Unique Determination of Quadratic Differentials by their Admissible Functions

by

Hye Seon, Kim

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Abstract

Let f be an analytic and one-to-one function on the unit disk such that $f(0) = 0$. Let $Q(w)dw^2$ be a quadratic differential. Suppose that f maps into the complex plane or the unit disk minus analytic arcs $w(t)$ satisfying $Q(w(t))(\frac{dw}{dt})^2 < 0$. We are interested in the question: if Q is unknown but of a specified form, does f determine the quadratic differential Q uniquely? Our main result is that for functions mapping into the unit disk and quadratic differentials with a pole of order 4 at the origin, the quadratic differential is uniquely determined up to exceptional cases. This question arises in the study of extremal functions for functionals over classes of analytic one-to-one maps.

Acknowledgements

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Dedication

To my parents, who I love the most in the world.

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Chapter 1

Introduction

One branch of complex analysis is the study of univalent functions on a fixed domain. One main way of studying the class of univalent functions is to study an extremal problem, which is the problem of maximizing or minimizing a specified functional over the class of functions. Much of the work on extremal problems for univalent functions was motivated by the Bieberbach conjecture, which has been studied by many mathematicians since it was posed in 1916.

It is a general principle that extremal functions are associated with quadratic differentials. We introduce some main concepts we should know before we discuss our main problem, namely quadratic differentials, admissibility, and extremality.

1) A quadratic differential is a formal expression $Q(w)dw^2$, where Q is meromorphic.

2) A one-to-one and analytic function f is admissible for a quadratic differential $Q(w)dw^2$ if f maps onto the complement of analytic arcs $w(t)$ satisfying $Q(w(t))\dot{w}(t)^2 < 0$.

3) For a functional ϕ over a family of analytic functions A, an extremal problem is the problem of finding the supremum of $Re\{\phi(f)\}$ over A. A function $f \in A$ is extremal if the supremum over A is attained by f. In this thesis, A is a class of one-to-one analytic functions, either

from the unit disk into itself or the unit disk into the complex plane.

Now consider the following two statements, where the above concepts appear.

A) A function is admissible for some quadratic differential.

B) A function is extremal for some functional.

There have been many studies about the relation between those two statements, especially by Schiffer, Teichmüller, Schaeffer and Spencer, and Pfluger. Schiffer's theorem says that an extremal function is admissible for some quadratic differential, and Teichmüller's theorem says that an admissible function is extremal for some functional. In either case, the relation of the admissibility and the extremality is not explicit. In the case of Schiffer's theorem, the quadratic differential depends on the initial coefficients of the unknown extremal function, and in the case of Teichmüller's theorem, the function is extremal only over a restricted class. Pfluger solved this problem in a certain case by combining Schiffer's and Teichmüller's methods. He showed: C) an admissible function is extremal for one and only one functional of order 3 with some exceptions. But it is not known whether Pfluger's results extend to all orders of functionals.

The above statement C is the general context for a specific problem that arises as part of the whole proof of C: does an admissible function determine a unique quadratic differential? This is the question we are mostly interested in. Demonstrating the uniqueness is a step towards understanding the general correspondence between A and B, and perhaps towards extending Pfluger's results.

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$, and let $\mathbb{C}$ denote the complex plane. There are two cases in our main problem, which we will state now. The first case, in which functions map into $\mathbb{D}$, will be called the unit disk case, and the second case, in which functions map into $\mathbb{C}$, will be called the complex plane case.

1) The main problem - Unit disk case:

For a one-to-one analytic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ such that $f(0) = 0$ and $f'(0) > 0$,

- if the function f is admissible for a quadratic differential on $\mathbb{D}$, does f determine the quadratic differential uniquely?

Until now, there was no known result about this problem in the unit disk case. We prove that the function in the main problem determines a unique quadratic differential with some exceptions in the case of order 3 (we will define the order of a quadratic differential precisely later, which depends on the order of the pole at the origin). We also give a partial result in the case of order 4 using the same method as one we use in the case of order 3.

2) The main problem - Complex plane case:

For a one-to-one analytic function $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that $f(0) = 0$ and $f'(0) = 1$,

- if the function f is admissible for a quadratic differential on $\mathbb{C}$, does f determine a unique quadratic differential?

It is known that this is true with some exceptions in some certain cases. The order 3 case is quite simple, and it has been proved by many mathematicians, e.g. Pfluger [6]. Schaeffer and Spencer [10] proved it for the order n case when $n - 1$ is prime number. However, it is not known if it is true for any order n. Although they are known, we applied our method (that we use in the unit disk case) to the case of order 3 and 4 hoping to generalize to higher orders, perhaps order 5. As a result, we give a new proof in the case of order 3 and a partial result in the case of order 4.

This thesis is divided into 4 chapters. In Chapter 2, we give some preliminaries that are essential to talk about the following chapters. Next Chapter 3 is devoted to background material and known results. In the last chapter, we give our main results.

Chapter 2

Preliminaries

To discuss our main problem, we need some preliminaries first. In the first section, we will state some of the classes of functions which appear in this thesis. In Section 2.2, we will define the trajectories of a quadratic differential. In Section 2.3, we will give definitions and properties of functionals and extremal problems, since our main problem is closely related to extremal problems.

2.1 Some classes of functions

Definition 1. *$\mathcal{A}$ is the class of analytic functions in the unit disk, endowed with the topology of uniform convergence on compact subsets of the unit disk.*

Thus from now on, when we speak of the class $\mathcal{A}$, we assume that the above topology is given on the class $\mathcal{A}$. We introduce a class that is a Banach space.

Definition 2. *$\mathcal{N}$ is the class of functions f analytic in the unit disk $\mathbb{D}$ for which*

$$\|f\| = \sum_{n=1}^{\infty} 2^{-n} max \left\{ |f(z)| : |z| \leqq 1 - \frac{1}{n} \right\} < \infty.$$

Note that convergence in $\mathcal{N}$ implies convergence in $\mathcal{A}$, but not the reverse.

Definition 3. *$\mathcal{S}$ is the class of functions f analytic and univalent (one-to-one) in the unit disk $\mathbb{D}$ with the conditions $f(0) = 0$ and $f'(0) = 1$.*

Thus if $f \in \mathcal{S}$, then f has the power series expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1.$$

The following theorem is very useful when we work over the class $\mathcal{S}$ (see Duren [1] for the proof).

Theorem 1 (Growth theorem)**.** *If $f \in \mathcal{S}$, then*

$$\frac{|z|}{(1+|z|)^2} \leqq |f(z)| \leqq \frac{|z|}{(1-|z|)^2}, \quad |z| < 1.$$

It follows from this theorem that the class $\mathcal{S}$ is contained in $\mathcal{N}$.

Now we will state and prove a theorem that says that the class $\mathcal{S}$ is a compact normal family, which assures the existence of extremal functions for continuous functionals (we will talk about functionals and extremal functions in the following sections).

Theorem 2. *The class $\mathcal{S}$ is normal, and compact in the $\mathcal{A}$-topology.*

Proof. It directly follows from the Growth theorem (Theorem 1) that $f \in \mathcal{S}$ is uniformly bounded on compact subsets of $\mathbb{D}$. In other words, $\mathcal{S}$ is locally bounded on $\mathbb{D}$; therefore $\mathcal{S}$ is normal by Montel's theorem (see Duren [1]). let us prove that $\mathcal{S}$ is compact. Assume that for a sequence $f_n \in \mathcal{S}$, f_n converges to f uniformly on compact subsets of $\mathbb{D}$. We claim that $f \in \mathcal{S}$. By the above assumption, f is either univalent or constant by Hurwitz's theorem (see Duren [1]). But if f is constant, it contradicts the fact that $f'(0) = 1$ since $f_n'(0) \to f'(0)$ and $f_n'(0) = 1$. Thus f is univalent. The conditions $f(0) = 0$ and $f'(0) = 1$ are preserved under uniform convergence. Therefore $f \in \mathcal{S}$, which implies that $\mathcal{S}$ is compact. $\square$

Now, we will give the Koebe function as an example of the function of the class $\mathcal{S}$ because it will appear several times in this thesis.

Example 1 (Koebe function)**.** *The Koebe function* $k(z) = \frac{z}{(1-z)^2}$ *clearly satisfies the conditions* $k(0) = 0,\ k'(0) = 1$ *and it is also analytic in the unit disk. let us check the univalence. For* $z_1,\ z_2 \in \mathbb{D}$*, assume* $k(z_1) = k(z_2)$*. Then*

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2} \quad \Leftrightarrow \quad (z_1 - z_2)(1 - z_1 z_2) = 0 \quad \Leftrightarrow \quad z_1 = z_2 \text{ or } z_1 z_2 = 1$$

But it is not possible to have $z_1 z_2 = 1$ *since* z_1 *is never equal to* $\frac{1}{z_2}$ *for* $z_1,\ z_2 \in \mathbb{D}$*. Thus* $z_1 = z_2$*. This shows the one-to-one property. Thus the Koebe function belongs to the class* $\mathcal{S}$*. To find the range of the Koebe function* $k(z) = \frac{z}{(1-z)^2}$*, we rewrite* $k(z)$ *as*

$$k(z) = \frac{1}{4}\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\}.$$

Then $k(z) = (f_4 \circ f_3 \circ f_2 \circ f_1)(z)$*, where* $f_1(z) = \frac{1+z}{1-z}$*,* $f_2(z) = z^2$*,* $f_3(z) = z-1$ *and* $f_4(z) = \frac{1}{4}z$*. It is easily shown that* $f_1(z)$ *is a linear fractional transformation that maps the unit disk onto the right half plane. Figure 2.1 shows the steps of getting the range of the Koebe function by composition. Therefore we see that the Koebe function* $k(z)$ *maps the unit disk onto the complement of the part of the real axis extending from* $-\frac{1}{4}$ *to* $-\infty$*.*

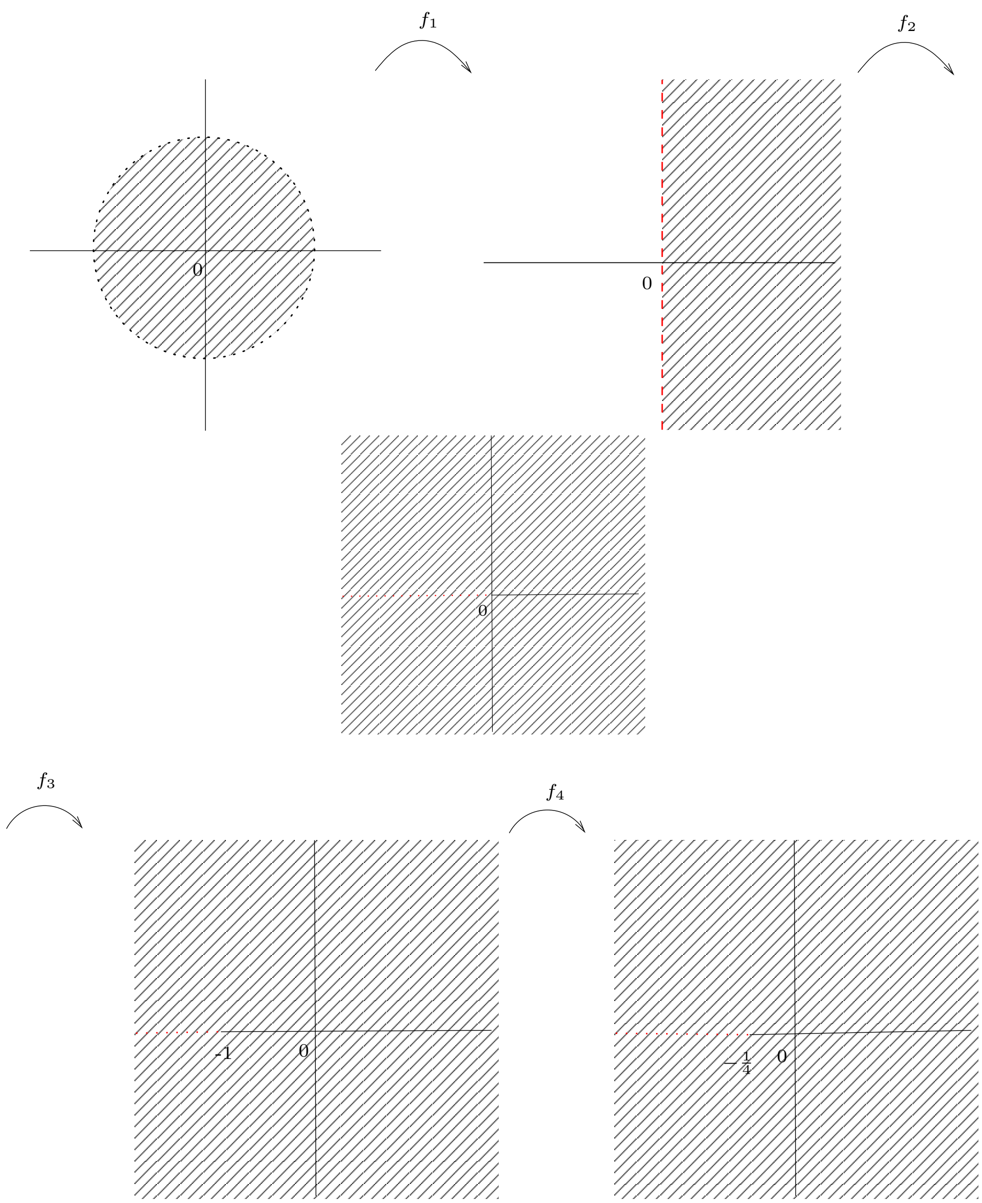
Example 2 (Rotation)**.** *It is easily verified that the rotation* $e^{-i\theta} f(e^{i\theta} z)$ *of the function* $f(z)$ *of the class* $\mathcal{S}$ *by* $-\theta$ *again belongs to the class* $\mathcal{S}$*.*

Definition 4. $\mathcal{B}$ *is the class of analytic and univalent (one-to-one) functions* $f : \mathbb{D} \longrightarrow \mathbb{D}$ *such that* $f(0) = 0$ *and* $f'(0) > 0$*.*

Example 3 (Pick function)**.** *The Pick function* $p_t(z)$ *defined by*

$$p_t(z) = (k_t^{-1} \circ k)(z), \quad \text{for some} \quad t \in (0, \infty),$$

Figure 2.1: Koebe function



where $k(z)$ is the Koebe function and $k_t(z) = e^t k(z)$, is analytic and univalent since the Koebe function $k(z)$ is analytic and univalent. To see the range of $p_t(z)$, we use Example 1. From that example,

$$k_t^{-1}(z) = (f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ f_{4,t}^{-1})(z),$$

$$\textit{where} \quad f_1^{-1}(z) = \frac{z-1}{z+1}, \; f_2^{-1}(z) = \sqrt{z}, \; f_3^{-1}(z) = z+1 \quad \textit{and} \quad f_{4,t}^{-1}(z) = \frac{4}{e^t} z.$$

Here, f_1^{-1} is a linear fractional transformation that maps the right half plane onto the unit disk. Figure 2.2 shows the steps of getting the range of the Pick function by composition $p_t = k_t^{-1} \circ k = f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ f_{4,t}^{-1} \circ k$. Therefore $p_t(z)$ maps the unit disk onto the unit disk minus a straight slit from -1 to $\frac{\sqrt{1-e^{-t}}-1}{\sqrt{1-e^{-t}}+1}$. In conclusion, the Pick function $p_t(z)$ belongs to the class $\mathcal{B}$.

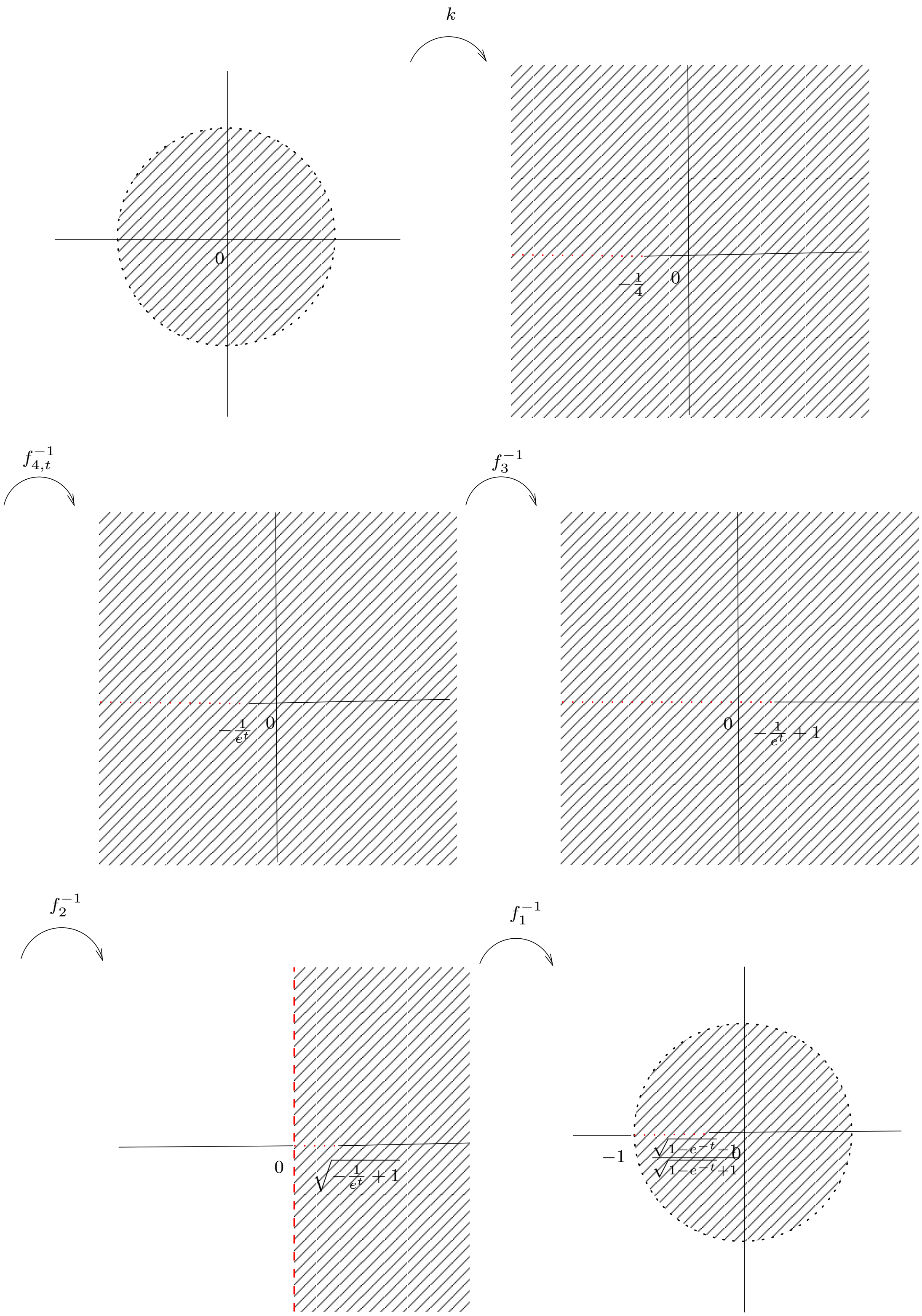
2.2 Trajectories of a quadratic differential

Quadratic differentials are important in function theory and they appear in many theorems such as Schiffer's theorem and Teichmüller's theorem (we will cover these in Chapter 3). The reason is that generally, extremal functions map onto the complement of the trajectories of some quadratic differential. Thus we will briefly study what a quadratic differential is in the beginning of this section, and then we will talk about the trajectories of a quadratic differential.

Definition 5. *A quadratic differential on a domain Ω is a formal expression $Q(z)dz^2$, where Q is meromorphic on Ω. In this case, we will sometimes call Q a quadratic differential for brevity.*

In particular, a quadratic differential on the extended complex plane $\overline{\mathbb{C}}$ is a formal expression $Q(z)dz^2$, where Q is rational simply because of the fact that meromorphic functions

Figure 2.2: Pick function



in the extended complex plane are rational (see Stein and Shakarchi [12]). In this thesis, all the quadratic differentials that we consider are rational.

It should be mentioned that the definition of a quadratic differential above can be made precise using differential geometry. However, for the purpose of this thesis, it suffices to precisely define the trajectories of a quadratic differential.

Definition 6. *A trajectory $\alpha : I \longrightarrow \Omega$ (I is an open interval of $\mathbb{R}$) of a quadratic differential $Q(z)dz^2$ is a C^1 function (i.e $\dot{\alpha}$ exists and continuous) such that*

$$Q(\alpha(t))\dot{\alpha}(t)^2 < 0, \quad \textit{for all} \ \ t \in I.$$

Remark 1. *1) We will also use the word "trajectory" to refer to the curve itself just as is done with the term "curve". 2) A trajectory of a rational differential must be an analytic curve (see Strebel [13]). 3) Sometimes, we will include the end points of a trajectory, and still call it a trajectory. We do this for a trajectory that terminates at a zero or pole of Q.*

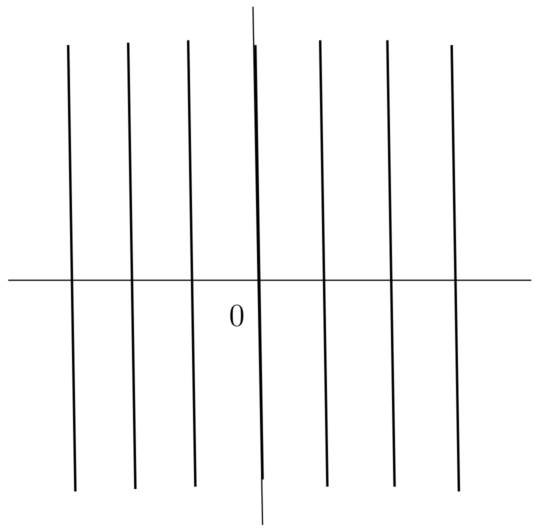
Example 4. *Consider the quadratic differential $Q(w)dw^2 = dw^2$. By the above definition, a trajectory of this quadratic differential is a curve $\alpha(t)$ such that $\dot{\alpha}(t)^2 < 0$. Let $\alpha(t) = x(t) + iy(t)$. Then $\dot{\alpha}(t) = \dot{x}(t) + i\dot{y}(t)$. Thus $\dot{x}(t)$ has to be 0 since $\dot{\alpha}(t)^2 < 0$. Therefore the trajectories are as in Figure 2.3.*

The following definition of the "pull back" of a quadratic differential defines a quadratic differential on E from a quadratic differential on Ω when a function $f : E \longrightarrow \Omega$ is given.

Definition 7. *Let $f : E \longrightarrow \Omega$ be one-to-one analytic, and let $Q(w)dw^2$ be a quadratic differential on Ω. The "pull back" of $Q(w)dw^2$ under f is a quadratic differential on E given by*

$$f^*(Q(w)dw^2) = Q(f(z))f'(z)^2dz^2.$$

Figure 2.3: Regular point



The following proposition motivates the above definition.

Proposition 1. *Given a one-to-one analytic function $f : E \longrightarrow \Omega$, if α is a trajectory of a quadratic differential $Q(w)dw^2$ on Ω, then $f^{-1} \circ \alpha$ is a trajectory of*

$$f^*(Q(w)dw^2) = Q(f(z))f'(z)^2dz^2.$$

Proof. By the assumption, $Q(\alpha(t))\dot{\alpha}(t)^2 < 0$. Let $\beta(t) = (f^{-1} \circ \alpha)(t)$. Then $f(\beta(t)) = \alpha(t)$ and $f'(\beta(t))\dot{\beta}(t) = \dot{\alpha}(t)$. Thus

$$Q(f(\beta(t)))f'(\beta(t))^2\dot{\beta}(t)^2 = Q(\alpha(t))\dot{\alpha}(t)^2 < 0.$$

Therefore $\beta = f^{-1} \circ \alpha$ is a trajectory of $Q(f(z))f'(z)^2dz^2$. □

Now we will talk about the trajectory structure of a quadratic differential. First, we need to define zeros and poles of a quadratic differential at a point $z_0 \in \mathbb{C}$ since the trajectory structure of a quadratic differential depends on zeros and poles (see Strebel [13]).

Definition 8. *A quadratic differential $Q(z)dz^2$ has a zero (or pole) of order n at z_0 if Q has a zero (or pole) of order n there.*

Definition 9. *A quadratic differential $Q(z)dz^2$ has a regular point at z_0 if Q has neither a zero nor a pole there.*

We now discuss how to find the local trajectory structure of a quadratic differential. The general theory is very involved, so we only sketch the idea. See Strebel [13] for details. If a quadratic differential Q has a regular point at z_0, Q has a square root near z_0. Thus we can define the following mapping.

$$w = \Phi(z) = \int_{z_0}^{z} \sqrt{Q(\xi)}d\xi \quad \Rightarrow \quad dw = \sqrt{Q(z)}dz. \tag{2.1}$$

Thus

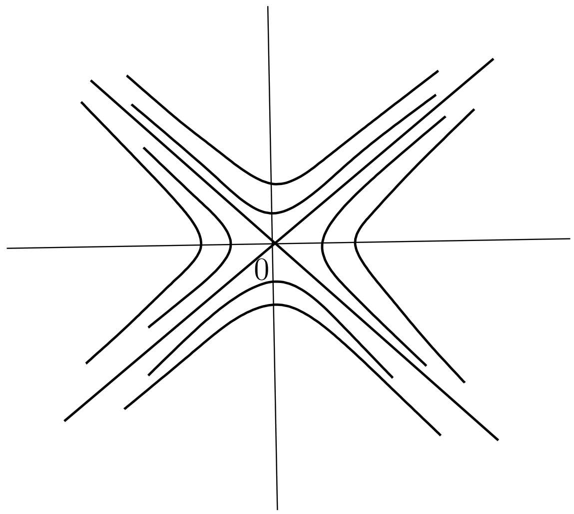
$$\Phi^*(dw^2) = Q(z)dz^2$$

This expression makes it easier to find the trajectories of $Q(z)dz^2$ by using the simple trajectory structure of dw^2 (Example 4) and the equation $w = \int_{z_0}^{z} \sqrt{Q(\xi)}d\xi$. This is because if $\alpha(t)$ is a trajectory of dw^2, then $\Phi^{-1}(\alpha(t))$ is a trajectory of $Q(z)dz^2$ by Proposition 1. The same idea would work when Q has a zero or pole if one is careful with branch cuts. We have already seen the trajectory structure of a quadratic differential with a regular point in Example 4 since $Q(w) = 1$ has a regular point at any point. We now give an example of the local trajectory structure of a quadratic differential Q with a zero of order 2.

Example 5. *Let $Q(z)dz^2 = z^2dz^2$. This quadratic differential has a zero of order 2 at the origin. From (2.1), $w = \Phi(z) = \int_0^z \sqrt{\xi^2}d\xi$. By taking a suitable branch of the square root, we have $w = \frac{1}{2}z^2$. We are able to find the trajectories of $Q(z)dz^2$ by using the equation $z^2 = 2w$ and the local trajectory structure of dw^2. Thus we have Figure 2.4 for the local trajectory structure of a quadratic differential with a double zero at the origin.*

In a similar way (though it is quite tricky for some cases), we can find the local trajectory structure of a quadratic differential with zeros and poles of any order. But our interest is

Figure 2.4: Local trajectory structure of z^2dz^2 at the origin



confined to certain cases, so we will give the figures only for the cases we will use later. Figure 2.5 shows the local trajectory structure of a quadratic differential near the point z_0 in which the quadratic differential has zeros and poles at z_0 (see Strebel [13]). In general, if a quadratic differential has a zero of order n at a point z_0, there are n+2 trajectories which radiate from the point z_0 with equal angles.

Remark 2. *One may use the general theory of trajectories (see Strebel [13]) to find the qualitative behavior of the trajectories in a neighborhood of any singularity as in Fig 2.5.*

Example 6. *Consider a quadratic differential $Q(w)dw^2 = \frac{1+Aw}{w^4}dw^2$ $(A \neq 0)$. Q has a simple zero at $-\frac{1}{A}$ and a pole of order 4 at 0. Thus the trajectories of Q near the zero and the pole locally look like Figure 2.6 (left) when $A < 0$ (A is real). It can be shown that the trajectories have the global structure as in Figure 2.6 (right) by explicit computation Schippers [11] or reasoning from general principles by Theorem 3.5 Jenkins [3]. Figure 2.7 shows the trajectories of Q when A is imaginary.*

Up to now, we gave the local trajectory structure of a quadratic differential near the point

Figure 2.5: Local trajectory structure near z_0

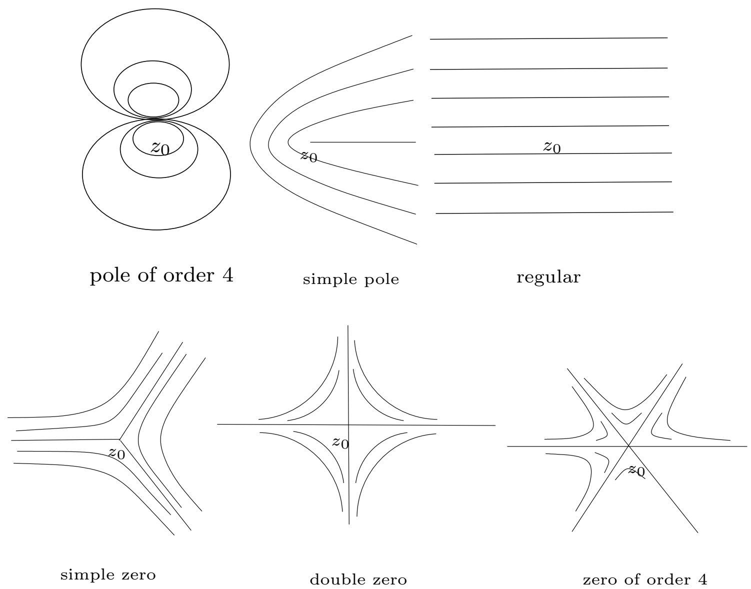


Figure 2.6: Trajectory structure of $\frac{1+Aw}{w^4}dw^2$, $A < 0$ (local structure-left, global structure-right)

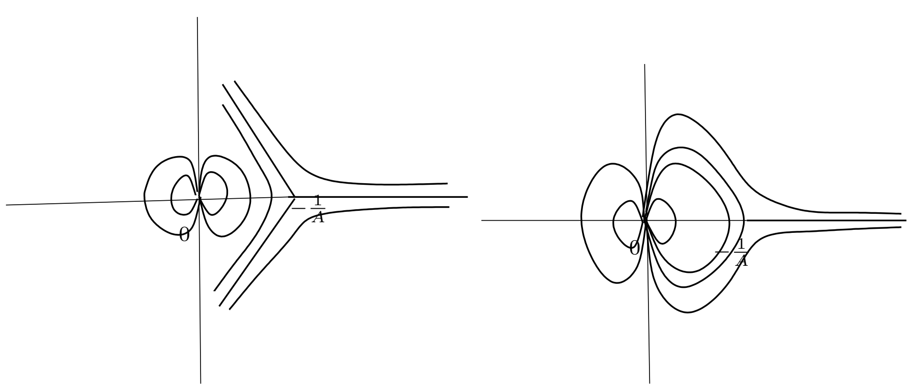
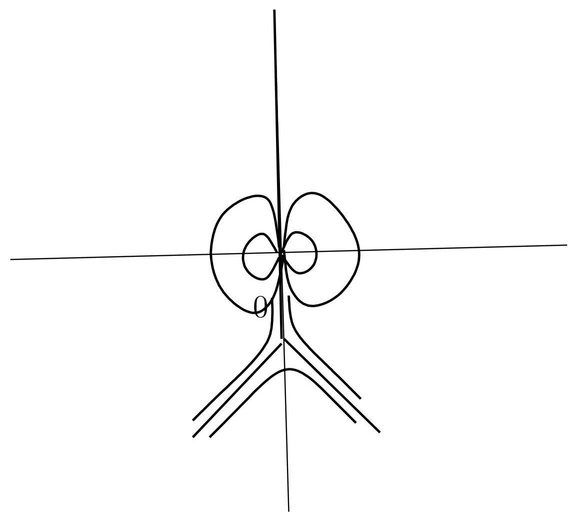


Figure 2.7: Local trajectory structure of $\frac{1+Aw}{w^4}dw^2$, A is imaginary



z_0 when z_0 is a finite point. We can also define the local trajectory structure of $Q(w)dw^2$ near ∞ by taking the transformation $w = f(z) = \frac{1}{z}$. To define this precisely, consider the quadratic differential $Q(w)dw^2$ on $\mathbb{C} \setminus \{0\}$. The function $f(z) = \frac{1}{z}$ pulls back the quadratic differential $Q(w)dw^2$ on $\mathbb{C} \setminus \{0\}$ to the quadratic differential $\frac{1}{z^4}Q(\frac{1}{z})dz^2$ on $\mathbb{C} \setminus \{0\}$. The new quadratic differential $\frac{1}{z^4}Q(\frac{1}{z})dz^2$ extends to $\mathbb{C}$ since we consider only rational quadratic differentials. Thus we can explore the local trajectory structure of $Q(w)dw^2$ near ∞ by looking at the local trajectory structure of $\frac{1}{z^4}Q(\frac{1}{z})dz^2$ near 0.

We define zeros and poles of a quadratic differential at infinity.

Definition 10 (zeros and poles at ∞)**.** *1) A quadratic differential $Q(w)dw^2$ has a zero of order n at ∞ if Q has a zero of order $n+4$ there. 2) A quadratic differential $Q(w)dw^2$ has a pole of order n at ∞ if Q has a pole of order $n-4$ there. 3) A quadratic differential $Q(w)dw^2$ has a regular point at ∞ if Q has a zero of order 4 there.*

In the course of this thesis, we will consider quadratic differentials of the following two types. Motivation for the specific form will be given in Section 3.3 and 4.1.

Definition 11. *$P(w)dw^2$ is a $\mathbb{C}$-type quadratic differential of order n if P is of the form*

$$P(w) = \sum_{k=1}^{n-1} \frac{\alpha_k}{w^{k+2}}.$$

$\frac{Q(w)}{w^2}dw^2$ is a $\mathbb{D}$-type quadratic differential of order n if Q is of the form

$$Q(w) = \sum_{k=-(n-1)}^{n-1} \beta_k w^k,$$

where

$$Im(\beta_0) = 0 \quad \textit{and} \quad \beta_k = \overline{\beta_{-k}}, \quad k = 1, 2, \cdots, n-1. \tag{2.2}$$

2.3 Functionals and extremal problems

In general, a functional is a map from a vector space to its field of scalars. Here we are interested in the vector space $\mathcal{A}$ (Definition 1) with field of scalars $\mathbb{C}$. We say that ϕ is a continuous functional on $\mathcal{A}$ if $\phi(f_n) \to \phi(f)$ whenever a sequence of functions $f_n \in \mathcal{A}$ converges uniformly on compact subsets of $\mathbb{D}$ to a function $f \in \mathcal{A}$. For instance, the coefficient functional $\phi_n(f) = a_n$ (n-th coefficient of f) is a continuous functional by the Cauchy integral formula. We introduce a special type of functional, which we are concerned with.

Definition 12. *ϕ is said to be a homogeneous functional of order n+1 over the class $\mathcal{S}$ if ϕ has the property*

$$\phi(e^{-i\theta} f(e^{i\theta} z)) = e^{ni\theta}\phi(f).$$

Example 7. *Consider a functional ϕ_λ ($\lambda \in \mathbb{C}$) defined by $\phi_\lambda(f) = a_3 - \lambda a_2^2$ for $f(z) = z + a_2 z^2 + \cdots$, and let g be the rotation of f. Then*

$$\begin{aligned} g(z) = e^{-i\theta} f(e^{i\theta} z) &= e^{-i\theta}(e^{i\theta} z + a_2 e^{2i\theta} z^2 + a_3 e^{3i\theta} z^3 + \cdots) \\ &= z + a_2 e^{i\theta} z^2 + a_3 e^{2i\theta} z^3 + \cdots. \end{aligned}$$

Thus

$$\begin{aligned}\phi_\lambda(g) &= a_3e^{2i\theta} - \lambda(a_2e^{i\theta})^2 \\ &= e^{2i\theta}(a_3 - \lambda a_2^2).\end{aligned}$$

Therefore ϕ_λ is a third order homogeneous functional.

The functional in the above example is one type of homogeneous functional. For the purpose of this thesis, we only consider the following type of homogeneous functionals. For a given function $f(z) = a_1z + a_2z^2 + a_3z^3 + \cdots$, $\alpha a_3 + \beta a_2^2$ ($\alpha,\ \beta \in \mathbb{C}$ and $\alpha \neq 0$) is called a third order homogeneous coefficient functional, and $\alpha a_4 + \beta a_3 a_2 + \gamma a_2^3$ ($\alpha,\ \beta\ ,\gamma \in \mathbb{C}$ and $\alpha \neq 0$) is called a fourth order homogeneous coefficient functional. Note that homogeneous coefficient functionals are continuous since coefficient functionals are continuous.

Now we introduce an important problem in function theory that is called an extremal problem. In general, an extremal problem is the problem of maximizing a continuous functional into $\mathbb{R}$. In this thesis, we only consider extremal problems of maximizing the real part of continuous complex functionals. We give the definition of extremal problems in this case. For a given continuous functional ϕ defined on $\mathcal{A}$, the extremal problem is a problem of finding the supremum of $Re\{\phi\}$ over $\mathcal{A}$. i.e finding $\sup\{Re\phi(f)\}$ for all $f \in \mathcal{A}$. A function $f \in \mathcal{A}$ is called an extremal function if the supremum over $\mathcal{A}$ is attained by f. In this thesis, the class $\mathcal{A}$ is $\mathcal{S}$ or $\mathcal{B}$ when we pose an extremal problem. The famous Bieberbach conjecture motivated and activated the study of extremal problems for univalent functions. In fact, most of the study of extremal problems was in connection to attempts to solve the Bieberbach conjecture. The Bieberbach conjecture was posed by Bieberbach in 1916 and proven by de Branges in 1984. One can find the proof in Rosenblum and Rovnyak [8].

Bieberbach conjecture If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in the class $\mathcal{S}$, then $Re\{a_n\} \leqq n$

for $n = 2, 3, \cdots$. Equality occurs if and only if f is the Koebe function.

Remark 3. *An extremal function for an extremal problem exists for any continuous functional over a compact normal family (see Duren [1]). Thus an extremal function is assured to exist when we pose an extremal problem for a continuous functional over the class $\mathcal{S}$ by Theorem 2.*

Chapter 3

Extremal functions and admissibility

3.1 Introduction

We should first introduce the concept of "admissible", which we will encounter a lot from now on.

Definition 13. *A one-to-one analytic function* $f : \mathbb{D} \longrightarrow \mathbb{C}$ *or* $\mathbb{D}$ *is admissible for a quadratic differential* $Q(w)dw^2$ *if* f *maps onto the complement in* $\mathbb{C}$ *or* $\mathbb{D}$ *of trajectories of* $Q(w)dw^2$.

We give two examples that demonstrate the definition of "admissible".

Example 8. *In Example 1, we have seen that the Koebe function* $k(z)$ *maps* $\mathbb{D}$ *onto the complement of a curve* $w(t) = t$, $t \leqq -\frac{1}{4}$*. let us consider a quadratic differential* $Q(w)dw^2 = \frac{1+Aw}{w^4}dw^2$*. The omitted arc* $w(t)$ *of* $k(z)$ *satisfies*

$$Q(w(t))\dot{w}(t)^2 = \frac{1+At}{t^4} < 0$$

if $A > 4$*. Thus* $k(z)$ *is admissible for* $\frac{1+Aw}{w^4}dw^2$ *if* $A \geqq 4$ *(see Remark 1-3). In a similar way,* *it can be shown that* $-k(-z)$ *is admissible for* $\frac{1+Aw}{w^4}dw^2$ *if* $A \leqq -4$, $-ik(iz)$ *is admissible for* $\frac{1+Aw}{w^4}dw^2$ *if* A *is negative imaginary (i.e* $A = ai$, $a < 0$*), and* $ik(-iz)$ *is admissible for* $\frac{1+Aw}{w^4}dw^2$ *if* A *is positive imaginary.*

Example 9. *In Example 3, we have seen that the Pick function* $p_s(z)$ *maps* $\mathbb{D}$ *onto* $\mathbb{D}$ *minus a straight slit from* -1 *to* $\frac{\sqrt{1-e^{-s}}-1}{\sqrt{1-e^{-s}}+1}$, $0 < s < \infty$. *Let us set* $\alpha(s) = \frac{\sqrt{1-e^{-s}}-1}{\sqrt{1-e^{-s}}+1}$, *and consider a quadratic differential*

$$\widetilde{Q}(w)dw^2 = \frac{(w+e^{ix})^2(w+re^{-ix})(w+\frac{1}{r}e^{-ix})}{w^4}dw^2, \quad 0 < r \leqq 1, \quad 0 \leqq x < 2\pi,$$

Then

1) $p_s(z)$ *is admissible for* $\widetilde{Q}$ *if* $x = 0$, $r \neq 1$, *and* $\alpha(s) \leqq -r$.

2) $-p_s(-z)$ *is admissible for* $\widetilde{Q}$ *if* $x = \pi$, $r \neq 1$, *and* $\alpha(s) \leqq -r$.

3) $ip_s(-iz)$ *is admissible for* $\widetilde{Q}$ *if* $x = \frac{\pi}{2}$.

4) $-ip_s(iz)$ *is admissible for* $\widetilde{Q}$ *if* $x = \frac{3\pi}{2}$.

let us see how we get these results.

1) $x = 0$, $r \neq 1$ $(e^{\pm ix} = 1)$, *and* $\alpha(s) \leqq -r$

We choose a parametrization of the omitted arc of $p_s(z)$ *as* $w(t) = t$, $-1 < t \leqq \alpha(s)$ $(\alpha(s) \leqq -r)$. *Therefore we need to show that* $w(t) = t$ $(-1 < t \leqq \alpha(s))$ *is admissible for*

$$\widetilde{Q}(w)dw^2 = \frac{(w+1)^2(w+r)(w+\frac{1}{r})}{w^4}dw^2, \quad 0 < r < 1.$$

Indeed,

$$\widetilde{Q}(t) = \frac{(t+1)^2(t+r)(t+\frac{1}{r})}{t^4} < 0, \quad 0 < r < 1, \quad -1 < t \leqq \alpha(s).$$

since $\alpha(s) \leqq -r$ *gives* $t + r \leqq 0$. *Observe that if* $r = 1$, $\widetilde{Q}(t) = \frac{(t+1)^4}{t^4}$, *which makes* $\widetilde{Q}(t)$ *positive. Thus Pick function* $p_s(z)$ *is admissible for* $\widetilde{Q}$ *when* $x = 0$, $r \neq 1$ $(e^{\pm ix} = 1)$, *and* $\alpha(s) \leqq -r$. *The proof of other parts is very similar to the one of part 1.*

2) $x = \pi$, $r \neq 1$ ($e^{\pm ix} = -1$), and $\alpha(s) \leqq -r$

Since $-p_s(-z)$ is a rotation of $p_s(z)$ by π, we take $w(t) = t$, ($-\alpha(s) \leqq t < 1$) as a parametrization of the omitted arc of $-p_s(-z)$. We claim that $-p_s(-z)$ is admissible for

$$\widetilde{Q}(w)dw^2 = \frac{(w-1)^2(w-r)(w-\frac{1}{r})}{w^4}dw^2, \quad 0 < r < 1.$$

Indeed,

$$\widetilde{Q}(t) = \frac{(t-1)^2(t-r)(t-\frac{1}{r})}{t^4} < 0, \quad 0 < r < 1, \quad -\alpha(s) \leqq t < 1.$$

since $\alpha(s) \leqq -r$ gives $t - r \leqq 0$. Like in the case (i), if $r = 1$, $\widetilde{Q}(t) = \frac{(t-1)^4}{t^4}$, which makes $\widetilde{Q}(t)$ positive. Thus the function $-p_s(-z)$ is admissible for $\widetilde{Q}$ when $x = \pi$, $r \neq 1$ ($e^{\pm ix} = -1$), and $\alpha(s) \leqq -r$.

3) $x = \frac{\pi}{2}$ ($e^{ix} = i$ and $e^{-ix} = -i$)

Since $ip_s(-iz)$ is a rotation of $p_s(z)$ by $\frac{\pi}{2}$, we take $w(t) = it$, ($-1 < t \leqq \alpha(s)$) as a parametrization of the omitted arc of $ip_s(-iz)$. We claim that $ip_s(-iz)$ is admissible for

$$\widetilde{Q}(w)dw^2 = \frac{(w+i)^2(w-ir)(w-i\frac{1}{r})}{w^4}dw^2, \quad 0 < r \leqq 1.$$

Indeed,

$$\begin{aligned}\widetilde{Q}(it)(i)^2 &= \frac{(it+i)^2(it-ir)(it-i\frac{1}{r})}{(it)^4}(i)^2 < 0 \\ &= -\frac{(t+1)^2(t-r)(t-\frac{1}{r})}{t^4} < 0, \quad 0 < r \leqq 1, \quad -1 < t \leqq \alpha(s).\end{aligned}$$

Thus $ip_s(-iz)$ is admissible for $\widetilde{Q}$ when $x = \frac{\pi}{2}$.

4) $x = \frac{3\pi}{2}$ ($e^{ix} = -i$ and $e^{-ix} = i$)

Since $-ip_s(iz)$ is a rotation of $p_s(z)$ by $-\frac{\pi}{2}$, we take $w(t) = it$, ($-\alpha(s) \leqq t < 1$) as a parametrization of the omitted arc of $-ip_s(iz)$. We claim that $-ip_s(iz)$ is admissible for

$$\widetilde{Q}(w)dw^2 = \frac{(w-i)^2(w+ir)(w+i\frac{1}{r})}{w^4}dw^2, \quad 0 < r \leqq 1.$$

Indeed,

$$\begin{aligned}\widetilde{Q}(it)(i)^2 &= \frac{(it-i)^2(it+ir)(it+i\frac{1}{r})}{(it)^4}(i)^2 < 0\\ &= -\frac{(t-1)^2(t+r)(t+\frac{1}{r})}{t^4} < 0, \quad 0 < r \leqq 1, \ \ -\alpha(s) \leqq t < 1\end{aligned}$$

Thus $-ip_s(iz)$ *is admissible for* $\widetilde{Q}$ *when* $x = \frac{3\pi}{2}$.

Our main problem arises when one studies the association of quadratic differentials and extremal functions. Thus we will outline some of the main results in the field regarding quadratic differentials and extremal functions, which are due to Schiffer, Teichmüller, Schaeffer and Spencer, and Pfluger. In Section 3.2, we state Schiffer's theorem that says an extremal function is admissible for some quadratic differential, and then we illustrate an application of Schiffer's theorem. In Section 3.3, we state Teichmüller's theorem that says an admissible function for a quadratic differential is extremal for some functional. This claims the opposite direction of the statement in Schiffer's theorem. We finish this section stating a remarkable theorem by Schaeffer and Spencer that gives the actual form of the functions admissible for more than one quadratic differential. In Section 3.4, we will talk about some of Pfluger's results in his two papers [6] and [7]. We will see how Pfluger improves Schiffer's theorem and Teichmüller's theorem by combining Schiffer's and Teichmüller's methods for 3rd order homogeneous functionals.

3.2 Schiffer's theorem

let us start with a theorem that is often applied to the problems concerning extremal functions.

We need to understand what a Fréchet differential is before stating Schiffer's theorem.

Definition 14. *Let* X *be a Banach space, and let* ϕ *be a complex-valued functional on a* *subset of X. Assume that ϕ is defined in a neighborhood of $x_0 \in X$. Then ϕ is said to have a Fréchet differential at x_0 if there is a continuous linear functional $\ell(\cdot; x_0)$ such that*

$$\ell(h; x_0) = \lim_{t \to 0^+} \frac{\phi(x_0 + th) - \phi(x_0)}{t},$$

uniformly on the unit sphere $\|h\| = 1$.

We state Schiffer's theorem (see Duren [1] for the proof).

Theorem 3 (Schiffer's theorem)**.** *Let $f \in \mathcal{S}$ be an extremal function for a continuous functional ϕ on the class $\mathcal{A}$. Suppose that ϕ has a Fréchet differential $\ell(\cdot; f)$ at f which is not constant on $\mathcal{S}$. Then f maps the unit disk onto the complement of a system of finitely many analytic arcs $w = w(t)$ satisfying the differential equation*

$$\frac{1}{w^2} \ell\left(\frac{f^2}{f - w}; f\right) \left(\frac{dw}{dt}\right)^2 > 0. \tag{3.1}$$

When we say that ϕ has a Fréchet differential at $f \in \mathcal{S}$, we mean that the functional restricted to $\mathcal{N}$ has a Fréchet differential at f. This is true because continuity on $\mathcal{A}$ implies continuity on $\mathcal{N}$ (The converse does not hold), and $\mathcal{S}$ is contained in $\mathcal{N}$.

In view of Definition 13, the conclusion of Schiffer's theorem actually shows that f is admissible for a quadratic differential $-\frac{1}{w^2}\ell\left(\frac{f^2}{f-w}; f\right) dw^2$. Thus this theorem shows why the quadratic differentials are important.

The following example illustrates an application of Schiffer's theorem showing explicitly that an extremal function is admissible for a quadratic differential.

Example 10. *If $f \in \mathcal{S}$ is extremal for $\Phi(g) = b_3 - \lambda b_2^2$ ($\lambda \in \mathbb{C}$), where $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, then f maps onto $\mathbb{C}$ minus trajectories of the quadratic differential*

$$\frac{1 + Aw}{w^4} dw^2, \quad \textit{where} \quad A = 2a_2(1 - \lambda) \quad (a_2 \;\; \textit{is the second coefficient of} \;\; f).$$

let us see how we get this result. By the assumption, f is extremal for Φ. Thus by Schiffer's theorem, f maps the unit disk onto the complement of a system of finitely many analytic arcs $w = w(t)$ satisfying the differential equation (3.1), where $\ell(\cdot, f)$ is Fréchet differential at f. Let $h = \frac{f^2}{f-w}$ and $h(z) = h_1 z + h_2 z^2 + h_3 z^3 + \cdots$ ($h_1,\ h_2, \cdots$ are constants).

Then

$$\begin{aligned}
h(z) &= \frac{f(z)^2}{f(z)-w} \\
&= \frac{(z + a_2 z^2 + \cdots)^2}{z + a_2 z^2 + \cdots - w} \\
&= \frac{z^2(1 + a_2 z + \cdots)^2}{-w\left(1 - \frac{z}{w} - \frac{a_2}{w} z^2 - \cdots\right)} \\
&= -\frac{z^2}{w}(1 + 2a_2 z + \cdots)\left(1 + \frac{z}{w} + \cdots\right) \\
&= -\frac{z^2}{w}\left(1 + (\frac{1}{w} + 2a_2) z + \cdots\right) \\
&= -\frac{1}{w} z^2 - \frac{1}{w}\left(\frac{1}{w} + 2a_2\right) z^3 - \cdots.
\end{aligned}$$

Thus

$$h_2 = -\frac{1}{w} \quad \textit{and} \quad h_3 = -\frac{1}{w^2} - \frac{2a_2}{w}. \tag{3.2}$$

let us first compute $\ell(h, f)$, where $h = \frac{f^2}{f-w}$.

By the definition of a Fréchet differential,

$$\ell(h; f) = \lim_{t \to 0^+} \frac{\Phi(f + th) - \Phi(f)}{t}. \tag{3.3}$$

Since

$$\begin{aligned}
(f + th)(z) &= f(z) + th(z) \\
&= (z + a_2 z^2 + a_3 z^3 + \cdots) + t(h_1 z + h_2 z^2 + h_3 z^3 + \cdots) \\
&= (1 + th_1) z + (a_2 + th_2) z^2 + (a_3 + th_3) z^3 + \cdots,
\end{aligned}$$

we get

$$\Phi(f + th) = (a_3 + th_3) - \lambda(a_2 + th_2)^2,$$

and we know that

$$\Phi(f) = a_3 - \lambda a_2^2.$$

Now from (3.3)

$$\begin{aligned}
\ell(h; f) &= \lim_{t\to 0^+} \frac{\Phi(f+th) - \Phi(f)}{t} \\
&= \lim_{t\to 0^+} \frac{\{(a_3 + th_3) - \lambda(a_2 + th_2)^2\} - \{a_3 - \lambda a_2^2\}}{t} \\
&= \lim_{t\to 0^+} \frac{th_3 - 2t\lambda a_2 h_2}{t} = h_3 - 2\lambda a_2 h_2.
\end{aligned}$$

Combining this with (3.2),

$$\begin{aligned}
\ell(h; f) &= -\frac{1}{w^2} - \frac{2a_2}{w} + \frac{2\lambda a_2}{w} \\
&= -\frac{1 + 2a_2(1-\lambda)w}{w^2}.
\end{aligned}$$

Therefore from (3.1), we conclude that f maps onto $\mathbb{C}$ minus the trajectories of the quadratic differential

$$\frac{1 + Aw}{w^4} dw^2, \quad \textit{where} \quad A = 2a_2(1-\lambda) \quad (a_2 \ \textit{ is the second coefficient of } \ f).$$

This example shows that the extremal function for the functional is admissible for the quadratic differential $\frac{1+Aw}{w^4}$ $(A = 2a_2(1-\lambda))$, which depends on the second coefficient a_2. Thus the quadratic differential depends on the extremal function.

3.3 Coefficient region and $\mathcal{D}_n$-functions

There is a close relation between the boundary of coefficient region and $\mathcal{D}_n$-functions, which are admissible for $\mathbb{C}$-type quadratic differentials. We give the definitions and related theorems, which we will use later.

Definition 15. *The coefficient region* V_n *is the subset of* $\mathbb{C}^{n-1}$ *consisting of all points* $A_n = (a_2, \cdots, a_n)$ *corresponding to the initial coefficients of some function* $f(z) = z + a_2 z + \cdots + a_n z^n + \cdots \in \mathcal{S}$.

The following theorem says that to each point $A_n \in \partial V_n$ (the boundary of V_n), there exists $f \in S(A_n)$ (the set of all functions $f \in \mathcal{S}$ whose initial coefficients is A_n) satisfying a differential equation (see Duren [1] for the proof).

Theorem 4. *For each point* $A_n \in \partial V_n$*, there exists a function* $f \in S(A_n)$ *which maps the unit disk* $\mathbb{D}$ *onto the complement of analytic arcs* $w = w(t)$ *satisfying*

$$\frac{P(w)}{w^2}\left(\frac{dw}{dt}\right)^2 < 0, \quad \textit{where} \quad P \quad \textit{is of the form} \quad P(w) = \sum_{k=1}^{n-1} \alpha_k w^{-k}. \tag{3.4}$$

Furthermore, f *satisfies a differential equation*

$$\left(\frac{zf'(z)}{f(z)}\right)^2 P(f(z)) = Q(z), \quad |z| < 1, \tag{3.5}$$

where

$$P(w) = \sum_{k=1}^{n-1} \frac{\alpha_k}{w^k}, \quad Q(z) = \sum_{k=-(n-1)}^{n-1} \beta_k z^k \tag{3.6}$$

with the properties

$$\beta_0 > 0, \ \beta_k = \overline{\beta_{-k}}, \ Q(z) \geqq 0 \ \textit{on} \ \partial\mathbb{D}, \ \textit{and} \ Q(z) = 0 \ \textit{for some} \ z \in \partial\mathbb{D}. \tag{3.7}$$

Remark 4. *In Theorem 4, one can observe a few things about the quadratic differentials, which we discussed in Section 2.2. First, the expression (3.5) is an example of pull back:* $\frac{Q(z)}{z^2}dz^2$ *is the pull back of* $\frac{P(w)}{w^2}dw^2$ *under* f *by looking at the modified expression* $\frac{P(f(z))}{f(z)^2}f'(z)^2 = \frac{Q(z)}{z^2}$ *from (3.5). Another observation is that* $\frac{P(w)}{w^2}dw^2$ *and* $\frac{Q(z)}{z^2}dz^2$ *in Theorem 4 are a* $\mathbb{C}$*-type and* $\mathbb{D}$*-type quadratic differential of order n, respectively (see Definition 11). Lastly, the* *property $Q(z) \geqq 0$ on $\partial\mathbb{D}$ in (3.7) shows that the boundary of $\mathbb{D}$ is a trajectory of $\frac{Q(z)}{z^2}dz^2$ since for $z = e^{it}$, we get*

$$\frac{Q(z)}{z^2}dz^2 = \frac{Q(e^{it})}{e^{2it}}(-e^{2it})dt^2 = -Q(e^{it})dt^2 < 0$$

from $Q(z) \geqq 0$ on $\partial\mathbb{D}$.

The following example illustrates Theorem 4.

Example 11. *It can be shown that a boundary point of V_3 is attained by the Koebe function $k(z)$ (Duren [1]). We have seen that the Koebe function $k(z) = \frac{z}{(1-z)^2}$ is admissible for a quadratic differential $\frac{P(w)}{w^2}dw^2 = \frac{1+Aw}{w^4}dw^2$, $A \geqq 4$ in Example 8. This $k(z)$ satisfies a differential equation*

$$\frac{Q(z)}{z^2}dz^2 = \frac{P(k(z))}{k(z)^2}k'(z)^2dz^2.$$

Simplifying, we get

$$\frac{Q(z)}{z^2}dz^2 = \frac{(z+1)^2(z^2+(A-2)z+1)}{z^4}dz^2.$$

Furthermore,

$$P(w) = \frac{A}{w} + \frac{1}{w^2} \quad \text{and} \quad Q(z) = \frac{1}{z^2} + \frac{A}{z} + (2A-2) + Az + z^2,$$

which shows that P and Q have the form of (3.6) for $n = 3$, and Q satisfies the properties (3.7) in Theorem 4.

Remark 5. *If $m-1$ is the largest index for which $\beta_k \neq 0$, then from the equation (3.5) and the fact that $f(0) = 0$, $f'(0) = 1$, we can see $\alpha_{m-1} = \beta_{m-1}$ and $\alpha_k = 0$ for $k = m, \cdots, n-1$.*

Definition 16. *A differential equation of the form (3.5) with the properties indicated in (3.7) and Remark 5 is called a $\mathcal{D}_n$-equation of degree m.*

Remark 6. *Now we can state the conclusion of Theorem 4 in two ways: 1) f satisfies a $\mathcal{D}_n$-equation, 2) f is admissible for a $\mathbb{C}$-type quadratic differential of order n. Thus 1) and 2) are equivalent.*

In view of admissibility, we can restate Theorem 4: every boundary point of V_n is attained by a function admissible for a quadratic differential of order n on $\mathbb{C}$. This shows the importance of quadratic differentials again.

If a function f is analytic near the origin and satisfies a D_n-equation of degree m, then $f(0) = 0,\ (f'(0))^{m-1} = 1$. We will make the normalization that $f'(0) = 1$.

Definition 17. *A function f is called a $\mathcal{D}_n$-function if it is analytic in $\mathbb{D}$ with $f'(0) = 1$, and it satisfies some $\mathcal{D}_n$-equation.*

It can be shown that every $\mathcal{D}_n$-function is one-to-one (see Schaeffer and Spencer [10]). Thus every $\mathcal{D}_n$-function belongs to the class $\mathcal{S}$.

In Theorem 4, we have seen that for each boundary point of V_n, there exists a function f satisfying a $\mathcal{D}_n$-equation, which implies that there exists a $\mathcal{D}_n$-function since f is analytic in $\mathbb{D}$ with $f'(0) = 1$. In fact, the converse is also true with the help of Teichmüller's theorem. Indeed, Schaeffer and Spencer [10] showed that there is a one-one correspondence between $\mathcal{D}_n$-functions and boundary points of the coefficient region V_n. One might ask if there is a one-one correspondence between $\mathcal{D}_n$-functions and $\mathcal{D}_n$-equations. The answer is negative. There are some $\mathcal{D}_n$-functions satisfying more than one equation and some $\mathcal{D}_n$-equations having more than one $\mathcal{D}_n$-function as solutions (see Schaeffer and Spencer [10] for details). The following theorem by Schaeffer and Spencer [10] actually gives the form of $\mathcal{D}_n$-functions which satisfy more than one equation.

Theorem 5. *Let $n - 1$ be a prime number, and let $f \in \mathcal{S}$ satisfy two different $\mathcal{D}_n$-equations* *(i.e. one is not a multiple of the other), one of degree n. Then f has the form*

$$f(z) = \frac{z}{(1 - e^{i\alpha}z)(1 - e^{i\beta}z)}.$$

This function is either a rotation of the Koebe function (if $\alpha = \beta$) or a mapping of the unit disk onto the complement of two rays situated on a common line through the origin.

The same assertion by contraposition shows that if a function $f \in S$ is not of the form $\frac{z}{(1-e^{i\alpha}z)(1-e^{i\beta}z)}$, then the function f satisfies only one $\mathcal{D}_n$ equation whenever $n-1$ is a prime number. Thus the function f uniquely determines the corresponding quadratic differential in the $\mathcal{D}_n$-equation.

Example 12. *In Example 11, we have seen that the Koebe function $k(z)$ satisfies different $\mathcal{D}_n$-equations $\left(\frac{zf'(z)}{f(z)}\right)^2 P(f(z)) = Q(z)$ depending on A ($A \geqq 4$), where*

$$P(w) = \frac{A}{w} + \frac{1}{w^2} \quad \textit{and} \quad Q(z) = \frac{1}{z^2} + \frac{A}{z} + (2A-2) + Az + z^2.$$

Thus n in Theorem 5 is 3, and so $n-1 = 2$, which is a prime number. Therefore the function $k(z)$ should be of the form $\frac{z}{(1-e^{i\alpha}z)(1-e^{i\beta}z)}$. Indeed, $k(z) = \frac{z}{(1-z)^2}$ is the case of $\alpha = \beta = 0$.

Let us state Teichmüller's theorem (see Schaeffer and Spencer [10] for the proof).

Theorem 6 (Teichmüller's theorem)**.** *Let a function $f \in \mathcal{S}$ map onto the complement of analytic arcs satisfying $\frac{P(w)}{w^2}dw^2 < 0$ (i.e $f \in \mathcal{S}$ maps onto the complement E of the trajectories of a quadratic differential $\frac{P(w)}{w^2}dw^2$), where P is of the form*

$$P(w) = \frac{\alpha_1}{w} + \cdots + \frac{\alpha_{n-1}}{w^{n-1}}, \quad \alpha_{n-1} \neq 0.$$

If $\psi(w) = w + \beta_n w^n + \beta_{n+1} w^{n+1} + ...$ is analytic and univalent in E, then $Re\{\alpha_{n-1}\beta_n\} \leqq 0$. Equality holds if and only if $\psi(w) = w$.

We will give a lemma that is verified easily (see Duren [1] for the proof) before illustrating an example of the application of Teichmüller's theorem.

Lemma 1. *If* $f(z) = z + a_2z^2 + \cdots$ *and* $g(z) = z + b_2z^2 + \cdots$ *are in the same set* $\mathcal{S}(A_{n-1})$, *then* $(g \circ f^{-1})(w) = w + (b_n - a_n)w^n + \cdots$.

Teichmüller's theorem can be used to demonstrate the uniqueness of the functions in Theorem 4 (Duren [1]). We give another application of Teichmüller's theorem, which shows that each admissible function is an extremal function within the subclass of all functions in $\mathcal{S}$ which have the same initial coefficients.

Example 13. *If* $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}$ *is admissible for* $\frac{1+Aw}{w^4}$ *for some* $A \in \mathbb{C}$, *then* f *is extremal for the functional* $Re\{a_3\}$ *over* $\mathcal{S}(a_2)$. *To see this, we use Teichmüller's theorem and Lemma 1. Set* $\frac{P(w)}{w^2} = \frac{1+Aw}{w^4}$, *and so* $n = 3$ *and* $\alpha_2 = 1$ *in Teichmüller's theorem. Let* $g(z) = z + b_2z^2 + \cdots$ *in* $\mathcal{S}(a_2)$. *Then by Lemma 1,* $(g \circ f^{-1})(w)$ *has the form* $(g \circ f^{-1})(w) = w + (b_3 - a_3)w^3 + \cdots$. *If we let* $\psi(w) = (g \circ f^{-1})(w)$, *then* $\psi(w)$ *is analytic and univalent in the complement of the trajectories of* $\frac{1+Aw}{w^4}$. *Thus by Teichmüller's theorem,* $Re\{b_3 - a_3\} \leqq 0$ *or* $Re\{b_3\} \leqq Re\{a_3\}$. *Therefore* f *is extremal for* $Re\{a_3\}$ *over* $\mathcal{S}(a_2)$. *Note that the functional is not uniquely determined because we can add any multiple of* a_2^2 *in the functional* $Re\{a_3\}$ *namely* f *is also extremal for* $Re\{a_3 - \lambda a_2^2\}$.

We end this section giving specific explanation about Schiffer's and Teichmüller's theorems. Schiffer's theorem shows that for a fixed functional, extremal functions are admissible for a quadratic differential. Here the quadratic differential is not known explicitly. It depends on the initial coefficients of unknown extremal functions. Teichmüller's theorem shows that for a fixed quadratic differential, admissible functions are extremal for a functional over a smaller class of functions. Thus it does not give the extremality over the entire class. Futhermore, the functional is not determined as we saw in Example 13.

3.4 Pfluger's results

In the previous sections, we have seen that Schiffer's and Teichmüller's theorems give necessary and sufficient conditions for extremality, respectively. In this section, we will give Pfluger's results, and briefly explain how he improved on the results of Schiffer's and Teichmüller's theorems for a special kind of third order functional.

The following two statements demonstrate that in order to understand homogeneous functionals and quadratic differentials of order 3, it suffices to consider those of the form

$$\phi_\lambda = a_3 - \lambda a_2^2 \quad \text{and} \quad Q_A(w)dw^2 = \frac{1 + Aw}{w^4} dw^2$$

The first statement follows directly from Example 7.

Statement 1. *A function $f \in \mathcal{S}$ is extremal for $a_3 - \lambda a_2^2$ if and only if $e^{-i\theta} f(e^{i\theta} z)$ is extremal for $e^{2i\theta}(a_3 - \lambda a_2^2)$.*

By this statement, to find extremal functions for any third order homogeneous coefficient functional, it suffices to show that the functions are extremal for $a_3 - \lambda a_2^2$. The following statement is easily verified by Definition 13 of admissibility.

Statement 2. *A function $f \in \mathcal{S}$ is admissible for $\frac{1+Aw}{w^4} dw^2$ if and only if $e^{-i\theta} f(e^{i\theta} z)$ is admissible for $\frac{e^{-2i\theta}(1+Ae^{i\theta}w)}{w^4} dw^2$.*

By this statement, to study admissible functions for quadratic differentials of the form $\frac{B+Cw}{w^4} dw^2$ ($\mathbb{C}$-type quadratic differential of order 3), it suffices to study quadratic differentials of the form $\frac{1+Aw}{w^4} dw^2$. It should be also mentioned that $\frac{1+Aw}{w^4} dw^2$ is the right form of quadratic differential, as Example 10 suggests.

There are two main results by Pfluger. We will state them, but not give the whole proof (see Pfluger [6] and [7] for details). The following is the first one that claims a one-one correspondence between a functional and a pair of extremal functions.

Theorem 7. *For each $\lambda \in \mathbb{C}$ ($\lambda \neq 1$), there exists exactly one pair of functions $f(z)$ and $-f(-z)$ (denoted by $f_-(z)$) that are extremal for $\phi_\lambda = a_3 - \lambda a_2^2$ over $\mathcal{S}$.*

Indeed, for $\lambda = 1$, functions of the form $\frac{z}{1+2icz-z^2}$, $|c| \leqq 1$ give the same value 1 to the functional ϕ_λ. Also, note that these functions are admissible for $\frac{1+Aw}{w^4}$ when $A = 2a_2(1-\lambda) = 0$.

We give the other main result in Pfluger's paper.

Theorem 8. *If $f \in \mathcal{S}$ is admissible for a quadratic differential Q_A, and f is neither the Koebe function $k(z)$ nor one of its rotations $-k(-z)$, $ik(-iz)$ or $-ik(iz)$, then f uniquely determines Q_A, and f is extremal for one and only one functional ϕ_λ over $\mathcal{S}$.*

We will not prove the two main theorems by Pfluger because the proofs are complicated and long. One can find the proofs in Pfluger's two papers [6], [7], which in turn rely on Jenkins [4]. We will only state the main ideas here. He combined the following 3 facts by Schiffer, Jenkins, and Teichmüller. 1) Schiffer's theorem (Example 10) gives a relation between A and a_2 ($A = 2a_2(1-\lambda)$), 2) It is known by Jenkins [4] that for given admissible functions for Q_A, A and a_2 satisfy another independent relation, and 3) Teichmüller's theorem shows that if f is admissible for Q_A, it is extremal for ϕ_λ (for any λ) over $\mathcal{S}(a_2)$ (a_2 is determined by f).

We summarize the key points from Pfluger's two papers [6], [7] and Jenkins [4] to see how A puts restrictions on the admissible function.

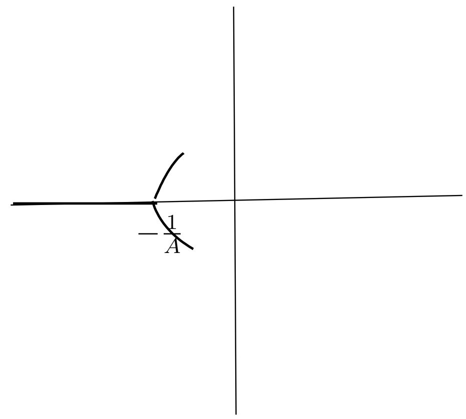
Theorem 9. *For a quadratic differential $Q_A = \frac{1+Aw}{w^4}$,*

Case 1) $A = 0$

The functions of the form $f(z) = \frac{z}{1+2icz-z^2}$, $|c| \leqq 1$ are the only admissible functions for $Q_0 = \frac{1}{w^4}$

Case 2) -4 $< A < 4$, $A \neq 0$

Figure 3.1: Slit with a fork



If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$ *is admissible for* Q_A*, then the second coefficient is given by*

$$a_2 = \frac{A}{2}(1 - log\frac{|A|}{4}) + 2it,$$

where

$$|t| \leqq sin\theta - \theta cos\theta, \quad |A| = 4cos\theta \quad and \quad 0 \leqq \theta \leqq \frac{\pi}{2}. \tag{3.8}$$

There is a bijection between (A, t) *satisfying (3.8) and* $f(z, A, t)$ *admissible for* Q_A*. ((*A, t*) and* $f(z, A, t)$ *are one parameter families by the parameter* t*). The omitted arc of this function has a fork. (see Figure 3.1)*

Case 3) $A \in \mathbb{C} \backslash (-4, 4)$

There is a unique function in $\mathcal{S}$ *admissible for* Q_A*. This function is a single slit map.*

Remark 7. *The above theorem explicitly shows the relation of the initial coefficients of a function admissible for a quadratic differential and the quadratic differential. In particular, for the functions of the type 2),* a_2 *is determined by* A *and parameter* t*. Thus each function* $f(z, A, t)$ *admissible for* Q_A *is determined by* A *and* t*. The parameter* t *determines the length of the fork (see Figure 3.1).*

3.5 Summary

We have seen some necessary conditions and some sufficient conditions for the function f to be extremal. Schiffer's theorem provides a necessary condition for f to be extremal; it states that extremal functions are admissible for some quadratic differential. But the quadratic differential depends on the initial coefficients of the function. On the other hand, Teichmüller's theorem provides a sufficient condition for f to be extremal; it states that admissible functions are extremal for some functional. But they are extremal only over some restricted class, and the functional is not determined. By combining Teichmüller's and Schiffer's methods, Pfluger showed that the functional determines a unique quadratic differential and has exactly two extremals; conversely, the admissible functions for a quadratic differential are extremal for a unique functional. However, Pfluger only considers a third order functional and the functions belonging to the class $\mathcal{S}$. We conclude this chapter raising some questions.

- Do Pfluger's results extend to homogeneous functionals of order 4?
- Do Pfluger's results extend to the class $\mathcal{B}$?

The answers to these questions are beyond the scope of the thesis, but they motivate the main problem of the thesis, which we will state in the following chapter.

Chapter 4

Results

We state our main problem :

- For a function f of the class $\mathcal{B}$ or $\mathcal{S}$, if f is admissible for a quadratic differential, does f determine the quadratic differential uniquely?

From the main problem, we have two cases: 1) when the function maps into the unit disk, and 2) when the function maps into the complex plane.

We show that if $f \in \mathcal{B}$ is admissible for a $\mathbb{D}$-type quadratic differential of order 3, then f uniquely determines the quadratic differential up to exceptional cases (Theorem 13). This is our main result in this thesis, which you will see in Section 4.4. Also, under the assumption that f is admissible for a $\mathbb{D}$-type quadratic differential of order 4, we give a partial result relating the coefficients of the quadratic differential to geometric properties of the curve omitted by f (Theorem 14). In Section 4.3, we give a new proof that for a $\mathbb{C}$-type quadratic differential of order 3, the admissible function $f \in \mathcal{S}$ uniquely determines the quadratic differential up to exceptional cases (Theorem 11). This section also includes a partial result for a $\mathbb{C}$-type quadratic differential of order 4 (Theorem 12).

Finally, we describe the method that we use to prove our theorems mentioned above. Our assumption is that a function f is admissible for a quadratic differential Q. To show that f uniquely determines Q, we first write the differential equation (that comes from admissibility) as a power series expansion, and equate coefficients. Then we write the coefficients of the quadratic differential in terms of the quantities that are invariant under reparametrization of the omitted curve of the function. We now give the reason why we use the invariant quantities. When we write the differential equation as a power series expansion, we reparametrize the curve so that $Q(w(t))\dot{w}(t)^2 = -1$ or $-t^2$. Thus this parametrization itself depends on Q, so the coefficients of $w(t)$ depend on Q (our claim is to show that Q is determined by $w(t)$). But the invariant quantities are unchanged under reparametrization.

4.1 The form of the quadratic differential

In the main problem, the function f belongs to either $\mathcal{S}$ or $\mathcal{B}$, but we have not discussed the exact form of quadratic differentials we consider. It can be shown by Schiffer's theorem that if f is extremal for a continuous functional of order n over the class $\mathcal{S}$, then f is admissible for a $\mathbb{C}$-type quadratic differential of order n (see Definition 11). Thus by Statement 2, for order three, we take the quadratic differential of the form

$$\frac{1 + Aw}{w^4} dw^2,$$

and by the same argument for order four, we take the quadratic differential of the form

$$\frac{1 + Aw + Bw^2}{w^5} dw^2.$$

let us move to the unit disk case. We need the following theorem of Roth [9] that gives the form of a quadratic differential for extremal functions of the class $\mathcal{B}$.

Theorem 10. *Let ϕ be a functional of finite degree n on $\mathcal{B}$ with complex derivative Λ. If*

$f \in \mathcal{B}$ is an extremal function for ϕ on $\mathcal{B}$ such that $\Lambda(f; f(z)) \neq 0$, then

$$\left(\frac{\xi f'(\xi)}{f(\xi)}\right)^2 Q(f(\xi)) = R(\xi), \quad \xi \in \mathbb{D}$$

Here the function Q is given by

$$Q(w) = \frac{1}{2}\Lambda\left(f; f(z)\frac{w+f(z)}{w-f(z)}\right) + \frac{1}{2}\overline{\Lambda}\left(f; f(z)\frac{1+\bar{w}f(z)}{1-\bar{w}f(z)}\right)$$
$$- Re\Lambda(f; f(z)) + maxRe\Lambda(f; f(z)(1-pf(z))), \quad (4.1)$$

where p is a function from $\mathbb{D}$ to $\mathbb{C}$ such that $p(0) = 1$ and $Rep(z) > 0$ for all $z \in \mathbb{D}$, and Q is rational of degree $\leqq 2n$.

Thus Q satisfies $\overline{Q(\frac{1}{\bar{w}})} = Q(w)$, so it satisfies the property (2.2). By using this theorem, it can be shown that if f is extremal for a continuous functional of order n over the class $\mathcal{B}$, then f is admissible for a $\mathbb{D}$-type quadratic differential of order n. Thus for order three, by using the property (2.2), the quadratic differential has the form

$$\frac{1}{w^4}(w+\rho e^{ix})(w+\frac{1}{\rho}e^{ix})(w+re^{-ix})(w+\frac{1}{r}e^{-ix})dw^2.$$

In the main problem, we assume that $f \in \mathcal{B}$ maps onto the unit disk $\mathbb{D}$ minus the trajectories of a quadratic differential Q. The omitted curve must touch the boundary of $\mathbb{D}$ unless f is the identity. Thus the quadratic differential Q must have at least a zero on the boundary of $\mathbb{D}$ by observing the local trajectory structure in Figure 2.5. Therefore $r = 1$ or $\rho = 1$. We choose $\rho = 1$ without loss of generality. Then we have the following form of a quadratic differential $\frac{Q(w)}{w^2}dw^2$ for order three :

$$\frac{(w+e^{ix})^2(w+re^{-ix})(w+\frac{1}{r}e^{-ix})}{w^4}dw^2, \quad 0 < r \leqq 1, \ x \in \mathbb{R}.$$

By the same argument, we have the following form of a quadratic differential $\frac{Q(w)}{w^2}dw^2$ for order four :

$$\frac{(w+e^{ix})^2(w+re^{iy})(w+\frac{1}{r}e^{iy})(w+pe^{-i(x+y)})(w+\frac{1}{p}e^{-i(x+y)})}{w^5}dw^2, \quad 0 < r, \ p \leqq 1, \quad x, \ y \in \mathbb{R}.$$

4.2 Motivation

Schiffer's theorem says that every extremal function f for some continuous functional over $\mathcal{S}$ is admissible for a $\mathbb{C}$-type quadratic differential Q, and Theorem 10 in Roth [9] says that every extremal function f for some continuous functional over $\mathcal{B}$ is admissible for a $\mathbb{D}$-type quadratic differential Q. Here one can ask if such function f uniquely determines the quadratic differential Q, which is our main interest. We ended Chapter 3 asking two questions about extension of Pfluger's results. In this thesis we only focus on one part of that problem: does the function uniquely determine the quadratic differential? It is a well-known fact that $f \in \mathcal{S}$ determines a unique $\mathbb{C}$-type quadratic differential Q of order 3. We ask two questions:

- Does this fact extend to a quadratic differential of order 4 (or higher)?
- Does this fact extend to the class $\mathcal{B}$?

We will try to answer these questions in the following sections.

4.3 The complex plane case

It is well known that for $\mathbb{C}$-type quadratic differentials of order 3, the admissible function $f \in \mathcal{S}$ determines a unique quadratic differential up to exceptional cases. The most widely known proof for this case is to use the linear fractional transformation, which results from the quotient of two quadratic differentials. But this method can not be used in other cases because for quadratic differentials of order greater than 3, the quotient is not linear fractional anymore. Thus this method is only for order 3.

Schaeffer and Spencer generalized the result to order n when $n-1$ is prime number. That is, for $\mathbb{C}$-type quadratic differentials of order n, the admissible function $f \in \mathcal{S}$ determines a unique quadratic differential whenever $n-1$ is prime number (Theorem 5). Though its proof is not that complicated, it still does not generalize to all order n.

We apply our method to order 3 and order 4 hoping to generalize to order 5, which is still open. We did not succeed in for the order 5 case because the expressions are too complex, and our main interest is the unit disk case. As a consequence, we can only give a new proof for order 3 and a partial result for order 4.

We need few lemmas first. The expressions in the following lemma are invariant quantities of the curve under reparametrization. We will need these when we prove our theorems in Section 4.3 and 4.4.

Lemma 2. *Let $w(t) : I \longrightarrow \mathbb{C}$ (I is an open interval) be an analytic curve. Then the unit tangent vector is*

$$T(t) = \frac{\dot{w}(t)}{|\dot{w}(t)|}, \tag{4.2}$$

the signed curvature is

$$K(t) = \frac{1}{|\dot{w}(t)|} Im\left(\frac{\ddot{w}(t)}{\dot{w}(t)}\right), \tag{4.3}$$

the derivative of $K(t)$ with respect to arc length function $s(t) = \int_a^t |\dot{w}(u)| du$, $a \in I$ is

$$\frac{dK}{ds} = \frac{1}{|\dot{w}|^2} Im\left(\frac{\dddot{w}}{\dot{w}} - \frac{3}{2}\frac{\ddot{w}^2}{\dot{w}^2}\right), \tag{4.4}$$

and the derivative of $\frac{dK}{ds}$ is

$$\frac{d^2K}{ds^2} = \frac{1}{|\dot{w}|^3}\left(-2Re\left(\frac{\ddot{w}}{\dot{w}}\right) Imp(t) + Imp'(t)\right), \quad where \;\; p(t) = \frac{\dddot{w}}{\dot{w}} - \frac{3}{2}\frac{\ddot{w}^2}{\dot{w}^2} \tag{4.5}$$

Proof. From the definition of the unit tangent vector of $w(t)$, $T(t) = \frac{\dot{w}(t)}{|\dot{w}(t)|}$. The signed curvature of the curve $w = w(t)$ is defined by $Im(\frac{\bar{\dot{w}}\ddot{w}}{|\dot{w}|^3})$ (see Minda [5]). This is invariant under reparametrization when the curve preserves its direction since for any reparametrization $z(t) = w(\alpha(t))$ of $w = w(t)$, we have

$$\dot{z}(t) = \dot{w}(\alpha(t))\dot{\alpha}(t) \quad \text{and} \quad \ddot{z}(t) = \ddot{w}(\alpha(t))\dot{\alpha}(t)^2 + \dot{w}(\alpha(t))\ddot{\alpha}(t).$$

Thus

$$Im\left(\frac{\bar{\dot{z}}\ddot{z}}{|\dot{z}|^3}\right) = Im\left(\frac{\bar{\dot{w}}\ddot{w}\dot{\alpha}^3}{|\dot{w}|^3|\dot{\alpha}|^3}\right) + Im\left(\frac{|\dot{w}|^2\dot{\alpha}\ddot{\alpha}}{|\dot{w}|^3|\dot{\alpha}|^3}\right) = \pm Im\left(\frac{\bar{\dot{w}}\ddot{w}}{|\dot{w}|^3}\right).$$

Therefore the signed curvature of $w(t)$ is

$$K(t) = \frac{1}{|\dot{w}(t)|} Im\left(\frac{\ddot{w}(t)}{\dot{w}(t)}\right).$$

Now to compute $\frac{dK}{ds}$, we need $\frac{d}{dt}|\dot{w}|^m$ (m is integer). let us compute first $\frac{d}{dt}|\dot{w}|$ as follows.

$$\begin{aligned} \frac{d}{dt}|\dot{w}| &= \frac{d}{dt}(\dot{w}\bar{\dot{w}})^{\frac{1}{2}} \\ &= \frac{1}{2(\dot{w}\bar{\dot{w}})^{\frac{1}{2}}}(\ddot{w}\bar{\dot{w}} + \dot{w}\bar{\ddot{w}}) \\ &= \frac{|\dot{w}|^2}{2(\dot{w}\bar{\dot{w}})^{\frac{1}{2}}}\left(\frac{\ddot{w}}{\dot{w}} + \frac{\bar{\ddot{w}}}{\bar{\dot{w}}}\right) \\ &= |\dot{w}| Re\left(\frac{\ddot{w}}{\dot{w}}\right). \end{aligned}$$

Thus

$$\frac{d}{dt}|\dot{w}|^m = m|\dot{w}|^{m-1}|\dot{w}| Re\left(\frac{\ddot{w}}{\dot{w}}\right) = m|\dot{w}|^m Re\left(\frac{\ddot{w}}{\dot{w}}\right). \tag{4.6}$$

By differentiating $K(t) = \frac{1}{|\dot{w}(t)|} Im\left(\frac{\ddot{w}(t)}{\dot{w}(t)}\right)$ and using (4.6), we have

$$\begin{aligned} \frac{dK}{ds} = \frac{dK}{dt}\frac{dt}{ds} &= \frac{1}{|\dot{w}|}\left\{-\frac{1}{|\dot{w}|} Re\left(\frac{\ddot{w}}{\dot{w}}\right) Im\left(\frac{\ddot{w}}{\dot{w}}\right) + \frac{1}{|\dot{w}|} Im\left(\frac{\dddot{w}\dot{w} - \ddot{w}^2}{\dot{w}^2}\right)\right\} \\ &= \frac{1}{|\dot{w}|^2}\left\{Im\left(\frac{\dddot{w}}{\dot{w}} - \frac{\ddot{w}^2}{\dot{w}^2}\right) - \frac{1}{2} Im \frac{\ddot{w}^2}{\dot{w}^2}\right\} \quad \left(Re\alpha Im\alpha = \frac{1}{2} Im\alpha^2, \quad \alpha \in \mathbb{C}\right) \\ &= \frac{1}{|\dot{w}|^2} Im\left(\frac{\dddot{w}}{\dot{w}} - \frac{3}{2}\frac{\ddot{w}^2}{\dot{w}^2}\right). \end{aligned}$$

Now let us compute $\frac{d^2K}{ds^2}$, which is the derivative of $\frac{dK}{ds} = \frac{1}{|\dot{w}|^2} Im p(t)$, where $p(t) = \frac{\dddot{w}}{\dot{w}} - \frac{3}{2}\frac{\ddot{w}^2}{\dot{w}^2}$.

$$\begin{aligned} \frac{d^2K}{ds^2} = \frac{1}{|\dot{w}|}\frac{d}{dt}\frac{dK}{ds} &= \frac{1}{|\dot{w}|}\frac{d}{dt}\left(\frac{1}{|\dot{w}|^2} Im p(t)\right) \\ &= \frac{1}{|\dot{w}|^3}\left(-2Re\left(\frac{\ddot{w}}{\dot{w}}\right) Im p(t) + Im p'(t)\right). \end{aligned}$$

□

We will need the following expressions for invariant quantities of the curve of the form $w(t) = b_1 t + b_3 t^3 + b_5 t^5 + \cdots$.

Lemma 3. *Let $w(t) = b_1 t + b_3 t^3 + b_5 t^5 + \cdots$ be an analytic curve on an interval containing 0. Then*

$$T(0) = \frac{b_1}{|b_1|}, \quad K(0) = 0, \quad \frac{dK}{ds}(0) = \frac{1}{|b_1|^2} Im\left(\frac{6b_3}{b_1}\right), \quad \frac{d^2K}{ds^2}(0) = 0, \tag{4.7}$$

and

$$\frac{d^3K}{ds^3}(0) = \frac{1}{|b_1|^4} Im\left(\frac{120b_5}{b_1} - \frac{180b_3^2}{b_1^2}\right). \tag{4.8}$$

Proof. Since $w(t) = b_1 t + b_3 t^3 + b_5 t^5 + \cdots$, $\dot{w}(t) = b_1 + 3b_3 t^2 + 5b_5 t^4 + \cdots$, $\ddot{w}(t) = 6b_3 t + 20b_5 t^3 + \cdots$, $\dddot{w}(t) = 6b_3 + 60b_5 t^2 + \cdots$, $w^{(4)}(t) = 120b_5 t + \cdots$, and $w^{(5)}(t) = 120b_5 + \cdots$. Thus $\dot{w}(0) = b_1$, $\ddot{w}(0) = 0$, $\dddot{w}(0) = 6b_3$, $w^{(4)}(0) = 0$ and $w^{(5)}(0) = 120b_5$, which implies that $T(0) = \frac{b_1}{|b_1|}$, $K(0) = 0$, $\frac{dK}{ds}(0) = \frac{1}{|b_1|^2} Im(\frac{6b_3}{b_1})$, and $\frac{d^2K}{ds^2}(0) = 0$ by Lemma 2. We need a bit more work to compute $\frac{d^3K}{ds^3}(0)$. We have $\frac{d^2K}{ds^2} = \frac{1}{|\dot{w}|^3}(-2Re(\frac{\ddot{w}}{\dot{w}})Imp(t) + Imp'(t))$, where $p(t) = \frac{\dddot{w}}{\dot{w}} - \frac{3}{2}\frac{\ddot{w}^2}{\dot{w}^2}$ by Lemma 2. Thus

$$\begin{aligned}
\frac{d^3K}{ds^3} &= \frac{dt}{ds}\frac{d}{dt}\left(\frac{dK}{ds^2}\right) \\
&= \frac{1}{|\dot{w}|}\frac{d}{dt}\left\{\frac{1}{|\dot{w}|^3}\left(-2Re\left(\frac{\ddot{w}}{\dot{w}}\right)Imp(t) + Imp'(t)\right)\right\} \\
&= \frac{1}{|\dot{w}|}\left\{\frac{-3}{|\dot{w}|^3}Re\left(\frac{\ddot{w}}{\dot{w}}\right)\left(-2Re\left(\frac{\ddot{w}}{\dot{w}}\right)Imp(t) + Imp'(t)\right)\right. \\
&+ \frac{1}{|\dot{w}|^3}\left(-2Re\left(\frac{\dddot{w}\dot{w} - \ddot{w}^2}{\dot{w}^2}\right)Imp(t) - 2Re\left(\frac{\ddot{w}}{\dot{w}}\right)Imp'(t)\right. \\
&+ \left.\left. Im\left(\frac{w^{(5)}\dot{w} - w^{(4)}\ddot{w}}{\dot{w}^2} - 4\frac{(\dddot{w}^2 + \ddot{w}w^{(4)})\dot{w}^2 - 2\dot{w}^2\dddot{w}\ddot{w}}{\dot{w}^4} + 9\left(\frac{\ddot{w}}{\dot{w}}\right)^2\frac{\dddot{w}\dot{w} - \ddot{w}^2}{\dot{w}^2}\right)\right)\right\}.
\end{aligned}$$

Simplifying this using $\ddot{w}(0) = w^{(4)}(0) = 0$,

$$\frac{d^3K}{ds^3}(0) = \frac{1}{|\dot{w}(0)|^4}\left\{-2Re\frac{\dddot{w}(0)}{\dot{w}(0)}Im\frac{\dddot{w}(0)}{\dot{w}(0)} + Im\left(\frac{w^{(5)}(0)}{\dot{w}(0)} - 4\frac{\dddot{w}(0)^2}{\dot{w}(0)^2}\right)\right\}.$$

Using $2Re\alpha Im\alpha = Im\alpha^2$ and substituting $\dot{w}(0) = b_1$, $\dddot{w}(0) = 6b_3$ and $w^{(5)}(0) = 120b_5$,

$$\frac{d^3K}{ds^3}(0) = \frac{1}{|\dot{w}(0)|^4} Im\left(\frac{w^{(5)}(0)}{\dot{w}(0)} - 5\frac{\dddot{w}(0)^2}{\dot{w}(0)^2}\right) = \frac{1}{|b_1|^4} Im\left(\frac{120b_5}{b_1} - \frac{180b_3^2}{b_1^2}\right).$$

□

The following lemma shows that we can choose a special parametrization when we have a trajectory of a quadratic differential with a regular point. This will simplify computations.

Lemma 4. *If $q(z)dz^2$ has a regular point at z_0, and an analytic curve $\alpha(t)$ with $\alpha(0) = z_0$ satisfies $q(\alpha(t))\dot{\alpha}(t)^2 < 0$ on an interval I containing 0, then there is a parametrization $w(t)$ on some interval I' containing 0 satisfying $q(w(t))\dot{w}(t)^2 = -1$, where $w(0) = z_0$.*

Proof. Since $q(z)dz^2$ has a regular point at z_0, and $\alpha(t)$ such that $\alpha(0) = z_0$ satisfies $q(\alpha(t))\dot{\alpha}(t)^2 < 0$, we can set

$$q(\alpha(t))\alpha'(t)^2 = -\psi(t), \quad \psi(0) \neq 0, \quad t \in \mathbb{C}, \tag{4.9}$$

where ψ is analytic at 0 and positive for real t (note that $\alpha(t)$ has an extension to complex t since α is an analytic curve). Let $h(z)$ be a square root of $q(z)$ near z_0, and then from (4.9) we get

$$h(\alpha(t))^2\alpha'(t)^2 = -\psi(t). \tag{4.10}$$

Let $g(t)$ be a square root of $\psi(t)$ near 0 (the square root exists because $\psi(0) \neq 0$) such that

$$h(\alpha(t))\alpha'(t) = ig(t), \quad g(0) \neq 0. \tag{4.11}$$

Define a function $H(z)$ as $H(z) = \int_{z_0}^{z} h(\xi)d\xi$. Substituting $H'(z) = h(z)$ into (4.11),

$$H'(\alpha(t))\alpha'(t) = ig(t) \quad \text{or} \quad (H \circ \alpha)'(t) = ig(t).$$

Integrating this,

$$(H \circ \alpha)(t) = i \int_0^t g(s)ds = itG(t), \quad \text{for some } G \text{ with } G(0) \neq 0. \tag{4.12}$$

Differentiating this,

$$(H \circ \alpha)'(t) = iG(t) + itG'(t),$$

which implies that

$$(H \circ \alpha)'(0) = iG(0) \neq 0.$$

Thus by Theorem 7.2 in Evgrafov [2], there exists a unique analytic function $\beta(t)$ in a neighborhood of 0 satisfying

$$(H \circ \alpha)(\beta(t)) = it, \quad \beta(0) = 0. \tag{4.13}$$

Differentiating this,

$$H'(\alpha(\beta(t)))\alpha(\beta(t))' = i \quad \text{or} \quad h(\alpha(\beta(t)))\alpha(\beta(t))' = i.$$

Squaring this and setting $\alpha(\beta(t)) = w(t)$,

$$h^2(w(t))w'(t)^2 = -1 \quad \text{or} \quad q(w(t))w'(t)^2 = -1, \quad w(0) = z_0.$$

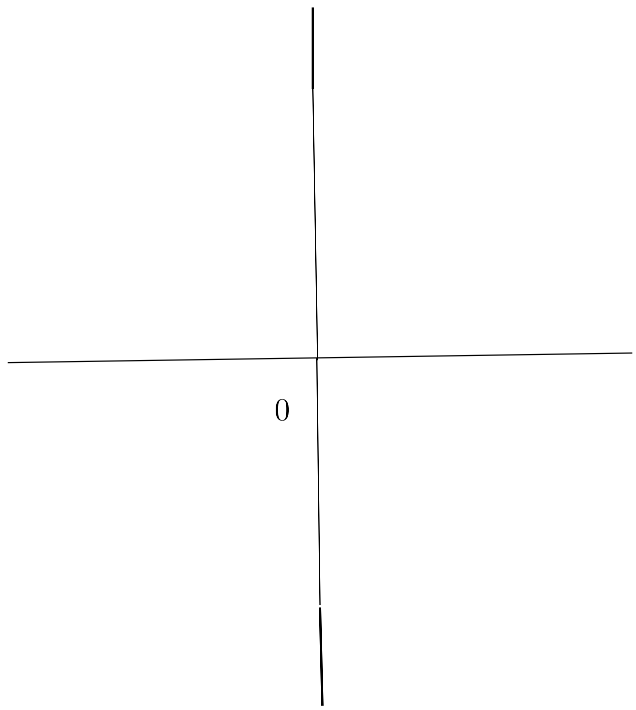
□

Lemma 5. *$f(z) = \frac{z}{1+2ciz-z^2}$, $-1 \leqq c \leqq 1$ maps $\mathbb{D}$ onto the complement of one or two rays extending to infinity and situated on the imaginary axis.*

Proof. To find the omitted curves of the function $f(z)$, let us substitute $z = e^{it}$ into $f(z)$ obtaining

$$\begin{aligned} f(e^{it}) &= \tfrac{e^{it}}{1+2cie^{it}-e^{2it}} \\ &= \tfrac{1}{e^{-it}+2ci-e^{it}} \\ &= \tfrac{1}{2ci-2isint} \end{aligned}$$

Figure 4.1: Omitted arc of $\frac{z}{1+2ciz-z^2}$, $-1 \leqq c \leqq 1$



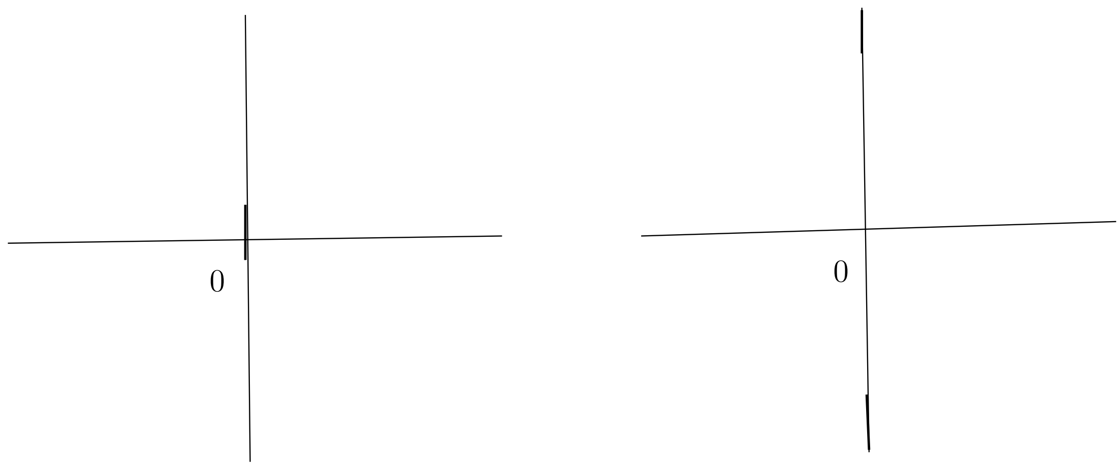
Since $2ci - 2isint$ is an interval containing 0 on the imaginary axis, the omitted arcs of $f(z)$ look like Figure 4.1. If $c = \pm 1$, the functions become rotations $ik(-iz)$ or $-ik(iz)$ of the Koebe function $k(z)$, whose omitted arc consists of one ray. □

Lemma 6. *1) If $-1 < c < 1$, $f(z) = \frac{z}{1+2ciz-z^2}$ is admissible for $P_A(w)dw^2 = \frac{1+Aw}{w^4}dw^2$, then $A = 0$.*

2) If $f \in \mathcal{S}$ is admissible for $\frac{1+Aw}{w^4}dw^2$, $A = 0$, then f is of the type $\frac{z}{1+2ciz-z^2}$, $-1 \leqq c \leqq 1$.

Proof. let us first prove part 1. From Lemma 5, the omitted curves of f look like Figure 4.1. Let $w(t)$ be the trajectory of P_A. Then $w(t) = it$, $t > a > 0$ or $t < b < 0$, for some $a, b \in \mathbb{R}$ (a and b depend on c). Since $w(t)$ is a trajectory of $P_A(w)dw^2$, $-\frac{1+itA}{t^4} < 0$ or $1 + itA > 0$. This only happens when $A = 0$. let us prove part 2. From the assumption, f is admissible for $\frac{1}{w^4}dw^2$. Take the transformation $w = \frac{1}{z}$. Then we have dz^2, which has a regular point at 0. Let $\alpha(t)$ be the trajectory of dz^2 such that $\alpha(0) = 0$. Then $\alpha(t)$ is uniquely determined

Figure 4.2: A trajectory of dz^2 at the origin and its inverted trajectory



by Theorem 5.5 in Strebel [13] that states that there exists a uniquely determined trajectory through every regular point. $\alpha(t)$ looks like Figure 4.2 (left). Thus the trajectory of $\frac{1}{w^4}dw^2$ looks like Figure 4.2 (right). By the uniqueness in the Riemann mapping theorem (Duren [1]), the function f is determined by the omitted arcs. From Lemma 5, we know that the functions of the type $\frac{z}{1+2ciz-z^2}$ has the omitted arc as in Figure 4.2 (right). It remains to show that $-1 \leqq c \leqq 1$. Since f belongs to the class $\mathcal{S}$, by the Koebe one quarter theorem (Duren [1]), c must be in the range $-1 \leqq c \leqq 1$. Thus f is of the type $\frac{z}{1+2ciz-z^2}$, $-1 \leqq c \leqq 1$.

□

Now, we are ready to give the following theorem in the third order case.

Theorem 11 (order3)**.** *Let $f \in \mathcal{S}$ be admissible for a quadratic differential $P(w)dw^2 = \frac{1+Aw}{w^4}dw^2$. Then f uniquely determines P unless f is one of the functions $k(z)$, $-k(-z)$, $ik(-iz)$ and $-ik(iz)$ ($k(z)$ is the Koebe function).*

Proof. Assume that f is not one of the functions $k(z)$, $-k(-z)$, $ik(-iz)$ and $-ik(iz)$.

<u>Case 1</u>

Suppose that f is of the type $\frac{z}{1+2ciz-z^2}$, $-1 \leqq c \leqq 1$. If $c = \pm 1$, then f is $ik(-iz)$ or $-ik(iz)$.

This contradicts the assumption. Thus $-1 < c < 1$. Now since the function $f(z) = \frac{z}{1+2ciz-z^2}$, $-1 < c < 1$, is admissible for $P(w) = \frac{1+Aw}{w^4}dw^2$, $A = 0$ by part 1 of Lemma 6. Thus P is determined by f.

Case 2

Suppose that f is not of the type $\frac{z}{1+2ciz-z^2}$, $-1 \leqq c \leqq 1$. By part 2 of Lemma 6, we conclude that $A \neq 0$. $w = \frac{1}{z^2}$ transforms $\frac{1+Aw}{w^4}dw^2$ to $4(z^2 + A)dz^2$. Let $Q(z)dz^2 = 4(z^2 + A)dz^2$, and $\alpha(t)$ be a trajectory of Q such that $\alpha(0) = 0$. Then $Q(z)dz^2$ has a regular point at 0, and $\alpha(t)$ such that $\alpha(0) = 0$ satisfies $Q(\alpha(t))\dot{\alpha}(t)^2 < 0$. Thus by Lemma 4, there is a parametrization such that

$$Q(w(t))\dot{w}(t)^2 = -1, \quad \text{where} \quad w(0) = 0.$$

We will show that $w(t)$ is a trajectory of only one $Q(w)dw^2$ satisfying

$$Q(w(t))\dot{w}(t)^2 = 4(w(t)^2 + A)\dot{w}(t)^2 = -1. \tag{4.14}$$

Since $w(0) = 0$, we can set $w(t) = b_1 t + b_2 t^2 + b_3 t^3 + \cdots$. Substituting this in (4.14),

$$4((b_1 t + b_2 t^2 + b_3 t^3 + \cdots)^2 + A)(b_1 + 2b_2 t + 3b_3 t^2 + \cdots)^2 = -1$$

$$\text{or} \quad 4(b_1^2 t^2 + 2b_1 b_2 t^3 + \cdots + A)(b_1^2 + 4b_1 b_2 t + (6b_1 b_3 + 4b_2^2)t^2 + \cdots) = -1$$

$$\text{or} \quad 4(Ab_1^2 + 4Ab_1 b_2 t + (A(6b_1 b_3 + 4b_2^2) + b_1^4)t^2 + \cdots) = -1$$

Note that $b_1 \neq 0$ since $b_1 = 0$ contradicts the above expression. Equating the first three coefficients,

$$A = -\frac{1}{4b_1^2}, \quad 4Ab_1 b_2 = 0, \quad \text{and} \quad 6Ab_1 b_3 + 4Ab_2^2 + b_1^4 = 0. \tag{4.15}$$

We have $b_2 = 0$ from the equation $4Ab_1 b_2 = 0$, and so two different expressions for A from (4.15) reduce to

$$A = -\frac{1}{4b_1^2} \quad \text{and} \quad A = -\frac{1}{6}\frac{b_1^3}{b_3}.$$

Equating these two,

$$b_3 = \frac{2}{3} b_1^5.$$

To show that f uniquely determines P, we claim that $w(t)$ is a trajectory of only one Q. Thus the key here is to express A in terms of $T(0)$ and $\frac{dK}{ds}(0)$ of $w(t)$. By Lemma 3,

$$T(0) = \frac{b_1}{|b_1|} \quad \text{and} \quad \frac{dK}{ds}(0) = \frac{1}{|b_1|^2} Im\left(\frac{6b_3}{b_1}\right). \tag{4.16}$$

Substituting $b_3 = \frac{2}{3}b_1^5$ into the equation $\frac{dK}{ds}(0) = \frac{1}{|b_1|^2} Im\left(\frac{6b_3}{b_1}\right)$ in (4.16),

$$\frac{dK}{ds}(0) = \frac{4}{|b_1|^2} Im b_1^4 = 4|b_1|^2 Im\left(\frac{b_1}{|b_1|}\right)^4 = 4|b_1|^2 Im T(0)^4. \tag{4.17}$$

Thus if $\frac{dK}{ds}(0) \neq 0$,

$$\frac{1}{|b_1|^2} = \frac{4 Im T(0)^4}{\frac{dK}{ds}(0)}.$$

Combining this with $A = -\frac{1}{4b_1^2}$ in (4.15) and using $b_1 = |b_1|T(0)$, if $\frac{dK}{ds}(0) \neq 0$,

$$A = -\frac{1}{4b_1^2} = -\frac{1}{4|b_1|^2 T(0)^2} = -\frac{Im T(0)^4}{\frac{dK}{ds}(0) T(0)^2}. \tag{4.18}$$

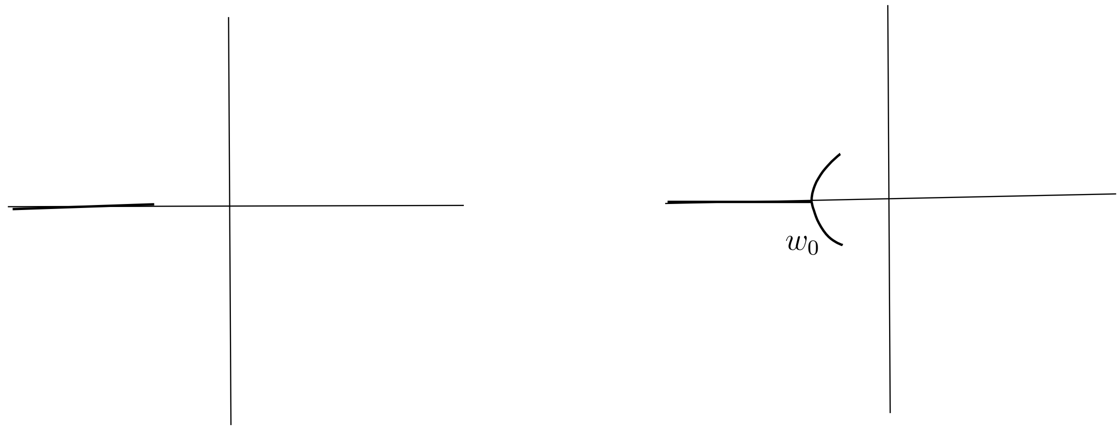
Thus if $\frac{dK}{ds}(0) \neq 0$, A is uniquely determined by $T(0)$ and $\frac{dK}{ds}(0)$; equivalently, P is determined by f. If $\frac{dK}{ds}(0) = 0$, from the equation (4.17),

$$\frac{dK}{ds}(0) = 0 \;\Rightarrow\; Im T(0)^4 = 0 \;\Rightarrow\; Im \frac{b_1^4}{|b_1|^4} = 0$$

$$\Rightarrow\; Im b_1^4 = 0 \;\Rightarrow\; b_1^4 \text{ is real } \;\Rightarrow\; b_1^2 \text{ is real or imaginary.}$$

Thus we can conclude that A is real or imaginary from the equation $A = -\frac{1}{4b_1^2}$ in (4.18). If A is real, the trajectories are either a ray on the real axis through infinity or a ray with a fork through infinity by observing the trajectory structure of the quadratic differential $\frac{1+Aw}{w^4}$ when A is real (see Figure 2.6). See Figure 4.3 to see the rough shape of the trajectories. If the trajectory is a ray with a fork, f determines the location w_0 of the fork (see Figure

Figure 4.3: Shape of the trajectories of $\frac{1+Aw}{w^4}$, A is real



4.3-right). Since $-\frac{1}{A} = w_0$, f determines A. If the trajectory is a ray, the admissible function is the Koebe function $k(z)$ or $-k(-z)$. But we assumed f was not one of these two functions. If A is imaginary, the trajectory is a single ray on the imaginary axis through infinity by observing the trajectory structure of the quadratic differential $\frac{1+Aw}{w^4}$ when A is imaginary (see Figure 2.7). Thus the admissible function is $ik(-iz)$ or $-ik(iz)$, which contradicts the assumption. Thus we can conclude that f uniquely determines P unless f is one of the functions $k(z)$, $-k(-z)$, $ik(-iz)$ and $-ik(iz)$ ($k(z)$ is the Koebe function). □

Let us give a partial result in the 4-th order case. The proof is similar to the one in Theorem 11.

Theorem 12. *Let $f \in \mathcal{S}$ be admissible for a quadratic differential $P(w)dw^2 = \frac{1+Aw+Bw^2}{w^5}dw^2$ ($B \neq 0$). Then the derivatives $\frac{dK}{ds}(0)$ and $\frac{d^3K}{ds^3}(0)$ of a parametrization of the omitted curve of f are*

$$\frac{dK}{ds}(0) = -Im\left(T(0)^2\frac{A}{B}\right) \quad \textit{and} \quad \frac{d^3K}{ds^3}(0) = 48T(0)^4 Im\left(\frac{A^2}{6B^2} - \frac{1}{4B}\right).$$

Proof. The transformation $\frac{1}{z^2}$ transforms $\frac{1+Aw+Bw^2}{w^5}dw^2$ to $4(z^4+Az^2+B)dz^2$. Let $Q(z)dz^2 = 4(z^4 + Az^2 + B)dz^2$, and $\alpha(t)$ be a trajectory of Q such that $\alpha(0) = 0$. Then $Q(z)dz^2$ has a regular point at 0, and $\alpha(t)$ such that $\alpha(0)=0$ satisfies $Q(\alpha(t))\dot{\alpha}(t)^2<0$. Thus by Lemma 4, there is a parametrization such that

$$Q(w(t))\dot{w}(t)^2=-1, \quad \text{where} \;\; w(0)=0,$$

which implies that

$$Q(w(t))\dot{w}(t)^2=4(w(t)^4+Aw(t)^2+B)\dot{w}(t)^2=-1. \tag{4.19}$$

We can set $w(t)=b_1t+b_2t^2+b_3t^3+\cdots$ since $w(0)=0$. Substituting this in (4.19),

$$4((b_1t+b_2t^2+b_3t^3+\cdots)^4+A(b_1t+b_2t^2+b_3t^3+\cdots)^2+B)$$
$$(b_1+2b_2t+3b_3t^2+4b_4t^3+5b_5t^4+\cdots)^2=-1$$

or

$$4(b_1^4t^4+\cdots+A(b_1^2t^2+2b_1b_2t^3+(b_2^2+2b_1b_3)t^4+\cdots)+B)$$
$$(b_1^2+4b_1b_2t+(4b_2^2+6b_1b_3)t^2+8b_1b_4t^3+(10b_1b_5+9b_3^2+16b_2b_4)t^4+\cdots)=-1$$

or

$$4(Bb_1^2+4Bb_1b_2t+(Ab_1^4+4Bb_2^2+6Bb_1b_3)t^2+(2Ab_1^3b_2+8Bb_1b_4)t^3$$
$$+(b_1^6+Ab_1^2b_2^2+2Ab_1^3b_3+10Bb_1b_5+9Bb_3^2+16Bb_2b_4)t^4+\cdots)=-1. \tag{4.20}$$

Equating the first five coefficients, we get

$$B=-\frac{1}{4b_1^2},\; 4Bb_1b_2=0,\; Ab_1^4+4Bb_2^2+6Bb_1b_3=0,\; 2Ab_1^3b_2+8Bb_1b_4=0$$
$$\text{and} \;\; b_1^6+Ab_1^2b_2^2+2Ab_1^3b_3+10Bb_1b_5+9Bb_3^2+16Bb_2b_4=0. \tag{4.21}$$

We have $b_2=0$ from the equation $4Bb_1b_2=0$. Substituting $b_2=0$ into the equation $2Ab_1^3b_2+8Bb_1b_4=0$, we also have $b_4=0$. Setting $b_2=b_4=0$ in (4.21),

$$B=-\frac{1}{4b_1^2},\;\; Ab_1^4+6Bb_1b_3=0, \quad \text{and} \;\; b_1^6+2Ab_1^3b_3+10Bb_1b_5+9Bb_3^2=0.$$

Combining these three equations,

$$B = -\frac{1}{4b_1^2}, \ \ A = \frac{3}{2}\frac{b_3}{b_1^5}, \quad \text{and} \quad b_5 = \frac{1}{10}\frac{39b_3^2 + 4b_1^8}{b_1}, \tag{4.22}$$

Now our purpose is to express $\frac{dK}{ds}(0)$ and $\frac{d^3K}{ds^3}(0)$ in terms of A and B. We have the following by Lemma 3,

$$T(0) = \frac{b_1}{|b_1|}, \quad \frac{dK}{ds}(0) = \frac{1}{|b_1|^2} Im\left(\frac{6b_3}{b_1}\right),$$

$$\text{and} \quad \frac{d^3K}{ds^3}(0) = \frac{1}{|b_1|^4} Im\left(\frac{120b_5}{b_1} - \frac{180b_3^2}{b_1^2}\right). \tag{4.23}$$

In (4.22), $A = \frac{3}{2}\frac{b_3}{b_1^5}$ or

$$\frac{b_3}{b_1} = \frac{2}{3}b_1^4 A. \tag{4.24}$$

Combining this with $B = -\frac{1}{4b_1^2}$ in (4.22) and using $T(0) = \frac{b_1}{|b_1|}$,

$$\frac{1}{|b_1|^2}\frac{b_3}{b_1} = \frac{T(0)^2}{b_1^2}\frac{b_3}{b_1} = \frac{T(0)^2}{b_1^2}\frac{2}{3}b_1^4 A = \frac{2}{3}T(0)^2 b_1^2 A = \frac{2}{3}T(0)^2\frac{-1}{4B}A = -\frac{1}{6}T(0)^2\frac{A}{B}.$$

Thus

$$\frac{dK}{ds}(0) = \frac{1}{|b_1|^2} Im\left(\frac{6b_3}{b_1}\right) = -Im\left(T(0)^2\frac{A}{B}\right). \tag{4.25}$$

Now substituting $b_5 = \frac{1}{10}\frac{39b_3^2+4b_1^8}{b_1}$ in (4.22) into the equation for $\frac{d^3K}{ds^3}(0)$ in (4.23) and using $T(0) = \frac{b_1}{|b_1|}$,

$$\begin{aligned}
\frac{d^3K}{ds^3}(0) &= \frac{48}{|b_1|^4} Im\left(6\frac{b_3^2}{b_1^2} + b_1^6\right) \\
&= \frac{48T(0)^4}{b_1^4} Im\left(6\frac{b_3^2}{b_1^2} + b_1^6\right) \\
&= 48T(0)^4 Im\left(6\frac{b_3^2}{b_1^6} + b_1^2\right).
\end{aligned}$$

In (4.24), $\frac{b_3}{b_1} = \frac{2}{3}b_1^4 A$, and so

$$\frac{b_3^2}{b_1^6} = \left(\frac{b_3}{b_1}\right)^2\frac{1}{b_1^4} = \frac{4}{9}b_1^8 A^2 \frac{1}{b_1^4} = \frac{4}{9}b_1^4 A^2 = \frac{4}{9}\frac{1}{16B^2}A^2 = \frac{A^2}{36B^2}$$

since $B = -\frac{1}{4b_1^2}$. Thus

$$\frac{d^3K}{ds^3}(0) = 48T(0)^4 Im\left(\frac{A^2}{6B^2} - \frac{1}{4B}\right). \tag{4.26}$$

□

Remark 8. *In the above proof, we have two expressions for* $\frac{dK}{ds}(0)$ *from (4.25) and* $\frac{d^3K}{ds^3}(0)$ *from (4.26), which include* A *and* B*. We have obtained these two expressions by combining the derivatives* $T(0)$, $\frac{dK}{ds}(0)$ *and* $\frac{d^3K}{ds^3}(0)$ *of a curve, which is a trajectory of the given quadratic differential, with the equations that we get from the first five coefficients of the power series (4.20) for (4.19). Thus one might show that* f *determines a unique* P *by using more equations from the power series (4.20) for (4.19) and higher derivatives of the curve, since the key problem of the uniqueness determination is to express* A *and* B *in terms of the derivatives of the curve.*

It should be mentioned that we did not keep trying to get more expressions because it seems too hard, and the order four case is already known. However, we state the theorem because the relations between the coefficients of the quadratic differentials and the geometric properties of the omitted curve are new.

4.4 The unit disk case

Until now, we only paid attention to the functions from $\mathbb{D}$ to $\mathbb{C}$. Now, we turn our attention to the functions from $\mathbb{D}$ to $\mathbb{D}$, which is our main interest. Here is the question again: If $f \in \mathcal{B}$ is admissible for a $\mathbb{D}$-type quadratic differential Q, does f uniquely determine Q? There are no known results about this question. We prove that such function uniquely determines the $\mathbb{D}$-type quadratic differential of order 3 up to exceptional cases, and give a partial result for a $\mathbb{D}$-type quadratic differential of order 4.

We give few lemmas first. The following lemma shows that we can choose a convenient parametrization when we have a trajectory of a quadratic differential with a double zero. The proof is very similar to that of Lemma 4.

Lemma 7. *If $q(z)dz^2$ has a double zero at z_0, and an analytic curve $w(t)$ with $w(0) = z_0$ satisfies $q(w(t))\dot{w(t)}^2 < 0$ on an interval I containing* 0*, then there is a parametrization $\beta(t)$ on some interval I' containing* 0 *satisfying*

$$q(\beta(t))\dot{\beta}(t)^2 = -t^2, \quad \textit{where} \ \ \beta(0) = z_0.$$

Proof. Since $q(z)dz^2$ has a double zero at z_0, there is a function $\phi(z)$ analytic at z_0 satisfying

$$q(z) = (z - z_0)^2\phi(z), \quad \phi(z_0) \neq 0. \tag{4.27}$$

Since $w(0) = z_0$, $w(t) = z_0 + a_1 t + a_2 t^2 + \cdots$, and so

$$w(t) - z_0 = tp(t), \quad p(0) \neq 0.$$

Combining this with (4.27),

$$\begin{aligned} q(w(t))\dot{w}(t)^2 &= (w(t) - z_0)^2\phi(w(t))\dot{w}(t)^2 \\ &= t^2 p(t)^2\phi(w(t))\dot{w}(t)^2. \end{aligned}$$

Since $q(w(t))\dot{w}(t)^2$ is negative, we can rewrite this as

$$q(w(t))w'(t)^2 = -t^2\psi(t), \quad t \in \mathbb{C}, \tag{4.28}$$

where $\psi(t)$ is positive for real t and $\psi(0) \neq 0$ (note that $w(t)$ has an extension to complex t since w is an analytic curve). Let $h(z)$ be a square root of $q(z)$ near z_0, and then $q(z) = h(z)^2$ near $z = z_0$. Substituting $q(z) = h(z)^2$ into (4.28),

$$h(w(t))^2 w'(t)^2 = -t^2\psi(t).$$

Let $g(t)$ be a square root of $\psi(t)$ near 0 (the square root exists because $\psi(0) \neq 0$) such that

$$h(w(t))w'(t) = itg(t), \quad g(0) \neq 0. \tag{4.29}$$

Now we want to integrate (4.29), and so we define the function $H(z)$ as

$$H(z) = \int_{z_0}^{z} h(\xi)d\xi.$$

Differentiating this, $H'(z) = h(z)$. Substituting this into (4.29),

$$H'(w(t))w'(t) = itg(t) \;\; or \;\; (H \circ w)'(t) = itg(t).$$

Integrating this,

$$(H \circ w)(t) = i\int_0^t sg(s)ds = it^2G(t), \;\; \text{for some } G \text{ with } \; G(0) \neq 0. \tag{4.30}$$

Define Φ to be

$$\Phi(t) = t\sqrt{iG(t)} \tag{4.31}$$

for some branch of square root in a neighborhood of $iG(0)$. Differentiating this,

$$\Phi'(t) = \sqrt{iG(t)} + t(\sqrt{iG(t)})',$$

which implies that

$$\Phi'(0) = \sqrt{iG(0)} \neq 0.$$

Thus by Theorem 7.2 in Evgrafov [2], there exists a unique analytic function $\alpha(t)$ in a neighborhood of 0 satisfying

$$\Phi(\alpha(t)) = \sqrt{\frac{i}{2}}t, \quad \alpha(0) = 0 \tag{4.32}$$

for either choice of $\sqrt{i}$. Combining this with (4.31),

$$(H \circ w)(\alpha(t)) = it^2G(\alpha(t)) = \Phi^2(\alpha(t)) = \frac{it^2}{2}.$$

Differentiating this,

$$H'(w(\alpha(t)))\{w(\alpha(t))\}' = it \quad \text{or} \quad h(w(\alpha(t)))\{w(\alpha(t))\}' = it.$$

Squaring this and setting $w(\alpha(t)) = \beta(t)$,

$$h^2(\beta(t))\beta'(t)^2 = -t^2 \quad \text{or} \quad q(\beta(t))\beta'(t)^2 = -t^2, \quad \beta(0) = 0.$$

□

Now we are ready to prove the following theorem for the third order case.

Theorem 13 (order3). *Let $f \in \mathcal{B}$ be admissible for a quadratic differential $\frac{Q(w)}{w^2}dw^2$ of the form*

$$\frac{Q(w)}{w^2}dw^2 = \frac{(w+e^{ix})^2(w+re^{-ix})(w+\frac{1}{r}e^{-ix})}{w^4}dw^2, \quad 0 < r \leqq 1, \; 0 \leqq x < 2\pi. \tag{4.33}$$

Then f uniquely determines $\frac{Q(w)}{w^2}$ unless f is one of the functions $p_s(z)$, $-p_s(-z)$, $ip_s(-iz)$ and $-ip_s(iz)$ ($p_s(z)$ is the Pick function).

Proof. Assume that f is not one of $p_s(z)$, $-p_s(-z)$, $ip_s(-iz)$ and $-ip_s(iz)$. Let $\alpha(t)$ be the parametrization of the omitted curve of f such that $\alpha(0) = -e^{ix}$. We separate into two cases.

<u>Case 1</u>

Assume that $\alpha(t)$ has a different angle from $\frac{\pi}{2}$ with the boundary of $\mathbb{D}$ at $-e^{ix}$. This is only possible if $r = 1$ and $e^{ix} = e^{-ix}$ (i.e $sinx = 0$). Thus $\alpha(t)$ uniquely determines $\frac{Q(w)}{w^2}dw^2$. Thus $\frac{Q(w)}{w^2}dw^2$ is uniquely determined by f.

<u>Case 2</u>

Assume that $\alpha(t)$ has an angle of $\frac{\pi}{2}$ with the boundary of $\mathbb{D}$ at $-e^{ix}$. Then $\frac{Q(w)}{w^2}dw^2$ has a double zero at $-e^{ix}$, and $\alpha(t)$ such that $\alpha(0) = -e^{ix}$ satisfies a differential equation $\frac{Q(\alpha(t))}{\alpha(t)^2}\dot{\alpha}(t)^2 < 0$. Thus by Lemma 7, there is a parametrization so that

$$\frac{Q(w(t))}{w(t)^2}\dot{w}(t)^2 = -t^2, \quad \text{where} \quad w(0) = -e^{ix}. \tag{4.34}$$

Therefore we can set $w(t) = -e^{ix} + b_1 t + b_2 t^2 + \cdots$, where $b_1, b_2, \cdots$ are complex constants. Letting $u = -e^{ix}$ and substituting $w(t) = u + b_1 t + b_2 t^2 + \cdots$ into the power series for $\frac{Q(w(t))}{w(t)^2} \dot{w(t)}^2 + t^2 = 0$, we get the following (We used 'Maple' to get the following power series expansion).

$$\begin{aligned} &\frac{-rub_1^4 + u^2 b_1^4 + r^2 u^2 b_1^4 - rb_1^4 - ru^6}{ru^6} t^2 \\ &+ \frac{b_1^3(-2ru^4 b_1^2 + 3u^2 b_1^2 + 3r^2 u^2 b_1^2 - 4rb_1^2 + 6ru^5 b_2 - 6u^3 b_2 - 6r^2 u^3 b_2 + 6rub_2)}{ru^7} t^3 + \cdots = 0. \end{aligned} \tag{4.35}$$

Thus we have all the coefficients of t in the above expression zero. In particular, we set the coefficient of t^3 zero obtaining

$$-2ru^4 b_1^2 + 3u^2 b_1^2 + 3r^2 u^2 b_1^2 - 4rb_1^2 + 6ru^5 b_2 - 6u^3 b_2 - 6r^2 u^3 b_2 + 6rub_2 = 0. \tag{4.36}$$

Keep in mind that we only need to show that r is uniquely determined by $T(0)$ and $K(0)$ of the curve $w(t)$ since the given function f determines $w(t)$, and this curve $w(t)$ determines the unit tangent vector at the origin $T(0)$ and the signed curvature at the origin $K(0)$. Also, notice that the function of r, $g(r) = r + \frac{1}{r}$ is a one-one function since $g'(r) = 1 - \frac{1}{r^2} < 0$ for $0 < r < 1$. Thus it suffices to manipulate the equation (4.36) so that $r + \frac{1}{r}$ is expressed in terms of $T(0)$ and $K(0)$. These are as follows by Lemma 2.

$$T(0) = \frac{b_1}{|b_1|} \text{ and } K(0) = \frac{2}{|b_1|} Im\left(\frac{b_2}{b_1}\right).$$

First, by dividing (4.36) by r, and then factoring left side by $r + \frac{1}{r}$, we get

$$\left(r + \frac{1}{r}\right)\left(3u^2 b_1^2 - 6u^3 b_2\right) = 2u^4 b_1^2 + 4b_1^2 - 6u^5 b_2 - 6ub_2.$$

Note that $b_1 \neq 0$ by the expression (4.35). Dividing each side of this equation by $b_1|b_1|$ observing the expression of $T(0)$ and $K(0)$, we get

$$\left(r + \frac{1}{r}\right)\left(3u^2 \frac{b_1}{|b_1|} - 6u^3 \frac{b_2}{|b_1|b_1}\right) = 2u^4 \frac{b_1}{|b_1|} + 4\frac{b_1}{|b_1|} - 6u^5 \frac{b_2}{|b_1|b_1} - 6u \frac{b_2}{|b_1|b_1}.$$

Next, dividing each side of this equation by u^3,

$$\left(r+\frac{1}{r}\right)\left(3\frac{b_1}{|b_1|}\frac{1}{u}-6\frac{b_2}{|b_1|b_1}\right)=2\frac{b_1}{|b_1|}u+4\frac{b_1}{|b_1|}\frac{1}{u^3}-6\frac{b_2}{|b_1|b_1}\left(u^2+\frac{1}{u^2}\right).$$

This equation is equivalent to the following since $u^2+\frac{1}{u^2}=e^{2ix}+e^{-2ix}=2cos2x\ (u=-e^{ix})$.

$$\left(r+\frac{1}{r}\right)\left(3\frac{b_1}{|b_1|}\frac{1}{u}-6\frac{b_2}{|b_1|b_1}\right)=2\frac{b_1}{|b_1|}u+4\frac{b_1}{|b_1|}\frac{1}{u^3}-6\frac{b_2}{|b_1|b_1}2cos2x. \tag{4.37}$$

Now we will take the imaginary part of each side of this equation by using the facts that

$$Im\alpha\beta = Re\alpha Im\beta + Im\alpha Re\beta, \quad \text{for} \quad \alpha,\ \beta\in\mathbb{C}$$

$$Imu=-sinx,\ Reu=-cosx$$

$$Im\frac{1}{u}=sinx,\ Re\frac{1}{u}=-cosx$$

$$Im\frac{1}{u^3}=sin3x,\ Re\frac{1}{u^3}=-cos3x$$

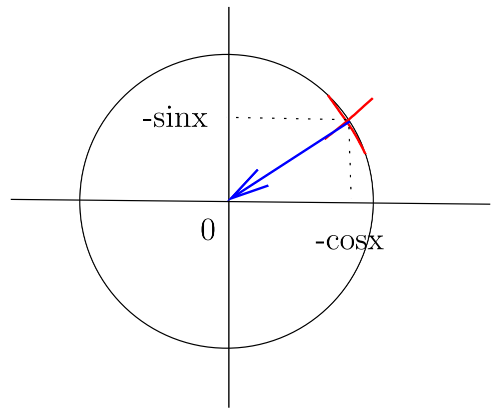
obtaining

$$\begin{aligned}&3\left(r+\frac{1}{r}\right)(ReT(0)sinx-ImT(0)cosx-K(0))\\&=2((2sin3x-sinx)ReT(0)-(2cos3x+cosx)ImT(0)-3cos2xK(0)).\end{aligned} \tag{4.38}$$

Notice that we actually know what $T(0)$ is by the trajectory structure as Figure 4.4 suggests. As you can see in Figure 4.4, the two trajectories intersecting at $-e^{ix}$ on the boundary of the unit disk are orthogonal each other because the quadratic differential has a double zero, and $-e^{ix}$ is the value of $w(t)$ when $t=0$. Thus

$$T(0)=e^{ix}=(cosx,sinx).$$

Figure 4.4: $T(0)$ by the trajectory structure



Therefore from the right hand side of the equation (4.38), we get

$$\begin{aligned} & (2sin3x - sinx)ReT(0) - (2cos3x + cosx)ImT(0) - 3cos2xK(0) \\ = & (2sin3x - sinx)cosx - (2cos3x + cosx)sinx - 3cos2xK(0) \\ = & -2sinxcosx - 2(sinxcos3x - cosxsin3x) - 3cos2xK(0) \\ = & -2sinxcosx - 2sin(x - 3x) - 3cos2xK(0) \\ = & -2sinxcosx + 4sinxcosx - 3cos2xK(0) \\ = & 2sinxcosx - 3cos2xK(0), \end{aligned}$$

and from the left hand side of the equation (4.38), we get

$$ReT(0)sinx - ImT(0)cosx - K(0) = cosxsinx - sinxcosx - K(0) = -K(0).$$

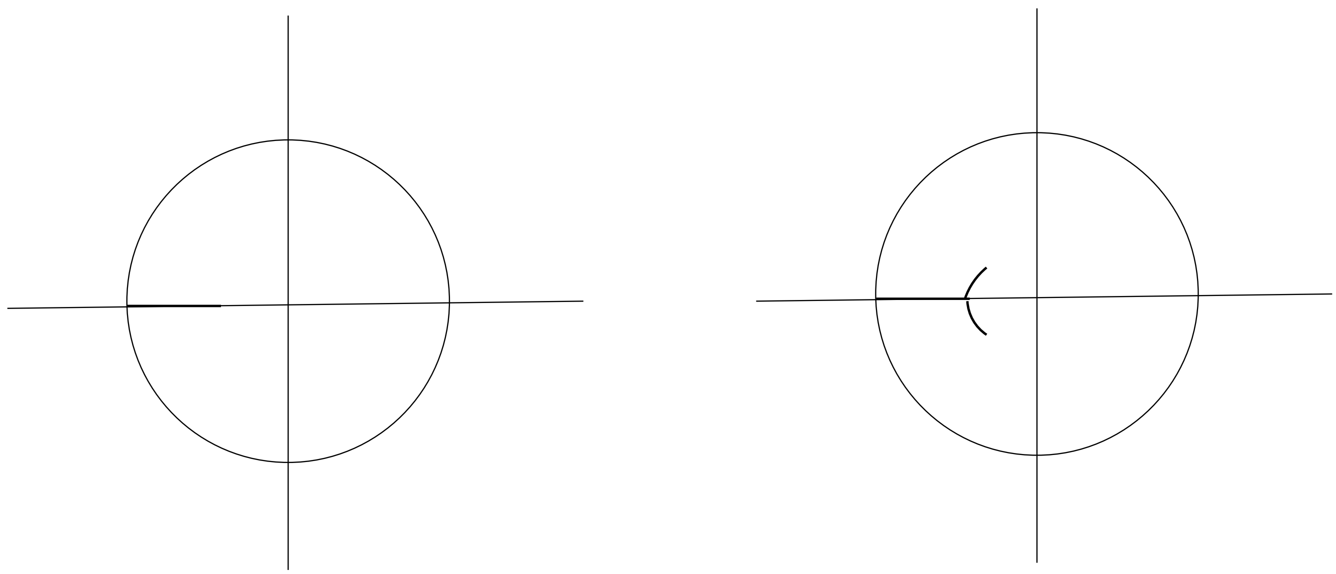
Thus the equation (4.38) reduces to

$$3\left(r + \frac{1}{r}\right)K(0) = 2(3cos2xK(0) - 2sinxcosx) \tag{4.39}$$

If $K(0) \neq 0$, $r + \frac{1}{r}$ is uniquely determined by $K(0)$. That is, $\frac{Q(w)}{w^2}dw^2$ is uniquely determined by f.

If $K(0) = 0$, either $sinx = 0$ or $cosx = 0$ from the equation (4.39). If $sinx = 0$ and $r = 1$,

Figure 4.5: Shape of the trajectory of $\frac{Q(w)}{w^2}dw^2$, $x = 0$ and $r \neq 1$



then the quadratic differential $\frac{Q(w)}{w^2}dw^2$ has a zero of order 4 at $-e^{ix}$. This contradicts the assumption of Case 2 that $\alpha(t)$ has an angle of $\frac{\pi}{2}$ with the boundary of $\mathbb{D}$ at $-e^{ix}$. If $sinx = 0$ and $r \neq 1$, then the trajectory of $\frac{Q(w)}{w^2}dw^2$ is either a straight slit or a straight slit with a fork by the trajectory structure of $\frac{Q(w)}{w^2}dw^2$. See Figure 4.5 to see the rough shape of the trajectories. If it is a fork, then the function determines the location of the zero, which is $-r$ $(x = 0)$ and r $(x = \pi)$. Thus $\frac{Q(w)}{w^2}dw^2$ is determined by f. If it is a straight slit, the admissible function is the Pick function $p_s(z)$ or $-p_s(-z)$ by the uniqueness in the Riemann mapping theorem. But we assumed f was not one of these two functions. If $cosx = 0$, then the trajectory is a straight slit. The admissible function is $ip_s(-iz)$ or $-ip_s(iz)$ by the uniqueness in the Riemann mapping theorem. Thus $\frac{Q(w)}{w^2}dw^2$ is determined by f unless f is one of the functions $p_s(z)$, $-p_s(-z)$, $ip_s(-iz)$ and $-ip_s(iz)$. $\square$

Example 9 shows that the hypothesis of Theorem 13 can not be weakened.

Let us give the partial result of the 4th order case. The proof is similar to the one in Theorem

13.

Theorem 14. *Let $f \in \mathcal{B}$ be admissible for a quadratic differential $\frac{Q(w)}{w^2}dw^2$ of the form*

$$\frac{Q(w)}{w^2}dw^2 = \frac{(w+e^{ix})^2(w+re^{iy})\left(w+\frac{e^{iy}}{r}\right)\left(w+pe^{-i(x+y)}\right)\left(w+\frac{1}{p}e^{-i(x+y)}\right)}{w^5}dw^2, \quad (4.40)$$

$$0 < r,\ p \leqq 1, \quad 0 \leqq x,\ y < 2\pi.$$

Then we have the relation

$$3K(0)\{2cos3x + 2cos(x+2y) - 2(r+\frac{1}{r})cos(2x+y) - 2(p+\frac{1}{p})cos(x-y) + (r+\frac{1}{r})(p+\frac{1}{p})\}$$

$$= (ReT(0)cosx + ImT(0)sinx)\{-2(r+\frac{1}{r})sin(2x+y) - 2(p+\frac{1}{p})sin(x-y) + 4sin3x\}$$

$$+ (-ReT(0)sinx + ImT(0)cosx)\{4(r+\frac{1}{r})cos(2x+y) + 4(p+\frac{1}{p})cos(x-y) - 3(r+\frac{1}{r})(p+\frac{1}{p})$$

$$+ 2(r+\frac{1}{r})cos(2x+y) + 2(p+\frac{1}{p})cos(x-y) - 2cos3x - 6cos(x+2y) - 4cos3x\} \quad (4.41)$$

between x, y, $r+\frac{1}{r}$, $p+\frac{1}{p}$, and $T(0)$, $K(0)$.

Proof. Let $\alpha(t)$ be the parametrization of the omitted curve of f such that $\alpha(0) = -e^{ix}$. $\frac{Q(w)}{w^2}dw^2$ has a double zero at $-e^{ix}$, and $\alpha(t)$ ($\alpha(0) = -e^{ix}$) satisfies a differential equation $\frac{Q(\alpha(t))}{\alpha(t)^2}\dot{\alpha}(t)^2 < 0$. Thus by Lemma 7, there is a parametrization so that

$$\frac{Q(w(t))}{w(t)^2}\dot{w}(t)^2 = -t^2, \quad \text{where} \quad w(0) = -e^{ix}. \quad (4.42)$$

Therefore we can set $w(t) = -e^{ix} + b_1 t + b_2 t^2 + \cdots$, where $b_1, b_2, \cdots$ are complex constants. Letting $u = e^{ix}$ and $v = e^{iy}$, and then substituting $w(t) = -u + b_1 t + b_2 t^2 + \cdots$ into the power series for $\frac{Q(w(t))}{w(t)^2}\dot{w}(t)^2 + t^2$=0, we get the following. (We used 'Maple' to get the following power series expansion.)

$$\begin{aligned}
&-\frac{1}{rpu^7v^2}b_1^4(rpu^6v^2-ru^4v-rp^2u^4v+rpu^2-pu^5v^3+u^3v^2+p^2u^3v^2-puv\\
&-r^2pu^5v^3+r^2u^3v^2+r^2p^2u^3v^2-r^2puv+rpu^4v^4-ru^2v^3-rp^2u^2v^3+rpv^2-rpu^7v^2)t^2\\
&-\frac{1}{rpu^8v^2}b_1^3\{6b_2(u^4v^2-pu^2v-ru^5v+r^2u^4v^2-ru^3v^3+rpu^3-pu^6v^3+p^2u^4v^2\\
&-r^2pu^2v+rpu^7v^2-rp^2u^5v-rp^2u^3v^3+rpu^5v^4+r^2p^2u^4v^2-r^2pu^6v^3+rpuv^2)\\
&+b_1^2(rpu^6v^2-4puv-2ru^4v+3r^2u^3v^2-4ru^2v^3+3rpu^2+5rpv^2-2pu^5v^3\\
&+3p^2u^3v^2+3u^3v^2-2r^2pu^5v^3-2rp^2u^4v-4r^2puv+3r^2p^2u^3v^2+3rpu^4v^4-4rp^2u^2v^3)\}t^3+\cdots.
\end{aligned}\tag{4.43}$$

Thus we have that all the coefficients of t in the above expression are zero. In particular, the following is from the coefficient of t^3.

$$\begin{aligned}
&\frac{1}{rpu^8v^2}b_1^3\{6b_2(u^4v^2-pu^2v-ru^5v+r^2u^4v^2-ru^3v^3+rpu^3-pu^6v^3+p^2u^4v^2\\
&-r^2pu^2v+rpu^7v^2-rp^2u^5v-rp^2u^3v^3+rpu^5v^4+r^2p^2u^4v^2-r^2pu^6v^3+rpuv^2)\\
&+b_1^2(rpu^6v^2-4puv-2ru^4v+3r^2u^3v^2-4ru^2v^3+3rpu^2+5rpv^2-2pu^5v^3\\
&+3p^2u^3v^2+3u^3v^2-2r^2pu^5v^3-2rp^2u^4v-4r^2puv+3r^2p^2u^3v^2+3rpu^4v^4-4rp^2u^2v^3)\}=0.
\end{aligned}\tag{4.44}$$

Now our purpose is to get the expression (4.41). We have the followings by Lemma 2.

$$T(0)=\frac{b_1}{|b_1|}\text{ and }K(0)=\frac{2}{|b_1|}Im\left(\frac{b_2}{b_1}\right).$$

Thus we will manipulate the equation (4.44) so that it becomes a function of $r+\frac{1}{r}$, $p+\frac{1}{p}$, v, $T(0)$ and $K(0)$. Multiplying the equation (4.44) by u^4,

$$\begin{gathered}
\tfrac{1}{rpu^4v^2}b_1^3\{6b_2(u^4v^2-pu^2v-ru^5v+r^2u^4v^2-ru^3v^3+rpu^3-pu^6v^3+p^2u^4v^2\\
-r^2pu^2v+rpu^7v^2-rp^2u^5v-rp^2u^3v^3+rpu^5v^4+r^2p^2u^4v^2-r^2pu^6v^3+rpuv^2)\\
+b_1^2(rpu^6v^2-4puv-2ru^4v+3r^2u^3v^2-4ru^2v^3+3rpu^2+5rpv^2-2pu^5v^3\\
+3p^2u^3v^2+3u^3v^2-2r^2pu^5v^3-2rp^2u^4v-4r^2puv+3r^2p^2u^3v^2+3rpu^4v^4-4rp^2u^2v^3)\}=0
\end{gathered}$$

or

$$b_1^3\{6b_2(\tfrac{1}{rp}-\tfrac{1}{r}\tfrac{1}{u^2v}-\tfrac{1}{p}\tfrac{u}{v}+\tfrac{r}{p}-\tfrac{1}{p}\tfrac{v}{u}+\tfrac{1}{uv^2}-\tfrac{1}{r}u^2v+\tfrac{p}{r}$$

$$-r\tfrac{1}{u^2v}+u^3-p\tfrac{u}{v}-p\tfrac{v}{u}+uv^2+rp-ru^2v+\tfrac{1}{u^3})$$

$$+b_1^2(u^2-4\tfrac{1}{r}\tfrac{1}{u^3v}-2\tfrac{1}{p}\tfrac{1}{v}+3\tfrac{r}{p}\tfrac{1}{u}-4\tfrac{1}{p}\tfrac{v}{u^2}+3\tfrac{1}{u^2v^2}+5\tfrac{1}{u^4}-2\tfrac{1}{r}uv$$

$$+3\tfrac{p}{r}\tfrac{1}{u}+3\tfrac{1}{rp}\tfrac{1}{u}-2ruv-2p\tfrac{1}{v}-4r\tfrac{1}{u^3v}+3rp\tfrac{1}{u}+3v^2-4p\tfrac{v}{u^2})\}=0$$

or

$$6b_2\{(u^3+\tfrac{1}{u^3})+(uv^2+\tfrac{1}{uv^2})-r(u^2v+\tfrac{1}{u^2v})-\tfrac{1}{r}(u^2v+\tfrac{1}{u^2v})$$

$$-p(\tfrac{u}{v}+\tfrac{v}{u})-\tfrac{1}{p}(\tfrac{u}{v}+\tfrac{v}{u})+(rp+\tfrac{1}{rp})+(\tfrac{r}{p}+\tfrac{p}{r})\}$$

$$=b_1^2\{4(r+\tfrac{1}{r})\tfrac{1}{u^3v}+4(p+\tfrac{1}{p})\tfrac{v}{u^2}-3(\tfrac{r}{p}+\tfrac{p}{r})\tfrac{1}{u}-3(rp+\tfrac{1}{rp})\tfrac{1}{u}$$

$$+2(r+\tfrac{1}{r})uv+2(p+\tfrac{1}{p})\tfrac{1}{v}-u^2-\tfrac{3}{u^2v^2}-\tfrac{5}{u^4}-3v^2\}$$

or

$$6b_2\{(u^3+\tfrac{1}{u^3})+(uv^2+\tfrac{1}{uv^2})-(r+\tfrac{1}{r})(u^2v+\tfrac{1}{u^2v})-(p+\tfrac{1}{p})(\tfrac{u}{v}+\tfrac{v}{u})+(r+\tfrac{1}{r})(p+\tfrac{1}{p})\}$$

$$=\tfrac{b_1^2}{u}\{4(r+\tfrac{1}{r})\tfrac{1}{u^2v}+4(p+\tfrac{1}{p})\tfrac{v}{u}-3(\tfrac{r}{p}+\tfrac{p}{r})-3(rp+\tfrac{1}{rp})$$

$$+2(r+\tfrac{1}{r})u^2v+2(p+\tfrac{1}{p})\tfrac{u}{v}-u^3-\tfrac{3}{uv^2}-\tfrac{5}{u^3}-3uv^2\}$$

$$=\tfrac{b_1^2}{u}\{2(r+\tfrac{1}{r})(u^2v+\tfrac{1}{u^2v})+2(p+\tfrac{1}{p})(\tfrac{u}{v}+\tfrac{v}{u})-3(r+\tfrac{1}{r})(p+\tfrac{1}{p})$$

$$+2(r+\tfrac{1}{r})\tfrac{1}{u^2v}+2(p+\tfrac{1}{p})\tfrac{v}{u}-(u^3+\tfrac{1}{u^3})-3(uv^2+\tfrac{1}{uv^2})-\tfrac{4}{u^3}\}.$$

Dividing each side of this equation by $|b_1|b_1$,

$$\frac{6b_2}{|b_1|b_1}\{(u^3+\frac{1}{u^3})+(uv^2+\frac{1}{uv^2})-(r+\frac{1}{r})(u^2v+\frac{1}{u^2v})-(p+\frac{1}{p})(\frac{u}{v}+\frac{v}{u})+(r+\frac{1}{r})(p+\frac{1}{p})\}$$

$$=\frac{b_1}{|b_1|}\frac{1}{u}\{2(r+\frac{1}{r})(u^2v+\frac{1}{u^2v})+2(p+\frac{1}{p})(\frac{u}{v}+\frac{v}{u})-3(r+\frac{1}{r})(p+\frac{1}{p}) \tag{4.45}$$

$$+2(r+\frac{1}{r})\frac{1}{u^2v}+2(p+\frac{1}{p})\frac{v}{u}-(u^3+\frac{1}{u^3})-3(uv^2+\frac{1}{uv^2})-\frac{4}{u^3}\}.$$

We are close to have the form $K(0) = \frac{2}{|b_1|} Im(\frac{b_2}{b_1})$. For the next step, we need the following information.

$$Im\alpha\beta = Re\alpha Im\beta + Im\alpha Re\beta, \quad Re\alpha\beta = Re\alpha Re\beta - Im\alpha Im\beta$$

$$u^3 + \frac{1}{u^3} = 2cos3x, \quad uv^2 + \frac{1}{uv^2} = 2cos(x + 2y)$$

$$u^2v + \frac{1}{u^2v} = 2cos(2x + y), \quad \frac{u}{v} + \frac{v}{u} = 2cos(x - y), \qquad u = e^{ix}, \quad v = e^{iy}$$

$$Re\frac{1}{u} = cosx, \quad Im\frac{1}{u} = -sinx$$

$$Re\frac{1}{u^2v} = cos(2x + y), \quad Im\frac{1}{u^2v} = -sin(2x + y)$$

$$Re\frac{v}{u} = cos(x - y), \quad Im\frac{v}{u} = -sin(x - y)$$

$$Re\frac{1}{u^3} = cos3x, \quad Im\frac{1}{u^3} = -sin3x$$

Taking the imaginary part of each side of the equation in (4.45) using the above information,

$$3K(0)\{2cos3x + 2cos(x + 2y) - 2(r + \tfrac{1}{r})cos(2x + y) - 2(p + \tfrac{1}{p})cos(x - y) + (r + \tfrac{1}{r})(p + \tfrac{1}{p})\}$$

$$= (ReT(0)cosx + ImT(0)sinx)\{-2(r + \tfrac{1}{r})sin(2x + y) - 2(p + \tfrac{1}{p})sin(x - y) + 4sin3x\}$$

$$+(-ReT(0)sinx + ImT(0)cosx)\{4(r + \tfrac{1}{r})cos(2x + y) + 4(p + \tfrac{1}{p})cos(x - y) - 3(r + \tfrac{1}{r})(p + \tfrac{1}{p})$$

$$+2(r + \tfrac{1}{r})cos(2x + y) + 2(p + \tfrac{1}{p})cos(x - y) - 2cos3x - 6cos(x + 2y) - 4cos3x\}.$$

□

Remark 9. *In the above proof, we have an expression that includes* y, $r + \frac{1}{r}$, $p + \frac{1}{p}$, $T(0)$ *and* $K(0)$. *Thus one might show that* f *uniquely determines* $\frac{Q(w)}{w^2}dw^2$ *by getting more expressions like (4.41) since the key point of the unique determination is to express* y, $r + \frac{1}{r}$ *and* $p + \frac{1}{p}$ *in terms of the derivatives of the curve.*

4.5 Summary

The following was our main question.

- When a function $f \in \mathcal{S}$ or $\mathcal{B}$ is admissible for a quadratic differential, does the function uniquely determine the quadratic differential?

To answer this question, we separated the question into two cases:

- complex plane case: the function $f \in \mathcal{S}$ is admissible for a quadratic differential on the complex plane.

- unit disk case: the function $f \in \mathcal{B}$ is admissible for a quadratic differential on the unit disk.

In each case, we proved that the answer for the question is positive for a quadratic differential of order 3 with some exceptions, and gave a partial result for a quadratic differential of order 4. As we mentioned in Remark 8 and 9, the strength of our method is that it is possible to extend our method to higher order cases.

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