000
3	0.476608391	0.478204077	0.478997126	0.479381061
4	0.479967311	0.479907987	0.479821847	0.479798151
5	0.480450023	0.480146346	0.479939835	0.479841771
10	0.480821185	0.480330928	0.480031893	0.479887744
50	0.480955487	0.480397908	0.480065343	0.479904459
100	0.480967829	0.480404065	0.480068419	0.47990600
500	0.480977188	0.480408734	0.480070751	0.47990716

However, we need to adjust the probability at 0 such that the mean and variance of the discrete random variable are 0 and $1/n$, respectively. Hence, if $\ell = 5$ then the distribution of $\hat{X}_j$ is modified and given by

$$P(\hat{X}_j = 0) = \frac{100}{110},$$

$$P(\hat{X}_j = \pm k\Delta x) = \frac{1}{110}, \quad k = \pm 1, \ldots, \pm 5, \tag{3.14}$$

and if $\ell = 10$ then the distribution of $\hat{X}_j$ is modified and given by

$$P(\hat{X}_j = 0) = \frac{750}{770},$$

$$P(\hat{X}_j = \pm k\Delta x) = \frac{1}{770} \quad k = \pm 1, \ldots, \pm 10. \tag{3.15}$$

By the same token, we can choose any value of ℓ and construct a sequence of discrete random variables with mean 0 and variance $1/n$. Again, the construction of finite Markov chains and the corresponding transition probability matrices can be done in a similar way in Section 3.1.1. The boundary crossing probabilities based on distributions in Eqs. (3.14) and (3.15), for $\ell = 5, 10$, are given in Table 3.4.

Table 3.4: The boundary crossing probabilities under uniform distributions.

Boundary	$\frac{1}{2} - t\log\left(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}})\right)$		$1+t$	
ℓ	5	10	5	10
$m = 1000$	0.090724	0.471973363	0.480594	0.088052401
2000	0.090566	0.475879864	0.480217	0.089181294
4000	0.090491	0.477799384	0.479975	0.089785615
8000	0.090454	0.478769728	0.479859	0.090098215

Table 3.5: Error (|exact - approximation|) for (a) one-sided Daniels' boundary $\frac{1}{2} - t\log(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}}))$ and (b) one-sided linear boundary $(1+t)$ under normal, $t(\nu)$ and uniform distributions.

(a)

m	Normal	$t(4)$	$t(10)$	Uniform(5)
2000	6.605×10^{-4}	1.5899×10^{-4}	5.8193×10^{-4}	4.6784×10^{-4}
4000	3.220×10^{-4}	7.2847×10^{-5}	2.8289×10^{-4}	2.5100×10^{-4}
8000	1.581×10^{-4}	4.9151×10^{-5}	1.3874×10^{-4}	1.1016×10^{-4}

(b)

m	Normal	$t(4)$	$t(10)$	Uniform(5)
2000	2.1141×10^{-4}	5.5880×10^{-5}	1.8589×10^{-4}	1.4833×10^{-4}
4000	1.0442×10^{-4}	2.4988×10^{-5}	9.1595×10^{-5}	7.2795×10^{-5}
8000	5.1902×10^{-5}	1.1710×10^{-5}	4.5489×10^{-5}	3.6085×10^{-5}

Table 3.5 provides a numerical comparison of rates of convergence among approximations based on distributions induced by normal, $t(\nu)$ and uniform distributions. There are several comments we can make from Table 3.5. The $t(\nu)$ and uniform distributions are considered distributions with heavy tails. We expect that the rate of convergence under the normal distribution is better than the rates of convergence under $t(\nu)$ or uniform distributions. However, in order to preserve the

variance structure, the discrete versions of $t(\nu)$ and uniform distributions actually have distributions with thin tails. We speculate that the approximation based on distributions with more weight (probability) on the point 0 would provide better convergence rate. In order to verify this hypothesis numerically, we reweight the discrete version of standard normal distribution to have higher probability on the point 0. The modified discrete distributions are give by, for $\jmath = 1, \dots, n$,

$$P(\hat{X}_\jmath = k\Delta x) = \begin{cases} C^{-1} w_k \phi(k) & \text{if } k \neq 0, \\ 1 - \sum_{k \neq 0} C^{-1} w_k \phi(k) & \text{if } k = 0, \end{cases} \tag{3.16}$$

where $C = \sum_{k \neq 0} w_k k^2 \phi(k)$ is the normalizing constant and w_k, $k = \pm 1, \pm 2, \dots$, are weights chosen to increase the probability on the point 0. For simplicity, we choose w_k the same values for odd and even values of k and denote the two values by w_{odd} and w_{even}, respectively. The errors for the approximation based on the distributions in Eq. (3.16) are given in Table 3.6 with $w_{odd} = 0.5$ and $w_{even} = 20$ or 50. As we compare the errors in Tables 3.5 and 3.6, the errors for the approximation based on distributions in Eq. (3.16) are smaller than that based on $t(\nu)$ or uniform distributions. Thus, we conclude that the approximation, based on the distribution with suitably more weight on the point 0, would generally provide better convergence rate.

3.2 Identity from Erdös and Kac

Erdös and Kac [31] have significant contribution on the theory of random walk and Brownian motion. In their paper in 1946, they gave an infinite series for the boundary crossing probability for Brownian motion to two-sided constant boundary

Table 3.6: Errors (|exact - approximation|) for approximations based on distributions in Eq. (3.16) for one-sided linear boundary $(1+t)$ under two different combinations of w_1 and w_2.

w_{odd}	0.5	
w_{even}	20	50
$m = 2000$	3.6283×10^{-5}	2.3676×10^{-5}
4000	1.6320×10^{-5}	9.8813×10^{-6}
8000	7.7331×10^{-6}	4.4800×10^{-5}

as the limiting boundary crossing probability for a random walk. The result is given as follows. Let $X_1, \dots, X_n$ be i.i.d. random variables with mean 0 and standard deviation 1. Let $W_n = \sum_{i=1}^{n} X_i$, then for $c > 0$,

$$\lim_{n\to\infty} P\left(\max_{1\le k\le n} |W_k| < cn^{1/2}\right) = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\exp\left(-\frac{(2i+1)^2\pi^2}{8c^2}\right). \tag{3.17}$$

The above probability is indeed the boundary crossing probability for Brownian motion to two-sided constant boundary $\pm c$, i.e.

$$\lim_{n\to\infty} P\left(\max_{1\le k\le n} |W_k| < cn^{1/2}\right) = P\left(\sup_{0\le t\le 1} |W(t)| < c\right).$$

The above result is the so-called invariance principle, since the limiting distribution does not depend on the underlying distribution. Here, we will show the strength of the FMCI technique by using it to derive Eq. (3.17) under the case of simple random walk.

Lemma 3.2.1. *Let $n = m^2/c^2$, then*

$$\cos^n\left(\frac{i\pi}{2m}\right) \le \frac{K}{i^2}, \quad i = 1, \dots, m-1,$$

for some positive constant K.

Proof. By Taylor expansion, we have, for $0 < \xi < \frac{i\pi}{2m}$,

$$\begin{aligned}\cos\left(\frac{i\pi}{2m}\right) &= 1 - \frac{1}{2}\left(\frac{i\pi}{2m}\right)^2 + \frac{\cos(\xi)}{4!}\left(\frac{i\pi}{2m}\right)^4 \\ &= 1 - \frac{i^2\pi^2}{8m^2}\left(1 - \frac{i^2\pi^2\cos(\xi)}{48m^2}\right) \\ &= 1 - \frac{i^2\pi^2}{8m^2}c_1,\end{aligned}$$

where $1 + \frac{\pi^2}{48} > c_1 = 1 - \frac{i^2\pi^2\cos(\xi)}{48m^2} > 1 - \frac{\pi^2}{48}$.

Thus,

$$\begin{aligned}\cos^n\left(\frac{i\pi}{2m}\right) &= \exp\left(n\log\cos\left(\frac{i\pi}{2m}\right)\right) \\ &= \exp\left(n\log\left(1 - \frac{i^2\pi^2}{8m^2}c_1\right)\right) \\ &\leq \exp\left(-n\frac{i^2\pi^2}{8m^2}c_1\right) \\ &= \exp\left(-\frac{i^2\pi^2c_1}{8c^2}\right) \\ &\leq \frac{1}{1 + \frac{i^2\pi^2c_1}{8c^2}} \\ &\leq \frac{8c^2}{i^2\pi^2c_1} = \frac{K}{i^2},\end{aligned}$$

where $K = 8c^2/\pi^2c_1$ is bounded above by $8c^2/\pi^2(1 - \frac{\pi^2}{48})$. The proof is completed.

$\square$

For one-sided constant boundary $c > 0$, the fundamental matrix $\boldsymbol{N}_c$ of the imbedded Markov chain is given in Eq. (2.8) and the boundary crossing probability is given by

$$P\left(\sup_{0\le t\le 1}|W(t)| < c\right) = \lim_{m\to\infty} \xi_0 \boldsymbol{N}_c^n \mathbf{1}^{'}.$$

The following theorem can be derived based on Lemma 3.2.1 and the results in Section 2.3.

Theorem 3.2.1. *For $c > 0$ and $n = m^2/c^2$, we have*

$$\lim_{m\to\infty} \xi_0 \boldsymbol{N}_c^n \mathbf{1}^{'} = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\exp\left(-\frac{(2i+1)^2\pi^2}{8c^2}\right).$$

Proof. From Eq. (2.9), we have

$$\begin{aligned}
\xi_0 \boldsymbol{N}_c^n \mathbf{1}^1 &= \sum_{i=1,odd}^{2m-1}\frac{(-1)^{\frac{i-1}{2}}\cos^n(\frac{i\pi}{2m})}{m'\tan(\frac{i\pi}{4m})}\\
&= \sum_{i=1,odd}^{m-1}\frac{(-1)^{\frac{i-1}{2}}\cos^n(\frac{i\pi}{2m})}{m'\tan(\frac{i\pi}{4m})} + \sum_{i=m+1,odd}^{2m-1}\frac{(-1)^{\frac{i-1}{2}}\cos^n(\frac{i\pi}{2m})}{m'\tan(\frac{i\pi}{4m})}\\
&= \sum_{i=1,odd}^{m-1}\frac{(-1)^{\frac{i-1}{2}}\cos^n(\frac{i\pi}{2m})}{m'\tan(\frac{i\pi}{4m})} + \sum_{i=1,odd}^{m-1}\frac{(-1)^{n+m-\frac{i-1}{2}}\cos^n(\frac{i\pi}{2m})}{m'\cot(\frac{i\pi}{4m})}\\
&= (1) + (2),
\end{aligned}$$

where $m^{'} = m - 1/2$. Let $u(m)$ be the positive integer such that

(i) $u(m) < m - 1$;

(ii) $u(m) = o(m)$, i.e. $\frac{u(m)}{m} \to 0$, as $m \to \infty$;

(iii) $u(m) \to \infty$, as $m \to \infty$.

Then,

$$(1) = \sum_{i=1,odd}^{u(m)} \frac{(-1)^{\frac{i-1}{2}} \cos^n(\frac{i\pi}{2m})}{m' \tan(\frac{i\pi}{4m})} + \sum_{i=u(m)+1,odd}^{m-1} \frac{(-1)^{\frac{i-1}{2}} \cos^n(\frac{i\pi}{2m})}{m' \tan(\frac{i\pi}{4m})}$$

$$= (3) + (4).$$

For (3), since $c_1 \to 1$ as $m \to \infty$, it follows from Taylor expansion,

$$\cos^n\left(\frac{i\pi}{2m}\right) = \left(1 - \frac{i^2\pi^2 c_1}{8m^2}\right)^{\frac{m^2}{c^2}} \to \exp\left(-\frac{i^2\pi^2}{8c^2}\right), \qquad \text{as } m \to \infty.$$

$$m^{'} \tan\left(\frac{i\pi}{4m}\right) \to \frac{i\pi}{4}, \qquad \text{as } m \to \infty.$$

Therefore, we have

$$(3) \to \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp\left(-\frac{(2i+1)^2\pi^2}{8c^2}\right), \qquad \text{as } m \to \infty.$$

It follows from Lemma 3.2.1,

$$|(4)| = \sum_{i=u(m)+1,odd}^{m-1} \frac{|(-1)^{\frac{i-1}{2}} \cos(\frac{i\pi}{4m}) \cos^n(\frac{i\pi}{2m})|}{m' \sin(\frac{i\pi}{4m})}$$

$$\leq \sum_{i=u(m)+1,odd}^{m-1} \frac{\cos^n(\frac{i\pi}{2m})}{m' \times \frac{2}{\pi} \times \frac{i\pi}{4m}}$$

$$\leq \sum_{i=u(m)+1,odd}^{m-1} \frac{4\cos^n(\frac{i\pi}{2m})}{i}$$

$$\leq \sum_{i=u(m)+1,odd}^{m-1} \frac{4}{i} \times \frac{K}{i^2} \to 0, \qquad \text{as } m \to \infty.$$

The first inequality holds, because $\sin(x) > 2x/\pi$, for $-\pi/2 < x < \pi/2$.

Next we show $(2) \to 0$, as $m \to \infty$.

$$
\begin{aligned}
|(2)| &= \sum_{i=1,odd}^{m-1} \frac{|(-1)^{n+m-\frac{i-1}{2}} \tan(\frac{i\pi}{4m}) \cos^n(\frac{i\pi}{2m})|}{m'} \\
&\leq \sum_{i=1,odd}^{m-1} \frac{\cos^n(\frac{i\pi}{2m})}{m'} \\
&\leq \frac{1}{m'} \sum_{i=1,odd}^{m-1} \frac{K}{i^2} \to 0, \qquad \text{as } m \to \infty.
\end{aligned}
$$

Hence, we have

$$
\boldsymbol{\xi}_0 \boldsymbol{N}_c^n \mathbf{1}' \to \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp\left(-\frac{(2i+1)^2 \pi^2}{8c^2} \right), \qquad \text{as } m \to \infty (n \to \infty).
$$

The proof is completed. □

3.3 Efficient approximation by eigenvalues and eigenvectors

We observe that under certain cases the fundamental matrix $\boldsymbol{N}$ is a Toeplitz matrix, the diagonals of which are of the same values. If the boundary is temporally homogeneous, we provide an efficient algorithm to approximate Eq. (3.9) by the eigenvalue-eigenvector decomposition of the matrix $\boldsymbol{N}$. Let $f(\theta)$ be the fourier series associated with the Toeplitz matrix $\boldsymbol{N}$. Our approximation requires the condition in Widom [89].

Condition A (Widom). Let $f(\theta)$ be continuous and periodic with period 2π. Let $\max f(\theta) = f(0) = M$ and let $\theta = 0$ be the only value of θ (mod 2π) for which this maximum is reached. Moreover, we assume that $f(\theta)$ is even, and has continuous derivatives up to the fourth order in some neighborhood of $\theta = 0$. Finally, let $\sigma^2 = -f''(0) \neq 0$.

Lemma 3.3.1 (Widom). *Let $\lambda_{[1]} \geq \lambda_{[2]} \geq \cdots \geq \lambda_{[\omega]}$ be the ordered eigenvalues of the symmetric Toeplitz matrix $\boldsymbol{N}$ of size ω. If condition A is satisfied, then we have*

$$\lambda_{[\nu]} = M - \frac{\sigma^2\pi^2\nu^2}{2\omega^2} + \mathcal{O}(\omega^{-3}), \text{ for some fixed } \nu, \tag{3.18}$$

where $M = f(0)$, $\sigma^2 = -f''(0)$ and $f(x)$ is the Fourier series associated with the Toeplitz matrix $\boldsymbol{N}$. f'' is the second derivative of f.

The proof can be found in Widom [89].

Lemma 3.3.2. *For temporally homogeneous boundaries, the symmetric Toeplitz matrix $\boldsymbol{N}$ defined by Eq. (3.3) is nonnegative definite, i.e. the smallest eigenvalue of $\boldsymbol{N}$ is nonnegative, and condition A is satisfied.*

Proof. The symmetric Toeplitz matrix $\boldsymbol{N}$ of size ω is of the form

$$\boldsymbol{N} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_\omega \\ c_1 & c_0 & c_1 & \cdots & c_{\omega-1} \\ c_2 & c_1 & c_0 & \cdots & c_{\omega-2} \\ \vdots & & \ddots & & \vdots \\ c_\omega & c_{\omega-1} & c_{\omega-2} & \cdots & c_0 \end{pmatrix}.$$

The associated Fourier series is $f(\theta) = \sum_{j=-\infty}^{\infty} c_j \exp(ij\theta)$. Let $m = \inf f(\theta)$ and it is known that $\lambda_{[\omega]} \geq m$, where $\lambda_{[\omega]}$ is the smallest eigenvalue of $\boldsymbol{N}$ (see, e.g., Kac et al. (1953)).

$$
\begin{aligned}
f(\theta) &= c_0 + 2\sum_{j=1}^{\infty} c_j \cos(j\theta) \\
&= \frac{C^{-1}}{\sqrt{2\pi}} \sum_{\ell\neq 0} (\ell^2 - 1)\exp\left(-\frac{\ell^2}{2}\right) + 2\sum_{j=1}^{\infty} \frac{C^{-1}}{\sqrt{2\pi}} \exp\left(-\frac{j^2}{2}\right)\cos(j\theta) \\
&= \frac{2C^{-1}}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (k^2 - 1 + \cos(k\theta)) e^{-\frac{k^2}{2}}.
\end{aligned}
$$

After differentiation, we know that the minimum value of $f(\theta)$ occurs at $\theta = \pi$. It is easy to see that $f(\pi) \geq 0$. In addition, It is easy to see that $f(\theta)$ reaches the maximum at $\theta = 0 \pmod{2\pi}$. It follows from the definition of $f(\theta)$ that Condition A is satisfied. □

The boundary crossing probabilities for temporally homogeneous two-sided boundaries can be efficiently computed by the following theorem.

Theorem 3.3.1. *Given the constant boundaries* $\pm h$ *and a large value* m ($\omega = 2m-1$), *we have*

$$
\begin{aligned}
\boldsymbol{\xi}_0 \boldsymbol{N}^{\lfloor m^2/h^2 \rfloor} \mathbf{1}' &= \sum_{i=1}^{2m-1} a_i \boldsymbol{\xi}_0 \boldsymbol{\eta}_{[i]}' \lambda_{[i]}^{\lfloor m^2/h^2 \rfloor} \\
&\leq \sum_{i=1}^{\ell-1} a_i \boldsymbol{\xi}_0 \boldsymbol{\eta}_{[i]}' \exp\left(-\frac{i^2\pi^2}{8h^2}\right) \\
&\quad + (2m-1)^2 (2m-\ell) \exp\left(-\frac{\ell^2\pi^2}{8h^2} + o(1)\right). \qquad (3.19)
\end{aligned}
$$

Proof. In our settings, it yields $f(0) = 1$ and $\sigma^2 = 1$. Since we have $(1 + x/n)^n = \exp(x)\,(1 - x^2/2n + \cdots)$ and substituting (3.18) into (2.7) yields, for large m and some fixed ℓ,

$$
\begin{aligned}
\sum_{i=\ell}^{2m-1} a_i \boldsymbol{\xi}_0 \eta_{[i]}^{'} \lambda_{[i]}^n &\leq (2m-1)^2 \sum_{i=\ell}^{2m-1} \lambda_{[i]}^n \\
&\leq (2m-1)^2 (2m-\ell) \lambda_{[\ell]}^n \\
&= (2m-1)^2 (2m-\ell) \left(1 - \frac{\ell^2 \pi^2 + o(1)}{2(2m-1)^2}\right)^n \\
&\leq (2m-1)^2 (2m-\ell) \left(1 - \frac{\ell^2 \pi^2 + o(1)}{8h^2 n}\right)^n \\
&\leq (2m-1)^2 (2m-\ell) \exp\left(-\frac{\ell^2 \pi^2}{8h^2} + o(1)\right).
\end{aligned}
$$

□

The cases for one-sided boundaries and asymmetric two-sided boundaries can be done in the same fashions with minor modifications of replacing the different boundaries. If we take $T = 1, h = 2$ and $m = 10000$, then we can estimate our boundary crossing probability by the first 30 largest eigenvalues and the associated eigenvectors with error smaller than 10^{-100}. Therefore, instead of multiplying the entire matrices, we speed up our calculation by using only the first $\ell - 1$ largest eigenvalues and the associated eigenvectors.

Chapter 4

Boundary Crossing Probability for One-dimensional Diffusion Processes

An Itô diffusion process is a solution of stochastic differential equation:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \tag{4.1}$$

where the drift $b(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ and diffusion coefficient $\sigma(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $W(t)$ is the standard Brownian motion. The solution to this stochastic differential equation exists uniquely under the following Lipschitz and growth conditions (Klebaner [43]): for some constants C_1 and C_2, and $x \in \mathbb{R}$, $t \in [0, T]$,

$$|b(t, x)| + |\sigma(t, x)| \leq C_1(1 + |x|),$$

and

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C_2(|x - y|).$$

4.1 Transformation to Brownian motion

In this section, we extend the method to a class of diffusion processes which can be transformed into functions of a Brownian motion. Two such examples are the Ornstein-Uhlenbeck (O-U) process and the Brownian bridge.

A well-known method for solving the boundary crossing problem for diffusion processes is to express them as functions of a Brownian motion, and the boundary crossing probability for diffusion processes is equivalent to a boundary crossing probability for Brownian motion with transformed time interval and boundaries. There are a number of papers in the literature concerning the transformation from diffusion processes to a Brownian motion. The one-to-one transformation of the transition probability density functions between diffusion processes described by Kolmogorov's backward equation was first posed by Kolmogorov [45]. Cherkasov [18] established a class of diffusion processes transformed into Brownian motion through one to one transformation of the transition probability density functions. Bluman [9] and Bluman and Shtelen [8] extended Cherkasov's [18] result to a wider class of diffusion processes. Wang and Pötzelberger [85] established a class of diffusion processes which can be expressed as functionals of Brownian motion. It is also known that any time-homogeneous diffusion process can be transformed into a Brownian motion by using random time change and change of variables (see Klebaner [43], page 208). Diffusion processes without drift are transformed using random time change by Watanabe [86].

Using the following well-known results, we give two examples, the O-U process and the Brownian bridge, to illustrate the transformations.

1. Itô's formula:

   Let $X(t)$ be an Itô process, $f(t,x)$ be a twice differentiable function on $[0,\infty)\times\mathbb{R}$ and $Y(t) = f(t, X(t))$. Then we have

$$dY(t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma(t, X(t))^2 dt.$$

2. Time change:

   Let $f(t)$ be a continuous function and let $X(t)$ be a process governed by

$$dX(t) = f(t)dW(t).$$

   Then a Brownian motion $\tilde{W}(\tau_t)$ is a weak solution, where $\tau_t = \int_0^t f^2(s)ds$ and $g^{-1}(t) = \tau_t$ given $f(t) > 0$. If $X(0) = x_0$ then $\tilde{W}(t)$ starts at x_0.

Ornstein-Uhlenbeck Processes

Let $X(t)$ denote the O-U process satisfying

$$dX(t) = -\mu X(t)dt + \sigma dW(t),\ X(0) = 0.$$

Let $Y(t) = X(t)e^{\mu t}$ so that, by Itô's formula, we have

$$\begin{aligned} dY(t) &= e^{\mu t}dX(t) + \mu e^{\mu t}X(t)dt \\ &= \sigma e^{\mu t}dW(t). \end{aligned}$$

In turn, by time change, we have

$$Y(t) = \tilde{W}(\tau_t),$$

for some Brownian motion $\tilde{W}(t)$ and $\tau_t = \sigma^2(e^{2\mu t} - 1)/2\mu$. It follows

$$X(t) = e^{-\mu t}\tilde{W}(\tau_t).$$

Brownian Bridge

The Brownian bridge is a solution of

$$dX(t) = \frac{c - X(t)}{S - t}dt + dW(t), \text{ for } 0 \le t \le S.$$

This process is a transformed Brownian motion on $[0, S]$ with $X(0) = x_0$ and $X(S) = c$. In viewing the above equation, we define

$$Y(t) = \frac{X(t)}{S - t} - \frac{t}{S(S - t)}c,\ Y(0) = \frac{x_0}{S}.$$

Again, by Itô's formula, we obtain

$$\begin{aligned} dY(t) &= \frac{1}{(S - t)^2}(X(t) - c)dt + \frac{1}{S - t}dX(t) \\ &= \frac{1}{S - t}\left[\frac{X(t) - c}{S - t}dt + dX(t)\right] \\ &= \frac{1}{S - t}dW(t). \end{aligned}$$

By time change,

$$Y(t) - Y(0) = \tilde{W}(\tau_t),$$

for $\tau_t = \frac{t}{S(S-t)}$ and $g(t) = \tau_t^{-1} = \frac{tS^2}{1+tS}$. It yields (see also Klebaner [43])

$$X(t) = (S - t)\tilde{W}(\tau_t) + \frac{S - t}{S}x_0 + \frac{t}{S}c. \tag{4.2}$$

Then, we can compute the boundary crossing probability

$$
\begin{aligned}
&P(a(t) < X(t) < b(t),\, 0 \le t \le 1) \\
&= P(a(t) < (S-t)\tilde{W}(\tau_t) + \frac{S-t}{S}x_0 + \frac{t}{S}c < b(t),\, 0 \le t \le 1) \\
&= P(a^{'}(t) < \tilde{W}(t) < b^{'}(t),\, 0 \le t \le \tau_1) \\
&= \lim_{m\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2\tau_1/h^2 \rfloor} \boldsymbol{N}_i \right) \mathbf{1}^{'}, \qquad (4.3)
\end{aligned}
$$

where $a^{'}(t) = a(g(t))/(S-g(t)) - \frac{(S-g(t))x_0+g(t)c}{S}/(S-g(t))$ and $b^{'}(t) = b(g(t))/(S-g(t)) - \frac{(S-g(t))x_0+g(t)c}{S}/(S-g(t))$.

4.2 Poisson processes and jump diffusion processes

Brownian motoin or a diffusion process driven by a Brownian motion have an important property "continuity" which means that their sample paths are a.s. continuous. Typically, the geometric Brownian motion is used to model a firm's asset or the price of a stock. In credit risk analysis, first passage models (FPM) were introduced by Black and Cox [6] to model that a firm can possibly default at any time before the maturity date. Classic models, such as the Black and Scholes model, assume continuity of sample paths, which implies the price of a stock behaves continuously. However, it is not the case in the real world that the price of a stock behaves continuously.

A jump diffusion model combines a diffusion process and a jump process, for example Poisson processes, to capture the behavior of price with jumps as a notion of sudden, unpredictable and extreme move (see Cont and Tankov [20]). We give the definition for a Poisson process.

Definition 4.2.1. *A counting process* $\{N(t), t \geq 0\}$ *is a Poisson process having rate* $\lambda > 0$ *if*

(i) $N(0) = 0$*,*

(ii) the process has independent increments,

(iii) $P(N(t+h) - N(h) = n) = (\lambda t)^n e^{-\lambda t}/n!$ *with moment generating function (m.g.f.), denoted by* $M_{N(t+h)-N(h)}(s) = e^{\lambda t(e^s - 1)}$.

Sometimes, it is useful to use the alternative definition as follows.

Definition 4.2.2. *A counting process* $\{N(t), t \geq 0\}$ *is a Poisson process having rate* $\lambda > 0$ *if*

(i) $N(0) = 0$*,*

(ii) the process has stationary and independent increment,

(iii) $P(N(h) = 0) = 1 - \lambda h + o(h)$*,*

$P(N(h) = 1) = \lambda h + o(h)$*, and*

$P(N(h) \geq 2) = o(h)$.

A jump diffusion process is of the form (see, e.g., Kou and Wang [46])

$$X(t) = \sigma W(t) + \mu t + \sum_{i=1}^{N(t)} Y_i,$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion, $N(t)$ is a Poisson process and the jump sizes $Y_1, Y_2, \ldots$ are i.i.d. random variables which are independent of $W(t)$ and $N(t)$. In general, $W(t)$ can be replaced by a diffusion process given in Eq. (4.1). Two well-known jump diffusion models are Merton's and Kou's models where the logarithm of jump sizes are assumed to be normally distributed and a so-called double exponential distribution, respectively (see, e.g., Bayraktar and Xing [4]). Specifically, a double exponential distribution is given by

$$f(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{\{y<0\}},$$

where $p, q \geq 0$, $p+q=1$ represent the probabilities of upward and downward jumps and $\eta_1, \eta_2 > 0$.

As there are not many closed form results for boundary crossing probabilities for Brownian motion except for certain limited boundaries, there are also only few closed form results for boundary crossing probabilities for some special jump diffusion processes, for example when the jump sizes have double exponential distribution (Kou and Wang [46]) or when the jump sizes can have only nonnegative values (Blake and Lindsey [7]). Hence, for general jump sizes, a simple and general method for computing the boundary crossing probabilities for jump diffusion processes is required, and we provide the details of our unified approach, the finite Markov chain imbedding technique, to calculate the boundary crossing probabilities for general jump sizes.

4.2.1 Finite Markov chain approximation for compound Poisson processes

Let $N(t)$ be a Poisson process with rate λ and $\{Y_i\}$ a sequence of i.i.d. continuous random variables with density function $f(y)$. A compound Poisson process is defined by

$$Z(t) = \sum_{i=1}^{N(t)} Y_i, \tag{4.4}$$

and $Z(t) = 0$ if $N(t) = 0$. We assume the m.g.f. $M_{Y_1}(s)$ of Y_1 exists and is continuous at $s = 0$. Thus, the m.g.f. of $Z(t)$ is given by

$$E[e^{sZ(t)}] = E[(M_{Y_1}(s))^{N(t)}] = e^{\lambda t(M_{Y_1}(s)-1)}. \tag{4.5}$$

For each i, we define a discrete version of Y_i:

$$P(\hat{Y}_i = k\Delta x) = \int_{(k-0.5)\Delta x}^{(k+0.5)\Delta x} f(y)dy, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{4.6}$$

and it follows that $\hat{Y}_i \xrightarrow{\mathscr{D}} Y_i$ as $\Delta x \to 0$.

Remark 4.2.1. *If the distribution of random variables Y_i is a lattice distribution then Eq. (4.6) is no longer needed.*

Given an integer n, let $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ be an equal-spaced partition of $[0, 1]$ and $\Delta t = 1/n$. We define a sequence of Bernoulli random variables $\{N_{in}\}_{i=1}^n$ with $P(N_{in} = 0) = 1 - \lambda\Delta t$ and $P(N_{in} = 1) = \lambda\Delta t$, $i = 1, \ldots, n$. Then, we construct a sequence of independent discrete random variables $\{\hat{Z}_{in}\}_{i=1}^n$ by

$$\hat{Z}_{in} = \hat{Y}_i N_{in}. \tag{4.7}$$

For simplicity, we suppress the subscript n. Given any $t \in [0,1]$, a partial sum is then defined by

$$\hat{Z}_n(t) = \sum_{i=1}^{\lfloor nt \rfloor} \hat{Z}_i, \tag{4.8}$$

which is a Markov chain having transition probabilities give by

$$\begin{aligned} p(k|j) &= P(\hat{Z}_n(t+\Delta t) = k\Delta x | \hat{Z}_n(t) = j\Delta x) \\ &= P(\hat{Z}_{\lfloor nt \rfloor + 1} = (k-j)\Delta x) \\ &= P(\hat{Z}_{\lfloor nt \rfloor + 1} = (k-j)\Delta x | N_{\lfloor nt \rfloor + 1} = 0) P(N_{\lfloor nt \rfloor + 1} = 0) \\ &\quad + P(\hat{Z}_{\lfloor nt \rfloor + 1} = (k-j)\Delta x | N_{\lfloor nt \rfloor + 1} = 1) P(N_{\lfloor nt \rfloor + 1} = 1) \\ &= \begin{cases} 1 - \lambda\Delta t + P(\hat{Y}_{\lfloor nt \rfloor + 1} = 0)\lambda\Delta t & \text{if } k - j = 0, \\ P(\hat{Y}_{\lfloor nt \rfloor + 1} = (k-j)\Delta x)\lambda\Delta t & \text{otherwise .} \end{cases} \end{aligned} \tag{4.9}$$

Theorem 4.2.1. *Let $\hat{Z}_n(t)$ be defined as above, we have*

$$\hat{Z}_n(t) \xrightarrow{\mathscr{D}} Z(t), \ \textit{as } n \to \infty.$$

Proof. We prove this by using Lévy's continuity theorem. From the definition of $\hat{Z}_n(t)$, we have

$$\begin{aligned} M_{\hat{Z}_n(t)}(s) &= E[e^{s\hat{Z}_n(t)}] \\ &= E[e^{s\sum_{i=1}^{\lfloor nt \rfloor} \hat{Z}_i}] \\ &= (E[e^{s\hat{Z}_1}])^{\lfloor nt \rfloor}. \end{aligned}$$

Since N_1 and $\hat{Y}_1$ are independent, it follows that

$$E[e^{s\hat{Z}_1}] = EE[e^{s\hat{Z}_1}|N_1] = P(N_1 = 0)E[e^{s\cdot 0}] + P(N_1 = 1)E[e^{s\cdot\hat{Y}_1}]$$
$$= 1 - \lambda\Delta t + \lambda\Delta t M_{\hat{Y}_1}(s)$$
$$= 1 + \lambda\Delta t(M_{\hat{Y}_1}(s) - 1).$$

We know that $M_{\hat{Y}_1}(s)$ is the discrete version of $M_{Y_1}(s)$. From the assumption of existence of the m.g.f. of Y_1 and the sandwich theorem, it is easy to show that

$$M_{\hat{Y}_1}(s) \to M_{Y_1}(s), \text{ as } \Delta x \to 0.$$

Then,

$$M_{\hat{Z}_n(t)}(s) = E[e^{s\hat{Z}_n(t)}] = (E[e^{s\hat{Z}_1}])^{\lfloor nt \rfloor}$$
$$= \left[1 + \lambda\Delta t \left(M_{\hat{Y}_1}(s) - 1\right)\right]^{\lfloor nt \rfloor}$$
$$= \left[1 + \frac{\lambda t}{nt} \left(M_{\hat{Y}_1}(s) - 1\right)\right]^{\lfloor nt \rfloor}$$
$$\to e^{\lambda t(M_{Y_1}(s)-1)} = M_{Z(t)}(s), \quad \text{as } \Delta t \text{ and } \Delta x \to 0. \tag{4.10}$$

By assumption that $M_{Y_1}(s)$ is continuous at 0, the proof is completed. □

Remark 4.2.2. *It is for the convenience of the proof in Theorem 4.2.1 that we assume the existence of the m.g.f. of Y_1. In fact, we can prove it using characteristic function.*

Let $a_i = \lfloor a(t_i)/\Delta x \rfloor$ and $b_i = \lfloor b(t_i)/\Delta x \rfloor$, where $a(t)$ and $b(t)$ are the lower and upper continuous boundaries, respectively. Then the induced discrete boundaries for $\hat{Z}_n(t_i)$ are $a^*(i/n) = a_i \Delta x$ and $b^*(i/n) = b_i \Delta x$, $i = 1, \ldots, n$. Using the same argument in the proof of Theorem 3.1.2, we have

$$\left(\min_{0 \le i \le n} (\hat{Z}_n(t_i) - a^*(i/n)), \max_{0 \le i \le n} (\hat{Z}_n(t_i) - b^*(i/n)) \right)$$

$$\xrightarrow{\mathscr{D}} \left(\inf_{0 \le t \le 1} (Z(t) - a(t)), \sup_{0 \le t \le 1} (Z(t) - b(t)) \right).$$

Define an imbedded Markov chain $\{Y_n(i)\}_{i=0}^n$ on the state spaces

$$\Omega_i = \{j : a_i < j < b_i\} \cup \{\alpha_i\}, \quad i = 1, \ldots, n, \tag{4.11}$$

by collapsing the values of $\hat{Z}_n(t_i)$ greater than $(b_i - 1)\Delta x$ or smaller than $(a_i + 1)\Delta x$ into an absorbing state α_i, i.e.

$$Y_n(i) = \begin{cases} \hat{Z}_n(t_i)/\Delta x & \text{if } a^*(t_i) < \hat{Z}_n(t_i) < b^*(t_i), \\ \alpha_i & \text{otherwise .} \end{cases}$$

Thus, $\{Y_n(i)\}_{i=0}^n$ forms a non-homogeneous Markov chain having transition probabilities given by

$$P(Y_n(i) = k | Y_n(i-1) = j) = \begin{cases} p(k|j) & \text{if } j \ne \alpha_{i-1}, k \ne \alpha_i, \\ p_i(\alpha_i|j) & \text{if } j \ne \alpha_{i-1}, k = \alpha_i, \\ 1 & \text{if } k = \alpha_i, j = \alpha_{i-1}, \\ 0 & \text{if } j = \alpha_{i-1}, k \ne \alpha_i, \end{cases} \tag{4.12}$$

where the initial probability is $P(Y_n(0) = 0) \equiv 1$, $p(k|j)$ is given in Eq. (4.9) and

$$p_i(\alpha_i|j) = \sum_{k \ge b_i} p(k|j) + \sum_{k \le a_i} p(k|j).$$

From the FMCI technique, it yields the following theorem.

Theorem 4.2.2. *Let $Z(t)$ be a compound Poisson process as defined in Eq. (4.4). Then we have*

$$P(Z(t) \leq a(t) \text{ or } Z(t) \geq b(t), \text{ for some } t \in [0,1])$$

$$= 1 - \lim_{n\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i \right) \mathbf{1}',$$

where $\boldsymbol{N}_i, i = 1, \ldots, n$, are the fundamental matrices of the imbedded Markov chain $\{Y_n(i)\}$ with transition probabilities given in Eq. (4.12).

4.2.2 Finite Markov chain approximation for jump diffusion processes

Now let us return to our main problem: the boundary crossing probabilility for a jump diffusion process which is given by

$$P(X(t) \leq a(t) \text{ or } X(t) \geq b(t), \text{ for some } t \in [0,1]),$$

where $X(t)$ is given by

$$X(t) = \sigma W(t) + \mu t + \sum_{i=1}^{N(t)} Y_i, \tag{4.13}$$

and $\sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process. Without loss of generality, let $\sigma = 1$ and $\mu = 0$.

We start with a simple case where $Y_i, i = 1, \ldots, n$, are i.i.d. random variables with distribution given by

$$P(Y_1 = \pm 1) = 0.5. \tag{4.14}$$

In the previous section, we know that

$$\hat{Z}_n(t) \xrightarrow{\mathscr{D}} Z(t), \text{ as } n \to \infty.$$

However, in order for $\hat{W}_n(t)$ and $\hat{Z}_n(t)$ to be defined on the same space, we need to rewrite the distribution of Y_i as follows. Given a large number m, we can express the distribution of Y_i as

$$P(Y_i = \pm m\Delta x) = 0.5,$$

where $\Delta x = 1/m$. Then we can adapt the setting ($\Delta x = 1/m$) to construct the Markov chain $\hat{W}_n(t)$ which converges to Brownian motion. Let $\hat{X}_n(t) = \hat{W}_n(t) + \hat{Z}_n(t)$. Since $W(t)$ and $Z(t)$ are independent, we have

$$\hat{X}_n(t) \xrightarrow{\mathscr{D}} X(t), \text{ as } n \to \infty.$$

By the same token, we can generalize the above construction to a lattice random variable Y_i which takes values on $\{kd, k = 0, \pm 1, \ldots\}$ for some $d \neq 0$. Given a large integer m, let $\Delta x = d/m$ and Y_i now takes values on $\{km\Delta x, k = 0, \pm 1, \ldots\}$. A Markov chain $\hat{W}_n(t)$ can then be constructed using such Δx, hence, the convergence of $\hat{X}_n(t)$ to $X(t)$ still holds for a sequence of lattice random variables $\{Y_i\}$. Furthermore, for continuous random variables $Y_i's$, we can discretize each Y_i as Eq. (4.6), then it becomes a lattice random variable. Therefore, the following theorem holds for continuous or lattice random variables $Y_i, i = 1, \ldots, n$.

Theorem 4.2.3. *Let $\hat{W}_n(t)$ and $\hat{Z}_n(t)$ be defined as above, and keep $\Delta x^2 = \Delta t$. Then we have*

$$\hat{X}_n(t) = \hat{W}_n(t) + \hat{Z}_n(t) \xrightarrow{\mathscr{D}} X(t) = W(t) + \sum_{i=1}^{N(t)} Y_i, \quad \text{as } n \to \infty.$$

Next, we construct an imbedded Markov chain with a series of absorbing states associated with the boundaries. We define a two-dimensional non-homogeneous Markov chain $\{\boldsymbol{Y}_n(i)\}$ on the state spaces

$$\Omega_i = \{(j_1, j_2) : a_i < j_1 + j_2 < b_i\} \cup \{\alpha_i\}, \tag{4.15}$$

where $a_i = \lfloor a(t_i)/\Delta x \rfloor$ and $b_i = \lfloor b(t_i)/\Delta x \rfloor$ and α_i is an absorbing state representing the area outside the boundaries at time t_i. Thus, the transition probabilities of the imbedded Markov chain $\{\boldsymbol{Y}_n(i)\}$ are given by, for $(j_1, j_2) \in \Omega_{i-1} \backslash \alpha_{i-1}$ and $(k_1, k_2) \in \Omega_i \backslash \alpha_i$,

$$P(\boldsymbol{Y}_n(i) = (k_1, k_2) | \boldsymbol{Y}_n(i-1) = (j_1, j_2)) \tag{4.16}$$
$$= P(\hat{W}_n(t_i) = k_1 \Delta x | \hat{W}_n(t_{i-1}) = j_1 \Delta x) \times P(\hat{Z}_n(t_i) = k_2 \Delta x | \hat{Z}_n(t_{i-1}) = j_2 \Delta x),$$

where $P(\hat{W}_n(t_i) = k_1 \Delta x | \hat{W}_n(t_{i-1}) = j_1 \Delta x)$ is given in Eq. (3.3) and $P(\hat{Z}_n(t_i) = k_2 \Delta x | \hat{Z}_n(t_{i-1}) = j_2 \Delta x)$ is given in Eq. (4.9). We do not provide the transition probabilities entering or starting from the absorbing states, since they are not required for computing the boundary crossing probability. The absorption probability of the imbedded Markov chain is equivalent to the boundary crossing probability of $\{\hat{X}_n(t)\}$. The following theorem follows from the FMCI technique.

Theorem 4.2.4. *Let X(t) be a jump diffusion process given in Eq. (4.13). Then we have*

$$P(X(t) \leq a(t) \textit{ or } X(t) \geq b(t), \textit{ for some } t \in [0,1])$$
$$= 1 - \lim_{n \to \infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i \right) \boldsymbol{1}',$$

where $\boldsymbol{N}_i, i = 1, \ldots, n$, are the fundamental matrices of the imbedded Markov chain $\{Y_n(i)\}$ with transition probabilities given in Eq. (4.16).

Remark 4.2.3. *The two-dimensional imbedding procedure shown above is simple and intuitive but is not the best in terms of computational speed. The size of sample space would increase rapidly when m gets larger. We can resolve this problem using one-dimensional imbedding procedure. We know that*

$$\begin{aligned}\hat{X}_n(t) &= \hat{W}_n(t) + \hat{Z}_n(t) \\ &= \sum_{i=1}^{\lfloor nt \rfloor} (\hat{X}_i + \hat{Z}_i).\end{aligned}$$

It then follows that $\hat{X}_n(t)$ itself is a Markov chain. Incorporating the boundary functions, the imbedded Markov chain $\{Y_n(i)\}$ can then be defined on the state spaces

$$\Omega_i = \{j : a_i < j < b_i\} \cup \{\alpha_i\},$$

i.e.

$$Y_n(i) = \begin{cases} \hat{X}_n(t_i)/\Delta x & \text{if } a^*(t_i) < \hat{X}_n(t_i) < b^*(t_i), \\ \alpha_i & \text{otherwise} , \end{cases}$$

and has transition probabilities as the convolution of distributions of $\hat{X}_i$ and $\hat{Z}_i$, i.e., for $k \in \Omega_i \backslash \alpha_i$ and $j \in \Omega_{i-1} \backslash \alpha_{i-1}$,

$$P(Y_n(i) = k | Y_n(i-1) = j) = P(\hat{X}_i + \hat{Z}_i = (k-j)\Delta x), \tag{4.17}$$

where distributions of $\hat{X}_i$ and $\hat{Z}_i$ are given in Eq. (3.2) and Eq. (4.9), respectively. Detailed examples are given in Chapter 6.

Chapter 5

Boundary Crossing Probability for Two or Higher-Dimensional Brownian Motion

In this chapter, we extend our results to the high-dimensional Brownian motion. If the high-dimensional Brownian motion is a standard one (components are independent), then our method can be extended directly. If it is not a standard but correlated one, then it can be transformed into a standard one. On the other hand, we can also approximate the boundary crossing probability by constructing a Markov chain for a high-dimensional correlated Brownian motion without transformation. We consider the two-dimensional Brownian motion first, and the higher dimensional one would follow in a similar fashion.

The problem is given as follows. Let $\{\boldsymbol{X}(t), t \geq 0\}$ be a two-dimensional correlated Brownian motion with drift $\boldsymbol{\mu}$ and covariance matrix

$$t\boldsymbol{\Sigma} = t \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

where ρ is the constant correlation. For a given boundary $\partial B(t)$ of some nonempty compact convex set $B(t)$ containing the starting point of the Brownian motion, the boundary crossing probability is defined by

$$P(\boldsymbol{X}(t) \in \partial B(t), \text{ for some } t \in [0,1]).$$

It is known that $\boldsymbol{W}(t) = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X}(t) - \boldsymbol{\mu} t)$ is a standard two-dimensional Brownian motion. We can transform the process and the boundary by multiplying $\boldsymbol{\Sigma}^{-1/2}$ to $\boldsymbol{X}(t) - \boldsymbol{\mu} t$ and $B(t) - \boldsymbol{\mu} t$, yielding

$$P(\boldsymbol{X}(t) \in \partial B(t), \text{ for some } t \in [0,1]) = P(\boldsymbol{W}(t) \in \partial \tilde{B}(t), \text{ for some } t \in [0,1]),$$

where $\partial \tilde{B}(t) = \{\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{b} - \boldsymbol{\mu} t) : \boldsymbol{b} \in \partial B(t)\}$ and $\tilde{B}(t)$ is still a compact convex set (see, e.g., Walsh [83]).

In the first section, we focus on the boundary crossing probabilities for the standard high-dimensional Brownian motion (perhaps after transformation), which is the direct extension of the results in Chapter 3. In the next section, without transformation, we directly construct a Markov chain to approximate the boundary crossing probabilities for two-dimensional correlated Brownian motion. We denote a vector or a matrix in high dimensional by the bold one, e.g. $\boldsymbol{W}(t)$ denotes a two or higher-dimensional Brownian motion.

5.1 Standard Brownian motion

Let $\{\boldsymbol{W}(t), t \geq 0\}$ be a standard two-dimensional Brownian motion with drift $\mathbf{0}$ and covariance matrix

$$t\boldsymbol{\Sigma} = t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given a boundary $\partial B(t)$ of a compact convex set $B(t) \subseteq \mathbb{R}^2$, we denote by $B^o(t)$ the interior of $B(t)$. Let $||\cdot||$ be the Euclidean norm in $\mathbb{R}^2$, and $h = \sup_{0\leq t\leq 1}\sup\{||\boldsymbol{b}|| : \boldsymbol{b} \in \partial B(t)\}$. Choose a large integer m, we define $\Delta x = h/m$ and discretize $\mathbb{R}^2$ as $\mathbb{R}^2_m = \{(k_1\Delta x, k_2\Delta x),\ k_1, k_2 = 0, \pm1, \pm2, \ldots\}$. The time interval $[0,1]$ is correspondingly partitioned into n equal sub-intervals, preserving the scale relationship $\Delta x^2 = \Delta t$, i.e. $n = m^2/h^2$, or $n = \lfloor m^2/h^2 \rfloor$ if n is not an integer.

Let $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be an equal-spaced partition of $[0,1]$ with $t_i = i\Delta t$. In Chapter 3, the partial sum

$$\hat{W}_n(t) = \hat{X}_1 + \hat{X}_2 + \cdots + \hat{X}_{\lfloor nt \rfloor},$$

induced by discretizing a standard one-dimensional Brownian motion , converges in distribution to a standard one-dimensional Brownian motion. Obviously, the convergence in distribution holds for standard two or d-dimensional ($d > 2$) Brownian motion whose components are independent. Since the components are independent, it follows from the construction in Chapter 3 that the transition probabilities of the two-dimensional $\hat{\boldsymbol{W}}_n(t)$ are given by

$$\begin{aligned} p((k_1,k_2)|(j_1,j_2)) &= P(\hat{\boldsymbol{W}}_n(t+\Delta t) = (k_1\Delta x, k_2\Delta x)|\hat{\boldsymbol{W}}_n(t) = (j_1\Delta x, j_2\Delta x)) \\ &= p(k_1|j_1)p(k_2|j_2), \end{aligned} \tag{5.1}$$

where

$$p(k_i|j_i) = \begin{cases} C^{-1}\exp\left(-\frac{(k_i-j_i)^2}{2}\right) & \text{if } k_i - j_i \neq 0, \\ C^{-1}\sum_{\ell\neq 0}(\ell^2-1)\exp\left(-\frac{\ell^2}{2}\right) & \text{if } k_i - j_i = 0, \end{cases}$$

and $C = \sum_{\ell\neq 0}\ell^2\exp(-\ell^2/2)$ is the normalizing constant. Note that the adjustment in the probability at 0 is used to preserve the variance structure of Brownian motion and is not unique. See the family of distributions indexed by p in Remark 3.1.1.

Theorem 5.1.1. *Given $t \in [0,1]$ and $\Delta t = \Delta x^2$ ($n = m^2/h^2$), we have*

$$\hat{\boldsymbol{W}}_n(t) \xrightarrow{\mathscr{D}} \boldsymbol{W}(t), \quad \text{as } n \to \infty.$$

Proof. We show that the characteristic function of $\hat{\boldsymbol{W}}_n(t)$ converges to that of $\boldsymbol{W}(t)$ for all t. From the proof of Theorem 3.1.1, we know that

$$\begin{aligned} E[e^{i\boldsymbol{s}\hat{\mathbf{X}}_1'}] &= E[e^{i(s_1,s_2)(\hat{X}_{11},\hat{X}_{12})'}] = E[e^{is_1\hat{X}_{11}}]E[e^{is_2\hat{X}_{12}}] \\ &= \left(1 - \frac{s_1^2h^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^4}\right)\right) \times \left(1 - \frac{s_2^2h^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^4}\right)\right) \\ &= 1 - \frac{s_1^2h^2 + s_2^2h^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^4}\right). \end{aligned}$$

Thus,

$$\varphi_{\hat{\boldsymbol{W}}_{n(t)}}(\boldsymbol{s}) = \left(1 - \frac{(s_1^2 + s_2^2)h^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^4}\right)\right)^{m^2t/h^2} \to \exp\left(-\frac{t\boldsymbol{s}\boldsymbol{s}'}{2}\right), \text{ as } m \to \infty,$$

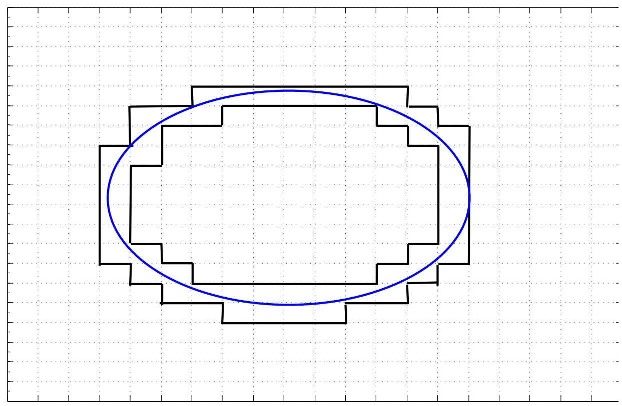
where $\exp\left(-t\boldsymbol{s}\boldsymbol{s}'/2\right)$ is the characteristic function of bivariate normal distribution with mean $\mathbf{0}$ and covariance matrix

$$t\boldsymbol{\Sigma} = t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

Imbedding procedure Given an open set $A \subseteq \mathbb{R}^2$, we introduce an oriented distance function $g(x,A) = d(x,A) - d(x,A^c)$, where $d(x,A) = \inf\{||x-y|| : y \in A\}$, A^c is the complement of A and $||\cdot||$ is the Euclidean norm in $\mathbb{R}^2$. The function g is continuous since d is continuous. It is easy to see that if $x \in A$ then $g(x,A) < 0$,

Figure 5.1: Inner and outer approximations.



and if $x \in A^c$ then $g(x, A) \geq 0$. In the sequel, we construct an imbedded Markov chain with absorbing states induced by the boundary $\partial B(t)$. For each t_i, we define the *inner* and *outer* approximations of $B(t_i)$ as follows. Let Q be the collection of all squares whose lengths are Δx and whose centres are in $\mathbb{R}^2_m$. Then, the inner and outer approximations of $B(t_i)$ are, respectively, given by

$$\underline{B}_i(\Delta x) = \cup\{q \in Q : q \subset B(t_i)\} \text{ and } \overline{B}_i(\Delta x) = \cup\{q \in Q : q \cap B(t_i) \neq \emptyset\},$$

and $\underline{B}_i(\Delta x) \subset B(t_i) \subset \overline{B}_i(\Delta x)$, $\underline{B}_i(\Delta x) \uparrow B(t_i)$ and $\overline{B}_i(\Delta x) \downarrow B(t_i)$ (see Figure 5.1). We can either use the inner or outer approximations. Here we choose the inner approximation and define the set $\hat{B}^o_i(t) = \{(\lfloor \frac{x_1}{\Delta x} \rfloor \Delta x, \lfloor \frac{x_2}{\Delta x} \rfloor \Delta x) : (x_1, x_2) \in \underline{B}_i(\Delta x)\}$. Thus, we can define a finite Markov chain $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ on the state spaces; $i = 1, \ldots, n$,

$$\Omega_i = \hat{B}^o_i(t) \cup \{\alpha_i\}, \tag{5.2}$$

where α_i stands for all values outside $\hat{B}^o_i(t)$. Then $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ forms a non-homogeneous

Markov chain having transition probabilities

$$P(\boldsymbol{Y}_n(i) = (k_1, k_2) | \boldsymbol{Y}_n(i-1)) = (j_1, j_2))$$

$$= \begin{cases} p((k_1,k_2)|(j_1,j_2)) & \text{if } (j_1,j_2) \in \Omega_{i-1}\backslash\alpha_{i-1},\ (k_1,k_2) \in \Omega_i\backslash\alpha_i, \\ p_i(\alpha_i|(j_1,j_2)) & \text{if } (j_1,j_2) \in \Omega_{i-1}\backslash\alpha_{i-1}, \\ 1 & \text{if } (j_1,j_2) = \alpha_{i-1},\ (k_1,k_2) = \alpha_i, \\ 0 & \text{if } (j_1,j_2) = \alpha_{i-1},\ (k_1,k_2) \in \Omega_i\backslash\alpha_i, \end{cases} \tag{5.3}$$

where $p((k_1,k_2)|(j_1,j_2))$ is given by Eq. (5.1), and $\Omega_0 = \{(0,0)\}$ and $P(\boldsymbol{Y}_n(0) = (0,0)) \equiv 1$. The probabilities $p_i(\alpha_i|(j_1,j_2))$ are not required for calculating the boundary crossing probabilities and are omitted. Again, all the transition probability matrices of the Markov chain $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ have the form

$$\boldsymbol{M}_i = \left(\begin{array}{c|c} p((k_1,k_2)|(j_1,j_2)) & p_i(\alpha_i|(j_1,j_2)) \\ \hline \boldsymbol{0} & 1 \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{N}_i & \boldsymbol{C}_i \\ \hline \boldsymbol{0} & 1 \end{array}\right),\ i = 1, 2, \ldots, n.$$

Together with Lemma 2.2.1, we are now in position to prove our main theorem.

Theorem 5.1.2. *Let $\boldsymbol{W}(t)$ be a standard two-dimensional Brownian motion. Given the boundary $\partial B(t)$ of a compact convex set $B(t)$, and assume $B^o(0)$ contains the starting point (origin) and $B^o(t)$ is not empty for all $t \in [0,1]$. Given $n = \lfloor m^2/h^2 \rfloor$, then*

$$P(\boldsymbol{W}(t) \in \partial B(t),\ \textit{for some } t \in [0,1]) = 1 - \lim_{m\to\infty} \boldsymbol{\xi}_0 \prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i \boldsymbol{1}', \tag{5.4}$$

where the transition probabilities in $\boldsymbol{N}_i$ are given in Eq. (5.3).

Proof. Let $h(x) = \sup_{0\le t\le 1} x(t)$. From the above definition of function g, it is not difficult to see that the following two sets are equal:

$$\left\{\max_{1\le i\le n} g(\hat{\boldsymbol{W}}_n(t_i), \underline{B}_i(\Delta x)) < 0\right\} \Leftrightarrow \left\{\max_{1\le i\le n} g(\hat{\boldsymbol{W}}_i(t), \hat{B}_i^o(t)) < 0\right\}.$$

Since $\boldsymbol{W}(t)$ is continuous and is starting within $B^o(0)$, $\sup_{0\le t\le 1} g(\boldsymbol{W}(t), B^o(t)) < 0$ means that $\boldsymbol{W}(t)$ stays inside $B(t)$ for all $t \in [0,1]$. Thus, due to the continuity of the probability measure and of the functions h and g, the boundary crossing probabilities can be obtained by

$$
\begin{aligned}
&P\left(\boldsymbol{W}(t) \in \partial B(t),\ \text{for some } t \in [0,1]\right) \\
&\quad = 1 - P\left(\boldsymbol{W}(t) \in B^o(t),\ \text{for all } t \in [0,1]\right) \\
&\quad = 1 - P\left(\sup_{0\le t\le 1} g(\boldsymbol{W}(t), B^o(t)) < 0\right) \\
&\quad = 1 - \lim_{m\to\infty} P\left(\max_{1\le i\le n} g(\hat{\boldsymbol{W}}_n(t_i), \underline{B}_i(\Delta x)) < 0\right) \\
&\quad = 1 - \lim_{m\to\infty} P\left(g(\hat{\boldsymbol{W}}_n(t_i), \hat{B}_i^o(t)) < 0, \text{ for all } 1 \le i \le n\right) \\
&\quad = 1 - \lim_{m\to\infty} P\left(\boldsymbol{Y}_n(1) \ne \alpha_1, \ldots, \boldsymbol{Y}_n(n) \ne \alpha_i\right) \\
&\quad = 1 - \lim_{m\to\infty} \boldsymbol{\xi}_0 \prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i \mathbf{1}'.
\end{aligned}
$$

For the second last equality, $g(\hat{\boldsymbol{W}}_n(t_i), \hat{B}_i^o(t)) < 0$ represents $\hat{\boldsymbol{W}}_n(t_i)$ stays inside $\hat{B}_i^o(t)$. In other words, it is equivalent to say $Y_i \ne \alpha_i$. The last equality follows from Lemma 2.2.1. The proof is completed. □

The above result is proved for two-dimensional Brownian motion, however, it is straightforward to extend to higher-dimensional Brownian motion. We leave the details to the reader. In fact, the results derived here are sufficient for general situations, even for the correlated Brownian motion (since it can be transformed into uncorrelated Brownian motion), however, we still provide some detail of the construction of a finite Markov chain for the correlated Brownian motion in the next section.

5.2 Correlated Brownian motion

The boundary crossing probabilities may also be obtained for high-dimensional correlated Brownian motion without transformation. A two-dimensional correlated Brownian motion $\boldsymbol{X}(t)$ has mean $\mathbf{0}$ and covariance matrix

$$t\boldsymbol{\Sigma} = t\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The construction of a finite Markov chain with absorbing states for a two-dimensional Brownian motion associated with boundaries is given. Following the same idea for one-dimensional Brownian motion, we discretize $\mathbb{R}^2$ and time interval $[0,1]$ in the same way and then construct a two-dimensional Markov chain preserving the variance and covariance structure of the two-dimensional correlated Brownian motion. Let $\hat{\boldsymbol{W}}_n(t) = \hat{\boldsymbol{X}}_1 + \cdots + \hat{\boldsymbol{X}}_{\lfloor nt \rfloor}$ where the distribution of the i.i.d. discrete random variables $\hat{\boldsymbol{X}}_\jmath = (\hat{X}_{1\jmath}, \hat{X}_{2\jmath})$, $\jmath = 1, \ldots, n$, is given by

$$P(\hat{\boldsymbol{X}}_1 = (x_1\Delta x, x_2\Delta x)) = \begin{cases} f_0 C^{-1} & \text{if } x_1 = 0 \text{ and } x_2 = 0, \\ f_1 C^{-1} & \text{if } x_1 = 0 \text{ and } x_2 = \pm 1, \\ f_2 C^{-1} & \text{if } x_1 = \pm 1 \text{ and } x_2 = 0, \\ f_3(x_1, x_2) C^{-1} & \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0, \end{cases} \tag{5.5}$$

where C is the normalizing constant, $f_3(x_1, x_2)$ is the density function of bivariate normal distribution with mean $\mathbf{0}$ and covariance $\boldsymbol{\Sigma}$, and f_0, f_1 and f_2 are given by

$$f_0 = \sum_{x_1 \neq 0} \sum_{x_2 \neq 0} f_3(x_1, x_2) \frac{x_1 x_2 (1 - \sigma_1^2 - \sigma_2^2)}{\sigma_1 \sigma_2 \rho} + \sum\sum_{x_1 \neq 0, \backslash\{(\pm 1, 0)\}} f_3(x_1, x_2) x_1^2$$

$$+ \sum\sum_{x_2 \neq 0, \backslash\{(0, \pm 1)\}} f_3(x_1, x_2) x_2^2 - \sum\sum_{\backslash\{(\pm 1, 0), (0, \pm 1), (0,0)\}} f_3(x_1, x_2),$$

$$f_1 = \sum_{x_1 \neq 0} \sum_{x_2 \neq 0} f_3(x_1, x_2) \frac{x_1 x_2 \sigma_1}{2 \sigma_2 \rho} - \sum\sum_{x_1 \neq 0, \backslash\{(\pm 1, 0)\}} f_3(x_1, x_2) \frac{x_1^2}{2},$$

$$f_2 = \sum_{x_1 \neq 0} \sum_{x_2 \neq 0} f_3(x_1, x_2) \frac{x_1 x_2 \sigma_2}{2 \sigma_1 \rho} - \sum\sum_{x_2 \neq 0, \backslash\{(0, \pm 1)\}} f_3(x_1, x_2) \frac{x_2^2}{2},$$

where $\backslash S = \{(x_1, x_2) : (x_1, x_2) \notin S \text{ and } x_1, x_2 \in \mathbb{Z}\}$ for some set S. Thus, from Eq. (5.5), we have $E[\hat{X}_{11}^2] = \Delta t \sigma_1^2$, $E[\hat{X}_{12}^2] = \Delta t \sigma_2^2$ and $E[\hat{X}_{11} \hat{X}_{12}] = \rho \sigma_1 \sigma_2$, and it follows that

$$Cov(\hat{\boldsymbol{W}}_n(t), \hat{\boldsymbol{W}}_n(t)) = \Delta t \lfloor nt \rfloor \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix} \to t \boldsymbol{\Sigma}, \quad \text{as } n \to \infty \text{ (or } m \to \infty).$$

The partial sum $\hat{\boldsymbol{W}}_n(t)$ is a Markov chain and the transition probabilities are given by

$$\begin{aligned} p((k_1, k_2) | (j_1, j_2)) &= P(\hat{\boldsymbol{W}}_n(t + \Delta t) = (k_1 \Delta x, k_2 \Delta x) | \hat{\boldsymbol{W}}_n(t) = (j_1 \Delta x, j_2 \Delta x)) \\ &= P(\hat{\boldsymbol{X}}_{\lfloor nt \rfloor + 1} = ((k_1 - j_1) \Delta x, (k_2 - j_2) \Delta x)), \end{aligned} \tag{5.6}$$

where $P(\hat{\boldsymbol{X}}_{\lfloor nt \rfloor + 1} = (x_1 \Delta x, x_2 \Delta x))$ is given by Eq. (5.5). Therefore, by such construction, we preserve the first two moments of the two-dimensional correlated Brownian motion.

The exact same imbedding procedure provided before can be used for the two-dimensional correlated dimensional Brownian motion. For a given boundary $\partial B(t)$

of a non-empty compact convex set $B(t)$, an imbedded finite Markov chain $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ is defined on the state spaces; $i = 1, \ldots, n$,

$$\Omega_i = \hat{B}_i^o(t) \cup \{\alpha_i\},$$

where $\hat{B}_i^o(t)$ is defined in the same way in Section 5.1 and α_i stands for all values outside $\hat{B}_i^o(t)$. Then the transition probabilities of $\{\boldsymbol{Y}_n(i)\}$ are given by

$$P(\boldsymbol{Y}_n(i) = (k_1, k_2) | \boldsymbol{Y}_n(i-1)) = (j_1, j_2))$$
$$= \begin{cases} p((k_1, k_2)|(j_1, j_2)) & \text{if } (j_1, j_2) \in \Omega_{i-1} \backslash \alpha_{i-1},\ (k_1, k_2) \in \Omega_i \backslash \alpha_i, \\ p_i(\alpha_i|(j_1, j_2)) & \text{if } (j_1, j_2) \in \Omega_{i-1} \backslash \alpha_{i-1}, \\ 1 & \text{if } (j_1, j_2) = \alpha_{i-1},\ (k_1, k_2) = \alpha_i, \\ 0 & \text{if } (j_1, j_2) = \alpha_{i-1},\ (k_1, k_2) \in \Omega_i \backslash \alpha_i, \end{cases} \tag{5.7}$$

where the probabilities, $p((k_1, k_2)|(j_1, j_2))$, are given in Eq. (5.6). Therefore, the boundary crossing probability for two-dimensional correlated Brownian motion to the boundary of a compact convex set can be approximated by our unified method

$$P(\boldsymbol{W}(t) \in \partial B(t), \text{ for some } t \in [0, 1]) = 1 - \lim_{m \to \infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i \right) \boldsymbol{1}', \tag{5.8}$$

where the transition probabilities in the fundamental matrices $\boldsymbol{N}_i$ are given in Eq. (5.7).

Remark 5.2.1. *We have shown how to construct a two-dimensional partial sum (Markov chain) which converges to a two-dimensional correlated Brownian motion. In general, the procedure can also be applied to high-dimensional Brownian motion but we have to adjust the distribution of the discrete partial sum to preserve the first two moments of the high-dimensional Brownian motion.*

Chapter 6

Examples and Numerical Results

6.1 One-dimensional processes

We give two examples to illustrate how to implement the Markov chain imbedding procedure for Brownian motion, one example in pricing corporate debt and one example where the eigenvalues and eigenvectors decomposition approximation is used. Numerical examples for three jump diffusion processes with ± 1, double exponential and standard normal jump sizes are also given.

Example 6.1.1 (Standard Brownian motion)**.** *Choose* $T = 1$, $b(t) = -a(t) = (1+t)$ *and* $m = 10$*, then we have the following quantities:*

$$h = 2, n = 25, \Delta x = 0.2 \text{ and } \Delta t = 0.04.$$

We have the state spaces from Eq. (4.11), for example for i from 4 to 5,

$$\Omega_4 = \{5, 4, \ldots, -4, -5, \alpha_4\},$$

$$\Omega_5 = \{5, 4, \ldots, -4, -5, \alpha_5\}.$$

Eq. (3.2) gives the transition probabilities, for example $j = 3$ and $k = 5$,

$$P(5|3) = \frac{C^{-1}}{\sqrt{2\pi}} e^{-\frac{2^2}{2}} = 0.054.$$

Therefore, we can obtain the transition probability matrices from Eq. (3.6) and calculate the boundary crossing probability by Theorem 3.1.2,

$$\begin{aligned} P(W(t) \leq -(1+t) \text{ or } W(t) \geq (1+t), \text{ for some } t \in [0,1]) &\approx 1 - \boldsymbol{\xi}_0 \left(\prod_{i=1}^{25} \boldsymbol{N}_i \right) \mathbf{1}' \\ &= 0.173057. \end{aligned}$$

Example 6.1.2 (Brownian Bridge)**.** *Let $T = 1$, $S = 5$, $c = x_0 = 0$ and $b(t) = -a(t) = \exp(t)$. Using the transformation in Eqs. (4.2) and (4.3), the new time interval is $[0, \tau_T = \frac{1}{20}]$, and boundaries $a'(t)$ and $b'(t)$ are given as follows:*

$$a'(t) = -\exp\left(\frac{tS^2}{1+tS}\right) / \left(S - \frac{tS^2}{1+tS}\right) = -\exp\left(\frac{25t}{1+5t}\right) / \left(\frac{5}{1+5t}\right),$$

$$b'(t) = \exp\left(\frac{tS^2}{1+tS}\right) / \left(S - \frac{tS^2}{1+tS}\right) = \exp\left(\frac{25t}{1+5t}\right) / \left(\frac{5}{1+5t}\right).$$

If we choose $m = 10$, then $n = 11$, $\Delta x = 0.067957$ and $\Delta t = 0.004545$. By our construction of the imbedded Markov chain, we have the state spaces, for example i from 1 to 2,

$$\Omega_1 = \{2, 1, 0, -1, -2, \alpha_1\},$$

$$\Omega_2 = \{3, 2, 1, 0, -1, -2, -3, \alpha_2\},$$

and the fundamental matrix is

$$\boldsymbol{N}_2 = \begin{pmatrix} 0.2420 & 0.4021 & 0.2420 & 0.0527 & 0.0042 & 0.0001 & 0.0000 \\ 0.0527 & 0.2420 & 0.4021 & 0.2420 & 0.0527 & 0.0042 & 0.0001 \\ 0.0042 & 0.0527 & 0.2420 & 0.4021 & 0.2420 & 0.0527 & 0.0042 \\ 0.0001 & 0.0042 & 0.0527 & 0.2420 & 0.4021 & 0.2420 & 0.0527 \\ 0.0000 & 0.0001 & 0.0042 & 0.0527 & 0.2420 & 0.4021 & 0.2420 \end{pmatrix}.$$

Then the boundary crossing probability is approximated by:

$$\begin{aligned} P(W(t) \leq a^{'}(t) \ \textit{or}\ W(t) \geq b^{'}(t),\ \textit{for some}\ t \in [0, \tfrac{1}{20}]) &\approx 1 - \boldsymbol{\xi}_0 \left(\prod_{i=1}^{11} \boldsymbol{N}_i\right) \mathbf{1}^{'} \\ &= 0.081654. \end{aligned}$$

The following Tables 6.1 to 6.3 are boundary crossing probabilities for Brownian motion and the O-U processes in the time interval $[0, 1]$ for various boundaries. Figure 6.1 is the plot of boundary crossing probabilities for the Brownian bridge with $T \in [0, 1]$, $x_0 = c = 0$, $S = 10$ and boundaries $\pm \exp(t)$.

Table 6.1: One-sided boundary crossing probabilities for Brownian motion.

Boundary	$m = 100$	500	1000	5000
$\exp(-t)$	0.558872	0.560512	0.560866	0.561233
$\frac{1}{2} - t\log\left(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}})\right)$	0.473422	0.478676	0.479266	0.479635
$\sqrt{t+1}$	0.193925	0.195480	0.195682	0.195935
$1+t^2$	0.145849	0.147430	0.147680	0.147900
$1+t-t^2$	0.253998	0.255644	0.255915	0.256153
$1+t$	0.088919	0.090061	0.090232	0.090379
$1+\sqrt{t}$	0.060808	0.061549	0.061664	0.061762
$\sin t + 1$	0.101419	0.102643	0.102824	0.102975

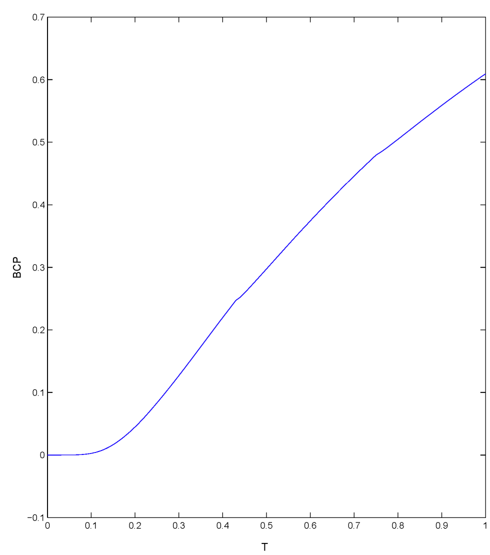
Table 6.2: Two-sided boundary crossing probabilities for Brownian motion.

Boundary	$m = 100$	500	1000	5000
$\pm(1+t)$	0.179427	0.180510	0.180656	0.180779
$\pm\exp(-t)$	0.984047	0.984366	0.984406	0.984439
$\pm(t^2+1)$	0.293783	0.295293	0.295512	0.295696
$\pm\sqrt{t+1}$	0.389771	0.391084	0.391259	0.391403
$\pm(1+t-t^2)$	0.509908	0.510977	0.511128	0.511254

Table 6.3: Two-sided boundary crossing probabilities for the O-U processes.

σ^2	0.5		2	
ρ	0.5	2	0.5	2
$\pm(1+t)$	0.006915	0.000185	0.445738	0.250645
$\pm\exp(-t)$	0.855233	0.745546	0.999629	0.999258
$\pm(t^2+1)$	0.029596	0.002957	0.603617	0.430781

Figure 6.1: Plot of boundary crossing probabilities for the Brownian bridge.



Next, we give a simple application to pricing corporate debt which requires the boundary crossing probability for geometric Brownian motion.

Example 6.1.3 (Pricing corporate debt)**.** *Let $X(t)$ be the asset value of a firm which follows geometric Brownian motion*

$$\frac{dX(t)}{X} = rdt + \sigma dW(t),$$

where r is risk-free interest rate. The firm issues corporate bond with face value F paid at maturity T. With some threshold $d(t)$, a firm is said to default if $X(t) \leq d(t)$ as shown in the Figure 6.2. The payoff at maturity is

$$P = F 1_{\{\tau > T, X(T) \geq F\}} + \alpha_1 X(T) 1_{\{\tau > T, X(T) < F\}} + \frac{\alpha_2 X(\tau)}{\nu(\tau, T)} 1_{\{\tau \leq T\}},$$

where $\nu(t,T) = e^{-r(T-t)}$. Hence, the price of the bond at time t is

$$\nu = E[e^{-r(T-t)} P].$$

For the threshold $d(t)$, Black and Cox (1976) suggested the boundary

$$d(t) = de^{-\alpha_0(T-t)} = 0.5e^{-0.5(1-t)}.$$

From our result, we have

$$P(\tau > T) = \lim_{m \to \infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2 T/h^2 \rfloor} \boldsymbol{N}_i \right) \mathbf{1}', \text{ and}$$

$$P(\tau > T, X(T) < F) = \lim_{m \to \infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{\lfloor m^2 T/h^2 \rfloor} \boldsymbol{N}_i \right) \boldsymbol{U}'(X(T) < F),$$

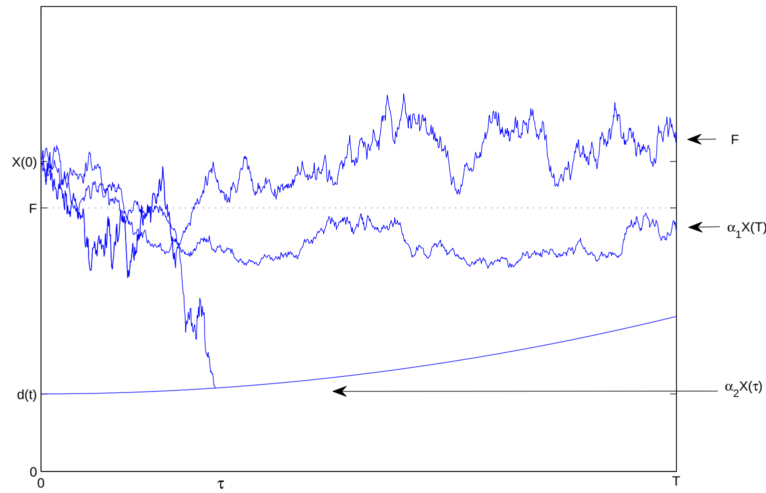


Figure 6.2: Diagram of the default of a firm.

where $\boldsymbol{U}'(X(T) < F)$ *is a column vector with ones corresponding the states associated with values of* $X(T)$ *less than* F*. Given* $m = 2000, T = 1, X(0) = 1, r = 0.1, \sigma = 0.6, F = 0.7$ *and* $d = \alpha_0 = \alpha_1 = \alpha_2 = 0.5$*, we obtain the price*

$$\nu \approx 0.498535,$$

which is close to the known exact price $\nu = 0.498695$*.*

Example 6.1.4 (Eigenvalues and eigenvectors)**.** *We consider the boundary crossing problem for a standard Brownian motion with one-sided boundary* $1 + t$ *and set* $m = 2000$*. Since the boundary is not time homogeneous, if we calculate the boundary crossing probability by Eq. (3.8), then we have to multiply the fundamental matrices one by one. However, it is equivalent that if we consider a diffusion process* $X(t) = W(t) - t$ *with one-sided boundary 1, then due to the time homogeneous boundary we can utilize Theorem 3.3.1 to efficiently approximate the boundary crossing probability by using the first 50 largest eigenvalues and the associated eigenvectors:*

$$P(W(t) \geq 1 + t, \textit{ for some } t \in [0,1]) \approx 1 - \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i \right) \mathbf{1}^{'} = 0.090322250840673,$$

$$P(W(t) - t \geq 1, \textit{ for some } t \in [0,1]) \approx 1 - \boldsymbol{\xi}_0 \tilde{\boldsymbol{N}}_i^n \mathbf{1}^{'} = 0.090365465385526$$

$$\approx \sum_{i=1}^{50} a_i \boldsymbol{\xi}_0 \boldsymbol{\eta}_{[i]}^{'} \lambda_{[i]}^n = 0.090361921767096.$$

Example 6.1.5 (Jump diffusion process with ± 1 jump size)**.** *Consider the following jump diffusion process:*

$$X(t) = W(t) + \sum_{i=1}^{N(t)} Y_i,$$

where $P(Y_i = \pm 1) = 0.5$.

For illustrative purpose, we choose $\lambda = 1$, $m = 4$ *and* $\Delta x = 1/4$, *then we have* $n = 16$. *Let the upper and lower boundaries be 1.5 and -1.5, respectively. We can rewrite the distribution of* Y_1 *as*

$$P(Y_i = \pm 4\Delta x) = 0.5.$$

We construct two partial sums, $\hat{Z}_n(t)$ *in Eq. (4.8) and* $\hat{W}_n(t)$ *in Eq. (3.1), to respectively approximate* $Z(t) = \sum_{i=1}^{N(t)} Y_i$ *and* $W(t)$. *Then, taking the boundaries* ± 1.5 *into account, the imbedded Markov chain* $\{Y_{16}(i)\}_{i=1}^{16}$ *is defined on the state spaces*

$$\Omega = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\} \cup \{\alpha\}.$$

From Eq.(4.17), the fundamental matrix is given by

$$\boldsymbol{N} = \begin{pmatrix} 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 & 0.0076 & 0.0017 & 0.0001 & 0.0000 & 0.0000 & 0.0000 \\ 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 & 0.0076 & 0.0017 & 0.0001 & 0.0000 & 0.0000 \\ 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 & 0.0076 & 0.0017 & 0.0001 & 0.0000 \\ 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 & 0.0076 & 0.0017 & 0.0001 \\ 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 & 0.0076 & 0.0017 \\ 0.0076 & 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 & 0.0076 \\ 0.0017 & 0.0076 & 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 & 0.0126 \\ 0.0001 & 0.0017 & 0.0076 & 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 & 0.0117 \\ 0.0000 & 0.0001 & 0.0017 & 0.0076 & 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 & 0.0523 \\ 0.0000 & 0.0000 & 0.0001 & 0.0017 & 0.0076 & 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 & 0.2270 \\ 0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0017 & 0.0076 & 0.0126 & 0.0117 & 0.0523 & 0.2270 & 0.3740 \end{pmatrix}.$$

The boundary crossing probability can be approximated using the unified formula and given by

$$1 - \boldsymbol{\xi}_0 \boldsymbol{N}^{16} \mathbf{1}' = 0.488740500953723.$$

Example 6.1.6 (Double exponential jump diffusion process)**.** *Let $X(t)$ be the double exponential jump diffusion process given as follows:*

$$X(t) = \sigma W(t) + \mu t + \sum_{i=1}^{N(t)} Y_i,$$

where $Y_i, i = 1, \ldots,$ have double exponential distribution and the density function is given by

$$f(y) = p\eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} \mathbf{1}_{\{y < 0\}}.$$

We shift the mean function (drift) of the Brownian motion to the one-sided boundary $b(t)$, and the boundary crossing probability becomes

$$P\left(W(t) + \sum_{i=1}^{N(t)} Y_i' > b'(t),\ \textit{for some } t \in [0,1] \right),$$

where $Y_i' = Y_i/\sigma$ *and* $b'(t) = (b(t) - \mu t)/\sigma$. *The density function of* Y_i' *is give by*

$$f(y) = \sigma(p\eta_1 e^{-\sigma\eta_1 y}\mathbf{1}_{\{y\geq 0\}} + q\eta_2 e^{\sigma\eta_2 y}\mathbf{1}_{\{y<0\}}).$$

Then the fundamental matrices of the imbedded Markov chain can be obtained by Eq. (4.16) and the boundary crossing probability is approximated by

$$1 - \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i \right) \mathbf{1}'.$$

Table 6.4 provides the boundary crossing probabilities with parameters $\mu = \pm 0.1$, $\sigma = 0.2$, $\lambda = 0.01, 3$, $p = q = 0.5$, $\eta_1 = 1/0.02$, $\eta_2 = 1/0.03$ *and* $b(t) = 0.3$ *selected from Kou and Wang [46].*

Table 6.4: Boundary crossing probabilities for the double exponential jump diffusion process.

	FMCI		Kou and Wang	
λ	$\mu = -0.1$	$\mu = 0.1$	$\mu = -0.1$	$\mu = 0.1$
0.01	0.0579	0.2601	0.0582	0.2606
3	0.0610	0.2553	0.0612	0.2558

Example 6.1.7 (Standard normal jump diffusion process)**.** *We consider the boundary crossing probability for a jump diffusion process with standard normal jump sizes, i.e.*

$$P\left(W(t) + \sum_{i=1}^{N(t)} Y_i > b(t), \ \textit{for some } t \in [0,1] \right),$$

where $Y_i, i = 1, \ldots,$ *are standard normal. We select* $\mu = 0$, $\sigma = 1$, $\lambda = 0.01, 1, 3, 5, 10$ *and* $b(t) = 1$. *Table 6.5 shows the performance and convergence rate of our method.*

It can be seen that our approximation provides satisfactory results even with small n. For comparison, we carry out a Monte Carlo simulation based on 50000 simulation runs using MATLAB®. Each simulated Brownian motion path consists of 10000 points between time interval $[0, 1]$.

Table 6.5: Boundary crossing probabilities for the jump diffusion process with standard normal jump sizes.

m	$\lambda = 0.01$	$\lambda = 1$	$\lambda = 3$	$\lambda = 5$	$\lambda = 10$
10	0.298587485	0.380280088	0.504065883	0.596313143	0.763681994
50	0.310852391	0.395245436	0.515662201	0.602644978	0.752653545
100	0.314353101	0.398456814	0.518330753	0.604913979	0.754009248
200	0.316266681	0.400165201	0.519746437	0.606143368	0.754874404
500	0.317467145	0.401222300	0.520619758	0.606908399	0.755449094
1000	0.317876022	0.401579934	0.520914649	0.607167696	0.755649529
Simulation	0.3137200000	0.399260000	0.519760000	0.606560000	0.752760000

6.2 Two-dimensional Brownian motion

Example 6.2.1 (Straight line). *Let $\{\boldsymbol{X}(t), t \geq 0\}$ be a two-dimensional correlated Brownian motion starting at $(-1, 0)$ with $\sigma_1 = 1, \sigma_2 = 2$ and $\rho = -0.5$, and $\partial B(t) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$ be the boundary. As it can be seen in Figure 6.3, the boundary crossing probability for two-dimensional correlated Brownian motion to the above boundary can be regarded as the boundary crossing probability for one-dimensional Brownian motion with another correlated one-dimensional Brownian motion being the one-sided boundary. Since the covariance matrix of $\boldsymbol{X}(t)$ is given by*

$$t\boldsymbol{\Sigma} = t \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix},$$

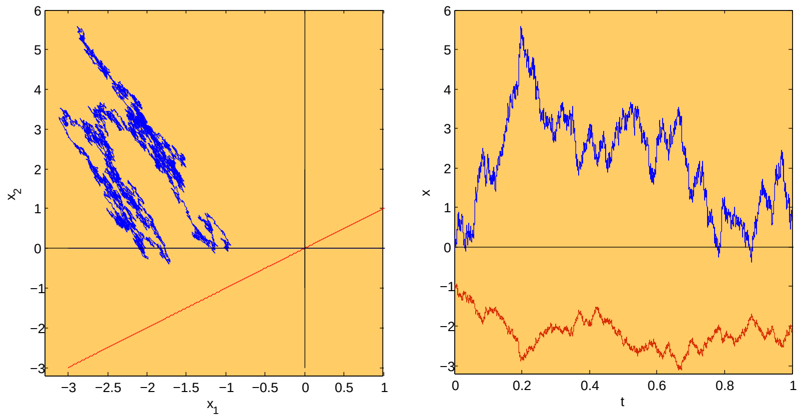


Figure 6.3: Linear boundary.

we transform $\boldsymbol{X}(t)$ *into* $\boldsymbol{W}(t)$ *by pre-multiplying* $\boldsymbol{\Sigma}^{-\frac{1}{2}}$, *and the boundary is accordingly transformed into* $\{(\tilde{x}_1, \tilde{x}_2)\}$ *satisfying*

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \doteq \begin{bmatrix} 1.1375 & 0.1984 \\ 0.1984 & 0.5422 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1.3360x_1 \\ 0.7406x_1 \end{bmatrix}.$$

Then the boundary after transformation becomes

$$\partial\tilde{B}(t) = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 : \tilde{x}_1 = \frac{1.3360}{0.7406}\tilde{x}_2 \right\}.$$

Therefore, choose $m = 100$, *by Theorem 5.1.2 we obtain the boundary crossing probability*

$$1 - \boldsymbol{\xi}_0 \boldsymbol{N}^n \mathbf{1}' = 0.690518396194389.$$

Example 6.2.2 (Cylinder)**.** *Let* $\boldsymbol{W}(t)$ *be a standard two-dimensional Brownian motion starting at the origin and* $\partial B(t) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ *be the boundary. Take* $m = 100$, *then we have* $n = 10^4$, $\Delta x = 0.01$ *and* $\Delta t = 10^{-4}$. *Since the boundary is temporally homogeneous, we can make use of Theorem 3.3.1 to* *estimate the boundary crossing probability using the first $\ell = 30$ largest eigenvalues and the associated eigenvectors. We define a finite Markov chain $\{Y_{10000}(i)\}_{i\geq 0}$ on the state space Ω_1 given by Eq. (5.2) and the transition probability matrix can be built according to Eq. (5.3). Hence, the estimated boundary crossing probability is*

$$P(W(t) \in \partial B(t),\ \textit{for some}\ t \in [0,1]) \approx \sum_{i=1}^{30} a_i \boldsymbol{\xi}_0 \boldsymbol{\eta}'_{[i]} \lambda^n_{[i]} = 0.908209579872515.$$

Nevertheless, the approximate boundary crossing probability by Eq. (5.4) is

$$P(\boldsymbol{W}(t) \in \partial B(t),\ \textit{for some}\ t \in [0,1]) \approx 1 - \boldsymbol{\xi}_0 \boldsymbol{N}^n \mathbf{1}' = 0.908209579872503,$$

which is very close to the above estimated boundary crossing probability.

Example 6.2.3 (Cone)**.** *Let $\boldsymbol{W}(t)$ be a standard two-dimensional Brownian motion starting at the origin and $\partial B(t) = \{(x_1, x_2) : x_1^2 + x_2^2 = 1 + t\}$ be the boundary. Since the boundary is a function of t, the imbedded Markov chain is not homogeneous, and the boundary crossing probability is obtained by multiplying the fundamental matrices which might not be square or might not be of the same sizes. Take $m = 50$, and for each t_i, collect the nodes inside the boundary as the states of the fundamental matrices $\boldsymbol{N}_i$ which can be constructed by Eq. (5.3). Then the boundary crossing*

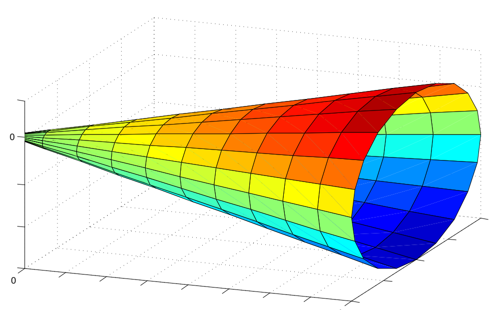


Figure 6.4: Cone boundary.

probability can be approximated by

$$P(\,\boldsymbol{W}(t) \in \partial B(t),\ \textit{for some}\ t \in [0,1]) \approx 1 - \boldsymbol{\xi}_0 \prod_{i=1}^{n} \boldsymbol{N}_i \mathbf{1}^{'} = 0.849959173588250.$$

Chapter 7

Summary and Discussion

In this thesis, we provide a unified approach to calculate the boundary crossing probabilities for one and high-dimensional Brownian motion and related stochastic processes including a class of diffusion processes and jump diffusion processes. We also introduce Y-channel boundaries which has not been studied yet in the literature. The entire content of this thesis is based on the sole idea that the crossing probability is treated as the absorption probability of a finite Markov chain; i.e. we consider the areas outside the boundary as a set of absorbing states, and the boundary crossing probability is cast as the limiting absorption probability. The method we use to calculate the absorption probability is the FMCI technique. Although the concept is simple, the construction of finite Markov chains is not trivial.

For the one-dimensional Brownian motion case, we construct a family of discrete distributions, based on which the partial sums converge to a Brownian motion. Interestingly, the partial sum in our family would reduce to a simple random walk when the parameter $p \to \infty$. Based on the results of Nagaev [61, 62] and Borovkov and Novikov [13], we show the error bound for our approximation of boundary crossing probability for Brownian motion to Lipschitz boundaries is with order $\mathcal{O}(1/m)$. Also we show that the error bound for boundary crossing probability to Y-channel boundary is with order $\mathcal{O}(1/m)$.

A well-known result is that the boundary crossing probability for Brownian motion for non-linear boundary $\frac{1}{2}-t\log(\frac{1}{4}(1+\sqrt{1+8e^{-1/t}}))$ is 0.479749 (see Daniels [22]). From our unified formula, we obtain the boundary crossing probability 0.47974239 using $m = 50000$. For one-sided linear boundary $(1+t)$, our approximate boundary crossing probability is 0.09041797, using $m = 50000$, which is close to the exact value 0.09041777 using the formula given in Robbins and Siegmund [69]. For two-sided linear boundary $\pm(1+t)$, the exact boundary crossing probability is 0.180812 by Anderson's [2] formula and our approximate boundary crossing probability is 0.180803 using $m = 20000$. The errors of those cases are negligible. Clearly, higher accuracy requires large m. The CPU times with PC for computing the boundary crossing probabilities using $m \leq 5000$ are negligible. There are still various practical ways to improve the accuracy. For example, instead of rounding down, the discrete boundaries $a_i = \lfloor a(t_i)/\Delta x \rfloor$ and $b_i = \lfloor b(t_i)/\Delta x \rfloor$, $i = 1, 2, \ldots, n$, can be round off to the nearest integer. This can generally increase the accuracy, especially for small m.

Diffusion and jump diffusion processes are considered in Chapter 4. Many known diffusion processes can be transformed into a function of a Brownian motion using Itô's formula and time change, for example the geometric Brownian motion, O-U processes and the Brownian bridge. After transformation, the boundary crossing problem for diffusion processes reduces to that for Brownian motion. Due to the rapid change in the market, the jump diffusion process is used to capture a sudden change. The jump diffusion process is the sum of a continuous-state Markov process and a jump process, such as Brownian motion and Poisson processes, thus it is approximated by the sum of two Markov chains. The easiest way to imbed the sum of two Markov chains is to form a 2-tuple of the two Markov chains. The 2-tuple is no doubt a Markov chain, but the sizes of the state spaces would increase rapidly. Instead, it is beneficial to imbed into a one-dimensional Markov chain in the consideration of computational speed. Since the sum of two partial sums can be viewed as another partial sum, hence the one-dimensional imbedded Markov chain can be defined and the sizes of the state spaces can be significantly reduced.

In general, the sum of two Markov chains is not necessarily a Markov chain. Although we did not mention that but, in fact, in our construction the sum of the two Markov chains in Theorem 4.2.3 is a Markov chain. The sizes of the state spaces can be dramatically reduced if we directly imbed the sum of the two Markov chains.

The results are extended directly to the higher-dimensional Brownian motion. The plausible difficulty is when the components of the higher-dimensional Brownian motion are correlated, but this can be resolved by transforming a correlated one to a standard one.

Our results require the multiplication of fundamental matrices, which is not difficult to compute even for large m using a modern computer. The sizes of transition probability matrices for high-dimensional cases are much larger than that for one-dimensional cases. We observe that the fundamental matrix of an imbedded Markov chain is Toeplitz if the boundary is time homogeneous. Hence, we provide an algorithm to calculate the boundary crossing probability by only using the first ℓ (≈ 30) largest eigenvalues and the associated eigenvectors. There are built-in functions in many computing softwares, for example MATLAB®, for finding the first few largest or smallest eigenvalues and eigenvectors. Rather than multiplying the entire matrices, the computation for finding the largest eigenvalues and eigenvectors of a matrix usually requires less CPU times when n is large.

One last thing we would like to mention is the flexibility of the FMCI technique. The FMCI technique are not only used to numerically calculate the boundary crossing probability, but it can be used to recover certain identities, such as Erdös and Kac's result given in Section 3.2. It can also be seen in pricing corporate debt example in Section 6.1 that other than default probability, we can also calculate the probability that Brownian motion stays at a specific region at time T given that it does not cross the boundaries, by simply replacing the column vector $\mathbf{1}'$ in Eq. (3.9) with a vector with ones in the positions of the states associated with the specific region and zeros elsewhere. Therefore, with minor modifications, the FMCI technique can be used to obtained various probabilities under different settings.

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