Shifting Attentions in Mathematics:

Developing Problem Solving Abilities

Through Problem-Solving Groups

by

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Abstract

The purpose of this study was to improve problem solving attitudes and abilities in students of mathematics through the exploration of John Mason's general problem solving strategy and the use of problem solving groups, and to document and understand this improvement process. The types of problems and tasks assigned to students as well as assessment practices were also examined. A Design-Experiment Research approach was used with thirty grade 9 students participating throughout the year-long study. A teacher-researcher journal, student problem-solving journals, and surveys were used.

The study showed that using a *general* problem solving strategy *with groups of students working together to solve problems* can improve problem solving attitudes and abilities. Students made significant improvements during initial engagement of problems, in specializing and generalizing, and in communication. Almost all students expressed a more positive attitude toward problem solving and their problem solving abilities. The study demonstrates how focusing on initial stages of the problem solving process like the understanding of the problem in a group context can reach multiple learning objectives and positively impact later stages of problem solving. In addition, recommendations for classroom teachers are provided concerning the roles within the groups, the nature of beneficial problem types and student tasks, and concerning the role of the teacher as researcher of his or her own teaching practice.

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Dedication

I would like to dedicate this project to Brad, Candy, David, Evan and Kristin. Though we don't work in the same building, we're often on the same page.

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Chapter 1: Introduction

1.1 Background

I had been teaching senior high school mathematics for about ten years when I first started to formulate this project. I was troubled by the large number of students who, when confronting a problem, would freeze up or immediately ask me (the teacher) or a peer "How do you do this?" This commonly asked question suggested to me possible student attitudes toward mathematics and their own abilities, as well as the past experiences that caused them. Asking for "the method" to solve a problem implies that mathematics is only an already-established set body of knowledge and procedures. While the majority of content of what we teach has been firmly established, it would seem that revealing – rather than discovering – of concepts is the method of choice for many students. It would also seem that students' own methods for solving problems are undervalued by the students themselves and perhaps teachers as well. This expectation for some outside agent to provide answers or algorithms suggests that students have little confidence in their own abilities to explore concepts on their own for the purpose of coming to a better understanding.

That students have such attitudes at the senior most levels of our public schooling system shows that many have gotten by without having to become good problem solvers. Perhaps the conclusion from students' experiences with mathematics is that they *can* wait for the teacher or a peer to tell them "how to do it" and that memorization of algorithms and facts is enough to demonstrate the level of understanding that is required by current assessment and evaluation practices. My observation is that many students are unwilling or poorly equipped to solve mathematical problems. What tools will such students have when confronting difficult and novel problems beyond the walls of our public education system?

At the opposite end of the spectrum are students who are highly adept at problem solving. While such students often have stronger mathematical abilities and understandings, they often also possess a more willing attitude to engage with problems. By the senior high level, skills and attitudes may have developed to such a point that some students appear to be "natural" problem solvers. Perhaps it is from observing the behaviours of such students that a pedagogical approach might be developed to strengthen the skills of the developing problem solver.

In one Pre-Calculus 40S class that I taught, I noticed that the majority of students were frequently reluctant to offer any suggestions on how to solve particular problems. Desperate for participation, I asked for students to offer even the most outlandish of suggestions so that we could start by exploring why the suggestions were ridiculous. Even then, few volunteered suggestions. It seemed to me that there was some sort of posing paralysis which prevented the students from offering – and perhaps even *making* – conjectures. One-on-one work with students would lead sometimes to only slightly better results, and in the large group students simply remained silent. I believed that this was likely due to several contributing factors:

1. Individual students are influenced by the large group. A need for acceptance within the larger group often influences the individual's behaviour in the larger group. Not answering a question or conjecturing could be for fear of being wrong (or looking stupid) or a fear of being right (and looking like a know-it-all). In this sense, the influence of the group is intimidating and negative.

2. Previous mathematics classes reinforced particular student behaviours. In this case, students in past classes were able to rely on being told (by the teacher or a peer) how to

solve particular problem-types, and conjecturing at any level was something that they were able to avoid. Perhaps in some classes teachers had not even provided students with the opportunities to seek their own methods of solution to problems. When the students were in small groups or together as a whole class, the importance of justifying and reasoning was not emphasized enough, and the development of the ability to communicate mathematical ideas was hindered. Thus, moving toward conceptual understanding through the making of conjectures or explaining reasoning was foreign to them, further reinforcing the group dynamics mentioned previously.

3. The group dynamics of the class. The class structure was very individualized in the sense that, given a choice, students generally worked on problems alone, sometimes looking to a partner. Partners were used to "show them how it's done" or to confirm methods they had already attempted. When individual students or loose partnerships could take a problem no further, they simply stopped working on the problem.

The students in a parallel running Pre-Calculus 40S Advanced class generally did not have such inhibitions. Students routinely offered conjectures and seemed to be at ease with suggesting ideas and conjectures that could be shown as wrong on closer inspection by the class. When asked to explain their thinking, students showed less reluctance to do so and were better able to vocalize their thoughts, even in front of the large group. When solving novel problems the Advanced students frequently moved into small groups – without encouragement – and worked together to solve the problems. In both the Advanced and Regular classes I had at various points encouraged students to form small groups to work together to solve particular problems, however, the exact structure of these groups or specific problem-solving behaviours or strategies they might use individually or as a group were not pre-planned or explicitly stated by me.

In the Advanced class, members of the small groups could routinely be heard freely offering ideas for the group to explore. Whether ideas were accepted or rejected, it seemed that the students viewed such acts as a natural part of the problem solving process, or at least a regular part of class routine. Further, it appeared that the understanding of concepts by the individual members of the group was valued by the other group members. This seems to contribute to what I have observed and mentioned previously:

1. Individual students are influenced by the large group. In this case, a culture had been created over several years (the Advanced students were in grade 12 and many would have shared several classes since grade 7) in which students voicing ideas, and making and suggesting conjectures had become commonplace. Knowing that offering ideas was accepted (i.e. potentially wrong ideas would not be ridiculed nor would conjectures that proved correct be met with similar scorn) resulted in a larger number of students offering conjectures.

2. Previous classes had reinforced particular student behaviours. For the Advanced class, exploration of concepts through the making conjectures and explaining their reasoning had become commonplace over time, and was now firmly established practice. The success of small groups and working together to come to understandings justified forming such groups.

3. The group dynamics of the class. While individuals were concerned with their own understanding, there seemed to be an increased concern for the understanding of others, and to use the understanding of others to build on personal understandings. Thus, problem solving or coming to a better understanding of some idea was a group process. While students as individuals would ultimately be responsible for their own understanding, there

was a relatively increased concern for or use of the understandings of the entire group. There was also an understanding that the conjectures made by individuals, even if eventually proven false, would improve the understanding for the other group members, thus benefiting the individual, the group, and ultimately the class as a whole – or perhaps the whole class, the groups, and ultimately the individual. Since many of these students would have shared classes for up to six years, this rationale for the types of interactions observed seems reasonable.

It seems that classes are self-reinforcing environments: some that lead to increased success and others that do not. The relationships between classroom culture, problem solving and conceptual understanding was becoming increasingly apparent to me, and the theoretical understanding I had gained through my master's program was starting to clearly mesh with my observations of my own classes.

1.2 What This Study Is About

Analysis of different classes I had had over the years had led me to believe that there were certainly different possible classroom cultures that could emerge. Within any given classroom culture, certain behaviours and types of interactions develop, however intentionally. It seemed to me that I could do a better job of intentionally developing the culture in my classroom. In particular, I sought to create an environment that was more conducive to developing problem skills in my students. Since I perceived a strong link between problem solving skills and attitude, it was also important for me to investigate how students could develop not only problem solving skills, but also positive attitudes toward problem solving and their abilities in this area. I wanted to intentionally go about helping my students to not only become better problem solvers, but to enjoy the process of solving problems. A rough sketch of how this might come about had been forming in my mind, and I sought to learn how my ideas fit in relation to existing research.

I found research describing problem solving as a process, as a mindset, as an educational outcome and as a method of learning. General problem solving strategies and their use in the classroom were explored, as well as the use of groups and other teaching practices, and how classroom norms are established. Much of the research I found confirmed my belief in the importance of developing problem solving skills; existing research also confirmed many of my views about the classroom as a social context. And while I did learn much from existing problem solving research, I found little research attempted to link problem solving and pedagogy. Where attempts were made, researchers often lamented the magnitude of the task and the hurdles to be overcome. While there appeared much research on the separate topics of the individual problem solver, pedagogy, classroom culture, and group work, there was little that attempted to link all of these areas to present a practical theory on the teaching of problem solving. That said, there exists considerable research on the roles of problem solving in education, mindset, and strategies, as well as classroom concerns faced by both students and teachers. While research into these areas often avoided overlap, or at least failed to offer implication in other areas, I was provided with many insights as to how I might bring about my goal of developing a problem solving culture. This research helped to provide me a framework for this project: *to improve problem solving attitudes and abilities through the use of problem solving groups.*

Chapter 2: Theoretical Framework

2.1 The Role of Problem Solving in Mathematics Education

As a teacher of mathematics and having taught mathematics for over a decade, one would probably assume that I had considerable training in the teaching and learning of problem solving, and that problem solving was a firmly established and a central aspect of mathematical teaching and learning in schools. Instead, the vast majority of both my training and past mandated curricula have focused on the teaching of specific mathematical content, with surprisingly little connection to problem solving either as a means or a goal in and of itself. Problem solving does appear in at the front of most mathematics curriculum documents as a "mathematical process" that should be emphasized throughout a course. For example, the following appears in the Pre-Calculus 30S Introduction (Manitoba Education and Training, 1999):

> Problem solving is to be the focus of mathematics at all levels of a student's mathematics education. The development of each student's ability to solve problems is essential. Prior to the Senior Years, it may be useful to distinguish between conceptual, procedural, and problem-solving goals for students. Once a student is in Senior Years, these distinctions begin to blur as a natural consequence of mathematical maturity. Mathematical problem solving serves to answer questions from daily life, the physical and social sciences, business, engineering, and mathematics itself.
>
> Within Senior 3 Pre-Calculus Mathematics, problems and applications should be used to introduce new ideas, to develop an

> understanding of concepts and procedures, to apply skills and processes previously learned, and to strengthen the connections among topics within and outside of mathematics. (p. 9)

This is followed by two example questions which supposedly represent problem solving. How a question is an example of problem solving, and indeed what problem solving actually entails is not elaborated on at any point in the remainder of the document. Throughout the rest of this curriculum document “Problem Solving” is a frequently used label for example questions, but how or why such questions would help develop problem solving is never explained. Subsequent discussion will document that while there seems to be some consensus as the importance of problem solving, there is little consensus as to what problem solving is or how it can be improved.

Kilpatrick (1969) observed that despite apparent agreement amongst educators that problem solving was highly valued, problem solving itself was not being investigated systematically by mathematics educators. Referencing Kilpatrick, Lester (1994) found that some progress had been made in “(a) determinants of problem difficulty, (b) distinctions between good and poor problem solvers, (c) attention to problem-solving instruction, and (d) the study of metacognition in problem solving” (p. 663). While Lester saw serious investigation on the horizon, 25 years later it would appear things had not improved. Lester, commenting on dismal performances by students at various grade levels, referred to student performance in problem solving as “desperate”:

> We may have learned quite a lot over the past 25 years or so about how students learn to solve problems and how problem solving can be taught, but we have not learned enough. And yet there are signs that problem solving has begun to receive

less attention from researchers. (p. 660)

The National Council of Teachers of Mathematics' (NCTM) influence on curricula throughout the past few decades has grown. The NCTM has had "special significance for recent problem solving curriculum development and research in North America" through their publication of *An Agenda for Action* (1980) and *Curriculum and Evaluation Standards for School Mathematics* (1989) (Lester, 1994, p. 660). While this influence led to a "decade of problem solving", Lester noted that while the NTCM called for problem solving to be the focus of school mathematics, it failed to provide suggestions as to how this was to happen. "One gets the impression from reading the *Standards* that by 1989 the mathematics education community had amassed a sizable body of knowledge about the learning and teaching of problem solving, although I maintain that this is not the case" (p. 262).

Lester (1994) went on to suggest that while many mathematics educators agree that problem solving should be a focus in school mathematics, "it has been the most written about, but possibly the least understood, topic in the mathematics curriculum in the United States" (p. 661). He wrote further:

> It is probably also safe to say that most mathematics educators agree that the development of students' problem-solving abilities is a primary objective of instruction. It is equally as evident that these same educators would admit that it is quite another matter to decide how this goal is to be reached (i.e., where to begin, what problems and problem-solving experiences to use, when to give problem solving particular attention, etc.). Thus, although acceptance of the notion that problem solving should play a prominent role in the curriculum has been widespread, there has been anything but widespread acceptance of how to make it

> an integral part of the curriculum. To date, no mathematics program has been developed that adequately addresses the issue of making problem solving the central focus of the curriculum. Instead of being given coherent programs with clear direction, teachers have had to be satisfied with a well-intentioned mélange of story problems, lists of strategies to be taught, and suggestions for classroom activities. Although we have made considerable progress during the past 25 years, there are many issues and questions dealing with learning, instruction, and assessment that we have only begun to address in our research. (p. 262)

Even in the NCTM publication *Teaching Mathematics through Problem Solving: Grades 6-12* (NCTM, 2003) where suggestions are explored on how problem solving might move to a central role in the mathematics classroom, authors admit that research into such practices is lacking (Stein & Boaler, 2003). Since 1994, there have been attempts by researchers to add to understandings of problem solving and possible pedagogical implications. A diverse collection of investigations – the diversity Lester (1994) thought might be contributing to a lack of investigation – has been and is being studied. Topics include implementation of famous strategies, implementation of personal strategies, problem solving as metaphor, student perspectives, teacher perspectives, types of problems, and characteristics of problem solving and problem solvers. In order to investigate such areas, a clear understanding of what constitutes problem solving is necessary, and there are many different views on this topic. The view one has of problem solving will obviously have significant influence on the strategies one might implement to develop or improve particular aspects of problem solving abilities in students. Further, if such strategies are to be implemented within the classroom context, classroom culture comes into play. In turn, the complex nature of the classroom, consisting of many different human beings interacting, is affecting possible strategies and perhaps refining the very definitions of problem solving. And, I believe, it is perhaps within this complicated environment that problem solving is best learned.

The importance of teaching problem solving, despite lacking a precise definition, stressed by the NCTM does not appear to be in dispute:

> The teaching of problem solving involves teaching or facilitating the learning of this process. This process, however, is being viewed not merely as 'the methods, procedures, strategies, and heuristics that students use in solving problems' (Branca, 1980: p. 4), but also a way of thinking in solving non-algorithmic problems. This way of thinking involves, 'the coordination of knowledge, previous experience, intuition, attitude, beliefs, and various abilities' (Charles, Lester and O'Daffer, 1987: p. 7). Thus, as Lester (1985) noted: '*The primary purpose of mathematical problem solving instruction is not to equip students with a collection of skills and processes, but rather to enable them to think for themselves. The value of skills and process instruction should be judged by the extent to which the skills and processes actually enhance flexible, independent thinking (p. 66).* (Chapman, 1997, p. 202)

The views expressed above are in line with my own views on problem solving and the teaching of mathematics. I view the development of problem solving and student independence primary goals of mathematics educators. While the NCTM and many curriculum documents state the importance of problem solving, I believe that descriptions of what problem solving is and how it can be developed can be made more concrete and more explicitly stated to those who wish to help their students improve in this area. By this, I do not mean that teachers need more example problems. I do believe that teachers could use more guidance in the processes of problem solving and the classroom practices by which goals in these areas might be addressed.

2.2 Role of Mindset & Emotion in Problem Solving

To the teacher-researcher, problem solving is the *seeking* of an understanding of something which is not clearly understood. It is the ownership of a problem that allows the problem solver to seek clarification, face difficulties and move toward an understanding. It is a risk-taking enterprise that requires that a person has the willingness to make mistakes and start over. In a mathematical context, a better understanding is brought about through the making and testing of conjectures until some satisfactory level of understanding has been reached. These processes, however, are riddled with complex issues. Before getting to making and testing conjectures, how does one initially engage with a problem? Are there ways to solve particular problems? Are there ways to solve problems in general? If so, are there different phases or stages to solving problems? Can one learn to make conjectures? How? What blocks might prevent someone from engaging a problem, from making conjectures, from testing conjectures? How does one deal with being stuck? How does one know if one has solved a problem? How does one encourage ownership of a problem?

Anyone who has ever attempted to solve a difficult problem, or witnessed others attempt to solve what is to them a difficult problem, is well aware that problem solving can be greatly helped or hindered by the mindset and emotions that accompany different phases of solving (or not solving) a problem. The word *accompany* is used to imply that while the emotions and possible phases of problem solving do go hand in hand, whether one causes the other might be unique to a particular person at some moment during the solving of a particular problem. It seems likely that a person's success in moving toward a solution will have a positive effect on the mindset or emotional state of the problem solver. However, it is equally if not more likely that it is the mindset or emotional state that allows someone to (or prevents someone from) engaging a problem effectively in the first place.

State of mind is not just a by-product or response to various phases of problem solving; rather, problem solving can be viewed as a state of mind. It is state of mind that allows you to enter, re-enter, get stuck, seek ways out of being stuck, offer and test conjectures, and expand or simplify the scope of a problem. Ironically, for some people not seeing "the solution" or path to a solution leads to frustration, and it is then this frustration that prevents the very processes that would lead to a solution. Those who are able to move beyond this frustration (or to use it positively as motivation for seeking solutions) are able to enter a very valuable phase of problem solving: conjecturing. The ability to throw out an idea, however wild, is a major signpost on the road to solving problems. It indicates a willingness to engage a problem and a willingness to make mistakes and move on.

Some students often assume that if their idea does not directly lead to a solution, then what they are doing is *wrong* – as opposed to a necessary step along the way to successfully solving the problem. This would seem consistent with the phenomenon of students failing to successfully engage with a problem. Initial steps to explore the problem are looked on as a waste of time, as being wrong, and the desire to not waste time or be wrong or have to face the frustration of being stuck may leave some students finished before they have even started.

In a classroom setting, the issues that all individual problem solvers must face are even more complex. Students are engaged in these problem solving processes in a social environment. Their successes and failures are public and in front of a most influential group: peers. There can be a strong emotional component tied to being part of a peer group, and this could have a strong impact on both students' desire to participate and how students participate. It seems reasonable to assume that successful problem solving strategies and problem solving pedagogy must take into account the mindset of students, and not only look at how specific skills can be developed but how student confidence and mindset relate to such developments.

2.3 Role of Strategies in Problem Solving

In order to develop problem solving strategies, one needs to have a firm grasp of what problem solving is. Many researchers when presenting their findings often start by focusing on their definition of problem-solving. Perhaps this is the case because there is by no means concordance among researchers on a particular definition.

> It is not yet the case that there is one, widely accepted theoretical model for the array of phenomena that come under the general term *problem solving.* Nevertheless, progress has been made to the point that we can say much is known and supported extensively by empirical research. (Goldin, 1992, p. 276)

According to Chapman, problem solving has "been interpreted as a goal, process and basic skill (Branca, 1980: p. 3); a method of inquiry (Charles et al., 1987); mathematical thinking (Baroody, 1993; Mason, Stacey & Burton, 1985) and a teaching approach (Baroody, 1993). The National Council of Teachers of Mathematics (NCTM) *Standards* (1989) use it to describe mathematics, i.e., Mathematics as problem solving" (Chapman, 1997, p. 220). Goldin (1992) also offers many suggested interpretations of the term "problem solving":

> It is generally understood that mathematical problem solving is not one thing. It involves a highly complex aggregate of internal psychological processes, which occur to varying degrees and in various combinations. Depending on one's theoretical orientation, these may include (but are not limited to) the establishment of goals and sub-goals; verbal and syntactic processing; visualization; spatial representation; kinaesthetic encoding; the use of a variety of complex heuristics; storage and retrieval in short- and long-term memory; algorithmic processing, algorithmic learning, and debugging; the use of mathematical notations; conceptual understanding (itself extremely complex); the detection of structures and structural similarities; change of representation; the experience of impasse; the "aha" experience; a variety of affective/emotional responses; metacognitive process, such as self-monitoring; belief systems about mathematics that come into play; the internal construction of meaning over short and long periods of time; and assimilation, accommodation, and equilibration as new constructions occur. Some of these processes are quite directly dependent on domain-specific, mathematical knowledge; others are less so. (p. 277)

The above quotation clearly indicates the complexity of the issue. The manner by which any of the constituent components of problem solving might be developed in the classroom – either in the sense of a teacher directly helping a student develop problem solving abilities or in the sense of a community of learners interacting and learning together – must be areas of some contention as well. Much research on problem solving and the processes it may involve explores the thought processes of a hypothetical individual problem solver: *What processes does a good problem solver use? What is she thinking? How is she relating this problem to past ones?* While the answers to these questions may be complex, the focus does simplify aspects of problem solving down to the internal processes of an individual solver engaging with a problem; the classroom as an environment in which these internal processes might be developed is ignored altogether. Thus descriptions of the nature of interactions between teacher and student, as well as between students and other students is ignored, as are descriptions of tasks in which students should engage and how students might be assessed or evaluated. Instead, exploration starts with the removal of all external contexts and assumes that problem solvers are already engaged in the problem (poorly to satisfactorily). Perhaps this approach is so common because so much research on problem solving has been greatly inspired by the works of George Polya. Problem solving research often refers to Polya's *problem solving heuristics* as well as his general problem solving strategy.

George Polya (December 13, 1887 – September 7, 1985) was a Hungarian mathematician and his landmark contribution to problem solving scholarship *How to Solve It* (1945) remains to this day a prominent focal point in any exploration of problem solving. This is in part because the book offers two different types of approaches to problem solving. The first and overriding approach is a general strategy that the expert problem solver should (does?) follow when attempting to solve any problem. The second type of approach is described by a set of strategies/heuristics one might attempt when encountering specific types of problems. While many authors suggest both specific and overall strategies for solving problems, Polya's glossary of problem solving heuristics that he (as a mathematician) developed through years of teaching and solving complex problems remains the one most referred to. Throughout the book, Polya appears to be addressing the would-be problem solver (or, perhaps he is writing to himself) with his own personal insights into what makes Polya such a successful problem solver. The book, and thus an abundant amount of later research on problem solving, focuses on internal processes of the engaged problem solver. Polya's general problem solving approach involves four key stages:

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look Back. (Review/Extend) (pp. xvi-xvii)

The logical nature – that is to say, seemingly straight-forward steps – of his approach have been adopted or adapted by many teachers, researchers and students. In particular, Polya's (1945) first stage on what it means to *understand a problem* has been stressed by many as the key step to solving a problem. Polya emphasizes the belief that knowing the answer or the steps to a solution cannot possibly occur until the solver has a firm understanding of what the problem is about. The solver is to ponder some basic questions:

- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

After developing an understanding of the problem, the solver is able to start devising a plan on how to solve the problem. Once again, inability to jump to a solution is not to be viewed as an insurmountable concern or even undesired position, but rather a common and often necessary spot on the road to resolution. In particular, Polya is famous for saying "*If you can't* *solve a problem, then there is an easier problem you can solve: find it*." It is with the very basic idea that one should start with taking the time to understand a problem – and the acceptance that answering an easier problem may be part of the solution process to a more difficult problem – that Polya tries to bring the solution to a difficult problem within reach. That Polya, an acclaimed solver of difficult problems, spends considerable time attempting to understand a problem before attempting to solve it, and then perhaps solves other easier problems before moving to the more difficult one, gives hope to the developing problem solver when facing a problem that is difficult to understand, and seemingly impossible to solve. This has certainly been an interpretation of Polya's approach by many problem solving teachers and researchers.

That said, Polya (1945) seems to address a mathematically adept student or teacher. Further, while he attempts to describe the very processes that he himself uses, he is clearly addressing *a well-motivated individual* about what thought processes that *individual* should engage in. While the book does offer analysis of common themes found in problems and the solving of problems, the organization of the book (it's presented alphabetically!) leaves one uncertain as to what Polya *really* does when he engages a problem. Perhaps his approach is intuitive or sporadic (while successful) enough to justify an alphabetical list. However considering that the work is by a professor of mathematics, the book does not suggest clear strategies by which a teacher might wish to improve problem solving abilities of an entire class of interacting students. In particular, Polya's idea that problem solvers critically review their potential solutions ("Be your own toughest critic!") seems logical, but seems more a description of what good problem solvers already can do than a legitimate offering of advice on how to get there. While the book might be better titled *How I Solve it*, Polya's offering remains an important work when investigating what it means to teach and learn problem solving.

In *Thinking Mathematically*, John Mason, Leone Burton and Kaye Stacey (1985) offer their take on a general problem solving strategy. Mason et al. state they aim to "show how to make a start on *any* question, how to attack it effectively and how to learn from the experience" (p. ix). Mason et al. describe problem solving as *thinking mathematically* and they suggest that mathematical thinking can be improved by:

- Tackling questions conscientiously;
- reflecting on this experience;
- linking feelings with action;
- studying the process of resolving problems; and
- noticing how what you learn fits in with your own experience (p. ix).

The book goes on to describe a generic approach to solving problems, describing various *phases* one may move through while solving problems: *Entry Phase* (Polya's (1945) Understanding the Problem), *Attack Phase* (Polya's Devise a Plan, and Carry it out), and *Review Phase* (Polya's Look Back/Review). While Polya seems more focused on explaining all the strategies he sometimes employs when solving problems – and the reader, it is assumed, should do likewise – Mason et al.'s (1985) work seems more concerned with the reader developing his own problem solving capabilities in a more general sense. As such, the problems are not domain-specific, i.e. they have a less algebraic bent and examples used are accessible by a novice. Each chapter is presented to allow the reader to focus on a specific phase of her problem solving model. The work is thus overall more approachable and, perhaps, the concepts presented are more transferable to the classroom.

Mason et al.'s (1985) work also explains the strategies of specializing and generalizing in problem solving. (Mason's *Learning and Doing Mathematics* (1998) examines the interplay between specializing and generalizing in greater detail and in relation to problems of a more algebraic nature.) In some ways this parallels Polya's (1945) "Find the easier problem" idea. However it allows the solver to have a better understanding of what it is that she has solved and a non-threatening vocabulary to describe the process. It also means that when the solver gets to the Reflection phase, she will have a better understanding of what it means to *extend the problem* (according to Mason and Polya) in both a specific and general sense.

An important aspect of Mason's work is his understanding of the significance of being "stuck" when solving a problem.

> Everyone gets stuck. It cannot be avoided, and it should not be hidden. It is an honourable and positive state, from which much can be learned. The best preparation for being stuck in the future is to recognize and accept being stuck now, and to reflect on the key ideas and key moments which begin a new useful activity. (Mason et al., 1985, p. 49)

As a teacher of mathematics and as a solver who has been stuck himself, Mason is well aware of the debilitating feelings of frustration that accompany being of stuck. Mason et al. (1985) suggests a full analysis of the being stuck state, starting with writing "*STUCK!*" "The act of expressing my feelings helps to distance me from my state of being stuck. It frees me from incapacitating emotions and reminds me of actions that I can take" (p. 49). Mason's Attack Phase is a cyclic phase moving from Polya's (1945) *Develop a plan/Carry out the plan* and Mason's honourable state of *being stuck*. This helps drive home the idea that there is no simple solution to a problem, and that being stuck is a natural part of the process. More importantly, Mason et al.

offer suggestions about what to do in such states. While Polya obviously knows how to jump around and try different heuristics, the boldly stated *Develop a plan / Carry out the plan* seems overly decisive; such steps suggest a domain-specific expertise that might seem far removed from the developing problem solver. The overall result is that Mason seems genuinely concerned with improving the problem solving abilities of the reader.

The work of Mason et al. (1985) seems a deliberate attempt to bring problem solving strategies to the masses. Contrasting this with Polya (1945), one is left feeling that Polya views his explanation sufficient for anyone who is trying (and too bad if it is not!), while Mason et al. comes across as being on your side cheering you on through the times when you are stuck. It seems that for Polya, there is an already-established culture of problem solvers, and that non-members had better learn the recipes. In contrast, it seems that Mason et al. view the *becoming* a problem solver as joining the culture. By *wanting* and trying to become a better problem solver, you are joining countless others who have embarked and are embarking on a similar journey. Again, such a culture has interesting possibilities when viewed as a culture to be developed in the classroom.

Neither Polya nor Mason specifically tackle the issue of pedagogy, i.e. how to teach problem solving, as it is not the purpose of their respective books. As mentioned, works such as *Thinking Mathematically* (Mason et al., 1985) and *How to Solve It* (Polya, 1945) address an individual, *mature and motivated* would-be problem solver and this is their greatest hindrance in bringing their ideas into classrooms at large. Both authors focus on the internal state of the individual, and while Mason et al. do move to offer some acknowledgment to the emotional states that may accompany various phases of problem solving, how teachers might take advantage of Mason et al.'s insights is, at least in this work, left to the creativity of the teacher. In

Mason's *Learning and Doing Mathematics* (1998), in which he addresses some hypothetical developing (school-aged) students, the difficulty of the content would suggest that the intended audience does already have significant domain-specific skills, or that the teacher is to be able to pick up the book and fill in the gaps for the classroom setting.

General problem solving strategies, though abstractions, can seem intuitively correct. That there are things people can do when solving problems, and steps that people can take to become better problem solvers offers hope to both the problem solvers and those who would teach them. The commonalities found in the two mentioned strategies suggest that there are phases to problem solving, and that these phases can be conscientiously developed by the individual. That the individual can improve by focusing on particular phases suggests that a teacher could develop tasks that would allow students to achieve similar improvements in the classroom setting.

2.4 Students as Problem Solvers

If there are in fact particular traits or behaviours of good problem solvers, as Polya and Mason would have us believe, then we would be on firmer ground when attempting to develop a pedagogy of problem solving. Both authors' works attempt to describe behaviours of a successful problem solver, and are no doubt informed by years of teaching and learning. Other researchers have pondered the differences in approach between successful problem solvers and those who are less successful. Lester (1994) comments:

> Today there is general agreement that problem difficulty is not so much a function of various task variables as it is of characteristics of the problem solver, such as traits (e.g., spatial visualization ability, ability to attend to structural features of

> problems), dispositions (i.e., beliefs and attitudes), ...and experiential background (e.g., instructional history, familiarity with types of problems). Perhaps what is most notable about the early attempts to identify what makes problems difficult for students was that they marked the beginning of efforts to make mathematical problem-solving research more systematic and analytic. (p. 664)

Lester (1994) summarizes findings of research up to 1994 that distinguished “good” problems solvers from “poor” ones:

> Good problem solvers know more than poor problem solvers and what they know, they know differently – their knowledge is well connected and composed of rich schemata.
>
> Good problem solvers tend to focus their attention on structural features of problems, poor problem solvers on surface features.
>
> Good problem solvers are more aware than poor problem solvers of their strengths and weaknesses as problem solvers.
>
> Good problem solvers are better than poor problem solvers at monitoring and regulating their problem-solving efforts.
>
> Good problem solvers tend to be more concerned than poor problem solvers about obtaining "elegant" solutions to problems. (p. 664)

Lester (1994) goes on to remind us that Lesh (1985) cautioned that “pinpointing the ways that experts solve problems and then trying to teach these ways to novices in a short amount of time may not result in desirable outcomes” (p. 665). Lesh suggested that the problem solving processes and heuristics “develop slowly over time in much the same way other mathematical ideas are known to develop", and any pedagogy of problem solving must take this into account. Nonetheless, the above suggests certain general differences between "good" and "poor" problem solvers, but the latter comments also suggest potential pitfalls when attempting to generalize research findings. Some studies involved students with particular abilities, and their findings were generalized if not by the researchers themselves then by creators or implementers of curriculum; it is these types of generalizations that compelled Sweller (1990) to caution (perhaps excessively) against the promotion of general strategies. Hart (1993) also warns against such generalizations, noting that considerable emphasis had been placed on teaching the methods of above-average or expert problem solvers to average or below average students. Hart cautions:

> As Lesh (1981) points out, 'the qualitatively different systems of thought used by gifted problem solvers may be ...inaccessible to ...average-ability children (p. 239). Lesh's comments indicate that it may be important to identify factors that enhance or impede the problem-solving progress of average-ability students, rather than focusing attention on experts. (p. 167)

Lawson (1990) notes that many studies regarding problem solving focus on "disabled or poorly performing students, often in non-mathematical contexts; there is need for more research" (p. 405). While research does seem to agree that there *are* characteristics of good problem solvers, that lack of direction from the NCTM to that time – aside from noting that problem solving should be the focus of school mathematics – suggested a need for further research.

With a better understanding of the characteristics of good problem solvers and the processes in which they engage, teachers will be more apt to develop tasks that will allow students to improve. Lester's (1994) description of good problem solvers offers to teachers a list of characteristics to look for, but more importantly, to develop. If good problem solvers are better at monitoring and regulating their efforts, teachers need to create tasks that will develop these skills. Teachers can also select problems and design tasks for students to complete that help connect knowledge to existing schemata while also encouraging students to understand the structural features of problems. Through the designing, implementation and analysis of tasks with specific problem solving goals, a teacher could gain a better understanding of how problem solving skills can be developed in the classroom.

2.5 The Teacher & Problem Solving

Many teachers and researchers have operated under the premise that there must be certain behaviours or internal processes shared by good problem solvers. However, lacking a firm theoretical model about how to go about developing such behaviours, teachers and researchers have been left to their own best guesses. Lester (1994) suggests that “we clearly have a long way to go before we will know all we need to know about helping students become successful problem solvers” (pp. 265-266). Lester notes that many mathematics or problem solving courses have been based “largely on the folklore of mathematics teaching, particularly the sage advice of master teacher and problem solver George Polya” (p. 262).

Polya’s (1945) general strategy can be broken down into stages, and the processes of each stage can be considered as skills to be developed by a teacher. Polya describes a basic process for approaching a problem, starting with *understanding the problem*. Many students who have difficulty solving problems often fail to adequately understand a problem before seeking solutions. This can lead to unintentionally finding solutions to the wrong (that is to say, a *different*) problem, or to students becoming frustrated with a problem and giving up before they have even really started. If teachers were to require students to state knowns and unknowns, draw diagrams, and even restate the problem in their own words, this would encourage students to spend time on understanding a problem before attempting to solve it, or, more importantly, see that doing such steps is in fact part of attempting to solve a problem.

Polya’s (1945) idea that there is an easier problem to solve (“Find it!”) seems like a difficult idea to bring across. It seems that both students and teachers would be somewhat opposed to the idea of students seeking partial solutions or modifying questions in order to produce a solvable problem. Understandably, students might feel uncomfortable presenting to their teacher a solution to a different problem than the one assigned. This issue could be compounded when students, after having solved a simpler problem, stop the process. Perhaps requiring some explanation as to what students have solved and what they require for the next part of a solution could help students see the value of the process, and to become more proficient at it. If there are certain phases to solving problems, perhaps students need to become aware of the value of reflection on the various phases of problem solving, and that such reflection is valued. Clearly, the types of problems typically assigned must be analyzed in terms of opportunity to develop and reflect on such concepts.

The “Devise a Plan” step, while clearly stated, certainly seems to imply that there is something that students can actually plan and then carry out. For students who explore problem solving in earnest for the first time, it seems likely that this will seem a most magical or mysterious “step” or a mere label for the very thing they are wishing to improve on. Teachers wishing to help students improve in this area will need carefully considered pedagogy; it will not just happen by telling students to make a plan. Also, the heuristics offered by Polya (1945) are numerous and not necessarily organized in a way which clearly identifies how a teacher might wish to realistically help a class of students engage in problem solving.

In contrast to Polya's (1945) work, the readability of the book by Mason et al. (1985) is such that the book or sections of it could be used by students and teachers in a secondary school setting. In the very least, teachers might want to explore Mason's ideas and terminology in the classroom. The generic nature of his approach increases the likelihood that teachers will be able apply at least some aspects to the classroom. Explanations by Mason et al. on specialization and generalizing more clearly emphasize the purpose and mathematical nature of Polya's "*Find the easier problem*" while also offering a more concrete approach to extension problems during the *Review Phase*. The concepts of specializing and generalizing emphasize two major aspects of mathematical thinking, suggesting people who attempt to work through such processes are not just learning *about* mathematics: they are *doing* mathematics. Further, in terms of pedagogy, there is nothing which really suggests that a particular problem needs to be solved. If the purpose of a class is to further develop aspects of problem solving, it could very well be that certain phases are explored with respect to a particular problem without necessarily following the problem to completion. This might allow for increased practice on the various phases without demanding the time required to completely solve problems. This would be a novel approach when regarding problem solving, and might also serve to help distance one's emotions from the actions that can be taken at various points of the problem solving process.

Like Polya's (1945) view to "*Be your own toughest critic!*", Mason et al. (1985) suggests that when you have a potential solution that you should "*convince yourself, convince a friend, convince an enemy*" (p. 97). While Mason et al. offers strategies for individuals, this process hints at possible classroom practice. Students that practice testing the conjectures of others seem likely to get better at testing their own conjectures, thereby developing the "internal enemy." Many students seem prepared to analyze the works of others more easily than turning a critical eye toward their own work. Teachers attempting to integrate more cooperative approaches to problem solving might find success in this area.

While many agree that good problem solvers do use heuristics or even general problem solving strategies, successful implementation of such a model in the classroom seems limited. There have also been many critics to both general methods and the implementation of such methods as attempted by various teachers and researchers. Sweller (1990), for instance, has argued that there are "few differences in strategies between experts and novices in general strategies" and that expertise in problem solving "consisted of the accumulation of a large store of domain-specific knowledge and strategies (that is why it takes so long to become an expert)" (, p. 411). This would suggest that, contrary to the NCTM's push for problem solving, that domain-specific content should be stressed rather than general strategies. The time required to develop expertise in any setting might further impede the development of problem solvers.

Sweller (1990), hoping to halt the gaining of prominence of general techniques in mathematics curricula goes so far as to state that "[t]here is very little evidence of successfully teaching general problem-solving techniques in mathematics education" (p. 414). He writes further that "transfer to unrelated domains is an essential tool of any study intended to test the effectiveness of general problem-solving strategies" (p. 414) and that such research is lacking. On the ineffectiveness of general strategies, Sweller mentions that attempts to create computer simulations such as chess programs which focused on general strategies failed; it was this failure that led to expert systems containing massive amounts of information – eventually used to create chess programs capable of beating human champions. He notes that experts often "revert" to using the same strategies as novices when dealing with unfamiliar material. As a result,

> it became reasonable to suggest that domain-specific rather than general strategies differentiated experts from novices and that teaching general strategies would be futile. Obviously, the best counter to this hypothesis would be evidence of the effectiveness of teaching general problem-solving strategies. (p. 412)

While domain-specific knowledge is going to be an obvious advantage in problem solving, Sweller's (1990) view of general strategies does not seem consistent with classroom observations of other researchers. In cases where students do not start a problem for whatever reason, it seems that a developed strategy for helping students start a problem (Mason et al.'s (1985) *Entry Phase* or Polya's (1945) *Understand the Problem* would have profound effects on students becoming better problem solvers regardless of mastery of domain-specific content. Sweller does acknowledge one limited study by Schoenfeld (1979) where "mathematics students [were trained] in the use of several heuristics and found both enhanced use and enhanced performance on unrelated problems" (p. 413). Nonetheless, Sweller's view on the teaching of general strategies is cautionary at best:

> Swing, Stoiber, and Peterson (1988) provided the other mathematics education experimental work cited by Lawson to support the introduction of general problem-solving instruction. In a large, relatively long-term study, Swing et al. found no generalized effect due to teaching general problem-solving strategies. High ability classes given thinking-skills training did better than classes not given such training. In contrast, low-ability classes did worse after thinking-skills training. Within classes, high-ability students did worse after thinking-skills training, but low-ability students did better. It is not clear to what extent the tests involved transfer. These results, although important from a theoretical

> perspective, do not provide evidence for the introduction of general problem solving into mathematics classrooms. The authors make no claims for curriculum changes in their report. (p. 413)

Lawson (1990) responds to complaints against the teaching of general strategies by suggesting a wider view of transferability than suggested by Sweller (1990):

> It is argued here (a) that it is important to distinguish among three different types of general problem-solving strategies, (b) that transfer needs to be viewed as a complex chain of processing rather than being treated as an afterthought to learning, and (c) that general problem-solving strategies can have an impact on the presence and extent of transfer and thus have a claim for inclusion in the mathematics curriculum. (p. 404)

Lawson (1990) goes on to describe three types of general strategies: *task orientation strategies* that "influence the dispositional state of the student, the broad affective, attitudinal, and attributional expectations held by the student about the task" (p. 404); *executive strategies* that are concerned with planning and monitoring cognitive activity and that are "argued to have a regulatory function across all cognitive domains – in everyday tasks and social tasks as well as in classroom problem solving" (p. 404); and *domain-specific strategies* which "include heuristics, such as means-ends analysis (Anderson, 1985), and other procedures developed by the individual for organizing and transforming knowledge" (p. 405). Lawson also sites Charles and Lester's mathematical problem solving program in which students were instructed in the use of domain-specific strategies "such as trying simple cases, creating a table, drawing diagrams, looking for patterns, or developing general rules", which have "applicability across many tasks within (and beyond) the broad field of mathematical problem solving" (p. 404**).**

While some such as Sweller (1990) question the validity of general problem solving techniques, others have argued against the teaching of the techniques on grounds that doing so is impractical. Chapman (1997) cites a situation in which a teacher tried to implement Polya's approach with a grade 6 class:

> One day, she questioned a group of her grade 6 students on what they were doing as they worked on a problem. They replied, "Carrying out the plan". She asked, "What is the plan?" They replied, "We don't have one." Similar behaviour was reflected by other groups of her students. This experience was an invitation for Lillian to draw on her tacit knowledge (Polanyi, 1958) to create a process that worked for her and allowed her teaching to make sense to her in a personally meaningful way. (p. 222)

The above example is an indicator that even the most famous of problem solving strategies has difficulties with classroom implementation. On the other hand, another teacher in Chapman’s (1997) study found interesting parallels between Polya’s general strategies and what students were actually doing:

> [One teacher's] change came as she realized that the students preferred to use their own strategies as opposed to being forced to use those stipulated by the textbook or her. She also observed that *they would naturally go through the Polya stages of problem solving informally* [italics added], but would encounter difficulty and resist identifying these stages or applying them in a formal, structured way. (p. 222)

This would seem to agree with many others in that there are, in fact, stages or processes that good problem solvers go through when solving problems. If this is the case, then it is obviously the goal of a teacher of problem solving to seek ways to develop these processes. Recent NCTM research indicates that a problem solving focus in the mathematics classroom *can* be successful. The NCTM cites the findings of many researchers – including Boaler (1997), Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000), Thompson and Senk (2001), Stein and Lane (1996), and Silver and Kenny (1999) – and compares and contrasts more traditional mathematics classes with those where problem solving is the means by which mathematical knowledge and understanding is gained. Overall trends in the studies showed that students' ability to perform specific skills/algorithms does not decrease, that performances on state/national tests did not decrease, that differences in performance between gender, race, socioeconomic backgrounds decreased, and that problem solving abilities increased (NCTM, 2003). Over time, whether or not the focus was on a particular problem solving method, clearly students were able to develop or improve on certain problem solving behaviours while maintaining or improving algorithmic abilities. While different research projects appeared to provide "ambiguous messages", Lester (1994) lists five findings:

> Students must solve many problems in order to improve their problem solving ability.
>
> Problem-solving ability develops slowly over a prolonged period of time.
>
> In order for students to benefit from instruction, they must believe that their teacher thinks problem solving is important.
>
> Most students benefit greatly from systematically planned problem solving instruction.
>
> Teaching students about problem-solving strategies and heuristics and phases of

> problem solving (e.g., Polya's [1973/1945] four-phase problem solving model) does little to improve students' ability to solve mathematics problems in general. (pp. 665-666)

If general strategies are inadequate for the teaching of problem-solving, what then should teachers do? Lester (1994) suggests looking at metacognition as “the driving force” in problem solving:

> Research on the role of metacognition in mathematical activity, especially mathematical problem solving, has been concerned with two related components: (a) knowledge of one's own thought processes, and (b) regulation and monitoring (also referred to as "control") of one's activity during problem solving... By the end of the 1980s, metacognition not only was regarded as a force driving cognitive behaviors, but also was being linked to a wide range of noncognitive factors – in particular, beliefs and attitudes (Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1987b). The degree to which metacognition influences problem-solving activity has not been resolved – indeed, we are only just beginning to understand the relationships involved – but three results have come to be generally accepted:
>
> 1. Effective metacognitive activity during problem solving requires knowing not only what and when to monitor, but also how to monitor. Moreover, teaching students how to monitor their behavior is a difficult task.
>
> 2. Teaching students to be more aware of their cognitions and better monitors of their problem-solving actions should take place in the context of

learning specific mathematics concepts and techniques (general metacognition instruction is likely to be less effective).

3. The development of healthy metacognitive skills is difficult and often requires "unlearning" inappropriate metacognitive behaviors developed through previous experience (Schoenfeld, 1992).

(Lester, 1994, p. 667)

For someone who does not advocate general strategies, Lester's (1994) comments on metacognition seem to echo Polya and, in particular, Mason, although he uses the more contemporary term of metacognition to describe the processes. Polya's reflection/looking back stages and Mason's more descriptive requirements for improving mathematical thinking (Tackling questions conscientiously / reflecting on this experience / linking feelings with action / studying the process of resolving problems / noticing how what you learn fits in with your own experience) do not seem at odds with what Lester is suggesting here. It is significant that there has at least been a shift in focus as to what teachers can do to help bring about an improvement in students' problem solving abilities.

If teachers can help create cultures where these metacognitive processes are valued and developed, it would have a profound impact on students' attitudes toward their own problem solving abilities and towards the field of mathematics. If classroom norms can be established where it is routine for students to engage in problems and deal with the many different emotions that might accompany different phases of problem solving, students might learn to better handle frustration and become better problem solvers as a direct result. What actions teachers might take to allow such development and just what place such emotional awareness might have in the classroom are areas to be explored.

Lester’s (1994) views on metacognition seem to reflect teacher actions more recently suggested by the NCTM to improve problem solving abilities in students:

> Teacher actions included (a) scaffolding of students’ thinking (b) a sustained press for students’ explanations; (c) thoughtful probing of students’ strategies and solutions; (d) helping students accept responsibility for, and gain facility with, learning in a more open way; and (e) attending to issues of equity in the classroom. (Stein & Boaler, 2003, p. 253)

Chapman (1997) has also looked at the teacher’s role in understanding problem solving. She notes that:

> It seems reasonable to conclude that teaching knowledge, regarding problem solving, cannot be packaged meaningfully into a set of prescriptions for practice. Therefore, how the teacher handles the dilemma of teaching such a complex process makes problem solving a process in which teachers have something important to say. (p. 202)

In one research project, Chapman (1997) analyzed three teachers’ classrooms to better understand root metaphors on problem solving. “The goal was to understand or gain insights into the teachers' perspectives of how, in the context of the classroom, they interpreted, organized and conducted the teaching of problem solving” (p.203). Chapman felt that regardless of existing theories on problem solving or teacher practice, that it is the “individual teacher who makes inferences about when and how particular practices are appropriate for use” (p.203).

2.6 The Social Context in Problem Solving

Teaching problem solving must consider more than just the internal processes of the individual problem solver. To teach problem solving must also be a more thoughtful and complex enterprise than the mere distribution of "good problems." The environment in which problem solving is to be learned must also be considered. In a mathematics classroom, a culture can encourage members to exhibit the behaviours of good problem solvers, while other cultures may severely limit growth. Some researchers have looked at the classroom environment as a place where problem solving skills be developed. In the classroom, the interactions between students and teacher add a new and complex dynamic to what some have tried to regard as isolated internal processes.

> Perhaps even more powerful [than any internal processes], there is the influence on problem solving of the human social and culture contexts in which the problems are posed – the perceived expectations associated with schools, classrooms, small peer groups, individual interviews, paper-and-pencil tests, real-life situations out of school and so on. *The study of problem solving is thus, fundamentally, the study of interactions between internal cognitive processes of the solver and the environment in which the solver functions.* (Goldin, 1992, p. 277, emphasis added)

Clearly noting the fallacy of treating problem solving as only involving internal process, Manoucheri and St. John (2006) analyzes the types of student cultures that need to be fostered to give the most opportunities for student growth. By describing such communities, they are also able to describe the responsibilities of a teacher hoping to foster such classroom communities. In such communities, teachers can help create contexts in which students "develop the belief that they are as individuals responsible for understanding and sharing mathematics" (p. 550). Further,

> by insisting that peers work together to try to figure things out, teachers can influence students' perceptions about their role in the classroom and their expectations of peers…. Teachers also need to help students learn how to talk with one another about mathematics in ways that are coherent and respectful by pointing out features of classroom conversations that are representative of the type of discourse they desire and by modeling for students those social and mathematical behaviors, including the norms of polite interaction, that are crucial to productive functioning of a learning community. (p. 550)

After examining students' mathematical discourses, Manoucheri and St. John (2006) describe a learning community as "a classroom environment that embodies a culture of learning in which everyone is involved in a collective effort of understanding" (p. 555). This certainly moves well beyond an individual's private attempts to learn (i.e., improve metacognition or problem solving abilities) since the individual is now regarded as a contributing member of a community, a community where each member can contribute to the learning of the other community members.

Manoucheri and St. John (2006) describe three key aspects of a community that shares such a collective effort: participation, commitment, and reciprocity. *Participation* deals with opportunities in dialogue for the "individuals to become engaged, question others, try out new ideas, and hear diverse points of view" (p. 545); *commitment* is used to describe student's willingness to be open to positions of others; while *reciprocity* is a "willingness to engage in an equilateral exchange with others" (p. 545). Such a description clearly suggests the importance of creating a classroom culture where the participation of individual members benefits all members of the group. If a goal is to improve performance of individual members, how such a culture is developed and how individual members relate to the culture will be key concerns. Thus, internal processes involving an individual's relationship to a particular problem must be considered in light of an individual's relationship to other group members via mathematical discourse:

> In this mode, the *structure* of discourse is multi-directional and responsive. The *content* of the dialogues is dynamic, connected, and unscripted. The *purpose* of the dialogue is to participate and engage others in deep inquiry into the meaning of things. This posture leads to significant changes in how individuals view the topic under consideration and their relationships with it. (p. 545)

Hart (1993) has also examined the role of classroom culture in enhancing problem solving performance, particularly through the creation of problem solving groups from members of a particular class. He found three key factors which affected student performance: group collaboration, group monitoring and social norms in small group problem solving:

> ***Group collaboration.*** The collective experience of a group often supplied background information that individual students did not possess, and in doing so countered lack of an experiential framework. For example, when none of the students in a group understood the term baseboard it was impossible to proceed with the problem. However, when one group member had experience to share, the group was able to create a more useful representation.
>
> ***Group monitoring.*** A second benefit of group work was that the challenge and disbelief of peers acted as a form of external monitoring when self-monitoring was not apparent. Although students seldom questioned their own strategies, they occasionally did challenge each other. Such encounters seemed to force them to

examine their own knowledge, strategies, and beliefs more closely.

Social norms in small-group problem solving. Social norms of the groups were somewhat different from those of a traditional classroom (cf. Yackel, Cobb & Wood, 1989). In some cases these differences were helpful in the problem-solving situation. For example, the group setting seemed to encourage taking time for reflection, an important component of problem solving that is often neglected in more traditional classroom settings. As with Noddings' (l985) students, group members seldom agreed upon an answer immediately. When answers were proposed, students often took several minutes to discuss them.

(Hart, 1993, p. 170)

Hart’s (1993) three factors could be extremely significant as it ties a particular teaching practice (the use of small-groups) to develop problem solving, to particular processes of good problem solvers (monitoring of strategies). This parallels the work of Rasmussen, Yackel and King (2003) with the NCTM where they discuss classroom norms – “expectations and obligations regarding class participation” - and what they call sociomathematical norms – the “emerging beliefs and dispositions specifically related to mathematics” (NCTM, 2003, p. 150). The problem solving classroom is to endeavour to develop and maintain norms by which students are to:

- Develop a personally meaningful solution
- Explain and justify their thinking to their peers and to the teacher
- Listen to, and attempt to make sense of, the explanations and justifications of others; and

- Ask questions and raise challenges if they did not understand or disagreed.

(NCTM, 2003, p. 148)

The development of sociomathematical norms seems to support aspects of some general problem solving strategies. Polya was of the belief that one had to be one's toughest critic. This was later to be paralleled by Mason's idea of developing an "internal enemy" (Mason et al., 1985, p. 98). Hart's (1993) work suggests that while it is often exceedingly difficult for students to be critical of their own work, being the critic of another's idea is attainable. Rasmussen et al. (2003) suggests that the cooperative processes of a classroom with appropriate sociomathematical norms will help the individual develop clearer mathematical thinking (NCTM, 2003, p. 150). If true, it would strongly suggest a pedagogical strategy for developing problem solving in students that – while developing internal strategies of the individual problem solver – is best fostered through group interactions. Perhaps such an approach can more easily address Goldin's (1992) concern that any "synthesis which avoids psychological theory is fundamentally flawed" (p. 276) as this could tie into child developmental theory. Here, children are encouraged to not only use their more egocentric nature to develop critical awareness via the critiquing of others' conjectures, but to turn that critical awareness inward as well.

If such abilities can indeed be developed through certain experiences and types of interaction, then perhaps the use of small problem solving groups is a way to develop individuals' problem solving abilities, or critical awareness, or metacognition, however one chooses to define boundaries of such concepts. In terms of developing young problem solvers, the development of classroom norms (re. metacognition) might have a profound effect on the way students initially engage a problem. If students' attitudes about their own abilities to solve problems or to develop conceptual understanding can be improved through development of certain routines or participation in particular activities, an increased number of students should feel more comfortable starting to solving problems. For some students this would be a significant shift in attitude, both toward mathematics and toward their own abilities. This shift would be a major checkpoint in the progress of a developing problem solver.

In such an environment, there must be a shift in a teacher's attention. Instead of merely attending to a student's successful or unsuccessful *demonstration* of understanding of a concept, the teacher must be concerned - perhaps, *more* concerned - with how students *engage* with a concept to come to that level of understanding. Teachers must become more aware of shifts in students' attention as they study particular problems and types of problems, and also how they attend problems in general. While some curricula point out possible misconceptions students may have about various topics, the means by which teachers can direct their own attention and that of their students to help develop conceptual understanding is vague at best. With a problem solving focus, teachers must have problems that will allow students to discover and explore the core of mathematical concepts in a manner which allows students to become aware of their shifts in attention. Mason actually describes the process of learning as consisting of shifts in attention. What problems should be used, and how teachers can help focus students' attentions while at the same time helping ensure that students are becoming aware of their shifts in attention will be key pedagogical concerns.

There appears to be a need for more research in problem solving around how such skills can be developed in the classroom. “Classroom processes have been largely ignored in problem solving research. This is particularly worrisome since without conscious attention to them, no reasonable theory of problem-solving instruction can evolve” (Chapman, 1997, p. 203). Even in the NCTM’s *Teaching Mathematics through Problem Solving: Grades 6 to 12* (2003), authors

Stein and Boaler (2003) admit "[providing a research perspective] is daunting because very little of this vast research base has explicitly investigated the kind of instructional approach advocated in this volume" (p. 246). Lester (1994) addresses what he sees as three key aspects of this very issue under the title "Needed: More Research on Problem-Solving Instruction":

> Issue I: *The Role of the Teacher in a Problem-Centered Classroom* - Simply put, I do not believe that any problem-centered mathematics curriculum has a chance of success unless the teacher's role in the curriculum is clearly and unambiguously spelled out…In my view, attention to the teacher's role should be the single most important item on any problem-solving research agenda.
>
> Issue II: *What Actually Takes Place in Problem-Centered Classrooms*? Good and Biddle (1988), Grouws (1985), and Silver (1985a) have noticed an absence of adequate descriptions of what actually happens in the classroom. In particular, there has been a lack of descriptions of teachers' behaviors, teacher-student and student-student interactions, and the type of classroom atmosphere that exists. It is vital that such descriptions be compiled if there is to be any hope of developing sound programs for teaching problem solving.
>
> Issue Ill: *Research Should Focus on Groups and Whole Classes Rather than Individuals* - Much of the research in mathematical problem solving has focused on the thinking processes used by individuals as they solve problems or as they reflect back on their problem-solving efforts. However, when our concerns are with classroom instruction, we should give attention to groups and whole classes. In order for the field to move forward, research on teaching problem solving needs to examine teaching and learning processes, not only for individuals, but

also for small groups and whole classes. (p. 272)

With these observations, Lester offers possible direction as to where the problem solving teacher should focus attention: the very environment in which problem solving instruction will take place.

2.7 Teaching Problem Solving

To conscientiously go about teaching problem solving, one must take into account existing research on a variety of areas. Problem solving strategies, the internal process of problem solvers, the context of the classroom and the role of the teacher must all be taken into account when developing problem solving pedagogy, though each of these areas have often only been considered separately in research. A goal of this project is to determine how a teacher can effectively use the classroom context to help students develop the internal processes of good problem solvers. General problem solving strategies attempt to break down these processes into smaller, more manageable stages which could provide focus for the classroom teacher. With careful planning and consideration of the classroom setting, it is believed that the internal processes of good problem solvers can not only be developed in the classroom setting, but that this very setting will allow such processes to best be developed.

Lester (1994) points to the need for study of the role of the teacher, the goings on in a classroom, and interactions of groups of students and the whole class at large. For this project, the major component is the linking of problem solving strategies to a group setting. Both Mason's and Polya's general strategies suggest development of internal processes that allow the individual to look at ideas from multiple angles. It is this skill that most needs to develop, and is most difficult to develop; the group setting offers hope to both of these issues. A problem solving group has several individuals to provide these multiple angles, and careful structure of the groups and the tasks they must perform can take advantage of these multiple viewpoints to help all the individual members of a group to develop skills in a variety of areas. *Group collaboration* and *group monitoring* (Manoucheri & St. John, 2006) speak to how groups have potential to improve individual students' abilities and abilities in a variety of areas: understanding the problem, suggesting possible methods of solution, critiquing ideas and verifying methods and calculations. More than that, it is through these multiple viewpoints that students will have opportunities to learn from one another and develop the confidence they need to become good problem solvers. As a teacher creates opportunities for students to learn to communicate mathematical ideas with one another and to negotiate meanings, a problem solving classroom culture can develop. Exploring the nature of such problem solving groups, the problems and tasks assigned to students, and the role of the teacher in such a context is the purpose of this project.

Chapter 3: Purpose and Design of the Study

3.1 Research Questions

While the NCTM and WNCP have broad guidelines encouraging teachers to explore and develop problem solving and group work, what is less understood are the specific means by which this can be done, the challenges that will arise in any attempt to do so, and the ways in which to address those challenges. The goal of this research project is to better understand how problem solving can be taught in the secondary classroom. Drawing on mathematics-education research that focused separately on relevant topics (problem solving strategies, the internal process of an individual, the use of groups in the classroom, and the role of the teacher), the research project is designed to inquire into how problem solving groups can be used to improve problem solving attitudes and abilities in a classroom setting. The project addresses the following questions:

1. How can Mason's problem-solving approach be used within a classroom setting to improve problem solving attitudes and abilities?
2. What types of tasks and problems benefit the development of problem solving abilities and conceptual understanding?
3. What assessment practices help promote development of problem solving attitudes and abilities in problem solving groups?
4. In what ways can problem solving groups improve students' attitudes toward and abilities in problem solving?

The environment of the classroom offers many challenges and opportunities in developing Mason's general strategy. The highly social and interactive nature of groups offers an opportunity to develop a "meta-individual" that could collectively develop the attitudes and abilities of Mason's individual would-be problem solver. How group members learn to problem solve and negotiate ideas with each other will have a profound effect on their problem-solving attitudes and abilities. To help develop such groups, students must not only be given carefully chosen problems, but carefully designed tasks that help students not only engage the problem but also learn how to work interactively with other group members. Here *task* is used to refer to any teacher-planned activity that students are required to participate in while attempting to solve a problem.

3.2 Methodology: Design Experiment Research

To explore these questions, a Design-Experiment approach is used. Design Experiment allows the researcher to develop practice and theory hand in hand, acknowledging the complex environment in which teaching and learning take place. Researchers have found that "traditional paradigms of educational and psychological experimentation were not especially helpful in informing them as they engaged in their work" (Schoenfeld, 2006, 194). As a result, researchers have turned to a methodology that allows researchers to explore practice and theory hand in hand in the very environment in which the practice takes place:

> To sum this story up in a nutshell, there are times when one has to create something to explore its properties. The act of creation is one of design. If the creation is done with an eye toward the systematic generation and examination of data and refinement of theory, the result may be considered a *design experiment*....

> design experiments in education are still evolving. Relevant methods for conducting such work have not been codified, and their theoretical underpinnings have not been settled. (p. 193)

Practitioner-researchers in education saw the limitations of past methods, and developed a method that would allow them to explore their fields in ways which reflected the complex nature of the phenomena they were studying:

> Specifically, (a) as a result of the 'cognitive revolution,' goals for instruction expanded significantly beyond the content mastery that had been the focus of earlier instruction and laboratory studies; (b) the conditions of laboratory studies are typically tightly controlled, and extrapolation to a classroom setting often requires significant changes and adaptations; and (c) many of the theoretical constructs and ways of characterizing them were 'emergent' - the issues themselves only became clear as one attempted to make things work in practice. (Schoenfeld, 2006, p. 194)

Acknowledging the classroom context, Cobb, Confrey, diSessa, Lehrer and Schauble (2003) note that "design experiments are conducted to develop theories, not merely to empirically tune 'what works'... Design experiments ideally result in greater understanding of a *learning ecology*" (p. 9). He goes on to describe five key *cross-cutting features* common to a variety of design experiments, which fit well with the proposed project. According to Cobb et al., the primary purpose of design experimentation is to "develop a class of theories about both the process of learning and the means that are designed to support that learning, be it the learning of individual students, of a classroom community, of a professional teaching community, or of a school or school district viewed as an organization" (p. 9). The development of such theories is supported by the collection of artefacts, be they created by teachers, students or some other stakeholder.

This approach fits well with the project's need to accumulate and analyze material artefacts that can demonstrate growth in students' problem solving abilities and attitudes as well as to accumulate and analyze data on how socio-mathematical norms are developed. Research teams "might focus simultaneously on the norms and practices of a professional teaching community, the participating teachers' pedagogical reasoning and instructional practices, and their students' reasoning in a particular content domain. A challenge that arises in such cases is therefore that of coordinating multiple levels of analysis" (Cobb et al., 2003, p. 10). While this is indeed a challenge, it is one necessitated by the simple fact that the environment studied is one which involves both teacher teaching and students learning.

Another criterion of Design Experiment research is that they "create conditions for developing theories yet must place those theories in harm's way" (Schoenfeld, 2006, p. 196). As Cobb et al. (2003) describes it:

> Design studies are typically test-beds for innovation. The intent is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them... The design developed while preparing for an experiment draws on prior research and attempts to cash in the empirical and theoretical results of that research. The process of engineering the forms of learning being studied provides the research team with a measure of control when compared with purely naturalistic investigation. Furthermore, in attempting to support a specified form of learning, the researcher is more likely to encounter relevant factors that contribute to the emergence of that form and to become

aware of their interrelations. (p. 10)

While for this project I attempt to draw on existing theories on problem solving and suggestions as to how problem solving might be brought into the classroom, it is the interactions with and observations of students that ultimately inform my practice and resulting theory. This process of coming to understand theory and practice is a cornerstone of design experimentation.

Another characteristic of design-experimentation is the iterative nature. This aspect is especially pertinent to this project. The daily interactions with students were informative to my understanding of theory and practice, both of which were ecological in nature. Interactions with groups or the class as a whole allowed me the opportunity to further design future tasks for the groups and class in subsequent cycles. It is this iterative, ecological aspect of Design Experiment research which perhaps most appeals to practitioner researchers.

Cobb et al.'s (2003) final characteristic of design research is that “they [practitioner researchers] are humble not merely in the sense that they are concerned with domain-specific learning processes, but also because they are accountable to the activity of design. The theory must do real work” (pp. 10-11). Isolated laboratory experiments, or theoretical models about what good problem solvers do might ultimately have little value for a teacher with a room full of students. “General philosophical orientations to educational matters – such as constructivism – are important to educational practice, but they often fail to provide detailed guidance in organizing instruction. The critical question that must be asked is whether the theory informs prospective design and, if so, in precisely what way?” (p. 11) Again, the iterative aspect of design theory provides a better way of developing and defending a theory through the cycles of Design Experiment research.

3.3 Design of the Study

3.3.1 Study Context

The participants of this study were the students in a grade 9 mathematics class of mine, in a grade 7-12 school of approximately 1100 students. The class originally had 30 students. Before the end of the first semester, one student moved away while another student switched classes. Of the remaining 28 students, 25 elected to become part of the study. Election to become part of the study took place near the end of the school year, which afforded students a greater understanding as to the nature of the project and their participation. Students were given letters of consent (See Appendix: Letters of Consent) to be completed and signed by both the students and their parents or guardians. These letters were returned to another staff member at school and held until the conclusion of the course.

3.3.2 Data Collection

The plan was to collect data for the purpose of assessing student's problem solving attitudes and abilities and how those evolved throughout the course. Furthermore, I wanted to collect data to determine how selected problems and designed tasks met or failed to meet my intentions as the students' mathematics teacher in regard to the development of the group structure and of problem solving strategies. At the outset of the project, the plan was to collect the following type of data, though not all were used as initially planned:

1. *Introductory and Final Surveys.* These provided students an opportunity to reflect on the nature of problem solving and mathematics, and their attitudes and abilities in these areas.

The *Exit Survey* in particular was to determine areas of growth and detect changes in students' attitudes.

Sample questions:

- *What is mathematics? What is problem solving?* Many students enrol in mathematics courses, yet are averse to problem solving. This question was to expose this view and allow for later comparisons.
- *How do you feel about your ability to solve problems?* This question allowed students to reflect on their feelings in a way was perhaps uncommon in some mathematics classrooms, while shedding light on the relationship between attitudes and abilities.
- *How do you feel about offering ideas in your group? In front of the class?* It was thought responses to this question would offer insight into possible confidence issues as well as communication difficulties between group members and during whole class discussions, and how such difficulties impact attitudes and abilities.
- *In what ways do you think solving problems in groups is different from solving problems on your own?* Responses to this question provided students' perceptions of the group experience. This helped show levels of awareness and provided further data on the nature of the group experience.
- *What skills do you think you learned or improved on by working with other members of your group?* Responses to this question were to reveal student perspectives and possibly support my observations and claims, or provide benefits that I did not initially consider.
- *What was the best part about working with other members of your group?* Responses were to point to aspects of the group experience that students found most enjoyable and

provide further insight into how groups can be structured to provide a positive problem-solving experiences for students.

- *What was the hardest part about working with other members of your group?* Responses were to point to difficulties that must be overcome in developing problem solving groups, and possible ways tasks might be structured to address such difficulties.

- *Do you think your attitude toward problem solving or confidence in your ability to problem solve has changed over the year? In what way?*

  What affect, if any, do you think these problem solving sessions had on your ability to problem solve? These questions allowed students an opportunity to comment directly on possible changes in problem solving attitudes and abilities.

- *What affect, if any, do you think these problem solving sessions had on your homework / test performances?* Responses to these questions were to acquire students' perspectives on transferability of possible skills and attitudes beyond the problem solving sessions.

- *How would you rate your problem solving ability before this course? How would you rate your problem solving ability now? (0 = very little ability 10=Very strong ability) Comments:* These questions allowed students to comment directly on perceived improvements in problem solving abilities.

- *If you had to solve a problem you've never seen before, what strategies might you try to solve it?* This question allowed students to demonstrate strategies, vocabulary and a willingness to tackle some hypothetical problem.

- *When trying to solve a problem, what do you look for?* This question allowed students to try to explain the strategies they use when facing a problem. At the beginning of the year this was to indicate starting points in terms of vocabulary or skills. At other points in the year this question was used to identify improvement in the Entry or Attack phase.

2. *Teacher-Researcher (my) observations of students*. During problem solving sessions, students were observed for time on task, and whether they were working individually, in partnerships or in groups. Observations on the nature of the interactions between students were recorded along with their use of particular terminology/phrases that related to mathematics and problem solving. Class discussions following group sessions provided further opportunities for me to observe and assess students as well as the success of the task design.

   The following types of observations were anticipated:

   - *Majority of students are showing multiple examples of specializing.* This could show a wide spread understanding of the concept, and point to the success of the selected problem and the design of the task.
   - *Tim is offering ideas more easily now in his group / in front of the class.* This comment could indicate possible growth in vocabulary, attitude or ability.
   - *Students appear more comfortable commenting on others' conjectures.* This could demonstrate a shift in ability to conjecture (ability) or in an attitude (offering now conjectures); or it could demonstrate the advantage of problem solving in a group setting (adding to / modifying others' conjectures).
   - *Daphne tried to explain her thinking to Alistair using a diagram and by explaining her thinking verbally. He added to her diagram, and they both noticed a pattern.* This could indicate an advantage of problem solving in a group setting, as well as changes or development in metacognition, and problem solving attitudes and abilities.

- *Zoe is not engaged with the task at hand. She did not offer any suggestions to her group, nor appear to pay attention to what they were doing. She was not trying to solve the problem on her own or as part of the group.* This could indicate a variety of things. First, it could just be an off day for any one of a several reasons unrelated to the class. Second, if this is one of several similar observations, it could indicate a need for shifting the structure of groups, or a need to modify tasks to help increase participation.

- *Students appear to be unable to apply knowledge about similar triangles in Problem X. Many students required teacher intervention.* Such an observation might indicate that the content of the problem is inappropriate, or that a possible change in sequencing is necessary.

- *Students in Group A appear to collaborate very well together. Individual students are offering ideas for the whole group to work with.* This could demonstrate an increase in confidence of particular students, trust in fellow group members, or improvements in communication skills.

- *Students in Group C appear to develop skills in analyzing other members' conjectures, however work must be done to ensure those who offer conjectures are not feeling rejected by the process.* This would indicate an improvement on what Hart (1993) referred to as group monitoring, which could suggest an advantage of developing Mason's problem solving model through group work. It would indicate a success in the group process (monitoring / analyzing conjectures) while showing a need for some direction in helping students critique each other in a respectful manner to ensure that they feel comfortable offering ideas to the group.

- *Students in Group B appear to be working independently for the most part. They are not sharing ideas, nor do they appear to be helping group members who are stuck.* This could indicate a need for me to perhaps modify the group structure, or to offer more guidance in regard to my expectations that group members collaborate at particular moments in the problem solving process. Depending on students' abilities to successfully solve a problem without collaboration, it could also indicate that the complexity and difficulty of the problems needs to be increased.

- *There is in an increase in the number of students volunteering responses during class discussions, and responses, and responses are longer and showing greater levels of thought.* This could demonstrate an improvement in students' communication skills and their conceptual understanding and possibly a greater willingness or comfort in sharing responses in front of the class. An increase in the number of students participating in this way could suggest that students at a variety of levels of understanding have improved with respect to their mathematical thinking and their willingness to share.

3. *Homework assignments.* I planned to take samples of individual students' homework assignments at various times throughout the term. Student work was to show how well students were developing and applying problem solving strategies on an individual level.

   The following types of observations were anticipated:

   - *Students are giving no response to a problem.* This could indicate possible low levels in attitude and ability.

   - *Students showing various levels of sophistication at the Entry Level, i.e. drawing diagrams, writing out knowns and unknowns, defining variables.* This could indicate an improvement in ability and possibly an improvement in attitude.

- *Problems are partially solved then students appear to have abruptly stopped.* This would show that a student knows how to engage a problem, but appears to require assistance in carrying out a problem, or dealing with being stuck. Again, this could indicate a need for developing attitude or ability.

- *Problems are partially solved, student wrote "Stuck – need to find slope."* This shows that the student is aware of why he or she is stuck and possibly points to the next step. This could, thus, indicate an improvement in metacognition and problem solving abilities. This could also indicate transfer from other classroom activities to individuals' homework assignments. For other students, stopping at this point might indicate a particular need for understanding of prior concepts.

- *Problem has been completely solved, student has checked answer using calculator.* This would indicate that the student appears to understand how to solve the problem, possibly showing an improvement in understanding, attitude or abilities. Perhaps it can be shown that group work had a positive effect on the students' performance.

4. *Students' Problem Solving Journals*. All student work from problem solving sessions was to be collected in individual problem solving journals. The journals were to include all work done on solving problems as well as reflection responses to teacher questions. Students' work on individual problems were to be examined to see improvement in particular aspects of problem solving strategies (attempts to show work, form equations, draw diagrams, specialize, generalize) and to help judge the success of the problems and associated tasks in regard to meeting teacher intentions. Journal responses were to offer insight into the development of students' problem solving skills, their understanding of a particular problem,

the development of their attitudes and their metacognition, as well as possible successes or challenges concerning the group structure.

The following types of observations were anticipated:

- *Many students are making attempts to specialize.* This could indicate that students understand the problem, and move on to the next step. If many students make such an attempt, it could indicate that conceptual understanding is starting to develop.
- *Little or no work is shown.* This could indicate a disconnection between students and teacher expectations or difficulties with the task. Tasks (verbal or written) might have to be emphasized in a way that helps ensure that teacher expectations are met. If it appears to be more an issue of understanding, the level of difficulty of problems and tasks might need to be re-examined.
- *Students appear to solve the problems but are putting little effort into answering reflection questions.* This could show a need to direct students' attention or give further instruction in an area that students might have traditionally undervalued. Again, teacher expectations may need to be emphasized in different ways.

Possible reflection questions for students to respond to in journals include:

- *Take a look at some of your homework assignments over the past few weeks. Has there been any change in*
    - *... how you do your homework?*
    - *... your ability to answer problems?*
    - *... how you deal with being stuck?*

These open-ended questions would offer students an opportunity to develop and demonstrate improvement in metacognition, and attitudes and abilities.

- *Was there a time it was difficult to understand what a group member was trying to say? How did you overcome this?* This would allow students an opportunity to reflect on how others communicate, thereby allowing them to think about what they need to do when working in the group setting in order to get their ideas across.
- *Was there a time when someone shared an idea that proved to be wrong, but was helpful in solving the problem?* The purpose of this question is to allow students to reflect on the necessity of following possible dead ends when solving a problem. This question could thus help bring about a change in attitude when attempting to solve novel problems, or in dealing with being stuck.
- *Give an example where you used specializing to help you understand a problem.* Responses would indicate a development in terminology, and, more importantly, indicate a method to test ideas that a student is using, which would suggest the development of problem solving abilities.
- *Give an example of where you had to work hard to explain your ideas to a group member. What did you do to explain yourself? Give an example of where you had to work hard to understand the ideas of a group member. What allowed you to understand them?*

These questions could help show some of the difficulties that group members need to overcome when working together, while also allowing for the individual member to reflect on his or her communication skills in a mathematics context. Reflections on past dialogues and possible growth could then be used as indicators of success for the group structure. The successes and failures as explained by students would also allow opportunity for the teacher-researcher to direct student attentions in future cycles.

- *What did you like / dislike about this problem?* This question would allow me to gain insight into what types of problems students enjoy or dislike solving. This could offer some insight as to the types of problems that might be selected and to possible topics that students might need to work on through other means to be more successful.

- *What did you find important or learn about this problem / today's problem-solving session?* This open-ended question, while possibly offering some insight into students' metacognition, also serves to inform me about unanticipated successes of the intervention. This information could be highly valuable for both determining the appropriateness of the problem and the group approach used in a particular cycle.

5. *Teacher-Researcher Journal.* This was a journal in which I planned to document many things: the planning processes of particular research cycles; intentions with regard to the development of the groups, the selection of problems and the designing of tasks; the rationale for these intentions and problems and tasks. Teacher observations of the students in groups and during whole class discussions were recorded, and reflections on a design's ability to meet a cycle's goals were documented in this journal. Further reflections were documented after an examination of students' problem solving journals. These reflections documented shifts in teacher intentions through the cycles and provided a rationale for intentions of and structures used in subsequent cycles.

3.3.3 The Problems

Over the course of several years of teaching high school mathematics and having researched problem solving and the nature of good problems for this project, I developed opinions as to what problem solving is and what constitutes a “good” problem. While a longer discussion of some characteristics of problem solving and good problems will follow in the Data Analysis section, the list of these characteristics is given at this time as many of the beliefs stated below were held during the initial design of the project and, in particular, the selection of problems and the design of tasks for the first few cycles. Thus, the initial designs, reflections and the analysis of the first few cycles were attempts to meet or develop the following criteria.

- Problem solving, in essence, is the attempt by a person to a solve problem for which (to the problem solver) there is no known solution or method of solution (non-standard problem). Thus, repetition of previously solved problems, slight modifications to quantities and similar adaptations do not qualify. As such, there is an element of “newness” to the problem itself, either in terms of presentation to the solver or in that the method(s) of solution is/are as yet unknown to the solver.

Generally, the teacher-researcher tried to use non-standard problems. While there may be several views as to what this might constitute, the following points attempt to demonstrate what is meant by this:

- *The problem is not for a specific situation or case* (i.e. the problem does not state how fast, how far, how much, etc.). This allows students to specialize by creating their own specific situations. I hypothesized that such an approach would allow students to understand that the specific situations are just examples of a larger overall concept. For students who are used to being provided with exactly and only the required information to answer a problem, such problems are recognizably different, however it is this lack of information in and of itself that provides an entry point for students.

- *The topics and/or presentation of a problem appear somewhat random*, at least in the eyes of students. This helps keep the process fresh and is more likely to show genuine development of problem solving skills, as opposed to understanding of a particular type of question. I hypothesized that multiple exposures to previously unseen problems would serve to build confidence in facing new types of problems, or to at least make having to face such types of problems a regular part of the routine. Since students would not have seen the topics or problems before in their problem solving groups, I hypothesized that students could view themselves on a more equal playing field with each other within the groups and thereby increase participation of students who viewed themselves as weaker than other group members; the newness of the problem would further necessitate discussion (and possible task structure) within the groups.

- Problems that allow *multiple points of entry* will allow students of varying types and degrees of ability to engage in the problem. In a group setting, this would mean students of varying degrees of ability can contribute meaningfully to the group. It was hypothesized that such exploration of such multiple forms of entry can provide a way to link multiple types of understanding such as forms of representation (i.e. numeric, algebraic and graphical) of concepts, helping solidify concepts even for students of higher abilities, while also validating the variety methods that students might use.

- For the most part, problems were to be selected that did not tie directly to specific learning outcomes of course content, but linked rather to larger aspects of mathematics such as number sense, problem solving, forms of representation, etc. Such problems would hopefully build understanding of these larger concepts, while providing students a genuine opportunity to improve in problem solving.

- For topics that relate directly to course content, problems and tasks were designed to allow students to discover concepts. This was a general intention at the outset of the project, and so problems were generally not used as extensions of what was already learned. This helps keep the problem a legitimate problem, while also helping the students continue to see the problems from each session as new. I hoped that such an approach would have a positive impact on the students' conceptual understanding of course content.
- Since the purpose of the entire project was in part to improve attitudes toward problem solving, problems (or at least the presentation of said problems) were engaging on their own as much as possible. Many problems were taken from Clifford A. Pickover's *The Mathematics of Oz: Mental Gymnastics from Beyond the Edge* (2002), in which the sinister alien of dubious morality, Dr. Oz, kidnaps Dorothy and her dog Toto. Successful completion of problems of various difficulties supposedly will lead to Dorothy's freedom. The problems in this book are presented in a short (albeit bizarre) story that forces students to search for key components of the problem and also helps to necessitate the need for discussion (and by extension, task design) amongst group members. It was hoped that students would actually find such presentation of problems engaging, adding to a more positive look at problem solving.

3.3.4 The Problem Solving Cycle

For this project, a design cycle constitutes the planning, implementation and analysis of a problem solving session. Each problem solving session consists of students working in their problem solving groups to solve a problem and complete teacher-given tasks, and a follow-up class discussion. *Problem solving groups are the vehicle for improving problem solving attitudes and abilities*. Each session has specific goals related to both the development of the group structure and of problem solving abilities and attitudes. A problem is selected and tasks are designed to help develop goals in each of these areas.

Following each session, data collected (observations of students, *Student Journals, Teacher-Researcher Journal*) is analyzed to see how the group structure can be used as a pedagogical tool for meeting specific problem solving-related goals, and a subsequent cycle begins. Goals for the next session are determined based on previous successes and areas of concern. The problem solving group structure, having been analyzed in the previous session, may be modified to better reach short term and long term goals. Tasks, as instruments for both developing problem solving and encouraging the development of the group structure, may be modified in terms of specificity and type. Through multiple cycles, the nature of the group structures will continue to provide insight as to how they can be used as a tool for improving problem solving attitudes and abilities.

Chapter 4: Data Analysis

4.1 Data Sources

Several data sources were used for the project. While many of these were used in anticipated ways, the importance or usage of some data sources was ultimately different than planned as the project evolved. The *Introductory Survey* was distributed as planned and provided me with some initial views from students on problem solving, mathematics and their own abilities in these areas. In their responses to this survey, students clearly confirmed a correlation between liking a topic or skill with their own abilities in that area. Almost all students said they liked a topic that they were strong in, that they liked a particular topic because they were strong in it, or that they disliked a topic because they did not feel they were very strong in that area (*TR Journal*, Intro Survey Reflection). The *Introductory Survey* responses were useful for identifying changes in students' attitudes about problem solving, mathematics and their abilities in these areas when contrasted with responses to the *Exit Survey* given at the end of the year. The *Exit Survey* proved to be an excellent source of data as it allowed students an opportunity to describe in their own words how they perceived the problem solving sessions affected both their attitudes and abilities. In their responses, students confirmed much of what I had observed, while still providing a few surprises. (In particular, I was surprised by student comments about the applicability of their problem solving skills to their daily assignments and tests. This is explained the response to Question 4.)

A data source that early on I chose not to use was student homework from regular day-to-day assignments. Initially I thought that student homework could be used to chart progress in students' attempts to problem solve (fewer questions left blank, more work shown on incomplete problems). However, the nature of the questions given during most homework assignments was usually quite different from the group problem solving sessions. Many of the regular homework assignments took the form of cumulative exercises; determining whether progress was made from any particular approach when all students would have been previously exposed to multiple approaches would be difficult to say the least. From a teacher-researcher point of view, determining to what extent improvement on homework could be attributed to the problem solving groups would be tenuous at best. That said, some students did have interesting insights into the relationship between the problem solving sessions, homework and test performance in their *Exit Survey Responses* that will be discussed later.

Students' *Problem Solving Journals* were used to collect work in which students were attempting to solve problems and also responding to reflection questions given by me. The manner in which these journals were used evolved throughout the project. I initially thought the written form of these journals would cause them to be a primary source of student work and ideas, and thus a primary data source for this project. As each student had his or her own personal journal, I anticipated that these journals would clearly document an individual's development of ideas in each session. In retrospect, this view clearly underestimated the *group* aspect of problem solving groups! Much of the work by groups of students working on problems was verbal and dynamic. Many steps toward solving a problem were verbal, or scattered throughout several students' journals, while other key thoughts were never written down at all (*TR Journal*, Cycle 4). At the time I believed a more rigorous documentation strategy would have several benefits. First, I hoped that it would better allow students an opportunity to reflect on their experiences, and thus become more aware of their own thinking and problem solving. Second, I hoped it would allow me to better understand individual and group processes and how these developed over time.

However, it seemed obvious to me from early on that many of the key moments (*Aha!* moments or moments when certain ideas were abandoned) were not being documented (*TR Journal*, Cycle 3-4).

I wondered whether this was because of students' oversight of the moments' significance, or perhaps because of a lack of ability by students to appreciate such moments. I spent considerable time contemplating how I could get students to better document all processes (both individual and group processes) however this left me wondering *why* I should want that. Would further written documentation about early false steps of a particular problem lead to a better understanding and enjoyment of problem solving? Such rigorous documentation seemed to be more for my own personal benefit, and less so for the students. As well, since I did not know precisely what I was after in having such a rigorous paper trail of ideas, my own ability to appreciate what was "significant" was probably just as unreliable as what students would have considered to be significant. My conclusion in this area was that full documentation was not possible, and that even if it was, it would negatively impact the spontaneity of group interactions and the students' enjoyment of the sessions. From a researcher stand point, it seemed to me that attempts to encourage such detailed documentation would only be shifting from the problem of having too little data to the problem of having too much data to possibly find meaningful insights. Instead, I decided to allow students to "go with the flow" and to rely on them to appreciate what was (to them) significant in each session. I encouraged this by designing tasks that would require students to reflect on key processes a couple of times each session (*TR Journal*, Cycle 4).

My initial concern with the *Student Journals* was around a lack of evidence demonstrating development of student thinking. However, I realized I was coming from a standpoint of needing to be able to *read* such evidence. My past experiences suggested that this was the quickest way – students can simultaneously write and I can later read at my leisure – while better assuring individual accountability to the work. Early on, however, I realized that that the individual growth that I could see in class through observations, discussions and students' general wherewithal in problem solving was not necessarily reflected in the individual student's writings. I realized this was not a problem with my students or even necessarily how I was structuring tasks, but rather one of my own expectations. All my thought processes at the time were bent on contemplating how to increase *written* documentation by students of their actions so that I could have evidence of growth. The disconnect between the (lack of) documentation and what it was to represent (i.e. growth) eventually allowed me to realize and accept that such growth could be evidenced in a variety of ways, not just in written form. In retrospect, it seems that past training and in-servicing served to reinforce in me the idea that all evidence of learning must be made as objective as possible, and that there must be an artefact (test, assignment, journal entry) that I could readily point to justify some mark eventually given. Direct observation was considered more subjective, and had potential to have a smaller paper trail. I think a parallel idea on having paper trail of student work occurred for me with this project and my ability to make conjectures and conclusions.

While I cannot say that I have developed any definite conclusions as to how my own observations of students in class can or should relate directly to a student's mark in math, I did learn to rely more on my observations of groups of students interacting with one another for assessing their development in a variety of areas. While I often tried to limit my own direct interaction with group members while they worked together to solve problems, I did have brief conversations with each group periodically to better understand their thinking. This had the consequence of increasing my reliance on my own *Teacher-Researcher Observations* as to what was actually going on in the groups and how students learn to solve problems together, while also impacting the structure of the tasks associated with the problems. I relied more on reflection questions given at various points in each session to allow students to reflect on and document their progress (*TR Journal*, Cycle 4). I used my *Teacher-Researcher Journal* to house my observations of session, my immediate reflections, and planning for subsequent cycles. Many verbal exchanges between students that would otherwise have been lost were documented in my *Teacher-Researcher Journal*. My journal was thus a key data source of documented student interactions, and I often copied selections from *Student Journals* directly into it.

It was the organization of reflections on important aspects of each session in my *TR Journal* that allowed me to more clearly think about possible causes of and hindrances to progress in each session, and to plan upcoming cycles. Observations and ideas were broken down and placed into various categories. During the planning stage of a cycle, I categorized my ideas under ***Teacher intentions*** (re. the development of groups, on problem solving, on course content, and on intended teacher-students interactions) and ***The Problem & Task*** (Description of the problem, characterizations of the problem, task design). After the problem solving session had taken place, I documented what had occurred in the session in terms of my own interactions with students and what I had seen under ***Interventions & Observations*** (During class, For next class [if session took place over multiple classes]). To help structure my analysis of each session, I used ***Teacher Reflections*** (re. Group Development, Problem Solving [Problem/Task/Mason's Strategy], Course Content, Intended Interactions); and ***Plans for Next Time*** (Group Development, Problem Solving, Course Content, Intended Interactions). Many sessions involved comments in all of these areas, while some focused on only some of the areas. My *TR Journal*, thus, became the central data source both during the cycles and after the completion of all cycles.

4.2 Responding to the Research Questions

In this section, responses will be given to each of the research questions. To better understand what took place in the classroom, a general description of the first few cycles is presented as well as a more detailed description of one of the cycles. After each research question is discussed, conclusions are presented. Restated below are the research questions.

1. How can Mason's problem-solving approach be used within a classroom setting to improve problem solving attitudes and abilities?
2. What types of tasks and problems benefit the development of problem solving abilities and conceptual understanding?
3. What assessment practices help promote development of problem solving attitudes and abilities in problem solving groups?
4. In what ways can problem solving groups improve students' attitudes toward and abilities in problem solving?

While determining how problem solving groups could be used to explore Mason's general problem solving strategy in the classroom was a primary concern of the project, it was ultimately the *tasks* that allowed the group structure to reach its potential. For each cycle, tasks were designed, given to students, and analyzed for their ability to bring about the broad goals of the project and the specific goals of each session. Useful problem types, the fit of Mason's general strategy with a group structure in the classroom, and tasks to explore these themes evolved ecologically during each cycle. To better understand how this project evolved, I will first respond to the second research question, which addresses tasks and problems.

4.2.1 Responding to Research Question 2

Building Up to Group Sessions

I wanted to establish both a group culture and a problem solving culture with certain characteristics early on. Because of this, I viewed the first few cycles of particular import and they required much more planning than many of the later cycles. It was in the first few cycles where I could see how some anticipated intentions, tasks and structures would have to be modified in subsequent cycles. For these reasons, I will discuss these early cycles in some detail.

I took care at the start of the course to establish a comfortable yet challenging environment for students in which to learn mathematics. Before the group cycles took place, I wanted to make sure that all students felt as comfortable as possible both with the problem solving component and the group structures. It was important for me to design the tasks of the first few sessions in a way which would allow students opportunities for success while clearly informing them of my classroom expectations. As a result, from very early on in my contemplation of this project it was my intention to have an *Introductory Survey*, a problem solving session where individual students or partners could successfully explore specializing and generalizing, and a session in which I could establish my expectations for the interactions between group members.

In the first session, the *Introductory Survey* was distributed. Students responded to this paper survey individually, working much of the class before handing it in. This survey provided students an opportunity to be reflective about their learning in a way which many had not previously experienced in a mathematics classroom. A recurring theme found in student responses to the survey – perhaps not too surprisingly – was that when asked what mathematics topics they most liked and why, students frequently cited simplicity as one of the chief factors (*TR Journal*, Cycle 0). The topics they liked where those that they found “easy” or that were such that students did not have to “work hard to understand.” Topics least liked were those that students felt were “hard”, “difficult” or “took a really long time to understand.” Since mathematics is frequently a subject where students are required to learn increasingly difficult concepts, such sentiments would be in opposition to making mathematics a likable subject for many. Problem solving by its very nature can be frustrating to those who are having difficulty searching for a solution. It is easy to see how students could describe problems as “hard” or “difficult” and so a major strand and method of mathematics has great potential to be disliked by a large number of students. Based on these responses to the *Introductory Survey*, it was very important to me to provide students with experiences that would have a positive impact on both their attitudes and abilities. This confirmed my early intentions to provide problems and tasks with which all students would be able to achieve some level of success, and to have small, attainable goals, particularly in the early sessions.

I wanted students to view the problem solving model as something that was larger and more far-reaching than what would be their group sessions, and I also did not want the introductory group session to have the added complication of being the introduction to the problem solving model. More importantly, I wanted to students to find success with the early stages of problem solving individually or in partnerships so that they would be able to carry that confidence with them into the group setting. The second cycle involved a session in which students were given a brief outline of Mason’s general strategy listing the key phases (*Entry*

Phase, Attack Phase, Review Phase), however it was emphasized that we would be focusing on what takes place during initial engagement of the problem (see *Appendix: Mason's Problem Solving Strategy*). Again, I wanted students to get an idea that there was some bigger picture that we were working toward, but that our goals would be relatively focussed.

A key aspect of Mason's *Entry Phase* is the ability to *specialize* and *generalize*. Such concepts proved to be of particular value in developing both problem solving skills and problem solving attitudes, as will be later discussed. Since some students have difficulties moving beyond reading the problem, I felt specializing was a crucial task both for the individual students and the groups. Specializing provides students with a strategy that allows them to initially engage with a respective problem. Individually, students can demonstrate development in problem solving ability and attitude when they specialize. Because specializing is a strategy that can eventually lead to a solution, specializing demonstrates problem solving ability. Because specializing is a process that might not *directly* lead to a solution (in that mistakes might be made along the way), students who specialize demonstrate an attitude of comfort with the unknown and a willingness to make mistakes.

Specializing also has advantages for students working together in groups. Group members can help check the logic or calculations of other students' specific examples, thereby helping build on particular mathematical understandings. Alternatively, group members can each specialize on their own creating multiple examples that could help confirm or question apparent patterns. Since developing algebraic understanding is a key aspect of the Grade 9 curriculum and problem solving, moving from a specific (say, numeric) understanding to a general (algebraic) understanding is also important. For these reasons, the first two problem solving sessions were designed to have two or three shorter problems in which students would have opportunities to explore specializing and generalizing.

The first problem solving session was the one session in which students were left to work individually. While some students did discuss what they were doing with a partner (students were sitting at tables of two), no formal groupings had yet been established. To help get a feel for the many forms that specializing can take, the three example problems included moving from a general equation to specialized examples, moving from specialized examples to general equations, and then a problem for which the concept of specializing would be difficult to apply or verbalize, and for which it was expected most students would be unable to express algebraically (see Appendix: *Problem #1: Specializing: roots – tax – paper clip*).

The first two problems were selected because of a reasonable degree of ease of entry to the problems so that they would hopefully provide positive experiences for students on which to build in future sessions (*TR Journal*, Cycle 1). This approach appeared to have strong benefits. Often students are given “problems” where they have to plug in numbers of a single case that is given to them, enforcing an idea that there is only one answer and often only one way to get there. In this session, the tasks given to students focused on students’ learning about the concept of specialization through explanations and creating their own examples. As with most future sessions, after students had time to work on the problem, a class discussion followed in which important aspects of the problem and the processes were reviewed.

It seemed that this session provided a new outlook for some students on what mathematics could be, and that students felt encouraged by the fact that their own examples – however different from other students – could lead to a better understanding and even a solution to a problem(*TR Journal*, Cycle 1). For example, though students were not working in a group structure, there were several instances of students sharing their examples with nearby students with a greater enthusiasm than I had previously seen with this class (*TR Journal*, Cycle 1). In particular, the fact that students were able to have different examples that nonetheless lead to similar conclusion freed students from a need to find “the answer”; instead students could generate unique and correct examples, and take pride that their own example led to an improved understanding of a concept. From student comments, many found this quite a fun and liberating experience (*TR Journal*, Cycle 1). This session helped solidify the concept of specialization, and served well in bringing across the idea that students were capable of making up and testing their own conjectures (*TR Journal*, Cycle 1).

Almost all students were able to answer the first problem, and most students were also able to explain a solution to the second problem (*TR Journal*, Cycle 1). The first two problems were selected in part because the first had a general algebraic expression, while the second one could lead to an algebraic solution. As a result of this, when some students went on to explain the second problem in terms of algebra during the follow-up class discussion, there was an opportunity to further develop the concepts of specializing and generalizing, and how one can lead to the other. Moreover, it allowed discussion as to how students *could* think about how they might go about these processes on their own. Having already successfully done so meant that students were aware – and were becoming more aware – of their ability to do so. Evidence of this occurred during the follow-up discussion in which numerous students volunteered comments. Many students showed a desire to explain their examples, while other students attempted to explain their examples generally (algebraically) (*TR Journal*, Cycle 1). The moderate level of difficulty of the problems used helped ensure success for the majority of students, while the lack of immediate solution provided a need for students to engage with the problem through specializing. I viewed this initial problem solving session as a success since so many students appeared to have had a positive experience specializing (*TR Journal*, Cycle 1). Encouraging this "new way" of looking at mathematics was something I wanted to continue, and this session confirmed for me that the first few cycles should be such that they would allow students to continue to explore specializing as a means of coming to a better understanding of a problem. This individual session provided a significant first step in developing problem solving skills and so the second session went ahead as planned (*TR Journal*, Cycle 1).

The second session was designed to introduce students to their problem solving groups and the group structure. Group membership had already been determined in hopes of best realizing some of the characteristics of good problem solvers as described by Mason. As previously mentioned, Mason's strategy provides a description of how to improve various aspects of problem solving to a well-motivated individual. Many aspects of the strategy rely on an individual being able to use or develop an ability to look at a problem from several different viewpoints – a condition that should easily be found in a group of problem solvers in which different group members are likely to hold different viewpoints than their fellow group members. Further, the group structure could also provide all group members with an opportunity to improve both their problem solving attitudes and abilities. With these ideas in mind, students were ranked according to final grades from grade 8 and distributed so that each group would have a range of rank positions represented. I also tried to maintain gender balance within each group (*TR Journal*, Cycle 2). Most groups usually had four students, while a couple had only three. Groups were modified only when class membership changed (two students left the course before its completion) and on occasions where some students missed individual sessions. In such cases groups were sometimes modified to maintain a group size of at least three students.

At the outset of the study I decided to attempt maintaining group membership as consistently as possible throughout the course. There were a few reasons for this. One, it would allow groups with a particular membership to evolve based on previous sessions in ways not possible if the memberships continually changed. Second, any emerging group traits would be difficult to attribute to particular factors. Third, consistent group membership was also to allow students in each group to become more familiar and (hopefully) more comfortable with each other. I hoped that such familiarity would allow students a more comfortable environment in which to share and critique ideas. Fourth, I expected that as groups evolved throughout the sessions, different groups might develop unique overall group personas or characteristics. These group characteristics, while differing between the groups, might yet lead to successful (or not so successful) problem solving groups and offer insight as to the nature of the tasks associated with the problems, i.e. what types and amounts of structure need be in place for the development of successful problem solving groups. From the outset, it was important to me to determine how much and what type of structure was needed for the tasks of each session to allow groups to develop to their potential while not interfering with their natural development.

To help establish acceptable conduct of students working in their problem solving groups, a “Working Together Brainstorm” preceded the first group session (see Appendix: *Working Together Brainstorm*). This brainstorm activity was to help students get in the habit of being reflective about their own actions and how their comments and actions might be perceived by others while also helping establish a standard of behaviour (not just an academic standard) in the classroom. Students had time to individually think about how to successfully interact with other students – particularly as disagreements would arise. This was followed by partner discussions and then a class discussion. The students took this very seriously – considerably more so than I had expected (*TR Journal*, Cycle 2). I had thought that students of this age might find this activity somewhat below grade level, however students took the topic seriously while still having fun with it (*TR Journal*, Cycle 2). The result of this process was that students of all abilities were allowed to participate freely in a discussion that was not directly related to mathematics content, but nonetheless was closely related to students' feelings and possible insecurities about discussing mathematical ideas in front of peers and the teacher. The "What to try to avoid saying…" question allowed a fairly light-hearted discussion about what not to say to people who are thought to be off the mark while still making classroom expectations clear, what would and would not be tolerated, and the reasons behind it all (*TR Journal*, Cycle 2).

That a significant portion of a class was set aside for this and the manner in which I dealt with the subject likely served to inform students that this subject was taken seriously by their teacher, and this likely had an impact on how students dealt with the topic. For example, students who disagreed with each were not seen to use any inappropriate comments or putdowns as discussed during this brainstorm. Students were seen to disagree with each other using language such as "I don't think it works that way because…" or "I think it works like this…." (TR Journal, Cycles 2, 3, 5).

I believe that the brainstorm and discussion had a significant impact on helping ensure that the students (as group members) had a level of confidence and comfort before the first group problem solving session, and that some protocols were established for disagreeing with other students before the first group session even took place (*TR Journal*, Cycle 2). Further, this session, combined with later reinforcements by me and the groups themselves helped to make disagreements between students more about the merit ideas under debate and less about the students themselves. Given the increased participation by students of lower academic standing demonstrated in subsequent cycles (examples are given in response to Question 4), I believe such an environment enabled students to take greater risks and allowed shyer students greater confidence to offer ideas in the group setting, and perhaps in front of the whole class.

I sought an overall group structure that would allow the different viewpoints within a group to benefit the group at large. I assigned roles to members within the groups that gave each a certain responsibility within the group in hopes of developing group cohesion, communication skills, leadership skills, and to further develop mathematical thinking and problem solving. At the outset, I contemplated roles like *Leader, Problem Solver* and *Recorder*. Given the small size of the groups and a desire to have all students participate in the solving of the problems and also record their own thoughts in their journals, the role of *Recorder* was not used.

The role of *Leader* was assigned to one student of the group for each session, allowing all group members at least two turns in this role. The *Leader's* primary responsibility was to help the group with the initial engagement of the problem: understanding the problem. The *Leader* was to determine how the problem should be read, facilitate a discussion on the group's interpretation of the problem, and attempt to ensure that all group members felt they understood the problem. All group members were considered to be *Problem Solvers* and they had the responsibilities of assisting the leader, attempting to solve the problem, and helping other group members.

Following the *Working Together* brainstorm, the roles of *Leader* and *Problem Solver* were described, which students wrote down on a worksheet. I tried to impress upon students that all of them would have an opportunity to be the *Leader* and that *Problem Solvers*' primary responsibility was to assist the leaders through their participation. The *Understanding the Problem* process was the extent of explicitly stated group-tasks at this time; how groups proceeded in attempting to solve the problem was left to the groups themselves. All individual group members were instructed to show their work, specializing for example, and attempt to generalize their results. Early cycles were quite explicit as to the responsibilities of each role, however in later sessions such responsibilities were either already established or left to the discretion of the group as to how they should be carried out (*TR Journal*, Cycle 2, 6).

The results of the first group session were mixed. While the *Working Together Brainstorm* and the initial engagement of the problem by the groups in terms of members allowing the *Leader* to decide how the problem would be read and discussed were quite successful, engagement with the problem beyond this stage was unexpectedly poor, particularly in terms of written documentation of students' attempts to solve the problem (*TR Journal*, Cycle 2). Several *Student Journals* had no examples of specializing, while a number of students had only partially formed examples (*TR Journal*, Cycle 2). The problem was selected because of some tie-ins to previously covered material, and was considered (both before and after the session) to be a problem that should have been accessible to all students of the class. Given the overwhelming success of the class in specializing individually in the previous session, I expected that students would have similar successes while making the transition into groups. Little or no work could be found in many students' journals, however most students did have thoughtful responses to questions about the interactions of group members and efforts at problem solving (*TR Journal* 2). These responses indicated that my emphasis on group interactions was taken very seriously by students, and that perhaps this focus detracted somewhat from efforts by students to record their attempts at solving the problem (i.e. in a written format).

Perhaps my emphasis on group interactions, combined with the previous success by students specializing and written documentation of their efforts meant that I did not perceive a need to emphasize this aspect as much as was required. This led to the creation of a quick rubric later returned to students giving them scores out of three for 1) specializing, 2) using of diagrams and 3) for generalizing their findings. Scores for the most part were very low and, while these marks were not used for summative evaluations, they were useful in reminding students (or informing, as the case may have been) what the expectations were for future sessions. Upon review of the problems the following day the general attitude expressed by students was that the problems were indeed well within their capabilities (*TR Journal*, Cycle 2). While this session showed that when introducing a new process there will be some bumps along the way, it did also show that the students could work well together in groups and that the roles of *Leader* and *Problem Solvers* had considerable potential in teaching problem solving.

Blood Water: A Look at One Cycle

Having gone through the individual and group trial runs, the third session in many ways felt like the first *real* group session. Indeed, this session had many characteristics that would be found in all subsequent sessions. In terms of my own interventions with students, I had planned to direct as many questions addressed to me by students back to their groups. Where necessary, I planned to answer questions with questions to help direct student attention and build independence. Students were placed in their groups at the beginning of class, and were to go through the process of *Understanding the Problem* as facilitated by the group *Leader*. Due to the nature of the problem selected, specializing would be necessary to proceed. Again, the majority of structure of the tasks given by me concerned only how students initially engaged with the problem; how they proceeded from there was left open. It was my intention to keep the focus on the *Entry Phase* and to observe how the work in the groups evolved and, where necessary, to redirect students' attention through questioning. Students at this time were to have a problem solving journal in which to store handouts and record their attempts at solving the problem and respond to reflection questions. Following attempts to solve the problem, there was to be a class discussion and time for students to reflect in their journals (*TR Journal*, Cycle 3).

As this session was quite representative of the thought processes that went into selection of a problem and the tasks that would be associated with it, I will describe it in detail. A “non-standard” problem called *Blood and Water* was chosen for this session (see Appendix: *Problem #3: Blood and Water*). A problem less directly related to specific course content was desired to help ensure progress made on the problem would indicate success in problem solving (particularly specializing), as opposed to understanding of specific curriculum content. The problem selected proved to be highly successful in both developing the roles within the groups and problem solving. The task of the *Leaders* was again to facilitate discussion about their group’s interpretation of the problem. Based on a disappointing lack of written record of student work during the previous session, I emphasized that all students were to show written work (i.e. attempts at specializing) in their problem solving journal (*TR Journal*, Cycle 3).

The *Blood and Water* problem involves a cup of water and a cup of blood. A spoonful of water is taken from the cup of water and poured into the blood. A spoonful from the cup of mostly blood is then poured in the cup of water. The question is which cup’s contents is more contaminated? This problem proved to be an excellent one for group development in terms of the interactions between group members and further developing the roles of *Leader* and *Problem Solvers*. The presentation of the problem is somewhat difficult, and many felt it ambiguous, thus it required group members to discuss what they thought the problem was about. This provided an opportunity for *Leaders* to facilitate such discussions and for various groups to develop their own methods, routines and group-cultures (*TR Journal*, Cycle 3). There was considerable energy in the room as students jumped from intuitive or reflex responses of “The answer is obvious” and “It’s a trick question” to attempts at solving the problem through more mathematical means (*TR Journal*, Cycle 3). Once again, a problem with no given values was chosen so that students could specialize and develop this key aspect of Mason’s problem solving strategy.

As anticipated, many students asked me, their teacher, for help interpreting the question or sought approval of their early answers/guesses. While the “go to the teacher” approach might be viewed as disappointing, I viewed such occurrences as opportunities to help develop the problem solving culture. As planned, I directed student questions back at the respective group whenever possible. Occasionally I tried to give some minimal guidance to queries with other questions to help direct focus to the approach to be developed – specializing. Interaction between groups was also discouraged to help groups realize that they had to rely on their own group members to come up with a solution.

By the end of the period, the class was pretty evenly split among those thinking the water was more contaminated, those thinking that the blood was more contaminated, those thinking that both were contaminated to the same level, and those that were “undecided.” The lack of resolution allowed a take-home assignment to try one (more) attempt at specializing and to come with a written argument to convince group members the following day. While most students did attempt one more example, few students attempted even the briefest of written explanations in their journals. Once again, there was a particular area with significantly less success than expected in an otherwise highly successful session. Some student examples were used at the board during the class discussion, and many students actively participated (*TR Journal* Cycle 3). As in many subsequent cycles, I gave students some questions to reflect on in their journals. Two questions focused on group interactions (what they thought their group did well or needed to improve on) and one question asked students to explain where they made an error either in their understanding of the problem or in their calculations. Most responses suggested that students were trying hard to work together, even if there were difficulties. I responded to these entries with comments, encouragement, or questions seeking further explanation, and in the next cycles most students responded in greater detail to reflection questions (*TR Journal*, Cycles 3, 5).

Students were given another example as homework to try to specialize. The following day, students returned to their groups, and there was a class discussion. I picked three students to put their work up on the board. All three had different approaches, and while two led to the correct conclusion, only one was error free. Since this was one of the first problems that the groups had tried, and I wanted to make sure we continued to emphasize the *Entry Phase*, we spent a good deal of time discussing what the problem was, and why it was difficult to interpret. This process allowed all students an opportunity to participate, since it did not require successful completion of the problem. The multiple representations used in the solutions allowed links between methods to be explored, and I think this served well in letting students know that in this class there is no “one way” to do a problem. I was impressed with the students’ ability to remain focussed for so long on a single problem, particularly during this discussion (*TR Journal*, Cycle 3). Again, I think the multiple entry points and difficultly of the problem actually helped with this, as many students had to think very hard about whether their method was actually already on the board, albeit in a slightly different form. This seems to represent a more open-ended view of mathematics in that solutions do not need to be of one specific form. Thus, students can perhaps see value in different approaches, and also value the possible uniqueness of their own approaches. This parallels the first problem solving session where students’ ability to generate their own examples led to increased participation in the problem solving process (*TR Journal*, Cycles 1, 3).

Many students used a diagram to assist in their understanding of the problem. Even where it was simply a cup with arrows going from one cup to the other, it clearly demonstrated how the use of a diagram as a strategy for understanding a problem can be beneficial. The following example shows one such diagram and an attempt to understand the problem through specializing with percentages. As this example shows, making the calculations required by specializing can lead to opportunities for learning beyond problem solving.

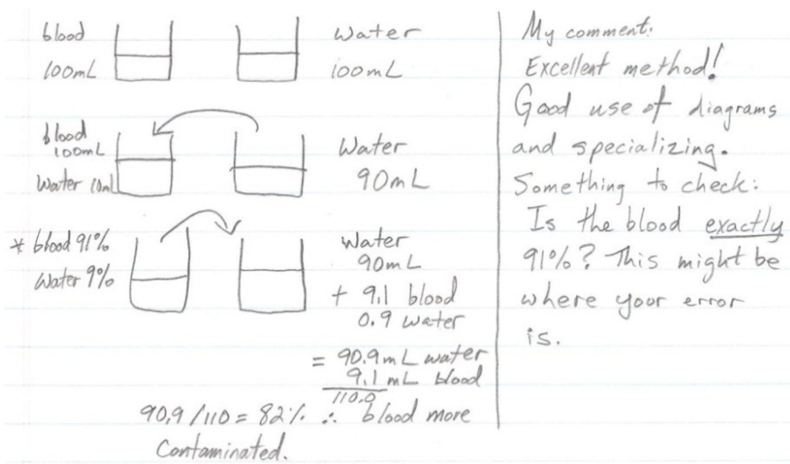
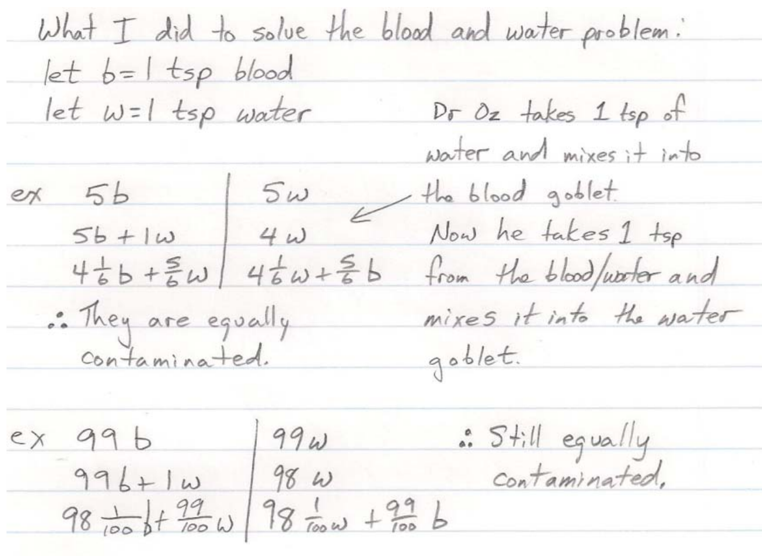


Figure 1: [Ryan]'s *PS Journal, Cycle 3*

Another student’s attempt at specializing shows an interesting way of representing the proportions of blood and water in each container. Here a student uses a variable as a unit of measurement:


What I did to solve the blood and water problem:
let $b = 1$ tsp blood
let $w = 1$ tsp water

Dr Oz takes 1 tsp of water and mixes it into the blood goblet.
Now he takes 1 tsp from the blood/water and mixes it into the water goblet.

ex $5b$	$5w$
$5b + 1w$	$4w$
$4\frac{1}{6}b + \frac{5}{6}w$	$4\frac{1}{6}w + \frac{5}{6}b$

∴ They are equally contaminated.

ex $99b$	$99w$
$99b + 1w$	$98w$
$98\frac{1}{100}b + \frac{99}{100}w$	$98\frac{1}{100}w + \frac{99}{100}b$

∴ Still equally contaminated.

Figure 2: [Carrie]'s *PS Journal*, Cycle 3

These are just two the many different ways students tried to represent this problem. I thought that the “messiness” of this problem (reading and interpreting the problem, difficulty in using fractions/percentages/ratios in a genuine context, groups coming to different conclusions) provided an excellent opportunity for the groups to develop their own cultures and for the teacher to direct/develop classroom and group cultures while providing a more genuine opportunity for students to explore core concepts of mathematics and problem solving (*TR Journal*, Cycle 3). For example, the lack of quantities in the problem meant that as various groups went on to use percentages, fractions and ratios for which some numbers were “nice”, while others were much more complicating. In some *Student Journals* specializing attempts could be seen to go from numbers like 25, 34, or 28 that quickly produced fractions with rather awkward denominators, to more advantageous numbers such as 10, 99 and 100 (*TR Journal*, Cycle 3). The problem is particularly good at developing specializing in that even those students who are quite strong in a more formalized algebraic approach, generalizing this problem algebraically is extremely difficult for students at this level. Specializing, on the other hand, is an approach that – however difficult – remains accessible to all students at this level.

I thought at the time (and still do think) that such “messy” problems, while still offering reasonably accessible entry points, can provide complex and genuine opportunities to meet and develop multiple goals. That many students came up with incorrect solutions (or rather, correct *methods* with errors leading to incorrect *answers*) and the manner with which students’ approaches were treated by other students and me likely helped to demonstrate that the attempt to solve the problem and the showing of work was valued over whether the final answer was correct or incorrect. Even at this relatively early point in the project, the group process of group problem solving followed by class discussion seemed to put the value on the process (and all that was learned along the way of *trying* to solve a problem) over whether the final answer was correct (*TR Journal*, Cycle 3). Students obviously still cared whether or not they were correct; however, if they were wrong, it did not mean that the entire attempt was a waste of time. From a mathematical standpoint, students’ use of ratios, percentage and fractions showed a genuine application and demonstration of understanding that cannot be demonstrated through simply giving students a ratio, fraction or percentage question (*TR Journal*, Cycle 3). The understanding required here is deeper, and it seems reasonable that such application provides students greater awareness and appreciation for particular mathematical concepts in ways that are unattainable through typical textbook questions.

Subsequent cycles followed a very similar overall structure. Typically, students would go into their groups near the beginning of class and problems and written tasks would be distributed. The main process the group as whole was to be involved in (as emphasized in every task given) was *Understanding the Problem.* Students worked on the problem(s) for most of the period, and a class discussion followed. There were roughly ten sessions throughout the course with students working together to solve problems in their groups. Problem solving sessions generally took place at least once every couple of weeks, lasting one or two classes. Tasks were designed for each problem to emphasize particular aspects of the problem or problem solving process. For example, early sessions focused on tasks to help develop the roles of *Leader* and *Problem Solver* as the group attempted to understand the problem. Leaders were given the tasks of determining how the problem should be read, and all group members were to participate in the subsequent discussion on the nature of the problem. In other cases, students were given questions, such as "What would specializing mean for this problem?" to help them develop concepts and communication skills (Problems A4-#1a Specializing: Slope & Square Roots, A4-#1b Specializing: roots – tax – paper clip). In some sessions, students were to use their specializing examples to help them make generalizations (*TR Journal*, Cycles 1-3). Tasks were designed in some sessions to help students move through increasing complex problems (*TR Journal*, Cycles 5, 6, 8). In almost all cycles, part of the task included a reflection piece designed to help students think about their thinking and about how certain processes or concepts might be applied in other situations (*TR Journal*, Cycles 1-7).

For any given cycle there were areas perceived as needing improvement, and this would influence the design of the session. Often this involved written or verbal emphasis on particular expectations. For example, the success of students in specializing led to a focus on using specialization to lead to generalizing (TR Journal, Cycles 1-2). When students were less successful in specializing, it led me to increase emphasis on this task in the subsequent sessions through the use of explanation or reflection questions (TR Journal, Cycles 2-5). As students appeared more competent in these areas, tasks would be designed to allow greater opportunity for written reflections (TR Journal, Cycles 3-7).

Reflection questions were typically given at the end of the session, however as the project went on it became more common for me to interrupt students while they were solving or stuck on a problem with pauses for reflection. Questions that interrupted student work generally focused on what students had tried to solve or needed to solve for the particular problem, and had dual purposes: 1) allowing students to reflect on their situation and 2) document their attempts and progress to me. (The need for such questions/reflections is discussed later in greater detail.)

It was during this (third) cycle that I developed a format for my Teacher-Researcher journal to better allow categorization of my thoughts and observations. I developed some headings that I would use for the remainder of the project to categorize my goals, tasks, observations, and reflections on the groups and on problem solving for each cycle. This method of organization allowed me to quickly categorize observations and comments for later expansion, while also facilitating examination of past cycles. It was through the organization and writings in my *Teacher-Research Journal* that I was starting to see a disconnect between what I was originally hoping to see in students' problem solving journals and what I was actually getting from those data sources. The result of this disconnect (explained in section 4.1) would ultimately result not in a restructuring of tasks to meet my original goals but in a shift in what I was trying to get out of the *Student Journals*. This shift in my perspective did result in a change of how the *Student Journals* would be used, and also the design of the tasks of each session. During the first few sessions I was very reluctant to interrupt the class as a whole during problem solving sessions. I did not want to disturb group processes nor did I wish to provide groups with hints before I gave them sufficient time to struggle with the problem. Nonetheless, I realized that part of the task had to be interruptions (planned or spontaneous) where students were given time to reflect on their progress. Expecting a complete written documentation of all steps, thoughts, mistakes, and *Stuck!* and *Aha!* moments was unreasonable, and would likely create an incomprehensible pile of information. Interruptions designed to allow students to reflect on their current state (how they got there, perceived needs, what they were trying to find) or what was to them an important idea or happening seemed a much better way of sifting through the many processes for important moments while also ensuring that the reflective processes were happening.

The use of such "interruptions" caused key shifts in focus or thinking for me. One was a freedom to interrupt students. I was extremely reluctant to interrupt students working on a problem because having students working on problems with minimal teacher input (in regard to the actual solving of the problem) was the whole purpose of the project! I did not want to be looked at as a provider of hints and I did not want to rob students of opportunities to make their own discoveries. Mistakenly, I was categorizing all interruptions together. Interruptions that allowed reflection on these processes had the benefits of allowing a pause in their work that could prove beneficial, improving problem solving abilities, allowing metacognition to develop, and providing a data source of these moments (*TR Journal*, Cycle 3). Thus, this represents a shift in how I viewed tasks, data sources, and assessment. I realized that while I could use the *Student Journals* to collect *some* of the student work, much of it would be lost. I would have to use the student reflections in the journals much more to understand key moments of the group process, and so tasks were altered to include written instructions that would allow for moments of reflection; alternatively, I interrupted group processes with a verbal instruction at some point(s) in the session (*TR Journal*, Cycle 4). An advantage of the verbal instructions was that my instructions could be based on what I was seeing in the room as opposed to something I had *anticipated* seeing. In either case, the reflective piece took on greater significance.

Overall, the potency of the tasks associated with each problem was somewhat unexpected. In the first problem solving session, the task centered around specializing, and students demonstrated their understanding of this concept through explanations and examples. Students performed exceedingly well. In the second session, where tasks focused more on the initial development of groups and understanding the problem, students did also very well in these areas, however students' demonstration of specializing did not carry over, at least not in written form in their *Student Journals*. In the following session (*Problem #3 Blood and Water*), I required more of students to attempt specializing. While students did excellent attempts at specializing, few went on to write conclusions or generalized statements. In *Problem #4 Are they Linear*, my goal was to increase the emphasis on moving to the conclusions or making generalized statements. While many journals showed such conclusions or generalizations, it appeared to be at the expense of showing some of the specialized examples. By the time *Problem #5 Factored Form to Expanded Form* came around, a sufficient balance in approach and emphasis appeared to have been reached.

Session Format

From the third cycle on, sessions took on a similar overall structure, regardless of current goals or how the current problem differed from those in previous sessions. I hoped that these similar structures would allow students to gain familiarity with such approaches both for subsequent sessions and for problem solving independently. The following structure became typical of each session from the third cycle on:

1. ***ENTRY PHASE* –** Students would join their groups, and the problems and associated tasks would be distributed. Students would remain in their groups while attempting to

solve the problems and often through the follow-up class discussion as well.

a) **Understanding the Problem –** As part of the task design for every problem solving session, there was a group check-stop after reading the problem (in a manner determined by the group or *Leader* of the group) to help ensure all group members understood the problem. This helped establish the role of the *Leader*, thereby providing some structure to the group and helping develop leadership skills. This process required dialogue, which in turn helped develop communication skills and group interaction. This also helped increased the likelihood that all group members understood the problem so they would be able to and feel more comfortable in trying to start solving it. Where such a process could not yet start, other students, perhaps in a similar situation, where there to support each other through this initial engagement of the problem.

b) **Direct or indirect instruction to specialize, and to share results.** At times specialization was done independently by group members, at other times group members worked together on common examples. In either case, students were instructed by me to share results, ideas and conjectures.

2. ***ATTACK PHASE* –** Students continue to work toward a solution in their groups. Teacher directed check-stops took place where students were instructed to just stop and think or discuss where they were at, where they needed to go, or what they needed to continue on a problem. Such check-stops were to allow students opportunities to reflect, to redirect the group and facilitate discussion, to develop metacognitive skills, and to provide written documentation from individual students of these processes to the teacher.

3. **Class Discussion.** Class discussions followed every session, lasting five to 20 minutes. The main topics of these discussions changed considerably as the groups developed and

different aspects of problem solving became the focus. Earlier sessions focused on working within a group and with specialization. As students' competencies in these areas developed, these areas were sometimes touched on only briefly or dropped altogether to move on to methods of solutions. In some cases, even solutions were dealt with only briefly to deal instead with applications of approaches or mathematical concepts.

4. **Teacher directed reflections.** Students were asked to write brief reflections on certain aspects of each session. With similar goals to the above mentioned check-stops, at times these reflections were to focus on group interactions, on mathematical ideas, and on problem-solving approaches.

Types of Problems

Many of the previously mentioned characteristics of good problems were confirmed or expanded upon throughout the project. While this may not be "the list" of characteristics required for a good problem, the following characteristics proved to be of significant value when developing problem solving in a group setting.

1) On the *TOPIC* and *TIMING* of a Problem

a) The problems have no consistent topic. By jumping around with the topics of problems, students came to expect (and want) to solve problems that they had never seen before. The very novelty of a problem, provided it is given in a comfortable learning environment, facilitated the development of group interactions while breaking down adverse feelings to dealing with new problems (*Exit Survey; Teacher Response Journal*, Exit Survey). This shows an improved attitude toward problem solving by students, while demonstrating a confidence in their ability to solve new problems. It seems that this increased confidence is linked with an improved ability to solve such types of problems, since it seems unlikely that students would feel better and more confident while continuing to fail to solve the problems. With each successive problem solving session, students continued to show such improvements in confidence.

Further, the novelty of each problem helped increased collaborative approach used by most groups as it necessitated engagement with the problem at a group level. This helped improve communication skills alongside other group interaction skills (*TR Journal*, Cycle 3, 4, 5). The novelty of a problem can allows focus on issues related to mathematics, problem solving, group interactions, thought processes, etc. in a much more purposeful and seamless manner. While many students were often apprehensive when given a new type of problem, the combined effects of this intervention led a majority of students to look forward to the problem solving sessions, and by extension, to solving new problems. A majority of students stated in their *Exit Survey* responses that they enjoyed the sessions and that their attitude toward problem solving had improved (*Exit Survey; TR Journal*, Exit Survey).

By keeping topics and presentations of problems new, students are also exposed to multiple types of problems. This seems to strengthen their adaptability to new situations, allows for methods to solve particular problem types to be learned, while also allowing students to see broad patterns across all types of problems. In terms of learning a general problem solving method, students can see how problem solving methods can be applied to a wide variety of problems. For example, specialization and the use of charts and graphs can be viewed as methods for engaging with a wide variety of problems. Over the course of several problem solving sessions, students showed considerable improvement in their ability to specialize and to apply this concept to new problems. Similarly, seeing specific methods used to finish solving a particular type might allow students to learn how to apply similar methods to future problems.

b) The problems are such that students do not perceive a direct tie-in to course content. (Problems #3 Blood & Water; #4 Toto Clone Puzzle; #9 Chamber if Death & Despair; #10 How tall is the flag pole?)

There are several advantages to this characteristic. An advantage of not having direct tie-ins to current course content is that students will not be biased to the topics or methods discussed in class. If problems are perceived by students to have no direct tie-in to course topics, then problem solving sessions will in fact be sessions on problem solving, and not an assessment of particular course content. This means progress in solving problems can be used to assess mathematical approaches, rather than assessment of correct repetition of an algorithm. Also, if students do tie in particular topics to a problem, it is more likely to be a genuine application of a concept or perhaps a more novel approach showing a greater level of understanding than would be the case where students simply apply a concept to an obviously related problem. Since students will not be biased by attempting to force links to (perhaps recently) covered content, students will be more likely to come up with a greater variety of approaches to the problem. (This feature leads to another characteristic discussed later: *multiple entry points*.)

c) Closely related to having problems appear to be unrelated to previously covered material is to use problems that *relate directly to course content not yet covered* – that is to say, using problems as a means to discover mathematical concepts. (*Problems #1 Specializing: roots – tax – paper clip; #2 Slope & Equation of a Line; #5 Are They Linear? #6 Factored Form Patterns; #7 Graphing Factored Form Equations; #8 Consecutive Numbers*)

In this way content/concepts are discovered by the students through their problem solving. This seems to be a much more powerful method of learning, with significant positive consequences, including a feeling of success that they had discovered something new to them, and a realization that they could in fact make mathematical discoveries on their own. Such successes might lead to greater self-confidence and increased independence when problem solving. In some cases when students solved problems that led to mathematical discoveries, students developed schema for mathematical concepts before formal terminology was introduced. For example, in *Graphing Factored Form Equations*, many students developed understandings of parabola, vertex and symmetry through their attempts at graphing prior to the introduction of such terminology (*TR Journal*, Cycle 7). This appeared to give students greater ownership of the concepts. Adding such terminology/details after students' exploration of the problems seemed a much better process – both in terms of ease and depth of understanding. Students were quick to pick up terminology, and the links between various concepts had already been made through the problem solving process (*TR Journal*, Cycle 7). Perhaps more importantly, some students cited in the *Exit Survey* that they "learned the material better" and that they "remembered it better" when they had discovered concepts through problem solving.

2) On the specificity of problems

Problems are not for a specific case, and therefore allow specializing. (*Problems #1 Specializing: roots – tax – paper clip; #2 Slope & Equation of a Line; #3 Blood & Water; #4 Toto Clone Puzzle; #6 Factored Form Patterns; #7 Graphing Factored Form Equations; #10 How tall is the flag pole?*)

For an individual, the ability to specialize provides an empowering opportunity to provide an example – or to "do math." This is not just a skill needed to solve a specific problem; it represents a means of engaging with many problems not about a specific case. That individuals can look at problems as something they can actively engage in can be a powerful step in developing problem solving skills, and for some students such an experience can represent a fundamental switch in their view of mathematics and of themselves as learners of mathematics.

The use of problems that allow specializing also serves to offer validity to "guess-and-check" approaches with which many students are very successful and provides a learning environment where such skills are valued. For example, [Mark] was strong at mental math and often quickly found solutions to problems where other students would use a more formalized algebraic approach. When he verbalized his rationale for selecting certain numbers he demonstrated a strong numerical understanding and algebraic thinking, however his ability to represent such thoughts using formal algebra was weak. Another student, [Lee] was very good at guess and check approaches to problems, however he seldom demonstrated verbally or in written form (algebraic or numerically) the rationale for his responses. In a problem solving context where specializing is not only valued but required, it helps value such an approach while also offering a bridge to the more formalized algebraic approaches that are to be developed in the course. In a group setting, such an advantage works in two directions. Students excelling at a more guess-and-check approach can try to explain their (pre-formalized algebraic-) reasoning to other students, who will benefit from hearing about a more intuitive algebraic understanding with strong ties to a numerical approach. On the other hand, some students appear to develop an ability to represent equations with formalized algebraic expressions while failing to demonstrate how such expressions relate to numerical concepts. Students with a more formalized algebraic approach can help other students see how a more numeric approach can be represented algebraically, and so both groups of students can benefit from such problem types.

3) Selecting problems with multiple entry points**.** (Problems #2 Slope & Equation of a Line; #3 Blood & Water; #5 Are They Linear? #6 Factored Form Patterns; #8 Consecutive Numbers; #9 Chamber if Death & Despair; #10 How tall is the flag pole?)

When students are only starting to develop links between related concepts, it is likely that students will view even slightly differing methods as unique. Through discussions in their groups, students learned to see how different methods are similar and developed a greater conceptual understanding of the topics. For example, *Problem #3 Blood and Water* had some groups working with percentages, fractions, and ratios. Groups that used two or more of such methods would learn how such concepts are closely related and that different representations can still yield a solution. As one student noted, “It can help you learn how to do things a different ways to do the problem. We shared our different ways of thinking and learned different ways to solve them.”

In cases where the methods are quite different from one another, students can learn different methods for solving a particular problem. This allows students an opportunity to see that there are multiple ways of solving a problem – as opposed to “the” way to solve a particular problem. This provides a more realistic view of mathematics, and hopefully a more empowering view as well since students will be in situations where they see other students of varying degrees of ability engage in problem solving. It thus frees students from psychological blocks when they cannot see a solution in some perceived “correct” way that one is “supposed to” solve a problem; rather they can look for *a* way to solve a problem that works for them. The former is restricting, the latter empowering. By providing problems that allow students to see that this is the case, students can develop a more positive attitude toward problem solving, and in particular, their own ability to problem solve.

Again, through discussion of methods groups can see how different approaches are related to each other, enforcing ideas on multiple levels through multiple forms of representation. In *Problem #8 Consecutive Numbers*, it was often the case that within a single group, several methods or approaches were present. Some students quickly went to an algebraic representation of the problem, while others used guess-and-check. The guess-and-check method in some cases showed a more intuitive understanding of the problem, and allowed for links to be made between the numeric representations and a more formalized algebraic approach. Further, in one group students came up with a couple of different algebraic representations which, though ultimately yielding the same answers, were thought by group members to be quite different. In another group, [Liz], [Michael] and [Carrie] all had slightly different concepts of average that were nonetheless consistent with each other, and they developed a more cohesive understanding through their group discussion as observed by me. It seems likely that such discussion and deep thinking about similarities and differences between ways of thinking and representing mathematical concepts will lead to deeper and longer lasting understandings.

4) Presentation – Closely tied to the previously mentioned characteristics of a good problem is the actual presentation of the problem. It is the presentation of the problem that determines the need for interpretation, that provides the information (too little, exactly the right amount, more than is required), ambiguity, and in some cases entertains. Problems with an appropriate amount of ambiguity are excellent in necessitating discussion, developing communication skills and building on mathematical thinking. Lacking specific information can allow for practice in specializing, while providing extraneous information allows students an opportunity to discern relevance. The inclusion or exclusion of diagrams can be used to clarify or to further provide ambiguity and a need for interpretation. Lastly, but not to be understated, is the importance to entertain. Many of the problems used in this project were taken from Pickover's *The Mathematics of Oz* (2002), not only for the nature of the problems, but their very presentation. In the book, the heroine, Dorothy, must outwit the evil Dr. Oz in order to win her freedom. Each problem solved is brings Dorothy and her faithful canine companion one step closer to freedom. In this way, what

could be (for some) a book of stagnant mathematical problems is a collection of stories with some inherent entertainment value while maintaining the mathematical integrity of the problems. Or, as one student stated in his *Exit Survey, "It made the math enjoyable without losing any logic."* Entertainment value can provide a hook while lessening the frustration due to the frequent ambiguity required for further growth. The presentation of such problems was so engaging that I created two problems for the sessions and presented them in a similar style to help students buy in.

Conclusion

There are several types of problems whose use can have many positive effects when developing problem solving skills provided that are presented with appropriate tasks. By structuring tasks so that all students can be part of the problem solving process, particularly when initially exploring the problem, a teacher can encourage a much greater buy-in to the process while also seeing the results of such efforts. While the above mentioned characteristics of good problems can allow much development in a group setting, a teacher must provide students with appropriate tasks to allow such benefits to be realized.

The single most successful aspect of this project, simple though it may seem, was the emphasis on *Understanding the Problem* as a *group process*. The emphasis on *group process* is used because while in any class there will always be some students who are already capable of understanding a problem once read, others will not. By structuring a group to have a *Leader* to lead the initial discussion with *Problem-Solvers* having the task of allowing the *Leader* to do so, a comfortable atmosphere with group cohesion was established quite quickly in almost all groups. The success in this entry level area allowed successes in multiple areas afterward, however unstructured these later stages might have been. Further, my main focus in the follow-up class discussion of almost every session was on *Entry Phase*: *Understanding the Problem, specializing* and focusing on what the groups did during their first attempts to come to a better understanding of the problem. Even after most or all groups had successfully solved the problem, the bulk of group discussions centered around initial engagement of the problem. The actual solution to any problem was secondary to how students initially engaged with the problem. I believe that this focus resulted in the largest growth in problem solving abilities of so many students. It also resulted in a shift in attitudes (discussed in later in greater detail) since a greater number of students could see success, allowing them to move forward with greater confidence.

A second major area of success resulted from my shift in focus from wanting an unnecessarily rigorous documentation of idea development, to a summary of methods and identification of important developments. This allowed me to modify tasks that freed students from such documentation, and allowed opportunity for more genuine and useful reflection. I allowed myself the freedom to "interrupt" students solving problems with a few (say 3 or fewer) verbal or written instructions allowing students to reflect on their work. I believe these moments of reflection had clear benefits in terms of development of problem solving, metacognition and assessing such developments. Such developments will be discussed in response to the forth research question.

4.2.2 Responding to Research Question 1

In this section I respond to research questions 1: How can Mason's problem-solving approach be used within a classroom setting to improve problem solving attitudes and abilities?

The use of problem solving groups proved to be an excellent way of allowing students to develop many aspects of Mason's general strategy in the classroom. The creation of the roles of *Leader* and *Problem Solver* allowed a structure that proved beneficial for classroom management and developing problem solving attitudes and abilities (*TR Journal*, Cycles 2-4). Further, I believe that the impact these roles had on classroom management facilitated the development of aspects of Mason' general strategy. For instance, the rotation of the role of *Leader* provided a distribution of leadership responsibilities amongst all group members over the course, while providing added incentive for students to cooperate with each other. The rotation of role of *Leader* helped make sure that no one student simply took over the group and also showed students that they did not have to be the strongest mathematics student to be an effective *Leader* (*TR Journal,* Cycle 2*)*. Because of their roles, group members had responsibilities to the entire group, and this helped keep organization within the groups as they attempted to complete tasks (*TR Journal,* Cycles 3, 4). While some students would from time to time work a fair bit ahead of other students, tasks with key check-stops where students were required to make a journal entry or discuss a topic with group members helped ensure that there were times for group members to reflect, share, clarify, and synthesize ideas as a group (*TR Journal*, Cycle 3,5,6). While more than one group member often facilitated such processes in some groups, the *Leader* of the group usually had a more dominant role at such times and *Problem Solvers* allowed the *Leader* to facilitate discussion (*TR Journal, Cycles 3-6)*.

With the roles of each group member defined prior to the first group problem solving session, I sought to bring in aspects of Mason's problem solving strategy into the classroom. While aspects of *Entry Phase* were always emphasized, many aspects of *Attack Phase* and *Review Phase* were developed, even if Mason's terminology in describing such phases was not used. To deal first with *Entry Phase*, the group structure was of great benefit in helping students learn to engage in a problem, particularly through the group task of *Understanding the Problem*. Having given group members opportunity to read the problem and attempt to come to a personal understanding, students then engaged in a discussion facilitated by the *Leader* about the nature of the problem (*TR Journal*, Cycles 2,3). Students learned to take into account others' viewpoints, and the communication skills demanded even at *Entry Phase* led to improvements in communication and clarity of thought (discussed later in greater detail), and provided a stronger position from which students could proceed in the problem. Structuring the groups through the roles of *Leader* and *Problem Solvers* appeared quite useful for exploration of this phase. According to students:

- "*[Carrie] was the leader in my group and I think she did a really good job. I had some trouble understanding the problem and she helped me out a lot.*" ([Michelle]'s *PS Journal*)
- "*We all did the same amount of work, but the group leader helped us to get organized. She explained how the methods work. We all understood it because we talked about it all together, and our group leader helped make sure we understood.*"([Patricia]'s *PS Journal*)
- "*We all helped each other equally, but our leader kept us organized.*" ([Kelly]'s *PS Journal*)

My own observations were that almost all class members seemed to be actively engaged in these important early processes (*TR Journal*, Cycles 2-5). The structure of the groups and tasks appeared to encourage students to spend a considerable amount of time making sure that they and fellow group members understood the problem. From early on, group members showed a willingness to spend more time on a problem than they would when working individually, concern for other group members' understanding, and a willingness to participate in discussion regardless of their standings (marks) in the class (*TR Journal*, Cycles 2-4). For instance, [Lindsay] and [Nelly] were typically shy students in math class with relatively poor marks, however in their group settings I often saw both girls offering ideas that were seriously considered by their respective group mates (*TR Journal,* Cycle 3).

The group structure and the group tasks provided students the comfort of a team atmosphere through the most critical steps of initially engaging with the problem. A number of student responses to the *Exit Survey* alluded to a "helpful" group approach where students could "bounce ideas off each other" and "come up with ideas [they] wouldn't have thought of on [their] own" (*TR Journal*, Exit Survey). Some students also indicated in the *Exit Survey* that they "worked longer on a problem" as a result of the problem solving sessions, and I believe that without such understanding or confidence at the early stages of solving a problem, some students would give up sooner (*TR Journal,* Exit Survey). This leads me to the conviction that through problem solving groups, all students have a greater opportunity to learn from other group members what it means to *Understand the Problem,* and can continue on. I believe that the high chances of success for all students participating in *Understanding the Problem* in a group context, regardless of their ability to eventually complete the problem, encourages important group processes and individual thinking processes that improve individuals' confidence, group interactions and mathematical thinking.

Specializing and *generalizing* are also key concepts described early in Mason's strategy. Problems were selected and tasks were designed to allow these concepts to be explored in a group setting. The very first few sessions (see Appendix Problems 1-3; Student Work: Cycle 1, Cycle 3) were all designed to allow students to explore the concept of specializing and generalizing. Someone specializing in isolation must correctly produce several examples in an attempt to understand a relationship. The group setting allows opportunity for learning about this concept in a genuine fashion and provides advantages over an individual working in isolation without taking away from the experience. In a group setting, two main advantages quickly come to the forefront: more than one student can try out and check the same special case, or they can try different cases (TR Journal, Cycles 2, 3, 4). In the former, students are able to check each other's calculations and verify if they are on a similar track. This results either in confirmation of a calculation (possibly building confidence) or exposes an error which can lead to an improved understanding. In cases where students specialize individually, multiple examples of a situation are created for the whole group to discuss. This situation still provides students the opportunity to specialize while limiting the time required to create multiple examples. Further, the multiple viewpoints provided by group members are more likely to result in exposure to special cases and examples that particular individuals would have missed if they had worked in isolation. This allows groups greater opportunity to pose conjectures, and provide firmer ground on which to confirm or reject conjectures posed by group members. Each member, through his or her own specializing has an opportunity then to contribute to the understanding of the group as a whole, and the group structure provides a greater opportunity for individual members to understand this aspect of Mason's strategy.

When solving problems, students will routinely become *stuck* and have to work to become *unstuck*. These are two naturally occurring states of *Attack Phase* according to Mason. In his strategy for the individual, Mason suggests writing "*STUCK!*" Doing so is to be a marker for the event, and to help the individual become reflective not just on one's mathematical understandings and needs, but on one's emotional state as well. Mason believes that this process might serve to allow the problem solver to relax, and reflect on both the emotional experience and the problem. Mason believes that it is through such reflection that problem solving skills will improve, along with an ability to deal with the possible frustration that arises from being stuck. Writing "*STUCK!*" beside work on a problem is thus a first step in this reflective process, and serves as visible acknowledgement of this state.

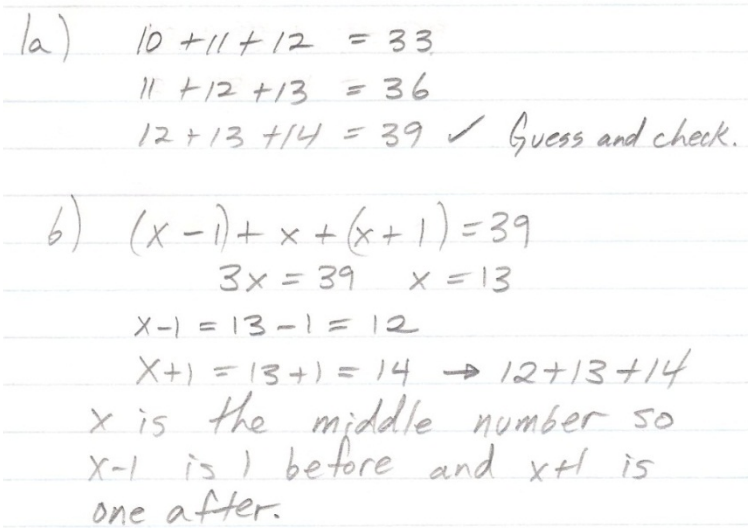
As teacher, I wanted to help students become more reflective about their work and thought processes. However, I believed that writing "*STUCK!*" beside work would not necessarily result in the reflective processes I desired, and that it would possibly appear to students as a tacked-on process. Also, for students usually working about one hour on problems, how many times would they become "legitimately" stuck? Because my main concern was *Entry Phase*, I was concerned that the word stuck might either lose meaning from overuse or that sorting through the documentation of such events would become a larger task than I desired. Fortunately, in the group problem solving sessions I found the practice of writing "*STUCK!*" to be unnecessary. Given the point of such practice is to provide a vehicle through which individual problem solvers can learn to deal with being stuck through reflection and to accept the emotional states of such situations, the group setting certainly allowed for such opportunities.

Because of the team atmosphere that developed within the groups, the group structure itself was often sufficient in creating an environment where students appeared to "naturally" review their work together when encountering road blocks, reflecting on both what had been tried and where they might want to go (*TR Journal*, Cycle 4, 5). Doing so in the group environment provided emotional backup and most provided some students with the wherewithal to continue with a much greater level of focus than might have been achieved through individual effort. In one session, upon seeing a false solution, another student was encouraged “not to worry” and to “try something else” by a fellow group member (*TR Journal*, Cycle 10). Such support offered by group members seems likely to lessen feelings of discouragement when solutions are not readily found.

To help students become more cognizant of such processes, tasks were designed to include reflections on key moments of problem solving processes – including moments of being stuck and how they were resolved (*TR Journal*, Cycle 4). I believe this allowed students opportunities to deal with being stuck and to participate on a group level in the reflective processes hinted at by Mason, and that mandating further documentation (i.e. consistently writing “*Stuck!*” and other reflections) would have only served to take away from the experience. [Isaac], along with previously mentioned students [Lindsay]and [Nelly], were three students who were much less likely to participate in early class discussions and who often had difficultly focusing for long on problems on their own. However, in the group context these students could be seen continuing to participate in mathematical dialogue within their respective groups – including moments where their groups appeared to be at an impasse (*TR Journal*, Cycle 6). In another situation, [Nelly] and [Ryan] demonstrated the reflective process of reviewing one’s work by going step by step through their work, “thinking out loud” their reasoning for each step in hopes of discovering possible mistakes (*TR Journal*, Cycle 5). Such instances suggest that in a group setting the reflective processes as proposed by Mason in the case of being stuck did not require the writing of “STUCK”.

Further, group members can provide another perspective from which to examine other group members’ work and also provide their own perspective of the problem. Group members can model different ways of thinking that can be of benefit to fellow group members in particular sessions that may also transfer to future mathematical thinking. For example, in *Problem #6: Consecutive Numbers*, I observed a few groups in which group members expressed their thinking about the problem in different, but nonetheless consistent ways. The problem involved knowing the sum of three consecutive numbers. In one group, students [Liz], [Carrie], [Michael] and [Ana] worked on the problem individually for a few minutes at a time before discussing ideas as a group. In my observation of one of their discussions, [Liz] explained how she thought a variable should be used for the first of the consecutive numbers. [Carrie] explained that while that approach would work, having the middle number be a variable (and the other two be x-1 and x+1) would result in cancelling that would make the calculation easier. This dialogue served to enforce [Michael]'s earlier idea of average that he initially used to approximate the solutions. By the end of their discussion, all group members agreed that each of their representations led to the same correct answer, and were "the same thing but different." (*TR Journal*, Cycle 6) In a later class, [Liz] referred to a similar type of problem as one where you similarly use a variable to represent any one of 3 unknown values "as long as you add or subtract the right amounts for the other ones." (*TR Journal*, Cycle 6 Follow up) This shows the potential for developing conceptual understanding through problem solving, and also demonstrates that the understandings or skills learned through problem solving can be transferred to future problems, at least of a similar type.

Student Journals from this session provided further insight into students' understandings and how they were developed. In some cases, students clearly approached the problem algebraically from the beginning. In others, a more numerical approach was first used and the tasks of the problem were designed to help students express their thinking in a more algebraic fashion. The following example shows a student's initial numeric approach followed by an algebraic approach. It is unknown to what extent fellow group members influenced the student in moving from the numeric approach the algebraic explanation that follows it:

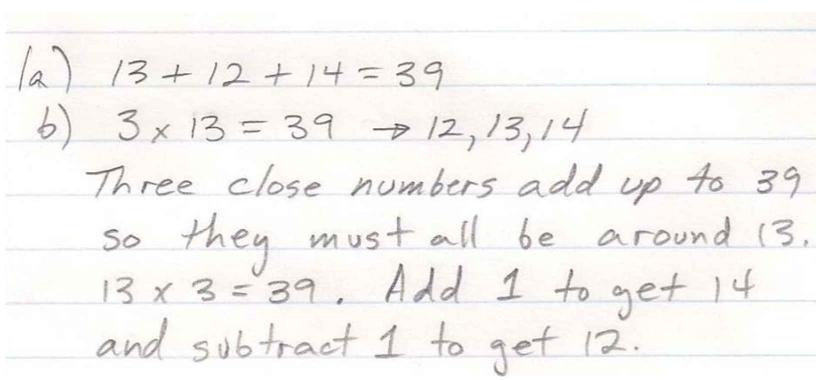


1a) $10+11+12 = 33$
$11+12+13 = 36$
$12+13+14 = 39$ ✓ Guess and check.

b) $(x-1)+x+(x+1)=39$
$3x = 39 \quad x = 13$
$x-1 = 13-1 = 12$
$x+1 = 13+1 = 14 \rightarrow 12+13+14$
x is the middle number so
x-1 is 1 before and x+1 is
one after.

Figure 3: [Mark]'s *PS Journal*, Cycle 8

Another student’s work shows an initial numerical approach, followed by a numerical approach to averaging. This is an example that suggests algebraic thinking, though as yet no formal algebraic notation is used:



1a) $13+12+14=39$
b) $3 \times 13 = 39 \rightarrow 12, 13, 14$
Three close numbers add up to 39
so they must all be around 13.
$13 \times 3 = 39$. Add 1 to get 14
and subtract 1 to get 12.

Figure 4: [Lee]'s PS Journal, Cycle 8

In the following example, a student presents similar thinking as the student in the above example, however a variable is introduced. Again the concept of average is present, even if a more rigorous justification for this approach is missing:

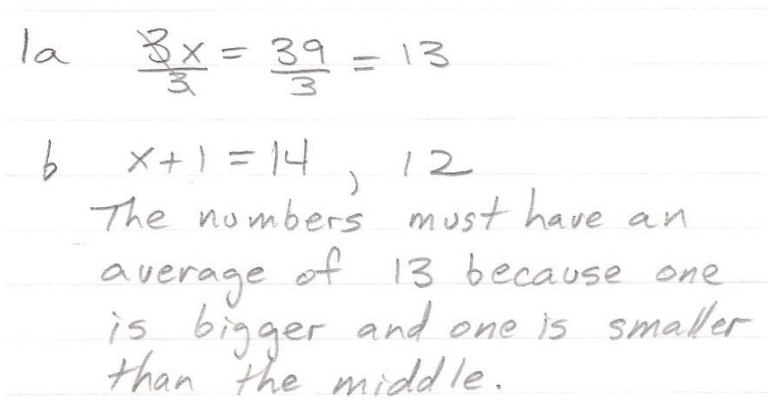


Figure 5: [Romi]'s *PS Journal*, Cycle 8

Other students showed that they were comfortable not only introducing a variable, but in using several. In the following example, a student introduces three variables and tries to explain the relationship between the variables and their total sum:

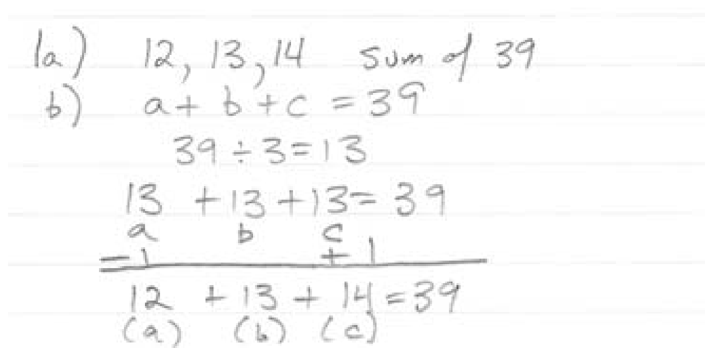


Figure 6: [Kelly]'s PS Journal, Cycle 8

[Michelle], a fellow group member to the student above used several variables as well and also shows the relationship between each variable. As in the above example, the student seems to start with a numeric approach, yet also shows a strong attempt to generalize the specific example through the use of variables. Again, the concept of average is used, however in this case the student is able to explicitly state the relationship between each variable (i.e. c + 1 =d). Also shown in response to question 2a, the student is showing a general result (2n for an even number) followed by specialized examples. Whether this is the actual testing of a conjecture, or the justification of the student's conclusions is unknown. This suggests that the tasks of the problem were successful in helping students move from a numerical approach to a more algebraic way of thinking.

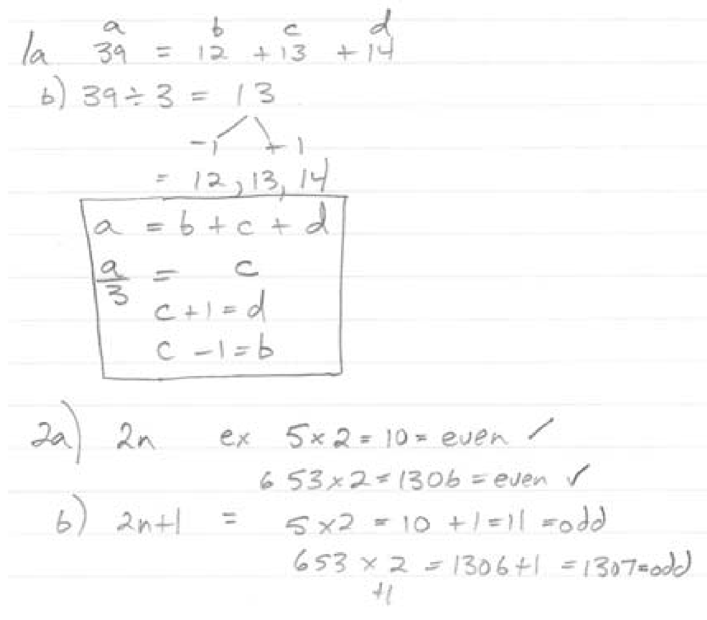
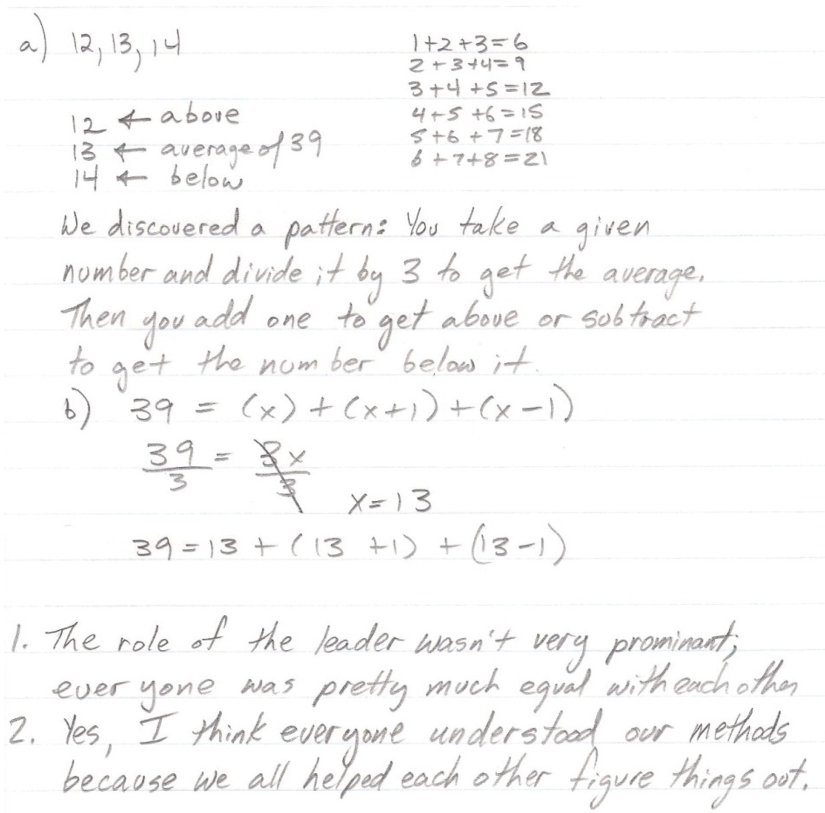


Figure 7: [Michelle]'s *PS Journal*, Cycle 8

In another student's journal, the student shows attempts at specializing and gives a written explanation to justify her approach. This written explanation is then followed by an algebraic representation showing the understanding that only one variable is needed to represent the situation:



a) 12, 13, 14

1+2+3=6
2+3+4=9
3+4+5=12
4+5+6=15
5+6+7=18
6+7+8=21

12 ← above
13 ← average of 39
14 ← below

We discovered a pattern: You take a given number and divide it by 3 to get the average. Then you add one to get above or subtract to get the number below it.

b) $39 = (x) + (x+1) + (x-1)$

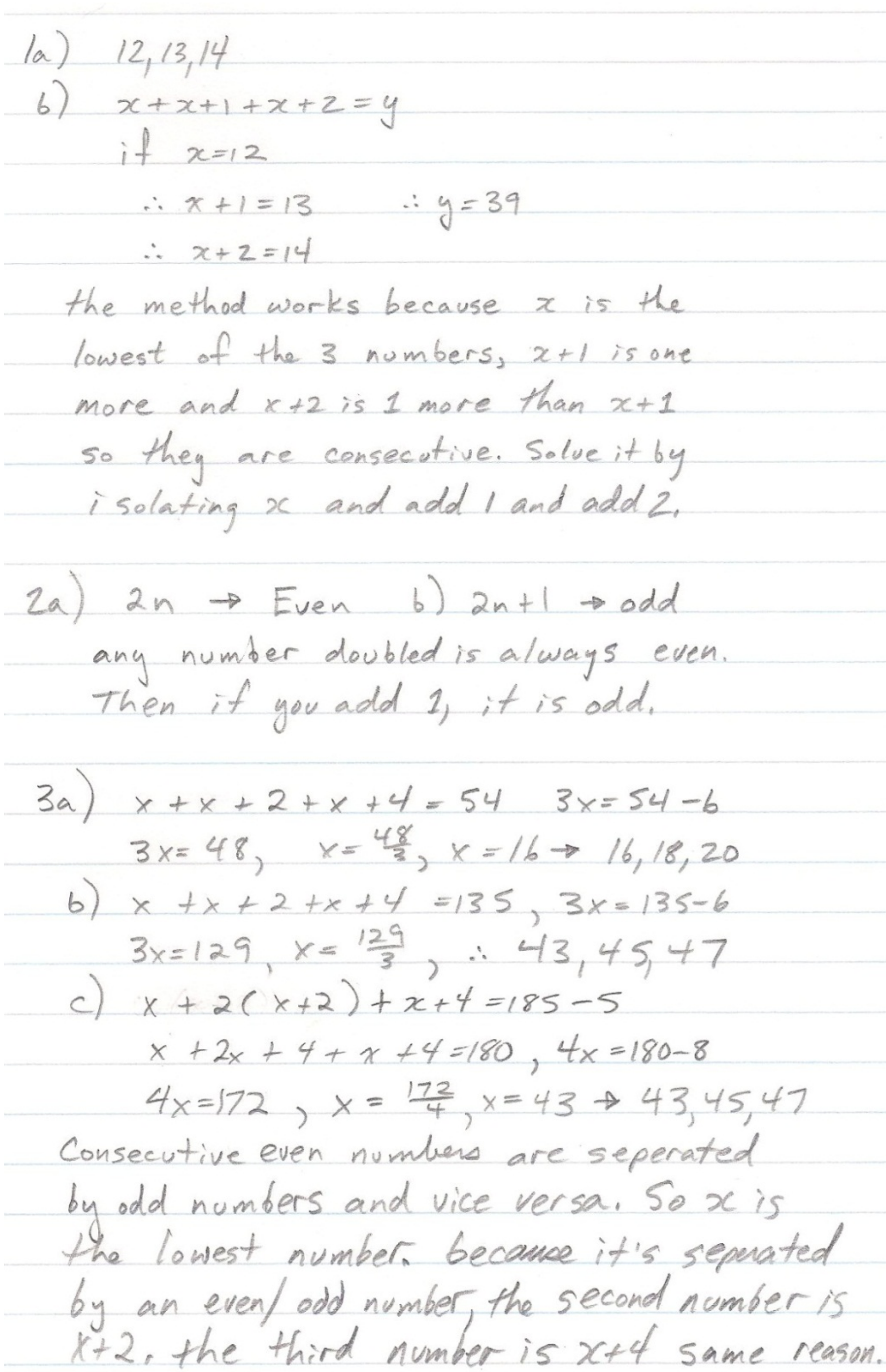
$\frac{39}{3} = \frac{\cancel{3}x}{\cancel{3}}$ $x = 13$

$39 = 13 + (13+1) + (13-1)$

1. The role of the leader wasn't very prominant; everyone was pretty much equal with each other
2. Yes, I think everyone understood our methods because we all helped each other figure things out.

Figure 8: [Patricia]'s *PS Journal*, Cycle 8

A last example shows one student who demonstrates strong understanding of the relationship between the numbers algebraically. Unlike the previous examples in which the variable was the average of the three values, here the student uses a variable to refer to the first number of the sequence:


1a) 12, 13, 14
b) $x+x+1+x+2=y$
if $x=12$
$\therefore x+1=13$ $\therefore y=39$
$\therefore x+2=14$
the method works because x is the lowest of the 3 numbers, $x+1$ is one more and $x+2$ is 1 more than $x+1$ so they are consecutive. Solve it by isolating x and add 1 and add 2.

2a) $2n \rightarrow$ Even b) $2n+1 \rightarrow$ odd
any number doubled is always even. Then if you add 1, it is odd.

3a) $x+x+2+x+4=54$ $3x=54-6$
$3x=48$, $x=\frac{48}{3}$, $x=16 \rightarrow 16, 18, 20$
b) $x+x+2+x+4=135$, $3x=135-6$
$3x=129$, $x=\frac{129}{3}$, $\therefore 43, 45, 47$
c) $x+2(x+2)+x+4=185-5$
$x+2x+4+x+4=180$, $4x=180-8$
$4x=172$, $x=\frac{172}{4}$, $x=43 \rightarrow 43, 45, 47$
Consecutive even numbers are seperated by odd numbers and vice versa. So x is the lowest number. because it's sepuated by an even/odd number, the second number is $x+2$, the third number is $x+4$ same reason.

Figure 9: [Jamie]'s *PS Journal*, Cycle 8

The above examples of student work help demonstrate how algebraic thinking can be encouraged by carefully designed tasks. It shows how students of varying degrees of ability can engage a problem and develop their thinking. A problem that only asks for the values would likely fail to do this. The actual task provided, however, required students to try to move their thinking from the specific (numerical) approach to a more general (algebraic) approach, as shown in the examples of [Michelle] and [Kelly]. The similarities as well as the differences are intriguing. Similarities hint at dialogues between group members that might have taken place, while differences point to unique ways of looking at the problem.

My observations of students in their groups and the written work in *Student Journals* show how the interactions of students solving problems in groups strongly parallel the internal processes of a “good problem solver” described in various general strategies. For this particular session, the task was designed to help students move from *specializing* to *generalizing* – that is, to develop algebraic thinking through a more numerical approach. In several cases, this process required considerable dialogue between group members (*TR Journal* Cycle 6). Whether the focus is on *Understanding the Problem* or on dealing with being stuck and other aspects of *Attack Phase*, there is much learning that happens as a result of students negotiating meanings and attempting to communicate their ideas. Both Mason and Polya encourage the development of an internal critic so that the individual can learn to attempt to analyze his or her own work critically. Mason speaks of this process as *convince yourself*, *convince a friend*, and *convince an enemy*. In a group setting, this very idea is realized without the need for the hostile confrontations. And while other group members can be viewed as external checkers, very often group members collaborated to create mathematical understandings where “convincing a friend” frequently involved offering partially formed ideas to be built on by others. In such cases, when students critique an idea that are in fact critiquing an idea that they helped generate. *Problem #3: Blood and Water* and *Problem #10 How tall is the flag pole?* in particular were two problems where I observed students forming, critiquing and reforming ideas that they themselves had helped to develop (*TR Journal* Cycles 3, 10).

To intentionally help bringing about reflection on such activities, tasks can be designed and problems selected to encourage groups to take advantage of their multiple viewpoints when examining work. In particular, students can be asked to explain what it is that they have tried to do to solve the problem thus far, or what has or has not worked (*TR Journal*, Cycle 11). Aside from helping improving students' ability to reflect on their experiences, it also allows an opportunity for the teacher to gain a better understanding of student work. Thus *Review Phase* of Mason's model in its strictest sense most often took place in the form of such written reflections, and during class discussions. These discussions helped ensure that students reflected on key aspects of the problem solving strategy, of the problem and tasks, and the group structure. The group structure and tasks were frequently setup so that Mason's CHECK *the Resolution* and *REFLECT on the key ideas and key moments* to a large extent had already taken place in the session prior to the follow up discussion. Students knew that they could not rely on me to tell them if their solution was correct or not while they were solving a problem; because they have seen me direct queries back to their group, group members got better at working to convince themselves as a group of the validity of their solutions (*TR Journal*, Cycle 7).

Mason's *EXTEND to a wider context* was typically not a concern for this project, as the focus was mostly on the early aspects of the strategy, particularly *Entry Phase*. (Exceptions to this were the graphing and factoring of quadratic functions during cycles 6 and 7, where extensions of the problem naturally flowed into course content.) Here again, though we did not explore this phase in any great detail, the multiple viewpoints available by a group and the class at large could offer opportunities unavailable to an individual working in isolation.

Conclusion

Many aspects of Mason et al.'s (1985) strategy (*Understanding the Problem*, specializing, generalizing, convincing yourself/friend/enemy, dealing with being stuck and getting unstuck, reflecting on the experience) can clearly be developed in a group setting. More than that, the group setting provides many opportunities to develop aspects of the general strategy through exposure to viewpoints and ways of thinking that would be unavailable to an individual working in isolation. Whether or not the aspects of a general problem solving strategy are explicitly stated or named, a teacher can conscientiously select problems and design tasks that will help students develop skills and conceptual understanding through their participation and reflections on their experiences. While I certainly felt freedom to design tasks and structure my groups in a manners that would allow the problem solving groups to develop in ways not necessarily described by Mason, his general strategy was a an excellent way of initially framing in my mind all that I wanted to accomplish with the problem solving groups, particularly as I tried to provide structure to what the groups should accomplish during the *Entry Phase*.

4.2.3 Responding to Research Question 3

In this section I respond to research questions 3: What assessment practices help promote development of problem solving attitudes and abilities in problem solving groups?

The tasks I designed, the group structure and interactions between students, and assessment practices all impacted one another. Their interrelation resulted in an evolution of practice throughout the course as changes were made each cycle to better achieve my goals.

The problem solving journals were highly valuable in assessing how individual students were progressing, and also assessing the successes of the group structure, problems and tasks. For individual students, they provided a collection of problems from each session, attempts by students to solve problems, student reflections, teacher responses and questions, and further student responses. Given that for almost every session I was the only teacher in the room, it was well understood that the bulk of group discussions would not be directly observed by me. The journals, as written documentation, were thus highly valuable in providing information to me. The problem solving journals provided evidence that students could make considerable improvements in mathematical thinking. That said, the group structure did provide some obstacles which resulted in refocusing attention in both the tasks and the use of the problem solving journals.

The journals did provide a space in which students could show their rough work, and in particular their attempts at specializing. Specializing is a fairly straightforward process, and for the most part attempts to do so were easily recognized by the teacher (*TR Journal*, Cycles 1-3). A teacher can thus assess a students' comprehension of a problem and specializing through students' examples. The quantity and difficulty of student examples could be seen, and I used class trends to determine the nature and specificity of tasks in future problem solving sessions. For example, in the first problem solving session, many students successfully managed multiple attempts at specializing, and were able to detect patterns from their examples. (*TR Journal*, Cycle 1) These successes showed me that students had a solid introductory understanding of specializing, and that future sessions could build on this foundation. Later, as students worked in groups, the *PS Journals* further helped document how groups were using their specializing to come to conclusions as a group:

> *"We did many examples to make sure that our way of answering the problem was right. After these examples we made a final answer decision as a group."* ([Tom]'s *PS Journal*).

Students' journals also provide a space in which students could respond to questions given to them by me to help them further reflect on their experiences. In some cases, students reflected solely on the problem or problem solving strategies, while in others the focus was more on group interactions.

> *"Our leader took initiative in the problem, and showed us how to begin the problem. He gave us a chance to solve the problem and then told us where we went wrong and the error in our equation. Everyone in our group understands the methods that were used because we each had a little bit of a different turn on the method used. We then found a method that pleased the equation and that we all understood." (*[Ryan]'s *PS Journal)*

> *"Something that we could improve on is going slower in our steps and explaining them better. This is because there is no point of jumping to step 3 while a group member is on step 1."* ([Ariel]'s *PS Journal*)

For students who were not immediately able to produce thorough reflections, I responded in their *PS Journals* with comments or questions seeking greater explanation, and this allowed some students to show improvement in this area. Such processes helped students to realize the importance of the reflection process (at least to me as their teacher), and thus increased their thoroughness in future reflections. Collectively, the journals showed that students were very capable at *Understanding a Problem*, *specializing*, as well as reflecting on the problem, problem solving strategies and the group experience. Such reflections allowed me to gauge the successes of various aspects of each session in relation to the goals set out, and to plan future sessions accordingly.

While many students had demonstrated an ability to specialize, to show work to solve a problem, make conclusions and write reflections, collectively theses journals also showed that during some sessions a number of students appeared to focus only on one or two aspects of the task and leave out written responses for other aspects. For example, some students showed specializing, but having established an understanding, failed to complete reflection tasks. Meanwhile, other group members having understood and solved the problem, presented only the conclusions and reflections in their *PS Journals* (*TR Journal,* Cycles 3-6).

To help make students more aware that several processes require written documentation of their efforts, a short scoring rubric was returned with *Student Journals* after the second session. I gave students marks out of 3 for each of 1) specializing, 2) use of diagrams and pictures assisting explanations, and 3) generalizing / conclusions. This tool was helpful in establishing that, regardless of completion of a problem, there were ways in which students could earn marks, and that they were expected to do so. A large number of the class did very poorly in all three areas during the second problem solving session. The rubrics and a class discussion served to highlight the importance of written work, and notable improvements were made in these areas in the third problem solving session (*TR Journal, Cycles* 2, 3)**.** While this rubric was not used in this form in future sessions, it did serve to indicate to students my expectations and it had a positive impact on their work. (Given the improvements made by almost all students in the third session, I chose to comment anecdotally on students' examples of specializing and generalizing rather than use the rubric. I continued this practice of commenting in subsequent sessions since it seemed more a more appropriate form of feedback.)

My observations of and interactions with students in their problem solving groups were most informative in allowing me to assess students and the various aspects of the group structure. My interactions with students during the solving of a problem both with particular groups and with the class as a whole were more numerous in earlier sessions. Many of the interactions, particularly in the first few sessions, were to redirect student questions back to fellow group members (*TR Journal, Cycles* 2, 3)**.** In particular, many students approached me with "Is this right?" questions after very little time on the problem. These types of interactions told me that students had not yet developed sufficient teacher independence, and that they were not yet ready to view themselves or other group members as capable of evaluating mathematical ideas. In cases where students had spent little time on a problem or done little to confirm their introductory conjectures, it also served to show that students were unaccustomed to problems requiring much more than a moment's thought to solve. By the fourth session, this type of interaction had dramatically decreased, with very few groups asking me if they had the correct answer, and several groups went the entire period without asking me a question while remaining engaged in the problem (*TR Journal*, Cycle 4).

As group dependence and teacher independence grew, teacher-directed questions decreased, and students became more reliant on themselves in interpreting words and problems, and in evaluating their mathematical ideas (*TR Journal*, Cycles 4-6). When students did ask me questions, the focus had changed from evaluation of a solution to requests for assistance on calculations that the entire group had already discussed (*TR Journal,* Cycle 5). Students were thus moving from seeing the teacher as one to offer validity to one person's solution to someone who could offer some insight into specific issues already addressed by the entire group. Again, I generally tried to direct questions back to the group, or offer questions that could help redirect attention of the students (*TR Journal,* Cycle 3-5). The evolution of the nature of questions directed at me, and my observations of groups working together to solve problems independently of teacher approval were indicators of growth, and in turn allowed me more opportunity to assess the groups through observation and the occasional directed question.

As direct interactions with students lessened, I was able to observe the groups in action more often, and observation became of greater importance in assessing student progress. Simple observation allowed me an opportunity to witness conversations and key moments in the problem solving process that would not otherwise be documented. For example, in one session a student, [Romi], said, "Oh this won't work then. I guess that was a waste of time. Let's try something else." Her group member, [Michelle], responded, "Maybe, but we might need it later." (*TR Journal*, Cycle 10) This clearly shows an ability to accept false starts and to try something new when an approach is realized to be inadequate. The second student's response demonstrates an awareness of possible future uses of an idea, however unusable at the moment. Since the statement did not need to be made at all, it also shows a concern for her fellow group member's feelings. Again, this was an interaction that was verbal, and of considerably more importance than the scrap of paper to which [Romi] was referring. That scrap of paper would also not necessarily have allowed me any insight as to how these two girls were able to develop ideas together. This is but one of several snippets of conversation that served to help me better assess individuals, the evolution of the group structure, and to assess the usefulness of particular problems and tasks.

In some cases, my observations of the groups in action resulted in tasks being created on the spot to help students better reflect on experiences or to help reengage group dialogue. For example, students were given a task to briefly review what they had tried so far with their group, and write a summary in their journal (*TR Journal*, Cycle 5). In other cases, I would interact with a specific group to assess their understandings or to help put them back on track. These observations and interactions with the students both in groups and at a class level helped to show that, at least in some sessions, understanding of the different aspects of the tasks given was quite strong, even though this would not necessarily be apparent from the problem solving journals. For example, in cycles 2 and 5 a number of students had successfully specialized to come to a conclusion, but many failed to complete follow up reflections though they did demonstrate understanding during the class discussions (*TR Journal,* Cycles 2, 5).

On the other hand, in those same cycles, some students wrote down their generalizations and reflections, however little work documented how they came to such conclusions. Several students from all the groups were able to discuss and add to ideas in class discussions that followed a group session. In such discussions students demonstrated an understanding of the mathematics, of problem solving strategies and working in groups. Students were able to verbalize their understandings, and demonstrate an ability to reflect on and apply their understandings. Such observations and interactions with students showed that, despite a lack of written evidence in some students' journals, the majority of students were successfully engaging in solving a problem and reflecting on the experience. An evaluation that only reflected the written piece would thus not reflect students' efforts or understanding.

As I have already mentioned, my thoughts on the disparity between the understandings demonstrated by written work and through group interactions and class discussions led me to rethink the use of the *PS Journals*. I began to see that emphasis by students on particular aspects of the tasks did not necessarily reflect a lack of understanding or laziness, but the possibility that students were concluding that certain aspects of a particular session were going to be of most value in the evaluation of their work. This could have perhaps been caused by an (accidental) overemphasis either in verbal or written instructions from me, or by students simply detecting patterns from one session to the next and seeing that there was an evolution in what was required from session to session. In any case, I began to see my own observations as a greater source for assessment, and tasks were modified to allow better use of the *PS Journals,* particularly through the use of summaries or reflections.

Class discussions followed every problem solving session, and these also were very useful in assessing the success of a session. Key areas on the specific problem, the general strategy, and the group structure were dealt with, and all discussions had some time spent focusing on the initial engagement of the problem. By the fourth and fifth sessions, the number of students who would voluntarily offer ideas had increased (*TR Journal,* Cycles 4-6). Students who typically would remain silent were showing comfort in offering ideas in front of the class. The emphasis on *Entry Phase* – how students initially engaged a problem regardless of their ability to reach a solution – allowed emphasis on this important aspect of problem solving while also allowing a greater number of students to participate. Their participation shows an increase in their confidence, both in their ideas and their ability to communicate their ideas in front of the class. As will be shown in the next section, many students spoke to this idea in the *Exit Survey*.

Conclusion

PS Journals were convenient to house students' written work while solving problems in groups however reliance on written work ignored the dynamic and verbal component of group problem solving. To better assess student understandings while also encouraging problem solving skills to develop, students must be given opportunities to reflect on important moments of each session, and to summarize their findings. To avoid canned responses and to help keep the reflections genuine, different areas can be targeted each session (i.e. the content of a problem, problems in communication, overcoming difficulties, working in a group, special cases of the problem, etc.).

I found my own observations as teacher of students to be the best way to assess students as they interact with other students. While the journals allowed me to have a data source from all students each session, it was generally my own observations which let me really understand where students were at in terms of their problem solving and group interactions, while also providing me further perspectives on the students' *PS Journals.* While much of the information I collected was rarely used for the evaluation of students (i.e. a grade), it was a very influential source when planning and evaluating each cycle.

4.2.4 Responding to Research Question 4

In this section I respond to research questions 4: In what ways can problem solving groups improve students' attitudes toward and abilities in problem solving?

The group structure provided many opportunities for improvement of both problem solving attitudes and abilities in students. I believe that many improvements in both of these areas only occurred because of the group settings in which students worked, and that such improvements could not have occurred if students were attempting to solve problems only individually.

In my teaching experience, I have not found it uncommon for some students who, on encountering difficulty when working alone on a problem, quickly give up or try to learn how to solve a problem *from* a fellow student. The group structure had an impact on both students' persistence in working on a problem and learning from others. The group structure was particularly beneficial in increasing the amount of time in which students would spend actively engaging in a problem. Encouraging the group process of *Understanding the Problem*, the group structure had two immediate benefits. One, students working together felt a level of comfort from the shared experience. Given the participation I observed of students in their groups – particularly the weaker students who had the tendency to give up more quickly than other students. Simply being in groups had some potential to increase comfort of students and the amount of time they spent on a problem; in each session I regularly saw students working up to an hour on a single problem – far longer a time than I witnessed for other assignments. In a sense, a problem can be viewed as a common enemy. Or, to use another euphemism, group members are all in the same boat. As mentioned earlier, the composition of the groups and the established class environment was such that even the weaker students felt comfortable in their small groups to offer ideas in their groups (*TR Journal*, Cycle 3, 6). Thus, simply having students in groups could provide an opportunity to make students feel more comfortable and engage in a problem longer than they would if they were left on their own.

It is safe to assume groups of students, left on their own, will not necessarily develop a group structure that will lead to successful participation of all group members, nor achieve the potential benefits of certain group structures. The group structure used in this project (i.e. the roles of *Leader* and *Problem Solver*) had particular benefits when trying to cause attitudinal changes. The role of *Leader* was a revolving one; students were not selected *Leader* based on their problem solving, communication, or leadership skills, nor was it a role that once attained was kept for any duration. This setup was useful for several reasons. One reason was reciprocity: being supportive of your group's *Leader* would be more likely to encourage that group member to support you during your turn. Because all students had at least two turns as *Leader*, many students who would not normally seek or achieve such a position had an opportunity to take leadership roles. The main responsibility of this role – to help students in *Understanding the Problem* – was one that all students showed some ability in; the role did not require that the *Leader* him- or herself understood the problem. Seeing themselves in such a position – and being seen by others in this position – no doubt contributed to students' confidence. One student noted that she learned how to express herself in groups and to "*take charge of a messy group situation*." (*Exit Survey*)

As well, the process of *Understanding the Problem* was one in which all group members were likely to find success in terms of their own understanding and participating in group discussions. Thus the process of *Understanding the Problem* provided an opportunity for students to engage – as part of a group – for some period of time in an activity that would help lead to a solution and in which most students would likely find some degree of success. For some students, their focus in group settings demonstrated an ability to maintain engagement for a longer period of time than they had previously demonstrated, and successes by the group served to bolster confidence. This confidence was readily seen through students' participation in group discussions (both during *Understanding the Problem* and after) where students of different levels of ability frequently engaged as equals (*TR Journal*, Cycles 3, 6). The sparks of confidence lit during the initial engagement of the problem through group design and task allowed students to continue to pursue solutions by working with fellow group members. Many students spoke positively about the sharing that took place in the group experience and about their own improvements in working in the group atmosphere:

- *First of all it's better I think, because you have people that can help you. Also everyone says their ideas, so you hear things you wouldn't normally think of yourself.*
- *I like that we could share our opinions and find answers.*
- *Everyone can contribute what they know.*
- *[I got better at] listening to other people and think[ing] more about the question.*
- *I learnt how to work in a group atmosphere better.*
- *I have improved in considering other people's thoughts in a group.*
- *[I got better at] working on problems with other people and agreeing on a solution. I learned it's good to share ideas because it makes it easier to work together.*
- *[I got better at] explaining math skills, teamwork skills.*
- *[I learned] to listen to my group's ideas and to do the problems.*
- *[I improved my] social skills and ability to suck it up and ask for help (though I'm still not very good at it!).*

(*Exit Survey*)

In the *Exit Survey*, students were also give questions where they were asked how their attitudes and skills changed from the beginning of the course. On a scale of 0 (not comfortable at all) to 10 (very comfortable), 18 of 23 students reported an improvement in comfort in offering ideas in front of their group, 5 students stated no change, and no students reported a negative change. The class average moved up from 5.3 for the beginning of the year to 7.5 at the end. Having to work with other group members to understand a problem necessitates communication, an area where both I and the students noticed considerable improvement. Several students commented in the *Exit Survey* that their greatest area of improvement was in communication. Through dialogue with fellow group members, students improved in their abilities to explain their thoughts and to understand the views of others. In the *Exit Survey*, many students spoke to this:

- *I think I improved at sharing and explaining my ideas to others.*
- *I think I'm able to explain what I'm doing better than I could before***.**
- *[I improved on] communication, explaining things.*
- *Problem solving in groups is different than solving problems on your own because you can bounce ideas off each other, and you have other people's ways of thinking to help you solve the problems.*
- *You talk to people about the problem and listen to how they solve it, therefore giving me more info on how to solve the problem.*
- *When you're stuck at a point, your group members can help you through that point. It can help you learn how to do things a different ways to do the problem. We shared our different ways of thinking and learned different ways to solve them.*
- *You learn different ways to solve the problem.*
- *People could tell me if my ideas were unintelligent or could expand on them.*
- *You can learn from other people's thoughts.*
- *We had a chance to state ideas and perspectives we wouldn't have thought of on our own.*

- *You get to hear many different opinions in the groups where as you only have one answer by yourself. You might agree with their way which could be more accurate.*
- *You get a variety of options, ideas and opinions as to what to do with the problem.*

(*Exit Survey*)

Through the need to communicate ideas to other group members, students were forced to clarify and develop their mathematical thinking. At times when students encountered difficulties expressing their ideas or being understood by others, they worked hard to verbalize their thinking and provided written support through diagrams, equations and calculations, and hand gestures. Some examples include:

- Problem #2: *Slope & Equation of a Line* – Many students created tables of values and/or diagrams for student generated equations, and discussed their ideas with fellow group members (*TR Journal*, Cycle 2).
- *Problem #3: Blood and Water* – Though the problem did not necessarily require it, many students drew diagrams with arrows to indicate liquid from one container being poured into another. Other students used hand gestures to convince or explain to group members what they thought was happening. In some groups, students discussed their calculations using a variety of forms including fractions, percentages and ratios using numbers that were generated by students through specializing (*TR Journal*, Cycle 3).
- *Problem #6 Factored Form Patterns* – In some cases, group members tried to draw graphs of the same equations to compare results, while in other groups students shared drawings of their own equations to find patterns (*TR Journal*, Cycle 6).

Such attempts to explain and develop ideas seemed to have clarified mathematical ideas. For example, in *Problem 10: How tall is the flag pole?*, student works from one group showed more than one diagram for the situation from a particular perspective with “final” diagrams from different members showing strong similarities that likely resulted from group discussion (*TR Journal*, Cycle 10). Intuitive understandings were developed and links were made between multiple representations of mathematical concepts. For example, in *Problem #8: Consecutive Numbers*, students in one group went from more intuitive understandings of averages, to numeric approaches and finally more formalized algebraic approaches (*TR Journal*, Cycle 8) In other cases, students learned specific mathematical content and procedures from other group members: in *Problem #7: Graphing Factored Form Equations* students learned about factored form equations, and characteristics of parabolas including x- and y-intercepts, vertices, and axis of symmetry; in *Problem #1: Consecutive Numbers* students developed an algebraic approach that could be used for a particular type of problem found in the course curriculum (*TR Journal,* Cycles 7-8).

The communication between group members is not just a giving and taking of information; rather, ideas are synthesized as students attempt to express and develop their thinking. This means that the strengths of any one group member can be added to those of other group members and, collectively, students can move beyond where anyone of them would have been able to reach on his or her own. As one student observed, “*You get to work together with other people so maybe their strength is your weakness and they can help out*” (*Exit Survey*). I observed many times dialogues between students, sometimes rapidly changing in direction as ideas were expressed, rejected, and reformed (*TR Journal*, Cycle 3, 6, 8). Students would write on each other’s papers or journals, drawing diagrams or jotting down calculations. While this obviously causes difficultly in being able to follow how a solution developed, time and again several students were simultaneously engaged for prolonged periods of time developing mathematical thinking. In such an environment all group members could claim ownership of their group's ideas and possible solutions. In terms of credit, I often heard, "*We did this*" or "*We tried that*", indicating a collective feeling of ownership of work (*TR Journal*, Cycle 6). At the same time, students were also quick to offer credit to those who came up with specific insights, which became part of the collective understanding. For example, in explaining how their group came to a solution in one session, Meg said to Matthew, "*You said it was like taking averages*." And he responded, "*Yah, but I didn't think of it as equations like you did*" (*TR Journal*, Cycle 8). This is a good demonstration of how students can develop ideas, of how students can see different ideas as the same thing (and vice versa), and how links can be made between different representations. Such developments should be much more common in a class where students are required to routinely construct meaning together. Several students spoke of not just learning from others, but developing ideas *with* other group members:

- *You can talk with other people about your ideas and listen to theirs and solve problems together.*
- *In groups, we combined several different ideas and ways to do the problem. We shared our different ways of thinking and learned different ways to solve the problem.*
- *You bounce ideas off each other to come up with solutions to problems, this makes it more effective.*
- *I think I learned to hear to people's ideas, and to combine ideas to make better ones.*

(*Exit Survey*)

I believe that it is these very opportunities to share and develop ideas, to hear and be heard by others that allow students to view these group sessions so positively. Being able to contribute to a group's ideas, and allow some part of other group members understanding to become part of their own has a powerful impact on students. Several students linked their attitudes toward problem solving to their level of ability to solve problems when asked if their attitude toward problem solving had changed from the beginning of the year:

- *My attitude towards problem solving has changed over the year because I guess I find solving problems more fun now.*
- *Yes, I think it has improved because now I'm better at it.*
- *I think my attitude and confidence in ability have changed in a positive manner, I feel better and sneakier being able to solve them.*
- *It's gotten a little better, I don't just give up as much.*
- *I have become more open to sharing my ideas.*
- *I'm more confident about solving problems now. Before, when I see some gross fraction questions, I didn't other trying to solve them. Now I try to solve them.*
- *I got better at solving a wider array of problems that I didn't know how to solve before. I improved my creative thinking for each problem.*
- *Yes. I don't dislike problem solving anymore and I think I can solve most problems as long as I try.*
- *I'm very confident in my problem solving skills and I think I've improved with my new confidence.*

(*Exit Survey*)

This improved confidence was not just brought about by a comfort level found within the groups, but by students' awareness that their skills were improving. Students were improving their problem solving skills in a number of ways: in reading and interpreting the question, working with others to understand the problem, specializing, drawing diagrams, making calculations, determining equations, thinking and communicating their thinking more clearly, synthesizing the groups ideas to form new ones, and showing willingness to spend an extended period of time working on a single problem, to restate a few. When students were asked how they would have rated their problem solving skills from the beginning of the year compared to the end on a scale of 0 (very little skill) to 10 (very strong ability), the class average at the start of the year was 5.3 and increased to an end-of-year average of 7.6, with 5 students stating no change and no students stating that they were worse off. When asked for comments about how their problem solving abilities had changed throughout the course, almost all students spoke of improvements:

- *I gained different perspectives of how to do questions. I probably improved my problem solving skills.*
- *I think I got better at factoring quadratic functions because of the problem solving. Also I got better at analyzing word problems.*
- *I improved on my problem solving skills and logic.*
- *Some people showed me how to think outside my box.*
- *Sharing my ideas. Thinking through the steps more when solving a problem.*
- *I improved how to justify my answer.*
- *My ability to solving problems has improved because I have learned to think of things in a different way.*

- *As I said before, it helped me because I had people that could help me and other ideas other than mine.*
- *I think my ability has gotten a little better because I listened to other people's ideas so I learned new ways to think about problems.*
- *I really do believe they made me at least a little bit better.*
- *I understand that I have to look closely into the problem to really know what the question is asking.*
- *They've made me better at solving in groups.*
- *It helped me learn different strategies.*
- *In certain areas, like how certain factors around you (ex. Shadows) can help solve problems, I've learned helpful techniques.*
- *I'm more confident about trying out a question.*
- *I got better.*
- *They have helped me a lot and I think I am much better at problem solving now.*
- *Made me more confident.*
- *My ability got better.*
- *I think I give myself a little more time to solve problems instead of giving up really fast.*
- *Looking at it from different people's points of view really helped and improved ability.*

(*Exit Survey*)

What is perhaps most important in the responses above is the emerging metacognition. Students throughout the year showed improvements in awareness of their understandings and thinking. Mason's emphasis on reflection throughout his strategy and Lester's (1994) emphasis on metacognition as the key to understanding suggest that such awareness of internal processes is a hallmark of good problem solving. Students offered a variety of statements indicating increasing awareness of their understandings, how they learn and how they think:

- *It helps you understand problems better because all of the people have different ways of thinking and solving problems.*
- *In the solving groups there was more input and verbal thinking.*
- *People have their own ways of doing things that you can learn by hearing them out.*
- *You get different ideas that you wouldn't have thought of on your own and it forces you to say your thoughts out loud which sometimes helps you make more sense of things.*
- *I remember what I learned more and it's more than just my view on how to do something.*

(*Exit Survey*)

Speaking to their metacognition, students also demonstrated awareness of their thought processes and understandings in the *Exit Survey* when they were asked how the problem solving sessions contributed to their understandings or success in homework and on tests. Regardless of whether they thought the sessions helped or not, an awareness of their thinking was demonstrated. First, there were a number of students who responded in the negative to "What affect, if any, do you think these problem solving sessions had on your homework / test performances?"

- *I don't think that these problem solving sessions have affected my homework or test performance because we didn't do much problem solving on homework or tests.*
- *Not that much.*
- *I personally don't think it had an effect on it.*
- *I don't think the problem solving sessions had any effect on my test etc. because they were different types of questions.*
- *I'm not really sure my test scores were below average all year.*
- *I don't think there was any affect.*

(*Exit Survey*)

Some clearly saw the homework and the problem solving sessions as two non-overlapping areas, citing that they saw little impact of the sessions on the homework since the nature of the questions was quite different. Written tests were used primarily for the testing of previously learned course content and were generally done individually; many homework questions were to practice or expand on previously covered content. Some students thus felt homework and tests were quite distinct from the problem solving sessions, and that their approaches to the different situations were distinct. On the other hand, there were students who felt that the problem solving sessions *did* lead to improvements in their homework and tests:

- *I remember more of the stuff that we did in groups.*
- *I think they really helped.*
- *I think it made me get a better mark on tests.*
- *It gave me the tools to learn how to deal with things I didn't know that well that I had to solve right away.*

- *Gave me new perspectives on the math questions I had.*
- *They have helped me on homework and tests because if I am not sure how to solve a question I think it through like a word problem.*
- *I now show all steps and all my work to show how I got an answer.*
- *Since I got better at solving I would finish my work faster.*
- *… I know how to look closely into problems in order to solve them.*

(*Exit Survey*)

Students thus saw a number of areas in which they felt the problem solving sessions had helped them on other types of assignments and tests: improved understandings of specific types of questions such as factoring and graphing (*TR Journal*, Cycles 6 and 7); willingness to spend more time on a problem; deeper understanding of concepts discovered through problem solving (*TR Journal*, Cycles, 6, 7 and 8); and willingness to "treat it like a problem" and try to solve it (*Exit Survey*)when they faced a question they did not know or remember how to solve.

The responses by both students who did and did not perceive a relationship between routine homework assignments and tests with the problem solving sessions show a well-developed awareness of understandings and skills. For those who perceived no difference, it shows an awareness of differences in the types of questions and the approaches and thinking used by students in facing those different types of questions. For those who did see benefit from the problem solving sessions, it shows a demonstration of transferability of skills and attitudes as hinted at by Lawson (1990). Since such transferability was actually offered by students as a benefit of the problem solving sessions, it also shows an awareness of their own thinking, skills and attitudes in complex situations – a significant development in metacognition.

The increased awareness by students of their own thinking, and improvements that I saw in terms of their mathematical thinking, communication, and understanding of concepts, combined with a group in which students were comfortably and confidently sharing and developing ideas served to – in some cases, dramatically – improve students' attitudes toward problem solving. Many students looked forward to upcoming sessions (asking when the next one would be occurring), and thoroughly enjoyed them even though the problems were challenging: "*Well I think that it was really fun but some of them were hard. But I was excited every time we did these*" (*Exit Survey*). When asked to single out what they enjoyed the most about the problem solving sessions, students offered a variety of opinions that collectively demonstrate an appreciation for what they learned, and an enjoyment of the process:

- *The best part about working with other members of my group was having people smarter than me in math explain things that I didn't understand.*
- *As I said before, being able to listen to other people's ideas.*
- *I learned how other people solve problems.*
- *Everyone helps one another while solving the problem and think together to solve one problem.*
- *Being able to work together to figure it out ☺ it's so cool.*
- *I saw different perspectives on what other people thought about a question.*
- *You had a mix of people who went about it in a different way.*
- *You get a different perspective and learn different ways to solve the same question.*
- *The part that there were 3 or 4 heads instead of one.*
- *It was interesting to share ideas on how to solve the problem.*
- *They each taught me to do different things.*

- *It made the math enjoyable without losing any logic.*
- *Other people had different thoughts than me, so I learned more about the questions and communicating.*
- *We had a chance to do word problems.*
- *Asking for help since no one learned it yet it made no difference.*
- *When they help you to understand a question or you help them understand one and working together to solve questions.*
- *I got different opinions of how to solve a problem.*
- *I got to know how [other people] worked.*
- *I was open to more ideas in the group.*
- *Get to know other people.*

(*Exit Survey*)

Conclusion

Using problem solving groups can improve students' attitudes and abilities on a variety of ways and levels. They can be used to help develop and clarify mathematical thinking while improving communication skills, conceptual understanding, problem solving skills and metacognition. Such changes coincide with an increase in student confidence, greater willingness to struggle with a problem, and increased enjoyment of problem solving. Problem solving groups can thus be used as a vehicle for improving mathematical understandings while also support a positive attitude toward the mathematics classroom. I believe that the comfort students experienced during the group approach in *Understanding the Problem* allowed confidence to develop as the group continued to work away at a problem. This confidence allowed improvements in skill and understanding to be made which in turn further improved confidence.

Chapter 5: Conclusions

5.1 Linking the Study Findings to Existing Research on Problem Solving

At the outset of the study, I did not know how much of Mason's *ATTACK PHASE* or *REVIEW PHASE* we would explore. Ultimately, the primary focus was Mason's *ENTRY PHASE* through the use of problem solving groups. Groups were designed with roles of *Leader* and *Problem Solver*, and a basic structure was developed to help students with *Understanding the Problem* and *specializing*. These ideas remained, for the most part, the only key and explicitly stated ideas of Mason's problem solving strategy that were used on a regular basis. However, as students worked together to understand and solve problems, key aspects of Polya's or Mason's models were regularly demonstrated in the students' work. Students improved in working in groups, mathematical thinking and metacognition, and their enjoyment of problem solving also increased (see chapter 4). For each session, the content of a problem was always new to the students and a wide range of problems were used. Progress made by students was thus improvement in general problem solving, and this was brought about by the classroom structure and the interactions of students working together to solve problems. How do these study findings relate to the research referenced in chapter 2 that provided for the theoretical framework of the study?

My desire to answer Question 1: "How can Mason's problem-solving approach be used within a classroom setting to improve problem solving attitudes and abilities?" was based on the fact that there was little existing research on the implementation of a general strategy inside the classroom. Nonetheless, the findings of this project do pertain to some previously mentioned research, particularly those involving the use of groups in a mathematics classroom. In particular, many of the findings of Manoucheri and St. John (2006) and Hart (1993) with regard to the development of classroom culture were supported. As Manoucheri and St. John noted, teachers "can influence students' perceptions about their role in the classroom and their expectations of peers" and that desired social and mathematical behaviours can be developed through modeling and interaction (p. 550). Their concepts of participation, commitment and reciprocity were also seen developing as a result of the problem solving group structure. Hart's key characteristics of groups of students (*group collaboration*, *group monitoring* and *social norms in a small-group problem solving*) were also evident in my study. Time and again, individual group members were observed offering ideas to allow the group to move forward when other members had reached an impasse (*group collaboration*); group members were often seen helping to critique the work of others (an example of Hart's *group monitoring*); and an environment developed in which a greater amount of time was spent working on problems and reflecting on ideas (Hart's *social norms in small-group problem solving*). Moreover, both Hart's and Manoucheri and St. John's terms can be viewed as alternative descriptions of processes that a group must undergo when problem solving and relate directly to key concepts of Mason's *ATTACK PHASE* and both Mason's and Polya's attempts to develop the internal questioner. Quite literally, students often had to "convince a friend" of the validity of an approach. It is this environment which allowed students to develop their language and thought hand in hand, to treat ideas as separate from the problem solver who came up with the idea, and to become more cognizant of their thought processes and those of others.

Chapman (1997) found in her study that when one teacher attempted to teach Polya's general strategy, students would naturally go through Polya's stages informally but resisted "applying them in a formal structure way" (p. 222). The terms *ENTRY PHASE*, and

Understanding the Problem were used frequently, particularly at the outset of the project, to help frame activities and student thought. The terms *conjecture*, *specialize*, and *generalize* were used as appropriate for they are mathematical terms with specific meanings to describe particular actions. Other aspects of Mason's or Polya's model (*ATTACK PHASE, REVIEW PHASE, Stuck, Unstuck*, *Devising a Plan*, etc.) were developed, though without labelling these processes. A major reason for this was that engaging with the problem was the focus; since this could be viewed as some students' first course in a formalized approach to problem solving, adding additional terminology to later processes would either take focus away from the initial phase or needlessly clutter and confuse dialogue. In future courses, it would perhaps make sense to move on to use the terminology of *ATTACK PHASE*. For this study, many of the processes of the different phases were modeled by me (the teacher), were facilitated by the group structure and tasks, or emerged ecologically from the group structure itself.

Do students develop problem solving skills on their own, or through careful instruction? The answer seems to be that significant gains *can* be made by carefully designed teaching practice. More than that, *by focusing on particular aspects of problem solving, many more aspects can be developed in tandem.* Whether or not all of these processes need to be given labels is beside the point. By giving students opportunities to develop particular problem solving skills, students will have to negotiate their way through a variety of thought processes, group interactions, mathematical concepts and misconceptions, problem solving strategies and awareness. With the support exemplified in this study, students can be helped to become more reflective of these processes; and by directing student attention to these processes, students can improve in all of these areas and in their metacognition – as the findings in this study suggest.

In attempting to answer Question 2, I believe I better understand the types of tasks and problems that can benefit the development of problem solving abilities and conceptual understanding. To have any chances of success in the teaching of or for problem solving, Lester (1994) believed that 1) students must solve many problems; 2) problem-solving ability develops slowly over a prolonged period of time; 3) students must believe that their teacher thinks problem solving is important; 4) students benefit greatly from systematically planned problem solving instruction; and 5) general strategies contribute little to improving students' abilities to solve mathematical problems in general. Let us review this list in relation to this study. Students had roughly ten problem solving sessions spaced throughout the year, with many other mathematics classes in between to further establish classroom culture. It would be safe to say that students of this study did benefit greatly from engaging with a few problems, though such engagement likely fails to meet Lester's requirement that students "must solve many problems." This perhaps offers hope that teaching for and of problem solving might not require students to solve "many problems", however it is certainly not an overnight process.

From the beginning of the course, I conscientiously emphasized my belief in the importance of problem solving, and I routinely attempted to demonstrate ways of approaching problems as one unaware of a method of solution. In routine classes, I often encouraged students to share partial solutions to problems in an attempt to value the process of coming to an answer, rather than focusing on the answer itself. During the problem solving sessions, there was always considerable time spent on the initial engagement with the problem, and later in the class we returned to reflect on the initial ideas and approaches of students. I believe such processes demonstrated, as Manoucheri and St. John (2006) suggested, that teachers can influence students' perceptions, and that through my actions students were well aware of my belief in the importance of these processes. I believe my attempts to discuss multiple methods of solution served to demonstrate that there is no “one way” of doing questions, and that the possibly unique approaches used by students would be something that I value. Comments I wrote in *Student Journals* and reflection questions I gave generally focused on processes and ideas of the moment, rather than on final answers. I believe all such actions would help meet Lester’s (1994) third criterion.

That students can benefit greatly from systematically planned problem solving instruction certainly appears to be true. From the start of the course, I set short term goals and developed tasks for problems that I thought would benefit both the abilities and attitudes of my students. Each session was carefully planned based on my analysis of the previous cycle. Each cycle had particular goals, and I set about to design tasks which I thought would help realize them. I believe that that many benefits that students received in terms of development of ability, improvement in attitude, improvements in language and other communication skills, and development of metacognition were results of this planning.

Lester’s (1994) fifth belief seems to refer to domain-specific heuristics. While some such heuristics were developed throughout the entire course, this study has demonstrated that carefully planned exploration of general aspects of problem solving can have significant impact on student learning and can meet many of the criteria required for development of metacognition as described by Lester. Indeed, many of Lester’s characteristics of good problem solvers (greater attention to structure features of problems, greater awareness of their own strengths, greater ability to monitor and regulate efforts) appear to have strong ties to metacognition. My attempts to get students to reflect on important aspects of their problem solving experiences through discussion and through their writings were to help develop such characteristics and I believe many improvements were made in terms of students’ metacognitive abilities. And, perhaps most importantly, such developments were made through exploration of problems using a *general problem strategy* – despite the warnings by Lester (1994) and Sweller (1990).

Problem solving sessions were carefully planned to allow students to reflect, often as they moved through increasingly complex aspects of a problem. Part of such planning, considerations was to answer Question 3: What assessment practices help promote development of problem solving attitudes and abilities in problem solving groups? My observations of students working together in their groups, my interactions with students on individual, group and whole class levels allowed me to better assess students' understandings. As well, *Student Journals* allowed me to review student work and also view written responses to reflection questions. Such forms of assessment – at times planned, often spontaneous – allowed me to assess while also meeting Stein and Boaler's (2003) recommendation that teacher actions include "(a) scaffolding of students' thinking (b) a sustained press for students' explanations; (c) thoughtful probing of students' strategies and solutions; (d) helping students accept responsibility for, and gain facility with, learning in a more open way" (p. 253). While many tasks, including reflection opportunities, were pre-planned, many were spontaneous and based on my observations of particular groups and the class as a whole. I believe such a combination of reflection processes are true examples of assessment for learning and helped achieve many the benefits for students mentioned in chapter 4.

In answering Question 4, several benefits to both problem solving attitudes and abilities from the use of problem solving groups were highlighted. Such benefits included increases in the time spent working on problems, improved communication skills, improved problem solving skills, and improved metacognition to name a few. Such findings contradict some existing research, much of which is rather pessimistic. Perhaps the researchers set their sights on loftier goals or on much higher levels of problem solving. Sweller (1990), in particular, believed that only domain-specific strategies are beneficial, that teaching general problem solving strategies does not work, and that he had not found research that indicating a transfer of skills to unrelated domains. A key concern of my study was to improve how students engage with a problem, and the group structure was designed to facilitate this. Significant improvements were made in this area and – with this introductory focus – many other aspects of any general problem solving strategy were developed even if aspects of the strategy were not explicitly labelled for students. Since the problems used were generally quite different from one another or involved topics yet not covered in the course, domain-specific strategies were more likely to be *developed* rather than simply implemented. Instead, general approaches such as specializing, drawing diagrams, and attempts to introduce equations were used by students.

The transferability of skills and attitudes demonstrated by students in this project seems more in line with Lawson's (1990) distinction between general strategies for *task orientation* ("broad affective, attitudinal, and attributional expectations of the student about the task"), *executive strategies* ("concerned with planning and monitoring cognitive activity"), and *domain-specific* (heuristics) (p. 404). Clearly students' attitudes demonstrated an awareness and expectation about problem solving (and doing so in groups) that was unique from other classroom experiences. More than that, some students were able to transfer such skills and attitudes beyond the group setting. Students demonstrated development of executive strategies through their class and group discussions, reflections and responses to teacher comments. Students commented in the *Exit Survey* that they had learned how to solve particular types of problems (or problems involving a particular content area), which indicates that exposure to domain-specific skills through problem solving can be transferred beyond the problem – as opposed to requiring large amounts of domain-specific knowledge in order to do the problem in the first place. The skills and attitudes developed by students (and their awareness of them) in this study seem to have transferability more in line with Lawson's views than with Sweller's (1990).

Student responses as to how their problem solving skills and attitudes transferred to tests and homework (spending more time, "treating it like a problem") demonstrated not only transferability, but awareness of such transferability. While not knowing specifically what Sweller (1990) refers to when he says there has been little evidence of transfer of skills, it would seem to be the case that some very pivotal first steps in engaging a problem can be made through the teaching of a general problem solving strategy, and that these skills and attitudes are transferable on at least some level. As the participants of my study were young students being introduced to a general problem solving strategy, this would show significant impact on the type of students one would most want to benefit. As Schoenfeld (2006) suggested, while teaching metacognition as a topic in and of itself is likely to fail, this study showed using problem solving groups with tasks designed to help students become more reflective can achieve improvements in metacognitive abilities. Again, Lester (1994) viewed such metacognition as the "driving force" in problem solving (p. 667).

5.2 Theoretical Framework for Teaching Problem Solving in Group Contexts

Mason et al. (1985) describes different phases the individual might go through when solving a problem, and also gives us useful labels for them. As an individual attempts to solve problems, the lines between phases may become somewhat blurred. For instance, a person in *Attack Phase* may discover that they did not fully understand what the problem was asking – an aspect of *Entry Phase*. Similarly, a person in *Review Phase* may discover his or her solution was only a partial one. People will often not follow a direct path to a solution, and so some grey area between problem solving phases should be expected. Nonetheless, *Entry Phase*, *Attack Phase*, and *Review Phase* are useful labels and allow Mason et al. to explore the processes of each phase, and how to deal with difficulties that arise.

In the classroom, I tried to structure group tasks such that the processes described by Mason et al. (1985) in each phase – particularly *Entry Phase* – could be explored. While *Entry Phase* remained the focus, many aspects of Mason's *Attack Phase* and *Review Phase* came about through the use the problem solving groups, whole-class discussions, and individual students' reflections. The following section compares and contrasts Mason et al.'s phases for the individual problem solver with that of problem solving groups.

Entry Phase

Mason et al. (1985) divides *Entry Phase* into a rubric consisting of *WHAT DO I KNOW?*, *WHAT DO I WANT?* and *WHAT CAN I INTRODUCE?* Further questions are given for the individual to ask herself, along with the instruction to specialize. With groups of students working together on the same problem, this set of processes was referred to as *Understanding the Problem* (as it was for Polya (1945)). Both Mason et al. and Polya encourage the individual problem solver to spend time carefully reading the problem. In a group context, reading the problem was facilitated by the group *Leader*, as was the discussion that followed. I believe there are several advantages brought on by the group setting:

1. The group structure helps ensure that the process of reading the question is done deliberately and carefully. Students are less likely to rush to solve a problem only

partially understood due to failing to adequately understand the problem. In a group context, understanding the problem generates considerable dialogue which also serves to demonstrate just how important this aspect of problem solving really is. Because the process is to help all group members to have some understanding of what the problem was about, more students are able to proceed with the problem than would be the case for students working in isolation.

2. Leadership skills and social skills are developed as students are given opportunity to participate as *Leaders* and *Problem Solvers.* Such skill improvement goes beyond those which any strategy designed for the individual might address.

3. Group discussion facilitates development of communication skills, both in terms of working with other members of the group socially and communicating mathematical ideas as well. Through the need for communication of ideas, students gain a greater understanding of their own ideas, and other students benefit from fellow group members' understandings.

4. Specializing in a group context has advantages over students doing so in isolation. One, students can try to do the same example and use each other to help check the logic of their ideas and to check calculations. Thus there are frequent opportunities for growth in understanding of mathematics that are perhaps tangential to the problem. Also, errors are resolved that might otherwise prevent students from continuing on with the problem.

   The second advantage is that students can pool their individual efforts to discover or confirm patterns and conjectures. This allows students to participate in the process of specializing while cutting down on the time required to produce a sufficient number of examples to make justified conjectures. This serves to meet the practical constraint of limited time in the classroom, and likely helps maintain students' interest in a process that could otherwise be very time consuming.

Attack Phase

According to Mason et al (1985), the *Attack Phase* is a stage filled with *Stuck!* and *Aha!* moments where the individual attempts to make and justify conjectures. As possible solutions arise, Mason et al. encourages the individual problem solver to develop an "internal enemy" to safeguard against partial or incorrect solutions. Many of the process of *Attack Phase* were seen as groups worked together on problems and, once again, the group setting proves beneficial:

1. In attempting to explain ideas to other group members, students can use verbal explanations, hand gestures, diagrams and different forms of representation in their calculations. Such a variety helps develop mathematical understanding and serves to show that similar ideas can be represented in multiple forms.

2. Students working together are able to build on each other's ideas, coming to understandings beyond what any of the individuals might achieve on their own. Moments of being stuck can be resolved by any member of the group. At times ideas are shared, at times students can give students time to make their own discoveries, and at times students can coach others to make discoveries. In cases where all group members are stuck, the experience is a shared one and students can work together to become unstuck. Thus the group context still allows students to experience being stuck and getting unstuck while reducing some of the stigma associated with being

stuck when working alone.

3. Both Mason et al.'s and Polya's models encourage development of the internal sceptic. In a group setting, group members are able to provide an external checker for other students' work. As students work together, this exemplifies to students how another person would regard their work. This might allow students to gain greater reliance on their own abilities to check the validity of conjectures and possible solutions.

Review Phase

The *Review Phase* of Mason et al. (1985) parallels Polya's (1945) final stage and consists of checking a problem's solution, reflecting on key moments, and extending the problem. Mason et al. is particularly concerned with reflection, which they believe is "probably the most important activity to carry out… REFLECTING on the key ideas and key moments intensifies the critical moments of an investigation and helps to integrate their resolution into your thinking repertoire" (p. 115). In this project, extending the problem as Mason et al. means it did not occur, although in a few cases problems were used to discover concepts that were expended on in subsequent classes. Reviewing key moments of the entire session was a priority, and time was taken at the end of each class to review key aspects of the problem solving experience – particularly how students initially engaged with the problem. This was generally done as a class discussion and provided an opportunity for students to learn more about what other groups had tried. While at times students were encouraged to discuss a reflection question as a group, generally written reflections were opportunities for the students to reflect individually on key moments.

Phases of Group Problem Solving

While many of the processes of Mason et al.'s (1985) general strategy were observed in the use of problem solving groups, the context of problem solving groups in the classroom suggests the following modified stages:

STAGE 1 – In Groups: *Understanding the Problem*

STAGE 2 – In Groups: *Trying to Find a Solution*

STAGE 3 – *Review:* Class Discussion & Individual Reflection

STAGE 1 – In Groups: *Understand the Problem*

Understand the Problem was such a central aspect in the classroom, and was so frequently viewed as the first step rather than a subset of *ENTRY PHASE* that it deserves to be acknowledged in its own right. *Understanding the Problem* provided a purpose for the groups while also suggesting the need for the structure provided by the roles of *Leader* and *Problem Solvers*. Without even becoming expert problem solvers, this stage provided students opportunities for success including seeing themselves as part of a team (both leadership and supportive roles) and learning about communicating, mathematics and mathematical thinking. And, as previously mentioned, the group context of attempting to understand the problem provides support to students during what is, to many, the most difficult aspect of problem solving.

STAGE 2 – In Groups: *Trying to find a solution*

While this parallels Mason et al.'s (1985) *Attack Phase*, it would also contain some elements of their *Entry Phase.* For example, Mason et al.'s *What Can I Introduce* and

Specializing often seemed to be more of an introductory concept to a secondary stage of working to solution rather than aspects of *Understanding the Problem*. At any rate, *Understanding the Problem* seemed to be a more clearly defined group process, and all the processes that took place after it would be better grouped under *Trying to Find a Solution*.

At times students were given problems which only stated a problem, at other times students were given structured problems with tasks to complete throughout the *Trying to Find a Solution* stage. As previously mentioned, this stage would sometimes have one or two tasks where students would write reflections. These were sometimes integrated as part of the written instructions, and sometimes given as verbal instructions.

STAGE 3 – *Review:* Class Discussion & Individual Reflection

The main aspect of the first two stages is that each of them involved *working in groups*. The main aspect of the third stage is that it is a *Review*, be it at individual, group or class levels. Typically, the *Review* usually took place toward the end of class. *Class Discussion* provided opportunities for individuals and groups to share ideas and learn from other groups. Following the *Class Discussion*, students were frequently given time for individual reflections in their *Problem Solving Journals*. I believe having such a structure for the final few minutes was very important in developing classroom culture and socio-mathematical norms. Such activities would remind students that they had participated as members of a group, members of an entire class, and ultimately as individuals responsible for themselves.

Comparing each of the stages used in the classroom with Mason et al.'s (1985) phases for the individual, the *Review* stages differ the most. For this project *Review* was much more focused on assessment, establishing culture, and individual accountability. The *Review Phase* as part of the general problem solving strategies usually involves checking solutions, seeking alternative solutions and extending the problem. Since groups are likely to encounter different methods of solving a problem, the *Class Discussion* can take care of checking multiple solutions, while extending the problem was generally beyond the scope of the study. Extending the problem seems likely to be an area that would require an extremely well-motivated group and, perhaps more significantly, the time to allow such exploration.

A key aspect of Mason et al.'s (1985) *Review Phase* is reflection. In the classroom context, reflections were eventually spread throughout the stages. An aspect of group problem solving that I significantly underestimated at the outset of the project was, ironically, how much group work would take place. Students' *Problem Solving Journals* did not necessarily document the details of the progress of a group trying to solve a problem. While I would still have students use *Problem Solving Journals* to help document their work and ideas, it is understood that the group collaboration of problem solving results in some ideas being lost. Journal entries (reflections) were thus used to help document student progress, to develop and support classroom expectations, and also house student reflections on key moments. Time was given during *Understanding the Problem, Trying to Solve the Problem* as well as during the *Review* for students to individually reflect.

5.3 Recommendations for Teachers

Below are some recommendations for teachers wishing to use a group structure to develop problem solving in their classrooms. Additions comments on what was used in the classroom during this project can be found in the appendices.

1. The roles of *Leader* and *Problem Solver* were highly effective in distributing and developing leadership and supporting roles, and providing the groups with some overall structure. While the role of *Recorder* is often used in group settings, it was not used in this project since all students were responsible for recording their own work and reflections in their *Problem Solving Journal*. This also meant that during the *Trying to Solve the Problem* stage that all group members had similar responsibilities to themselves and the group. Given the small size of the group, having a *Recorder* role might cause an imbalance to this unless they had minimal and specific responsibilities.

2. The *Problem Solving Journals* were quite useful as they helped keep students' work organized and together in one location. Perhaps more important than the showing of work, these journals were a location in which students could write reflections. An important discovery for me was that students' reflections often provided greater insight to their thinking and efforts than other written documentation of the work on the problem. This discovery led changes in the tasks of problems and how the *Student Journals* were used.

3. The use of reflections throughout the session, not just during the *Review* stage, meets a variety of purposes. During *Trying to Find a Solution* reflections can be particularly useful in helping a group that is stuck take stock of what they think they know, and where they want to go. Reflections during the *Review* stage allow students to reflect on moments of any part of the entire process, and can be used to point out to the teacher what were to them key moments.

4. The developments that took place during *Understanding the Problem* were the most important. Careful planning in the first few stages allowed the processes of this stage, the roles of *Leader* and *Problem Solver*, and the cooperation of group members to be

developed. Well defined tasks, particularly during *Understanding the Problem* helped introduce and develop the structure of the groups and encouraged a cooperative atmosphere which carried over into the next stages. While progress was often made in a variety of areas, each session had only a few particular goals so as to not overwhelm students. For example, specializing was introduced in a session where students worked individually *before* the session in which the group structure was introduced. In general, though improvements may have been made in several areas, each session generally had a limited focus to make greater progress in particular areas.

5. The stage of *Trying to Find a Solution* is sufficiently inexplicit and will no doubt be filled with numerous sub-stages or processes that students discover throughout the course. Determining or describing such processes at the outset of the year would be overwhelming and also would deny students from making the very discoveries that empowered them in this project. Instead, developing a class *Problem Solving Toolkit*, filled with processes as they are introduced by the teacher or discovered by students would have greater meaning. As per this project, *specializing* would be a good first entry into such a *Problem Solving Toolkit*.

6. The *Review* stage at the end of class usually consisted of a *Class Discussion* and a time for *Individual Reflection*. Both served to better assess student understanding, develop classroom culture and ensure individual accountability.

7. I believe that there are certain characteristics of problems that may make them more useful in developing problem solving ability (see Chapter 4). However, provided that problems have such characteristics, the use of any particular problem is somewhat arbitrary. Rather, it is the set of *tasks* a teacher gives with a problem that will develop a

problem solving culture in the classroom, and improve problem solving skills and attitudes of an individual. Determining how a problem is read, how group members are encouraged to interact and explain ideas, how and when students will reflect on key moments in their group interactions, their thinking and the problem itself are areas for which teachers can design tasks to meet their students' needs. While these tasks should not be so numerous as to take away from the challenges and ambiguity of the problem solving experience, tasks can provide structure and allow the experiences to have a more lasting impact.

8. An important characteristic of problems is there level of difficulty. The more difficult the problem, the more time is required to go through the stages. Time was always an unknown variable to me when selecting problems. While some problems were accurately gauged to require most of a class to go through all three stages, others required more or less time. A problem of sufficient difficulty will necessitate time spent communicating with other students, thinking mathematically, and dealing with being stuck. There were only a few sessions in which I came to the session with only one problem for students to work on; in most cases I had a second (shorter) problem ready to go in case students solved the first one more quickly than anticipated. While shorter problems could allow multiple exposures in a single class to the group structures and the problem solving stages, the prolonged attention required by more difficult problems led to the multiple areas of improvement in students' attitudes and abilities found in this study.

9. Problem solving in groups provides rich experiences that can simultaneously achieve multiple goals. Even when the focus was on *Understanding the Problem*, improvements occurred on a variety of levels. Similarly, when assessing student work, teachers can limit

the focus of assessment made during any particular problem solving session while still establishing standards and achieving improvements in a variety of areas.

An area of future exploration could be possible evaluative schemes for students' individual participation within a group and at a classroom level. Students were well aware that their problem solving journals were being evaluated, however in future classes I would attempt to use peer assessment and teacher observation more often in providing students with feedback in regard to their participation – that is to say, his or her problem solving behaviours in a group setting. I hope such assessments would show that not only written work by students is valued, but also their working towards a solution and the competency they show in their communication with their group members. The main goals of the K-12 Mathematics Curriculum in Manitoba identify communication skills as one of the goals for mathematics education in schools.

5.4 Teachers as Researchers

I started this project with a perceived need to improve the attitudes and abilities of my students in regard to problem solving. While I had taught mathematics for many years, my experience and training to that point seemed unable to provide me with the answers I needed to make significant improvements to my teaching practice. This led me to delve into existing research on the teaching of mathematics, the role of problem solving in mathematics education, problem solving behaviours and strategies, and the social context of problem solving. I learned a great deal from this research, however I was rather surprised to find, despite prominence of problem solving as a part of mathematics education, that there seemed little in the way of specific guidance to teachers as to how to intentionally go about improving attitudes and abilities in this important aspect of mathematics. Having failed to find satisfactory solutions in the works of others, I decided to learn what I could through a design-experiment approach in my own classroom.

While I had often reflected on the successes or failings of particular lessons or units I had taught, this project went beyond such analyses in almost every way. To start with, in attempting to improve my practice I was frequently considering the manner in which I teach or students learn specific content. Problem solving is something which transcends particular content, and thus the manner in which I approached the topic was fundamentally different from previous analyses. I had to examine almost every aspect of my teaching practice and what I knew about problem solving in order to even begin contemplating what I might do. Even by this time however, I had reached a point of acceptance that I would have to rely on myself to find the answers to the questions I had, and that this would require an on-going analysis at a much deeper level than I had previously undertaken. With problem solving transcending specific content, I frequently found my examining of my teaching and my students' learning as a sort of meta-approach to teaching and learning. The scope of this was at times somewhat overwhelming however it was exciting and new to me as well. The newness was itself exciting and gave me an energy that helped sustain me throughout this project.

This project affected my teaching in ways that went far beyond any changes brought on by any other form of professional development. The inquiry into my own practice necessitated careful thought into the planning, implementation and analysis on many aspects of my teaching that would otherwise likely have gone unexplored. The project spanned a year-long course, with modifications continually being made to both how and what I planned, implemented, and reflected. This paralleled changes in how and what I assessed from students. Such changes were dynamic and always based on what was going on in the classroom. It was such an inquiry-based approach that allowed me to better understand the effects of my interactions with students and the activities I had designed to influence classroom culture and student learning. Without such a level of inquiry and analysis, I would have been unaware of many of the effects of my actions, or left with only partially-substantiated views. Thus, the decision to improve my practice through inquiry allowed me to have greater impact on student learning and for me to be more aware of the nature of such impacts.

For teachers wishing to make inquiries into their own teaching practice, I have some recommendations. First, a personal journal is a key data source for any teacher-researcher. The *Teacher-Researcher Journal* that I developed in this project was excellent for documenting my intentions, observations, and immediate analyses. The format of the journal evolved over time; however, even the changing format itself allows one to become more aware of how attentions shift over time. Writing observations as soon as possible following a class helped ensure that as many observations and initial ideas were recorded as possible and provided data for later more in-depth analysis. Doing so on a session-by-session basis allowed me to plan sessions based on previous ones. Perhaps more importantly, examination of my journal allowed to me to see trends over several cycles. It was this feature which helped me to realize where more significant changes needed to be made. In particular, analyzing my journal entries helped develop my thinking as to how both reflection questions and *Student Journals* would be used. This resulted in a different outlook on reflection questions (impacting the nature of tasks given to students) and on the purpose of *Student Journals* (impacting how students would be assessed).

As the purpose of such inquiry is to learn how to improve one's practice, I recommend to teachers to be ready to make changes when a need for change is perceived. This is perhaps much more easily said than done. For me, this meant an increasing reliance on my direct observation of students and lessening my reliance on the written works of students. This change also resulted in a change in focus for written work, where students' reflections gained in importance while the importance of students' actual work on problems lessened. Such a change did not come easily for me; a considerable amount of time was spent contemplating how to make students' work coincide with my expectations when I should have been altering my expectations based on how the students were actually working. I wanted improvements in student learning, and while such improvements were being made, I was focused on a demonstration of such learning, which did not fit what was going on in the classroom. It was only when I accepted this that I was able to change my expectations and shift my focus that the tasks I designed and the work of students better matched my intentions.

Something that worked very well for this project was keeping the focus rather limited despite an overall process that could be rather all-encompassing. Though the project was about improving problem solving, understanding the problem remained a focus throughout. I believe this to be significant for teachers who may wish to inquire into aspects of their practice that may be quite large in scope. A smaller focus within some topic may not only allow for the inquiry to become more manageable, but for significant gains to be made in a variety of areas within that topic despite continued focus on one aspect.

Aside from limiting the focus of the inquiry, the number of data sources used was also limited. The nature of group work requires much interaction between group members. If audio or video recordings are not made, most of such interactions will be lost unless directly observed by the teacher-researcher. I did not use audio or video recording devices. While I can say that I wish I did have some of the problem solving sessions recorded, I believe that the amount of data that would have been collected would have become unmanageable. The ratio of benefits for my teaching practice to hours spent analyzing would have become far too small. Instead, I accepted that some great data would be lost and I tried to make the most of my observations and the written data I collected from students. I believe that this decision made my project manageable, and perhaps more significantly, this reflects the general circumstances of my teaching practice. I never record my classes and I usually rely on my own observations for understanding what is taking place in my classroom, and to make immediate and longer-term classroom decisions. The decision to not make audio or video recordings required me to improve how I structure lessons, how I use observation, and how I attempt to structure and assess students' written work. Thus my primary data sources were those that I will continue to routinely use beyond the scope of this project and are the ones that I now use more effectively as a result of the project.

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Appendices

A-0 Mason's Problem Solving Strategy

Comments:

I gave the following handout, which shows a modified version of Mason's general strategy, to students at the beginning of the project to give the impression that problem solving was something that could be viewed as being at least somewhat structured. I felt this was important to do at the outset of the project because I wanted the students to know that there was some structure to what we were going to be doing and that we had some goals to reach throughout the course – as opposed to leaving students with the impression that I expected them to improve on problem solving by me giving them difficult problems with little or no guidance. I think that acknowledging some overall structure while emphasizing how we would break down the problem solving phases into manageable chunks allowed a greater number of students to feel comfortable right from the beginning of our exploration of problem solving.

A-0 Mason's Problem Solving Strategy

Key Processes (p24)

SPECIALIZING
- randomly
- systematically
- artfully

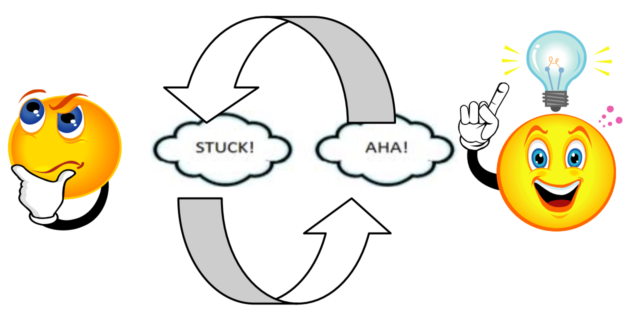
GENERALIZING
- what seems likely
- why it seems likely
- where it seems likely

Problem Solving Strategy (p29)

1) Entry Phase
- What do I KNOW?
- What do I WANT?
- What can I INTRODUCE?

Pictures
Diagrams
Graphs
Variables
Equations

2) Attack Phase
- **Stuck!**
  - Specialize
  - What do you KNOW? Are the conditions clear?
  - What do you WANT?
  - What can you INTRODUCE?
  - Describe WHAT YOU NEED to continue.
- **Aha!**



3) Review Phase
- CHECK the resolution
- REFLECT on the key ideas and key moments
- EXTEND to a wider context

Taken from John Mason's *Thinking Mathematically*

A-1 Introductory Survey

Comments:

Being asked to complete such a survey as the one that follows – particularly at the beginning of the course – was a new experience for most of the students. Students took it seriously and many were quite reflective on their attitudes and abilities. While these were compared with the *Exit Surveys* to see how attitudes had changed as a result of the project, the purpose of these surveys was primarily for me to learn more about the people I was going to be teaching. I did gain some insights into my students' attitudes toward their abilities from the survey, and I believe this survey was one of many things that contributed to a positive classroom culture from early on in the year.

A-1 Introductory Survey (cont'd)

Name: __________________________ Course: ___________________ Date: ___________________

Intro Math Survey

This survey is to help me learn a little bit more about you and your attitudes and beliefs on math, problem solving and just being in a math class. There aren't any "correct" answers, but please answer these as honestly as you can.

Part A

1. What topics in math do you like the most? Why?

2. What math topics do you like the least? Why?

3. What would you say are your strengths in math? Can you explain why these areas are strengths, or how these areas came to be strengths?

4. What would you say are your weaker areas in math? Can you explain why or how these areas came to be difficult for you?

5. Why are you taking this particular course?

6. What are your goals for this math course? (You may wish to comment on homework completion, attendance, understanding, marks, etc.) How do you plan to achieve them?

A-1 Introductory Survey (cont'd)

Name: ________________________ Course: __________________ Date: __________________

Survey on Problem Solving

1. What does the word "mathematics" mean to you?

____

____

____

____

2. What is "problem solving" to you?

____

____

____

____

3. How do you feel about your ability to solve problems?

____

____

____

____

4. How do you feel about offering ideas in your group? In front of the class?

____

____

____

____

5. If you had to solve a problem you've never seen before, what strategies might you try to solve it?

____

____

____

____

6. When trying to solve a problem, what do you look for?

____

____

____

____

A-1 Introductory Survey (cont'd)

7. How would you rate your enjoyment of previous math classes?

   1 = I hated them!   10 = I loved them!

   0 1 2 3 4 5 6 7 8 9 10

8. How would you rate your enjoyment of mathematics?

   1 = I hate it!   10 = I love it!

   0 1 2 3 4 5 6 7 8 9 10

9. How do you feel about your ability to solve problems?

   1 = very little ability   10 = Very strong ability

   0 1 2 3 4 5 6 7 8 9 10

10. How do you feel about offering ideas in your group?

    1 = Uncomfortable   10 = Very Comfortable

    0 1 2 3 4 5 6 7 8 9 10

11. How do you feel about offering ideas in front of the class?

    1 = Uncomfortable   10 = Very Comfortable

    0 1 2 3 4 5 6 7 8 9 10

12. When encountering a problem you've never seen before, which of the following are things you consider yourself most likely to do?
    - ☐ Ask a friend what to do.
    - ☐ Look up in a text book what you're supposed to do.
    - ☐ Try to solve the problem on your own.
    - ☐ Try to solve the problem with a partner.
    - ☐ Wait for the teacher to explain the problem.

A-2 Working Together Brainstorm

Comments:

Before students worked together in groups, I wanted students to feel comfortable and be well aware of behavioural expectations. Students had time to work on this individually and then in partners prior to a class discussion. This had several benefits. Students enjoyed the activity – particularly the question asking what students should avoid doing – while managing to generate all the expectations that I held as the classroom teacher. Thus students had an opportunity to create the rules for the classroom and to understand a code of conduct that would be in place while they worked together with other students. Some phrases that came up in the discussion about how to respectfully disagree with the students were actually used by students during the problem solving sessions. The brainstorm and follow up discussion served well to further establish classroom culture and is a precursor to group work that I would use again.

A-2 Working Together Brainstorm (cont'd)

Working Together Brainstorm

Name: _______________ Group #____
Date: _______________

Explaining one's thinking can be very difficult, especially mathematical thinking. The goal in working as a group is to have all members feel that they can contribute to the group and also seek clarification from other group members. This brainstorm is to help you see that there are many options available to us when we try to explain our mathematical thinking. It is also to help make it clear that there are things you can say and do to work well with people – even those you that you might not agree with.

1. What are some ways group members can try to explain themselves?

□ □
□ □
□ □
□ □
□ □

2. What expressions can a group member use to ask for help / clarification?

□ □
□ □
□ □
□ □
□ □

3. What can group members say or do to disagree in a respectful manner?

□ □
□ □
□ □
□ □
□ □

4. What should group members try to avoid when disagreeing with a group member?

□ □
□ □

A-3 Handout: Entry Phase & Group Roles

Comments:

The following is a handout given to students prior to the first group session. It lists Mason's phases, and gives particular emphasis to aspects of *Entry Phase*. This sheet introduced the roles of *Leader* and *Problem Solver*, and outlines their roles. Of particular importance is the list of responsibilities of the *Leader* as the group attempts to understand the problem. This is the one aspect of the problem solving group structure that would remain consistent in all subsequent sessions. This structure likely had the greatest impact on establishing the roles of the *Leader* and *Problem Solver* while also developing a positive atmosphere within the groups.

Because of its anticipated importance, all students received a copy of this handout, and we went through it as a class. I wanted to be sure to emphasize at this early stage how important it was for students to work together, and to support their *Leader* in doing his or her job of helping all group members understand the problem. It was emphasized that all students would have turns as *Leader* in future sessions. We discussed some ideas from the *Working Together Brainstorm* and it was discussed how building a cooperative atmosphere right from the start was important. Such a deliberate approach to the group structure and behavioural expectations resulted in a successful introduction to the group structure and positive interactions between group members during the introductory sessions.

A-3 Handout: Entry Phase & Group Roles (cont'd)

Focus: Entry Phase

Problem Solving Strategy (Simplified Version)

1) ENTRY PHASE
2) ATTACK PHASE
3) REVIEW PHASE

Special Focus: Entry Phase

- Goals of Entry Phase
  - To better understand what the problem is / what the question is asking.
  - To be able to answer the following questions in relation to the problem:
    - What do I KNOW?
    - What do I WANT?
    - What can I INTRODUCE?

Last class, we discussed some different options people can use when trying to start a problem. Refer to those notes to help you to answer the What do I KNOW/WANT/ can INTRODUCE?

Group Roles

GROUP LEADER

- Throughout the PROBLEM SOLVING STRATEGY, a primary goal of the leader will be to FACILITATE DISCUSSION.
- It is not necessary for the leader to be able to answer everyone's questions.
- Instead, the leader facilitates discussion by calling on individuals to help explain their thoughts or what they are doing.

- TODAY: The role of the leader today will be to help people focus on ENTRY PHASE. This can be done by...
  1. Having group members read the problem. This can be done individually or as a group.
  2. By giving people some time to try to understand on their own what the problem is asking.
  3. Discussing with the group what they think the problem is asking.

- The above tasks can be done in a variety ways, however it is important for the leader to help ensure that by the end of this process, ALL group members feel as if they understand what the problem is asking.
- From here, the group can continue by working to solve the problem.

PROBLEM SOLVERS

- All members of the group are PROBLEM SOLVERS, including the LEADER.
- The goals of PROBLEM SOVLERS include:
  1) Allowing the LEADER to facilitate the discussion by speaking / listening / assisting as required.
  2) Assisting others in solving the problem.
  3) Trying to solve the problem on your own.

A-4 The Problems

A4-#1a Specializing: Slope & Square Roots

A4-#1b Specializing: roots – tax – paper clip

A4-#2 Slope & Equation of a Line

A4-#3 Blood & Water

A4-#4 Toto Clone Puzzle

A4-#5 Are They Linear?

A4-#6 Factored Form Patterns

A4-#7 Graphing Factored Form Equations

A4-#8 Consecutive Numbers

A4-#9 How tall is the flag pole?

A4-#10 Chamber if Death & Despair

A4-#11 Cavern Number Patterns

A4-#1a Specializing: Slope & Square Roots

Problem Solving – Starting a Problem #1 Name: ________________________

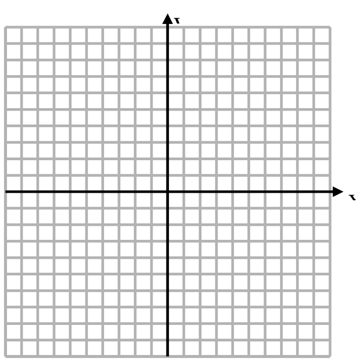
Specializing is a step during problem solving in which you simply try numbers / starting values / examples in an equation or problem to see what happens. The purpose is not to come up with "the answer", but to learn more about what is going on. This is hopes that through trying out different values, you can better understand the problem. If you don't know what numbers / examples to try – just pick one and see what happens!

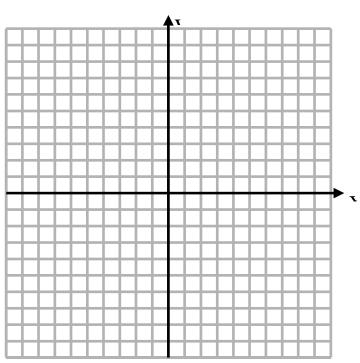
Problem 1: Determining Slope

If you were given any two points on a linear function, could you determine the slope of the line?

a) Specializing: What would specializing mean for this problem?

b) Do it! Specialize using a few examples, and see if you can determine the slope of a line. Can you determine other ways of calculating slope?





#1a Specializing: Slope & Square Roots (cont'd)

Problem 2: Square Roots

Here are a couple of equations. Are they always, sometimes, or never true?

i) $\sqrt{a*b} = \sqrt{a}*\sqrt{b}$ ii) $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$

a) Specializing: What would specializing mean for this problem?

b) Do it! Specialize using a few examples, and see if you can determine if the expressions are always, sometimes, or never true.

A4-#1b Specializing: roots – tax – paper clip

Problem Solving: Starting a Problem Name: ________________

Specializing is when you choose some initial conditions (starting values, an example) to see what happens to a particular equation or problem. If you don't have any starting values, just make some up!

1) With our problem where you had to figure out how to determine slope, what would the specializing have been?

2) With our problems of determining if $\sqrt{a*b}=\sqrt{a}*\sqrt{b}$ and $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ were true statements, what would the specializing have been?

3) Discount & Tax Problem: In a warehouse you obtain a 20% discount but you must pay a 15% sales tax. Which would you prefer to have calculated first: the discount or the tax?

   a) What would specializing mean for this problem?

   b) Specialize: Do it! Try three examples.

A4-#1b Specializing: roots – tax – paper clip (cont'd)

4) Paper Strip Problem: Imagine a long thin strip of paper stretched out in front of you, left to right. Imagine taking the ends in your hands and placing the right-hand end on top of the left. Now press the strip flat so that it is folded in half and has a crease. Repeat the whole operation on the new strip two more times. How many creases are there? How many creases will there be if the operation is repeated 10 times in total?

a) What would specializing mean for this problem?

b) Can you do this question without actually folding a piece of paper? Could you draw what is going on? Explain.

c) Specialize: Determine the number of creases after each fold. (Do it for several, but not all ten!)

A4-#2 Slope & Equation of a Line

Introduction to Problem Solving
Slope &The Equation of a Line Name: ______________________

- Note: You are NOT allowed to consult your textbook while attempting to solve the following problems. You may refer to your notes.

- All group members are to attempt to solve the problem and to try to work together as a group. All group members are to record their own work.

- Do not erase mistakes or false starts – just stroke them out and keep on going. These false starts will help document your attempt to solve a problem, and can be valuable learning experiences.

- The goal for your group is to solve the following problems.
- These problems should be solved in order.

Problem 1: How do you determine the slope of a straight line running through two points?

a) *Specialize:* Pick a couple of points, and try to determine the slope of a line running through them. What points make this question easier? More difficult?

b) *Generalize:* Can you determine the slope of a straight line passing through *any* two points? Determine the slope of a line passing through points (a, b) and
(c, d).

Problem 2: If you are given the slope of a line, and how far you are to move left or right from a point on that line, can you determine how far up or down you must move? (i.e. If the slope of a line is 2)

Problem 3: Can you determine the equation of a straight line passing through 2 points?

a) *Specialize:* Pick a couple of points, and try to determine the equation of a line running through them. What points make this question easier? More difficult?

b) *Generalize:* Can you determine the equation of a straight line passing through *any* two points? Determine the equation of a line passing through points (a, b) and
(c, d).

A4-#3 Blood & Water

Problem 78: Blood and Water from Pickover's (2002) "The Mathematics of Oz."

Dr. Oz presents Dorothy a problem to solve. There are two goblets, one filled with blood, the other water. A teaspoon of liquid from one is then added to the other. A teaspoon of liquid is then taken from the second goblet and placed in the first. Dorothy's task is to determine which goblet is more contaminated.

A4-#4 Toto Clone Puzzle

Problem 27: Toto Clone Puzzle from Pickover's (2002) "The Mathematics of Oz."

Dr. Oz presents Dorothy a problem to solve. She is asked how three Toto clones could be arranged so that they are all equidistant from each other. On solving this problem, she is presented with the problem of arranging four Toto clones so that they are all equidistant from each other.

A4-#5 Are They Linear?

Problem – Table of Values – Are They Linear? Name: ___________________

Main Problem:

If you are given a table of values, can you determine if it is a linear function?

Sub-Problems:

The following are questions you should think about while solving the main problem. You are to EXPLAIN the answers to each of these questions either as you go or after you have investigated the problem.

- How many ways can your group come up with to do this? **Explain** each method.
- How many points are need to determine if a problem is linear? **Explain** your reasoning.
- If the function IS linear, can you determine the equation for the line? How?

Specializing:

- Here are 3 examples.
  BE SURE TO SPECIALIZE ON YOUR OWN AND COME UP WITH OTHER TABLES!

Table 1

X	y
-2	-3
-1	1
0	5
1	9
2	13

Table 2

X	y
4	-4
6	-10
8	-16
10	-22

Table 3

x	y
5	7.25
12	9
14	10

A4-#6 Factored Form Patterns

Binomials & Trinomials: Factored Form Patterns

- Recall: **Factors** are numbers that go (evenly) into a number.
- i.e. 3 and 4 are factors of 12.
- (3)(4) is a factored form of 12.
- (-24)(-1/2) is also a factored from (even though both aren't integers).

The Problem: To determine patterns between factored form and expanded form.

- For factored form examples, keep the first term as x, and fill in the blanks with whatever numbers you wish. Determine the expanded form for your examples, and simplify.

  Example:
  Factored form Expanded from
  $(x + \underline{\ 3\ })(x + \underline{\ 7\ }) =$ ____________________

Goal 1: To go from factored form to expanded form quickly based on patterns.

Goal 2: To go from expanded form to factored form. You will be given expanded form expressions, and your job will to bring them back to factored form. Start with your examples and see if you can figure out how to go from expanded back to factored form.

A4-#7 Graphing Factored Form Equations

Graphing Factored Form Equations *Name: ___________________*

1. You have already graphed linear equations in slope-intercept form ($y = mx + b$). Write down the characteristics of the graph of such equations, and how the constants of the equation affect the graph.

2. Today you will be working with a new type of function which has its own special characteristics. As with slope-intercept form, there will be certain relationships between the equations and the resulting graphs. It will be your job to find as many patterns as you can in the graphs, and between the graphs and the equations.

   Individual Submission:
   Your individual work will consist of your submission of your Problem Solving Journal showing all the equations and the graphs that you attempted, along with your personal explanation/thoughts about the patterns you have observed. This is *your* explanation, and does not need to necessarily reflect the findings of the group. Do not leave your explanations to the end as a summary, rather after every couple of graphs, comment on what you have learned or think might be going on.

   Group Submission:
   Your group product will be a poster explaining your group's discoveries and could include verbal explanations, calculations, graphs and equations. You will be given time for this next class.

3. The following equations are only *examples* of quadratic functions. You should also create your own equations, and graph them using a table of values. Remember, the purpose of this exercise is *not* necessarily to draw these particular equations but rather to *understand the relationships* between the equations and graphs of *any* quadratic function in a similar form.

   a) $y = x^2$
   b) $y = (x+2)^2$
   c) $y = (x-3)^2$
   d) $y = (x+4)(x-2)$
   e) $y = (x+2)(x+8)$
   f) $y = (x-5)(x-3)$
   g) $y = x^2 + 3$
   h) $y = x^2 - 4$

   Make sure that in your booklet it is clear which table of values, equations and graphs go together.

A4-#8 Consecutive Numbers

Your Name: ____________________

Group Members: ____

Group Problem Solving: Consecutive Numbers

Leader: Your role today will be to lead a discussion with everyone about what the problem is. It is up to you to decide how you want to do this. Your role is to guide other group members and help everyone be part of the process and to feel included.

Problem Solvers (and Leader): Your role is to solve the problem, and to help others in your group to solve the problem. As with the Toto Clone Problem, part of your task is to help others without giving the problem away.

Your solutions today must be integers. If you solve for a variable and it is not a variable, you should look to see if you might have made a mistake.

The problems should be attempted in order.

Problem 1: Three consecutive numbers have a sum of 39.

a) Can you determine the numbers?
b) Can you determine these numbers algebraically? Show your work.

Explain how your method(s) work.

Problem 2: Even Numbers & Odd Numbers

a) Can you determine an equation using an integer *n* that will always return an even number?
b) Can you determine an equation using an integer *n* that will always return an odd number?

Explain how your method(s) work.

Problem 3: Consecutive Even Numbers & Odd Numbers

a) Three consecutive even numbers have a sum of 54. Determine the number algebraically.
b) Three consecutive odd numbers have a sum of 135. Determine the number algebraically.
c) There are three consecutive odd numbers such that the sum of the first, twice the second , and the third number is 5 less than 185.

Explain how your method(s) work.

A4-#9 How tall is the flag pole?

243 Raise the Flag

"How do you tell when you run out of invisible ink?"

Dr. Oz tentacled Dorothy a metre stick. "Here. You must determine the height of that flag pole."

"I don't want to climb a flag pole!"said Dorothy.

"Good! Because I've hooked it up to a powerful generator, and you will be electrocuted if you touch it."

"How am I supposed to determine the height of the flag pole?"

"Is there an echo in here? That is your problem. I want a thorough explanation of the process. Your last diagram looked like Toto drew it."

"Are you going to give me any help?" asked Dorothy.

"Do it while the sun is still up!" said Dr. Oz with a sly grin.

"Is that to help me?" asked a confused Dorothy.

"Yes. Do it while the sun is still up."

Difficulty Level: ✱✱

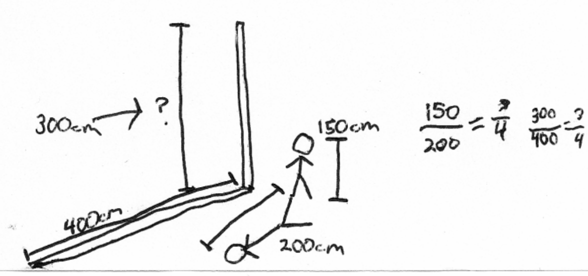
You measure the length of your shadow and compare it to your own height.

You measure the length of the shadow and compare it to the ratio of ~~the~~ $\frac{\text{h of human}}{\text{h of shadow}}$

ex. h of human = 150 cm
h of shadow = 200 cm

$\frac{3}{4}$ $\frac{300}{400}$ pole = 300cm

h of pole = ?
h of shadow = 400 cm



This was a problem I made to follow the style of Pickover's (2002) *The Mathematics of Oz*, and shows the work of one student to solve the problem.

A4-#10 Chamber if Death & Despair

Problem 32: The Chamber of Death and Despair from Pickover's (2002) "The Mathematics of Oz."

This is a spin on the classic Monty Hall problem. In this version, Dorothy must choose from one of three tunnels to walk down, however two of the three tunnels are coated with poison. On making a choice, Dr. Oz, who knows what path should be chosen, truthfully tells her which of the two tunnels she did not choose is poisoned. If Dorothy is given an option to change her choice, should she?

A4-#11 Cavern Number Patterns

Problem 79: Cavern Probem from Pickover's (2002) "The Mathematics of Oz."

Dorothy is given a table of numbers. Half of the cells are shaded, half are not. Her task it to determine which shaded number and which non-shaded number should be switched so that all shaded numbers have a common property and so all non-shaded numbers have a common property.

A-5 Exit Survey

Comments:

The *Exit Survey* that follows provided me with some very valuable insight into what impact the problem solving sessions had on students' attitudes and abilities. Some questions required written responses which allowed students an opportunity to express themselves in their own words, while some scale questions provided students with an easy way to describe how their attitudes had changed from the beginning of the year. I found this source of data to be an excellent summative data source on the impact the project had on students.

A-5 Exit Survey (cont'd)

9 Advanced Math **End of Year Reflections** Name: Anonymous

So the year is coming to a close.... Time to do a little reflecting.

Please provide some explanations to help me understand your thoughts and opinions.

1. What topics from this year did you like the most? Why?

___

___

___

___

2. What math topics do you like the least? Why?

___

___

___

___

3. In what areas do you think you've improved on the most over the year? What do you think caused this?

___

___

___

___

4. In what area(s) do you think you need to work on the most for next year?

___

___

___

5. At the beginning of the year you had some goals for this math course (homework completion, attendance, understanding, marks, etc.) These may have changed over the course of the year. How close would you say you got in achieving your goals?

___

___

___

___

A-5 Exit Survey (cont'd)

6. What are two things that **you did** that helped your learning this year?

7. What are two things **you could have done** that would have helped you learn better?

8. What things did **your teacher** do that helped your learning this year?

9. What are some things **your teacher could have done** that would have helped you learn better?

10. This year we had several problem-solving group sessions.

    In what ways do you think solving problems in groups is different from solving problems on your own?

11. What skills do you think you learned or improved on by working with other members of your group?

A-5 Exit Survey (cont'd)

12. What was the best part about working with other members of your group?

____

____

____

____

____

13. What was the hardest part about working with other members of your group?

____

____

____

____

____

14. Do you think your *attitude toward* problem solving or *confidence in your ability* to problem solve has changed over the year? In what way?

____

____

____

____

____

15. What affect, if any, do you think these problem solving sessions had on your *ability* to problem solve?

____

____

____

____

____

____

16. What affect, if any, do you think these problem solving sessions had on your homework / test performances?

____

____

____

____

____

A-5 Exit Survey (cont'd)

17. How did comfortable were you in offering ideas to your group at the *beginning* of the year?

    0 = not comfortable at all   10 = very comfortable

    0 1 2 3 4 5 6 7 8 9 10

18. How did comfortable were you in offering ideas to your group *now*?

    0 = not comfortable at all   10 = very comfortable

    0 1 2 3 4 5 6 7 8 9 10

    If you wish to add a comment, please do:

    ____

    ____

    ____

15. How would you rate your problem solving ability before this course?
    0 = very little ability   10 = Very strong ability

    0 1 2 3 4 5 6 7 8 9 10

16. How would you rate your problem solving ability now?
    0 = very little ability   10 = Very strong ability

    0 1 2 3 4 5 6 7 8 9 10

17. How would you rate your enjoyment of *previous* math courses?

    0 = I hated them!   10 = I loved them!

    0 1 2 3 4 5 6 7 8 9 10

18. How would you rate your enjoyment of *this* math course?

    0 1 2 3 4 5 6 7 8 9 10

    0 = I hate it!   10 = I loved it!

18. What do you think was the most important thing you learned this year?

    ____

    ____

    ____

    ____

19. Anything else you'd like to pass along:

    ____

    ____

    ____

    ____

Thank you for taking the time to fill this out!
Mr. McIntosh

A-6 Parent Consent Forms

Dear Parents/Guardians:

Your son or daughter was a student of mine in Grade 9 Advanced Mathematics. During the course there were times where students attempted to solve problems as a part of small groups. I am currently working on a project centered around how students can improve problem solving attitudes and abilities by working with each other in small groups. This research is part of the requirements for a Master of Education degree through the Department of Education: Curriculum and Instruction at the University of Manitoba. It is being conducted under the supervision of Dr. Thomas Falkenberg. You may contact my advisor at (xxx) xxx-xxxx or xxxxx@xxxxx regarding this study.

The purpose of this research project is to investigate the development of problem solving attitudes and abilities through participation in problem solving groups. Problem solving is an important strand throughout Manitoba's mathematics curricula. Manitoba Education, Citizenship and Youth states its tops three goals for students in mathematics education are that students should:

- use mathematics confidently to solve problems
- exhibit a positive attitude toward mathematics
- communicate mathematically

In the written report of this project, my thesis, I intend to include examples of students' writings and problem solving, along with observations that I collect when watching the students during instructional times, and my own personal reflections. The collection of anecdotal observations of children's learning is a regular part of ongoing assessment in the classroom and no additional tests or tasks beyond what is usually done for the purposes of assessment and reporting have been conducted. The University and Winnipeg School Division require that permission be sought for the use of any information for the purposes of research. Therefore, I am requesting your and your child's permission to include selected examples of comments, observations, writing samples, and interpretations that I have collected in the classroom in my Masters Thesis Report.

In accordance with the University of Manitoba's standards for ethical research, the identities of all students will be protected. Any examples of student work, responses, or comments used will be anonymous and pseudonyms will be used for the children and the school in the written report. All of my observational notes, personal reflections, and other data will be kept secure in a locked cabinet in my classroom. At the conclusion of the research, no one other than me will have access to any information which includes the identity of any participating students.

Your permission for your child's works to be used as data in this research project must be voluntary and I want to assure you that there are no consequences that arise from giving or withholding your permission. The course is coming to a close, and the data to be used as a part of the project arose naturally from routine classroom instruction. In order to alleviate any pressure you might feel because I am your child's teacher, I have asked that all returned consent forms be returned in a sealed envelope to either Mrs. Tran or to myself. The names of students participating in the study will not be revealed to me until the conclusion of the semester. I have also informed the school principal, Mr. Yale Chochinov, and Winnipeg School Division's Director of Research, Planning and Systems Management, Doug Edmond, of my intended research, which they have granted permission for me to complete. Should you feel that there are pressures or unanticipated consequences as a result of participating or not, you are free to contact my research advisor, Dr. Thomas Falkenberg (Phone: xxx-xxxx or e-mail xxxxx@xxxxx, or the human ethics secretariat at the University of Manitoba, Mrs. Margaret Bowman (Phone: xxx-xxxx or email xxxxx@xxxxx) to have your concerns addressed. If you decide to withdraw your consent you are free to do so by July 30th by emailing me at xxxxx@xxxxx. If permission is not given or is withdrawn, no work samples, observations, questionnaire comments, or my own personal reflections regarding your child will be used in my thesis report.

There are no known or anticipated risks to your child associated with giving consent for information to be used in my research study, and as mentioned, the course is already coming to a close. Potential benefits for students include improving problem solving attitudes and abilities, improving communication and mathematical thinking, improving mathematical dialogue between students, and increasing communication between students and teachers. This research may also benefit my own professional practice and provide information on problem solving groups for other teachers. A summarized version of the results of my completed thesis can be made available, should you be interested.

I will be available at your convenience to answer any questions you may have. I may be reached at the school (xxx-xxxx) or via e-mail xxxxx@xxxxx). In addition to contacting me or my supervisor, you may verify the ethical approval of this study or raise any concerns you might have by contacting the human ethics secretariat at the University of Manitoba (xxx-xxxx or xxxxx@xxxxx).

Please discuss this letter with your child and determine whether he or she agrees to give consent Your signatures below, yours and your child's, indicate that you understand the above conditions of participation in this study and agree to allow your child to participate. You are free to withhold permission without prejudice or consequence. Please return the YELLOW COPY of the signed consent form in the attached envelope directly to Mrs. Tran or myself and keep the other for your records. Thank you for your time and consideration.

Sincerely,

Mr. Blaine McIntosh

Grant Park High School

Consent Form – Researcher Copy

Please check one of the following:

□ I give my consent for **anonymous** examples of my child's classroom work to be included in Blaine McIntosh's Masters Thesis for the Department of Education: Curriculum and Instruction, at the University of Manitoba.

□ I do not give my consent.

________________________ ______________________________ _______________

Name of Participant's Parent/Guardian Signature Date

Please talk about this with your child and if they consent, have them sign the form

I have asked my child, ___________________, who has indicated consent to have anonymous examples based on their work used in Blaine McIntosh's Masters Thesis for the Department of Education: Curriculum and Instruction, at the University of Manitoba.

________________________ ______________________________ _______________

Name of Participating Student Student's Signature Date

________________________ _______________

Researcher's Signature Date

Please return to Mrs. Tran or Mr. McIntosh.

Thank you.

RESEARCHER COPY – PLEASE SEAL IN THE PROVIDED ENVELOPE AND RETURN TO

MRS. TRAN OR MR. MCINTOSH AT YOUR EARLIEST CONVENIENCE.

Consent Form – Personal Copy

Please check one of the following:

☐ I give my consent for **anonymous** examples of my child's classroom work to be included in Blaine McIntosh's Masters Thesis for the Department of Education: Curriculum and Instruction, at the University of Manitoba.

☐ I do not give my consent.

______________________	___________________________	_____________
Name of Participant's Parent/Guardian	Signature	Date

Please talk about this with your child and if they consent, have them sign the form

I have asked my child, ___________________, who has indicated consent to have anonymous examples based on their work used in Blaine McIntosh's Masters Thesis for the Department of Education: Curriculum and Instruction, at the University of Manitoba.

______________________	___________________________	_____________
Name of Participating Student	Student's Signature	Date

______________________	_____________
Researcher's Signature	Date

Please return to Mrs. Tran or Mr. McIntosh.

Thank you.

- -

This consent form, a copy of which will be left with you for your records and reference, is only part of the process of informed consent. It should give you the basic idea of what the research is about and what your participation will involve. If you would like more detail about something mentioned here, or information not included here, you should feel free to ask. Please take the time to read this carefully.

This research has been approved by the Winnipeg School Division and by the Education & Nursing Research Ethics Board (ENREB). If you have any concerns about this project you may contact any of the above-named persons or the Human Ethics Secretariat at xxx-xxxx, or e-mail xxxxx@xxxxx, or me, at xxxxx@xxxxx .

PERSONAL COPY - TO BE KEPT FOR YOUR RECORDS