ON CONSTRAINED MARKOV-NIKOLSKII AND BERNSTEIN TYPE INEQUALITIES

by

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Abstract

This thesis is devoted to polynomial inequalities with constraints. We present a history of the development of this subject together with recent progress. In the first part, we solve an analog of classical Markov's problem for monotone polynomials. More precisely, if $\triangle_n$ denotes the set of all monotone polynomials on $[-1,1]$ of degree n, then for $P_n \in \triangle_n$ and $x \in [-1,1]$ the following sharp inequalities hold:

$$|P_n'(x)| \leq 2\max(S_k(x), S_k(-x))\|P_n\|,$$

for $n = 2k+2$, $k \geq 0$, and

$$|P_n'(x)| \leq 2\max(F_k(x), H_k(x))\|P_n\|,$$

for $n = 2k+1$, $k \geq 0$, where

$$S_k(x) := (1+x)\sum_{l=0}^{k}(J_l^{(0,1)}(x))^2;$$

$$H_k(x) := (1-x^2)\sum_{l=0}^{k-1}(J_l^{(1,1)}(x))^2;$$

$$F_k(x) := \sum_{l=0}^{k}(J_l^{(0,0)}(x))^2,$$

and $J_l^{(\alpha,\beta)}(x)$, $l \geq 1$ are the Jacobi polynomials.

Let $\triangle_n^{(1)}$ be the set of all monotone nonnegative polynomials on $[-1,1]$ of degree n. In the second part, we investigate the asymptotic behavior of the constants

$$M_{q,p}^{(1)}(n,1) := \sup_{P_n \in \triangle_n^{(1)}} \frac{\|P_n'\|_{L_q[-1,1]}}{\|P_n\|_{L_p[-1,1]}},$$

in constrained Markov-Nikolskii type inequalities. Our conjecture is that

$$M_{q,p}^{(1)}(n,1) \asymp \begin{cases} n^{2+2/p-2/q}, & \text{if } 1 > 1/q - 1/p, \\ \log n, & \text{if } 1 = 1/q - 1/p, \\ 1, & \text{if } 1 < 1/q - 1/p. \end{cases}$$

We prove this conjecture for all values of $p, q > 0$, except for the case $0 < q < 1$, $1/2 \le 1/q - 1/p \le 1$, $p \ne 1$.

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Contents

0.1 Notation

The following notation is used throughout this thesis.

$\|A\|$	Lebesgue measure of the set A
$A_n \asymp B_n$	$C_1 A_n \le B_n \le C_2 A_n$ with positive constants C_1, C_2 independent on n
$\mathbb{C}$	set of all complex numbers
$C(A)$	space of all continuous functions f on A
$C[a,b]$	space of all continuous functions f on $[a,b]$
$C(\alpha,\dots,\beta)$	real constants, which depend on parameters $\alpha,\dots,\beta$
$c,\ C$	real constants
$\mathbb{D}$	closed unit disk of the complex plane
$\triangle_n$	set of all nondecreasing polynomials of degree n on $[-1,1]$
$\triangle_n^{(k)}$	set of all absolutely monotone polynomials of order k on $[-1,1]$
$\\|f\\|_{C[a,b]}$	$\max_{x\in[a,b]} \|f(x)\|$
$\\|f\\|$	$\\|f\\|_{C[-1,1]}$
$\\|f\\|_{L_p(\Omega)}$	$\left(\int_\Omega \|f(x)\|^p dx\right)^{1/p}$, $p>0$
$J_k^{(\alpha,\beta)}(x)$	Jacobi polynomial of degree k associated with weight $(1-x)^\alpha(1+x)^\beta$
$M_{q,p}(n,k)$	$\sup_{P_n\in\mathbb{P}_n} \\|P_n^{(k)}\\|_{L_q[-1,1]}/\\|P_n\\|_{L_p[-1,1]}$, $0\le k\le n$
$M_{q,p}^{(k)}(n,m)$	$\sup_{P_n\in\triangle_n^{(k)}} \\|P_n^{(m)}\\|_{L_q[-1,1]}/\\|P_n\\|_{L_p[-1,1]}$, $0\le m,k\le n$
$n!!$	$n(n-2)(n-4)\dots$
$\mathbb{P}_n$	set of all algebraic polynomials of degree $\le n$
$\mathbb{P}_n^+$	set of all polynomials from $\mathbb{P}_n$ nonnegative on $[-1,1]$
$\mathbb{P}_{n,1}^+$	subset of $\mathbb{P}_n^+$ of polynomials with coefficients between -1 and 1,
	with at least one coefficient equal to 1
$\mathbb{P}_{n,k}(A)$	subset of $\mathbb{P}_n$ of polynomials with at most k zeros in A
$\mathbb{R}$	set of all real numbers
$\mathbb{T}_n$	set of all trigonometric polynomials of degree $\leqslant n$ on $[-\pi,\pi]$
$T_n(x)$	$\cos(n\arccos x)$, $x\in[-1,1]$

Chapter 1

Markov and Bernstein type inequalities

1.1 Some history of polynomial inequalities

Weierstrass (1885) was the first to prove that an arbitrary continuous function which is defined over a closed finite interval may be uniformly approximated by a sequence of polynomials. A more difficult problem of best approximation by polynomials had been initiated earlier by Chebychev ([8], 1854). Fifty years later, in 1905, de la Vallee Poussin raised the following question of best approximation: is it possible to approximate every continuous piecewise linear function by polynomials of degree n with an error of $o(1/n)$ as n becomes large? He had proved that the approximation can be carried out with an error of $O(1/n)$. Serge Bernstein [3] gave a negative answer in a prize-winning essay on problems of best approximation (Prize of Paris Academy of Science, 1912). In this paper, Bernstein proved and made use of an inequality concerning the derivatives of polynomials. These inequalities have supplied one approach to questions concerning the derivatives of quasi-analytic functions [12]. A generalization of Bernstein's theorem has been applied to almost periodic functions [16]. In Approximation Theory, Bernstein's and Markov's inequalities play a key role in the proofs of so-called *Inverse theorems* (*i.e.*, characterization of classes of functions via their approximation properties). For instance, Telyakovskii writes

[40]: "Among those, that are fundamental in approximation theory are the extremal problems connected with inequalities of the derivatives of polynomials... The use of inequalities of this kind is a fundamental method in proofs of inverse problems of approximation theory. Frequently further progress in inverse theorems has depended on first obtaining a corresponding generalization or analog of Markov's and Bernstein's inequalities."

In some sense, it all started with a question asked and answered by the celebrated Russian chemist D. Mendeleev [26], the inventor of the periodic table of elements. He made a study of the specific gravity y of a solution as a function of percentage x of the dissolved substance. The graph representing y in terms of x could be closely approximated by successions of quadratic arcs, but he wondered whether the corners where the arcs joined were genuine or just caused by errors of measurement. The answer lays in knowing how large the derivative of the quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0$ on $[a, b]$ can be if $\max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x) = 2M$. Indeed, two adjacent arcs could not come from the same quadratic function if the slope of one exceeds the largest possible slope of the other. Mendeleev himself was able to solve this mathematical problem. He mentioned his result to a famous Russian mathematician, A. Markov, who made a great contribution to many branches of mathematical research (such as Probability theory, Theory of Functions, Differential equations, etc.). In 1889, A. Markov [24] generalized Mendeleev's result for all polynomials of degree n. In particular, he proved that if

$$P_n(x) = \sum_{i=0}^{n} a_i x^i,$$

and

$$\max_{x\in[a,b]} P_n(x) - \min_{x\in[a,b]} P_n(x) = 2M,$$

then $\max_{x\in[a,b]} |P_n'(x)| \le 2Mn^2/(b-a)$. This is now known as Markov's inequality.

1.2 The inequalities of Markov and Bernstein: classical results

We begin this section by considering the following extremal problem:

> *For a given norm $\|\cdot\|_X$, determine the best constant C such that the inequality*
> $$\|P_n'\|_X \le C\|P_n\|_X$$
> *holds for all $P_n \in \mathbb{P}_n$, i.e., determine*
> $$A_n := \sup_{P_n\in\mathbb{P}_n} \frac{\|P_n'\|_X}{\|P_n\|_X}.$$

The first result in this area appeared in 1889. It is the well-known A. Markov's inequality, namely:

Theorem 1.2.1 (A. Markov [24], 1889)**.** *For every polynomial $P_n \in \mathbb{P}_n$, the following inequality holds:*

$$\|P_n'\| \le n^2\|P_n\|. \tag{1.1}$$

The equality holds only at ± 1 (in other words, if $\|P_n'\| = n^2\|P_n\|$ and $|P_n'(x_0)| = \|P_n'\|$ for some point $x_0 \in [-1,1]$, then $x_0 = \pm 1$, $|P_n'(\pm 1)| = n^2|P_n(\pm 1)|$), and only when $P_n = cT_n$, where T_n is the Chebyshev polynomial of the first kind, that is $T_n(x) = \cos(n\arccos x)$.

The proof given by A. Markov was quite hard and technical. During the last hundred years, mathematicians discovered a huge number of different proofs of this remarkable result. Here, we sketch the idea of one of them [31] that, in some sense, demonstrates a general approach to many types of sharp polynomial inequalities (*i.e.*, inequalities where equality occurs). The trick is very simple and contains two steps:

1. Suppose that inequality fails for some polynomial P_n of degree n and denote by $\tilde{P}_n$ an extremal polynomial (a polynomial for which our inequality becomes equality). For instance, for the inequality (1.1), any polynomial of the form cT_n, $c \in \mathbb{R}$, is extremal.

2. If we can show that the difference $P_n - \tilde{P}_n$ has “too many” (more than n) zeros on the given interval, this would imply that $P_n = \tilde{P}_n$ (by the Fundamental Theorem of Algebra, a non-constant polynomial of degree n has at most n real zeros).

Arguments of such type were first used by Chebyshev in the proof of his famous Alternance Theorem [9].

Corollary 1.2.2. *For every $a < b$ and $P_n \in \mathbb{P}_n$,*

$$\|P_n'\|_{C[a,b]} \leq \frac{2n^2}{b-a}\|P_n\|_{C[a,b]}. \tag{1.2}$$

Proof. This immediately follows from Theorem 1.2.1 after the linear transformation

$$l(x) = \frac{(b-a)x + b + a}{2}.$$

Indeed, consider the polynomial

$$P_n(l(x)) = Q_n(x),\ \ x \in [-1, 1].$$

Observe, that

$$\frac{b-a}{2}P_n'(l(x)) = Q_n'(x).$$

By Theorem 1.2.1,

$$\|P_n'(l(x))\| \leq n^2 \, \|P_n(l(x))\| \, ,$$

which implies (1.2). □

From now on, we only consider the interval $[-1, 1]$. However, all of our results can be generalized to any interval $[a, b]$ of the real line in the same way as in the proof of Corollary 1.2.2.

A question that naturally arises after A. Markov’s result (Theorem 1.2.1) is how to get an upper bound for the k-th derivative of P_n. Iterating the inequality from Theorem 1.2.1 k times, one can get an estimate for the k-th derivative:

$$\|P_n^{(k)}\| \leq n^2 \cdot (n-1)^2 ... (n-(k-1))^2 \|P_n\| = \left[\frac{n!}{(n-k)!}\right]^2 \|P_n\|.$$

However, this inequality is not sharp and can be significantly improved. The best possible constant was found by a younger brother of A. Markov, Vladimir Markov [25] in 1892.

Theorem 1.2.3 (V. Markov [25], 1892)**.** *For every polynomial* $P_n \in \mathbb{P}_n$,

$$\|P_n^{(k)}\| \le \frac{(n^2-1)(n^2-2^2)(n^2-3^2)...(n^2-(k-1)^2)}{(2k-1)!!}\|P_n\| = |T_n^{(k)}(1)| \cdot \|P_n\|. \quad (1.3)$$

Equality is achieved if and only if $P_n = cT_n$.

Remark 1.2.4. For $k = 1$, this is precisely Theorem 1.2.1.

The right-hand side of (1.3) equals to $|T^k(1)|\|P_n\|$. V. Markov's proof of this result is based on a variational method. We now sketch the main idea of this deep and difficult proof (used frequently for proofs of many polynomial inequalities).

By homogeneity of (1.3), it is enough to prove it for $\|P_n\| = 1$. We denote by $\tilde{P_n}$ a polynomial that satisfies the following extremal property:

$$\tilde{P_n} = \underset{P_n:\ \|P_n\|=1}{\text{argmax}} \|P_n^{(k)}\|$$

(it is easily shown that this polynomial exists by using sequential compactness of the set of polynomials of degree n with bounded uniform norm). V. Markov used a variational method to show that $\|\tilde{P_n}(x)\|$ must be equal to 1 at either n or $n+1$ different points in the interval $(-1, 1)$. In the latter case $\pm\tilde{P_n}(x)$ is the n-th Chebyshev polynomial, whose derivatives are easily shown to satisfy our inequality. In the former case, it is possible to show that $\tilde{P_n}(x)$ satisfies a differential equation of the form

$$1 - (\tilde{P_n}(x))^2 = \frac{(1-x^2)(x-b)(x-c)}{n^2(x-a)^2}(\tilde{P}_n'(x))^2. \quad (1.4)$$

Here, a, b, c are real constants which depend on n. V. Markov was then able to show that the derivatives of this class of polynomials satisfy the inequality (1.3), but the proof is quite difficult. The differential equation (1.4) appeared in the paper [8] of Chebyshev in 1854, but the polynomial solutions are often named after Zolotarev (he was a student of Chebyshev), who investigated their properties extensively at a later date. Zolotarev [42] showed that (1.4) admits a polynomial solution if and only if the constants a, b, c satisfy certain equations, which involve elliptic integrals. These elliptic integrals were not used, however, by V. Markov in the proof of his result.

V. Markov [25] also investigated a more general problem:

if $m_0, m_1, ..., m_n$ are given constants and $P_n(x) = \sum_{i=0}^{n} a_i x^i$ satisfies $\|P_n\| = 1$, what is the precise bound for the linear form $\sum_{i=0}^{n} a_i m_i$?

The best bound will, of course, depend on the constants m_i as well as on the degree of the polynomial. By choosing the constants m_i the linear form can be made equal to any derivative of $P_n(x)$ at any preassigned point. However, this general problem was not investigated so far as much as the special problem concerning derivatives of $P_n(x)$ (the solution of which is given by Theorem 1.2.3).

Simple proofs of Theorem 1.2.3 were given by Bernstein [2], Schaffer and Duffin [35], and an elementary proof was published by Mohr [27]. Recently, Shadrin [36] gave another elegant and short proof of this inequality.

As was mentioned at the beginning of this thesis, another type of these inequalities goes back to Bernstein [1], who considered the following problem:

Let $P(z) : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n and $|P(z)| \leq 1$ in the unit disk $|z| \leq 1$. Determine how large can $|P'(z)|$ be for $|z| \leq 1$.

In other words, if we define the norm

$$\|f\|_{\mathbb{D}} := \sup_{|z|\leq 1} |f(z)|,$$

this problem can be reduced to the inequality of the form

$$\|P_n'\|_{\mathbb{D}} \leq A_n \|P_n\|_{\mathbb{D}}.$$

Bernstein [1] proved the following:

Theorem 1.2.5 (Bernstein [1], 1912)**.** *For every polynomial $P_n \in \mathbb{P}_n$, the following inequality holds:*

$$\|P_n'\|_{\mathbb{D}} \leq n \|P_n\|_{\mathbb{D}}.$$

Since every polynomial is an analytic function, it attains maximum on the boundary of the unit disk, which is the unit circle $|z| = 1$. Therefore, it is enough to consider only values $z = e^{i\theta}$, $0 < \theta \leq 2\pi$, and

$$\|f\|_{\mathbb{D}} = \max_{|z|\leq 1} |f(z)| = \max_{0<\theta\leq 2\pi} |f(e^{i\theta})|,$$

if f is a polynomial.

Recall, that $\mathbb{T}_n$ denotes the set of all trigonometric polynomials of order n on $[-\pi, \pi]$, that is polynomials of the form

$$t_n(x) = \sum_{k=0}^{n} (a_k \sin kx + b_k \cos kx).$$

It's easy to see that $P_n(e^{i\theta})$ is a trigonometric polynomial of degree n, and we arrive at Bersntein's theorem for trigonometric polynomials:

Theorem 1.2.6 (Bernstein [1], 1912)**.** *For every trigonometric polynomial $t_n \in \mathbb{T}_n$ the following inequality holds:*

$$\|t_n'\|_{C[-\pi,\pi]} \leq n\|t_n\|_{C[-\pi,\pi]}. \tag{1.5}$$

Equality is achieved if and only if $t_n(\theta) = c\cos(n\theta - \theta_0)$, where c and θ_0 are real constants.

On the other hand, if we consider an algebraic polynomial $P_n(t)$ of degree n, then $P_n(\cos\theta)$ is a trigonometric polynomial of degree n. Bernsteins's inequality now implies

$$|(-\sin\theta)P_n'(\cos\theta)| \leq n\|P_n\|.$$

Denoting $x = \cos\theta$ we get the most "standard formulation" of Bernstein's result, that we are going to use throughout this thesis:

Theorem 1.2.7 (Bernstein [1], 1912)**.** *For every polynomial $P_n \in \mathbb{P}_n$ the following inequality holds:*

$$|P_n'(x)| \leq \frac{n}{\sqrt{1-x^2}}\|P_n\|, \quad x \in (-1,1). \tag{1.6}$$

Bernstein's proof of Theorem 1.2.7 was based on a variational method. Simpler proofs of this theorem have been obtained by M. Riesz [33], F. Riesz [32] and de la Vallee Poussin [11].

Using similar methods one can show the validity of a sharper inequality

$$n^2 t_n(\theta)^2 + t_n'(\theta)^2 \leq n^2,$$

where t_n is assumed to be real trigonometric polynomials of degree n. This was first explicitly stated by Schaake and Van Der Corput [34], although it is implicit in an earlier inequality due to Szegő [38].

Note, that one cannot get an analog of Bernstein's inequality for higher derivatives of algebraic polynomials by just iterating inequality (1.6). However, it is possible to iterate inequality (1.5). We get

$$\|t_n^{(k)}\|_{C[-\pi,\pi]} \leq n^k \|t_n\|_{C[-\pi.\pi]},$$

the equality holding if $t_n(\theta) = c\cos(n\theta - \theta_0)$, where c and θ_0 are real constants.

Combining Bernstein's and Markov's inequalities we get the following inequality for $P_n \in \mathbb{P}_n$:

$$|P_n'(x)| \leq \min\left(n^2, \frac{n}{\sqrt{1-x^2}}\right) \|P_n\|, \quad x \in [-1,1]. \tag{1.7}$$

Using Cauchy's integral formula and some auxiliary inequalities T. Erdelyi proved the following analog of inequality (1.7) for higher order derivatives:

Theorem 1.2.8 (Erdelyi [7]). *There exists a constant $c(m)$, such that, for every polynomial $P_n \in \mathbb{P}_n$, the following inequality holds:*

$$|P_n^{(m)}(x)| \leq \min\left(n^2, \frac{c(m)}{\sqrt{1-x^2}}\right)^m \|P_n\|, \quad x \in [-1,1]. \tag{1.8}$$

Remark 1.2.9. The exact value of $c(m)$ is unknown for all $m \geq 2$. Theorem 1.2.7 implies that $c(1) = n$.

Remark 1.2.10. For the detailed proofs of theorems in this section see, for example, [7].

1.3 Polynomial inequalities with constraints

Throughout his life, Paul Erdős, a famous Hungarian mathematician, showed a particular interest in inequalities for constrained polynomials. In a paper [14] in 1940, Erdős found a class of polynomials for which the Markov factor n^2 improves to cn. He proved that

$$|P_n'(x)| \le \min\left(\frac{en}{2}, \frac{c\sqrt{n}}{(1-x^2)^2}\right)\|P_n\|,$$

for all polynomials $P_n \in \mathbb{P}_n$ that have all zeros in $\mathbb{R}\setminus(-1,1)$. This result motivated a number of people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov-Bernstein type inequality of Erdős were extended later in many directions. In this section, we introduce examples of polynomial inequalities with constraints. Here, the word "constraints" means that we are looking for the analogs of classical polynomial inequalities, when we restrict our attention to some special classes. Of course, in such cases, one can expect to get better constants (because the supremum is taken over "smaller" classes of polynomials).

Recall that, for integers $0 \le k \le n$,

$$\mathbb{P}_{n,k}(\mathbb{D}) := \{P \in \mathbb{P}_n : P_n \textit{ has at most } k \textit{ zeros in } \mathbb{D}\}.$$

In 1981, J. Szabados [37] proposed the following:

Conjecture (Szabados [37], 1981).

> *If* P_n *is a polynomial of degree* n *and* P_n *has at least* $n-k$ *zeros in* $\mathbb{R}\setminus(-1,1)$ *then there is a constant* c $(c<9)$ *so that*
>
> $$\|P_n'\| \le c(k+1)n\|P_n\|.$$

A few years later, P. Borwein [6] proved that this conjecture is indeed true.

In [37], J. Szabados constructed a polynomial P_n of degree n with $n-k$ roots in $\mathbb{R}\setminus(-1,1)$ so that

$$\|P_n'\| \ge \frac{kn}{2}\|P_n\|.$$

Therefore, Theorem **??** provides a sharp (up to the constant) Markov type inequality for the class $\mathbb{P}_n^k([-1,1])$.

1.4 Orthogonal polynomials

For our future purposes, we briefly recall some facts from the theory of orthogonal polynomials (see [39] for their details).

For a given general weight function $w(x) : \mathbb{R} \to \mathbb{R}^+$, we define orthonormal polynomials $p_0, p_1, ...$ such that $p_n(x) = p_n(w; x) = \gamma_n x^n + ...,\ \gamma_n > 0$ and such that

$$\int_{\mathbb{R}} p_n(x)p_m(x)w(x)dx = \delta_{mn},$$

where δ_{mn} denotes Kronecker's δ-function, *i.e.*, $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{nn} = 1$.

Note that each p_n has exactly n simple real zeroes. Indeed, otherwise, p_n has at most $n-1$ distinct odd order zeroes y_i, $1 \leq i \leq n-1$. Then the polynomial $P(x) := \prod_{i=1}^{m}(x - y_i)$ is of degree less than n and $P(x)p_n(x) \geq 0$ for all $x \in \mathbb{R}$. Hence, $\int_{\mathbb{R}} p_n(x)P(x)w(x)dx > 0$, on the other hand, it is well known that the system of orthonormal polynomials $p_0, p_1, ...p_{n-1}$ forms a basis in $\mathbb{P}_{n-1}$. This, together with

$$\int_{\mathbb{R}} p_n(x)p_m(x)w(x)dx = \delta_{mn},$$

implies that

$$\int_{\mathbb{R}} p_n(x)P(x)w(x)dx = 0.$$

This contradiction yields that p_n has n simple zeros. Denote them by

$$x_{n,n} < x_{n-1,n} < ... < x_{1,n}.$$

One can easily prove the following useful identity, which is called the Christoffel-Darboux formula:

$$\sum_{k=0}^{n} p_k(x)p_k(y) = \frac{\gamma_{n+1}}{\gamma_n} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x - y}. \qquad (1.9)$$

For the case when $w(x) := w(\alpha, \beta, x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$ for $x \in [-1, 1]$ and $w(x) = w(\alpha, \beta, x) = 0$ otherwise, polynomials $p_n(x) := J_n^{(\alpha,\beta)}(x)$ are called Jacobi polynomials associated with the weight $(1-x)^\alpha(1+x)^\beta$.

We would like to emphasize that there are different ways to normalize orthogonal polynomials. For example, in [39], Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n \geq 0$, are defined as

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\delta_{mn},$$

with normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

where $\Gamma(\cdot)$ denotes the gamma function.

Throughout this thesis, we consider Jacobi polynomials with $\{\alpha,\beta\} = \{0,1\}$. For such polynomials, the last formula can be rewritten as follows:

$$\int_{-1}^{1}(1+x)P_n^{(0,1)}(x)P_m^{(0,1)}(x)dx = \frac{2}{n+1}\delta_{mn}, \qquad P_n^{(0,1)}(1) = 1,$$

which implies

$$J_n^{(0,1)}(x) = \frac{\sqrt{n+1}}{\sqrt{2}}P_n^{(0,1)}(x). \tag{1.10}$$

Using the relation

$$J_l^{(1,0)}(x) = (-1)^l J_l^{(0,1)}(-x) \tag{1.11}$$

together with (1.10), we get

$$J_l^{(0,1)}(1) = (-1)^l J_l^{(1,0)}(-1) = \frac{\sqrt{l+1}}{\sqrt{2}}. \tag{1.12}$$

1.5 Markov's and Bernstein's inequalities for monotone polynomials

In 1926, S. Bernstein [4] pointed out that Markov's inequality for monotone polynomials is not essentially better than for all polynomials, in the sense, that the order of $\sup_{P_n\in\triangle_n}\|P_n'\|/\|P_n\|$ is n^2. He proved his result only for odd n. In 2001, Qazi [30] extended Bernstein's idea to include polynomials of even degree. The next theorem contains their results:

Theorem 1.5.1 (Bernstein [4], Qazi [30]).

$$\sup_{P_n \in \triangle_n} \frac{\|P_n'\|}{\|P_n\|} = \begin{cases} \frac{(n+1)^2}{4}, & \text{if } n = 2k+1, \\ \frac{n(n+2)}{4}, & \text{if } n = 2k. \end{cases}$$

Let us recall V. Markov's problem that has been discussed in the previous section:

V. Markov's problem.

Let $x_0 \in [-1,1]$ be a fixed point. For $0 \le k \le n$, find the maximum value of $|P^{(k)}(x_0)|$ over all $P_n \in \mathbb{P}_n$ such that $\|P_n\| = 1$.

The problem was studied more completely and in a considerably shorter way by Gusev ([17], 1961) with the help of a method developed by Voronovskaja, who solved this problem for the case $k = 1$, ([41], 1960).

We give a solution of an analogous problem for the case of monotone polynomials and $k = 1$, namely the following problem is considered:

Let $x_0 \in [-1,1]$ be a fixed point. Find the maximum value of $|P'(x_0)|$ over all monotone polynomials $P_n \in \mathbb{P}_n$ such that $\|P_n\| = 1$.

In order to formulate the main result of this section the following three types of polynomials are needed :

$$\begin{aligned} S_k(x) &:= (1+x)\sum_{l=0}^{k} (J_l^{(0,1)}(x))^2; \qquad (1.13) \\ H_k(x) &:= (1-x^2)\sum_{l=0}^{k-1} (J_l^{(1,1)}(x))^2; \\ F_k(x) &:= \sum_{l=0}^{k} (J_l^{(0,0)}(x))^2. \end{aligned}$$

Theorem 1.5.2. *Let x_0 be a fixed point in the interval $[-1,1]$. Then, for every $P_n \in \triangle_n$, $n \ge 1$, the following sharp inequality holds:*

$$|P_n'(x_0)| \le 2\max(S_k(x_0), S_k(-x_0))\|P_n\|,$$

for $n = 2k+2$, $k \ge 0$, and

$$|P_n'(x_0)| \le 2\max(F_k(x_0), H_k(x_0))\|P_n\|,$$

for $n = 2k+1$, $k \ge 0$.

Proof. We start with the solution of the following problem. Fix $x_0 \in [-1,1]$, find the maximum value of

$$S(P,x_0) := \frac{P(x_0)}{\int\limits_{-1}^{1} P(x)dx},$$

over $P \in \mathbb{P}_n^+$, where $\mathbb{P}_n^+$ denotes the set of all nonnegative polynomials of degree at most n on $[-1,1]$ (see Section 1). In what follows, we assume that $x_0 \in (-1,1)$. All the results can be extended to $x_0 = 1$ and $x_0 = -1$ by continuity.

Note, that this maximum value is attained because of the sequential compactness of $\mathbb{P}_n^+$. Indeed, by the classical Markov inequality applied to the polynomial $G(x) = \int_{-1}^{x} P(t)dt$ of degree $\le n+1$, we get

$$P(x_0) = G'(x_0) \le (n+1)^2\|G\| = (n+1)^2 \int_{-1}^{1} P(x)dx,$$

hence, $S(P,x_0)$ is bounded by $(n+1)^2$. Note, that by homogeneity $S(\alpha P, x_0) = S(P,x_0)$, we can assume that all coefficients of P are between -1 and 1, and the maximal coefficient is 1. We denote a subset of such polynomials by $\mathbb{P}_{n,1}^+$. Then, our extremal problem can be rewritten as

$$\begin{aligned} &\textit{maximize } S(P,x_0) \\ &\textit{subject to } P \in \mathbb{P}_{n,1}^+. \end{aligned} \tag{1.14}$$

Note that $\mathbb{P}_{n,1}^+$ is a compact subset of a finite dimensional space and the function $S(P,x_0)$ is continuous on $\mathbb{P}_{n,1}^+$. Therefore, by Weierstrass theorem it attains its maximum there.

Let us denote by $P^*(x)$ an extremal polynomial for the problem (1.14) from $\mathbb{P}_{n,1}^+$ with the largest degree and the maximal number of zeros inside the interval $[-1,1]$. In other words,

$$P^*(x) = \operatorname*{argmax}_{P \in \mathbb{P}_{n,1}^+} S(P,x_0)$$

and if

$$Q(x) = \operatorname*{argmax}_{P \in \mathbb{P}_{n,1}^+} S(P,x_0)$$

is another extremal polynomial, then $\deg P^* \geq \deg Q$, and the number of zeros of Q inside $[-1, 1]$ is not more than the number of zeros of P^* there.

We first prove that $\deg(P^*) = n$. Indeed, if $\deg(P^*) \leq n - 1$ consider two polynomials:

$$P_1(x) = (1 - x)P^*(x),$$
$$P_2(x) = (1 + x)P^*(x).$$

None of them can be extremal, hence,

$$\frac{P^*(x_0)}{\int\limits_{-1}^{1} P^*(x)dx} > \frac{(1 - x_0)P^*(x_0)}{\int\limits_{-1}^{1} (1 - x)P^*(x)dx},$$

and

$$\frac{P^*(x_0)}{\int\limits_{-1}^{1} P^*(x)dx} > \frac{(1 + x_0)P^*(x_0)}{\int\limits_{-1}^{1} (1 + x)P^*(x)dx}.$$

Multiplying both inequalities by common denominators and adding the results up we get

$$\int_{-1}^{1} P^*(x)dx > \int_{-1}^{1} P^*(x)dx,$$

that provides a contradiction, and so $\deg(P^*) = n$.

The next step is to show, that all zeros of $P^*(x)$ lie in the interval $[-1, 1]$. Suppose that this is not the case and write $P^*(x) = P_1(x)P_2(x)$, where all zeros of P_1 lie in $[-1, 1]$ and $P_2(x) > \delta > 0$, for all $x \in [-1, 1]$, and $\deg(P_2) \geq 1$. Note, that for every fixed polynomial h with $\deg(h) \leq \deg(P_2)$ and sufficiently small t, all polynomials of the form $Q(x) = P^*(x) + th(x)P_1(x)$ belong to $\mathbb{P}_n^+$. Hence, $t = 0$ should be a point of local minimum of the function

$$g(t) = \frac{P^*(x_0) + th(x_0)P_1(x_0)}{\int\limits_{-1}^{1} (P^*(x) + th(x)P_1(x))dx}.$$

This implies that $g'(0) = 0$, where

$$g'(0) = \frac{P_1(x_0)h(x_0)\int\limits_{-1}^{1} P^*(x)dx - P^*(x_0)\int\limits_{-1}^{1} P_1(x)h(x)dx}{\left(\int\limits_{-1}^{1} P^*(x)dx\right)^2},$$

and so

$$\int\limits_{-1}^{1} P_1(x)(P_2(x)h(x_0) - P_2(x_0)h(x))dx = 0,$$

for all polynomials h with $\deg(h) \le \deg(P_2)$. Observe, that this equality implies that if $l(x)$ is such that

$$l(x)(x - x_0) = P_2(x)h(x_0) - P_2(x_0)h(x),$$

ans if $h(x)$ runs over all polynomials of degree $\le \deg(P_2)$, then $l(x)$ runs over all polynomials with $\deg(l) \le \deg(P_2) - 1$. Therefore,

$$\int\limits_{-1}^{1} P_1(x)(x - x_0)l(x)dx = 0 \tag{1.15}$$

holds for all polynomials $l(x)$ of degree $\le \deg(P_2) - 1$.

If $\deg(P_2) \ge 2$ take $l(x) = x - x_0$ to get a contradiction (integral of a nonnegative non-zero function cannot be equal to 0). Now, suppose that $\deg(P_2) = 1$. Then

$$\int\limits_{-1}^{1} P_1(x)(x - x_0)dx = 0$$

and one can write $P^*(x) = (a - x)P_1(x)$ where $a > 1$ or $P^*(x) = (b + x)P_1(x)$ for some $b > 1$. In both of these cases it is easy to see that $S(P^*, x_0) = S(P_1, x_0)$. Indeed, in the first case

$$\begin{aligned} S(P^*, x_0) &= \frac{(a - x_0)P_1(x_0)}{\int_{-1}^{1}(a - x)P_1(x)dx} \\ &= \frac{(a - x_0)P_1(x_0)}{\int_{-1}^{1}(x_0 - x)P_1(x)dx + (a - x_0)\int_{-1}^{1} P_1(x)dx} = S(P_1, x_0). \end{aligned}$$

The second case can be done in the same way. But then, taking

$$P_3(x) = (1+x)P_1(x)$$

and

$$P_4(x) = (1-x)P_1(x)$$

and repeating all arguments from the beginning of the proof, one gets that either $S(P_3, x_0)$ or $S(P_4, x_0)$ is not less then $S(P^*, x_0) = S(P_1, x_0)$ and all zeros of P_3 and P_4 lie in the segment $[-1, 1]$, which contradicts our assumption that P^* maximizes $S(P, x_0)$ and has the maximal number of zeros. Hence, all zeros of $P^*(x)$ lie in the interval $[-1, 1]$.

We distinguish between two cases depending on the parity of n.

If $n = 2k+1$, $k \geq 0$, an extremal polynomial can be expressed in one of the following ways: $P^*(x) = (1+x)g^2(x)$ or $P^*(x) = (1-x)g^2(x)$. If $n = 2k$, then an extremal polynomial can be expressed as $P^*(x) = (1-x^2)g^2(x)$ or $P^*(x) = g^2(x)$. In general, we can write an extremal polynomial as

$$P^*(x) = w(x)g^2(x),$$

where $w(x)$ is one of the functions $1-x, 1+x, 1-x^2, 1$.

For any fixed polynomial $h(x)$ with $\deg(h) \leq \deg(g)$ consider the function

$$\psi(t) = \frac{w(x_0)(g(x_0)+th(x_0))^2}{\int\limits_{-1}^{1} w(x)(g(x)+th(x))^2 dx}.$$

Since P^* is extremal, this function has a local maximum at $t = 0$, and so $\psi'(0) = 0$, *i.e.*,

$$\psi'(0) = 2w(x_0) \cdot \frac{g(x_0)h(x_0)\int\limits_{-1}^{1} w(x)g^2(x)dx - g^2(x_0)\int\limits_{-1}^{1} w(x)g(x)h(x)dx}{\left(\int\limits_{-1}^{1} w(x)g^2(x)dx\right)^2} = 0. \quad (1.16)$$

Since $g(x_0) \neq 0$ (otherwise, $\psi(0) = 0$, which contradicts to maximality of P^*) the last equality implies

$$h(x_0)\int_{-1}^{1} w(x)g^2(x)dx - g(x_0)\int_{-1}^{1} w(x)g(x)h(x)dx = 0$$

or

$$\int_{-1}^{1} w(x)g(x)\left[h(x_0)g(x) - h(x)g(x_0)\right] dx = 0 \tag{1.17}$$

for all polynomials $h \in \mathbb{P}_k$ if $w(x) = 1$, $1-x$ or $1+x$, and for all polynomials $h \in \mathbb{P}_{k-1}$ if $w(x) = 1 - x^2$. We first consider the case $w(x) = 1, 1-x, 1+x$. Repeating the same argument as we used to prove (1.15) we can deduce that (1.17) implies that for all $l \in \mathbb{P}_{k-1}$ we have

$$\int_{-1}^{1} w(x)g(x)(x - x_0)l(x)dx = 0.$$

Denote

$$G(x) = (x - x_0)g(x)$$

and consider the sequence of polynomials p_k orthonormal on $[-1, 1]$ with respect to the weight $w(x)$. Since $\deg(G) = k + 1$ and orthonormal polynomials of degree $\le k + 1$ form a basis (over $\mathbb{R}$) of $\mathbb{P}_{k+1}$, one can write

$$G(x) = \sum_{m=0}^{k+1} c_m p_m(x)$$

for some real constants c_m. Taking $l(x) = p_i(x)$ for $0 \le i \le k-1$ we obtain that $c_i = 0$ for $0 \le i \le k - 1$. Indeed, if $l(x) = p_i(x)$, $0 \le i \le k - 1$ then

$$0 = \int_{-1}^{1} w(x)G(x)p_i(x)dx = \sum_{m=0}^{k+1} c_m \int_{-1}^{1} w(x)p_m(x)p_i(x)dx = c_i.$$

Thus,

$$G(x) = (x - x_0)g(x) = c_{k+1}p_{k+1}(x) + c_k p_k(x).$$

Letting $x = x_0$, we get $c_{k+1}p_{k+1}(x_0) + c_k p_k(x_0) = 0$, and so

$$g(x) = g_{extr}(x) := c\frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0},$$

for some real constant c, if $w(x) = 1$ or $1 \pm x$. For the case where $w(x) = 1 - x^2$, we have to take $k - 1$ instead of k. It gives us a polynomial g, of the form

$$g(x) = g_{extr}(x) := c\frac{p_k(x)p_{k-1}(x_0) - p_k(x_0)p_{k-1}(x)}{x - x_0}, \quad k \ge 1.$$

Now, using the Christoffel-Darboux formula (1.9), $S(P, x_0)$ can be computed explicitly. Indeed,

$$\int_{-1}^{1} w(x) \left(\frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0} \right)^2 dx$$

$$= \frac{\gamma_k^2}{\gamma_{k+1}^2} \int_{-1}^{1} w(x) \left(\sum_{l=0}^{k} p_l(x_0)p_l(x) \right)^2 dx$$

$$= \frac{\gamma_k^2}{\gamma_{k+1}^2} \int_{-1}^{1} w(x) \sum_{l=0}^{k} (p_l(x_0)p_l(x))^2 dx$$

$$= \frac{\gamma_k^2}{\gamma_{k+1}^2} \sum_{l=0}^{k} p_l^2(x_0),$$

and

$$\left. \frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0} \right|_{x=x_0} = \frac{\gamma_k}{\gamma_{k+1}} \sum_{l=0}^{k} p_l^2(x_0).$$

Hence, in the case where $w(x) = 1$ or $1 \pm x$, we have

$$S(P, x_0) = w(x_0) \frac{(g_{extr}(x_0))^2}{\int_{-1}^{1} w(x)(g_{extr}(x))^2 dx}$$

$$= w(x_0) \frac{\left(\frac{\gamma_k}{\gamma_{k+1}}\right)^2 \left(\sum_{l=0}^{k} p_l^2(x_0) \right)^2}{\int_{-1}^{1} w(x) \left(\frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0} \right)^2 dx}$$

$$= w(x_0) \left(\sum_{l=0}^{k} p_l^2(x_0) \right).$$

Similarly, if $w(x) = 1 - x^2$, we get

$$S(P, x_0) = w(x_0) \left(\sum_{l=0}^{k-1} p_l^2(x_0) \right).$$

Now, if $n = 2k + 1$ and $w(x) = 1 + x$, then

$$p_k(x) = J_k^{(0,1)}(x),$$

where $J_k^{(0,1)}(x)$ is the Jacobi polynomial defined in Section 1.4. Hence,

$$S(P, x_0) = (1 + x_0) \sum_{l=0}^{k} (J_l^{(0,1)}(x_0))^2 = S_k(x_0).$$

By analogy, if $w(x) = 1 - x$, then

$$S(P, x_0) = (1 - x_0) \sum_{l=0}^{k} (J_l^{(1,0)}(x_0))^2 = S_k(-x_0),$$

where we used (1.11).

Therefore, if $n = 2k + 1$, we get the following sharp pointwise inequality

$$P_{2k+1}(x_0) \le \max\left(S_k(x_0), S_k(-x_0)\right) \int_{-1}^{1} P_{2k+1}(x)dx, \tag{1.18}$$

for all $P_{2k+1} \in \mathbb{P}_{2k+1}^+$.

In the case $n = 2k$, $w(x) = 1$ or $w(x) = 1 - x^2$ and

$$S(P, x_0) = F_k(x_0),$$

or

$$S(P, x_0) = H_k(x_0),$$

respectively, where H_k and F_k are defined in (1.13). We arrive at the following sharp pointwise inequality:

$$P_{2k}(x_0) \le \max\left(F_k(x_0), H_k(x_0)\right) \int_{-1}^{1} P_{2k}(x)dx, \tag{1.19}$$

for all $P_{2k} \in \mathbb{P}_{2k}^+$.

Let P_n be a polynomial of degree n, that is nondecreasing on $[-1, 1]$, *i.e.*, $P_n \in \Delta_n$ Then, P_n' is a nonnegative polynomial on $[-1, 1]$. Note, that

$$\int_{-1}^{1} P_n'(x)dx = P_n(1) - P_n(-1) \le 2\|P_n\|.$$

Combining this with (1.18) and (1.19) we get

$$P'_{2k+2}(x) \le 2 \cdot \max\left(S_k(x), S_k(-x)\right)||P_{2k+2}|| \tag{1.20}$$

and

$$P'_{2k+1}(x) \le 2 \cdot \max\left(H_k(x), F_k(x)\right)||P_{2k+1}||, \tag{1.21}$$

for all monotone polynomials P_n. This completes the proof. □

Using Theorem 1.5.2, one can give an alternative proof of Qazi's recent result, that is Theorem 1.5.1 for polynomials of even degree. The following fact about orthogonal polynomials is needed.

Theorem 1.5.3 (Szegő [39], 1919). *Let $w(x)$ be a weight function which is non-decreasing (non-increasing) in the interval $[a, b]$, b and a are finite. If $\{p_n\}$ is the set of the corresponding orthogonal polynomials, the functions $w(x)p_n^2(x)$ attain their maxima in $[a, b]$ at $x = b$ ($x = a$).*

Proof of Theorem 1.5.1. We consider an even case $n = 2k + 2$, $k \ge 0$. Using Szegő's theorem for non-decreasing weight $w(x) = 1+x$ and for non-increasing weight $w(x) = 1 - x$ together with (1.12) we get:

$$(1 + x)(J_l^{(0,1)}(x))^2 \le 2(J_l^{(0,1)}(1))^2 = l + 1$$

and

$$(1 - x)(J_l^{(1,0)}(x))^2 \le 2(J_l^{(1,0)}(-1))^2 = l + 1,$$

for all $l \ge 0$ and $x \in [-1, 1]$. Summing these inequalities for $0 \le l \le k$ and using Theorem 1.5.2 we get

$$P'_n(x) \le 2\sum_{l=0}^{k}(1 + l)\|P_n\| = (k + 1)(k + 2)\|P_n\| = \frac{n(n + 2)}{4}\|P_n\|,$$

and the proof is complete for monotone polynomials of even degree. □

Multiplying both sides of (1.20) and (1.21) by $\sqrt{1 - x^2}$ and taking supremum over all $x \in [-1, 1]$ we get the following

Corollary 1.5.4 (Sharp Benstein-type inequality for monotone polynomials)**.**

$$\max_{P_n \in \triangle_n} \frac{\|P_n'(x)\sqrt{1-x^2}\|}{\|P_n\|} \tag{1.22}$$
$$= \begin{cases} 2\|\sqrt{1-x^2}S_k(x)\|, & n = 2k+2, \\ 2\max(\|\sqrt{1-x^2}H_k(x)\|, \|\sqrt{1-x^2}F_k(x)\|), & n = 2k+1. \end{cases}$$

Remark 1.5.5. From the proof of Theorem 1.5.2 it follows that equality in (1.22) holds for one of the following polynomials

$$s_k(x) := -\frac{1}{2}\int_{-1}^{1} S_k(t)dt + \int_{-1}^{x} S_k(t)dt,$$
$$h_k(x) := -\frac{1}{2}\int_{-1}^{1} H_k(t)dt + \int_{-1}^{x} H_k(t)dt,$$
$$f_k(x) := -\frac{1}{2}\int_{-1}^{1} F_k(t)dt + \int_{-1}^{x} F_k(t)dt.$$

Chapter 2

Markov-Nikolskii type inequalities

2.1 Markov-Nikolskii type inequalities: classical results

In 1951, a famous Russian mathematician S. Nikolskii (he is now 106 years old) in [29] proved an inequality which compares different norms of P_n. Namely, in the inequalities of the type

$$C_1\|P_n\|_{L_p[a,b]} \leq \|P_n\|_{L_q[a,b]} \leq C_2\|P_n\|_{L_p[a,b]},$$

where $p \leq q \leq \infty$, we want to find out how the constants depend on n. Of interest is, of course, only the second relation, since the first one is an immediate consequence of Hölder's inequality. S. Nikolskii [29] proved:

Theorem 2.1.1 (S. Nikolskii [29], 1951)**.** *For $0 < p \leq q \leq \infty$ and $P_n \in \mathbb{P}_n$,*

$$\|P_n\|_{L_q[-1,1]} \leq C(p,q)n^{\frac{2}{p}-\frac{2}{q}}\|P_n\|_{L_p[-1,1]}.$$

A natural question is: can we "compare" the L_q-norms of derivatives of a polynomial with the L_p-norm of the polynomial itself?

Let us recall the following:

Definition 2.1.2. Two variables $A_n, B_n > 0$ are asymptotically equivalent with respect to n, $A_n \asymp B_n$, if and only if there exist constants $C_1, C_2 > 0$ which do not depend on n and such that:

$$C_1 A_n \leq B_n \leq C_2 A_n.$$

For $n \geq k \geq 0$, we denote

$$M_{q,p}(n,k) := \sup_{P_n \in \mathbb{P}_n} \frac{\|P_n^{(k)}\|_{L_q[-1,1]}}{\|P_n\|_{L_p[-1,1]}}.$$

In paper Glazyrina [15], complete information about the orders of $M_{q,p}(n,k)$ for all values $p > 0,\ q > 0$ is given.

Theorem 2.1.3 ([15])**.** *For $0 < p, q \leq \infty$ and $P_n \in \mathbb{P}_n$ we have:*

$$M_{q,p}(n,k) \asymp \begin{cases} n^{2k+2/p-2/q}, & \text{if } k > 2/q - 2/p, \\ n^k (\log n)^{1/q-1/p}, & \text{if } k = 2/q - 2/p, \\ n^k, & \text{if } k < 2/q - 2/p. \end{cases} \tag{2.1}$$

Here, the asymptotics are taken when $n \to \infty$ and $k \geq 0$ is fixed.

The first contributor to this direction was Pafnutii Lvovich Chebyshev ([8], 1854), who proved that if $\|P_n\| = 1$ then $\|P(n)_n\| \leq 2^{n-1} n!$ (this is one way to formulate the statement of his remarkable theorem about polynomial of least deviation from 0), which corresponds to the case $p = q = \infty$, $k = n$. A few decades later, in 1873, famous Russian mathematicians Korkin and Zolotarev [20] found the exact constant $M_{1,1}(n,n)$. As mentioned in the first part of this thesis, A. Markov ([24], 1889) and V. Markov ([25], 1892) investigated $M_{\infty,\infty}(n,1)$ and $M_{\infty,\infty}(n,k)$, respectively. In 1937, Szegő, Tamarkin and Hille [18] found the order of $M_{p,p}(n,1)$ for $p \geq 1$. It is worth to note that using a technique from Matrix Analysis they also found the exact value of $M_{2,2}(n,1)$. In 2006, A. Kroo [22] gave a short and elegant alternative proof (using a variational approach). Here, we mention some main contributors:

Year	Order	Author
1937	$M_{p,p}(n,1)$, $p \geq 1$	Szegő, Tamarkin, Hille [18]
1951	$M_{q,p}(n,0)$, $0 < p, q \leq \infty$	Nikolskii [28]
1972	$M_{q,p}(n,k)$, $1 \leq p, q \leq \infty$	Daugavet, Rafalson [10]
1975	$M_{q,p}(n,k)$, $0 < p, q \leq \infty$	Ivanov [19]

Exact values of $M_{q,p}(n,k)$ are known only for a few cases (see table below).

Year	Exact constant	Author
1873	$M_{1,1}(n,n)$	Korkin, Zolotaryov [20]
1889	$M_{\infty,\infty}(n,1)$	A. Markov [24]
1892	$M_{\infty,\infty}(n,k)$	V. Markov [25]
1937	$M_{2,2}(n,1)$	Szegő, Tamarkin, Hille [18]
1969	$M_{\infty,2}(n,k)$	Labelle [23]
1982	$M_{q,\infty}(n,k)$, $q \geq 1$	Bojanov [5]

2.2 Markov-Nikolskii inequality with constraints

We start with the following

Definition 2.2.1. We will say that a function $f : [a,b] \to \mathbb{R}$ is *absolutely monotone of order* k if, for all $x \in [a,b]$,

$$f^{(m)}(x) \geq 0,$$

for all $0 \leq m \leq k$, and denote by $\triangle_n^{(k)}$ the set of all absolutely monotone polynomials of order k on $[-1,1]$.

For example, absolutely monotone functions of order 0 are just nonnegative functions on $[a,b]$, and $\triangle_n^{(1)} = \triangle_n \cap \triangle_n^{(0)}$ is the set of all nonnegative monotone polynomials of degree n on $[-1,1]$.

A natural modification of $M_{q,p}(n,k)$ for $\triangle_n^{(k)}$ is

$$M_{q,p}^{(k)}(n,m) = \sup_{P_n \in \triangle_n^{(k)}} \frac{\|P_n^{(m)}\|_{L_q[-1,1]}}{\|P_n\|_{L_p[-1,1]}},$$

for $0 \leq m \leq n$, $0 \leq k \leq n$.

In 2009, J. Szabados and A. Kroo [21] found the exact constants for Markov-Nikolskii inequalities in L_1 and L_∞. Note, that J. Szabados and A. Kroo referred to absolutely monotone polynomials of order k as "k-monotone polynomials", which is not quite correct. Usually, k-monotonicity of a function f means that only its kth derivative (more precisely, kth divided difference) is nonnegative.

The next theorem contains their results:

Theorem 2.2.2 (Kroo and Szabados [21], 2009). *For* $2 \le k \le n$, $m = \left\lfloor \frac{n-k}{2} \right\rfloor + 1$, $\beta = \frac{1-(-1)^{n-k}}{2}$*:*

$$M_{\infty,\infty}^{(k)}(n,1) = \frac{k-1}{1 - x_{1,m}^{(k-2,\beta)}},$$

$$M_{1,1}^{(k)}(n,1) = M_{\infty,\infty}^{(k+1)}(n+1,1),$$

where $x_{1,m}^{(k-2,\beta)}$ *is the largest zero of the Jacobi polynomial* $J_m^{(k-2,\beta)}$, *i.e., associated with the weight* $(1-x)^{k-2}(1+x)^{\beta}$.

T. Erdelyi [13] found the order of $M_{q,p}^{(k)}(n,m)$ in the case $q \ge p$. He was interested in how this order depends on k.

Theorem 2.2.3 (Erdelyi [13], 2009). *For* $0 \le m \le k/2$, $1 \le k \le n$, $0 < p \le q \le \infty$, *we have*

$$M_{q,p}^{(k)}(n,m) \asymp \left(n^2/k\right)^{m+1/p-1/q} \asymp M_{q,p}(n,m).$$

The second asymptotic is used when k *is fixed.*

Note that Theorem 2.2.3 does not answer the question about the order for the case of monotone nonnegative polynomials, in other words, when $k = 1$. Indeed, if $k = 1$, then $m = 0$ and Theorem 2.2.3 simplifies to the classical Nikolskii inequality. It also follows from Theorem 2.2.3 that whenever $q \ge p$ the order of constants in Markov-Nikolskii type inequalities in the case of absolutely monotone polynomials of order k is essentially the same as for ordinary polynomials (when k is fixed). It is natural to ask if these inequalities are still valid for all p, q? The next trivial example shows that some improvements are possible. Indeed, we have seen that

$$\|P_n'\|_{L_1[-1,1]} \le 2\|P_n\|,$$

for all $P_n \in \triangle_n$. Note that, from [15], it follows that in the case of all polynomials for $q = 1, p = \infty$ the order is n. Our main goal in this section is to construct an analog of the table for $k = 1$ (monotone nonnegative polynomials).

We conjecture that the correct order is given by the following:

Conjecture.

$$M_{q,p}^{(1)}(n,1) \asymp \begin{cases} n^{2+2/p-2/q}, & \text{if } 1 > 1/q - 1/p, \\ \log n, & \text{if } 1 = 1/q - 1/p, \\ 1, & \text{if } 1 < 1/q - 1/p. \end{cases}$$

The next theorem is useful for the proof of many polynomial inequalities. The main point here is that a polynomial of degree n cannot have a large part of its L_q-norm over $[-1,1]$ concentrated in the interval of length less than $(10n^2)^{-\max(q,1)}$ (in other words, the integral over a "small" interval can not be too "large"). The proof follows from Remez's inequality [7]. Here, we present a short direct proof of this result.

Theorem 2.2.4. *If $q > 0$, then for every $n \in \mathbb{N}$ and an interval $[a,b] \subset [-1,1]$ such that $0 < b-a \le (10n^2)^{-\max(q,1)}$, the following inequality holds for all $P_n \in \mathbb{P}_n$:*

$$\|P_n\|_{L_q[a,b]} \le C(q)\|P_n\|_{L_q[-1,1]}. \tag{2.2}$$

Proof. For every $P_n \in \mathbb{P}_n$ and $-1 \le a < b \le 1$ we have

$$\left(\int_a^b |P_n(x)|^q dx\right)^{1/q} \le (b-a)^{1/q}\|P_n\|_{C[a,b]} \le (b-a)^{1/q}\|P_n\|.$$

Now, if $G_n(x) = \int_{-1}^{x} P_n(t)dt$, then

$$|G_n(x)| = \left|\int_{-1}^{x} P_n(t)dt\right| \le \int_{-1}^{x} |P_n(t)|dt \le \int_{-1}^{1} |P_n(t)|dt,$$

and Theorem 1.2.1 implies

$$(b-a)^{-1/q}\left(\int_a^b |P_n(x)|^q dx\right)^{1/q} \le \|P_n\| = \|G_n'\| \le (n+1)^2\|G_n\| \le (n+1)^2 \int_{-1}^{1} |P_n(t)|dt.$$

If $q \ge 1$, take $b-a \le \left(\frac{1}{10n^2}\right)^q$. Applying Hölder's inequality we get

$$\begin{aligned} \|P_n\|_{L_q[a,b]} = \left(\int_a^b |P_n(x)|^q dx\right)^{1/q} &\le \frac{(n+1)^2}{10n^2}\int_{-1}^{1} |P_n(x)|dx \\ &\le \frac{1}{2}\|P_n\|_{L_1[-1,1]} \\ &\le C(q)\|P_n\|_{L_q[-1,1]}. \end{aligned}$$

If $q < 1$, take $b - a \le \frac{1}{10n^2}$ and apply Theorem 2.1.1. We get

$$\|P_n\|_{L_q[a,b]} = \left(\int_a^b |P_n(x)|^q dx \right)^{1/q} \le (b-a)^{1/q}(n+1)^2 \int_{-1}^{1} |P_n(x)| dx$$
$$\le C(q)(1/10n^2)^{1/q} n^{2/q-2}(n+1)^2 \|P_n\|_{L_q[-1,1]}$$
$$\le C_1(q)\|P_n\|_{L_q[-1,1]}$$

□

From the proof of Theorem 2.2.4 it follows that for the case $q = 1$, the constant $C(q)$ can be found explicitly. To simplify the calculations in the future, we state the case $q = 1$ as a separate lemma.

Lemma 2.2.5. *For every $n \in \mathbb{N}$ and an interval $[a, b] \subset [-1, 1]$ such that $0 < b-a \le \frac{1}{10n^2}$, the following inequality holds for all $P_n \in \mathbb{P}_n$:*

$$\|P_n\|_{L_1[a,b]} \le \frac{1}{2}\|P_n\|_{L_1[-1,1]}.$$

Lemma 2.2.5 and Theorem 2.2.4 give a tool to handle the so-called endpoints problem.

Theorem 2.2.6. *Let $a, b \in \mathbb{R}$ and $p, q > 0$. The following inequality holds for all nonnegative measurable functions $f \ge 0$ if and only if $\frac{1}{q} - \frac{1}{p} > 1$:*

$$\left(\int_a^b (f(x))^q dx \right)^{\frac{1}{q}} \le C(p, q, b-a) \left(\int_a^b \left(\int_a^x f(t)dt \right)^p dx \right)^{\frac{1}{p}}. \tag{2.3}$$

Proof. We start with the "only if" part and show that if

$$\frac{1}{q} - \frac{1}{p} \le 1,$$

then (2.3) does not hold for all nonnegative f. The characteristic function of a short interval near b gives a counterexample for $\frac{1}{q} - \frac{1}{p} < 1$. Indeed, if

$$f(x) = \chi_{[b-\varepsilon,b]} = \begin{cases} 1, & \text{if } b - \varepsilon \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\frac{\left(\int\limits_a^b (f(x))^q dx\right)^{\frac{1}{q}}}{\left(\int\limits_a^b \left(\int\limits_a^x f(t)dt\right)^p dx\right)^{\frac{1}{p}}} = C(p)\varepsilon^{\frac{1}{q}-\frac{1}{p}-1} \to \infty,\ \varepsilon \to 0^+.$$

To handle the case $\frac{1}{q} - \frac{1}{p} = 1$, consider the function $g(x) = (b+\varepsilon-x)^{-1/q}$ with small $\varepsilon > 0$. We have:

$$\begin{aligned}
\frac{\left(\int\limits_a^b (g(x))^q dx\right)^{\frac{1}{q}}}{\left(\int\limits_a^b \left(\int\limits_a^x g(t)dt\right)^p dx\right)^{\frac{1}{p}}} &= \frac{p\left(-\ln\varepsilon + \ln(b+\varepsilon-a)\right)^{\frac{1}{q}}}{\left(\int\limits_a^b \left((b+\varepsilon-x)^{-\frac{1}{p}} - (b+\varepsilon-a)^{-\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}} \\
&\geq \left(\frac{1}{2}\right)^{\frac{1}{q}} \frac{p|\ln\varepsilon|^{\frac{1}{q}}}{\left(\int\limits_a^b (b+\varepsilon-x)^{-1}dx\right)^{\frac{1}{p}}} \\
&= \left(\frac{1}{2}\right)^{\frac{1}{q}} \frac{p|\ln\varepsilon|^{\frac{1}{q}}}{(-\ln\varepsilon + \ln(b+\varepsilon-a))^{\frac{1}{p}}} \to \infty
\end{aligned}$$

as $\varepsilon \to 0^+$.

We now prove (2.3) for $\frac{1}{q} - \frac{1}{p} > 1$ by using a standard discretization method. Let $E_k \subset [a,b]$ be the set on which f is "about 2^k". In other words, for $k \in \mathbb{Z}$,

$$E_k := \left\{x \in [a,b] \,|\, 2^k \leq f(x) < 2^{k+1}\right\},$$

and let $m_k := |E_k|$. Then,

$$\left[\sum_k 2^{kq} m_k\right]^{1/q} \leqslant \left(\int\limits_a^b (f(x))^q dx\right)^{\frac{1}{q}} \leqslant 2\left[\sum_k 2^{kq} m_k\right]^{1/q}. \tag{2.4}$$

To estimate the right hand side of (2.3) from below, we note that for all $k \in \mathbb{Z}$, there exists $x_k \in [a,b]$ such that

$$|E_k \cap [x_k, b]| = \frac{1}{2} m_k.$$

Denote $A_k := E_k \cap [x_k, b]$. Then, for all $x \in A_k$ (in fact, for all $x \in [x_k, b]$) we have

$$\int_a^x f(t)dt \geq \int_{E_k \setminus A_k} f(t)dt \geq 2^k |E_k \setminus A_k| = 2^{k-1} m_k.$$

So the right hand side of (2.3) can be estimated as follows:

$$\left(\int_a^b \left(\int_a^x f(t)dt\right)^p dx\right)^{\frac{1}{p}} \geq \left[\sum_k 2^{(k-1)p} m_k^p \cdot |A_k|\right]^{1/p} = 2^{-1-1/p}\left[\sum_k 2^{kp} m_k^{p+1}\right]^{1/p}, \tag{2.5}$$

Observe that if we have several k with $m_k \in [2^n, 2^{n+1})$ for some $n \in \mathbb{Z}$, then the contribution of the one with the largest k to the sum $\sum 2^{kq} m_k$ dominates all other terms with k from such interval (this boils down to the fact that sum of a geometric progression is comparable to its largest term). More precisely, if $m_{k_1}, m_{k_2}, ...m_{k_l} \in [2^n, 2^{n+1})$ and the $k_1 < k_2... < k_l$, then

$$\sum_{i=1}^{l} 2^{k_i q} m_{k_i} \leq \sum_{i=1}^{l} 2^{k_i q} 2^{n+1} \leq C(q) 2^{k_l q + n}.$$

Therefore, if we denote for each $n \in \mathbb{Z}$

$$K_n := \left\{k \in \mathbb{Z} \,|\, 2^n \leqslant m_k < 2^{n+1}\right\} \quad \text{and} \quad \kappa_n := \max(K_n),$$

then $\mathrm{card}(K_n) < \infty$ (since $\sum_k m_k = b - a$), $K_n \cap K_m = \emptyset$ if $n \neq m$, $K_n = \emptyset$ if $n > \log_2(b-a)$, and

$$\sum_{k \in K_n} 2^{kq} m_k \leqslant C(q) 2^{q\kappa_n + n}.$$

Taking into account that, for all $\delta > 0$,

$$\sum_{n \leqslant \log_2(b-a)} 2^{n\delta} \leqslant C(\delta, b-a),$$

and choosing

$$\delta := q\left(\frac{1}{q} - 1 - \frac{1}{p}\right) > 0,$$

we have

$$\begin{aligned}
\left[\sum_k 2^{kq} m_k\right]^{1/q} &= \left[\sum_n \sum_{k\in K_n} 2^{kq} m_k\right]^{1/q} \leqslant C(q)\left[\sum_{n\leqslant \log_2(b-a)} 2^{q\kappa_n+n}\right]^{1/q} \\
&\leqslant C(q)\left[\sum_{n\leqslant \log_2(b-a)} 2^{q\kappa_n+n(1-\delta)}\cdot 2^{n\delta}\right]^{1/q} \\
&\leqslant C(q,\delta,b-a)\left[\max_{n\leqslant \log_2(b-a)} 2^{q\kappa_n+n(1-\delta)}\right]^{1/q} \\
&\leqslant C(p,q,b-a)\left[\max_{n\leqslant \log_2(b-a)} 2^{p\kappa_n+np+n}\right]^{1/p} \\
&\leqslant C(p,q,b-a)\left[\sum_{n\leqslant \log_2(b-a)} 2^{p\kappa_n+np+n}\right]^{1/p}.
\end{aligned}$$

Now, inequalities (2.4) and (2.5) together with

$$\sum_k 2^{kp} m_k^{p+1} = \sum_n \sum_{k\in K_n} 2^{kp} m_k^{p+1} \geq \sum_{n\leqslant \log_2(b-a)} 2^{p\kappa_n+np+n}$$

complete the proof of the theorem. □

Take $P_n \in \triangle_n^{(1)}$. Then

$$P_n(x) = \int_{-1}^{x} P_n'(t)dt + P_n(-1),$$

P_n' is a nonnegative function on $[-1,1,]$, $P(-1)\geq 0$. Applying Theorem 2.2.6 with $f = P_n'$ we get we get

$$\|P_n'\|_{L_q[-1,1]} \leq C(p,q)\left(\int_{-1}^{1}(P_n(x)-P_n(-1))^p dx\right)^{1/p} \leq C(p,q)\|P_n\|_{L_p[-1,1]},$$

for $\frac{1}{q}-\frac{1}{p}>1$. On the other hand, it is clear that the polynomial $P_n(x)=x+1$ gives a bound for the ratio $\|P_n'\|_{L_q[-1,1]}/\|P_n\|_{L_p[-1,1]}$ from below.

Hence, we get the following

Corollary 2.2.7. *For* $\frac{1}{q}-\frac{1}{p}>1$, *we have:*

$$M_{q,p}^{(1)}(n,1) \asymp 1.$$

We now find the asymptotic behavior of $M^{(1)}_{q,p}(n,1)$ in the case $q \geq 1$, and prove that an upper bound is given by the following:

Theorem 2.2.8. *For* $1 \leq q \leq \infty,\ 0 < p \leq \infty$ *we have:*

$$M^{(1)}_{q,p}(n,1) \leq C(p,q)n^{2+2/p-2/q}. \tag{2.6}$$

Proof. Observe, that for any function $f \in C^1[-1,1]$ which is monotone on $[-1,1]$, we have that $\|f'\|_{L_1[-1,1]} = \int\limits_{-1}^{1} f'(x)dx = f(1) - f(-1) \leq 2\|f\|$. By Theorem 2.1.1,

$$\|P_n'\|_{L_q[-1,1]} \leq Cn^{2-\frac{2}{q}}\|P_n'\|_{L_1[-1,1]}.$$

Combining the latter with our observation and applying Theorem 2.1.1 again we get:

$$\|P_n'\|_{L_1[-1,1]} \leq 2\|P_n\| \leq Cn^{\frac{2}{p}}\|P_n\|_{L_p[-1,1]}.$$

and so

$$\|P_n'\|_{L_q[-1,1]} \leq Cn^{2-\frac{2}{q}+\frac{2}{p}}\|P_n\|_{L_p[-1,1]}.$$

□

Remark 2.2.9. It follows from Theorem 2.1.3 that, if $2/q - 2/p < 1$, then (2.6) is also valid.

To prove that the order $n^{2+2/p-2/q}$ is sharp, it is enough to construct a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, $P_n \in \triangle_n^{(1)}$, such that

$$\frac{\|P_n'\|_{L_q[-1,1]}}{\|P_n\|_{L_p[-1,1]}} \geq C(p,q)n^{2+2/p-2/q}.$$

In order to do that, we modify an example taken from [13].

Example 2.2.10 (Erdelyi [13])**.** Let

$$P_N(x) := \left[\int_{-1}^{x}\left(\frac{1}{2} + \sum_{j=1}^{m} T_j(t)\right)^8 dt\right]^{\nu}, \tag{2.7}$$

where $T_j(t)$ are Chebyshev polynomials of the first kind,

$$m := \left\lfloor \frac{n-2}{8} \right\rfloor, \qquad \nu := \left\lfloor \frac{1}{p} \right\rfloor + 1, \qquad N := (8m+1)\nu.$$

Then P_N is a polynomial of degree N and $N \approx (n-1)\nu$, $P_n \in \triangle_n^{(1)}$ and

$$\frac{\|P_n'\|_{L_q[-1,1]}}{\|P_n\|_{L_p[-1,1]}} \asymp n^{2+\frac{2}{p}-\frac{2}{q}}.$$

We now prove that for $p=1$, the asymptotic behavior of $M_{q,1}^{(1)}(n,1)$ coincides with what we conjectured at the beginning of this section.

Theorem 2.2.11. *For $p=1$, $q>0$ we have:*

$$M_{q,1}^{(1)}(n,1) \asymp \begin{cases} n^{4-2/q}, & \text{if } q > 1/2, \\ \log n, & \text{if } q = 1/2, \\ 1, & \text{if } q < 1/2. \end{cases}$$

Proof. For $q < 1/2$, the result follows from Corollary 2.2.7. For $q > 1$, it follows from Theorem 2.2.8 and Example 2.2.10. Let $1/2 \le q < 1$. We first prove an upper bound. For a given polynomial $P_n \in \triangle_n^{(1)}$, let $G_n = P_n'$. Using integration by parts we get

$$\|P_n\|_{L_1[-1,1]} \ge \int_{-1}^{1}\int_{-1}^{x} G_n(t)dtdx = \int_{-1}^{1}(1-x)G_n(x)dx,$$

so it is sufficed to prove the following inequalities:

$$\left(\int_{-1}^{1}(G_n(x))^q dx\right)^{1/q} \le Cn^{4-2/q}\int_{-1}^{1}(1-x)G_n(x)dx, \tag{2.8}$$

for $q > \frac{1}{2}$, and

$$\left(\int_{-1}^{1}(G_n(x))^{1/2} dx\right)^{2} \le C\log n\int_{-1}^{1}(1-x)G_n(x)dx, \tag{2.9}$$

for $q = \frac{1}{2}$. Applying Hölder's inequality

$$\|fg\|_{L_r[a,b]} \le \|f\|_{L_t[a,b]}\|g\|_{L_s[a,b]}, \qquad \frac{1}{r} = \frac{1}{s} + \frac{1}{t},$$

for $f(x) = G_n(x)(1-x)$ and $g(x) = (1-x)^{-1}$, $r = q$, $t = 1$, $s = \frac{q}{1-q} > 0$, together with Theorem 2.2.4, we get

$$\begin{aligned}\int_{-1}^{1}(1-x)G_n(x)dx &\geq \int_{-1}^{1-\frac{1}{10n^2}}(1-x)G_n(x)dx \\ &\geq \frac{\left(\int_{-1}^{1-\frac{1}{10n^2}}(G_n(x))^q dx\right)^{1/q}}{\left(\int_{-1}^{1-\frac{1}{10n^2}}(1-x)^{-s}dx\right)^{1/s}} \\ &\geq C(q)\frac{\left(\int_{-1}^{1}(G_n(x))^q dx\right)^{1/q}}{\left(\int_{-1}^{1-\frac{1}{10n^2}}(1-x)^{-s}dx\right)^{1/s}}.\end{aligned}$$

Now, if $s \neq 1$ (in other words, $q \neq \frac{1}{2}$), then

$$\left(\int_{-1}^{1-\frac{1}{10n^2}}(1-x)^{-s}dx\right)^{1/s} = \left(\frac{1-q}{2q-1}\right)^{\frac{1-q}{q}}(1-x)^{\frac{1-2q}{q}}\Big|_{-1}^{1-\frac{1}{10n^2}} \leq C(q)n^{4-2/q},$$

and (2.8) follows.

If $s = 1$, then $q = \frac{1}{2}$ and we have

$$\int_{-1}^{1-\frac{1}{10n^2}}(1-x)^{-1}dx \leq C\log n,$$

which implies (2.9).

To prove the lower bounds, we observe that for $q > \frac{1}{2}$, the result follows from Example 2.2.10. For $q = \frac{1}{2}$, consider

$$F_n(x) := \left(\sum_{k=0}^{n} x^k\right)^2 = \left(\frac{1-x^{n+1}}{1-x}\right)^2,$$

and take

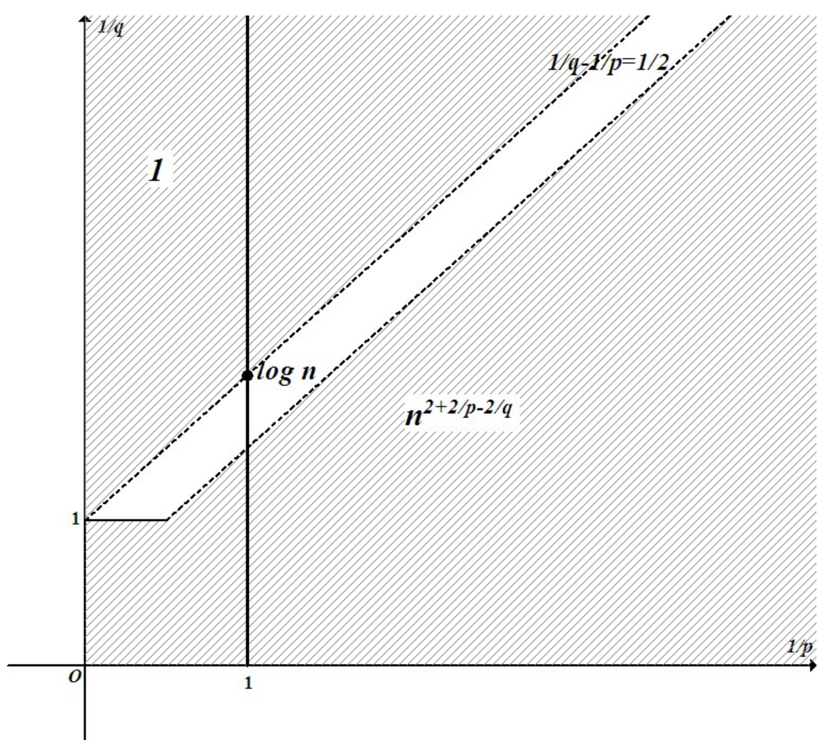
$$P_n(x) = \int_{-1}^{x} F_n(t)dt.$$

Then $P_n \in \triangle^{(1)}_{2n+1}$, and

$$\frac{\|P'_n\|_{L_{1/2}[-1,1]}}{\|P_n\|_{L_1[-1,1]}} = \frac{\left(\int_{-1}^1 \left(\sum_{k=0}^n x^k\right) dx\right)^2}{\int_{-1}^1 (1-x)\left(\sum_{k=0}^n x^k\right)^2 dx}$$
$$\geq 2\frac{\left(\int_0^1 \left(\sum_{k=0}^n x^k\right) dx\right)^2}{\int_{-1}^{1-\frac{1}{10n^2}} (1-x)\left(\sum_{k=0}^n x^k\right)^2 dx}$$
$$\geq 2\frac{\left(\int_0^1 \left(\sum_{k=0}^n x^k\right) dx\right)^2}{\int_{-1}^{1-\frac{1}{10n^2}} (1-x)\frac{1}{(1-x)^2} dx}$$
$$\geq C \cdot \frac{(\log n)^2}{\log n} = C \log n.$$

This completes the proof. □

Combining the results of Theorems 2.2.8, 2.2.11 and Corollary 2.2.7 we get the following range of parameters p, q for which the orders of $M^{(1)}_{q,p}(n,1)$ are known (in the picture, white region and punctured lines correspond to the orders of $M^{(1)}_{q,p}(n,1)$ which are still unknown).



2.3 Conclusions

In this thesis, new constrained polynomial inequalities in the L_p-metric and the uniform norm are obtained. In particular, we proved an analog of the classical Bernstein inequality for monotone polynomials and found the order of constants $M_{q,p}^{(1)}(n, 1)$ in constrained Markov-Nikolskii type inequalities. The results were obtained using a variational approach using the theory of orthogonal polynomials.

It is worth mentioning that there are still values of p, q for which the order of $M_{q,p}^{(1)}(n, 1)$ is unknown. In particular, it would be interesting to generalize the results of this thesis for higher order derivatives and absolute monotonicity of order $k \geq 2$. Results of this nature were obtained in two recent papers [21] and [13], but only for the cases where $p \leq q$.

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