Graph theory notes

for MATH 2070 and MATH 3370

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Contents

0.1 Preface

This manuscript began as a collection of notes made for the second course in graph theory, MATH 3370, at University of Manitoba. Basic graph theory concepts were then added, serving to both establish notation and to make some discussions self-contained. This manuscript now also contains a solid core of notes for the first course in graph theory, MATH 2070.

Even in the early chapters I give notes on topics that are appropriate for a second course in graph theory, or on areas that some students might want to know more about.

In an appendix (Chapter 19) are virtually all basic notations for sets and functions are given, along with all definitions used in a first course in graph theory. For the reader using this text as an advanced text, these definitions not only serve as one way to review their first graph theory course, but also to establish uniformity in notation, since various texts may differ slightly. For the first graph theory course students, this list of definitions might be useful as a study guide, as they might follow the order of topics in a first year course.

In many sections I give far more information than is needed for a first or second course in graph theory, so please do not be overwhelmed. I can only hope that some of these extras are at least interesting. Many famous algorithms are not covered in detail (although some basic ones and references are given)

This document has 552 exercises and nearly 100 pages of solutions Nearly 90% of exercises have solutions or hints. I have also added appendices on some basic mathematical tools, including some matrix theory.

An attempt has been made to reference all individual results (I have surely missed a few, though); there are over 1000 references. If a particular topic is not covered here, I recommend looking at some of the popular graph theory texts (*e.g.*, [94], [125], [143], [190], [255], [256], [429], [584], and [977]). All of these texts are written by leaders in the field, and between them, the knowledge contained therein is extensive. This text is not close to being comprehensive, and I apologize for omissions. On the other hand, some topics here are expanded upon so much that trying to cover all the details in lectures will use up far too many class-hours. So, both the instructor and the student need to be selective about topics covered and those that a student is responsible for. (Nearly every textbook I have seen on graph theory seems to stress a different collection of results or areas for research, and this textbook is no different—just to cover adequately all areas that can be taught in an elementary graph theory course might take a thousand pages.)

There is an extensive index—however, there is a peculiarity that I can not fix; namely, if an author uses special characters in their name, the index may make separate entries for the text and for the bibliography references. Items in the bibliography are back-referenced (with page numbers indicating at least one place the item was used in this document). The pdf file is cross-linked. Most of the spelling used here is

Canadian/British, except that I use "center", since "centre" just doesn't sit well with me.

0.2 Sample schedule of topics for first course

Each instructor has ideas on how to present the first course in graph theory. Here is one schedule that I followed for MATH 2070 in the W2020 semester given on Tuesdays and Thursdays. Some lectures deliver a number of topics, and such can be accomplished if slides are used and distributed (so students need not take detailed notes, enabling more content each lecture).

Lecture	Tentative topics (timing probably not precise)	Sections
L1	Course outline, intro, review notation in discrete math and MI, origins of graph theory, bridges of Königsberg	19.1.1 19.1.2, 1,
L2	Types of graph, multigraph, digraph, drawing graphs, definition of graph, vertices, edges, incidence, adjacency, degree, walks, trails, paths, cycles, circuits, K_n	1.2, 1.3
L3	Isomorphism	1.5
L4	Classes of graphs: bipartite, stars, trees, planar, polyhedral, multipartite, cube and hypercube, Petersen graph, intersection graphs, social networks, named graphs	1.6
L5	New graphs from old (subgraphs, complements, products, line graphs, deletion, subdivisions,	1.7
L6	Handshaking lemma, regular graphs, Havel–Hakimi	1.8
L7	Connected graphs, distance, computing distances in graphs (brief) Diameter, radius, eccentricity Graphs as matrices, adjacency and incidence matrices	1.9,1.10, 1.11, 1.12
L8	Cycles and circuits: Eulerian circuits, Fleury's algorithm, applications (*e.g.*, figure tracing)	2.1, 2.2
L9	Hamiltonian cycles, Dirac's theorem, Ore's theorem, Hamiltonian cycles in a cube graph, TSP	2.3, 2.4 2.4.2
L10	Gray codes, knight's tours, instant insanity	2.5, 2.6, 2.10
L11	Trees and forests, basic properties of trees, counting non-isomorphic trees, Cayley's formula, Prüfer sequences	3.1, 3.2
L12	Minimum spanning trees for weighted graphs, Prim's algorithm, Kruskal's theorem	3.3, 3.4
L13	Trees and bracing networks, chemical trees, review	3.5, 3.6
L14	Midterm, covering material up to MST	
L15	rooted trees, plane trees, binary trees, counting binary trees, parentheses	3.7, 3.7.2, 3.7.3, 3.7.6
L16	Connectivity, vertex-connectivity, edge-connectivity, Menger's theorem (proof optional)	4
L17	Planar graphs, face degrees, planar graphs and polyhedra, Euler's formula, duals, Fary's theorem	7.1, 7.2, 7.3, 7.4, 7.6
L18	Graph colouring, independent sets, independence number, chromatic number, greedy colouring, Brooks' theorem, Gaddam–Nordhaus theorem	6.1, 6.2
L19	Colouring planar graphs and the four colour theorem	7.7
L20,	Planar graphs cont'd, edge-colourings, Vizing's theorem	7.7, 6.11
L21	Ramsey's theorem (brief)	9.1, 9.2, 9.3, 9.4, 9.5, 9.9
L22	Extremal GT, Mantel's theorem,Turan's theorem	10.1, 10.2, 10.3.1,10.5
L23	Digraphs, De Bruin graphs, Eulerian circuits in digraphs, rotating drum problem, Flows (brief)	12.1, 12.2, 12.3, 12.4, 12.5, 5.11
L24	Strongly connected digraphs, orientable graphs, Robbins' theorem. tournaments, Hamiltonian paths and cycles in tournaments, kings	12.2, 12.7
L25	Last class, review	

I have since re-ordered the sections on planar graphs so that edge colouring is done first.

0.3 Possible topics for a second course in graph theory

I have taught MATH 3370 a number of times and the content differed slightly from year to year. The following is the schedule I used for the W2019 semester in MATH 3370 with 50 minute lectures. (I also added extra projects)

Brief review of elementary graph theory, including Eulerian trails, degree sequences, trees, connectivity, Hamiltonian cycles, the 4CT, planar graphs, Euler's formula, and basic definitions. (3–4 lectures)

Matchings and covers: Gallai's theorem, Hall's theorem, the König-Egerváry theorem, stable marriages, and related theorems (*e.g.*, by Birkhoff–Neuman, Berge, Tutte, Menger, Petersen, and Dilworth). (8 lectures)

Flows and networks: Ford–Fulkerson algorithm, integer flows, max-flow min-cut theorem, squaring the square. (2 lectures)

Extremal graph theory: Mantel's theorem, Edward's theorem, Turan's theorem, Zarankiewicz numbers. (2–3 lectures)

Graph colouring: vertex colourings, chromatic polynomial, Brook's theorem, edge colourings, Vizing's theorem, König's theorem, graceful tree conjecture. (3–4 lectures)

Perfect graphs. (2 lectures)

Cages, Moore graphs, friendship theorem. (2 lectures)

Basic Ramsey theory, including the Erdős-Szekeres recursion, bounds on Ramsey numbers. (3 lectures)

Algebraic graph theory, spectrum of a graph. (5–7 lectures)

Depending on time available or individual interests, other topics may be introduced (*e.g.*, geometric graphs, crossing numbers, infinite graphs, expanders, hypergraphs, probabilistic methods in graph theory).

This text also contains the roots of topics for many graduate courses in graph theory. Perhaps the next version of these notes will have a few chapters added on other recent directions in graph theory.

0.4 Acknowledgements

I thank the Department of Mathematics at University of Manitoba for letting me teach many first courses in graph theory and for letting me and my students develop a second course, as well as many graduate courses in graph theory.

Thanks to Michael Doob for many helpful conversations. Thanks also go to Andrii Arman, Michelle Davidson, Xiaohong Zhang, Jasmin Bissonnette, Megan Grant, Max Gutkin, Amanda Helliar, Matthew Markham, Maria Uehara, Colin Desmarais, Juliana Felix, Alex Penner, Krista Reimer, and Suraj Srivinasan, for comments on early versions of this document. Thanks also to comments from a number of students in the class the first time this text was used for MATH 2070, including Adrien Dinzey, Jared Gobin, Dana Kapoostinsky, Fengyi Liu, Taylor Roy, Benjamin Schneider. In the summer of 2013, Vanessa Reimer produced the code for drawings of some graphs here. Thanks also to Michael Szestopalow for helpful comments while using this text for his class.

My deepest thanks go to my wife, Karen Gunderson. I don't really know how to thank her enough. Some sections here were written by her, and she proofread many sections and provided wonderful feedback while using these notes in her class. Karen also provided help with Tikz (*e.g.*, diagrams for Section 2.10.6, Heawood's counterexample, and Figures 9.3 and 10.6). It was absolutely wonderful to be able to ask her questions at any time of the day or night and to always have her reply with gentleness and just the right answers. She also took beautiful care of me, even when she was ridiculously busy with her life. She refused being named a co-author of this version, but I hope she claims authorship soon (it could only improve the result!).

DSG, 2022

To my beloved Karen, Thor, and Christine

Chapter 1

Graph theory basics

1.1 Origins of graph theory

It is often said that "graph theory" began in 1736 with a problem regarding tours through what was then known as Königsberg (now Kaliningrad).

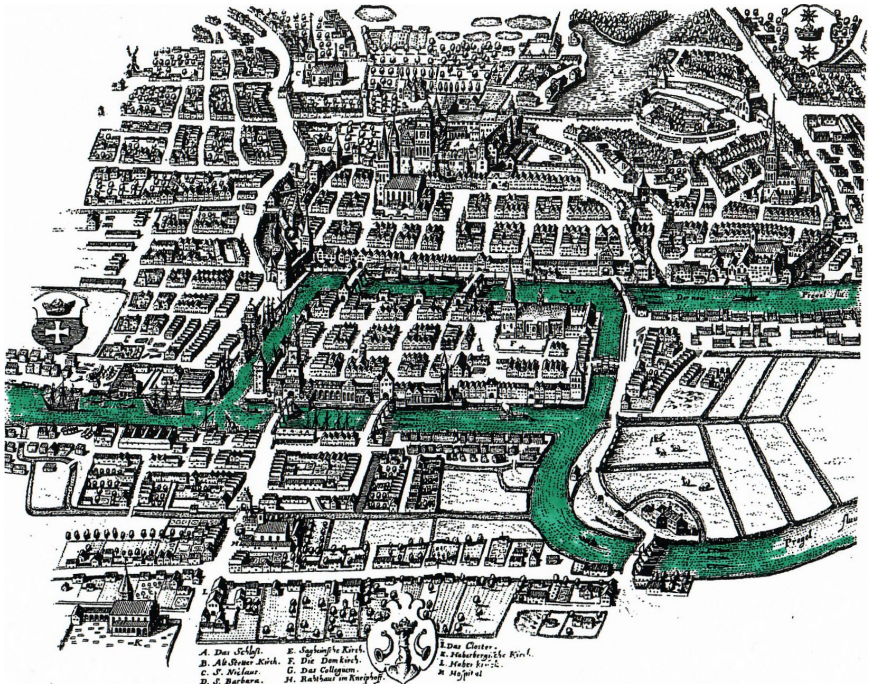


Figure 1.1: The seven bridges of Königsberg

As shown in Figure 1.1 there were seven bridges in the city that crossed the river Pregel, and the river divided the city into four land masses. (Figure 1.1 is from the British Museum, reproduced in [105], and with the river coloured by DSG; there are

two bridges from the center rectangle to the land mass above it, one of which is barely discernible.)

Problem 1.1.1. *Is it possible to go for a walk in Königsberg, leaving home and returning home, that crosses each of the seven bridges precisely once? Is there a walk that crosses all bridges precisely once where the walk finishes in a different location?*

In 1758, Leonhard Euler (1707–1783) (pronounced "Oiler") gave the answer "no" [349] to both questions in Problem 1.1.1. His method was to model the four land masses and the seven bridges by a schematic (that is now known as a "multigraph"). Let A, B, C, D represent the four land masses, and draw an arc between land masses for each bridge as in Figure 1.2.

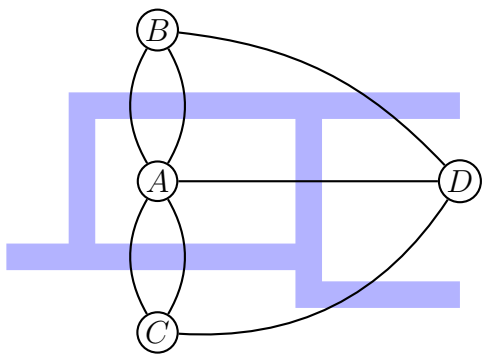


Figure 1.2: A schematic modelling of the seven bridges problem.

Euler argued that anyone entering and leaving an arbitrary land mass X (which is one of A, B, C or D) uses two of the "lines" from X. Hence, in any complete tour, an even number of "lines" is used for each land mass—but in this case, each land mass has an odd number of "lines" from it. So no complete tour is possible. Even if someone starts a tour at one mass and ends in a different mass, at most two masses will have an even number of lines to it. So his answer is that neither type of tour is possible.

Euler never used the words "graph" or "multigraph", but what Euler proved, in modern parlance, is that in order for any complete tour (using all edges and each edge is used only once) of a graph to exist, each vertex must have even degree, and if an open ended tour exists, at most two vertices have an odd number of edges touching them. However, in the multigraph created for the Königsberg bridge problem, each of the four vertices have an odd number of edges from it.

According to Chung and Sternberg [215], the first time that "graph" was used to describe the relational structure now known by that name is in an article about chemical structures by J. J. Sylvester in 1878:

> Every invariant and covariant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph. [896]

In the same article is the use of "valence", which is now called the "degree".

1.2 Informal introduction to graphs

In many areas of science and mathematics, the word "graph" is used in various contexts. The "graph of a function" $f(x) : \mathbb{R} \to \mathbb{R}$ is defined to be the set $\{(x, y) : x \in \mathbb{R}, y = f(x)\}$ of points in the plane. For example, the graph of the function f defined by $f(x) = x^2$ is the set of points in the Euclidean plane known as a parabola. In statistics, "Bar graphs" are also used to show distribution of data. "Graph paper" is used to denote pages with some kind of grid system upon which certain graphs can be drawn.

In the area of mathematics called "graph theory", yet another kind of "graph" is used, a kind of relational structure that consists of points called "vertices" (singular is "vertex"), and "edges" that record whether or not points are related. The set of vertices is often denoted by V and the collection of edges is often denoted by E.

For example, one might be interested in a network structure among which only some pairs of computers can communicate directly with each other. In this case, V is the set of computers, and E represents connections between computers. Another example of this new kind of "graph" can be a collection of places on the map and a drawing of roadways between certain locations. Just as in computer networks or traffic systems, sometimes there are multiple connections between two computers (or cities), and some of these connections might have direction (*e.g.*, computer A can send data to computer B, but not the other way around, or there may be one-way streets). In chemistry, molecules are often depicted as graphs whose vertices are the atoms and bonds are the edges.

In graph theory, structures are "discrete", in that points (or vertices) can be easily separated or distinguished, as opposed to "continuous", as in curves used for a graph of a function.

There are many types of "graphs", some with labels on vertices or edges, some with multiple edges between two vertices, or some with directions on the edges. Before giving precise definitions, see Figure 1.3 and 1.4 for examples of five kinds of "graphs".

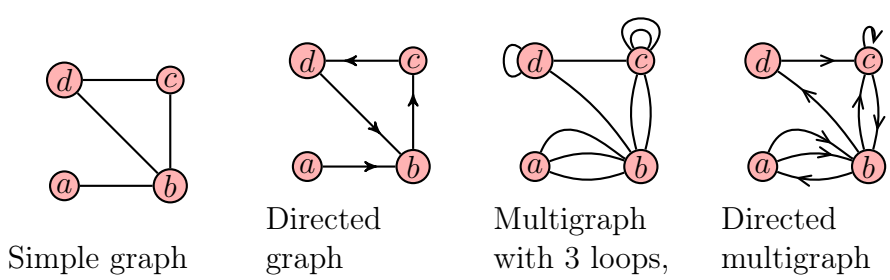


Figure 1.3: Various kinds of "graphs"

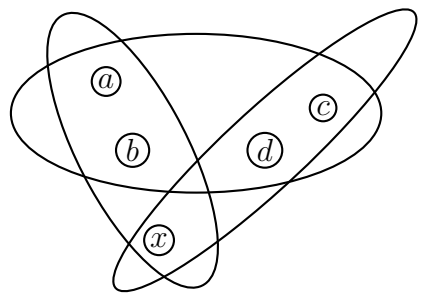


Figure 1.4: A hypergraph with hyperedges $\{a, b, c, d\}$, $\{c, d, x\}$, $\{a, b, x\}$

In Figure 1.5 are four drawings of the same graph on $V = \{a, b, c, d\}$ with $E = \{\{a, b\}, \{b, c\}, \{b, d\}, \{d, c\}\}$. Vertices can be represented by black dots (with the labels

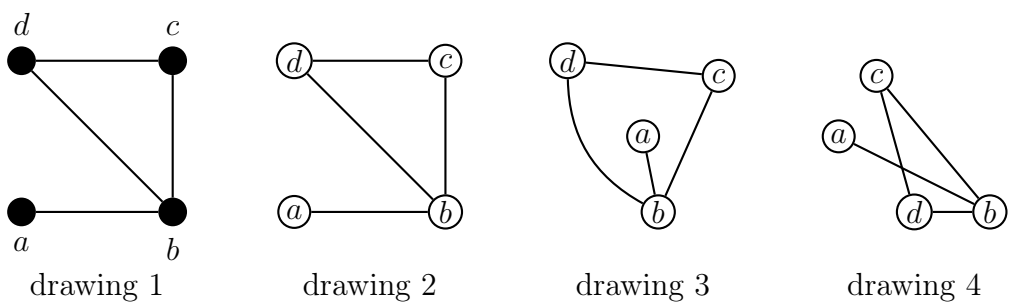


Figure 1.5: Four drawings of the same simple graph

nearby) or by small circles with the vertex labels inside. An advantage of using small circles for vertices is that in graphs with many vertices, sometimes there is no room for outside labels. The third drawing shows that the position of the vertices is irrelevant, and curved edges are allowed. The fourth drawing shows that if vertices are placed somewhat randomly, then edges may cross (at which point there is no vertex).

In graphs, digraphs, and multigraphs, each edge consists of two points (or in the case of a loop, just one point). In structures called "hypergraphs" a "hyperedge" can be any subset of the vertices, not restricting to just one or two vertices per hyperedge. If X is a set, let $\mathcal{P}(X)$ denote the power set of X, namely, the collection of all subsets of X.

Definition 1.2.1. *A hypergraph is an ordered pair $(X, \mathcal{E})$, where X is a non-empty set and $\mathcal{E} \subset \mathcal{P}(X)$. Elements of X are called vertices and elements of $\mathcal{E}$ are called hyperedges.*

A hypergraph $H = (X, \mathcal{H})$ is sometimes referred to as simply $\mathcal{H}$ (if there are no isolated vertices, the set of hyperedges determines the hypergraph). For any $r \in \mathbb{Z}^+$,

if all hyperedges in a hypergraph contain r vertices, then the hypergraph is called r-uniform. A 2-uniform hypergraph is a simple graph. Drawing hyperedges can often be done with ovals. For example, if the hypergraph in Figure 1.4 is denoted by $H = (V, \mathcal{H})$, then $V = \{a, b, c, d, x\}$ is the vertex set and the set of hyperedges $\mathcal{H} \subset \mathcal{P}(V)$ is given by $\mathcal{H} = \{\{a, b, c, d\}, \{a, b, x\}, \{c, d, x\}\}$.

As with ordinary graphs, for $k \in \mathbb{Z}^+$, a hypergraph $H = (V, \mathcal{H})$ is called k-partite if and only if there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ so that for any $i = 1, \ldots, k$ and any hyperedge $E \in \mathcal{H}$, $|V_i \cap E| \leq 1$.

The study of graph theory is now of central importance to many areas of science, including information flow, logistics, scheduling, chemical structure, data compression, crystallography, epidemiology, printed circuit design, CNC routers, 3D printing, and even structural engineering.

1.3 Basic concepts and definitions

The word "graph" has many interpretations and various definitions in the literature. The definition given here is often given as what is called a "simple graph". Other types of "graph" are discussed below.

Definition 1.3.1 (graph)**.** *A* graph *is an ordered pair $G = (V, E)$, where V is a non-empty set and E is a set of unordered pairs of distinct elements from V. An element of V is called a vertex and an element of E is called an* edge.

For a graph $G = (V, E)$, it is often convenient to write $V = V(G)$ and $E = E(G)$, a notation that is particularly useful when more than one graph is being discussed.

Two vertices x, y in a graph G are said to be *adjacent* if and only if $\{x, y\} \in E(G)$. An edge $e = \{x, y\}$ is said to be *incident* with both vertices x and y, and conversely, the vertices x and y are incident with the edge e.

A common notation helps to rewrite the definition of a simple graph. For any set S and positive integer k, define

$$[S]^k = \{T \subseteq S : |T| = k\}.$$

Using this notation, $[V]^2$ is the set of all 2-element subsets of V. So the definition of a (simple) graph can be rewritten:

Definition 1.3.2. *A graph is an ordered pair (V, E) such that V is a non-empty set and $E \subseteq [V]^2$.*

In notation used for sets, elements are not repeated. For example, the set $\{a, b, b\}$ is the same as the set $\{a, b\}$.; in a multi-set, repetitions are allowed. For example, the multi-set $\{a, b, b\}$ indeed has three elements.

Definition 1.3.3 (multigraph)**.** *A* multigraph *is an ordered pair $G = (V, E)$, where V is a non-empty set and E is a multiset of pairs (not necessarily distinct) of vertices in V.*

In the literature, a multigraph is sometimes called a pseudograph. An edge in a graph consists of a pair of *distinct* vertices, and in a graph, E is a set, no such pair of vertices is repeated in E. in a multigraph, a pair of the form $\{x, x\}$ is allowed, and is called a *loop*, which is usually drawn as a small loop at the vertex x. Also note that in Definition 1.3.1, edges form a set, and so no one edge is repeated; in multigraphs, a pair $\{x, y\}$ may be repeated in the list E of edges. In drawing multiple edges in a graph, roughly parallel lines are used. Letting H denote the multigraph in Figure 1.3, then $V(H) = \{a, b, c, d\}$ and

$$E(H) = \{\{a, b\}, \{a, b\}, \{a, b\}, \{b, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d, d\}, \{c, c\}, \{c, c\}\},$$

Exercise 1. *Show that the following is equivalent to Definition 1.3.1: A (simple) graph is a non-empty set V together with an irreflexive, symmetric binary relation on V.*

In some texts, the word "graph" denotes any of a varied collection of structures. For example, "graph" may be used to indicate a multigraph (with multiple edges and loops allowed), or a *weighted graph* (where "weights" on vertices and/or edges are given), or a *hypergraph* (where edges are allowed to have more than two vertices). So-called "directed graphs" have ordered pairs for edges (as opposed to the unordered pairs in Definition 1.3.1) and so in directed graphs, edges have a direction ("digraph" is an abbreviation for "directed graph"). In these notes, many theorems given for graphs are also true for multigraphs or hypergraphs. Chapter 12 is devoted to directed graphs and Chapter 13 is on hypergraphs.

Remark 1.3.4. *Since there are so many different kinds of "graph", one often identifies a graph as in Definition 1.3.1 as a* simple graph*.* **Unless otherwise stated, all graphs in this text are simple.**

Definition 1.3.5. *Let G be a graph or a multigraph and let $v \in V(G)$ be a vertex. The degree of v in G, denoted $\deg_G(v)$, is the number of edges in G that are incident with v, where loops count as two degrees.*

Definition 1.3.6. *A vertex in a graph G is called* isolated *if and only if $\deg_G(x) = 0$.*

In general, a vertex in a graph, multigraph, hypergraph, or digraph is called isolated if and only if the vertex is not contained in any edge or hyperedge.

When discussing degrees in only one graph, the subscript G can be omitted, using the simpler notation $\deg(v)$. In the first graph in Figure 1.3, $\deg(a) = 1$, $\deg(b) = 3$, $\deg(c) = 2$ and $\deg(d) = 2$. In the multigraph in Figure 1.3, $\deg(a) = 3$, $\deg(b) = 6$,

$\deg(c) = 7$, and $\deg(d) = 4$. In Section 1.8, degrees in a graph are studied in more detail.

When drawing graphs, especially unlabelled ones, vertices are represented by dots (or small circles) and edges are represented by lines or arcs (or other curves) connecting the dots; it is often necessary to cross lines and at such a crossing, no vertex is drawn.

The *order* of a graph G is the number of vertices, and the *size* is the number of edges. Often a graph order p and size q is called a (p, q)-graph (however convenient this usage is, it is seldom used here). The terms "order" and "size" are not employed so often here because these words have meanings in so many different contexts. (For example, the phrase "size of a vertex set" indicates cardinality, nothing to do with the number of edges.)

Exercise 2. *Prove that any (simple) graph G on n vertices has at most $\binom{n}{2}$ edges.*

The graph on n vertices with all possible $\binom{n}{2}$ edges is denoted by K_n, called the *complete* graph on n vertices (see Figure 1.6).

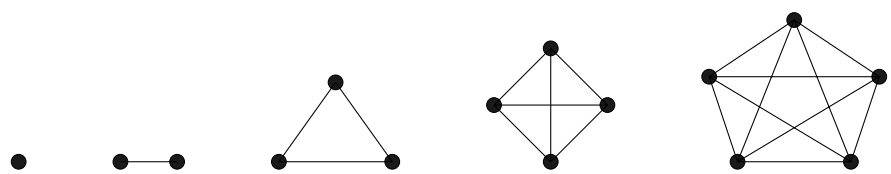


Figure 1.6: Small complete graphs: K_1, K_2, K_3, K_4, K_5

Exercise 3. *How many edges does K_{101} have?*

Recall from Definition 1.3.1, a graph has a non-empty set of vertices, but the edge set might be empty. A graph with no edges is called an *empty graph.* .

Exercise 4. *Let n be a positive integer and $V = \{v_1, \ldots, v_n\}$ be a set of n distinct elements. How many graphs have V as its (labelled) vertex set? For example, how many graphs are there on 5 labelled vertices?*

1.4 Walks, trails, paths, cycles, and circuits

Definition 1.4.1. *Let $G = (V(G), E(G))$ be a graph (or multigraph). For a non-negative integer m, a* walk of length m *in G is an alternating sequence of vertices (not necessarily distinct) and edges*

$$w_0, e_1 w_1, e_2, w_2, \ldots, e_m, w_m,$$

so that for each $i = 1, \ldots, m$, $e_i = \{w_{i-1}, w_i\} \in E(G)$. A walk on vertices $w_0, w_1, \ldots, w_m$ is called closed *if and only if $w_0 = w_m$, and is* open *otherwise.*

Definition 1.4.2. *A graph is $G = (V, E)$ is* connected *if and only if for every $x, y \in V$, $x \neq y$, there is a walk from x to y.*

For example, any graph on at least two vertices containing an isolated vertex (see Definition 1.3.6) is not connected. Conditions that guarantee when a graph is connected are examined in Section 1.9.

Remark 1.4.3. *Some authors misuse the expression "connected" by saying, e.g., "two vertices are connected" instead of saying that "two vertices are adjacent". However, expressions like " two vertices are* connected by an edge*" or " two vertices are* connected by a walk *of length 3" are common.*

Definition 1.4.4. *A* trail *in a graph or multigraph is a walk with no edge repeated.*

Definition 1.4.5. *A* path *in a graph or multigraph is a walk with no vertex repeated. For each $k \geq 2$, a path of length k is denoted P_k.*

Since no vertex in a path is repeated, a path is an open walk that never intersects itself. Also, since no vertices in a path are repeated, it follows that no edges are repeated in a path either.

One can say that a path of length 0 is a single vertex, and a path of length 1 is a pair of adjacent vertices (and the corresponding edge). The single vertex and the single edge are usually denoted K_1 and K_2 respectively (instead of P_0 and P_1, resp.). For $k \geq 2$, a the path P_k has $k + 1$ vertices.

Remark 1.4.6. *Some authors (e.g., [22]) prefer to denote a path of length k by P_{k+1} since it has $k+1$ vertices, and after all, both K_k and C_k have k vertices, The definition of P_k used in this text is often repeated to reduce such (natural) confusion.*

Exercise 5. *Let G be a connected graph. Show that any two longest paths share a vertex in common.*

See [486, p. 24] for an example on 25 vertices due to Walther, where the common point from Exercise 5 is not the same for any two longest paths.

Definition 1.4.7. *In a graph or multigraph, a* cycle *is a closed walk with no vertex repeated (except the first and last). A cycle on n vertices is denoted by C_n.*

In a multigraph, a cycle of length 1 is a vertex with a loop, and a cycle of length two is a pair of vertices with two (multi)edges incident with both. In simple graphs, the smallest cycle is $C_3 = K_3$. For drawings of some small cycles, see Figure 1.7.

There are many ways to draw or label cycles. For example, Figure 1.8 shows two common ways to draw C_5.

As another example, by adding 2 modulo 9, a directed C_9 is given in Figure 1.9

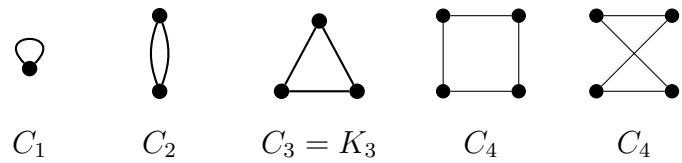


Figure 1.7: Some small cycles, with two drawings of C_4

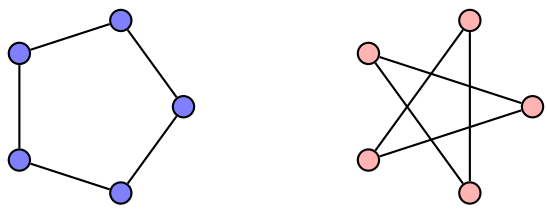


Figure 1.8: Two drawings of C_5

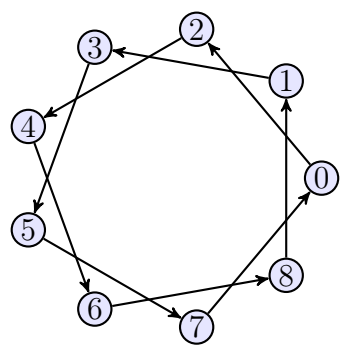


Figure 1.9: Drawing a 9-cycle with modular arithmetic.

1.5 Graph isomorphism

When are two graphs "the same"?

Definition 1.5.1. *An* isomorphism *between two graphs G and H is a bijection $f : V(G) \to V(H)$ such that $\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} \in E(H)$. Two graphs G and H are called* isomorphic *(written $G \cong H$) if and only if there exists an isomorphism between them.*

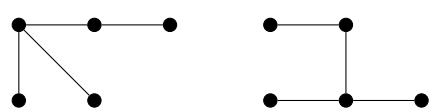
If two graphs are isomorphic, they have essentially the same (*iso*) form (*morphe*) or structure. If two graphs are isomorphic, there may be many different isomorphisms between them. Isomorphic graphs have the same number of vertices, the same number of edges, and corresponding vertices have the same degree. Sometimes it is easy to show that two graphs are not isomorphic by showing that one has a feature the other one does not. For example, if two graphs have different numbers of vertices or edges,

or, say, one has a vertex of degree 4 and the other does not, or one has a triangle and the other does not, then the graphs are not isomorphic.

To check whether or not two graphs are isomorphic is, in general, a difficult problem. If one were to ask a computer to check whether or not two n-vertex graphs are isomorphic, one guarantee for a correct answer is available using brute force, checking each bijection between vertex sets. For example, for two graphs on n vertices, there are $n!$ possible bijections between the respective vertex sets. For each such bijection, there are $\binom{n}{2}$ pairs of vertices in the first graph to see if adjacency or non-adjacency is preserved in the range of the bijection. Such a process may take as many as $n! \cdot \binom{n}{2}$ steps, and as n grows, this is much larger than any polynomial in n.

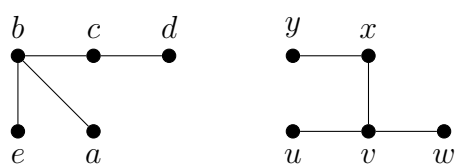
The complexity of the isomorphism problem is unknown (see [410]). In 2016, Laszlo Babai announced some progress on the problem by showing that it is "quasipolynomial". (It is widely believed his proof is correct but as of 2017, it does not seem to have published yet—see `https://valuevar.wordpress.com/2017/01/04/graph-isomorphism-in-subexponential-time/` for a blog reporting on the result.) In general, it is easier to show that two graphs are not isomorphic by showing some structural difference, whereas to absolutely guarantee isomorphism, the bijection from the definition needs to be exhibited.

To be pedantic, here is a simple example. Consider two graphs G and H given by



G H .

A first step in proving that these graphs are isomorphic is to label the vertices of each:



G H .

If $f : V(G) \to V(H)$ is a bijection that needs to be an isomorphism, some obvious pairings are needed. In G, the vertex b has degree 3, and in H there is only one such vertex, namely v, so let $f(b) = v$. Similarly, in G, vertex c is incident with only two edges, and in H, the only such vertex with this property is x, so let $f(c) = x$. In G, since d is the remaining vertex adjacent to c, in H, the remaining vertex adjacent to $f(c) = x$ is y, so set $f(d) = y$. The remaining two vertices e and a in G are both adjacent to b, and otherwise, they are indistinguishable, so try pairing up e and a in

G with the two remaining vertices u and w in H by arbitrarily setting $f(e) = u$ and $f(a) = w$.

Summarizing, the bijection $f : \{a, b, c, d, e\} \to \{u, v, w, x, y\}$ created above is given by

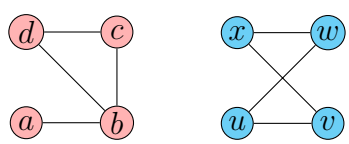
$$f(a) = w, f(b) = v, f(c) = x, f(d) = y, f(e) = u.$$

To verify that f is also an isomorphism, check all $\binom{5}{2} = 10$ pairs of vertices in G and see whether or not the corresponding pairs in H determine the same adjacency relations. IRecording this process in a chart, $\binom{5}{2} = 10$ rows are needed:

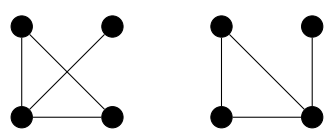
$\{s, t\}$ in $V(G)$	adjacent?	$\{f(s), f(t)\}$	adjacent in H?
$\{a, b\}$	yes	$\{w, v\}$	yes
$\{a, c\}$	no	$\{w, x\}$	no
$\{a, d\}$	no	$\{w, y\}$	no
$\{a, e\}$	no	$\{w, u\}$	no
$\{b, c\}$	yes	$\{v, x\}$	yes
$\{b, d\}$	no	$\{v, y\}$	no
$\{b, e\}$	yes	$\{v, u\}$	yes
$\{c, d\}$	yes	$\{x, y\}$	yes
$\{c, e\}$	no	$\{x, u\}$	no
$\{d, e\}$	no	$\{y, u\}$	no

To save on the number of pairs one checks, the four pairs in $V(G)$ that are edges in G can be checked first, and if the corresponding pairs in $V(H)$ are all edges, then one can stop since H has exactly four edges (but it is usually advised to check a few non-adjacent pairs in $V(G)$ as well).

Exercise 6. *Show whether or not the following two graphs are isomorphic.*



Exercise 7. *Show whether or not the two graphs are isomorphic:*



Exercise 8. *The two graphs in Figure 1.10 both have 12 vertices and 18 edges and each vertex has degree 3. Are they isomorphic?*

For the next exercise, see Chapter 19 for the definition of "equivalence relation".

Exercise 9. *Show that graph isomorphism " $\cong$ " is an equivalence relation.*

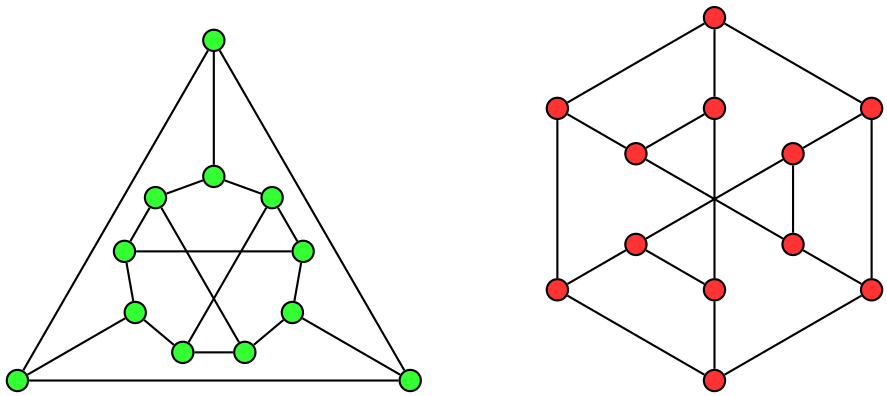


Figure 1.10: Two graphs for Exercise 8

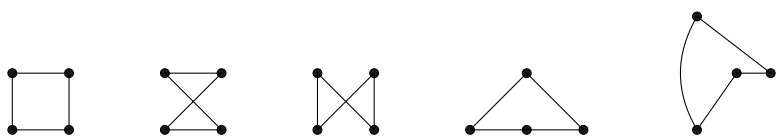


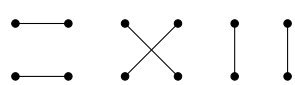
Figure 1.11: Five isomorphic graphs.

Unless specific labels on vertices are considered, when one says "the graph G", one is usually talking about any graph from the equivalence class containing G. A graph G can be drawn or presented in many different ways (see Figure 1.11).

As was seen in Exercise 4, the number of graphs on n (labelled) vertices is $2^{\binom{n}{2}}$. For large n, counting the number of *non-isomorphic* graphs on n vertices involves more complicated computation. Even if one asks for the number of non-isomorphic graphs on n vertices and some specified number m of edges, there is work to do.

For example, how many graphs on four (labelled) vertices have exactly two edges? Since there are $\binom{4}{2} = 6$ possible pairs of vertices, and two of these pairs are chosen to be edges, there are $\binom{6}{2} = 15$ ways to do this.

How many non-isomorphic graphs on four vertices have two edges? There are only two, a pair of disjoint edges, or two edges sharing a vertex. Of the first type, there are three ways to draw this graph on four labelled vertices, namely



and there are 12 ways to draw those of the second type:

So there are $3 + 12 = 15$ labelled graphs on 4 vertices with 2 edges, which confirms the previous calculation, with 3 of the labelled graphs of the first type being isomorphic and 12 of the second type being isomorphic.

In counting all non-isomorphic graphs on 4 vertices, it might help to list, in order, the graphs with 0 edges (only one), 1 edge, 2 edges, ..., to 6 edges (which is K_4).

Although the next exercise might seem tedious, it might help the reader to enhance skills in spotting non-isomorphic pairs of graphs.

Exercise 10. *Find all 11 non-isomorphic graphs on 4 vertices.*

Note the difference in the number of labelled graphs and the number of non-isomorphic graphs for 4 vertices—11 versus $2^{\binom{4}{2}} = 2^6 = 64$. It is common to say that the number of non-isomorphic graphs is the number of unlabelled graphs on n vertices.

Exercise 11. *Find all non-isomorphic graphs on 5 vertices with 5 edges.*

The values in the first row in the following chart are easy to calculate.

n	1	2	3	4	5	6	7	8
labelled graphs	1	2	8	64	1024	32768	2097152	268435456
unlabelled graphs	1	2	4	11	34	156	1044	12346

To find the number of non-isomorphic graphs on n vertices, a technique now known as "Pólya counting" (developed by George Pólya [756] in the 1930s) can be used. Pólya counting involves looking at the number of "symmetries" of various structures or configurations and applying methods from an area in mathematics called "group theory". (See Chapter 19 for the definition of a group.) An introduction to Pólya counting is given in many textbooks on combinatorics (see, *e.g.*, [117], [244], [629, Ch. 5], or [796, Ch. 8]).

For some early applications of Pólya counting to counting non-isomorphic graphs of various types, see [482], [483], or [484], and for counting matrices from graphs, see [499]; for more recent work and further references, the reader might look at [61].

Counting multigraphs is far more difficult than counting simple graphs. For example, how many multigraphs are there on three vertices with exactly three edges? (There is only one such simple graph.)

Exercise 12. *Find all non-isomorphic multigraphs on 3 vertices with 3 edges (with unlabelled vertices or edges).*

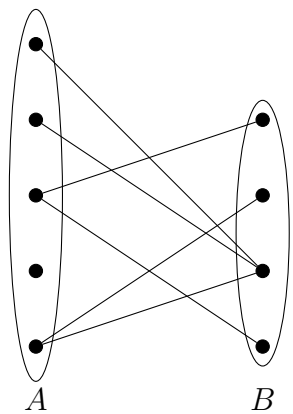
1.6 Some standard graphs or classes of graphs

Some of the most widely used classes of graphs include complete graphs, cycles, and paths. These graphs and many others naturally fall into other categories of graphs. In this section, only a few other such categories are given, with many more classes of graphs to be identified later in this text. Some other common individually named graphs are given for future reference.

1.6.1 Bipartite graphs, stars, and trees

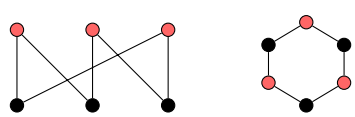
Definition 1.6.1. *A graph G is* bipartite *if and only if there exists a partition $V(G) = A \cup B$ (where $A \cap B = \emptyset$) of its vertex set so that every edge in G is of the form $\{x, y\}$, where $x \in A$, $y \in B$. The sets A and B are called* partite sets.

To show that a graph G is bipartite, it suffices to find a partition $V(G) = A \cup B$ so that neither A nor B induce any edges (i.e., $[A]^2 \cap E(G) = \emptyset$ and $[B]^2 \cap E(G) = \emptyset$.



no edges inside A or inside B

Bipartite graphs can be drawn with the partite sets beside each other (as above), or with one partite set above the other (as in Figure 1.13). An equivalent way to show that a graph is bipartite is to colour vertices with 2 colours so that no edge contains vertices of the same colour. For example, the following two drawings of C_6 show the bipartition, each partite set a different colour: medskip



Bipartite graphs can arise from many applications, including matching people to acceptable jobs/universities, exams to time slots, or jobs to machines, to name but a very few.

Observe that K_3 is not bipartite, and so any graph containing a K_3 is not bipartite. (A restatement of this fact is that any bipartite graph is "triangle-free".)

A cycle C_m is called *odd* [or *even*] if and only if m is odd [even, resp.]. Note that $C_4 \cong K_{2,2}$ and C_6 are bipartite, and, in general, for any integer $k \geq 2$, the even cycle C_{2k} is bipartite (listing vertices in order around C_{2k}, put the even numbered vertices in one partite set and the odds in the other). For each $k \geq 1$, an odd cycle C_{2k+1} is not bipartite (the first $2k$ vertices might alternate among two sides, and then the $(2k+1)$th vertex is adjacent to vertices of either side, so cannot be placed in a bipartite graph).

Note that the condition of being triangle-free is not sufficient to guarantee a graph is bipartite since, for example, C_5 is not bipartite.

Exercise 13. *Show that a graph G is bipartite if and only if G contains no odd cycles.*

Definition 1.6.2. *For positive integers a and b, the* complete bipartite graph $K_{a,b}$ *is a bipartite graph with a partition $V(K_{a,b}) = A \cup B$, where $|A| = a$, $|B| = b$, and $E(K_{a,b}) = \{\{x, y\} : x \in A, y \in B\}$.*

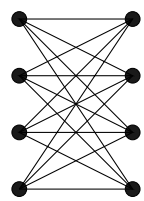


Figure 1.12: *Standard drawing of $K_{4,4}$.*

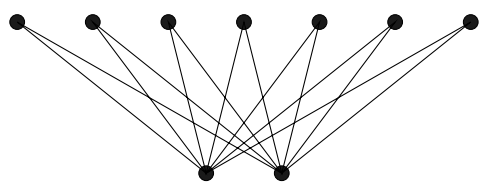


Figure 1.13: *$K_{2,7}$, drawn vertically.*

The proof of the following lemma is left as an (easy) exercise.

Lemma 1.6.3. *Let a and b be positive integers. Then $K_{a,b}$ has $a + b$ vertices and ab edges.*

Exercise 14. *Prove Lemma 1.6.3.*

Observe that $K_{a,b}$ is isomorphic to $K_{b,a}$.

Exercise 15. *Find all non-isomorphic bipartite (simple) graphs with 4 vertices.*

Exercise 16. *Find the five (non-isomorphic) connected bipartite graphs on 5 vertices.*

In general, counting all non-isomorphic bipartite graphs on n vertices is a difficult problem (again, Pólya counting can apply). For articles about such counting and further references, see [61], [480], [484], [496], or [499]. Exact formulae for the number of bipartite graphs are cumbersome. Incidentally, bounds for the number of bipartite graphs with a vertices in one part and $b > a$ vertices in the other were given [61] to be bounded below and above by $\frac{1}{a!}\binom{2^a+b-1}{b}$ and $\frac{2}{a!}\binom{2^a+b-1}{b}$.

A graph G is called a *star* if and only if for some n, $G \cong K_{1,n}$ (see Figure 1.14 for an example). Some authors use S_n to denote a star with $n+1$ vertices, *i.e.*, $S_n = K_{1,n}$;

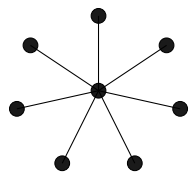


Figure 1.14: The star $S_7 = K_{1,7}$

other authors use S_n to denote a star with n vertices, *i.e.*, $S_n = K_{1,n-1}$. The graph $K_{1,3}$ is called a *claw*.

A graph that contains no cycles is called *acyclic*. Recall from Definition 1.4.2 that a graph is *connected* if and only if there is a walk from any vertex to any other vertex.

Definition 1.6.4. *A* tree *is a connected acyclic graph.*

For reference, this definition is repeated in Chapter 3 on trees. Paths and stars are examples of trees. Three other examples of trees occur in Figure 1.15. As is observed in Chapter 3, trees are bipartite graphs.

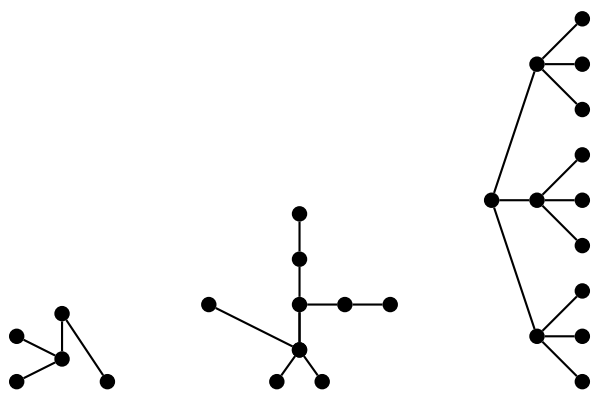


Figure 1.15: Three more trees

If an acyclic graph is not connected, each of its connected "pieces" (called "components"; see Definition 1.9.1) are trees. The following definition is then somehow "natural".

Definition 1.6.5. *A* forest *is an acyclic graph.*

Trees are studied in more detail in Chapter 3. Trees arise in many applications, including searching through or sorting data. Trees are often oriented so that one vertex is used as a starting point (called a "root"—see Section 3.7). Trees are usually drawn so that no edges cross (such drawings are called "planar", as defined in Section 1.6.2).

1.6.2 Planar graphs, an introduction

A drawing (in the Euclidean plane) of a graph so that no edges cross or touch except at the incident vertices is called a *plane drawing* (or planar drawing), and a graph is called *planar* if it has a plane drawing. For example, paths, cycles, stars, and wheels, are evidently planar. In Exercise 136 it is asked to show that all trees are planar.

Some graphs might not at first seem planar, but with a little effort, a planar drawing can be found. For example, the reader might verity that K_4 and $K_{2,3}$ are both planar (these facts are considered in Figure 7.1 and Exercise 325). To prove that a particular graph G is planar, it suffices to find one planar drawing (curved edges are allowed). For example, a graph and a planar drawing is given in Figure 1.16.

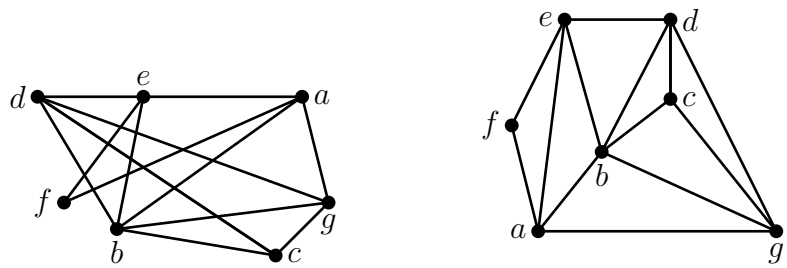


Figure 1.16: A graph together with a plane drawing

Is $K_{3,3}$ planar? This question is often introduced in graph theory texts as "the utilities problem"—a problem that also occurs in many puzzle books. Is it possible to connect three utilities stations (say, power, water, and gas) to three houses so that none of the supply lines cross? See Figure 1.17 for an attempt.

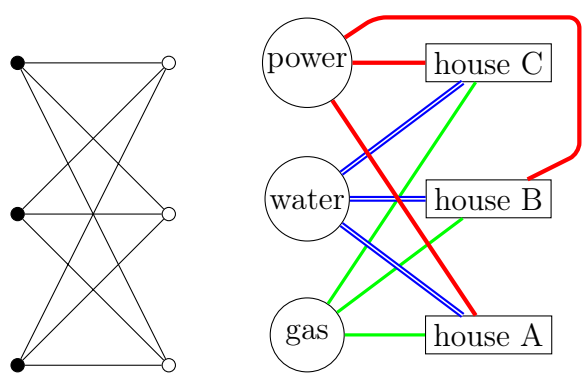


Figure 1.17: $K_{3,3}$, the utilities graph

If you give up on the answer to the utilities problem, the answer is given in Theorem 7.1.4. Can you find a planar drawing of K_5? K_6? As is seen in Chapter 7, $K_{3,3}$ and K_5 are of central importance in the theory of planar graphs.

Planar graphs are used in the analysis and design of electrical circuits, where vertices correspond to components of the circuit or connection points and edges might correspond to interconnections (like wires) between components. For example, if the graph of a circuit is planar, then it can be printed on one side of a circuit board (with some kind of conductive paint or foil, say). With the advent of VLSI chips, planarity plays an increasing role in electrical engineering. If an electronic circuit does not have a planar drawing, then one goal might be to limit the number of crossings (and hence the number of jumpers or shunts required in a circuit). See Section 7.11 for an introduction to such "crossing numbers".

Planar graphs played a central role in the history of graph theory, driven by the question of whether or not the regions of a map can be 4-coloured so that no two regions sharing a common border have the same colour. (This question was known as the 4-colour conjecture, which was finally proved in 1977—see Chapter 7 for a more detailed look at the 4 Colour Theorem.)

1.6.3 Polyhedral graphs

Some of the origins of many theorems in graph theory can be traced back to the study of polyhedral graphs and planar graphs and colourings thereof. A significant class of planar graphs arise from geometry and polyhedra.

There are many definitions of a polyhedron, but for present purposes, suppose that a polyhedron is a 3-dimensional shape with flat faces that are polygons. *A Platonic solid* is a convex (no indentations) polyhedron whose faces are all the same sized regular n-gon, and the same number of faces meet at each vertex. Using graphs (see Chapter 7) gives a proof that there are precisely five platonic solids, shown in Figure 1.18 (thanks to KG for providing the *Sage* files with coordinates). The graph of each Platonic solid is regular (and planar).

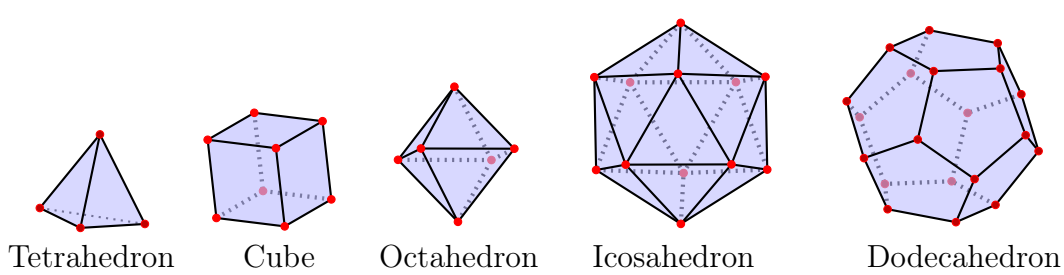


Figure 1.18: The Platonic solids

Archimedean solids are allowed to have different (but still regular) polygons for faces, but the pattern of polygons at each vertex is constant. For example, the truncated icosahedron has the face pattern of a soccer ball. Other "regular" solids include prisms and anti-prisms. Using graph theory, one can show that there are 13

Archimedean solids. For more on polyhedral graphs, see Chapter 7. Here, only a few examples are considered.

For each polyhedron, there is a graph associated. Corners of a polyhedron are called *vertices* and borders between faces are called *edges*. As one might guess, the terminology for graphs arises largely from terms in geometry.

In a polyhedron, each edge is incident with just two vertices, and each vertex is incident with the edges eminating from that vertex. So, if P is a polyhedron with vertex set V and edge set E, the graph for P is $G = (V, E)$. A polyhedral graph retains only the vertex-edge incidence structure of the associated polyhedron.

A common example is the graph that arises from a cube; see Figure 1.19 for two representations of its graph; the first graph is obtained by projecting onto the plane, and the second graph in Figure 1.19 can be obtained by a "stereographic projection", the view looking into the front face, and the edges of the front face then become the outside border in the view.

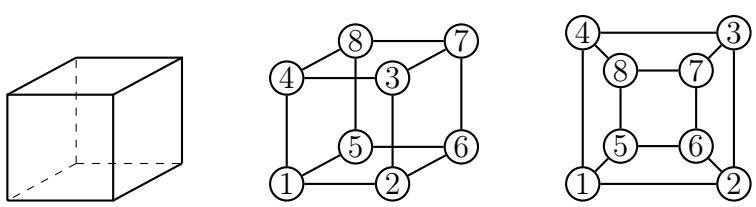


Figure 1.19: A cube and two drawings of its graph

Using stereographic projection, graphs of all convex polyhedra have planar drawings. For example, see Figure 1.20 for drawings of the graphs for the five Platonic solids. In Chapter 7, a proof is given that there are only five Platonic solids—a simple proof that uses graph theory!

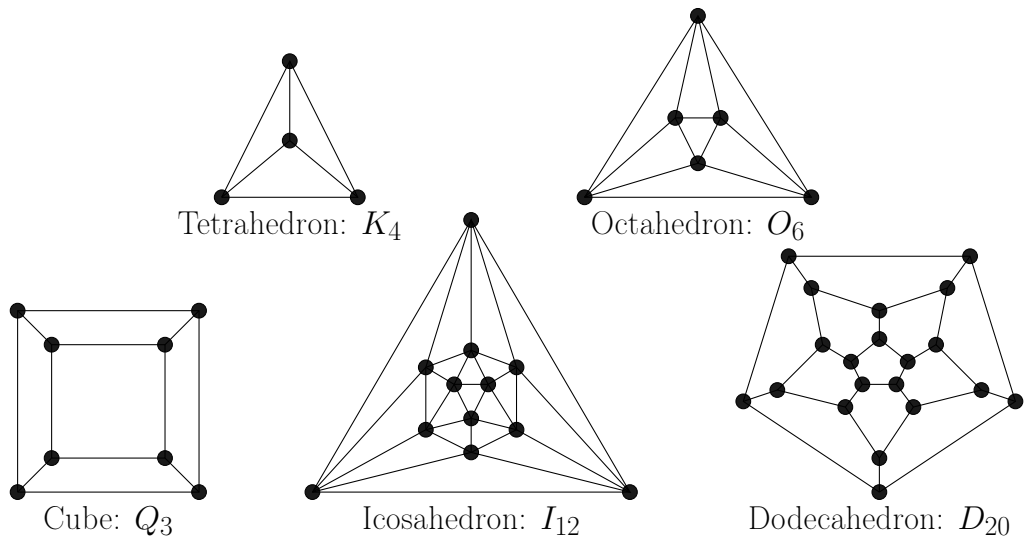


Figure 1.20: Planar graphs of the five Platonic Solids

Two infinite classes of polyhedra are called *prisms* and *anti-prisms*. For example, Figure 1.21 shows a hexagonal prism and its graph (drawn as a planar graph). An "anti-prism" is formed from a prism by slightly rotating one end of the prism and adding edges from each point on one end to the two closest points on the other end. The 6-anti-prism and its graph are represented in Figure 1.22. (The octahedron is a 3-anti-prism.)

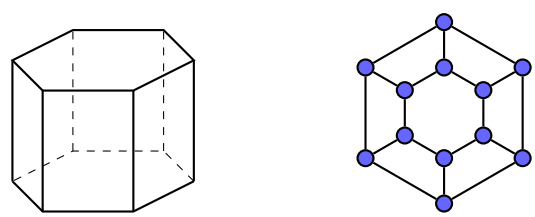


Figure 1.21: A hexagonal prism and its graph

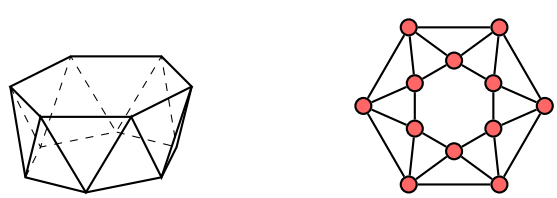


Figure 1.22: The 6-anti-prism and its graph

Polyhedral graphs are considered again in Chapter 7.

1.6.4 Multipartite graphs

The definition for bipartite graphs can be extended to more than two partite sets.

Definition 1.6.6. *Let $k \geq 2$ be an integer. A graph $G = (V, E)$ is k-partite if and only if there exists a partition $V = X_1 \cup X_2 \cup \cdots \cup X_k$ so that for each edge $\{x, y\} \in E$, $|\{x, y\} \cap X_i| \leq 1$.*

In some contexts, the definition of a k-partite graph has the added condition that each partite set is non-empty; if this condition is not added, one can consider a 3-partite graph also as a 4-partite graph, say. If one allows empty partite sets, a graph is k-partite if and only if there is a k-colouring of its vertices so that no two vertices forming an edge are the same colour.

Definition 1.6.7. *Let $k \geq 2$ and let $t_1, t_2, \ldots, t_k$ be positive integers. The complete k-partite graph $K_{t_1,t_2,\ldots,t_k}$ is the graph whose vertex set is a union of k disjoint sets $V(K_{t_1,t_2,\ldots,t_k}) = X_1 \cup X_2 \cup \cdots \cup X_k$, where for each $i = 1, \ldots, t$, $|X_i| = t_i$, and whose edge set is*

$$E(K_{t_1,t_2,\ldots,t_k}) = \{\{x, y\} : x \in X_i, y \in X_j, i \neq j\}.$$

The sets $X_1, \ldots, X_k$ in Definition 1.6.7 are called *partite sets*. In $K_{t_1,t_2,\ldots,t_k}$, the graph formed by any two of the partite sets is a complete bipartite graph. For example, the graph $K_{2,2,2}$ is drawn in two ways in Figure 1.23.

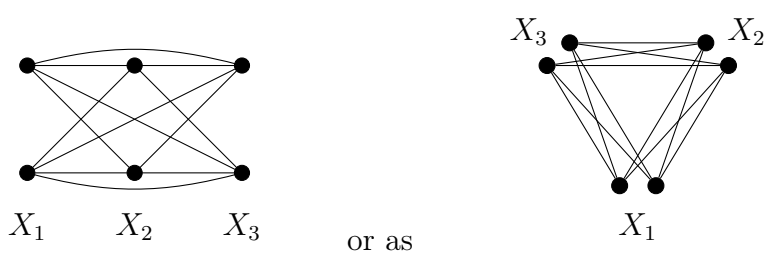


Figure 1.23: Two drawings of $K_{2,2,2}$

Exercise 17. *Show that the graph of the octahedron is isomorphic to $K_{2,2,2}$.*

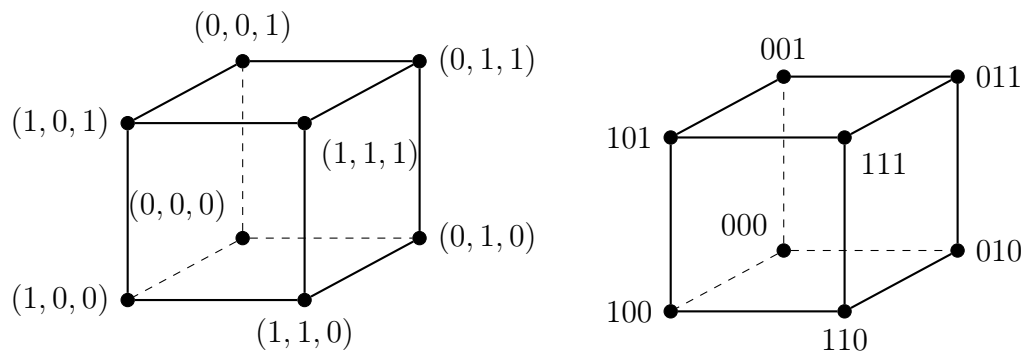
Exercise 18. *How many vertices and edges does $K_{2,3,7}$ have?*

The result in Exercise 18 has a generalized form.

Exercise 19. *Let $k \geq 2$ and let $t_1, \ldots, t_k$ be positive integers. Find the number of vertices and edges in $K_{t_1,t_2,\ldots,t_k}$.*

1.6.5 Cube and hypercube graphs

If a unit cube is placed at the origin of a 3-dimensional axes system with its vertices having coordinates either 0 or 1, the vertices can be labelled with the coordinates for the point, or with the associated binary word:



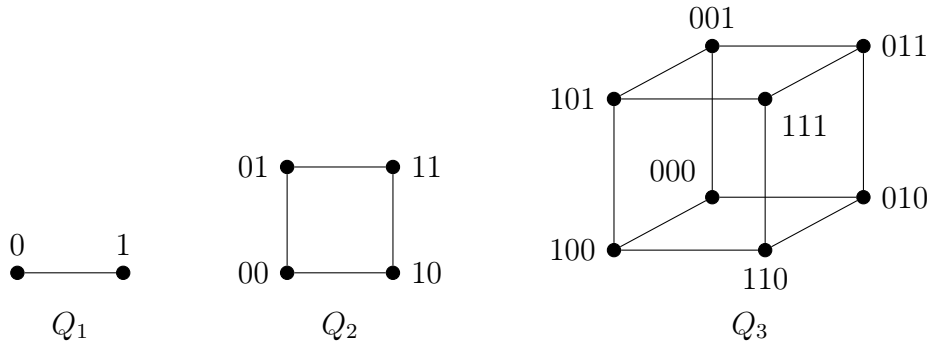
All binary sequences of length n can be viewed as the numbers from 0 to $2^n - 1$ in binary (base 2). Another way to write all such binary strings is the n-fold cartesian product of the set $\{0, 1\}$,

$$B_n = \{0, 1\}^n = \{(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) : \forall i, \epsilon_i \in \{0, 1\}\}.$$

In practise, binary words are written without the parenthesis and commas, so B_n can be considered as the set of all 0-1 sequences of length n or as the set of all binary words of length n.

Definition 1.6.8. *For each positive integer n, Q_n is the graph whose vertices are the n-bit binary words, and two words form an edge if and only if the words differ in exactly one bit.*

For example, in Q_4, words 0001 and 1001 are adjacent, but 0100 and 0010 are not. A graph Q_k is often called a "cube graph", or more descriptively, Q_k is the "unit hypercube graph of dimension k" or simply, the "k-cube". The word 'unit" is sometimes added to stress that edges of the hypercube in $\mathbb{R}^k$ indeed have side length 1; however, just as a graph, distances are usually ignored. The first three cube graphs can be drawn as follows:



As was observed already, Q_3 can also be drawn without crossing edges. What does Q_4 look like? How does one visualize the 4-th dimension? There are a number of approaches. Perhaps the simplest way is to form two copies of Q_3 and add a 0 on the right end of each label in the first graph and add a 1 to the right end of each label in the second graph, then join corresponding vertices (for example, 0100 is then adjacent to 0101) with eight more curved edges. A drawing capturing this technique is given in Figure 1.24.

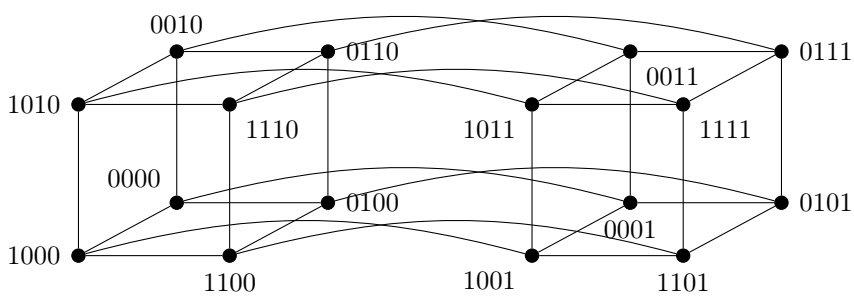


Figure 1.24: Two copies of Q_3 make a copy of Q_4

The drawing in Figure 1.24 is helpful when analyzing paths in Q_4 and the same technique also helps for larger Q_k. Two other drawings for Q_3 and Q_4 in Figure 1.25 are more symmetric.

Just as an ordinary cube has vertices, edges, and faces, the 4-dimensional analogue, called the "tesseract" has Q_4 as its "skeleton" and is an example of a higher dimensional shape called a *polytope* (where certain convexity requirements are added). The interested reader can find more on the general theory of polytopes in, *e.g.*, [448] or [1014].

Lemma 1.6.9. *For each positive integer n, the cube graph Q_n has 2^n vertices and $\frac{1}{2}2^n n = n \cdot 2^{n-1}$ edges.*

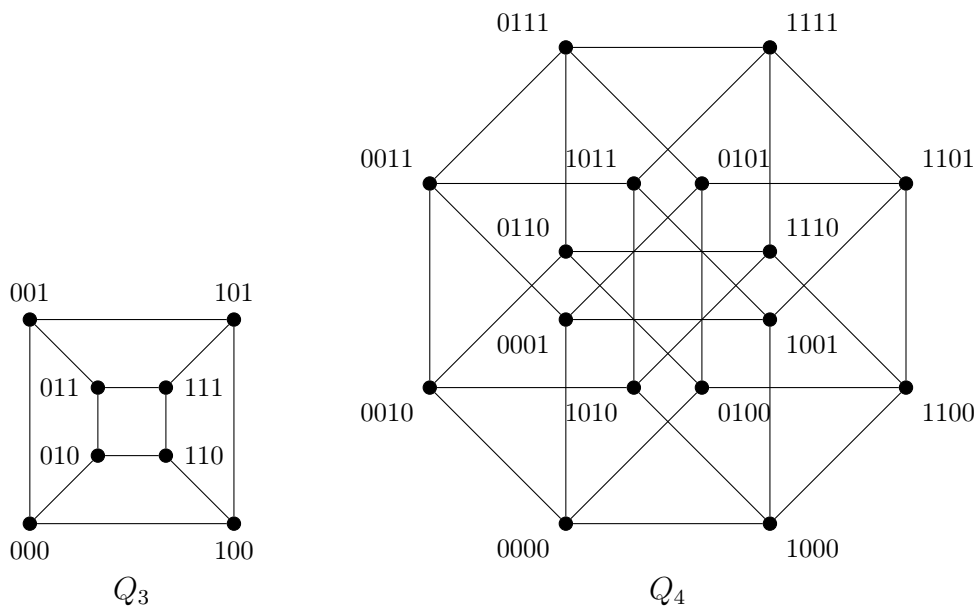


Figure 1.25: Symmetric drawings of Q_3 and Q_4.

Proof: For a positive integer n, the vertex set $V(Q_n)$ is the set of all binary strings of length n. In any such string, there are 2 choices for the bit in the first position, 2 choices for the bit in the second position, ..., and 2 choices for the nth position. So by the product rule, there are

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n} = 2^n$$

binary strings of length n.

The simple proof given here that $|E(Q_n)| = n \cdot 2^{n-1}$ is by mathematical induction (MI) on n. Another simple proof (see Exercise 48) for the number of edges in Q_n follows from the "handshaking lemma" (Lemma 1.8.1), a counting technique that relates the number of edges in a graph to the number of vertex-edge incidences. However, the proof given here by MI has its advantages.

For each positive integer n, let $A(n)$ be the assertion that $|E(Q_n)| = n \cdot 2^{n-1}$.

Base step: When $n = 1$, Q_1 has just one edge, namely $\{0, 1\}$, and since $1 \cdot 2^{1-1} = 1$, $A(1)$ is true.

Inductive step: Let $k > 1$ and suppose that $A(k-1)$ is true, that is, Q_{k-1} has $(k-1)2^{k-1-1} = (k-1)2^{k-2}$ edges. Since Q_k can be formed (as in the above construction for Q_4 from two copies of Q_3) by taking two copies of Q_{k-1} and joining each of the 2^{k-1} vertices of the first copy to the corresponding vertex in the second copy, the number

of edges in Q_k is

$$\begin{aligned}
&|E(Q_{k-1})+2^{k-1} + |E(Q_{k-1})| \\
&\qquad = (k-1)2^{k-2} + 2^{k-1} + (k-1)2^{k-2} \qquad \text{(by } A(k-1)\text{, twice)} \\
&\qquad = 2(k-1)2^{k-2} + 2^{k-1} \\
&\qquad = (k-1)2^{k-1} + 2^{k-1} \\
&\qquad = k2^{k-1},
\end{aligned}$$

and so $A(k)$ is also true. This finishes the inductive step $A(k-1) \to A(k)$, and so by mathematical induction, for every positive integer n, the statement $A(n)$ is true. □

Verifying this for $n = 3$, Q_3 has $12 = \frac{1}{2}2^3 3 = 3 \cdot 2^{3-1}$ edges.

Lemma 1.6.10. *For each positive integer k, the graph Q_k is bipartite.*

Proof: Let k be a positive integer. Define the bipartition $V(Q_k) = L \cup R$ by putting those vertices (binary words of length k) with an even number of 1s in L and those with an odd number of 1s in R. It remains to show that neither L nor R induce any edges.

Since "even minus even is even", any two vertices in L differ in an even number of positions, that is, in at least two positions; but adjacent vertices differ in precisely one position, so no two vertices in L are adjacent. Similarly, since "odd minus odd is even", no two vertices in R are adjacent. □

For more on cube graphs and their application in computing, see Section 2.5 on Gray codes.

1.6.6 The Petersen graph

A famous graph is called the Petersen graph [747]. In Figure 1.26 are three drawings of this graph.

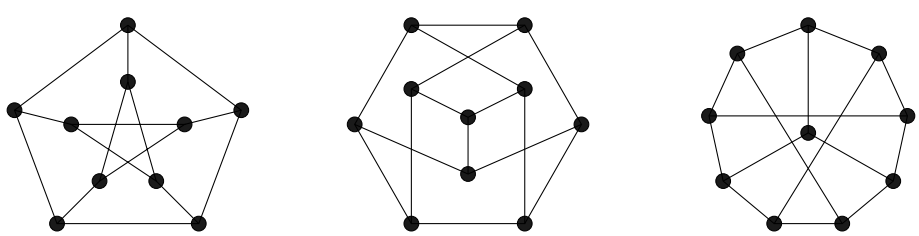


Figure 1.26: Three drawings of the Petersen graph

The Petersen graph has many remarkable properties; in fact, an entire book [525] has been written about this graph.

The Petersen graph P contains odd cycles (*e.g.*, the 5-cycle around the outside of the first drawing), and so P is not bipartite. Since every vertex of the Petersen graph is incident with the same number of edges, it may be non-trivial to find isomorphisms between any two of the graphs given in Figure 1.26.

Exercise 20. *Show that the first two graphs in Figure 1.26 are isomorphic by giving a labelling of the vertices of the first one, and then using these same labels, find the corresponding labels of the second.*

The Petersen graph was not first discovered by Petersen. In 1886, (12 years before Petersen's paper), Kempe [565] first published a drawing of the Petersen graph (the third drawing in Figure 1.26). graph related to something called the "Desargues' configuration" in geometry (see Figure 1.27 for one drawing).

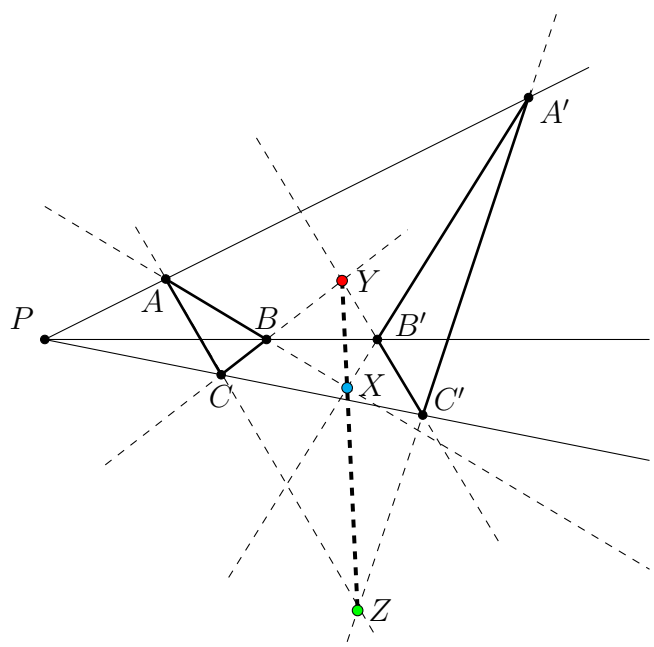


Figure 1.27: The Desargues' configuration with 10 points and 10 lines

The ten vertices of the Petersen graph correspond to the ten lines in the Desargues' configuration, and two vertices are adjacent if and only if the corresponding lines intersect at a point outside of the ten points in the Desargues' configuration. (Interchanging the roles of points and lines in the Desargues' configuration gives the same graph.)

The Petersen graph was then rediscovered by Julius Petersen, who, in 1898, published a short paper [747] showing that this graph is a (smallest) counterexample to a "theorem" by Tait [903]. Petersen's drawing of this graph appears in a paper by Martyn Mulder [702] on Petersen's work on regular graphs; see Figure 1.28 for my attempt at duplicating that drawing. [Mulder tells me [703] that he had to go to a library in

Denmark to see Petersen's paper, where he was required to wear gloves; the drawing was in an appendix that had been coloured by hand.]

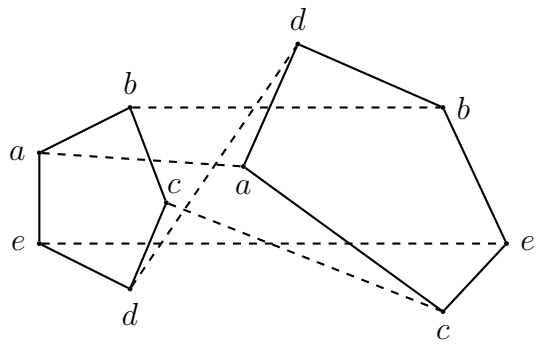


Figure 1.28: Approximate version of original drawing of Petersen's graph

Tait (1831–1901) thought that he had proved that every cubic bridgeless graph is 3-edge-colourable (see Section 6.11), i.e., had a decomposition into 1-factors (see Section 8.2), thereby giving "another" proof of the 4 colour theorem (see Section 7.7). Petersen's drawing uses two 5-cycles side by side with edges added between them. Since then, the Petersen graph has been used as an example (or counterexample) for many other graph properties. See Appendix 16 for a summary of some of its properties. The first drawing of the Petersen graph given in Figure 1.26 is featured in the cover design of two journals, *Journal of Graph Theory* and *Discrete Mathematics*; it also is one of four graphs on the cover of Harary's 1969 graph theory textbook [486]. The second drawing appears on the cover of Doug West's text [977]. For more information on the Petersen graph, see [102], [191] or the book "The Petersen graph" by Holton and Sheehan [525].

1.6.7 Intersection graphs

If A and B are two sets, $A \cap B = B \cap A$, so the relation "intersects" (have non-empty intersection) is symmetric. So if sets correspond to vertices and pairs of sets with non-empty intersection correspond to edges, then a graph can capture this information.

Definition 1.6.11. *For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of sets, the* intersection graph *for $\mathcal{F}$ is the graph whose vertices are the S_is and for $i \neq j$, the pair of vertices $\{S_i, S_j\}$ form an edge if and only if $S_i \cap S_j \neq \emptyset$.*

The following theorem has a simple proof , which is left as Exercise 21:

Theorem 1.6.12 (Marczewski, 1945 [653])**.** *For any graph G, there exists a family of sets whose intersection graph is G.*

Exercise 21. *Prove Theorem 1.6.12*

One of the earliest non-trivial theorems regarding intersection graphs gives a bound on the family of sets needed to produce an arbitrary graph as an intersection graph.

Theorem 1.6.13 (Erdős–Goodman–Pósa, 1966 [319])**.** *For any graph G on n vertices, there exists a set S with $|S| \leq \frac{n^2}{4}$ and a family $\mathcal{F}$ of subsets of S whose intersection graph is G.*

For a closely related theorem that is used to prove Theorem 1.6.13, see Theorem 8.1.6.

Various classes of graphs arise when the sets allowed in the definition of intersection graphs are restricted to particular types.

In Section 6.12.5, a particular kind of intersection graph is studied, namely when the sets are intervals of real numbers, in which case such graphs are called "interval graphs". When sets are restricted to arcs of a circle, the resulting graphs are called "circular arc graphs"; such graphs can be used to analyze timing of traffic lights (see, *e.g.*, [795]).

The sets used to form an intersection graph may be restricted to certain subsets of the Euclidean plane. If $\mathcal{F}$ is a set of curves in the plane (strings), no three of which are concurrent, the intersection graph for $\mathcal{F}$ is called a *string graph*. Gojan, Kratochvĭl, and Kučera [423] proved that all graphs with at most 11 vertices are string graphs, but that for 12 vertices, there are two graphs (see Figure 1.29) that are not string graphs.

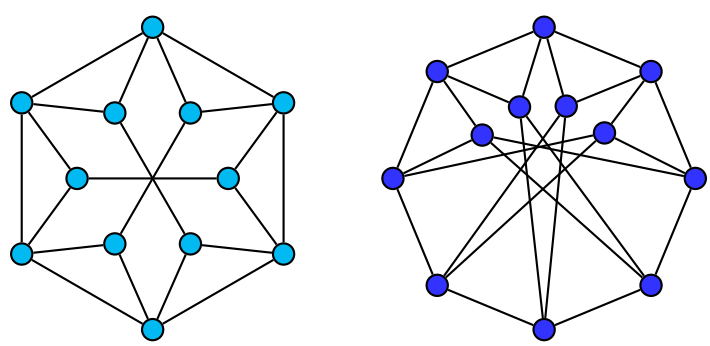


Figure 1.29: Two graphs on 12 vertices that are not string graphs; the one on the left was called [423] a "propeller graph".

The study of string graphs arose from different areas in science, from circuit layouts to genetic structures (see, *e.g.*, [89] and [867]). One of the first appearances of string graphs in combinatorics was in 1976 (see [432]). For more on classifying intersection graphs, see [831].

Somewhat related to intersection graphs are graphs that are obtained by intersecting lines or curves (called "pseudolines"), but then the vertices are given by intersection

points (not sets) and edges are formed by intervals of a curve between consecutive crossing points. For examples, see Theorem 7.8.1 and subsequent material.

1.6.8 Social network graphs

Graphs can be used to model relations between people, companies, or countries. For example, consider three people, A, B, C, represented by the vertices of the complete graph on 3 vertices (K_3). Consider the relation "can work together" (which is symmetric), and its opposite "can't work together". (Suppose that there is no middle ground.) For each pair of people, assign a + or - to the corresponding edge depending on whether or not these two people can work together, or can't work together, respectively. The graph with edges labelled either + or - is called a *signed graph*. (If there are more people, use a larger complete graph.) The four types of signed graphs on three vertices are of the forms (taken from [795, p. 80]) given in Figure 1.30.

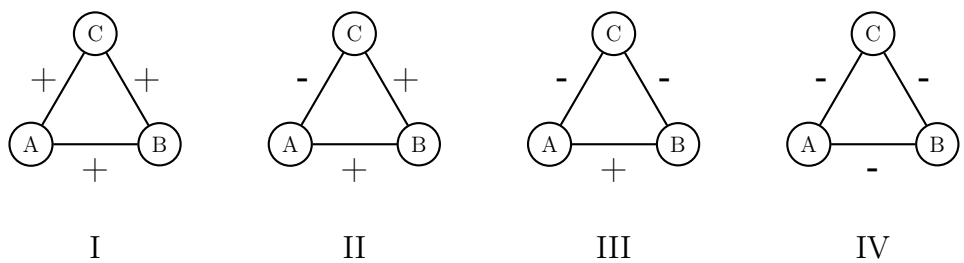


Figure 1.30: Four types of signed graphs on 3 vertices

In a type I graph, everyone can work together. In a type III graph, C is fine working alone. However, in types II and IV, there might be tension in the work place. Observe that types I and III have an even number of negative edges.

A signed graph is called *balanced* if every cycle contains an even number of negative labels.

Theorem 1.6.14 (Harary, 1954 [481])**.** *A signed graph is balanced if and only if the vertices can be partitioned into two classes so that every edge joining within a class is labelled + and every edge joining vertices between classes is labelled -.*

Harary [481, p. 144] writes

> "A psychological interpretation of Theorem 1 is that a "balanced" group consists of two highly cohesive cliques which dislike each other."

In Theorem 1.6.14, Harary neglected to add the possibility that all can work together, perhaps since such a situation is not that interesting. The reader can verify the partitions guaranteed by Theorem 1.6.14 for types II, III, and IV from Figure 1.30.

Above was only a cursory introduction to signed graphs. For more details, see, *e.g.*, [22] or [795] (signed digraphs are also used to consider other "social" situations). For other analysis of group social structure concentrating more on matrices, see, *e.g.*, [649].

Instead of using signs + and -, one could model a network by colouring relevant edges blue and red, say, or use "edge" versus "non-edge". Later in this text, "signed graphs" are rarely labelled with +/-, nor are they called "signed graphs". In graph theory, a maximal complete subgraph is often called a "clique", perhaps from the meaning of a "clique" of mutual friends.

Results in Ramsey theory for graphs (see Chapter 9) or in extremal graph theory (see Chapter 10) are also relevant in social theory. The interested reader might also read about *percolation theory* (*e.g.*, see [131]), which is a very active research area overlapping with graph theory, physics, and random structures concerning infection and information (or rumour) flow in a social network. There are a vast number of social, economic, biological, or technology situations modelled by graphs, and sometimes only simple graph theoretic results yield critical knowledge. [This text does not emphasize such applications, but there are many texts that do; see, *e.g.*, [202]. For but one more application to sociology, see Section 7.11.3.]

1.6.9 Some other named graphs

Only a few simple graphs with natural names are given here. Some small graphs are given names to match their drawings. In Figure 1.31 are, in order, the "bowtie graph" (also known as the "butterfly graph"), a "barbell graph", the "house graph", and the "bull graph". (Barbell graphs need not have two triangles at the end of the "bar", for $n \geq 3$, a "barbell" graph can have two copies of K_n joined by an edge.)

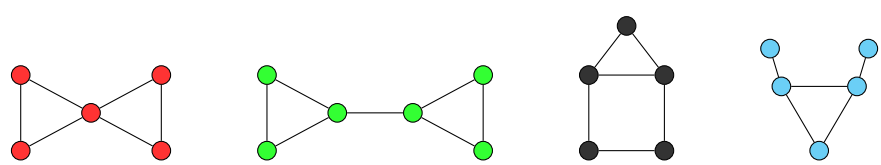


Figure 1.31: The bowtie graph, a barbell graph, the house graph, and the bull graph

A "ladder graph" looks just like one might expect. A *circular ladder* is the graph of a prism (see Figure 1.21).

For $n \geq 3$, define the *wheel graph* W_n by either adding edges between consecutive pairs of end vertices of a star S_n, or by starting with a cycle C_n and adding a new (central) vertex adjacent to all vertices of C_n. See Figure 1.32 for the wheel W_5. The edges coming from the center are sometimes called spokes.

A Möbius graph is formed from a prism graph (or circular ladder graph) with a twist in the same way a Möbius band is formed by a long strip of paper; see Figure 1.33 for an example on 20 vertices.

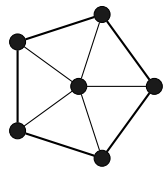


Figure 1.32: The wheel W_5.

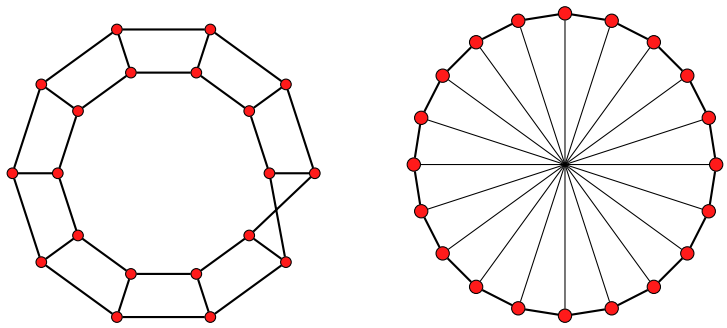


Figure 1.33: Two drawings of a Möbius ladder

Exercise 22. *Show that the two graphs in Figure 1.33 are isomorphic. (In the second drawing, there is no vertex at the center.)*

1.7 New graphs from old

1.7.1 Subgraphs

IF G is a graph, a graph H is a (weak) subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq (E(G) \cap [V(H)]^2)$ and is an *induced subgraph* if $E(H) = E(G) \cap [V(H)]^2$. See Figure 1.34 for examples.

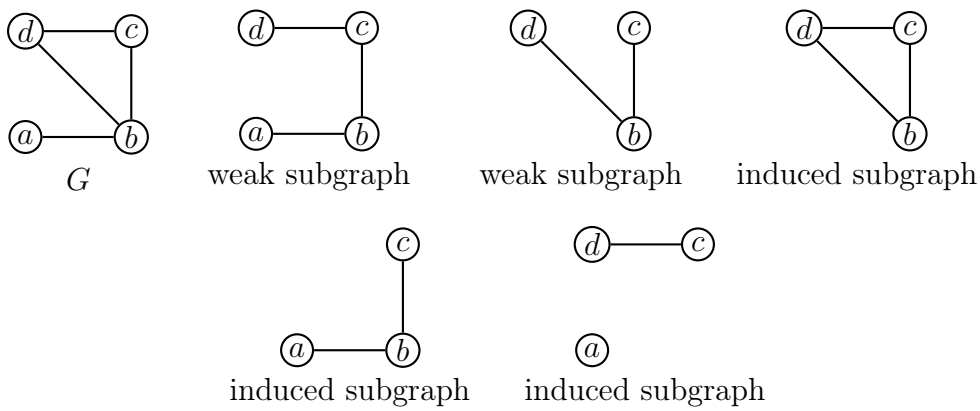


Figure 1.34: A graph G with some of its subgraphs

If the term "subgraph" is used, it is usually taken to mean "weak subgraph". An unlabelled graph H is said to be a subgraph of G if there exists a subgraph F of G so that $H \cong F$.

Exercise 23. *Let r be a positive integer. How many non-isomorphic connected (weak) subgraphs of $K_{1,r}$ are there?*

Remark 1.7.1. *Later, in Sections 2.3 (Exercise 117) and 2.5 it is shown that for $n \geq 2$, the (hyper)cube graph Q_n contains a cycle using all 2^n vertices; however, for $n \geq 3$, such cycles are not induced (since all vertices induce all edges). As seen in Section 2.3, a cycle that uses all vertices in a graph is called a "Hamiltonian cycle", and a graph containing a hamiltonian cycle is called "Hamiltonian". So for $n \geq 2$, Q_n is Hamiltonian.*

Exercise 24. *Show that the longest induced cycle in the cube graph Q_3 has length 6; find such an example. How many induced 6-cycles are there?*

Danzer and Klee [240] used the term "snake" for an induced cycle in a cube graph and studied maximum lengths of such snakes (the title of their article is "Lengths of snakes in boxes"). For $n \geq 2$, let $S(n)$ denote the maximum length of an induced cycle

in Q_n. Danzer and Klee found that $S(2) = 4$, $S(3) = 6$, $S(4) = 8$, and $S(5) = 14$, and gave a recursive construction that shows that for $c, d \geq 2$,

$$S(c + d) \geq 2^{c-1} S(d).$$

If G is a graph, a *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. If X is a subset of vertices of a graph G, the notation $G[X]$ means the subgraph of G induced by X; in other words, $V(G[X]) = X$ and $E(G[X]) = [X]^2 \cap E(G)$.

Definition 1.7.2. *For a graph $G = (V, E)$, a subset $S \subseteq V$ is called* independent *if and only if no two vertices of S are adjacent, i.e., S is independent if and only if $E(G[S]) = \emptyset$.*

An independent set of vertices is also called a *stable set*. The cardinality of a largest independent set in a graph G is denoted $\alpha(G)$ and is called the "independence number" of G; this concept is formally defined again in Definition 5.5.1). The independence number is a well-studied parameter of a graph and is of central interest when studying graph colourings (see, *e.g.*, Chapter 6).

1.7.2 Complements

Definition 1.7.3. *If $G = (V(G), E(G))$ is a graph, the* complement *of G is the graph $\overline{G}$ defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = [V]^2 \backslash E(G)$; that is, edges are replaced by non-edges and non-edges are replaced by edges.*

An example of a graph G and its complement $\overline{G}$ is given in Figure 1.35.

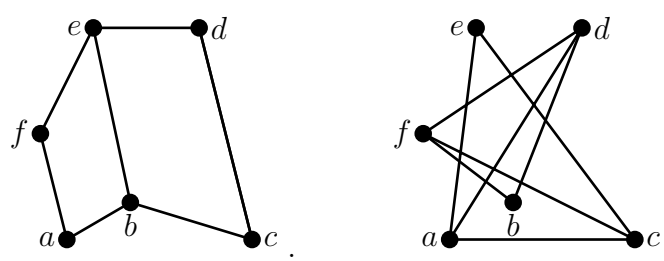


Figure 1.35: A graph and its complement

A graph G is called *self-complementary* if and only if G is isomorphic to $\overline{G}$. If G is a self-complementary graph on n vertices, then $|E(G)| = \frac{1}{2}\binom{n}{2}$.

Exercise 25. *If G is a self-complementary graph on n vertices, show that either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.*

When $n = 1$, the single vertex graph K_1 is self-complementary. By checking cases, one can verify that the path P_3 is the only self-complementary graph on four vertices.

For $n = 1, 2, 3, \ldots$, the number of self-complementary graphs on n vertices is given by the sequence

$$1, 0, 0, 1, 2, 0, 0, 10, 36, 0, 0, 720, 5600, 0, 0, 703760, 11220000, \ldots;$$

more terms in this sequence are found at `http://oeis.org/A000171`.

Exercise 26. *Find the two self-complementary graphs with 5 vertices.*

Exercise 27. *Find a self-complementary graph on 8 vertices.*

For each prime power q congruent to 1 modulo 4, there exists a self-complementary graph, called a *Paley graph*, on q vertices. Paley graphs are defined by a simple number theoretic construction, (see Definition 9.6.2). The fact that Paley graphs are self-complementary follows directly from a simple property of quadratic residues (see Lemma 9.6.3). The graph C_5 is the smallest Paley graph. The next smallest Paley graph is on $n = 9$ vertices, which can also be defined by a "product" construction (see Exercise 30).

Since self-complementary graphs have the same number of edges as non-edges, they are of interest as possible examples for lower bounds on Ramsey numbers (see Chapter 9).

1.7.3 Unions and products

For two graphs G and H, there are many graphs that can be defined by "joining" G and H in some way. One way to join graphs is to simply take unions. If G and H are graphs, their union, denoted $G \cup H$, is the graph on vertex set $V(G) \cup V(H)$ with edge set $E(G) \cup E(H)$. When $V(G)$ and $V(H)$ are disjoint, the graph $G \cup H$ can be formed by drawing G and H side by side. For a positive integer m and graph G, let mG denote m vertex disjoint copies of G. For example, $3K_2$ is a triple of disjoint edges.

A *matching* is a graph consisting of pairwise disjoint edges; if a matching M contains m edges, then a convenient notation is $M = mK_2$. There has been considerable study of when graphs contain certain matchings as subgraphs; see Chapter 5.

For a positive integer m, the graph $mK_1 = \overline{K_m}$, the empty graph on m vertices, sometimes denoted by E_m.

Another way to construct a graph from G and H is to take disjoint copies of each and join all vertices in G with all vertices in H. The notation for this "graph sum" varies, but one such notation is $G \vee H$ (called the join of G and H), defined by $V(G \vee H) = V(G) \cup V(H)$ and

$$E(G \vee H) = E(H) \cup E(H) \cup \{\{x, y\} : x \in V(G), y \in V(H)\}.$$

Another common notation for $G \vee H$ is $G \oplus H$.

For example, for $n \geq 4$, the wheel graph W_n is formed by adding one common neighbour to the vertices of a cycle C_n, and so. $W_n = K_1 \vee C_n$.

There many ways to define a "product" of two graphs, and notation in the literature seems to vary for each. One natural product for graphs G and H is called the "weak product" in [143] or "tensor product".

Definition 1.7.4. *For graphs G and H, define the weak (or tensor) product $G \times H$, by $V(G \times H) = V(G) \times V(H)$, where (x_1, y_1) is adjacent to (x_2, y_2) if and only if both $\{x_1, x_2\} \in E(G)$ and $\{y_1, y_2\} \in E(H)$.*

Exercise 28. *Draw the graph $C_5 \times C_3$ and show that this graph contains a cycle of length 15.*

Definition 1.7.5. *For graphs G and H, define the* Cartesian product *$G \square H$ to be the graph on vertex set $V(G) \times V(H)$, where $\{(x_1, y_1), (x_2, y_2)\}$ is an edge in $G \square H$ if and only if $x_1 = x_2$ and $\{y_1, y_2\} \in E(H)$ or $y_1 = y_2$ and $\{x_1, x_2\} \in E(G)$.*

One can check that the cartesian product of two paths is a "grid graph"(as in Figure 1.36) perhaps motivating the $\square$ notation.

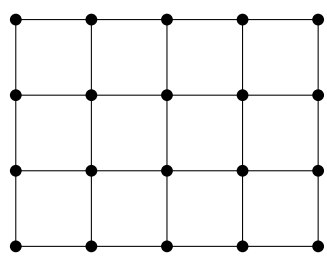


Figure 1.36: The graph $P_4 \square P_3$ (where P_n denotes a path of length n)

In the game of chess, a rook (also called a "castle") is a piece that can move horizontally along rows or vertically along columns (and not both at the same time). The cartesian product $K_m \square K_n$ is called a *rook's graph* (see Figure 1.37 for $K_4 \square K_5$) because on an $m \times n$ chessboard, the moves possible for a rook are given by the edges in such a graph.

Exercise 29. *Draw $K_2 \square K_3$ and show that its complement is isomorphic to C_6.*

The graph in the following exercise is an example of a special class of graphs mentioned above, called Paley graphs (see Definition 9.6.2), any of which are self-complementary (by Lemma 9.6.3). .

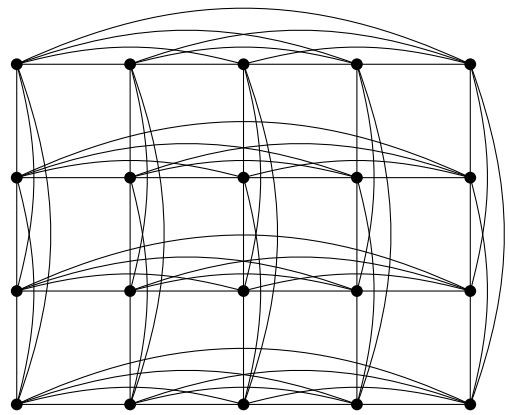


Figure 1.37: The rook's graph $K_5 \square K_4$

Exercise 30. *Show directly (without using Lemma 9.6.3) that the* 3×3 *rook's graph* $K_3 \square K_3$ *is self-complementary.*

Exercise 31. *Show that for any graphs* G *and* H*, the cartesian product* $G \square H$ *is isomorphic to* $H \square G$*.*

Another type of product (defined in, *e.g.*, [422]) is called the "strong product", denoted by $G * H$, where (x_1, y_1) is adjacent to (x_2, y_2) if and only if x_1 is equal to or adjacent to x_2 and y_1 is equal to or adjacent to y_2.

Exercise 32. *Let* $G = P_3$ *be the path of length 3 (on 4 vertices) and let* $H = P_2$ *be the path of length 2 (on 3 vertices). Draw the product graphs* $G \times H$*,* $G \square H$*, and* $G * H$*.*

One more type of "product" uses only one graph. Let G be a simple graph. For each positive integer k define the *kth power of* G, denoted by G^k, to be the graph formed by adding an edge between any two vertices that are joined by a path of length at most k. For example, the square of C_5 is K_5.

There are various applications of different kinds of products in Ramsey theory, extremal theory, graph colouring, and constructions of highly symmetric graphs that exhibit certain properties. However, not too many results with graph products are used in this text—and when they do occur, the definition of the particular product used is reviewed, especially since notation for graph products varies in the literature.

Another type of product is called a "blow-up". Recall (see Definition 1.7.2) that a set S of vertices in a graph is called independent if and only if S induces no edges. For a graph G and an integer $t > 1$, let $G(t)$ denote the graph obtained by replacing each vertex $x \in V(G)$ with an independent set $S(x)$ of t vertices, and if $\{x, y\} \in E(G)$, then for each $x' \in S(x)$ and $y' \in S(y)$, add the edge $\{x', y'\}$. In each $S(x)$, no edges occur. Thus, in $G(t)$, each vertex is replaced by an independent set of vertices, and each edge

is replaced by a complete bipartite graph. For example, $K_3(4)$ is the complete 3-partite graph $K_{4,4,4}$ (see Definition 1.6.7).

1.7.4 Line graphs

One might study graphs from the perspective of which pairs of vertices form edges. One could also look at a graph as a collection of edges, some pairs of which are incident (share a vertex).

Definition 1.7.6. *If $G = (V, E)$ is a graph, define its* line graph *$L(G)$ to be the graph whose vertices are E and edges in $L(G)$ are pairs of edges in G that share a vertex.*

For example, $L(K_2)$ is a single isolated vertex. If P_2 denotes a path with two edges (on 3 vertices), then $L(P_2) = K_2$. One can also verify that $L(K_3) = K_3$.

Exercise 33. *Find the line graphs $L(K_4)$ and $L(K_{2,3})$.*

Exercise 34. *Show that $L(K_{3,3}) = K_3 \square K_3$ (which is a rook's graph—see Definition 1.7.5 and comments following).*

Exercise 35. *Show that the line graphs of $K_{1,3}$ and K_3 are the same.*

Exercise 36. *Show that the line graph of a cycle is another cycle of the same length.*

A graph G is called *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph. Exercise 37 shows that line graphs are claw-free. (For a survey on claw-free graphs, see [355].)

Exercise 37. *Show that for any graph G, its line graph does not contain an induced copy of $K_{1,3}$.*

Exercise 38. *Let G be a graph. Find expressions (in terms of parameters for G) for the number of vertices and the number of edges in the line graph $L(G)$.*

Let G be a graph and let $H = L(G)$ be its line graph. Then viewing each edge in G as a set, these sets are a witness to the fact that $L(G)$ is an intersection graph (see Section 1.6.7).

See Exercise 52 for another simple exercise regarding line graphs.

1.7.5 Deleting, contracting, and subdividing

Let G be a graph and let $x \in V(G)$. The graph $G - x$ is the graph formed by deleting the vertex x and all edges incident with x. (The notation $G - x$ is sometimes replaced with $G \backslash \{x\}$, however set subtraction is not entirely correct.) Similarly, if $e \in E(G)$, the graph $G - e$ is the graph obtained by simply deleting the edge e (and not the vertices

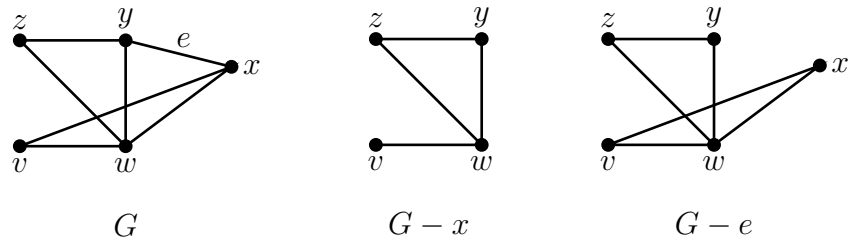


Figure 1.38: Deleting a vertex or an edge

incident with e). The notation $G - e$ is often denoted $G \backslash \{e\}$, which is consistent if one thinks of a graph only as a collection of edges. See Figure 1.38 for an example.

If G is a graph and $e = \{x, y\} \in E(G)$ is an edge, form the graph G/e by "contracting" the edge, where x and y are identified, and any multiple edges or loops thereby formed are deleted. The new graph G/e has one less vertex than G (see Figure 1.39, where x and y become a new vertex u).

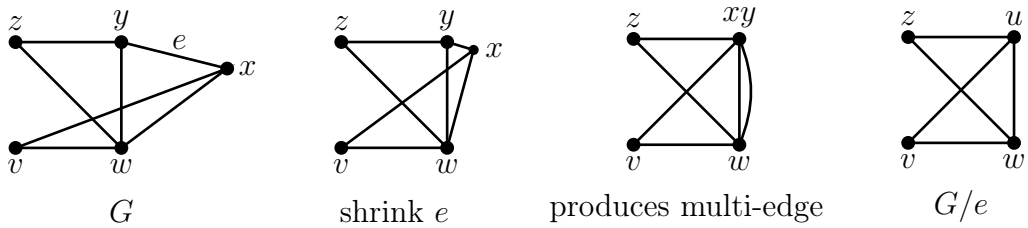


Figure 1.39: Contracting the edge e

The following concept can be used to describe certain structural properties of graphs (including colouring properties—*e.g.*, see Section 6.11 or Chapter 7).

Definition 1.7.7. *A graph H is a minor of G if and only if H can be obtained from G by deleting vertices and/or edges and by contracting edges.*

For example, for each $n \geq 3$, the graph K_3 is a minor of the cycle C_n.

Exercise 39. *Show that K_5 is a minor of the Petersen graph.*

A minor of a graph is sometimes called a *graph minor*. It is known [799] that for graphs H and G, there is a polynomial time algorithm that tests whether or not H is a minor of G. In the 1980s and 1990s, Robertson and Seymour gave a sequence of roughly 20 papers about minors; their work seems to have begun a whole new field of study using minors to classify certain graph structures. For a short survey and more references for graph minors and their applications, see [641].

If G is a graph, a new graph H can be constructed from G by "subdividing" edges, that is, by adding more vertices along an edge. In Figure 1.40, the graph H is formed from G by subdividing the edges $\{v, w\}$ and $\{z, y\}$.

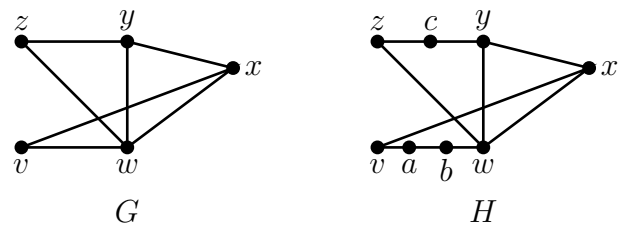


Figure 1.40: Subdividing with 3 new vertices

1.8 Degrees

The degree of a vertex in a graph is a graph notion that corresponds to valence in chemistry.

Let $G = (V, E)$ be a (simple) graph and let $x \in V$ be a vertex. By Definition 1.3.5, the degree of x, denoted $\deg(x)$, (or $\deg_G(x)$, if needed), is the number of edges incident with x.

In a multigraph (which may arise from, say, a chemical structure), the degree of a vertex v is the number of edges incident with v, where loops contribute 2 to the degree. For example, for the multigraph in Figure 1.3, $\deg(a) = 3$, $\deg(b) = 6$, $\deg(c) = 4$, and $\deg(d) = 4$. (Degrees for directed graphs are discussed in Section 12.1.)

For a graph G and a vertex $x \in V(G)$, the *neighbourhood of* x in G is

$$N(x) = N_G(x) = \{y \in V(G) : \{x, y\} \in E(G)\}.$$

So for simple graphs, $\deg(x) = |N(x)|$, but this fails for multigraphs (since x may be adjacent to a vertex y with three multi-eddges, but have only one vertex y in its neighbourhood). The neighbourhood $N(x)$ of a vertex x is sometimes called the *open neighbourhood*, and the *closed neighbourhood* is $N[x] = N(x) \cup \{x\}$.

If more than one graph is under discussion, the notation $\deg_G(x)$ indicates the degree of x as a vertex of G. A vertex x with $\deg(x) = 0$ is said to be an *isolated vertex*. A vertex x with $\deg(x) = 1$ is called an *end vertex*, or a *pendant vertex.*, and the sole edge containing x is called a pendant edge.

Exercise 40. *For any graph G on $n \geq 2$ vertices, are there always (at least) two vertices with the same degree? If so, prove it; if not, give a counterexample.*

See Exercise 108 for a similar problem (about Eulerian graphs; see Definition 2.2.1).

Exercise 41. *For each $n \geq 2$, construct a graph on n vertices that have only one pair of vertices with the same degree.*

A graph that has only one pair of vertices with the same degree is sometimes (*e.g.*, [190]) called "nearly irregular". If a graph G is nearly irregular, then so is $\overline{G}$. If a graph on n vertices has an isolated vertex, then there is no vertex of degree $n-1$, and conversely. So if a nearly irregular graph is found on n vertices, there is always another. In fact, for each n, there are only two nearly irregular graphs—the graph found in Exercise 41 and its complement.

Exercise 42. *Prove that for each n, there are precisely two nearly irregular graphs. Hint: Use induction.*

Degrees can be used to count edges.

Lemma 1.8.1 (Handshaking lemma)**.** *For any graph or multigraph G,*

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|.$$

Proof: Let G be a multigraph, Since each edge in G contributes 2 to the total of degrees (one from each end, loops contribute 2), the lemma follows. □

"Handshaking lemma" is often abbreviated by "HSL".

Exercise 43. *Prove Lemma 1.8.1 by mathematical induction in two ways: inducting on the number of vertices and inducting on the number of edges.*

Exercise 44. *Let G be connected graph where every vertex has degree 10. Show that deletion of any edge does not disconnect the graph.*

The result in the next lemma might be considered as one of the first interesting theorems regarding graph structure, but it has a surprisingly simple proof using the handshaking lemma.

Lemma 1.8.2. *In any graph or multigraph, there are an even number of vertices with odd degree.*

Exercise 45. *Using Lemma 1.8.1, prove Lemma 1.8.2.*

The result in Lemma 1.8.2 is the key to many proofs in graph theory; for example, see Theorem 2.2.4. Cameron and Edmonds [177] outline more examples of non-trivial theorems that are proved about a given graph G by producing a related graph H and applying Lemma 1.8.2 to H. (An example of such a non-trivial result that can be proved by this technique is Smith's theorem, given here as Theorem 2.3.16 or Thomason's generalization thereof, given here as Theorem 2.3.19.)

The smallest degree in a graph G is denoted by $\delta(G)$ and the largest degree by $\Delta(G)$.

Exercise 46. *Classify all graphs G with $\Delta(G) \leq 2$.*

If every vertex in a graph G has degree k, then G is called *k-regular* (in which case $\delta(G) = \Delta(G) = k$). A 3-regular graph is called *cubic* (not to be confused with "cube graphs"—see Section 1.6.5).

A 0-regular graph is a collection of isolated vertices, and a 1-regular graph is a collection of disjoint edges. The 2-regular graphs are collections of disjoint cycles. There is no such short description of 3-regular (cubic) graphs. For $n \geq 1$, the graph K_n is $(n-1)$-regular.

Remark 1.8.3. *Recall that a "cube graph" is not necessarily cubic! A cubic graph is 3-regular, so the only cube graph that is cubic is Q_3.*

Lemma 1.8.4. *For any non-negative integer k, a k-regular graph on n vertices has $\frac{nk}{2}$ edges.*

Exercise 47. *Use the handshaking lemma (Lemma 1.8.1) to prove Lemma 1.8.4.*

A simple application of Lemma 47 with $k = 3$ is worth noting for later application.

Lemma 1.8.5. *Any cubic graph has an even number of vertices.*

In a graph on n vertices, each vertex can have at most $n-1$ neighbours, and so the following elementary result holds:

Lemma 1.8.6. *For integers k and n satisfying $0 \leq k \leq n-1$, if a graph G is k-regular, then its complement $\overline{G}$ is $(n-1-k)$-regular.*

Recall from Section 1.6.5 that Q_k denotes the (hyper)cube graph whose vertices are binary strings of length k, where two strings are adjacent if and only if they differ in precisely one coordinate. The number of edges in Q_k was shown in Lemma 1.6.5 to be $k \cdot 2^{k-1}$. The proof given for Lemma 1.6.5 was by induction. Counting degrees gives another short proof.

Exercise 48. *For a positive integer k, use Lemma 1.8.4 to show that the cube graph Q_k has $k \cdot 2^{k-1}$ edges.*

Exercise 49. *Show that there are precisely two 4-regular graphs on 7 vertices.*

Theorem 1.8.7. *For positive integers k and n, there exists a k-regular graph on n vertices if and only if $0 \leq k \leq n-1$ and k and n are not both odd.*

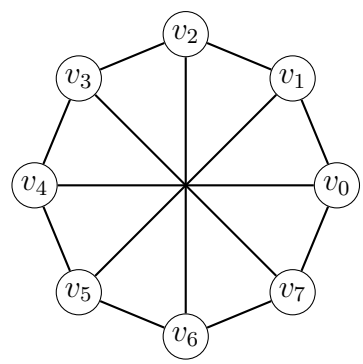
Proof: Suppose that k and n are so that there exists a k-regular graph on n vertices. By Lemma 1.8.2, if k and n are both odd integers, there is no k-regular graph on n vertices. Also, in a graph on n vertices, the smallest possible degree is 0 and the largest is $n-1$.

To prove the other direction, let k and n be integers, not both odd, so that $0 \leq k \leq n-1$. Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$. (Indices $0, \ldots, n-1$ are used instead of $1, \ldots, n$ because modular arithmetic is used for the constructions.)

First suppose that k is even. If $k = 0$, the empty graph on n vertices is 0-regular, so suppose that for some $j \geq 1$, $k = 2j$. Define G by, for each $i = 0, \ldots, n-1$, let v_i be adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{i+j}$ and $v_{i-1}, v_{i-2}, \ldots, v_{i-j}$ (where subscripts are computed modulo n). Then G is k-regular.

When k is odd, say $k = 2j+1$, for a graph to exist, n is even, say $n = 2\ell$. In this case, define G by, for each i, let v_i be adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{i+j}$, $v_{i-1}, v_{i-2}, \ldots, v_{i-j}$ and to $v_{i+\ell}$. □

Example 1.8.8. *By the construction given in the proof of Theorem 1.8.7, a 3-regular graph on 8 vertices is*



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Exercise 50. *Using the construction in Theorem 1.8.7 to draw a 4-regular graph on 9 vertices and a 5-regular graph on 12 vertices.*

Exercise 51. *Show that for any graph G, if $\Delta(G) = k$ then there is a k-regular graph H containing G as a subgraph.*

Exercise 52. *Let G be a k-regular graph. Show that its line graph $L(G)$ is $(2k-2)$-regular.*

Lemma 1.8.9 (Folklore)**.** *If $G = (V, E)$ is a simple graph, then*

$$\sum_{x \in V} \deg(x)^2 = \sum_{\{a,b\} \in E} (\deg(a) + \deg(b)). \tag{1.1}$$

Proof: For a fixed $x \in V$, the expression $\deg(x)$ occurs in the right hand sum precisely when $\{x, b\}$ is an edge, which occurs $\deg(x)$ times. □

One proof of the following lemma uses the Cauchy–Schwarz inequality (see Section 18.3).

Lemma 1.8.10 (Folklore)**.** *If G is a graph with average degree d, then G contains an edge $\{x, y\} \in E(G)$ for which* $\deg(x) + \deg(y) \geq 2d$.

Proof: Let G have n vertices and m edges. Then $d = 2m/n$, and

$$\begin{aligned}
\sum_{\{x,y\}\in E(G)} (\deg(x) + \deg(y)) &= \sum_{x\in V(G)} \deg(x)^2 && \text{(by Lemma 1.8.9)} \\
&\geq \frac{1}{n}\left(\sum_{x\in V(G)} \deg(x)\right)^2 && \text{(by Cauchy–Schwarz)} \\
&= n\left(\frac{2m}{n}\right)^2 && \text{(by Lemma 1.8.1).}
\end{aligned}$$

Thus the average of $\deg(x) + \deg(y)$ (over all edges) is at least $4m/n = 2d$. □

Exercise 53 (The handshake problem)**.** *At a party with n married couples, some pairs of people shake hands. No pair of people shake hands more than once, no person shakes their own hand, and no person shakes their own partner's hand. Of these n couples, one couple, say, Jack and Jill, hosts the party. Prove that if the people not including Jack all shake a different number of hands, then Jill shook $n - 1$ hands.*

Definition 1.8.11. *If G is a graph on n vertices, a degree sequence for G is a sequence $(d_1, \ldots, d_n)$ of non-negative integers so that there exists a labelling of $V(G) = \{v_1, \ldots, v_n\}$ so that for each $i = 1, \ldots, n$,* $\deg(v_i) = d_i$.

A sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of non-negative integers is called *graphic* if and only if there is some (simple) graph G (and an ordering of its vertices) with degree sequence $\mathbf{d}$. Any simple graph with degree sequence $\mathbf{d}$ is said to realize the sequence $\mathbf{d}$; some say that a graphic sequence is *realizable*.

Note that a sequence is graphic if and only if it is realizable by a simple graph—one can extend such definitions for multigraphs, but if so, this needs explicit mention. There are many sequences realizable as a multigraph but not as a simple graph. By the handshaking lemma, the sum of the degrees in a multigraph is even. This is the only constraint on a sequence of non-negative integers in order to be the degree sequence of a multigraph!

Exercise 54. *Prove that a sequence of non-negative integers $d_1, \ldots, d_k$ represents the degree sequence of a multigraph if $\sum_{i=1}^{k} d_i$ is even.*

Exercise 55. *Find a sequence that is the degree sequence of a multigraph but is not the degree sequence of a simple graph.*

Depending on the author or application, often a degree sequence is written as either a non-decreasing sequence or a non-increasing sequence. (For this section, non-decreasing is used, that is, largest first, smallest last.)

Remark 1.8.12. *A graphic sequence can sometimes be realized by different graphs. For example, the sequence* $(2, 2, 2, 2, 2, 2, 2)$ *is realized by* C_7 *and by the disjoint union of* C_4 *and* C_3*.*

Not every (non-decreasing or not) sequence is graphic (for example, (5,0,0,0) is not). By the handshaking lemma, if entries in a sequence sum to an odd number, then the sequence is not graphic. There are other tests to see if a particular sequence is indeed graphic; one simple test was proved independently by Havel and Hakimi:

Theorem 1.8.13 (Havel, 1955 [500], Hakimi, 1962 [476])**.** *Let*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n)$$

be a non-increasing sequence of non-negative integers. Define the sequence

$$\mathbf{d}^* = (\underbrace{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1}_{d_1}, d_{d_1+2}, \ldots, d_n)$$

of $n - 1$ *numbers defined by deleting* d_1 *and subtracting 1 from the next* d_1 *entries.The sequence* $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}^*$ *is graphic.*

Proof: If the sequence

$$\mathbf{d}^* = (\underbrace{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1}_{d_1}, d_{d_1+2}, \ldots, d_n)$$

is realizable by a graph G^* on vertices $v_2, \ldots, v_n$, then the graph G formed by adding a new vertex v_1 that is adjacent to $v_2, \ldots, v_{d_1+1}$, is a witness to $\mathbf{d}$ being graphic. So, if $\mathbf{d}^*$ is graphic, then $\mathbf{d}$ is graphic.

So suppose that $\mathbf{d}$ is graphic with witness G on vertices $v_1, \ldots, v_n$. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_{d_1}}$ be the neighbours of v_1, where $i_1 < i_2 < \cdots < i_n$. If $i_1 = 2$, $i_2 = 3$, $\ldots$, $i_{d_1} = d_1 + 1$, then $\mathbf{d}^*$ is realized by the subgraph $G - v_1$. Suppose that v_1 is not adjacent to the next d_1 vertices. Let $1 < i < j$ be so that v_1 is adjacent to v_j but not adjacent to v_i. If $d_j = d_i$, then swapping the labels for v_j and v_i gives the same graph G with the same degree sequence, so suppose, without loss of generality, that $d_i > d_j$. Then there exists a vertex v_k so that $\{v_i, v_k\} \in E(G)$ but $\{v_j, v_k\} \notin E(G)$. Create a new graph H by deleting the edges $\{v_1, v_j\}$ and $\{v_j, v_k\}$ and inserting edges $\{v_1, v_i\}$ and $\{v_j, v_k\}$. Then H has the same degree sequence as G, and v_1 has neighbours that are "closer" to v_1. Continue this process of interchanging two edges to maximize the degree of the neighbours of v_1, producing a graph with the same degree sequence of G and where

v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$. Now deletion of v_1 creates the sequence $\mathbf{d}^*$. Thus $\mathbf{d}^*$ is realizable. $\square$

For example, if $\mathbf{d} = (5, 4, 3, 2, 2, 1, 1)$, then $\mathbf{d}^* = (3, 2, 1, 1, 0, 1)$, and after re-arranging, becomes $(3, 2, 1, 1, 1, 0)$. Repeated application of $*$ gives gives $(1, 0, 0, 1, 0)$, and after re-arranging, $(1, 1, 0, 0, 0)$, which is realizable with five vertices, only one pair of which is an edge. So by Theorem 1.8.13, each of the sequences is realizable; in particular, the original $\mathbf{d}$ is realizable.

If one tests a given sequence by applying Theorem 1.8.13 repeatedly, and the sequence $(3, 1, 0, 0)$ is arrived at, then this sequence is not graphic because the vertex with degree 3 sends out three edges, so three 1s (or bigger) are necessary. Then (by the Havel–Hakimi process) the original is not graphic. However, the sequence $(3, 1, 0, 0)$ is realized by a *multigraph*: one loop and edge emanating from the same vertex.

On the other hand, if at some point in the Havel–Hakimi process, a graphic sequence is produced (for example, if a sequence of all 0s is derived, which is realized by an empty graph) then going back up the list of sequences obtained, create a graph for each sequence by adding a new vertex each time.

Note: When testing to see if a sequence is graphic (before applying the Havel–Hakimi process), a preliminary check (mentioned above) that the sum of the degrees is even might save on some unnecessary calculations.

Exercise 56. *Show that the degree sequence* $(5, 4, 4, 4, 3, 2, 1)$ *is not realizable.*

Exercise 57. *Use the Havel–Hakimi theorem (Theorem 1.8.13) to show whether or not the sequence* $(7, 6, 5, 4, 4, 4, 2, 1, 1)$ *is graphic. If this sequence is graphic, find a graph that realizes it.*

Exercise 58. *Either find a graph realizing the degree sequence* $(6, 4, 4, 2, 2, 1, 1)$ *or prove that no such graph exists.*

Another (more direct) test for realizability was proved by Erdős and Gallai:

Theorem 1.8.14 (Erdős–Gallai, 1961 [317])**.** *Let* $\mathbf{d} = (d_1 \geq \cdots \geq d_n)$ *be a non-increasing sequence of non-negative integers. Then* $\mathbf{d}$ *is graphic if and only if* $\sum_{i=1}^{n} d_i$ *is even and for each* $k \in \{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

There are many proofs of Theorem 1.8.14; for example, proofs have appeared by Berge (using network flows and Ferrers diagrams, see, *e.g.*, [94]), Harary [486] (a long inductive proof), Choudum [210], Aigner and Triesch [17] (using ideals in a dominance order), Tripathi and Tyagi [930] (indirect), and by Tripathi, Venugopalan, and West [931] (the shortest proof I have found).

1.9 Connected graphs and distance

Recall (from Definition 1.4.2) that a graph is $G = (V, E)$ is *connected* if and only if for every $x, y \in V$, $x \neq y$, there is a walk from x to y.

Exercise 59. *Show that in a graph G with $u, v \in V(G)$, if there exists a walk from u to v, then there exists a path from u to v.*

In virtue of Exercise 59), some authors might define "connected" by saying that between any two vertices, there is a path.

Definition 1.9.1. *A* component *of a graph is a maximal connected subgraph (i.e., a connected subgraph not contained in any larger connected subgraph).*

If a graph G is not connected, say that G is *disconnected.*

Exercise 60. *Let G be a graph on at least 3 vertices. Prove that at least one of G or $\overline{G}$ is connected.*

Lemma 1.9.2. *If G is a connected graph on n vertices, then $|E(G)| \geq n - 1$.*

Proof: One proof uses strong induction on n. For each positive integer n, let $S(n)$ be the statement that if a graph G on n vertices is connected, then $|E(G)| \geq n - 1$.
Base step: When $n = 1$, there is only one graph with 1-1=0 edges, so $S(1)$ is true.

Inductive step: Let $p \geq 2$ and suppose that each of $S(1), \ldots, S(p-1)$ is true. Let G be a connected graph on p vertices and let $x \in V(G)$. Let $G - x$ have k components ($k = 1$ is possible), say $G_1, G_2, \ldots, G_k$. Suppose that the numbers of vertices in the components are, respectively, $p_1, \ldots, p_k$, where $\sum_{i-1}^{k} p_i = p - 1$. For each $i = 1, \ldots, k$, there is an edge from x to G_i, so

$$\begin{aligned}
|E(G)| &\geq |E(G - x)| + k \\
&= \sum_{i=1}^{k} |E(G_i)| + k \\
&\geq \left(\sum_{i=1}^{k} (p_i - 1) \right) + k \qquad\qquad \text{(by } S(p_1), \ldots, S(p_k)\text{)} \\
&= p - 1 - k + k \\
&= p - 1.
\end{aligned}$$

So $|E(G)| \geq p - 1$, showing that $S(p)$ holds, completing the inductive step.

By mathematical induction, for all $n \geq 1$, $S(n)$ holds. □

Exercise 61. *Does the statement of Lemma 1.9.2 also hold true for multigraphs?*

Exercise 62. *Let G be a graph on n vertices with m edges and k (connected) components. Show that $m + k \geq n$.*

Exercise 63. *Prove that if G is a graph on n vertices and $\delta(G) \geq \frac{n-1}{2}$, then G is connected.*

(The upcoming Lemma 2.3.2 uses minimum degree to guarantee long cycles.)

Exercise 64. *Let $d_1 \leq d_2 \leq \cdots \leq d_n = \Delta(G)$ be the degree sequence of a graph G. Show that if for each $j \leq n - 1 - d_n$, $d_j \geq j$ holds, then G is connected.*

For connected graphs, a result that gives a lower bound of the length of a longest path in terms of minimum degree appears in Exercise 94.

Exercise 65. *Prove that the maximum number of edges in a disconnected simple n-vertex graph is $\binom{n-1}{2}$, with equality only for $K_1 \cup K_{n-1}$.*

Exercise 66. *Suppose that G is a graph with exactly two vertices of odd degree. Show that these two vertices are in the same component (i.e., there is a path from one to the other).*

Definition 1.9.3. *The* distance *between vertices v and w in a connected component of a graph is the length (number of edges) of a shortest walk between them.*

Note that by Exercise 59, the distance between vertices is the length of shortest path between them.

The distance between u and v is usually denoted by $d_G(v, w)$, or simply $d(v, w)$ when clear. If v and w lie in different components of G, define their distance to be infinite and write $d(u, v) = \infty$.

If two vertices v, w have distance $d(v, w) = D$ any v–w path of length D is called a *geodesic*; in some texts, the less familiar term "diametral path" is used.

Distance in graphs satisfies some of the usual properties of distance in a metric space (*e.g.*, in a Euclidean geometry).

Theorem 1.9.4. *Let G be a graph (or mulitgraph). Then for any three vertices $u, v, w \in V(G)$, (i) $d(u, v) \geq 0$, and $d(u, v) = 0$ if and only if $u = v$. (ii) $d(u, v) = d(v, u)$. (iii) $d(u, w) \leq d(u, v) + d(v, w)$, called the "triangle inequality".*

The proofs of parts (i) and (ii) are immediate by definition; a proof of the triangle inequality is left as Exercise 67. (One reason for the name "triangle inequality" is that the sum of lengths of any two sides of a triangle is at least the length of the third side.)

Exercise 67. *Prove the triangle inequality in Theorem 1.9.4.*

Exercise 68. *Suppose that G is a graph and let $f : V(G) \to V(G)$ be a bijection. Show that if f is distance preserving (that is, if $d(x, y) = s$, then $d(f(x), f(y)) = s$) then f is an isomorphism.*

Exercise 69. *Let v be a vertex in a connected graph G. Prove that the sum of the distances from v to all other vertices of G, $\sum_{w \in V(G)} d(v, w)$, is at most $\binom{n}{2}$.*

1.10 Computing distances

Algorithms have been developed that find distances between points in graphs, weighted graphs (graphs whose edges have non-negative weights, like distances between cities in highway networks) or weighted digraphs. In this section is only a brief introduction to four such algorithms based on the Gould text [429] (see the original for discussions of complexity and pseudocode for each).

1.10.1 Distances in simple unlabelled graphs—breadth first search

In a simple undirected graph (with no labelling on the edges), distance between two vertices is measured by the number of edges in a shortest walk (which is a path) joining the two vertices. In such graphs, two adjacent vertices have distance 1, and the distance between any two vertices is either some positive integer or ∞. Finding distances in such graphs can be accomplished by what might be called a *breadth first search* (BFS) algorithm. (*e.g.*, in 1957, Moore [698] used such a BFS algorithm).

The idea is rather simple: to find the distances from a vertex x to all other vertices in a graph G, begin by labelling x with 0. Then label the neighbours of x with 1. If an unlabelled vertex is adjacent to some vertex labelled 1, label that vertex with 2. If an unlabelled vertex is adjacent to some vertex labelled 2, then label such a vertex with a 3. Continue until there are no unlabelled vertices that are adjacent to any of the labelled vertices; then all vertices reachable from x are labelled with their distance from x. Vertices not labelled by the end of the algorithm are then labelled ∞. Then all vertices are labelled with their distance from x.

A proof that Moore's BFS algorithm does the intended job is a straightforward induction argument, and so is left to the reader.

1.10.2 Distances in graphs/digraphs with positive weighted edges/arcs—Dijkstra's algorithm

Let $G = (V, E)$ be a graph (or digraph) and for each $e \in E$, let $w(e) > 0$ be a label on e indicating the length or "weight" of e. (The same assumption is made for directed graphs.) Let x and y be distinct vertices in G so that there exists an x–y path (or directed path). The "length" of a path in G is the sum of the weights of its edges. The

distance $d(x, y)$ between vertices x and y is the minimum length of any (directed) x-y path. Similar notions hold for undirected graphs.

How can one find a "shortest" path between two vertices x and y? A brute-force method might be to list all possible x–y paths and compute the "length" of each. However, such a technique might not be feasible for large graphs. The BFS algorithm works for graphs, but not for digraphs. One algorithm handles either (where distance in a digraph is "directed distance").

An algorithm given by Edsger W. Dijkstra [258] in 1956 (published in 1959) computes the distance in a graph or digraph G between a specified vertex x and all other vertices in G by using labels for vertices that are updated in rounds; in each round, the label of a vertex z is $L(z)$, the length of shortest x–z path found so far. The number of rounds is also considerably smaller than in any brute force attack—essentially one round for each vertex, and in each round, the number of calculations depends only linearly upon the degree (or outdegree).

The version presented here finds a specific distance between x and some y—but it works for any y.

Dijkstra's algorithm for graphs or digraphs

INPUT: A simple graph $G = (V, E)$ (or digraph) with a weight function $w : E \to \mathbb{R}^+$ and $x, y \in V$.

OUTPUT: $d(x, y)$.

Step 1: Put $L(x) = 0$ and for each other vertex v, put $L(v) = \infty$. Put $T = V$.

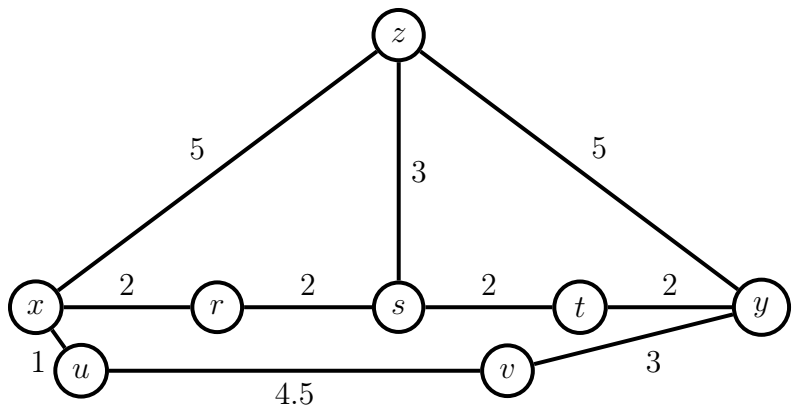
Step 2: Find a vertex $v \in T$ with minimum label $L(v)$.

Step 3: If $v = y$, stop.

Step 4: For every edge (or arc) $e = vw$, if $v \in T$ and $L(v) + w(e) < L(w)$ then put $L(w) = L(v) + \ell(e)$.

Step 5: Update T by removing v, that is, let the new T be $T \backslash \{v\}$. Go to step 2.

Here is an example for a simple graph. (The reader might benefit from running the algorithm on some version of this graph with directed edges.)



Solution:

Put $L(x) = 0$, and label all other vertices with ∞; put $T = V = \{r, s, t, u, v, x, y, z\}$.

The vertex in T with the smallest label is x, with $L(x) = 0$. So examine the three edges from x, namely xu, xr, and xz into T.

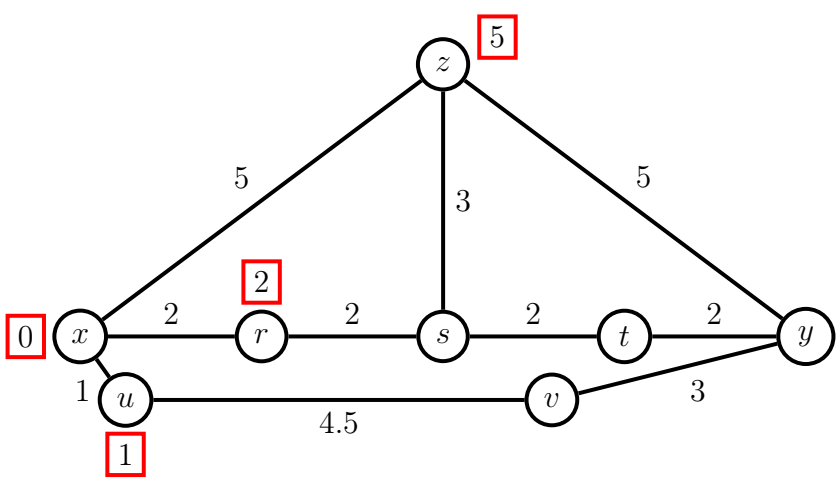
(i) For xu, $L(x) + w(xu) = 0 + 1 = 1 < \infty = L(u)$, so update $L(u) = 1$.

(ii) For xr, $L(x) + w(xr) = 0 + 2 = 2 < \infty = L(r)$, so update $L(r) = 2$.

(iii) For xz, $L(x) + w(xz) = 0 + 5 = 5 < \infty = L(z)$, so update $L(z) = 5$.

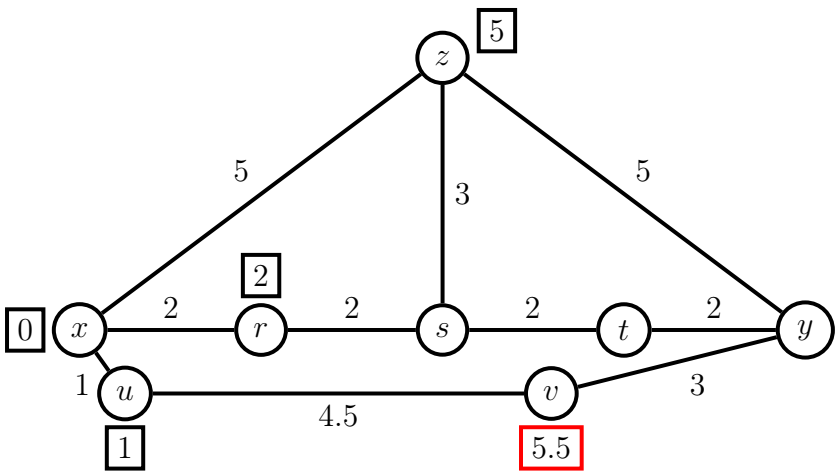
Delete x from T, leaving $T = \{r, s, t, u, v, y, z\}$.

Updated vertex labels are in red:



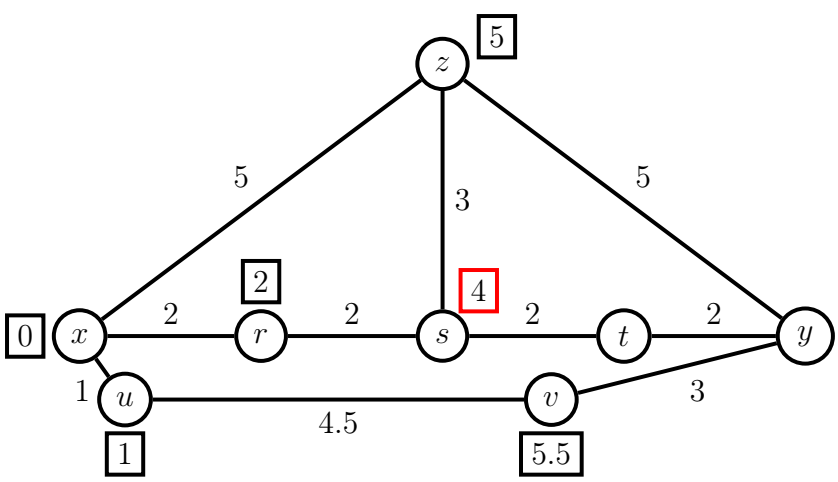
Since u is the vertex in T with the smallest label, consider edges leaving u going into T—there is only one, namely uv. Since $L(u) + w(uv) = 1 + 4.5 = 5.5 < \infty = L(v)$, put $L(v) = 5.5$.

Delete u from T, leaving $T = \{r, s, t, v, y, z\}$.



The vertex in T with the smallest label is r, with $L(r) = 2$. There is only one edge from r to T, namely rs. Since $L(r) + w(rs) = 4 + 4 < \infty = L(s)$, relabel $L(s) = 4$.

Remove r from T, leaving $T = \{s, t, v, y, z\}$.

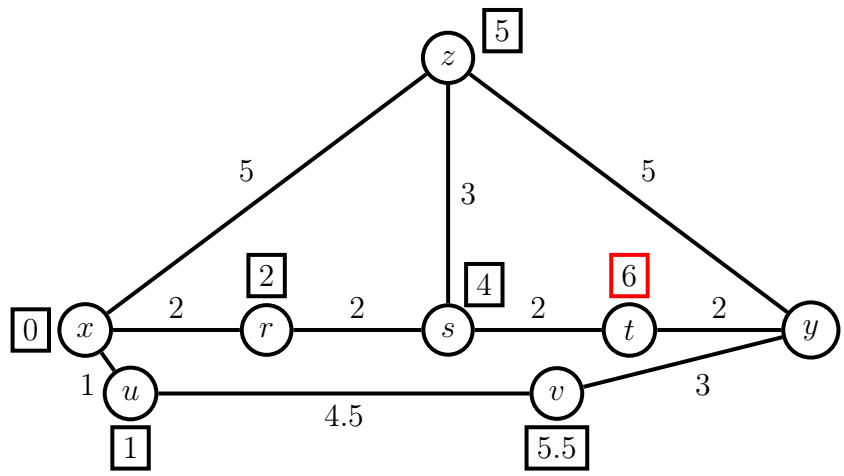


The vertex with a smallest label in T is s, with $L(s) = 4$. Examine the edges from s into T, namely, sz and st.

(i) For sz, since $L(s)+w(sz) = 4+3 = 7$ is greater than $L(z)$, the label of z remains unchanged at $L(z) = 5$.

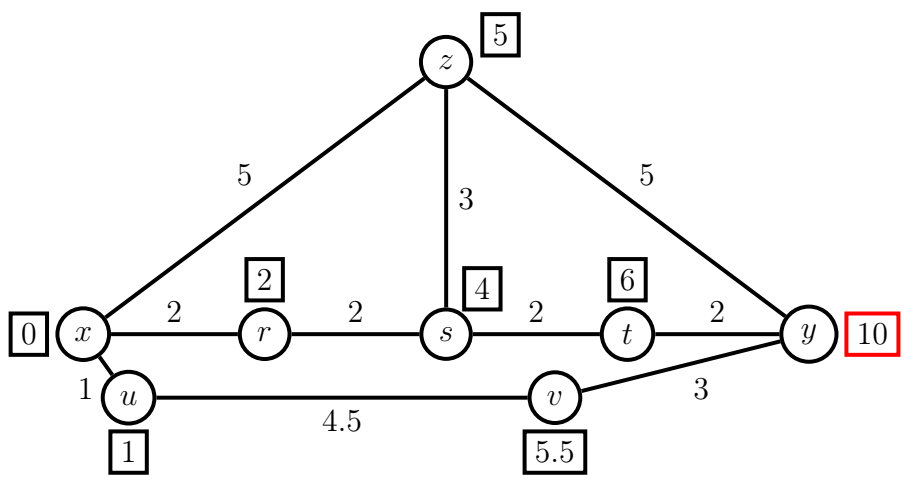
(ii) For st, since $L(s) + w(st) = 4 + 2 < \infty = L(t)$, put $L(t) = 6$.

Remove s from T, leaving $T = \{t, v, y, z\}$.



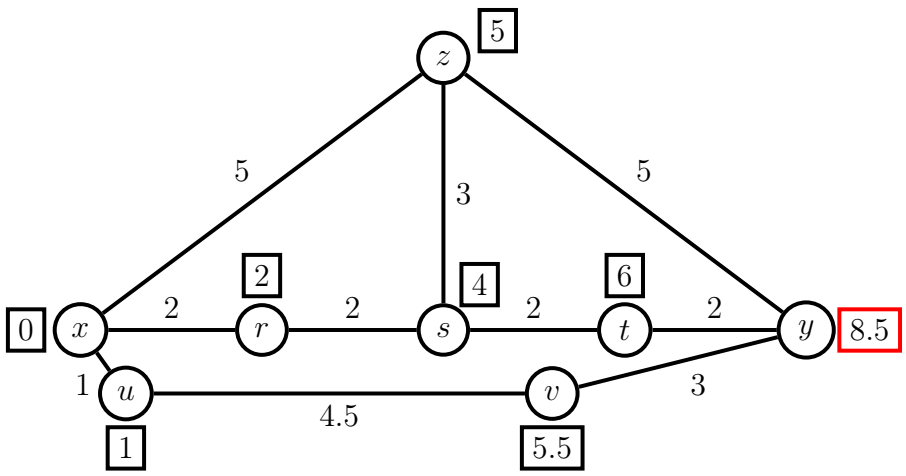
Among all those vertices in T, z has a smallest label, $L(z) = 5$. The only edge from z to T is zy. Since $L(z) + w(zy) = 5 + 5 < \infty = L(y)$, put $L(y) = 10$.

Remove z from T, leaving $T = \{t, v, y\}$.

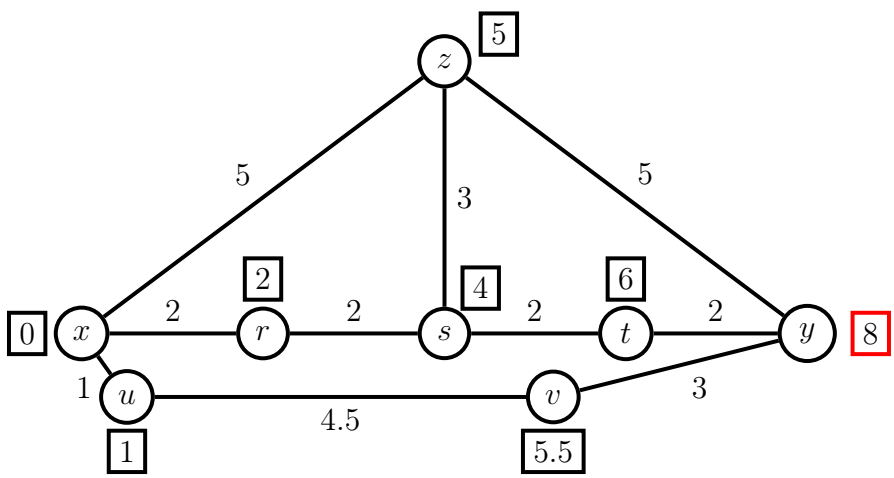


The vertex in T with the least label is now v, with $L(v) = 5.5$. The only edge from v to T is vy. Since $L(v) + w(vy) = 5.5 + 3 = 8.5 < 10 = L(y)$, relabel $L(y) = 8.5$.

Delete v from T, leaving $T = \{t, y\}$.



The vertex in T with a smallest label is t, with $L(t) = 6$. The only edge from t to T is ty. Since $L(t) + w(ty) = 6 + 2 = 8 < 8.5$, update $L(y) = 8$.



Delete y from T, leaving only y, so stop. □

Observe that the last drawing gives the distance from x to any other vertex, not just y.

Dijkstra's algorithm also works for digraphs. For another fully worked example (for digraphs), see [429].

1.10.3 Distances in digraphs with arbitrarily weighted arcs—Ford's and Floyd's algorithms

In a digraph $D = (V, A)$ suppose that each arc $e \in A$ is labelled with a real number $w(e)$, called its weight, possibly negative. One algorithm, given by Ford [373] in 1962, computes the distance from a vertex to all other vertices. The process is by labelling

and updating labels. A directed cycle with total negative weight (called a negative cycle) causes the algorithm to not to halt. Ford's algorithm for weighted digraphs also works for weighted simple graphs. Another similar algorithm was given by Floyd [370] also in 1962 that finds distances in digraphs (without negative cycles). (See [429, pp. 42–43] for the details and an example worked out.)

1.11 Diameter, radius, eccentricity

The diameter of a circle is the maximum distance between two points on that circle. An analogous definition holds for graphs.

Definition 1.11.1. *The* diameter *of a graph G is the maximum distance between vertices in G, that is,*

$$diam(G) = \max_{v,w \in V(G)} d(v, w).$$

For example, the diameter of a complete graph is 1. If two vertices in a graph lie in different components (that is, there is no path between them), define their distance to be infinite. So if a graph is disconnected, its diameter is infinite; hence, discussion of diameter is usually restricted to connected graphs. A path with infinitely many vertices has infinite diameter as well (but the infinite complete graph K_ω has diameter 1). In this section, all graphs are finite.

Exercise 70. *Show that for any $a, b \geq 2$, $diam(K_{a,b}) = 2$.*

Deleting edges from a graph can increase its diameter.

Exercise 71. *For $m, n \geq 2$, let G be a connected spanning proper subgraph of $K_{m,n}$ Show that the diameter of G is at least 3.*

Exercise 72. *Find the diameter of the Petersen graph.*

Exercise 73. *Find a graph G with three vertices, x, y, z so that $d(x, y) = d(y, z) = d(z, x) = diam(G) = 3$, or prove that none exists.*

Exercise 74. *Let G be a graph with diameter greater than 3. Show that the complement $\overline{G}$ has diameter less than 3.*

Exercise 75. *Show that every non-trivial self-complementary graph has diameter 2 or 3.*

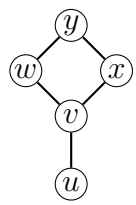
Exercise 76. *Let G be a simple graph with diameter 2 and maximum degree $\Delta(G) = |V(G)| - 2$. Show that $|E(G)| \geq 2|V(G)| - 4$.*

Definition 1.11.2. *The* radius *of a connected graph G is*

$$rad(G) = \min_{v \in V} \max_{y \in V} d(v, y).$$

So the radius is the smallest distance r for which there exists a vertex v so that all other vertices are within distance r from v. For example, K_7 has radius 1 and the cycles C_6 and C_7 have radius 3. Quite unlike ordinary circles, the diameter of a cycle C_n is the same as its radius.

Exercise 77. *Find the diameter and radius of the graph*



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Exercise 78. *Find the radius of the Petersen graph.*

Exercise 79. *Show that for any graph G, $rad(G) \leq diam(G) \leq 2 \cdot rad(G)$.*

Definition 1.11.3. *For a connected graph G, the* eccentricity *of a vertex v is*

$$\varepsilon(v) = \max_{w \in V(G)} d(v, w),$$

the maximum distance from v to any other vertex.

So the radius of G is the minimum eccentricity of any vertex in G.

Definition 1.11.4. *A vertex of minimum eccentricity is called a* central vertex*. The* center *of a graph is the subgraph induced by the central vertices.*

Some authors (*e.g.*, [429]) say that the center of a graph is simply the collection of central vertices, not necessarily the graph induced by them.

The next result is sometimes called Jordan's lemma, not to be confused with Jordan's lemma regarding contour integrals.

Lemma 1.11.5 (Jordan, 1869 [545])**.** *The center of a tree is either a vertex or an edge.*

Exercise 80. *Prove Lemma 1.11.5. Hint: one proof is by induction by removing leaves in rounds, and after each round the center remains the same.*

Definition 1.11.6. *The girth of a graph G is the length of a shortest cycle in G.*

The girth of G is denoted here by $\text{girth}(G)$ (although some texts use $g(G)$). Observe that the girth of an acyclic graph is not defined.

Lemma 1.11.7. *If G is a graph containing a cycle, then $girth(G) \leq 2 \cdot diam(G) + 1$.*

Exercise 81. *Prove Lemma 1.11.7.*

Lemma 1.11.8. *Let G be a connected graph with radius at most r. Then G contains at most $1 + r(\Delta(G))^r$ vertices.*

Proof: Let v be a central vertex in G, and for each $i = 0, 1, \ldots, r$, let V_i be the set of vertices with distance i from v (so $V_0 = \{v\}$). Then $|V(G)| = \sum_{i=0}^{r} |V_i|$. For each $i = 1, \ldots, r$,

$$|V_i| \leq \Delta(G)|V_{i-1}|.$$

By a simple induction argument, each $|V_i| \leq \Delta(G)^i$. Hence, using the trivial upper bound of $\Delta(G)^i \leq \Delta(G)^r$,

$$|V(G)| \leq 1 + \sum_{i=1}^{r} \Delta(G)^i \leq 1 + r(\Delta(G))^r.$$

□

The result in Lemma 1.11.8 can be improved in certain cases (*e.g.*, when $r \geq \Delta(G) > 1$) by using the identity $1 + \Delta + \Delta^2 + \cdots + \Delta^r = \frac{\Delta^{r+1}-1}{\Delta-1}$.

The diameter of K_3 is 1 and its girth is 3; similarly, the diameter of C_5 is 2 and its girth is 5. Observe that both of these graphs are regular. The following exercise might be considered challenging.

Exercise 82. *Let $d \in \mathbb{Z}^+$ and let G be a graph with diameter d and girth $2d+1$. Prove that G is regular.*

1.12 Adjacency and incidence matrices

1.12.1 Adjacency matrix of a graph

An *adjacency matrix* of a graph G on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ 0-1 matrix $A = [a_{ij}]$ with $a_{ij} = 1$ if and only if $\{v_i, v_j\} \in E(G)$. [If needed, the adjacency matrix of G is denoted by $A(G)$.] Note that a different ordering of the vertices yields a different adjacency matrix.

Observe that an adjacency matrix for an undirected graph is symmetric (i.e., $A^T = A$).

The following result is quite useful, and has many applications (*e.g.*, see Chapter 15).

Lemma 1.12.1. *Let $A = [a_{ij}]$ be the adjacency matrix of a graph G with vertices labelled $v_1, \ldots, v_n$. Then for each $k = 1, 2, \ldots$, the (i, j)-entry of A^k is the number of walks of length k from v_i to v_j.*

Proof: For $k \geq 1$, let $S(k)$ be the statement that the (i,j)-entry of A^k is the number of walks of length k from v_i to v_j. The proof of the general statement $S(k)$ is given here by induction on k.

Base step: For $k = 1$, $A^1 = A$ and $a_{ij} = 1$ if and only if $\{v_i, v_j\} \in E(G)$, and $a_{ij} = 0$ otherwise, and since a walk of length 1 is simply an edge, $S(1)$ holds.

Inductive step: Let $w \geq 1$ and suppose that $S(w)$ holds. Fix a graph on vertices $\{v_1, v_2, \ldots, v_n\}$ with adjacency matrix A. It remains to show $S(w+1)$, that is, the (i,j) entry of A^{w+1} is the number of walks of length $w+1$ from v_i to v_j.

Put $A^w = B = [b_{ij}]$. Then $A^{w+1} = A^w A = BA$ and so the (i,j) entry of A^{w+1} is $\sum_{\ell=1}^{n} b_{i,\ell} a_{\ell,j}$. A walk of length $w+1$ consists of a walk of length w first, to some v_ℓ, then one final edge from v_ℓ to v_j. For each $\ell = 1, \ldots, n$, by induction hypothesis, the number of walks of length w from v_i to v_ℓ is $b_{i,\ell}$. Such a walk can be completed to a walk of length $w+1$ to v_j if and only if $\{v_\ell, v_j\} \in E(G)$, that is, if and only if $a_{\ell,j} = 1$. Counting all walks of length $w+1$ from v_i to v_j then shows the total to be precisely the sum above, the (i,j) entry of A^{w+1}, so $S(w+1)$ holds. This completes the inductive step.

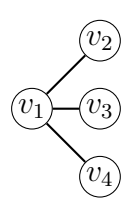
By MI, for each $k \geq 1$, $S(k)$ holds. □

Corollary 1.12.2. *Let G be a graph on n vertices with adjacency matrix A. Then G is connected if and only if every non-diagonal entry of $B = A + A^2 + \cdots + A^{n-1}$ is non-zero.*

Proof: If G is connected, between any distinct vertices v_i and v_j, there is a walk with length of most $n-1$ from v_i to v_j; hence for some $k \in \{1, \ldots, n-1\}$, the (i,j) entry of A^k is non-zero. Thus, when G is connected, every off-diagonal entry of B is non-zero. [In fact, when G is connected and $k \geq 2$, the diagonal entries of B are non-zero as well, since there is always some walk from v_i to some other v_j, then back to v_i.]

Let $B = (b_{ij})$. If for some $i \neq j$, $b_{ij} = 0$, then the (i,j) entry of each of $A, A^2, \ldots, A^{n-1}$ is also zero, and so there is no walk from v_i to v_j. Thus G is not connected. □

Example 1.12.3. *Let $G = K_{1,3}$ with vertices $\{v_1, v_2, v_3, v_4\}$ labelled in the diagram:*



With the vertices in natural order, let A be the associated adjacency matrix for G. Then

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}.$$

The $(1,2)$ entry of A^3 is 3, and there are three walks beginning at v_1 and ending at v_2, namely

$$v_1v_2v_1v_2, \quad v_1v_3v_1v_2, \quad v_1v_4v_1v_2.$$

Example 1.12.3 used only a very simple graph, one whose matrix is fairly easy to raise to arbitrary high powers (*e.g.*, by using the fact that the matrix satisfies $A^3 = 3A$, making computations elementary). For complicated graphs with more vertices and longer walks, computing A^k is often best done using one of the many matrix calculators available on-line. In larger more complicated graphs, finding the number of walks of some specified length between two vertices can be rather difficult to verify by tracing walks in a diagram, so the matrix approach is more reliable.

Exercise 83. *In the house graph in Figure 2.4, how many walks of length 4 are there from x to b (the apex and one vertex on the ground)?*

Using precisely the same arguments as for Lemma 1.12.1 and Corollary 1.12.2 yields a characterization for directed graphs being what is called "strongly connected"—see Section 12.6.

The adjacency matrix of a graph can be used to analyze many properties of a graph. For example, if A is an adjacency matrix for a graph G and the main diagonal of A^3 is all zeros, then the graph is triangle-free (contains no triangles), since a walk of length 3 from a vertex to itself is a triangle.

Exercise 84. *For a positive integer n, let J_n denote the $n \times n$ all 1s matrix. Show that a graph G is regular if and only its adjacency matrix A commutes with J_n (i.e., $AJ_n = J_nA$).*

The study of graphs by looking at an adjacency matrix (or other matrices associated with a graph) is sometimes called *algebraic graph theory*, or *spectral theory* (since the spectrum of eigenvalues of an adjacency matrix is very closely associated with many graph properties, *e.g.*, being bipartite, or regular). See Chapter 15 for an introduction to the theory.

1.12.2 Incidence matrix of a graph or hypergraph

Let G be a graph on n vertices $v_1, \ldots, v_n$ with m edges $e_1, \ldots, e_m$. The *incidence matrix* for G with respect to the given orderings of vertices and edges is an $n \times m$ 0-1 matrix $B = (b_{i,j})$, where each $b_{i,j} = 1$ if and only if $x_i \in e_j$.

This definition works also for hypergraphs, and an incidence matrix fully describes the structure of any hypergraph. If a hypergraph is k-uniform, then each row of B contains precisely k 1s, and the number of 1s in column j is the degree of v_j.

The incidence matrix for a (hyper)graph is used in various methods of counting (*e.g.*, see Theorem 15.3.14). Another feature of an incidence matrix is that sometimes its transpose gives another graph that is "dual" to the original (where vertices and edges change their roles in incidence); for example, see comments following Conjecture 6.13.5.

Exercise 85. *Let $k \geq 3$. Show that the transpose of an incidence matrix for the cycle C_k is an incidence matrix for C_k as well.*

See Definition 12.2.4 for an incidence matrix of a digraph.

Certain finite geometries can be interpreted as hypergraphs; in the case of finite projective planes, the transpose of an incidence matrix gives another, perhaps non-isomorphic, finite projective plane, called the *dual plane*. Incidence matrices have substantial use in the theory of designs (which are often viewed as uniform hypergraphs).

Chapter 2

Cycles and circuits

2.1 Existence of cycles

As given in Definition 1.4.7, a *cycle* in a simple graph (or multigraph) is a closed walk on at least three vertices with no vertex repeated (except the first and last). (Since no vertex is repeated, neither can any edge be repeated, so a cycle is an example of a closed trail.) A cycle with k vertices is denoted by C_k. In simple graphs, it is implicit that $k \geq 3$; however, in multigraphs, C_2 is sometimes defined as the graph with two vertices and two edges joining them; C_1 can also be used to denote a single vertex with a loop. For a more general definition of a cycle that applies to graphs, multigraphs, hypergraphs, and multihypergraphs, see Definition 13.1.1.

A graph with no cycles is called *acyclic*.

Definition 2.1.1. *A circuit in a graph or multigraph is a closed trail.*

For example, a closed trail in the shape of a figure eight is a circuit; in a circuit, vertices are allowed to be used more than just once.

Note: In some older texts and journal articles, the word "circuit" is used to denote what is defined here as a cycle, and the word "cycle" to denote what is called a circuit here. It seems that many papers written in the 1960s or before used "circuit" to mean what is defined here as a cycle. I can not be sure, but it seems that the switch occurred soon after the 1960s, but I can not seem to recall any "first" paper using the new (modern) definition for cycle. In the early 1960s, both Ore [728] and Kotzig [602] use "circuit", as does Woodall [994] in 1972 and Sheehan [857] in 1975. In 1967, Brian Alspach [39] used "cycle" in a tournament setting. One of the first textbooks on graph theory (by Harary [486, p. 13]) published in 1969 defines a cycle as is done here. Then Bondy [135], Bondy and Simonovits [144], Faudree and Schelp [356], Thomason [913], and Nešetřil and Rödl [717] used "cycle" in 1971, 1974, 1974, 1978, and 1979, respectively. Still in the 1970s, the word "polygon" was used to indicate a cycle. My guess is that over 99% of the papers written in the last two decades use the definitions given here.

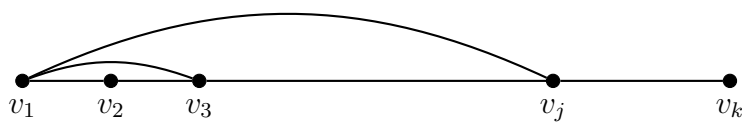
The result in the next exercise can be proved using facts about trees, but such facts are certainly not necessary for its solution.

Exercise 86. *Let G be a graph with n vertices. Give a direct proof (without using facts about trees) that if G has at least n edges, then G contains a cycle.*

The result in Exercise 86 doesn't yield any information on how large of a cycle is guaranteed (since, *e.g.*, a graph on n vertices with n edges could be a triangle attached to a long path or a cycle that uses all the vertices). The next result guarantees a cycle with a certain minimum length.

Lemma 2.1.2 (Dirac, 1952 [261, Thm 2])**.** *If G is a finite graph with minimum degree $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G) + 1$.*

Proof: Let G be a graph with $\delta(G) \geq 2$, and let $P = v_1, v_2, \ldots, v_k$ be a maximal path in G (i.e., the path cannot be extended). Since P is maximal, $N(v_1) \subseteq V(P)$. Let v_j be the farthest neighbour of v_1.



Since $\deg(v_1) \geq \delta(G)$, $j \geq \delta(G) + 1$. Then a cycle is formed joining $v_1, v_2, \ldots, v_j, v_1$, a cycle on at least $\delta(G) + 1$ vertices. □

Dirac's lemma (and its proof) is an essential tool in graph theory.

Exercise 87. *Does Dirac's lemma extend to multigraphs? Explain.*

Exercise 88. *Show that if a graph G has $n \geq 4$ vertices and $n + 1$ edges, then G contains at least two cycles.*

Exercise 89. *Let G be a connected graph containing exactly one cycle. Show that G has average degree 2.*

A graph containing only one cycle is sometimes called *unicyclic*.

The following exercise generalizes both Exercise 86 and 88:

Exercise 90. *Let G be a graph on n vertices and $m \geq n$ edges. Show that G contains at least $m - n + 1$ cycles.*

Exercise 91. *Show that if a connected graph G on $n \geq 4$ vertices has at least $2n - 2$ edges, then G has two cycles of the same length.*

Exercise 92. *For each positive integer n, find a graph on $2n+1$ vertices and $3n$ edges that contains no even cycle (a cycle on an even number of vertices).*

Exercise 93. *Show that if a graph G on n vertices has at least $\lfloor 3(n-1)/2 \rfloor + 1$ edges, then G contains a cycle of even length.*

The following generalizes Lemma 2.1.2 and, in a sense, Exercise 88.

Theorem 2.1.3 (Corrádi–Hajnal, 1963 [230]). *If G is a graph with $\delta(G) \geq 2k$, then G contains at least k vertex-disjoint cycles.*

A special case of Theorem 2.1.3 is worth noting separately:

Theorem 2.1.4 (Corrádi–Hajnal, 1963 [230]). *If n is a multiple of 3 and G is a graph with $\delta(G) \geq \frac{2}{3}n$, then G contains $n/3$ vertex-disjoint triangles.*

Theorem 2.1.3 was strengthened [207] to a condition that (for large enough graphs) either G is one of a few examples, or G has k vertex-disjoint *even cycles*.

If G is not connected, the bound in Lemma 2.1.2 is sharp (take the disjoint union of complete graphs, each with $\delta(G)+1$ vertices). The proof of Lemma 2.1.2 also gives a lower bound on the length of a path; if one knows that G is connected, much longer paths can be found (but not necessarily long cycles).

Although the result in the next exercise does not deal with cycles, it is in this section because to prove it, an idea similar to the proof of Theorem 2.3.2 can be used.

Exercise 94. *Prove that if G is a connected graph on $n \geq 3$ vertices, then G contains a path with least $\min\{n, 2\delta(G)+1\}$ vertices.*

Exercise 95. *Show that Lemma 2.1.2 is false for infinite graphs.*

Exercise 96. *Prove that the edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.*

Theorem 2.1.5 (Toida, 1973 [927]). *Let G be a graph with every vertex having even degree. Then each edge of G is contained in an odd number of cycles in G.*

The proof of Toida's theorem uses a particular sequence of edges (which Toida calls an "edge train"), and is less trivial than one might think; the proof is not given here. A proof of Toida's theorem is also discussed in [177], where it is considered as, in a sense, a corollary of Lemma 1.8.2.

An equivalent formulation of Toida's theorem is:

Theorem 2.1.6 (Toida, 1973 [927, Thm 3]). *Let G be a graph with every vertex having even degree. Then the number of paths between any two vertices is even.*

Toida's theorem also says that if u and v are the only two vertices of odd degree, then the number of u–v paths is odd.

Exercise 97. *Show why Theorem 2.1.5 is equivalent to Theorem 2.1.6.*

Later, McKee showed that a converse of Toida's theorem is also true.

Theorem 2.1.7 (McKee, 1983 [669]). *If a graph G has every edge in an odd number of cycles, then all vertices of G have even degree.*

Together, the theorems of Toida and McKee give a characterization of those graphs with even degrees. In Section 2.2, connectedness and even degrees guarantee that the graph is "Eulerian", that is, that the graph contains a circuit containing each edge precisely once—so the Toida–McKee result then gives another characterization of Eulerian graphs.

For other proofs of Toida's and McKee's theorems and other characterizations of even-degree graphs, see [368].

Exercise 98. *Show that the complement of C_6, denoted $\overline{C_6}$, is the graph of the triangular prism.*

Exercise 99. *Show that for any integer $k \geq 2$, if G is a k-regular graph with girth 5 then G contains at least $k^2 + 1$ vertices. Are there such graphs with exactly $k^2 + 1$ vertices?*

Exercise 100. *Show that if G is a graph on n vertices with $n+1$ edges, then girth$(G) \leq 2(n+1)/3$.*

As mentioned in [125, p. 105], for graphs on n vertices, if there are $n + 2$ edges, then the girth is at most $(n+2)/2$ and for $n+3$ edges, the girth is at most $4(n+3)/9$. There are many theorems about how the minimum degree guarantees certain types of cycles. A few of these results are reviewed here without proof (see [29] for more details). Some of these theorems also add a condition on "connectivity" (see Chapter 4 for more details on connectivity). There are various definitions of what it means for a graph to be 2-connected, but for present purposes, say that G is 2-connected if between any two distinct vertices x and y, there exists two internally disjoint x–y paths.

As defined already, the girth of a graph is the length of a shortest cycle. The length of a longest cycle in a graph G is called the circumference of G, often denoted by $c(G)$—here the circumference is abbreviated with the more informative notation circum(G).

Theorem 2.1.8 (Dirac, 1952 [261]). *If G is a 2-connected graph, then*

$$\textit{circum}(G) \geq \min\{|V(G)|, 2\delta(G).$$

A cycle is called even [odd] if the number of vertices in the cycle is even [odd].

Theorem 2.1.9 (Voss–Zuluaga, 1977 [965]). *Let G be a 2-connected graph on at least $2\delta(G)$ vertices. Then there exists an even cycle of length at least $2\delta(G)$. If G is not bipartite, then there exists an odd cycle of length at least $2\delta(G) - 1$.*

As mentioned in [29], three papers [32], [129], and [278] regarding minimum degree and circumference give a result that drops the requirement of being 2-connected:

Theorem 2.1.10. *For a positive integer k, if G is a graph on n vertices with $\delta(G) \geq n/k$, then circum$(G) \geq \lceil \frac{n}{k-1} \rceil$.*

For more on such problems, see [29] and [721]. Two more related theorems of possible interest (also found in [29]) are as follows:

Theorem 2.1.11 (Häggkvist, 1981 [471])**.** *Let $\ell \geq 2$ be an integer, and let $n \geq \binom{\ell+1}{2}(2\ell+1)(3\ell-1)$. If G is a graph on n vertices with $\delta(G) \geq \frac{n}{2\ell+1}$, then either G contains a copy of $C_{2\ell-1}$ or G contains no odd cycle on more than $\ell/2$ vertices.*

To state the next theorem, some notation (almost as in [29]) is handy: let ecircum(G) [or ocircum(G)] denote the length of a largest even [or odd] cycle in G.

Theorem 2.1.12 (Gould–Haxell–Scott, 2002 [430])**.** *For any constant $c > 0$, there exists another constant $K = K(c)$ so that if $n > 45Kc^{-4}$ and G is a graph on n vertices with $\delta(G) \geq cn$, then for every even $t \in [4, \text{ecircum}(G) - K]$, G contains C_t, and for every odd $t \in [K, \text{ocircum}(G) - K]$, G contains C_t.*

For a recent (2019) result on minimum degree conditions to guarantee cycles of all lengths modulo k, see [208].

2.2 Eulerian graphs and Eulerian circuits

Definition 2.2.1. *Let G be a graph or multigraph. An* Eulerian circuit *is a closed walk in G that uses each edge of G precisely once. A graph or multigraph G is called* Eulerian *if and only if G contains an Eulerian circuit.*

A closed walk that never repeats edges is a closed trail, and so sometimes an Eulerian circuit is called an *Eulerian trail*. It follows from the above definition that an Eulerian graph with no isolated vertices is connected. (Some texts define only connected graphs to be Eulerian.)

Exercise 101. *Find all five connected simple graphs on at most 5 vertices that are Eulerian.*

If $G = (V, E)$ is a multigraph and $v \in V$, the degree of v is the number of edges incident with v, where a loop at v counts as 2 degrees.

One direction of the next theorem (now called Euler's theorem) was proved by Euler, and the other direction was stated by Euler, but it was over a century before a complete proof was written by Hierholzer [511], who also gave an algorithm to find an Eulerian circuit (see [105] for history).

Theorem 2.2.2 (Euler)**.** *A connected multigraph G is Eulerian if and only if every vertex has even degree.*

Proof: There are two directions to this proof. If G is Eulerian, then every vertex has even degree, because when traversing an Eulerian circuit, two edges of the circuit are used each time a vertex is passed.

To see the other direction, induct on $m = |E(G)|$. For $m \geq 0$, let $S(m)$ be the statement that if G is a connected multigraph with m edges and all degrees even, then G is Eulerian. It is slightly easier to prove the same theorem for multigraphs, in which case the induction can start with either 0 or 1 edges (a single vertex with a single loop is connected, degree 2, and Eulerian). If one carries out the following proof for only simple graphs, perhaps the base case needs to consider the cases with $m \leq 3$ edges separately.

Base step: For $m = 0$, the only connected (multi)graph with even degrees and no edges is a single vertex, which is trivially Eulerian.

Inductive step: Let $\ell \geq 0$, and suppose that for each $i = 0, 1, \ldots, \ell$, $S(i)$ holds, that is, assume that the theorem is true for all multigraphs with at most ℓ edges. Let G be a connected multigraph, with all degrees even and with $\ell + 1$ edges. To prove $S(\ell + 1)$, it remains to show that G is Eulerian.

Since G is connected (and has at least two vertices), $d(x) = 0$ is impossible, so $\delta(G) \geq 2$. Then by Lemma 2.1.2, G contains a cycle C. Delete edges of C from G, giving a graph H that still has even degrees. Let $H_1, \ldots, H_k$ be the components of H. Since each H_i has fewer than ℓ edges, apply the induction hypothesis $S(|E(H_i)|)$ to each to get an Eulerian circuit C_i in H_i. Splice these together with C to get an Eulerian circuit for G as follows: Traverse C (in either direction) and for each i, let x_i be the first point on C contained in H_i. Then travel along all of C_i returning to x_i, then continue along C until the next H_j is intersected and repeat the process. The resulting circuit then uses all edges of G and so $S(\ell + 1)$ is true, completing the inductive step.

By MI, the theorem holds for graphs (or multigraphs) with any number of edges. □

As a result of Theorem 2.2.2, some authors (*e.g.*, see [927]) use the term "Eulerian graph" or "Euler graph" to simply mean that all degrees are even, with no concern for connectivity (other authors call such graphs "even").

How difficult is it to find an Eulerian circuit in an Eulerian graph? One obvious way is to duplicate the inductive proof given—start by finding any cycle, remove the cycle, leaving (perhaps) components. Then in each component, find an Eulerian circuit (if such a circuit is not obvious, apply the same algorithm to the component, first finding a cycle...) and splice each such circuit into the original cycle. This method can also be interpreted by simply splicing cycles together as they are found, since finding an Eulerian circuit in a component can be also found by first finding a cycle (and splicing

smaller circuits, which can also be found by first finding a cycle, and so on). Hierholzer's algorithm is based on splicing together cycles as they are found.

However, there is a "conceptually simpler" way to construct an Eulerian circuit in an Eulerian graph—found by Fleury [369] (also see [648, vol. IV]) in the late 1880s. To describe Fleury's algorithm, say that a *bridge* in a connected graph is an edge removal disconnects the graph.

Fleury's algorithm: In an Eulerian graph, start at any vertex and walk with the rule that in the graph forming the edges not yet used, don't use a bridge unless there is no other option. The resulting walk is an Eulerian circuit.

Fleury's algorithm is not so simple from a computational viewpoint since determining whether or not an edge is a bridge is a somewhat difficult computation.

Exercise 102. *In each of the graphs in Figure 2.1, find an Eulerian circuit by two methods, first by undoing the proof given for Theorem 2.2.2, and second, by Fleury's algorithm.*

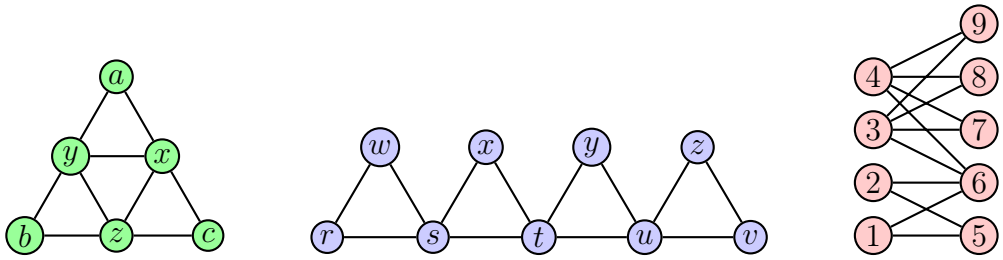


Figure 2.1: Eulerian graphs for Exercise 102.

Some graphic images can be drawn with a single stroke, with the starting point and end point being the same. Such images often have an obvious underlying graph. In Figures 2.2 and 2.3 are puzzles known to Lewis Caroll and Thomas O'Beirne (see [408, Ch. 10, Graph theory] for references). The solution to the Caroll puzzle given in [408] does not use graph theory, but instead uses black and white regions.

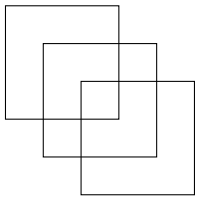


Figure 2.2: A puzzle used by Lewis Caroll—can you draw this in one stroke?

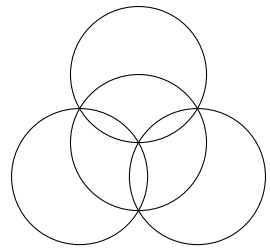


Figure 2.3: O'Beirne's four circle puzzle—can you draw this in one stroke?

Exercise 103. *Find drawings of the images in Figures 2.2 and 2.3 that are done in one stroke (without lifting the pencil, and retracing not allowed), with the beginning and end point the same. (Graph theory is not really required to spot one of many solutions, but the theory of Eulerian graphs can be used.)*

Exercise 104. *Prove that if G is a connected graph (not necessarily Eulerian), then there exists a closed walk on G that uses each edge precisely twice.*

See Theorem 12.4.1 for the analogous result to Theorem 2.2.2 for digraphs.

Recall that if not all vertices in a graph have even degree then, by Lemma 1.8.2, the number of odd degree vertices is even.

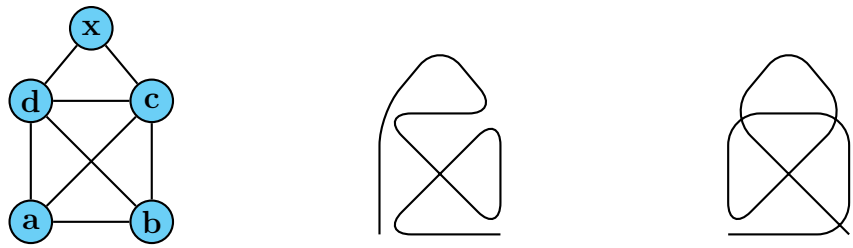
Theorem 2.2.3. *Let k be a positive integer, and let G be a connected multigraph with precisely $2k$ vertices with odd degree. Then there exist k open trails that together use each edge of G exactly once.*

Proof: Let $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ be the vertices of odd degree. Form a new multigraph H by adding the edges $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}$. If some of these pairs are already edges in G, the addition of these new edges produce multi-edges. Then vertices in G with even degree have no new edges added and so such vertices in H still have even degree, and since the degree of odd vertices in G is increased by 1 in H, all vertices of H have even degree. So by Theorem 2.2.2, H has an Eulerian trail, call it F. Removing the new edges from F breaks F into k open trails which, together, cover all edges of G precisely once. □

When $k = 1$ and there is only one pair of odd degrees, the one open trail guaranteed by Theorem 2.2.3 is called an *open Eulerian trail*, and so a graph with only two odd vertices is called *semi-Eulerian* (or sometimes, "pseudo-Eulerian"). So semi-Eulerian graphs can be drawn with a single stroke. See Figure 2.4 for an example.

Restating Theorem 2.2.3:

Theorem 2.2.4 (Graph drawing)**.** *Let $k \geq 1$ and let G be a connected multigraph with exactly $2k$ vertices of odd degree. Then drawing G can be done using k (and no fewer) strokes, each stroke drawn without lifting the pencil and retracing not allowed.*



Only vertices a and b have odd degree,... ...so start at a, going upward, say. ...or start at a, going to the right, say.

Figure 2.4: A semi-Eulerian graph can be drawn in one stroke, an open Eulerian trail.

The result in Theorem 2.2.4 can be used in industry. For example, a decomposition of a drawing into as few strokes as possible can be advantageous for laser etching devices, 3D printers, or CNC routers, since fewer strokes means fewer times the head of the machine needs to be switched on/off or lifted from the surface being machined. For example, it is not difficult to verify that the corresponding graph for the pattern in Figure 2.5 has only two vertices of odd degree, and so can be traced or etched by one continuous pass (no lifting of the stylus, no portion gone over twice).

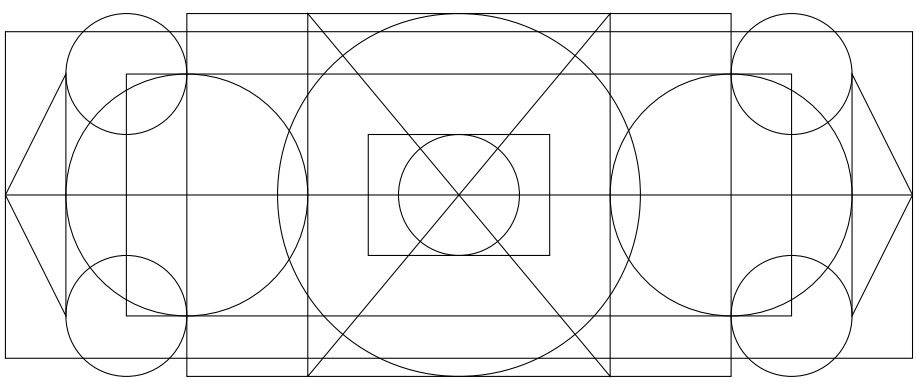
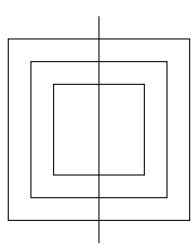


Figure 2.5: A pattern that can be etched using one continuous stroke, where no line is gone over twice

Exercise 105. *Find a way to draw*



using as few strokes as possible.

Exercise 106. *How many strokes are required to draw the pattern formed by the borders of squares in a chess board?*

Exercise 107. *Let G be a regular graph so that both G and $\overline{G}$ are connected. Prove that one of G or $\overline{G}$ is Eulerian.*

As in the solution to Exercise 40, any non-trivial graph has two vertices with the same degree; the following exercise is related.

Exercise 108. *Let G be an Eulerian graph on at least 3 vertices. Show that at least three vertices have the same degree.*

For another characterization of Eulerian graphs, the Toida's and McKee's theorems (Theorems 2.1.5 and 2.1.7), which, when taken together, say that a connected graph is Eulerian if and only if each edge is in an odd number of cycles. Also see Fleischner's paper [368] for more characterizations.

Another class of problems regarding tours can be solved with the above techniques. One classic instance of such a problem occurs as the "Crossing the lines" puzzle in Dudeney's *536 Puzzles & curious problems* [268, No. 414]. Another name for this problem might be "the five room puzzle".

Problem 2.2.5. *In Figure 2.6 is the floor plan of a house with five rooms and doorways in every wall. Is it possible to enter the house through some doorway and continue walking through each doorway precisely once, returning to the original starting point outside the house? If one loosens the requirements by allowing the starting point and terminal point to be different, can such a tour be found?*

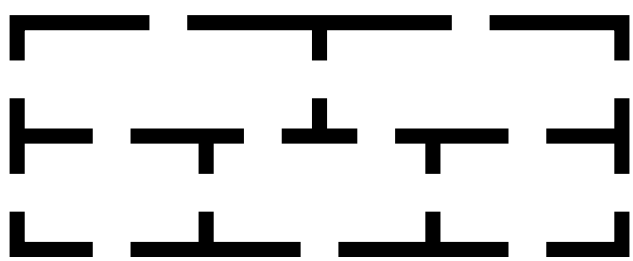


Figure 2.6: Can you go through each doorway exactly once and (maybe) return to your starting point?

One way to approach Problem 2.2.5 is draw the floor plan in ink, and attempt to find the tour by pencilling in a tour. Another approach is to make a physical model with wood and string as in Figure 2.7.

Figure 2.7: A model of the five room puzzle, ca. 2003. Photo by Y. Won

The first part of the puzzle asks if one starts with a loop of string, can the string be placed appropriately going through each doorway once? After many attempts, it seems as if this is not possible. So maybe if the string is not looped, a placement of the string might show a positive answer to the second part. Again, after many attempts one might convince oneself that again, this is impossible. However, one might not be certain that all possible placements have been tried, so a more mathematical solution is desired.

Solution to Problem 2.2.5: From the floor plan, construct a graph on six vertices. Represent each of the five rooms by a vertex, and let "outside the house" be represented by a sixth vertex z. Two vertices form an edge every time the corresponding rooms share a doorway, giving the multigraph in Figure 2.8. Then Problem 2.2.5 first asks if there is an Eulerian circuit for this graph, and then asks if there is an open Eulerian trail.

Then $\deg(u) = \deg(w) = 4$, $\deg(v) = \deg(x) = \deg(y) = 5$, and $\deg(z) = 9$, giving four vertices of odd degree. By Theorem 2.2.3 or 2.2.4, the graph is neither Eulerian, nor is it semi-Eulerian, and so requires at least 2 strokes to draw. Thus, the answer to Problem 2.2.5 in either case is "it can't be done". □

Exercise 109. *Modify Problem 2.2.5 by locking (or removing) as few doors as possible so that (a) a closed tour is possible, or (b) an open tour is possible.*

Exercise 110. *Twenty seven unit cubes are arranged to make one* 3×3 *cube Between any two unit cubes that share a face, there is a portal through which a nanobot can travel between adjacent cubes. Can a nanobot start in some unit cube and find a tour through the cubes using each portal exactly once, returning to the original cell? If the condition of returning to the original cell is dropped, can the nanobot find a tour that uses every portal?*

For (directed) Eulerian circuits in digraphs, see Section 12.4.

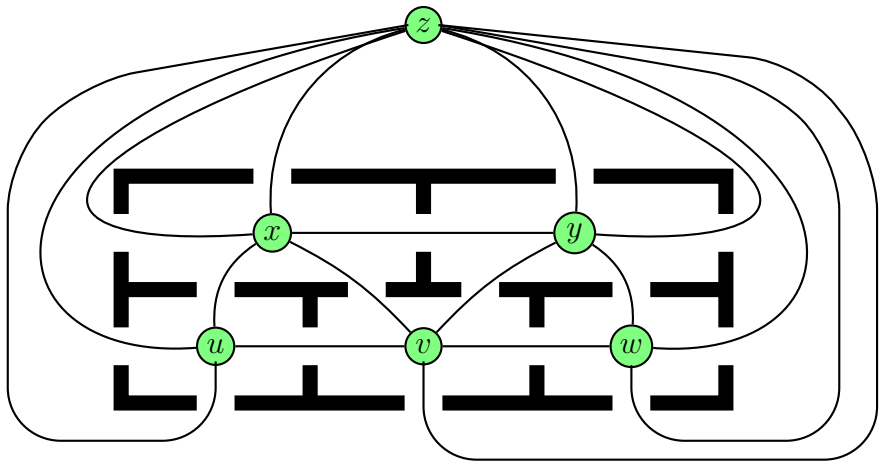


Figure 2.8: Each of the six regions (5 rooms and "outside the house") are vertices of a multigraph.

2.3 Hamiltonian cycles and paths

A *Hamiltonian cycle* in a graph is a cycle containing all vertices (each exactly once). A *Hamiltonian path* is a path containing all vertices precisely once. A graph G is called *Hamiltonian* if and only if G contains a Hamiltonian cycle. A graph G is called *traceable* if and only if G contains a Hamiltonian path.

In 1855, [572] Reverend Thomas Penyngton Kirkman (1806–1895) noted [572] that the graph of the dodecahedron is Hamiltonian (his results were published in 1856). According to popular literature, very soon after (1856 or 1857), Sir William Rowan Hamilton then described a game on the 20 faces of the icosahedron, or equivalently, on the 20 vertices of the dodecahedron graph (see Section 7.6 about planar duals). The dodecahedron version became the most popular, where cities around the world were interpreted as vertices.

See [104] or [105] for historical details. According to Martin Gardner [406], in 1859, Hamilton sold the game to a London game dealer for 25 pounds and it appeared across Europe in a number of forms. [I remember reading somewhere that this was one of the earliest mathematical games to make money.] See the "Icosian game" on the *Puzzle Museum* website [238] for a photo of an original game board made from wood and plugs made from bone or ivory. (apparently, only four of the original games survive today) and a photo of a newer version, called "The traveller's dodecahedron" which is also in the Hordern-Dalgety Collection.

Let D_{20} denote the graph of the dodecahedron and let the vertices be labelled with letters as in Figure 2.9.

The first player picks a path in D_{20} on five vertices. The second player is then challenged to complete this path into a Hamiltonian cycle. In the game, vertices

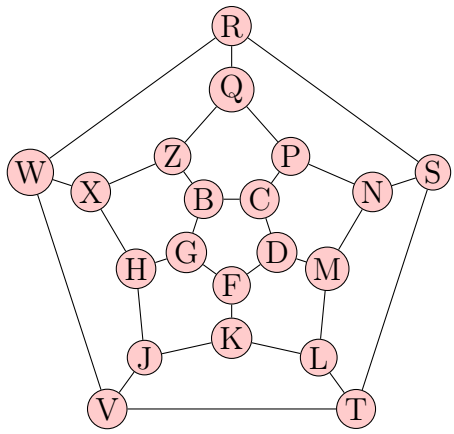


Figure 2.9: The Icosian Game by Hamilton

represent cities and edges represent direct routes between cities. The example provided by Hamilton has the first player pick the partial route BCDFG (see Figure 2.10), and one way for the second player to finish is to use the remaining vertices in alphabetical order, or to use HXWRSTVJKLMNPQZ, ending back at B.

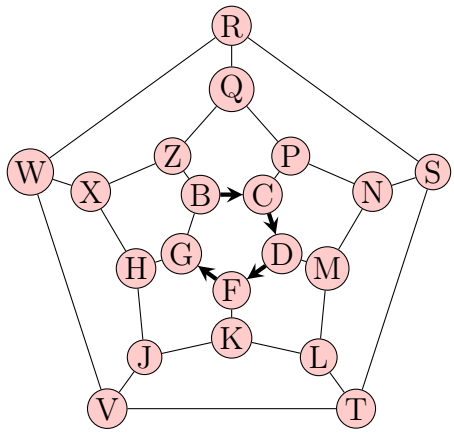


Figure 2.10: The Icosian Game, beginning with path B C D F G

Hamilton convinced himself that any path on 5 vertices in the dodecahedron graph can be extended to a Hamiltonian cycle, which is a rather strong property. (In a Hamiltonian graph, one might ask if there is a Hamiltonian cycle going through any *one* edge, never mind four given edges.)

Exercise 111. *In the Icosian Game with diagram with vertices as given in Figure 2.9, suppose that the first player starts with the path XWVTL. Extend this path into a Hamiltonian cycle.*

Exercise 112. *A Hamiltonian cycle on the dodecahedron (in 3D) can be described by a sequence of left turns (L) and right turns (R). Show that any such Hamiltonian cycle avoids the six subsequences LLLL, LRRL, LRLRLRL, LLRLRLL, LLRLRR, LLRLL and the six more obtained by interchanging L and R. Conclude that, essentially, there is only one Hamiltonian cycle. Note: This result is not true for a planar graph of the dodecahedron since left-right distinction differs for vertices in the background of a drawing of the dodecahedron.*

Exercise 113. *Decide whether or not the graph of the octahedron (see Figure 1.20) is Hamiltonian.*

Exercise 114. *Prove that I_{12}, the graph of the icosahedron (see Figure 1.20), is Hamiltonian.*

Exercise 115. *Show that for $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if $m = n$.*

Extending the idea behind Exercise 115, the following result has a similar proof.

Lemma 2.3.1. *Let G be a bipartite graph with vertex partition $V(G) = A \cup B$. If $|A| \neq |B|$, then G is not Hamiltonian.*

An example of a graph that is not Hamiltonian but that is traceable (has a Hamiltonian path) is the Herschel graph on 11 vertices, depicted in Figure 2.11.

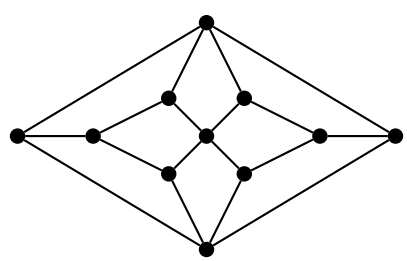


Figure 2.11: The Herschel graph

Exercise 116. *Show that the Herschel graph is not Hamiltonian. Hint: The graph is bipartite. Find a Hamiltonian path in this graph.*

Exercise 117. *Show that for each $n \geq 2$, the graph of the n-dimensional cube (see Section 1.6.5) is Hamiltonian.*

The fact that Q_n is Hamiltonian also follows by interpreting vertices as binary words of length n, and producing an ordered list of such words (see Section 2.5) called "cyclic Gray codes".

Exercise 118. *Find examples of graphs, one that is Hamiltonian but not Eulerian, and one that is Eulerian but not Hamiltonian.*

Exercise 119. *Let G be an Eulerian graph. Show that the line graph $L(G)$ (see Section 1.7.4 for definition) is both Eulerian and Hamiltonian.*

No "nice" condition (like that for Eulerian circuits) has been found that characterizes Hamiltonian graphs. Two sufficient conditions for Hamiltonicity were given by Dirac and Ore.

Theorem 2.3.2 (Dirac, 1952 [261])**.** *For any simple graph G with $|V(G)| \geq 3$, if $\delta(G) \geq \frac{1}{2}|V(G)|$, then G is Hamiltonian.*

Proof: The proof is nearly identical to that found in the solution to Exercise 94. Let G be a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$. Since the neighbourhoods of any two vertices intersect, G is connected. If $n = 3$ then $G = K_3$, which is Hamiltonian, so assume that $n \geq 4$.

Let $P \subset G$ be a path of maximum length, say on vertices $x_1, x_2, \ldots, x_m$ (where for each $i = 1, \ldots, m-1$, $\{x_i, x_{i+1}\} \in E(G)$). Since P has maximum length, P is maximal, and so all neighbours of x_1 lie on P; similarly, all neighbours of x_m lie on P.

Since x_1 and x_m each have at least $\frac{n}{2}$ neighbours on P, by the pigeon hole principle (see solution to Exercise 94), there is $j \in \{1, \ldots, m-1\}$ so that both $\{x_1, x_{j+1}\} \in E(G)$ and $\{x_j, x_m\} \in E(G)$. Then the walk

$$C = x_1, x_2, \ldots, x_j, x_m, x_{m-1}, \ldots, x_{j+1}, x_1$$

is a cycle containing all vertices of P. If $n = m$, then C is a Hamiltonian cycle of G.

If $m < n$, since G is connected, then there exists an outside vertex adjacent to some vertex in P, which, together with a path around C gives a path longer than P, which is impossible, and so $m = n$. □

Theorem 2.3.3 (Ore, 1960 [728])**.** *For a simple graph G, if for any non-adjacent vertices u and v, $d(u) + d(v) \geq |V(G)|$, then G is Hamiltonian.*

Proof idea: Suppose the theorem is false and that G is a graph satisfying the hypothesis but is not Hamiltonian. Furthermore, suppose that G has maximally many edges, so adding any edge produces a Hamiltonian circuit. Thus assume G has a Hamiltonian path $P = v_1 v_2 \cdots v_n$. Since v_1 and v_n are not adjacent, $d(v_1) + d(v_n) \geq n$. This implies (with a little vision) that there exists i, with $1 < i < n$ so that $v_{i+1} \in N(v_1)$ and $v_i \in N(v_n)$. The Hamiltonian cycle now can be seen as: $v_1, v_2, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1}, v_1$. □

Remark: Ore's theorem (Theorem 2.3.3) implies Dirac's theorem (Theorem 2.3.2).

Exercise 120. *Show that if a graph G on n vertices contains at least $\frac{n^2-3n+6}{2}$ edges, then G is Hamiltonian.*

(The result in Exercise 120 is used later in Exercise 409.)

The following theorem gives a sufficient condition for the degree sequence of a graph for it to be Hamiltonian and gives a necessary condition for a degree sequence to have *every* graph realizing the sequence to be Hamiltonian.

Theorem 2.3.4 (Chvátal, 1972 [216])**.** *Let $n \geq 3$ and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of non-negative integers satisfying $d_1 \leq d_2 \leq \cdots \leq d_n < n$. Then* every *graph with degree sequence $\mathbf{d}$ is Hamiltonian if for every $i < \frac{n}{2}$, at least one of $d_i \leq i$ or $d_{n-i} \geq n - i$ holds. If there exists an $i \leq n/2$ that fails this condition, then there exists a graph that is not Hamiltonian with degree sequence $\mathbf{d}'$ majorizing $\mathbf{d}$ (i.e., for each $j = 1, \ldots, n$, $d'_j \geq d_j$).*

The proof of Chvátal's theorem is omitted here.

A single cycle C_n (which is Hamiltonian) has degree sequence 2,2,..., 2, which violates the condition in Chvátal's theorem, but not all graphs with such a degree sequence are Hamiltonian (*e.g.*, when $n \geq 6$, the disjoint union of two cycles has the same degree sequence).

Exercise 121. *Let G be a graph on $n \geq 3$ vertices, and suppose that for each d with $1 \leq d < n/2$, the number of vertices with degree at most d is less than d. Show that G is Hamiltonian.*

Recall from Section 1.7.3 that for any graph G and positive integer k, the k-th power of G, denoted G^k, is the graph formed by adding an edge between every pair of vertices at distance at most k.

Conjecture 2.3.5 (Pósa (see [297]))**.** *If G is a graph on n vertices with $\delta(G) \geq \frac{2}{3}n$, then G contains the square of a Hamiltonian cycle.*

After a great deal of work and partial advances on Pósa's conjecture, Komlós, Sárkozy, and Szemerédi [591] proved that Pósa's conjecture is true for n sufficiently large. (In that paper, the authors use their "blow-up lemma", together with the "regularity lemma" and so "sufficiently large" was astronomical.) See [193] for more history of this problem and techniques used (and a lower bound of 2×10^8 vertices).

The following exercise is related to Exercise 107.

Exercise 122. *Let $n \geq 4$ be an even number, and let G be a regular graph on n vertices. Show that at least one of G or $\overline{G}$ is Hamiltonian.*

Another famous theorem regarding Hamiltonicity arose from noticing that Ore's proof does not use all of the consequences of the degree sum condition.

Theorem 2.3.6 (Bondy–Chvátal, 1976 [140]). *If G is a graph with two vertices x and y satisfying $d(x) + d(y) \geq |V(G)|$ then G is Hamiltonian if and only if $G + \{x, y\}$ is.*

Although the next theorem is not about Hamiltonian cycles, the statement is similar to that of Ore's theorem (and a proof uses Ore's theorem).

Theorem 2.3.7 (Faudree–Schelp, 1974 [357]). *Let G be a graph on $n \geq 5$ vertices so that for each pair u, v of non-adjacent vertices, $d(u) + d(v) > n$. Then between every pair of distinct vertices u, v, for every $5 \leq i \leq n$, there exists a u–v path on i vertices.*

The original proof is straightforward, but has many steps. Another simpler proof [1000] was given in 1984.

The following result has a fairly simple proof:

Theorem 2.3.8. *For any graph $G = (V, E)$ there exists disjoint (possibly empty) sets of vertices U, W of equal size such no edges are between U and W, and the remaining vertices $V \backslash (U \cup W)$ induce a graph with a Hamiltonian path.*

Exercise 123. *Prove Theorem 2.3.8.*

Some Hamiltonian graphs have a strong property.

Definition 2.3.9. *A graph G is* Hamiltonian-connected *if and only if for any two vertices x, y in G, there is a Hamiltonian x-y path.*

If a Hamiltonian-connected graph has at least three vertices, then the graph is Hamiltonian. However, a Hamiltonian graph need not be Hamiltonian-connected. For example, the cube graph Q_3 is Hamiltonian, but, for example, for two vertices on the same face but opposite corners, there is no Hamiltonian path with these as endpoints.

Theorem 2.3.10 (Ore, 1963 [731]). *Let G be a graph on $n \geq 4$ vertices. If for every pair u, v of non-adjacent vertices,*

$$\deg(u) + \deg(v) \geq n + 1,$$

then G is Hamiltonian-connected.

Recall from Section 1.7.3 that if G is a graph and $k \geq 2$ is an integer, then G^k is the graph formed by adding edges to G between any two vertices with distance at most k.

Theorem 2.3.11 (Sekanina,1960 [848]). *If G is a connected graph, then G^3 is Hamiltonian-connected.*

Corollary 2.3.12. *If T is a tree, then T^3 is Hamiltonian-connected and hence Hamiltonian.*

Recall from the discussion just before Theorem 2.1.8, a graph is 2-connected if and only if between any two vertices there are two internally disjoint paths joining them (for another definition of "2-connected", see Chapter 4).

According to [190, p. 139], the following theorem was conjectured independently by Crispin Nash-Williams and Mike Plummer.

Theorem 2.3.13 (Fleischner, 1974 [367]). *If a graph G is 2-connected, then G^2 is Hamiltonian.*

In the same year, a considerable strengthening of Theorem 2.3.13 was published.

Theorem 2.3.14 (Chartrand–Hobbs–Jung–Kapoor–Nash-Williams, 1974 [189]). *If a graph G is 2-connected, then G^2 is Hamiltonian-connected.*

For a proof of Theorem 2.3.14, see also [190].

There are many other results for Hamiltonian cycles. For example, see Section 7.9 regarding Hamiltonian cycles in planar graphs. The following result might be rather surprising:

Theorem 2.3.15 (Robinson–Wormald, 1994 [802]). *Almost all regular graphs are Hamiltonian.*

In other words, Theorem 2.3.15 says that as $n \to \infty$, the probability that a regular graph on n vertices is Hamiltonian tends to 1.

A famous problem regarding Hamiltonian cycles is called the "middle levels problem". Let n be an odd positive integer. In the Boolean lattice formed by subsets of an n-element set, the middle levels represent sets with either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ elements. In the bipartite graph formed by the two middle levels in the lattice, does there exist a Hamiltonian cycle on these vertices? The general feeling was that such a cycle always exists, and such was conjectured in the 1980s (by many authors, including Havel, Buck and Wiederman, and Erdős—see [706] for references). In 2014, a proof of this conjecture was posted by Torsten Mütze and was published in 2016 [706].

This section is concluded with a brief look at when a graph has more than one Hamiltonian cycle. The next theorem was known to Smith, who then told it to Tutte; when Tutte published a paper [936] that gave his example of a planar 3-connected (and so polyhedral) cubic graph that is not Hamiltonian (disproving Tait's conjecture, see Figure 7.13), he also included Smith's theorem and a short proof (apparently, different from Smith's original).

Theorem 2.3.16 (Smith, pre-1946, see [936]). *Let G be a cubic graph. For any edge $e \in E(G)$, the number of Hamiltonian cycles containing e is even.*

Tutte also added the following observation.

Corollary 2.3.17 (Tutte, 1946 [936]). *If a cubic graph G is Hamiltonian, then G contains (at least) three Hamiltonian cycles.*

Exercise 124. *Show why Corollary 2.3.17 follows from Theorem 2.3.16.*

Conjecture 2.3.18 (Sheehan, 1975 [857]). *Every 4-regular Hamiltonian graph has at least two Hamiltonian cycles.*

A short proof of Theorem 2.3.16 uses the fact that a graph with odd degrees has an even number of vertices—see Exercise 45. Andrew Thomason [913] showed a stronger result (Thomason's proof is sometimes called the "lollipop" proof, which can be given as an algorithm to find the second Hamiltonian cycle).

Theorem 2.3.19 (Thomason, 1978 [913]). *Let G be a Hamiltonian graph and let $e = \{x, y\}$ be an edge in G. If all vertices other than x and y have odd degree (x and y may also have odd degree, but this is not necessary), then the the number of Hamiltonian cycles containing e is even.*

So if all degrees in a Hamiltonian graph are odd, Theorem 2.3.19 says that there are an even number of Hamiltonian cycles through any given edge. In 1998, Carsten Thomassen [921] showed that for $r > 72$, every r-regular Hamiltonian graph has at least two Hamiltonian cycles. For the next conjecture, recall from Section 1.7.5 the definitions of $G - e$ and G/e.

Conjecture 2.3.20 (Thomassen 1996 [920]). *For every Hamiltonian graph G with $\delta(G) \geq 3$, there exists an edge $e \in E(G)$ so that both $G - e$ and G/e are Hamiltonian.*

Belak [86] showed that Conjecture 2.3.20 holds for claw-free graphs. For more on counting Hamiltonian cycles, see [380], [912], or [966]. For more on Hamiltonian cycles in triangle-free graphs or claw-free graphs, see [846]. One recent result (see Theorem 2.9.9) is an upper bound on the number of Hamiltonian cycles in triangle-free graphs.

2.4 The travelling salesperson problem

2.4.1 The problem and the combinatorial explosion

Note: The travelling salesperson problem is more commonly known as the travelling "salesman" problem, but here the non-gender version is used.

A salesperson is responsible to visit each of n cities for a sales call. The goal is to plan a trip that visits each city precisely once and return home (which is one of the n cities). In what order does the salesperson visit the cities so that the total cost of her/his trip is minimized? Rather than minimizing costs, one might want to minimize total distance travelled or total travelling time. Each cost/distance/time can

be considered as a "weight" on an edge drawn between vertices of a graph. Here is a reformulation of the problem:

The "travelling salesperson problem (TSP) : Let G be a Hamiltonian graph with weighted edges. How does one find a Hamiltonian cycle with the least total weight?

Without going into much detail here, the TSP is a "hard" problem; so far, some solutions guaranteed to be within a fraction of the optimal use (oddly enough) a minimum weight spanning trees. For more on the TSP, see, *e.g.*, [227]. One reason for the problem being so hard is that the number of possible routes to check can be huge even for a few cities.

For example, if one has to check all possible routes among 26 cities (one of which is the home base) there are $25! = 1.55 \times 10^{25}$ different orderings for the cities, and since a route and its reverse have the same weight, this number can be cut in half.

To get an idea as to how large this number is, if the earth is 2 billion years old, this time is approximately 6.3×10^{25} nano-seconds, so if each route can be checked in a nano-second, it might take about 500 million years to compute. Raising the bar to just 65 stops, there are $\frac{1}{2}65! \approx 4 \times 10^{90}$ possible routes. Some people might describe the pattern in such numbers as a "combinatorial explosion". It turns out that there is no reasonable algorithm to find a minimum weight Hamiltonian cycle, even though this problem has been seriously looked at by many authors or companies. (In computing language, the TSP is NP-complete.)

2.4.2 An example: TSP for Manitoba cities/towns

This section is taken from Karen Gunderson's notes [457].

The graph in Figure 2.12 is the graph for distances between 44 cites or towns in Manitoba, based on data from from the Travel Manitoba website [929].

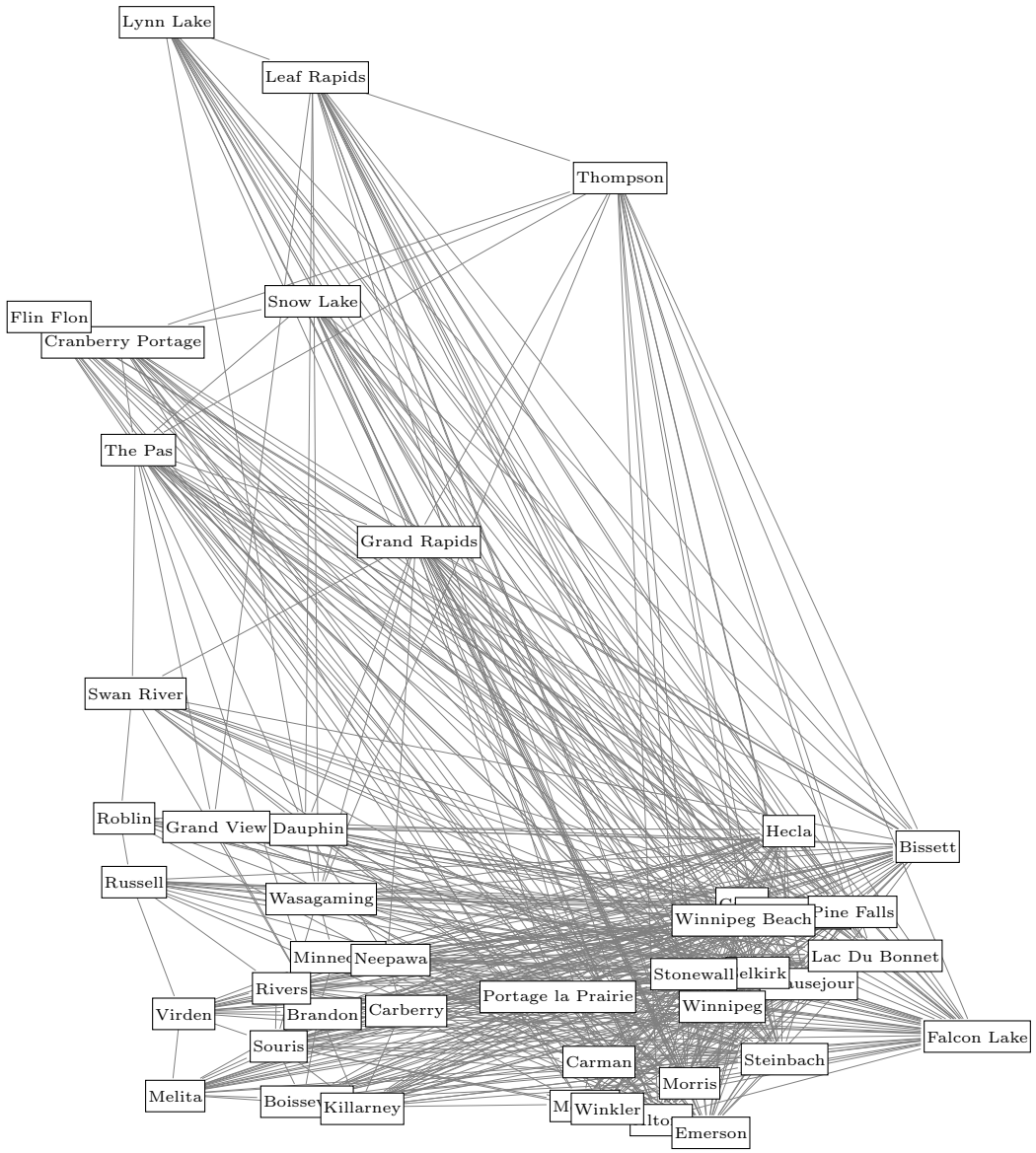


Figure 2.12: A connected graph for 44 towns/cities in Manitoba, with edges weighted by distance

Consider the above graph to have edges weighted by distance. Is the graph Hamiltonian, and if so, what is a solution to the Travelling Salesperson Problem for this

weighted graph? If there were direct routes between every pair of cities, there would be $\frac{1}{2}44! \approx 3 \times 10^{52}$ possible tours to check,.

Using *Sage*, the route depicted in Figure 2.13) was found to have the smallest total distance of of 5181km. (The cities are listed in order below, since the graph is a little cluttered.)

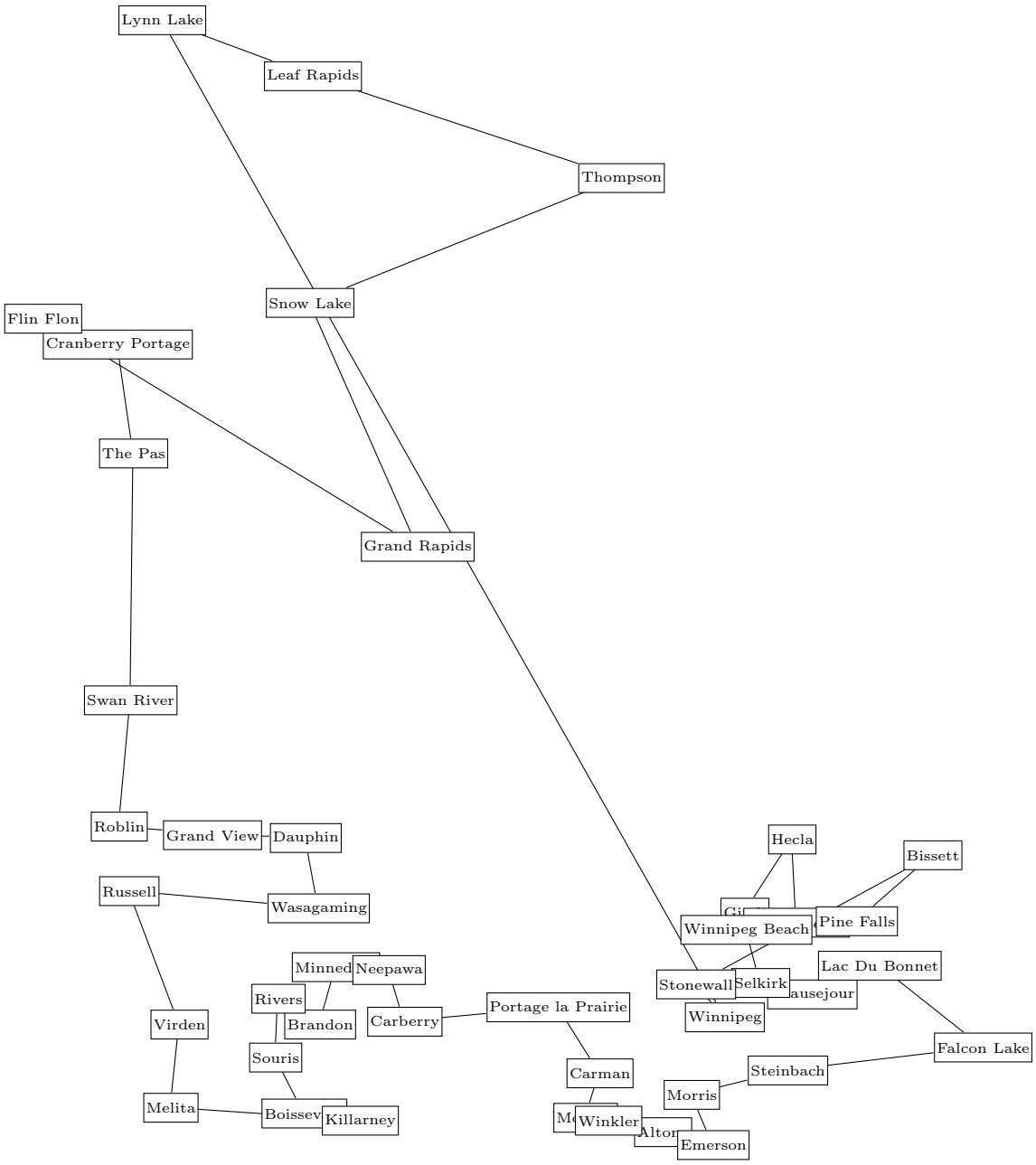


Figure 2.13: A solution to the TSP for 44 cities/towns in Manitoba

The order of the cities in the minimum weight Hamiltonian cycle is;

Winnipeg – Stonewall – Bisset – Pine Falls – Grand Beach – Hecla – Gimli – Winnipeg Beach – Selkirk – Beausejour – Lac Du Bonnet – Falcon Lake – Steinbach – Morris – Emerson – Altona – Winkler – Morden – Carman – Portage la Prairie – Carberry – Neepawa – Minnedosa – Brandon – Rivers – Souris – Boissevain – Killarney – Melita – Virden – Russell – Wasagaming – Dauphin – Grand View – Roblin – Swan River – The Pas – Cranberry Portage – Flin Flon – Grand Rapids – Snow Lake – Thompson – Leaf Rapids – Lynn Lake – Winnipeg.

2.4.3 An example: The TSP and UPS

The travelling salesperson problem is of considerable importance to UPS (United Parcel Service of America). As of 2013 or so, an average UPS driver had over 100 stops per day, so trying to solve the TSP just for one driver might be at best an approximation.

Sometime early this century, UPS had 55,000 routes to cover, including over 200 million addresses in its database. Each route has, on average, 100–140 stops. Some years ago, UPS claimed (see [725]) that if they could reduce the length of each route by just one mile, they could save approximately 1 million dollars per day!

Until a few years ago, UPS used one system that allowed for a fair amount of human judgement. In 2003, UPS launched *Package Flow Technology* (PFT), the foundation upon which a newer system sits. In 2016, after nearly a decade of beta trials, UPS finished the change over to a new tool (developed by UPS) called ORION, short for *On Road Integrated Optimization and Navigation* [947]. (See also [991] for an older article about ORION.) This project took over ten years, used 700 people, and thousands of pages of code were written.

Not only does ORION succeed in finding routes for pickup and delivery that might be close to optimal, but it does so subject to many other constraints, such as promised delivery times, the size and number of packages, or extra time for parking and loading (which may vary according to if the address is business or residential). Other factors to consider while attempting to reduce total time per delivery include average traffic volumes and (reducing) the number of left turns.

As a bonus, ORION plans the trip and the order to load packages most efficiently. The program can even shut off the delivery vehicle so that fuel is not wasted by idling unnecessarily. Not only does the new system improve overall efficiency (so drivers can also make more stops in a day), but also improves drivers' job satisfaction. Rather than worry about where to go next and how to find a particular package, a driver can concentrate more on safety and consumer satisfaction.

For each of the routes, ORION does not necessarily find the optimum TSP tour, but it seems to do rather well by using an algorithm that breaks stops into clusters of five, and then organizes the clusters in groups of five. For a brief video about ORION and its algorithm, see an episode of *NOVA* [725].

In 2016, UPS finished implementing ORION for all 55,000 routes, annually saving 10 million gallons of fuel, and reducing CO_2 emissions by 100,000 tons. Together with optimizing costs for drivers, ORION saves UPS between 300 and 400 million dollars per year. For this project, UPS received the INFORMS Franz Edelman Award for Achievement in Operations Research and the Management Sciences (see [537] for details on the award and other references).

2.5 Gray codes

Definition 2.5.1. *A* Gray code *of order n is a listing of the 2^n n-bit binary words so that adjacent words in the list differ in precisely one bit; furthermore, such a Gray code is called* cyclic *if and only if the first and last words also differ in one bit.*

The Gray code given by the following outline is called a "binary reflected Gray code" (BRGC), which also turns out to be *cyclic*, that is, the first and last word differ in precisely one bit as well. For each $n \geq 1$, let B_n denote the set of all binary strings of length n.

Since $B_1 = \{0, 1\}$, the ordered list $C_1 = (0, 1)$ is a Gray code for B_1. Let $k \geq 1$, and suppose that a Gray code has been constructed for B_k, say $C_k = (\mathbf{w}_1, \ldots, \mathbf{w}_{2^k})$.

Create a new code C_{k+1} by listing two consecutive copies of C_k, the second copy reversed, and affixing a 0 in front of every word in the first copy of C_k, and then affixing a 1 in front of every word in the reversed copy of C_k, and juxtapose these two new lists (see Table 2.1 for the first few examples). By induction hypothesis, no element in B_{k+1} is repeated, and there are $2^k + 2^k = 2^{k+1}$ words in C_{k+1}. By construction, the

$$\begin{array}{|c|}\hline 0\\ 1\\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 0 & 0\\ 0 & 1\\ \hline 1 & 1\\ 1 & 0\\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 1\\ 0 & 1 & 0\\ \hline 1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 0 & 1\\ 1 & 0 & 0\\ \hline\end{array}$$

Table 2.1: Constructing binary reflected Gray codes C_1, C_2, and C_3

first three binary reflected Gray codes (BRGCs) are:

$$\begin{aligned} &C_1 : 0, 1;\\ &C_2 : 00, 01, 11, 10;\\ &C_3 : 000, 001, 011, 010, 110, 111, 101, 100. \end{aligned}$$

Observe that a Gray code can also be formed by the above construction idea but by adding a bit at the end of each word instead of at the beginning of each word. If the construction above is used (adding bits at the front), to create a code for n-bit words, the code is unique, denoted here by BRCG(n).

Exercise 125. *Find BRCG*(4).

Gray codes are named after Frank Gray, who published a paper [441] in 1953 where these codes are developed and applied in computing. For other information on Gray codes, see [100], [829], [956], or [967]; for an unexpected application in combinatorial (polytope) geometry, see [110].

The ordered list of vertices of any Hamiltonian path in the cube graph Q_n can be seen as a Gray code (where each word in the code is the binary string associated with a vertex). A Hamiltonian cycle in Q_n then corresponds to a cyclic Gray code for n-bit words.

2.6 Knight's tours

In the game of chess, a knight is a piece (usually shaped like a horse) that can move in an L pattern. From any square, it can move to a square that is one row and two columns over or two rows and one column over (see Figure 2.14).

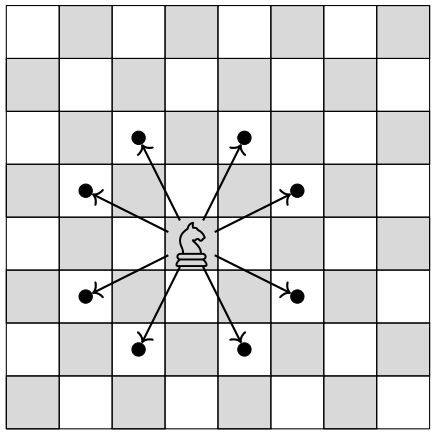


Figure 2.14: Possible moves for a knight

Definition 2.6.1. *For positive integers m and n, a knight's tour on an $m \times n$ board is a sequence of mn moves by a knight that starts in some square, travels through all other squares exactly once each, and then returns to the starting square.*

For positive integers m and n, let the *knight's graph* have mn vertices, each vertex corresponding to a square in an $m \times n$ board, and two vertices are adjacent if and only if in one move a knight can go from one to the other. In other words, two vertices with positions (a, b) and (c, d) are adjacent if and only if either $|a-c| = 1$ and $|b-d| = 2$, or $|a - c| = 2$ and $|b - d| = 1$. Observe that a knight moves between squares of different colour, so the knight's graph associated with any board is bipartite.

For example, the knight's graph for a 3×4 board is given in Figure 2.15.

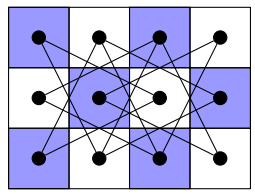


Figure 2.15: The 3×4 knight's graph

Some sources say that a knight's tour may also be a Hamiltonian path on a knight's graph; others call such "open knight's tours". Here, a knight's tour is closed (a Hamiltonian cycle).

The next simple lemma was known to Euler and Vandermonde.

Lemma 2.6.2. *If both m and n are odd numbers, there is no knight's tour on an $m \times n$ board.*

Proof: If both m and n are odd, then mn is also odd. Since the knight's graph G for an $m \times n$ board is bipartite with an odd number of vertices, G is not Hamiltonian. □

Exercise 126. *Show that there is no knight's tour for a 4×4 board. Hint: Look at edges from two opposite corners.*

Use reasoning similar to that in Exercise 126 solves some other small examples.

Exercise 127. *Show that there is no knight's tour for the 3×4 board. In other words, show that the graph in Figure 2.15 is not Hamiltonian.*

In fact, for any $n < 10$, there is no knight's tour on a $3 \times n$ board. However, there are 16 different knight's tours on a 3×10 board.

Exercise 128. *Find a knight's tour on a 3×10 board.*

A knight's tour for a standard 8×8 board is given in Figure 2.16.

For what values of m and n does there exist a knight's tour for an $m \times n$ board? Before stating the theorem, one case is given as an exercise.

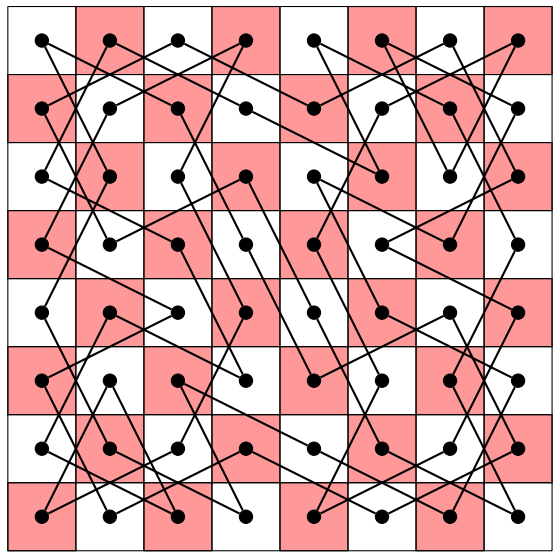


Figure 2.16: A knight's tour on a standard 8×8 chessboard

Exercise 129. *Show that for any positive integer n, there is no knight's tour for a $4 \times n$ board.*

Exercise 130. *Find a knight's tour on a 5×6 board.*

The complete classification of those m and n for which there is a knight's tour on an $m \times n$ board, was apparently known to Euler 1759 and independently by Vandermonde 12 years later.

Theorem 2.6.3. *For positive integers $m \leq n$, the only values for which there do not exist knight's tours on an $m \times n$ board are: (i) both m and n are odd; (ii) $m \in \{1, 2, 4\}$; (iii) $m = 3$ and $n \in \{4, 6, 8\}$.*

For a proof, see [840]; also see [840] or [973] for more details and history). Euler also considered knight's tours that gave magic squares (or nearly so) by placing numbers that count the steps in the tour; see [22] for an example.

The number of knight's tours for an 8×8 board is astronomical (a number 13 or 14 digits long!), and counting or constructing such tours has occupied many a mathematician and computer scientist. See Wolfram's page [976] for extensive references.

2.7 The Erdős–Gallai theorem for long cycles

An application of Dirac's theorem (Theorem 2.3.2) provides the base case for an inductive proof of the following theorem. The proof then uses a common tactic: find a set $W \subseteq V(G)$ that has few edges incident, delete W, obtain a smaller graph with a higher concentration of edges, and apply the inductive hypothesis.

Theorem 2.7.1 (Erdős–Gallai, 1959 [316]). *Let $3 \leq c \leq n$. If G is a graph on n vertices with more than $(c-1)(n-1)/2$ edges, then G contains a cycle of length at least c.*

Proof: The proof here is due to Woodall [994] (which also appears in, *e.g.*, [977, p. 416]).

Fix $c \geq 3$ and use strong induction on n; for each $n \geq c$, let $S(n)$ be the statement of the theorem.
BASE STEP: $n = c$. Since $\frac{(n-1)^2}{2} = \binom{n}{2} - (n-1)/2$, any graph with more than $\frac{(n-1)^2}{2}$ edges has $\delta(G) \geq n/2$, and so G is Hamiltonian by Dirac's theorem. Thus $S(c)$ is true.

INDUCTIVE STEP: Let $m \geq c$ and suppose that each of $S(c), S(c+1), \ldots, S(m)$ is true. The inductive step shows that $S(m+1)$ follows. To this end, let G be a graph on $m+1$ vertices with $|E(G)| > (c-1)(m+1-1)/2 = (c-1)m/2$ edges.

Case 1: $\delta(G) \leq (c-1)/2$. Let $x \in V(G)$ with $d(x) \leq (c-1)/2$. Deleting x produces a graph with $|E(G \backslash x)| \geq (c-1)m/2 - (c-1)/2 = (c-1)(m-1)/2$; by induction hypothesis, $G \backslash x$ contains a cycle of length c, hence so does G.

Case 2: $\delta(G) > (c-1)/2$. Without loss of generality, assume that G is connected, for if G is disconnected, then some component has more than the average number of edges, in which case the induction hypothesis applies to that component. (One should check this.)

Among all longest paths in G pick a path $P = v_1 \ldots v_\ell$ with $d = \deg(v_1)$ maximum. Observe that if $\ell < m+1$ then $\{v_1, v_\ell\} \notin E(G)$ because otherwise, any edge leaving P would be in a longer path, and since G is connected (and $\ell < m+1$), such an edge exists.

Let $W = \{v_i : \{v_1, v_{i+1}\} \in E(G)\}$. Then $|W| = d$. Since P is maximal, every neighbour of v_1 lies in P and so $\deg(v_1) \leq \ell - 1$. If for some $v_k \in W$ and $j > k+1$, $j \geq c$, the edge $\{v_k, v_j\}$ is present, then the cycle $v_1 v_2 \ldots v_k v_j v_{j-1} \ldots v_{k+1} v_1$ has length at least c, satisfying the theorem. So assume that if $v_k \in W$ and $j > k+1$, $j \geq c$, then $\{v_k, v_j\} \notin E(G)$.

For any $v_k \in W$, the path $v_k v_{k-1} \ldots v_1 v_{k+1} v_{k+2} \ldots v_\ell$ also has ℓ vertices, so $\deg(v_k) \leq d$ and as this new path is maximum length ℓ, this path is maximal, and so $N(v_k) \subseteq V(P)$.

Let $r = \min\{\ell, c-1\}$ and put $X = \{v_1, \ldots, v_r\}$. For each $v_k \in W$, $N(v_k) \subseteq X$, and $\deg_G(v_k) \leq d$. For the moment, suppose that $r = c - 1 \leq \ell$.

The number of edges in $G[W]$ (the graph induced by W) is at most $\frac{1}{2}\sum_{v \in W} \deg(v) \leq \frac{1}{2}d^2$. Let H be the bipartite subgraph consisting of edges between W and $X \backslash W$. Then $|E(H)| \leq |W| \cdot |X \backslash W| = d(r-d)$. For notational convenience below, let d_W and d_X denote the degrees in the graphs induced by W and X, respectively, and put $d_H = \deg_H$.

Then

$$\begin{aligned}
|E(G[W])| &= \frac{1}{2}\sum_{w\in W} d_W(w) \\
&= \frac{1}{2}\sum_{w\in W} d_W(w) + \frac{1}{2}\sum_{w\in W} d_H(w) - \frac{1}{2}\sum_{w\in W} d_H(w) \\
&= \frac{1}{2}\sum_{w\in W} (d_W(w) + d_H(w)) - \frac{1}{2}\sum_{w\in W} d_H(w) \\
&= \frac{1}{2}\sum_{w\in W} d_X(w) - \frac{1}{2}\sum_{w\in W} d_H(w) \\
&= \frac{1}{2}\sum_{w\in W} d_X(w) - \frac{1}{2}|E(H)| \\
&\leq \frac{1}{2}d^2 - \frac{1}{2}|E(H)|.
\end{aligned}$$

Adding $|E(H)|$ to each side of above shows that the number of edges incident with W is

$$\begin{aligned}
|E(G[W])| + |E(H)| &\leq \frac{1}{2}d^2 + \frac{1}{2}|E(H)| \\
&\leq \frac{1}{2}d^2 + \frac{1}{2}d(r-d) \\
&= \frac{1}{2}dr.
\end{aligned}$$

Upon deleting W and all edges incident with W, one obtains a graph on $m-d$ vertices with more than

$$\frac{1}{2}(m-1)(c-1) - \frac{1}{2}dr \geq \frac{1}{2}(m-d-1)(c-1)$$

edges. By the induction hypothesis $S(m-d)$, this remaining graph contains a cycle of length at least c, hence so does G, proving $S(m+1)$.

When $r = \ell < c-1$, even fewer edges are incident with W, and the induction hypothesis again applies.

By (strong) mathematical induction, for all $n \geq c$, $S(n)$ is true. □

2.8 Cycle lengths

For $n \geq 3$ the complete graph K_n contains cycles of all possible lengths from 3 to n. Any bipartite graph contains no odd cycles (cycles of odd length), but for $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ contains cycles of length 4.

Definition 2.8.1. *A graph G on n vertices is called* pancyclic *if and only if G contains a cycle of every possible length ℓ (where $3 \leq \ell \leq n$). If for every vertex v, there are cycles through v with all possible lengths, then G is called* vertex pancyclic.

For example, complete graphs are pancyclic. Another example is the Frucht graph on 12 vertices (see Figure 15.5), with cycle lengths 3,4,..., 12.

In particular, a pancyclic graph is Hamiltonian. One early result about pancyclic graphs is by Bondy [135], who showed that if G on n vertices has at least $n^2/4$ edges and is not (for n even) $K_{n/2,n/2}$, then G is pancyclic. Another related result by Bollobás and Thomason [133] says that if G is non-bipartite on n vertices and has at least $\lfloor \frac{n^2}{4} \rfloor - n + 59$ edges, then G contains all cycle lengths from 4 to the length of its longest cycle. (Such graphs are called "weakly pancyclic".)

However, when graphs have far fewer edges, cycle lengths are harder to predict. Counting the number of lengths of cycles for certain classes of "sparse" graphs is an area of extensive research (see, *e.g.* [893] for some references, results, and conjectures not covered here).

Graphs with minimum degree 3 have many cycles, and an attractive conjecture says that for such graphs, there is always a cycle length of a particular type:

Conjecture 2.8.2 (Erdős–Gyárfás, 1993, see [313])**.** *Any (simple) graph with minimum degree 3 contains a cycle of length that is a power of 2.*

A special case of Conjecture 2.8.2 says that any cubic (3-regular) graph contains a cycle whose length is a power of 2.

In one of the last papers Erdős co-authored, it was shown [241] that if G is a counterexample to Conjecture 2.8.2, then $\frac{|E(G)|}{n} < \frac{n}{4}$. The same authors then state that they believe the conjecture is false, and so Erdős [241, p. 245] offered \$100 for a proof but only \$50 for a counterexample. As mentioned in the same paper [241], another reference given for an announcement of Conjecture 2.8.2 was in a talk by Erdős in March 1995 given at the SE Conference on Graph Theory, Combinatorics, and Computing, Boca Raton Florida.

According to a webpage by Doug West [978], Yair Caro suggested a weaker version of Conjecture 2.8.2, where every such graph contains a cycle whose length is some power of a natural number.

Royle [814] checked that Conjecture 2.8.2 holds for all C_4-free graphs on at most 16 vertices. Markström [655] checked all cubic graphs on at most 28 vertices.

It is known [955] that as $n \to \infty$, there exists $S \subset \{3, 4, \ldots, n\}$ with $|S| = O(n^{.99})$ so that all graphs with average degree 10 contains a cycle whose length is in S. In 2008, Sudakov and Verstraëte [893] proved that the conjecture holds for graphs whose average degree is an iterated logarithm of $|V(G)|$.

In 1998, Shauger [856] settled the conjecture for (induced) $K_{1,m}$-free graphs G that satisfy $\delta(G) \geq m + 1$ or $\Delta(G) \geq 2m - 1$. In the same paper [856, p. 61], Shauger

states "The conjecture is believed not to be true, but the problem is still open." In 2001, Daniel and Shauger [239] proved the conjecture for claw-free planar graphs (a claw-free graph is one containing no induced copy of $K_{1,3}$). In 2013, Conjecture 2.8.2 was shown to be true for 3-connected cubic planar graphs [505].

For a recent (2019) result on minimum degree conditions to guarantee cycles of all lengths modulo k, see [208].

2.9 The number of cycles

Various theorems regarding the number of cycles in a graph have already been introduced. For example, the result in Exercise 88 shows that $n+1$ edges in an n-vertex graph guarantees two cycles, and the Corrádi–Hajnal theorem (Theorem 2.1.3) is a result that shows if the minimum degree is high enough, many vertex-disjoint cycles exist.

Just as a simple example, the reader can verify that K_4 has 7 cycles (four C_3s and three C_4s). Here are some warm-up exercises that might give the reader an idea as to how difficult it might be to count cycles in a graph.

Exercise 131. *How many cycles are in the graph K_5?*

Exercise 132. *How many cycles are there in $K_{3,4}$?*

For a graph G, let $C(G)$ be the number of cycles in G. In a sense, the number of cycles in a graph depends upon how many more edges than vertices there are, as well as the structure of G, but in general, bounds on $C(G)$ are generally quite loose. Only statements and references are given here—without proofs.

Theorem 2.9.1 (Ahrens, 1897 [14])**.** *If G is a graph on n vertices with m edges and k components, then*

$$m - n + k \leq C(G) \leq 2^{m-n+k} - 1. \tag{2.1}$$

The lower bound in (2.1) is tight, which is achieved by any disjoint union of cycles or trees.

Theorem 2.9.2 (Mateti–Deo, 1976 [658])**.** *The upper bound in (2.1) is tight, and the only examples achieving equality are K_3, K_4, $K_{3,3}$, and $K_4 - e$.*

For connected graphs, the Ahrens upper bound is improved slightly:

Theorem 2.9.3 (Aldred–Thomassen, 2008 [26])**.** *If G is a connected graph with n vertices and m edges, then*

$$C(G) \leq \frac{15}{16} 2^{m-n+1}.$$

For a graph G on n vertices and m edges, the value $p = n - m + 1$ is called the *cyclomatic* number of G. Entringer and Slater studied the number of cycles for graphs with a fixed cyclomatic number.

Theorem 2.9.4 (Entringer–Slater, 1981 [287])**.** *For graphs on n vertices and m edges with cyclomatic number $r = m - n + 1$, if $\phi(r)$ denote that maximum number of cycles in such graphs, then*

$$2^{r-1} \leq \phi(r) \leq 2^r.$$

Entringer and Slater also conjecture that their lower bound is asymptotically correct. In the same paper, they also show that for $r \geq 3$, there exists a cubic (3-regular) graph on $2r - 2$ vertices with $\phi(r)$ cycles. They also mention that the upper bound for $\phi(r)$ could be improved if another of their conjectures is true—that all cubic graphs except K_4 and $K_{3,3}$ have an edge contained in at most half of the cycles.

Harary and Manvel [491] used adjacency matrices to count 3-, 4-, and 5-cycles (*e.g.*, in a complete graph). Bounds for the number of cycles in various classes of graphs have been studied; here is a partial list (most in this list are taken directly from [59] or [60]):

- planar graphs [26], [43], [174].
- outerplanar graphs and series-parallel graphs [249].
- graphs with large maximum degree without a specified odd cycle [70].
- graphs with specified minimum degree [963].
- graphs with a specified cyclomatic number or number of edges [14], [287], [450], [571] (see also [597, Ch. 4,10]).
- cubic graphs [205] (counting long cycles).
- 2-connected cubic graphs [77] (minimum number of cycles).
- 3-connected cubic graphs [25].
- graphs with fixed girth [655].
- k-connected graphs [579].
- Hamiltonian graphs [774], [859], [963].
- Hamiltonian graphs with a fixed number of edges [458].
- factors of the binary de Bruijn graph [380].
- graphs with a cut-vertex [963].

- complements of trees [523], [783], [1012].
- random graphs [905].

In 2015, Durocher, Gunderson, Li, and Skala [271] asked which triangle-free graphs on n vertices have the most number of cycles? (This question arose from a question in computer science.) A natural guess might be the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ since (by Mantel's theorem, Theorem 10.3.1 and Turán's theorem, Theorem 10.5.2) this graph is the unique triangle-free graph on n vertices with the most edges. Since bipartite graphs have no odd cycles whatsoever, it might be surprising that the graph with the most cycles misses every second possible cycle size. Small graphs (and many cases to check) seemed to confirm that the complete balanced bipartite graph does indeed have the most cycles among all triangle-free graphs. Various other candidates all seemed to fall short.

In 1973, Erdős, Kleitman, and Rothschild [333] showed that for $r \geq 3$, as $n \to \infty$, the number of K_r-free graphs on n vertices is

$$2^{(1-\frac{1}{r-1}+o(1))\binom{n}{2}}.$$

It follows then that the number of triangle-free graphs (with $r = 3$) is very close to the number of bipartite graphs, and so almost all triangle-free graphs are bipartite.

Conjecture 2.9.5 (Durocher–Gunderson–Li–Skala, 2015 [271])**.** *For each $n \geq 4$, there is only one triangle-free graph on n vertices that has the maximum number of cycles, namely the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.*

The authors of [271] were able to confirm the conjecture for $n \leq 13$, and for some special cases. For example, among all *regular* triangle-free graphs on n vertices, when n is even, $K_{n/2,n/2}$ has the most cycles. (Thus, if one could show that a cycle-maximal graph is always regular, then the conjecture would be answered.) An extension of this result includes when a graph is "nearly regular", that is, when the maximum and minimum degrees differ only by some constant, in which case the result holds for sufficiently large n. It was also shown [271] that among all bipartite graphs on n vertices, $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ has the most cycles.

To help in the proofs, the actual number of cycles in this example is calculated.

Lemma 2.9.6 (Durocher–Gunderson–Li–Skala, 2015 [271])**.** *For $n \geq 4$, the number of cycles in $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is*

$$C\left(K_{\lceil n/2\rceil,\lfloor n/2\rfloor}\right) = \sum_{k=2}^{\lfloor n/2\rfloor} \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2k(\lfloor n/2\rfloor-k)!(\lceil n/2\rceil-k)!} \tag{2.2}$$

Exercise 133. *Prove equation (2.2).*

The expression in (2.2) is a bit cumbersome, so it was simplified to an approximate version in [60]. For this, the authors used "modified Bessel functions" (see [3]) and the constants

$$2.27958 \leq \sum_{i=0}^{\infty} \frac{1}{(i!)^2} = I_0(2) \leq 2.279586; \tag{2.3}$$

$$1.5906 \leq \sum_{i=0}^{\infty} \frac{i}{(i!)^2} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) \leq 1.59064. \tag{2.4}$$

Theorem 2.9.7 (Arman–Gunderson–Tsaturian, 2016 [60]). *Using $I_0(2)$ and $I_1(2)$ as defined in (2.3) and (2.4), as $n \to \infty$,*

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = (1 + o(1)) \begin{cases} I_0(2)\pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is even} \\ I_1(2)\pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is odd} \end{cases}.$$

Using the above calculations and a result of Andrásfai [52] (see Theorem 10.3.23) showed that a (sufficiently large) cycle-maximal triangle-free graph is indeed bipartite, thereby giving a partial proof of Conjecture 2.9.5:

Theorem 2.9.8 (Arman–Gunderson–Tsaturian, 2016 [60]). *For $n \geq 141$, among all triangle-free graphs on n vertices, the balanced complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is the unique triangle-free graph with the maximum number of cycles.*

As a consequence of their calculations, a result with perhaps independent interest was achieved:

Theorem 2.9.9 (Arman–Gunderson–Tsaturian, 2016 [60]). *Any triangle-free graph on n vertices contains at most*

$$e^2 \left(\frac{n}{2e}\right)^n$$

Hamiltonian cycles (where e is the base of the natural logarithm).

2.10 Application: Instant Insanity

2.10.1 The puzzle

A puzzle, now called "Instant Insanity" (see Figure 2.17) consists of four cubes, each of the same size, and each side of each cube is one of four colours, red, blue, green, and white (below, I replaced white by yellow for cosmetic purposes). The goal is to find an arrangement of the cubes into a $1 \times 1 \times 4$ pattern (for example, all four side-by-side in a row, or all four stacked—I use the horizontal row here) so that on each of the four 1×4 faces, all four colours appear (so there are no repeated colours on any face). Figure 2.18 shows variants of the same puzzle. See Jaap's puzzle page [539] for more references and examples, complete with Java programs to encode and solve such puzzles.

Figure 2.17: The Instant Insanity puzzle. Photo by Yeyoung Won

2.10.2 The number of arrangements

In how many ways can one place a cube on the table so that one side faces you? There are 6 choices for the face, which can be rotated in any of four ways; so there are 24 ways to place a single cube. There are also $4! = 24$ ways to order the cubes. So in all, there are

$$4! \times 24^4 = (4 \cdot 3 \cdot 2)(2^3 \cdot 3)^4 = 2^{15} \cdot 3^5 = 7062624$$

possible arrangements. However, for each such placement, there are many that are equivalent; for example, one can order the cubes in any of 4! ways, and one can rotate the entire row to any of four positions, so one must divide by $24 \cdot 4 = 2^5 \cdot 3$, leaving $2^{10}3^4 = 82,944$ "different" arrangements. (The six-cube version has $2^{15}3^6 = 23,887,872$ different arrangements.) Note that flipping the entire row end for end does not create a different pattern, since this is equivalent to flipping each cube, then putting them in reverse order.

2.10.3 History

This puzzle was patented (patent US646463A) by Frederick A. Schossow in 1900 under the name "The Katzenjammer puzzle" (see also [95, p. 784]). Somewhere around 1940 (at least according to popular sources), it was released under the name "The Great Tantalizer" (4 wooden cubes), produced in England (see [248] for picture), and referred to by that name in a 1961 article by O'Beirne [726].

In 1967, Franz (Frank) Owen Ambruster released it in the United States through Parker Brothers under the name "Instant Insanity", with four plastic cubes (see Figure 2.17).

According to [248], the name is now trademarked by Winning Moves, Inc.

Similar puzzles have been released; here are a few (some of these are mentioned in the popular website [539], but a much more complete list, together with many fine

Figure 2.18: Other versions of Instant Insanity. Photos by Yeyoung Won

pictures and history is on the website [892] of J. A. Storer (at Brandeis); I think that it is incredible, and highly recommend the reader to that source):

- Groceries
- Buvos Golyok: Hungarian version with balls instead of cubes;
- Cube-4;
- Trikki 4;
- Watch It!: has pictures of clocks on each cube face; this puzzle has more solutions than Instant Insanity;
- Mutando: a German version marketed by Logika Spiele, which also has a solution in the form of a $2 \times 2 \times 1$ pattern (it uses a different colour pattern than Instant Insanity, see [628] for discussion of solution for the $2 \times 2 \times 1$ version). It comes in a simple plastic case, with small cubes, perhaps 3/4 inch (see the second picture in Figure 2.18).
- Drive Ya Crazy: a six-cube version (and hence six colours). The oldest source for this that I could see so far is [620, p. 78], a book written in 1946 (where this version is not called Drive Ya Crazy, if I remember correctly).
- Flags of the allies: a five cube version by O'Beirne using the flags of Belgium, France, Japan, Russia, and the United Kingdom. (This probably appeared in his

article that he wrote for *New Scientist* [726] in the early 1960s, but I have not yet found the source; it might have appeared in his 1965 book [727].)

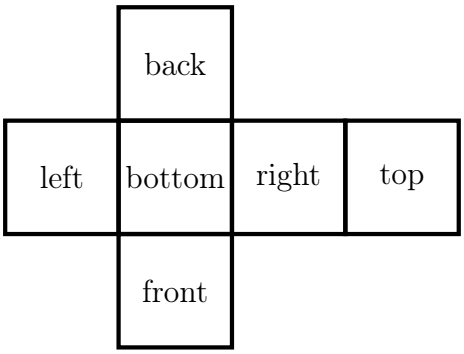
- Dorobo: a five cube version made in Japan sold by the company Hanayama; it uses symbols rather than colours.

- In the article [248] is a picture of a similar puzzle with octagonal pieces, also only using four colours; I have not dug up the details.

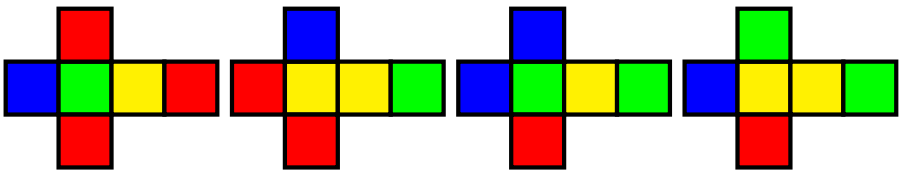
This puzzle (or variants thereof) has appeared in numerous puzzle books (just to name a very few: [95], [620], [727], [874]). It is now used as an example in many discrete math and graph theory texts (*e.g.*, [58], [187]) because one solution uses cycles in multigraphs..

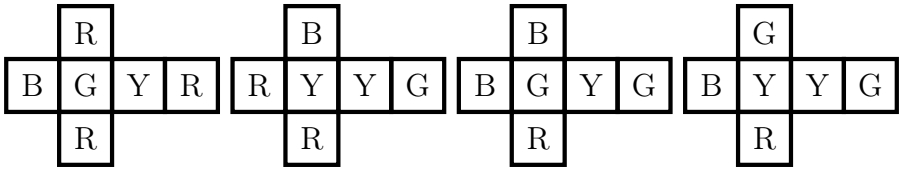
2.10.4 Solution using a multigraph decomposition

The solution outlined in this section uses multigraphs, and according to Berlekamp, Conway, and Guy, [95, p 785] this solution was first published by the puzzleist T. H. O'Beirne in [726] and then in his book [727, 112–129]. (O'Beirne is the author of the famous "melting box puzzle".) Encode each cube as follows, as if one unfolded the cube taking the top to the right:

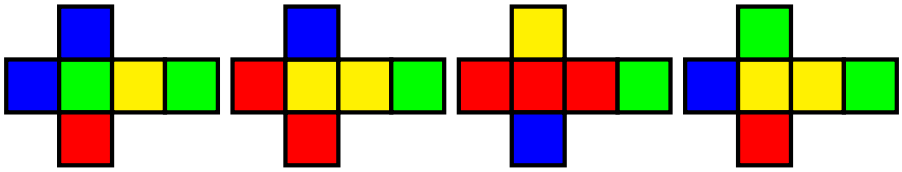


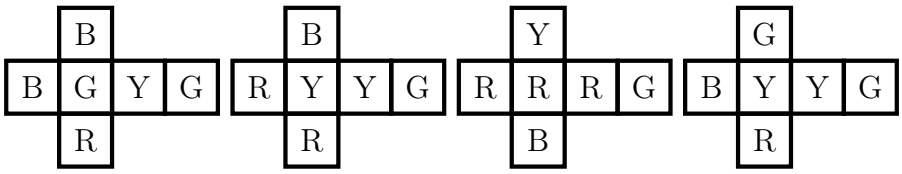
Line up the four cubes, left to right, calling them cube 1, cube 2, cube 3, and cube 4 on the right. Throughout the whole solution, remember to keep the cubes in the same order.





So that this example does not precisely duplicate the "standard one", suppose that cube 1 was spun and interchanged with cube 3, arriving at:



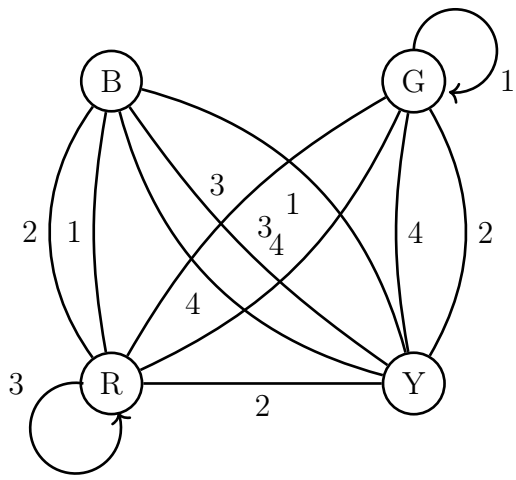


Call the above chart **Step 1**.

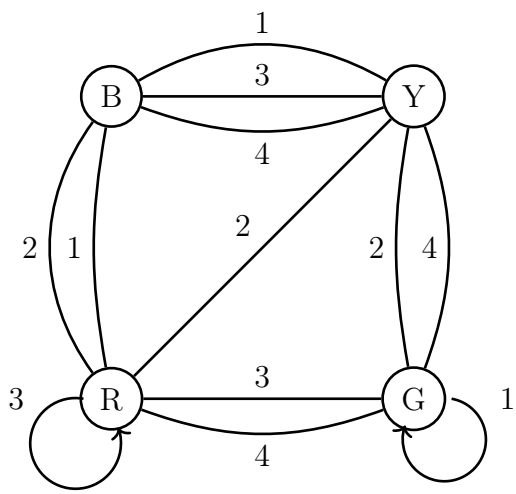
It does not matter what the original position of your cubes is; the process explained below gives the same multigraph (but with possibly the edge labels permuted) and the same set of possible solutions. However, be aware that there are many such puzzles out there, and some are completely different (with a different multigraph and a different set of solutions, even after checking all possible permutations of colours).

In fact (after this exercise), you can create your own puzzle. If you are not careful, you may create a set with more solutions than you want—increasing the chances that someone can stumble upon one.

Step 2: From the chart in Step 1, create the labelled multigraph G where $V(G) = \{R, G, B, Y\}$, and two vertices (colours) X, Y, are adjacent by an edge labelled i if and only if X and Y are colours of opposite sides of cube i. On a first attempt at drawing the multigraph (for the second arrangement of cubes given in Step 1, not the first one!), one might arrive at something like:



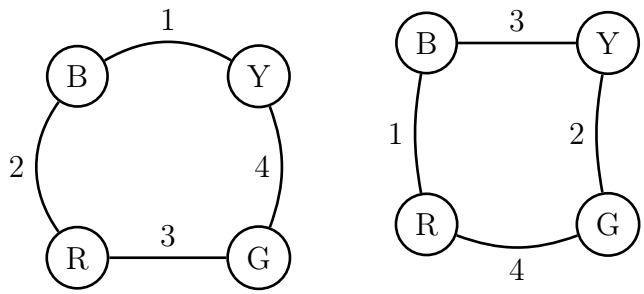
Step 3: If necessary, redraw the graph. The graph above is hard to read —to be of any use, the inner labels need to be shifted so matching edges with labels is more evident. However, swapping locations of the colours in this multigraph gives a drawing with no crossing edges, much easier to read.



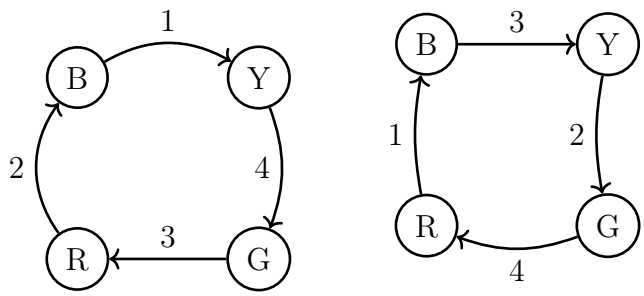
Step 3: Find two sub(multi)graphs of the above multigraph; each subgraph satisfying:
(a) All four cubes are used (so there are four edges, each with a different label from 1–4);
(b) all four colours are used.
(c) All vertices are of degree 2 (so you get a 4-cycle or a 3-cycle and a loop disjoint from the 3-cycle, or in some cases, two disjoint 2-cycles, or maybe even a 2-cycle and two loops, or even stranger, four loops?). This is because the colour of the front of one subcube may be linked to the colour of the back of the next cube, ensuring that each colour is used once for each 1×4 face.

Furthermore, the two subgraphs must not share any edges, because one subgraph is used to find front/back pairs, and the other is used to find top/bottom pairs.

The only two such subgraphs are:



Step 4: Put arrows on each so that the arrows follow cycles.



The first (multi)graph indicates how to place the front/back faces, and the second indicates the top/bottom faces. In the first graph, use $F \to B$ for front/back, and in the second, use $T \to B$ for top/bottom. Now place the cubes according to your two charts. Do the front/back sequence first, and then by spinning each cube, maintain the front/back orientation and get the tob/bottom pairs lined up, too.

Note that there are only two solutions to this puzzle since one can choose the cycles to go in different directions.

2.10.5 Another solution

The following solution uses numbers rather than colours. I first learned this from a video [989] by the well-known mathematician Robin Wilson (Gresham Professor of Geometry, also famous for work in graph theory). Soon after, while looking through the bibliography in [95], I discovered that this solution was published by T. A Brown in 1968 [168]. Here is the outline.

Assign numbers to the colours, say $R = 1$, $B = 2$, $Y = 3$ and $G = 5$. Since each face in a solution has all colours, the product on each side is 30; therefore, the product for the front-back faces is 900, and the product for the top/bottom faces is also 900.

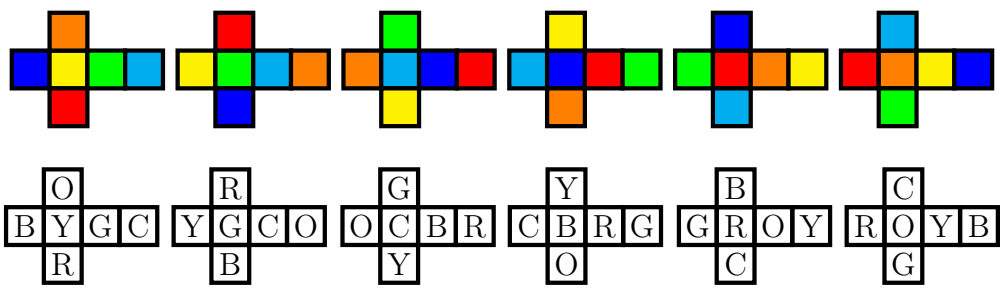
Work out all possible products between a face-pair in cube 1 and 2. Do the same for cubes 3 and 4. Of all these products, only three match up (two from cubes 1,2 and two from cubes 3,4) to multiply to 900. Of these, only two are "compatible"—they do not share faces. Use one for front/back, and use the other for top/bottom. □

Exercise 134. *Suppose that a set of four cubes has opposite pairs as follows:*
Cube 1: RG, BB, RY; Cube 2: RR, BG, BY; Cube 3: RY, BB, RG; Cube 4: YY, BG, BY.
Using the multigraph method, show that this set has more than one solution.

Exercise 135. *Suppose that a set of four cubes has opposite pairs as follows: Cube 1: GR, YR, GB; Cube 2: GB, GR, RY; Cube 3: RG, YY, YB; Cube 4: YG, BB, BR. Use the multigraph method to show that this set of cubes has no solution.*

2.10.6 Drive ya crazy

Here are the six cubes in a version of "Drive ya crazy", presented in the pattern with solution, both in colour and in black and white.



2.10.7 Making the puzzles

I have made hundreds of sets of cubes and given them away; for some (including the six cube version, "drive ya crazy"), I made wooden cases. Most of them were just painted cubes. I found that buying cheap cubes (at a dollar store, for example) and using bingo markers is a fast way to generate sets (but colours tend to fade).

One need not use colours; for example, one could pick four different symbols, including numbers or letters. If using symbols, it might be preferable to use symbols that are centrally symmetric (with the full permutation group of 24 symmetries), like $\cdot, \times, +, \circ, \square$.

One model (by RainTree) has a wooden case with three sides (and two ends) and a string running through the end that can be pulled tight across the open face to hold the cubes in the case. I was looking for a small clear plastic case that might fit a cube set, but I have not yet found one. For mass production, I just put the four cubes in a

plastic bag. However, I wanted something with a nice presentation, perhaps one that contains the cubes in a (visible) solution position.

I designed a case that has all four sides the same, (each with holes to see the cubes) but where one panel is held in only by magnets and can be removed; the removable panel is not obvious, and it often baffles people trying to find out how to open it. I once made a six cube version out of ebony and bloodwood, (see my website [453]), with a small compartment inside for the solution, and a few more with a slightly better design. Here is an ebony model:

I have made many others using purpleheart, tulipwood, walnut, or cherry, and these have been given as gifts or prizes.

Chapter 3

Trees and forests

3.1 Basic properties of trees and forests

Trees were defined in Definition 1.6.4; for convenience, this definition is repeated here.

Definition 3.1.1. *A* tree *is a connected acyclic graph.*

A *forest* is an acyclic graph; so a forest is a graph whose connected components are trees.

The results in the next exercise can be proved by a simple induction on the number of vertices.

Exercise 136. *Prove that trees and forests are both bipartite and planar.*

A *leaf* in a graph is a vertex of degree 1; the edge joined to a leaf is called a *pendant edge*.

Lemma 3.1.2. *Every tree on at least two vertices contains at least one leaf.*

Proof: Let T be a tree on at least two vertices. If every vertex has degree at least two, then by Lemma 2.1.2, T contains a cycle, contrary to T being a tree, so there exists at least one vertex with degree either 0 or 1. Since T is connected and has at least two vertices, there are no isolated vertices, that is, T contains no vertices of degree 0. Hence, T contains at least one vertex of degree 1. □

Lemma 3.1.3. *Between any two distinct vertices in a tree, there is a unique path.*

Proof outline: If two vertices x and y are connected by two different paths, a cycle results. □

A *bridge* in a connected graph is an edge whose removal disconnects the graph. (Some authors apply this definition to graphs that are not necessarily connected, in which case a bridge is an edge whose removal increases the number of components.)

Lemma 3.1.4. *Every edge in a tree is a bridge.*

Proof: Let T be a tree. By Lemma 3.1.3, for any $x, y \in V(T)$, there is precisely one path joining x and y. If $\{x, y\} \in E(T)$, there is no other x-y path than $\{x, y\}$ itself, hence its removal produces a graph with no x-y path, that is, a disconnected graph. □

The following exercise characterizes trees on n vertices.

Exercise 137. *For a graph on $n \geq 1$ vertices, prove that the following statements are equivalent:*

*(a) G is connected and acyclic (*i.e.*, G is a tree);*
(b) G is connected and has $n - 1$ edges;
(c) G is acyclic and has $n - 1$ edges.

The next lemma extends that of Lemma 3.1.2:

Lemma 3.1.5. *Every tree on at least two vertices has at least two leaves.*

Proof: Let T be a tree on $n \geq 2$ vertices. A tree is connected, so $\delta(T) \geq 1$. By Exercise 137, T has $n - 1$ edges, and so by the handshaking lemma (Lemma 1.8.1),

$$\sum_{x \in V(T)} d(x) = 2|E(T)| = 2(n - 1). \tag{3.1}$$

If at most one vertex has degree 1, all other $n - 1$ vertices have degree at least 2, and so the sum of all degrees is at least $2(n - 1) + 1 = 2n - 1$, which violates equation (3.1). □

Exercise 138. *Let T be a tree on $n \geq 3$ vertices. If each vertex has degree either 1 or 3, how many leaves does T have?*

Exercise 139. *Show that if T is a tree on $n \geq 3$ vertices, then the average degree of vertices in T is not an integer.*

Exercise 140. *Find all trees T so that the complement of T is also a tree.*

The next exercise seems to be a strengthening of Exercise 140:

Exercise 141. *Find all trees that are isomorphic to their complement.*

Exercise 142. *Let G be a graph on n vertices with $n - 1$ edges. Show that at least one component of G is a tree.*

Exercise 143. *Let k be a positive integer. Prove that if G is a graph with minimum degree $\delta(G) = k$, then G contains every tree with k edges as a (weak) subgraph.*

Exercise 144. *Show that any tree on n vertices is a subgraph of $\overline{C_{n+2}}$.*

From Exercise 143, it is a short journey to the result in the next exercise:

Exercise 145. *Show that for any positive integer k, if a graph G has average degree $2k$, then G contains every tree on k vertices.*

In many cases, the result in the following exercise gives an improvement on Lemma 3.1.2 or Lemma 3.1.5.

Exercise 146. *Let T be a tree with maximum degree $\Delta(T)$. Show that T contains at least $\Delta(T)$ leaves.*

Exercise 147. *Let T be a tree on n vertices with no vertices of degree 2. Show that T has more than $n/2$ leaves.*

Exercise 148. *Let $d_1, \ldots, d_n \in \mathbb{Z}^+$. Prove that there exists a tree on n vertices with degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.*

Exercise 149. *Let T be a tree on n vertices with degree sequence $d_1, d_2, \ldots, d_n$, in non-increasing order. Show that for each $i = 1, \ldots, n$, $d_i \leq \lceil \frac{n-1}{i} \rceil$.*

Exercise 150. *Let T be a tree on n vertices. Show that if $T \neq K_{1,n-1}$, then T is isomorphic to a subgraph of $\overline{T}$.*

Exercise 151. *Show that C_4 is the only graph on four or more vertices so that every three vertices induce a tree.*

3.2 Counting non-isomorphic trees

When vertices are unlabelled, counting different trees on n vertices means counting the number of non-isomorphic trees on n vertices. As seen in Figure 3.1, for $n = 1, 2, 3, 4, 5, 6$, there are, respectively, 1, 1, 1, 2, 3, 6 non-isomorphic trees.

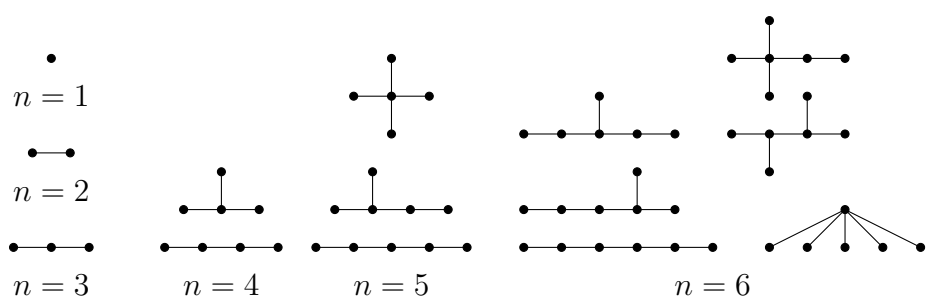


Figure 3.1: Non-isomorphic trees on n unlabelled vertices

Exercise 152. *Find 11 non-isomorphic trees on 7 vertices.*

Counting the number of non-isomorphic unlabelled trees on n vertices in general requires some work—there seems to be no simple formula for such a number. The first few values are:

$$1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, 7741, 19320, 48629, 123867, 317955, 823065, ...$$

For more terms in this sequence, see *The On-Line Encyclopedia of Integer Sequences* (OEIS) `https://oeis.org/A000055`.

However, counting trees with labelled vertices is much easier.

Theorem 3.2.1 (Cayley's tree formula)**.** *For $n \geq 2$, there are n^{n-2} trees on n labelled vertices.*

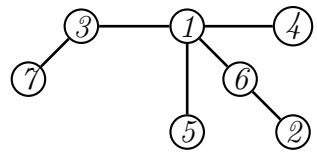
Theorem 3.2.1 was first proved in 1857 by J. J. Sylvester (see [18]) and again in 1860 by Carl Wilhelm Borchardt [146] by using determinants. In 1889, Arthur Cayley published a paper [186] extending the theorem (while acknowledging Borchardt) to spanning forests and using graph theoretic terms; the result is now eponymous with Cayley. There are now many proofs of Cayley's theorem; for example, one "standard" proof is to derive it from a stronger theorem called the *Matrix-Tree Theorem* (see Section 15.8). See [18] for four proofs of Cayley's tree formula, not including the one given here.

The proof given below is due to Heinz Prüfer [765] in 1918. His idea was to establish a one-to-one correspondence between each spanning tree T on a vertex set $[n] = \{1, 2, \ldots, n\}$, and a unique sequence $P(T) \in \{(a_1, a_2, \ldots, a_{n-2}) : \forall i, a_i \in [n]\}$. If this correspondence is established, then Cayley's theorem is proved because there are precisely n^{n-2} such sequences.

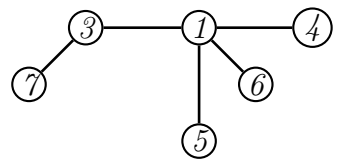
Let T be a tree on n labelled vertices; for convenience, let $V(T) = [n] = \{1, 2, \ldots, n\}$. The *Prüfer sequence* (or *Prüfer code*) for T is a sequence of the form $P = P(T) = (a_1, \ldots, a_{n-2})$, where each $a_i \in [n]$, defined as follows.

Let v_1 be the leaf with the smallest label, and let i_1 be the label of v_1's neighbour. Partially define the sequence $P = (i_1, ?, \ldots, ?)$. Let T_1 be the tree with v_1 deleted. Let v_2 be the leaf of T_1 with the smallest label, say i_2, and set $P = (i_1, i_2, ?, ..., ?)$. In general, suppose this process has been completed for j steps, giving a tree T_j formed by deleting $v_1, v_2, \ldots, v_j$ and the partial sequence $P = (i_1, \ldots, i_j, ?, ..., ?)$ has been defined; if v_{j+1} is the leaf of T_j with the smallest label and i_{j+1} be the label of the neighbour of v_{j+1} in T_j, put $P = (i_1, i_2, \ldots, i_j, i_{j+1}, ...)$. Continue the process until there are only two vertices left, giving the sequence $P = (i_1, \ldots, i_{n-2})$.

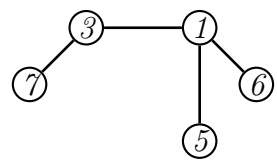
Example 3.2.2. *Consider the following tree on 7 vertices:*



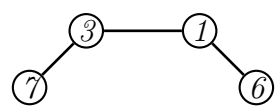
The Prüfer sequence P for this tree has 5 entries, defined iteratively as follows. The smallest leaf is 2, with neighbour 6. So $P = (6, \ , \ , \ , \ )$. Deleting 2 gives



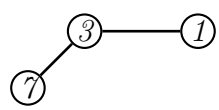
The smallest leaf is 4, with neighbour 1. So $P = (6, 1, \ , \ , \ )$. Deleting 4 gives



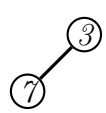
The smallest leaf is 5, with neighbour 1. So $P = (6, 1, 1, \ , \ )$. Deleting 5 gives



The smallest leaf is 6, with neighbour 1. So $P = (6, 1, 1, 1, \ )$. Deleting 6 gives



The smallest leaf is 1, with neighbour 3. So $P = (6, 1, 1, 1, 3)$. Deleting 1 gives



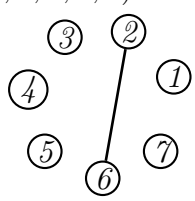
Since there are only two vertices remaining, the process stops, and the Prüfer sequence is $(6, 1, 1, 1, 3)$.

Lemma 3.2.3. *For each $i = 1, \ldots, n-2$, let $a_i \in \{1, \ldots, n\}$. There exists a unique tree on n vertices with $P = (a_1, \ldots, a_{n-2})$ as its Prüfer sequence.*

Proof: Let $P = (a_1, \ldots, a_{n-2}) \in [1, n]^{n-2}$ be a Prüfer sequence. Using the following algorithm, construct a tree on vertices labelled 1 through n as follows: Begin with the empty graph on vertices labelled $1, 2, \ldots, n$, and let $L = 1, 2, \ldots, n$ be a "list" (which is modified at each step). The first edge to add is between a_1 and the least label not used in P. Call this new graph G_1. Delete this least label from the list, giving a list L_1 of $n-1$ labels, and shorten the sequence to $P_1 = (a_2, \ldots, a_{n-2})$. Supposing that for some $j < n$, G_j has been created and L_j is the reduced list and P_j is the shortened sequence, create G_{j+1} by adding the edge between a_j and the least label in the list L_j that is not in P_j. Continue until the list has only two labels left, and join these two last labels. □

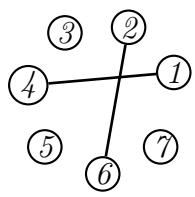
Example 3.2.4. *Let* $P = (6, 1, 1, 1, 3)$ *and put* $L = \{1, 2, 3, 4, 5, 6, 7\}$.

The least label not in P *is 2, so add the edge* $\{6, 2\}$*:*



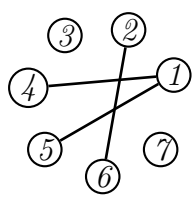
Then $P_1 = (1, 1, 1, 3)$ *and* $L_1 = \{1, 3, 4, 5, 6, 7\}$.

The least label in L_1 *not in* P_1 *is 4, so add the edge* $\{1, 4\}$*:*



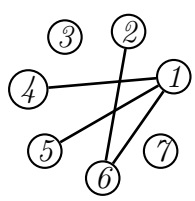
Then $P_2 = (1, 1, 3)$ *and* $L_2 = \{1, 3, 5, 6, 7\}$.

The least label in L_2 *not in* P_2 *is 5, so add the edge* $\{1, 5\}$*:*



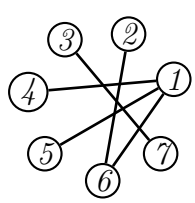
Then $P_3 = (1, 3)$ *and* $L_3 = \{1, 3, 6, 7\}$.

The least label in L_3 *not in* P_3 *is 6, so add the edge* $\{1, 6\}$*:*



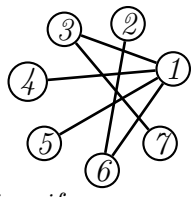
Then $P_4 = (3)$ *and* $L_4 = \{1, 3, 7\}$.

The least label in L_4 *not in* P_4 *is 7, so add the edge* $\{3, 7\}$*:*



Then P_5 *is empty and* $L_5 = \{1, 3\}$.

Finally, draw the edge $\{1,3\}$*, the two labels left in* L_5*:*



Then confirm (by redrawing if necessary) that this tree is the one given in Example 3.2.2 □

Exercise 153. *Find the labelled tree whose Prüfer sequence is*

(a) $(3,3,3)$.

(b) $(1,3,2,5)$.

(c) $(3,3,1,6,2)$.

Proof of Cayley's tree formula: By construction, to each spanning tree on n vertices, there is a unique Prüfer sequence of length $n-2$, and by Lemma 3.2.3, to each of the n^{n-2} possible Prüfer sequences, there is a tree. Thus the desired bijection holds, proving Cayley's formula. □

To confirm Cayley's formula, it straightforward to check that a vertex-labelled K_3 has only $3 = 3^{3-2}$ spanning trees, and that a labelled K_4 has $4^2 = 16$ spanning trees. Cayley's formula then says that a labelled K_5 has 125 spanning trees.

Exercise 154. *Let* T *be a tree with Prüfer sequence consisting of a single digit repeated* $n-2$ *times. Show that* T *is a star* $K_{1,n-1}$.

Exercise 155. *Let* T *be a tree with Prüfer sequence having no repeated entries. Show that* T *is a path.*

Exercise 156. *List all of the spanning trees of a labelled* $K_{2,3}$.

Exercise 157. *For* $n \geq 2$*, find (with proof) the number of spanning trees of a labelled* $K_{2,n}$.

The result (shown by Scoins [841] in 1962) in the following exercise generalizes the result in Exercise 157.

Exercise 158. *Show that for positive integers* m *and* n*, the number of spanning trees of a labelled* $K_{m,n}$ *is* $m^{n-1}n^{m-1}$.

In 1966, Alfréd Rényi (1921–1970) extended the result in Exercise 158 to k-partite graphs ($k \geq 3$); here is the result when $k = 3$.

Theorem 3.2.5 (Rényi, 1966 [788])**.** *For positive integers* ℓ, m, n*, the number of spanning trees of a labelled* $K_{\ell,m,n}$ *is*

$$(\ell+m)^{n-1}(\ell+n)^{m-1}(m+n)^{\ell-1}(\ell+m+n).$$

3.3 Minimum spanning trees

Recall that for a graph G, a subgraph $H \subset G$ is a *spanning subgraph* if and only if $|V(G)| = |V(H)|$; in other words, a spanning subgraph consists of all of the vertices of G and a collection (perhaps empty) of the edges in G. A spanning tree of G is any spanning subgraph that happens to be a tree (or any tree that happens to be a spanning subgraph of G).

Exercise 159. *Prove that every connected graph contains a spanning tree.*

Let G be a graph, and let $w : E(G) \to [0, \infty)$ be called a *weight function*, assigning non-negative real numbers to the edges in G. [One can and does also entertain vertex weights, but such is not used here.] For example, one might use a graph G to indicate a certain network of highways between cities; if there is a highway joining two particular cities, the corresponding edge in B might be weighted with mileage, volume of traffic it can support, or cost of transport. One might say then that such a graph is weighted by w. The weight of any subgraph of an edge-weighted graph is the total weight of all its edges. In notation, if H is a subgraph of a weighted graph G, define

$$w(H) = \sum_{e \in E(H)} w(e).$$

In an edge-weighted connected graph G, a *minimum* spanning tree (MST) (or *minimum weight* spanning tree) of G is a spanning tree T of G with minimum weight, where the minimum is taken over all spanning trees of G.

MSTs in connected graphs are not, in general, unique, but they always exist (the set of weights for all spanning trees is finite, and so has a minimum, but this minimum might be witnessed by different spanning trees). The *minimum spanning tree problem* is to find a MST. See [436] for a history of the problem and [911] for additional references. The MST problem is also known as the *minimum connector problem*. Is there an "effective" procedure to find a MST (rather than exhaustively listing of all spanning trees and then checking the weight of each)?

There are two simple algorithms that produce MSTs, one given by Kruskal [610] in 1956 and another by Prim, [757] in 1957. Both algorithms are "greedy" algorithms (at each step, take the "cheapest" option available). These algorithms were discovered independently and earlier by other authors, but somehow it is these names that survive.

For Kruskal's algorithm, edges are added according to their weight, providing that the cheapest edge available does not form a cycle.

Kruskal's algorithm:
INPUT: a non-empty graph $G = (V, E)$ on $|V| = n$ vertices with a weight function $w : E \to [0, \infty)$, and edges labelled $e_1, e_2, \ldots, e_{|E|}$, so that $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_{|E|})$, i.e., weights are in nondecreasing order.

BASE STEP: Pick e_1, a least weight edge, and set G_1 to be the graph consisting of just e_1 on vertex set V.

RECURSIVE STEP: Suppose that a subgraph $G_i = (V, E_i)$ has been formed by the previous step. Select an edge $e \in E \setminus E_i$ of least weight that does not does not form a cycle if added to G_i, and set $E_{i+1} = E \cup \{e\}$ and $G_{i+1} = (V, E_{i+1})$.

Terminate the algorithm after $n - 1$ recursive steps.

OUTPUT: $G_n = (V, E_n)$.

Exercise 160. *Prove that when Kruskal's algorithm terminates, G_n is a minimum spanning tree.*

Prim's algorithm "grows" a tree starting at any vertex by recursively attaching available edges of least weight without forming a cycle.

Prim's algorithm:
INPUT: a non-empty graph $G = (V, E)$ on $|V| = n$ vertices with weighted edges (given by a weight function $w : E \to [0, \infty)$).

BASE STEP: Pick $x_1 \in V$ arbitrarily. Set $V_1 = \{x\}$ and G_1 to be the graph consisting of just x.

RECURSIVE STEP: Suppose that for some $i \geq 1$, a subgraph $G_i = (V_i, E_i)$ has been formed, by the previous step. Among all edges not in E_i, but having at least one vertex in V_i, select an edge $\{x, y\}$ (say, with $x \in V_i$) with least weight so that the addition of $\{x, y\}$ does not form a cycle together with edges of G_i (so $y \notin V_i$). Set $V_{i+1} = V_i \cup \{y\}$, $E_{i+1} = E_i \cup \{x, y\}$, and $G_{i+1} = (V_{i+1}, E_{i+1})$.

Terminate the algorithm only when $V_n = V$, that is, after $n - 1$ steps.

OUTPUT: $G_n = (V, E_n)$.

Theorem 3.3.1. *The output of Prim's algorithm is a minimum weight spanning tree.*

Exercise 161. *Prove Theorem 3.3.1.*

The complexity of Kruskal's algorithm is $O(|E| \ln(|V|)$ and the complexity of Prim's algorithm is $O(|V|^2)$ (see, *e.g.*, [872]). So Kruskal's algorithm is more efficient for sparse graphs and Prim's algorithm is better for dense graphs.

An MST solution can give an answer for the TSP that is within 2 times the optimal (see, *e.g.*, [460] for one of many references possible).

3.4 MST example—Manitoba roads

This is a continuation of notes by K. Gunderson [457] from Section 2.4.2 on the TSP for a network of 44 cities and towns in Manitoba. A connected graph for Manitoba roads was given in Section 2.4.2, each edge weighted by distance. A minimum weight spanning tree (found by computer) for that graph is given in Figure 3.2; the total weight is 3173km.

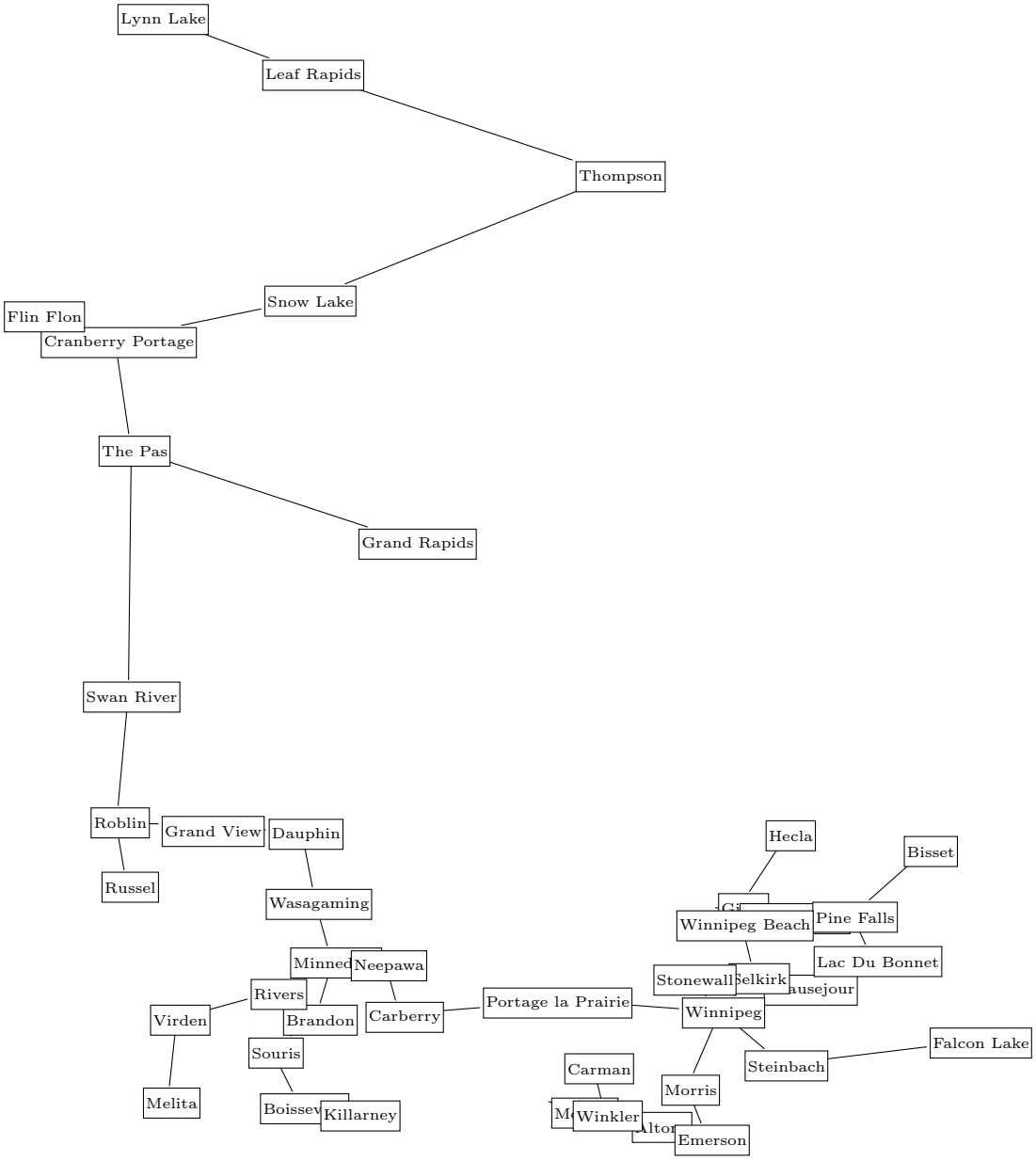


Figure 3.2: A minimum spanning tree with for 44 Manitoba cities/towns

3.5 Trees and bracing rectangular frameworks

In any connected graph, there exists a spanning tree (a subgraph that is a tree using all vertices in the graph). This simple fact is applied to solve a problem regarding rigidity of certain structural frameworks. The presentation here is based on [22] and [973] (texts once used at University of Manitoba). For further information on the math behind this engineering problem, its solution, and generalizations thereof, see, *e.g.*, [119].

If four bars of steel are bolted together at their ends to form a square, with enough pressure, the square can be deformed into a parallelogram (as in Figure 3.3), but adding a brace (the red bar) between opposite corners stabilizes the square. In many bookcases, a thin sheet of hardboard is attached to the back (like the blue square in Figure 3.3) in order to maintain the right angles between the shelves and the uprights.

Figure 3.3: A square without bracing is not rigid, but attaching a brace (in red) or a solid plate (in blue) makes the structure rigid.

A brace between opposite corners of a square (or any rectangle) creates two right angle triangles, and triangles are rigid (*e.g.*, by the cosine law). Triangular bracing is essential in many areas of engineering.

Consider a 2-dimensional rectangular framework of squares constructed by bolting together bars of steel at their ends as in Figure 3.4. How many of the square cells need to be braced before the structure is rigid? A similar question for 3-dimensional frameworks can be asked, but the only case considered here is where all struts are in the same plane. (Also, squares can be replaced by rectangles, but for simplicity, only square cells are used here.)

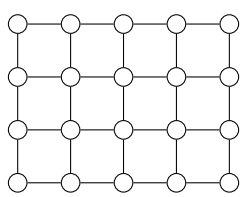


Figure 3.4: A 3×4 framework, where small circles indicate bolted joints

If all of the cells in a rectangular framework are braced, then the entire structure is then rigid. However, not all cells need to be braced to give stability; for example, the braced framework in Figure 3.5 is rigid.

Not all braces in Figure 3.5 are needed for overall rigidity. For example, in Figure 3.6, both braced frameworks are rigid.

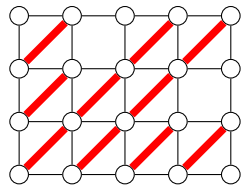


Figure 3.5: Bracing 10 of 12 cells may make a stable structure

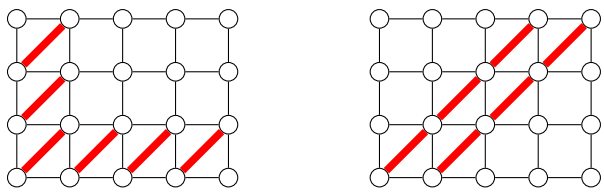


Figure 3.6: Six well-placed braces give a rigid framework

In the first example in Figure 3.6, because the cells in positions $(1,1)$, $(2,1)$ and $(1,2)$ are braced, the cell in position $(2,2)$ is rigid. Since the cells $(1,2)$, $(2,2)$, $(1,3)$ are rigid, so is the cell in position $(3,2)$. Continuing in like manner, all remaining squares are also seen to be rigid. A similar recursive argument shows that the second bracing in Figure 3.6 is also rigid.

Call a placement of braces a *rigid bracing* if the braces produce a rigid framework. What is the minimum number of braces required for a rigid bracing? In Figure 3.6, only 6 of the 10 braces from Figure 3.5 are needed; does there exist a rigid bracing using fewer than 6 braces? In a rigid bracing, how can one tell whether or not all of the braces are needed (i.e., when a bracing is minimal)? What are conditions on bracing placement assure rigidity? To help answer these questions, an elementary observation is first made:

Lemma 3.5.1. *In any rigid bracing of a rectangular framework, a brace is required in at least one cell of each row and in at least one cell in each column.*

The idea behind the proof of Lemma 3.5.1 is apparent from the example in Figure 3.7, where the third column has no braces, thereby allowing the framework to be deformed.

However, the condition in Lemma 3.5.1 is not sufficient for stability. In Figure 3.8, each column and each row has at least one braced cell, but the framework is not stable.

In order to tell whether or not a particular bracing of a framework is stable, an auxiliary graph is used to encode the braces. For positive integers m and n, let F be a framework with m rows and n columns of cells, where some subset of the mn cells are braced. Construct a bipartite graph G on $m+n$ vertices, where the two partite sets are labelled $r_1, r_2, \ldots, r_m$ (for rows) and $c_1, c_2, \ldots, c_n$ (for columns), and a pair $\{r_i, c_j\}$

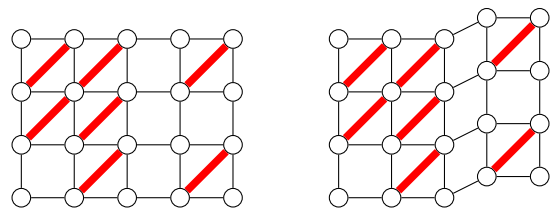


Figure 3.7: The third column has no braces, so the structure is not stable.

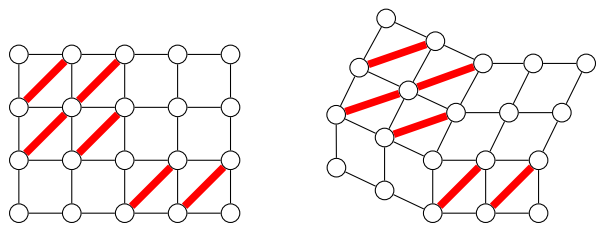


Figure 3.8: Each row and column contains a brace, but the framework is not rigid.

is an edge if and only if the cell in position (i, j) is braced. (So the number of braces used in the framework is the number of edges in the associated bipartite graph.

For example, the associated bipartite graphs for the two bracings in Figure 3.6 are given in Figure 3.9.

Theorem 3.5.2 (see [973])**.** *For positive integers m and n, an $m \times n$ rectangular framework with braces is rigid if and only if the associated bipartite graph on $m + n$ vertices is connected.*

To confirm Theorem 3.5.2, the graphs in Figure 3.9 are connected, and the corresponding bracings are rigid.

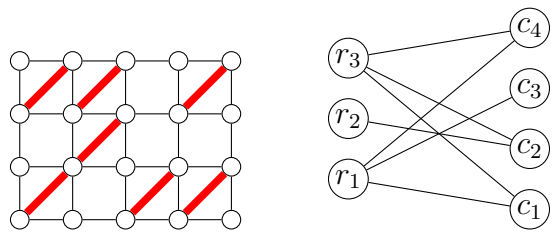
Corollary 3.5.3. *For positive integers m and n, a rigid bracing uses at least $m+n-1$ braces.*

Proof: Any connected graph on $m + n$ vertices contains a spanning tree. By Exercise 137, such a spanning tree has $m + n - 1$ edges. □

Figure 3.9: The graphs associated with the bracings in Figure 3.6.

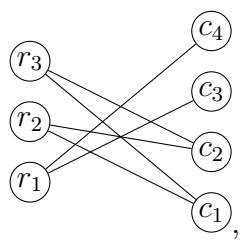
The result in Lemma 3.5.1 says that each vertex in the associated bipartite graph for a rigid bracing has degree at least 1, and if some row or column is not braced, then some vertex in the graph has degree 0, in which case the graph is not connected.

Example 3.5.4. *Consider the following bracing and its associated bipartite graph:*



The graph is connected, so the bracing is rigid. In fact, since the graph contains a cycle, (say, on r_1, c_1, r_3, c_4, r_1), the bracing is not minimal (any edge of this 4-cycle can be deleted while still preserving connectedness). Any bracing with 7 braces is also not minimum since a minimum connected spanning tree has $m + n - 1 = 3 + 4 - 1 = 6$ edges, corresponding to 6 braces.

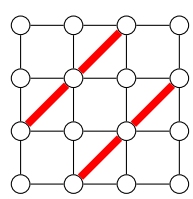
To give yet another confirmation of Theorem 3.5.2, the (non-rigid) bracing given in Figure 3.8 has the associated graph



,

which is not connected, even though the graph has 6 edges—the minimum number of edges to be connected.

Exercise 162. *Show that the bracing*



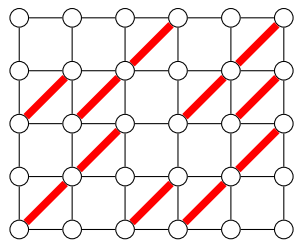
is not rigid in two ways: first find a deformation, and second, use Theorem 3.5.2. Show that the addition of just one more brace in any of the empty cells gives a rigid bracing.

The result in the next exercise shows that any rigid bracing contains a minimum bracing; in other words, any rigid bracing can modified by only deleting braces, and not adding any new braces to obtain a minimum bracing.

Exercise 163. *Let m, n be positive integers. Show that if an $m \times n$ framework has a rigid bracing that uses more than $m + n - 1$ braces, some collection of braces can be removed to give a minimum bracing.*

The result in Exercise 163 can also be restated to say that any minimal rigid bracing is also a minimum rigid bracing.

Exercise 164. *Determine whether or not the bracing in the framework*



is rigid. If the bracing is rigid, find (if possible) braces that can be removed to produce a minimum rigid bracing.

3.6 Chemical trees

If a chemical formula is known (for example, ethylene has formula C_2H_4), can one determine the structure of its molecules? For many compounds, the structure of its molecule follows from simple counting and a little graph theory.

Atoms join together to form molecules. What keeps atoms together in a molecule are called "bonds", which are viewed as some kind of sharing of electrons by two adjacent atoms. In the classical model for an atom, electrons are arranged in orbital shells; the innermost shell can have at most two electrons, the second shell can have 8, the third shell can have up to 18, and in general, the nth shell can have up to $2n^2$ electrons. To complicate matters, some shells can have "subshells" that are less likely to "share" electrons in a molecule. For example, electrons in groups of eight seem to have some kind of stability (look for the "octet rule" in a chemistry book, in particular, with reference to argon).

A precise definition or mechanical explanation of a "bond" is avoided here; instead, the simplest kind of bond is described, namely a "covalent bond". A covalent bond between two atoms is formed by sharing (a pair of) electrons from respective outer shells. For example, hydrogen has one of a possible two electrons in its outer (and only) shell. Carbon has atomic number 6, and in the Bohr model, there are 2 electrons completing its inner orbital shell, and 4 in its outer shell. To "complete" the outer shell of carbon, as many as 4 more electrons could be added. So, for example, if a carbon molecule "bonded" with four hydrogens, each hydrogen could "loan" one electron, giving four electrons to complete the outer shell around carbon; on the other hand, carbon can "lose" the four it has so that each hydrogen can share one, completing the outer shell for the hydrogens. So the four bonds in the molecule essentially arise from sharing four pairs equally.

There are other types of bonds (*e.g.*, divalent) where the balance of sharing electrons is unequal, and perhaps not just two electrons are shared. The interested reader can consult nearly any chemistry textbook for more details.

The *valence* of an atom is the number of hydrogen atoms it can bond with, or the number of "bonds" the atom can make. (The degree of a vertex in a graph used to be called the *valence* of that vertex—terminology derived from chemistry.) Bonds are not completely understood, so for present purposes, the approach here is a simplification of reality. There are many examples in chemistry that demonstrate variations that are not totally understood. (A few of these are mentioned below.)

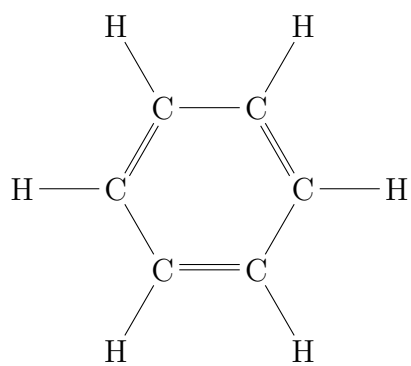
Hydrogen (H) has valence 1, oxygen (O) has valence 2, and carbon (C) has valence 4.

Molecules are modelled by diagrams that can be viewed as multigraphs. Atoms are represented by vertices and bonds are represented by edges. A "double bond" is represented by two (multi)edges between a pair of vertices. So vertices for hydrogen have degree (valence) 1, vertices representing oxygen have degree 2, and vertices for

carbon have degree 4.

The molecule H_2O is modelled by the path on three vertices, H — O — HThe molecule O_2 (with its double bond) can be represented by O ═ O. Carbon dioxide CO_2 is given by O ═ C ═ O. In giving a diagram for a molecule, the actual placement of the atoms is not critical; so bent molecules, like CO_2 or H_2O are sometimes drawn in a straight line.

Some molecules form cycles; for example, benzene C_6H_6 has a ring of six carbons (with alternating single and double bonds), and a hydrogen joined to each carbon.



benzene, C_6H_6

Cyclopentadienyl anion has formula C_5H_5, and is similar in structure to benzene, but in the ring of five carbons, only two double and three single bonds are used. In this molecule, bonds are not "balanced", and the result is a molecule with a charge; such molecules are not considered here. Another molecule that seems to defy the rules is carbon monoxide CO.

For a positive integer n, a molecule of the form C_nH_{2n+2} is called an *alkane*. When $n = 1$, the alkane is CH_4, methane. For $n = 2$, C_2H_6 is butane, for $n = 3$, C_3H_8 is propane. (See below for diagrams.)

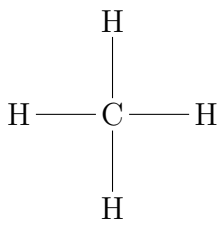
Lemma 3.6.1. *The graph of an alkane is a tree.*

Proof: Let n be a positive integer and let G be a multigraph representing the alkane C_nH_{2n+2}. The number of atoms in C_nH_{2n+2} is $n+2n+2 = 3n+2$, so $|V(G)| = 3n+2$. The total number of valences is $n \cdot 4 + 1 \cdot (2n+2) = 6n+2$, and so by the handshaking lemma, the number of edges in G is $\frac{1}{2}(6n+2) = 3n+1$. Since $|E(G)| = |V(G)| - 1$ and G is connected, G is a tree. □

In any hydrocarbon, often a structure is represented only by its "backbone" or "skeleton" of carbons; from its backbone, it is usually clear how the remaining hydrogens are attached. Here, all atoms are shown.

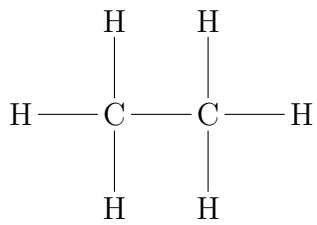
The same collection of atoms can be bonded in different patterns. Different chemicals with the same number of each type of atom but different structure are called *isomers*.

$n = 1$



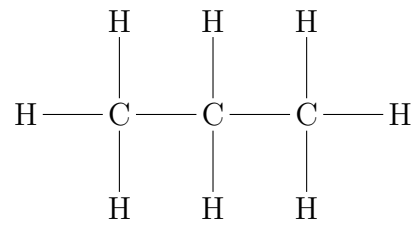
methane

$n = 2$



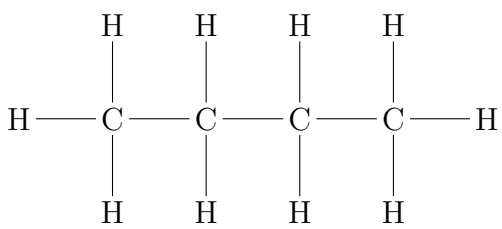
ethane

$n = 3$,

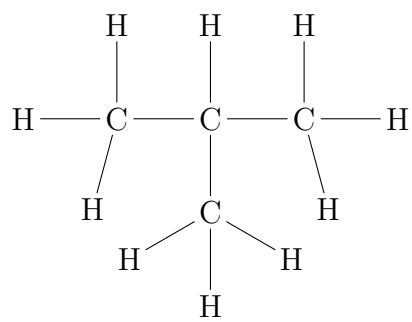


propane

$n = 4$, two isomers:

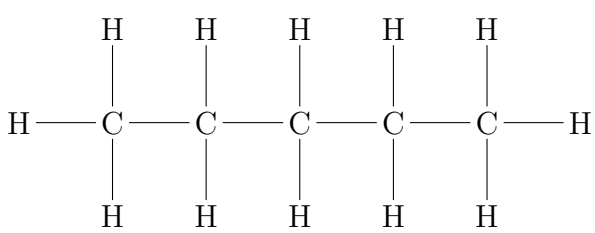


n-butane

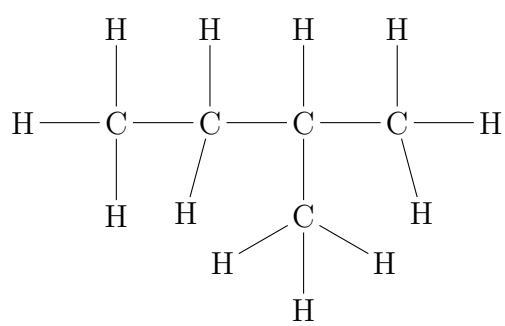


isobutane

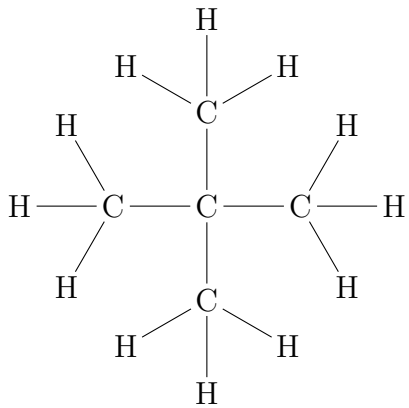
$n = 5$, three isomers:



pentane



2-methylbutane



2,2 dimethyl propane

How many isomers are there of an alkane with arbitrary n? The answer is that there are as many as there are non-isomorphic trees on n vertices. Each such tree represents the carbon skeleton of the molecule. For $n = 1, 2, 3, \ldots$, the number of non-isomorphic trees on n vertices is 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, ... Counting non-isomorphic trees on n vertices is, in general, a challenging problem, not discussed here.

Exercise 165. *Is any graph of ethylene C_2H_4 a tree?*

Exercise 166. *Is a molecule of caffeine tree-like? (Use the valence of nitrogen to be 3.) One formula for caffeine is $C_8N_4O_2H_{10}$.*

3.7 Rooted trees

3.7.1 Terminology

Definition 3.7.1. *A* rooted tree *is a tree with one vertex specified as its root.*

Any vertex in a tree can be treated as a root. If T is a tree and $r \in V(T)$ is designated as the root of T, one might use (T, r) to denote such a rooted tree. The notation $T(r)$ is used by Bondy and Murty [143, p. 104].

Rooted trees are often drawn with the root at the bottom and branches extending upward with no edges crossing. Computer scientists often draw the root at the top, with branches going downward—they probably started doing so because line printers only printed in one direction, but they continue this practice nevertheless. In a sense, a computer scientist's rooted tree diagram looks more like a root system, not a tree. One feature of drawing rooted trees with the root at the top is that in "family trees", "descendants" occur below "parents".

In modern mathematics, many documents are typeset using a program called LaTeX, and in such documents, pictures or drawings can be produced using a program called

"Tikz". In the manual for Tikz [906, p. 213] is a paragraph that summarizes the difference between trees drawn upward or downward:

> You can tell whether the author of a paper is a mathematician or a computer scientist by looking at the direction their trees grow. A computer scientist's trees will grow downward while a mathematician's tree will grow upward. Naturally, the *correct* way is the mathematician's way,...

Recall that in a tree, any two vertices are connected by a unique path. So, in a rooted tree, there is only one path from any vertex to the root.

Definition 3.7.2. *The* rank *of a vertex in a rooted tree is its distance from the root. The* height *of a rooted tree is the maximum rank of its vertices.*

In a rooted tree, a *child* (or an *immediate descendant*) of a vertex v is a neighbour x of v with $\text{rank}(x) = \text{rank}(v) + 1$. The *descendants* of a vertex v include the children of v, the children of children (grandchildren) of v, and so on.

For any vertex v that is not the root, define the parent of v to be the (unique) vertex adjacent to v on the path from v to the root. For any vertex v in rooted tree (T, r), the *ancestors* of v are the vertices on the unique path from v to r.

Comment: In a rooted tree, any vertex has only one parent.

Rooted trees are used in many different settings, such as to represent classification schemes (*e.g.*, ancestry and evolution, library catalogues), sequences of outcomes (as in spread of disease, games of strategy or probability of consecutive events), or even grammatical interpretations of words. In computer science, trees are used in sorting, data structures, searches, and various algorithms.

For example, in Figure 3.10 is a photograph of a drawing of a tree [625] prepared by Thomas Jefferson, circa 1900, for use by the Library of Congress (in U. S. A.) to sort catalogue holdings.

Figure 3.10: Photograph of drawing (circa 1900) by Thomas Jefferson of tree used for classification of Library of Congress holdings

3.7.2 Plane trees

A *plane tree* is a rooted tree together with its drawing having a left-right orientation (where the root is drawn either at the top or at the bottom).

In Figure 3.11 are the five plane trees with 3 edges:

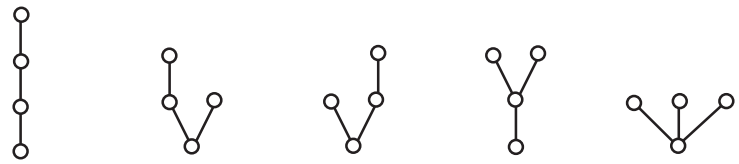


Figure 3.11: The five plane trees with 3 edges (with roots at the bottom).

Exercise 167. *Find all plane trees with 4 edges.*

3.7.3 Binary trees

A *binary tree* is a plane rooted tree where each vertex has at most two children, and each child is designated "left" or "right". If a vertex has a single descendant, it is either left or right. The five binary trees with two edges (or three vertices) are given in Figure 3.12.

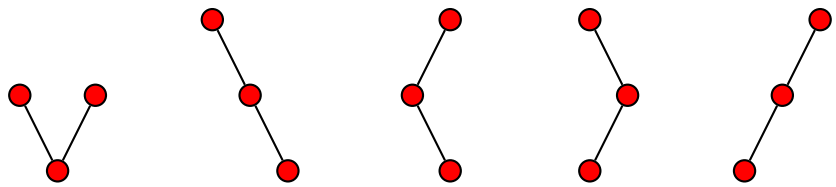


Figure 3.12: The five binary trees on 3 vertices

Exercise 168. *Show that there are 14 binary trees on 4 vertices.*

A *full binary tree* is a rooted plane tree whose every non-leaf has precisely two children. (So every vertex has either 0 or 2 children.) See Figure 3.13 for two examples of full plane binary trees of height 3, that without left-right orientation, are otherwise isomorphic.

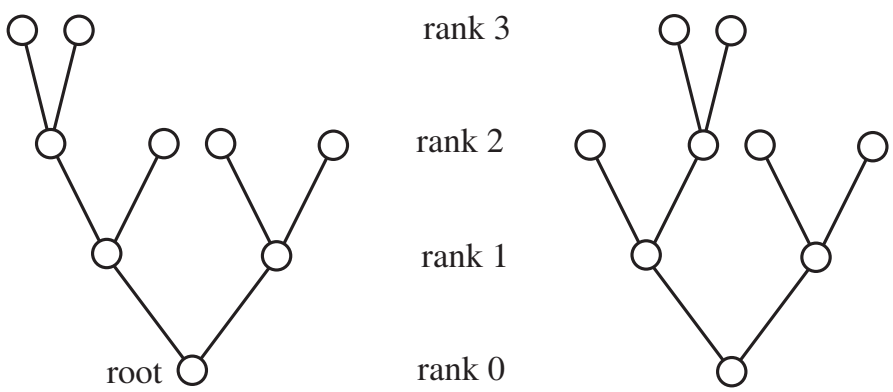


Figure 3.13: Two full plane binary trees of height 3

A plane binary tree is called *complete* if and only if every leaf has rank equal to the height of the tree. [Caution: some authors use "full" to mean "'complete".] So a full complete plane binary tree has one vertex of rank 0, two vertices of rank 1, four vertices of rank 2, eight vertices with rank 3, and so on. Thus, a binary tree of height h has at most $1 + 2 + \cdots + 2^h = 2^{h+1} - 1$ vertices. This simple fact can also be proved by induction.

Exercise 169. *Use induction on h to show that if T is a plane binary tree with height h, then $|V(T)| \leq 2^{h+1} - 1$. Hint: In the inductive step, delete the root.*

3.7.4 Increasing trees

This topic is optional in that increasing trees are rarely studied in a first course in graph theory.

If a rooted tree T has vertices labelled $0, 1, 2 \ldots, n$, where 0 is the root and every path starting at 0 consists of an increasing sequence of vertices, then T is called *increasing*. For example, ignoring left-right orientation, there are six increasing trees on vertices $\{0, 1, 2, 3\}$, as listed in Figure 3.14.

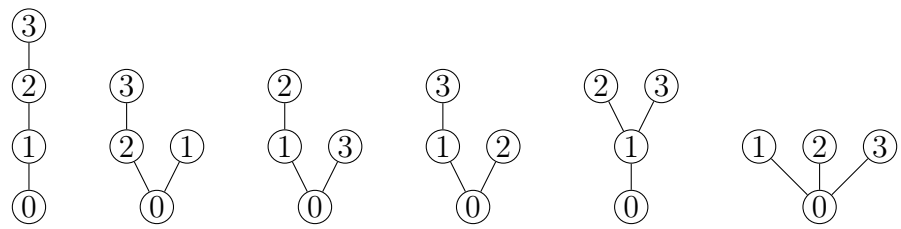


Figure 3.14: The six increasing trees on $\{0, 1, 2, 3\}$

To answer the next exercise, instead of deleting the root as in the solution to Exercise 169), delete (or attach) a leaf.

Exercise 170. *Prove by induction that the number of increasing (rooted) trees on vertex set $\{0, 1, \ldots, n\}$ is $n!$. Consider two trees with precisely the same edge set to be equal, regardless of what orientation is used to draw them.*

The Eulerian number $E_{n,k}$ is the number of permutations on $\{1, 2, \ldots, n\}$ that have precisely k ascents (adjacent pairs $(\sigma(i), \sigma(i+1))$ with $\sigma(i) < \sigma(i+1)$). It is known that the Eulerian numbers are given explicitly by

$$E_{n,k} = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} (k-j)^n;$$

however, this formula is not required for the next exercise.

Exercise 171. *Prove that the number of increasing trees on $n+1$ vertices with precisely k leaves is the Eulerian number $E_{n,k}$.*

See [883, Vol 1, pp. 24–25] for other exercises on increasing trees.

3.7.5 Red-black trees

This subject is considered optional, as this topic is rarely given in a first year course in graph theory (but may occur in a low-level computer science course). Hence, trees in this section have the root at the top. In computer science, a vertex in a tree is often called a node.

Red-black trees on n vertices are a class of binary search trees with height $O(\log_2 n)$.

Definition 3.7.3. *A* red-black tree *is a binary search tree T with nodes (vertices) 2-coloured, say red and black, so that*

1. *Every node is either red or black.*
2. *The root is black.*
3. *Every leaf node is black.*
4. *If a node is red, then both its children are black.*
5. *For each node, all paths from that node to descendant leaves contain the same number of black nodes.*

Figure 3.15 shows a red-black tree based on a diagram from [228, p. 275].

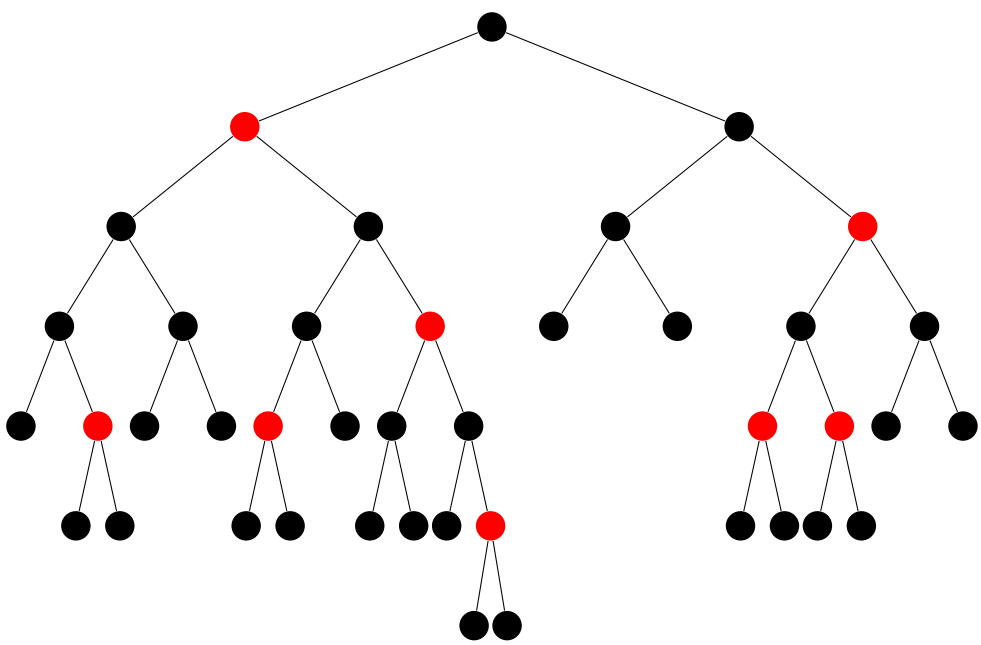


Figure 3.15: A red-black tree with 20 internal vertices and 21 leaves

The *height* of a node v in a red-black tree is the length of a longest path from v to any descendant leaf. (For trees or other partial orders drawn with a root at the bottom, often the height of a node is defined to be the length of the path to the root, but for this section, trees are drawn in the computer science fashion.) Define the *height* of a red-black tree to be the height of the root. To confirm ideas, the height of the red-black tree in Figure 3.15 is 6.

By property 5 above, for any node v, the number of black nodes on any descending path to a leaf is the same. For each node v in a red-black tree, define the *black height* of

v, denoted $\mathrm{bh}(v)$, to be the number of black nodes, not including v, on any descending path from v to a leaf. So all leaves have black height 0. Define the black height of a red-black tree to be the black height of the root. The black height of the tree in Figure 3.15 is 3.

Lemma 3.7.4. *A red-black tree with n internal nodes has height at most*

$$2\log_2(n+1).$$

Proof: The proof is in two parts: first a claim (involving black height) is made and proved inductively, and then some calculations follow the claim.

Claim: For any red-black tree T and any node v of T, the subtree rooted at v contains at least $2^{\mathrm{bh}(v)} - 1$ internal nodes.

Proof of Claim: This is achieved by induction on the height of a node.

Base step: If the height of v is 0, then v is a leaf (with $\mathrm{bh}(v) = 0$). In this case, the subtree rooted at v contains no internal nodes, and $2^{\mathrm{bh}(x)} - 1 = 2^0 - 1 = 0$.

Induction Step: Let $r > 0$ and suppose that for all nodes w of height less than r, the subtree rooted at w contains at least $2^{\mathrm{bh}(w)} - 1$ internal nodes. Let v have height $r > 0$; then v is an internal node with 2 children. If v is black, then each of its children has black height $\mathrm{bh}(v) - 1$; if v is red, each of these children has black height $\mathrm{bh}(v)$. By the induction hypothesis (where each of these two children play the role of w above), the subtrees rooted at these two children each have at least $2^{\mathrm{bh}(v)-1} - 1$ internal children. In all, the tree rooted at v has at least

$$2(2^{\mathrm{bh}(v)-1} - 1) + 1 = 2^{\mathrm{bh}(v)} - 2 + 1 = 2^{\mathrm{bh}(v)} - 1$$

internal nodes, satisfying the claim for any vertex at height r, thereby completing the inductive step.

By mathematical induction on height, for all v, the claim is true.

Returning to the proof of Lemma 3.7.4, let T be a red-black tree of height h and with n internal nodes.

Because of property 4, where v is the root of the tree, along any downward path from the root to a leaf, for every red, there is a black, and so the number of non-root blacks is at least the number of reds on that same path. Hence the black height of the root of T is at least $h/2$. By the claim above, since the tree rooted at v is T itself, $n \geq 2^{h/2} - 1$. Upon solving for h,

$$h \leq 2\log_2(n+1),$$

as desired, completing the proof of Lemma 3.7.4. □

3.7.6 Parentheses and rooted trees

One way to count some rooted trees uses the concept of "parenthetization". A sequence of parentheses is called "legal" or "balanced" if each parenthesis has a matching parenthesis (on the appropriate side). So (( ))( ) is legal, whereas (( )))) and)) ((are not.

In English, the phrase "hot pink flamingo" may be taken to indicate a flamingo that is coloured hot pink, or it could mean a pink flamingo that is hot. To clearly indicate which of the two meanings is intended, the expressions "(hot pink) flamingo" or "hot (pink flamingo)" could be used. So parentheses can be used to parse sentences.

In the real numbers, multiplication is both commutative ($xy = yx$) and associative ($(xy)z = x(yz)$). So evaluating a product of three real numbers x, y, z can be done by a computer in various ways. In all, there are 12 ways the product xyz can be computed, namely

$$(xy)z, x(yz), (xz)y, x(zy), (yx)z, y(xz), (yz)x, y(zx), (zx)y, z(xy), (zy)x, z(yx),$$

where the order of operations is given by evaluating inside parenthesis first.

Exercise 172. *For $n \geq 0$, let $f(n)$ be the number of ways to multiply n numbers in any order. Show that $f(n+1) = \frac{(2n)!}{n!}$.*

If numbers x, y, z are given in order, there are only two ways to compute the product:

$$xyz = (xy)z = x(yz).$$

For four real numbers w, x, y, z given in order, there are five ways to evaluate the product $wxyz$ by grouping with two matching pairs of parentheses:

$$wxyz = w(x(yz)) = w((xy)z) = (w(xy))z = (w(xy))z = (wx)(yz),$$

where in the last product, the expressions wx and yz can be computed in either order. An outside pair of parentheses is not really needed, but is sometimes included for clarity. The following has a simple proof by induction.

Lemma 3.7.5. *A product of $n+1$ numbers given in order can be written with $n-1$ pairs of matching parentheses.*

To each set of n matching parentheses pairs, there exists a corresponding rooted tree (as in Figure 3.16) with n vertices. (The outside set of parenthesis is sometimes omitted by some authors, as they are implicit, corresponding to the root.)

The tree in Figure 3.16 can also represent a sorted set system, for example,

$$\{\{a\}, \{d, \{i, j\}\}, \{g, h\}\}.$$

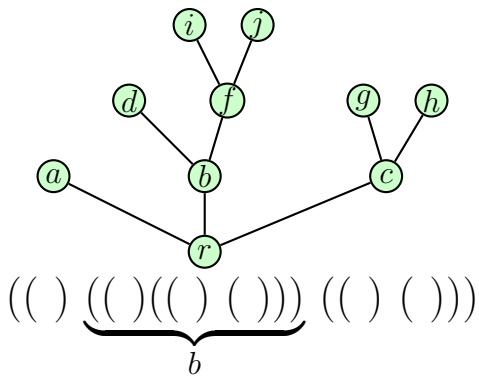


Figure 3.16: Parentheses associated with a rooted tree

The same tree is also a model for a catalogue or a book, as in Figure 3.17.

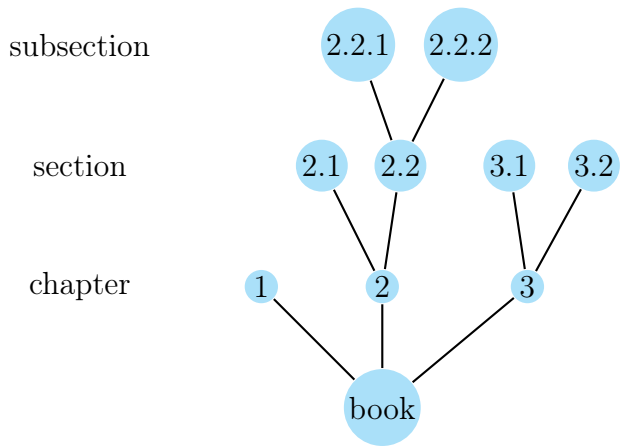
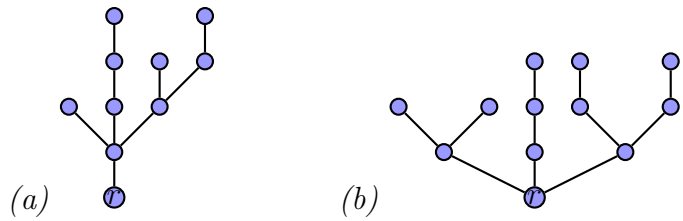


Figure 3.17: A catalogue

Exercise 173. *For the following rooted plane trees, give the associated sequence of parentheses:*



Exercise 174. *Find the rooted plane tree associated with the string of parenthesis* (( ) (( ) (( ))) (( ))), *where the outside pair of parentheses represents the root.*

The tree in Figure 3.16 is not binary; with binary trees, the parentheses are more "balanced", and so can be used to define binary operators.

3.7.7 Catalan numbers and counting rooted trees

Catalan numbers, definition and history

Definition 3.7.6. *For $n = 0, 1, 2, \ldots$, define the nth Catalan number by*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The first few are $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$, and $C_9 = 4862$.

Catalan numbers are named after Eugéne Charles Catalan (1814–1894), but in fact they arose much earlier from work by Euler in the mid 1700s.

For $n \geq 1$, let $f(n)$ be the number of ways can a convex $(n+2)$-gon be triangulated, that is, in how many ways can $n-1$ non-intersecting chords (joining vertices) be added so that the regions thereby formed are triangles. For example, $f(1) = 1$, and since a square is triangulated by either of its two diagonals, $f(2) = 2$, and as shown in Figure 3.18, there are five ways to triangulate a pentagon, so $f(3) = 5$.

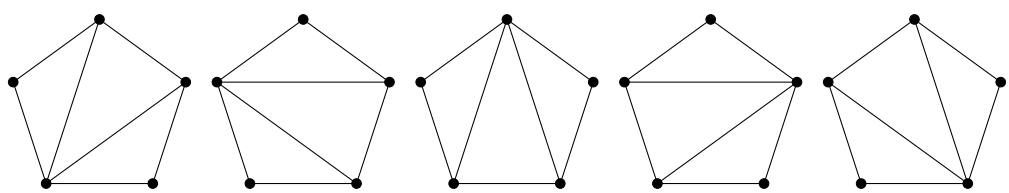


Figure 3.18: Five ways to triangulate a pentagon

In 1751 letter (see [739] for a copy), Euler gave the first few values of $f(n)$ and guessed that

$$f(n) = \frac{1}{n+1}\binom{2n}{n}. \tag{3.2}$$

(Euler did not use binomial coefficients in his letter.) In 1758, the Hungarian mathematician Johann Andreas von Segner [847] gave the recursion

$$f(n+1) = f(n)+f(1)f(n-1)+f(2)f(n-2)+\cdots+f(n-2)f(2)+f(n-1)f(1)+f(n).$$

Using Segner's recursion (and generating functions), Euler was able to prove equation (3.2); his result [350] was published in 1761.

In 1839, Catalan [182] showed that the number of ways to parenthetize a product is an Euler-Segner number. It is for this work that someone much later (in the 20th century) called the numbers "Catalan numbers". According to Igor Pak [738], the earliest use of the name "Catalan number" was by John Riordan (1903–1988) in *Math reviews* dated 1948 and 1964, and it was only after Riordan used the term in his 1968

textbook [793, p. 82] did the name catch on. See [739] for a website with many facts and references for Catalan numbers (including scans of original documents).

The modern definition is simply the algebraic one given above, rather than the number of ways to triangulate a polygon.

Trees counted with Catalan numbers

Stanley [884] lists 26 types of trees that can be counted using Catalan numbers. Only two types of rooted trees are examined here, namely binary trees and plane trees.

Recall from earlier in this section, in order to show that some sequence $s_0, s_1, s_2, \ldots$ is a Catalan sequence, it suffices to show an initial value $s_0 = 1$ and that these numbers satisfy the Segner recursion,

$$s_{n+1} = s_0 c_n + s_1 s_{n-1} + s_2 s_{n-2} + \cdots + s_{n-1} s_1 + s_n s_0. \tag{3.3}$$

In application, sometimes the variables need to be shifted, especially when, say, the first in the sequence is not 0. (For example, sometimes working with trees on n vertices, it might be easier to find the sequence depending on trees with $m = n + 1$ edges.)

For example, for $n \geq 1$, let $b(n)$ denote the number of plane binary trees on n vertices, and put $b(0) = 1$. It is not difficult to see that $b(1) = 1$, $b(2) = 2$, and from Figure 3.12, that $b(3) = 5$ (Exercise 168 shows that $b(4) = 14$.) Suppose T is a binary rooted tree on $n + 1$ vertices. For each $k = 0, 1, \ldots, n$, there are $b(k)b(n - k)$ binary trees with k children of the root on the left and $n - k$ on the right. So for $n \geq 0$,

$$b(n + 1) = b(0)b(n) + b(1)b(n - 1) + b(2)b(n - 2) + \cdots + b(n)b(0). \tag{3.4}$$

So $b(n)$ is a Catalan number.

Let $f(n)$ be the number of rooted plane trees on n vertices. Then $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, and as shown in Figure 3.11, $f(4) = 5$. Is every $f(n)$ a Catalan number?

Exercise 175. *Prove that the number of (rooted) plane trees with n edges is the Catalan number C_n.*

Exercise 176. *Prove that the number of full binary trees with n internal nodes is the Catalan number C_n.*

For more relations between Catalan numbers and trees, see [209], [252], [573], [883], or [884].

3.8 Treewidth

This section is a very brief introduction of the "treewidth" of a graph G—a kind of measure of how close G is to being a tree. Treewidth is not a subject commonly given in a first course in graph theory.

The notions of tree decompositions and treewidth (under the name "dimension") were introduced by Halin [478] in 1976, and independently, later by Seymour and Thomas (see [853]) in in their study of graph minors.

Definition 3.8.1. *A* tree decomposition *of a graph G is a pair $(T, \mathcal{V})$, where T is a tree and $\mathcal{V} = \{V_t \subseteq V(G) : t \in V(T)\}$ is a family of subsets of $V(G)$ indexed by vertices in T so that three conditions hold: (i) $\cup_{i \in V(T)} = V$; (ii) for every edge $\{x, y\} \in E(G)$, there exists $V_t \in \mathcal{V}$ containing both x and y; (iii) if $t_i t_j t_k$ is a path in T, then $V_i \cap V_k \subseteq V_j$. The width of a tree decomposition is $(T, \mathcal{V})$ is* $\max\{|V_t| - 1 : t \in V(T)\}$ *and the* treewidth *of G, denoted $tw(T)$ is the minimum width of any tree-decomposition of G.*

There are different characterizations of treewidth, some of which are described in [256]. For example, Seymour and Thomas [854] characterized treewidth in terms of "brambles". Another characterization of treewidth uses chordal graphs and clique numbers. For a graph H, let $\omega(H)$ denote the clique number of H, the order of the largest complete subgraph of H.

Lemma 3.8.2 (see [256, Cor. 12.3.12])**.** *For any graph G,*

$$tw(G) = \min\{\omega(H) - 1 : G \subseteq H, H \text{ chordal}\}.$$

So the treewidth of a tree is 1, the treewidth of a cycle is 2, and the treewidth of the complete graph K_n is $n - 1$. As noted in Section 7.10, outerplanar graphs have treewidth at most 2.

Chapter 4

Connectivity and Menger's theorems

Recall that a graph is connected if and only if for any two vertices in the graph, there is a walk from one to the other. The "connectivity" of a connected graph is a measure of the "connectedness" (*e.g.*, if between any two vertices there are many walks from one to the other, then one might say that the graph has higher connectivity). There are two versions of connectivity, one for vertices and one for edges.

4.1 Vertex connectivity

Definition 4.1.1. *If a graph G is connected, a set $S \subseteq V(G)$ of vertices is called a* cut-set *if the removal of S (and all edges using these vertices) disconnects the graph. A* cut-vertex *is a cut-set consisting of a single vertex.*

A cutset, as defined in Definition 4.1.1, is often referred to as a "vertex cut-set", to differentiate from an "edge cut-set", as given in Definition 4.2.1.

Observe that K_n has no vertex cut-set. A particular kind of (vertex) cut-set is useful to define:

Definition 4.1.2. *Let $G = (V, E)$ be a graph (or multigraph), and let $x, y \in V$ be vertices in the same component of G. An x–y separating set is a subset of vertices $S \subseteq V \backslash \{x, y\}$ so that x and y lie in different components of $G - S$.*

A connected graph with a cut-vertex is sometimes called *separable*. For example, in a tree, any vertex that is not a leaf is a cut-vertex.

Exercise 177. *Show that if G is a connected graph with $|V(G)| \geq 2$, then there exist at least two vertices, neither of which is a cut-vertex.*

Theorem 4.1.3 (Ramachandra Rao, 1968 [772])**.** *If G is connected and has r cut-vertices, then*

$$|E(G)| \leq \binom{n-r}{2} + r.$$

Exercise 178. *Prove Theorem 4.1.3 and show that this bound is tight by exhibiting a graph for which equality is attained.*

Exercise 179. *Show that for any (simple) graph G, at least one of G or its complement $\overline{G}$ is connected.*

Exercise 180. *Let $d_1 \leq \cdots \leq d_n$ be the degrees of an n-vertex graph G. Show that if for each $j \leq n-1-d_n$, $d_j \geq j$ holds, then G is connected.*

Exercise 181. *Describe all graphs with the property that any connected subgraph is an induced subgraph.*

In a connected computer network, machines talk to each other via some kind of link. Imagining each link to be an edge of a graph, the associated network could crash by the underlying graph becoming disconnected. If certain machines fail or certain links fail, and the remaining network remains connected, then the network can be rerouted to keep things running smoothly while repairs are made to certain components.

Below, two notions are introduced, "vertex-connectivity", which corresponds to the minimum number of machines (vertices) that might bring down the network, and "edge-connectivity", corresponding to the minimum number of critical links (edges).

This next definition turns out to be one of two equivalent definitions.

Definition 4.1.4 (First definition of k-connected)**.** *A graph is called* k-connected *if and only if every vertex cut-set contains at least k vertices.*

Definition 4.1.4 agrees with that in, *e.g.*, [190] and [429]. For each $k \geq 1$, Definition 4.1.4 says that a connected graph G is k-connected if and only if the removal of any $k-1$ vertices fails to disconnect G.

Another way to define k-connectivity is in terms of paths. Say that two x–y paths are *internally disjoint* if and only if they share no vertices other than x and y.

Definition 4.1.5 (Second definition of k-connected)**.** *A graph G is k-connected if and only if for any two vertices $x, y \in V(G)$, there exist k internally disjoint x–y paths.*

Menger showed that these two definitions (Definitions 4.1.4 and 4.1.5) of "k-connected" are the same.

Theorem 4.1.6 (Menger, 1927 [676], undirected vertex form)**.** *Let G be a graph and $x, y \in V(G)$ be non-adjacent vertices. The minimum number of vertices in an x–y separating set is the maximum number of internally disjoint x–y paths.*

Proof of Menger's theorem (undirected vertex version): For any graph G and non-adjacent vertices $x, y \in V(G)$, if any minimum x–y separating set has k vertices, then there are at most k internally disjoint x–y paths (since each path uses at least one point in a x–y separating set). It remains to show that if a minimum separating set

has k vertices, there are at least k (and hence exactly k) internally disjoint x–y paths. The proof of this direction is by induction on the number of edges in G.

For each $m \geq 0$, let $S(m)$ be the statement that for any graph G with m edges then for any nonadjacent vertices $x, y \in V(G)$, if the minimum number of vertices in an x–y separating set is k, then there exists k internally disjoint x–y paths.
BASE STEP: If a graph has $m = 0$ edges, there are no vertices in an x–y separating set, and there are no x–y paths. So $S(0)$ is true.

INDUCTION STEP: Let $m \geq 1$ and suppose that for each $m' < m$, $S(m')$ is true. Let G be a graph on m edges, let x and y be non-adjacent vertices, and let k be the minimum number of vertices in a x–y separating set.

If x and y are in different components of G, then $k = 0$ and there are no x–y paths. So assume that x and y are in the same component. For $k = 1$, the component containing x and y is connected, and so there is at least one x–y path; so assume that $k \geq 2$.

If there is a vertex z adjacent to both x and y, then z is in every minimum x–y separating set. Thus, in $G - z$, a minimum x–y separating set has $k - 1$ elements, and so by the induction hypothesis, there are $k - 1$ disjoint x–y paths in $G - z$. Together with the path xzy in G, there are k disjoint x–y paths in G. So suppose that there is no vertex adjacent to both x and y.

CASE 1: Suppose that S is a minimum x–y separating set so that x is not adjacent to every vertex in S and y is not adjacent to every vertex in S. Let G_x denote the component of $G - S$ containing x. Since x is not adjacent to all of S, G_x has, in addition to x, a neighbour of x (if not, some element of S is superfluous). Form a new graph G_x^* formed by replacing G_x with a new vertex x' and making x' adjacent to every vertex in S. The graph G_x^* has fewer edges than G since at least one edge in G_x collapsed, and for each $s \in S$, there is an edge to G_x. Similarly, define G_y, also a non-trivial component, and G_y^* with vertex y', which also has fewer edges than G.

In G_x^*, S is an x'–y separating set so by the induction hypothesis, G_x^* contains k internally disjoint x'–y paths. Similarly, in G_y^*, there are k internally disjoint x–y' paths. Splicing these paths together at S produces k internally disjoint x–y paths.

CASE 2: Suppose that S is a minimum x–y separating set so that either every vertex of S is adjacent to x and not to y, or every vertex of S is adjacent to y and not to x. Then $d(x, y) \geq 3$. Let $P = xu_1u_2...y$ be a shortest x–y path, and let $e = u_1u_2$. Both of u_1 and u_2 are not in S (since otherwise, P is not a shortest path). So in $G - e$, every x–y separating set contains at least $k - 1$ vertices. In fact, every x–y separating set has at least k vertices, as is now shown.

In hope of a contradiction, let $W = \{w_1, \ldots, w_{k-1}\}$ be an x–y separating set in $G - e$. Then $W \cup \{u_1\}$ is an x–y separating set in G with k elements, and so is a minimum x–y separating set in G. Since x is adjacent to u_1, by assumption x is adjacent to every w_i as well and y is not adjacent to any element of $W \cup \{u_1\}$.

Similarly, $W \cup \{u_2\}$ is a minimal x–y separating set in G, and since x is not adjacent to u_2 (otherwise P is not a shortest x–y path), by assumption, y is adjacent to u_2, and hence to every element in W as well. Since $k \geq 2$, w_1 is adjacent to both x and y, contrary to assumption. (A contradiction can also be reached by noting that since x is adjacent to w_1, then x is adjacent to all of $W \cup \{u_2\}$, in particular, adjacent to u_2, contrary to P being shortest.)

So any x–y separating set in $G - e$ has at least k vertices, and so exactly k vertices. By induction hypothesis applied to $G - e$, there exist k internally disjoint x–y paths, concluding this case, and hence the proof. □

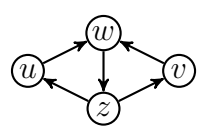
It may be noted that there are four versions of Menger's theorem, two for deleting vertices (one for graphs and one for digraphs) and two for deleting edges (one for graphs and one for digraphs). The version above is often simply called "Menger's theorem". The other three versions are stated here for reference; these are discussed later, including proofs and relationships with the maxflow-mincut theorem in Section 5.11.2.

Theorem 4.1.7 (Menger's theorem, undirected edge form)**.** *Let $G = (V, E)$ be a graph and let $x, y \in V$. The minimum number of edges whose deletion separates x from y is equal to the maximum number of edge-disjoint x-y paths.*

Theorem 4.1.8 (Menger's theorem, directed graph vertex form)**.** *Let $G = (V, A)$ be a digraph and let $x, y \in V$ where $(x, y) \notin A$. Then the minimum number of vertices in a x-y separating set (of vertices) is equal to the maximum number of internally disjoint directed x–y paths.*

Theorem 4.1.9 (Menger's theorem, directed graph edge form)**.** *Let $G = (V, A)$ be a digraph and let $x, y \in V$. The minimum number of (directed) edges whose deletion separates x from y is equal to the maximum number of directed edge-disjoint x-y paths.*

Exercise 182. *Using the directed version of Menger's theorem for edges (Theorem 4.1.9), prove the undirected version of Menger's theorem for edges (Theorem 4.1.7). Hint: Let G be an undirected graph. Create a directed graph D by replacing each edge $\{u, v\} \in E(G)$ with the following "gadget":*



Menger proved a stronger version of Theorem 4.1.6, where the two vertices are replaced by disjoint sets; see Theorem 4.1.13.

There are many proofs of Menger's theorem(s); for example, West [977, p. 167] mentions that over 15 proofs have been published (of the first version) and gives a proof

by induction on the number of vertices that uses the König-Egeváry theorem (Theorem 5.6.1). In a remark, West[977, p. 181] also outlines how the maxflow-mincut theorem (Theorem 5.11.7) proves the directed vertex version; virtually the same proof also proves the digraph edge version of Menger's theorem (see Theorem 5.11.13). Bollobás [125, p. 75–76] gives two proofs, one short one from a vertex capacity version of the maxflow-mincut theorem (Theorem 5.11.14), and the above proof, which relies only on first principles (induction). Bondy and Murty [143, pp. 208–209] mentions a proof that uses the maxflow-mincut theorem, and gives a short proof due to Göring [427], also by induction on the number of edges.

Not only does Menger's theorem follow from the König–Egeváry theorem or the maxflow-mincut theorem, but versions of Menger's theorem can be used to prove these theorems, so the proof of Menger's theorem given above is from first principles (as found in [125, pp. 75–76]) so as to avoid any possible circular arguments.

A proof nearly identical to the above proof of Theorem 4.1.6 gives the directed version of Menger's theorem for vertices (Theorem 4.1.8).

So Theorem 4.1.6 says that one can safely define the *connectivity* of a connected graph G in either of two ways.

Definition 4.1.10. *The* vertex-connectivity *of a graph G, denoted by $\kappa(G)$, is the minimum number (if it exists) of vertices required to delete in order to disconnect the graph; for each $n \in \mathbb{Z}^+$, define $\kappa(K_n) = n - 1$. If G is disconnected, define $\kappa(G) = 0$.*

Definition 4.1.11. *A connected graph with a cut-vertex is called separable. A 2-connected graph is called nonseparable.*

Exercise 183. *For $n \geq 3$, find $\kappa(K_n)$, $\kappa(C_n)$, and $\kappa(Q_n)$. For integers $1 \leq a \leq b$, find $\kappa(K_{a,b})$.*

Exercise 184. *Let $d \geq 2$ and let G be a bipartite d-regular graph. Show that $\kappa(G) \neq 1$.*

In Menger's theorem (Theorem 4.1.6), if x and y are adjacent, there is no x–y separating set; Whitney extended Menger's theorem to give a characterization of k-connected graphs.

Theorem 4.1.12 (Whitney, 1932 [981])**.** *For $k \in \mathbb{Z}^+$, a graph is k-connected (under the first definition) if and only if every pair of vertices are connected by k internally disjoint paths.*

Proof: The proof here is only for $k = 2$; the general case is given as Exercise 185. (Another proof based on Menger's theorem for general k can be found in [190].)

For one direction, suppose that any two vertices x and y in a graph G are connected by two disjoint paths. Then x and y can not be separated by removing a single vertex, and so G can not be disconnected by the removal of a vertex, and so is 2-connected.

For the other direction, suppose that G is a graph with $|V(G)| \geq 3$, and that G is 2-connected. Since G is 2-connected, G is connected, and so for any $u, v \in V(G)$, the distance $d(u, v)$ is finite. The proof here is by induction on $d(u, v)$ that there are two internally disjoint u–v paths.

Base step: When $d(u, v) = 1$, the pair $\{u, v\}$ is an edge of G, and since removal of either u or v does not disconnect the graph, removing the edge uv does not disconnect G. Hence there is a u–v path in G that does not use the edge $\{u, v\}$; thus are there two disjoint u–v paths (one of which is $\{u, v\}$).

Inductive step: Let $m \geq 1$ and suppose that for any 2-connected graph H (with at least 3 vertices), if two vertices $x, y \in V(H)$ have $d(x, y) \leq m$, then there exists two internally disjoint x–y paths. Fix a 2-connected graph G and two vertices u, v with $d_G(u, v) = m + 1$, and let P be a (shortest) u–v path with length $m + 1$. Let v_1 be the vertex in P next to v. Then $d_G(u, v_1) \leq m$, and so by induction hypothesis, there are two disjoint u–v_1 paths R_1 and R_2. Since $G \backslash v_1$ is connected (G is 2-connected), there exists a u–v path S not containing v_1. If S is disjoint from either R_1 or R_2, then there are two disjoint u–v paths (S and the R_i extended by $\{v_1, v\}$). So suppose that S intersects both R_1 and R_2 in an internal vertex. If y is the last vertex on S (closest to v) that lies on $R_1 \cup R_2$, say, on R_1, then the u–v path that starts with R_1 and continues with $\{y, v\}$ is disjoint from the path $R_2 \cup \{v_1, v\}$. This completes the inductive step, showing two internally disjoint u–v paths.

By mathematical induction, for any 2-connected graph G and any two vertices $u, v \in V(G)$, (regardless of their distance) there are two internally disjoint u–v paths. □

As previously mentioned, according to [375, pp. 54–55] (and other sources), Menger proved a far stronger theorem, of which Theorem 4.1.6 is a special case:

Theorem 4.1.13 (Menger, 1927 [676])**.** *Let $G = (V, E)$ be a graph. For any two disjoint subsets of vertices S and T, the maximum number of internally disjoint paths from S to T is equal to the minimum number of vertices in an S-T separating set.*

For a short proof of Theorem 4.1.13, see [427].

In the case $k = 2$, Whitney's theorem (Theorem 4.1.12) can be restated to say that a graph is 2-connected if and only if every pair of vertices lies on a cycle.

Exercise 185. *Complete the proof of Whitney's theorem for k-connected graphs.*

Exercise 186. *Let $u = (0, 0, \ldots, 0)$ and $v = (1, 1, \ldots, 1)$ be two vertices in the unit cube graph Q_n. Give an example of a minimum u–v separating set, and give an example of a maximum collection of internally disjoint u–v paths.*

The following definition is not used immediately, but is recorded here for later reference.

Definition 4.1.14. *If G is a graph, a* block *is a maximal 2-connected subgraph of G.*

For example, the bowtie graph (see Figure 1.31) consists of two blocks (each block is a triangle) sharing a common vertex. (See also "cactus graphs" mentioned in Section 7.10—graphs whose blocks are cycles.) Many proofs in graph theory rely on looking at block-subgraphs (*e.g.*, Brooks' theorem, Theorem 6.2.10), but such concepts are not of immediate concern here.

4.2 Edge-connectivity

The notions of vertex cut-set and vertex separating set have direct analogues for edges.

Definition 4.2.1. *Let G be a connected graph. A collection C of edges in G is called an* edge cut-set *if the removal of C disconnects G.*

An edge cut-set is sometimes referred to as simply an edge cut.

Definition 4.2.2. *Let G be a connected graph and let x and y be vertices in G. A collection C of edges in G is called an x-y separating set if the removal of C gives a graph with no x-y paths.*

Definition 4.2.3. *For a connected graph G on at least two vertices, the* edge-connectivity *of G, denoted $\lambda(G)$, is the minimum number of edges whose removal disconnects the graph; define $\lambda(K_1) = 0$. If G is disconnected, define $\lambda(G) = 0$.*

If every edge of a graph G is on a cycle, then $\lambda(G) \geq 2$.

Exercise 187. *Show that if $n \geq 2$, then $\lambda(K_n) = n - 1$.*

Recall that $\delta(G)$ denotes the minimum degree of vertices in G.

Theorem 4.2.4 (Whitney, 1932 [981])**.** *For any graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof: The second inequality follows from the fact that deleting all edges incident to some vertex of minimum degree disconnects the graph.

For the first inequality, some special cases are handled easily. If G is disconnected, then $\kappa(G) = \lambda(G) = 0$, so assume that G is connected. If for some n, $G = K_n$, then $\kappa(K_n) = n - 1$ and by Exercise 187, $\lambda(K_n) = n - 1$ (and $\delta(K_n) = n - 1$). So, assume that G is connected and not complete; put $|V(G)| = n \geq 2$. Since G is not complete, $\lambda(G) \leq \delta(G) \leq n - 2$.

The idea is to take a minimal edge-cut and use one vertex per edge from that edge-cut to form a cut-set of vertices, and since this vertex set has at most the cardinality

of the edge cut-set, the inequality is proved. However, one needs to pick the vertex cut-set so that there are components left.

Let C be a minimum edge-cut (so $|C| = \lambda(G) \leq n-2$) and let G_1 and G_2 be the components of $G - C$.
CLAIM: There exists $x \in V(G_1)$ and $y \in V(G_2)$ so that $\{x, y\} \notin E(G)$.
PROOF OF CLAIM: Put $|V(G_1)| = k$ and $|V(G)| = n - k$. If the claim fails, then $|C| = k(n-k)$. Since $k \geq 1$ and $n - k \geq 1$,

$$0 \leq (k-1)(n-k-1) = k(n-k) - (n-1),$$

and so $|C| = k(n-k) \geq n-1$, which contradicts $\lambda(G) \leq n-2$, finishing the proof of the claim.

By the claim, let $x \in V(G_1)$ and $y \in V(G_2)$ be so that $\{x, y\} \notin E(G)$. Define

$$K = \{y' \in V(G_2) : \{x, y'\} \in C\} \cup \{x' \in V(G_1) : x' \neq x, \{x', y'\} \in C\}.$$

Since K contains a vertex from every edge in C, $|K| \leq |C|$. (Inequality can happen if two edges in C meet at a vertex $x' \neq x$ in G_1.) Since x and y are not vertices in K, the graph $G - K$ still has (at least) two components, so indeed K is a cut-set and thus $|K| \geq \kappa(G)$. □

Exercise 188. *Find a graph G on at least 3 vertices so that $\kappa(G) < \lambda(G)$, and find a graph H so that $\lambda(H) < \delta(H)$. (Challenge problem: Find a graph that satisfies both inequalities.)*

It is known (see [190, p. 93] for this and other examples) that if G is a cubic graph then $\kappa(G) = \lambda(G)$.

Chapter 5

Matchings, covers

The theory of matchings is rather well-developed, and an interested reader could consult some graph theory texts or the book *Matching theory* by Lovász and Plummer [643]. Here is only a very brief introduction, introducing the language and touching on only a few highlights. Only finite sets and finite graphs are studied here; for cases involving infinite graphs, see the survey by Aharoni [12].

5.1 Motivation

Suppose that there are four workers, say, 1,2,3,4, and four jobs, J_1, J_2, J_3, J_4, where each worker i is suitable only for certain jobs: 1 can do J_1, 2 can do J_1,J_2, 3 can do J_1 and J_2, and 4 can do jobs J_1, J_3, J_4. Is there a job assignment (one job per person) that matches people with jobs? There are many ways to argue that this can not be done; for example, J_3 and J_4 can only be done by 4; another more complicated argument might be that since J_4 can only be done by 4, in what remains, J_1 and J_2 are done collectively by 1 and 2, leaving 3 with no job possible.

Rather than finding a sequence of deductions to show whether or not a desired job assignment exists, there are some general theoretical concerns that help to decide if such an assignment exists. One technique involves representing above type problems as bipartite graphs, with people forming one partite set, and jobs the other, connecting p to J_j if and only if p can do J_j. Expanding on this idea requires some formal definitions.

5.2 Some definitions and basics

A set of edges in a graph is called *independent* if no two share a vertex. A *matching* in a graph G is a collection of independent edges $M \subseteq E(G)$. The empty set can be considered as an "empty matching", though such is seldom used. A *perfect matching* M in a graph G is a matching that uses all vertices (so a graph with a perfect matching has

an even number of vertices). A matching M in a graph G is called a *maximal* matching if and only if there does not exist a larger matching containing M. A matching M is *maximum* if and only if M is of maximum cardinality among all possible matchings.

If $X \subset V(G)$ and $Y \subset V(G)$, where $X \cap Y = \emptyset$, a matching M in G *matches* X *to* Y if for every $x \in X$, there exists a $y \in Y$ so that $\{x, y\} \in M$. (Such a matching is often called X*-saturated.*)

Exercise 189. *Find (with proof) the number of perfect matchings in* $K_{n,n}$.

Exercise 190. *Let* G *be a subgraph of* $K_{n,n}$ *with at least* $n^2 - n + 1$ *edges. Show that* G *contains a perfect matching.*

Exercise 191. *For* $n \geq 1$*, how many perfect matchings are there in* K_{2n}*? Prove your answer. (Once you find the formula, at least one proof is by induction and one proof is more direct.)*

For the next exercise, the cube graph Q_n is as in Definition 1.6.8.

Exercise 192. *For* $n \geq 1$*, show that the* n*-dimensional cube graph* Q_n *contains a perfect matching.*

The solution given for Exercise 192 gives some of the ways perfect matchings in Q_n can be constructed. (There are many more constructions of perfect matchings in Q_k.) How can one count the total number of matchings in Q_n? In fact, there is a fairly simple way (however, it may take many computations) to find the total number of perfect matchings in any bipartite graph. One idea is to use the *permanent* of a matrix:

Definition 5.2.1. *Let* $A = (a_{ij})$ *be an* $n \times n$ *matrix with real entries. An elementary product from* A *is a product of* n *entries, no two from the same row or column. (So for each row/column, exactly one entry is chosen.) The permanent of* A*, denoted* perm(A)*, is the sum of all* $n!$ *elementary products from* A*.*

Note that the permanent of a matrix is similar to the determinant, except that in the determinant, an elementary product is either added or subtracted depending on the positions of the elements in the elementary product. An elementary product can also be written in terms of permutations.

Let $\sigma : [n] \to [n]$ denote a permutation, and let S_n be the set of all permutations on $[n]$. For an $n \times n$ matrix A,

$$\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

If $G = (X \cup Y, E)$ is a bipartite graph with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, define the $n \times n$ 0–1 matrix $B(G) = (b_{ij})$ by $b_{ij} = 1$ if and only if $\{x_i, y_j\} \in E$. This

matrix can be seen as one quarter of the standard adjacency matrix, and is called the "bipartite adjacency matrix". The following simple observation was published in 1961 by two authors independently.

Theorem 5.2.2 (Fisher, 1961 [366], Kasteleyn, 1961[555]). *If G is an equibipartite graph with bipartite adjacency matrix B, then the number of perfect matchings in G is perm(B).*

Proof: If some elementary product in B is 1, then all entries forming this product are 1, and these entries correspond to a perfect matching. If M is a perfect matching, the corresponding entries in the adjacency matrix determine an elementary product with all 1s. So the sum of the elementary products in B is the number of matchings. □

In particular, using permanents, the number of perfect matchings in Q_n can be computed (see [431] for more details).

Exercise 193. *Show that for each positive integer $k \geq 2$, the cube graph Q_k has at least $2^{2^{k-2}}$ perfect matchings. Hint: use induction on k.*

Exercise 194. *Show that the Petersen graph has six perfect matchings.*

As is asked for in Exercise 382, at most two of the six perfect matchings asked for in Exercise 194 are pairwise disjoint. (In other words, P does not have a 1-factorization—see Section 8.2.)

5.3 Existence of matchings and applications

Recall that for $x \in V(G)$, $N_G(x)$ is the neighbourhood of x; when clear, only $N(x)$ is written. For any $S \subset V(G)$, define $N(S) = \cup_{s \in S} N(s) \backslash S$. An obvious necessary condition for a bipartite graph $G = (X \cup Y, E)$ to have a matching from X into Y is

$$\forall S \subseteq X, \quad |N(S)| \geq |S|.$$

In the example with people and jobs, $N(\{1, 2, 3\}) = \{J_1, J_2\}$, so there is no possible way to provide jobs for 1,2, and 3 (since only two are available to all). The following theorem shows that the above condition (called "Hall's condition") is also sufficient! (Marshall Hall was a major figure in combinatorics, but the following is due to Philip Hall.)

Theorem 5.3.1 (P. Hall's matching theorem, 1935 [479]). *A bipartite graph $G = (X \cup Y, E)$ has a matching from X into Y if and only if*

$$\forall S \subseteq X, \quad |N(S)| \geq |S|. \tag{5.1}$$

One standard proof of Theorem 5.3.1 relies on the concept of "augmenting path" (a path is M-augmenting if and only if the path has odd length, and every second edge of the path is in M, with the first and last edge not in M—see Section 5.8 for definitions and a reference for the details). Another proof of Hall's theorem follows from Menger's theorem (*e.g.*, see [125]). The proof given here is an inductive proof, apparently due to Halmos and Vaughn, as found in [125] (though with different terminology):
Proof of Hall's theorem: The condition (5.1) is necessary, so it remains to so show sufficiency. The proof of sufficiency is by induction on $|X|$. For each $m \in \mathbb{Z}^+$, let $A(m)$ be the assertion that for any $G = (X \cup Y, E)$ with $|X| = m$, if condition (5.1) holds, then there exists a matching of X into Y.

Base step: When $|X| = 1$, condition (5.1) says that there is at least one edge leaving X, and this one edge is a matching from X into Y. So $A(1)$ is true.

Inductive step: Let $k \geq 1$ and suppose that for all $m = 1, \ldots, k$, the statement $A(m)$ is true. Let $G = (X \cup Y, E)$ be bipartite with $|X| = k + 1$. Suppose that for G, Hall's condition (5.1) holds.

Case 1: Suppose that for each $j \in \{1, 2, \ldots, k\}$, any j elements from X are adjacent to at least $j+1$ elements of Y. Then pick one $x \in X$ and $y \in N(x)$ and delete the edge $\{x, y\}$ together with its vertices; the remaining graph $G' = ((X \backslash \{x\}) \cup (Y \backslash \{y\}), E')$ has $|X \backslash \{x\}| = k$ and satisfies (5.1), so by $A(k)$, $X \backslash \{x\}$ can be matched into $Y \backslash \{y\}$. The k edges in such a matching together with $\{x, y\}$ form a matching of X into Y.

Case 2: Suppose that for some j the property in Case 1 fails and so pick j, $1 \leq j \leq k$, and a set $S \subset X$ with $|S| = j$ and $|N(S)| \leq j$. Then by (5.1), $|N(S)| \geq |S| = j$, so $|N(S)| = j$. By $A(|S|)$, these j vertices can be matched into Y. Remove these j vertices from X and their matches in Y, producing a smaller graph G^*.

Claim: G^* satisfies condition (5.1). One simple way to see this claim is to argue by contradiction. If some $\ell \leq k + 1 - j$ of the remaining vertices in X are adjacent to less than ℓ remaining vertices in Y, then these ℓ vertices in X together with the original j vertices in X have less than $j + \ell$ neighbours in Y, violating the condition. This proves the claim. So by $A(k + 1 - j)$, there is a matching in G^* from the remaining $k + 1 - j$ vertices in X to the remaining vertices in Y. These two matchings match all of X.

So in either case, a matching from X to Y exists, and so $A(k+1)$ is true, concluding the inductive step.

By mathematical induction, for all $m \geq 1$, $A(m)$ is true. □

Exercise 195. *Use Menger's theorem for sets (Theorem 4.1.13) to prove Hall's theorem. Hint: To a given bipartite graph $G(X, Y, E)$, add a vertex x adjacent to each vertex in X and a vertex y adjacent to each vertex in Y.*

Hall's theorem is often stated in terms of marriages. Here "marriage" is of the traditional sense, one man to one woman. [There are theorems that cover various

forms of polygamy (see, *e.g.*, [65]) and similar results for same-sex marriages seem to be limited (see, *e.g.* [618, p. 220]).]

Theorem 5.3.2 (Hall's Marriage theorem)**.** *Let X be a set of women, and Y a set of men. To each woman $w \in X$, let $S(w) \subseteq Y$ be the set of men that w finds "suitable" for marriage. It is possible to marry off all women if and only if for each subset $C \subset X$, $|\cup_{w \in C} S(w)| \geq |C|$.*

See Section 5.4 for an extension of Hall's marriage that produces "stable marriages".

Hall [479] did not originally prove his theorem in terms of marriages or graphs. He proved an equivalent theorem about sets, Theorem 5.3.5 below; for this version, a definition is needed.

Definition 5.3.3 (SDR)**.** *For a collection of non-empty sets $\mathcal{S} = \{S_1, \ldots, S_k\}$, a set*

$$\{x_1, \ldots, x_k\} \subseteq \bigcup_{i=1}^{k} S_i$$

is called a system of distinct representatives *(SDR) for $\mathcal{S}$ if and only if the x_i's are all different, and for each j, $x_j \in S_j$.*

It might be interesting to note that Hall [479, p. 26] used the notation "C.D.R", which stood for "complete set of distinct representatives". [I don't know who first coined SDR.]

Example 5.3.4. *Suppose that $S_1 = \{b, c, d\}$, $S_2 = \{a, b, c, e\}$, $S_3 = \{a, c\}$. Then the union of all S_is, is $S_1 \cup S_2 \cup S_3 = \{a, b, c, d, e\}$. A SDR for this family could be: (b, a, c), where b "represents" S_1, a represents S_2, and c represents S_3.*

In Example 5.3.4, since $c \in S_1$, $c \in S_2$, and $a \in S_3$, one might say that (c, c, a) is a system of representatives, but not a system of *distinct* representatives.

Exercise 196. *Find another SDR for the Example 5.3.4, thereby showing that if SDRs exist, they need not be unique.*

In Definition 5.3.3, the sequence $\{x_1, x_2, ..., x_k\}$ is often written an ordered k-tuple $(x_1, \ldots, x_k)$, but the indexing of the x_is is meant to capture the ordering so that $x_1 \in S_1$, $x_2 \in S_2$, $\ldots$, $x_k \in S_k$.

The following is the original form of Hall's theorem, which was proved by induction.

Theorem 5.3.5 (SDRs, Hall, 1935 [479])**.** *Let B be a set and let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a family of subsets of B. Then $\mathcal{S}$ has a system of distinct representatives if and only if*

$$\forall I \subseteq \{1, \ldots, k\}, \quad \left| \bigcup_{i \in I} S_i \right| \geq |I|. \tag{5.2}$$

First proof: Consider the bipartite graph $G = (\mathcal{S} \cup B, E)$, where, for any $x \in B$, the pair $\{S_i, x\} \in E$ if and only if $x \in S_i$, and apply Hall's matching theorem (Theorem 5.3.1). $\square$

Second proof: The proof given here is due to Rado [769] does not use Hall's theorem. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a family of sets satisfying Hall's condition (5.2). If each A_i contains only one element, then $\mathcal{A}$ is itself an SDR. So suppose that the A_is are not all singletons, and let x_1 and x_2 be distinct elements of A_1. Put $B_1 = A \backslash \{x_1\}$ and $B_2 = A_1 \backslash \{x_2\}$.

Claim: At least one of $\{B_1, A_2, \ldots, A_n\}$ or $\{B_2, A_2, \ldots A_n\}$ satisfies Hall's condition (5.2) for SDRs.

Proof of claim: Suppose condition (5.2) fails, and let $J, K \subseteq \{2, 3, \ldots, n\}$ be so that $|B_1 \cup \bigcup_{j \in J} A_j| \leq |J|$ and $|B_2 \cup \bigcup_{k \in K} A_k| \leq |K|$. Put $S = B_1 \cup \bigcup_{j \in J} A_j$ and $T = B_2 \cup \bigcup_{k \in K} A_k$. Then $S \cup T = A_1 \bigcup_{i \in J \cup K} A_i$. By condition (5.2), $|S \cup T| \geq |J \cup K| + 1$. Also,

$$\bigcup_{i \in J \cap K} A_i \subseteq \left(\bigcup_{j \in J} A_j \right) \cap \left(\bigcup_{k \in K} A_k \right) \subseteq S \cup T,$$

and so by condition (5.2), $|J \cap K| \leq |S \cap T|$. Thus

$$|J| + |K| \geq |S| + |T| = |S \cup T| + |S \cap T| \geq |J \cup T| + 1 + |J \cap K| = |J| + |K| + 1,$$

which is a contradiction, finishing the proof of the claim.

Removing one vertex at a time, arrive at a system of singleton sets; these singletons are representatives of the A_is. $\square$

By identifying sets with vertices in a bipartite graph as in the first proof of Hall's SDR theorem (Theorem 5.3.5) above, the reader might easily observe that the SDR statement also implies the matching version (Theorem 5.3.1). In this sense, the SDR version is equivalent to the matching version.

Note: Many theorems in graph theory, network flow theory, and matrix theory are now known to be equivalent to Hall's theorem; see Section 5.12 for a summary of such equivalent theorems examined here. Below (Theorem 5.3.10) is such an example about stochastic matrices.

In Example 5.3.4, to see that Hall's condition is met, there are 7 non-empty subsets $I \subseteq \{1, 2, 3\}$, and in each case, here are the unions:

I	union over I	$\lvert I\rvert$	$\lvert\cup_{i\in I} S_i\rvert$
$\{1\}$	$\{b,c,d\}$	1	3
$\{2\}$	$\{a,b,c,e\}$	1	4
$\{3\}$	$\{a,c\}$	1	2
$\{1,2\}$	$\{a,b,c,d,e\}$	2	5
$\{1,3\}$	$\{a,b,c,d\}$	2	4
$\{2,3\}$	$\{a,b,c,e\}$	2	4
$\{1,2,3\}$	$\{a,b,c,d,e\}$	3	5

Since in each case, the size in the last column is at least the third column, the conditions for Hall's theorem are satisfied. Note also that, in a sense, finding a SDR is often easy; when the last column numbers are closer to the third column, finding a SDR might be a little more challenging.

If, in some example of $S_1, S_2, S_3, ..., S_k$, there exists an I, say $I = \{5, 6, 7\}$ so that $S_5 \cup S_6 \cup S_7$ has *fewer* than $|I| = 3$ elements, then the system has no SDR. In general, however, to check all the unions can be time-consuming, but at least there is a simple method (listing all such) that one can check to guarantee a SDR (or not).

Recall that a graph is k-regular if every vertex has degree k. A simple application of Hall's theorem shows a seemingly strong result about regular bipartite graphs (also known to König many years before Hall's theorem—see Corollary 6.11.3):

Theorem 5.3.6. *Let $k > 0$ and let $G = (X \cup Y, E)$ be a k-regular bipartite graph. Then $|X| = |Y|$ and G has a perfect matching.*

Proof: Counting edges from X, $|E| = k|X|$, and counting from Y, $|E| = k|Y|$. Since $k > 0$, it follows that $|X| = |Y|$.

Let $S \subseteq X$. If one can show that $|N(S)| \geq |S|$, then by Hall's theorem, there is a matching from X into Y, but since $|X| = |Y|$, such a matching is perfect, and the proof would be done. But showing $|N(S)| \geq |S|$ is straightforward: The number of edges from S to $N(S)$ is at most the number of edges leaving $N(S)$, so $k|S| \leq k|N(S)|$, from which the desired inequality follows. □

Exercise 197. *Let $G = (X \cup Y, E)$ be a bipartite graph on disjoint sets X and Y with $|X| = |Y| = n$ and $\delta(G) \geq \frac{n}{2}$. Show that G has a perfect matching.*

The following lemmas about matchings in bipartite graphs (which were given as exercises in [125, 21, 22 p. 94]) are of independent interest, but are also used later for Theorem 6.11.2, a result that relates maximum degree and edge-colourings. The first of these uses counting like that used in the proof of Theorem 5.3.6.

Lemma 5.3.7. *Let $G = (X \cup Y, E)$ be a bipartite graph and let $A \subset V(G)$ be the set of all vertices with maximum degree (that is, $A = \{v \in X \cup Y : \deg(v) = \Delta(G)\}$). Then there exists a matching of $X \cap A$ into Y.*

Proof: Let $\Delta = \Delta(G)$. Let $S \subseteq X \cap A$. Then the number of edges from S to Y is $|S|\Delta$. On the other hand, the number of edges from $N(S)$ to X is at most $|N(S)|\Delta$ (since every vertex in Y has degree at most Δ), and since the first set of edges is contained in the second set,

$$|S|\Delta \leq |N(S)|\Delta.$$

Thus $|S| \leq |N(S)|$. Since this relationship holds for any $S \subseteq X \cap A$, by Hall's theorem (Theorem 5.3.1), the desired matching exists. □

Lemma 5.3.8. *Let $G = (X \cup Y, E)$ be a bipartite graph and let A be the set of vertices with maximum degree. Then there exists a matching whose edges use all vertices of A.*

Proof outline: By Lemma 5.3.7, let M_1 be a matching from $X \cap A$ into Y, and applying Lemma 5.3.7 let M_2 be a matching from $Y \cap A$ into X. The union of M_1 and M_2 creates a bipartite graph with maximum degree 2, and so is a disjoint union of paths and (even) cycles. By taking every second edge from each of these paths and cycles, construct the desired matching. □

Hall's theorem can be applied in many areas of mathematics. A *latin square* of order n is an $n \times n$ array of n distinct symbols, (usually $\{1, 2, \ldots, n\}$) so that each symbol appears exactly once in any row or any column. For $m < n$, an $m \times n$ latin rectangle is an array with all n symbols in each row and no symbol repeated in any column.

Theorem 5.3.9. *For $m < n$, any $m \times n$ latin rectangle can be completed to an $n \times n$ latin square.*

Proof: Let M be an $m \times n$ latin rectangle with $m < n$. By mathematical induction, it suffices to show that one can add another row to M producing an $(m+1) \times n$ latin rectangle. Put $X = \{1, 2, \ldots, n\}$, and for each $j \in \{1, 2, \ldots, n\}$, let $T_j \subset X$ be those elements of X not in column j of M, and put $\mathcal{T} = \{T_1, \ldots, T_n\}$. To find one more row to add to M, it suffices to show that there is a matching (in the obvious bipartite graph) from X into $\mathcal{T}$.

By Hall's theorem (Theorem 5.3.1) or its equivalent form for SDR's (Theorem 5.3.5), it suffices to show that the union of *any* k of the T_i's contains at least distinct k elements. Such a union contains, with repetition, $k(n-m)$ elements, and if such a union were to contain fewer than k distinct elements, then one element $x \in X$ is repeated more than $n - m$ times—meaning that x appears in less than m rows, contrary to M being a latin rectangle. Hence, M can be extended by adding one more row to an $(m+1) \times n$ latin rectangle. □

For the next theorem, a *permutation matrix* is a square 0-1 matrix with one 1 in each row and one 1 in each column. (Such matrices are called permutation matrices because multiplying such a matrix by a vector permutes coordinates of the vector.)

Either of the next two theorems is called "the Birkhoff–von Neumann theorem" (due to Garrett Birkhoff and independently by John von Neumann), however an early version appeared in the first book on graph theory by König [596] (which was published in the same year as Hall's theorem!) The first of these next two theorems is a special case of the second.

Theorem 5.3.10 (König, 1935 [596]; Birkhoff, 1946 [106]; von Neumann, 1953 [964])**.** *For any $k \in \mathbb{Z}^+$, a square matrix with non-negative integer entries can be expressed as a sum of k permutation matrices if and only if all row sums and all column sums are k.*

Exercise 198. *Use Hall's theorem to prove Theorem 5.3.10.*

For a more general version of Theorem 5.3.10, two definitions are needed.

Definition 5.3.11. *A matrix $A = (a_{i,j})$ is called doubly stochastic if each $a_{i,j} \geq 0$, and all entries in any row sum to 1 and all entries in any column sum to 1.*

So a permutation matrix is a doubly stochastic matrix. It follows from the definition that a doubly stochastic matrix is square since the sum of all entries counted according to rows or according to columns is the same.

Definition 5.3.12. *A* convex combination *of vectors $\mathbf{x_1}, \ldots, \mathbf{x_m}$ in a real vector space is an expression of the form*

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_m \mathbf{x}_m,$$

where each $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Theorem 5.3.13 (Birkhoff, 1946 [106]; von Neumann, 1953 [964])**.** *Any doubly stochastic matrix is a convex combination of permutation matrices.*

Theorem 5.3.13 follows directly by scaling the result in Theorem 5.3.10. For another proof (and references for more), see [535], and for more references to linear programming and the travelling salesperson problem, see, *e.g.*, [1014, p. 20–21].

The set of all doubly stochastic matrices is sometimes called the Birkhoff–von Neumann (convex) polytope, with vertices being the permutation matrices; in other words, the set of $n \times n$ doubly stochastic matrices is the "convex hull" of the $n \times n$ permutation matrices. The Birkhoff-von Neumann theorem can be viewed as a result for the vector space consisting of the set of all $n \times n$ real valued matrices (and so is of dimension n^2).

So far, three versions of Hall's theorem have been examined. A "deficient" version of Hall's theorem is possible as a simple corollary:

Corollary 5.3.14. *Let d be a positive integer and let $G = (X \cup Y, E)$ be a bipartite graph so that for every $S \subset X$,*

$$|N(S)| \geq |S| - d.$$

Then G contains $|X| - d$ independent edges.

Proof: From G create a new graph $\hat{G}$ by adding d new vertices to Y, and connect each new vertex to all of X. Then for any $S \subseteq X$, $|N_{\hat{G}}(S)| \geq |S|$, and so Hall's theorem gives a perfect matching of X in $\hat{G}$. All but d of these edges in the perfect matching occur in G. □

5.4 Stable marriages

As above, let X denote a set of men, and let Y denote a set of women. For now, consider only the case where men and women are equinumerous ($|X| = |Y|$). A perfect matching $M \subseteq X \times Y = \{(m, w) : m \in X, w \in Y\}$ is called a *marriage*, and an edge $(m, w) \in M$ is called a married couple, or a couple married by M. It is convenient to drop the parentheses and comma and denote a married couple by simply mw.

Suppose that each person ranks the members of the opposite sex, and in the case of a tie, breaks it arbitrarily; in other words, each man provides a linear order on the women, and each woman provides a linear order on the men. Note that in contrast to marriages in Hall's theorem, all members of the opposite sex are acceptable, just some are "more acceptable" than others.

How does one decide what marriage is best? What can "best" mean? Consider the following example (which extends an example from [582]), with men m_1, m_2, m_3, women w_1, w_2, w_3, and preference lists:

Man	first	second	third
m_1	w_2	w_1	w_3
m_2	w_1	w_3	w_2
m_3	w_1	w_2	w_3

Woman	first	second	third
w_1	m_1	m_3	m_2
w_2	m_3	m_1	m_2
w_3	m_2	m_1	m_3

If $M_0 = (m_1w_1, m_2w_2, m_3w_3)$ is a marriage, the last two couples might want to divorce and swap partners, because in doing so, each of the four people m_2, m_3, w_2, w_3 would get a mate higher on their list. In trying to find a "good" marriage, one might want to avoid having two couples, each man preferring the other's wife, and each woman preferring the other's husband. If there exist two couples where each of the two men and two women prefer the mate of the other, one might say that this pair of couples is "unstable". Some authors (see, *e.g.*, [909]) say that "a marriage is unstable" if and only if there exists an "unstable" *pair* of couples (and *stable* otherwise). In swapping among such an "unstable" pair of couples, nobody gets hurt. The now-standard definition of a stable marriage precludes even more:

Definition 5.4.1. *For a given marriage M, a man-woman pair $(m, w) \notin M$ is called* unstable *if and only if m prefers w to his mate in M and w prefers m to her mate in M. A marriage is called* unstable *if and only if there exists an unstable pair, and* stable *otherwise.*

For example, the marriage $\{m_1w_1, m_2w_3, m_3w_2\}$ is stable since each woman gets their first pick, and hence there are no unstable pairs. When two women have the same first choice and two men have the same first choice, another strategy to find a stable marriage is needed.

Can an unstable marriage be made stable by marrying unstable pairs (and marrying the remaining partners)? In each of the following marriages, an unstable pair is identified in boldface, and the subsequent marriage is created by marrying the marked unstable pair:

$$\begin{aligned}
M_1 =& \{m_1w_3, m_2\mathbf{w_1}, \mathbf{m_3}w_2\}\\
M_2 =& \{\mathbf{m_1}w_3, m_2w_2, m_3\mathbf{w_1}\}\\
M_3 =& \{\mathbf{m_1}w_1, m_2\mathbf{w_2}, m_3w_3\}\\
M_4 =& \{m_1\mathbf{w_2}, m_2w_1, \mathbf{m_3}w_3\}\\
M_5 =& \{m_1w_3, m_2\mathbf{w_1}, \mathbf{m_3}w_2\} = M_1,
\end{aligned}$$

arriving back at the original marriage M_1.

So it seems that marrying unstable pairs might not be a good strategy to find a stable marriage (of all men and women). Is there a strategy to find a stable marriage? The answer to this question is "yes", and the proof is in an algorithm that finds a stable marriage.

Theorem 5.4.2 (Stable marriage theorem, Gale–Shapley, 1962 [399])**.** *For any collection of n men and n women, each person with a ranking of all members of the opposite sex, there exists a stable marriage marrying all men and women.*

An algorithm that proves the stable marriage theorem is called the *Gale–Shapley algorithm*, due to David Gale and Lloyd Stowell Shapley in 1962 [399]. First the algorithm is described (often called a "deferred acceptance algorithm"), and the proof that it works is left as an exercise. It might be interesting to know that the Gale–Shapley paper was titled "College admissions and the stability of marriage". (I am told that this algorithm is still used for certain colleges to match applications to colleges, but I do not have first-hand evidence at my fingertips now.)

Gale–Shapley algorithm: Men propose to women in rounds.

Round 1: Each man proposes to the favourite woman on his list. If every woman receives a proposal, then stop, and have each woman accept that proposal; this produces a perfect marriage (and easily seen to be stable, because all men got their first choice). If some woman has not been proposed to, proceed to the next round. If any woman receives one or more proposals, she gives her favourite of these a "maybe" (like a tentative engagement?) and rejects all others.

Suppose that $j > 1$ and that the algorithm has not terminated in round $j - 1$.

ROUND j: Every man rejected in Round $j-1$ proposes to the next woman on his list. Again, any woman receiving more than one proposal keeps the highest proposer (among all rounds) as a "maybe" and rejects all others. Any man with "maybe" status can be rejected later if some "more-preferred" man proposes to her on any later round and she upgrades.

At the end of any round, if every woman has received at least one proposal, (in the course of all rounds) terminate the algorithm and marry each woman to her "maybe".

Exercise 199. *Prove that the Gale–Shapley algorithm terminates and produces a stable marriage.*

It might be interesting to note that the Gale–Shapley algorithm can be extended to the case where the number of men and women are different (see, *e.g.*, [672]), and to the case where some men or women have equal preferences for some other women or men. Another (solved) problem asks to find a stable marriage where each person gets a partner as high on his/her list as possible, a "minimum regret" problem.

Exercise 200. *Prove that the stable marriage provided by the Gale–Shapley algorithm is optimal for men, that is, in any other stable marriage any man could not get a woman preferable to the one he was paired up with. Hint: Use strong induction.*

See [451] for a (perhaps easy-to-read) chapter with examples regarding stable matchings and the Gale–Shapley algorithm. For more references and a much more theoretical examination of the problem, (also providing many higher level connections to other problems, abstractions, including probability, hashing, the coupon collector problem, the shortest path, and data structures), see the 74-page book by Knuth [582]. For more on this aspect, see also [673]. Another more recent reference that Knuth recommends is [358].

Remark: Lloyd Shapley and Alvin Roth won the 2012 Nobel Prize in Economics "for the theory of stable allocations and the practice of market design" [813].

5.5 Four parameters for matchings and covers

Recall (see Definition 1.7.2) that for a graph G, a subset of vertices $I \subset V(G)$ is called *independent* if and only if no two vertices in I are adjacent. An independent set of vertices is also called a *stable* set. Similarly, a collection of edges is called independent if and only if no two edges in the collection share a vertex. Independent sets of edges are also called "matchings" (an expression perhaps first used by Berge [91] in 1957).

Definition 5.5.1. *For a graph G, define the following four parameters:*

- $\alpha(G)$ *is the* independence number *(also called the* stability number*), the order of a largest independent set of vertices;*
- $\nu(G)$ *is the* matching number *of G, the number of edges in a maximum matching;*
- $\rho(G)$ *is the* edge covering number*, the minimum number of edges required to cover $V(G)$, that is, the edges are incident with all vertices;*
- $\tau(G)$ *is the* vertex covering number*, the smallest number of vertices incident with all edges.*

Notes on notaton: In many texts, an "edge cover" is a set of vertices covering the edges—here an edge cover is a set of edges. Also, the notations $\alpha_0, \alpha_1, \beta_0, \beta_1$ are commonly used for independence numbers for vertices and edges, and covering numbers for vertices and edges respectively. The notation used here follows that of Lovász and Plummer [643]. A *vertex cover* is a set of vertices so that every edge is incident with at least one vertex in the cover. (It was, perhaps, J. P. Roth [811] who, in 1958, first used "cover" in this context.) In set theory, a *transversal* in a set system (or hypergraph) $(X, \mathcal{F})$ is a collection $T \subseteq X$ so that for each $F \in \mathcal{F}$, $F \cap T \neq \emptyset$, and since a graph is a simple set system, some authors refer to a vertex cover as a transversal and τ as the transversal number. Perhaps this is why the notation τ is used for the cardinality of a smallest vertex cover.

When dealing with perfect matchings, the following terminology is often used. For a graph G and positive integer k, a *k-factor* of G is a k-regular spanning subgraph of G. Thus, a perfect matching is a 1-factor. (A 2-factor is then a collection of disjoint cycles.)

Exercise 201. *Give an example of a cubic graph with no 1-factor.*

Exercise 202. *Let $k \geq 1$ and let G be a connected $2k$-regular graph with an even number of edges. Show that G contains a k-factor.*

Exercise 203. *For each $n \geq 1$, find $\alpha(K_n)$, $\rho(K_n)$, $\tau(K_n)$, and $\rho(K_n)$. For $n \geq 3$, do the same for C_n.*

Exercise 204. *If P is the Petersen graph, find $\alpha(P)$.*

Remark: Many questions regarding colourings and independence number are answered by probabilistic means, and such techniques are also very useful for many areas of graph theory, game theory, computing, data science, and even number theory, and so in most areas of science today.

Exercise 205. *Let G be a graph and let H be an induced subgraph of G. Show that $\alpha(G) \geq \alpha(H)$. Give an example where this result is false when the word "induced" is dropped.*

The next theorem gives a bound on the independence number α in terms of a graph's degree sequence:

Theorem 5.5.2. *For a simple graph $G = (V, E)$,*

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v) + 1}.$$

Exercise 206. *Prove Theorem 5.5.2. Hint: try induction on $|V|$ (although other proofs are available).*

Remark: Theorem 5.5.2 was used [181] (see also [36]) to give yet another proof of a major result in extremal graph theory called Turán's theorem (see Theorem 10.5.2); see also Theorem 10.5.6. For a result on the size of α for planar graphs, see Theorem 7.1.8. For a result regarding the matching number for planar graphs, see Theorem 7.1.9.

Lemma 5.5.3. *For any graph G, $\alpha(G) + \tau(G) = |V(G)|$.*

Proof: Fix G and put $\alpha = \alpha(G)$ and $\tau = \tau(G)$. The proof is by showing both inequalities $\alpha + \tau \leq |V(G)|$ and $\alpha + \tau \geq |V(G)|$. Let $I \subseteq V(G)$ be an independent set with $|I| = \alpha$. The remaining vertices cover all edges (because I contains no edges) so $\tau \leq |V(G)| - \alpha$, giving $\alpha + \tau \leq |V(G)|$.

Let $S \subseteq V(G)$ be a vertex cover (of $E(G)$) with $|S| = \tau$. The remaining set $I = V(G) \backslash S$ is independent (because if I contained an edge, S didn't cover it). So $|V(G)| - \tau \leq \alpha$, giving $\alpha + \tau \geq |V(G)|$. □

Exercise 207. *Let G be a triangle-free graph. Show that $|E(G)| \leq \alpha(G)\tau(G) \leq n^2/4$.*

Tibor Gallai (1912–1992), König's only doctoral student, proved the following fundamental result:

Theorem 5.5.4 (Gallai, 1959 [401])**.** *Let G be a graph with no isolated vertices. Then $\nu(G) + \rho(G) = |V(G)|$.*

Proof: The proof is divided into two parts, showing the two inequalities that give equality.

First show $\nu(G) + \rho(G) \geq |V(G)|$: Let $C \subset E(G)$ be a minimum edge cover with $|C| = \rho$. If C were to contain a path of length three (3 edges), then the middle edge would be redundant, contrary to C being minimal, so C does not contain a path of

length three. Let $G[C] = (V(C), C)$ be the graph restricted to the vertices and edges of C. Since $G[C]$ contains no paths of length 3, it is a union of vertex disjoint stars. Let s be the number of components in $G[C]$.

Since, for each star S, $|V(S)| = |E(S)| + 1$, and since $V(G) = \cup_S V(S)$, $|V(G)| = (\sum_S |E(S)|) + s$. The sum is simply $|C| = \rho$, hence, $|V(G)| = \rho + s$.

Taking an edge from each component shows that $\nu(G) \geq s = |V(G)| - \rho$, and so $\nu(G) + \rho(G) \geq |V(G)|$.

To prove the reverse inequality, let M be a maximum matching of size $\nu(G)$. Let U be the vertices in G not covered by M. Then U is an independent set (since if there were an edge, it could be added to M to produce a larger matching). Also,

$$|U| + 2\nu(G) = |V(G)|.$$

Let S be an edge cover of the vertices in U. Then $M \cup S$ is an edge cover for $V(G)$, so

$$\begin{aligned}
\rho(G) &\leq |M \cup S| \\
&\leq |M| + |S| \\
&= |M| + |U| \\
&= \nu(G) + |V(G)| - 2\nu(G) \\
&= |V(G)| - \nu(G),
\end{aligned}$$

and so the desired inequality $\nu(G) + \rho(G) \leq |V(G)|$ follows, completing the proof. □

Remark 5.5.5. *Lemma 5.5.3 and Theorem 5.5.4 are sometimes collectively known as "Gallai's theorem".*

Lemma 5.5.6. *For any graph G,*

$$\nu(G) \leq \tau(G). \tag{5.3}$$

Proof: Let M be a largest matching in G. Any vertex cover of G uses at least one vertex per edge of M. □

For the next exercise, recall the difference between *minimal* and *minimum*; the first indicates that the example can not be reduced, and the second indicates smallest overall.

Exercise 208. *Let G be a graph. Show that a minimal edge-cover $L \subset E(G)$ is minimum if and only if L contains a maximum matching. Then show that a maximal matching $M \subset E(G)$ is maximum if and only if M is contained in a minimum edge-cover.*

Exercise 209. *Prove that any tree T has a perfect matching (1-factor) if and only if for any $x \in V(T)$, the graph $T \backslash x$ has exactly one odd component.*

Exercise 210. *Show that if G is a graph with $2n$ vertices has exactly one perfect matching, then $|E(G)| \leq n^2$ edges. Furthermore, for each $n \geq 1$, construct a graph on $2n$ vertices with exactly n^2 edges that has exactly one perfect matching.*

Exercise 211. *Let M_1 and M_2 be perfect matchings in a graph G. What types of graphs are the components (maximal connected subgraphs) of the graph given by $M_1 \cup M_2$?*

5.6 The König–Egerváry theorem

For bipartite graphs, the inequality (5.3) is strengthened to equality. This was shown independently by Dénes König (1884–1944) and, in the more general case of weighted graphs, by Jenő Egerváry (1891–1958). Both results were published in 1931 in the same volume of the same journal (with Egerváry's appearing first). In 1955, Kuhn translated Egerváry's paper [611] (and in another paper of that year, gave an algorithm to implement Theorem 5.6.1—see Section 5.7). The following form is sometimes called simply "König's theorem", but here it is called the "König–Egerváry theorem" (do differentiate from König's other theorems (*e.g.*, Theorem 6.11.2, a line-colouring theorem, or Lemma 6.14.2 on paths in infinite trees). A minimum set of vertices in a graph that is incident with all edges of a graph is often called a *vertex cover*.

Theorem 5.6.1 (König, 1931 [595], Egeváry, 1931 [279])**.** *Let $G = (X \cup Y, E)$ be a bipartite graph. Then $\nu(G) = \tau(G)$, that is, the size of a maximum matching is equal to the minimum number of vertices incident with all edges.*

First proof of Theorem 5.6.1—using Hall's theorem: By equation (5.3), $\tau(G) \geq \nu(G)$, so it remains to show that $\tau(G) \leq \nu(G)$.

Fix a minimum vertex cover $C \subseteq V(G)$ (so $|C| = \tau(G)$). To show that $\tau(G) \leq \nu(G)$, it suffices to construct a matching M with $|M| = |C|$. This is done by applying Hall's theorem applied to certain subgraphs.

Let $R = C \cap X$ and $T = C \cap Y$. Let H_1 be the graph induced by $R \cup (Y \backslash T)$ and let H_2 be the graph induced by $(X \backslash R) \cup T$. Since $C = R \cup T$, it follows that G has no edge between $X \backslash R$ and $Y \backslash T$.

To find an R-saturated matching in H_1, by Hall's theorem, it suffices to show that for any $S \subseteq R$, $|N_{H_1}(S)| \geq |S|$. If $|N_H(S)| < |S|$, then since $N_{H_1}(S)$ covers all edges incident with S that are not covered by T, one could substitute $N_{H_1}(S)$ for S in the vertex cover C to obtain a smaller cover. Thus $|N_{H_1}(S)| \geq |S|$, and so by Hall's theorem, an R-saturated matching M_1 in H_1 exists. Similarly, there is a T-saturated matching M_2 in H_2. These matchings are disjoint because H_1 and H_2 are disjoint. Since $|M_1| = |R|$ and $|M_2| = |T|$, $|M_1 \cup M_2| = |R| + |T| = |C|$, so the matching $M = M_1 \cup M_2$ is as desired. $\square$

Second proof of Theorem 5.6.1—using Menger's theorem: Let s, t be vertices not in $V(G)$, and construct a new graph G^* on $X \cup Y \cup \{s, t\}$ by letting s be adjacent to each vertex of X, and t be adjacent to each vertex of Y (and all edges of G remain). By Menger's theorem (Theorem 4.1.6), the maximum number of internally disjoint s–t paths is equal to the minimum number of vertices in an s–t separating set.

For each matching M in G, there is a collection of internally disjoint s–t pathsin G^*, so the maximum number of internally disjoint s–t paths is $\nu(G)$. On the other hand, if $C \subset (X \cup Y)$ is a minimum set of vertices that separate s and t, then C is a vertex cover for G; thus $|C| = \tau(G)$. □

Exercise 212. *Give an example of a graph that is not bipartite for which $\tau \neq \nu$, showing that "bipartite" is necessary in the statement of the König–Egervary theorem (Theorem 5.6.1).*

Exercise 213. *Use the König–Egerváry theorem to show that if $G = (X \cup Y, E)$ is bipartite with $|X| = |Y| = n$ and $|E(G)| > kn$, then G contains at least $k + 1$ independent edges.*

Exercise 214. *Show that the König–Egerváry theorem implies Hall's theorem.*

The König–Egerváry theorem (Theorem 5.6.1) relates ν and τ; a dual form relates α and ρ.

Theorem 5.6.2 (König–Rado, 1942 [768])**.** *Let $G = (X \cup Y, E)$ be a bipartite graph with no isolated vertices. Then $\alpha(G) = \rho(G)$; in other words, the maximum number of vertices in an independent set is equal to the minimum number of edges that cover all vertices.*

Exercise 215. *Use Gallai's theorem (Theorem 5.5.4) to prove Theorem 5.6.2.*

5.7 Hungarian method

In 1955, Harold Kuhn [612] showed how to apply the weighted version of Theorem 5.6.1 with an algorithm, now called "the Hungarian method." For more history and an application to communication satellites, see [863]. [Note: There is another famous Kuhn, namely Thomas Samuel Kuhn, who wrote *The structure of scientific revolutions*, first published in 1962.]

Basically, the Hungarian method uses Berge's technique (see Theorem 5.8.1), with a tree and a complicated labelling technique. Since the details are a little difficult, the reader is invited to see an example in [429].

5.8 Berge and matchings

Claude Berge was one of the pioneers in matching theory and graph theory in general. Here is only an introduction of Berge's most famous theorem regarding matchings. (More on Berge's work can be found in many popular texts.)

If $G = (V, E)$ is a graph, and $M \subseteq E$ is a matching, say that an edge $e \in E$ is *weak with respect to* M if and only if $e \notin M$. A vertex is said to be *weak* (or "unsaturated") with respect to M if and only if it is not incident with any edge in M.

An M-*alternating path* in G is a path where only alternating edges are in M. If an M-alternating path P has both end vertices not incident with any edge of M, then P is called an *augmenting path with respect to* M, or simply, an M-*augmenting path*. If P is an augmenting path, then M can be made larger by taking the alternate edges of P.

Theorem 5.8.1 (Berge, 1957 [91])**.** *Let G be a graph and let $M \subseteq E(G)$ be a matching. Then M is maximum if and only if there is no M-augmenting path with respect to M.*

Proof: (The proof here is standard, and can be found in, *e.g.*, [585, p. 171].)

Above it was noted that if a matching M has an M-augmenting path, M is not maximum.

For the reverse implication, suppose that G has no M-augmenting paths. Suppose, in hope of contradiction, that M is not maximum. Let M' be a maximum matching (with $|M| < |M'|$). Consider the shape of the graph H formed by $M \cup M'$. (For a similar problem with perfect matchings, see Exercise 211.) In H, each vertex is incident to at least one edge and is incident to at most one M-edge and at most one M'-edge (i.e.,for every $v \in V(H)$, $1 \leq \deg_H(v) \leq 2$). So (connected) components of H are either paths or cycles (see Exercise 46). Edges in any path or cycle in H alternate between M-edges and M'-edges. So any cycles are even with same number of each type, and since $|M'| > |M|$, there exists a path with more M'-edges than M-edges. This means that there is an alternating path with M'-edges on each end, and so is M-augmenting, providing the desired contradiction. □

Note that Berge's theorem can be used to give a fairly simple proof of Hall's theorem; see, *e.g.*, [585, p. 171] for details.

5.9 Tutte's 1-factor theorem

Recall that a spanning subgraph of a graph G is a subgraph H with $V(H) = V(G)$, that is, one that uses all of the vertices of G.

Definition 5.9.1. *For any graph G, a* factor *of G is any spanning subgraph of G For any positive integer k, a k-factor is a k-regular spanning subgraph.*

So a 1-factor is a perfect matching (and so can occur only in even order graphs). A 2-factor is either a Hamiltonian cycle, or a disjoint union of cycles (that together use all vertices).

Exercise 216. *Show that any tree can have at most one 1-factor.*

An *odd component* of a graph is a (connected) component with an odd number of vertices. For any graph H, let $o(H)$ denote the number of odd components in H. Also, if S is a subset of vertices of a graph G, let $G-S$ denote the graph obtained by deleting S (elsewhere, this may be denoted by $G\backslash S$).

William Thomas Tutte (in 1947) gave a necessary and sufficient condition for a graph to contain a 1-factor. In 1954, Tutte [938] published a short inductive proof (using Hall's theorem), which is given here. [I am not aware of the original proof!]

Theorem 5.9.2 (Tutte's 1-factor theorem, 1947 [937])**.** *A non-trivial graph G has a 1-factor if and only if for any proper subset $S \subset V(G)$, $o(G-S) \leq |S|$.*

Proof: Say that a graph G satisfies Tutte's condition if for any proper subset $S \subset V(G)$, $o(G-S) \leq |S|$.

($\Rightarrow$) Assume that G has a 1-factor $M \subseteq E(G)$. Let $S \subseteq V(G)$. If $o(G-S) = 0$, then trivially, $o(G-S) \leq |S|$. So suppose that $o(G-S) = k \geq 1$ with odd components $G_1, \ldots, G_k$. Since any 1-factor of any G_i uses an even number of vertices, for each $i = 1, \ldots, k$, one edge of M joins S to G_i; these k edges are vertex disjoint, so $|S| \geq k$.

($\Leftarrow$) Assume that G satisfies Tutte's condition. In particular, using $S = \emptyset$, $o(G-S) = o(G) \leq 0$ and so G has all components even. Without loss of generality, one can assume that G is connected—for if not, apply the following proof to each component.

For each (even) integer $n \geq 2$, let $S(n)$ be the statement that if a graph on n vertices satisfies Tutte's condition, then the graph has a 1-factor.

Base step: If $n = 2$, K_2 is the only graph with all components even (and is the only connected graph on two vertices), and K_2 trivially has a 1-factor, so $S(2)$ is true.

Inductive step: Fix an even $\ell \geq 4$ and suppose that each of $S(2)$, $S(4)$, ..., $S(\ell-2)$ holds. Let G be a graph on ℓ vertices that satisfies Tutte's condition. It remains to show that G has a 1-factor.

Pick a vertex $z \in V(G)$ that is not a cut-vertex (which exists by Exercise 177). Then $o(G - \{z\}) = 1$. Let $S \subset V(G)$ be maximal so that $o(G-S) = |S|$, say with $|S| = k$ (at least one such set exists, so a maximum is achieved).

Claim: $G - S$ has no even components.

Proof of claim (by contradiction): Suppose that $G-S$ has an even component H. Again by Exercise 177, let $v \in V(H)$ be a vertex that is not a cut-vertex of H, and put $S' = S \cup \{v\}$. Since H is even, deletion of a vertex makes it odd, and so

$$o(G-S') \geq o(G-S) + 1 = k + 1.$$

On the other hand, by Tutte's condition,

$$o(G - S') \leq |S'| = k + 1,$$

and so $o(G - S') = k + 1$, contradicting the maximality of S, proving the claim.

Let $G_1, \ldots, G_k$ be the (odd) components of $G - S$, and for each $i = 1, \ldots, k$, let $S_i \subseteq S$ be the set of those vertices in S with a neighbour in G_i.
Claim: For every $\ell = 1, \ldots, k$, the union of ℓ different S_i's contains at least ℓ vertices.
Proof of claim (by contradiction): Suppose, in hopes of a contradiction that, some union of ℓ many S_is has fewer than ℓ vertices. Without loss, suppose that $|\cup_{i=1}^{\ell} S_i| < \ell$. Since $G_1, \ldots, G_\ell$ are odd components of $G - (S_1 \cup \cdots \cup S_\ell)$,

$$o(G - (S_1 \cup \cdots \cup S_\ell)) \geq \ell > |S_1 \cup \ldots \cup S_\ell|,$$

violating Tutte's condition, thereby proving the claim.

By Hall's theorem (Theorem 5.3.1), there exist distinct $u_1, \ldots, u_k \in S$ so that each $u_i \in S_i$; for each i, let $v_i \in V(G_i)$ be a neigbour of u_i.

Claim: Each $G_i - v_i$ has a 1-factor.

Proof of Claim: To see the claim, it suffices to show that each $G_i - v_i$ satisfies Tutte's condition (and then use the induction hypothesis). If $G_i - v_i$ fails Tutte's condition, then there exists a proper subset $W \subset V(G_i - v_i)$, with $o(G_i - v_i - W) > |W|$, and since both $(G_i - v_i - W)$ and W are both even or odd, then $o(G_i - v_i - W) \geq |W| + 2$. Putting $X = S \cup W \cup \{v_i\}$,

$$\begin{aligned} |X| &= |S| + |W| + 1 \\ &= |S| + |W| + 2 - 1 \\ &\leq o(G - S) + o(G_i - v_i - W) - 1 \\ &= o(G - X) \\ &\leq |X|, \end{aligned}$$

and so $o(G - X) = |X|$; but $|X| > |S|$, contradicting the maximality of S. This completes the proof of the claim.

The 1-factors in each of the G_i together with all edges $\{u_i, v_i\}$ form 1-factor of G. So $S(\ell)$ is true, completing the inductive step.

By (strong) mathematical induction, for all even $n \geq 2$, $S(n)$ is true. □

See [50] for another (but similar) proof of Tutte's theorem. Also see Chapter 14 of Hornsberger's *Mathematical Gems II* [529, pp. 147–157] for the article "Lovász's proof of a theorem of Tutte".

Using Tutte's theorem, one can derive an older theorem by Petersen. Recall that a *bridge* is an edge in a connected graph whose removal disconnects the graph. (Recall also that a 3-regular graph is also called "cubic".)

Theorem 5.9.3 (Petersen, 1891 [746]). *Every bridgeless 3-regular graph has a 1-factor.*

Proof: The proof given here relies on Tutte's 1-factor theorem (Theorem 5.9.2) and can be found in, *e.g.*, Bondy and Murty [143, p. 431].

Let G be a cubic bridgeless graph. Without loss of generality, assume that G is connected (for if not, apply the proof below to each component). Let $S \subsetneq V(G)$, and let $G_1, \ldots, G_k$ be the odd components of $G \backslash S$; for each $i = 1, \ldots, k$, let m_i be the number of edges from S to G_i, and put $n_i = |V(G_i)|$.

For each $i = 1, \ldots, k$, $\sum_{v \in V(G_i)} \deg_G(v) = 3n_i$, and $\sum_{v \in S} \deg(v) = 3|S|$. Also for each i,

$$m_i = \sum_{v \in V(G_i)} \deg(v) - 2|E(G_i)| = 3n_i - 2|E(G_i)|,$$

which is an odd number. Since G has no bridges, each $m_i > 1$, and so each $m_i \geq 3$. Thus $\sum_{i=1}^{k} m_i \geq 3k$. Then

$$\begin{aligned} o(G \backslash S) =& k \\ \leq& \frac{1}{3} \sum_{i=1}^{k} m_i \\ \leq& \frac{1}{3} \sum_{v \in S} \deg(v) \\ =& |S| \qquad\qquad \text{(since } \deg(v) = 3\text{)}, \end{aligned}$$

and so by Tutte's theorem, G has a 1-factor. □

Note that Theorem 5.9.3 implies that any cubic bridgeless graph can be decomposed into a 1-factor and a 2-factor. Petersen's result was improved somewhat.

Theorem 5.9.4 (C. Chen, 1990 [197]). *indexChen, Ciping Every 3-regular graph with at most two cut-edges has a 1-factor.*

Incidentally, Petersen also proved another famous theorem for 2-factors;

Theorem 5.9.5 (Petersen, 1891 [746]). *For $k \geq 1$, every $2k$-regular graph has a 2-factor.*

Proof: Let $k \geq 1$ and let G be a $2k$-regular graph. Without loss of generality, let G be connected (since otherwise, apply the proof below to each component). Then G is Eulerian, with some circuit C. Let $V(G) = \{v_1, \ldots, v_n\}$, and let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be two disjoint copies of $V(G)$. Define a bipartite graph $B = (X \cup Y, E^*)$ by $\{x_i, y_j\} \in E^*$ if and only if v_j immediately follows v_i in C (fix one direction for C).

Since G is $2k$-regular, the circuit C passes through each vertex of G k times, so B is k-regular. By Theorem 5.3.6, B contains a 1-factor. This 1-factor then corresponds to a 2-factor in G. □

A slight generalization of Theorem 5.9.5 is postponed to Section 8.4 (which is about decompositions into 2-factors).

Exercise 217. *Apply the proof of Theorem 5.9.5 to the 4-regular graph G on six vertices a, b, c, d, e, f with edges*

$$ab, ad, ae, af, bc, be, bf, cd, ce, cf, de, df, da, ea, fa$$

to find a 2-factor in G.

Exercise 218. *For each odd $k > 1$, find a k-regular graph with no 1-factor.*

Exercise 219. *Show that the removal of any perfect matching in the Petersen graph leaves two disjoint 5-cycles.*

For a result including regular graphs of odd degree, a variant of Theorem 5.9.5 for certain graphs was shown by Bäbler:

Theorem 5.9.6 (Bäbler, 1937 [63])**.** *For every positive integer k, Every 2-edge-connected $(2k+1)$-regular graph (or multigraph) contains a 2-factor.*

In 1985, Bollobás, Saito and Wormald [132] classified the types of k-factors in an r-regular ($r \geq 3$) graph with edge-connectivity λ (which includes Theorem 5.9.6 for the case $r \geq 3$, r odd, and $\lambda = 2$). Their proof is based on Tutte's 1-factor theorem (Theorem 5.9.2).

The following result strengthens Petersen's theorem.

Theorem 5.9.7 (Schönberger, 1934 [837])**.** *Let G be a bridgeless cubic graph. Any edge in G is in a perfect matching.*

Theorem 5.9.7 was generalized:

Theorem 5.9.8 (Tutte, 1947 [937])**.** *If G is a k-regular graph on an even number of vertices with edge-connectivity at least $k-1$, then every edge is in a 1-factor.*

For a proof of Theorem 5.9.8, see [94, pp. 160–162]. For more on theorems related to Petersen's factorization results and their impact, see [962]. For a comprehensive survey on graph factors and factorizations (focussing on the years 1985–2003, but with extensive coverage of historic results), see [754]. There are hundreds of other references for factors; one is the PhD thesis of Chen [198].

Exercise 220. *Show that if a graph G has a vertex incident with two pendant edges, then G does not have a 1-factor.*

Exercise 221. *Show that for $s \geq 2$, every $(s-1)$-regular connected simple graph on $2s$ vertices has a 1-factor.*

Exercise 222. *Let $n \geq 6$ be an even integer and let $e_1, \ldots, e_{n-1}$ be edges in K_n. Show that $K_n - \{e_1, \ldots, e_{n-1}\}$ has a 1-factor unless these $n-1$ edges share a common vertex.*

Theorem 5.9.9 (Sumner, 1974 [894, Cor. 2])**.** *Let G have an even number of vertices. If G is claw-free (*i.e.*, has no induced $K_{1,3}$) and connected, then G has a 1-factor.*

Since a cubic graph whose every vertex is in a triangle is claw-free, the following is an immediate consequence of Theorem 5.9.9

Corollary 5.9.10 (Sumner, 1974, [894])**.** *Let G be a cubic (3-regular) graph so that every vertex is contained in a triangle. Then G has a 1-factor.*

Sumner [894, p. 10] also gave a small example of such a cubic graph and one of its 1-factors (see Figure 5.1).

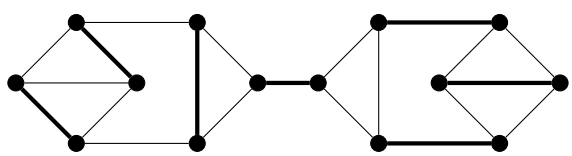


Figure 5.1: A cubic graph whose every vertex is in a triangle and a 1-factor indicated by thick edges

For complete decompositions of a graph into 1-factors, see Section 8.2.

5.10 Dilworth's theorem

A partially ordered set (poset) is an ordered pair $(P, \leq)$, where P is a non-empty set and $\leq$ is a binary relation $\leq$ om P that is reflexive, antisymmetric, and transitive. When clear, a poset $(P, \leq)$ is denoted by simply P.

Remark 5.10.1. *In many applications, a poset can be viewed as a "strict partial order", where a relation $<$ is irreflexive, antisymmetric and transitive; the two notions are equivalent.*

Two elements x, y in a poset are said to be *comparable* if and only if $x \leq y$ or $y \leq x$, and *incomparable* otherwise. A *chain* in a poset is a collection of elements any two of which are comparable, and an *antichain* is a collection of elements no two of which are comparable.

Perhaps the most famous result on antichains is now eponymous with Robert P. Dilworth [260] —even though the result was, according to Tverberg [945], discovered earlier by Tibor Gallai in 1936.

Theorem 5.10.2 (Dilworth, 1950 [260]). *Let $(P, \leq)$ be a finite poset. The minimum number of disjoint chains necessary to cover P is equal to the maximum number of elements in an antichain.*

First proof: This proof is due to Tverberg [945]. For any poset $(P, \leq)$, let $c(P)$ denote the minimum number of disjoint chains required to cover P, and let $\alpha(P)$ be the number of elements in a largest antichain. Since any chain contains at most one element from any antichain, $\alpha(P) \leq c(P)$. It remains to prove the reverse inequality, and this is achieved by strong induction on $|P|$. For each $n \geq 1$, let $S(n)$ denote the statement "for any poset $(P, \leq)$ with $|P| = n$, P can be covered by at most $\alpha(P)$ disjoint chains".
Base step: If $|P| = 1$, both $\alpha(P) = c(P) = 1$, so $S(1)$ holds.

Inductive step: Let $k > 1$ and assume that $S(1), \ldots, S(k-1)$ are all true. Let $(P, \leq)$ be a poset with $|P| = k$ elements and let $\alpha = \alpha(P)$ be the cardinality of a largest antichain in P. It remains to show that P can be covered by (at most) α chains.

Let C be any maximal chain in P.

Case 1: If every antichain in $P \setminus C$ contains less than α elements, then by $S(|P \setminus C|)$, $P \setminus C$ can be covered by at most $\alpha - 1$ disjoint chains, and so together with C, P can be covered by at most α disjoint chains, and $S(n)$ is satisfied.

Case 2: Suppose there exists an antichain $A = \{a_1, a_2, \ldots, a_\alpha\}$ in $P \setminus C$ with α elements. Larger antichains do not exist by the definition of α, and so every element in P (and hence in $P \setminus C$) is comparable to some element of A. Define

$$U = \{p \in P : \exists a_i \in A \text{ with } \ a_i \leq p\}$$

and

$$L = \{p \in P : \exists a_i \in A \text{ with } \ p \leq a_i\}.$$

Then $U \cup L = P$. The maximum element of C is not in L, for if it were, C could be extended by adding some a_i; but C is maximal. Similarly, the minimum element of C is not in U. Thus $U \neq P$ and $L \neq P$. By induction hypotheses $S(|U|)$ and $S(|L|)$, each of U and L can be covered by at most α disjoint chains, say C_i^U's and C_j^L's. Since A is an antichain with α elements, each $a_i \in A$ is contained in precisely one chain in U and one in L; relabelling the C_i^U's and C_j^L's if necessary, suppose that for each i, $a_i \in C_i^U$ and $a_i \in C_i^L$.

Observe that for each i, both $a_i = \min C_i^U$ and $a_i = \max C_i^L$, for if, say, $a_i \neq \min C_i^U$, that is, for some $x \in C_i^U$, $x < a_i$, then for some a_j, $a_j \leq x < a_i$ contradicts A being an antichain. Thus for each i, form the chain $C_i = C_i^U \cup C_i^L$; the chains $C_1, \ldots, C_\alpha$ are disjoint and cover P, showing that $S(|P|) = S(k)$ holds, completing the inductive step.

By strong mathematical induction on $|P|$, for each finite poset P, Dilworth's theorem holds. $\square$

Second proof of Dilworth's theorem: This proof is due to Fred Galvin [405] and is also by induction.

For each $n \geq 1$, let $S(n)$ be the statement that for any partial order on n vertices, the minimum number of disjoint chains in a cover is the same as the maximum number of elements in an antichain. The statement $S(1)$ is trivially true.

Fix $k \geq 1$ and suppose that $S(1), \ldots, S(k)$ are true. Let $(P, \leq)$ be a finite poset on $|P| = k+1$ vertices, and let $m \in P$ be maximal (that is, there does not exist $x \in P$ with $m < x$). Delete m and consider the subposet on $P^* = P \backslash \{m\}$. Let $C_1, C_2, \ldots, C_w$ be a minimum set of disjoint chains that cover $(P^*, \leq)$. By the induction hypothesis $S(k)$, the largest antichain in $(P^*, \leq)$ has w elements, one from each C_i. For each $i = 1, \ldots, w$, let a_i be the maximal element in C_i that appears in some w-element antichain.

Claim: The set $A = \{a_1, \ldots, a_w\}$ is an antichain.

Proof of claim: In hope of a contradiction, suppose that A is not an antichain; in other words, suppose that there are a_j and a_k in A so that $a_j < a_k$. Let $X = \{x_1, \ldots, x_w\}$ be an antichain with, for each i, $x_i \in C_i$, and $x_k = a_k$. Since a_j is maximal, $x_j \leq a_j$. By transitivity, $x_j \leq a_j < a_k$ shows $x_j < a_k = x_k$, contradicting X being an antichain, concluding the proof of the claim.

If $A \cup \{m\}$ is an antichain (with $w+1$ elements), then P can be covered with $w+1$ chains, (the original C_is and a trivial chain $\{m\}$). If $A \cup \{m\}$ is not an antichain, then for some i, $a_i < m$, in which case $K = \{m\} \cup \{x \in C_i : x \leq a_i\}$ is a chain, and $P \backslash K$ has no w-element antichain, so by induction hypothesis, can be covered by at most $w-1$ chains, again, giving a covering of P with at most w chains. In any case, $S(k+1)$ is true, finishing the inductive step, and hence the proof. □

For other proofs of Dilworth's theorem and related information, see [686], [743], or [782]. Dilworth's theorem can be considered as the beginning of what is now called "dimension theory" for posets, and has far-reaching consequences in many aspects of combinatorics and set theory. (See [118] for more information.)

Dilworth's theorem is also valid for infinite sets; however, a proof is beyond what is intended for this chapter (one such proof uses a compactness argument—see [686, pp. 61–62] for a proof attributed to Rado).

Theorem 5.10.3 (Dilworth's theorem, infinite)**.** *Let P be poset (not necessarily finite) whose largest antichain contains $\alpha < \infty$ elements. Then P can be covered by a union of α chains.*

The following theorem is a "dual" to Dilworth's theorem.

Theorem 5.10.4 (Mirsky, 1971 [685])**.** *If a longest chain in a poset $(P, <)$ has length m, then P can be covered by at most m antichains.*

Proof: For each $i = 1, 2, \ldots, m$, define the set L_i to be the set of all $x \in P$ such that x is the maximal element of a chain of length i but is not the maximum of any longer chain.

CLAIM: Each L_i is an antichain.
PROOF OF CLAIM: If L_i is not an antichain, there exists two elements, say $x, y \in L_i$ with $x < y$. In this case, a chain of length i ending at x can be extended by adding y, giving a chain of length $i + 1$, contrary to the definition of L_i, proving the claim.

By definition, every element of P is in some L_i, and so $\{L_1, \ldots, L_m\}$ is a set of m antichains as required. □

Exercise 223. *Prove Mirsky's theorem (Theorem 5.10.4) by induction on m. For $m > 1$, let M be the set of all maximal elements in S. Then $S \setminus M$ has no chains of length greater than $m - 1$, and M is an antichain.*

See Exercises 308 and 309 for relationships between Mirsky's and Dilworth's theorems with the facts that comparability graphs and their complements are perfect, giving another proof of Dilworth's theorem—using Mirsky's theorem.

Corollary 5.10.5. *Let m and n be positive integers and let P be a poset with $mn + 1$ elements. Then either P contains a chain of at least $m + 1$ elements or P contains an antichain with at least $n + 1$ elements.*

Proof: Suppose that every antichain has at most n elements. Then by Dilworth's theorem, there are at most n disjoint chains that cover P. If every chain has at most m elements, then P has at most mn elements. □

The next result is originally by Erdős and Szekeres, which has a proof using the pigeonhole principle; however, this same result has a short proof by Dilworth's theorem. In the literature, this theorem is often stated for the case $m = n$, but the proof for the more general result comes at no extra cost.

Theorem 5.10.6 (Erdős–Szekeres, 1935 [346])**.** *Let m and n be positive integers. Any sequence of $mn + 1$ distinct real numbers contains either an increasing subsequence of length $m + 1$ or a decreasing sequence of length $n + 1$.*

Original proof: Put $N = mn + 1$ and let $(x_1, x_2, \ldots, x_N)$ be a sequence of N distinct reals. For each $i = 1, \ldots, N$, define a_i to the be length of a longest increasing subsequence starting with x_i, and let b_i be the length of a longest decreasing subsequence starting with x_i. So to each x_i, there is a pair (a_i, b_i). These pairs are all distinct because for $i < j$, if $x_i < x_j$ then $a_i > a_j$, and if $x_i > x_j$ then $b_i > b_j$.

In hope of a contradiction, assume that there is no increasing subsequence of length $m + 1$ and no decreasing subsequence of length $n + 1$. Then each $a_i \in [m]$ and each

$b_i \in [n]$, giving only mn possible (different) pairs; but there are $mn+1$ such pairs, the desired contradiction. (In other words, the $mn+1$ pairs (a_i, b_i) are placed into at most mn holes, so by PHP, two pairs agree, contradicting the fact shown above that all pairs are distinct.)

So the assumption above is abandoned, and the desired conclusion holds. □

Proof using Dilworth's theorem: Put $N = mn+1$ and let $(x_1, x_2, \ldots, x_N)$ be a sequence of real numbers. Define a partial order $(P, \prec)$ on $P = \{x_1, \ldots, x_n\}$ by $x_i \prec x_j$ if and only if $i < j$ and $x_i \leq x_j$. Any increasing subsequence of the x_is is a chain in $(P, \prec)$, and any decreasing subsequence is an antichain in $(P, \prec)$. Corollary 5.10.5 finishes the proof. □

Dilworth's theorem can be used to give another proof of Hall's theorem (Theorem 5.3.1):

Proof of Hall's theorem from Dilworth's: Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall's condition: $\forall S \subseteq X$, $|S| \leq |N(S)|$. Let $P = V(G) = X \cup Y$ and define the poset $(P, <)$ by $x < y$ if and only if $x \in X$, $y \in Y$, and $\{x, y\} \in E$.

Let $A = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ be a maximum antichain in $(P, <)$, and put $r+s = t$. Since A is an antichain,

$$N(\{x_1, \ldots, x_r\}) \subseteq (Y \backslash \{y_1, \ldots, y_s\}).$$

Since Hall's condition is assumed, $|N(\{x_1, \ldots, x_r\})| \geq r$, and so $|Y| - s \geq r$, giving $|Y| \geq t$.

By Dilworth's theorem, $P = V(G)$ can be covered by t disjoint chains. However, these chains are either edges (which form a matching) or singletons in either X or Y; if these t disjoint chains contain a matching of size m, then counting chains, $t = m + (|X| - m) + (|Y| - m) \geq |X| - m + t$, and so $m \geq |X|$. Since each edge of a matching uses a vertex from X, $m \leq |X|$, giving $m = |X|$, and so all of X is matched (into Y). □

5.11 Flow networks

5.11.1 Terminology and basics

The word "network" might bring to mind any number of different structures. A computer network, a pipeline network, a social network, a highway system, or an electrical circuit are all types of a network. "Network" is also used as verb (for example, in business, networking can be making a list of contacts). For the purposes of this chapter, the term "network" is a noun used as a short form of "flow network".

Definition 5.11.1. *A network is a 5-tuple of the form $N = (V, D, s, t, c)$, where*

- *V is a non-empty set whose elements are called vertices or "nodes";*
- *$D \subset V \times V$, and so D corresponds to a set of directed edges on V.*
- *s and t are designated vertices in V; s is called the* source *and t is called the* sink.
- *$c : D \to \mathbb{R}^+$ is a function, where for each (directed) edge $e \in D$, $c(e)$ is called the* capacity *of e.*

Comments on definition of network:

- Multiple edges in D are (by definition) not allowed. This might make sense, because if, for example, two parallel flows between two vertices can be replaced with a single flow along just one edge. If there are two vertices x, y so that both $(x, y) \in D$ and $(y, x) \in D$, one might replace them with a single edge whose capacity is the difference between the two. Flow going both ways between a pair of vertices might be represented one way as positive and the other way as negative, again in just one edge. So for now, at least, assume no multiple edges.
- It is safe to assume that there are no loops (directed edges of the form (x, x)), since loops are of no help in transporting across a network.
- The capacity of each edge is a finite positive number. If one were to allow 0 capacity for some edge, then that edge is of no use in moving material, information, or automobiles, so simply delete it. For now, each edge has a finite capacity—infinite capacity is a useful condition in some cases, but for the moment, finite capacities are needed (so that certain sequences used in proofs have upper bounds).
- By the definition, one might think that all edges incident with s are going away from s, but this is not necessarily so. A network can have edges into the source or edges leaving the sink.

Some notation helps for the next definition. For a vertex x in a digraph, let $\text{in}(x)$ denote the set of edges into x, and $\text{out}(x)$ denote the set of edges leading out of x. So the indegree of x is $d^-(x) = |\text{in}(x)|$ and the outdegree is $d^+(x) = |\text{out}(x)|$.

Definition 5.11.2. *A* flow *on a network $N = (V, D, s, t, c)$ is a function $f : D \to [0, \infty)$ that satisfies:*

(i) For each $e \in D$, $f(e) \leq c(e)$;

(ii) For each vertex $x \in V(D)\backslash\{s,t\}$,

$$\sum_{e \in in(x)} f(e) = \sum_{e \in out(x)} f(e).$$

Part (i) of Definition 5.11.2 is called the "capacity restraint" and (ii) is called the "conservation law"; a similar assumption to (ii) is made in electronics ("current in equals current out"), which is sometimes referred to as one of Kirchoff's laws. Some authors call a flow as in Definition 5.11.2 a "feasible flow"; here all flows are considered to be feasible. A zero flow is a flow where every edge has a flow of 0.

Note that in some settings, negative flows are allowed, but in this section, negative flows are not needed. In other applications, flows are developed on graphs where undirected edges and negative flows might be helpful. Instead of capacities on edges, capacities can be placed on vertices (*e.g.*, on hubs in a communication network). The reader is invited to compare treatments in other popular texts (*e.g.*, [125], [429], [547], [584] or [977].)

Definition 5.11.3. *For any flow f on a network N, let the* value *of the flow be*

$$val(f) = \sum_{e \in in(t)} f(e) - \sum_{e \in out(t)} f(e).$$

(One might say that the flow is measured as the "net value" of the flows on edges surrounding t.)

Definition 5.11.4. *A* cut *in a network $N = (V, D, s, t, c)$ is a partition $V = S \cup T$ (where $S \cap T = \emptyset$)) with $s \in S$ and $t \in T$. Such a cut is denoted by (S, T), or $(S, \overline{S})$. The capacity (or "value") of a cut (S, T) is the sum of the capacities of all arcs from S to T, denoted by*

$$cap(S, \overline{S}) = \sum_{e \in out(S)} c(e).$$

Note that "reverse" edges from T to S are not counted since they do not add to the value of any flow. Since any flow from s to t passes across any cut, any such flow has value at most the capacity of any cut, and so at most that of any minimum cut.

The next lemma shows how to measure the value of a flow by using a cut.

Lemma 5.11.5. *Let $N = (V, D, s, t, c)$ be a network and let f be a flow on N. For any cut $(S, \overline{S})$, let $out(S)$ denote the set of edges from S to $\overline{S}$ and let $in(S)$ denote the set of edges from $\overline{S}$ to S. Then the value of f is measured across the cut $(S, \overline{S})$ with*

$$val(f) = \sum_{e \in out(S)} f(e) - \sum_{e \in in(S)} f(e). \tag{5.4}$$

Proof: Let $(S, \overline{S})$ be a cut with $s \in S$ and $t \in \overline{S}$. Starting with the definition of $\mathrm{val}(f)$, for all $u \in \overline{S} \backslash \{t\}$, add the equation for conservation at u,

$$0 = \sum_{e \in \mathrm{in}(x)} f(e) - \sum_{e \in \mathrm{out}(x)} f(e),$$

giving

$$\begin{aligned} \mathrm{val}(f) &= \sum_{e \in \mathrm{in}(t)} f(e) - \sum_{e \in \mathrm{out}(t)} f(e) \\ &= \sum_{e \in \mathrm{in}(t)} f(e) - \sum_{e \in \mathrm{out}(t)} f(e) + \sum_{u \in \overline{S} \backslash \{t\}} \left[\sum_{e \in \mathrm{in}(u)} f(e) - \sum_{e \in \mathrm{out}(u)} f(e) \right]. \end{aligned}$$

Consider an arbitrary arc $a = (x, y) \in D$, and the four cases:

- If both x and y are in S, this arc is not counted in the sum.
- If both x and y are in $\overline{S}$, then the $f(a)$ is counted once positively and once negatively, and so they cancel each other.
- If $x \in S$ and $y \in \overline{S}$, then $f(a)$ is counted positively (just once in summand when $u = y$).
- If $x \in \overline{S}$ and $y \in S$, then $f(a)$ is counted once negatively (in the summand when $u = x$).

Hence the above expression becomes equation (5.4), as desired. □

The value of a flow is bounded by the capacity of a cut:

Lemma 5.11.6. *Let $N = (V, D, s, t, c)$ be a network with flow f. For any cut $(S, \overline{S})$,*

$$\mathit{val}(f) \leq \mathit{cap}(S, \overline{S}).$$

Proof: By Lemma 5.11.5,

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(S)} f(e) - \sum_{e \in \mathrm{in}(S)} f(e).$$

For each $e \in D$, $0 \leq f(e) \leq f(c)$ and so the first sum above is at most $\sum_{e \in \mathrm{out}(S)} c(e)$, and the expression is maximized when the second term is 0, so

$$\mathrm{val}(f) \leq \sum_{e \in \mathrm{out}(S)} c(e) = \mathrm{cap}(S, \overline{S}).$$

□

5.11.2 The Ford-Fulkerson maxflow-mincut theorem

The following is now considered as a fundamental result in network theory; it was published independently by two groups in the same year.

Theorem 5.11.7 (Ford–Fulkerson, [374], and Elias–Feinstein–Shannon [282], 1956)**.** *(***Maxflow-Mincut Theorem***) The maximum flow in a network is the minimum capacity over all cuts.*

The proof of the maxflow-mincut theorem given by Ford and Fulkerson is based on a simple algorithm, loosely described as follows:

Ford-Fulkerson algorithm [374]

INPUT: A network N.

OUTPUT: A maximum flow f on N and a cut $(S, \overline{S})$ whose capacity is minimal (and matches the value of f).

INITIALIZATION: Let f be the zero flow on N (for each $e \in D$, $f(e) = 0$).

RECURSION: Given a flow f, pick an s-t path P in the underlying undirected graph. If the flow f on each of the edges in P is less than its capacity (such a path is called an "augmenting path".), then increase (augment) the flow (as much as possible) along P, creating a new flow $f*$ with a larger value. Then reduce capacities of each edge in P by the amount augmented, and return the new network as N.

TERMINATION: If there are no augmenting s-t paths remaining, stop and output f. In this case, let S be the set of all vertices v for which there is a s-v path that can have its flow increased.

The Ford-Fulkerson algorithm was not guaranteed to terminate if capacities are transcendental; a more specific algorithm was given by Edmonds and Karp [273] that was guaranteed (in any case) to terminate in $O(|V| \cdot |D|^2)$ time. Ford and Fulkerson used a modified breadth-first search to find augmenting paths and updated labelling to locate S.

Remark 5.11.8. *The expression "augmenting path" is also used in other areas of graph theory to indicate a path that can be extended (augmented). (Perhaps it was Berge [91] who first used the expression in work on matchings. See Section 5.8.)*

Theorem 5.11.7 is now proved by showing that the Ford-Fulkerson algorithm produces a flow with maximum value along with a corresponding cut of minimum capacity.

Lemma 5.11.9. *Let $N = (V, D, s, t, c)$ be a network with flow f and let $(S, \overline{S})$ be a cut. Then $val(f) \leq cap(S, \overline{S})$. If $val(f) = cap(S, \overline{S})$, then f is a maximum flow and $(S, \overline{S})$ is a cut with minimum capacity (a "minimum cut").*

Proof: By Lemma 5.11.6 for any cut $(S, \overline{S})$ (with $s \in S$, $t \notin S$), $\mathrm{val}(f) \leq \mathrm{cap}(S, \overline{S})$.

Suppose $(S, \overline{S})$ is a cut with $\mathrm{val}(f) = \mathrm{cap}(S, \overline{S})$. Let g be a maximum flow. If $(R, \overline{R})$ is a minimum cut, then by Lemma 5.11.5,

$$\mathrm{val}(f) \leq \mathrm{val}(g) \leq \mathrm{cap}(R, \overline{R}) \leq \mathrm{cap}(S, \overline{S}),$$

but since $\mathrm{val}(f) = \mathrm{cap}(S, \overline{S})$, all terms in the above expression are equal, and so f is a maximum flow and $(S, \overline{S})$ is a cut with minimum capacity. □

Let $N = (V, D, s, t, c)$ be a network with flow $f : D \to [0, \infty)$. Let $P = x_1 x_2 \cdots x_k$ be a path in the undirected underlying graph. An edge $\{x_i, x_{i+1}$ in P corresponds to either $(x_i, x_{i+1}) \in D$ or $(x_{i+1}, x_i) \in D$. If $\{x_i, x_{i+1}\}$ corresponds to $(x_i, x_{i+1}) \in D$, call this edge in P *forward*, and *backward* otherwise.

For each edge $e = \{x, y\}$ in P, define the slack (or residual capacity, or tolerence) of e to be

$$\mathrm{slack}(e) = \begin{cases} c(e) - f(e) & \text{if } e \text{ is a forward edge;} \\ f((y, x)) & \text{if } e \text{ is a backward edge.} \end{cases}$$

The slack of a backward edge says how much f can be decreased along that edge. Define $\mathrm{slack}(P)$ to be the minimal slack of any of the edges in P. Say that P is *f-saturated* if and only if $\mathrm{slack}(P) = 0$ (if $\mathrm{slack}(P) > 0$, P is called *unsaturated*).

If P is an f-unsaturated s-t "path" then define f^* on D by

$$f^*(e) = \begin{cases} f(e) & \text{if } e \text{ is not on } P; \\ f(e) + \mathrm{slack}(P) & \text{if } e \text{ is a forward edge in } P; \\ f(e) - \mathrm{slack}(P) & \text{if } e \text{ is a backward edge in } P. \end{cases}$$

If P is an s-t path, then it is not difficult to verify that f^* is indeed a flow and that its value is $\mathrm{val}(f) + \mathrm{slack}(P)$. (For example, see [585, p. 197] for details.)

If P is an s-t path with $\mathrm{slack}(P) > 0$, the flow along the path P is "augmented" by f^*, and so is called an *augmenting path*, or perhaps more descriptively, an *f-augmenting path*.

Theorem 5.11.10. *Let $N = (V, D, s, t, c)$ be a network with flow f. Then f is maximal if and only if N contains no f-augmenting path.*

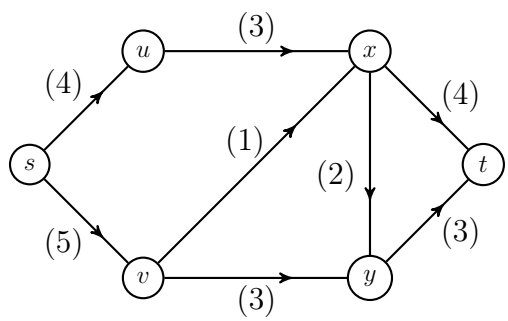
Proof: If f is a maximal flow, there is no f-augmenting path in N, for if there was, a larger flow can be created.

Assume that there is no f-augmenting path. Let $S \subseteq V$ be the set of all vertices u such that there is an unsaturated s-u path. (Such vertices are sometimes called "s-reachable" in the literature.) So, for example, $s \in S$. Because there are no augmenting paths, $t \notin S$ (and so $t \in \overline{S}$). Consider the cut $(S, \overline{S})$. For any edge $(u, v) \in D$ with $u \in S$ and $v \in \overline{S}$, $\mathrm{slack}(u, v) = 0$, otherwise v determines an unsaturated s-v path.

So all such edges, have a flow equal to its capacity; that is, $f(u,v) = c(u,v)$, and so the flow out of S is equal to $\text{cap}(S,\overline{S})$. If $u \in S$ and $v \in \overline{S}$ but is a backward edge (i.e., $(v,u) \in D$), then $f(v,u) = 0$ (since otherwise $v \in S$ by using the backward edge). Hence the value of the flow into S is 0. So $\text{val}(f) = \text{cap}(S,\overline{S})$, and then by Lemma 5.11.9, f is a maximum flow and $(S,\overline{S})$ is a minimum cut. □

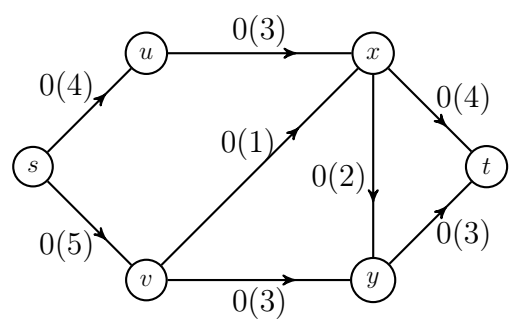
So the Ford-Fulkerson algorithm produces a maximum flow f and a cut $(S,\overline{S})$ with $\text{val}(f) = \text{cap}(S,\overline{S})$. This finishes the proof of the maxflow-mincut theorem (Theorem 5.11.7). □

To demonstrate the Ford-Fulkerson algorithm on a small network, consider the following network N with capacities in parentheses.

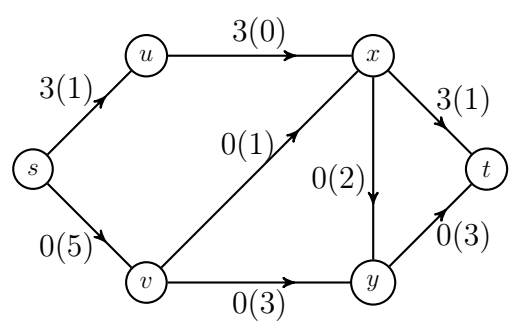


Applying the FF algorithm, flows along edges are denoted by numbers in front of the capacities.

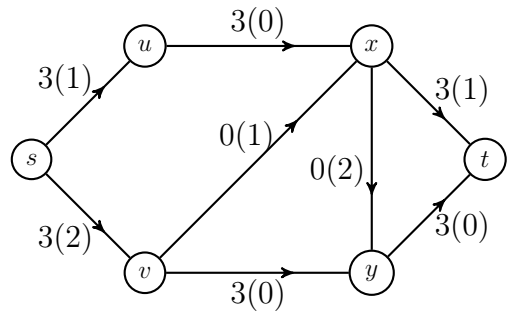
The first step is to assign the flow of 0 to each edge:



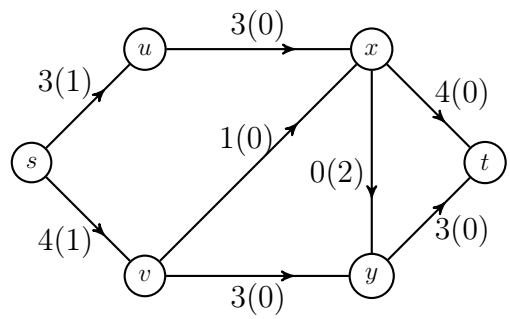
The path *suxt* has slack 3, so push 3 units of flow along this path and reduce the corresponding (residual) capacities:



The path $svyt$ has slack 3, so push 3 units of flow along this path and reduce the corresponding (residual) capacities:



The path $svxt$ has slack 1, so push 1 unit of flow along this path and reduce the corresponding (residual) capacities:



There are no more augmenting s-t paths. By looking at the edges into t verifies that the value of the resulting flow is 7. In the last diagram, the only vertices reachable from s by non-zero capacity paths are $S = \{s, u, v\}$, and it is easy to verify that the cut $(S, \overline{S})$ has value 7 as well. □

The Ford-Fulkerson (FF) algorithm has a useful consequence:

Theorem 5.11.11 (Integrality theorem on flow networks)**.** *In a flow network where all edge capacities are integers, there exists a maximum flow with all edge-flows being integers.*

Proof: Apply the above algorithm to such an network. At each step, all values of slack are integers so flows on edges are also integers at each step. □

(In [782], the integrality result is called the "integrity" theorem.)

Theorem 5.11.11 does not say that in integer capacity networks, every maximum flow uses integer flows on each edge—only that there exists such a flow.

Exercise 224. *Give an example of network with integer capacities and a maximum flow f with not all edges having integer flow.*

Remark 5.11.12. *Certain statements for graphs can be proved using networks with capacities 0 and/or 1 and the integrality theorem.*

Theorem 5.11.13. *Menger's theorem for edge-disjoint paths in digraphs (Theorem 4.1.9) follows from the maxflow-mincut theorem (Theorem 5.11.7).*

Proof: Let $G = (V, D)$ be a digraph and let $x, y \in V$. Consider a network $N = (V, D, x, y, c)$ where the capacity c of each arc is 1. Then units of flow from x to y correspond to edge disjoint x-y paths, so a maximum value flow f corresponds to the maximum number m of such paths. On the other hand, a cut separates x from y, and since each edge in a cut has capacity 1, the capacity of a cut (S, T) is the number of edges that can be deleted to separate x from y. By the maxflow-mincut theorem, the minimum capacity of a cut is m, so the minimum number of edges deleted to separate x from y is also m. □

Flows can also be used to prove the vertex version of Menger's theorem for graphs (Theorem 4.1.6). In particular, a version of the maxflow-mincut theorem where vertices (and not edges) have capacities.

Theorem 5.11.14 (Vertex capacity version of the maxflow-mincut theorem)**.** *Let G be a directed graph with two specified vertices s and t with capacities on all vertices except s and t. Then the maximum value of a flow on G is equal to the minimum capacity of a vertex-cut.*

Proof outline: Split each vertex x into two vertices x^- x^+, where x^- is incident with edges going into x and x^+ incident with edges leaving x, and form the edge (x^-, x^+) with capacity the same as the capacity of x. See, *e.g.*, [125] for the remaining details. □

Exercise 225. *Use the maxflow-mincut theorem (Theorem 5.11.7) to prove the König–Egervary theorem. Hint: If $G = (X, Y, E)$ is bipartite, create a flow by adding two vertices, say u and v and add directed edges from u to each vertex in X and directed edges from all vertices in Y to v. Also, direct all edges from X to Y.*

In 1962, Ford and Fulkerson wrote a book [375] on flow networks, describing in more detail algorithms and their relations to other areas of mathematics or computer science (*e.g.*, linear programming).

5.12 Many equivalent theorems

This section contains only a few ideas that can be more fully developed by the interested student.

Many of the following theorems are related (the order in which these appear may not be the most natural):

- Berge's theorem (augmenting paths, Theorem 5.8.1);
- Hall's theorem (Theorem 5.3.1) for matchings;

- Hall's SDR theorem (Theorem 5.3.5);
- The König–Egerváry theorem (Theorem 5.6.1);
- The Birkhoff–von Neumann theorem (Theorem 5.3.13) for doubly stochastic matrices;
- Tutte's 1-factor theorem (Theorem 5.9.2);
- Gallai's theorem (Theorem 5.5.4);
- Five versions of Menger's theorem (see Theorem 4.1.6 and following).
- Dilworth's theorem (Theorem 5.10.2);
- Ford-Fulkerson's max-flow/min-cut theorem (Theorem 5.11.7).

Each of the above theorems can be proved by assuming some other theorem in the list; in this sense, the theorems above have been said to be "equivalent". (Many of the above theorems are first proved here directly—not just by using another theorem in the list.) For many of the implications not proved here, see the book *The equivalence of some combinatorial matching theorems* by Philip Reichmeider [782].

For example, in Section 5.3, the Birkhoff–von Neumann theorem is proved using Hall's theorem. Menger's theorem (Theorem 4.1.6) is used to prove Hall's theorem in Exercise 195. In Section 5.6, either Hall's theorem or Menger's theorem is used to prove the König–Egerváry theorem. In Exercise 225, the maxflow-mincut is used to prove König–Egervary. In Exercise 214 the König–Egerváry theorem is used to prove Hall's theorem. In Section 5.9, Hall's theorem is used to prove Tutte's theorem. In Chapter 4, the max-flow/min-cut theorem is used to prove versions of Menger's theorem. In Section 5.10, Dilworth's theorem is used to prove Hall's theorem. Trivially, Hall's theorem is equivalent to the SDR theorem.

Chapter 6

Graph colouring

It might be said that the serious study of graph colouring began with a map colouring question, but since then, the theory of graph colouring is applied in many areas in industry, including scheduling, chemical storage, and job assignments.

6.1 Introduction

Classic applications of graph theory are in scheduling, and in such applications, one often looks for an optimal assignment of "labels" or "colours" to vertices of some graph. Before giving the formal definitions used for vertex colourings, two similar problems are examined, the first, about scheduling exams is discussed in more detail than the second example regarding adhering to a given schedule.

Scheduling exams

Suppose that exams need to be scheduled for seven different courses, and each exam takes the same amount of time. If there is a student in two or more courses, these courses need different exam time slots. What is the least number of time slots that all courses can be tested? Certainly 7 time slots will do the job, but in most situations, this number can be optimized. Construct a graph on 7 vertices, each vertex corresponding to a course, and let two vertices be adjacent if and only if the two corresponding courses conflict because they share one or more students. For example, consider the "conflict graph" in Figure 6.1 for four elementary math courses (DM=discrete math, GT=graph theory, LA=linear algebra, EG=Euclidean geometry, NT=number theory, ST= set theory, and HM=History of math).

If each exam had its own time slot, 7 different time slots would suffice. In fact, having 7 different time slots makes programming quite easy, but this is inefficient—fewer than 7 time slots are required.

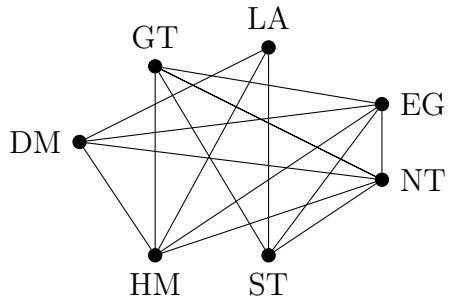
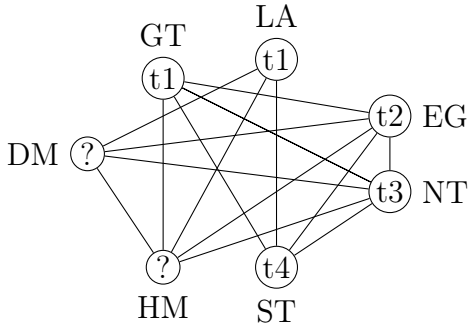
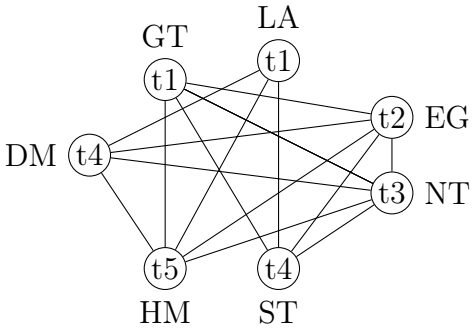


Figure 6.1: A conflict graph for seven exams

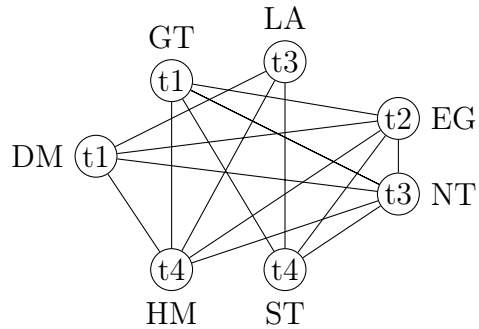
Any two of the four courses GT, EG, NT, and ST have a conflict, so these four courses require different time slots, say t1, t2, t3, t4. (So the minimum number of time slots required is at least 4.) Since there is no student in both GT and LA, one might first suggest to put LA in the same time slot as GT, namely t1. The situation is now



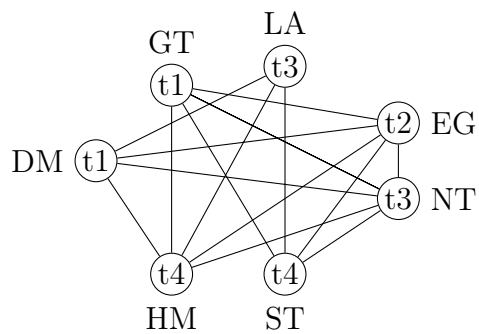
Since DM is in conflict with LA, EG, and NT, either DM runs in t4 or in a new time slot; suppose that DM runs in t4. But then HM is in conflict with LA, EG, NT, and DM, and so a new time slot, say t5 is needed for HM, giving the picture



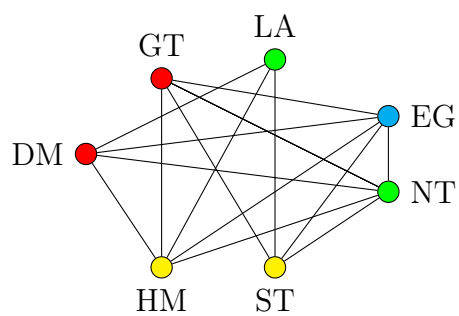
So 5 exam time slots are sufficient. However, had one chosen t3 for LA, one could put DM in t1 and HM in t4, so four time slots is sufficient:



As already observed, four time slots are required, so the assignment using four slots above is optimal.



The labels t1, t2, t3, t4 could have just as easily been a, b, c, d or $1, 2, 3, 4$ or red, green, blue, and yellow:



The labels above can be permuted and the same information is found. (For example, if the four exam times above are assigned according to the conflict graph, one could interchange any two of the times and the exam assignment would still work.

The colouring of the vertices in the last graph is said to be "proper" or "good" since no two vertices of the same colour are adjacent. (The 5-colouring given in the previous picture is also proper.) This last colouring is also optimal, since 4 is the least number of colours for which a proper colouring can be found.

It might now be obvious to the reader that the scheduling of final exams for an entire university with tens of thousands students and thousands of courses is a hard problem. If the busiest student is enrolled in 7 courses, at least 7 time slots are needed, but since there are thousands of courses and countless conflicts possible, the number of time slots needed might be in the hundreds. Some factors bring down this number (like the fact that science students will likely have conflicts with only a few courses outside of science, or the fact that some courses have very small enrolments). However other factors may complicate matters bringing numbers up (*e.g.*, large calculus courses may have 2000 students and finding rooms for 2000 all at the same time might make things very challenging).

Other problems universities face regarding scheduling include teaching assignments, room assignments, and scheduling of committee meetings. In many of these situations, an optimal solution simply requires too many resources, so some tolerance is worked into the program where as few as possible i ndividual cases might be handled by other methods (like a student with 3 exams scheduled the same day might defer an exam).

6.2 Vertex colouring, basics

Let k be a positive integer (usually $k \geq 2$) and let C be a set of k different labels, also called colours. For a graph $G = (V, E)$, a function $f : V \to C$ is called a k-colouring of the vertices of G, or simply a k-colouring of G. (In a later chapter, edge-colourings are also considered, but for now, a k-colouring is a vertex colouring.) Since a k-colouring does not depend upon what the colours actually are, it is standard to use $[k] = \{1, 2, \ldots, k\}$ as the set of colours.

Note: A k-colouring need not use all k colours, just as a function $f : A \to B$ need not use all of B in the range of f.

Definition 6.2.1. *For a positive integer k and a graph G, a vertex k-colouring $f : V(G) \to [k]$ is called* proper *if and only if for any $\{x, y\} \in E(G)$, $f(x) \neq f(y)$. A graph is called vertex k-*colourable *if and only if there exists a proper k-colouring of G.*

Remarks: In some texts, the term "colouring" already means "proper colouring", however in most texts, a colouring is simply a labelling function. In this text, if a colouring is meant to be a proper colouring, to avoid confusion, either the expression "proper colouring" or "good colouring" is used. Often the word "vertex" is omitted when the context is clear, and one says simply that a graph is "k-colourable". (Edge-colourings are considered in Section 6.11.) The set $[k]$ above is called a set of colours; in fact, any k-element set can form the set of colours. When k is small, often the colours are given names like "red" and "blue".

Let $c : V(G) \to [k]$ be a k-colouring of $V(G)$. For each $i \in [k]$, the set of vertices $c^{-1}(i)$ is called the ith colour class. Note that for a proper colouring, each colour class is an independent set of vertices.

An *injective vertex colouring* of a graph is a colouring where no two vertices receive the same colour. (Sometimes, injective colourings are called "rainbow colourings".)

For example, the triangle K_3 is 3-colourable but not 2-colourable. Any cycle with an odd number of vertices is 3-colourable but not 2-colourable. All bipartite graphs (including even cycles and trees) are 2-colourable. [In fact, any k-colourable graph is, in a sense, k-partite, where some "partite sets" may be considered empty.]

Exercise 226. *Let G be a copy of C_5 with labelled vertices. How many proper 3-colourings are there of $V(G)$ (using just one set of three colours)?*

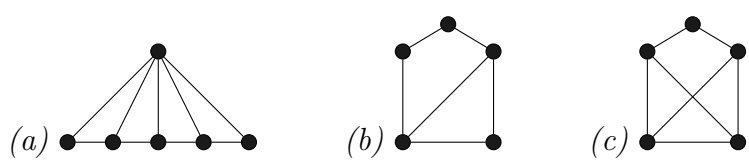
Definition 6.2.2. *For a graph G, the* chromatic number *of G, denoted $\chi(G)$, is the least integer k for which G is k-colourable.*

To show that a particular graph has chromatic number k, one needs to exhibit (or prove the existence of) a proper k-colouring of G and then show that no $(k-1)$-colouring is proper. For example, it was shown in Section 6.1 it was shown that the chromatic number of the conflict graph in Figure 6.1 is 4.

Some graphs have chromatic numbers that are easy to compute. For example, for $n \geq 1$, $\chi(K_n) = n$, $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$, $\chi(\overline{K_n}) = 1$. All non-trivial bipartite graphs have chromatic number 2, and so for any tree T on at least two vertices, $\chi(T) = 2$.

Exercise 227. *Show that for each $n \geq 1$, the n-dimensional cube graph Q_n satisfies $\chi(Q_n) = 2$.*

Exercise 228. *Find the chromatic number of each of the following graphs:*



Exercise 229. *Let F be a graph formed by two graphs G and H that share exactly one common vertex (so this vertex is a cut-vertex for F). Show that $\chi(F) = \max\{\chi(G), \chi(H)\}$.*

The following exercise is useful for the proof of Theorem 6.14.4, a theorem about the chromatic number of the unit-distance graph in the plane (compare with Moser's spindle in Figure 6.7).

Exercise 230. *Show that the graph in Figure 6.2 has chromatic number 4.*

Exercise 231. *Let P denote the Petersen graph (see Figure 1.26). Show that $\chi(P) = 3$.*

Exercise 232. *Let $r \geq 3$. Show that the only r-regular graph G with $\chi(G) = r + 1$ is K_{r+1}.*

From the definition of chromatic number, if H is a subgraph of G (denoted $H \subseteq G$), then $\chi(H) \leq \chi(G)$. If $\omega(G)$ denotes the clique number of G (the order of a largest complete subgraph), then

$$\chi(G) \geq \omega(G).$$

For example, the conflict graph in Figure 6.1 contains a K_4, which was used as a starting point to find a proper colouring, and in that case, a proper 4-colouring was found.

One might think that the clique number is the "driving force" behind the chromatic number; however, by constructions given in Section 6.8, graphs without triangles (K_3s) can have arbitrarily high chromatic number!

Exercise 233. *Find a graph whose chromatic number exceeds its clique number. Hint: such a graph is given somewhere in this chapter.*

Exercise 234. *For each positive integer n, show that there is a graph on n vertices with $\lceil n/2 \rceil$ cliques of different sizes.*

Exercise 235. *For each $m \geq 3$, construct a graph G on $m + 2$ vertices so that K_m is not a subgraph of G and G is not $(m - 1)$-colourable.*

The following theorem does not seem to be well-known, except that its proof is given as an exercise ("neither easy nor straightforward") by Harary [486, 12.2, p. 148].

Theorem 6.2.3 (Erdős–Hajnal, 1966 [327])**.** *Let $\ell \geq 3$ be the length of a longest odd cycle in a graph G. Then $\chi(G) \leq \ell + 1$.*

Another related theorem is also given as an exercise in [486, 12.28, p. 149]:

Theorem 6.2.4 (Gallai, 1968 [402])**.** *If ℓ is the length of a longest path in a graph G, then $\chi(G) \leq \ell + 1$.*

In Section 5.5, the fundamental notions of "independent sets " and "independence numbers" were given. For convenience, they are repeated here.

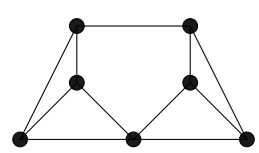


Figure 6.2: The Hajós graph

Definition 6.2.5. *Let G be a graph. A set $S \subseteq V(G)$ of vertices is called an* independent set *if and only if S induces no edges in G; in other words, S is an independent set if and only if $E(G) \cap [S]^2 = \emptyset$. The* independence number *of G, denoted $\alpha(G)$, is the largest cardinality of an independent set of vertices in G.*

So $\alpha(K_7) = 1$ and $\alpha(K_{3,7}) = 7$. Note that for any simple graph G, $\alpha(G) = \omega(\overline{G})$.

Exercise 236. *Find $\alpha(G)$ when G is (i) an odd cycle C_{2n+1}; (ii) the Petersen graph; (iii) $K_{4,5}$.*

The next lemma follows directly from the definition of a proper colouring of the vertices of a graph.

Lemma 6.2.6. *Let G be a graph and k be a positive integer. If $f : V(G) \to [k]$ is a proper k-colouring, then for each $i \in [k]$, the colour class $f^{-1}(i)$ is an independent set.*

Using Lemma 6.2.6 gives a lower bound for the chromatic number is in terms of $\alpha(G)$.

Exercise 237. *Prove that for any graph G,*

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}. \tag{6.1}$$

Exercise 238. *Use the result in Exercise 237 to show that for $k \geq 2$,*

$$\chi(\overline{C_{2k+1}}) = k + 1.$$

Exercise 239. *Show that for any simple graph G,*

$$|E(G)| \geq \binom{\chi(G)}{2}. \tag{6.2}$$

What is a bound that follows for the chromatic number in terms of the size (number of edges) of the graph?

Exercise 240. *Let $G = (V(G), E(G))$ be a graph with the property that for any subgraph $H \subseteq G$, $\alpha(H) \geq \frac{1}{2}|V(H)|$. Prove that $\chi(G) \leq 2$.*

Regarding Exercise 240, Erdős and Hajnal [328] showed that for $0 < c < \frac{1}{2}$, graphs with the property that any subgraph H satisfies $\alpha(H) \geq c|V(H)|$ can have arbitrarily high chromatic number.

Exercise 241. *Show that for any (simple) graph $G = (V, E)$,*

$$\chi(G) \geq \frac{|V|^2}{|V|^2 - 2|E|}. \tag{6.3}$$

Exercise 242. *Let G be a graph where any two odd cycles share a vertex. Prove that $\chi(G) \leq 5$.*

Exercise 243. *Let G be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Show that $\chi(G) \leq \max_i \min\{d_i + 1, i\}$.*

Exercise 244. *Show that for any graph G, there are at least $\chi(G)$ vertices, each of which has degree at least $\chi(G) - 1$.*

One of the most used upper bounds for the chromatic number comes from an elementary greedy colouring:

Lemma 6.2.7. *For any simple graph G, $\chi(G) \leq \Delta(G) + 1$.*

Proof: Let $x_1, x_2, \ldots, x_n$ be the vertices of G. Colour the vertices consecutively, starting with x_1, using at most $\Delta(G) + 1$ colours. Suppose that at some point, all of $x_1, \ldots, x_i$ have been properly coloured using at most $\Delta(G) + 1$ colours. Since x_{i+1} has at most $\Delta(G)$ neighbours, x_{i+1} has at most $\Delta(G)$ neighbours to the left, and so one colour remains to properly colour x_{i+1}. (To be formal, an inductive proof can be given.) □

In the greedy algorithm used to show Lemma 6.2.7, to colour any one vertex x_i, one only needs that the number of previous neighbours is small. In the case that G is not regular (of degree $\Delta(G)$), this fact can be used to show a slight improvement.

Lemma 6.2.8 (Szekeres–Wilf, 1968 [899])**.** *Let G be a graph and let k be the maximum, taken over all induced subgraphs H of G, of $\delta(H)$. Then $\chi(G) \leq k + 1$.*

Proof: An ordering $x_1, x_2, \ldots, x_n$ of the vertices of G is constructed so that the number of edges from each x_i going to the left is at most k.

Since G has a vertex of degree k, let x_n be such a vertex, and consider $H_{n-1} = G - x_n$. By assumption, H_{n-1} also has a vertex of degree at most k; let such a vertex be x_{n-1}. Put $H_{n-2} = H_{n-1} - x_{n-2}$. Continuing in this manner, enumerate all vertices of G, producing the desired ordering mentioned above.

So in the greedy algorithm, for each $i = 2, \ldots, n$, the vertex x_i has at most k neighbours previous in the list, and so each x_i can be properly coloured with an additional $(k+1)$-st colour. □

One consequence of Lemma 6.2.8 is when $\delta(G) < \Delta(G)$:

Corollary 6.2.9. *If G is a graph that is not regular, then $\chi(G) \leq \Delta(G)$.*

Exercise 245. *Prove that every graph G has a vertex ordering for which the greedy colouring algorithm uses precisely $\chi(G)$ colours.*

Greedy colouring algorithms can sometimes fail badly:

Exercise 246. *For every $n \geq 2$, find a bipartite graph $G \subset K_{n,n}$ and an ordering of the vertices of G so that the greedy colouring algorithm requires n colours—not just the needed two.*

Note that for complete graphs and odd cycles, $\chi(G) = \Delta(G) + 1$; R. L. Brooks proved that these are the only examples (of connected graphs) where the bound in Lemma 6.2.7 is achieved.

Theorem 6.2.10 (Brooks, 1941 [163])**.** *Let G be a connected graph. If G is neither the complete graph nor a cycle with an odd number of vertices, then $\chi(G) \leq \Delta(G)$.*

Proof: The proof given here (see [125] or [637]) is separated into cases depending on the connectivity of G. For another proof, see [674].

Assume that G is neither a complete graph nor an odd cycle. If G is not regular, then Corollary 6.2.9 shows $\chi(G) \leq \Delta(G)$, so assume that G is regular of degree $\Delta = \Delta(G)$.

If $\Delta(G) = 1$, then G is a single edge, which is a complete graph, and so $\Delta(G) \geq 2$. If $\Delta(G) = 2$, then G is a cycle; in this case, either G is an odd cycle (which is assumed not to be the case) or G is an even cycle, which has chromatic number 2, in which case the theorem holds. So, suppose that $\Delta = \Delta(G) \geq 3$.

Suppose that G has a cut-vertex x, and $G_1, G_2, ...$ be the components of $G - x$. Then the number of edges from x into each G_i is less than Δ, so each graph H_i induced by $V(G_i) \cup \{x\}$ has a vertex of degree less than Δ, and so by Lemma 6.2.8, each H_i graph is Δ-colourable. In each H_i, relabel the colours so that x receives the same colour in each H_i. Together, these individual colourings give a colouring of G with at most Δ colours.

So suppose that G has no cut-vertex; that is, suppose that G is 2-connected (and Δ-regular).

If G is 3-connected, let x_1, x_2, x_n be vertices so that x_n is adjacent to both x_1 and x_2, but x_1 is not adjacent to x_2; such a triple exists because G is not complete. In this case, $G - \{x_1, x_2\}$ is connected.

If G is not 3-connected, (but still 2-connected), let x_n be a vertex so that $G - x_n$ has a cut vertex, and so contains at least two blocks, say $B_1 \subset G$ and $B_2 \subset G$. Since $G - x_n$ is connected, there exists two non-adjacent vertices $x_1 \in V(B_1)$ and $x_2 \in V(B_2)$ both adjacent to x_n. In either case, x_1, x_2 are not adjacent, and both are adjacent to x_n.

Let $x_{n-1} \in V \backslash \{x_1, x_2, x_n\}$ be a neighbour of x_n; similarly, let x_{n-2} be a neighbour of either x_n or x_{n-1}. Continue so that each of $x_3, x_4, \ldots, x_{n-1}$ (and x_1 and x_2) all have a neighbour to the right. In the greedy colouring, x_1 and x_2 receive the same colour, and for each $i = 3, \ldots, n-1$, the vertex x_i has at most $\Delta - 1$ neighbours before it. Since x_n is adjacent to vertices with at most $\Delta - 1$ different colours, x_n can receive one of Δ (or fewer) colours, and the greedy algorithm uses at most Δ colours. $\square$

Brooks' theorem can be used to find precisely the chromatic number of various graphs—without having to find an optimal proper vertex colouring. For example, in

Exercise 231 it was asked to show that the Petersen graph has chromatic number 3; by giving a proper 3-colouring (as in Figure 10.4), the chromatic number is at most 3.

Exercise 247. *Show how Brooks' theorem implies that the Petersen graph P has chromatic number 3 (without explicitly finding a good 3-colouring).*

As another simple application of Brooks' theorem, the graph G in Figure 6.3 contains a K_4 (the four central vertices) and so $\chi(G) \geq 4$. However, G is not complete and has $\Delta(G) = 4$, and so by Brooks's theorem, $\chi(G) \leq 4$. The fact that $\chi(G) \leq 4$ also follows from the Four Colour Theorem (given here as Theorem 7.7.5) since the graph is planar.

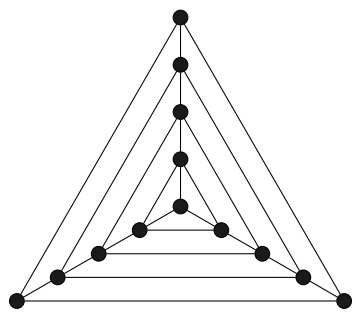


Figure 6.3: An example for Brooks' theorem

There are a number of strengthenings of Brooks' theorem involving $\omega(G)$, the clique number of G, (the number of vertices in a largest clique in G).

Theorem 6.2.11 (Borodin–Kostochka, 1977 [150]). *If G is a graph with $\omega(G) \leq \frac{\Delta(G)}{2}$, then $\chi(G) < \Delta(G)$.*

Theorem 6.2.12 (Kostochka, 1980 [598]). *If G is a graph with $\omega(G) \leq \Delta(G) - 29$, then $\chi(G) < \Delta(G)$.*

Theorem 6.2.13 (Mozhan, 1987 thesis). *If G is a graph with $\omega(G) \leq \Delta - 4$ and $\Delta(G) \geq 31$, then $\chi(G) < \Delta(G)$.*

As the clique number gets closer to the maximum degree $\Delta(G)$, the chromatic number can still drop below $\Delta(G)$.

Theorem 6.2.14 (Reed, 1999 [781]). *There exists $k \in \mathbb{Z}^+$, so that for every graph G with $\Delta(G) \geq k$ and $\omega(G) < \Delta(G)$, then $\chi(G) < \Delta(G)$.*

Reed also showed that $k = 10^{14}$ is sufficiently large for the result in Theorem 6.2.14 to hold.

Exercise 248. *Show that any triangle-free graph G on n vertices satisfies $\chi(G) \leq 2\sqrt{n}$. Hint: Try induction.*

Definition 6.2.15. *For positive integers k, n, where $2k \leq n$, define the Kneser graph $KG(n,k)$ to be the graph (V,E), where $V = [n]^k$ and $E = \{\{S,T\} : S \cap T = \emptyset\}$.*

So $KG(n,k)$ has $\binom{n}{k}$ vertices. (Another common notation for the Kneser graphs is $KG_{n,k}$.) See Figure 6.4 for the case $n = 5$ and $k = 2$.

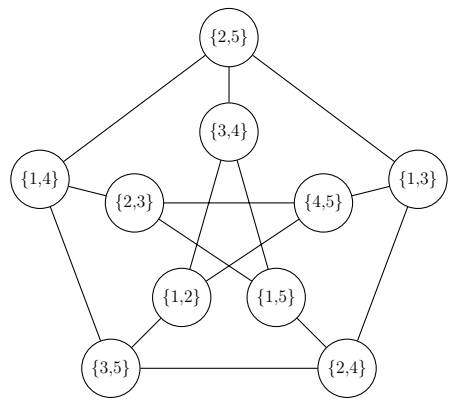


Figure 6.4: The Kneser graph $KG(5,2)$

Exercise 249. *What is the Kneser graph $KG(4,2)$?*

Exercise 250. *How many edges does the Kneser graph $KG(n,k)$ have?*

Exercise 251. *Show that the chromatic number of the Kneser graph $KG(n,k)$ is at most $n - 2k + 2$.*

Note that the Kneser graph $KG(5,2)$ is the Petersen graph, which is 3-chromatic, so in some cases at least, the bound in Exercise 251 is tight. In fact, in 1955, Kneser [578] conjectured that the chromatic number of $KG(n,k)$ is precisely $n - 2k + 2$. In 1978, Lovász [638] proved Kneser's conjecture using topology, and Bárány [75] gave, in the same volume of the journal, a simpler proof (using a variation of what is now called the Borsuk–Ulam theorem [151] and a theorem by Gale [398] regarding polytopes). In 2002, Greene [442] gave another proof of Kneser's conjecture (although I have not looked at it yet). In 2004, Matoušek [660] gave a (simple) combinatorial proof of Kneser's conjecture.

Lemma 6.2.16. *For graphs G and H on the same vertex set, $\chi(G \cup H) \leq \chi(G)\chi(H)$.*

Proof: Let $\chi(G) = k$ and $\chi(H) = \ell$, and suppose that f is a good k-colouring of G and g is a good ℓ-colouring of H. Define the $k\ell$-colouring α of $G \cup H$ by $\alpha(v) = (f(v), g(v))$. Then α is a proper colouring. $\square$

Exercise 252. *Show that Lemma 6.2.16 applies even when the graphs are not on the same vertex set.*

For the next results, recall that $\overline{G}$ denotes the complement of a graph G.

Exercise 253. *Let G be a bipartite graph on $n \geq 3$ vertices with at least one edge. Prove that $\frac{n}{2} \leq \chi(\overline{G}) \leq n-1$.*

Theorem 6.2.17 (Gaddum–Nordhaus, 1956 [396])**.** *For any graph G,*

$$\chi(G) + \chi(\overline{G}) \leq |V(G)| + 1,$$

and

$$\chi(G)\chi(\overline{G}) \geq |V(G)|.$$

Proof: (The proof of this theorem appears as an exercise in [977, 5.1.41, p. 202].) Only the first inequality is proved here; showing the second inequality is given below as Exercise 254. For each $n \geq 1$, let $S(n)$ be the statement that if G is a graph with $|V(G)| = n$ vertices, then $\chi(G) + \chi(\overline{G}) \leq n+1$.

Base step: When $n = 1$, both G and $\overline{G}$ consist of a single vertex, each with chromatic number 1, so $S(1)$ is true.

Induction step: Let $k \geq 1$, suppose that $S(k)$ is true, and let H be a graph on $|V(H)| = k+1$ vertices. To show that $S(k+1)$ holds, it suffices to show that

$$\chi(H) + \chi(\overline{H}) \leq (k+1) + 1.$$

Select a vertex $x \in V(H)$ and let $G = H - x$, the graph formed by deleting x (and all edges incident with x) from H. Then $|V(G)| = k$, and so by $S(k)$,

$$\chi(G) + \chi(\overline{G}) \leq k+1.$$

Since H is only one vertex larger than G, both $\chi(H) \leq \chi(G) + 1$ and $\chi(\overline{H}) \leq \chi(\overline{G}) + 1$ hold. If either $\chi(H) = \chi(G)$ or $\chi(\overline{H}) = \chi(\overline{G})$ (or both), then

$$\begin{aligned} \chi(H) + \chi(\overline{H}) &\leq \chi(G) + \chi(\overline{G}) + 1 \\ &\leq (k+1) + 1 \qquad\qquad \text{(by } S(k)\text{)}, \end{aligned}$$

confirming $S(k+1)$.

So assume that both $\chi(H) = \chi(G) + 1$ and $\chi(\overline{H}) = \chi(\overline{G}) + 1$. Let $d = \deg_H(x)$ and $\overline{d}$ be the number of edges in $\overline{H}$ between x and $\overline{G}$. Since $|V(G)| = k$, $d + \overline{d} = k$.

Since $\chi(H) = \chi(G) + 1$, then $d \geq \chi(G)$ (since otherwise, colouring of G can be extended to a colouring of H by colouring x with one of the colours not used by neighbours of x). Similarly, $\overline{d} \geq \chi(\overline{G})$. Hence,

$$k = d + \overline{d} \geq \chi(G) + \chi(\overline{G}),$$

and so $\chi(G) + \chi(\overline{G}) \leq k$. Then

$$\chi(H) + \chi(\overline{H}) = \chi(G) + \chi(\overline{G}) + 2 \leq k + 2$$

again confirming $S(k+1)$. This completes the inductive step $S(k) \to S(k+1)$.

By mathematical induction, for all $n \geq 1$, the statement $S(n)$ is true. □

Exercise 254. *Show that $\chi(G)\chi(\overline{G}) \geq |V(G)|$, the second inequality in Theorem 6.2.17.*

In 1969, Stewart [889] showed that the bounds in Theorem 6.2.17 are attainable in that if a, b, and $n = |V(G)|$ are positive integers satisfying $a + b \leq n + 1$ and $ab \geq n$, then there exists an n-vertex graph G with $\chi(G) = a$ and $\chi(\overline{G}) = b$.

Exercise 255. *Let G be a graph on n vertices. Show that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$.*

As an example, the graph C_5 is self-complementary, and so $\chi(C_5)+\chi(\overline{C_5}) = 3+3 = 6$, which is indeed larger than $2\sqrt{5}$.

6.3 Equitable colourings

For any graph G and positive integer k, a k-colouring $f : V(G) \to [k]$ is called *equitable* if and only if f is a proper colouring and the numbers of vertices in any two of the k colour classes differ by at most one.

The study of equitable colourings arises in many applications, including garbage truck routes [932], partitions and elliptic partial differential equations [875], and scheduling processes [112].

For a graph G, Meyer [682] defined the *equitable chromatic number* $\chi_=(G)$ to be the least k for which there is an equitable k-colouring of $V(G)$. For any graph G, by definition, $\chi(G) \leq \chi_=(G)$. Often, the chromatic number and the equitable chromatic number are far apart. For example, for $n \geq 1$, the star $K_{1,n}$ has chromatic number 2, but the equitable chromatic number is much higher.

Exercise 256. *For each $n \geq 2$, find $\chi_=(K_{1,n})$.*

Recall that Lemma 6.2.7 says $\chi(G) \leq \Delta(G) + 1$. The following is a significant strengthening of Lemma 6.2.7 (settling a question that Erdős [300] asked six years earlier).

Theorem 6.3.1 (Hajnal–Szemerédi, 1970 [474])**.** *For any graph G, if $k \geq \Delta(G) + 1$ then there is an equitable k-colouring of $V(G)$.*

Hajnal and Szemerédi proved their result in the following form, from which Theorem 6.3.1 follows directly:

Theorem 6.3.2 (Hajnal–Szemerédi, 1970 [474]). *For positive integers k and ℓ, if G is a graph on $k\ell$ vertices with $\Delta(G) < k$, then G has an equitable k-colouring (with each colour class having ℓ vertices).*

A few special cases of Theorem 6.3.2 were known previously. If either $k = 1$ or $\ell = 1$, then Theorem 6.3.2 holds trivially. It is left as exercises to show that Dirac's theorem (Theorem 2.3.2) shows the case $k \geq 1$ and $\ell = 2$ and that the Corrádi–Hajnal theorem (Theorem 2.1.3) gives the case $k \geq 1$ and $\ell = 3$. According to [474], the cases $k = 2, 3$ and $\ell \geq 1$ were proved by Zelinka [1010], and the case $k = 4$ and $\ell \geq 1$ was shown by Grünbaum [447].

Exercise 257. *Show that Dirac's theorem proves the case $k \geq 1$ and $\ell = 2$ of Theorem 6.3.2.*

As a consequence of either Dirac's theorem (as in the solution to Exercise 257) or Theorem 6.3.1, the following fact is noted:

Corollary 6.3.3. *Let n be an even positive integer. If G is a graph on n vertices with $\delta(G) \geq \frac{n}{2}$, then G contains a perfect matching.*

Exercise 258. *Show that the Corrádi–Hajnal theorem proves the case $k \geq 1$ and $\ell = 3$ of Theorem 6.3.2.*

In a paper published in 2008, Kierstead and Kostochka [569] gave a proof of Theorem 6.3.1 that is simpler than the original; a polynomial time algorithm (on the order $O(|V(G)|^5)$) to construct the colouring was also given.

Recall that Brooks' theorem (Theorem 6.2.10) states that if G is neither a complete graph or an odd cycle, then $\chi(G) \leq \Delta$. In 1973, Meyer conjectured that Theorem 6.3.1 could be strengthened:

Conjecture 6.3.4 (Meyer, 1973 [682]). *If G is a connected graph that is neither complete nor an odd cycle, then $\chi_=(G) \leq \Delta(G)$.*

In 1994, Chen and Lih [194] showed that Conjecture 6.3.4 holds for trees. Chen and Lih also found, for any tree T, the values k for which T is equitably k-colorable (answering a question from Bollobás and Guy). In 1996, Lih and Wu [627] proved that Conjecture 6.3.4 is true for connected bipartite graphs. In 2000, Wang and Zhang [971] showed that Conjecture 6.3.4 holds for line graphs.

In 1973, Meyer [682] claimed (incorrectly) a proof that any tree T is equitably $(\lceil \Delta(T)/2 \rceil + 1)$-colourable.

Also in 1973, Bollobás and Guy [128] showed that any tree T satisfying either $|V(T)| = 3\Delta(G) - 10$ or $|V(T)| \geq 3\Delta(G) - 8$ has an equitable 3-colouring.

In 1975, Guy [466, pp. 998–999] reports that Eggleton fixed the faulty proof of Meyer, showing that for any tree T, $\chi_=(T) \leq \lceil \Delta(T)/2 \rceil + 1$.

Which graphs G have an equitable $\Delta(G)$-colouring? If G is an odd cycle or a complete graph, then $\chi(G) > \Delta(G)$, and so such graphs have no proper $\Delta(G)$-colourings and hence no equitable $\Delta(G)$-colourings.

Exercise 259. *Let $n \geq 2$. Show that the graph $K_{n,n}$ has an equitable n-colouring if and only if n is even.*

So by Exercise 259, $K_{3,3}$ has no equitable 3-colouring.

Exercise 260. *Let G be the disjoint union of two copies of $K_{3,3}$. Find an equitable 3-colouring of G.*

The following, if true, would be a joint strengthening of both Brooks' theorem and the Hajnal–Szemerédi theorem:

Conjecture 6.3.5 (Equitable Δ-colouring conjecture, Chen–Lih–Wu, 1994 [195])**.** *Let G be a connected graph. If for any positive integer n, if $G \notin \{K_n, C_{2n+1}, K_{2n+1,2n+1}\}$, then G has an equitable $\Delta(G)$-colouring.*

Chen, Lih, and Wu also showed that Conjecture 6.3.5 is true when $\Delta(G) \leq 3$. The conjecture has also been shown for a variety of different classes of graphs (*e.g.*, bipartite graphs [196], outerplanar graphs, interval graphs, d-degenerate graphs, and planar graphs with maximum degree at least 9—see, *e.g.*, [599] for some references). In 2005, Kostochka and Nakprasit [599] showed that Conjecture 6.3.5 is true for graphs G for with average degree at most $\Delta(G)/5$. In 2012, Kierstead and Kostochka [570] showed that Conjecture 6.3.5 is true for graphs G with $\Delta(G) \leq 4$. [599]. In the same year, Chen and Yen showed the cases when $\Delta(G) \geq \frac{|V(G)|}{3} + 1$ or $\Delta(G) \leq 3$. (See [570] for more references.)

The result in Exercise 260 is an example of a disconnected graph G with an equitable $\Delta(G)$-colouring. However the condition in Conjecture 6.3.5 that G is connected is necessary by the example given in Exercise 261.

Exercise 261. *Let G be the disjoint union of $K_{3,3}$ and K_3. Show that G has no proper equitable 3-colouring.*

6.4 Graph homomorphisms and colouring

Definition 6.4.1. *A* homomorphism *from a graph G to a graph H is a map $f : V(G) \to V(H)$ so that $\{x, y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(H)$.*

So homomorphisms preserve edges (but not necessarily non-edges, as in isomorphisms). One reason to study graph homomorphisms is that proper colourings of vertices of a graph are homomorphisms into complete graphs. For example, if a graph

G has chromatic number 3, and $f : V(G) \to \{a, b, c\}$ is a proper 3-colouring, then with H being a copy of K_3 on $\{a, b, c\}$, then f is a homomorphism from G to H.

Work on chromatic numbers, in particular, on Hedetniemi's conjecture (see Section 6.5) seemed to be driving research on graph homomorphisms. For much more on such concepts, see work done by Hell, Nešetřil, Sauer, and Xhu, much of which was work done at Simon Fraser University for the last few decades of the 1900s—see references given in the surveys mentioned in Section 6.5.

The collection of all homomorphisms from a graph G to itself is a semigroup.

6.5 Hedetniemi's conjecture

Conjecture 6.5.1 (Hedetniemi, 1966 [506])**.** *For finite simple graphs G and H, $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.*

For surveys on Hedetniemi's conjecture, see [825], [908], or [1013] for results known up to 2008.

Theorem 6.5.2 (El-Zahar–Sauer, 1985 [285])**.** *If $G \times H$ is 3-colourable, then one of G or H is 3-colourable.*

Exercise 262. *Prove that Theorem 6.5.2 shows that Hedetniemi's conjecture holds if either G or H is 4-colourable.*

Hedetniemi's conjecture was proved false in 2019 by Yaroslav Shitov [860].

In 1985, Hajnal [472] gave an example of two infinite graphs that are each not $\aleph_0$-colourable, but their product is. This result was generalized in 2013 to other cardinals [792].

6.6 Critical graphs

During a visit to London in 1949, Erdős was introduced to the following notion by Dirac:

Definition 6.6.1. *For a positive integer k, a graph G is k-critical (or critically k-chromatic, or colour-critical, or sometimes, just critical) if and only if $\chi(G) = k$ but for any proper subgraph $H \subsetneq G$, $\chi(H) < k$.*

For example, any odd cycle is critically 3-chromatic (removing any vertex or edge produces a 2-chromatic graph).

Note: Various authors (*e.g.*, [257, p. 143], [429, p. 224]) say that a graph is critical if the removal of a vertex lowers the chromatic number; the definition above (which

agrees with, *e.g.*, [125, p. 173], [977]) says that a graph is colour critical if the removal of any vertex *or edge* decreases the chromatic number. In any graph without isolated vertices, removal of a vertex also removes an edge. So under the present definition, to prove that a graph with no isolated vertices is critical, it suffices to show that the removal of any edge lowers the chromatic number. For more on graphs that are critical under one of either vertex or edge removal, see Section 6.7.

A proof of the following result is left as an exercise.

Lemma 6.6.2. *For $k \in \mathbb{Z}^+$, every k-chromatic graph contains a k-critical subgraph.*

Exercise 263. *Prove Lemma 6.6.2.*

Exercise 264. *Show that 3-critical graphs are odd cycles.*

Recall from Lemma 6.2.7 that a greedy argument proved that for any graph G, $\chi(G) \leq \Delta(G) + 1$. The same idea proves something stronger for critical graphs.

Exercise 265. *Show that if G is k-critical, then*

$$k \leq 1 + \max_{H \subset G} \delta(H),$$

where the maximum is taken over all induced subgraphs of G.

In 1952, Dirac [261] observed (with no proof given) that a critically chromatic graph has no cut-vertex; this fact is the first case of a more general statement:

Exercise 266. *Show that if G is a critically chromatic graph, then no vertex cut-set induces a clique.*

Exercise 267. *Let G be a critically chromatic graph. Show that for any two vertices x and y, $N(x) \not\subseteq N(y)$. Conclude that no k-critical graph has exactly $k+1$ vertices.*

Exercise 268. *Let $k \geq 1$ and let $G = (V, E)$ be a graph with a partition of its vertices $V = X \cup Y$, where the two induced graphs $G[X]$ and $G[Y]$ are both k-colourable. Show that if the number of edges between X and Y is at most $k-1$, then G is also k-colourable.*

Exercise 269. *Use the result in Exercise 268 to show that any k-critical graph is $(k-1)$-edge-connected.*

6.7 Vertex-critical and edge-critical

Definition 6.7.1. *A graph G is edge-critically k-chromatic (or simply, k-edge-critical) if $\chi(G) = k$ and the deletion of any edge reduces the chromatic number. A k-chromatic graph is k-vertex-critical if the removal of any vertex decreases the chromatic number.*

So, for $k \geq 2$, k-vertex-critical graphs have no isolated vertices.

Remark 6.7.2. *Some authors (even Dirac [262]) use "k-critical" to mean "k-vertex-critical", however sometimes k-critical means "k-edge-critical". To avoid confusion, each type is identified here. For connected graphs, k-edge-critical implies k-vertex-critical; if a graph is simply k-critical, then it is both vertex-critical and edge-critical.*

It is known that there are vertex-critical graphs that are not edge-critical (see, e.g., Jason Brown's 1992 paper [169] for references, including his example of a 5-vertex-critical graph that is not edge-critical. In 1996, Jensen [541] showed that for each $k \geq 5$, and $m \geq 1$, there exists a k-vertex-critical graph whose chromatic number is not decreased even after deleting any m edges all incident with a common vertex.

For $n \geq 2$, K_n is n-edge-critical, and for $t \geq 1$, C_{2t+1} is edge-critically 3-chromatic.

Exercise 270. *If G is a graph with $\chi(G) = k$, and H is a subgraph of G formed by deleting just one edge of G, show that $\chi(H) \geq k - 1$. (In other words, by removing an edge, the chromatic number cannot drop by 2.)*

Recall that that the minimum degree in a graph G is denoted $\delta(G)$.

Theorem 6.7.3 (Dirac, 1952 [262])**.** *If a graph G is k-vertex-critical, then $\delta(G) \geq k-1$.*

Proof: See Exercise 265. □

Erdős then asked Dirac:

Question 6.7.4 (Erdős, 1949 (see [309]))**.** *For each $2 \leq k \leq n$, what is the largest number $f(n;k)$ of edges in a k-edge-critical graph on n vertices?*

Dirac responded quickly with some answers: Dirac [262] noted the "obvious" bounds

$$\frac{n(k-1)}{2} \leq f(n;k) \leq \binom{n}{2},$$

with equality when $n = k$.

Exercise 271. *Why is the lower bound $\frac{n(k-1)}{2} \leq f(n;k)$ obvious?*

Dirac also noted that, by Turán's theorem (see Theorem 10.5.2),

$$f(n,k) < \frac{k-2}{2(k-1)}n^2.$$

Lemma 6.7.5. *Let G be k-vertex-critical and let H be ℓ-vertex-critical, where $V(G) \cap V(H) = \emptyset$. Form the graph $F = G \vee H$ by joining every vertex of G to every vertex of H with new edges. Then F is $(k+\ell)$-vertex critical, and each new joining edge is chromatically critical in F. Thus, if G and H are also both edge-critical, then so is F.*

The following theorem was written under the definition of "critical" as "vertex-critical" (see [262, p. 86], about 3/4 the way down the page). However, after a close look at the constructions used in its proof, and using Lemma 6.7.5, one soon sees that the theorem is also true when "critical" means "edge-critical". Recall that $f(n,k)$ is the maximum number of edges in a k-edge-critical n-vertex graph.

Theorem 6.7.6 (Dirac, 1952 [262]). *For each $k \geq 4$, there exists a_k so that for each n with the same parity as k, there exists a k-edge-critical graph with $a_k n$ edges. For $k \geq 6$, there exists a constant A_k so that for $n \geq 6 \cdot 2^{\lfloor (k-6)/3 \rfloor} + 2$, of the same parity as k, there exists a k-critical graph with at least*

$$\frac{n^2}{4} \sum_{i=0}^{\lfloor (k-6)/3 \rfloor} \frac{1}{4^i} + A_k n$$

edges. In particular, for $n \equiv 2 \pmod 4$, $f(n,6) \geq \frac{n^2}{4} + n$ and when $n \equiv 0 \pmod 4$, $f(n,6) \geq \frac{n^2}{4} + n - 1$.

Proof outline: To see the first statement of the theorem, for $k \geq 4$, let G be the graph formed by joining all vertices of a K_{k-3} to all vertices of an odd cycle C_s; the resulting graph has $n = k-3+s$ vertices, $s(k-2) + \frac{1}{2}(k-3)(k-4)$ edges and is easily seen to be k-edge-critical. Then

$$|E(G)| = n(k-2) - \frac{k(k-3)}{2}.$$

To see the last statement of the theorem, first let $n \equiv 2 \pmod 4$. Form the graph by adding to two disjoint copies of $C_{n/2}$ (an odd cycle) all edges between these copies; this graph is 6-edge-critical, and contains $\frac{n^2}{4} + n$ edges. Now let $n \equiv 0 \pmod 4$. Form the graph from disjoint copies of $C_{(n-2)/2}$ and $C_{(n+2)/2}$ (both odd cycles) by adding all edges between. This graph has $\frac{n-2}{2} + \frac{n+2}{2} + \frac{(n-2)(n+2)}{4} = \frac{n^2}{4} + n - 1$ edges and is 6-edge critical. This proves the last statement of the theorem.

The construction for higher values of k is only outlined. First construct a 9-edge-critcal graph by the join of a 6-edge-critical graph (with the optimal number of edges above) and another cycle with one less vertex. (Note that the resulting graph is edge-critical.) To get a 12-vertex-critical graph, take a 9-critical graph on n (which is odd) vertices and adjoin a cycle with n or $n+2$ vertices. Continuing this process, get a sequence of graphs G_v that are $(6+3v)$-vertex-critically chromatic on the correct number of edges. To any of these last graphs, form a $(6+3v+1)$-vertex-critical graph by adding one more vertex joined to all vertices of a $(6+3v)$-critical graph. Repeat the process similarly to create a $(6+3v+2)$-critical graph. □

According to [213], Erdős asked the following (which I think is still outstanding, but I could not find this question in [309]):

Question 6.7.7 (Erdős). *Is it true that $f(n,6) = \frac{n^2}{4} + n$ for $n \equiv 2 \pmod 4$?*

The problem for $k = 4, 5$ remained open for many years. Both Erdős and Simonovits [865] showed that $f(n;4) < \frac{n^2}{4} + cn$. Toft [923] showed $f(n,4) > \frac{n^2}{16} + cn$. Erdős asked if there is a k-critical graph G with $\delta(G) > cn$ (Dirac's proof for $k = 6$ has such a property).

Independently, Toft [924] and Simonovits [865] found a 4-critical graph with $\delta(G) > cn^{1/3}$. For a while, the only known examples on cn^2 edges contained a $K_{t,t}$ for $t > cn$. Erdős [309] reported that Rödl (unpublished) constructed an example on cn^2 edges where the largest $K_{t,t}$ has $t < c \log n$. Here is Rödl's elegant construction:

Theorem 6.7.8 (Rödl, 1985, 2013 [804]). *For each $k \geq 4$, there are positive constants c and c_1 so that there exists a k-critical graph H on n vertices with cn^2 edges so that for $t = c_1 \log n$, H contains no $K_{t,t}$. In fact, there are $O(n^2)$ such graphs.*

Proof: Let G be a k-edge-critical graph with n vertices and cn^2 edges. (There are various constructions of such graphs.)

Let $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$ be two copies of the vertex set of G, where for each i, v_i and w_i correspond to the same vertex in G. Construct a graph H on $V \cup W$ as follows: Consider a random 2-colouring of edges of G by red and blue such that neither colour contains $K_{t,t}$ with $t = (2 + o(1)) \log n$. (There are many such colourings.) Put the red edges in V and blue edges in W.

For each pair of "twin" vertices v_i and w_i, insert one copy of K_{k-2} joined to both v_i and w_i. (These copies of K_{k-2} are to be pairwise disjoint.) This way, $(k-2)n$ new vertices are added and for each i, two copies of K_{k-1} share a K_{k-2}, one containing v_i and one containing w_i. Call all edges of all such K_{k-1}s green.

The resulting graph H contains no large $K_{t,t}$ (due to the red–blue colouring) and H is k-colourable. Also, H is not $(k-1)$-colourable, since if it were, each of the twins v_i, w_i get the same colour, which would mean that G was $(k-1)$-chromatic.

Since G is k-critical, no red or blue edge can be deleted without dropping the chromatic number; deleting any such edge would result in a graph for which both V and W could be coloured with $k-1$ colours (with twins coloured by same colour). Hence any k-critical subgraph of H contains all red and all blue edges and so has cn^2 edges and no $K_{t,t}$. □

Question 6.7.9 (Erdős, 1985 [309]). *Letting $f(n,k)$ denote the maximum number of edges in k-edge-critical graph on n vertices, what is*

$$\lim_{n\to\infty} \frac{f(n,k)}{n^2} = c_k?$$

I think that this question is (as of 2013) unanswered for $k \geq 4$. Erdős [311, p. 65] offered \$100 for the first c_k, $k \geq 4$.

As of 1985 (see [309, p. 208]) it was not known whether or not there exists a 4-edge-critical graph with c_1n^2 edges that can be made bipartite only by the omission of c_2n edges. According to Erdős [309], in 1984 at a meeting in Kalamazoo, Toft posed the following:

Question 6.7.10 (Toft, 1984)**.** *Is there a 4-chromatic edge-critical graph with c_1n^2 edges that can be made bipartite only by the omission of c_2n^2 edges?*

I am not aware of the present status of either question.

According to Chung [213], Erdős posed the next problems on k-edge-critical graphs and degrees in [309] and [312]; the second paper is the proceedings of the 1988 Kalamazoo conference, published in 1991. This topic did not appear in his 1991 paper [312], and, as far as I could see, these questions were not in the 1985 paper, either. I shall assume that the questions were given in 1988 or earlier.

Define $g(n;k)$ to be the maximum value of $\delta(G)$ so that there exists a k-edge-critical graph G on n vertices with $\delta(G) = g(n;k)$.

Question 6.7.11 (Erdős, 1988?)**.** *What is the magnitude of $g(n,k)$? Is it true that $g(n,k) \geq cn$ for some constant c?*

As mentioned above, the constructions by Simonovits [865] and Toft [924] show $g(n,4) \geq cn^{1/3}$.

6.8 The Mycielski construction

The following construction (due to Jan Mycielski [707]) takes a triangle-free graph and constructs a triangle-free graph with chromatic number one larger. In fact, if G is a triangle-free vertex critical k-chromatic graph, the construction produces a triangle-free vertex critical $(k+1)$-chromatic graph H with $G \subseteq H$.

Construction: [Mycielski, 1955 [707]]] Let G be a graph on $V(G) = X = \{x_1, x_2, \ldots, x_n\}$. Let $Y = \{y_1, y_2, \ldots, y_n\}$ and z be distinct vertices not in X. Define the graph H on vertex set $X \cup Y \cup \{z\}$, where $H[X] = G$, (so $X = V(G)$) and for each $i = 1, \ldots, n$, let y_i be adjacent to every neighbour of x_i and to z.

Theorem 6.8.1 (Mycielski, 1955 [707])**.** *Let G be a graph and let H be the graph given by the above construction. Then $\chi(H) = \chi(G) + 1$. Furthermore, if G is triangle-free, then so is H.*

Proof: Suppose that $\chi(H) = \ell$, and let $h : V(H) \to [\ell]$ be a proper ℓ-colouring. Without loss, suppose that $h(z) = \ell$. Then no vertex of Y receives colour ℓ. If for some i, $h(x_i) = \ell$, then since y_i is adjacent to all neighbours of x_i, a new colouring where x_i receives the same colour as y_i is also a proper colouring. Repeating this for

every x_i with $f(x_i) = \ell$ produce a proper colouring where $V(G)$ is coloured by only $1, \ldots, \ell - 1$. Thus, $\chi(G) < \chi(H)$.

On the other hand, if $\chi(G) = k$ with proper k-colouring f, extend f to a colouring of all of $V(G)$ as follows: for each $i = 1, \ldots, n$, define $f(y_i) = f(x_i)$ and $f(z) = k + 1$. Then f is a proper $(k + 1)$-colouring of $V(G)$. So $\chi(H) \leq \chi(G) + 1$. □

Suppose that G has no triangles and, in hope of a contradiction, suppose that H contains a triangle. This triangle does not use z, and uses at most one y_i, so two of its vertices are in $V(G)$. Let x_i, x_j, y_k be the vertices of a triangle. Then x_k is a neighbour of both x_i and x_j, contrary to G being triangle-free. □

Starting with G as the path on two vertices, the construction first produces C_5, then the Grötzsch graph (see Figure 6.5).

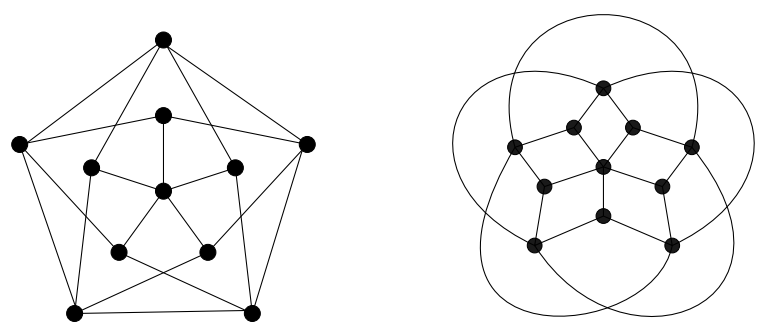


Figure 6.5: Drawings of the Grötzsch graph, 4-chromatic, girth 4, 11 vertices

Exercise 272. *Prove that if a graph G is vertex colour-critical, then the graph produced by the Mycielski construction is also vertex colour-critical.*

Is the result in Exercise 272 still true if "vertex critical" is replaced with "edge critical" (and so with just "critical")?

Chvátal [217] studied the minimality of the Mycielski graphs, and found a 4-regular triangle-free example (with one more vertex than the Grötzsch graph—see Figure 6.6).

Exercise 273. *Prove that Chvátal's graph (see Figure 6.6) is 4-critical.*

It might be interesting to note that Mycielski's construction was not the first construction of a highly-chromatic graph with no triangles. For each $k \geq 2$, In 1954, a collection of mathematicians calling themselves "Blanche Descartes" [253] gave a construction of critical graphs with girth at least six (in response to the question of finding

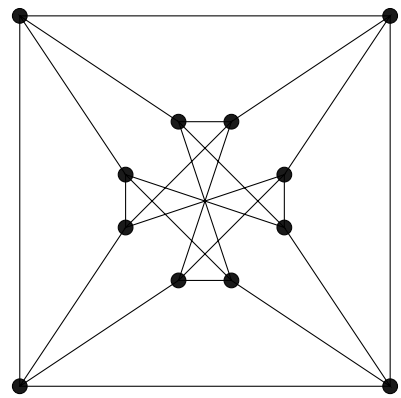


Figure 6.6: Chvátal's graph, 4-critical, 4-regular, girth 4, 12 vertices

triangle-free graphs with arbitrarily high chromatic number). This result is often just attributed to Tutte. Here is their construction:

Let G be a k-critical graph on n vertices with girth at least 6. Let S be a set of nk elements (which are "new" vertices). To each n-subset $T \in [S]^n$, let G_T be a corresponding copy of G, where all such copies of G are vertex disjoint (and disjoint from S). For each such n-subset $T \subseteq S$, add a matching between T and G_T. The set S has no edges added. The new graph (on $nk + \binom{nk}{n}n$ vertices) thereby created has chromatic number at least $k+1$ and girth at least 6. □

Exercise 274. *Show that if f G is Hamiltonian, then the Mycielski graph obtained from G is also Hamiltonian.*

6.9 Large chromatic number and large girth

Using Mycielski's construction (see Section 6.8), a triangle-free graph with arbitrarily large chromatic number can be constructed. That construction shows that the clique number is not the sole force that drives chromatic numbers up.

Intuitively, if a graph has few short cycles, it might be easier to find a proper k-colouring of its vertices, and so if a graph has many short cycles, its chromatic number might be driven up (short cycles might be seen as a hinderance in finding a proper colouring). One might think that since, in a sense, short cycles force the chromatic number up, short cycles are necessary to do so. In fact, quite the opposite is true. According to [327] and [717], the study of such problems began with Tutte and Zykov (see, *e.g.*, [253]) in the 1940's. In 1959, Erdős [292] showed that graphs with both arbitrarily high girth and arbitrarily high chromatic number exist. The Erdős proof, given below, may be seen as one of the highlights of the "probabilistic deletion" method. In 1966, Erdős and A. Hajnal [327] also used probabilistic methods to generalize the

Erdős result to k-uniform hypergraphs. (Also in [327], various versions for infinite graphs are studied, and details of the history of such problems is given.) See also [293] and [303] for other works by Erdős on probability applied to graphs.

The first constructive proof (of graphs with high girth and high chromatic number) was given by Lovász [633]. I am not familiar with Lovász's construction; the construction of Nešetřil and Rödl [717] (given below) is apparently an extension of an idea of Tutte [253]. See also [656] for another construction (and more references).

For Erdős' probabilistic proof alluded to above, only basic facts regarding probability and expectation are used.

Lemma 6.9.1 (Markov's inequality)**.** *Let X be a non-negative random variable. If $t > 0$ then*

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}. \tag{6.4}$$

The following simple lemma follows from Markov's inequality.

Lemma 6.9.2. *For a random variable X with each $X(\omega) \in [0, n]$, if $\mathbb{E}[X] = o(n)$, then as $n \to \infty$, $\mathbb{P}\left[X \geq \frac{n}{2}\right] = o(1)$.*

Theorem 6.9.3 (Erdős, 1959 [292])**.** *For all $k \geq 2$ and $\ell \geq 3$, there exists a graph G with girth$(G) > \ell$ and $\chi(G) > k$.*

Proof: The proof given here follows [36, pp. 41–42]. Let $k \geq 2$ and $\ell \geq 3$. Let $\theta \in \mathbb{R}$ satisfy $0 < \theta < \frac{1}{\ell}$. Putting $p = n^{\theta-1}$, let $G \in G(n, p)$ be a random graph on n vertices, where each pair of vertices is chosen to be an edge randomly and independently with probability p. Let $X = X(G)$ be the random variable counting the number of cycles in G with length of at most ℓ.

For any set S of $i \geq 3$ vertices, the number of possible cycles on S is $\frac{i!}{2i}$, where division by $2i$ is necessary because there are i possible starting points for a C_i, and two directions to proceed from any starting point. Since each possible C_i appears with probability p^i,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=3}^{\ell} \binom{n}{i} \frac{i!}{2i} p^i \\
&< \sum_{i=3}^{\ell} \frac{n^i}{2i} p^i \\
&= \sum_{i=3}^{\ell} \frac{n^{i\theta}}{2i} \\
&< \frac{\ell}{6} n^{\ell\theta} \\
&= o(n) && \text{(since } \theta\ell < 1\text{).}
\end{aligned}$$

By Lemma 6.9.2 (applied with a much larger n)

$$\mathbb{P}[X \geq n/2] = o(1). \tag{6.5}$$

To bound the chromatic number, examine independent sets. Setting $x = \lceil (3/p) \ln n \rceil$, and using bounds $\binom{n}{x} < n^x$ and $1 - p < e^{-p}$,

$$\mathbb{P}[\alpha(G) \geq x] \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left(ne^{-p(x-1)/2}\right)^x = o(1). \tag{6.6}$$

Let n be so large that the $o(1)$ terms in each of equations (6.5) and (6.6) are less than $1/2$. Then there is a graph G for which both $X(G) < n/2$ and $\alpha(G) < x$, that is, G has less than $n/2$ cycles of length at most ℓ, and $\alpha < 3n^{1-\theta} \ln(n)$.

Create the graph G^* from G by removing one vertex from each cycle of length at most ℓ. Then $\text{girth}(G^*) > \ell$ and $|V(G^*)| \geq n/2$. Also, $\alpha(G^*) \leq \alpha(G)$, and so by the result in Exercise 237,

$$\chi(G^*) \geq \frac{|V(G^*)|}{\alpha(G^*)} \geq \frac{n/2}{3n^{1-\theta}\ln(n)} = \frac{n^\theta}{6\ln(n)}.$$

Letting n be so large that this last expression exceeds k completes the proof. □

The next proof of the existence of highly-chromatic large-girth graphs is a constructive method called *partite amalgamation*, a technique used by Nešetřil and Rödl to settle a number of questions. For a survey of this technique, see [715]. The general partite amalgamation technique is a process of 'gluing' partite graphs together but only at certain parts or coordinates; the process used here consists of gluing only at a single part.

In the following theorem, the variables have been switched around from Theorem 6.9.3. One may also note that the following construction of a high-chromatic high-girth graph, because of the inductive argument, uses partite hypergraphs (see comments after Definition 1.2.1 for what a partite hypergraph is). . (Lovász's construction also uses hypergraphs; a construction using only graphs is not known.) For present purposes, a k-uniform hypergraph is a pair $H = (V, E)$, where V is a non-empty set and $E \subseteq [V]^k$ (when $k = 2$, this is just a graph). Such a graph is called a-partite if and only if there is a partition of V into sets $X_1 \cup \cdots \cup X_a$ so that each hyperedge intersects each X_i in at most one vertex.

Theorem 6.9.4 (Erdős–Hajnal, 1966 [327]). *For positive integers $k \geq 2$, $n \geq 3$, and $g \geq 1$ there exists a k-uniform hypergraph G so that girth$(G) > g$ and $\chi(G) > n$.*

Proof: The original proof duplicated the Erdős 1959 probabilistic proof for graphs. The constructive proof given here is by Nešetřil and Rödl [717].

The proof uses a special type of "amalgamation" or "gluing together" of hypergraphs (perhaps many at once). Let $a \geq 2$ and let $G = ((V_i)_{i\in[a]}, E)$ be an a-partite graph. For some $r \in [a]$ and $\ell \geq 2$, let $|V_r| = \ell$. For any ℓ-uniform hypergraph $H = (X, D)$ define $H *_r G = ((V_i^{'})_{i\in[a]}, E')$, an a-partite k-uniform hypergraph as follows.

For $i \neq r$, set $V_i^{'} = V_i \times D$ and set $V_r^{'} = X$. For each $d \in D$ fix an injection $\psi_d : \cup_{i\in[a]} V_i \to \cup_{i\in[a]} V_i^{'}$ taking V_r to $d \subset V_r^{'}$ and for $i \neq r$, $\psi_d(V_i) = \{(v, d) : v \in V_i\}$. Define

$$E' = \{\{\psi_d(v_1), \ldots, \psi_d(v_k)\} : \{v_1, \ldots, v_k\} \in E, d \in D\}.$$

An edge $e \in E'$ is denoted $\psi_d(e)$ for some $e \in E$ and $d \in D$. So $H *_r G$ is formed by taking $|D|$ copies of G and identifying the copies of V_r with edges of H. Copies of G are said to be *amalgamated* together along the r-th part, using H as a 'template' for the new r-th part.

The theorem is proved by induction on g. For $g = 1$, observe that any loopless hypergraph G satisfies $\text{girth}(G) \geq 2$, and for each k, trivial examples of k-graphs exist with $\chi(G) \geq n$. So fix $g \geq 2$ and suppose that the theorem holds for every q satisfying $1 \leq q < g$ and for all edge sizes k'. Put $a = (k-1)n + 1$ (note that $a \longrightarrow (k)_n^1$), and let

$$G^0 = ((V_i^0)_{i\in[a]}, E^0)$$

be such an a-partite k-uniform hypergraph so that for every set $A \in [a]^k$ there is an edge $e \in E^0$ with $e \cap V_i^{'} \neq \emptyset$ for every $i \in A$ and has $\text{girth}(G^0) > g$. (One can take G^0 to be a collection of disjoint k-edges.)

Define inductively a-partite k-graphs $G^j = ((V_i^j)_{i\in[a]}, E^j)$, $1 \leq j \leq a$, as follows. Having defined $G^m = ((V_i^m)_{i\in[a]}, E^m)$, $(m < a)$, put $|V_m^m| = l_m$ and let $H^m = (X^m, D^m)$ be an l_m-graph with $\text{girth}(H^m) \geq g$ and $\chi(H^m) > n$. (Such a hypergraph exists by induction hypothesis using a different value for k.) Put

$$G^{m+1} = H^m *_m G^m = ((V_i^{m+1})_{i\in[a]}, E^{m+1}).$$

Claim: The graph $G^a = ((V_i^a)_{i\in[a]}, E^a)$ satisfies the theorem.

Proof of claim: For each $j \leq a$, $\text{girth}(G^j) > g$, induct on j: suppose that $\text{girth}(G^j) > g$ for a fixed j. In E^{j+1}, pick a sequence of vertices

$$C = \{\psi_{d_0}(v_0), \ldots, \psi_{d_{q-1}}(v_{q-1})\}$$

of minimal length q that determine a cycle. If all the d_i, $i \in q$ are equal, by the induction hypothesis there are no small cycles in a copy of G^j and so $q > g$ would hold as desired. So suppose that not all the d_i are equal. Then in this case, the only way that C can be a cycle is if it uses vertices from V_j^{j+1}, the j-th part of G^{j+1}. Now use the fact that, by induction hypothesis, $\text{girth}(H^j) \geq g$ and conclude that $q > g$ (in fact, q would in this case be at least $2g$) as desired, ending the proof of the claim.

To see that $\chi(G^a) > n$, let a colouring $\Delta : V(G^a) \longrightarrow n$ be given. The restriction of Δ to X_a^a, the last part of G^a, imposes a colouring on X^{a-1}, the vertices of H^{a-1}.

By the inductive hypothesis, $\chi(H_{a-1}) > n$, and so there exists a monochromatic edge $d_{a-1} \in D_{a-1}$. Setting $Z_{a-1} = d_{a-1} \in [X^a_{a-1}]^{l_{a-1}}$ to be the last part of a new graph $F^{a-1} \preceq G^{a-1}$, i.e.,

$$F^{a-1} = ((X^{a-1}_i)_{i\in a-1}, Z_{a-1}, E^{a-1} \cap [\cup_{i\in a-1} X^{a-1}_i \cup Z_{a-1}]^k),$$

look only at how Δ colours $V(F^{a-1})$. By design, Δ is constant on Z_{a-1}. Repeat in this manner, using the vertices of a monochromatic edge of H^{a-2} as a new part, a subset of X^{a-2}_{a-1}, create F^{a-2}, Δ being constant on the second last part thereof. Continuing, get $F^0 = ((Z^0_i)_{i\in a}, E(F^0))$, a copy of G^0, the vertices of which have colours depending only on the part whence they came. By the choice of a, there exist k parts all coloured the same. By the design of G^0, there exists a k-edge determined by those parts, guaranteed now to be monochromatic. □

It might be noted that Nesetril and Rödl used a generalized version of the above amalgamation technique to prove some rather major results in Ramsey theory.

6.10 Chromatic polynomials

For a graph G and $k \geq \chi(G)$, how does one count the number of good k-colourings of $V(G)$? (Consider $V(G) = \{v_1, \ldots, v_n\}$ to be labelled, so two good k-colourings $f, g : V(G) \to [k]$ are the same if and only if for each i, $f(v_i) = g(v_i)$.) This number is denoted differently in different texts: in [429] this number is denoted by $c_k(G)$; in [125] it is denoted by $p_G(k)$; in [256] it is denoted by $P_G(k)$; in [190] and [422] it is denoted by $P(G, k)$. Below, it is shown that this number is indeed a polynomial in k, so it might make sense to stick with the notation $p_G(k)$. In 1912, these "chromatic polynomials" were developed by George David Birkhoff (1884–1944), the father of Garrett Birkhoff) with planar graphs and the four colour conjecture in mind [107]. Whitney [982, 983] continued the study of chromatic polynomials for graphs in general (Birkhoff was Whitney's PhD supervisor). For more on chromatic polynomials, see [103], [422], [777], or [778].

Two chromatic polynomials are easy to find: $p_{K_n}(k) = k(k-1)\cdots(k-n+1)$ and $p_{\overline{K_n}}(k) = k^n$.

As defined in Section 1.7.5, for a graph G and edge $e \in E(G)$, let the graph $G - e$ denote the graph formed by deleting e (and not its endpoints), and let G/e be the graph formed by contracting e to a single vertex and then removing any loops or multi-edges thereby formed.

Lemma 6.10.1. *Let G be a graph and let $e \in E(G)$,*

$$p_{G-e}(k) = p_G(k) + p_{G/e}(k). \tag{6.7}$$

Proof: Let $e = \{x, y\}$. A proper k-colouring of $G - e$ for which x and y are coloured the same gives a proper k-colouring of G/e; a proper k-colouring of $G - e$ for which x and y are coloured differently is also a proper k-colouring of G. □

Rewriting equation (6.7) gives a form that can be used recursively each time $e \in E(G)$:

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k). \tag{6.8}$$

So to find the chromatic polynomial of a graph, use the formula (6.8) recursively, using smaller and smaller graphs until small complete graphs are used (or other small graphs whose polynomial is obvious).

The result in Lemma 6.10.1 can be turned around by replacing $G - e$ with H, and so $H + e = G$. Then $e = \{u, v\}$ is not an edge of H, and $(H + e)/e$ is the graph formed by identifying vertices u and v in G. So equation (6.7) becomes, whenever $e = \{u, v\}$ is not an edge in H,

$$p_H(k) = p_{H+e}(k) + p_{H+e)/e}(k). \tag{6.9}$$

One advantage of using the "add edge, then contract" equation (6.9) is that if the graph has a subgraph that is nearly complete, consecutive applications produces chromatic polynomials of complete graphs fairly quickly. The first method of delete then contract instead reduces everything to a rather large collection of polynomials for small graphs, and so calculations are relatively simple for graphs with few edges.

Theorem 6.10.2 (Birkhoff, 1912 [107]). *For $n = |V(G)|$, the chromatic polynomial $p_G(k)$ has degree n, the coefficient of k^n is 1, the coefficient of k^{n-1} is $-|E(G)|$, and the constant term in $p_G(k)$ is 0. Furthermore, the coefficients alternate in sign (where 0 counts as either sign).*

Proof: The proof is by induction on $|E(G)|$.

Base step: As noted above, $p_{\overline{K_n}}(k) = k^n$, so the theorem is true for graphs with 0 edges.

Inductive step: Let $m \geq 1$ and suppose that the theorem is true for all graphs with $m - 1$ edges. Let G have m edges and let $e \in E(G)$. Then both $G - e$ and G/e have $m - 1$ edges, and so the induction hypothesis applies to both, giving some non-negative $a_1, \ldots, a_{n-1}$, and $b_1, \ldots, b_{n-2}$, with

$$p_{G-e}(k) = k^n + \sum_{i=1}^{n-1} (-1)^{n-i} a_i k^i.$$

and

$$p_{G/e}(k) = k^{n-1} + \sum_{i=1}^{n-2} (-1)^{n-i-1} b_i k^i.$$

By Lemma 6.10.1,

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k),$$

and so

$$p_G(k) = k^n - (a_{n-1} + 1)k^{n-1} + \sum_{i=1}^{n-2}(-1)^{n-i}(a_i + b_i)k^i.$$

These coefficients satisfy the conclusion of the statement, finishing the inductive step.

By MI, the conclusion holds for any number of edges in G. □

Exercise 275. *Let P_2 denote the path of length 2 (with three vertices). Use Lemma 6.10.1 to show that*

$$p_{P_2}(k) = k^3 - 2k^2 + k.$$

Lemma 6.10.3. *In $p_G(k)$, the coefficient of k is non-zero if and only if G is connected.*

When k-colouring a path, starting at an end point, there are k choices for a colour, and for each subsequent vertex along the path, there are $k-1$ choices. Thus:

Lemma 6.10.4. *If a path has n vertices, there are $k(k-1)^{n-1}$ possible proper k-colourings.*

The same argument used for paths works for trees in general:

Exercise 276. *Show that the chromatic polynomial of a tree on $n \geq 2$ vertices is $k(k-1)^{n-1}$.*

Exercise 277. *Show that for $n \geq 3$, the chromatic polynomial of the cycle C_n is*

$$p_{C_n}(k) = (k-1)^n + (-1)^n(k-1).$$

Exercise 278. *Verify the result in Exercise 277 for C_5 by finding the 30 proper 3-colourings of C_5.*

Exercise 279. *Let G be the graph on four vertices and four edges formed by adding a pendant edge to a triangle. Find the chromatic polynomial $p_G(k)$. Verify that there are indeed $p_G(3)$ proper 3-colourings of $V(G)$.*

Exercise 280. *Find $p_{K_{2,2,2}}(k)$, and give the value for $k = 3$. Show directly that this value is correct.*

Using Lemma 6.10.1 recursively, the chromatic polynomial of any graph can be computed. For example, the interested reader can confirm that the chromatic polynomial of the Petersen graph is

$$k(k-1)(k-2)(k^7 - 12k^6 + 67k^5 - 230k^4 + 529k^3 - 814k^2 + 775k - 352).$$

In 1930, Birkhoff [108] showed that for planar graphs on n vertices with chromatic number $k \geq 5$,

$$p_G(k) \geq k(k-1)(k-2)(k-3)^{n-3}. \tag{6.10}$$

In 1946, Birkhoff and Lewis [109] conjectured that (6.10) also holds for $k = 4$, which is now known to be true by the Four Colour Theorem for planar graphs (since from any proper 4-colouring, permuting colours alone generates 4! good colourings).

Conjecture 6.10.5 (Tomescu, 1971 [928]). *For $k \geq 4$, connected graphs on n vertices with chromatic number k,*

$$p_G(k) \leq k!(k-1)^{n-k}. \tag{6.11}$$

Observe that Tomescu's conjecture fails with $k = 3$ on odd cycles. For example, in Exercise 277, it was shown that $p_3(C_5) = 30$; however, the right side of (6.11) is 24. Similarly, any graph formed by attaching trees to C_5 also violate (6.11).

Exercise 281. *Show that the Tomescu bound (6.11) is true if G contains a copy of K_k.*

In 2017, Knox and Mohar [580] proved that Tomescu's conjecture holds for $k = 4$; in fact, they proved a little more:

Theorem 6.10.6 (Knox–Mohar, 2017 [580]). *Let G be a graph on n vertices with $\chi(G) \geq 4$. Then for every k,*

$$p_G(k) \leq k(k-1)^{n-3}(k-2)(k-3),$$

where equality holds only when G is formed by starting with K_4 and adding leaves.

According to the 2017 paper [580], Knox and Mohar, together with help from Azaruija, had a paper in preparation that shows Tomescu's conjecture is also true for $k = 5$. A preprint of that paper (later in 2017) can be found at [581].

In 2019, Fox, He, and Manners [376] proved Tomescu's conjecture for all $k \geq 4$. (In the Fox et al. paper, [376], the authors acknowledged that Mohar et al. had also, independently, found a proof.)

Together the papers [580] and [376] provide many references for developments surrounding chromatic polynomials, including applications (including physics and graph limits) and algorithmic aspects.

A polynomial in two variables, called the "Tutte polynomial" generalizes the chromatic polynomial and others, and from the Tutte polynomial, many other graph parameters can be read off. For more information on the Tutte polynomial, see, *e.g.*, [422] or [643].

There are other polynomials associated with a graph (for example, the "Jones polynomial" for knots, or the "flow polynomial") not examined in this text.

6.11 Edge colouring

A colouring of the edges of a graph is called an edge colouring. When k colours are used, an edge colouring is called an edge k-colouring or sometimes a k-edge-colouring. As with vertex colouring, the colours used are often $[k] = \{1, 2, \ldots, k\}$, and so a k-edge-colouring is often denoted by a function $c : E(G) \to [k]$.

A k-edge-colouring $c : E(G) \to [k]$ is called *proper* if and only if for any two incident edges e and f, $c(e) \neq c(f)$. In other words, a proper edge colouring gives different colours to any two edges that contain the same vertex.

Definition 6.11.1. *For any graph (or multigraph), define the* edge-chromatic number $\chi'(G)$ *(or* chromatic index*) to be the minimum number k of colours so that there exists a proper k-edge-colouring of G.*

Recall from Definition 1.7.6 that for any simple graph G, its *line graph* $L(G)$ is the graph whose vertices are $E(G)$, and two vertices in $L(G)$ are adjacent if and only if the two associated edges in G are incident. It follows that $\chi(L(G)) = \chi'(G)$.

In any graph G, since there exists a vertex with $\Delta(G)$ edges incident,

$$\chi'(G) \geq \Delta(G). \tag{6.12}$$

In some cases, equality in (6.12) holds:

Theorem 6.11.2 (König, 1916 [592, 593])**.** *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof: Let G be bipartite. By Lemma 5.3.8 there is a matching that uses all vertices of maximum degree. Delete this matching to produce a graph G_1, and then apply Lemma 5.3.8 to those vertices of maximum degree in G_1. Delete this second matching to produce G_2. Continue this process until the graph is empty, and the Δ matchings thereby found are the Δ colour classes desired. □

For another proof of Theorem 6.11.2, see [643, pp. 37–38]; the idea is simple: embed the graph G into a regular bipartite graph of degree $\Delta(G)$, and then (as above) use the perfect matchings obtained from Theorem 5.3.6 to give an edge-colouring.

An immediate consequence of Theorem 6.11.2 also follows by repeating the corollary to Hall's theorem that says that any k-regular bipartite graph has a 1-factor.

Corollary 6.11.3 (König)**.** *Let G be bipartite and k-regular. The edges of G can be decomposed into k 1-factors.*

Proof: By Theorem 6.11.2, $\chi'(G) = k$, and in any proper k-colouring of $E(G)$, each colour class is a perfect matching. □

König is famous for a number of theorems (*e.g.*, see Theorem 5.6.1, the König–Egeváry theorem), so some authors refer to Theorem 6.11.2 as König's "line colouring theorem". See Theorem 6.12.6 for an application of Theorem 6.11.2 regarding "perfect graphs".

Exercise 282. *Find the edge-chromatic number (chromatic index) of the graphs, K_4, K_5, C_5, $K_{3,7}$, and the graph of the octahedron (see Figure 1.20).*

The result in the next exercise is related to Corollary 2.3.17.

Exercise 283. *Suppose that G is a cubic graph (3-regular) with, up to permutation, only one $\chi'(G)$-edge-colouring. Show that $\chi'(G) \leq 3$ and G has precisely three Hamiltonian cycles.*

It might seem incredible that the edge-chromatic number is always one of only two possible values; this result was first proved by Vizing, and independently, two years later by Gupta, and is now known as "Vizing's theorem":

Theorem 6.11.4 (Vizing, 1964 [957]; Gupta, 1966 [459])**.** *For any simple graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Proof: The proof given here is by induction on the number of edges in a graph. For each $m \geq 0$, let $S(m)$ be the statement that for any graph G with m edges, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Base step: When $m = 0$, there are no edges to colour, and so $\chi'(G) = 0$, and so $S(0)$ holds. When $m = 1$, only one edge needs a colour, so $S(1)$ is also true.

Inductive step: Let $t > 1$ and for all $m < t$, suppose that $S(m)$ holds. Let G be a graph with t edges. Without loss of generality, suppose that G is connected (if G has isolated vertices, they have no effect on $\chi'(G)$, and if G has more than one non-trivial component, apply the inductive hypothesis to each component).

Put $\Delta = \Delta(G)$. (For convenience, let any edge $\{x, y\}$ in G be written xy.) Let $e = uv_1$ be an edge in G, and, by $S(|E(G)| - 1)$, suppose that all edges of G except e are properly $(\Delta + 1)$-coloured. For any vertex $v \in V(G)$, the edges incident with v receive at most Δ colours, so there is at least one colour c not used at v; in this case, say that v misses colour c.

Let b be a colour missing at u and let c_1 be a colour missing at v_1. If $b = c_1$, then $e = uv_1$ can be recoloured $b = c_1$, thereby giving a proper edge-colouring of G, so suppose that $c_1 \neq b$.

Construct a sequence $v_1, v_2, \ldots, v_k$ of (distinct) neighbours of u and a sequence $c_1, c_2, \ldots, c_{k-1}$ of distinct colours as follows: If $y_1, \ldots, y_i$ and $c_1, \ldots, c_i$ have been constructed, pick (if possible) $v_{i+1} \notin \{v_1, \ldots, v_i\}$ so that uv_{i+1} is coloured c_i, and let c_{i+1} be missing at v_{i+1}.

The sequence $y_1, y_2, ...$ stops (because, *e.g.*, x has at most Δ neighbours). Suppose that the sequence stops at stage k and c_k is missing at v_k. Then either (1) there is no $v_{k+1} \notin \{v_1, \ldots, v_k\}$ so that uv_{k+1} is coloured c_k, or (2) the colour c_k has already appeared, for some $j < k$ as c_j.

Case (1): Suppose that there is no uv_{k+1} with colour c_k. Then since v_k is missing colour c_k, let uv_k be coloured c_k, let vw_{k-1} be re-coloured c_{k-1}, and so on, shifting all colours up so that, finally, c_1 becomes available to colour uv_1.

Case (2): Let $j \leq k$ be so that the colour c_k missing at v_k is c_j (the colour of the edge uv_{j+1}). To begin, colour e with c_1 and recolour the edges $uv_2, \ldots, uv_j$ so that for each $i \leq j$, uv_i receives colour c_i, and leave uv_{j+1} uncoloured. Let H be the graph formed by all edges of colour b (missing at u) or c_j. Then $\Delta(H) \leq 2$, and so components of H are either paths or (even) cycles. Since $\deg_H(u) \leq 1$, $\deg_H(u_j) = 1$ and $\deg_H(u_k) \leq 1$, the three vertices u, v_j, v_k can not all be in the same component of H.

If u and v_j are in different components, continue the recolouring so that for each $i < k$, uv_j is coloured c_j, leaving uv_k uncoloured. Interchange colours in the component containing v_j; then b can be used to colour uv_k.

If u and v_k are in different components, continue shifting colours so that for every $i < k$, uv_i receives colour c_i, leaving uv_k uncoloured. Then reverse the colouring in the component of H containing v_k. Then b is missed at v_k (as well as at u), so uv_k can be coloured b. □

A graph G is said to be of Class 1 if and only if $\chi'(G) = \Delta(G)$ and is of Class 2 if and only if $\chi'(G) = \Delta(G) + 1$. For example, with $G = K_3$, $\chi'(K_3) = 3$ and $\Delta(K_3) = 2$, and so K_3 is of Class 2.

Exercise 284. *Show that for each $k \in \mathbb{Z}^+$, the odd cycle C_{2k+1} is of Class 2.*

Exercise 285. *Let H be the graph on five vertices formed by adding a pendant edge to K_4. Decide which class H is.*

Exercise 286. *Show that the Petersen graph is of Class 2.*

Note that not every bridgeless cubic graph is 3-edge-colourable, with the Petersen graph as an example. In fact, Tutte (see [940, Ch. 8]) showed that the Petersen graph is the smallest such example. By Vizing's theorem (Theorem 6.11.4), all cubic graphs are 4-edge-colourable (bridgeless or not).

Exercise 287. *Let G be a cubic (i.e., 3-regular) graph with a Hamiltonian cycle. Show that G is of Class 1, i.e., show that $\chi'(G) = 3$.*

A cubic bridgeless graph with edge-chromatic number 4 is called a *snark*. It is known [538] that there are an infinite number of snarks. A conjecture of Tutte says that a snark contains the Petersen graph as a minor (see Definition 1.7.7).

Note that K_3 is of Class 2, yet K_4 is of Class 1. Is there a pattern? Before answering that question, a few elementary observations are made.

Recall that $\nu(G)$ is the maximum size of a matching in G. In any proper edge-colouring of G, any colour class is an independent set and so has most ν edges. Since all such colour classes make up $E(G)$, it follows that

$$\chi'(G) \geq \left\lceil \frac{|E(G)|}{\nu} \right\rceil. \tag{6.13}$$

If equality holds in (6.13), then in any proper edge colouring, each colour class has exactly ν edges.

For $n \geq 2$, $\Delta(K_n) = n - 1$, and so by Vizing's theorem (Theorem 6.11.4), either $\chi'(K_n) = n - 1$ or $\chi'(K_n) = n$. When n is even, $\nu(K_n) = n/2$, and so by equation (6.13),

$$\chi'(K_n) \geq \frac{n(n-1)/2}{n/2} = n - 1,$$

which Vizing's theorem already says. However, when n is odd, $\nu(K_n) = (n-1)/2$, and so equation (6.13) gives

$$\chi'(K_n) \geq \frac{n(n-1)/2}{(n-1)/2} = n,$$

and so by Vizing's theorem, $\chi'(K_n) = n$. So if n is odd, K_n is of Class 2. Is it true that if n is even, then K_n is of Class 1?

A *1-factorization* of a graph G is a decomposition of $E(G)$ into edge-disjoint 1-factors. A k-regular graph G is of Class 1 if and only if G has a 1-factorization, (in which case G has an even number of vertices). So, for example, K_5 is 4-regular, but has an odd number of vertices, so $\chi'(K_5) = \Delta(K_5) + 1 = 5$.

Theorem 6.11.5. *For $n \geq 2$, K_n is of Class 1 if and only if n is even.*

Proof: From the above discussion, if K_n is Class 1, then n is even. So let n be even; it remains to show that $\chi'(K_n) = n - 1$. There are different ways to show this. Perhaps the easiest way is to give a proper edge-colouring of K_n with $n - 1$ colours, that is, to give a 1-factorization of K_n. One such 1-factorization is given in Lemma 8.2.1. □

Exercise 288. *Let $1 \leq a \leq b$. Show that $\chi'(K_{a,b}) = b$. Hint: If $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ are the vertices, colour the edge $\{x_i, y_j\}$ with colour $j - i + 1 \pmod b$.*

Exercise 289. *Using Brooks' theorem (and not Vizing's theorem), show that if G is a (simple) graph with $\Delta(G) = 3$, then $\chi'(G) \leq 4$.*

For the next result, recall (see Section 1.6.5) that Q_k is the graph of the k-dimensional cube, whose vertices are the binary words of length k, and two such words form an edge if and only if they differ in precisely one coordinate.

Lemma 6.11.6. *For each positive integer k, $\chi'(Q_k) = k$, and so Q_k is of Class 1.*

Exercise 290. *Prove Lemma 6.11.6 by exhibiting an appropriate edge-colouring of Q_k.*

Exercise 291. *Show that for k odd, if a k-regular graph G has a cut-vertex, then $\chi'(G) = k + 1$ and so G is of Class 2.*

It is an NP-complete problem to decide if a graph is of Class 1 or 2 (see [584, p. 262] for more details). Compared to Class 1 graphs, Class 2 graphs are rare. In 1973, Beineke and Wilson [83] showed that among the 143 connected graphs on at most six vertices, only eight are of Class 2. In 1977, Erdős and Wilson [347] showed that as $n \to \infty$, the probability that a graph on n vertices is of Class goes to 0. The Erdős–Wilson result was refiined:

Theorem 6.11.7 (Frieze–Jackson–McDiarmid–Reed, 1988 [383])**.** *If p_n is the probability that a graph on n vertices is of Class 2 then as $n \geq \infty$,*

$$n^{-(\frac{1}{2}+o(1))n} \leq p_n \leq n^{-(\frac{1}{8}+o(1))n}.$$

Vizing [958] showed that planar graphs with maximum degree 10 are of Class 1, and then showed [959] that planar graphs with maximum degree 8 are of Class 1 (the first of these two results is fairly easy to prove; both proofs are given in [363]). In 2001, Sanders and Zhao [821] showed that all planar graphs with maximum degree 7 are of Class 1.

Adding edges to a graph can never decrease its edge-chromatic number, so intuition says that a graph with more edges has a better chance of being of Class 2. For the next result, recall that $\nu(G)$ is the matching number of G, the size of the largest set of independent edges.

Theorem 6.11.8 (Beineke–Wilson, 1973 [83])**.** *Let G be a graph. If*

$$|E(G)| > \Delta(G)\nu(G), \tag{6.14}$$

then G is of Class 2.

Proof: Let $k \in \{\Delta(G), \Delta(G) + 1\}$. For any proper edge k-colouring of G, each colour class is an independent set of edges and so contains at most ν edges. Thus $|E(G)| \leq k\nu$. If $k = \Delta(G)$, then equation (6.14) is violated, so $k = \Delta(G) + 1$. □

Exercise 292. *Let G be a regular graph on an even number of vertices. Show that If H is obtained from G by subdividing any edge of G, then H is of Class 2.*

Exercise 293. *Show that if G is a graph obtained from an odd cycle C_{2k+1} by adding at most $k-2$ independent edges, then G is of Class 2.*

Exercise 294. *Prove that an outerplanar (see Section 7.10) graph G has chromatic index 3 if and only if G is an odd cycle.*

A version of Vizing's theorem also holds for multigraphs. Note that the degree of a vertex in a multigraph is the number of edges incident with that vertex (where loops count as 2 degrees).

Theorem 6.11.9 (Vizing, 1965 [958])**.** *For a multigraph G without loops and with maximum multiplicity μ, then $\chi'(G) \leq \Delta(G) + \mu$.*

When $\mu = 1$, Theorem 6.11.9 reduces to Vizing's theorem (Theorem 6.11.4). Another theorem about the chromatic index for multigraphs was known long before Vizing's theorem:

Theorem 6.11.10 (Shannon, 1949 [855])**.** *If G is a multigraph without loops, then $\chi'(G) \leq \frac{3}{2}\Delta(G)$*

See [584, p. 263] for a proof by Ore [730]; see [429, p. 238] for a proof (essentially by Shannon), but using Theorem 6.11.9.

As with vertex-colouring, in edge-colouring, there is a notion of "critical".

Definition 6.11.11. *If G is of Class 2 and has chromatic index $k+1 = \Delta(G)+1$, if the removal of any edge lowers the chromatic number, then say that G is critically edge-chromatic. In this case, say that G is k-critical, or simply* critical.

Exercise 295. *The definition of "critical" in Definition 6.11.11 is only for class 2 graphs. Why are not class 1 graphs included in this definition?*

For example, any odd cycle C_{2k+1} is (with respect to edge colourings) 2-critical. Also, since the removal of any edge from K_5 gives a 4-critical graph, but K_5 itself is not critical.

Exercise 296. *Show that under edge-colourings, the Petersen graph is not critical. Does the removal of any vertex reduce the chromatic index?*

Without going into too many details, some results for critical graphs for edge-colourings are listed below; see [363] for these results and many more.

Theorem 6.11.12 (Vizing)**.** *Let G have maximum degree k and $\chi'(G) = k+1$. If G is a k-critical graph (with respect to edge-colourings—see Definition 6.11.11) and u and v are adjacent vertices in G, then* $\deg(u) + \deg(v) \geq k+2$.

Theorem 6.11.13 (Vizing). *A critical graph (with respect to edge-colourings, see Definition 6.11.11) has no cut-vertex.*

Theorem 6.11.14 (Vizing, 1965 [958]). *Let G have maximum degree $\Delta(G) = k$ and let G be of Class 2 (so $\chi'(G) = k + 1$). Then for each $j \in \{1, 2, \ldots, k\}$, G contains a j-critical subgraph.*

Exercise 297. *Let G be a critical graph with respect to edge colouring. If $I \subset E(G)$ is a set of independent edges, show that there exists a $(\Delta(G) + 1)$-colouring of the edges so that I is a colour class; furthermore, show that $\chi'(G - I) = \chi'(G) - 1$.*

Theorem 6.11.15 (See [363]). *For $k \geq 3$, there are no k-regular graphs that are critical with respect to edge-colourings.*

Thus, the only regular graphs that are critical with respect to edge colourings are the odd cycles.

Exercise 298. *Prove that a graph is critical with respect to edge colourings if and only if its line graph is vertex critical (in the Dirac sense for vertex colourings).*

Exercise 299. *Let G be a graph of Class 2 that is critical for edge-colourings, where $\chi'(G) = \Delta(G) + 1$. Show that $|E(G)| \leq \Delta(G)\nu(G) + 1$.*

The following result by Vizing might be interesting.

Theorem 6.11.16 (Vizing). *Let G be a graph that is critical with respect to edge colourings. Then every vertex of G is adjacent to at least two vertices of maximum degree. In particular, there are at least 3 vertices of maximum degree.*

See [363] for an extension of Theorem 6.11.16 to what is called "Vizing's adjacency lemma", and more theorems regarding the structure of critical (with respect to edge colourings) graphs, including some bounds that seem rather interesting on the number of edges in a graph critical with respect to edge colourings. Here is just a sample:

Theorem 6.11.17 (Vizing). *Let B be a graph critical with respect to edge colourings (see Definition 6.11.11) with maximum degree Δ and minimum degree δ. Then*

$$|E(G)| \leq \begin{cases} \frac{1}{2}(|V(G)| - 1)\Delta + 1 \text{ if } |V(G)| \text{ is odd,} \\ \frac{1}{2}(|V(G)| - 2)\Delta + \delta - 1 \text{ if } |V(G)| \text{ is even.} \end{cases}$$

It is known (see the surveys [85] or [363] for details) that there are no critical graphs (using Definition 6.11.11) with 8 or 10 vertices, and there are no 3-critical graphs of order 12 or 14.

If the vertices of an n-vertex graph are labelled injectively with the first n primes, then any two adjacent vertices have relatively prime (coprime) labels. The theory of coprime labelling is rather extensive (see [403] for a survey), and only one non-trivial result is considered here in Exercise 300 for "coprime" edge labellings.

Exercise 300. *Let G be a connected graph with m edges. Show that it is possible to (injectively) label the edges $1, 2, \ldots, m$ so that if a vertex is incident with two or more edges, then the greatest common divisor (gcd) of the labels of these edges is 1.*

6.12 Perfect graphs

6.12.1 Introduction

A *clique* in a graph G is a maximal complete subgraph of G. Let the *clique-number* of G, denoted by $\omega(G)$, be the maximum number of vertices forming a complete subgraph. Trivially, $\chi(G) \geq \omega(G)$. Also note that the clique number of a graph is the independence number of its complement (and vice-versa).

Definition 6.12.1. *A graph G is called* perfect *if and only if for every induced subgraph H, $\chi(H) = \omega(H)$.*

Perfect graphs were introduced by Claude Berge in 1963 [93]. Perfect graphs are of interest in computing; for example, it is known that certain algorithms on perfect graphs have polynomial run-times (see [445]).

The study of perfect graphs seems to have been driven mostly by two major conjectures by Berge (see Section 6.12.6) that had hoped to completely classify perfect graphs; both of these have since been found to be correct (the latter of which took roughly 40 years). Some theorems for perfect graphs are now known to be, in a sense, equivalent to other well-known theorems (*e.g.*, Dilworth's theorem, Theorem 5.10.2).

6.12.2 Some examples

The triangle K_3 is perfect, since all of its induced subgraphs (each of which is either K_3, an edge, or a single vertex) have identical clique numbers and chromatic numbers (either 3, 2 or 1, respectively).

Theorem 6.12.2. *Complete graphs are perfect.*

Proof: Let $n \geq 1$. Then $\omega(K_n) = n$ and $\chi(K_n) = n$. Since any induced subgraph of K_n is also complete, the theorem follows. □

The next result is rather trivial, however is recorded for later reference.

Theorem 6.12.3. *For $k \geq 2$, the odd cycle C_{2k+1} is not perfect.*

Proof: If $k \geq 2$, then $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$. □

Theorem 6.12.4. *Let $k \geq 2$. Then $\omega(\overline{C_{2k+1}}) = k$ and $\chi(\overline{C_{2k+1}}) = k+1$, and so complements of odd cycles of length at least 5 are not perfect.*

Proof: In Exercise 238 it is shown that $\chi(\overline{C_{2k+1}}) = k+1$. Together with the observation that $\omega(\overline{C_{2k+1}}) = k$, the theorem follows. □

Exercise 301. *(Easy) Show that bipartite graphs are perfect.*

Although a direct proof shows that paths are perfect, this result also follows from Exercise 301.

Exercise 302. *Let P_3 denote a path of length 3 (on 4 vertices). Show that if G is a graph with no copies of P_3 as a subgraph, then G is perfect.*

Exercise 303. *Let P_3 denote a path of length 3 (on 4 vertices). Show that if G is a graph with no induced copies of P_3 as a subgraph, then $\chi(G) = \omega(G)$. Are such graphs perfect?*

The next theorem requires some preparation; recall that $\nu(G)$ is the maximum size of a matching in G, and that $\alpha(G)$ is the cardinality of a largest independent set of vertices. If $\{x, y\} = e$ is an edge in a graph, say that e "covers" both x and y; a vertex is said to cover itself (and no other vertex). The following lemma can be found in [125, p.79, Cor.10]:

Lemma 6.12.5. *Let $G = (X \cup Y, E)$ be a bipartite graph and let k be the minimum number of edges and vertices required to cover all vertices. Then*

$$k = \alpha(G) = |X| + |Y| - \nu(G).$$

Proof: Let $C = V^* \cup E^*$ be a minimal set of k vertices and edges that cover all vertices. If $e, f \in E^*$ share a vertex, then one can replace f by its other end-vertex, so one may assume that E^* is a matching. In this case, E^* is indeed a maximum matching since if it were not, one could reduce the size of C by replacing two vertices with an edge. That is, $|E^*| = \nu$, and so $k = |X| + |Y| - \nu$. Similarly, V^* is an independent set.

If $I \subset X \cup Y$ is an independent set, then I contains at most one vertex from each edge in E^*, $|I| \leq |C|$, and so $\alpha(G) \leq k$. To show that $\alpha(G) \geq k$, first observe that V^* is an independent set. For each edge $\{x, y\} \in E^*$, at least one endpoint is also independent of V^*, for if not, say $\{x, v_1\}, \{y, v_2\} \in E(G)$, then $v_1 \neq v_2$, and replacing $\{x, y\}$ and vertices v_1, v_2 from C with these two edges gives a smaller cover. This gives an independent of size k, so $\alpha(G) \geq k$. Hence, $\alpha(G) = k$. □

Note that in the proof given for Lemma 6.12.5, there are a number of ways to finish the proof of $\alpha(G) = k$. For example, one can apply König's theorem (as in [640, 7.2]) or apply Corollary 5.3.14 (as in [125, p.79, Cor.10]). The idea for the above proof came from [190], in the proof of the following theorem:

Theorem 6.12.6 (Gallai, 1958 [400]). *The complement of a bipartite graph is perfect.*

Proof: Let $G = (X \cup Y, E)$ be bipartite and let $\overline{G}$ denote its complement. Any induced subgraph of $\overline{G}$ is the complement of some bipartite graph, so it remains only to show that $\chi(\overline{G}) = \omega(\overline{G})$. Put $k = \chi(\overline{G})$, and let $f : V(G) \to [k]$ be a proper k-colouring of $\overline{G}$. Each colour class $f^{-1}(i)$ contains either one or two vertices (since both X and Y induce complete graphs in $\overline{G}$). If a colour class has two vertices, these two are adjacent in G. So $\chi(\overline{G})$ is the minimum number of edges and vertices that cover every vertex in G. By Lemma 6.12.5, $\chi(\overline{G}) = \alpha(G)$, which is the same as $\omega(\overline{G})$. □

6.12.3 Comparability graphs

Other classes of graphs are known to be perfect. To describe one such class, recall the definition of a partially ordered set (also called a "poset"):

Definition 6.12.7. *A partially ordered set is a set P together with a binary relation $\leq$ on P that satisfies:*

- *For all $x \in P$, $x \leq x$. (Reflexivity)*
- *For all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry)*
- *For all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity)*

The partially ordered set is denoted by $(P, \leq)$, and $\leq$ is called a partial order.

Some authors define a partial order that instead uses an *irreflexive* binary relation, denoted $<$ rather than $\leq$; the two descriptions are equivalent, the second often called a "strict partial order". The three axioms for a (strict) partial order $(P, <)$ then read:

- For all $x \in P$, $x \not< x$. (Irreflexivity)
- For all $x, y \in P$, at most one of $x < y$ and $y < x$ holds. (Antisymmetry)
- For all $x, y, z \in P$, if $x < y$ and $y < z$, then $x < z$. (Transitivity)

Two elements x, y in a partially ordered set $(P, \leq)$ are called *comparable* if and only if either $x \leq y$ or $y \leq x$, and *incomparable* otherwise.

A *linearly ordered set* is a partially ordered set with the additional property that for every x, y, either $x \leq y$ or $y \leq x$.

Definition 6.12.8. *A graph G is a* comparability graph *if and only if there is a partial order $\leq$ on $V(G)$ so that $\{x, y\} \in E(G)$ if and only if x and y are comparable in the poset $(V(G), \leq)$.*

For a survey on comparability graphs, see the article by David Kelly [559] (a professor in the Mathematics department at University of Manitoba who retired in 2010 or so)..

Exercise 304. *Show that bipartite graphs are comparability graphs.*

Showing that a graph is comparability graph is often done by exhibiting a corresponding partial order.

Exercise 305. *Show that all graphs on at most 4 vertices are comparability graphs.*

To show that a graph is not a comparability graph often reduces to checking a number of cases for any potential corresponding partial orders.

Exercise 306. *Show that an odd cycle of length at least 5 is not a comparability graph.*

In fact, one can check directly that C_5 is the smallest non-comparability graph.

The next exercise can be solved in a manner similar to that used in Exercise 306.

Exercise 307. *Let G be the graph on 7 vertices by attaching 4 triangles in a row (by edges). Show that G is not a comparabilty graph.*

Theorem 6.12.9. *Comparability graphs are perfect.*

Exercise 308. *Prove Theorem 6.12.9 by using Mirsky's theorem (Theorem 5.10.4).*

Theorem 6.12.10. *The complement of a comparability graph is perfect.*

Exercise 309. *Using Dilworth's theorem. prove Theorem 6.12.10.*

Remark: Recall that Theorem 6.12.25, (the weak perfect graph theorem) says that complements of perfect graphs are perfect, and so Theorems 6.12.9 and 6.12.10 together with Theorem 6.12.25 give another proof of Dilworth's theorem (Theorem 5.10.2) using Mirsky's (Theorem 5.10.4).

6.12.4 Chordal graphs

Definition 6.12.11. *In a cycle of at least 4 vertices, a* chord *is an edge between non-consecutive vertices. A graph G is* chordal *if and only if every cycle in G contains a chord.*

A chordal graph is one whose every induced cycle is a triangle, and so sometimes chordal graphs are called *triangulated*; Dirac called chordal graphs "rigid circuit graphs".

Lemma 6.12.12. *Let G be a chordal graph, and let H be an induced subgraph of G. Then H is chordal.*

Proof: Let $H = (X, E(G) \cap [X]^2)$ be an induced subgraph of G. If C is a cycle in H, then all edges induced by C in G are also edges in H; so if C has a chord in G, then C has a chord in H. □

Definition 6.12.13. *For a graph G, a* separating set *for G is a set S of vertices whose removal separates some two vertices (that is, if u and v are connected in G, in $G \backslash S$ they are no longer connected and thus lie in different components). If furthermore, no proper subset of S separates the same two vertices, then S is a* minimal separating set. *If a set S separates vertices u and v, call S a u–v separating set.*

Note that a (vertex) cut-set is a separating set.

Theorem 6.12.14 (Dirac, 1961 [264, Thm 1])**.** *A graph G is chordal if and only if every minimal vertex separating set induces a complete graph.*

Proof: Let G be a chordal graph. A graph is complete if and only if it contains no cut-sets, so assume that G is not complete. Also assume that G is connected, since a minimal separating set occurs in one component, and so it suffices to prove the theorem for that component.

Let S be a minimal cut-set. Let G_1 and G_2 be connected components of $G - S$. Since S is minimal, there is an edge from every vertex of S to each G_i. If $s, t \in S$ are not adjacent, for each $i = 1, 2$, (since G_i is connected) there is a shortest s–t path whose other vertices reside in G_i; the union of these two paths is a cycle without a chord. So every pair of vertices in S forms an edge in G.

Suppose G is a graph whose every minimal cut-set induces a complete graph. Let C be a cycle of length at least 4. Suppose that x, y are non-adjacent vertices on C and let Γ_1 and Γ_2 be the two x–y paths that make up C. Let S be a (minimal) cut-set that separates x and y (*e.g.*,start with only neighbours of x, and then prune this set to make it minimal). Then S contains at least one vertex from Γ_1 and one vertex from Γ_2. Since S induces a complete graph, these two vertices are adjacent, and so C has a chord. □

Corollary 6.12.15 (Dirac, 1961 [264, p.72])**.** *If G is a chordal graph, then any minimal cut-set induces a complete graph.*

Proof: Any minimal cut-set is also a minimal separating set. □

Dirac also pointed out that the converse of Corollary 6.12.15 is not true: Consider the graph G on 8 vertices formed by adding a pendant edge to each vertex of a C_4; then each minimal cut-set contains a single vertex, but G is not chordal. (Note that opposite vertices of the C_4 form a minimal separating set—for the other two—and these vertices do not induce a complete graph.)

Theorem 6.12.16 (Dirac, 1961 [264, Thm 2]). *Let G_1 and G_2 be chordal graphs. If $S = V(G_1) \cap V(G_2)$ induces a complete subgraph (or empty subgraph), then the graph $G = G_1 \cup G_2$ formed by their union is also chordal.*

Proof: Let C be a cycle in G that contains at least 4 vertices. If $C \subseteq G_1$ or $C \subseteq G_2$, then C has a chord since each G_i is chordal. If $S = \emptyset$, then G is the disjoint union of two chordal graphs, and so is chordal. So suppose that $S \neq \emptyset$ and that C contains vertices that are contained in each $V(G_i) \backslash S$. Then there are two vertices x and y that are separated by S, and so by the second part of Theorem 6.12.14, C has a chord. $\square$.

If G_1 and G_2 are graphs and both graphs contain a subgraph isomorphic to the same graph H, the *amalgamation* of G_1 and G_2 along H is the graph G formed by identifying vertices in each copy of H.

Corollary 6.12.17. *If a graph G is not a complete graph, G is chordal if and only if G is the amalgamation of two chordal graphs along a complete graph.*

To aid in the proof of the next theorem is an observation regarding chromatic numbers:

Lemma 6.12.18. *Let G_1, G_2 and H be graphs, where H is a complete subgraph of both G_1 and G_2, and let G be the amalgamation of G_1 and G_2 along H. Then*

$$\chi(G) = \max\{\chi(G_1), \chi(G_2)\}.$$

Proof outline: Without loss of generality, suppose that $k = \chi(G_1) \geq \chi(G_2)$. Then $\chi(G) \geq k$, and to show equality, define a good k-colouring of G by first properly k-colouring vertices of G_1. Vertices of H receive $|V(H)| \leq k$ different colours; extend this colouring of H to a good k-colouring of G_2. $\square$.

Theorem 6.12.19 (Berge, 1961 [92]). *If G is a chordal graph, then $\chi(G) = \omega(G)$.*

Proof: The proof given here is by induction on $n = |V(G)|$, roughly following the idea in [264, Thm. 3] by Dirac.
Base step: When $|V(G)| \leq 3$, G is chordal, and $\chi(G) = 3$ if and only if $G = K_3$.

Induction step: Fix $m \geq 3$, and suppose that the theorem is true for all graphs on at most m vertices. Let G be a graph on $m + 1$ vertices. If G is complete, then $\chi(G) = \omega(G) = m + 1$, so suppose that G is not a complete graph. Let $x, y \in V(G)$ be non-adjacent vertices, and let S be any minimal set separating x and y.

By Theorem 6.12.14, S induces a complete graph. [Note: one could have simply used Corollary 6.12.17 here.] Consider the two subgraphs G_1 and G_2, where $x \in V(G_2)$ and $y \in V(G_2)$, and $V(G_1) \cap V(G_2) = S$. By Lemma 6.12.18, one of G_1 or G_2 has chromatic number the same as G; say $\chi(G_1) = \chi(G) = k$. By the induction assumption,

$\omega(G_1) = k$, so G contains a clique with k vertices (and G contains no larger clique since otherwise, its chromatic number is larger than k), and so $\omega(G) = k$, completing the inductive step.

By mathematical induction, the theorem holds for chordal graphs on any number of vertices. □

Dirac's [264] proof of Theorem 6.12.19 was also by induction, using contradiction and minimal counterexample.

Corollary 6.12.20. *Chordal graphs are perfect.*

Proof: By Lemma 6.12.12, any induced subgraph of a chordal graph is again chordal, and so by Theorem 6.12.19, the proof is complete. □

A vertex x in a simple graph G is *simplicial* if and only if its neighbours, $N_G(x)$, induce a complete subgraph of G. The following result, given here without proof, was shown by Dirac (also see [387] concerning simplicial vertices).

Theorem 6.12.21 (Dirac, 1961 [264, Thm 4])**.** *Every chordal graph contains a simplicial vertex, and the removal of any simplicial vertex produces another chordal graph. Furthermore, if a chordal graph is not complete, there exist at least two non-adjacent simplicial vertices.*

Proof outline: Let G be a chordal graph. If G is complete, every vertex is simplicial, and the result is trivial. So only consider graphs that are not complete. For each $n \geq 2$, let $S(n)$ be the statement that for any graph G on n vertices that is not complete, then G contains two non-adjacent simplicial vertices.
Base step: If G is a non-complete graph on at most 3 vertices, in each case two simplicial vertices are easily identified, and in removing either, a chordal graph remains, so $S(2)$, $S(3)$ hold.

Inductive step: Let $k \geq 3$ and suppose that $S(2), \ldots, S(k)$ are all true. Let G be a non-complete chordal graph on $k+1$ vertices. Let x, y be non-adjacent vertices and let S be a minimal x–y separating set. Let G_x and G_y be components of $G \backslash S$, where $x \in V(G_x)$, $y \in V(G_y)$. Put $X = V(G_x)$ and $Y = V(G_y)$.

Suppose that $G[X \cup S]$ is not complete. Then by the induction hypothesis, $G[X \cup S]$ contains at least two non-adjacent simplicial vertices, say u and v. By Corollary 6.12.15, $G[S]$ is complete and so at least one of u or v is in X. If $u \in X$, then u is simplicial in G because u has no neighbours in Y. Now suppose that $G[X \cup S]$ is complete. Then any vertex in X is simplicial. In either case, X contains at least one simplicial vertex in X.

Similarly, Y also contains at least one simplicial vertex. Since no element of X is adjacent to any element of Y, the simplicial vertex found in X is not adjacent to the simplicial vertex found in Y, completing the inductive step.

By MI, for all $n \geq 2$, $S(n)$ is true. □

For examples of chordal graphs or comparability graphs that are not both, see [691]. Here are two: the graph C_6 is not chordal, but it is a comparability graph (see Exercise 304), and the graph on 7 vertices formed by attaching 4 triangles in a row (by edges) is chordal, but is not a comparability graph (see Exercise 307).

Exercise 310. *Prove that any maximal outerplanar graph (see Section 7.10) is chordal.*

See [473] for an early result about chordal graphs (finite or infinite) and their independence numbers.

6.12.5 Interval graphs

In this section, an interval of real numbers can be open, closed, or half-open.

Definition 6.12.22. *A graph G is called an* interval graph *if and only if there exists a set $\{I_v : v \in V(G)\}$ of intervals (of real numbers) so that $\{v, w\} \in E(G)$ if and only if $I_v \cap I_w \neq \emptyset$.*

From the definition, an interval graph is kind of intersection graph (see Section 1.6.7).

For example, K_3 is an interval graph because if the vertices of K_3 are labelled u, v, w, for example, the intervals $I_u = [1, 5]$, $I_v = [1, 3]$, and $I_w = [2, 4]$ have pairwise non-empty intersection (*e.g.*, 3 is in each). In fact, in this case, one can use the same interval to represent each vertex.

Apparently (see [795, p. 17]), interval graphs "arose from purely mathematical considerations" [475], and from a problem in genetics from Seymour Benzer [89, 90], when comparing subsequences of two genes. (See also [794] for more discussion.)

Exercise 311. *Show that for $n \geq 4$, C_n is not an interval graph.*

Exercise 312. *Let G be the graph on 5 vertices formed by adding a pendant edge to each of two vertices in K_3 (called the "bull graph"—see Figure 1.31). Prove that G is an interval graph.*

Exercise 313. *Show that an interval graph is chordal.*

Exercise 314. *Show that the complement of an interval graph is a comparability graph.*

Exercise 315. *Prove that the only trees that are interval graphs are caterpillars (pendant edges attached to a path).*

The next result follows from the fact that interval graphs are chordal and Corollary 6.12.20:

Theorem 6.12.23. *Interval graphs are perfect.*

See [691] and [795] for more information about interval graphs and their applications which include scheduling problems, finding chronological orders of artifacts (a process called "seriation"), or traits in genetics.

6.12.6 Berge's perfect graph conjectures

In 1963, Claude Berge [93] gave two conjectures regarding perfect graphs: the "weak perfect graph conjecture" and the "strong perfect graph conjecture".

Conjecture 6.12.24 (Weak perfect graph conjecture, Berge, 1963 [93])**.** *A graph is perfect if and only if its complement is perfect.*

In 1972, Lovász [635] proved the weak perfect graph conjecture (see also [639]; the result is often now called "the perfect graph theorem").

Theorem 6.12.25 (Perfect graph theorem, Lovász 1972 [635])**.** *The weak perfect graph conjecture is true. In other words, a graph is perfect if and only if its complement is perfect.*

By Theorems 6.12.3 and 6.12.4, an odd cycle of length at least 5 or its complement is not perfect.

Conjecture 6.12.26 (Strong perfect graph conjecture, Berge, 1963 [93])**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains any odd cycle of length 5 or more as an induced subgraph.*

It is common to call an induced odd cycle an "odd hole". Conjecture 6.12.26 was finally proved to be true by Chudnovsky, Robertson, Seymour, and Thomas [211];

Theorem 6.12.27 (Strong perfect graph theorem, Chudnovsky et al., 2006 [211])**.** *The strong perfect graph conjecture is true. In other words, a graph G is perfect if and only if neither G nor $\overline{G}$ contains any odd cycle of length 5 or more as an induced subgraph.*

Exercise 316. *Show that the strong perfect graph theorem (Theorem 6.12.27) implies the (weak) perfect graph theorem (Theorem 6.12.25).*

6.13 The Erdős–Faber–Lovász conjecture

One of the most famous problems in graph theory is notorious because the problem is easy to state and understand, and uses only n copies of K_n. Thus, one might first

think that there is an elementary solution. Such a solution has resisted discovery for at least four decades.

According to Erdős [308], the following conjecture was formulated at a party in Boulder, Colorado in September 1972. The authors of the conjecture were Paul Erdős, Vance Faber, and László Lovász. Early published versions of this conjecture appeared in 1976 (see [306, 307]).

Conjecture 6.13.1 (Erdős–Faber–Lovász, 1972, see [308, Problem 2]). *For a positive integer n, if G is a graph formed by n copies of K_n, any two copies intersecting in at most one vertex, then $\chi(G) = n$.*

In the above conjecture, certainly $\chi(G) \geq n$ since any copy of K_n requires n colours to be properly coloured. Observe that a graph satisfying the conditions of the conjecture has at least $n(n+1)/2$ vertices (and at most n^2 when all copies of K_n are disjoint).

Erdős offered money for settling the conjecture. In [306], 50 British pounds was offered; later that year in [307], \$100 (in USD) was offered, and a few years later, Erdős [308] offered \$500. I think that he later offered \$1000, but I do not have a reference. Erdős gave this problem as one of his favourites in many lectures.

Erdős also gave a version of the Erdős–Faber–Lovász (EFL) conjecture without using the word "graph".

Conjecture 6.13.2 (Erdős–Faber–Lovász, 1972, see [308]). *Let $A_1, \ldots, A_n$ be n-element sets and for each $1 \leq i < j \leq n$, suppose that $|A_i \cap A_j| \leq 1$. Prove that one can colour the elements of $\cup_{i=1}^n A_i$ by n colours so that each A_i contains elements of all the colours.*

One can translate the EFL conjecture into "hypergraphs". See Chapter 13 for more notation and definitions for hypergraphs. The following short description is sufficient for the present discussion:

Definition 6.13.3. *A hypergraph is a pair $H = (V, \mathcal{H})$ where V is a (non-empty) set of elements called vertices, and $\mathcal{H}$ is a collection of subsets of V; each $H \in \mathcal{H}$ is called a hyperedge. If for some $k \geq 2$,*

$$\mathcal{H} \subseteq [V]^k = \{E \subseteq V : |E| = k\}$$

then H is called k-uniform.

Definition 6.13.4. *A hypergraph is called* almost disjoint *if and only if for any two hyperedges E_1 and E_2, $|E_1 \cap E_2| \leq 1$.*

In the literature, sometimes almost disjoint hypergraphs are called "linear" or "loose". The condition that two hyperedges intersect in at most one point avoids cycles with only two edges.

Recall that the chromatic number of a graph is the minimum number of colours used to colour the vertices so that no edge is monochromatic; the same definition holds for the chromatic number of a hypergraph (not necessarily uniform). There is another kind of chromatic number for hypergraphs: for a hypergraph H, define the *strong chromatic number* $\chi_s(H)$ to be the least k so that there is a k-colouring of $V(H)$ so that any hyperedge E with at least two vertices, all vertices in E receive different colours. Note that the strong chromatic number of a simple graph is (simply) its chromatic number.

So the EFL conjecture says that any n-uniform almost disjoint hypergraph H with n hyperedges satisfies $\chi_s(H) = n$.

Hindman observed that by interchanging the roles of elements (vertices) and sets (hyperedges, or copies of K_n), Conjecture 6.13.2 is equivalent to a statement regarding chromatic index. Recall that the chromatic index of a graph G, denoted $\chi'(G)$, is the least k so that there exists a proper k-colouring of its edges (any two edges that share an incident vertex receive different colours). This definition holds precisely for hypergraphs, too.

Conjecture 6.13.5 (EFL, Hindman, 1981 [514])**.** *Any almost disjoint hypergraph H (not necessarily uniform) with $n = |V(H)|$ satisfies $\chi'(H) \leq n$.*

Lemma 6.13.6. *Conjectures 6.13.2 and 6.13.5 are equivalent.*

Proof: Let M be the incidence matrix for the set system in 6.13.2 (see Section 1.12.2) then the transpose M^T is the incidence matrix for the hypergraph in Conjecture 6.13.5. The condition for intersecting hyperedges becomes "any two elements are in at most one set". A proper colouring of the hyperedges corresponds to a colouring of the vertices in the set system. □

Hindman [514] then proved that Conjecture 6.13.5 is true for $n \leq 10$, thereby showing that the original EFL Conjecture 6.13.2 is also true for $n \leq 10$.

In 1992, Jeff Kahn [549] gave a remarkable proof that the number of colours required to colour any n-uniform almost disjoint hypergraph H with n hyperedges is, as $n \to \infty$, at most $n(1 + o(1))$. Kahn's proof used Rödl's "nibble method", a certain style of probabilistic proof introduced in the now classic papers [378, 803], and also used in a famous paper by Pippenger and Spencer [752]. (These four papers give approximate answers to a number of challenging packing or covering problems in hypergraphs, but these results are mostly beyond the scope of what is intended here.)

Addendum In 2021, the EFL conjecture was finally solved for sufficiently large n by Dong Yeap Kang, Tom Kelly, Daniela Kühn, Abhishek Methuku, and Deryk Osthus, *A proof of the Erdős–Faber–Lovász conjecture*, arXiv:2101.04698, 39 pages,12 January 2021. A not-too-technical article about how this discovery was made, appeared in *Quanta Magazine* shortly thereafter; see `https://www.quantamagazine.`

org/mathematicians-settle-erdos-coloring-conjecture-20210405/ (Thanks to Juliana Felix for this link.)

6.14 Colouring vertices of infinite graphs

Any general theory of colouring infinite graphs is beyond the scope of these notes, but two notable concepts are examined here. The first concept relates the chromatic number of an infinite graph and its finite subgraphs. The second concept concerns colouring infinite graphs whose vertices are the points in the Euclidean plane.

6.14.1 Chromatic number of countably infinite graphs

If an infinite graph is k-colourable, then any finite subgraph is also k-colourable. However, if a graph G with a countably infinite number of vertices has the property that every finite subgraph is k-colourable, then is G also k-colourable?

The version of the following theorem is for graphs on countably many vertices, a simple case of a more general theorem.

Theorem 6.14.1 (De Bruijn–Erdős, 1961 [246]). *Let G be a graph on countably infinite many vertices and let $k \geq 1$ be an integer. If each finite subgraph of G is k-colourable, then G is k-colourable.*

Besides the original proof of Theorem 6.14.1, (which was by transfinite induction), many other proofs have since been found. Two proofs are given here, the first using trees (probably first observed by Nash-Williams [712]) and the second using compactness in topology (following ideas of Gottschalk [428], using Tychonoff's theorem; see also [640, Prob. 9.14]). The proof using trees uses only elementary notions, whereas the second proof assumes some basic knowledge of topology.

For a proof with trees, a lemma is required. Say that a tree T (or any graph) is *locally finite* if and only if the degree of every vertex is finite. The following result is sometimes called "König's infinity lemma":

Lemma 6.14.2 (König, 1927 [594]). *Any locally finite infinite rooted tree has an infinite path beginning at the root.*

Proof: Let $(T, \leq)$ be a locally finite tree with root v, and let $v_0^1, v_1^1, \ldots, v_i^1, \ldots$, $i \in I^1$ be a labelling of the successors of v. By the infinite pigeonhole principle, one of the trees

$$\{(T_i^1, \leq) : T_i^1 = \{x \in T : v_i^1 \leq x\} : i \in I^1\}$$

is infinite, say T_0^1 (with root v_0^1) is one such. Repeat this idea using trees having roots which are successors of v_0^1 to obtain another infinite tree T_0^2. Continue in this manner to

get an infinite number of trees, $T_0^1, T_0^2, T_0^3, \ldots$. Then the vertices $v, v_0^1, v_0^2, \ldots$ determine an infinite path. □

First proof of Theorem 6.14.1: Linearly order the vertices of G, say $V(G) = \{v_1, v_2, v_3, \ldots\}$. Construct a tree T as follows: let the ith level of T be the set of all proper k-colourings of the graph induced by $\{v_1, \ldots, v_i\}$, (and the root is the empty colouring of the empty set). To each proper k-colouring $c : \{v_1, \ldots, v_i\} \to [k]$, the descendants of c are the proper k-colourings of $\{v_1, \ldots, v_i, v_{i+1}\}$ that extend c. (One can think of a colouring as a set of pairs $(v, c(v))$ and so a colouring c' extends c if and only if the set of pairs for c' contains the set of pairs for c.) Since the upward degree of the tree is at most k, the tree is locally finite. Since an infinite graph has infinitely many finite subsets of the form $\{v_1, \ldots, v_s\}$, each with a good k-colouring, the tree is infinite. Now apply König's infinity lemma. □

Second proof of Theorem 6.14.1: Suppose that for every subgraph H of G is k-colourable. Let X be the set of all k-colourings (not necessarily proper) of G. Then X can be viewed as the product space $[k]^{V(G)}$. By Tychonoff's theorem, X is compact in the product topology.

For each subgraph H of G, let $X_H \subseteq X$ be the set of all k-colourings of $V(G)$ that properly k-colour H. Then the system

$$\{X_H : H \text{ is a finite subgraph of } G\}$$

is a family of closed sets with the finite intersection property, and so by the Riesz intersection theorem, has a non-empty intersection. Any element of this intersection is indeed a proper k-colouring of $V(G)$. □

6.14.2 Unit-distance graphs

For any positive integer n, let $\mathbb{E}^n$ denote the vector space $\mathbb{R}^n$ with the standard Euclidean metric. A graph $G = (V, E)$ is called a *unit-distance graph* in $\mathbb{E}^2$ if there is an embedding of V into $\mathbb{E}^2$ so that two vertices from V form an edge if and only if their distance is 1. For example, C_4 is a unit-distance graph shown by the four points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

It is fairly easy to see that the Petersen graph is a unit-distance graph, a result that was observed in, *e.g.*, [414]:

Exercise 317. *Show that the Petersen graph is a unit-distance graph.*

Define the *unit-distance graph of the plane* to be the graph G whose vertices are points in $\mathbb{E}^2$, and two vertices are adjacent in G if and only if the corresponding two points are distance 1 apart. Let $\chi(\mathbb{E}^2)$ denote the chromatic number of the unit-distance graph of $\mathbb{E}^2$.

Problem 6.14.3. *Determine* $\chi(\mathbb{E}^2)$.

Problem 6.14.3 is often known as "the Hadwiger-Nelson problem", and was well known by the 1950s. The origin of this problem is a rather complicated story; Soifer [876] gives a detaled analysis about the history and different players behind the problem. The problem has been attributed to various authors at various times. Some of the characters involved were Hugo Hadwiger (1908–1981), Edward Nelson (1932–2014), who is said to have formulated the problem in 1950, and John R. Isbell (1930–2005). The problem became more popular after Martin Gardner [407] discussed it in the column he wrote for *Scientific American* in 1960.

Until very recently (2018), the following gave the best bounds on this chromatic number.

Theorem 6.14.4. *If G is the unit-distance graph of the plane, then* $\chi(G) \in \{4, 5, 6, 7\}$.

To see that the chromatic number is at least four, a unit-distance graph called the Moser graph (due to the Moser brothers [699], William and Leo, 1961) or "Moser spindle" (see Figure 6.7) is not 3-colourable. A planar drawing of the Moser spindle is the Hajós graph given in Figure 6.2); in Exercise 230 it was asked to show that the chromatic number of the Hajós graph is 4.) (Golomb also gave a 10-vertex unit-distance graph that is not 3-colourable—see [876] for details.)

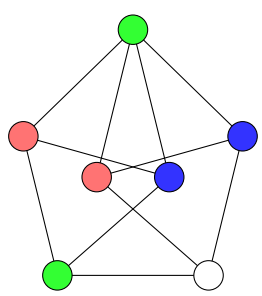


Figure 6.7: Moser's unit-distance graph, not 3-colourable

To show that at most 7 colours are required, a hexagonal tiling pattern by John R. Isbell (see [876]) given in Figure 6.8 is used. (In order to have a proper 7-colouring, points on the borders of the hexagons are assigned to just one hexagon.)

Exercise 318. *Show that a* 3×3 *patch of 9 squares can be used to give a 9-colouring of* $\mathbb{E}^2$ *so that no unit-length edge is monochromatic.*

In 1998, Pritikin [758] showed that if a 7-chromatic unit-distance graph exists, then it has at least 6198 vertices (i.e., all unit-distance graphs on at most 6197 vertices are 6-colourable).

In 2018, the amateur mathematician Aubrey de Grey created a unit distance graph by adding two vertices to the Moser graph, giving a 9 vertex graph, then splicing

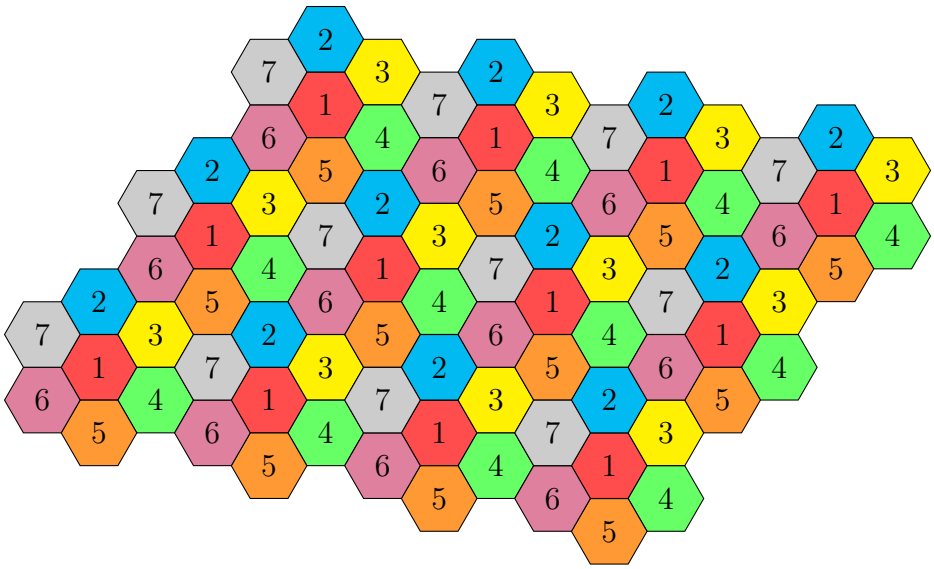


Figure 6.8: Isbell's 7-colouring of the plane

together many copies of these 9-vertex graphs, producing a graph on 1581 vertices that is not 4-colourable.

Theorem 6.14.5 (de Grey, 2018 [247]). $\chi(\mathbb{E}^2) \geq 5$.

The number 1581 was originally 1567 (see the blog [550] for more information). After de Grey's announcement of his construction, a *Polymath16* (see [688]) project reduced the number of vertices to 553.

6.14.3 The Hadwiger-Nelson problem over other fields

What is the effect on the chromatic number of the unit-distance graph of the plane if only rational points in the plane (points (x, y) with $x, y \in \mathbb{Q}$) are allowed? The answer might come as a surprise to some:

Theorem 6.14.6 (Woodall, 1973 [995]). *Let G be the unit distance graph on $\mathbb{Q}^2$ (i.e., $V(G) = \mathbb{Q} \times \mathbb{Q}$, and $\{\mathbf{v}, \mathbf{w}\} \in E(G)$ if and only if $d(\mathbf{v}, \mathbf{w}) = 1$). Then $\chi(G) = 2$.*

To streamline a proof of Theorem 6.14.6 an elementary lemma is convenient. For integers x and y, the notation $x \mid y$ means that x divides y.

Lemma 6.14.7. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be rationals in reduced form satisfy*

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 1. \tag{6.15}$$

Then $b = d$, which is odd, and exactly one of a or c is even.

Proof of Lemma 6.14.7: Rewriting equation (6.15),

$$a^2d^2 + c^2b^2 = b^2d^2. \tag{6.16}$$

Then $d^2 \mid c^2b^2$ and since d and c are relatively prime, $d^2 \mid b^2$, and so $d \mid b$. Similarly, $b^2 \mid a^2d^2$, and so $b \mid d$. Hence $b = d$. Rewriting equation (6.16) and simplifying gives,

$$a^2 + c^2 = b^2. \tag{6.17}$$

If $b = d$ is even, then both a and c are odd (because of reduced forms), and so counting modulo 4, it follows that equation (6.17) has no solution, so both $b = d$ is odd. Also from equation (6.17), since b is odd, it follows that exactly one of a or c is odd. □

One might observe that the conclusion $b = d$ in Lemma 6.14.7 is not needed below.
Proof of Theorem 6.14.6: Define a binary relation on rational points by $(r_1, r_2) \sim (s_1, s_2)$ if and only if both $s_1 - r_1$ and $s_2 - r_2$ have odd denominators in their reduced forms. Since 0 in its reduced form is $\frac{0}{1}$, the relation $\sim$ is reflexive. Also, $\sim$ is trivially symmetric. The sum of two rationals with odd denominators yields another rational with odd denominator (since the sum put over a common denominator has only a product of odd numbers even before simplification). Since $t_1 - r_1 = (t_1 - s_1) + (s_1 - r_1)$ and $t_2 - r_2 = (t_2 - s_2) + (s_2 - r_2)$, it follows that $\sim$ is also transitive. Therefore, $\sim$ is an equivalence relation and thus $\sim$ partitions the rational plane into equivalence classes. Letting C denote the equivalence class containing $(0, 0)$, each equivalence class is a translate of C.
Claim: Two points in different equivalences do not have distance 1.
Proof of claim: If two points (r_1, r_2) and (s_1, s_2) are in different equivalent classes, at least one of $s_1 - r_1$ or $s_2 - r_2$ has an even denominator in its reduced form. By Lemma 6.14.7, $(s_1 - r_1)^2 + (s_2 - r_2)^2 \neq 1$, and so the claim follows.

So to prove the theorem, it suffices to give a good 2-colouring of the class C. Since $(0, 0) \in C$, any other point in C has coordinates that are rationals with odd denominators in their reduced forms.

Define a colouring $f : C \to \{\text{red, blue}\}$ as follows: $f(\frac{w}{x}, \frac{y}{z}) = \text{red}$ if and only if w and y are either both odd or both even (and x, z are odd), and blue otherwise.

To see that f is indeed a good 2-colouring, some cases need checking. Let o and e denote odd and even. If two points are coloured red, the distance vector is one of the reduced forms:

$$\left(\frac{o}{o}, \frac{o}{o}\right) - \left(\frac{o}{o}, \frac{o}{o}\right) = \left(\frac{e}{o}, \frac{e}{o}\right)$$
$$\left(\frac{o}{o}, \frac{o}{o}\right) - \left(\frac{e}{o}, \frac{e}{o}\right) = \left(\frac{o}{o}, \frac{o}{o}\right)$$
$$\left(\frac{e}{o}, \frac{e}{o}\right) - \left(\frac{e}{o}, \frac{e}{o}\right) = \left(\frac{e}{o}, \frac{e}{o}\right)$$

If two points are coloured blue, there are essentially two cases:

$$\left(\frac{o}{o}, \frac{e}{o}\right) - \left(\frac{o}{o}, \frac{e}{o}\right) = \left(\frac{e}{o}, \frac{e}{o}\right);$$
$$\left(\frac{o}{o}, \frac{e}{o}\right) - \left(\frac{e}{o}, \frac{o}{o}\right) = \left(\frac{o}{o}, \frac{o}{o}\right).$$

In each of the above expressions, the right side $(\frac{a}{b}, \frac{c}{d})$ does not have exactly one of a and c odd, and so by Lemma 6.14.7, identically coloured points are not distance 1 apart. □

Note: In the proof of Theorem 6.14.6, one might ask how many equivalence classes there are. Among only points of the form $(\frac{1}{2^n}, 0)$, no two are in the same class (since the denominator of the difference is a power of 2, not odd). So there are a countably infinite number of classes.

The Hadwiger–Nelson problem has also been looked at for fields other than $\mathbb{R}$ or $\mathbb{Q}$. Until 2009, it seems that not much was done; Soifer [876] posed the problem of finding $\chi(\mathbb{Q}(\sqrt{2})^2)$, (where $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$ is the field extension of $\mathbb{Q}$ formed by adding $\sqrt{2}$).

For example, a recent (2015) paper by David Madore [650] shows that $\chi(\mathbb{Q}(\sqrt{2})^2) = 2$, $\chi(\mathbb{Q}(\sqrt{3})^2) = 3$, $\chi(\mathbb{Q}(\sqrt{7})^2) = 3$, and $4 \leq \chi(\mathbb{Q}(\sqrt{3}, \sqrt{11})^2) \leq 5$. (Thanks to Danylo Radchenko for this reference, who also found that $\chi(\mathbb{Q}(\sqrt{2})^2) = 2$ in 2012, but his results were never published.) Madore gives some references for results in other fields (*e.g.*, p-adic numbers, finite fields), and points out that the only known bounds for the complex numbers are the trivial ones of $4 \leq \chi(\mathbb{C}^2) \leq \infty$. Unfortunately, it seems that Madore was unaware that in 1990, Fischer [364] proved that $\chi(\mathbb{Q}(\sqrt{3})^2) = 3$, Fischer also proved $\chi(\mathbb{Q}(\sqrt{11})^2) \leq 4$.

6.15 Specialized colourings

In practice, colourings of graphs may have certain restrictions or additional properties. This section is an introduction to just four special colourings, namely "list colourings", "fractional colourings", "total colourings", and "harmonious colourings". The literature on these colourings is immense, and the list of "real-world" problems that are solved using them seems nearly endless. However, such topics are most often taught in higher level courses in pure math, applied math, or computer science, so this section is very brief, giving only an introduction, some basics, and a few references for further reading.

6.15.1 List colouring

In the 1976, Vizing [961] and, independently, in 1979, Erdő, Rubin, and Taylor [340] considered colourings of graphs where at each vertex v the colour of v is restricted

to some specified "list" $L(v)$ of colours. (For ordinary vertex colourings and chromatic numbers, each vertex has the same list.)

Definition 6.15.1. *For an integer $k \geq 2$, a graph $G = (V, E)$ is k-list-colourable (or k-choosable) if and only if for any set of lists $\{L(v) : |L(v)| \geq k, v \in V(G)\}$, there is a proper colouring $f : V \to \cup_{v \in V} L(v)$ so that for each v, $f(v) \in L(v)$. The list chromatic number $\chi_\ell(G)$ of G is the least k so that G is k-list-colourable; The list chromatic number is all called the "choosability" number, denoted $ch(G)$.*

The graph with lists in Figure 6.9 (as given in [340]) is 2-colourable (bipartite), but not 2-choosable (e.g, if x is coloured 1, then any proper colouring forces y to be coloured 2, then z to be coloured 3, then w to be coloured 1, leaving no colour to colour v).

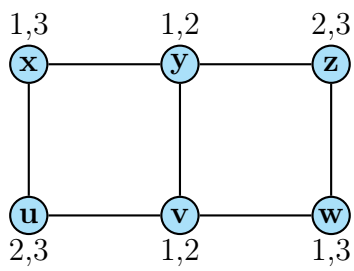


Figure 6.9: A graph that is 2-colourable but not 2-choosable

Since a k-choosable graph G has a proper colouring when each list is the same k elements, it follows that $\mathrm{ch}(G) \geq \chi(G)$. Vizing [961] observed (similar to Vizing's Theorem 6.11.4 for edge-chomatic numbers)) that for any graph G,

$$\mathrm{ch}(G) \leq \Delta(G) + 1.$$

List colouring numbers satisfy an inequality analogous to the Gaddum–Nordhaus theorem (Theorem 6.2.17)

Theorem 6.15.2 (Erdős–Rubin–Taylor, 1979 [340])**.** *For any graph G,*

$$ch(G) + ch(\overline{G}) \leq n + 1.$$

The same authors showed that for each n, there exists a least m so that $K_{m,m}$ is not n-choosable. (using hypergraphs with property B—see Section 13.6), thereby showing that the gap between $\chi(G)$ and $\mathrm{ch}(G)$ can be arbitrarily large.

Exercise 319. *Show that $K_{3,3}$ is not 2-choosable by using the lists $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ on each partite set.*

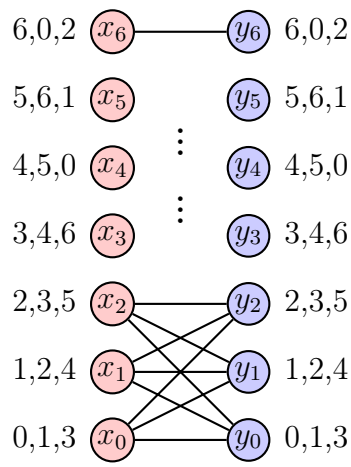


Figure 6.10: $K_{7,7}$ is 2-colourable but not 3-choosable

As another example, the graph $K_{7,7}$ is not 3-choosable, evident by the lists given in Figure 6.10.

Carsten Thomassen proved a conjecture made in 1979 by Erdős, Rubin, and Taylor [340, p. 153], thereby giving a different proof of the 5-colour theorem for planar graphs (Theorem 7.7.4).

Theorem 6.15.3 (Thomassen, 1994 [919])**.** *Every planar graph is 5-choosable.*

Also in [340], it was asked if there are any bipartite planar graphs that are not 3-choosable.

Theorem 6.15.4 (Alon–Tarsi, 1992 [37])**.** *Every bipartite planar graph is 3-choosable.*

Erdős et al. [340] gave an example of a planar graph on 14 vertices that is 3-colourable but not 3-choosable.

For a survey on list colourings, see [997].

6.15.2 Fractional colouring

According to [234], fractional chromatic numbers were introduced in the early 1970s in hope of either proving or disproving the Four Colour Conjecture. Fractional chromatic numbers (and the edge variant, fractional chromatic indexes) have since been used to prove or improve results in graph theory (e.g, finding large independent sets and matchings), and are related to polytopes, linear programming, and probabilistic methods.

This section is only a very brief introduction to fractional chromatic numbers; to see applications, the interested reader might look at the textbook [832] on fractional

colouring or the early paper by Jack Edmonds [272] from a linear programming perspective; see also [693, Ch. 21] for modern applications and further references. For a relationship between fractional colourings and the density of sets in the plane with no unit-distances, see [87].

There are different ways to interpret what is meant by a fractional colouring; here is one way:

Definition 6.15.5. *For any graph $G = (V, E)$ and positive integers $\ell \leq k$, a proper $\frac{k}{\ell}$-colouring is a colouring with k colours, that assigns to each vertex a collection of ℓ distinct colours in such a way that adjacent vertices receive disjoint sets of ℓ colours. The fractional chromatic number of G is defined as*

$$\chi_f(G) = \inf\left\{\frac{k}{\ell} : \textit{there exists a proper } \tfrac{k}{\ell}\textit{-colouring of } V(G)\right\}.$$

The notation $\chi^*(G)$ is also used. Note that using $\ell = 1$ shows that $\chi_f(G) \leq \chi(G)$.

Another way to interpret a (proper) fractional colouring is given in [143, p. 389]: A proper k-colouring of the vertices a graph G is a (ordered) partition $V(G) = S_1 \cup \cdots \cup S_k$, where each colour class S_i is an independent set (also called a "stable set", hence the S_is). If $G = (V, E)$ has n vertices, fix some order of vertices, say $V = \{v_1, \ldots, v_n\}$. To each S_i, associate the characteristic 0-1 vector $\mathbf{s}_i$ (of length n), also called an incidence vector. Then a proper k-colouring of G is then seen as a collection of 0-1 stable set vectors $\mathbf{s}_1, \ldots, \mathbf{s}_k$ so that $\sum_{i=1}^k \mathbf{s}_i$ is the all 1s vector $\mathbf{1}$ (of length n).

Generalizing, a fractional colouring of a graph G is a linear combination of the form

$$\mathbf{1} = \lambda_1 \mathbf{s}_1 + \cdots + \lambda_k \mathbf{s}_k.$$

where the λ_is are non-negative rational numbers and the $\mathbf{s}_i$s are incidence vectors of stable sets in G. The fractional chromatic number $\chi_f(G)$ is the least sum $\sum_{i=1}^k \lambda_i$ of coefficients λ_i (also called "weights") taken over all collections of stable sets and coefficients.

For the relationship between fractional colourings and list colourings (choosability), see [38].

Recall that $\alpha(G)$ is the order of a largest independent set of vertices in G and, from Exercise 237, that $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. A similar result holds for fractional chromatic numbers:

Exercise 320. *Let G be a graph on n vertices. Show that $\chi_f(G) \geq \frac{n}{\alpha G}$.*

Example 6.15.6. *Let a copy of C_5 be given on vertices u, v, w, x, y , in order. There are five non-trivial stable sets, $S_1 = \{u, w\}$, $S_2 = \{v, x\}$, $S_3 = \{w, y\}$, $S_4 = \{x, u\}$, $S_5 = \{y, v\}$. Each vertex is in exactly two of the S_is, so one might try setting each $\lambda_i = \frac{1}{2}$. Then the total weight at each vertex is 1, as desired, giving a proper 2.5-fractional colouring. Thus, $\chi_f(C_5) \leq 2.5$. However, by Exercise 320, $\chi_f(C_5) \geq \frac{5}{2}$, so $\chi_f(C_5) = 2.5$.*

Let $\chi_f(\mathbb{R}^2)$ denote the fractional chromatic number of the unit-distance graph in the plane. Recall that the Moser spindle (see Figure 6.7) is a unit-distance graph with chromatic number 4. In Figure 6.11 is a proper $\frac{7}{2}$-colouring given by Fisher and Ullman [365] in 1992, which shows the graph has fractional chromatic number at least 3.5.

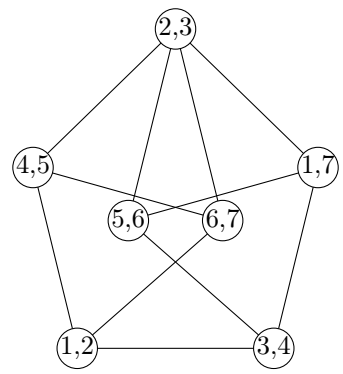


Figure 6.11: A proper $\frac{7}{2}$-colouring of Moser's spindle, with colours 1,2,... 7.

Fisher and Ullman [365] showed that the Moser spindle indeed has fractional chromatic number 3.5, and so $3.5 \leq \chi_f(\mathbb{R}^2)$. They also gave a colouring that showed $\chi_f(\mathbb{R}^2) \leq \frac{8\sqrt{3}}{\pi} \sim 4.4106$. The best bounds as of 2017 were

$$3.6290 \sim \frac{76}{21} \leq \chi_f(\mathbb{R}^2) \leq 4.3599,$$

where the lower bound is due to Cranston and Rabern [234] and the upper bound was proved by Hochberg and O'Donnell [517] in 1993. In 2020, the lower bound was improved by Bellitto, Pĉher, and Sédillot to $\chi_f(\mathbb{R}^2) \geq \frac{1999983}{512933} \sim 3.8991$; the authors used a graph given by de Grey used in showing $\chi(\mathbb{R}^2) \geq 5$ (see Theorem 6.14.5).

6.15.3 Total colouring

Definition 6.15.7. *A total colouring of a graph G is a labelling of both vertices and edges so that adjacent vertices are labelled differently, incident edges are labelled differently, and each vertex is labelled differently from all edges incident with that vertex. The total colouring number $\chi''(G)$ is the least number of colourings required in a total colouring of G.*

Remark 6.15.8. *Total colourings are tacitly assumed to be "proper" colourings, whereas vertex colourings (or edge colourings) are not always assumed to be proper (although some authors do use "colouring" to mean "proper colouring").*

For a survey on total colourings, see [203] or [1005]. Trivially, for any graph G, $\chi''(G) \geq \chi(G)$ and $\chi''(G) \geq \chi'(G)$, and so $\chi''(G) \geq \omega(G)$ and $\chi''(G) \geq \Delta(G)$. Also,

for any graph G, a total colouring is available by using $\chi(G)$ colours for vertices and a different set of $\chi'(G)$ colours for edges, and so $\chi''(G) \leq \chi(G) + \chi'(G)$. Another trivial lower bound is given in Exercise 321, the analogue to equation (6.12) for chromatic index.

Exercise 321. *Show that* $\chi''(G) \geq \Delta(G) + 1$.

Exercise 322. *Show that* $\chi''(K_3) = 3$.

Exercise 323. *Find* $\chi''(K_4)$.

It was suspected [81] that the bound $\chi''(G) \geq \Delta(G)+1$ from Exercise 321 is nearly optimal (as in Vizing's theorem).

Conjecture 6.15.9 (Bezhad, 1965 [81]; Vizing, 1968 [960])**.** *For any simple graph G,*

$$\chi''(G) \leq \Delta(G) + 2.$$

Theorem 6.15.10 (Behzad–Chartrand–Cooper, 1967 [82])**.** *If n is odd, then $\chi''(K_n) = n$; if n is even, then $\chi''(K_n) = n + 1$. For any n, $\chi''(K_{n,n}) = n + 2$. For $m \neq n$, $\chi''(K_{m,n}) = \max\{m, n\} + 1$.*

Hence, the upper bound of $\Delta(G) + 2$ in Conjecture 6.15.9 is tight since for n even, $\chi''(K_n) = n + 1 = \Delta(K_n) + 2$.

There are a number of results that approach Conjecture 6.15.9. For example, in 1998, Molloy and Reed [692] showed that for sufficiently large Δ, if a simple graph G has $\Delta(G) = \Delta$, then $\chi''(G) \leq \Delta + 10^{26}$ (the authors mentioned that their techniques can reduce the constant 10^{26} to around 500, but probably not lower than 10). See [692] for related results and more references.

In 1990, McDiarmid [663] (and later McDiarmid and Reed [664]) showed that graphs G having total chromatic number more than $\Delta(G)+1$ are very rare (similar to Theorem 6.11.7 for chromatic index).

Definition 6.15.11. *A (proper) total colouring of a graph G is called "adjacent vertex distinguishing" if for every edge $\{u, v\}$ the set of colours used at u is distinct from those at v. The minimum number of colours in an adjacent vertex distinguishing total colouring of G is denoted χ_{at}.*

In a sense, adjacent vertex distinguishing total colourings do not require significantly more colours than just for a total colouring.

Theorem 6.15.12 (Coker–Johannson, 2012 [223])**.** *There exists a universal constant C so that for every graph G,*

$$\chi_{at} \leq \Delta(G) + C.$$

For the above theorem and more on adjacent vertex distinguishing total colourings, see Johannson's thesis [543].

6.15.4 Harmonious colourings

In the early 1980s, the notion of harmonious colouring was introduced in two slightly different ways—by Frank, Harary, and Plantholt [377] in 1982 and, independently, by Hopcroft and Krishnamoorthy in 1983.

Frank, Harary, and Plantholt [377] published the notion of "line-distinguishing chromatic numbers" for various classes of graphs, based on a question by Pierre Duchet. Let G be a graph, possibly with loops (but no multiple edges). For a positive integer k, and let $f : V(G) \to \{1, 2, \ldots, k\}$ be an onto colouring. Then f induces a colouring of each $e = \{x, y\} \in E(G)$ by $f(e) = \{f(x), f(y)\}$. Such a k-colouring is called *line-distinguishing* (or in modern language, *edge distinguishing*) if for every pair e_1, e_2 of distinct edges, $f(e_1) \neq f(e_2)$. The *line distinguishing chromatic number* of a graph G is the least k for which there exists a line distinguishing k-colouring. Frank et al. [377] used the notation $\lambda(G)$ for the line-distinguishing number of G, but since this notation is already taken, the notation $LDCN(G)$ is used here.

As noted in [377], in any line-distinguishing colouring, two vertices with the same colour have disjoint neighbourhoods. As a trivial example,

$$LDCN(K_n) = \begin{cases} n & \text{if } n \neq 2 \\ 1 & \text{if } n = 2. \end{cases}$$

Exercise 324. *Show that* $LDCN(K_{m,n} = m + n - 1$.

Using Eulerian trails, Frank et al. also showed that for a path P_n (on n vertices, not the usual here with length n)

$$LDCN((P_n) = \min\left\{2\left\lceil\sqrt{\frac{n-2}{2}}\right\rceil, 2\left\lceil\frac{1+\sqrt{8n-7}}{4}\right\rceil - 1\right\}.$$

The same authors showed a similar result for cycles:

$$LDCN((C_n) = \min\left\{2\left\lceil\sqrt{\frac{n}{2}}\right\rceil, 2\left\lceil\frac{1+\sqrt{8n+1}}{4}\right\rceil - 1\right\}.$$

Frank et al. continued their paper by looking at general bounds, digraphs, and relations to block designs.

In 1983, Hopcroft and Krishnamoorthy [526] called a vertex-colouring of a simple graph to be *harmonious* if no two edges received the same (unordered) colour pair. (Note, the vertex colouring need not be proper). An example given in [526] appears in Figure 6.12.

Hopcroft and Krishnamoorthy mention that harmonious colourings are related to harmonious labellings (see [425] or [437]), graceful labellings (see, *e.g.*, [23], [805], [277],

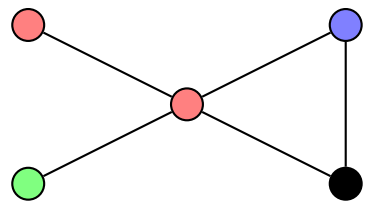


Figure 6.12: A graph with a harmonious 4-colouring

or Section 8.5.2 on graceful labellings for more references), and minimal perfect hash functions (see [221]).

If a graph G has a harmonious r-colouring then $|E(G)| \leq \binom{r}{2} + r = \binom{r+1}{2}$. For any graph G, let $HC(G)$ denote the smallest r for which G has a harmonious colouring; $HC(G)$ is called the harmonious colouring number of G. (In [687], $HC(G)$ is denoted by $h'(G)$.) Trivially, $HC(G)$ is at least the maximum degree in G.

Only for a few classes of graphs G is $HC(G)$ known; for example, for a path P, $HC(P)$ is known (see [526] for the proof—using an Eulerian path in some K_n; a similar proof also occurs in [377]). Miller and Pritikin [684] found $HC(G)$ for a few other classes of graphs (including grids) and gave bounds for others (*e.g.*, the cube graph Q_n). Even when G is a tree, finding $HC(G)$ is already difficult.

The "harmonious colouring problem" is to determine whether or not for a given number r of colours, if a graph has a harmonious r-colouring. Hopcroft and Krishnamooorthy [526] showed that the harmonious colouring problem is NP-complete. (These authors also note that David S. Johnson also attained the same result, a fact first conjectured by Frank et al. [377] even for trees.)

In 1989, Mitchem [687] defined a slightly stronger type of harmonious colouring. For a graph G, define the harmonious chromatic number $h(G)$ to be the least number of colours r so that there is a *proper* r-colouring of $V(G)$ so that no two edges receive the same pair of *distinct* colours. Note that $h(G) \geq HC(G)$. Similar to above, $E(G)| \leq \binom{h(G)}{2}$.

Mitchem [687] studied harmonious colourings for various trees (including complete binary trees) and regular graphs.

Chapter 7

Planar graphs

7.1 Basics

A graph (or multigraph) G is called *planar* if and only if it can be drawn in the Euclidean plane with no edges crossing; such a drawing is called a *plane drawing* (or *planar embedding*) of G. Removal of a plane drawing of G from the plane leaves connected regions called *faces* of G; the outside infinite region is counted as a face.

As an example, Figure 7.1 shows K_4 drawn in a "standard" way along with three planar drawings. Each of the planar drawings have four points ($v = 4$), six edges ($e = 6$), and four faces ($f = 4$) including the outer face.

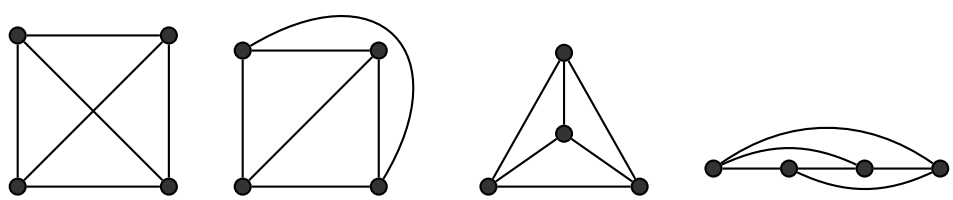


Figure 7.1: Standard drawing of K_4 and three planar drawings of K_4

Exercise 325. *For each $n \geq 2$, show that the graph $K_{2,n}$ is planar by exhibiting a plane drawing.*

The result in the next exercise was proved by Leonhard Euler [349] in 1758.

Exercise 326 (Euler's formula for planar graphs)**.** *Let G be a connected simple planar graph, with v vertices and e edges. Then in any plane drawing of G, the number f of faces is the same, and*

$$v + f = e + 2 \quad \textit{(Euler's formula).}$$

Hint: Induct on e.

Exercise 327. *Show that Euler's formula for planar graphs also holds for connected planar multigraphs.*

Since Euler's formula $v + f = e + 2$ also holds for planar multigraphs, many of the subsequent results also hold for multigraphs. However, the reader is reminded that in this text, "graph" means "simple graph", and so questions here about graphs are almost always only questions about simple graphs.

Exercise 328. *Show by induction that if a simple planar graph has v vertices, f faces, e edges, and $k \geq 1$ components, then*

$$v + f = e + k + 1.$$

The five platonic solids (tetrahedron, cube, octahedron, icosahedron, dodecahedron, (see Figure 1.18) give rise to regular planar graphs (see Figure 1.20); respectively; call them K_4, Q_3, O_6, I_{12}, and D_{20}.

The graph of any convex polyhedron is planar (by the method of projection outlined in Section 1.6.3). As shown in Figure 1.20, each of the platonic solids has a plane drawing. In a plane drawing of a convex polyhedron, each face corresponds to a face of the polyhedron (similarly for each vertex and edge). Many planar graphs are indeed graphs of polyhedra; a characterization of such graphs was given by Ernst Steinitz (1871–1928).

Theorem 7.1.1 (Steinitz, 1922 [888])**.** *Every 3-connected (simple) planar graph is the graph of a polyhedron, and every polyhedron has a 3-connected (simple) planar graph.*

The proof of the second part of Steinitz's theorem seems quite natural, however the first part is harder. For more on Steinitz's theorem, see, *e.g.*, [1014, Chapter 4].

The next lemma gives an upper bound on the number of edges in a planar graph, and is used in many other theorems regarding planar graphs.

Lemma 7.1.2. *If G is a connected planar graph or multigraph on $v \geq 3$ vertices with e edges, then*

$$e \leq 3v - 6. \tag{7.1}$$

Proof: Let G be a connected planar graph on $v \geq 3$ vertices, with e edges and f faces. The number k of edge-face incidences satisfies $k \leq 2e$ since each edge is adjacent to at most two faces. On the other hand, since every face has at least 3 edges, $3f \leq k$. Thus

$$3f \leq 2e. \tag{7.2}$$

Multiplying Euler's formula by 3 gives $3v + 3f = 3e + 6$, and using (7.2) yields

$$3v + 2e \geq 3e + 6,$$

from which the result follows. □

Exercise 329. *Show that Lemma 7.1.2 holds even for planar graphs that are not necessarily connected but whose every component has at least 3 vertices. Does this result extend to* any *planar graph?*

Theorem 7.1.3. *The complete graph K_5 is not planar.*

Proof: In K_5, there are $\binom{5}{2} = 10$ edges, so put $v = 5$, and $e = 10$; these values violate Lemma 7.1.2. [Note: Another proof is also available—draw C_5 and examine the possible placement of the other five edges.] □

Exercise 330. *Find a connected planar multigraph with 5 vertices and 10 edges. How many faces does any planar drawing of this multigraph have?*

Is the graph $K_{3,3}$ planar? This graph is sometimes called the "Thomsen graph"or the "utilities graph" (see Figure 7.2).

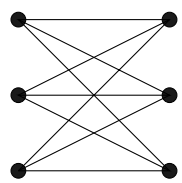


Figure 7.2: A standard drawing of $K_{3,3}$

Theorem 7.1.4. *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof: Suppose for the moment that $K_{3,3}$ is planar, with $v = 6$, $e = 9$. Euler's formula $v + f = e + 2$ then becomes $6 + f = 9 + 2$ and so $f = 5$. Since $K_{3,3}$ is bipartite, any shortest cycle has length at least 4, and therefore each face has at least 4 edges. Counting face-edge incidences, the equation analogous to (7.2) is $4f \leq 2e$, which yields $20 \leq 18$, a contradiction. [Note: Again, a direct proof is also possible; draw C_6 and try to place the remaining three edges to produce $K_{3,3}$.] □

Exercise 331. *Give a drawing in the plane of $K_{3,3}$ with only one pair of edges crossing. Does there exist such a drawing using only straight lines?*

Exercise 332. *Find (with proof) a formula analogous to (7.2) for planar graphs whose smallest cycle length (girth) is g.*

Exercise 333. *Let G be a connected planar (simple) graph with $v \geq 3$ vertices and e edges. If the girth (length of the smallest cycle) of G is g, prove that*

$$e \leq \frac{g(v-2)}{g-2}.$$

Lemma 7.1.5. *Every planar graph contains a vertex of degree at most 5. In other words, if G is planar, then $\delta(G) \leq 5$.*

Proof: Let G be planar. One need only consider when G is connected. (Why?) The proof is by contradiction. If every vertex has degree at least 6, then by the handshaking lemma (Lemma 1.8.1) and Lemma 7.1.2,

$$6v \leq \sum_{x \in V(G)} \deg(x) = 2e \leq 2(3v - 6),$$

from which the desired contradiction is obtained. □

Exercise 334. *Show that a connected planar graph G on at least 3 vertices has at least 3 vertices with degree at most 5.*

Exercise 335. *Show that a connected planar graph G on at least 4 vertices has at least 4 vertices with degree at most 5.*

Exercise 336. *Either find a simple planar graph with 8 vertices, 10 faces, and 19 edges, or prove that one does not exist.*

Exercise 337. *Either find a simple planar graph that has each vertex incident with 4 faces, and each face has four edges, or prove that such a graph does not exist.*

If a graph has n vertices, there are algorithms with run time $O(n)$ that decide whether or not a graph is planar. One of the first and most popular algorithms is the Hopcroft–Tarjan algorithm [527].

The next two theorems characterize planar graphs. The first was proved by Kazimierz Kuratowski (1896–1980), a Polish topologist (see [190, p. 237] for a discussion of other authors who also proved the same theorem, some perhaps earlier).

Recall that a subdivision of a graph G is one obtained by inserting vertices (of degree 2) into edges of G. (Technically, if one does not insert any new vertices, still say that G is a subdivision of G.)

Theorem 7.1.6 (Kuratowski, 1930 [613])**.** *A graph G is planar if and only if G contains no subgraph that is a subdivision of either $K_{3,3}$ or K_5.*

The next theorem was proved by Klaus Wagner (1910–2000). Recall that an edge-contraction in a graph G is achieved by identifying two adjacent vertices and removing any multiple edges or loops thereby formed. A graph H is a *minor* of G if and only if H can be obtained by a sequence (in any order) of edge removals, vertex removals, or edge-contractions in G.

Theorem 7.1.7 (Wagner, 1937 [969])**.** *A graph G is planar if and only if G contains no minor isomorphic to either $K_{3,3}$ or K_5.*

The next three exercises regarding degree sequences in planar graphs appear in [125, p. 33–34]; solutions are not provided here.

Exercise 338 ([125, 60, p. 33]). *Let G be a planar graph on $n \geq 3$ vertices with degree sequence $(d_1, \ldots, d_n)$. Show that*

$$\sum_{d_i \leq 6} (6 - d_i) \geq \sum_{i=1}^{n} (6 - d_i) \geq 12.$$

Exercise 339 ([125, 61, p. 33]). *Let G be a planar graph on $n \geq 3$ vertices with degree sequence $(d_1, \ldots, d_n)$. Show that for each $k \geq 3$,*

$$\sum_{i=1}^{k} d_i \leq 2n + 6k - 16.$$

Exercise 340 ([125, 67, p. 34]). *Let G be a planar graph on n vertices with degree sequence $(d_1, \ldots, d_n)$. Show that if $\delta(G) \geq 4$ then*

$$\sum_{i=1}^{n} d_i^2 < 2(n+3)^2 - 62.$$

Prove by induction on n that if $n \geq 4$ then

$$\sum_{i=1}^{n} d_i^2 \leq 2(n+3)^2 - 62,$$

and that equality can hold for every $n \geq 4$.

The next two theorems are given for general interest; proofs are not given here. Recall (see Definition 1.7.2) that an independent set is a collection of vertices, no two adjacent. (Some authors call an independent set a *stable set* of vertices.)

Theorem 7.1.8 (See [206, Thm 1.6]). *In a connected planar graph on n vertices, with minimum degree 4, a maximum independent set has less than $n/2$ vertices.*

Theorem 7.1.9 (See [206, Thm 1.8]). *In a connected planar graph on n vertices, with minimum degree at least 3, the maximum size of a matching is at least $\lceil (n+2)/3 \rceil$.*

As a corollary to Theorem 7.1.9, a minimum vertex cover for a connected planar graph has at least $\lceil (n+2)/3 \rceil$ vertices.

7.2 Face degrees and platonic solids

A platonic solid (also called a *uniform* or *regular* solid) is a polyhedron with every face being the same regular polygon (all edge lengths are the same, and all interior angles are the same), and each vertex is incident with the same number of faces. For example, the regular tetrahedron is the simplest platonic solid.

A somewhat amazing fact (already known to the ancient Greeks) is that there are precisely five platonic solids (see Figure 1.20). This fact has a surprisingly simple proof using Euler's formula.

Theorem 7.2.1. *There are only five platonic solids, namely the tetrahedron, octahedron, icosahedron, cube, and dodecahedron.*

Proof: Consider a polyhedron P formed by regular s-gons, ($s \geq 3$) with $d \geq 3$ such s-gons meeting at a vertex. There are only five choices for the pair (s, d), and this fact is seen by looking at how polygons can surround a point in the plane.

If $s = 3$, then since 6 equilateral triangles can meet at a point in the plane, $d \in \{3, 4, 5\}$ are the only possibilities.

If $s = 4$, since 4 squares surround a point in the plane, $d = 3$ is the only choice.

If $s = 5$, only three regular pentagons (each with internal angle 108°) can surround a point (with a little room to spare), so $d = 3$ is the only possibility.

Thus, $(s, d) \in \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}$. Next, it is shown that for each pair (s, d), there is precisely one polyhedron satisfying the constraints. This is proved by Euler's formula and some "double counting".

Let P have v vertices, f faces, e edges, (with s sides per face, and d s-gons meeting at a vertex). Note that d is also the number of edges that meet at any vertex (called the *degree* of a vertex).

Count the number of pairs (vertex, edge), where the vertex and edge are incident. (Such a pair is often called a "flag", but this terminology is not used here.) Counting from the vertices, since each vertex is incident with d edges, there are vd pairs. Counting from the edges, since each edge is incident with two vertices, there are $2e$ such pairs (this is a special case of the handshaking lemma, , Lemma 1.8.1). Thus,

$$vd = 2e. \tag{7.3}$$

Similarly, count the pairs (edge, face), where the edge is on the border of the face. Since each edge is on the border of two faces, the number is $2e$. Since each face has s edges on its border (sometimes this number s is called a "face-degree"), the number is fs. Thus,

$$fs = 2e. \tag{7.4}$$

Putting $v = \frac{2e}{d}$ and $f = \frac{2e}{s}$ into Euler's formula $v + f = e + 2$,

$$\frac{2e}{d} + \frac{2e}{s} = e + 2.$$

Multiplying through by $\frac{1}{2e}$,

$$\frac{1}{d} + \frac{1}{s} = \frac{1}{2} + \frac{1}{e}.$$

The key idea is that for each pair (s, d), the number e of edges is uniquely determined. In fact, one can solve for e:

$$\frac{1}{e} = \frac{s+d}{sd} - \frac{1}{2} = \frac{2s + 2d - sd}{2sd},$$

and so

$$e = \frac{2sd}{2s + 2d - sd}.$$

Case 1: When $s = 3$ and $d = 3$, get $e = 6$; by equation (7.3), get $v = 4$ and by equation (7.4), get $f = 4$. These parameters agree with those for the tetrahedron.

Case 2: When $s = 3$ and $d = 4$, get $e = 12$; by equation (7.3), get $v = 6$ and by equation (7.4), get $f = 8$. These parameters agree with those for the octahedron.

Case 3: When $s = 3$ and $d = 5$, get $e = 12$; by equation (7.3), get $v = 12$ and by equation (7.4), get $f = 20$. These parameters agree with those for the icosahedron.

Case 4: When $s = 4$ and $d = 3$, get $e = 12$; by equation (7.3), get $v = 8$ and by equation (7.4), get $f = 6$. These parameters agree with those for the cube.

Case 5: When $s = 5$ and $d = 3$, get $e = 12$; by equation (7.3), get $v = 20$ and by equation (7.4), get $f = 12$. These parameters agree with those for the dodecahedron.

To prove that each of these shapes is unique (for the given v, f, e, d, s), one need only experiment with putting together pieces and retaining convexity. Note that convexity is critical, since otherwise, one could put a dent in the icosahedron. □

Name	v	f	e	s	d
Tetrahedron	4	4	6	3	3
Cube	8	6	12	4	3
Octahedron	6	8	12	3	4
Icosahedron	12	20	30	3	5
Dodecahedron	20	12	30	5	3

Table 7.1: Data for Platonic solids

7.3 Archimedean solids

An Archimedean solid is a convex polyhedron whose faces are all regular polygons (all with the same edge length), not all the same, so that at each vertex, the pattern of polygons surrounding the vertex is the same. There are 13 Archimedean solids, two of

which also have different mirror reflections (called enantiomorphs). See Figure 7.3 for pictures of each, and see Table 7.3 for details (given in the order in Figure 7.3).

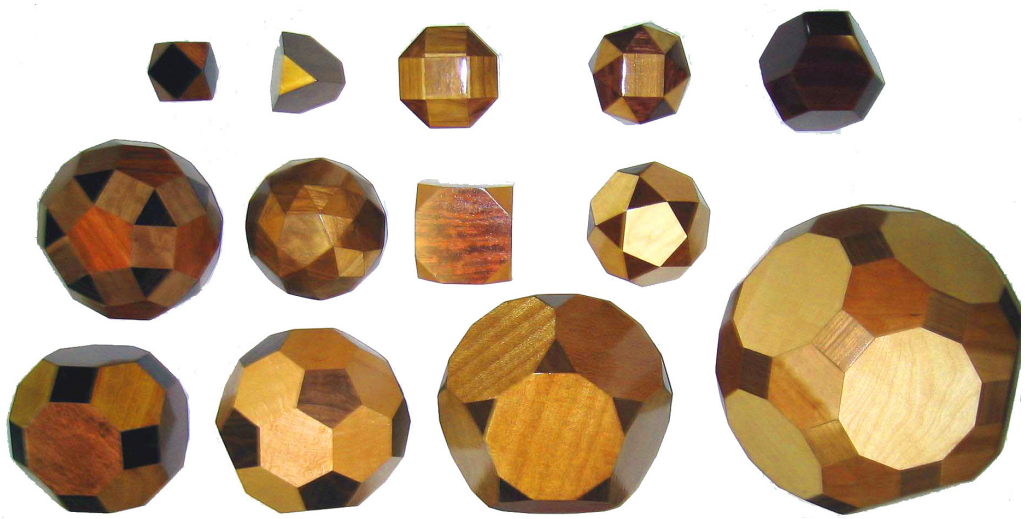


Figure 7.3: The Archimedean solids; hollow wood by DSG, 2003

Name	v	e	f	face count (type)	d
Cuboctahedron (Dymaxion)	12	24	14	8(3), 6(4)	4
Truncated tetrahedron	12	18	8	4(3), 4(6)	3
(Lesser) Rhombicuboctahedron	24	48	26	8(3), 18(4)	4
Snub cube	24	60	38	32(3),6(4)	5
Truncated octahedron (Mecon)	24	36	14	6(4),8(6)	3
Lesser rhombicosidodecahedron	60	120	62	20(3),30(4),12(5)	4
Snub dodecahedron	60	150	92	80(3), 12(5)	5
Truncated cube	24	36	14	8(3), 6(8)	3
Icosidodecahedron	30	60	32	20(3),12(5)	4
Great rhombicuboctahedron (Truncated cuboctahedron)	48	72	26	12(4),8(6),6(8)	3
Truncated icosahedron	60	90	32	12(5),20(6)	3
Truncated dodecahedron	60	90	32	20(3),12(10)	3
Great rhombicosadodecahedron (Truncated icosidodecahedron)	120	180	62	30(4),20(6),12(10)	3

Table 7.2: Table for Archimedean solids

To prove that only 13 Archimedean solids exist, one can repeat the ideas in the

proof of Theorem 7.2.1, but first by considering how different polygons can surround a vertex. The calculations take a few pages, but they are elementary.

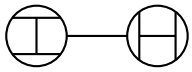
7.4 Fáry's theorem

The following is named after István Fáry, even though it was proved by Klaus Wagner [968] 12 years earlier, and later, independently by S. K. Stein [886] in 1951.

Theorem 7.4.1 (Fáry, 1948 [354]). *Any planar graph has a plane drawing with all edges as straight line segments.*

One simple proof of Theorem 7.4.1 is by induction (see, *e.g.*, [190, pp. 259–260]).

Exercise 341. *Give a drawing of the following 3-regular graph (where vertices are the intersection points) due to Tait [903, Fig.2, facing p.410], using straight lines:*



(This graph is repeated in Figure 7.12.)

Harborth *et al.* [498] later conjectured that any planar graph can be drawn with straight line segments having *integer* lengths. So far, this conjecture has been confirmed for only cubic planar graphs [412].

7.5 Trees and planar graphs

The following result is not ordinarily given in a graph theory course, but I am told that it is used by those studying graph drawing or computer graphics. A "triangulated" planar graph is a plane drawing of a planar graph whose every face has a triangular boundary (including the outer face). The following theorem was proved by Nash-Williams [711] in 1961, and was rediscovered twice later (see [553] and [835, 836]).

Theorem 7.5.1 (Nash-Williams, 1961 [711]). *Every (connected) triangulated planar graph can be decomposed into three spanning trees.*

I am told that the proof is essentially a greedy algorithm. Thanks to Jyoti [694] for bringing this to my attention.

The result in the next exercise has to do with planar trees and geometric graphs. A geometric graph is a graph whose vertices are points in the plane and edges are straight line segments; each edge is incident with only two vertices.

Exercise 342. *Prove by strong induction on $n \geq 2$ that if the edges of a complete geometric graph on n vertices are 2-coloured, there exists a planar spanning tree (i.e., not self-intersecting) that is monochromatic.*

The result in Exercise 342 extends Exercise 60 since connected graphs have spanning trees.

For another topic regarding trees and some planar graphs, see Section 3.8 on "treewidth".

7.6 Dual of a planar graph

To each drawing of a planar graph G, there is another graph determined by adjacency of faces in G.

Definition 7.6.1. *Let $G = (V, E)$ be a planar graph with a given planar drawing with faces $F_1, F_2, \ldots, F_f$ (including the outer face). Define the* dual, *or* planar dual *of G, the dual of G, denoted $G^* = (V^*, E^*)$, with vertices $V^* = \{F_1, \ldots, F_f\}$, and edge set defined by $\{F_i, F_j\} \in E^*$ if and only if faces F_i and F_j share a (non-trivial) common border.*

See Figure 7.4 for an example.

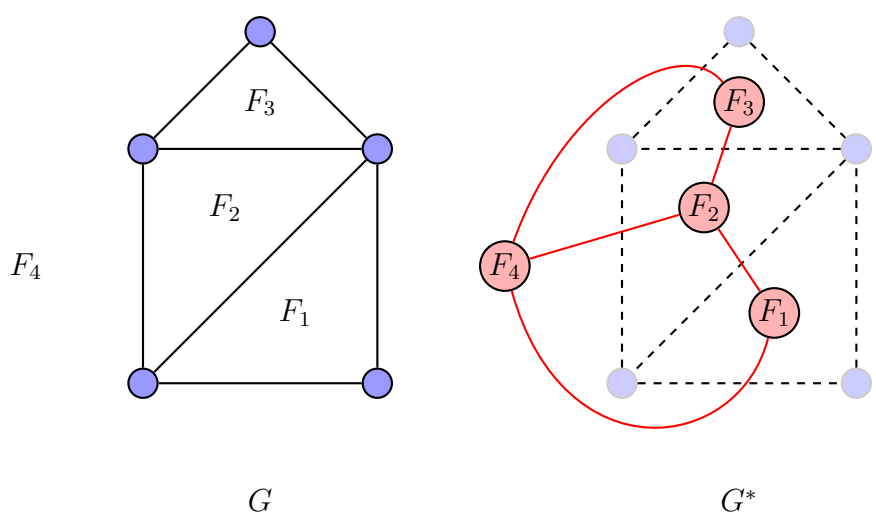


Figure 7.4: A planar graph and its dual in red

As is seen in Section 7.7, questions regarding colouring faces of planar graphs (or maps) are translated to colouring vertices of the dual.

Note: Some authors define the dual of a planar graph G to be a multigraph, where if two regions F_i and F_j share multiple edges of G, then G^* has multiple edges between

F_i^* and F_j^*. (For example, in Figure 7.4, such a dual would have two edges between F_4 and F_1 and two edges between F_4 and F_3.) Since such a multigraph has the same vertex-colouring properties as the simple dual graph described above, the simple dual graph definition given here is not restrictive.

Exercise 343. *Let G be a planar graph with a fixed planar drawing. Show that the dual G^* is also planar (regardless of which definition of dual is used, the one for simple graphs or the one for multigraphs).*

If a planar graph has two different plane drawings, the multigraph duals of each need not be isomorphic; such an example occurs in Exercise 344.

Exercise 344. *Show that the images in Figure 7.5 (given in [22, p. 265]) are different plane drawings of the same planar graph. Using the alternative definition of planar dual that gives a multigraph (one edge in the dual for every edge in G that separates two regions), show these two drawings have non-isomorphic multigraph duals.*

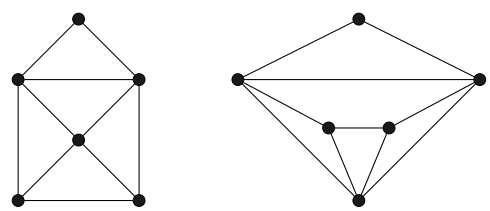


Figure 7.5: Isomorphic graphs with different plane drawings and different mulitigraph duals.

Exercise 345. *Let G be a planar drawing of K_4. Show that the dual of G is isomorphic to K_4.*

Exercise 346. *Show that the dual of the octahedron graph is the cube graph Q_3, and that the dual of the cube graph is the octahedron graph.*

Exercise 347. *Show that planar graphs of the dodecahedron and the icosahedron are duals of each other.*

The result in the following exercise might seem surprising.

Exercise 348. *Let G be a connected planar graph. Show that G is bipartite if and only if the dual of any plane drawing of G is Eulerian.*

Exercise 349. *If a planar graph G is Hamiltonian, is its dual G^* also Hamiltonian?*

For more on duals and how they relate to colouring planar graphs, see Section 7.7.

7.7 Colouring planar graphs and the four colour theorem

Colouring planar graphs is an area of mathematics with a rich history, much of which started with the four colour conjecture for colouring maps:

Conjecture 7.7.1 (Four colour map conjecture)**.** *Any map with contiguous countries can be coloured with at most four colours so that any two neighbouring countries (sharing a non-trivial border) are coloured differently.*

In certain cases, as least four colours must be used since, for example, Luxembourg is surrounded by France, Germany, and Belgium, and each pair of these countries is adjacent (see Figure 7.6). In other words, the dual of the map for these four countries is K_4, which is not (properly) 3-colourable.

Remark 7.7.2. *When looking at maps as planar graphs, the outer region is often not coloured, and so the "dual" graph formed by letting faces be vertices does not always have a vertex for the outer region of the map. In many cases, this causes no difficulties since a colour for the outer region can also be found that differs from that of the countries on the perimeter. One might also imagine an outer region as ocean, which gets its own colour or is not coloured at all. So, technically, the "dual" of a planar graph for a map does not always have the extra vertex for the outer region. It turns out that the four colour conjecture is true in either case, with or without the outer region being coloured.*

Figure 7.6: Luxembourg needs a fourth colour

The fact that at least four colours might be required to properly colour a map was observed by Francis Guthrie in the mid 1800s; the schematic diagram in Figure 7.7 was given by his brother Frederick Guthrie in the 1880 *Proceedings of the Royal Society* (of Edinburgh), page 728. (See below for more history of the four colour map conjecture.) Note that the outer region in Figure 7.7 is uncoloured, but it can also be coloured with colour 1.

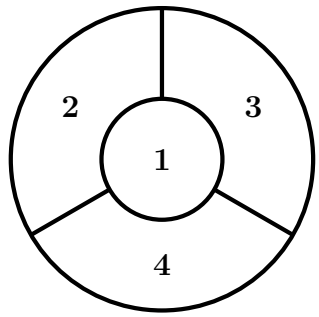


Figure 7.7: One type of map where four colours are needed

Exercise 350. *Find four countries in South America of the pattern in Figure 7.7 that shows that the map of the South America requires at least four colours.*

The result in the following exercise was noted by Tait [901, p. 501], although I can not confirm that Tait was the first to make this observation.

Exercise 351. *Let G be a connected planar graph with all vertex degrees even. Prove that for any planar drawing of G, the faces can be 2-coloured so that faces sharing a common edge receive different colours.*

Exercise 352. *Let G be a connected planar (simple) graph with a drawing so that every face is a triangle, and $\chi(G) = 3$. Show that the faces can be 2-coloured so that faces sharing a common edge are coloured differently.*

Instead of "face colourings", vertex or edge colourings are also studied for planar graphs.

Exercise 353. *Find an example of a triangulated planar graph (every face is a triangle) whose vertices are not 3-colourable.*

The four colour conjecture for map colouring can also be stated in the dual form (where countries become vertices, and two vertices are adjacent if and only if the corresponding two countries share a common border):

Conjecture 7.7.3 (Four colour conjecture (4CC))**.** *If G is a planar graph, then $\chi(G) \leq 4$.*

The four colour map conjecture was entertained by the British mathematician Francis Guthrie, who then talked to his brother Frederick about it sometime around 1852. Frederick then told Augustus De Morgan about the conjecture, who then (according to [871]) shared it with William R. Hamilton in a letter (23 October 1852). It is believed that De Morgan also published the conjecture (14 April 1860, in a book review for Whewell's *Philosophy of discovery* in the journal *Athenaeum*). On 13 June 1878,

Cayley [184] revived interest in the problem while at the London Mathematics Society meeting and the following year published a short paper [185] on the topic.

A London lawyer (and former student of Cayley) Alfred Bray Kempe (1849–1922) thought that he had settled the four colour map conjecture and in 1879, his "proof" was announced in *Nature* [562], and later that year he published [563] a "proof", one that was acclaimed for over a decade before an error was found. He also wrote an article for *Nature* [564] in 1880. Very shortly after the announcement of Kempe's "proof", in 1880 and 1884, Tait [901, 902, 903, 904] announced that he had several new proofs, and outlined some his ideas, each time making false assumptions. However, Tait did correctly provide a way to look at 4-colourings of faces/regions as 3-colourings of the edges, giving an equivalent form of the 4CC in terms of edge-colourings; see Theorem 7.7.8.

In 1890, Percy John Heawood (1861–1955) discovered an error in Kempe's "proof" [503] and provided an example (see Figure 7.8) where Kempe's approach failed. However, as Heawood observed, the Kempe proof idea indeed proves "the 5-colour theorem".

Before giving the proof of the 5-colour theorem, observe that there is simple proof of a "6-colour theorem":

Exercise 354. *Use induction and Lemma 7.1.5 to prove that any planar graph is 6-colourable.*

Theorem 7.7.4 (Kempe–Heawood)**.** *Any planar graph G is 5-vertex-colourable, i.e., $\chi(G) \leq 5$.*

Proof sketch: The original proof was in terms of maps and regions. Instead, here the proof is sketched for the dual problem for graphs. The proof is by induction on the number of vertices. After eliminating some trivial cases, let $n \geq 6$ and suppose that any graph on $n-1$ vertices is 5-colourable. Let G be a planar graph on n vertices.

By Lemma 7.1.5 let $x \in V(G)$ have degree at most 5, and by the induction hypothesis, suppose that a proper 5-colouring of $H = G - x$ is given using colours, say 1,2,3,4,5. If $\deg(x) \leq 4$, the colouring of H can be extended by using any colour for x that is not used by its neighbours. If $\deg(x) = 5$ and not all colours are used for neighbours of x, then again, an unused colour can be used to colour x.

Remaining is the case when $\deg(x) = 5$ and where all five colours are used among the neighbours of x. Suppose that y_1, y_2, y_3, y_4, y_5, are the neighbours of x, where each y_i is coloured i.

For each pair $1 \leq i < j \leq 5$ let H_{ij} be the subgraph of H induced by vertices coloured i or j. If y_1 and y_3 are in different components of H_{13}, then in the component containing y_1, the colours 1 and 3 can be switched, thereby colouring y_1 with 3, leaving colour 1 free for x.

So suppose that y_1 and y_3 are in the same component of H_{13}, that is, there is a path P (called a Kempe chain) joining y_1 and y_3 that contains only vertices coloured 1

or 3. Similarly, consider colours 2 and 4; if y_2 and y_4 are different components of H_{24}, colours in one of these components can be switched, leaving either colour 2 or 4 for x. Finally, suppose that y_2 and y_4 are in the same component of H_{24}. Then there exists a path Q from y_2 to y_4 using vertices coloured only 2 and 4. Since Q and P intersect, this final case is impossible. □

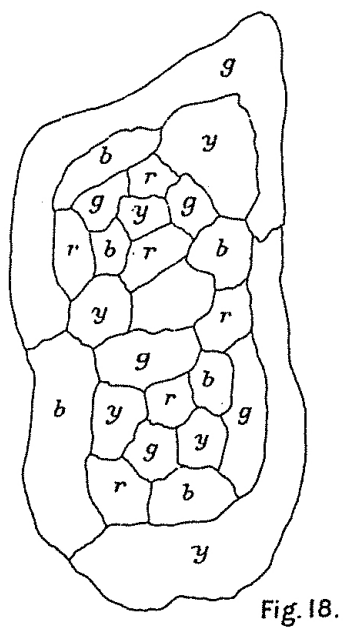


Figure 7.8: Heawood's example of regions where Kempe's proof fails

In Figure 7.9 is the partially 4-coloured Heawood's example, with touching blue-green and blue-yellow chains identified (thanks to K. Gunderson for this diagram and the next).

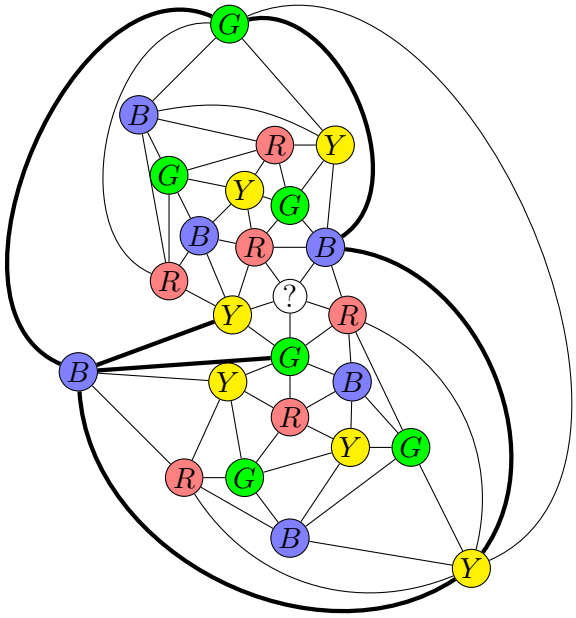


Figure 7.9: Partially coloured Heawood example

Soifer [876] found a 9-vertex example (probably the smallest such) showing how Kempe's proof can go wrong.

As Heawood observed, not all partial 4-colourings can be extended. In Figure 7.10 is a 4-colouring of the Heawood example.

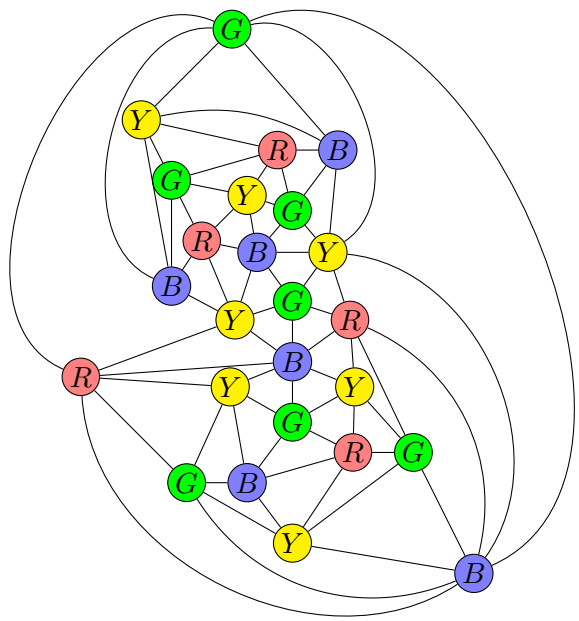


Figure 7.10: Heawood's example 4-coloured

By switching to the dual graph, proving that any planar graph is 4-vertex-colourable would then solve the four colour map conjecture. Using a method called "discharging" developed by Heesch, Kenneth Appel and Wolfgang Haken (with help on computers from Koch), finally proved the four colour conjecture in 1976, and published it in 1977. Their proof took 1200 hours of computing time, and there were 1936 configurations to check.

Theorem 7.7.5 (Four colour theorem (4CT) Appel–Haken–Koch, 1977 [56, 57])**.** *If G is a planar graph, then $\chi(G) \leq 4$.*

Since a proper face colouring of a planar graph with four colours is equivalent to a 4-colouring of its dual, which is also planar, it follows that:

Corollary 7.7.6. *The four colour map conjecture is true.*

The Appel–Haken proof was aided by computer, and there was much controversy surrounding such proofs. For a 2002 memoir on the four colour theorem (4CT), see Robin Wilson's *Four colours suffice* [987]. Another substantial (but earlier) reference for four colouring problems is a book by Oystein Ore [730]. For more on the history of the 4CT, related theorems, and many references, see [876].

Remark 7.7.7. *Apparently, at about the same time or very soon after the Appel–Haken proof was announced, the 4CT was also proved right here at the University of Manitoba (!) by Allaire and Swart [27], [28]. Sorry, I don't yet know any more details.*

The 4CT was proved again in 1997 by Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas [798], again by computer, but with only 633 configurations.

Shortly after Kempe's "proof" of the 4CC, Peter Guthrie Tait (1831–1901) claimed to give other "proofs" in a presentation to the Royal Society of Edinbourgh on 15 March 1880 (the paper was later withdrawn, but the abstract of this presentation remains in his *Scientific papers* [901, pp. 501–503]), where four "proofs" are mentioned, one of which was sketched. Tait's proofs were later found to be lacking (see below for more details). However, Tait (correctly) formulated an equivalent form of the 4CC that used *edge colourings*. See [816] for a list of thirteen conjectures equivalent to the 4CC, along with many historical details.

Tait showed (Theorem 7.7.8 below) that if a planar graph G was 4-face-colourable then the edges of G are 3-colourable. In fact, with some reading between the lines, Tait showed that these conditions were equivalent!

Recall (from Section 2.2, Fleury's algorithm) that a *bridge* in a graph is an edge whose removal disconnects the graph (or increases the number of components). A graph with no bridges is called "bridgeless" (or "2-edge-connected"). If a map, viewed as a graph, has a bridge, then the colours of the regions on either side of the bridge are the same, so to simplify the problem, only bridgeless graphs are considered.

Let G be a bridgeless planar graph of a map (with at least four countries). Then there are no vertices of degree 1. Also, a graph from a map has no vertices of degree 2 (such vertices correspond to a vertex along a border that serves no purpose). Transform G into a cubic (3-regular) graph G' by "blowing up" each vertex of degree $d > 3$ by replacing the vertex with a small d-gon (as in Figure 7.11), thereby giving a cubic graph.

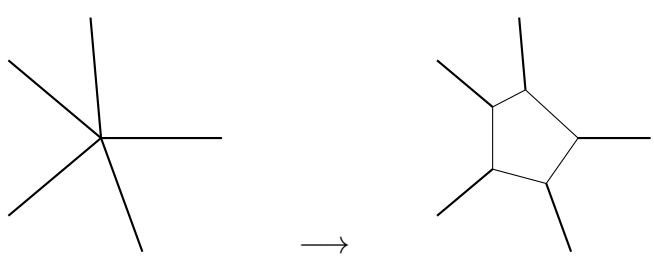


Figure 7.11: Replacing a high degree vertex

If one can face colour G' with four colours then one can 4-colour the faces of G (ignoring the colours of the newly added regions). Thus, it suffices to prove the 4CC for planar bridgeless cubic graphs.

Theorem 7.7.8 (Tait, 1880 [901, 902, 903]). *Let G be a planar bridgeless cubic graph. The faces of G are 4-colourable if and only if the edges of G are 3-colourable.*

Proof ($\rightarrow$): Let the faces of G be properly 4-coloured with the colours A, B, C, D.

Since G is bridgeless, the faces on either side of an edge are coloured differently. For each edge in G, there are $\binom{4}{2} = 6$ possible pairs for colours of faces on either side, namely (dropping extra notation) AB, CD, AC, BD, AD, BC. For each edge, if the two incident faces are coloured

- either A and B or C and D, colour the edge α;
- either A and C or B and D, colour the edge β;
- either A and D or B and C, colour the edge γ.

It remains to verify that the above colouring is a proper 3-colouring of $E(G)$. Indeed, verification is seen by showing that at each vertex, all three colours are used. For example, consider a vertex w, with the three faces surrounding w coloured A, B, and C, respectively. The edge (from w) separating A and B is then coloured α; the edge separating B and C is coloured γ; and the edge separating C and A is coloured β. The similar situation occurs for any 3 colours.

($\leftarrow$): Let $c : E(G) \rightarrow \{1, 2, 3\}$ be a proper 3-colouring of the edges of G. For each $i, j \in \{1, 2, 3\}$, $i \neq j$, let E_{ij} denote the set of edges coloured i or j. Since each vertex is incident with all colours, each E_{ij} induces a 2-factor (a spanning 2-regular subgraph) of G, say $G_{ij} \subset G$, and so is a union of cycles. In each such cycle, the two colours alternate and so each such cycle is even. (Note: Since G is cubic, Lemma 1.8.5 says that G has an even number of vertices.) Thus, the faces of each G_{ij} can be two coloured (one colour inside cycles, a second colour outside). Note that a face in G_{ij} is a union of faces in G. Let the faces of G_{12} be coloured A and B, and let the faces of G_{13} be coloured 1 and 2.

If a face in G is contained in an A-coloured face and a 1-coloured face, then colour this face with A1. Similarly, all faces are coloured with one of A1, A2, B1, B2. This colouring is a proper 4-colouring of the faces of G. □

The first part of the proof above appears in [902], but a variant appears in [903]. It might be interesting to note that Tait [903, item 10], just after defining this colouring, writes only "... and the thing is done." Some authors (*e.g.*, see [190, p. 464], [977]) prefer to define the colouring by using two bit binary words that obey certain addition conditions.

The second part of the above proof has many variations. The proof above is modelled after the one given in [143, 290–291]. Some of these variations use the dual graph (*e.g.*, [876, p. 185]). Tait's papers are challenging to read, since some proofs are only

sketched. For example, in [901, p. 502], Tait gives a sketch of one proof of the second part that begins by translating to the dual (perhaps with dual edges receiving the same colours as the original edges they cross). Then Tait triangulates the graph, and turns all triangles into 4-cycles by adding degree 2 vertices. Thus all cycles are even, and two sets of cycles are used for the 4-colouring. Then remove all the extra edges added.

Tait then claimed that

Claim 1 (Tait)**.** *Every cubic planar 3-connected graph is 3-edge-colourable.*

So, by Theorem 7.7.8, he claimed that he had proved the four colour conjecture. Where did he go wrong? Although it is now known (by the 4CT and Theorem 7.7.8) that Claim 1 is true, Tait failed to prove this as he claimed.

In the second last paragraph of Tait's first paper [901, pp. 502–503], just after a proof sketch of Theorem 7.7.8 Tait says:

> This mode of treating the question shows incidentally that in a map where only three boundaries meet at each point, the boundaries may be coloured with *three* colours, so that no two of the same colour are conterminous.

It is not clear if he is claiming that planar cubic graphs are 3-edge-colourable independent of Kempe's "proof". However, in his (one page) second paper [902], Tait writes:

> In a paper read to the Society on 15th March last (*ante*, p. 501), I gave a series of proofs of the theorem that four colours suffice for a map.

Tait mentions that he has since withdrawn the paper, and wishes instead to give a proof based on the remark in the second last paragraph of the first paper. In his third paragraph, Tait [902] writes

> The remark referred to is to the effect that, if an even number of points be joined, so that three (and only three) lines meet in each, these lines may be coloured with *three* colours only, so that no two conterminous lines shall have the same colour.

Tait then mentions that if a graph has a bridge (the "excepted case"), then the theorem is not true. (He only proves this in a later paper where he gives a counterexample—see below.) This restriction to bridgeless cubic graphs is repeated in [903, p. 408] , where Tait writes:

> The difficulty of obtaining a simple proof of this theorem originates in the fact that it is not true without limitation.

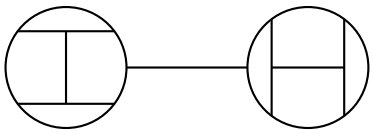


Figure 7.12: Tait's example of a cubic planar graph that is not 3-edge-colourable

Tait [903, Fig.2, facing p.410], then gives an example (see Figure 7.12, also previously given in Exercise 341) of a cubic planar graph with a bridge that is not 3-edge-colourable:

Exercise 355. *Show that the example in Figure 7.12 is not 3-edge-colourable.*

Tait [902] (second last paragraph) claimed "an elementary theorem", namely that any cubic bridgless graph is 3-edge-colourable (with no mention of planarity) and the proof "is given easily by induction". However, as Julius Petersen [747] pointed out in 1898, the Petersen graph (see Section 1.6.6) is a 3-regular bridgeless graph that is not 3-edge-colourable (nor Hamiltonian).

In the last paragraph of [902], Tait remarks that "We escape the excepted case [of having a bridge] by taking the points as the summits of a polyhedron." Thus by Steinitz's theorem, Tait implicitly restricts the bridgeless (2-connected) property to "3-connected" for cubic planar graphs. (It might be of independent interest that "3-connected" can indeed be replaced by "2-connected", as is shown in [190, pp. 466–468].)

In [904, p.94, (16)] Tait repeats that cubic graphs discussed earlier "may be regarded as mere distorted *plane projections of polyhedra*" and mentions "two obvious classes of exceptions, which will be at once understood from the simple figures 7 and 8." In [904, Figs. 7, 8, plate III], Tait gives two 3-regular multigraphs, the first with a bridge and no good 3-edge-colouring, and the second with a good 3-edge colouring but only 2-connected; it is not clear what his second example is meant to exclude—perhaps only multigraphs or 2-connected graphs.

In summary, Tait asserts Claim 1 with no proof. Tait outlines a proof idea, based on the following simple notion. If G is a cubic planar 3-connected graph, and if one can find a 2-factor consisting of even cycles (*e.g.*, a Hamiltonian cycle), then these even cycles can be edge-2-coloured and the remaining 1-factor receive a third colour. If there are odd cycles, Tait [903, p.410] gives only an example that can be transformed into even cycles (see Figures 6a,b,c,d, Plate 10), but no general proof. He says (note the use of "must").

> Such solutions must evidently be possible in all cases, with the exception of that already excluded.

It seems as if he hinted at the possibility that all graphs under consideration were Hamiltonian; again, the Petersen graph is a (non-planar) counterexample.

Oddly, in his 1984 paper, Tait [904, p. 93, (14)] yet again asserts Claim 1:

> But, if $2n$ points in a plane be joined by $3n$ lines, no two of which intersect, (i.e., so that *every* point is a terminal of 3 different lines), the figure requires n separate pen-strokes. It has been shown that in this case (unless the points be divided into two groups, between which there is but *one* connecting line, fig.7) the $3n$ lines may be divided into 3 groups of n each, such that one of each group ends at each of the $2n$ points.

It is noteworthy that in various places in the four Tait papers, Tait mentions that he did not know how to prove certain claims (relying only on exhaustive searches and examples), while at the same time asserting Claim 1 as truth. For example, in [904, pp. 94–95], Tait writes:

> Now in *every one* of the great variety of cases which I have tried (where the figure was, like fig. 6, a projection of a true polyhedron) I have found that a *complete* circuit of edges, alternately of two of these groups (such as $\alpha\beta\alpha\beta$ & c.) can be found, usually in many ways, so as to exhaust both groups and pass once through each of the angles. ... But, in the words of the extraordinary mathematician Kirkman, whom I consulted on the subject, "the theorem.....has this provoking interest, that it mocks alike at doubt and proof[1]" Probably the proof of this curious proposition has (§11) hitherto escaped detection from its sheer simplicity. Habitual stargazers are apt to miss the beauties of the more humble terrestrial objects.

The above sentiment was also mentioned in [903, p. 409, 2.], "But on this point, I am not yet certain; and I pass it by for the present, as it is not of importance to the proposition, ...". So, it seems as if Tait formulated the following:

Conjecture 7.7.9 (Tait, 1980 [903], 1884 [904, (16)])**.** *Every polyhedral (3-connected) cubic graph is Hamiltonian.*

He also seems to says that the truth of this conjecture is not necessarily needed for his proof of Claim 1.

Remark 7.7.10. *As mentioned above, if Conjecture 7.7.9 is true, then Claim 1 is true, however it seems to me that Tait believed he had proved Claim 1 and solving Conjecture 7.7.9 would just give an easier proof. In some texts (see, e.g., [977, p. 302]), "Tait's conjecture" is that Claim 1 is true. After thoroughly reading Tait's papers, he never seemed to abandon the notion that he had proved Claim 1 and hence had proved the 4CT. I would not be so generous as to call this one of Tait's conjectures (even though it follows from Conjecture 7.7.9).*

If indeed Conjecture 7.7.9 were to be true, then every 3-connected cubic planar graph is 3-edge-colourable, from which it would follow that every bridgeless planar graph is 3-edge-colourable, and so by Tait's main theorem (Theorem 7.7.8), the 4CC would be proved.

In 1946, Tutte [936] found a counterexample to Conjecture 7.7.9, given in Figure 7.13.

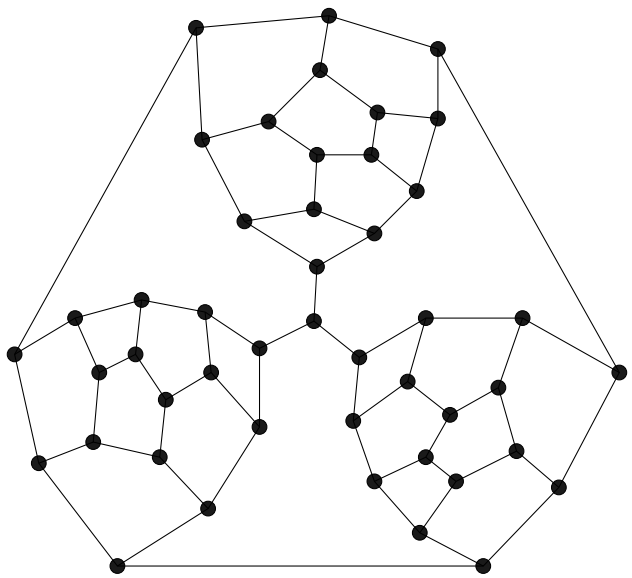


Figure 7.13: The Tutte graph, 46 vertices, 69 edges, 25 faces

The Tutte graph is planar, 3-regular, and 3-connected, and so (by Theorem 7.1.1) is a cubic polyhedral graph.

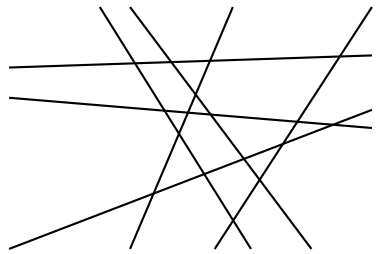
Exercise 356. *Show that Tutte's graph (in Figure 7.13) is indeed a counterexample to Tait's claim that every planar polyhedral cubic graph is Hamiltonian.*

Note that by the 4CT and Tait's theorem (Theorem 7.7.8), the Tutte graph is 3-edge-colourable. For more parameters for the Tutte graph, see Appendix 16.

For more on Hamiltonian cycles in planar graphs, see Section 7.9.

7.8 3-colourable planar graphs

For some planar graphs, a vertex 3-colouring is easy to find. Consider a family of lines in the plane, no three concurrent (passing through one vertex).

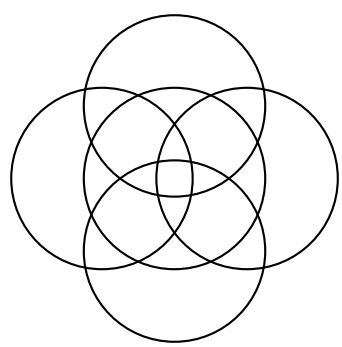


Create the associated (planar) graph by letting the intersection points be vertices and edges are pairs of vertices that are adjacent (on some line).

Theorem 7.8.1. *Let G be planar graph induced by a finite number of lines, no three concurrent, Then G is (vertex) 3-colourable, i.e., $\chi(G) \leq 3$.*

Proof: Perturb the lines (if necessary) so that no vertex is directly above another. Colour the vertices from left to right using a greedy algorithm. Each vertex being coloured has at most two coloured neighbours to the left, and so at worst, a third colour is needed. □

A similar theorem for intersecting circles does not hold. One counterexample is a 4-regular graph given by Koester [587, 588] (in the mid 1980s) arising from the five circles below; it is not too difficult to check that the associated graph is not 3-colourable (but since it is planar, it is 4-colourable).



Some maps are 2-colourable.

Exercise 357. *Show that if G is a planar graph, its faces can be properly 2-coloured (that is, its dual is 2-colourable) if and only if every vertex in G has even degree.*

Theorem 7.8.2 (Grötzsch, 1959 [446])**.** *Any triangle-free planar graph is 3-colourable.*

Grötzsch's theorem has been strengthened to allow 3 triangles. For a recent proof of Grötzsch's theorem, see [918]; also see [926, p. 260] for further references. In 1993, an extensive survey of 3-colourability problems was completed by Steinberg [887].

Steinberg [887] conjectured that every planar graph without 4-cycles or 5-cycles is 3-colourable. Answering a weaker conjecture by Erdős (see [614]), Borodin [148], and independently, Sanders and Zhao [820] proved that every planar graph not containing any cycles of length 4, 5, 6, 7, 8, or 9 is indeed 3-colourable. In 2000, Salavatipour [819] first showed that planar graphs not containing cycles of lengths 4, 5, 6, 7, or 8 are 3-colourable; the proof given relies on reducible configurations and yields an $O(n^2)$ time algorithm for finding a good 3-colouring. [Salavatipour was a student of Mike Molloy at Toronto.] Then, in 2005, Borodin, Glebov, Raspaud, and Salavatipour [149] showed that forbidding only cycles of lengths 4, 5, 6, and 7 is sufficient.

Exercise 358. *Prove that if G is a planar Eulerian graph (on at least 3 vertices) with a planar embedding so that all faces are triangles, then $\chi(G) = 3$.*

Theorem 7.8.3 (Heawood, 1898 [504])**.** *A maximal (triangulated) planar graph is 3-colourable if and only if all degrees are even.*

In fact, a much stronger theorem is known; Steinberg [887] attributes the following result to Heawood as well, but since then (according to [259]), it has also been proved by Król [608, 609] and Martinov [657].

Theorem 7.8.4 (Heawood, see [887])**.** *A planar graph G is 3-colourable if and only if it is a subgraph of a maximal (triangulated) planar graph with all degrees even. Any proper 3-colouring of a planar graph can be extended to that of some triangulation with all even degrees.*

7.9 Planar graphs and Hamiltonian cycles

By adding edges, any planar graph can be triangulated (so that every region is a triangle).

Theorem 7.9.1 (Whitney, 1931 [980])**.** *Any triangulated planar graph is Hamiltonian.*

One reason to study Hamiltonian cycles in planar graphs is that such cycles can be used either to 3-colour the edges or to 4-colour the faces. The solution to the following exercise has been discussed earlier in this chapter (this also appears as Exercise 287).

Exercise 359. *Show that if G is a cubic Hamiltonian graph (not necessarily planar), then G is 3-edge-colourable.*

The result in the following exercise was noted by Tutte [939, p. 116], where he says "It is a commonplace of the theory of map-colourings that any map defined by a planar graph having a Hamiltonian circuit can be coloured in four colours." Tait [901] made a similar observation (but in the dual with even cycles) , and in 1931, Whitney [980] also observed that a cubic planar graph with a Hamiltonian circuit is 4-face-colourable, (with no mention of Tait's work, even though this follows directly from Tait's theorem (Theorem 7.7.8) and the result in Exercise 287, another observation from Tait).

Exercise 360. *Show directly (not using the 4CT) that a planar Hamiltonian graph is 4-face-colourable. Hint: Use two colours for regions outside a Hamiltonian cycle and two other colours for the inside regions.*

A Hamiltonian graph is 2-connected (every point lies on a cycle). In 1970, Barnette and Jucovič [78] showed that the minimum number of vertices in a 3-connected planar non-Hamiltonian graph is 11. The Herschel graph in Figure 2.11 attains this bound. (Recall, that any bipartite graph on an odd number of vertices is not Hamiltonian.)

Recall that a *polyhedral graph* is a graph for some polyhedron, and by Steinitz's theorem (Theorem 7.1.1), polyhedral graphs are precisely the planar 3-connected graphs.

In 1988, Holton and McKay [524] showed that all 3-connected cubic planar graphs on at most 36 vertices are Hamiltonian, and showed that there are precisely six counterexamples on 38 vertices, which had previously been found by Lederberg, Barnette, and Bosák.

In 1961, Balinski [69] conjectured that every planar cubic 3-connected graph contains a Hamiltonian path. The following year, Grünbaum and Motzkin [449] provided a counterexample on 944 vertices whose longest path contained no more than 939 vertices. In an unpublished note by Thomas A. Brown in 1960 (see [167]) another counterexample was given. As of 1973, the smallest counterexample contains 88 vertices. See [94, pp. 223–225] for more references and details.

However, increasing the connectivity in planar graphs from 3 to 4 indeed guarantees Hamiltonicity.

Theorem 7.9.2 (Tutte, 1956 [939])**.** *Every 4-connected planar graph is Hamiltonian.*

For more on paths and cycles in planar graphs, see, *e.g.*, [206] or [917]. For counting the number of hamiltonian cycles in planar graphs (and outerplanar graphs, see Section 7.10) and for additional references, see [630] or [477].

7.10 Outerplanar graphs

This section is only a very brief introduction to outerplanar graphs, which were introduced and studied by Chartrand and Harary [188] in 1967.

Definition 7.10.1. *A graph G is called* outerplaner *if and only if G is planar and G has a plane drawing so that every vertex is incident with the outer region.*

So trees and cycles are outerplanar. A graph satisfying the condition that any two cycles intersect in at most one vertex is called a *cactus graph*. (So trees are cactus graphs and a tree where some edges have been replaced by empty cycles is a cactus graph.) It is not difficult to see that cactus graphs are also outerplanar.

The graph $K_4 - e$ is an outerplanar graph that is not a cactus graph.

Exercise 361. *Show that if G contains a copy of K_4, then G is not outerplanar.*

Exercise 362. *Prove that if G contains a copy of $K_{2,3}$, then G is not outerplanar.*

The examples in Exercises 361 and 362 essentially characterize outerplanar graphs.

Theorem 7.10.2 (Chartand–Harary, 1967 [188])**.** *A graph G is outerplanar if and only if G does not contain a subdivision of either K_4 or $K_{2,3}$.*

A planar graph containing at least three cycles all sharing a single edge is not outerplanar. So, essentially, outerplanar graphs look like trees made out of edges and empty cycles. It is known (see, *e.g.*, [115], [116], or [224]) that outerplanar graphs have treewidth (see Section 3.8) at most 2.

An outerplanar graph is called *maximal* if and only if the addition of any edge gives a graph that is no longer outerplanar. In Section 6.12.4, a *chordal graph* is a graph whose every cycle of length at least 4 has a chord. In Exercise 310, it is asked to show that a maximal outerplanar graph is chordal.

Exercise 363. *Let G be an outerplanar graph on n vertices. Show that $|E(G)| \leq 2n-3$.*

Exercise 364. *Show that any outerplanar graph has a vertex of degree at most 2.*

Exercise 365. *Prove that if G is outerplanar, then $\chi(G) \leq 3$.*

7.11 Crossing numbers

This section is adapted from my notes on combinatorial geometry [454].

7.11.1 Definitions and examples

In this section, unless otherwise stated, all graphs are simple. Recall that a graph is called planar if and only if it can be drawn in the plane with no edges crossing (or touching) except at vertices of the graph. Such a drawing is often called a "plane drawing" of a planar graph or a "planar embedding". For the moment, let a "drawing" D of a graph $G = (V, E)$ in the plane be a set of $|V|$ points in $\mathbb{R}^2$ given by an injective embedding $\phi : V \to \mathbb{R}^2$, together with, for each $\{x, y\} \in E$, a curve (or arc, or "edge") joining $\phi(x)$ and $\phi(y)$ but containing no other vertex $\phi(z)$.

Two edges in a drawing of a graph are said to *cross* if they share a common point other than an endpoint. An *edge crossing* is an instance of a cross, namely, a triple $(e_1, e_2, \mathbf{x})$, where e_1 and e_2 are distinct edges both containing the point $\mathbf{x}$, and a crossing point is a point $\mathbf{x}$ that witnesses a crossing. *A priori*, in a drawing of a graph, two edges may cross arbitrarily many times (even edges drawn from the same vertex), and a crossing point can be a witness to more than one crossing. If two edges in a drawing

touch at a point but do not actually cross (when they are tangent), another drawing can be made where these two edges do not touch (at that point), so it is tacitly assumed that a pair of edges form a "crossing" if and only if one cuts through the other.

For a non-planar graphs G, one measure of the "non-planarity" is the minimum number of edge crossings required in any drawing of G.

Definition 7.11.1. *The* crossing number *of a graph G, denoted by $cr(G)$, is the minimum number of edge crossings among all drawings of G in the plane (or sphere).*

The crossing number is often denoted in the literature by $\nu(G)$, but in graph theory, $\nu(G)$ is often used to denote the "matching number" of a graph (see, *e.g.*, [125]), so here, the more descriptive notation $\mathrm{cr}(G)$ is used. For an introduction to crossing numbers, see the 1973 article by Erdő and Guy [323], or the blog post by Terence Tao [907]. For an extensive survey (including 409 references), see Marcus Schaefer's 2014 article [830]. There are numerous other surveys (*e.g.*, [789]).

There are many variations on this "crossing number" (for example, see [736] or [830]). Only a very few variations are considered here. One might restrict drawings to straight lines (giving "rectilinear crossing numbers", defined below), or one might restrict drawings where edges can intersect only an odd number of times, or one might ask for only drawings where each edge is crossed by at most some number of times. If more than two edges are allowed to cross at a point, one might ask how few crossing points (not pairs of edges) are required (see "degenerate crossing numbers", defined below). One might also count the minimum number of crossings in a graph drawn on some surface S, in which case the notation cr_S (or $\nu_S(G)$) might be used. If no subscript is used, it is assumed that the surface is the plane (or equivalently, a sphere). There are also "spherical crossing numbers", where drawings are restricted to spheres but edges are geodesics. The survey by Schaefer [830] contains a rather extensive description of different kinds of crossing number, only a few of which are touched upon here.

To give an upper bound for the crossing number of a particular graph, it suffices to give a drawing with few crossings, but then to check that such a drawing is optimal can be non-trivial. In general, Garey and Johnston [411] showed that computing crossing numbers is NP-complete. (To see the complexity of other types of crossing numbers, see [830].) In 2013, Cabello and Mohar [176] showed that just among the class of planar graphs with one extra edge added, it is NP-hard to decide whether or not a graph has crossing number 1, even if one assumes an upper bound on the degree! This result might seem somewhat surprising, since determining planarity can be done in linear time, *e.g.*, by the Hopcroft–Tarjan algorithm [527]. (Thanks to Stephane Durocher [270] for pointing out this reference.)

Definition 7.11.2. *The* rectilinear crossing number *of a graph G, denoted by $\overline{cr}(G)$ (or $\overline{\nu}(G)$ in some texts), is the minimum number of crossings for drawings of G in the plane that use only straight line segments.*

Trivially, $\overline{\mathrm{cr}}(G) \geq \mathrm{cr}(G)$, and there are examples for which the rectilinear crossing number is larger than the crossing number. (For example, as noted below, $\mathrm{cr}(K_8) < \overline{\mathrm{cr}}(K_8)$.)

Since graphs can represent electrical circuits, those who study integrated circuits (both simple, or more complex VLSI circuits) are also interested in crossing numbers. In very large circuits, one might want to minimize the number of layers of integrated circuit boards necessary, or the minimum number of "jumper" connections needed on one board or between boards.

If a drawing D of a graph G has $\mathrm{cr}(G)$ crossings, then D is called an *optimal* drawing (as far as crossing numbers go). It was observed (*e.g.*, in [323]) that optimal drawings satisfy a simple property.

Lemma 7.11.3. *Let G be a graph and D be an optimal drawing for $cr(G)$. Then in D, any two edges sharing a common vertex (in G) do not cross.*

So for the present discussion of crossing numbers, assume that all "good drawings" satisfy the condition that any two edges incident with the same vertex (sometimes called neighbouring edges) do not cross. This condition is not always mentioned explicitly in papers about crossing numbers. (When looking at only rectilinear drawings, this condition is automatic.) Also note that if two edges touch at a point other than at a vertex, these edges must actually cross, not just touch and then "bounce off" or be tangent.

7.11.2 Some examples

Since K_1, K_2, K_3 and K_4 are planar, for $1 \leq k \leq 4$, $\mathrm{cr}(K_k) = 0$, and by a standard drawing of K_4 with one vertex in the middle, $\overline{\mathrm{cr}}(K_4) = 0$. Other common planar graphs are trees, cycles, and the cube graph Q_3.

Recall that the graphs K_5 and $K_{3,3}$ are not planar; however, each can be drawn with precisely one pair of edges crossing.

Exercise 366. *Give drawings of K_5 and $K_{3,3}$ that each have exactly one pair of edges crossing, thereby showing $cr(K_5) = cr(K_{3,3}) = 1$. Show that such drawings exist using only straight line segments for edges, thereby showing that also $\overline{cr}(K_5) = \overline{cr}(K_{3,3}) = 1$.*

The drawing given in Figure 7.14 shows that $\mathrm{cr}(K_6) \leq 3$.

Exercise 367. *Show that $cr(K_6) = 3$.*

A complete tripartite graph $K_{a,b,c}$ is a graph whose vertex set is a disjoint union $V = A \cup B \cup C$, where $|A| = a$, $|B| = b$, $|C| = c$, and all edges go between either A and B, A and C, or B and C. For example, the graph $K_{3,2,2}$ with 7 vertices and 16 edges is given in Figure 7.15.

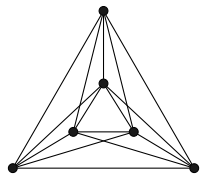


Figure 7.14: Drawing for $\mathrm{cr}(K_6) \leq 3$

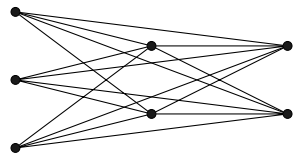


Figure 7.15: The graph $(K_{3,2,2})$

Exercise 368. *Show that $cr(K_{3,2,2}) = 2$. Hint: Apply Euler's formula to an optimal drawing considered as a planar graph on the original 7 vertices plus crossing points.*

For more general results on tripartite graphs, see, *e.g.*, [979, Thms 6-67, 6-68]. Also noted in [979] are results by Beineke and Ringeisen for crossing numbers of cartesian products of cycles and/or complete graphs.

A famous graph is the Petersen graph; it is often drawn with 5 crossings, as in Figure 7.16.

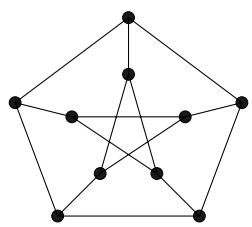


Figure 7.16: A standard drawing of the Petersen graph with 5 crossings

Exercise 369. *Find a drawing of the Petersen graph (see Figure 7.16 that shows its crossing number is at most 3 (such a drawing also exists with straight lines).*

Exercise 370. *Find a drawing of the Petersen graph with only 2 crossings. Can you find such a drawing that uses only straight line segments?*

Exercise 371. *Show that any optimal drawing of the Petersen graph has at least two crossings.*

The Heawood graph (see Figure 11.1) is the point-line incidence graph for the Fano plane, and so is a 3-regular bipartite graph on 14 vertices. The shortest cycle in the

Heawood graph has length 6 (a 4-cycle would correspond to two points on two lines, contrary to the definition of a finite projective plane). The number of crossings in Figure 11.1 is 14, but the crossing number of the Heawood graph is 3; this bound is witnessed by different rectilinear drawings, one given in Figure 7.17.

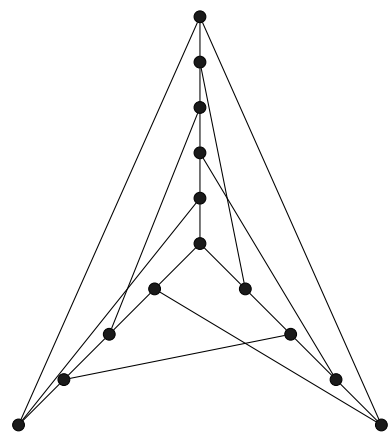


Figure 7.17: The Heawood graph with 3 crossings

Each of K_4, $K_{3,3}$, the Petersen graph, and the Heawood graph (see Figure 11.1) is 3-regular (called cubic). These are the smallest cubic graphs with respective crossing numbers 0,1,2,3. The smallest cubic graphs with crossing numbers up to 8 have been found (see Sloane's sequence A110507). It was shown [515] that finding crossing numbers just for cubic graphs is still a hard problem.

Exercise 372. *Let Q_4 denote the 4-dimensional cube graph (where vertices are binary words of length 4 and edges are between words that differ in exactly one position). Give a drawing that shows $cr(Q_4) \leq 8$.*

Eggleton and Guy [280] once thought that

$$\mathrm{cr}(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2},$$

but according to [323], the construction was found to have a gap; equality is still conjectured (also see [323]).

7.11.3 Crossing numbers for complete bipartite graphs

It is said that the study of crossing numbers began with Turán, who considered the number of crossings of tracks used in a brick factory (see [935] for Turán's description of the problem, which he thought about while in a work camp during WWII). The system of tracks formed a complete bipartite graph between kilns and storage sites.

See [84] for a history of the brick factory problem. A similar problem arose in a 1944 paper in sociology [162] in representing relationships by graphs (sociograms), where the partite sets were boys and girls, and the aim was to draw the graph with fewest crossings so that information was more visibly apparent.

For any positive integer k, both $K_{1,k}$ and $K_{2,k}$ are planar, so $\mathrm{cr}(K_{1,k}) = \mathrm{cr}(K_{2,k}) = 0$. In 1953, Zarankiewicz [1007] gave a general construction that gives an upper bound on the crossing numbers for complete bipartite graphs. For each $m, n \geq 3$, define a drawing of $K_{m,n}$ by putting the m vertices balanced on points $(\pm i, 0)$ on the x-axis, and similarly putting the other n vertices balanced on the y-axis, and draw all edges with straight lines. For example, see Figure 7.18 (which also appears in [323]) for such a drawing of $K_{7,7}$ showing $\mathrm{cr}(K_{7,7}) \leq 81$.

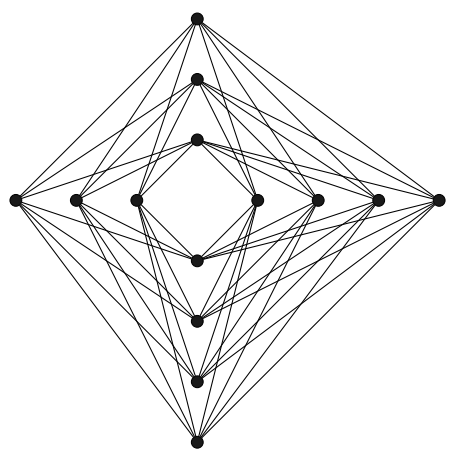


Figure 7.18: $\mathrm{cr}(K_{7,7}) \leq 81$

With a bit of work, one can show that the Zarankiewicz construction gives

$$\begin{aligned}
\mathrm{cr}(K_{2k,2\ell}) &\leq (k^2 - k)(\ell^2 - \ell), \\
\mathrm{cr}(K_{2k,2\ell}) &\leq (k^2 - k)\ell^2, \\
\mathrm{cr}(K_{2k,2\ell}) &\leq k^2\ell^2,
\end{aligned}$$

or equivalently (as observed by Urbanik, and Rényi and Turán, see [490]),

$$\mathrm{cr}(K_{m,n}) \leq \frac{1}{4}\left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lfloor \frac{m-1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{7.5}$$

In his 1953 paper, Zarankiewicz [1007] announced that his construction was optimal, and he provided details in 1954 [1008]. However, it was observed (see [462]) that Zarankiewiicz's proof was not complete; it now remains as a conjecture (*e.g.*, see [323]) that the inequality in equation (7.5) is in fact equality.

Conjecture 7.11.4. *For all positive integers m and n,*

$$cr(K_{m,n}) = \frac{1}{4}\left\lfloor\frac{m}{2}\right\rfloor \cdot \left\lfloor\frac{m-1}{2}\right\rfloor \cdot \left\lfloor\frac{n}{2}\right\rfloor \cdot \left\lfloor\frac{n-1}{2}\right\rfloor.$$

Since the drawing proposed by Zarankiewicz uses only straight lines, the bound in (7.5) is also a bound for the corresponding rectilinear number $\overline{\mathrm{cr}}(K_{m,n})$.

In 1970, Kleitman [574] showed that for each $n \geq 3$,

$$\mathrm{cr}(K_{5,n}) = 4\left\lfloor\frac{n}{2}\right\rfloor \cdot \left\lfloor\frac{n-1}{2}\right\rfloor, \tag{7.6}$$

and

$$\mathrm{cr}(K_{6,n}) = 6\left\lfloor\frac{n}{2}\right\rfloor \cdot \left\lfloor\frac{n-1}{2}\right\rfloor,$$

confirming Conjecture 7.11.4 whenever $\min\{m,n\} \leq 6$. Kleitman [575] revised and clarified one part of his proof (a parity argument) a few years later. In 1993, Woodall [996] showed that the conjecture is true also when $m \in \{7,8\}$ and $n \in \{7,8,9,10\}$.

7.11.4 Crossing numbers for complete graphs

Since K_1, K_2, K_3 and K_4 are planar, for $1 \leq k \leq 4$, $\mathrm{cr}(K_k) = 0$, and by a standard drawing of K_4 with one vertex in the middle, $\overline{\mathrm{cr}}(K_4) = 0$. In Exercises 366 and 367, it was shown that $\mathrm{cr}(K_5) = 1$ and $\mathrm{cr}(K_6) = \overline{\mathrm{cr}}(K_6) = 3$.

One idea for a drawing of K_n with fewest crossings was given by Anthony Hill in 1959 (see [84, p. 45] for a brief story of how Hill's result was picked up by Richard Guy); see [4] for a brief description of Hill's construction. Also see [323], [461], or [490]. The basic idea is to put $\lfloor n/2 \rfloor$ vertices equally spaced around the base of a cylinder and $\lceil n/2 \rceil$ vertices around the top of the cylinder. For each circle, connect all vertices with straight lines, and for edges between the circles, use geodesics along the cylinder. According to Andrii Arman (personal communication, 2017), the similar construction on a sphere also gives the same bounds. ("Cylindrical crossing numbers" have since been well-studied—see [4] or [830].) (Such constructions were also given by Blažek and Koman [111] and by Guy and Jenkyns [467].) Such a "cylindrical" construction gives (with a bit of work) the following bound:

Theorem 7.11.5 (Hill 1959, see [461])**.**

$$cr(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor \cdot \left\lfloor\frac{n-1}{2}\right\rfloor \cdot \left\lfloor\frac{n-2}{2}\right\rfloor \cdot \left\lfloor\frac{n-3}{2}\right\rfloor.$$

The drawing of K_6 in Figure 7.14 is based on a cylindrical construction.

Exercise 373. *Find a drawing of K_7 with 9 crossings. Hint: Use the Hill idea of putting 4 points in a square surrounded by 3 points in a triangle.*

In the Harary–Hill paper [490] are two drawings of K_8 that show $\mathrm{cr}(K_8) \leq 18$; the first of these drawings is formed with the 8 vertices on a circle; the second is based on the cylindrical method of Hill, and is given here in Figure 7.19.

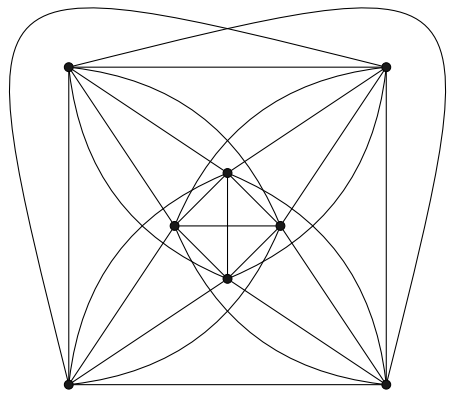


Figure 7.19: $\mathrm{cr}(K_8) \leq 18$

Crossing numbers for a few other small complete graphs have been computed (see [464] or [465]): $\mathrm{cr}(K_7) = 9$, $\mathrm{cr}(K_8) = 18$, $\mathrm{cr}(K_9) = 36$, and $\mathrm{cr}(K_{10}) = 60$. Based on these values, the following conjecture (probably first due to Hill; see [461]) became popular:

Conjecture 7.11.6 (see, *e.g.*, [490])**.**

$$cr(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor .$$

According to [323], if Conjecture 7.11.6 is true for some odd n, then it is also true for $n+1$. Eggleton and Guy [280] showed that for n odd, $\mathrm{cr}(K_n)$ and $\binom{n}{4}$ have the same parity.

In 2007, Pan and Richter [741] confirmed that Conjecture 7.11.6 is true for $n = 11, 12$ as well, by showing that $\mathrm{cr}(K_{11}) = 100$, from which it follows that $\mathrm{cr}(K_{12}) = 150$.

To support Conjecture 7.11.6, a simple lower bound argument shows that $\mathrm{cr}(K_n)$ is of the order n^4. (In [84], this argument is credited to Guy.)

Lemma 7.11.7 (Guy, 1969 [462])**.** *For $n \geq 5$,*

$$cr(K_n) \geq \frac{1}{120} n(n-1)(n-2)(n-3).$$

Proof: Consider some drawing of K_n. For each of the $\binom{n}{5}$ copies of K_5 there is at least one pair of edges crossing. Each crossing uses 4 points, so each crossing is in at most $n-4$ copies of K_5. Thus $\mathrm{cr}(K_n) \geq \frac{\binom{n}{5}}{n-4}$, from which the result follows. □

In 2017, Balogh, Lidický, and Salazar [72] used flag algebras to show that, asymptotically,

$$\mathrm{cr}(K_n) \geq 0.985 \cdot \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor.$$

The same authors [72] give a similar result for spherical crossing numbers, but with .996 replacing .985.

7.11.5 Rectilinear crossing numbers

Recall from Definition 7.11.2 that the rectilinear crossing number of G, $\overline{\mathrm{cr}}(G)$, is the least number of crossings when drawings are restricted to those whose edges are straight-line segments. Rectilinear linear crossing numbers appear to be introduced in the 1963 paper by Harary and Hill [490] (where they are denoted $\overline{c}(G)$).

Recall that Fary's theorem (Theorem 7.4.1) says that any planar graph has a plane drawing with all edges as straight line segments and hence has rectilinear crossing number 0.

By 2001, rectilinear crossing numbers for complete graphs of order $n \leq 10$ were found. As was shown above, $\overline{\mathrm{cr}}(K_4) = 0$, $\overline{\mathrm{cr}}(K_5) = 1$ (see Figure 20.10), and $\overline{\mathrm{cr}}(K_6) = 3$ (see Figure 7.14). Harary and Hill [490] conjectured that $\overline{\mathrm{cr}}(K_8) = 19$ (and gave a drawing like in Figure 7.20 witnessing the upper bound). Harary and Hill also showed $\overline{\mathrm{cr}}(K_9) = 36$ (they gave a drawing as in Figure 7.21, which shows that the rectilinear crossing number agrees with crossing number). Harary and Hill conjectured that $\overline{\mathrm{cr}}(K_{10}) = 63$; even though Harary and Hill did not accurately predict $\overline{\mathrm{cr}}(K_{10})$, their conjecture that for $n = 8$ and $n \geq 10$, $\overline{\mathrm{cr}}(K_n) > \mathrm{cr}(K_n)$ was indeed correct.

Exercise 374. *Find a drawing that shows $\overline{cr}(K_7) \leq 9$. Note that since $cr(K_7) = 9$, this inequality is equality. Hint: Does removing a vertex from the drawing in Figure 7.20 work? Can the drawing in Figure 20.15 be modified to give a rectilinear drawing?*

In 1971, David Singer [868] confirmed that $\overline{\mathrm{cr}}(K_8) = 19 > 18 = \mathrm{cr}(K_8)$ and

$$\overline{\mathrm{cr}}(K_{10}) \in \{61, 62\}.$$

Singer also showed that

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{n^4} \leq \frac{5}{312}.$$

It wasn't until 2001 when Brodsky, Durocher (now a professor at U. Manitoba), and Gethner [159] showed that $\overline{\mathrm{cr}}(K_{10}) = 62$. See also [5], [6], or [160] for more recent developments on evaluating $\overline{\mathrm{cr}}(K_n)$. In [5], it is claimed that

$$\overline{\mathrm{cr}}(K_n) \geq \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor.$$

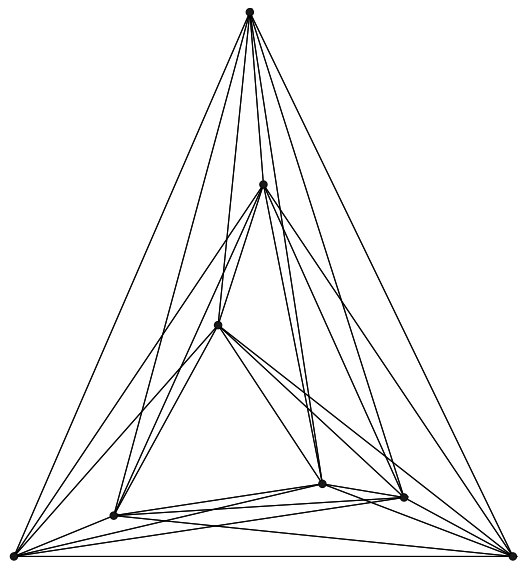


Figure 7.20: $\overline{\mathrm{cr}}(K_8) \leq 19$, drawing from [490, Fig. 6].

According to [6], for all $n \leq 27$, the rectilinear numbers for K_n have been found. See [15] for a page with more recent results, including examples of optimal drawings, and bounds for $n \leq 100$.

In 1993, Bienstock and Dean [101] showed that if G is a simple graph with $\mathrm{cr}(G) \leq 3$, then $\overline{\mathrm{cr}}(G) = \mathrm{cr}(G)$. Bienstock and Dean also show that there are graphs with crossing number 4 but arbitrarily large rectilinear crossing number.

7.11.6 Degenerate crossing numbers

In calculating crossing numbers, it is required that only one pair of edges cross at any one point, or that any crossing point has precisely two edges through it. If a drawing is given with $k \geq 3$ lines all crossing at one common point, this one point is witness to $\binom{k}{2}$ (pairwise) crossings, and by perturbing the edges slightly, all such crossings can occur at $\binom{k}{2}$ distinct points. If one allows more than a pair of edges to cross at any one point, one might be able to reduce the total number of "crossing points" by allowing more than two edges at a crossing point.

Definition 7.11.8. *The* degenerate crossing number *of G, denoted $cr^*(G)$, is the minimum number of crossing* points *(where more than two edges may pass through the same point in $\mathbb{R}^2$).*

By definition, for any G, $\mathrm{cr}^*(G) \leq \mathrm{cr}(G)$. There are examples of graphs that have a degenerate crossing number that is smaller than the crossing number; for example, by Kleitman's above result (7.6), $\mathrm{cr}(K_{5,5}) = 16$; however, $\mathrm{cr}^*(K_{5,5}) = 15$ (see [737]).

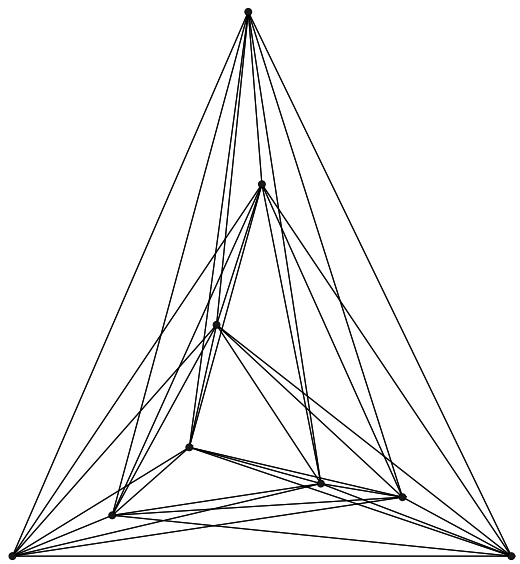


Figure 7.21: $\overline{\mathrm{cr}}(K_9) \leq 36$, drawing from [490, Fig. 7].

Exercise 375. *What is the degenerate crossing number for K_6?*

When limiting the definition of degenerate crossing numbers to drawings where edges intersect at most once, a lower bound is obtained.

Theorem 7.11.9 (Pach–Toth, 2009 [737])**.** *There exists a constant $c^* > \frac{1}{40^4}$ so that for every graph $G = (V(G), E(G))$ with $|E(G)| \geq 4|V(G)|$,*

$$cr^*(G) \geq c^* \frac{|E(G)|^4}{|V(G)|^4}.$$

7.11.7 General bounds for crossing numbers

Lemma 7.11.10. *Let D be a drawing of a graph on n vertices with m edges and $cr(D)$ crossings. Then $cr(D) \geq m - 3n + 6$.*

Proof: Create a planar drawing H (with no crossings) by removing one edge from each crossing in D; at most $\mathrm{cr}(D)$ edges need be removed (less may be required if there are edges in D with multiple crossings). So H has at least $m - \mathrm{cr}(D)$ edges and by Lemma 7.1.2) for planar graphs,

$$m - \mathrm{cr}(D) \leq 3n - 6,$$

from which the desired result follows. □

The following theorem (which was conjectured by Erdős and Guy [323, p. 56]) is sometimes (*e.g.*, [659, p. 55]) called "the crossing number theorem", first published in

1982 by Ajtai, Chvatál, Newborn, and Szemerédi [19] (and independently a year later by Leighton [623] in work on VLSIs—see also [622] for the relations between crossing numbers and VLSIs).

Theorem 7.11.11 (Crossing number theorem)**.** *There exists a constant* $0 < c < 1$ *so that for every simple graph* $G = (V, E)$ *with* $|E| \geq 4|V|$,

$$cr(G) \geq c \cdot \frac{|E|^3}{|V|^2} - |V|.$$

Proof of Theorem 7.11.11: An original proof used $c = \frac{1}{100}$ and was complicated. The proof given here is a simple probabilistic one by Chazelle, Sharir, and Welzl (as found in [18] or [659]), that shows $c \geq \frac{1}{64}$.

Let $G = (V, E)$ have $n = |V|$ vertices and $m = |E|$ edges. Let D be a drawing of G with a minimal number of crossings. Let $p \in (0, 1]$ be a probability (to be identified later), and let H be a subgraph of D formed by selecting vertices of D independently and randomly with probability p (and deleting vertices of D with probability $1 - p$). Then $\mathrm{cr}(H)$ is a random variable (as are $|E(H)|$ and $|V(H)|$). The probability that an edge of D remains in H is p^2, and since each crossing in D uses four vertices, the probability that a crossing in D survives in H is p^4. By linearity of expectation,

$$\begin{aligned} p^4\mathrm{cr}(G) &= \mathbb{E}[\mathrm{cr}(H)] \\ &\geq \mathbb{E}[|E(H)| - 3|V(H)| + 6] && \text{(by Lemma 7.11.10)} \\ &= p^2 m - 3pn + 6, \end{aligned}$$

and so

$$\mathrm{cr}(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}.$$

Using $p = \frac{4n}{m}$ (which is at most 1 because $m \geq 4n$),

$$\mathrm{cr}(G) \geq \frac{1}{64}\left(\frac{4m^3}{n^2} - \frac{3m^3}{n^2}\right) = \frac{m^3}{64n^2}.$$

□

For a short history of Theorem 7.11.11 and related facts, see [659, pp. 52–54]. Original bounds for c were apparently quite small (one review of the paper [19] gives $c = \frac{1}{100}$, although I have not seen the original). The value for c in the above proof is $\frac{1}{64} \sim .0156$. In 1997, Pach and Tóth [735] showed that $c \leq .09$, and for $m \geq 7.5n$, $c \geq \frac{1}{33.75} \sim .0296$. This last result was improved in 2006 for $m \geq \frac{103}{16}n$ by Pach, Radoičić, Tardos, and Tóth [733] to $c \geq \frac{1024}{31827} \sim .0321$.

In 2013, Ackerman [8] showed that if $m \geq 6.95n$, then a lower bound for c can be improved to $\frac{1}{29}$ if each edge is crossed at most 4 times.

The bound in Theorem 7.11.11 is asymptotically tight because for $5n \le m \le \binom{n}{2}$ there exists a graph with n vertices, m edges, and crossing number $O\left(\frac{m^3}{n^2}\right)$ (see [659, Ex. 1, p. 58]).

A multigraph version of Theorem 7.11.11 has also been shown by a number of authors. (One proof follows the above one. See, for example, [897].)

Theorem 7.11.12 (Various authors)**.** *Let $G = (V, E)$ be a multigraph on n vertices with edge-multiplicity at most k. Then for $|E| \ge 4k|V|$,*

$$cr(G) \ge \frac{1}{64}\frac{|E|^3}{|V|^2k^3}.$$

For degenerate crossing numbers, Pach and Tóth [737] showed that $\mathrm{cr}^*(G) = \Omega(\frac{m^4}{n^4})$. Ackerman and Pinchasi [9] showed that for a graph G with $m \ge 4n$, $\mathrm{cr}^*(G) = \Omega(\frac{m^3}{n^2})$.

7.11.8 Applications in combinatorial geometry

In 1997, Székely [897] (see also [898]) gave several (perhaps surprising) applications of the crossing number theorem (Theorem 7.11.11) to famous problems in combinatorial geometry, only two of which are given here. Also, Scheinerman and Wilf [833] showed a relation between rectilinear crossing numbers and a famous problem in convex geometry.

The Szemerédi–Trotter theorem

For positive integers p and q, let $I(p, q)$ be the maximum number of incidences in any configuration of p points and q lines in $\mathbb{R}^2$.

Theorem 7.11.13 (Szemerédi–Trotter, 1983 [900])**.** *For $p, q \ge 1$, $I(p, q) = O\left((pq)^{2/3} + p + q\right)$.*

The original proof of the upper bound in Theorem 7.11.13 used a complicated argument. In 1997, Székely [897] gave a simple proof that uses crossing numbers.

Székely's proof of upper bound in Theorem 7.11.13: Suppose that a configuration of p points and q lines are given in the plane. Define a graph G whose vertices are points and whose edges are consecutive pairs of points on a line. Since any two lines cross at most once, $\mathrm{cr}(G) \le \binom{q}{2}$. In each line, the number of vertices is one more than the number of edges. So, counting over all lines, the number of incidences is at most $|E| + q$. Thus, $I(p, q) \le |E| + q$, and so $|E| \ge I(p, q) - q$. Then either $|E| \le 4|V|$ (in which case, $I(p, q) \le 4p + q$, satisfying the theorem), or $|E| \ge 4|V|$ and

$$\binom{q}{2} \ge \mathrm{cr}(G)$$

$$\geq \frac{1}{64}\frac{|E|^3}{|V|^2} \qquad \text{(by Thm. 7.11.11)}$$
$$\geq \frac{1}{64}\frac{(I(p,q)-q)^3}{p^2},$$

and so

$$(I(p,q)-q)^3 \leq 64\binom{q}{2}p^2.$$

Hence, there is a constant c so that $I(p,q) \leq c(pq)^{2/3} + q$. The two cases give the desired result. □

Maximum number of unit distances between n points

One of Paul Erdős' favourite problems was to find bounds of the number of unit distances that can occur among n points.

Theorem 7.11.14 (Erdős, 1946 [289]). *For $n \geq 2$, let $g(n)$ denote the maximum number of unit distances between n points in the plane. There exists a constant c_1 so that for any n,*

$$n^{1+c_1/\log\log n} \leq g(n) \leq O(n^{3/2}).$$

His proof of the lower bound used an integer lattice and some number theory, and the upper bound followed easily from a theorem in extremal graph theory. Erdős thought (see also [293]) that the lower bound in Theorem 7.11.14 was closer to being correct. There has been a number of improvements on Theorem 7.11.14 (see, *e.g.*, [659]). One such notable result reduces the upper bound:

Theorem 7.11.15 (Spencer–Szemerédi–Trotter, 1984 [879]). *As $n \to \infty$, the maximum number of unit-distances in a geometric graph in the plane on n vertices is $O(n^{4/3})$.*

The original proof of Theorem 7.11.15 was rather complicated. Székely used the crossing number theorem (Theorem 7.11.11) to give a much simpler proof:

Proof by Székely [897]: Let X be a set of n points in the plane. Around each point in X, draw a unit circle. One of these unit circles may be incident with other points. Delete those circles that pass through less than two points. If a circle centered at some point P contains $m \geq 2$ points, then the circle is divided into m arcs, and these m points determine m unit distances. Since two circles intersect in at most 2 points, there may be at most two arcs between a pair of points that are adjacent on some circle.

Form the geometric multigraph H with vertex set X and the set of edges are the arcs drawn between consecutive vertices on a circle. If two vertices are connected by

two arcs, arbitrarily delete one of these arcs thereby creating a simple graph G (still with vertex set X).

If N is the number of unit distances between points of X, then N is the number of point-circle incidences, and removing the circles with only one such incidence gives deletes at most n incidences, and since at most half of the arcs have been deleted, it follows that $|E(G)| \geq \frac{N-n}{2}$. For this value to be positive, one can assume that, say, $N > 5n$ (for if not, the bound on N is proved).

Now count the number of crossings in two ways. By the crossing number theorem (Theorem 7.11.11)

$$\mathrm{cr}(G) \geq \frac{|E(G)|^3}{64n^2} \geq \frac{(N-n)^3}{8 \cdot 64n^3}. \tag{7.7}$$

On the other hand, since any two circles intersect at most twice, there are at most $2\binom{n}{2} < n^2$ crossings. Thus, by (7.7)

$$n^2 > \frac{(N-n)^3}{512n^2},$$

form which it follows that $(N-n)^3 < 512n^4$, and so $N < 8n^{4/3} + n$, completing the proof. □

Sylvester's four point problem

In 1864, Sylvester [895] posed his (now famous) four point problem, which is to find the probability that among four "random" points in the plane, one is in the triangle determined by the other three. It turns out that Sylvester's problem has many different answers, depending on what "random" means, and in what domain the problem is considered. See Pfiefer's survey [748] for details, history of the problem, and references for facts stated below. Also see the paper by Imre Bárány [76].

For example, the answer of 1/4 was given by both Cayley and Sylvester (using a triangular domain), 1/2 by DeMorgan, 1/3 by Wilson, and $35/12\pi^2 \sim .296$ by Woolhouse [998] (using a circular domain). (For higher dimension analogues and random polytopes, the interested reader might consult [80] or [948].)

In 1994, Scheinerman and Wilf [833] showed how rectilinear crossing numbers are related to Sylvester's four point problem (see also [986]). To describe this relation, a folklore result (found in [833, Thm. 2]) that extends Lemma 7.11.7 is first given.

Theorem 7.11.16 (Folklore)**.** *There exists a positive finite constant L so that*

$$L = \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} = \sup_n \frac{\overline{cr}(K_n)}{\binom{n}{4}}.$$

Proof: Let $4 \le m < n$, and let D be an optimal rectilinear drawing of K_n (with $\overline{\mathrm{cr}}(K_n)$ crossings) on an n-element set V of vertices. Each crossing C in D is determined by 4 vertices, and so is contained in $\binom{n-4}{m-4}$ copies of K_m in D. For any subset A of m vertices, let $\mathrm{cr}(A)$ denote the number of crossings induced in D of the complete graph on A. Counting pairs (crossing C, copy of K_m containing C) in two ways,

$$\begin{aligned}\overline{\mathrm{cr}}(K_n)\binom{n-4}{m-4} &= \sum_{A\subset V(D);|A|=m} \mathrm{cr}(A)\\ &\ge \binom{n}{m}\overline{\mathrm{cr}}(K_m),\end{aligned}$$

where the inequality above is given in case any of the copies of K_m have more than the necessary number of crossings Rearranging,

$$\frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{m}} \ge \frac{\overline{\mathrm{cr}}(K_m)}{\binom{n-4}{m-4}}.$$

Since $\frac{\binom{n}{m}}{\binom{n-4}{m-4}} = \frac{\binom{n}{4}}{\binom{m}{4}}$,

$$\frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}} \ge \frac{\overline{\mathrm{cr}}(K_m)}{\binom{m}{4}}.$$

Thus the sequence $\{\overline{\mathrm{cr}}(K_n)/\binom{n}{4}\}$ is increasing, bounded above by 1 and below by $\overline{\mathrm{cr}}(K_5)/\binom{5}{4} = \frac{1}{5}$. □

Since $\overline{\mathrm{cr}}(K_{10}) = 62$, $L \ge \frac{31}{105}$. By a result of Singer [868], when n is a power of 3, $\overline{\mathrm{cr}}(K_n) \le \frac{5n^4-39n^3+91n^2-57n}{312}$. So $L \le \frac{5}{312}\cdot 24 = \frac{5}{12}$, and thus

$$\frac{31}{105} \le L \le \frac{5}{12}.$$

□

Let q be the probability that in a convex open set R with finite area, if four points are chosen at random, they form a convex quadrilateral). Scheinerman and Wilf [833] proved that $q = L$. (The proof is short, but omitted here; essentially, the proof relies on the correspondence between four points in convex position and a crossing.)

7.11.9 Crossing numbers on other surfaces

This section contains only a few facts and references on crossing numbers for surfaces other than the plane or sphere.

The complete graph K_7 is not planar and $\mathrm{cr}(K_7) = 9$. However, K_7 can be drawn on a torus with no crossings; that is, if $\mathrm{cr}_T(G)$ denotes the "toroidal crossing number"

Figure 7.22: K_7 embedded on torus, by K. Gunderson, circa 2008

of a graph G drawn on a torus, then $\mathrm{cr}_T(K_7) = 0$. See Figure 7.22 for a torus with K_7 embedded with no crossings. See [468] for early work on the toroidal crossing numbers for the complete graph, and [467] for early work on toroidal crossing numbers for $K_{m,n}$. Much more work has since been done on embedding graphs on tori; for example, see [397]. For early work on crossing numbers for projective planes or Klein bottles, see Koman's 1969 work [590]. Crossing numbers for a "torus" with g holes (and many other problems) were considered by Pach, Spencer, and Tóth in [734].

For more on embedding graphs on surfaces, see one of the standard references for topological graph theory, *e.g.*, [444] (this book appears in Figure 7.22), [690] (where crossing numbers are not considered), or [979] (where crossing numbers are briefly introduced on pp. 76ff, including crossing numbers for graphs on surfaces with higher genus).

Chapter 8

Decompositions, factorizations, and graceful trees

8.1 Some famous decomposition questions

The literature on graph decompositions is very extensive; only a few select problems are considered here. A general kind of problem asks if it is possible to partition the edges of a particular type of graph into (disjoint) copies of some smaller graph (or smaller graphs of a particular type); such a problem is a "decomposition problem". One can ask to decompose a large graph G (usually the complete graph) into factors (subgraphs F with with $V(F) = V(G)$). For example, Petersen's theorem (Theorem 5.9.3) says that a cubic (3-regular) simple graph can be decomposed into a 1-factor and a 2-factor. For a rather large collection of results regarding factors and factorizations, see [21]. Only a few results are given here.

Finding decompositions into 1-factors is considered separately in Sections 8.2 and 8.3, and decompositions into 2-factors are studied in Section 8.4. Decompositions of the complete graph into trees is considered in Section 8.5. Some other famous miscellaneous decomposition questions are given in this section.

Very much related to questions concerning decomposition are those of "packing" (how many edge-disjoint copies of a small graph can be embedded in the larger graph) but such questions are often too general of a nature to be adequately discussed here. A packing that covers all edges of the larger graph is simply a decomposition. However, packings are not the focus of this chapter.

A weaker type of question is to ask when a large graph can be covered by smaller graphs of a particular type (where copies of graphs used for the cover need not be disjoint). Graph coverings are also not the focus of this chapter, but some selected results are mentioned.

For some $n \geq 2$, if K_n has a decomposition into paths, since each path has at most $n-1$ edges, then at least $|E(K_n)|/(n-1) = n/2$ paths are required. When n is even,

one such decomposition is given by "zig-zag" paths; for $n = 8$, see the example given in Figure 8.1.

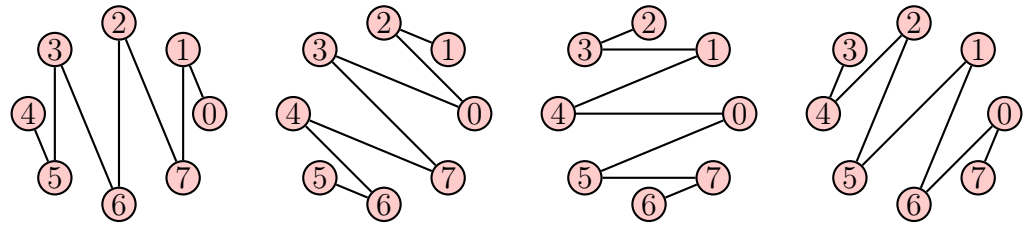


Figure 8.1: Decomposition of K_8 into Hamiltonian paths

Exercise 376. *Show that the zig-zag pattern given in Figure 8.1 for $n = 8$ extends to prove that for any even $n \geq 2$, a decomposition of K_n into Hamiltonian paths exists.*

Conjecture 8.1.1 (Gallai, see [634])**.** *Every connected graph on n vertices can be decomposed into at most $\lceil n/2 \rceil$ paths.*

The following theorem is well-known (see, *e.g.*, [125, Thm. 11, p. 16]).

Theorem 8.1.2. *For $n \geq 2$, the graph K_n can be decomposed into Hamiltonian paths if and only if n is even. For $n \geq 3$, the graph K_n can be decomposed into Hamiltonian cycles if and only if n is odd.*

Proof: If K_n has a decomposition into Hamiltonian paths (each with $n - 1$ edges), there are $\frac{n}{2}$ such paths and so n is even. Any Hamiltonian cycle in K_n has n edges, so if K_n has a decomposition into Hamiltonian cycles, there are $\frac{n-1}{2}$ such cycles, and so n is odd. Thus the parity conditions on n are necessary in each case.

Let $n \geq 2k$ be even and let $V(K_n) = \{0, 1, \ldots, 2k-1\}$. By Exercise 376 there exist edge-disjoint Hamiltonian paths $G_0, G_1, \ldots, G_{k-1}$ be edge-disjoint Hamiltonian paths, where for each $j = 0, \ldots, k$, $E(G_j)$ has end points j and $j + k$. So no two of the G_is share an end point. (As noted in [125], for any decomposition into Hamiltonian paths, not just the one given by Exercise 376, since each vertex in K_{2k} has odd degree, at each vertex, at least one Hamiltonian path ends at that vertex. Thus each vertex in K_{2k} is an endpoint of just one of the Hamiltonian paths.)

Having a decomposition of K_{2k} into Hamiltonian paths $G_0, \ldots, G_{k-1}$, an additional vertex in K_{2k+1} extends each G_i into a Hamiltonian cycle H_i; the H_is are edge-disjoint because no two of the G_is have common endpoints. □

One idea in trying to prove Gallai's conjecture might be to start with Eulerian graphs, then chop an Eulerian circuit into paths; however, this approach can get complicated depending on the nature of the Eulerian circuit. It is well known that Eulerian graphs can be decomposed into cycles, but how many?

Conjecture 8.1.3 (Hajós, 1968, see [634]). *Any graph with all degrees even can be decomposed into at most $\lfloor n/2 \rfloor$ cycles.*

Hajós' conjecture has been proved for planar graphs [851] and for graphs with maximum degree 4 [439]. In a recent (2017) paper [508], the conjecture is shown to be true for all graphs with at most 12 vertices.

One approach to Gallai's conjecture was to first loosen the requirements somewhat by also allowing cycles.

Theorem 8.1.4 (Lovász, 1968 [634]). *Any graph on n vertices can be decomposed into at most $\lceil n/2 \rceil$ paths and cycles.*

Lovász proved Gallai's conjecture for those graphs whose every vertex has odd degree.

The *path decomposition number*, or simply the path number, of a graph G is the minimum number $\mathrm{pn}(G)$ of paths that G can be decomposed into. This number was introduced by Harary, and studied in, *e.g.*, [232], [497], and [744]. Theorem 8.1.2 shows that $\mathrm{pn}(K_{2k}) = k$ and $\mathrm{pn}(K_{2k+1}) = k + 1$. (See also [40].)

Definition 8.1.5. *A collection of graphs $G_1, G_2, \ldots, G_N$ is said to* cover *a graph G if $G = G_1 \cup G_2 \cup \cdots \cup G_N$.*

Note that in a cover $G = G_1 \cup \cdots \cup G_N$, the G_is need not be edge-disjoint; i.e., a particular edge in G may be covered by more than one G_i.

The last author of the next result, Louis Pósa, was still a student at Michael Fazekas HIgh School in Budapest when the paper was submitted.

Theorem 8.1.6 (Erdős–Goodman–Pósa, 1966 [319]). *The edges of any connected graph G on $n \geq 2$ vertices can be covered by at most $\lfloor \frac{n^2}{4} \rfloor$ complete graphs. In fact, the complete graphs used can be chosen from only K_2s and K_3s.*

Proof: The proof is a simple induction using a double jump.

For each $n \geq 2$, let $A(n)$ denote that assertion that any connected graph on n vertices can be covered with at most $\lfloor \frac{n^2}{4} \rfloor$ complete graphs using only edges and triangles.

Base step: The only connected graph on 2 vertices is K_2, which is trivially covered by one K_2, and $\frac{2^2}{4} = 1$, so $A(2)$ holds. When $n = 3$, the only two connected graphs on 3 vertices are two edges joined at a vertex or K_3, which can be covered by either two K_2s or a single K_3, and $\lfloor \frac{3^2}{4} \rfloor = 2$, so $A(3)$ holds.

Inductive step $[A(k) \to A(k+2)]$: Let $k \geq 2$ and suppose that $A(k)$ is true. Let G be a connected graph on $k + 2$ vertices $\{v_1, \ldots, v_k, x, y\}$, where $\{x, y\} \in E(G)$, and let H be the induced subgraph of G obtained by removing x and y. By assumption $A(k)$, H can be covered by $\lfloor \frac{k^2}{4} \rfloor$ copies of K_2 and K_3. Each v_i is adjacent to neither

x or y, one of x and y or to both x and y. For those v_i that are adjacent to one of x or y, exactly one K_2 covers the extra edge, and if v_i is adjacent to both x and y, a K_3 covers the subgraph induced by v_i, x, y. In any case, at most k new complete graphs cover the edges from H to x and y. Finally, one more edge may be required to cover the edge $\{x, y\}$. In all, the maximum number of complete graphs required to cover G is

$$\frac{k^2}{4} + k + 1 = \frac{(k+2)^2}{4},$$

and so $A(k+2)$ holds, completing the inductive step.

By an alternate form of MI, for each $n \geq 2$, the statement $A(n)$ is true. □

Exercise 377. *Show that the value $\lfloor \frac{n^2}{4} \rfloor$ in Theorem 8.1.6 is optimal.*

In [319], it was mentioned that Lóvasz was also aware of Theorem 8.1.6. In the same paper, a stronger result was also obtained:

Theorem 8.1.7 (Erdős–Goodman–Póza, 1966 [319])**.** *The edges of any connected graph on $n \geq 2$ vertices can be decomposed into edge-disjoint copies of K_2 and K_3.*

Exercise 378. *Prove Theorem 8.1.7 using induction.*

Lovász proved two more theorems of interest regarding coverings rather than decomposition, but they might be considered as starting points for certain decomposition questions.

Theorem 8.1.8 (Lovász, 1968 [634])**.** *Any graph on n vertices can be covered by $\lceil 2n/3 \rceil$ trees of diameter at most 3.*

In 1958, Hajnal and Surányi [473] showed that if G is a chordal graph (finite or infinite) with $\alpha(G) = k < \infty$, then G can be covered by at most k complete graphs.

Theorem 8.1.9 (Lovász, 1968 [634])**.** *Let G be a graph on n vertices, and put $k = \binom{n}{2} - |E(G)|$. Then G can be covered by $k + \lceil (1 + (1+4k)^{1/2})/2 \rceil$ complete subgraphs.*

Complete graphs have diameter 1, and stars have diameter 2. If G is a graph so that for any set S of s vertices in G, there exists a subgraph H that is a star, then G has *star number* s. (This definition occurs in [486].) Let $f(s, k)$ be the smallest number so that the complete graph on $f(s, k)$ vertices can be decomposed into k factors $F_1, \ldots, F_k$ (each is a spanning subgraph) with each F_i having star number s. In [152], it was proved that $f(2, k) \geq 4k - 1$. For similar work but with higher star numbers, see [826]. For other results on decompositions of complete graphs into factors with diameter 2, see [153] or [824].

Other natural questions arise for decompositions of complete graphs. When is it possible to decompose K_n into identical copies of some graph H? A trivial requirement

is that $|E(H)|$ divides $\binom{n}{2}$, and not much more is obvious. As an example, by Theorem 8.1.2, the complete graph K_6 (which has 15 edges) can be decomposed into 3 paths of length 5, but is it possible to decompose the edges of K_6 into 5 paths of length 3? In general, if P is a path with k edges and k divides $\binom{n}{2}$, is there a decomposition of K_n into copies of P? It was noted in [137] that Kotzig has shown that any connected graph with an even number of edges has a decomposition into paths of length 2.

Exercise 379. *For which n can K_n be decomposed into paths of length 2?*

Exercise 380. *Show that the edges of the cube graph Q_3 can be decomposed into four copies of $K_{1,3}$.*

As a consequence of Petersen's 1-factor theorem (Theorem 5.9.3), the proof of the following is relatively simple.

Theorem 8.1.10. *The edge set of a simple 2-edge-connected cubic graph can be partitioned into paths of length 3.*

Proof: Let G be a cubic bridgeless graph, and by Petersen's theorem, suppose that G is decomposed into a 1-factor M and a 2-factor H. Orient all cycles in H. For each $e = \{x, y\}$ in M, let f_1 and f_2 be the two edges from H that end in x and y respectively. Then f_1, e, f_2 are the edges of a path with three edges. Repeating this process, create a path through each edge of M. □

For decomposing arbitrary graphs (with a connectivity condition) into paths of length 3 or 4, this problem has been addressed by Thomassen and others; it was subsequently settled by Thomassen [922] when the path has length that is a power of 2 (see the same paper for background and more valuable references).

Some questions regarding decompositions into regular bipartite factors have also been studied; I give only one such result as a sample, due to Anton Kotzig (1919–1991), a Slovak mathematician who spent a year at the University of Calgary (in 1969–1970) and later moved to the University of Montreal.

Theorem 8.1.11 (Kotzig, 1973 [605])**.** *For positive integers n and r, there exists a decomposition of K_n into r regular spanning bipartite graphs if and only if $n > 4$ and n is even and there exists $k \le r$ such that $2^{k-1} < n \le 2^k$.*

One other famous decomposition question is by P. J. Kelly (given around 1968—see [916]): Can every regular tournament be decomposed into (directed) Hamilton cycles? According to Bondy [139] this has been solved by Häggkvist (unpublished) for very large tournaments.

There is also extensive literature about "packing" small graphs into large graphs. If G is a graph and $H_1, \ldots, H_k$ are subgraphs of G, a packing of the H_is into G is a collection of embeddings $\sigma_i : H_i \to G$ so that the embedded copies of the H_is are

pairwise edge-disjoint, but do not necessarily use all the edges of G. For some theorems about packing, the reader is invited to look at [827] and [1004], and for packing trees into a complete graph, see [469], [516], [568], [828], or [999], just for a beginning. In Section 8.5.1, a special case of the tree-packing (or decomposition) problem is concentrated on (namely, Ringel's conjecture, given here as Conjecture 8.5.1).

8.2 1-factorizations

A *factor* of a graph is a spanning subgraph; a partition of the edges of a graph into factors is called a *factorization*. A perfect matching is called a *1-factor*, short for "1-regular factor". A decomposition of the edges of a graph into perfect matchings is called a *1-factorization.*. If a graph G has a 1-factorization then G has an even number of vertices. For general work on 1-factorizations, see the book [970] by Wallis. For a 1985 survey (with 146 references) on 1-factorizations, see [675]. Also see [776].

In Exercise 192, a 1-factor from the cube graph Q_n was asked for. In the solution given, one kind of 1-factor was given. These different kinds make up a 1-factorization.

Exercise 381. *Show that for each positive integer k, the cube graph Q_k has a 1-factorization.*

In Exercise 194, it was asked to show that the Petersen graph has exactly 6 perfect matchings. Each pair of such matchings has at least one common edge.

Exercise 382. *Show that the Petersen graph has no 1-factorization.*

When n is even, does K_n have a 1-factorization? In other words, can the edges of the graph K_n be decomposed into $n-1$ disjoint perfect matchings? If so, each matching is then of size $n/2$.

The construction that proves the following lemma can be found in, *e.g.*, König's 1935 book [596, pp. 155–6, Satz 2], perhaps the earliest textbook on graph theory. (In that book, one page earlier there is mention of Hilbert, Petersen, and Sylvester; I could not determine the original source—according to Seah [842], this construction goes back as far as at least 1859, given by Reiss [787].)

Lemma 8.2.1. *For a positive integer n, there exists a decomposition of K_n into $n-1$ perfect matchings if and only if n is even.*

Proof: If n is odd, K_n has no perfect matchings (since each edge in a matching uses two vertices), so suppose that $n \geq 2$ is even. Let $V(K_n) = \{0, \ldots, n-1\}$ and draw the vertices with 0 in the center of a circle, and the remaining vertices on the circle in order and equally spaced. Starting with an arbitrary $i \neq 0$, the edge $\{0, i\}$ together with all edges at right angles to this edge indeed form a perfect matching.

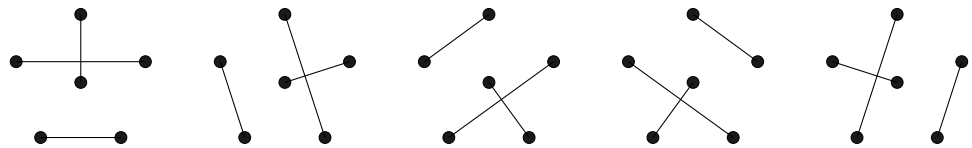


Figure 8.2: A 1-factorization of K_6

To describe each matching more algebraically, for each $i = 1, \ldots n-1$, put

$$M_i = \{\{j,k\} : j + k = i \pmod{n-1}\}$$

is a matching. To see that these matchings are indeed disjoint, suppose that two, say M_a and M_b (where $a \neq b$ and $a, b \in [1, n-1]$) have a common edge, say $\{x, y\}$. Then $x + y = a$ and $x + y = b \pmod{n-1}$, which forces $a = b$.

Another common way to describe the above 1-factors of K_{2n} is to label the vertices $w, v_0, v_1, \ldots, v_{2n-2}$ (imagine w is the center vertex, and the remaining vertices are distributed equally around a circle centered at w). For each $i = 0, 1, \ldots, 2n-2$, let the ith matching be

$$M_i = \{w, v_i\} \cup \{\{v_{i-x}, v_{i+x}\} : x \in \{1, 2, \ldots, n-1\}\},$$

where arithmetic in the indices is done modulo $2n-1$. □

The construction applied to K_6 is given in Figure 8.2. The result in Lemma 8.2.1 is used in the proof of Theorem 6.11.5, which characterizes $\chi'(K_n)$.

Theorem 8.2.2 (Chetwynd–Hilton, 1986 [204]). *Let G be an r-regular graph on n vertices. If n is even and $r \geq \frac{\sqrt{7}-1}{2}n$, then G has a decomposition into 1-factors.*

8.3 Perfect 1-factorizations

At a 1963 conference in Smolenice (in present day Slovakia), many problems were proposed, one of which is of particular interest here. Kotzig [602] called a regular graph with an even number of vertices a "Hamilton graph" if there exists a decomposition into 1-factors so that the union of any two of the 1-factors form a Hamiltonian cycle. Today, such a 1-factorization of a complete graph K_{2n} is called a "perfect 1-factorization" of K_{2n} (*e.g.*, see [47]). (A perfect 1-factorization is not to be confused with a perfect matching, which is just a 1-factor.)

Question 8.3.1 (Kotzig, 1963 [602]). *Does there exist an integer $n > 1$ so that K_{2n} has no perfect 1-factorization?*

This question is still open, but many cases have been solved. For a survey until 1991, see [842]. A more current survey was given by Rosa in a number of talks since 2013 (*e.g.*, see [807]); a survey by Rosa [808] appeared in 2019. The analogous question for complete bipartite graphs is also outstanding (see [172]). Both questions have positive answers for certain values of n, *e.g.*, for K_n when n is twice an odd prime or a prime plus one, or for $K_{n,n}$ when n is the square of an odd prime [172].

Kotzig proved the following, perhaps in an earlier paper with a Russian title, but the result was also presented at a 1963 conference, so I give the proceedings of that conference as the reference (where the proof is in English):

Theorem 8.3.2 (Kotzig, 1963 [602])**.** *If $2n-1$ is prime, there exists a perfect 1-factorization of K_{2n}.*

The proof that Kotzig used [602, Thm 15] for the case when $2n-1$ is prime is based on a standard decomposition given in [596] (see Lemma 8.2.1), which, according to [842], was known as far back as at least 1859 by Reiss [787].

Exercise 383. *Let p be a prime, and let $2n = p+1$. Show that the 1-factorization given in the solution to Lemma 8.2.1 is indeed a perfect 1-factorization.*

The result in Exercise 383 then shows that the complete graph K_{2n} can be decomposed into $n-1$ Hamiltonian cycles and one perfect matching. See [40] for Walecki's construction that shows the same result. (For a survey on decomposing complete graphs into Hamiltonian cycles, or Hamiltonian cycles and one perfect matching, see [41].)

Early in 1973, Bruce Anderson [44] also gave a proof of Theorem 8.3.2 and another theorem (see Theorem 8.3.3 below) using topologies, perhaps totally unaware of the work on Kotzig's problem. According to [45], Alex Rosa then introduced Anderson to the literature and people involved. Later in 1973, Anderson [45, p. 142] wrote that Kotzig claimed (in [603]) to have a proof for the following, but he had not seen it in print. (In some of Kotzig's papers, there are no bibliographic references, so one can only guess that he gave the proof in an earlier work written in Slovak or Russian, most likely [600], or perhaps [601].) This theorem was also proved independently by Nakamura.

Theorem 8.3.3 (Kotzig, 1958? [600]?; Anderson, 1973 [44], Nakamura, 1975 [710])**.** *If p is an odd prime, then K_{2p} has a perfect 1-factorization.*

A proof of Theorem 8.3.3 can be found in [776, Thm 2.1.8]; that proof takes roughly five pages of case checking, and so is omitted; however, the starting point of that proof is similar to that of Theorem 8.3.2, and can be easily described: Let p be an odd prime and let the vertices of K_{2p} be given as $v_0, v_1, v_2, \ldots, v_{2p-1}$. Let F_0 be the 1-factor given by the edges

$$\{v_0, v_p\}, \{v_1, v_{2p-1}\}, \{v_2, v_{2p-2}\}, \ldots, \{v_{p-1}, v_{p+1}\}.$$

For each $i = 1, 2, \ldots, p-1$, let F_i be the 1-factor given by permuting the labels in order (modulo $2p$); in other words, if $\{v_a, v_b\}$ is an edge in F_0, then $\{v_{a+i}, v_{b+i}\}$ is an edge in F_i. It is easy to check that the p 1-factors $F_0, F_1, \ldots, F_{p-1}$ are pairwise disjoint and contain all edges of K_{2p}. Showing that the factors combine properly is another story. It seems as if the necessary math was also given by Anderson, although somewhat disguised by the topologies.

Anderson [48] showed that for $n \leq 5$, K_{2n} has just one perfect 1-factorization (he also showed, by looking at symmetry groups, some conditions that guarantee at least two different perfect 1-factorizations). Since 14 is twice a prime, the existence of a perfect 1-factorization of K_{14} is guaranteed by Theorem 8.3.3; Seah and Stinson [843] found 20 different perfect 1-factorizations of K_{14} (and cite one additional one).

The first few values (for $2n$) not covered by Theorems 8.3.2 or 8.3.3 are 16, 28, 36, 40, 50, 52, 56, 64, 66, 70, 76, 78, 88, 92, 96, 100. Anderson [45] found perfect 1-factorizations for K_{16} and K_{28}. (Anderson also reports that both Kotzig and independently, Ehrenfucht et al. [281] also found found perfect 1-factorizations of K_{16}). See also [46] for details on K_{28}. Anderson and Morse [49] showed that K_{244} and K_{344} have perfect 1-factorizations. The cases $2n = 36, 40$ were handled by Seah and Stinson [844, 845], and the case $2n = 50$ was settled by Ihrig, Seah, and Stinson [536]. Adam Wolfe [992] solved the case $2n = 52$. As of 2013, Rosa said that the first value for $2n$ for which the result was not known is 56. However, in 2019, David Pike [749] claimed a proof of the case $2n = 56$. (See Pike's paper for many more recent values for which the perfect 1-factorization conjecture has been solved.) For some recent results and more references about perfect 1-factorizations, the reader might begin with [173], [808], or [972].

Instead of asking for 1-factors in perfect 1-factorizations that pairwise combine to form a Hamiltonian cycle, decompositions into 1-factors so that any two forbid some k-cycle have also been studied [681]. Decompositions of K_{2n+1} into "almost" 1-factors so that any two of the almost 1-factors is a Hamiltonian path was also studied by Kotzig [603] using groupoids and quasigroups.

Kotzig and others also studied perfect 1-factorizations for cubic graphs; see [601], or for further references, see [145]. Work on the bipartite versions of the perfect 1-factorization is also mentioned in many papers (*e.g.*, [172], [605], or [617]) and is not further discussed here.

8.4 Decompositions into 2-factors

Recall that a k-factor of a graph G is a k-regular subgraph that spans all of $V(G)$. For convenience, Petersen's 2-factor theorem is repeated here. Recall that the second theorem by Petersen (Theorem 5.9.5) says that a $2k$-regular graph has a 2-factor. Removing such a 2-factor leaves a $2(k-1)$-factor, and so the theorem applies again,

continuing until a 2-factorization is found. The proof of Petersen's theorem givien earlier used Hall's theorem. The proof given here for the (slightly) stronger theorem about 2-factorizations uses König's line colouring theorem.

Theorem 8.4.1 (Petersen, 1891 [746]). *For each positive integer k, any $2k$-regular graph has a decomposition into k edge-disjoint 2-factors.*

Proof: Let k be a positive integer and let G be a $2k$-regular graph. Without loss of generality, assume that G is connected (for if not, apply the following proof to each component).

First examine the case $k = 2$, that is, assume that G is 4-regular, and create two 2-factors as follows. Since G is connected and has all degrees even, there exists (by Euler's theorem, Theorem 2.2.2) an Eulerian trail C. Colour consecutively the edges of C alternating red and blue. Since C passes through each vertex twice, at each vertex there are two red and two blue edges. So the red edges form a 2-factor and the blue edges form a 2-factor.

A similar, but slightly tricky method is used for the general case. Let $k \geq 1$, G be $2k$-regular, and let C be an Eulerian circuit. Assign a direction to follow around C, thereby creating a digraph D where every vertex has indegree k and outdegree k. If G has n vertices, Put $V(G) = \{x_1, x_2, \ldots, x_n\}$, and create two disjoint copies of $V(G)$, say $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$. Form a (undirected) bipartite graph H with bipartition $V(H) = Z \cup Y$, where

$$E(H) = \{\{y_i, z_j\} : (x_i, x_j) \in E(D)\}.$$

Then H is k-regular and bipartite. By König's line colouring theorem (Theorem 6.11.2, or Corollary 6.11.3), there exists a decomposition of H into k 1-factors. Since no pair of the form $\{y_i, z_i\}$ is an edge of H, recover G by identifying corresponding pairs $\{y_i, z_i\}$ with x_i, in which case each 1-factor of H becomes a 2-factor in G. □

One particular kind of 2-factor is a Hamiltonian cycle. Note that for any $k \geq 1$, K_{2k+1} is $2k$-regular; is it possible that all 2-factors guaranteed by Petersen's theorem (Theorem 8.4.1) can be Hamiltonian cycles? (To learn more about this and related problems, the article by Alspach [40] might be valuable; also see [975].)

Eduoardo Lucas (1842–1891) asked this question in the second volume of his four volumes set [648] but using a different setting.

Problem 8.4.2 (problèm de ronde, Lucas [648], 1882–1894). *Is it possible to seat $2n+1$ people around a table (that seats $2n+1$ people) on n successive nights so that any two people sit adjacent (on either side) on at most one night?*

Translating Problem 8.4.2 to graph theory, consider K_{2n+1}, which has $2n+1$ labelled vertices (people) and $n(2n+1)$ edges (where an edge is formed by adjacent diners).

Each arrangement of $2n+1$ people around the table corresponds to a Hamiltonian cycle in K_{2n+1}. Since any two people are adjacent at most once, the question asks for n edge-disjoint Hamiltonian cycles in K_{2n+1}. In fact, since $|E(K_{2n+1})| = n(2n+1)$, and each Hamiltonian cycle has $2n+1$ edges, the problem is to find a decomposition of K_{2n+1} into (edge-disjoint) Hamiltonian cycles.

The solution given by Lucas in [648] was attributed to Walecki, and is based roughly on the decomposition of K_{2n+1} into 1-factors as shown in Lemma 8.2.1

Theorem 8.4.3 (Walecki, see [648])**.** *Let k be a positive integer. Then K_{2k+1} can be decomposed into k edge-disjoint Hamiltonian cycles.*

Proof: Let $V(K_{2k+1} = \{v_0, v_1, \ldots, v_{2k}\}$. Consider the Hamiltonian cycle H_1 given by

$$v_0, v_1, v_2, v_{2k}, v_3, v_{2k-1}, v_4, v_{2k-2}, \ldots, v_{k+2}, v_{k+1}, v_0. \tag{8.1}$$

(See Figure 8.3 for a pictorial representation of the case $k = 5$, where H_1 is in black.) .

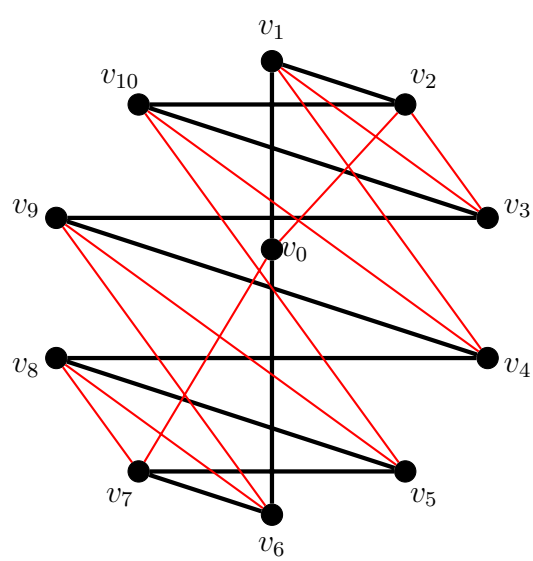


Figure 8.3: Decomposing K_{11} into Hamiltonian cycles. The starter cycle H_1 is in black, H_2 is in red

For $i = 2, \ldots, k$ create H_i from H_1 by cyclically permuting labels of all vertices except v_0 in (8.1) to the right by $i-1$ steps. With a bit of work, it can be shown (see [40] for details) that the Hamiltonian cycles H_1, H_2, $\ldots$, H_k are pairwise edge-disjoint and so partition the edges of K_{2k+1}. (Relabeling the center vertex with w and the remaining vertices $x_0, \ldots, x_{2k-1}$ can make the algebra behind the proof a bit easier.) □

The following corollary can be found in [94] (or as an exercise in *e.g.*, [125, p. 16]), where it is given without the attribution to Walecki.

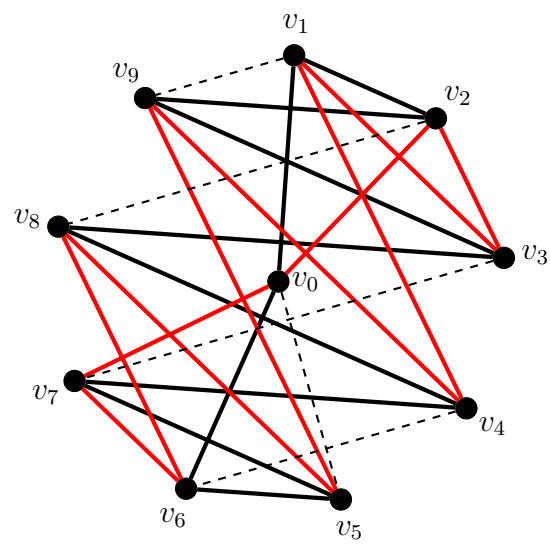


Figure 8.4: K_{10} minus a matching (dashed) can be decomposed into Hamiltonian cycles. The first two cycles of Walecki's are shown in black and red, respectively.

Corollary 8.4.4 (Walecki)**.** *For $n \geq 3$, K_n is decomposable into edge-disjoint Hamiltonian cycles if and only if n is odd and K_n is decomposable into edge disjoint Hamiltonian paths if and only if n is even.*

Proof outline: The first part is shown in Theorem 8.4.3. To see the second part, delete v_0 in the construction from Theorem 8.4.3. □

Theorem 8.4.5 (Walecki)**.** *For each positive integer n, if M is a matching in K_{2n}, then the graph $K_{2k} - M$ can be decomposed into $k-1$ Hamiltonian cycles.*

Proof sketch: Let $V(K_{2n}) = \{v_0, \ldots, v_{2n-1}\}$ and let

$$M = \{\{v_0, v_n\}, \{v_1, v_{2n-1}\}, \{v_2, v_{2n-2}\}, \ldots, \{v_{n-1}, v_{n+1}\}\}.$$

The construction is (for $n = 5$) is depicted in Figure 8.4. In general, the starter cycle is given by

$$v_0, v_n, v_{n-1}, v_{n-2}, v_{n+2}, \ldots, v_1, v_{2n-1}$$

and each subsequent cycle is obtained by cyclically permuting the indices $1, 2, \ldots, 2n-1$. □

The next problem was given by Ringel at a conference in Oberwolfach (Germany) in 1967, and was first published in [463]; it is now known simply as "The Oberwolfach problem":

Problem 8.4.6 (Oberwolfach problem, Ringel, 1967)**.** *Is it possible to seat an odd number $2n+1$ of people at s round tables $T_1, T_2, \ldots, T_s$ (where T_i can accommodate exactly $k_i \geq 3$ people and $\sum k_i = 2n+1$) for m different meals so that each person has every other for a neighbour exactly once?*

The Oberwolfach problem is equivalent to asking for a 2-factorization of complete graph K_{2n+1} where all 2-factors are isomorphic and are of prescribed cycle lengths. If there is only one table, the Oberwolfach problem is solved by a decomposition of K_{2n+1} into Hamiltonian cycles, as in Problem 8.4.2.

A number of cases for the Oberwolfach problem have been solved. For example, for the case when all cycles are of the same odd length m, it was shown in 1989 [42] that if m divides $2n+1$, then such a decomposition exists. For a survey of the Oberwolfach problem, see [530].

8.5 Decomposing K_{2n+1} into copies of a tree

8.5.1 Ringel's conjecture

For a positive integer n, the graph K_{2n+1} has $\binom{2n+1}{2} = (2n+1)n$ edges. Since n divides the number of edges in K_{2n+1}, a natural question might ask if, for some graph H with n edges, can K_{2n+1} be decomposed into $2n+1$ copies of H?

Exercise 384. *Let P_2 be a path of length 2 (on three vertices). Find a decomposition of the edges of K_5 into 5 edge-disjoint copies of P_2.*

One might also ask if F is a graph with $2n+1$ edges, then is there a decomposition of K_{2n+1} into n copies of F? For example, K_5 can be easily decomposed into two copies of C_5. In this section, only the first question is addressed (where H has n edges).

Conjecture 8.5.1 (Ringel, 1963 [791])**.** *Let $n \in \mathbb{Z}^+$. For any tree T with n edges, K_{2n+1} can be decomposed into $2n+1$ copies of T.*

In 1973, Kotzig refined Ringel's conjecture:

Conjecture 8.5.2 (Kotzig, 1973 [604])**.** *For any tree T with n edges, K_{2n+1} can by cyclically decomposed into $2n+1$ copies of T.*

Kotzig's conjecture inspired a great deal of work, in particular, work begun by Alex Rosa on what is now known as "graceful labelling", the topic of the next section.

8.5.2 Approaching Ringel's conjecture: graceful labellings

Definition 8.5.3. *Let T be a tree on n vertices. A* graceful labelling *of T is a bijection $f : V(T) \to \{1, 2, \ldots, n\}$ so that*

$$\{|f(x) - f(y)| : \{x, y\} \in E(T)\} = \{1, 2, \ldots, n-1\}.$$

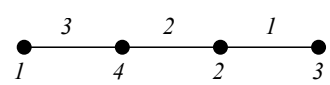
A tree T is called graceful *if and only if there exists a graceful labelling of T.*

The reader can easily extend the definition of graceful labellings for other graphs, not just trees, but in this section, only trees are considered.

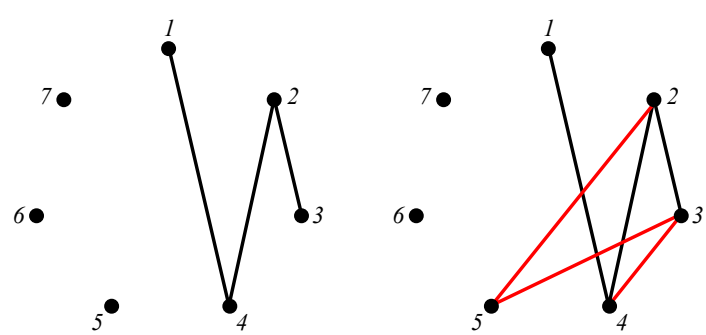
Sometimes it is convenient to use a labelling of n vertices with the numbers $0, 1, \ldots, n-1$, in which case the edges are still "coloured" from 1 to $n-1$.

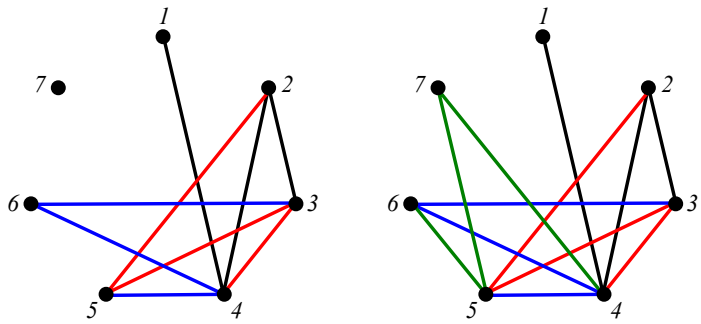
Graceful colourings were introduced by Rosa in 1966 [805] (although they were not called "graceful colourings" until Golomb [425] coined the phrase in 1972; Rosa called them "β-valuations"). Rosa observed that trees with graceful colourings "pack" cyclically into a complete graph.

For example, let T be a path of length 3, say, with vertices u, v, w, x in order. Abusing notation, let 1423 denote a colouring of $V(T)$. Then the differences formed by colours of adjacent pairs of vertices are, respectively, 3,2,1, and so the colouring 1423 is graceful.



Since the number of edges in K_7 is 21 and T has 3 edges, if possible, it takes 7 edge-disjoint copies of T to create K_7. Put the first copy on vertices, say 1,2,...,7, of K_7 according to the graceful labelling 1423, and then put additional copies rotated one vertex clockwise. Here are the first four steps:





Since each edge of T has its unique "distance", there is no collision between different copies of T. Since the path has a graceful colouring, K_7 indeed has a cyclic decomposition as asked for in Kotzig's conjecture. (For more on graceful colourings of paths, see Section 8.5.5.)

Using precisely the same idea as in the above example, if every tree is graceful, then Kotzig's Conjecture 8.5.2 is correct, and hence also Ringel's conjecture. So far, no non-graceful trees have been found.

8.5.3 The graceful tree conjecture

Conjecture 8.5.4 (Graceful tree conjecture (GTC), Rosa, 1966 [805])**.** *Every tree has a graceful labelling.*

As of 2021, the GTC is still open. See the dynamic survey by Gallian [403] in *The Electronic Journal of Combinatorics*.

It was shown by Rosa and Širáň [809] that all trees are $\frac{5}{7}$ graceful, that is, for any tree T, there exists an injective labelling (with $1, \ldots, |V(T)|$) so that there are at least $\frac{5}{7}|E(T)|$ different edge-labels (in fact, they used α-labellings—see below).

Some references call the GTC the "Ringel-Kotzig conjecture", since it arose from an attempt to solve the conjectures by Ringel and Kotzig.

Rosa considered four different types of labellings (including graceful). Two such labellings had a "balance" condition. One was called an "α-labelling". An α-labelling satisfies: $f : V \to \{0, 2, \ldots, 2n\}$, and there exists k so that for every edge $\{x, y\} \in E(T)$, one of $f(x)$ or $f(y)$ is at most k, and the other is greater than k. (So an α-labelling is also graceful). In 1973, Kotzig showed that for any tree T and any fixed $e \in E(T)$, there are only finiltely many i so that if e is replaced by a path of length i, then the resulting tree has no α-labelling.

In 2006, a conjecture stronger than GTC was proposed (see [277]), with the added condition that for adjacent vertices x and y, then $f(x) + f(y) = n + 1$.

8.5.4 Classes of trees satisfying GTC

The Graceful Tree Conjecture has been verified for many classes of trees (many of these and more can be found in the survey by Robeva [800]):

- Paths (see Exercise 388).

- Trees with bounded number of vertices.

  In 1978, Huang and Rosa [534] showed that all trees with at most 9 vertices are graceful.

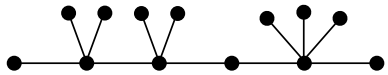
  In 1998, Aldred and Mckay [23] verified the GTC for trees with at most 27 vertices (of which there are 751,065,460).

  In 2003, Horton [531] showed that the GTC holds for trees with at most 29 vertices.

  In 2010, Fang [353] claimed a proof for trees with at most 35 vertices.

- Trees with at most 4 leaves, by Huang, Kotzig, and Rosa [533], 1982.

- Caterpillars (pendant edges attached to a path), by Rosa [805].



- Symmetric trees (vertices at the same level have the same degree), by Bermond and Sotteau, 1976 [97].

- Trees with bounded diameter.

  Rosa first showed that GTC is true for trees whose diameters are at most 3.

  In 1989, Zhao [1011] did diameter 4.

  In 2001, Hrnčiar and Haviar [532] did diameter 5 (this proof is also in [277]).

- Spiders (only one vertex of degree more than two).

- Lobsters (add leaves to a caterpillar, defined in [534]), conjectured by Bermond [96] in 1979, but according to Robeva [800], not all classes of lobster trees have been proved to be graceful.

- Firecrackers (stars all joined at one vertex) by Chen, Lu, and Yeh [201] in 1997 (with independent proof in [800]).

- Banana trees (disjoint stars, and a single vertex adjacent to one point each star), conjectured to be graceful in [201], with proof claimed by Sethuraman and Jesintha [850] in 2009.

See [800] for more examples, including "regular bamboo trees" (*e.g.*, [849]), "olive trees", and "spraying pipes".

Much of the above work uses smaller graceful trees to make larger ones, a general technique used by many authors; *e.g.*, Stanton and Zarnke [885] and Koh, Rogers, and Tan [589] glued trees together. In 1998, Burzio and Ferrarese [175] generalized the Koh–Rogers–Tan work and also proved that if a tree is graceful, then so is any subdivision of that tree.

Using the "nibble method" (a certain style of probabilistic proof invented by Rödl [803]), a limited version of GTC was shown to be almost true by A. Adamaszek, Allen, Grosu, and Hladký [10] (this result was announced by 2011, but the paper seems to have appeared only in 2016). They showed that trees on n vertices with maximum degree $o(n/\ln n)$ are "almost" graceful. (As mentioned before, Rosa and Širáň [809] showed that all trees are $5/7$ graceful.) The paper [10] also gives references for previous developments in trying to show that all trees of bounded degree are graceful, ending with the success by Joos, Kim, Kühn, and Osthus [544] (they used a number of high-powered tools, like Szemerédi's "regularity lemma") and "Doob martingales" (topics not covered in this text).

Applications of graceful colourings of trees are reported (see, *e.g.*, [24] and [114]) to include

- Radar pulse codes;
- Convolution codes;
- X-ray crystallography;
- Topology and transitive triangulations of K_n.

Exercise 385. *Show that every tree of diameter at most 2 is graceful.*

Exercise 386. *Let G be the tree on 7 points formed by a center vertex from which there are three disjoint paths of length 2. (Such a graph is a spider graph, but in this case, the spider has only three short legs.) Show that T is graceful. Hint: There are at least two very different labellings that work.*

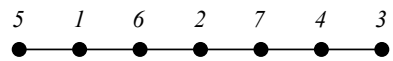
8.5.5 Graceful paths

An injective labelling of a path on n vertices with labels $1, 2, \ldots, n$ can be thought of as a permutation of $(1, \ldots, n)$. (Round parentheses are often used to emphasize one natural order, even though a permutation of an unordered set X is defined to be a bijection from X to X.)

Exercise 387. *Find all graceful colourings of a path on vertices u, v, w, x, taken in order along the path. Hint: There are at least 4.*

Exercise 388. *Show that any path is graceful.*

In one solution to Exercise 388, an alternating pattern begins $1, n, 2, n-1, \ldots$. However, other graceful colourings exist. For example, consider the following colouring of a path on 7 vertices:



As noted below, the number of graceful labellings of long paths is exponential in the number of vertices in a path, although there is something to be said for the alternating pattern.

Exercise 389. *Let $(b_1, b_2, \ldots, b_{2n})$ be a permutation of $(1, 2, \ldots, 2n)$ such that $(|b_2 - b_1\,|b_3 - b_2|, \ldots, |b_{2n} - b_{2n-1}|)$ is a permutation of $(1, 2, \ldots, 2n-1)$. Show that*

$$\{b_2, b_4, \ldots, b_{2n}\} = \{1, 2, \ldots, n\}$$

if and only if $b_1 = b_{2n} + n$.

Rosa [806] considered graceful colourings of paths (which he called "snakes") and showed that the lowest label can appear in many places. In 1990, Abrhám and Kotzig [7] showed that the number of graceful labellings of a path of length n is asymptotically at least $(1.3953)^n$. In 2003, Aldred, J. Širáň and M. Širáň [24] improved this to $(5/3)^n$. In 2013, this was improved further (see Section 18.1 for definition of $\Omega(\cdot)$) to $\Omega(2.37^n)$ by Michał Adamaszek [11].

Can the GTC be approached used probabilistic methods? Balister and Gunderson [71] proved that for the case of paths on n vertices, as $n \to \infty$, in a random bijective vertex-colouring (permutation) σ, the expected number of distinct differences (in absolute value) between consecutive elements

$$|\{|\sigma(i) - \sigma(i+1)| : i = 1, \ldots, n-1\}|$$

tends to $\frac{e^{-2}+1}{2}n \sim (0.56766)n$.

Chapter 9

Ramsey theory

9.1 Introduction

This chapter serves as a modest introduction to some Ramsey theory as it applies to graphs. For a more comprehensive survey of Ramsey theory see, *e.g.* [435], [716], or [763].

Frank Plumpton Ramsey was a logician, economist, philosopher, and mathematician, born in 1903. He died at 26 years of age. Despite his early demise, he was a man of many accomplishments. (See, for example, [435], [759], or [762], for personal history and further references.) In 1930, "On a problem of formal logic" appeared [773], where Ramsey proved some partition theorems that extend the pigeonhole principle.

In the early 1930s, three young Hungarians (Paul Erdős, Esther Klein, and George Szekeres) were working on a problem regarding convex sets, and in doing so, duplicated some of Ramsey's results, but with completely different proofs and more reasonable bounds. The motivations behind these two papers were completely different—the first was an attempt to prove a general result regarding model theory, whereas the second was the outcome of pursuing a combinatorial geometry problem. Today, there are a number of different results called "Ramsey's theorem". To help state some of these results, a "Ramsey arrow" notation was developed by Erdős and Rado [336] in 1950; a simple version of this arrow is given first, with more general notation following.

9.2 Simple partitioning results

Let X be a set. A collection of subsets $X_1, X_2, \ldots, X_k$ of X is called a *partition of* X if and only if (i) $\cup_{i=1}^{n} X_i = X$, and (ii) for each $i \neq j$, $X_i \cap X_j = \emptyset$. For some $k \in \mathbb{Z}^+$, a k-colouring of X is a function from X to some set of k colours. Commonly, the set $[k] = \{1, \ldots, k\}$ is used for colours and so a k-colouring might be denoted by $f : X \to [k]$; in this case, f is equivalent to the partition $X = X_1 \cup \cdots \cup X_k$, where

the colour classes (or partition classes) are $X_1 = f^{-1}(1), \ldots, X_k = f^{-1}(k)$. Instead of using $[k] = \{1, 2, \ldots, k\}$ for the set of colours, some authors prefer the ordinal notation $\mathbf{k} = \{0, 1, \ldots, k-1\}$. In many cases, the order of the colours is important, hence the ordinal notation can be useful.

For some modern authors, it is tradition to use Δ instead of f (or c or θ or...) for a colouring function (only when there can be no confusion with the maximum degree of a graph, $\Delta(G)$). The tradition of using Δ for colourings seems to have arisen in Germany in the 1980s (or earlier?); see, *e.g.*, [759]. Many partitioning problems can be viewed as colouring problems since a partition class can be viewed as the set of all elements that were given some specific colour. Elements of the same colour class are often called "monochromatic".

One version of the pigeonhole principle (PHP) says that if $n = kr+1$, and a set with n elements (pigeons) is partitioned into k sets (holes), then at least one set contains at least $r+1$ elements. This version of the PHP stated in terms of colourings says: if an n-element set has its elements coloured by k colours, then one colour class contains at least $r+1$ elements.

The pigeonhole principle is about partitioning (single) elements of a set. Much of Ramsey theory deals with partitioning pairs, triples, or in general, k-sets. For the next definition, recall that for a set X and a positive integer k,

$$[X]^k = \{Y \subseteq X : |Y| = k\},$$

the collection of k-element subsets of X. [Note: some authors use the notations $\binom{X}{k}$ or $X^{(k)}$ instead of $[X]^k$.]

Definition 9.2.1. *For positive integers $k, \ell \geq 2$, and n, write*

$$n \to (k, \ell)^2 \tag{9.1}$$

if and only if for any colouring $\Delta : [n]^2 \to \{0, 1\}$, there exists either $K \in [n]^k$ so that the pairs $[K]^2$ are all coloured 0, or $L \in [n]^\ell$ so that all pairs $[L]^2$ are coloured 1.

Note that the 2 in "$(k, \ell)^2$" indicates the magnitude of the items being coloured, in this case, pairs (or edges). For colouring points, the notation $n \longrightarrow (k, \ell)^1$ might indicate a PHP problem (in this case, $n = k + \ell - 1$ works). Often this superscript 2 is omitted when no confusion can arise, in which case one writes simply $n \to (k, \ell)$.

Often, names of colours (like red and blue) take the place of 0 and 1, say. Note that if $n \longrightarrow (k, \ell)^2$, one can say that for any subgraph G of K_n, either G contains a K_k or a set of ℓ independent vertices (since "edge" can be considered as one colour, and "non-edge" is the other). In this case, one might write the Ramsey arrow notation as

$$K_n \longrightarrow (K_k, K_\ell)_2^{K_2}. \tag{9.2}$$

A similar notation for more colours is also available. For $r \geq 1$, let the notation

$$n \longrightarrow (k_1, k_2, \ldots, k_r)^2 \tag{9.3}$$

denote the fact that for any r-colouring of $E(K_n)$, there is some i and k_i vertices all of whose pairs are coloured i. The bulk of this chapter is devoted to $r = 2$ colours, in part because much less is known for more colours; see Section 9.7 for Ramsey numbers

When all k_is in (9.3) are equal to m, the notation is abbreviated to

$$n \longrightarrow (m)_r^2,$$

where the subscript r denotes the number of colours.

The slightly more general notation

$$n \longrightarrow (m)_r^k$$

means that for every colouring $\Delta : [n]^k \to [r]$, there is a set $M \in [n]^m$ so that $[M]^k$ is monochromatic.

Definition 9.2.2. *For positive integers k, ℓ, let $R(k, \ell)$ be the least n, if it exists, so that $n \to (k, \ell)$. Such a $R(k, \ell)$ is called a Ramsey number.*

Theorem 9.2.3. *The Ramsey number $R(3, 3)$ exists and is equal to 6.*

Proof: To prove the theorem, one needs to show that $6 \longrightarrow (3, 3)$ but $5 \not\longrightarrow (3, 3)$.

To see the first part, suppose that K_6 is on vertices A,B,C,D,E,F and its edges are coloured either red or blue. Consider A; from A there are five edges, and by the PHP, there are three of which that are the same colour, say AC, AD, and AE are red (see Figure 9.1).

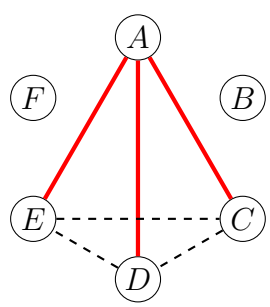


Figure 9.1: $R(3, 3) \leq 6$.

If any of CD, CE or DE are red, a red triangle is formed; if none of these are red, then CDE forms a blue triangle. If the first three edges are blue, the same argument holds. In any case, a monochromatic triangle is produced showing $6 \longrightarrow (3, 3)$.

To see the second part, let the vertices of a K_5 be A, B, C, D, E; consider the red-blue colouring of $E(K_5)$ given by colouring all edges of the cycle AB, BC, CD, DE, EA red and the remaining five edges blue (see Figure 9.2).

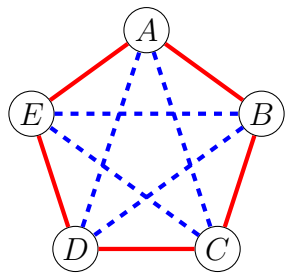


Figure 9.2: $R(3,3) > 5$.

There is no triple all of whose pairs (edges) are the same colour, so $5 \not\longrightarrow (3,3)$. $\square$

Theorem 9.2.3 is often stated as a "party problem": if six people are at a party, show that either there are 3 that all know each other or there are 3 that are all strangers to one another (assuming that "know" is symmetric and two people who don't know each other are strangers to each other). Theorem 9.2.3 appeared in 1935 in a seminal paper (that marked one of two beginnings of the study of Ramsey numbers) by Erdős and Szekeres [346]; this result again appeared 1955 in a paper by Greenwood and Gleason [418] (although they did not mention the first paper). The party problem also appeared as a question in the 1953 Putnam exam.

Exercise 390. *(a) Show that under any 2-colouring of the edges of K_6, two monochromatic triangles exist (perhaps of different colours). (b) Find such a 2-colouring so that exactly two monochromatic triangles exist, one of each colour.*

Goodman (see Theorem 10.3.11) gave an extension of the result from Exercise 390. The minimum number of monochromatic copies of a desired graph in Ramsey theory is called the "Ramsey mulitiplicity".

For $k \geq 2$ and $\ell \geq 2$, $R(k,2) = k$ and $R(2,\ell) = \ell$ since any colouring that avoids one pair (the 2) in one colour has all edges of the opposite colour.

The following is a special case of a theorem first proved by Ramsey (published in 1930), and independently by Erdős and Szekeres five years later. Since the second proof is is so elegant, it is given first.

Theorem 9.2.4 (Ramsey's theorem, simple case)**.** *For each $k \geq 2$ and $\ell \geq 2$, $R(k,\ell)$ exists.*

One proof of Theorem 9.2.4 is based on the following:

Theorem 9.2.5 (Erdős–Szekeres recursion, 1935 [346]). *Let $s, t \geq 3$. If both $R(s-1,t)$ and $R(s,t-1)$ exist, then so does $R(s,t)$, with*

$$R(s,t) \leq R(s-1,t) + R(s,t-1). \tag{9.4}$$

Proof: Suppose that $a = R(s-1,t)$ and $b = R(s,t-1)$ exist, and put $n = a+b$. It remains to show that $n \to (s,t)$. Let $\Delta : E(K_n) \to \{0,1\}$ be given. Let $x \in V(K_n)$ and put $A = \{y : \Delta(\{x,y\}) = 0\}$ and $B = \{y : \Delta(\{x,y\}) = 1\}$. By the pigeonhole principle, one of $|A| \geq a$ or $|B| \geq b$ holds.

If $|A| \geq a$, then in the graph induced by A, there is either a K_{s-1} all of whose edges are coloured 0 or a K_t all of whose edges are coloured 1. In the first case, this K_{s-1} together with x form a K_s, all of whose edges are coloured 0. In either case, either a 0-coloured K_s or a 1-coloured K_t exists.

If $|B| \geq b$, a similar argument shows the existence of a monochromatic K_s or a monochromatic K_t, concluding the proof. □

Proof of Theorem 9.2.4: For $k \geq 2$ and $\ell \geq 2$, $R(k,2) = k$ and $R(2,\ell) = \ell$ since any colouring that avoids one pair (the 2) in one colour has all edges of the opposite colour. Theorem 9.2.4 now follows by induction (on $k+\ell$) from the Erdős–Szekeres recursion (equation (9.4)). □

By interchanging roles of colours, the next statement is rather obvious; however, it is stated formally for the record.

Lemma 9.2.6. *If $R(s,t)$ exists, then $R(t,s)$ exists, and they are equal.*

Proof: Put $n = R(s,t)$. Fix any colouring $\Delta : E(K_n) \to 2$, and define $\Delta' : E(K_n) \to 2$ by $\Delta'(e) = 1 - \Delta(e)$ for each edge in K_n. Then any K_s that is 0-monochromatic under Δ is 1-monochromatic under Δ', and any K_t that is 1-monochromatic under Δ is 0-monochromatic under Δ'. Thus $R(t,s)$ exists and $R(t,s) \leq n$. By reversing roles of s and t, $n \leq R(t,s)$, proving the result. □

Numbers of the form $R(k,k)$ are called *diagonal Ramsey numbers*; unfortunately, the only diagonal Ramsey numbers known are $R(3,3) = 6$ and $R(4,4) = 18$. Other known Ramsey numbers (for two colours) are: $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$), $R(4,5) = 25$.

It is known that

$$43 \leq R(5,5) \leq 48,$$

where the upper bound was improved from 49 only in 2017 [55], and

$$102 \leq R(6,6) \leq 165.$$

The lower bound for $R(6,6)$ comes from the Paley graph for $q = 101$ (see Definition 9.6.2 below for details).

See the dynamic survey by Stanislaw Radziszowski [770] for other bounds on Ramsey numbers.

9.3 Some small Ramsey numbers

The Erdős–Szekeres recursion shows that

$$R(3,4) \leq R(2,4) + R(3,3) = 4 + 6 = 10,$$

however by a closer look at the recursion, one can do a bit better.

Exercise 391. *Prove that* $R(3,4) \leq 9$.

Exercise 392. *Give a 2-colouring of* $E(K_8)$ *that proves* $R(3,4) > 8$.

Putting these last two exercises together:

Theorem 9.3.1 (Gleason–Greenwood, 1955 [418])**.** $R(3,4) = 9$.

Corollary 9.3.2. $R(3,5) \leq 14$.

Proof: By the Erdős–Szekeres recursion, $R(3,5) \leq R(2,5) + R(3,4) = 5 + 9 = 14$. □

The result in the next exercise was also found by Gleason and Greenwood [418]:

Exercise 393. *Show that* $R(3,5) = 14$ *by finding a red-blue colouring of the edges of* K_{13} *that has no red triangle and no blue* K_5*. Hint: look at the cubes in* $\mathbb{Z}_{13}$.

Corollary 9.3.3. $R(4,4) \leq 18$.

Proof: By Theorem 9.3.1 and the Erdős-Szekeres recursion (Theorem 9.2.5),

$$R(4,4) \leq R(3,4) + R(4,3) = 18.$$

□

Lemma 9.3.4. $R(4,4) > 17$.

Proof: In Figure 9.3, a red-blue colouring of $E(K_{17})$ is given. One can verify (by exhaustive case-checking) that no K_4 contains monochromatic edges. □

By Corollary 9.3.3 and Lemma 9.3.4, the Ramsey number $R(4,4)$ is found:

Corollary 9.3.5. *R(4,4)=18.*

The two different graphs (one for each colour) given in Figure 9.3 are self-complementary and play the same role in proving $R(4,4) > 17$ as the role C_5 plays in the proof of $R(3,3) > 5$. These two graphs are examples from a family of self-complementary graphs called "Paley graphs". Due to their significance in Ramsey theory, Section 9.6.1 is devoted to their construction (using quadratic residues) and their properties.

One can check by inspection that the red-blue colouring of K_{17} given in Figure 9.3 yields no monochromatic K_4. (This fact can also be shown algebraically—see Exercise 399.)

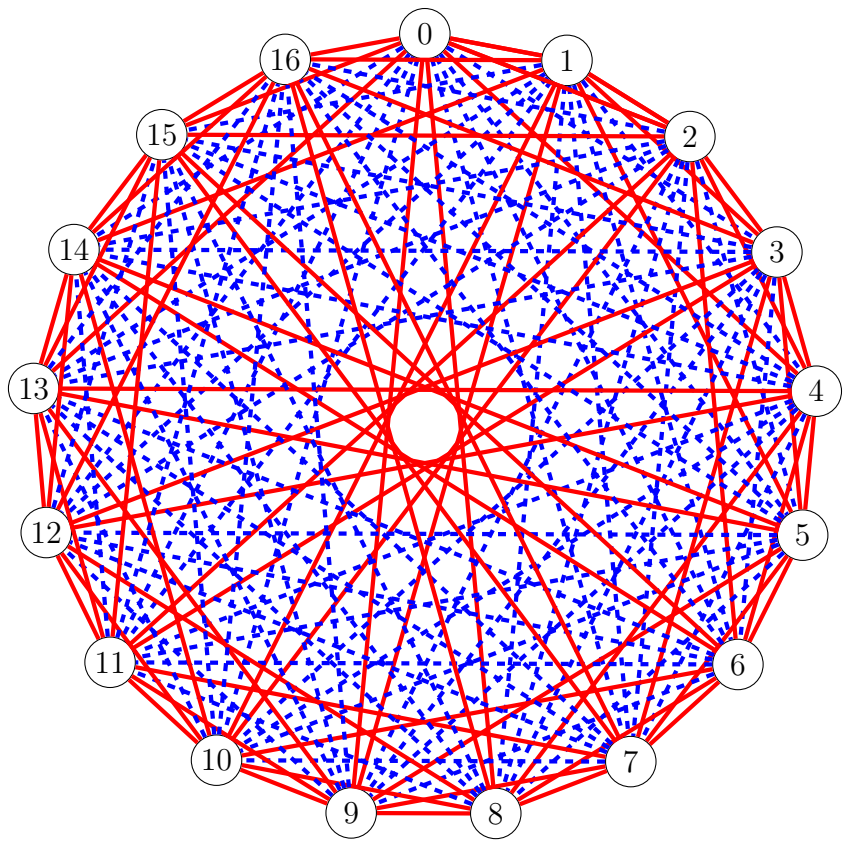


Figure 9.3: A 2-colouring of the edges of K_{17} with no monochromatic K_4, thereby showing $R(4,4) > 17$. (Each colour forms a Paley graph.)

Theorem 9.3.6 (Greenwood–Gleason, 1955 [418])**.** $R(3,5) = 14$.

Proof outline: By Corollary 9.3.2, $R(3,5) \leq 14$. To see that $R(3,5) > 13$, it suffices to give a red-blue colouring of $E(K_{13})$ so that there are no red K_3's and no blue K_5s.

Let $V = \{0, 1, 2, \ldots, 12\}$ be the vertex set of a copy of K_{13}. In the field $\mathbb{Z}_{13} = \mathbb{Z}/13\mathbb{Z}$, the cubes of the non-zero numbers are, respectively, $1^3 = 3^3 = 9^3 = 1$, $2^3 = 5^3 = 6^3 = 8$, $4^3 = 10^3 = 12^3 = 12$, and $7^3 = 8^3 = 11^3 = 5$, so the "cubic residues" are 1,5, 8,12=-1 (modulo 13). Define a colouring $c : E(K_{13}) \to \{\text{red, blue}\}$ by $c(\{i,j\}) =$ red if and only if $i - j$ is a (non-zero) cubic residue modulo 13. Since -1 is a cubic residue, it does not matter in which order i and j appear.

It is not difficult to check that there are no red triangles. It remains to see show that there is no blue K_5, that is, that among any five vertices, at least two differ by a cubic residue.

Choose any five vertices $u, v, w, x, y \in V$. By symmetry, assume that $u = 0$. If either 1, 3, 8, or 12 occurs as another vertex, then a red edge appears, so assume that none of these is chosen. The remaining four vertices are among 2, 3, 4, 6, 7, 9, 10,

11. By the PHP, two of these are in one of the sets $A = \{2, 3, 4\}$, $B = \{6, 7\}$, or $C = \{9, 10, 11\}$.

If these two are consecutive, a red edge is formed (since 1 is a cubic residue), so assume (by symmetry again) that 2 and 4 are chosen. Since 3, 7, and 10 all differ from 2 by a cubic residue, and 9 differs from 4 by a cubic residue, the only choices left are 6 and 11—which themselves differ by a cubic residue, yet again creating a red edge. □

Gleason and Greenwood also showed that $16 < R(3, 6) \leq 19$.

According to [770], Kalbfleisch receives credit for $R(3, 6) = 18$ (in his 1966 dissertation [551], which I have not seen). However, Sós [877] informed me that in her 1963–1964 class on Combinatorics (and graph theory) at Eötvös Loránd University (ELTE, Budapest), one of her students, Gerzson Kéry, first showed $R(3, 6) = 18$; this result was published in 1964 [566]. If I remember correctly, at the time of his discovery, Kéry was a 16-year old first year student. [Comment: I found it interesting that Sós said that in 1961, ELTE was the first university to make discrete math compulsory.] Graver and Yackel [440] are responsible for $R(3, 7) = 23$.

See Figure 9.4 for a summary of known Ramsey numbers $R(s, t)$ (for $3 \leq s, t \leq 10$) or bounds on such as of 2017. (See [770] for a dynamic survey with references for each bound.)

$s\backslash t$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40–42
4		18	25	36–41	49–61	59–84	73–115	92–149
5			43–48	58–87	80–143	101–216	133–316	149–442
6				102–165	115–298	134–495	183–780	204–1171
7					205–540	217–1031	252–1713	292–2826
8						282–1870	329–3583	343–6090
9							565–6588	581–12677
10								798–23556

Figure 9.4: Known bounds on small Ramsey numbers $R(s, t)$

9.4 Upper bounds on diagonal Ramsey numbers

Upper bounds for $R(k, k)$ follow from the Erdős-Szekeres recursion.

Exercise 394. *Use the Erdős–Szekeres recursion and induction to show that*

$$R(s, t) \leq \binom{s + t - 2}{t - 1}$$

.

Using Stirling's formula (see Section 18.2) for approximating factorials, the last corollary yields an upper bound for diagonal Ramsey numbers.

Theorem 9.4.1. *There exists a universal constant c so that for every k,*

$$R(k,k) < \frac{c}{\sqrt{k}} 2^{2k}.$$

Exercise 395. *Prove Theorem 9.4.1.*

In 1972, Yackel [1001] improved the upper bound in Theorem 9.4.1 by showing that there exists a constant $c > 0$ so that for every $k \geq 3$,

$$R(k,k) < \frac{c \log\log k}{\log k} 2^{2k}.$$

Restating these bounds with $k = \log_2(n)/2$, one obtains that $n \to (\log_2(n)/2)^2_2$. (Paul Erdős reviewed Yackel's paper for *Math Reviews*, and reported that the proof is "surprisingly difficult".)

Subsequently, Rödl (see [434]) improved the upper bound for diagonal Ramsey numbers to $R(k,k) = o(\frac{4^k}{\sqrt{k}})$, and Thomason [914] further improved the bound to $O(\frac{4^k}{k})$. As of 2017, the best known upper bounds are due to David Conlon:

Theorem 9.4.2 (Conlon, 2009 [226])**.** *There exists a constant C so that*

$$R(k+1,k+1) \leq k^{-C\frac{\log k}{\log\log k}} \binom{2k}{k}.$$

In particular, for all $s > 0$, there exists a constant C_s so that

$$R(k+1,k+1) \leq \frac{C_s}{k^s} \binom{2k}{k}.$$

9.5 Lower bounds on diagonal Ramsey numbers

To find lower bounds for $R(k,k)$, it suffices to find a 'large' n, and give a 'bad' colouring of the edges of K_n, or at least show that such a bad colouring exists. As seen while trying to prove a lower bound for $R(4,4)$, 'bad' constructions on a large number number of vertices may be increasingly difficult to find. However, due largely to Erdős, one may enjoy more success in proving lower bounds by proving that a bad colouring or bad graph merely exists (even if only on a somewhat smaller than optimal number of vertices). If one interprets a 2-colouring of $E(K_n)$ as "edge-nonedge" then Ramsey's theorem gives that any graph on $R(k,k)$ vertices contains either a K_k or its complement $\overline{K_k}$, a collection of k independent vertices.

Theorem 9.5.1 (Erdős, 1947 [290]). *If*

$$\binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}+1} < 2^{\binom{n}{2}} \tag{9.5}$$

then $R(k,k) > n$.

Proof: Let n satisfy (9.5). Call a graph on n vertices 'good' if it contains a K_k or k independent vertices. The number of good graphs is at most

$$\binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}+1}$$

and since there are only $2^{\binom{n}{2}}$ graphs on n vertices, equation (9.5) shows there exists a graph that is not good. □

To obtain a lower bound for $R(k,k)$ from Theorem 9.5.1, one needs some facility for estimating factorials and binomial coefficients.

The following approximation (given as Lemma 18.2.1 in the appendix Chapter 18) for the binomial coefficient $\binom{n}{k}$ is often applied for a fixed k and n very large, however, it also applies for variable k, as long as n is "much larger" than k. As $n \to \infty$, for $k = o(n^{3/4})$,

$$\binom{n}{k} = (1+o(1)) \frac{1}{\sqrt{2\pi k}} \left(\frac{en}{k}\right)^k.$$

The basic facts are now in place to give an asymptotic lower bound for $R(k,k)$ derived from Theorem 9.5.1.

Corollary 9.5.2. $(1+o(1))\frac{k}{e\sqrt{2}}2^{k/2} < R(k,k)$ *where* $o(1)$ *tends to zero as* k *increases.*

Exercise 396. *Prove Corollary 9.5.2.*

Erdős also couched the proof of Theorem 9.5.1 in the language of probability (rather than explicit counting). This new viewpoint has often been said to be the beginnings of a powerful technique in Ramsey theory (and many other areas in combinatorics) called the "probabilistic method" (for example, see [36], [344] or [878] for an introduction). Only a very basic understanding of probability is assumed in the proof below.

Theorem 9.5.3 (Erdős, 1947 [290]). *If* $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, *then* $R(k,k) > n$.

Proof: Let G be a random graph on n vertices, where the probability of an edge is $p = 1/2$. [One can consider the sample space as the $2^{\binom{n}{2}}$ graphs on n labelled vertices.] The probability that a fixed set of k vertices induces a K_k or a $\overline{K_k}$ is $2^{1-\binom{k}{2}}$. So the probability that some set of k vertices does so is at most

$$\sum_{\text{all } k\text{-sets}} 2^{1-\binom{k}{2}} = \binom{n}{k} 2^{1-\binom{k}{2}}$$

which is less than one. So there exists a graph on n vertices which does not contain any K_k or $\overline{K_k}$. □

With a little extra work, one can give a slight improvement on this lower bound of roughly $\frac{k}{e\sqrt{2}}2^{k/2}$ by a factor of 2. The technique used for this improvement is called the *Lovász Local Lemma* (LLL), proved in 1975 [335], which is beyond the scope of these notes—see [36] for details.

Lower bounds for $R(k,k)$ are of the form $2^{\frac{k}{2}}$ and upper bounds are of the form 2^{2k}. It is not even known whether or not $\lim_{k\to\infty}(R(k,k))^{1/k}$ exists (if it does exist, it is between $\sqrt{2}$ and 4).

9.6 Constructive lower bounds for diagonal Ramsey numbers

To prove a lower bound for the Ramsey function $R(k,k)$, it suffices to find an n and show that there exists a 2-colouring of $E(K_n)$ so that no K_k-subgraph is monochromatic; in some cases, a colouring is given explicitly by construction. Such a construction then yields $n < R(k,k)$. It seems that most general constructive lower bounds are not even close to those obtained by probabilistic methods.

9.6.1 Paley graphs

In the proof of $R(3,3) > 5$ (see Theorem 9.2.3), the graph K_5 is decomposed into two copies of C_5. For the proof of $R(4,4) > 17$, a 17-vertex graph, called a "Paley graph", was used that was self-complementary. To describe the class of graphs called Paley graphs, some terminology is required.

For any prime power q, there is a unique field of order q, usually denoted by $\mathbb{F}_q$. A non-zero element $x \in \mathbb{F}_q$ is called a *quadratic residue* if there exists $y \in \mathbb{F}_q$ so that $y^2 = x$. For example, when $q = 5$, $0^2 = 0$, $1^2 = 1$, $2^2 = 4 = -1$, $3^2 = 4$, and $4^2 = 1$, and so the quadratic residues in $\mathbb{F}_5$ are 1 and 4. If x is not a quadratic residue, say that x is a *quadratic non-residue*.

The following can be found in many number theory texts.

Lemma 9.6.1. *Let q be an odd prime power. Then -1 is a quadratic residue in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod 4$. Furthermore, if q is an odd prime, exactly half of the non-zero elements in $\mathbb{F}_q$ are quadratic residues.*

For example, in $\mathbb{F}_7$ (which is the same as $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$ with arithmetic done modulo 7), the non-zero squares are 1, $2^2 = 4$, $3^2 = 2$, $4^2 = 2$, $5^2 = 4$ and $6^2 = 1$, so $-1 = 6$ is not a quadratic residue.

It is a trivial observation that if x and y in $\mathbb{F}_q$ are quadratic residues, then so is xy. When $x = -1$, together with the above lemma, this shows that when $q \equiv 1 \pmod 4$, y is a quadratic residue if and only if $-y$ is also a quadratic residue.

The following graphs were defined [740] by Raymond E. A. C. Paley (1907–1933), a MIT mathematician killed in an avalanche while skiing (he is buried in the cemetery in the town of Banff, Alberta). His graphs arose from the study of Hadamard matrices. For some history and generalizations, see [284].

Definition 9.6.2. *Let q be a prime power with $q \equiv 1 \pmod 4$. Let $G = (V, E)$ be the graph on $V = \mathbb{F}_q$ defined by*

$$\{a, b\} \in E \quad \textit{if and only if } a - b \textit{ is a quadratic residue in the field } \mathbb{F}_q.$$

Since $q \equiv 1 \pmod 4$, -1 is a quadratic residue, $a - b$ is a quadratic residue if and only if $(a - b)(-1) = b - a$ is, so the colour does not depend on the order subtraction of the endpoints. (If working with $p \equiv 3 \pmod 4$, the order of subtraction gives rise to a digraph version of the Paley graph.)

Exercise 397. *Show that the graph C_5 is the Paley graph for $p = 5$.*

Lemma 9.6.3. *Let $q \equiv 1 \pmod 4$ be a prime power. The Paley graph on q vertices is self-complementary.*

Proof: Let $G = (V, E)$ be the Paley graph on vertices $V = \mathbb{F}_q$. Define a function $f : V \to V$ by letting $b \in \mathbb{F}_p \backslash \{0\}$ be a quadratic non-residue and for each $x \in \mathbb{F}_q$, setting $f(x) = bx$. Let $x, y \in V$, $x \neq y$. If $\{x, y\} \in E$, then $x - y$ is a quadratic residue. Since b is not a quadratic residue and $f(x) - f(y) = b(x - y)$, then $f(x) - f(y)$ is not a quadratic residue. Hence, $\{f(x), f(y)\}$ is not an edge in G. Similarly, f sends non-edges to edges. Using the properties of the field $\mathbb{F}_q$, it is direct to show that f is a bijection, and hence induces an anti-isomorphism. $\square$

By Exercise 30, the 3×3 rook's graph is self-complementary; one can check this fact by multiplying all elements of $\mathbb{F}_9$ by a non-square element. It might be an interesting exercise to actually construct $\mathbb{F}_9$ with vertices $\{0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2\}$ with operations done modulo $x^2 + 1$, and show that the non-zero quadratic residues are $1, 2, x, 2x$; so multiplication by $x^2 + 1$ gives the complement.

If $q \equiv 1 \pmod 4$, and $q = p^2$ is the square of a prime, it is not difficult to see [161] that the elements of the prime subfield $\mathbb{F}_p$ in $\mathbb{F}_q$ (and each translate thereof) form a clique in the Paley graph.

Exercise 398. *Let G be the Paley graph for $q = 9$. Show that G is the 3×3 rook's graph.*

For a strong result on the maximal cliques and independent sets in the Payley graphs for $q = p^2$, see [113].

Exercise 399. *If G is the Paley graph for $p = 17$, prove algebraically that neither G nor $\overline{G}$ contains a K_4. Conclude that $R(4,4) > 17$, and so $R(4,4) = 18$.*

Paley graphs are highly "symmetric" (see Theorem 15.11.12). Due to the symmetry of a Paley graph, it is sometimes thought that indeed they are optimal examples in Ramsey theory. In 1979, Evans, Pulham and Sheehan [351] showed that the Paley graph on 17 vertices is the only such graph with no K_4 or $\overline{K_4}$.

The energetic reader might check that using $p = 101$, the resulting Paley graph has no K_6 nor $\overline{K_6}$, and so $R(6,6) \geq 102$, a lower bound that has endured for decades as the best known.

9.6.2 Other constructions

Constructive lower bounds for $R(k,k)$ by Frankl and Wilson [379] are on the order of $e^{\log^2 k/4 \log\log k}$, rather weak compared to $2^{k/2}$. Here is a simple construction giving a lower bound on the order of k^3, due to Zsigmond Nagy in 1972:

Theorem 9.6.4 (Nagy, 1972 [709]). *For $k \geq 3$, $R(k+1,k+1) > \binom{k}{3}$.*

Proof: Let $k \geq 3$ and let G be the graph whose vertices are triples from $\{1,2,\ldots,k\}$. Connect two triples with red if they share exactly one element, and connect them with blue otherwise.

Claim: The graph G has no monochromatic K_{k+1}.

Proof of claim: Suppose that m triples determine a red complete graph; each triple has a characteristic 0-1 vector of length k (containing exactly three 1's). Form the $m \times k$ 0-1 matrix M with these vectors as rows. It is not difficult to compute that $MM^T = J + 2I_m$, where J is the $m \times m$ matrix consisting entirely of 1's, and I_m is the $m \times m$ identity matrix. (So MM^T has 3's on the main diagonal and 1's elsewhere.) Then $m = \text{rank}(MM^T) \leq \text{rank}(M) \leq k$.

Suppose that m triples form a blue complete graph. If $\mathbf{v}$ and $\mathbf{w}$ are characteristic vectors corresponding to vertices (triples) in this blue graph, then (using the Euclidean inner product, or dot product) $\mathbf{v} \bullet \mathbf{w} = 0$ or $2 \equiv 0 \pmod 2$. Letting $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the vectors corresponding to the m triples of the blue subgraph. Suppose that $a_1, ..., a_m \in \{0,1\}$ are constants so that

$$a_m\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = \mathbf{0}. \tag{9.6}$$

For each $i = 1, \ldots, m$, taking the dot product of each side of (9.6) with $\mathbf{v}_i$ gives $a_i(\mathbf{v}_i \bullet \mathbf{v}_i) = \mathbf{0}$ and so $a_i = 0$. This shows that the vectors $\mathbf{v}_1, \ldots \mathbf{v}_m$ are linearly independent over the field $\mathbb{F}_2 = \{0,1\}$. However, the dimension of the vector space of all binary vectors of length k has dimension k, so $m \leq k$. This proves the claim.

Another way to see the second part of this proof is by setting M to be the matrix whose rows are the characteristic vectors, and modulo 2; then $MM^T = I_m$, and so $m = \text{rank}(MM^T) \leq \text{rank}(M) \leq k$. □

The aforementioned construction of Frankl and Wilson is fairly easy to describe; it is left to the reader to confirm that the bound follows from the construction:

Theorem 9.6.5 (Frankl–Wilson, 1981[379]). *Let p be a prime and let G be a graph whose vertex set is $\binom{p^3}{p^2-1}$ and edge set is $\{(V,W) : |V \cap W| \not\equiv -1 \pmod p\}$. Then G contains no complete or empty subgraph on $\binom{p^3}{p-1} + 1$ vertices.*

The proof of Theorem 9.6.5 is omitted; one half uses the following result by Ray-Chaudhuri and Wilson (and the other half employs a modular form of this theorem proved in [379]):

Theorem 9.6.6 (Ray-Chaudhuri–Wilson, 1975 [775]). *Let X be a set and let $\mathcal{F}$ be a family of subsets of X, all of the same size. If the number of sizes of pairwise intersections in $\mathcal{F}$ is s, then $|\mathcal{F}| \leq \binom{|X|}{s}$.*

See the article by Peter Frankl [718] for more information on constructive lower bounds for some Ramsey functions.

9.7 More colours

For any $r \geq 2$, the number $n = R(s_0, s_1, \ldots, s_{r-1})$ is defined to be the least number so that for every $\Delta : E(K_n) \to r$, there exists a colour $i \in r$ and a K_{s_i} which is i-monochromatic. The r-colour generalization of Lemma 9.2.6 holds.

Lemma 9.7.1. *If $(t_0, t_1, \ldots, t_{r-1})$ is a permutation of $(s_0, s_1, \ldots, s_{r-1})$, then*

$$R(s_0, s_1, \ldots, s_{r-1}) = R(t_0, t_1, \ldots, t_{r-1}).$$

Proof: A proof identical to that of Lemma 9.2.6 yields

$$R(s_0, s_1, \ldots, s_{r-1}) = R(s_1, s_0, s_2, s_3, \ldots, s_{r-1}),$$

and so apply this for each transposition determining the permutation.

Another way to prove this lemma is to merely interchange the roles of the colours. To be precise, let $n = R(s_0, s_1, \ldots, s_{r-1})$, and let $\sigma : r \to r$ be a permutation, where $t_i = s_{\sigma(i)}$. For $\Delta : E(K_n) \to r$, define $\Delta^* : E(K_n) \to r$, by $\Delta^*(e) = \sigma(\Delta(e))$. Then a copy of K_{s_i} is i-monochromatic under Δ if and only if K_{t_i} is $\sigma(i)$-monochromatic under Δ^*, proving the result. □

Duplicating the proof of Theorem 9.2.5, but only with more colours gives a version of the Erdős-Szekeres recursion (for example, this can be found in [143]).

Theorem 9.7.2. *For $r \geq 2$, and $s_0, s_1, \ldots, s_{r-1}$, each $s_i > 2$,*

$$R(s_0, s_1, \ldots, s_{r-1}) \leq R(s_0 - 1, s_1, \ldots, s_{r-1}) + R(s_0, s_1 - 1, \ldots, s_{r-1}) + \ldots \\ + R(s_0, s_1, \ldots, s_{r-2}, s_{r-1} - 1) - r + 2.$$

Proof: For each $i \in r$, put $n_i = R(s_0, s_1, \ldots, s_{i-1}, s_i - 1, s_{i+1} \ldots, s_{r-1})$, and set $n = (\sum_{i \in r} n_i) - r + 2$. Let $\Delta : E(K_n) \to r$ be given, and fix $x \in V(K_n)$. For each $i \in r$, set $X_i = \{y : \Delta(x, y) = i\}$. There exists an i so that $|X_i| \geq n_i$, since if not, $n = 1 + \sum_{i \in r} |X_i| = 1 + \sum_{i \in r}(n_i - 1) = \sum_{i \in r} n_i - r + 1$, a contradiction. Fix such an i with $|X_i| \geq n_i$. If for every $j \neq i$, X_i does not induce a j-monochromatic copy of K_{s_j}, then by the choice of n_i, there exists $Y_i \subset X_i$ that induces an i-monochromatic copy of $K_{s_i - 1}$, and, in this case, $\{x\} \cup Y_i$ induces an i-monochromatic copy of K_{s_i}. □

As of 2017 [770], the only known 3-colour Ramsey number is $R(3, 3, 3) = 17$; the upper bound follows from the Erdős–Szekeres recursion and the lower bound is due to Gleason and Greenwood [418] (and rediscovered in [551]). There are only two 3-colourings of K_{16} that show $R(3, 3, 3) > 16$, one of which is indicated in Figure 9.5.

Both colourings were found by Kalbfleisch and Stanton [552], and were shown to be the only such colourings (modulo relabelling of K_{16} or interchanging colours). See also [619] for another proof of this fact that only two colourings exist. These two 3-colourings are called the "twisted" and the "untwisted" colourings, and in each case, each colour class is isomorphic to what is known as the "Clebsch graph" (also called the "Greenwood–Gleason graph" [779]). There are many ways to define the Clebsch graph. For example, the Clebsch graph is formed by starting with the 4-dimensional hypercube graph Q_4 and adding edges between opposite corners (giving a 5-regular triangle-free graph on 16 vertices). Another (perhaps the standard definition) is to let its vertex set be the set of all even cardinality subsets of $\{1, 2, 3, 4, 5\}$ and join two vertices if and only if the symmetric difference of the underlying subset has cardinality 4. Another way to define the Clebsch graph is to identify opposite vertices of the 5-dimensional hypercube. The Clebsch graph is also the "Keller graph of order 2" (see [242]); the Keller graph of order n has vertex set $V = \{0, 1, 2, 3\}^n$ (a collection of 4^n ordered n-tuples), where two n-tuples are adjacent if and only if they differ in at least two coordinates, and in at least one coordinate, the difference is 2. See Figure 9.6 for the case $n = 2$.

Exercise 400. *Show that for each vertex v of the Clebsch graph, the non-neighbours of v (not including v itself) induce a copy of the Petersen graph..*

One of the ways to describe a required 3-colouring of the edges of K_{16} is to use the Galois field on 16 elements as vertices, and colour a pair vertices according to their difference, modulo perfect cubes (in $\mathbb{F}_{16}$). (In this setting, each colour class is called a "16-cyclotomic graph", yet one more way to define the Clebsch graph.)

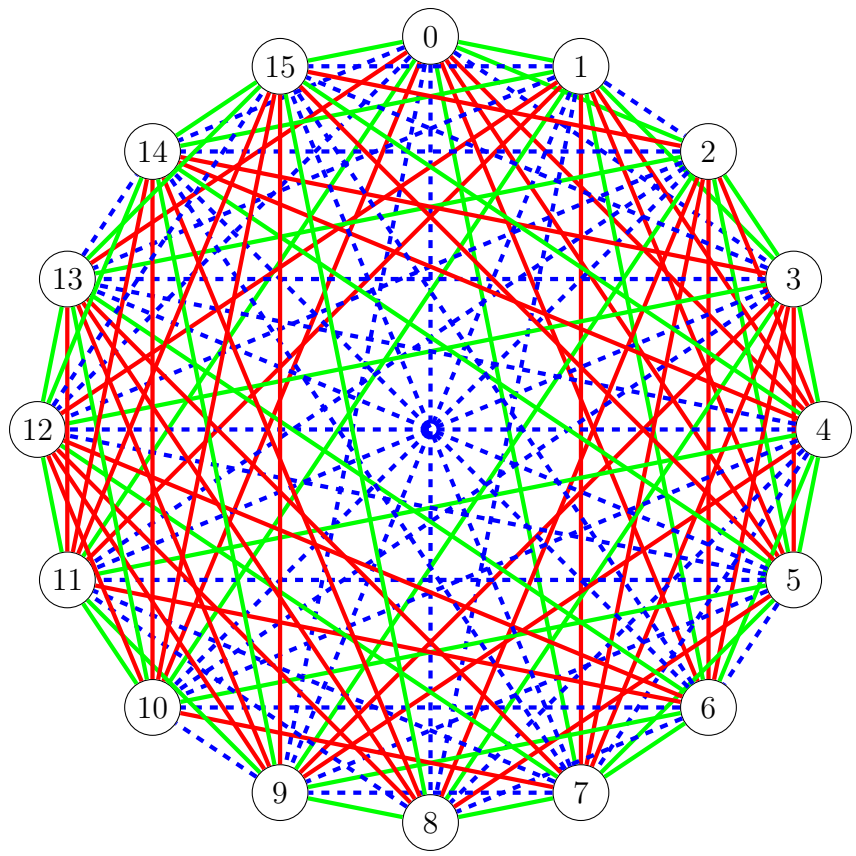


Figure 9.5: A 3-colouring of $E(K_{16})$ with no monochromatic triangles

Recall that the notation $R(m;r)$ is the Ramsey number for colouring edges with r colours and finding a monochromatic K_m. In other words, $R(m;r) = R(\underbrace{m, m, \ldots, m}_{r})$.

Exercise 401. *For $k, \ell, m \in \mathbb{Z}^+$, show that*

$$R(m; k+\ell) \geq (R(m;k)-1)(R(m;\ell)-1)+1.$$

9.8 Ramsey's theorem: other proofs

Although the existence of the Ramsey numbers follows from the Erdős–Szekeres recursion, it may be of interest to see one of Ramsey's original proof ideas. First is given what is known as the "infinite version of Ramsey's theorem". The proof given here is an adaptation of that found in [759].

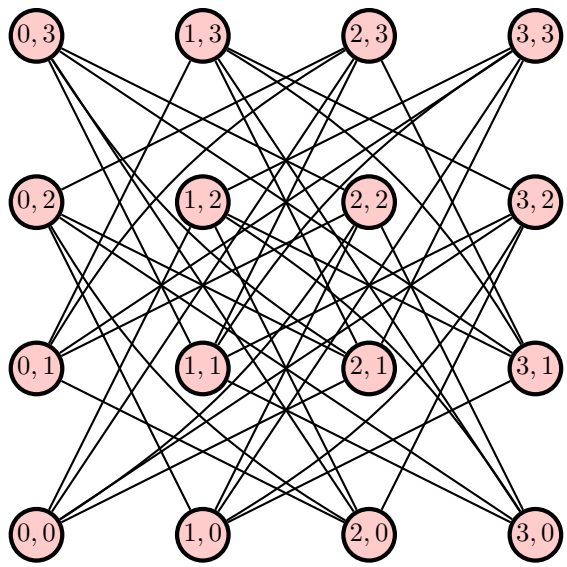


Figure 9.6: The Clebsch graph as a Keller graph with $n = 2$.

Theorem 9.8.1 (Ramsey, 1930 [773]). *For positive integers r and k,*

$$\omega \longrightarrow (\omega)^k_r,$$

that is, for any countably infinite set S and any r-colouring $\Delta : [S]^k \longrightarrow [r]$, there exists an infinite set $X \subset S$ so that Δ is constant on $[X]^k$.

Proof: The proof is by induction on k. The case $k = 1$ is trivial by the pigeonhole principle, so assume the theorem is true for some $k \geq 1$. Let $\Delta : [S]^{k+1} \longrightarrow r$ be a colouring of a countable set S, and pick an arbitrary $x_0 \in S$. Then Δ induces a colouring $\Delta_0 : [S \backslash \{x_0\}]^k \longrightarrow r$ by $\Delta_0(H) = \Delta(H \cup \{x_0\})$. By the induction hypothesis, there exists an infinite set $A_0 \subset S$ so that Δ_0 is constant on $[A_0]^k$, and hence Δ is constant on

$$\{x_0\} \times [A_0]^k = \{\{x_0, y_1, \ldots, y_k\} : \{y_1, \ldots, y_k\} \in [A]^k\} \subset [S]^{k+1},$$

say $\Delta(\{x_0\} \times [A_0]^k) = r_0 \in r$. Now pick any element $x_1 \in A_0$. Repeating the same argument, there exists an infinite set $A_1 \subset A_0$ so that Δ is constant on $\{x_1\} \times [A_1]^k$, say $\Delta(\{x_1\} \times [A_1]^k) = r_1 \in r$. [Note that r_0 and r_1 may be different, while still $\Delta(\{x_0\} \times [A_1]^k) = r_0$.] Continuing inductively, get a set $X = \{x_i : i \in \omega\} \subseteq S$ so that for H, $H' \in [X]^{k+1}$, $\Delta(H) = \Delta(H')$ whenever $\min(H) = \min(H')$. [If $i = \min\{j : x_j \in H\}$ say $\min(H) = x_i$.] This induces an r-colouring Δ^* of X by $\Delta^*(x_i) = \Delta(H)$ for any $H \in [X]^{k+1}$ satisfying $\min(H) = x_i$.

By the pigeonhole principle, there is an infinite set $Y \subset X$ so that Δ^* is constant on Y, and hence Δ is constant on $[Y]^{k+1}$, since

$$[Y]^{k+1} \subseteq \{H \in [X]^{k+1} : \min(H) \in Y\}.$$

□

Remark: It is sometimes convenient to totally order the set S at the beginning of the above proof, and then one need only pick the lowest element in the order for the subsequent element in the formation of X. Such a device also allows one to pick an ordering of some other type rather than just an ω-ordering and obtain extensions of this result. Similar results are also obtained for other infinite cardinals. The infinite version of Ramsey's theorem was first generalized to all cardinals by P. Erdős and R. Rado [337], [331]. There is an extensive bibliography on the subject of infinite generalizations; for other work see [329], [330], or [858].

Ramsey gave an inductive construction for the proof of the finite version of his theorem; however, another proof is given here. One way to approach the finite Ramsey theorem is to derive it from the infinite version using König's infinity lemma (Lemma 6.14.2), a method referred to in [346]—perhaps for the first time.

Theorem 9.8.2 (Ramsey, 1930 [773]). *For any positive integers m, k, r, there is a smallest (finite) number n so that $n \to (m)^k_r$.*

Proof: Assume, in hope of a contradiction, that the theorem does not hold; that is, assume that for every $n \in \omega$ there exists a 'bad' colouring $\Delta : [n]^k \longrightarrow r$ so that for every $A \in [n]^m$, Δ is not constant on $[A]^k$. The restriction of a bad colouring $\Delta : [n]^k \longrightarrow r$ to a colouring $\Delta^* : [n-1]^k \longrightarrow r$ is again bad.

Order all such bad colourings by restriction to form a tree $(T, \leq)$ with the colouring of the empty set as the root. $(T, \leq)$ is locally finite since any colouring of $[n-1]^k$ can only have finitely many 'extensions' to a colouring of $[n]^k$. Thus by König's Lemma (Lemma 6.14.2), $(T, \leq)$ contains an infinite branch, corresponding to a bad colouring of ω. This contradicts Theorem 9.8.1. □

With virtually the same proof, one obtains an extension of Theorem 9.8.2:

Theorem 9.8.3 (Paris–Harrington, 1977 [742]). *For any positive integers $k, r, s \geq 2$, there exists $n_0 = PH(k, s; r)$ so that for any $n \geq n_0$, for every r-colouring*

$$\Delta : [n]^k \longrightarrow r,$$

there exists $S \subseteq [n]$ such that
(1) Δ is constant on $[S]^k$, and
(2) $|S| \geq \max\{s, \min S\}$.

Proof sketch: Define a set $S \subseteq [n]$ to be *large* if and only if $|S| \geq \min S$. Fix s, k, r. Let $\mathcal{F}$ be the family of large sets in $[s, \infty)$. If $[s, \infty)$ is r-coloured, by the infinite Ramsey theorem, there exists an infinite $T \subseteq [s, \infty)$ so that $[T]^k$ is monochromatic. Let S be the set consisting of the first $\min T$ elements of T. Then $S \in \mathcal{F}$.

Now, either by compactness (or by using restricted bad colourings get a locally-finite infinite tree and hence an infinite branch) get an infinite bad colouring. □

Note that the condition $|S| \geq s$ in the statement of Theorem 9.8.3 shows that Theorem 9.8.3 implies Theorem 9.8.2. The Paris-Harrington result is an example of a true result not provable in first order arithmetic. See [435] for further details.

9.9 Graph Ramsey theory

Ramsey numbers, as discussed earlier in this chapter, can be interpreted as results about graphs. For example, the fact that $R(3, 4) \leq 9$ can be written as $K_9 \longrightarrow (K_3, K_4)$, where the "Ramsey arrow" notation extends the Ramsey arrow notation in (9.1) and (9.2).

Definition 9.9.1. *Let F, G, H be graphs. The notation*

$$F \to (G, H)$$

means that for every red-blue colouring of $E(G)$, there exists either a copy of G (as a weak subgraph of F) with all edges red, or a copy of H with all edges blue. For any integer $r \geq 2$, and graphs $G_1, G_2, \ldots, G_r$, the notation

$$F \to (G_1, G_2, \ldots, G_r)$$

says that for any colouring of $E(F)$ with colours $1, 2, \ldots, r$ there is some $i \in \{1, 2, \ldots, r\}$ so that there is a copy of G_i with all edges coloured i.

For example, if P_2 denotes a path of length 2 (on three vertices), then $C_5 \to (P_2, P_2)$ since any red-blue colouring of $E(C_5)$ yields a monochromatic copy of P_2.

Remark 9.9.2. *The Ramsey arrow for graphs given above is but one of many; there are versions where all subgraphs are* induced *subgraphs, or when graphs are restricted to ordered graphs, directed graphs, or even partial orders. The area of "graph Ramsey theory" as studied here might be more properly called "weak graph Ramsey theory" and had its beginnings with a group of authors including Burr, Chvátal, Erdős, Harary, Faudree, Gyárfás, Schelp, and more from Louisianna and Tennessee. Graph Ramsey theory for infinite graphs or homogeneous structures is also extensive, but not discussed here.*

One possible reason for extending the notion of Ramsey statements to arbitrary graphs is that sometimes such results give bounds on the (harder) problems for complete graphs. For example, if one can find an n so that $K_n \to (K_6 - e, K_6 - e)$, this could provide information on $R(6,6)$. Another possible reason for the popularity of graph Ramsey theory is that they generalize the theory for complete graphs, and so "natural" questions are plentiful.

Only a few of the famous results in graph Ramsey theory are mentioned; the interested reader is recommended to see, for example, [314], [348], [487], [489], or the Master's thesis [147] of Rob Borgersen (which is available electronically). For current status of the most popular graph Ramsey problems, see the dynamic survey by Radziszowski [770].

Definition 9.9.3. *For graphs G and H, the Ramsey number $R(G,H)$ is the least n so that $K_n \longrightarrow (G,H)$. When $G = H$ let $R(G)$ denote $R(G,G)$.*

When both G and H are complete, the graph Ramsey numbers $R(K_a, K_b)$ are simply the Ramsey numbers $R(a,b)$. As already mentioned just before Goodman's theorem (Theorem 10.3.10), a 2-colouring of $E(K_n)$ can be considered as an n-vertex graph and its complement. For graphs G, H and integer n, the notation $K_n \longrightarrow (G,H)$ says that if an n-vertex graph F (think red) does not contain a copy of G, then its complement $\overline{F}$ (think blue) contains a copy of H.

Often the order of the colours is not important since for graphs G and H, $R(G,H) = R(H,G)$ (just rename the colours).

For any $m \geq 2$, if the edges of K_m are red-blue coloured, the statement $R(K_2, K_m) = m$ says that if no edge of K_m is red, then all edges are blue (and no smaller complete graph accomplishes this).

For the next exercise, recall that P_k denotes a path of length k (on $k+1$ vertices).

Exercise 402. *Show that for all $k, \ell \geq 1$, $R(P_k, P_\ell) \leq k + \ell$.*

Considering the result in Exercise 402, the following improvement might seem unexpected:

Theorem 9.9.4 (Gerensér–Gyárfás, 1967 [415])**.** *For $k \geq \ell$, $R(P_k, P_\ell) = k + \lfloor \frac{\ell+1}{2} \rfloor$.*

To see that $R(P_k, P_\ell) \geq k + \lfloor \frac{\ell+1}{2} \rfloor$, consider the graph $G = K_k \cup K_{\lfloor \frac{\ell+1}{2} \rfloor - 1}$ (a disjoint union); then G contains no P_k and $\overline{G}$ contains no P_ℓ. The original proof of the harder direction was by induction on k, and is omitted here.

Exercise 403. *Show that for any positive integer m,*

$$R(K_{1,m}) = \begin{cases} 2m & \text{if } m \text{ is odd} \\ 2m-1 & \text{if } m \text{ is even.} \end{cases}$$

As mentioned in Section 7.5, a *geometric graph* is a graph drawn with vertices in the Cartesian plane and edges that are straight line segments. (To eliminate confusion, assume that if three points in a geometric graph form a triangle, then these three points are not collinear.) The result of the next exercise is proved by induction in [554]; no solution is given here, but the proof idea is similar to that from Exercise 342.

Exercise 404. *Prove that if the edges of a complete geometric graph on $3n-1$ vertices are 2-coloured, there exist n pairwise disjoint (not crossing and sharing no endpoints) edges of the same colour.*

On the other hand, for graphs that are not necessarily geometric, Theorem 9.9.4 says that for any 2-colouring of $E(K_{3n-1})$ there exists a monochromatic path of length $2n-1$, which contains n independent edges. In 1967, Gerensér and Gyárfás [415] showed that for any 3-colouring of $E(K_n)$, a path of length $\lfloor \frac{n+1}{2} \rfloor$ is guaranteed to be monochromatic.

Theorem 9.9.5 (Chvátal, 1977 [218])**.** *Let T be a tree on $p \geq 2$ vertices. Then for any $n \geq 2$,*

$$R(T, K_n) = (p-1)(n-1) + 1.$$

Theorem 9.9.6 (Rosta, 1973 [810]; Faudree–Schelp, 1974 [356])**.** *Two-coloured Ramsey numbers for cycles are:*

(i) If $3 \leq r \leq s$, r odd, and $(r,s) \neq (3,3)$ then $R(C_r, C_s) = 2s-1$.

(ii) If $4 \leq r \leq s$, r and s both even, $(r,s) \neq (4,4)$, then $R(C_r, C_s) = s + \frac{1}{2}r - 1$.

(iii) If $4 \leq r < s$, r even and s odd, then $R(C_r, C_s) = \max\{s + \frac{1}{2}r - 1, 2r-1\}$.

Exercise 405. *Without using Theorem 9.9.6, directly find $R(C_4, C_4)$.*

Exercise 406. *Letting sK_2 denote s disjoint copies of K_2, show that for any $p \geq 3$, $R(sK_2, K_p) = 2s + p - 2$.*

As was pointed out in [125], the result in Exercise 406 implies that for any graph H on h vertices,

$$R(sK_2, H) \leq R(sK_2, K_h) = 2s + h - 2.$$

Exercise 407. *Find $R(K_3, C_5)$. Hint: Use $R(3,4) = 9$.*

One more powerful result regarding graph Ramsey numbers is given here for interest sake (its proof uses an advanced tool called "the regularity lemma", not covered in this text). Recall that the ordinary Ramsey numbers grow exponentially; however many of the bounds for graph Ramsey numbers $R)G, H)$ considered in this section have bounds that are linear combinations of $|V(G)|$ and $|V(H)|$. It might come as a surprise to find that for graphs G with bounded maximum degree, the numbers $R(G, G)$ have linear upper bounds!

Theorem 9.9.7 (Chvátal–Rödl–Szemerédi–Trotter, 1983 [220]). *For each positive integer d, there exists a constant $c = c(d)$ so that for any graph G with $\Delta(G) \leq d$,*

$$R(G, G) \leq c|V(G)|.$$

The field of graph Ramsey theory includes many other generalizations and related results. For example, one might restrict a Ramsey-type problem to only bipartite graphs, or to induced subgraphs. One can also ask Ramsey-type questions about graphs when only vertices are coloured, or when small K_k's are coloured (rather than just edges—which are copies of K_2), or even when other small (induced) subgraphs are coloured. For some of these results, see [147], [435], [455], or [760].

Ramsey theory also extends to geometric graphs, that is, graphs whose vertices are drawn in some Euclidean space; this area is now called "Euclidean Ramsey theory". Two such results have already been mentioned (see Exercises 342 and 404 for geometric versions of Exercise 60 and Theorem 9.9.4.) For more on Euclidean Ramsey theory, I recommend [320, 321, 322, 433, 438] for an introduction to the area, the first three of which, in a sense, started the field, and the last two contain many more current references. Some problems regarding colouring of a unit-distance graph in the plane can be interpreted as Euclidean Ramsey results (see Section 6.14.2).

Chapter 10

Extremal graph theory

10.1 Introduction

This chapter is only an introduction to extremal graph theory. There are many active research areas not covered here (*e.g.*, Szemerédi's regularity lemma, hypergraphs, stability, saturation, super-saturation, random graphs, expanders, concentrators, topological subgraphs, percolation, directed graphs).

Note to instructor: Even though this chapter forms only an introduction to the field, there are far too many results in this chapter to be included in most undergraduate courses in graph theory. If the topic of extremal graph theory is limited to only three or four lectures, the following topics, along with a few of the related exercises, cover what might be considered as the highlights or "standard" material

- Basic terminology and elementary examples, Section 10.2;
- Mantel's theorem (Theorem 10.3.1) and at least one or two of its proofs, and Exercise 420;
- Turán's theorem (Theorem 10.5.2), at least one of its proofs, and its dual (Theorem 10.5.6);
- Statement of Goodman's theorem (Theorem 10.3.11);
- Lemmas 10.4.1 and 10.4.3;
- Definition of Zarankiewicz numbers (Section 10.4.4), Theorems 10.4.6, and 10.4.8;
- Statement of the Erdős–Stone theorem (Lemma 10.7.1);
- Statement of the Erdős–Simonovits theorem (Theorem 10.7.4);
- Statement of the Bondy–Simonovits theorem (Theorem 10.8.5) for even cycles.

Some theorems in extremal graph theory were proved over a century ago, but in the early to mid 20th century, Paul Erdős and a few other Hungarians popularized the field; some extremal graph theory problems turned out to have analogues in number theory and geometry, also increasing their popularity.

In 1979, Bollobás wrote the definitive work *Extremal graph theory* [121]. Since then, there seems to have been an explosion in research publications. In addition to the Bollobás book, many (more recent) surveys are recommended, including those by Bollobás [124] in the *Handbook of combinatorics*, Simonovits (*e.g.*, [866]), Füredi (*e.g.*, [390]), Nikiforov [720] in *Surveys in Combinatorics 2011*, and the extensive survey on degenerate extremal problems by Simonovits and Füredi [395] appearing in the 2013 Erdős Centennial proceedings. Jukna's textbook [546] also contains many extremal results. Many open problems remain in extremal graph theory; for a collection of such problems by Paul Erdős, see the Chung and Graham book [214]. [I have started my own notes [452] on extremal graph theory. Much of this chapter was pulled from those notes in some form or another, then slightly reworked.]

What is extremal graph theory? Essentially, a result in graph theory may be considered as an "extremal" result if it examines how large (or small) some parameter needs to be in order to force some structural property. Many basic graph-theoretic concepts or results could be considered as "extremal" (*e.g.*, minimal cut-sets). The discussion here is limited more to problems of a "Turán–type": For a graph F, and a number of vertices n, how many edges in a graph G on n vertices guarantee a copy of F as a subgraph of G, and what do the graphs look like that have as many edges as possible while not containing F?

One can think of such extremal graph theory questions as asking about the density (the portion of edges out of all possible) of a graph needed to guarantee certain subgraphs.

10.2 Basics

Definition 10.2.1. *For graphs F and G, if G contains no (weak) subgraph isomorphic to F, then G is said to be F-free. For any family of graphs $\mathcal{F}$,* the extremal number *$ex(n;\mathcal{F})$ is the maximum number of edges in a graph G on n vertices so that for each $F \in \mathcal{F}$, G is F-free. Graphs in $\mathcal{F}$ are called "forbidden subgraphs". If $\mathcal{F} = \{F\}$ consists of a single graph, write $ex(n;F)$, dropping the extra set of braces. The set*

$$EX(n;F) = \{G : |V(G)| = n, |E(G)| = ex(n;F), G \text{ is } F\text{-free}\}$$

is called the collection of n-vertex extremal graphs forbidding F.

An extremal number $\mathrm{ex}(n;\mathcal{F})$ is often called a "Turán number" for $\mathcal{F}$, named after Paul Turán, who proved a central theorem (Theorem 10.5.2, below) in extremal graph theory.

Note: In this text, the notation $\mathrm{ex}(n; H)$ uses a semicolon; however, many authors simply use a comma.

In order to determine precisely when some $\mathrm{ex}(n; H) = m$, one needs to show that any graph on n vertices with $m+1$ edges contains a copy of H, and then one needs to show that there is a graph G on n vertices and m edges that contains no copy of H; in this case, such a G is called an *extremal graph* (for H).

Letting $\mathcal{C} = \{C_i : i \geq 3\}$ denote the set of all graph cycles, the result from Exercise 86 says $\mathrm{ex}(n; \mathcal{C}) \leq n-1$, and any tree on n vertices indeed shows this bound is tight. So, $\mathrm{ex}(n; \mathcal{C}) = n-1$ and the set $\mathrm{EX}(n; \mathcal{C})$ of extremal graphs is the set of all trees on n vertices.

Exercise 408. *Let $2K_2$ denote the graph on 4 vertices formed by 2 disjoint copies of K_2. Find $ex(n; 2K_2)$ and $EX(n; 2K_2)$.*

Recall that a graph G is *Hamiltonian* if and only if G contains a cycle using all vertices of G, or in other words, G is a Hamiltonian graph on n vertices if and only if G contains a C_n. The following gives the extremal number for being Hamiltonian.

Lemma 10.2.2 (Ore, 1961 [729])**.** *For any $n \geq 3$, $ex(n; C_n) = \binom{n-1}{2} + 1$.*

Exercise 409. *Prove Lemma 10.2.2.*

For the moment, consider extremal numbers for paths. Letting P_2 denote a path of length two (on three vertices), then

$$\mathrm{ex}(n; P_2) = \lfloor n/2 \rfloor.$$

A maximum matching is P_2-free, with the correct number of edges, so $\mathrm{ex}(n; P_2) \geq \lfloor n/2 \rfloor$; the opposite inequality is left as an exercise:

Exercise 410. *Letting P_2 denote a path of length 2, show that $ex(n; P_2) \leq \lfloor n/2 \rfloor$. In other words, show any graph on n vertices with more than $\lfloor n/2 \rfloor$ edges contains a P_2 (a path of length 2).*

Lemma 10.2.3 (Folklore)**.** *If P_3 is the path with three edges, then*

$$ex(n; P_3) = \begin{cases} n & \text{if } 3 \mid n \\ n-1 & \text{if } 3 \nmid n. \end{cases}$$

Proof: Consider any P_3-free graph G. Then G contains no cycles of length 4 or greater. Also, G does not contain any triangle with an edge attached. Therefore, any component of G is either a K_3 or a tree. Trees with no P_3s are stars(of the form $K_{1,k}$). So, to maximize $|E(G)|$, if n is divisible by 3, let all components be K_3s; if n

is not divisible by 3, at least one component is a tree so either use one big star, or a combination of triangles and a star. □

Notes: Extremal graphs for P_3 are not unique when n is not divisible by 3. Since P_3 contains a copy of $2K_2$, the extremal number for $2K_2$ is at most the extremal number for P_3.

Theorem 10.2.4 (Erdős–Gallai, 1959 [316])**.** *If P_k denotes a path with k edges, for $3 < k < n$,*

$$ex(n; P_k) \leq \left(\frac{k-1}{2}\right) n,$$

with equality if and only if k divides n, in which case the extremal graph is the disjoint union of copies of K_k.

Exercise 411. *Prove Theorem 10.2.4.*

After briefly considering paths, it may seem natural to consider trees. A simple case is to consider when the forbidden tree is a star $K_{1,t}$. This simple case is given as lemma for future reference.

Lemma 10.2.5. *For any positive integer t, $ex(n; K_{1,t}) \leq \frac{n(t-1)}{2}$.*

Proof: Fix t. The $K_{1,t}$-free graphs are precisely those G with $\Delta(G) < t$. By the handshaking lemma, if $G = (V, E)$ is $K_{1,t}$-free, then

$$|E| = \frac{1}{2}\sum_{v \in V} \deg(v) \leq \frac{1}{2}\sum_{v \in V} \Delta(G) \leq \frac{1}{2}n(t-1),$$

finishing the proof. □

Lemma 10.2.6. *If T is a tree with k edges, then*

$$\frac{k-1}{2} \cdot n \leq ex(n; T) \leq (k-1)n.$$

Proof outline: For the lower bound, the disjoint union of K_ks contains no tree with k edges. For the upper bound, reduce to the case that for any graph with minimum degree at least $k-1$ embeds T. □

The result in Exercise 143 (which says if $\delta(G) = k$, then G contains every tree with k-edges as a weak subgraph) can also be interpreted as an extremal result for trees, but instead of one more edge producing a single forbidden graph, it produces a large collection of subgraphs, all trees on n vertices.

Much work on extremal numbers for trees seems to be inspired by the following conjecture.

Conjecture 10.2.7 (Erdős–Sós, 1963 [298]). *For any tree T with k edges,*

$$ex(n;T) \leq \left(\frac{k-1}{2}\right) n,$$

with equality holding if and only if k divides n.

In 2013 (perhaps a year or two earlier) Conjecture 10.2.7 was proved by Ajtai, Komlós, Simonovits, and Szemerédi by a sequence of three technical papers that are soon to be published (as of July 2013, it is reported that the papers occupy 180 pages). An outline of the proof was given jointly by Simonovits and Szemerédi on the second day of the Erdős centennial conference, held at the Hungarian Academy of Sciences, 1–5 July 2013. See [395, pp. 225–226] for a sketch of the proof.

Many special cases of Conjecture 10.2.7 had previously been shown to be true. Here are a few:

Theorem 10.2.8 (Sidorenko, 1989 [862]). *The Erdős–Sós conjecture is true for k-vertex trees that have a vertex adjacent to at least $k/2$ leaves (vertices of degree 1).*

Theorem 10.2.9 (McLennan, 2005 [670]). *The Erdős–Sós conjecture is true for trees of diameter at most 4.*

Theorem 10.2.10 (Brandt–Dobson, 1996 [157]). *Let G be a graph with girth at least 5 and let T be a tree with k edges. If $\delta(G) \geq k/2$ and $\Delta(G) \geq \Delta(T)$, then G contains T as a subgraph.*

It follows from Theorem 10.2.10 that every graph on n vertices and girth at least 5 with more than $n(k-1)/2$ edges contains every tree with k edges as a subgraph. See [157] for more observations and references surrounding such problems. Saclè and Woźniak [818] also discussed the Erdős–Sós conjecture in relation to C_4-free graphs.

10.3 Extremal results for triangles

10.3.1 Mantel's theorem

The following result is one of the fundamental results in extremal graph theory.

Theorem 10.3.1 (Mantel, 1907 [652]). *If a (simple) graph G on $n \geq 1$ vertices has more than $\frac{n^2}{4}$ edges, then G contains a triangle.*

There are many proofs of Mantel's theorem; four are given here (two are variants of a standard inductive proof). For these and others, see [18].

A common tactic in inductive proofs for graphs is to find a set $W \subseteq V(G)$ that has few edges incident with it, delete W, obtain a smaller graph with a higher concentration of edges, and apply the inductive hypothesis to the smaller graph.

Inductive proofs of Mantel's theorem:

For each $n \geq 1$, let $S(n)$ be the statement that if a simple graph G on n vertices has more than $n^2/4$ edges, then G contains a triangle.

BASE STEP: Since any graph on one or two vertices is triangle-free, both $S(1)$ and $S(2)$ hold. When $n = 3$, $n^2/4 = 2.25$, so $S(3)$ says that any graph on three vertices with at least three edges has a triangle, which is a true statement.

There are (at least) two proofs of the inductive step. One (rather elegant) proof is based on $S(k) \to S(k+2)$ and uses two base cases. The second proof uses $S(k) \to S(k+1)$ and needs only a bit more care; such a proof is often the first proof taught to (or discovered by) students.

INDUCTIVE STEP $S(k) \to S(k+2)$: Suppose that for some $k \geq 1$, $S(k)$ holds, and let H be a graph on $k+2$ vertices with no triangle. Consider some edge $e = \{x, y\}$ in H, and the graph $G = H \backslash \{x, y\}$. There are at most k edges from e to G, for otherwise a triangle is formed. By $S(k)$, G has at most $k^2/4$ edges, so H has at most $k^2/4 + k + 1 = (k+2)^2/4$ edges, thereby confirming $S(k+2)$.

INDUCTIVE STEP $S(k) \to S(k+1)$: Suppose that for some $k \geq 3$, $S(k)$ is true. Let H be a graph on $k+1$ vertices. The idea is to delete a vertex in H with smallest possible degree to create G on k vertices, and then show there are still lots of edges left, enough to apply $S(k)$.

Let H have more than $(k+1)^2/4$ edges. Note that if H has a triangle, then any graph with additional edges will also, so suppose, without loss of generality, that H has as few edges as possible, but still more than $(k+1)^2/4$. It is convenient to break the proof that H has a triangle into two cases, k even, and k odd.

First suppose that $k = 2m$. Then $\frac{(k+1)^2}{4} = \frac{4m^2+4m+1}{4}$, and so assume that H has $m^2 + m + 1$ edges. The average degree of vertices in H is $\frac{2(m^2+m+1)}{2m+1} = m + \frac{m+2}{2m+1}$, and so there is a vertex $x \in V(H)$ with degree at most m. Delete x (and all edges incident with x) to give a graph G on $k = 2m$ vertices and with at least $m^2 + m + 1 - m = m^2 + 1 = k^2/4 + 1$ edges. Thus by $S(k)$, G contains a triangle, and hence so does H.

Suppose that $k = 2m + 1$. Then $\frac{(k+1)^2}{4} = \frac{(2m+2)^2}{4} = (m+1)^2$, so assume that H has $(m+1)^2 + 1 = m^2 + 2m + 2$ edges. Then vertices in H have average degree $\frac{2(m^2+2m+2)}{2m+2} = m + 1 + \frac{2}{2m+2}$, so there is a vertex x with degree at most $m+1$. Delete x to give G with k vertices and at least

$$m^2 + 2m + 2 - (m+1) = m^2 + m + 1 > m^2 + m + \frac{1}{4} = \frac{(2m+1)^2}{4} = \frac{k^2}{4}$$

edges remaining. Thus by $S(k)$, G contains a triangle, and hence H also.

So in either case, k even or odd, $S(k+1)$ is true. This completes the inductive step $S(k) \to S(k+1)$.

Thus by mathematical induction, for all $n \geq 3$, $S(n)$ is true. $\square$

For the next proof of Mantel's theorem, a simple form of the Cauchy–Schwarz inequality (see Section 18.3 for general form) is needed:

Theorem 10.3.2 (Cauchy–Schwarz)**.** *For* $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

In particular, when each $x_i = 1$,

$$\left(\sum_{i=1}^{n} y_i\right)^2 \leq n \sum_{i=1}^{n} y_i^2. \tag{10.1}$$

C-S proof of Mantel's theorem: Let $G = (V, E)$ be a graph on $|V| = n$ vertices with $|E| = m$ edges. Assume that G has no triangles. Then adjacent vertices have no common neighbours, so for each edge $\{x, y\} \in E$, $d(x) + d(y) \leq n$. Summing over all edges of G, and applying Lemma 1.8.9,

$$\sum_{x \in V} \deg(x)^2 = \sum_{\{x,y\} \in E} (\deg(x) + \deg(y)) \leq mn.$$

On the other hand, by the Cauchy-Schwarz inequality (10.1) and Euler's identity (the handshaking lemma $\sum_{v \in V} \deg(v) = 2|E|$),

$$\sum_{x \in V} \deg(x)^2 \geq \frac{1}{|V|} \left(\sum_{x \in V} \deg(x)\right)^2 = \frac{(2m)^2}{n}.$$

These two inequalities show that $m \leq n^2/4$. □

The next proof relies upon the "AM-GM inequality" (see Section 18.3), which says that for positive real numbers $a_1, \ldots, a_n$,

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

In particular, when $n = 2$, this says that $ab \leq (a + b)^2/4$.

AM-GM proof of Mantel's theorem: Let $G = (V, E)$ be a graph on $|V| = n$ vertices and assume that G has no triangles. Let $A \subseteq V$ be a largest independent set. Since G is triangle-free, the neighbours of any vertex $x \in V$ form an independent set, and so for each vertex x, $\deg(x) \leq |A|$.

The set $B = V \backslash A$ meets every edge of G, and so

$$|E| \leq \sum_{x \in B} \deg(x) \leq \sum_{x \in B} |A| \leq |B| \cdot |A|. \tag{10.2}$$

By the AM-GM inequality,

$$|E| \leq (|A| + |B|)^2/4,$$

and since $|A| + |B| = n$, G has at most $n^2/4$ edges. □

Comment: Among all triangle-free graphs on n vertices, there is only one with the most number of edges, namely the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, where the partite sets are chosen as equal as possible in size. This fact follows from a more general theorem, Turán's Theorem (Theorem 10.5.2), but can also be extracted from the above proofs. For example, in order for the first inequality of (10.2) to be an equality, B is necessarily an independent set (and to have equality in AM-GM, $a = b$ is required).

Exercise 412. *For $n \geq 4$, show that if G has more than $\frac{n^2}{4}$ edges, then G contains at least two triangles.*

(See Theorem 10.3.4 for a significant strengthening of the result in Exercise 412.)

Exercise 413. *Show that any triangle-free graph G on n vertices satisfies*

$$\sum_{x \in V(G)} \deg(x)^2 \leq \frac{n^3}{4},$$

where equality holds if and only if n is even.

Mantel"s theorem can be used to bound degrees in claw-free graphs (graphs with no induced $K_{1,3}$).

Theorem 10.3.3 (Kloks–Kratsch–Müller, 2000 [576])**.** *If G is a claw-free graph with m edges, then $\Delta(G) \leq 2\sqrt{m}$.*

Exercise 414. *Prove Theorem 10.3.3. Hint: In a claw-free graph, for each vertex $v \in V(T)$, $\overline{G[N(v)]}$ is triangle-free.*

Mantel's theorem is the first of many of its kind; a slight generalization may also be fairly easy to prove by induction (a more general version is given later):

Exercise 415. *Prove that a graph on $n \geq 4$ vertices with more than $\lfloor n^2/3 \rfloor$ edges contains the graph of a tetrahedron (a K_4).*

Exercise 416. *Show that a graph with n vertices and more than $\frac{3}{2}(n-1)$ edges contains two vertices joined by three internally disjoint paths. (Such a graph is called a θ graph.)*

Exercise 417. *Show that a graph on n vertices with $n+1$ edges has girth at most $\lfloor \frac{2}{3}(n+1) \rfloor$. Show also that if a graph on n vertices has $n+2$ edges, then the graph has girth at most $\lfloor \frac{n+1}{2} \rfloor$.*

10.3.2 Many triangles

Recall that one case of Corrádi–Hajnal theorem (Theorem 2.1.4) says that if n is a multiple of 3, and the minimum degree in a graph G on n vertices is greater than $\frac{2}{3}n$, then G contains $n/3$ vertex-disjoint triangles. Graphs G for which $\delta(G) \geq \frac{2}{3}n$, are rather "dense", with roughly two-thirds of all possible edges. As seen below, many more triangles (not vertex disjoint, of course) appear in graphs with only roughly half the possible edges—the threshold at which Mantel's theorem (Theorem 10.3.1) guarantees only one triangle.

By the slight strengthening of Mantel's theorem given in Exercise 412, for graphs on $n \geq 4$, at least two triangles are guaranteed when the number of edges is more than $\frac{n^2}{4}$. According to Erdős [294], the following theorem was proved by Rademacher in 1941, but went unpublished until Erdős [291] gave a simple proof in a 1955 paper.

Theorem 10.3.4 (Rademacher, 1941 (see [291])). *Any graph on $n \geq 3$ vertices with more than $\lfloor \frac{n^2}{4} \rfloor$ edges contains at least $\lfloor n/2 \rfloor$ triangles.*

Exercise 418. *Prove Rademacher's theorem (Theorem 10.3.4). Hint: try induction. Show that the value $\lfloor n/2 \rfloor$ is the best possible.*

In the 1955 paper [291], Erdős claimed slightly more, that for $t \leq 3$ (and $n > 2t$), if G has $\lfloor \frac{n^2}{4} \rfloor + t$ edges, then G contains at least $t\lfloor n/2 \rfloor$ triangles (but only gave a proof for $t = 1$). Erdős later extended Rademacher's theorem and his previous claim:

Theorem 10.3.5 (Erdős, 1962 [294]). *There is a constant $c > 0$ so that for every $t < cn$, if G is a graph on n vertices with $\lfloor \frac{n^2}{4} \rfloor + t$ edges, then G contains at least $t\lfloor \frac{n}{2} \rfloor$ triangles, and $t\lfloor \frac{n}{2} \rfloor$ is best possible.*

In [294], Erdős conjectured that the above result holds for all $c < \lfloor \frac{n}{2} \rfloor$, which was proved by Lovász and Simonovits [644] in 1983.

The next theorem was written by Erdős (around the time of his visit to the University of Manitoba in 1963). First a lemma is given:

Lemma 10.3.6 (Erdős, 1964 [299]). *Let G be a graph on vertices $x_1, \ldots, x_n$, and suppose that x_1, x_2, x_3 form a triangle. Further assume that there are $n + r$ edges, each incident to at least one of x_1, x_2, x_3. Then G contains at least r triangles of the form (x_i, x_j, x_k) where $1 \leq i < j \leq 3 < k$.*

Proof: Induct on r. For each $r \geq 1$, let $S(r)$ be the statement of the lemma.

Base step: When $r = 1$, there are at least $n - 2$ edges from $x_4, \ldots, x_n$ to x_1, x_2, x_3, and so there is at least one x_k, $k > 3$ adjacent to two of x_1, x_2, x_3, creating a triangle, so $S(1)$ holds.

Inductive step: Fix $s \geq 2$ and suppose that $S(s-1)$ holds. Just as in the case $r = 1$, there exists at least one triangle of the desired form x_i, x_j, x_k. In the graph H

induced by $V(G)\backslash\{x_i, x_j\}$, there are $n+s-1$ edges incident to x_1, x_2, x_3, and so by $S(s-1)$, H contains $s-1$ triangles. Together with x_i, x_j, x_k, there are s triangles.

By MI, for each $r \geq 1$, the statement $S(r)$ is true.

Using Lemma 10.3.6, Erdős proved the following:

Theorem 10.3.7 (Erdős, 1964 [299]). *Let $\ell \geq 0$. If G is a graph on n vertices with $\lfloor \frac{n^2}{4} \rfloor - \ell$ edges, and contains a triangle, then G contains at least $\lfloor \frac{n}{2} \rfloor - \ell - 1$ triangles.*

Proof idea: Without loss of generality, let $n > 5$. Argue two cases, depending on whether or not the graph formed by deleting the triangle x_1, x_2, x_3 (and its vertices) has at most $\lfloor (n-3)^2/4 \rfloor$ edges or more than $\lfloor (n-3)^2/4 \rfloor$ edges. In the first case, there are at least

$$\lfloor n^2/4 \rfloor - \lfloor (n-3)^2/4 \rfloor - \ell = n + \lfloor n/2 \rfloor - 2 + \ell$$

edges, and by Lemma 10.3.6, contains at least $\lfloor n/2 \rfloor - 2 - \ell$ triangles; together with the deleted triangle, In the second case, by Rademacher's theorem, there are at least $\lfloor (n-3)/2 \rfloor$ triangles that are not incident with x_1, x_2, x_3; together with the first triangle, there are at least $\lfloor (n-1)/2 \rfloor - \ell - 1$ triangles (where $\ell \geq 0$). □

In the same paper, Erdős [299] suggested that a proof by induction also works (inducting from $n-3$ to n).

Some of the next few facts are from C. S. Edwards [275] and [276]. [Note: My handwritten copy of [276] (given to me by Erdős in 1993) is missing the bibliography, so I might miss some citation specifics.] According to Edwards [275], at a lecture given at the University of Reading (April 1975) Bollobás and Erdős conjectured the following:

Conjecture 10.3.8 (Bollobás and Erdős, 1975, see [275]). *Let G be a graph on n vertices with m edges. If $\frac{2m}{n} \geq \frac{2n}{3}$, then G contains a triangle for which the sum of the degrees is at least $6m/n$.*

Edwards settled Conjecture 10.3.8:

Theorem 10.3.9 (Edwards, 1977 [275]). *Let G be a graph on n vertices with m edges. If $\frac{2m}{n} \geq \frac{2n}{3}$, then G contains at least one triangle with degree sums at least $\frac{6m}{n}$, and this is best possible if and only if G is regular.*

At the end of the paper [275], Edwards points out that he believes that Bollobás and Erdős had an example showing that the bound in Conjecture 10.3.8 is best possible.

Edwards also mentions that A. J. W. Hilton has shown that for any $\epsilon > 0$ and n sufficiently large, there exists a graph for which there is an *edge* with degree sums greater than $2(\frac{2}{3} - \epsilon)n$, and triangle degree sum strictly less than $6m/n$, again showing that the result is best possible.

In Ramsey theory, one of the most basic results says that under any 2-colouring of the edges K_6, there exists a monochromatic triangle. Viewing the two colours as "edge" and "non-edge", this says that any graph G on six vertices has either a triangle, or its complement $\overline{G}$ has a triangle. In general, for a graph G on n vertices, how many triangles are in either G or $\overline{G}$? This question is answered by a corollary of Goodman's theorem (Theorem 10.3.11); before presenting this result, a preliminary theorem by Goodman is required:

Theorem 10.3.10 (Goodman, 1959 [426]). *If G is a graph on n vertices with m edges, then the number of triangles in G or $\overline{G}$ is at least*

$$\binom{n}{3} - m(n-1) + \frac{2m^2}{n}.$$

Proof: Let G be a graph on n vertices with m edges, and let $t(G, \overline{G})$ be the number of triangles in either G or $\overline{G}$. Let $V(G) = \{v_1, \ldots, v_n\}$, and for each $i = 1, \ldots, n$, let $d(i) = \deg(v_i)$.

For each v_i, let $t(i)$ be the number of pairs $\{v_j, v_k\}$ so that v_i is adjacent to exactly one of v_j or v_k. In a triple contributing to the count of $t(i)$, v_i is connected to one neighbour and one non-neighbour, and so $t(i) = d(i)(n-1-d(i))$.

Suppose that a fixed triple $\{v_i, v_j, v_k\}$ of vertices do not form a triangle in G or in $\overline{G}$; then this "non-trivial" triple induces either one or two edges. If, say, v_iv_j is the only edge, then this triple is counted in both $t(i)$ and $t(j)$. If v_iv_j and v_jv_k are the only edges from this triple, then this triple is counted in both $t(i)$ and $t(k)$. In both cases, each such non-trivial triple is counted twice in the sum of all $t(i)$s, and so

$$\begin{aligned}
t(G, \overline{G}) &= \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} t(i) \\
&= \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} d(i)(n-1-d(i)) \\
&= \binom{n}{3} - \frac{n-1}{2}\sum_{i=1}^{n} d(i) + \frac{1}{2}\sum_{i=1}^{n} d(i)^2 \\
&= \binom{n}{3} - \frac{n-1}{2}(2m) + \frac{1}{2}\sum_{i=1}^{n} d(i)^2 \\
&\geq \binom{n}{3} - m(n-1) + \frac{1}{2n}\left(\sum_{i=1}^{n} d(i)\right)^2 \qquad \text{(by Cauchy-Schwarz)} \\
&= \binom{n}{3} - m(n-1) + \frac{1}{2n}(2m)^2 \\
&= \binom{n}{3} - m(n-1) + \frac{2m^2}{n},
\end{aligned}$$

as desired. □

The above proof of Theorem 10.3.10 is based on an argument outlined in [546, Ex. 4.17]. A slightly different proof (*e.g.*, see [429]) uses the fact that there are $\sum_{i=1}^{n}\binom{d(i)}{2}$ pairs of edges in G that share a vertex (say that such edges are "adjacent") and there are $\sum_{i=1}^{n}\binom{n-1-d(i)}{2}$ pairs of adjacent edges in $\overline{G}$. Using this idea, it follows that the number of triangles in G and its complement $\overline{G}$ is at least

$$\binom{n}{3}-(n-2)m+\sum_{i=1}^{n}\binom{d(i)}{2}=\binom{n}{3}-(n-2)(\binom{n}{2}-m)+\sum_{i=1}^{n}\binom{n-1-d(i)}{2}.$$

The number m of edges in Theorem 10.3.10 is then optimized, and examples are given (proofs omitted) that show:

Theorem 10.3.11 (Goodman, 1959 [426])**.** *If G is a graph on n vertices, the total number of triangles in a graph and its complement is at least*

$$\begin{cases}\frac{u(u-1)(u-2)}{3} & \text{if } n=2u;\\ \frac{2u(u-1)(4u+1)}{3} & \text{if } n=4u+1;\\ \frac{2u(u+1)(4u-1)}{3} & \text{if } n=4u+3;\end{cases},$$

which translates to

$$\begin{cases}\frac{n(n-2)(n-4)}{24} & \text{if } n\equiv 0 \pmod 2;\\ \frac{n(n-1)(n-5)}{24} & \text{if } n\equiv 1 \pmod 4;\\ \frac{(n+1)(n-3)(n-4)}{24} & \text{if } n\equiv 3 \pmod 4;\end{cases}.$$

The following is often known as "Goodman's theorem" (see, *e.g.*, [429] for statement and proof):

Corollary 10.3.12. *If G is a graph on n vertices, then the total number of triangles in G and $\overline{G}$ is at least*

$$\frac{n(n-1)(n-5)}{24}$$

Goodman's proof has since been simplified, and strengthened.

Exercise 419. *Find a graph on 9 vertices that shows the bound in Goodman's theorem is sharp.*

10.3.3 Many intersecting triangles

Two triangles sharing a single vertex is often called a "bowtie" (as seen in Figure 1.31).

Having one more edge than is necessary to force a triangle forces a bowtie:

Exercise 420. *Prove that if a simple graph G on $n \geq 5$ vertices has more than $\frac{n^2}{4} + 1$ edges, then G contains two triangles joined at a single vertex. Hint: An inductive proof may have one case that requires special treatment.*

Exercise 420 has been generalized in 1995, to the graph consisting of $k \geq 2$ triangles sharing only one common vertex (see Figure 10.1). This graph is called a *friendship graph*, the unique graph on $2k + 1$ vertices so that any two vertices have a unique common neighbour—see Section 15.5. Some authors call such a graph a "fan graph", for its obvious similarity to a fan propeller and denote it by F_k.

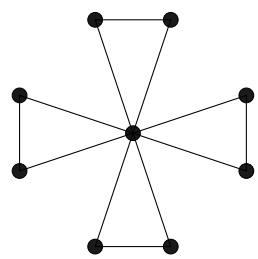


Figure 10.1: The friendship graph F_4

Theorem 10.3.13 (Erdős–Füredi–Gould–Gunderson, 1995 [315])**.** *For each $k \geq 1$ and for $n \geq 50k^2$, if a graph on n vertices has more than*

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even} \end{cases}$$

edges, then G contains k triangles sharing a single vertex. Furthermore, this number of edges is best possible.

Theorem 10.3.13 was further generalized [200] from triangles to copies of K_r with one common vertex.

According to Edwards [276], Bollobás and Erdős also posed the following:

Conjecture 10.3.14 (Bollobás–Erdős)**.** *There exists a constant c so that if G is a graph on n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, then G contains cn triangles all sharing a common edge.*

For each $t \geq 2$, let B_t be the graph consisting of triangles sharing a common edge. The graph B_t is often called a book graph with t pages—see Figure 10.2.

Note: Some other authors define a *book graph* to be a collection of C_4s sharing a common edge. Defined with C_4s, at least "pages" are rectangular, not triangular, so there is an argument for this alternate definition.

Erdős [294] conjectured that $c = 1/6$ is the correct value for the constant in Conjecture 10.3.14; in 1975, Bollobás and Erdős [127] claimed that $c \geq 1/6$.

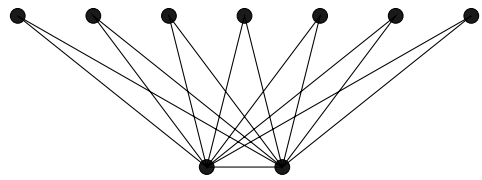


Figure 10.2: The book graph B_7

Conjecture 10.3.15 (Bollobás–Erdős, 1975 [294], [127])**.** *If a graph on n vertices with $\lfloor n^2/4 \rfloor + 1$ edges has at least $n/6$ triangles sharing a common edge.*

Conjecture 10.3.15 was proved by Edwards and independently, by Khadziivanov and Nikiforov.

Theorem 10.3.16 (Edwards, circa 1977 [276])**.** *If G is a graph on n vertices with more than $n^2/4$ edges, then G contains at least $n/6$ triangles all sharing an edge.*

Theorem 10.3.17. *Khadziivanov–Nikiforov, 1979 [567]] Let G be a graph on $n \geq 2$ vertices with m edges. If $m \geq \lfloor \frac{n^2}{4} \rfloor$ and G is not a complete bipartite graph, then G contains at least $\frac{n}{6}$ triangles sharing a common edge, with equality only if G is regular of degree $\frac{n}{2}$. If $m > \lfloor \frac{n^2}{4} \rfloor$, then G contains more than $\frac{n}{6}$ triangles sharing an edge.*

The proof of Theorem 10.3.16 given in the unpublished manuscript [276] begins in a similar manner to the proof of Theorem 10.3.9, so I suspect that the Edwards' result was also written in the late 1970s. (In fact, Bollobás and Nikiforov [130] give it a date of 1977.) The Edwards result confirmed Conjecture 10.3.15 and the Khadziivanov–Nikiforov paper proved a little more. The following example (appearing in, *e.g.* [720, p. 250]) shows that Theorems 10.3.16 and 10.3.17 are essentially best possible.

Example 10.3.18. *Let $k \geq 2$. Let A_1, A_2, A_3 be disjoint sets of $k-1$ vertices each, and let B_1, B_2, B_3 be disjoint sets of $k+1$ vertices each. Define V to be the union of these six sets, giving $|V| = 6k$. For each pair A_i, A_j, $i \neq j$, join all vertices of A_i to all in A_j; similarly join all B_is. Finally, for each i, join all vertices of A_i to all in B_i. The resulting graph G has more than $\frac{n^2}{4} + 1$ edges and any edge is in at most $k + 1 = \frac{n}{6} + 1$ triangles.*

For more recent work on extremal problems involving book graphs (of the kind defined here), see, *e.g.*, [130].

10.3.4 Triangle-free graphs

If a graph G on n vertices is triangle-free, then by Mantel's theorem (Theorem 10.3.1), G has at most $n^2/4$ edges, and by Turán's theorem (Theorem 10.5.2) the extremal graph for triangles is the complete bipartite graph with parts "balanced" (the same

order or within one of each other). What are the largest triangle-free graphs on n vertices that are not bipartite? The following appears as an exercise without proof in [125] and in [143], where it is attributed to Erdős without a specific citation; the same result is stated and proved without reference in [155]. In fact, this result occurs in a 1962 paper by Erdős [294, Lemma 1] on Rademacher's theorem.

Lemma 10.3.19 (Erdős 1962 [294])**.** *Let G be a non-bipartite graph on n vertices with more than $\frac{(n-1)^2}{4}+1$ edges. Then G contains a triangle.*

Proof: Since G is non-bipartite, G contains an odd cycle. Let C be a shortest odd cycle, say of length c. In hope of a contradiction, let $c \geq 5$. Due to the minimality of c, C is an induced cycle and every vertex in $G - C$ has at most two neighbours in C. Since $G - C$ is triangle-free, Mantel's theorem applies, and so

$$|E(G)| \leq c + 2(n-c) + \frac{(n-c)^2}{4} \leq \frac{(n-1)^2}{4} + 1,$$

contradicting the assumption on the number of edges in G. □

Exercise 421. *Show that for odd n, there exists a non-bipartite triangle-free graph on n vertices with exactly $\frac{(n-1)^2}{4}+1$ edges.*

Brandt proved a much stronger version of Lemma 10.3.19:

Theorem 10.3.20 (Brandt, 1997 [155])**.** *Every non-bipartite graph on n vertices with more than $\frac{(n-1)^2}{4}+1$ edges contains cycles of every length from 3 to the length of the longest cycle.*

Lemma 10.3.21. *If G is triangle-free, then $\sum_{v \in V(G)} \deg(v)^2 \leq n^3/4$, with equality if and only if n is even and $G = K_{n/2,n/2}$.*

Exercise 422. *Prove Lemma 10.3.21. Hint: Use Lemma 1.8.9.*

Exercise 423. *If G is maximal triangle-free (adding any missing edge produces a triangle), then any two non-adjacent vertices have a common neighbour.*

The result in the next exercise follows directly from Exercise 423:

Exercise 424. *Show that a triangle-free graph G on at least 3 vertices is maximal triangle-free if and only if G has diameter 2.*

Any odd cycle is triangle-free but not bipartite. In general, "almost all" triangle-free graphs are bipartite, which is a consequence of the following seminal theorem:

Theorem 10.3.22 (Erdős–Kleitman–Rothschild, 1976 [333]). *For $r \geq 3$, as $n \to \infty$, the number of K_r-free graphs on n vertices is*

$$2^{(1-\frac{1}{r-1}+o(1))\binom{n}{2}}.$$

Using $r = 3$, the number of triangle-free graphs on n vertices (as $n \to \infty$) is $2^{(\frac{1}{2}+o(1))\binom{n}{2}} = 2^{(\frac{1}{4}+o(1))n^2}$. On the other hand, the number of labelled equibipartite graphs on n vertices (n even) is $2^{n^2/4}$.

For an analysis of the structure of triangle-free non-bipartite graphs, see [761], where it is shown that almost all triangle-free non-bipartite graphs can be made bipartite by removing just one vertex. The proof by Erdős, Kleiteman, and Rothschild used a "clever" induction. A newer proof uses the "hypergraph container method" (see [73] for a survey of the method and application to this situation).

According to Sloane's *On-Line Encyclopedia of Integer Sequences*, [873] sequence A006785, for $n = 1, 2, 3..., 10$, the numbers of triangle-free graphs on n vertices are 1, 2, 3, 7, 14, 38, 107, 410, 1897, 12172. According to sequence A033995, the numbers of bipartite graphs on $n \leq 10$ vertices is 1, 2, 3, 7, 13, 35, 88, 303, 1119, 5479. (This last list, up to $n = 20$, also occurs in [779, p. 190].) It seems as if these two lists diverge instead of approaching a ratio of 1, so perhaps n must be very large before a pattern emerges. (Brendan McKay [666] says that this is a known example where the convergence is rather slow.)

Exercise 425. *Confirm that there are exactly 7 non-isomorphic triangle-free graphs on 4 vertices, and that each such graph is bipartite.*

The only triangle-free graph on 5 vertices that is not bipartite is C_5.

10.3.5 Triangle-free graphs with large minimum degree

In general, increasing the number of edges in a graph lessens the likelihood that a graph is bipartite. Therefore, the following result may be somewhat surprising.

Theorem 10.3.23 (Andrásfai, 1964 [52]). *Any triangle-free graph G on n vertices with $\delta(G) > \frac{2n}{5}$ is bipartite.*

Theorem 10.3.23 is sharp because of C_5 (or a blow-up of C_5). Ten years later, Andrásfai, Erdős, and Sós [53] also published a generalization of Theorem 10.3.23; perhaps since that paper was in English, it is often the one cited for this result.

Theorem 10.3.24 (Brandt, 1999 [156]). *Every triangle-free graph G on n vertices with $\delta(G) > n/3$ is homomorphic with C_5 or contains a Möbius ladder.*

One of the tools Brandt used was that if G is triangle-free and $\delta(G) \geq n/3$, then G has a unique maximal triangle-free supergraph.

In 1973, Erdős and Simonovits [343] asked the following: if G is a triangle-free graph on n vertices with $\delta(G) > n/3$, is $\chi(G) \leq 3$? Based on a construction using the Kneser graph ($KG(n,r)$ has vertex set $[n]^r$, with vertices adjacent if and only if corresponding r-sets are disjoint; $KG(5,2)$ is the Petersen graph—see Figure 10.3) Hajnal (see [343]) showed that for every $\epsilon > 0$, a triangle-free graph on n vertices with $\delta(G) > (\frac{1}{3} - \epsilon)n$ can have arbitrarily high chromatic number (since, by a result of Lovász [638], Kneser graphs can have arbitrarily high chromatic number). Note that the Petersen graph is 3-colourable (see Figure 10.4).

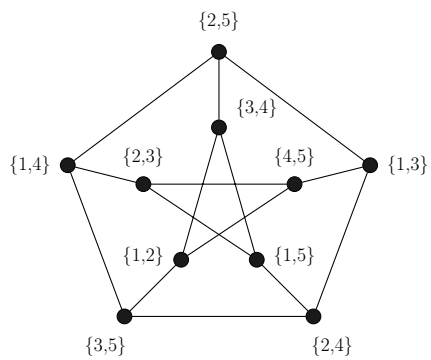


Figure 10.3: Kneser graph $KG(5,2)$

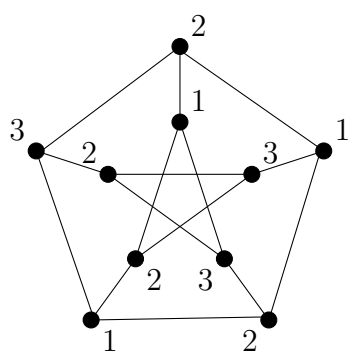


Figure 10.4: A proper 3-colouring of vertices of the Petersen graph

Erdős and Simonovits [343] conjectured that every triangle-free graph on n vertices with chromatic number 4 has minimum degree at most $(1 + o(1))n/3$. Häggkvist [471] gave a counterexample to the Erdős–Simonovits conjecture by presenting a triangle-free graph, based on the Grötzsch graph (see Figure 10.5), with chromatic number 4 and is regular of degree $(10/29)|V(G)|$.

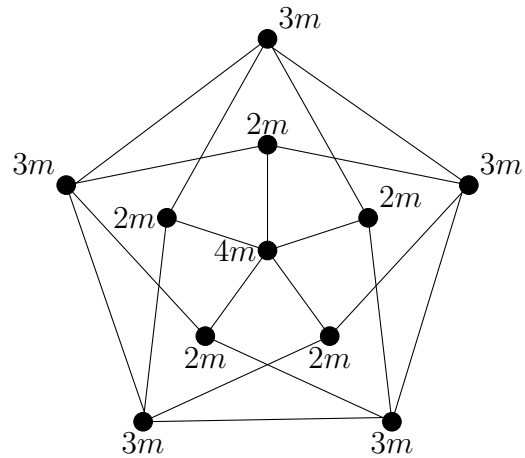


Figure 10.5: Häggkvist's example: blown-up Grötzsch graph on $29m$ vertices

Häggkvist also showed that if G is a triangle-free graph on n vertices with $\delta(G) > (3/8)n$, then $\chi(G) \leq 3$. In 1993, Jin [542] was able to replace the $3/8$ with $10/29$, the best possible (in view of Häggkvist's construction).

In the following decade, many authors produced results in an effort to answer the Erdős–Simonovits conjecture. Much of this work was related to previous work by Pach.

Theorem 10.3.25 (Brandt–Thomassé, 2004 [158])**.** *If G is triangle-free on n vertices and has $\delta(G) > n/3$, then G is 4-colourable.*

See, for example, [156] or [199] for more on classifications of triangle-free graphs with large degree.

10.4 Forbidding complete bipartite graphs

Forbidding the complete bipartite graph $K_{1,t}$ was considered in Lemma 10.2.5. The next simplest case (for complete bipartite graphs) is when the forbidden graph is $K_{2,2} = C_4$.

10.4.1 Forbidding $C_4 = K_{2,2}$

In 1938, Erdős and Klein [288] (see [298]) proved that $\mathrm{ex}(n; K_{2,2}) = \Theta(n^{3/2})$. It might be interesting to note that finding the extremal number for C_4 arose in a number theoretic context. Erdős asked for the maximum cardinality of a subset of $[1, n]$ so that no two pairwise products are the same. He calls such a set a "B-cequence" [sic] and shows that such a sequence has less than $\pi(n) + O(n^{3/4})$ elements, where the error

term cannot be better than $O(n^{3/4}/(\log n)^{3/2})$. For example, the set of primes is a B-sequence. He mentions [288, p. 79] that the construction of a $K_{2,2}$-free graph was communicated to him by "Miss E. Klein" (see Lemma 10.4.3 below for the construction).

Note: B-sequences are often called "multiplicative Sidon sets".

In 1966, Erdős, Rényi, and Sós [339] and Brown [170] showed $\mathrm{ex}(n; K_{2,2}) = \frac{1}{2}n^{3/2} + O(n^{4/3})$. It might be noted that in 1954, Kővári, Sós, and Turán [606] proved that $\mathrm{ex}(n; K_{2,2}) \leq 1 + n + \frac{1}{2}n^{3/2}$, so it is interesting to see the "$O(n^{4/3})$" term in a later result (Sós was a common author)

The following upper bound on $\mathrm{ex}(n; K_{2,2})$ is proved by counting pairs of vertices in neighbourhoods. The idea is now common (*e.g.*,, see [640, 10.36(a)]), and is used again for a similar proof for bipartite graphs (Lemma 10.4.7, regarding Zarankiewicz numbers).

Lemma 10.4.1. *For every $n \geq 4$,*

$$ex(n; K_{2,2}) \leq \frac{n}{4}(1 + \sqrt{4n-3}).$$

Proof: Let G be a graph on n vertices and m edges, and assume that G does not contain a copy of $K_{2,2}$. Count pairs of vertices $\{x, y\}$ that are adjacent to a third point z. For each z, there are $\binom{\deg z}{2}$ such pairs. Each pair $\{x, y\}$ can be counted only once, since if this pair were adjacent to z_1 and z_2, a copy of $K_{2,2}$ is formed. Thus

$$\sum_{z \in V(G)} \binom{\deg z}{2} \leq \binom{n}{2}.$$

By Jensen's inequality,

$$\sum_{z \in V(G)} \binom{\deg z}{2} \geq n\binom{2m/n}{2} = m\frac{2m-n}{n}.$$

Thus

$$m^2 - \frac{n}{2}m \leq \frac{n^3 - n^2}{4},$$

and so by the quadratic formula,

$$e \leq \frac{n}{4} + \sqrt{\frac{n^3}{4} - \frac{3n^2}{16}} = \frac{n}{4}(1 + \sqrt{4n-3}),$$

as desired. □

In a 1964 paper, Erdős [297] proved an upper bound for the extremal number of a special class of d-uniform d-partite hypergraphs (see also [390, eq'n. 4.2]). This upper bound was proved by induction, and the following was the base case for that induction (with proof similar to that of Lemma 10.4.1).

Theorem 10.4.2. *For any $\epsilon > 0$, and n sufficiently large, if a graph G on n vertices contains at least $(1/2+\epsilon)n^{3/2}$ edges, then G contains a copy of $K_{2,2}$, that is,*

$$ex(n; K_{2,2}) < (1/2+\epsilon)n^{3/2}.$$

Proof: Let $c > 1/2$ and let G have n vertices and $cn^{3/2}$ edges. The average degree of vertices in G is $2cn^{1/2}$. For any vertex x, there are $\binom{\deg(x)}{2}$ pairs of vertices in $N(x)$, the neighbourhood of x. Then counting vertices and pairs of neighbours,

$$\begin{aligned}
|\{(x,\{y,z\}) : \{y,z\} \subset N(x)\}| &= \sum_{x\in V(G)} \binom{\deg(x)}{2} \\
&= n \cdot \frac{1}{n} \sum_{x\in V(G)} \binom{\deg(x)}{2} \\
&\geq n\binom{\text{avg deg}}{2} && \text{(by Jensen's ineq.)} \\
&= n\binom{2cn^{1/2}}{2} \\
&> \binom{n}{2},
\end{aligned}$$

for n large enough.

By the pigeonhole principle, there is a pair of vertices in both neighbourhoods of two different vertices, that is, G contains a copy of $K_{2,2}$. □

For certain choices of n, the upper bound in Theorem 10.4.2 is nearly tight:

Lemma 10.4.3 (Reiman, 1959 [786])**.** *If $n = q^2+q+1$ for some prime power q, then*

$$ex(n; K_{2,2}) > (1/2)(n^{3/2} - n^{1/2}).$$

Proof idea: Let $\mathcal{P}$ be a finite projective plane of order q generated by the field $GF(q)$. Assign homogeneous coordinates (x,y,z) to the points in the plane and $[a,b,c]$ to the lines in the plane. Define a graph G on the vertices consisting of the point set of $\mathcal{P}$. Define the edge set of G by connecting the vertex (x,y,z) to every vertex in its polar $[x,y,z]$. Then G is C_4-free and has the right number of edges. □

In 1983, Füredi [388] determined the exact values of $\text{ex}(n; C_4)$ when n is a power of 2. In 1996, Füredi [391] proved that for $q \leq 15$ and $n = q^2+q+1$,

$$\text{ex}(n : C_4) \leq \frac{1}{2}q(q+1)^2,$$

and when $q \geq 13$ is a prime power and $n = q^2 + q + 1$, then

$$\mathrm{ex}(n; C_4) = \frac{1}{2}q(q+1)^2.$$

Precise values of $\mathrm{ex}(n; C_4)$ for all $n \leq 21$ were found by McCuaig [662]. In 1989, Clapham, Flockhart, and Sheehan [222] found the extremal graphs for $n \leq 21$. Then, in 1992, these values were reproved by Yuansheng and Rowlinson [1006], and for $22 \leq n \leq 31$, the precise values together with the associated extremal graphs were given.

10.4.2 Bounds for $\mathrm{ex}(n; K_{2,t})$

Elementary bounds for extremal numbers $K_{2,t}$ are given in this section; a more thorough discussion of extremal numbers for complete bipartite graphs occurs in Section 10.4.4 on Zarankiewicz numbers.

Exercise 426. *Let G be a graph on n vertices, and let $t \geq 2$. Show that if*

$$\sum_{v \in V(G)} \binom{\deg(v)}{2} > (t-1)\binom{n}{2},$$

then G contains $K_{2,t}$. Deduce that if G contains more than $\frac{(t-1)^{1/2}n^{3/2}}{2} + \frac{n}{4}$ edges, then G contains $K_{2,t}$.

In 1996, Füredi [392] showed that

$$\mathrm{ex}(n; K_{2,t}) = \frac{\sqrt{t-1}}{2}n^{3/2} + O(n^{4/3}).$$

10.4.3 Upper bound for $\mathrm{ex}(n, K_{t,t})$

Theorem 10.4.4 (Kövári–Sós–Turán, 1954 [606])**.** *Let $t \geq 2$ be an integer. Then for n sufficiently large,*

$$ex(n; K_{t,t}) < \frac{1}{2}(t-1)^{1/t}n^{2-1/t}.$$

Note that when $t = 2$, Theorem 10.4.4 is simply Theorem 10.4.2.

Exercise 427. *Prove Theorem 10.4.4 by duplicating the proof of Theorem 10.4.2.*

10.4.4 Zarankiewicz numbers

For positive integers $t < n$, the Zarankiewicz number $z(n,t)$, was first defined as the minimum number of 1's required in an $n \times n$ matrix so that there is an $t \times t$ prinicipal submatrix with all 1's. Letting the large matrix be the adjacency matrix of a bipartite graph, finding an all 1's submatrix is tantamount to finding a complete bipartite subgraph. A slightly more general question is when non-square matrices are also considered.

Definition 10.4.5. *For positive integers m, n, s, t, define the* Zarankiewicz number *$z = z(m, n; s, t)$ to be the least number z so that for any bipartite graph $G = (X_1, X_2, \mathcal{E})$, where $|X_1| = m$, $|X_2| = n$, if $|\mathcal{E}| \geq z$, then G contains a copy of $K_{s,t}$ with the s vertices in X_1 and the t vertices in X_2.*

So Zarankiewicz numbers are very similar to extremal numbers, except that all "host" graphs under consideration are bipartite. In fact, the notation "$\mathrm{ex}(m, n; s, t)$ might easily replace $z(m, n; s, t)$. There is a simple relationship between these two numbers, as given in [125, p. 114].

Theorem 10.4.6. *For positive integers n, $s \geq 2$ and $t \geq 2$,*

$$ex(n; K_{s,t}) \leq \frac{1}{2} z(n, n; s, t).$$

.

Proof: Let G be a $K_{s,t}$-free graph on n vertices. Construct a bipartite graph $H = (V_1 \cup V_2, E(H))$ as follows: Let V_1 and V_2 be copies of $V(G)$. If $x' \in V_1$ and $y' \in V_2$ denote vertices corresponding to vertices x and y, respectively, in $V(G)$, then $\{x', y'\} \in E(H)$ if and only if $\{x, y\} \in E(G)$. Then H has twice as many edges as G, and H is $K_{s,t}$-free (and $K_{t,s}$-free). Hence $|E(H)| \leq z(n, n; s, t)$, and so $2|E(G)| \leq z(n, n; s, t)$. □

Essentially repeating the proof of Lemma 10.4.1 gives the following:

Lemma 10.4.7. *If G is an equibipartite graph on n vertices ($n/2$ in each partite set), and $|E(G)| > \frac{n}{4}(1 + \sqrt{2n-3})$, then G contains a $K_{2,2}$.*

Proof: Write $G = (X_1, X_2, E)$, where $|X_1| = |X_2| = n/2$, and let D be the average degree of vertices in X_1 (which is the same as in X_2).

If $\sum_{x \in X_1} \binom{\deg(x)}{2} > \binom{n/2}{2}$, one pair of vertices in X_2 is in two different neighbourhoods of vertices from X_1. By Jensen's inequality,

$$\sum_{x \in X_1} \binom{\deg(x)}{2} \geq \frac{n}{2} \binom{D}{2},$$

and so it suffices to have

$$\frac{n}{2}\binom{2|E|/n}{2} > \binom{n/2}{2} \tag{10.3}$$

to guarantee a $K_{2,2}$. Simple calculations shows that $|E| > \frac{n}{4}(1+\sqrt{2n-3})$ is sufficient to guarantee (10.3). □

Theorem 10.4.8 (Kövári–Sós–Turán, 1954 [606]). *Let $G = (X_1, X_2, E)$ be a bipartite graph with $|X_1| = m$ and $|X_2| = n$, and let $s \le m$ and $t \le n$. If $|E| \ge t^{\frac{1}{s}} m n^{1-\frac{1}{s}}$, then G contains*

$$\binom{m}{s}\binom{\frac{n}{\binom{m}{s}}\binom{\frac{|E(G)|}{n}}{s}}{t}$$

copies of $K_{s,t}$ where the $s \le m$ vertices occur in X_1 and $t \le n$ vertices occur in X_2. Consequently,

$$z(m,n;s,t) < t^{\frac{1}{s}} m n^{1-\frac{1}{s}}.$$

Proof: The number of copies of $K_{s,t}$ in G (where the $s \le m$ vertices occur in X_1 and $t \le n$ vertices occur in X_2) is

$$\begin{aligned}
\sum_{S\in[X_1]^s}\binom{\deg(S)}{t} &\ge \binom{m}{s}\binom{\text{avg deg}(S)}{t} && \text{(by Jensen's ineq.)}\\
&= \binom{m}{s}\binom{\frac{\sum_{S\in[X_1]^s}\deg(S)}{\binom{m}{s}}}{t}\\
&= \binom{m}{s}\binom{\frac{1}{\binom{m}{s}}\sum_{x\in X_2}\binom{\deg(x)}{s}}{t}\\
&\ge \binom{m}{s}\binom{\frac{1}{\binom{m}{s}}n\binom{\text{avg}_{X_2}\deg(x)}{s}}{t} && \text{(by Jensen's ineq.)}\\
&= \binom{m}{s}\binom{\frac{n}{\binom{m}{s}}\binom{\frac{|E(G)|}{n}}{s}}{t}.
\end{aligned}$$

[Checking, if G is complete bipartite with mn edges, the number of oriented copies of $K_{s,t}$ works out to be $\binom{m}{s}\binom{n}{t}$, which is true.]

For the number $\binom{m}{s}\binom{\frac{n}{\binom{m}{s}}\binom{\frac{|E(G)|}{n}}{s}}{t}$ to be at least one, the following is needed:

$$\frac{n}{\binom{m}{s}}\frac{|E(G)|^s}{n^s s!} \ge t.$$

With a little basic algebra, the above condition is met if

$$|E(G)| \geq t^{\frac{1}{s}} mn^{1-\frac{1}{s}}.$$

With $m = n$, this becomes $|E(G)| \geq t^{\frac{1}{s}} n^{2-\frac{1}{s}}$. □

As a corollary to the above proof, by symmetry, if

$$|E(G)| \geq \min\{t^{\frac{1}{s}} mn^{1-\frac{1}{s}}, s^{\frac{1}{t}} m^{1-\frac{1}{t}} n\},$$

then at least one $K_{s,t}$ is assured. So if

$$\begin{aligned} |E(G)| &> \frac{1}{2}(t^{\frac{1}{s}} mn^{1-\frac{1}{s}} + s^{\frac{1}{t}} m^{1-\frac{1}{t}} n) \\ &= \frac{1}{2} mn(t^{\frac{1}{s}} n^{-\frac{1}{s}} + s^{\frac{1}{t}} m^{-\frac{1}{t}}) \\ &= \frac{1}{2} mn \left(\left(\frac{s}{m}\right)^{1/t} + \left(\frac{t}{n}\right)^{1/s} \right) \end{aligned}$$

then G contains at least one $K_{s,t}$. By adding the two terms together and taking the average, one loses at most a constant $1/2$, but the symmetry in the variables might be easier to reproduce in the larger dimension cases.

Note that Theorem 10.4.8 also gives the following bound on the Zarankiewicz numbers:

$$z(m, n; s, t) \leq t^{\frac{1}{s}} mn^{1-\frac{1}{s}},$$

or the more elegant looking bound:

$$z(m, n; s, t) \leq \frac{1}{2} mn \left(\left(\frac{s}{m}\right)^{1/t} + \left(\frac{t}{n}\right)^{1/s} \right).$$

An interested reader might also have a look at [125, Thm. 11, p. 113], which says

$$\mathrm{ex}(n; K_{s,t}) \leq \frac{1}{2}(s-1)^{1/t} n^{2-1/t} + \frac{1}{2}(t-1)n.$$

The proof for this is based on similar double counting and convexity, however counts from the bottom up, that is, begins by assuming that G has no forbidden subgraph.

10.5 Forbidding complete graphs: Turán's theorem

Both Mantel's theorem (Theorem 10.3.1) and the result in Exercise 415 are special cases of a theorem proved by P. Turán (1910–1976). Turán was also known for his work in number theory and analysis. To state this theorem, a few definitions are given.

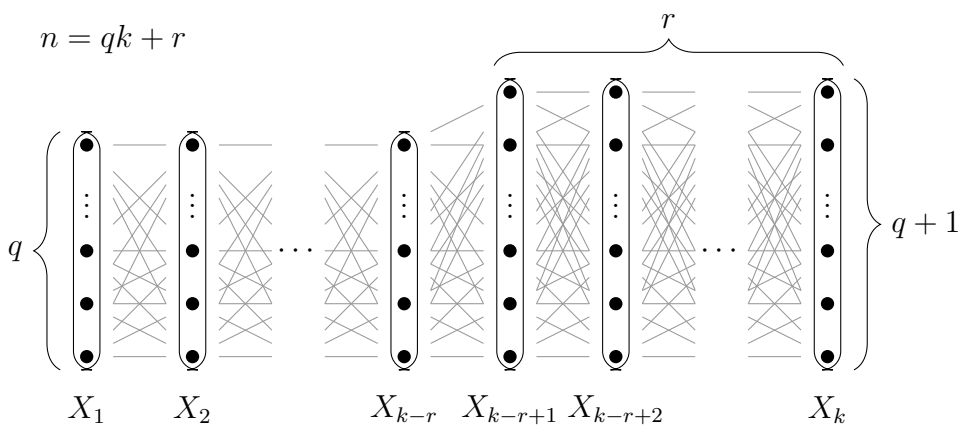


Figure 10.6: The Turán graph $T(n,k)$. For $i \neq j$, the pair X_i, X_j forms a complete bipartite graph (in this diagram, edges between partite sets more than one apart are not drawn)

For positive integers n and k, let $T(n,k)$ be the complete k-partite graph on n vertices whose partite sets have sizes that are as nearly equal as possible. The graph $T(n,k)$ is called "the Turán graph"; see Figure 10.6 for a sketch.

Denote the number of edges in $T(n,k)$ by $|E(T(n,k))| = t(n,k)$. If $n = qk + r$, where q and r are non-negative integers with $0 \leq r < k$, then r of the partite sets in $T(n,k)$ have $q+1 = \lceil n/k \rceil$ vertices, and the remaining $k-r$ have $q = \lfloor n/k \rfloor$ vertices. Hence

$$t(n,k) = \binom{r}{2}(q+1)^2 + r(k-r)(q+1)q + \binom{k-r}{2}q^2.$$

There are many ways to count the edges in $T(n,k)$; here is one convenient approximation:

Lemma 10.5.1. *For a fixed $k \geq 2$, as n increases,*

$$t(n,k) = (1+o(1))\frac{n^2}{2}\left(1 - \frac{1}{k}\right).$$

Proof outline: Assume that k divides n (the remaining case is similar and is omitted). The number of "missing edges" in the Turán graph $T(n,k)$ is $k\binom{n/k}{2}$, so

$$\begin{aligned} t(n,k) &= \binom{n}{2} - k\binom{n/k}{2} \\ &= \frac{1}{2}\left[n(n-1) - k\frac{n}{k}\left(\frac{n}{k} - 1\right)\right] \\ &= \frac{n}{2}\left[n - 1 - \frac{n}{k} + 1\right] \end{aligned}$$

$$= \frac{n^2}{2}\left(1 - \frac{1}{k}\right).$$

□

In 1940, Turán was in a labour camp when he discovered his famous theorem (Theorem 10.5.2 below), and so one story goes, without the use of pencil and paper. [I do not have complete details of how he got his result published the following year, but a short account of the story can be found in [935], in the first issue of the *Journal of Graph Theory*.] This result was also published in 1954 [934] in English.

Theorem 10.5.2 (Turán, 1941 [933, 934])**.** *For positive integers k and n,*

$$ex(n; K_{k+1}) = t(n, k).$$

Furthermore, $T(n, k)$ is the unique extremal K_{k+1}-free graph on n vertices.

There are at least six different proofs of the first sentence (or equation (10.4)) in the statement of the theorem (see, for example, [16], or [18, pp. 183–187] for just five). The first proof given here (due to Turán) is by induction on n (this proof also yields the second statement), and is sometimes called a *chopping off* proof. A second proof provided below is due to Zykov [1015], which uses "symmetrization". As noted in [866], another proof, due to Motzkin and Strauss [700], can be seen as a variation of Zykov's proof.

Turán's Proof: Let $k \geq 1$ and for each $n \geq 1$, let $S(n)$ be the statement that if G is a K_{k+1}-free graph on n vertices with $\mathrm{ex}(n; K_{k+1})$ edges, then $G = T(n, k)$.

Base cases: For each $i = 0, 1, \ldots, k$, the graph with the most edges on i vertices is $K_i = T(i, k)$, so $S(i)$ holds.

Inductive step: Fix some $m \geq k$ and suppose that $S(m - k)$ holds. Let G be a K_{k+1}-free graph on m vertices with $\mathrm{ex}(m, K_{k+1})$ edges. As G is extremal for K_{k+1}, G contains a copy of K_k, call it H, on vertices $A = \{a_1, \ldots, a_k\}$. Put $B = V(G)\backslash A$, and let G^* be the graph induced on B. (In Figure 10.6, this corresponds to chopping off the bottom row of the diagram.)

Since G is K_{k+1}-free, each vertex in B is adjacent to at most $k - 1$ vertices of A. Then

$$\begin{aligned}
|E(G)| &\leq \binom{k}{2} + |B|(k-1) + |E(G^*)| \\
&\leq \binom{k}{2} + (m-k)(k-1) + \mathrm{ex}(m-k; K_{k+1}) && (G^* \text{ is } K_{k+1}\text{-free}) \\
&\leq \binom{k}{2} + (m-k)(k-1) + t(m-k, k) && (\text{by IH, } S(m-k))
\end{aligned}$$

$$= t(m,k) \qquad \text{(structure of } T(m,k)\text{)}.$$

Hence $|E(G)| \leq t(m,k)$. Also, since $T(m,k)$ is K_{k+1}-free and G has an extremal number of edges, $t(m,k) \leq |E(G)|$. Thus $|E(G)| = t(m,k)$, forcing equality in the equations above. Then each vertex in B is joined to exactly $k-1$ vertices of A.

For each $i = 1, \ldots, k$, put $W_i = \{x \in V(G) : \{x, a_i\} \notin E(G)\}$. Note that $a_i \in W_i$ and the W_i's partition $V(G)$ since every vertex in B is not adjacent to one of the a_i's. Each W_i is an independent set, since if some $x, y \in W_i$ were adjacent, x, y and $A \backslash \{a_i\}$ form a K_{k+1}. Hence, G is k-partite.

Since $T(m,k)$ is the unique k-partite graph with as many edges as possible, $G = T(m,k)$. This completes the inductive step $S(m-k,k) \to S(m,k)$.

By mathematical induction, for all $n \geq 0$, $S(n)$ is true. □

Zykov's proof: Let $G = (V, E)$ be a graph on n vertices, and suppose that G does not contain a copy of K_{k+1} as a subgraph. Let $v_1 \in V$ have maximum degree. For each non-neighbour w of v_1, remove all edges incident with w and add all edges $\{\{w, v\} : \{v_1, v\} \in E\}$ to create a new graph G_1. Put $X_1 = \{v_1\} \cup \{w \in V : \{v_1, w\} \notin E\}$. Then each vertex in X_1 is now adjacent to all remaining vertices $V \backslash X_1$ and X_1 is an independent set. Also, G_1 has at least as many edges as does G, and more importantly, since G contains no K_{k+1}, neither does G_1.

Repeat this process of "symmetrization" as follows: pick $v_2 \in V \backslash X_1$ with maximum degree in G_1. For each non-neighbour w of v_2 in $V \backslash X_1$ remove all edges incident with w and add all edges $\{\{w, v\} : \{v_2, v\} \in E\}$ to create a new graph G_2. Put $X_2 = \{v_2\} \cup \{w \in V : \{v_2, w\} \notin E$. Again, X_2 is an independent set, with each vertex of X_2 adjacent to all vertices in $V \backslash X_2$, and G_2 is K_{k+1}-free, and $|E(G_2)| \geq |E(G_1)| \geq |E(G)|$. Continue this process creating, for some r, the graph $G_r = T(n,r)$. Since G_r is K_{k+1}-free, $r \leq k$.

Finally, observe that if $|E(G)| = |E(T(n,k))|$, no edges were added, and so G was originally $T(n,k)$. □

Exercise 428. *Use Turán's theorem to show that if $r \geq 2$ and G is a K_r-free graph on n vertices, then*

$$|E(G)| \leq \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}. \tag{10.4}$$

Exercise 429. *What is the maximum number of edges in a graph on 10 vertices containing no copy of K_4?*

The following is a trivial consequence of Turán's theorem or equation (10.4):

Corollary 10.5.3 (Zarankiewicz, 1954 [1008])**.** *For an integer $r \geq 2$, if G is a graph on n vertices with no copy of K_r, then*

$$\delta(G) \leq \left(1 - \frac{1}{r-1}\right) n = \frac{r-2}{r-1} n.$$

In particular, if G is triangle-free on n vertices, then $\delta(G) \leq n/2$. Corollary 10.5.3 was improved upon:

Theorem 10.5.4 (Andrásfai–Erdős–Sós, 1974 [53]). *Let G be a graph on n vertices. If $\chi(G) \geq r$ and G is K_r-free, then*

$$\delta(G) \leq \frac{3r-7}{3r-4}n.$$

For example, if G is triangle-free and not bipartite, then $\delta(G) \leq (2/5)n$, as was previously stated in Theorem 10.3.23.

The following theorem was given by Erdős in 1970 (and appears in many texts now, *e.g.*, [121, Thm. 4.1, pp. 295–6]) The reader may note that Turán's theorem (Theorem 10.5.2) follows from Theorem 10.5.5.

Theorem 10.5.5 (Erdős, 1970 [305]). *Let $G = (V, E)$ be a graph containing no K_{r+1}. Then there exists an r-partite graph H on vertex set V so that for each $z \in V$, $d_G(z) \leq d_H(z)$. If G is not a complete r-partite graph, then there exists at least one z for which $d_G(z) < d_H(z)$.*

Proof: For each $r \geq 1$, let $A(r)$ be the assertion (both statements) in the theorem. The proof is by induction on r.

Base step: When $r = 1$, G has no edges, so is "1-partite".

Inductive step: Suppose that $s \geq 2$ and that $A(s-1)$ is true. Let G contain no K_{s+1}. Pick a vertex $x \in V(G)$ of maximal degree in G, and put $Y = N_G(x)$, the neighbourhood of x, and put $X = V \setminus Y$ (so $x \in X$). Then the graph $G^* = G[Y]$ induced by vertices of Y is K_s-free (otherwise $G^* + x = K_{s+1}$). Applying $A(s-1)$, get an $(s-1)$-partite graph H^* on vertex set Y, where for every $y \in Y$, $d_{G^*}(y) \leq d_{H^*}(y)$, and if G^* is not complete $(s-1)$-partite, there exists a vertex in Y with strict inequality. Form the graph H by adding the vertices of $V \setminus Y$ to H^*, connecting all vertices in W to all in $V \setminus Y$.

For $v \in X$, $d_G(v) \leq d_G(x) = |Y| = d_H(x) = d_H(v)$. For $z \in Y$, $d_G(v) \leq d_{G^*}(v) + |X| \leq d_{H^*} + |X| = d_H(v)$. In any case, $d_G(v) \leq d_H(v)$, as required.

To show the second statement in $A(s)$, it suffices to show that if $d_G(z) < d_H(z)$ never holds, then G is a complete s-partite graph. So assume that for every $v \in V$, $d_G(v) = d_H(v)$. Counting degrees in Y,

$$\sum_{v \in Y} d_{H^*}(v) + |X| \cdot |Y| = \sum_{v \in Y} d_H(v) = \sum_{v \in Y} d_G(v) \leq \sum_{v \in Y} d_{G^*}(v) + |X| \cdot |Y|, \tag{10.5}$$

and so $\sum_{v \in Y} d_{H^*}(v) \leq \sum_{v \in Y} d_{G^*}(v)$. However, H^* was chosen so that for each $y \in Y$, $d_{G^*}(y) \leq d_{H^*}(y)$, so $d_{G^*}(y) = d_{H^*}(y)$ and equality holds in (10.5). Hence

$$\sum_{v \in Y} d_G(v) = \sum_{v \in Y} d_{G^*}(v) + |X| \cdot |Y|,$$

which says that the number of edges leaving Y in G is maximized, that is, each vertex in Y is adjacent to all of X. Since $|E(G)| = |E(H)|$, $|E(G^*)| = |E(H^*)|$, and all edges in H contain at least one vertex in Y, it follows from equality in (10.5) that $|E(G[X])| = 0$, and so G is a complete s-partite graph, proving the second statement in $A(s)$. The inductive step is complete.

By mathematical induction, for each $r \geq 1$, $A(r)$ is true. □

The following may be thought of as a "dual" (or "complemented") version of Turán's theorem but the earliest formulation I could find in the literature was one that follows from a theorem about $\tau(G)$.

Theorem 10.5.6 (Erdős–Gallai, 1961 [318]). *Let k be a positive integer. If G is a graph with n vertices and at most $nk/2$ edges then $\alpha(G) \geq n/(k+1)$.*

Exercise 430. *Prove Theorem 10.5.6.*

For the next exercise, recall that for a graph G, the order of a largest independent set of vertices in G is denoted by $\alpha(G)$ and $\nu(G)$ denotes the size of a largest matching in G.

Exercise 431. *Let G be a graph on n vertices. Suppose that $\alpha(G) < n - k$ and $\nu(G) \leq k$. Prove that $|E(G)| \geq k + 2$.*

10.6 Extremal numbers and geometric graphs

In this section, "distance" means "Euclidean distance", not "graph distance" (length of shortest path between two vertices). If a graph G on n vertices is drawn in the plane, $\binom{n}{2}$ distances are realized between vertices. If the drawing of G occurs in a circle of diameter 1, and $0 < d < 1$, one can ask how many pairs of vertices have distance d (or greater than d, or less than d).

For example, when $n = 6$ vertices occur as vertices of a regular hexagon (with diameter 1), one can check that three pairs have distance 1, six pairs have distance $1/2$, and six pairs have distance $\sqrt{3}/2$. Thus nine pairs have distance greater than $1/\sqrt{2}$. However, if six vertices are formed by splitting the three vertices of an equilateral triangle (where each split pair remain very close), then all pairs except the split pairs have distance greater than $1/\sqrt{2}$, giving 12 such pairs. Turán's theorem shows that such a result is optimal:

Theorem 10.6.1 (see [142, pp. 114–115]). *If $\{v_1, \ldots, v_n\}$ is a set of points in the plane with Euclidean diameter 1, then the maximum number of pairs at distance greater than $1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$; furthermore, there exists such a placement of the n points so that exactly $\lfloor n^2/3 \rfloor$ pairs have distance greater than $1/\sqrt{2}$.*

Proof outline: For any points u and v in the plane, let $d(u, v)$ denote their Euclidean distance. Consider the graph G on $V(G) = \{v_1, \ldots, v_n\}$, where v_i and v_j are adjacent if and only if $d(v_i, v_j) > 1/\sqrt{2}$.

Claim: G contains no K_4.
Proof of claim: Any four points in the plane determine at least one angle with measure at least $\pi/2$. (This can be seen since any four points have a convex hull that is either a line, a triangle, or a quadrilateral, and in each case, three points can be found that have a large angle.) Consider four points, say, v_1, v_2, v_3, v_4, and suppose that $v_1v_2v_3$ form a large angle. Both distances $d(v_1, v_2)$, $d(v_2, v_3)$, cannot be larger than $1/\sqrt{2}$ (since otherwise, $d(v_1, v_3) > 1$). Thus any 4-tuple of points has at least one pair that is not connected in G, proving the claim.

From Turán's theorem, $|E(G)| \leq t(n; 3) = \lfloor n^2/3 \rfloor$, which shows the first part of this theorem.

To see the second statement, form the arrangement of n vertices by separating them into three groups with sizes as close as possible. For each group, put the vertices in a very small circle, and arrange the three groups at the vertices of an equilateral triangle. □

Exercise 432. *Use the result in Exercise 426 to show that if n points are placed in the plane, the number of pairs of points with distance exactly 1 is at most*

$$\frac{n^{3/2}}{\sqrt{2}} + \frac{n}{4}.$$

The next two exercises can be solved by applying Turán's theorem.

Exercise 433. *Suppose a flat circular city has radius 6 miles, and is patrolled by 18 police cars, each with a two-way radio. If the range of a radio is 9 miles, show that at all times there are at least two cars that can communicate with at least five other cars.*

Exercise 434. *A flat circular city with radius 4 miles will get 18 cell-towers, each tower with transmission range of 6 miles. Prove that no matter the placement of the towers in the city, at least two will each be able to transmit to at least 5 others.*

10.7 The Erdős–Stone–Simonovits theorems

For integers p and t, let $K_p(t)$ denote the complete p-partite graph where each part has t vertices; the notation is consistent with indicating a K_p whose vertices are "blown up" to sets of t independent vertices. (One could also use $T(pt, p)$, denoting the Turán graph.)

Lemma 10.7.1 (Erdős–Stone 1946 [345]). *Let $\epsilon > 0$, and let $k, t \in \mathbb{Z}^+$. For n sufficiently large, if an n-vertex graph G satisfies $\delta(G) \geq (1 - \frac{1}{k} + \epsilon)n$, then G contains a copy of $K_{k+1}(t)$.*

Proof: For each $k = 1, 2, \ldots$, let $S(k)$ be the statement that for any ϵ and for any t, the second sentence in the statement is true.

Base step: $S(1)$ says that if n is large enough and $\delta(G) > \epsilon n$, then there is a $K_{t,t}$, which is true by Theorem 10.4.4. So the base case holds.

Induction step: Fix ϵ and $k \geq 1$; suppose that $S(k)$ is true; in particular, let $S(k)$ be true with t replaced with $s = \lceil \frac{t}{\epsilon} \rceil$.

Then for large enough n, there are (pairwise disjoint) sets $V_1, \ldots, V_k$, each with s vertices, so that any two vertices from distinct V_is are adjacent. Let $Y = V(G) \backslash (V_1 \cup \cdots \cup V_k)$ and let $X \subseteq Y$ be the set of all vertices in Y that are adjacent to at least t vertices in each V_i.

The number of edges missing between $Y \backslash X$ and $V_1 \cup \cdots \cup V_k$ is at least

$$(|Y| - |X|)(s - t + 1) \geq (n - ks - |X|)(1 - \epsilon)s.$$

Counting from the V_i's, for each $v \in V_i$, since $\delta(G) \geq (1 - \frac{1}{k} + \epsilon)n$, from v to X there are at most $(\frac{1}{k} - \epsilon)n$ edges missing; so from $V_1 \cup \cdots V_k$ to X there are at most $ks(\frac{1}{k} - \epsilon)n = s(1 - k\epsilon)n$ edges missing. Hence

$$(n - ks - |X|)(1 - \epsilon)s \leq s(1 - k\epsilon)n,$$

from which it follows that

$$|X| \geq \frac{n(k-1)\epsilon}{1 - \epsilon} - ks.$$

Thus, by increasing n, X can be made arbitrarily large. In particular, for n large enough, let

$$|X| > \binom{s}{t}^k (t - 1);$$

then by the pigeonhole principle, there exist $x_1, \ldots, x_t \in X$ so that for each $i = 1, \ldots, k$, there is $T_i \subset V_i$ with $|T_i| = t$ so that each x_j is adjacent to all vertices in $T_1 \cup \ldots \cup T_k$. Then $T_1, \ldots, T_k$, and $\{x_1, \ldots, x_t\}$ satisfy $S(k+1)$, completing the inductive step.

By mathematical induction, for each $k \geq 1$, $S(k)$ holds. □

Theorem 10.7.2 (Erdős–Stone, 1946 [345]). *Let $k, t \in \mathbb{Z}^+$ and let $0 < \epsilon < 1/k$. For n sufficiently large, if G is a graph on n vertices with at least $(1 - \frac{1}{k} + \epsilon)\frac{n^2}{2}$ edges, then G contains a copy of $K_{k+1}(t)$, the complete $(k+1)$-partite graph with t vertices in each part.*

Proof: Let G be a graph on n vertices with at least $(1-\frac{1}{k}+\epsilon)\frac{n^2}{2}$ edges. If there is $x \in V(G)$ with $\deg_G(x) < (1-\frac{1}{k}+\frac{\epsilon}{2})|V(G)|$, delete it, producing G_1. In G_1, if there exists a vertex y with $\deg_{G_1}(y) < (1-\frac{1}{k}+\frac{\epsilon}{2})|V(G_1)|$, delete y, producing G_2. Continue this process, until one arrives at (if possible) a graph H where $\delta(H) \geq (1-\frac{1}{k}+\frac{\epsilon}{2})|V(H)|$. Put $|V(H)| = N$. The next goal is to show that N can be made large by making n large.

The number of edges removed to produce H is at most

$$\begin{aligned}\sum_{j=N+1}^{n}(1-\frac{1}{k}+\frac{\epsilon}{2})j &= (1-\frac{1}{k}+\frac{\epsilon}{2})\left[\binom{n+1}{2}-\binom{N+1}{2}\right]\\ &= (1-\frac{1}{k}+\frac{\epsilon}{2})\left[\binom{n}{2}+n-\binom{N}{2}-N\right]\\ &\leq (1-\frac{1}{k}+\frac{\epsilon}{2})\left[\binom{n}{2}-\binom{N}{2}\right]+n-N.\end{aligned}$$

Since the number of edges in G is the number of edges removed plus the number of edges in H,

$$(1-\frac{1}{k}+\epsilon)\binom{n}{2} \leq (1-\frac{1}{k}+\frac{\epsilon}{2})\left[\binom{n}{2}-\binom{N}{2}\right]+n-N+\binom{N}{2}.$$

Thus

$$\frac{\epsilon}{2}\binom{n}{2} \leq (\frac{1}{k}-\frac{\epsilon}{2})\binom{N}{2}+n-N,$$

from which it follows that as $n \to \infty$, so also $N \to \infty$.

Applying Lemma 10.7.1 with ϵ replaced by $\epsilon/2$ finishes the proof. □

One might note that in the proof of Theorem 10.7.2, one can compute the relation between N and n; I think that one gets

$$N > (1+o(1))\left(\frac{\epsilon}{1-\frac{1}{k}+\frac{\epsilon}{2}}\right)^{1/2} n.$$

Using the approximation $t(n,k) \sim \frac{n^2}{2}(1-\frac{1}{k})$, (see Lemma 10.5.1) Theorem 10.7.2 says:

Corollary 10.7.3. *Let $\epsilon > 0$ and $t \geq 1$. For n sufficiently large, if G is a graph on n vertices with at least $t(n,p-1)+\epsilon n^2$ edges, then G contains a copy of $T(pt,t) = K_p(t)$.*

Exercise 435. *Show that for any graph G, the limit*

$$\lim_{n\to\infty}\frac{ex(n;G)}{n^2}$$

exists and is at most $\frac{1}{2}$.

A proof of the following uses Theorem 10.7.2.

Theorem 10.7.4 (Erdős–Simonovits, 1966 [342]). *Let G be a (simple) graph with $\chi(G) \geq 2$. Then*

$$\lim_{n\to\infty} \frac{ex(n;G)}{n^2} = \frac{1}{2}\left(1 - \frac{1}{\chi(G)-1}\right). \tag{10.6}$$

Proof: Let G have t vertices, and let $\epsilon > 0$ be fixed. Letting $r = \chi(G)$, it follows that for any n, G is not a subgraph of $T(n; r-1)$, and so (by Lemma 10.5.1)

$$\mathrm{ex}(n;G) > |T(n;r-1)| = (1+o(1))\frac{n^2}{2}\left(1 - \frac{1}{r-1}\right).$$

By Theorem 10.7.2, there exists n_0 so that for every $n \geq n_0$, any graph H with at least n vertices and at least $(1 - \frac{1}{r-1} + \epsilon)\frac{n^2}{2}$ edges, then H contains a copy of $K_r(t)$ and hence a copy of G,

$$\mathrm{ex}(n;G) \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1} + \epsilon\right).$$

The result follows. □

Note that when G is bipartite, Theorem 10.7.4 says that for any constant $c > 0$, as $n \to \infty$, the value $\mathrm{ex}(n;G)$ is much smaller than cn^2; some bipartite graph cases appear in Sections 10.4.1, 10.8.2, 1.6.5, and 10.4.4.

Another stronger version of Theorem 10.7.4 is also true. For a collection of graphs $\mathcal{G}$, let $\mathrm{ex}(n;\mathcal{G})$ denote the maximum number of edges in a graph on n vertices that contain no $G \in \mathcal{G}$ as a subgraph.

Theorem 10.7.5 (Erdős–Simonovits, 1966 [342]). *Let $\mathcal{F}$ be a collection of (simple) graphs where for each $G \in \mathcal{F}$, $\chi(G) \geq 2$, and let $k = \min_{G\in\mathcal{F}} \chi(G)$. Then*

$$\lim_{n\to\infty} \frac{ex(n;\mathcal{F})}{n^2} = \frac{1}{2}\left(1 - \frac{1}{k-1}\right). \tag{10.7}$$

Exercise 436. *Prove Theorem 10.7.5.*

As noted in [342], the lowest chromatic number of any graph in $\mathcal{F}$ is not the sole determiner of the extremal number, as seen in the two examples from that paper; in the following, let W_4 denote the (3-chromatic) wheel graph on five vertices obtained by adding a vertex adjacent to all vertices of a C_4.

Exercise 437. *Use mathematical induction to show that for n sufficiently large,*

$$ex(n;W_4) = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor + 1,$$

but for the forbidden family $\{W_4, K_4\}$,

$$ex(n;W_4,K_4) = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor + 1,$$

.

10.8 Forbidding cycles

Even though cycles are very simple structures, finding extremal numbers for cycles is challenging. Since odd cycles have chromatic number 3 and even cycles have chromatic number 2, by the Erdős–Simonovits theorem (Theorem 10.7.4), $\mathrm{ex}(n; C_{2k+1}) \sim n^2/4$, and $\mathrm{ex}(n; C_{2k}) = o(n^2)$, which might suggest that the even and odd cases might need radically different approaches.

One can also consider forbidding more than one cycle; even though there is a vast amount of literature in this direction, such results are not focussed on here, but one seems worthy of mention (the proof is by induction, and is omitted).

Theorem 10.8.1 (Pósa, 1967; see [302])**.** *If G is a graph on n vertices and at least $3n-5$ edges, then G contains two disjoint cycles.*

Only selected results are given here for forbidding a single cycle, many given without proofs.

10.8.1 Forbidding an odd cycle

Lemma 10.8.2. *For positive integers k and $n \geq 2k+1$, $ex(n; C_{2k+1}) \geq \lfloor \frac{n^2}{4} \rfloor$.*

Proof: Bipartite graphs contain no odd cycles, and so $\mathrm{ex}(n; C_{2k+1})$ is at least as large as the number of edges in any bipartite graph on n vertices, which is maximized at $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$, and so $\mathrm{ex}(n; C_{2k+1}) \geq \frac{n^2}{4}$. □

How does one find an upper bound for $\mathrm{ex}(n; C_{2k+1})$? From Mantel's theorem (or Turan's theorem), when $k = 1$, the inequality in Lemma 10.8.2 is indeed equality. For $k > 1$, it turns out that for large n, equality still holds!

To prove an upper bound on $\mathrm{ex}(n; C_{2k+1})$, it suffices to show that any graph with more than $n^2/4$ edges has a copy of C_{2k+1}; by Mantel's theorem, such a graph contains a triangle, but does it also contain other odd cycles? Is the complete equibipartite graph the only extremal graph when $k > 1$? Much work has been done in answering these questions. Up to recently, it was known (proved by Simonovits, using his famous "colour-critical" theorem—details omitted here) that for any odd cycle and for n sufficiently large, the extremal graph is indeed the complete equibipartite graph. One recent result answers the question completely, including a complete description of all extremal graphs.

Some notation is given first. For $n \geq 1$ and $2k+1 \geq 5$, let $B(n; 2k, n-2k+1)$ be the graph formed by joining a copy of K_{2k} and K_{n-2k+1} at a single vertex. When $2k < n \leq 4k$, write $n = 2k + r$; then the number of edges in $B(n; 2k, n-2k+1)$ is $g(n,k) = \binom{2k}{2} + \binom{r}{2}$. For $n > k$, define the graph $H_1(n,k)$ to be one with k vertices of degree $n-1$ and all other vertices having degree k (and so $H_1(n,k)$ is the union of a K_k and $K_{k,n-k}$); the number of edges in $H_1(n,k)$ is $\binom{k}{2} + k(n-k)$.

Theorem 10.8.3 (Füredi–Gunderson, 2014 [393]). *For any $n \geq 1$ and $2k+1 \geq 5$,*

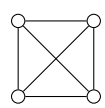
$$ex(n; C_{2k+1}) = \begin{cases} \binom{n}{2} & \text{for } n \leq 2k, \\ g(n,k) & \text{for } 2k+1 \leq n \leq 4k-1, \quad \text{and} \\ \lfloor n^2/4 \rfloor & \text{for } n \geq 4k-2. \end{cases}$$

Furthermore, the extremal graphs are

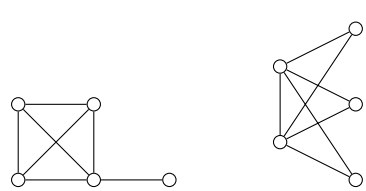
- K_n *for* $n \leq 2k$;
- $B(n; 2k, n-2k+1)$ *for* $2k+1 \leq n \leq 4k-1$;
- $H_1(n,k)$ *for* $n \in \{3k-1, 3k\}$; *and*
- $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ *for* $n \geq 4k-2$.

The proof is omitted. Examples are given below for forbidding C_5 and one for forbidding C_{15}; in each case, the bound of $\lfloor n^2/4 \rfloor$ is given for comparison.

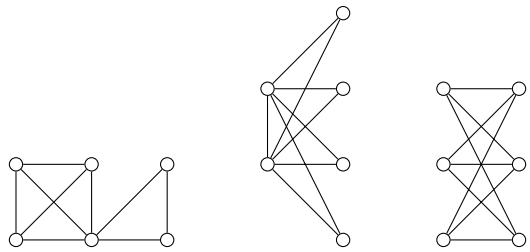
When $n = 4$, $\frac{n^2}{4} = 4$; $\mathrm{ex}(4; C_5) = 6$. In this case, the complete graph is the only extremal graph for C_5:



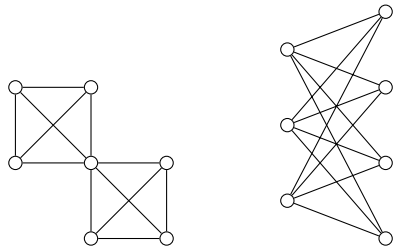
When $n = 5$, $\lfloor n^2/4 \rfloor = 6$. In this case, $\mathrm{ex}(5; C_5) = 7$ and there are two extremal graphs with 7 edges:



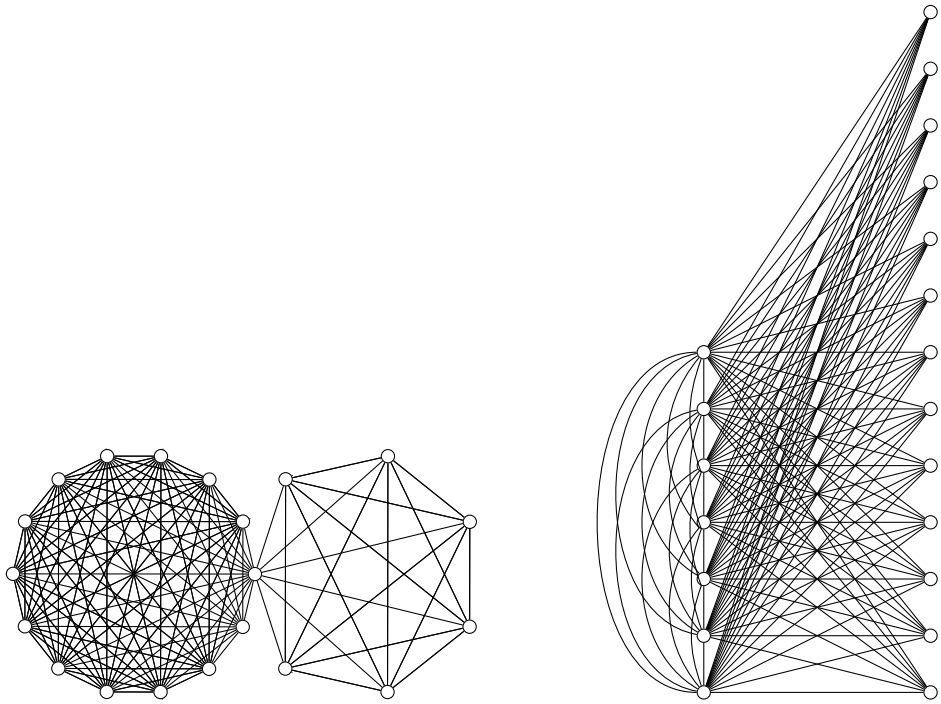
When $n = 6$, $\lfloor n^2/4 \rfloor = 9 = \mathrm{ex}(6; C_5)$, and there are three extremal graphs with 9 edges:



When $n = 7$, $\lfloor n^2/4 \rfloor = 12 = \mathrm{ex}(7; C_5)$, and there are two extremal graphs with 12 edges:



When $n = 20$, $n^2/4 = 100$, however $\mathrm{ex}(20, C_{15}) = 112$, and there are two extremal C_{15}-free graphs on 20 vertices with 112 edges (the $n = 3k - 1$ case):



10.8.2 Forbidding an even cycle

The case for forbidding even cycles is more complicated than for odd cycles. Forbidding any or all even cycles gives one quadratic bound. If one forbids all even cycles, then a simple linear bound holds, but forbidding only one specific even cycle is, in general, difficult.

Exercise 438. *Show that a graph on n vertices with no even cycles has at most $\lfloor \frac{3}{2}(n-1) \rfloor$ edges.*

In 1964, Erdős suggested the following:

Conjecture 10.8.4 (Erdős, 1964 [298])**.** *For $k = 4$ and $k \geq 6$, there exists a universal constant c so that*

$$ex(n; C_{2k}) \geq cn^{1+1/k}.$$

Erdős [298], stated without proof, that for each k, there exists c_k so that

$$\mathrm{ex}(n; C_{2k}) \leq c_k n^{1+1/k}.$$

It wasn't until 1974 when Bondy and Simonovits [144] proved Erdős' statement.

Theorem 10.8.5 (Bondy–Simonovits, 1974 [144])**.** *There exists a universal positive constant $c \leq 100$ so that for each integer $k \geq 2$,*

$$ex(n; C_{2k}) \leq ckn^{1+1/k}. \tag{10.8}$$

Furthermore, if G is a graph on $ckn^{1+1/k}$ edges, then for every integer ℓ, $k \leq \ell \leq kn^{1/k}$, G contains a copy of $C_{2\ell}$.

The constant 100 in the statement of Theorem 10.8.5 has since been improved to nearly 1.

Theorem 10.8.6 (Pikhurko, 2012 [750])**.** *For all $k \geq 2$ and n,*

$$ex(n; C_{2k}) \leq (k-1)n^{1+1/k} + 16(k-1)n.$$

A proof [344] by the probabilistic deletion method gives a lower bound for $\mathrm{ex}(n; C_{2k})$:

Exercise 439. *Use the probabilistic deletion method to show that for each $k \geq 2$, there is a constant $c = c_k$ so that*

$$ex(n; C_{2k}) \geq cn^{1+1/(2k-1)}.$$

There are very few k for which lower bounds for $\mathrm{ex}(n; C_{2k})$ of the form $\Omega(n^{1+1/k})$ are known. All such constructions of these densest C_{2k}-free graphs come from certain geometries, which exist only for the cases $k = 2, 3, 5$; see [394] for references.

10.9 Large partite subgraphs

The material in this section might not normally occur in an undergraduate text. However, since the first theorem below is a classic extremal result with a surprisingly elegant proof, at least an introduction seems warranted.

The first result has a proof that demonstrates the power of a simple probabilistic proof using only expectation.

Theorem 10.9.1 (Erdős, 1963 [295], and 1967 [302]). *Every graph $G = (V, \mathcal{E})$ contains a bipartite subgraph (on V) containing at least half the edges of G.*

Proof: Let G be a graph on V and partition V into two sets R and L at random (that is, 2-colour the vertices of V independently and uniformly at random, where for each $v \in V$, $\mathbb{P}[v \in R] = \frac{1}{2} = \mathbb{P}[v \in L]$). Let $\mathcal{E} = \{e_1, \ldots, e_m\}$, and for each $i = 1, \ldots, m$, let X_i be the 0-1 indicator random variable for e_i containing precisely one vertex from each of L and R. Set $X = X_1 + \cdots + X_m$, the number of edges that cross between L and R. Then

$$\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = m\mathbb{E}[X_1] = m\frac{1}{2}.$$

Thus there exists a partition so that $X \geq \frac{m}{2} = \frac{|\mathcal{E}|}{2}$, as desired. □

Exercise 440. *Prove Theorem 10.9.1 using a simple counting argument.*

Exercise 441. *Find a similar (probabilistic) proof that shows that for any $r \geq 2$, every r-uniform hypergraph H contains an r-partite subgraph with at least $\frac{r!}{r^r}$ times the number of hyperedges in H.*

The result in Theorem 10.9.1 has since been improved in a number of ways.

Theorem 10.9.2 (Edwards [274], 1973). *Any graph with m edges contains a bipartite subgraph with at least*

$$\frac{m}{2} + \frac{1}{8}(\sqrt{8m+1} - 1)$$

edges.

Theorem 10.9.3 (Edwards [274], 1973). *Let G be a graph with n vertices and m edges. If G has no isolated vertices, then G contains a bipartite subgraph with at least $\frac{1}{2}(m + \frac{1}{3}n)$ edges.*

For more on large bipartite (or k-partite) subgraphs, see [324], which contains another proof of Theorem 10.9.2 and references for related work by Alon, Hofmeister, and Lefmann.

10.10 Dominating sets

Definition 10.10.1. *Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ of vertices is called a* dominating set *if and only if for every $v \notin D$, there is a vertex in D adjacent to v. The domination number of G, denoted here by $dom(G)$, is the cardinality of a smallest dominating set.*

Trivially, $V(G)$ is a dominating set for G. The next lemma has a fairly direct proof, which is left as an exercise. Recall that an independent set of vertices in a graph G induces no edges and the cardinality of a largest independent set in $V(G)$ is denoted $\alpha(G)$.

Lemma 10.10.2. *If $G = (V, E)$ is a graph and $I \subset V$ is a maximum independent set, then I is a dominating set. Thus $dom(G) \leq \alpha(G)$.*

Exercise 442. *Prove Lemma 10.10.2.*

Exercise 443. *For any $n \geq 1$, let P_n denote a path on $n+1$ vertices. For each n, find $dom(P_n)$.*

Exercise 444. *For each $n \geq 3$, find the domination number for the cycle C_n.*

Exercise 445. *Find a graph with a smallest dominating set that is not an independent set.*

Exercise 446. *Let G be a connected graph with spanning tree T. Show that the set of vertices of T that are not leaves forms a dominating set.*

By the result in Exercise 446, to give an upper bound on the domination number of a graph, find a spanning tree with the greatest number of leaves.

For more on domination in graphs, see [502].

Chapter 11

Cage graphs

11.1 Basics

Recall that for a graph G with cycles, girth(G) is the length of a shortest cycle in G and diam(G) is the diameter of G (the maximum distance between pairs of vertices).

Definition 11.1.1. *For integers $r \geq 2$ and $g \geq 3$, an (r, g)-*cage *is an r-regular graph with girth g that has minimum order (number of vertices).*

For example, a cycle C_g is a $(2, g)$-cage, K_r is an $(r-1, 3)$-cage, and $K_{r,r}$ is an $(r, 4)$-cage.

Theorem 11.1.2 (Erdős–Sachs, 1963 [341])**.** *For each $r \geq 3$ and $g \geq 2$, an (r, g)-cage exists. If multiple edges are allowed, for each $r \geq 3$, a $(r, 2)$ cage exists.*

The proof of the first part of Theorem 11.1.2 was reproduced in Tutte's book on connectivity [940, pp. 81–83] just three years later, but for a proof in modern graph language, the reader might see, *e.g.*, [190, pp. 500–502].

Lemma 11.1.3 (Erdős–Sachs, 1963 [341])**.** *If G is an (r, g)-cage, then G has diameter at most g.*

Exercise 447. *Prove Lemma 11.1.3. Hint: if two vertices are at distance greater than g, modify the graph to obtain a smaller r-regular graph with girth g.*

The following bound is one of two named after Edward Forrest Moore (1925–2003; worked at IBM Research 1952–1966):

Theorem 11.1.4 (Moore bound)**.** *Let G be a graph with minimum degree $\delta \geq 3$ and girth $g \geq 3$. Defining*

$$M(r, g) = \begin{cases} 1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^{d-1} & \text{if } g = 2d + 1, \\ 2[1 + (\delta - 1) + (\delta - 1)^2 + \cdots + (\delta - 1)^{d-1}] & \text{if } g = 2d, \end{cases}$$

then $|V(G)| \geq M(r, g)$.

Proof: The result arises from a simple BFS (breath-first-search). Consider first the case when $g = 2d + 1$. Let $x \in V(G)$; there is no vertex y at distance d with two edge-disjoint x–y paths, since otherwise a $2k$-cycle exists. So vertices with distance at most d from x are counted. There are at least δ neighbours of x, say $y_1, \ldots, y_\delta$. For each y_i, there are at most $\delta - 1$ neighbours (not including x), so there are at most $\delta(\delta - 1)$ vertices at distance 2 from x. Similarly, for each $i \leq d$, there are $\delta(\delta - 1)^{i-1}$ neighbours at distance i from x.

A similar argument works for g even, but by starting with an edge, not a single vertex. □

Since cage graphs are regular, the Moore bound gives a lower bound on the order of a cage graph. In general, determining the order of cages is an open problem. Norbert Sauer found upper bounds for the orders of cages.

Theorem 11.1.5 (Sauer, 1967 [822, 823])**.** *Letting $n(r, g)$ denote the order of a (r, g)-cage, then for $g \geq 3$,*

$$n(3, g) \leq \begin{cases} \frac{4}{3} + \frac{29}{12}2^{g-2} & \text{if } g \text{ is odd} \\ \frac{2}{3} + \frac{29}{12}2^{g-2} & \text{if } g \text{ is even} \end{cases}$$

For $r \geq 4$,

$$n(r, g) \leq \begin{cases} 2(r-1)^{g-2} & \text{if } g \text{ is odd} \\ 4(r-1)^{g-3} & \text{if } g \text{ is even} \end{cases}$$

11.2 3-regular cages

In 1947, Tutte [937] investigated the order of 3-regular cages. Tutte determined that the orders for $(3, g)$ for $g = 2, 3, 4, 5, 6, 8$ are 2, 4, 6, 10, 14, and 30, respectively. For $g = 7$, by Theorem 11.1.4, a $(3, 7)$-cage has at least $1 + 3 + 3 \cdot 2 + 3 \cdot 2^2 = 22$ vertices. However, a 3-regular graph with precisely 22 vertices has a cycle of length at most 6 (this takes some checking), so a $(3, 7)$-cage contains at least 23 vertices. Since there is always an even number of odd-degree vertices (see Exercise 45), a $(3, 7)$-cage then contains at least 24 vertices. In 1960, McGee [665] showed that the order for a $(3, 7)$-cage is indeed 24 (see Figure 11.2).

It is now known that for $3 \leq g \leq 8$, $(3, g)$-cage graphs are unique (see, for example, Tutte's 1966 book *Connectivity in graphs* [940, pp. 72–81] for proofs).

The $(3, 3)$-cage is K_4. The $(3, 4)$-cage is $K_{3,3}$ (also called the *Thomsen graph*). The only $(3, 5)$-cage is the Petersen graph. The unique $(3, 6)$-cage is called the Heawood graph (see Figure 11.1), which is the incidence graph for the Fano plane (a bipartite graph with 7 vertices in each part, where (P_i, ℓ_j) is an edge if and only if the point P_i lies on line ℓ_j).

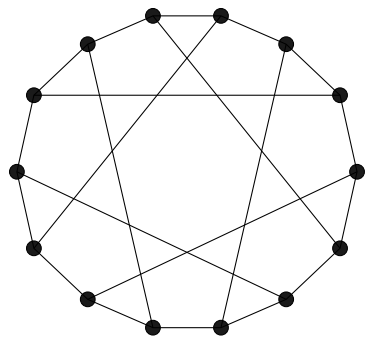


Figure 11.1: Heawood Graph, $(3,6)$-cage, 14 vertices; incidence graph for Fano plane

Exercise 448. *Prove that the Heawood graph is the only* $(3,6)$*-cage.*

Remark 11.2.1. *In general, if there exists projective plane (FPP) of order* $r-1$*, then its incidence graph is a* $(r,6)$*-cage. This fact might be now called folklore, but it occurred at least as early as 1966, when Singleton [869] showed that any* $(r,6)$*-cage with* $2(r^2-r+1)$ *vertices is indeed the incidence graph of a FPP of order* $r-1$*.*

For $g = 8, 12$, Benson [88] showed that existence of a (r,g)-cage is related to the existence of certain projective geometries.

The McGee graph (see Figure 11.2) is the unique $(3,7)$-cage, and has 24 vertices (as pointed out earlier in this section). The McGee graph has 32 automorphisms (see [940, p. 77] for a description of the automorphisms).

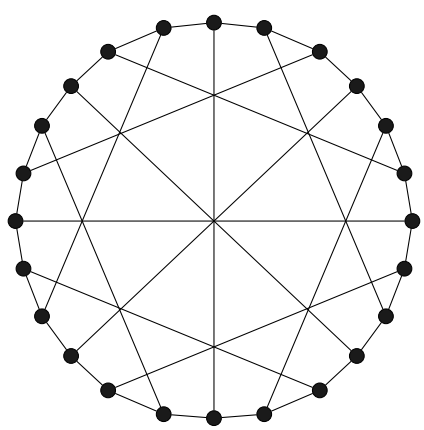


Figure 11.2: McGee (3,7)-cage graph, 24 vertices

The unique $(3,8)$-graph (see Figure 11.3) is often called the Tutte–Coxeter graph, or the Levi graph. This graph was (of course) named after Bill Tutte and H. S. M. Coxeter.

Note: The Tutte-Coxeter graph is not to be confused with the Coxeter graph, given in Figure 15.6.

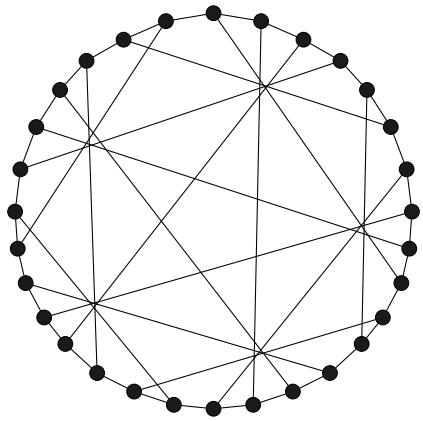


Figure 11.3: Tutte–Coxeter (3,8)-cage graph, or Levi graph, on 30 vertices

There are 18 different $(3, 9)$-cages. There are three $(3, 10)$-cages on 70 vertices (due to Balaban, Harries, and Harries–Wong). In 1998, McKay, Myrvold, and Nadon [668] showed that the order of a $(3, 11)$-cage is 112.

11.3 Higher degree cages

Not too many $(4, g)$-cages are known. The graph K_5 is a $(4, 3)$-cage, and $K_{4,4}$ is a $(4, 4)$-cage. The unique $(4, 5)$-cage graph, the Robertson graph (see Figure 11.4), was discovered by Neil Robertson [797] in 1964; it has 19 vertices and its automorphism group is the dihedral group of order 24. A (4,6)-cage on 26 vertices exists. In 2007, Exoo, McKay, and Myrvold discovered a $(4, 7)$-cage (see [975]).

There are four $(5, 5)$-cages on 30 vertices, one reported by Wong [993] in his 1982 survey, the Robertson–Wegner graph (see Figure 11.5; this graph is called simply "Robertson's graph" in some sources), Foster's cage, and the Meringer graph [677] discovered in 1999 (see [975], where it explains that in his survey [993] Wong had claimed there were only three, until Meringer found a fourth and proved that the list was complete).

Gordon Royle (see [975]) found 5-regular cages for $g = 6, 7, 8, 12$, (with 42, 152, 170, and 2730 vertices, respectively) but at the time of this writing, uniqueness is yet to be shown for these values.

The $(6, 3)$-cage is K_7, and the $(6, 4)$-cage is $K_{6,6}$. Royle (see [975]) also found one 6-regular cage for each of $g = 7, 8, 12$, but uniqueness had not yet been proven by the time of this writing.

Among 7-regular cages, only three are known. A $(7, 3)$-cage is K_8, a $(7, 4)$-cage is $K_{7,7}$, and a $(7, 5)$-cage, called the Hoffman–Singleton graph (discovered in 1960 [522];

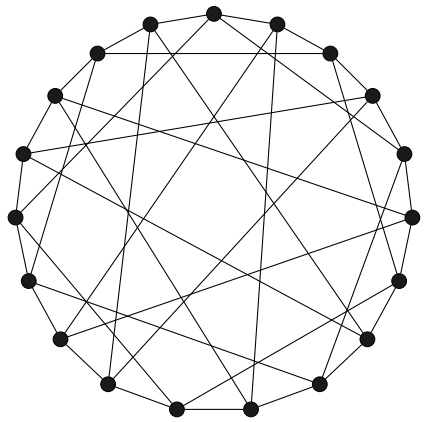


Figure 11.4: Robertson's (4,5)-cage graph, 19 vertices

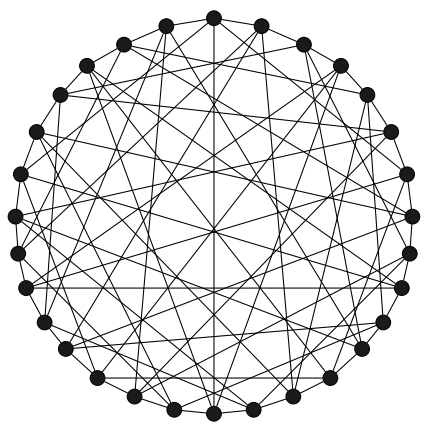


Figure 11.5: The Robertson–Wegner (5,5)-cage graph.

see Figure 11.6, a representation based on the drawing in Bondy and Murty's *Graph Theory* [143, Fig. 3.1, p. 84]). The Hoffman-Singleton graph is notable because it is the only $(7, 5)$-cage that achieves the Moore bound (see Section 11.4) of 50 vertices.

Read and Wilson's *An atlas of graphs* [779, pp. 271–273] gives drawings of a few more cages, but a comprehensive (and perhaps current) list of cages (and references) is given by Wolfram's *Mathworld* [975] (there are listings up to $(14, 8)$-cages, and many incredible drawings done using *Mathematica*).

11.4 Moore graphs

In a 1960 paper [522] by Alan Jerome Hoffman (1924–2021, founding editor of *Linear Algebra and its Applications*) and Robert R. Singleton (from IBM) the following simple observation was attributed to Moore:

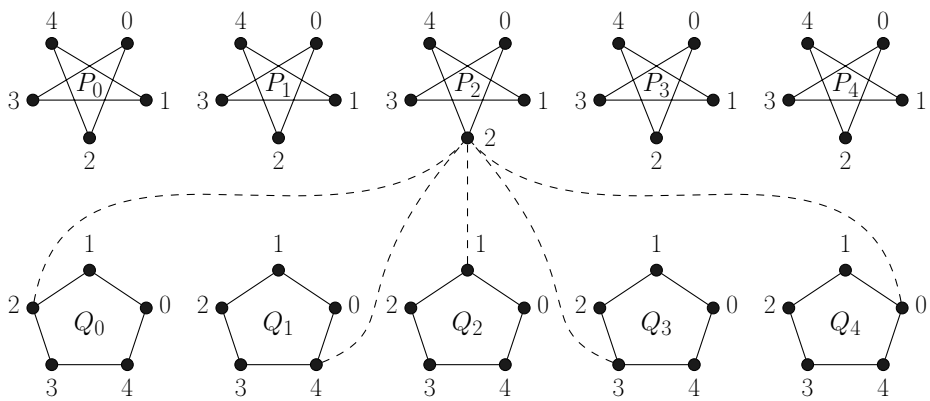


Figure 11.6: The Hoffman–Singleton (7,5)-cage graph; vertex i of P_j is joined to vertex $i + jk$ $(mod\ 5)$ of Q_k.

Theorem 11.4.1 (Moore bound with diameter)**.** *For $r, d \in \mathbb{Z}^+$, if $G = (V, E)$ is an r-regular graph with diameter d, then*

$$|V| \leq 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{d-1}. \tag{11.1}$$

Proof: Let $x \in V(G)$ be an arbitrary vertex and for each $i = 0, 1, 2, \ldots, d$, let V_i be the set of vertices at distance i from x (so $V_0 = \{x\}$). Just as the proof of Theorem 11.1.4, terms in the sum in equation (11.1) are the sizes of the V_is. qed.

Using the same proof, equation (11.1) also holds when d is the maximum degree in a graph.

Hoffman and Singleton mention that Moore asked for which graphs equality holds in equation (11.1) and named such graphs "Moore graphs". For later reference, this definition is made formal:

Definition 11.4.2. *For positive integers r and d, an (r, d)-Moore graph (or a Moore graph of type (r, d)) is an r-regular graph with diameter d for which the inequality in (11.1) is equality.*

Remark 11.4.3. *Godsil and Royle [422, p. 90] define a* Moore graph *to be a graph with diameter d and girth $2d + 1$; they then use a later result by Singleton to show that Moore graphs are regular. Bondy and Murty [143, p. 83] define a Moore graph of diameter d to be a regular graph of diameter d and girth $2d + 1$. Below, it is shown that these other definitions of a Moore graph are equivalent. (Some authors define a Moore graph to be a regular graph of diameter 2 and girth 5.)*

The only graphs of diameter 1 are the complete graphs (which are regular), so K_n is a Moore graph of type $(n - 1, 1)$. For $d = 2$, both C_{2k} and C_{2k+1}) are 2-regular and

have diameter k. However, in C_{2k}, for any vertex x in a C_{2k} the set V_k (vertices at distance k from u) has less than the optimal number of vertices.

The next result is implicit in [522], and is credited to Moore.

Lemma 11.4.4 (Moore, $\leq$ 1960, see [522])**.** *For $r, d \geq 2$, a Moore graph of type (r, d) (as in Definition 11.4.2) is an $(r, 2d+1)$-cage.*

Proof: Let $r, d \geq 2$ and let G be a (r, d)-Moore graph, i.e., G is r-regular, has diameter d, and equality holds in (11.1). If G has a cycle of length less than $2d+1$, then for some x and $i \leq d$, the set V_i (as defined in the proof of Theorem 11.4.1) has fewer than the required number of vertices. If G has a cycle longer than $2d+1$, then for some vertex x, the V_is do not contain all the vertices. Hence, G has girth $2d+1$. Finally, so see that G has the minimal number of vertices for a r-regular graph with girth $2d+1$, equality in equation (11.1) implies equality in the first equation in Theorem 11.1.4. □

Lemma 11.4.5 (Singleton, 1968 [870])**.** *For each $d \geq 1$, if G is a graph with diameter d and girth $2d+1$, then G is regular.*

Proof: Let G be a graph with diameter d and girth $2d+1$. If $d = 1$, then G is complete (in which case $\text{girth}(G) = 3$) and G is regular. So assume that $d > 1$.

Let u and v be vertices at distance d, and let P be a u–v path of length d. Let w be a neighbour of u not on P. Since the diameter of G is d, $d(w, v) \leq d$. If $d(w, v) < d$, then a cycle of length at most $2d$ is formed (contradicting $\text{girth}(G) = 2d+1$), so $d(v, w) = d$. Let P' be a w–v path of length d. Then P' contains a neighbour of v not on P. Repeating this reasoning for every neighbour of u shows that $\deg(v) \geq \deg(u)$. Reversing the roles of u and v shows that $\deg(u) \geq \deg(v)$, and so $\deg(u) = \deg(v)$. Thus, any two vertices at distance d have the same degree.

Let C be a cycle of length $2d+1$ (which has no chords). For any vertex x on C, there are two (adjacent) vertices at distance d from x, and so these two vertices have the same degree as x. Arguing back and forth between vertices of C at distance d shows that all vertices of C have the same degree.

Finally, consider any vertex y not on C, and let z on C be the vertex closest to y; then y is at distance d from a vertex on C (namely, one at distance $d - d(y, z)$ from z) and so by the first part of the proof, y has the same degree as vertices on C. □

In 1960, Hoffman and Singleton [522] showed that there are at most four Moore graphs of diameter 2: C_5 (2-regular), the Petersen graph (3-regular), the Hoffman–Singleton graph (7-regular; see Figure 11.6), and possibly one of regular degree 57 (if it exists, it has been nicknamed the "monster"). This result is formally stated later in Theorem 15.6.1, where it is proved by considering adjacency matrices and their eigenvalues. (The original proof also uses the adjacency matrix.) Hoffman and Singleton [522] also showed (using matrices) that for diameter 3, the only $(2, 3)$-Moore graph is C_7.

The next exercise is closely related to Exercise 99:

Exercise 449. *Show that if G is a k-regular Moore graph of diameter 2, then $|V(G)| = k^2 + 1$.*

By the result in Exercise 449, if the monster exists, it has 3250 vertices.

How many edges can a graph of order n and girth g have? For d-regular graphs, a bound can be found by the Moore bound (Theorem 11.1.4). The case for irregular graphs was studied by Alon, Hoory, and Linial [33].

Chapter 12

Digraphs and Tournaments

12.1 Directed graphs

The term *digraph* is an abbreviation for "directed graph", a graph with directed edges.

Definition 12.1.1. *A* digraph $D = (V, E)$ *is an ordered pair* $D = (V, E)$*, where* V *is a non-empty set, and* E *is a set of* ordered pairs *of distinct vertices from* V*. Elements of* V *are called vertices and elements of* E *are called* directed edges *or* arcs.

Alternatively, a directed graph is a set V together with an irreflexive binary relation on V.

Directed edges are usually drawn with an arrow; if $(x, y) \in E$, the arrow usually goes from x to y, in which case some texts say that x *dominates* y. The head of arrow is y and the tail is x. Notation for digraphs varies; some authors prefer to put arrows above either the name of the digraph or above the name of the edge set just to highlight the directed aspect (*e.g.*, $\overrightarrow{G} = (V, E)$ or $D = (V, \overrightarrow{E})$). If (s, t) is an arc in some digraph, say that s and t are adjacent, but that s is adjacent to t and t is adjacent *from* s.

A digraph D is *simple* if and only if for every $x, y \in V$, at most one of (x, y) or (y, x) is in E; any ordered pair occurs as a directed edge. So a simple digraph D can be obtained from a graph $G = (V, E)$ by simply 'orienting' each edge in E; that is, if $\{x, y\} \in E$, choosing either (x, y) or (y, x) to be a directed edge in D. Another way to say this is that a simple digraph D is an *orientation* of some graph G. A path in a digraph follows the arrows, so one might say, for example, that there is a directed path from v_4 to v_1, but there might not be one from v_1 to v_4.

In the above definition of a digraph (Definition 12.1.1), there are no "parallel arcs" (pairs (x, y) repeated as an arc) or "loops" (arcs of the form (x, x)). (Repeated arcs and loops are allowed in a class called "multidigraphs", but often authors include loops in their definition of digraph—in which case the phrase "distinct vertices" in the above definition is dropped. To further confuse things, some authors define a digraph to be a

graph whose edges have been directed (so both (x, y) and (y, x) can not be arcs), but such structures are called "simple digraphs" here.

If $x \in V$ is a vertex of some digraph D, then the outdegree of x, denoted $d^+(x)$, is the the number of edges of the form (x, z), that is, in the case of a simple digraph,

$$d^+(x) = |\{z \in V : (x, z) \in E(D)\}|.$$

Similarly define the *indegree* $d^-(x)$. Since the letter d can be overused (for distance, degree, diameter, average degree, ...) to be more readable, the notations $\deg^+(x)$ and $\deg^-(x)$ for indegree and outdegree are common. Some authors (*e.g.*, [190]) use "id" and "od".

Recall that for simple graphs, the notation $N(x)$ or $N_G(x)$ denotes the neighbourhood of a vertex x in a graph G; the analogous notations for digraphs are used.

Definition 12.1.2. *If $D = (V, A)$ is a digraph and $x \in V$, define the out-neighbourhood of x to be $N^+(x) = \{y \in V : (x, y) \in A\}$, and the in-neighbourhood of x to be $N^-(x) = \{y \in V : (y, x) \in A\}$.*

The notations Γ^- and Γ^+ are also still quite common to indicate in- and out-neighbourhoods, resp. (these are convenient when too many Ns are floating around).

Notions of walk, trail, path, and cycle all have the obvious directed versions in digraphs, so I will not belabour the point with all new definitions. Even the chromatic number has a natural analog for digraphs.

Exercise 450. *Show that if D is a digraph where each vertex has outdegree at least 1, then D contains a directed cycle.*

Exercise 451. *Let D be a digraph without (directed) cycles and with no (directed) path of length k. Prove that $\chi(D) \leq k$.*

Exercise 452. *Let $n \geq 2$, and define a digraph $D = (V, E)$ on vertex set $V = [n] = \{1, 2, \ldots, n\}$ by (x, y) is a directed edge if and only if $x \neq y$ and $x(y-1) \equiv 0 \pmod{n}$. Show that D contains no cycles.*

12.2 Strongly connected digraphs and orientable graphs

Definition 12.2.1. *A digraph G is* strongly connected *if and only if for any ordered pair of distinct vertices $x, y \in V(G)$, there exists a directed path from x to y.*

If G is a strongly connected digraph, by interchanging the roles of x and y in Definition 12.2.1, for every pair of distinct vertices x and y, there is a directed x-y path and a directed y-x path.

An algorithm, due to Tarjan [910], is known that efficiently finds all strongly connected components of a digraph, but such algorithms are not studied here. For an explanation of how Tarjan's algorithm works, please see [429, pp 53–54].

An "orientation" of an edge $\{x, y\}$ in a graph is a replacing of the edge by one of the ordered pairs (x, y) or (y, x), so essentially, an edge is replaced by an arc (directed edge).

Definition 12.2.2. *A graph is* orientable *if there is an orientation of its edges so that the resulting digraph is strongly connected.*

Herbert Ellis Robbins (1915–2001) was a student of Hassler Whitney (see Whitney's theorem, Theorem 4.1.12) at Harvard. The following theorem is sometimes known as "Robbins' one-way street theorem", because it identifies what kind of street systems can be made into one-way streets so that every intersection is reachable from any other.

Theorem 12.2.3 (Robbins, 1939 [801])**.** *Let G be a (simple) graph. Then G is orientable if and only if G is connected and has no bridges.*

Proof: outline Let G be orientable witnessed by an orientation D that is strongly connected. Since D is strongly connected, D contains no "one-way bridges", that is, for any non-trivial subset $S \subset$V(G)$= F(D)$, there is at least one arc leaving S and one arc entering S, and so G contain no bridges. (This is another way of saying that if G has a bridge, no orientation of G is strongly connected because vertices on one end of the bridge are not reachable from the other end.)

Next assume that G is connected with no bridges. Then G contains a cycle. Orient this cycle to produce a directed cycle. Then look at a vertex not on the cycle that has a path through it connecting to the cycle in two points (this path was called an ear by Robbins). Then orient this "ear" in one direction matching the ordering on the first cycle. Continue to break off ears and orient them one at a time. It only then remains to observe that any 2-edge-connected graph has such an ear decomposition, and so induction finishes the proof. □

It might be interesting to note that Robbins also co-authored the famous book *What is mathematics?* [231] with Richard Courant in 1941. (I highly recommend this book to anyone even mildly interested in mathematics.)

The orientation of edges in a graph can also be captured in an incidence matrix by setting one of the two 1s in each column to -1.

Definition 12.2.4. *Let G be an oriented graph on vertices $v_1, \ldots, v_n$ and let $e_1, \ldots, e_m$ be an enumeration of $E(G)$. Define the* oriented incidence matrix $Q = (q_{i,j})$ *(corresponding to the linear orders given on vertices and edges) to be the $n \times m$ matrix with entries -1,0, 1 defined as follows. For each $j = 1, \ldots, m$, if $e_j = (x_i, x_k)$ then define $q_{i,j} = 1$, $q_{k,j} = -1$, and for all other $\ell \notin \{i, k\}$, set $q_{\ell,j} = 0$.*

Exercise 453. *Let G be an oriented graph on vertices $v_1, \ldots, v_n$ and let $e_1, \ldots, e_m$ be an enumeration of $E(G)$. Let A be the adjacency matrix for G (using the given order of the vertices) and let Q be the corresponding oriented incidence matrix for G. Let $D = (d_{i,j})$ be the $n \times n$ diagonal matrix defined by $d_{i,i} = \deg(v_i)$. Show that*

$$QQ^T = D - A.$$

The matrix $D - A$ in Exercise 453 is called the *combinatorial Laplacian* matrix (or simply, the Laplacian) or *Kirchoff matrix* of the graph G. Such matrices are the center of much study in algebraic graph theory (some algebraic graph theory is given in Chapter 15).

12.3 De Bruijn digraphs

In 1946, Nicolaas Govert de Bruijn (1918–2012) used a particular digraph to solve a problem regarding circular sequences that contain all binary words of a specified length (in fact, he answered the problem for words over any finite alphabet, not just $\{0, 1\}$). Later in that century, it was found that his techniques were discovered much earlier (see [371], [245]), but it is his name that is attached to these digraphs. (See Section 12.5 for more on the problem and its history.) Often, these digraphs (with loops) are referred to as de Bruijn "graphs". (De Bruijn [243] originally called these digraphs "T-sets".)

Let $n \geq 1$, and let B_n be the set of all binary words of length n. For any binary word $b_1 b_2 \cdots b_n$, eliminating b_1, shifting the bits one spot to the left, and adding a bit to the right end of the word produces a word either of the form $b_2 b_3 \cdots b_n 0$ or $b_2 b_3 \cdots b_n 1$; such an operation is called a "left shift".

The de Bruijn digraph for n-bit words (and a left shift) is the digraph (perhaps with loops) whose vertex set is B_n and whose directed edges are of the form

$$(b_1 b_2 \cdots b_n, b_2 \cdots b_{n-1} 0) \text{ or } (b_1 b_2 \cdots b_n, b_2 \cdots b_{n-1} 1).$$

Then each vertex has indegree 2 and each vertex has outdegree 2. (So, there are 2^n directed edges.) The de Bruijn digraph for 3-letter binary words is given in Figure 12.1.

Exercise 454. *Draw the left shift de Bruijn digraph for 2-bit binary words.*

For a recent paper on de Bruijn graphs (with more references) , see [815]. For a paper about the number of factors in a de Bruijn graph, see [380].

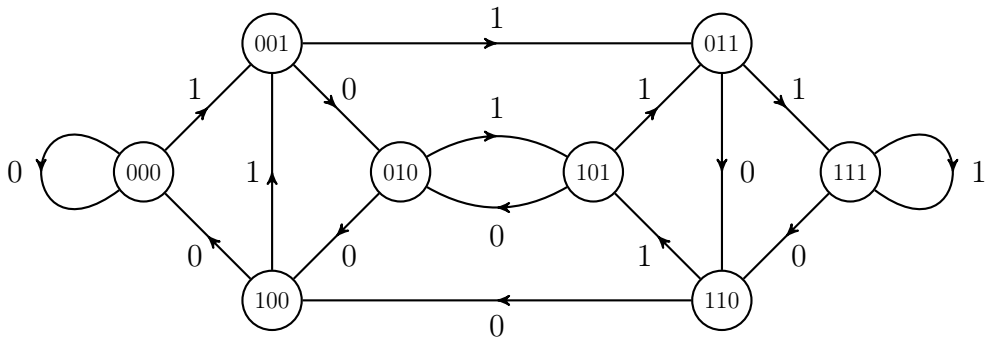


Figure 12.1: *The left-shift de Bruijn digraph for 3-bit words with edges labelled by the new rightmost bit*

12.4 Eulerian circuits in digraphs

A digraph (or multi-digraph) D is called *Eulerian* if and only if there is a closed (directed) trail (called an *Eulerian circuit*) containing all the directed edges of D. Repeating the proof of Theorem 2.2.2 to directed graphs gives:

Theorem 12.4.1. *A multi-digraph D with at least one arc is Eulerian if and only if D is strongly connected and for each vertex v, $d^+(v) = d^-(v)$.*

Exercise 455. *Give the details to the proof of Theorem 12.4.1.*

For example, a de Bruijn digraph (see Figure 12.1) is Eulerian since each vertex has indegree 2 and outdegree 2 (and de Bruijn digraphs are strongly connected). A directed Eulerian circuit for the de Bruijn digraph for 3-bit words is given in Figure 12.2.

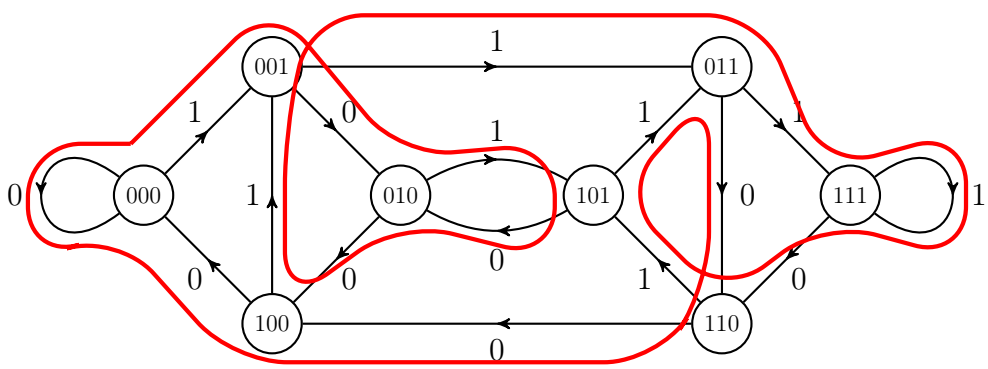


Figure 12.2: *An Eulerian circuit in left-shift de Bruijn digraph for 3-bit words*

For more work on counting Eulerian circuits in digraphs and random digraphs, see [235], including the "BEST" theorem (due to de Bruijn, Aardene-Ehrenfest, Smith,

and Tutte), a count of Eulerian tours calculated by counting arborescences, structures not developed in these notes (an arborescence in a digraph is a rooted spanning tree with all arcs directed to the root.) The interested reader might also see [456] for more references and the application of "interlace polynomials".

12.5 Application: the rotating drum problem

In 1894, de Rivièra [251] asked if there exists a circular arrangement of 2^n zeros and ones so that every n-bit binary word appears exactly once (as a substring of consecutive bits).

For a positive integer n, concatenating all n-bit binary words takes $n2^n$ bits (n bits for each word, and 2^n words). For example, with $n = 3$, one might list the words by

$$000 \mid 001 \mid 010 \mid 011 \mid 100 \mid 101 \mid 110 \mid 111,$$

which takes $3 \cdot 2^3 = 24$ bits.

Consider a rotating drum (or wheel, or plate) where on the edge of the drum there are 0s and 1s, and a sensor can detect a small sequence of consecutive bits, say n bits, and the drum rotates one step at a time, (say) clockwise.

For example, in Figure 12.3, the sensors can read $n = 4$ consecutive bits at a time. If the rotating drum is large enough to contain all n-bit words, how large must the drum be? If one naively puts all n-bit words in a consecutive manner, then $n \cdot 2^n$ positions are required.

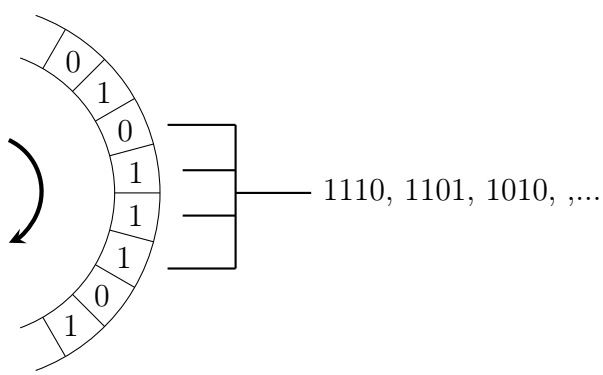


Figure 12.3: Rotating drum with sensor reading 4-bit words

However, far fewer bits around the edge of the drum are necessary. For example, in Figure 12.3, the sensor detects the words 1110, 1101, and 1010 (so 3 words are encoded by only 6 bits instead of 12). A somewhat surprising fact is that if all n-bit words are put in a particular circular order, only 2^n bits on the edge of the drum are required. So the drum in Figure 12.3 need only have 16 cells around the outside if an appropriate way to embed the 16 bits is found.

Theorem 12.5.1. *Let n be a positive integer. There is a circular sequence of 2^n bits so that if this sequence is put on a circular drum, then reading n consecutive bits at a time, the sensor detects all 2^n words.*

Proof: There are two proofs based on de Bruijn graphs, one using (directed) Eulerian circuits, and one using (directed) Hamiltonian cycles. Here is the proof using Eulerian circuits.

If $n = 1$, only two bits are necessary, so assume that $n \geq 2$. Let G be the de Bruijn digraph for $n-1$, whose vertices are the binary words of length $n-1$ and has $m = 2^n$ edges Label each arc $e \in E(G)$ by either $\ell(e) = 0$ or $\ell(e) = 1$, depending upon the choice for the new most right bit (see Figure 12.1 for the labelling in the case $n = 4$).

By Theorem 12.4.1, let C be an Eulerian circuit in G with edges $e_1, \ldots, e_m$ in order, starting at, say, the all 0s word. Then the sequence

$$\ell(e_1)\ell(e_2)\cdots\ell(e_m)$$

is a (circular) list of 2^n bits so that each of the 2^n n-bit words occur as a continuous substring (with wrapping around from back to front). □

In practise, to find the desired circular sequence, one can begin with the all 0s word and add bits on the right (the last $n-1$ labels $e_{m-n+2}, \ldots, e_m$ are 0s):

$$\underbrace{00\cdots0}_{n-1}\ell(e_1)\ell(e_2)\ldots\ell(e_{m-n+1})$$

As an example, for the sensors to read all sixteen 4-bit words on a drum, only sixteen bits need be arranged on the drum. To find one such arrangement, consider the Eulerian circuit C in the de Bruijn graph for 3-bit words given in Figure 12.2. The vertices (in order) with labels highlighted are: **000**- 00**0**- 00**1**- 01**0**- 10**1**- 01**0**- 10**0**- 00**1**- 01**1**- 11**1**- 11**1**- 11**0**- 10**1**- 01**1**- 11**0**- 10**0**- 00**0**. The highlighted bits then give the arrangement

$$0000101001111011,$$

(where the last three 0s are repeated in the first word 0000).

Exercise 456. *Let G be the de Bruijn graph for $n = 2$ (as asked for in Exercise 454). By finding an Eulerian circuit in G, give a circular listing of 8 bits that contain every 3-bit word.*

If B_n denotes a de Bruijn digraph for n, observe that an optimal solution to the rotating drum problem for $(n+1)$-bit words not only corresponds to a an Eulerian circuit in B_n but also to a Hamiltonian cycle in B_{n+1}.

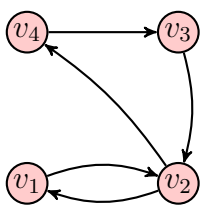
12.6 Digraphs, connectedness, and adjacency matrices

In Section 1.12.1 the adjacency matrix of a graph was used to analyze walks and to find if a graph is connected. In this section, similar results are given for digraphs.

A digraph (or multi-digraph) is called *connected* if and only if the underlying undirected graph is connected. As already mentioned in Definition 12.2.1, a digraph is called *strongly connected* if and only if for any two distinct vertices v and w, there is a directed path from v to w (and from w to v). The same definition holds for multi-digraphs.

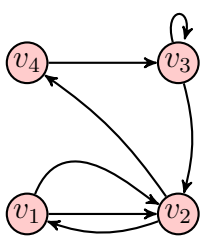
If D is a digraph on vertices $v_1, \ldots, v_n$, an adjacency matrix for D is a 0-1 $n \times n$ matrix $B = (b_{ij})$ satisfying $b_{ij} = 1$ if and only if (v_i, v_j) is an arc in D. As in graphs, an adjacency matrix for a digraph also depends upon the ordering of the vertices. (Usually, A is used to denote an adjacency matrix; however, B is used here to avoid confusion with the notation A for the set of arcs in a digraph.)

For example, using the obvious ordering of vertices,



has adjacency matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

The adjacency matrix $B = (b_{ij})$ for a multi-digraph captures the number of arcs from one vertex to another by putting b_{ij} to be the number of arcs from v_i to v_j. For example,



has adjacency matrix $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Using the same arguments as for Lemma 1.12.1 and Corollary 1.12.2 yields the following characterization of strongly connected digraphs—details are left to the reader.

Theorem 12.6.1. *Let D be a directed graph on n vertices, and let B be an adjacency matrix for D. Then D is strongly connected if and only if every off-diagonal entry of*

$$B + B^2 + \cdots + B^{n-1}$$

is non-zero.

Note: Some authors prefer to say that D is strongly connected if and only if *every* entry of

$$I_n + B + B^2 + \cdots + B^{n-1}$$

is non-zero. With a little thought, one sees that Theorem 12.6.1 holds even when D is a multi-digraph.

Corollary 12.6.2. *Let D be a directed graph on n vertices, and let B be an adjacency matrix for D. Then D is strongly connected if and only if every entry of*

$$(I_n + B)^{n-1}$$

is positive (written $(I_n + B)^{n-1} > 0$). Similarly, a graph G is connected if and only if any of its adjacency matrices B satisfies $(I_n + B)^{n-1} > 0$.

Proof outline: Expand $(I_n + B)^{n-1}$ using the binomial expansion. For each i, j, the (i, j) entry of $(I_n + B)^{n-1}$ is zero if and only if it is zero in every term (since entries of B are non-negative). □

12.7 Tournaments

12.7.1 Definitions of a tournament

If $G = (V, E)$ is a graph, an *orientation* of G is a digraph $H = (V, D)$ where each edge (which is an unordered pair) in G is replaced by exactly one directed edge (or "oriented edge").

A digraph H is called a *tournament* if and only if H is an orientation of some complete graph:

Definition 12.7.1. *A digraph $T = (V, D)$ is a* tournament *if and only if for every two vertices $a, b \in V$, precisely one of $(a, b) \in D$ or $(b, a) \in D$ holds (and for any $x \in V$, $(x, x) \notin D$).*

See Figure 12.4 for the non-isomorphic tournaments on 4 vertices.

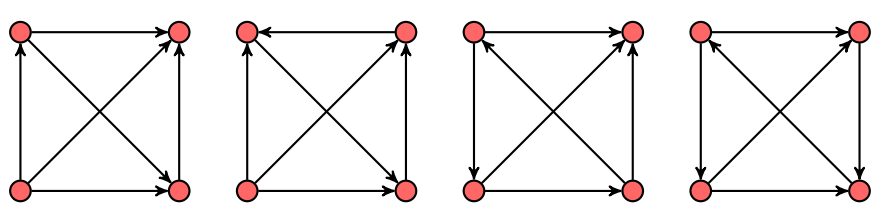


Figure 12.4: The four non-isomorphic tournaments on 4 vertices.

The reason for calling such digraphs "tournaments" is that in a "round-robin tournament" with n players (where every player meets every other in a match precisely once), players are vertices and the winner of each match can be recorded by an appropriate arrow. There are arguments for which direction the arrow should go. For example, often a directed edge (x, y) with the arrow pointing to y in a tournament indicates that x defeats y, or x "dominates" y, but one might also argue that the arrow should point to the winner. In this text, if x and y are vertices (players) in a tournament, let the arrow direction $x \longrightarrow y$ indicate that x defeats y.

Recall that a binary relation $\mathcal{R}$ on a set X is *transitive* if and only if for every $a, b, c \in X$, $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ implies $(a, c) \in \mathcal{R}$. Since a tournament can be viewed as a binary relation on some vertex set, the following definition is natural.

Definition 12.7.2. *A tournament $T = (V, D)$ is* transitive *if and only if for any $x, y, z \in V$, $(x, y) \in D$ and $(y, z) \in D$ implies $(x, z) \in D$.*

A tournament T (or any other digraph) is called *acyclic* if and only if T contains no directed cycles.

Theorem 12.7.3. *A tournament is transitive if and only if it is acyclic.*

Exercise 457. *Prove Theorem 12.7.3.*

Recall (Definition 12.2.1) that a digraph $G = (V, D)$ is *strongly connected* if and only if for every pair of vertices $x, y \in D$, there is a (directed) path from x to y and there is a directed path from y to x.

Exercise 458. *Which of the tournaments in Figure 12.4 are strongly connected? For those (if any) that are not strongly connected, find two vertices so that there is no directed path from one to the other.*

Exercise 459. *Let $T = (V, D)$ be a strongly connected tournament. Is it true that for any $x \in V$, the induced tournament on $V \backslash \{x\}$ is also strongly connected?*

Remark 12.7.4. *If a tournament T contains a (directed) Hamiltonian cycle, then T is strongly connected (since any vertex can be reached from another by following the Hamilton cycle). On the other hand, any strongly connected tournament on at least 3 vertices contains a cycle (select two vertices x, y, and look at the walk formed by a x-y path and a y-x path—such a walk is closed, and hence induces a cycle). It also follows that for any vertex v in a strongly connected tournament, there is a cycle containing v.*

12.7.2 Hamiltonian paths in tournaments

One basic property of tournaments was published by László Rédei (1900–1980).

Theorem 12.7.5 (Rédei, 1934 [780]). *Every tournament contains a (directed) Hamiltonian path.*

Exercise 460. *Prove Rédei's theorem (Theorem 12.7.5). Hint: A simple proof exists by (strong) induction on the number of vertices.*

So Rédei's theorem (Theorem 12.7.5) says that in any round-robin tournament among n players there is a listing of all the players $a, b, c, \ldots$, so that a beats b, b beats c, and so on, continuing until the last player. Such a listing of the players is sometimes called a "ranking" of the players; however, such a "ranking" might be deceptive since the last ranked person might then beat player a.

Remark 12.7.6. *The term "ranking" is often used in a different sense; one could define a* ranking *of any n-element set V to be simply a bijection $\sigma : V \to [n]$ (for example, see [878, p. 6]). The ordering of vertices in the present context might be more aptly called a "chain ranking", or some such. The difference between these two types of ranking is that in the bijective labelling, transitivity is implicit, whereas in a ranking here, transitivity is not guaranteed.*

Redei's theorem is now a special case of a more general theorem:

Theorem 12.7.7 (Roy 1967 [812], Gallai 1968 [402]). *Any digraph D has a directed path of length at least $\chi(D) - 1$.*

(See [142] for a proof.)

Since any tournament T on n vertices has $\chi(T) = n$, the Roy–Gallai theorem says that there exists a directed path of length $n - 1$.

It is known (see, *e.g.*, [640, 5.20]) that every tournament has an odd number of Hamiltonian paths, but it is easy to construct a tournament that is not Hamiltonian. For example, transitive tournaments (see Definition 12.7.2) have no Hamiltonian cycles; an example is the tournament on three vertices with arcs $(a, b), (a, c), (b, c)$, which has no Hamiltonian cycle. As is seen below, as tournaments get larger, they are more likely to have a Hamiltonian cycle.

In the next section, it is shown that if a tournament is strongly connected, then cycles of all possible lengths exist, including a Hamiltonian cycle (in which case, a "ranking" mentioned above is circular, and hence useless when awarding trophies).

12.7.3 Cycles in tournaments

The following exercise has a simple solution by induction; the choice of notation is about the hardest part of the exercise.

Exercise 461. *Let T be a tournament on $n \geq 3$ vertices. Show that if T contains any cycle, then T contains a cycle of length 3.*

Exercise 462. *Let $T = (V, D)$ be a tournament, and suppose that $(a, b) \in D$. Show that if the outdegree of a is less than the outdegree of b, then the arc (a, b) lies on a directed cycle of length 3.*

If a tournament is strongly connected, much more can be said about possible cycle lengths. The first (and perhaps most famous) such result is the following.

Theorem 12.7.8 (Camion, 1959 [179])**.** *Every strongly connected tournament (on at least 3 vertices) contains a Hamiltonian cycle.*

Rather than prove Camion's theorem here, some preliminaries are discussed that prepare for stronger theorems that include Camion's, the strongest of which is proved below (Theorem 12.7.10). The proof of the stronger version is not much more difficult than a proof of the original.

A natural choice for a proof of Camion's theorem might be by induction on the number of vertices, but another choice is to fix the number of vertices throughout, show that there are directed triangles, then directed 4-cycles, and inductively, find longer cycles, until there are Hamiltonian cycles.

As was pointed out earlier in Remark 12.7.4, each vertex in a strongly connected tournament is on some cycle.

Lemma 12.7.9. *Let $n \geq 3$, and let $T = (V, D)$ be a strongly connected tournament on $n = |V|$ vertices. Then every vertex $x \in V$ is contained in at least one directed 3-cycle.*

Proof: Let $x \in V$. Consider the out-neighbourhood of x,

$$A = N^+(x) = \{y \in V : (x, y) \in D\},$$

and the in-neighbourhood of x,

$$B = N^-(x) = \{w \in V : (w, x) \in D\}.$$

If A is empty, then there is no directed path from any vertex in B to x, contradicting that T is strongly connected, so $A \neq \emptyset$. Similarly, $B \neq \emptyset$. If for every $a \in A$ and $b \in B$, $(b, a) \in D$, then there is no directed path from any vertex in A to any vertex in B, again contradicting that T is strongly connected, so there exists at least one $a \in A$ and one $b \in B$ so that $(a, b) \in D$. In this case, x, a, b, x form a directed C_3. □

Recall from Definition 2.8.1, a graph G on n vertices is *pancyclic* if G contains cycles of all possible lengths $3 \leq \ell \leq n$, and is *vertex pancyclic* if and only if there are, through each vertex, cycles of all possible lengths. The same definitions apply to digraphs, where cycles are directed cycles.

One way (see, *e.g.*, [190]) to prove Camion's theorem (Theorem 12.7.8) is to show that in a strongly connected tournament, any non-Hamiltonian cycle can be extended to a larger cycle. The following theorem implies Camion's theorem (as well as a strengthening by Harary and Leo Moser [492] that says strongly connected tournaments are pancyclic).

Theorem 12.7.10 (Moon, 1966 [696]). *Let $T = (V, D)$ be a strongly connected tournament on $n \geq 3$ vertices. Then T is vertex pancyclic.*

Proof: According to [190, p. 168], the proof given here is due to Carsten Thomassen.

Let $v \in V$; the goal is to show that there are directed cycles of all possible lengths through v, and this is achieved by induction on the cycle lengths. For each $\ell = 3, \ldots, n$, let $S(\ell)$ be the statement that there exists a cycle of length ℓ passing through v.

Base case, $\ell = 3$: By Lemma 12.7.9, v is in a (directed) triangle, so $S(3)$ holds.

Inductive step: Let $3 \leq k < n$ and assume that $S(k)$ holds, with a witness cycle $C = (v = v_0, v_1, \ldots, v_{k-1}, v_0)$.

If a vertex $y \notin C$ is both adjacent to some vertex in C and adjacent from a vertex in C, then there exists some $i = 0, 1, \ldots, k$ so that $(v_i, y) \in D$ and $(y, v_{i+1}) \in D$ (because in the cyclic order (with indices modulo k), arcs between y and C change from going toward y to going toward C). In this case,

$$C' = (v_0, v_1, \ldots, v_i, y, v_{i+1}, \ldots, v_{k-1}, v_0)$$

is a cycle of length $k + 1$, satisfying $S(k + 1)$.

So assume that no vertex $z \in V \backslash C$ has arcs both going to and coming from C. Then each $z \in V \backslash C$ either has only arcs directed from C to z, or all arcs directed from z to C. Let $W \subset V \backslash C$ be the set of vertices that are adjacent from all vertices in C, and let $X \subset V \backslash C$ be the set of vertices that are adjacent to all vertices in C. So $V \backslash C = W \cup X$ (and $W \cap X = \emptyset$).

If $W = \emptyset$, then there are no paths from vertices in C to vertices in X, contradicting strongly connected, so $W \neq \emptyset$. Similarly, $X \neq \emptyset$. Also, strongly connected implies that there is a $w \in W$ and $x \in X$ so that $(w, x) \in D$ (otherwise, if all arcs between W and X go from X to W, then there is no path into X). In this case, the cycle

$$C^* = (w, x, v = v_0, v_1, \ldots, v_{k-1}, w)$$

has length $k + 1$, as desired, completing the inductive step.

Hence, by mathematical induction, for each $\ell \in \{3, \ldots, |V|\}$, T contains a cycle of length ℓ containing $v = v_0$. □

The proof of the next result uses only very elementary probability. The meaning of "almost all" is that as $n \to \infty$, the probability that a tournament on n vertices is strongly connected tends to 1. The proof idea below is modelled after a 1963 proof by Erdős [295] of a theorem for tournaments that says that for any $k \geq 1$, there is an n large enough so that there is a tournament on n vertices with the property that for every set of k vertices, there is one vertex that "dominates" every vertex in that set (see [36, pp. 3–4]). [Note: Almost surely, the following was discovered by Erdős, but I have yet to find the earliest source.]

Theorem 12.7.11. *Almost all (finite) tournaments are strongly connected (and hence, by Camion's theorem, are Hamiltonian).*

Proof: Create a random tournament $T = (V, D)$ on n vertices by choosing the direction of an arc between two vertices x and y by $\mathbb{P}[(x, y) \in D] = \mathbb{P}[(y, x) \in D] = \frac{1}{2}$. Instead of showing strong connectivity in a general sense, the proof below only is based on the fact that with high probability, between any two vertices x and y, there is a directed path from x to y of length at most 2! [Note: This is a classic instance of proving a far stronger condition with a simple proof.]

Let $x, y \in V$. If $(x, y) \in D$, then a directed x-y path exists with length 1. So assume that $(y, x) \in D$. Let $z \in V \backslash \{x, y\}$. The probability that x, z, y form a directed x-y path is $\mathbf{P}[((x, z) \in D) \wedge ((z, y) \in D)] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. So the probability that x, z, y is not a directed path of length 2 is $\frac{3}{4}$. Thus, the probability that no z forms such a desired path is $\left(\frac{3}{4}\right)^{n-2}$. Repeating this for each pair of vertices x, y, the probability that sum pair fails to be connected by a path of length 2 is at most (by the loose upper bound found from just summing probablities) $\binom{n}{2}\left(\frac{3}{4}\right)^{n-2}$, which goes to zero as $n \to \infty$. $\square$

Exercise 463. *Show that if two vertices in a tournament do not lie on a cycle, then they have different out-degrees.*

Exercise 464. *Let $p_1, p_2, \ldots, p_n$ be integers with $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$, and for each $k = 1, \ldots, n$, denote the partial sums by $s_k = \sum_{i=1}^{k} p_i$. Prove that there exists a tournament with outdegrees $p_1, \ldots, p_n$ if and only if for each $k < n$, $s_k \geq \binom{k}{2}$ and $s_n = \binom{n}{2}$. Hint: Use induction on $\sum_{k=1}^{n}\left(s_k - \binom{k}{2}\right)$.*

A regular tournament is one whose every vertex has the same outdegree. Some authors call the outdegree of a vertex in a tournament its *score*. If T is a tournament on n vertices is regular with outdegree k, then each vertex has indegree $n - k - 1$.

Recall that for simple (undirected) graphs on n vertices, there are various values of k so that k-regular graphs exist. For example, C_n is 2-regular and K_n is $(n-1)$-regular. However, for tournaments on n vertices, there is only one k for which k-regular tournaments exist.

Exercise 465. *Show that if T is a k-regular tournament on n vertices, then $n = 2k+1$, and so each vertex also has indegree k.*

Vertex-pancyclic tournaments occurred in Theorem 12.7.10. The next result for is for "edge-pancyclic" tournaments.

Theorem 12.7.12 (Alspach, 1967 [39])**.** *In a regular tournament on $n = 2k + 1$ vertices, through any arc there is a directed cycle of every possible length from 3 to n.*

Exercise 466. *Show that Theorem 12.7.12 is true for $n = 3$ and $n = 5$.*

So every regular tournament has at least one Hamiltonian cycle passing through any given edge. P. J. Kelly (see [697]) conjectured that any regular tournament can be decomposed into Hamiltonian cycles. As of 2008 (see [40]), this conjecture was still open.

Theorem 12.7.13 (Thomassen, 1982 [916]). *If T is a tournament on n vertices where the outdegrees and indegrees of all vertices are at least $\frac{n}{2} - (n/1000)^{1/2}$, then T is hamiltonian.*

Alspach gave an example of an "almost regular" tournament that is not edge-pancyclic:

Example 12.7.14 (Alspach, 1967 [39]). *Let $T = (V, A)$ be a tournament on $2k$ vertices $v_0, v_1, \ldots, v_{2k-1}$, where A consists of the following arcs:*

$$\forall i = 0, 1, \ldots, k-1,\ (v_i, v_{i+1}), (v_i, v_{i+2}), \ldots, (v_i, v_{i+k}) \in A.$$

and, using addition modulo $2k$ for subscripts,

$$\forall i = k+1, \ldots, 2k-1,\ (v_i, v_{i+1}), \ldots, (v_i, v_{i+n-1}) \in A.$$

Then (v_{k-1}, v_k) is not in any 3-cycle.

12.7.4 Landau's theorem on kings

Unless otherwise stated, all tournaments in this section are finite.

Definition 12.7.15. *If T is a tournament, a vertex $v \in V(T)$ is a* king *if and only if for every other vertex $y \in V(T) \backslash \{v\}$, there is a directed path from v to y with at most two edges.*

So a king can be thought of a person in a tournament so that for those whom the king didn't beat directly, the king beat someone who beat them. The following was discovered by Landau while studying the pecking order of chickens.

Theorem 12.7.16 (Landau, 1953 [616]). *Every tournament has a king.*

Landau's theorem (often in disguise) appears in many graph theory texts and puzzle books. For example, in [286, 8.3, p. 207], it appears as a problem on one-way streets in Sikinia (wherever that is!). There are three ways to prove Theorem 12.7.16. The first proof given here consists of a procedure to identify a king,

Proof: Let $T = (V, E)$ be a tournament with $|V|$ finite, and consider some vertex x. If x is not a king, then there is some other vertex y so that there is no path of length at most two from x to y. This means that $(y, x) \in E$ and for every other $z \in V$, if $(x, z) \in E$, then $(y, z) \in E$. But this implies that $d^+(y) > d^+(x)$. So repeat this

argument replacing x with y, a vertex with higher outdegree. This process can not continue *ad infinitum*, and so when it stops, a king is located. □

Observe that in the above proof, the king produced has maximum outdegree. In fact, every maximum outdegree vertex is a king and this stronger result has an even easier proof (which then gives a second proof of a king's existence).

Exercise 467. *Show that in a tournament, any vertex with maximum outdegree is a king.*

Exercise 468. *Find a proof for Landau's theorem (Theorem 12.7.16) that uses induction on the number of vertices.*

Remark: The inductive proof asked for in Exercise 468 does not immediately reveal the much stronger statement of Exercise 467.

There may be kings that do not have maximum degree.

Exercise 469. *Give an example of a tournament with a king that is not a vertex of maximum outdegree.*

There exist tournaments with only one king; for example, if there is a vertex v so that all arcs containing v leave v, then v is the only king (since v is not reachable from any other vertex). If G is a tournament on n vertices, a vertex with $\deg^+(v) = n-1$ is called an *emperor*. An emperor in a tournament is trivially a king. (In the literature, an emperor is sometimes called a *transmitter*.)

Lemma 12.7.17. *In a tournament on at least 2 vertices, every vertex that is not an emperor is dominated by a king.*

So Lemma 12.7.17 says that in a tournament, for any vertex v with $\deg^-(v) \geq 1$, then v is dominated by a king.

Exercise 470. *Prove Lemma 12.7.17.*

Exercise 471. *Let T be a tournament on at least three vertices with no emperor (i.e., every vertex is dominated by at least one other). Show that T contains at least two kings.*

Exercise 472. *Show that no tournament has exactly two kings.*

As a result of Exercises 471 and 472, it follows that:

Theorem 12.7.18 (see [864] or [493, p. 295])**.** *Let T be a tournament on $n \geq 3$ vertices with no emperor (i.e., for every vertex, $\deg^-(v) > 0$). Then T contains at least 3 kings.*

In a tournament on 2 vertices, exactly one vertex is a king. In a cyclic tournament on 3 vertices, every vertex is a king. One can check Figure 12.4 to observe that there are no tournaments on 4 vertices where every vertex is a king. The following general theorem by Maurer says just how many kings are possible.

Theorem 12.7.19 (Maurer, 1980 [661])**.** *If $1 \leq k \leq n$, there is a tournament on n vertices with precisely k kings—except when $k = 2$ or $n = k = 4$.*

For an inductive proof of Theorem 12.7.19, see [785].

Corollary 12.7.20. *For every $n \geq 5$, there are tournaments on n vertices where every vertex is a king.*

In fact, Reid [785] first proved Corollary 12.7.20 as kind of a base case to show Theorem 12.7.19, starting off with a "all kings" tournament on k vertices, then adding $n - k$ new vertices (none of which are kings) to produce the desired tournament with exactly k kings.

Exercise 473. *Find a tournament on 5 vertices so that every vertex is a king. Hint: K_5 is Eulerian.*

In a tournament, a *serf* [661] is a vertex v so that v is reachable from any other vertex by a directed path of length at most 2. So if v is a king in a tournament T, and T^{-1} denotes the tournament obtained by reversing the arcs in T, then v is a serf in T^{-1}, so every tournament contains a serf. So theorems about kings have analogous results for *serfs*. There are results about how many kings *and* serfs are possible in a tournament; the reader is referred to [784] for an introduction to such problems.

Chapter 13

Hypergraphs

13.1 Introduction

The first major textbook covering the basics of hypergraphs is by Berge, *Graphs and hypergraphs* [94], published in 1970 (in French), with the English translation in 1973. Since then, notations and definitions surrounding hypergraphs have evolved. The notations used here have many variants in the literature. For those interested in advanced applications of hypergraphs (including much of what is in this chapter), I highly recommend a small (but rich) book by Bollobás [122]. For a survey of extremal problems for hypergraphs, the reader might begin with a classic paper by Füredi [390].

As already given in Definition 6.13.3 a *hypergraph* G is a pair $G = (V(G), E(G))$ where $V(G)$ is a non-empty set (elements of which are called *vertices*) and $E(G) \subseteq \mathcal{P}([V(G)])$ is the set of *hyperedges* (often just called *edges*, but that term is more commonly reserved for 2-element hyperedges). Here, hypergraphs are usually finite, *i.e.*, $|V(G)|$ is finite. Sometimes both G and $E(G)$ are written in script style (to emphasize that, *e.g.*, $\mathcal{E}$ is a set of sets). One might refer to a hypergraph by $\mathcal{H}$ instead of $(X, \mathcal{H})$ or one might write $H = (X, \mathcal{E}(H))$.

By this definition, no hyperedge appears more than once in $E(G)$ (*i.e.*, for any $X \subseteq V(G)$, the set X determines at most one hyperedge). This is often expressed by saying that a hypergraph has no *multiple edges*, or that the hypergraph is *simple*. Unless otherwise specified, all hypergraphs here are considered to be simple. "Hypergraphs" that are not simple can be called "multihypergraphs".

By the definition of a hypergraph, any collection of distinct sets is a hypergraph, where the vertex set is the union of all the sets, and the hyperedges are the sets. Therefore, many (maybe most?) results in combinatorics can be expressed in terms of hypergraphs. The limited number of results contained in this chapter are only meant to introduce some theorems in combinatorics that might be easily thought of as hypergraph results and perhaps not ordinarily looked at in an elementary combinatorics course.

In a hypergraph, each hyperedge is considered to be an unordered set, quite unlike the case of directed graphs. If all edges in a hypergraph G contain k vertices, then G is called k-uniform. Occasionally, the expression "k-graph" is short for "k-uniform hypergraph". A *simple graph* is a 2-uniform hypergraph, or a 2-graph.

The theory of uniform hypergraphs applies to many finite geometries (where each line is a hyperedge and each point is a vertex), although this connection is only sparingly mentioned here. A few remarks here are made only to establish some parallels between graph theory and hypergraph theory. Since a graph is a 2-uniform hypergraph, many theorems for hypergraphs take into account what is known for graphs. In general, theorems for hypergraphs are harder to find and harder to prove, but some are mere duplications of those found in graphs.

As seen already in Section 6.13, for a hypergraph $H = (X, \mathcal{H})$, the chromatic number of H, denoted $\chi(H)$, is the least k such that there is a k-colouring of X so that no $H \in \mathcal{H}$ has all its vertices the same colour. The strong chromatic number $\chi_s(H)$ is the least k so there is a k-colouring so that every $H \in \mathcal{H}$ has all of its vertices coloured differently. The strong chromatic number of a graph is simply the chromatic number.

For $r \geq 2$ and $n \geq 1$, let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices. For this graph to have any hyperedges, $n \geq r$ is required. Then $K_n^{(r)}$ can be seen as $[n]^r$, the collection of all subsets of $[n]$ of cardinality r. Observe that $K_4^{(3)}$ has only four hyperedges, and $\chi(K_4^{(3)}) = 3$, whereas $\chi_s(K_4^{(3)}) = 4$.

Cycles in hypergraphs can be of many different types, but one definition applies to all.

Definition 13.1.1. *Let $H = (V, \mathcal{H})$ be a hypergraph, perhaps with multiple edges. For $k \geq 1$, a cycle of length k in H is an alternating sequence of vertices and edges of the form*

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = v_0,$$

where $v_0, v_1, \ldots, v_{k-1}$ are distinct vertices, $e_1, \ldots, e_k$ are distinct (hyper)edges, and for each $i = 1, \ldots, k$, both $v_{i-1}, v_i \in e_i$.

Under Definition 13.1.1, if two hyperedges e and f have two vertices x, y in their intersection, then x, e, y, f, x is a cycle of length 2. Recall from Definition 6.13.4 and the comments following, a hypergraph is called almost disjoint (or linear) if and only if any two hyperedges intersect in at most one vertex, so almost disjoint (simple) hypergraphs have no 2-cycles.

13.2 The Erdős–Ko–Rado theorem

In Section 6.13, linear hypergraphs are introduced, where any two hyperedges intersect in at most one vertex. In this section, hypergraphs are studied that have each pair of hyperedges intersecting in *at least* one vertex.

Definition 13.2.1. *A hypergraph is called* intersecting *if any two hyperedges intersect (in at least one vertex).*

For example, the uniform hypergraph given by the vertices and lines of a finite projective plane (FPP) is both linear and intersecting since in a FPP, any two lines intersect in a unique point.

If $k \geq 2$ and $n < 2k$, any two k-subsets of an n-set intersect, so any k-uniform hypergraph on n vertices is intersecting.

If $n = 2k$ and $H = (V, \mathcal{F})$ is a k-uniform hypergraph, if H is intersecting, then for each k-set $F \in [V]^k$, at most one of F or $V \backslash F$ is in $\mathcal{F}$, so $|\mathcal{F}| \leq \frac{1}{2}\binom{n}{k}$.

If $n > 2k$, what is the maximum number of hyperedges in an intersecting k-uniform hypergraph on n vertices? When $n > 2k$, one obvious example of an intersecting k-uniform hypergraph is formed by taking all k-sets that contain some particular vertex. In such a hypergraph, the number of hyperedges is $\binom{n-1}{k-1}$. The next theorem states that this value is optimal (which was, according to [310], published 23 years after its discovery). The original proof was by induction; however an easier proof is given here.

Theorem 13.2.2 (Erdős–Ko–Rado (EKR), 1938, 1961 [334])**.** *Let $k \geq 2$ and $n > 2k$. If $H = (V, \mathcal{H})$ is an intersecting k-uniform hypergraph on n vertices, then*

$$|\mathcal{H}| \leq \binom{n-1}{k-1}, \tag{13.1}$$

where equality holds if and only if there exists a vertex $x \in V$ such that $\mathcal{H}$ is the collection of all k-sets containing x.

Proof: One short proof of the inequality is due to Katona [556], which is given here. The proof when equality holds is left as an exercise (but follows from the Katona proof). This proof is also reproduced in, *e.g.*, [122] or [546].

Let $H = (V, \mathcal{H})$ be an intersecting k-uniform hypergraph, where $|V| = n$. Label the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$, and for each $i = 0, \ldots, n-1$ define the $I_i = \{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k-1}\}$ (where addition in subscripts is modulo n). Each such interval is a k-set, and so could be a hyperedge. How many intervals can be hyperedges? If, say, $I_0 \in \mathcal{H}$, then the intervals (other than I_0) that intersect I_0 are only the $2k-2$ intervals

$$I_{n-k}, I_{n-k+1}, \ldots, I_{n-1}, \text{ and } I_1, I_2, \ldots, I_{k-1}.$$

However, for each $i = n-k-1, \ldots, n-1$, the pair of intervals I_i, I_{i+k} does not intersect so at most one from each pair is a hyperedge; there are $k-1$ such pairs, and so at most k intervals can be hyperedges.

Let N be the number of pairs (σ, i), where $\sigma : V \to V$ is a permutation and $\sigma(I_i) = \{\sigma(v_i), \sigma(v_{i+1}), \ldots, \sigma(v_{i+k-1}\} \in \mathcal{H}$. Since any permutation has at most k intervals as hyperedges, $N \leq k \cdot n!$. Instead, counting from vertices, for each vertex

v_i, there are $k!$ ways to order $\sigma(I_i)$, and $(n-k)!$ ways to order remaining vertices, so $N = |\mathcal{H}| \cdot n \cdot k!(n-k)!$. Putting the two bounds on N together,

$$|\mathcal{H}| \leq \frac{kn!}{nk!(n-k)!} = \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

□

Exercise 474. *Complete the proof of Theorem 13.2.2 in the case of equality.*

For a recent (2016) book on algebraic aspects of EKR, see [421]. For some other aspects of intersecting hypergraphs, see [122, Sections 7, 11].

There are many results about bounds on the sizes of intersection; only a few are included here. For such results, a few preliminaries are useful. Recall that in a hypergraph $(X, \mathcal{F})$, for any vertex $x \in X$, the *degree* of x is $\deg(x) = |\{F \in \mathcal{F} : x \in F\}|$.

Lemma 13.2.3. *Let $\mathcal{F}$ be a finite family of subsets of some ground set X. Then*

(i) $\displaystyle\sum_{x \in X} \deg(x) = \sum_{F \in \mathcal{F}} |F|$.

(ii) For any $Y \subset X$, $\displaystyle\sum_{x \in Y} \deg(x) = \sum_{F \in \mathcal{F}} |F \cap Y|$.

(iii) $\displaystyle\sum_{x \in X} (\deg(x))^2 = \sum_{F \in \mathcal{F}} \sum_{x \in F} \deg(x) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|$.

The proof of Lemma 13.2.3 is left as Exercise 475.

Exercise 475. *Prove Lemma 13.2.3.*

The following can be stated in terms of r-uniform hypergraphs with N hyperedges where the cardinality of the intersection of any two hyperedges is bounded from above. It says (roughly) that if a r-uniform hypergraph has many edges, any two of which have small intersection, then the vertex set is large.

Theorem 13.2.4 (Corrádi, 1969 [229])**.** *Let $r, k, N \in \mathbb{Z}^+$ with $k \leq r$. Let X be a set and let $S_1, \ldots, S_N \in [X]^r$ so that for all $i \neq j$, $|S_i \cap S_j| \leq k$. Then*

$$|X| \geq \frac{r^2 N}{r + (N-1)k}.$$

Proof: For each $i = 1, \ldots, N$,

$$\begin{aligned}
\sum_{x \in S_i} \deg(x) &= \sum_{j=1}^{N} |S_i \cap S_j| && \text{(by Lemma 13.2.3)} \\
&= |S_i| + \sum_{i \neq j} |S_i \cap S_j| \\
&\leq r + (N-1)k.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^{N} \sum_{x \in S_i} \deg(x) &= \sum_{x \in X} (\deg(x))^2 && \text{(by Lemma 13.2.3)} \\
&= |X| \sum_{x \in X} (\deg(x))^2 \cdot |X| \cdot \frac{1}{|X|^2} \\
&= |X| \sum_{x \in X} (\deg(x))^2 \cdot \sum_{x \in X} \frac{1}{|X|^2} \\
&\leq \left(\sum_{x \in X} \deg(x) \frac{1}{|X|} \right)^2 && \text{(by Cauchy–Schwarz)} \\
&= \frac{1}{|X|} \left(\sum_{x \in X} \deg(x) \right)^2 \\
&= \frac{1}{|X|} \left(\sum_{i=1}^{N} |S_i| \right)^2 && \text{(by Lemma 13.2.3)} \\
&= \frac{1}{|X|} (Nr)^2.
\end{aligned}$$

Combining both above results,

$$|X| \geq \frac{(Nr)^2}{\displaystyle\sum_{i=1}^{N} \sum_{x \in S_i} \deg(x)} \geq \frac{(Nr)^2}{\displaystyle\sum_{i=1}^{N} (r + (N-1)k)} = \frac{Nr^2}{r + (N-1)k},$$

as desired. □

13.3 Sperner's lemma, LYM inequality

Definition 13.3.1. *A family $\mathcal{F}$ of sets is called a Sperner family if and only if no set in $\mathcal{F}$ is contained in another.*

If V is a non-empty set and $\mathcal{P}(V)$ denotes the collection of all subsets of V, then $(\mathcal{P}(V), \subseteq)$ is a partial order, called the inclusion lattice for subsets of V. A Sperner family of subsets of V is then an antichain in this partial order.

If $\mathcal{F}$ is a collection of sets all with the same cardinality, then no set is contained in any other, and so $\mathcal{F}$ is Sperner. Since the binomial coefficient $\binom{n}{k}$ is maximized when $k = \lfloor n/2 \rfloor$, the maximum size of a Sperner family of subsets of an n-element set is at least $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$. Sperner's theorem says that this natural bound for a Sperner family is indeed optimal (even when sets in $\mathcal{F}$ are possibly of different sizes).

Theorem 13.3.2 (Sperner, 1928 [880]). *If V is a set of n elements, and $\mathcal{F}$ is a Sperner family of subsets of V, then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

There have been many proofs of Sperner's theorem (also called Sperner's lemma). One short proof of Sperner's theorem is derived immediately from a slightly more general result (Theorem 13.3.3, below) called the "LYM inequality", proved independently by four authors, Bollobás (1965, [120]), Lubell (1966, [647]), Meshalkin (1963, [678]), and Yamamoto (1954, [1002]).

Theorem 13.3.3 (LYM inequality). *Let $\mathcal{A}$ be a Sperner family of subsets of an n-element set. Then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Proof: The proof given here is one that Lubell published [647] in a single page paper in the first year of *Journal of Combinatorial Theory* (this paper was also reprinted in *Classic papers in combinatorics*).

Let $\mathcal{A}$ be a Sperner family of subsets of S, where $|S| = n$. Consider the inclusion lattice $L = (\mathcal{P}(S), \subseteq)$ for subsets of S. The number of maximal chains in L (from $\emptyset$ to S) is $n!$ (where each such chain corresponds to a permutation of elements of S).

Since $\mathcal{A}$ is Sperner, each maximal chain contains at most one $A \in \mathcal{A}$. For any fixed $A \in \mathcal{A}$, there are $|A|!(n - |A|)!$ maximal chains containing A, and thus

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!.$$

Multiplying both sides by $\frac{1}{n!}$ finishes the proof. □

Exercise 476. *Show that the LYM inequality implies Sperner's theorem.*

By using authors in alphabetical order, the LYM inequality might be better called the BLMY inequality (or, using order of publication, the YMBL inequality). One

reason that the LYM inequality is not called the" BLMY inequality" might be that (according to Tuza [944, p. 494]), nobody seemed aware that a lemma in the Bollobás 1965 paper [120] (given below as Lemma 13.3.4) implied the LYM inequality, until this was observed by Tuza [943] in 1984. In Bollobás' wonderful 1986 book *Combinatorics: set systems, hypergraphs, families of vectors and combinatorial probability* [122], Lemma 13.3.4 (below) appears as "Theorem 2" (pp. 63–64); after its proof, Bollobás then writes [122, p. 66]:

> In the original paper (Bollobás (1965)) Theorem 2 appears as a lemma since its statement is not too attractive and, as above, it was used to prove Theorem 1. Because of its numerous applications, in the presentation above we upgraded it to a theorem. In fact, Theorem 2 is not as esoteric as it seems at first glance—just the contrary, it belongs to the mainstream of the theory of set systems. It not only contains Sperner's theorem but it is also a considerable extension of the celebrated LYM inequality! Indeed, on putting $B_i = \emptyset$ for all i in Theorem 2 the condition becomes that $\{A_i : i \in I\}$ is a Sperner system and we obtain precisely the LYM inequality, Theorem 3.2.

Here is the lemma that Bollobás gave in 1965 (using the original notation) that was used to prove a result regarding saturated hypergraphs (or τ-critical hypergraphs).

Lemma 13.3.4 (Bollobás, 1965 [124]). *Let I denote an index set. For every $i \in I$, A_i and B_i are subsets of a set P with p elements satisfying the following conditions:*

1. $A_i \cap B_i = \emptyset$.
2. $A_i \not\subset A_j \cup B_j$, *if* $i \neq j$.

If there are a_i and b_i elements in A_i and B_i, respectively, then

$$\sum_{i \in I} \frac{1}{\binom{p-b_i}{a_i}} \leq 1,$$

with equality if and only if $B_i = B$ for all $i \in I$ and the sets A_i are the q-tuples of the set $P - B$ for some value of q.

The proof of Lemma 13.3.4 given by Bollobás was by induction. The following version of Lemma 13.3.4 appeared in, *e.g.*, [943].

Theorem 13.3.5 (Bollobás, 1965 [120]). *Let X be a set. Let $A_1, \ldots, A_k \in [X]^a$ be distinct a-subsets of X and let $B_1, \ldots, B_k \in [X]^b$ be distinct b-element subsets of X. If for each $i = 1, \ldots, k$,*

$$A_i \cap B_i = \emptyset, \tag{13.2}$$

and

$$\text{for every } j \neq i, \quad A_i \cap B_j \neq \emptyset, \tag{13.3}$$

then $k \leq \binom{a+b}{a}$.

Proof: (Another proof of this theorem can be found in [557].) Let (13.2) and (13.3) hold, and let $X = [n]$ be the union of all the A_i's and B_i's. For any two subsets $C \subseteq X$ and $D \subseteq X$, and any permutation $\pi : X \to X$, write $\pi(C) < \pi(D)$ if and only if every element of C appears before any element of D in π. Let $\mathcal{S}_n$ denote the set of all permutations on $[n]$. For each $i = 1, \ldots, k$, define classes of permutations on $[n]$,

$$\Pi_i = \{\pi \in \mathcal{S}_n : \pi(A_i) < \pi(B_i)\}.$$

Claim: Each permutation $\pi \in \mathcal{S}_n$ is in at most one class.
Proof of Claim: Suppose for the moment that this is not the case, say with $\pi \in \Pi_i \cap \Pi_j$, that is, both $\pi(A_i) < \pi(B_i)$ and $\pi(A_j) < \pi(B_j)$. If $\max \pi(A_i) \leq \max \pi(A_j)$, say, then $\pi(A_i) < \pi(B_j)$; but by (13.3), $A_i \cap B_j \neq \emptyset$, a contradiction. So the classes are disjoint as claimed.

Thus

$$\begin{aligned} n! &\geq \sum_{i=1}^{k} |\Pi_i| \\ &= \sum_{i=1}^{k} \binom{n}{a+b} a!b!(n-a-b)! \\ &= \frac{n!}{\binom{a+b}{a}}, \end{aligned}$$

and so

$$1 \geq \sum_{i=1}^{k} \frac{1}{\binom{a+b}{a}}. \tag{13.4}$$

from which it follows that $k \leq \binom{a+b}{a}$. □

Following the same proof as above, a version of Theorem 13.3.5 is available when the A_is and B_is are of possibly different sizes:

Theorem 13.3.6. *Let I be a finite index set, and let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be collections of finite sets so that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i \in I} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1,$$

with equality if and only if there is a set Y and integers $0 \leq a \leq a + b \leq |Y|$ and $\{(A_i, B_i) : i \in I\}$ is the collection of all ordered pairs of disjoint subsets of Y where for each i, $|A_i| = a$ and $|B_i| = b$.

In Theorem 13.3.6, the size of the ground set is not used in the conclusion.

The pairs of sets $\{(A_i, B_i) : i \in I\}$ satisfying $A_i \cap B_j = \emptyset$ if and only if $i = j$ are sometimes called "strong intersecting pair systems" (see [944]). (Such families are not examples of "cross-intersecting families" because for each A_i, there is one B_j, namely B_i, that does not intersect A_i.) The introduction of such systems by Bollobás has led to many powerful techniques in solving extremal hypergraph problems; the reader is strongly recommended to see [122] (or [389], [944]) for more details.

13.4 Packing or covering uniform sets

For $2 \le r < k < n$, let $m(n, k, r)$ be the size of a maximal family $\mathcal{F}$ of k-element subsets of $[n]$ so that each r-set in $[n]$ is contained in at most one k-set in $\mathcal{F}$. Evidently,

$$m(n, k, r) \le \frac{\binom{n}{r}}{\binom{k}{r}}.$$

Conjecture 13.4.1 (Erdős–Hanini, 1963 [332])**.**

$$\lim_{n\to\infty} \frac{m(n, k, r)}{\frac{\binom{n}{r}}{\binom{k}{r}}} = 1.$$

In 1985, Rödl [803] introduced a method, now called "the Rödl nibble", to prove the Erdős–Hanani conjecture.

For parameters n, q, r, an (n, q, r)-Steiner system is collection S of q-subsets of an n-set X so that every r-subset of X belongs to exactly one element of S. Without going into too many details, the cases when $r = 2$ are studied in the theory of balanced incomplete block designs. For a chosen set of parameters n, q, r certain divisibility conditions are required for an (n, q, r)-Steiner system to exist. In 2014, Peter Keevash [558] gave a remarkable proof that, almost always, these divisibility conditions are sufficient, answering a question of Steiner posed in 1853.

For more on the history of this problem, see [988] and the many references in Keevash's paper [558]. Keevash formulated the existence problem in terms of clique decompositions of certain hypergraphs.

13.5 Ryser's conjecture

A hypergraph $H = (X, \mathcal{H})$ is called k-partite if and only if there is a partition $X = X_1 \cup X_2 \cup \cdots \cup X_k$ so that for each $i = 1, \ldots, k$ any hyperedge in $\mathcal{H}$ intersects X_i in at most one vertex. Recall that $\tau(G)$ is the transversal number of H, the minimum number of vertices that intersect all hyperedges, and $\nu(H)$ is the matching number of

H, the maximum number of pairwise disjoint hyperedges. The following first appeared in the 1971 PhD thesis of Henderson, a student of Herbert John Ryser.

Conjecture 13.5.1 (Ryser)**.** *Every r-partite hypergraph H satisfies*

$$\tau(H) \leq (r-1)\nu(H).$$

In particular, Ryser's conjecture says that if H is an intersecting hypergraph, then $\tau(H) \leq r-1$.

Note that the case $r = 2$ is answered by König's theorem (here called the König–Egeváry theorem), Theorem 5.6.1, where equality is shown. Aharoni [13] proved Ryser's conjecture for $r = 3$ (and any ν), but aside from his result, only isolated cases have been solved. For $r = 4, 5$, approximate versions of Ryser's conjecture were shown to be true by Haxell and Scott [501] in 2012. For more information on Ryser's conjecture and related problems, see [942] or [389].

13.6 Property B

Definition 13.6.1. *For a set X, a family $\mathcal{F}$ of subsets of X has Property B if and only if there exists $S \subset X$ so that for each $F \in \mathcal{F}$,*
$S \cap F \neq \emptyset$ and
(ii) $S \not\subset S$.

In 1937, Miller [683] coined the expression "Property B", where B was in honour of Felix Bernstein who showed in 1908 [98] that perfect sets (closed and no lsolated vertices) have Property B. Using one colour for S and another for $X \backslash S$, the definition can be restated:

Definition 13.6.2. *A family $\mathcal{F}$ of subsets of a set X has* Property B *iff there exists a 2-colouring of X so that no $F \in \mathcal{F}$ is monochromatic.*

If $\mathcal{F}$ has Property B and $\mathcal{E} \subseteq \mathcal{F}$, then $\mathcal{E}$ also has property B. Adding more sets to $\mathcal{F}$ might destroy Property B.

A family of sets, each of which has cardinality r is called r-uniform. In particular, an r-uniform family $\mathcal{F}$ can be interpreted as an r-uniform hypergraph H with vertex set $\cup_{F \in \mathcal{F}} F$, where each $F \in \mathcal{F}$ is a hyperedge. The thrust of the research regarding property B is for uniform families; see, *e.g.*, [79] for other families. Many of the results below were originally given using set systems language, but they have direct analogues in r-uniform hypergraph language.

By definition, an r-uniform hypergraph $H = (V, \mathcal{E})$ has property B if there is a partition of V into two colour classes so that no hyperedge is monochromatically coloured. In a sense, an r-uniform hypergraph with Property B is "bipartite".

Sometimes Ramsey results can be stated in terms of Property B. For example, the famous "van der Waerden's theorem" [953] states that for positive integers k and r, there exists a least number $n = W(k : r)$ so that if the integers $\{1, 2, \ldots, n\}$ are partitioned into r colour classes, then some arithmetic progression of length k is monochromatic. The existence of $W(k; 2)$ shows that the family of k-term arithmetic progressions does not have Property B.

Erdős and Hajnal [326] studied Property B, (even for infinite families), and defined $m(r)$ to be the minimum number of sets in an r-uniform family that does not have Property B. Trivially, $m(1) = 1$. The set system $\{\{1, 2\}, \{1, 3\}, \{2, 3\}$, shows that that $m(2) = 3$. Erdős and Hajnal also found that the set system (using addition in $\mathbb{Z}_7$) $\{\{i, i + 1, i + 4\} : i \in \mathbb{Z}_7\}\}$

$$= \{\{0, 1, 4\}, \{1, 2, 5\}, \{2, 3, 6\}, \{3, 4, 0\}, \{4, 5, 1\}, \{5, 6, 2\}, \{6, 0, 3\}\}$$

which is a Steiner triple system, does not have Property B, and so $m(3) \leq 7$. They then checked cases to show that $m(3) > 6$, and so concluded that $m(3) = 7$. It turns out that $m(4)$ is difficult to compute. As a general (but weak) bound, Erdős and Hajnal [326] observed that $m(r) \leq \binom{2r-1}{r}$ (since the r-subsets of a $(2r-1)$-set has property B). They also worked on the problem where additional restrictions on intersections were given.

In a series of papers written in the 1960s, Erdős [296], [301], [304] added more understanding of $m(r)$. In [296] (which was written while at the University of Alberta, Edmonton) Erdős showed that for $r \geq 2$, $m(r) \geq 2^{r-1}$, (see Theorem 13.6.3 below) and for each $\epsilon > 0$, there exists $r_0 = r_0(\epsilon)$ so that for $r \geq r_0$, $m(r) > (1 - \epsilon)2^r \ln 2$. Erdős used a very simple probabilistic argument to show that for uniform families if a family has "too few" members, then the family has Property B.

Theorem 13.6.3 (Erdős, 1963 [295])**.** *Let $\mathcal{F}$ be a family of m subsets of some set S, and for each $F \in \mathcal{F}$, $|F| = k$. If $m < 2^{k-1}$, then $\mathcal{F}$ has property B.*

Proof: Let the vertices of S be independently and uniformly 2-coloured at random. For each $F \in \mathcal{F}$, let A_F be the event that F is monochromatic. Then $\mathbb{P}[A_F] = 2 \cdot (\frac{1}{2})^k = 2^{1-k}$, and so by the union bound,

$$\mathbb{P}\left[\vee_{F \in \mathcal{F}} A_F\right] \leq \sum_{F \in \mathcal{F}} \mathbb{P}[A_F] = m \cdot 2^{1-k}.$$

Thus, if $m < 2^{k-1}$,

$$\mathbb{P}\left[\vee_{F \in \mathcal{F}} A_F\right] < 1,$$

or

$$\mathbb{P}\left[\overline{\vee_{F \in \mathcal{F}} A_F}\right] > 0,$$

and so there exists a colouring so that no $F \in \mathcal{F}$ is monochromatic, witnessing that $\mathcal{F}$ has Property B. $\square$

As noted in [301] in 1964, Schmidt [834] showed that $m(r) > 2^r(1 + 4r^-1)^{-1}$, the best lower bound at that time. Around the same time, Abbott and Moser [2] showed

$$m(a \cdot b) \leq m(a) \cdot m(b)^a.$$

Also in [301], Erdős used the probabilistic method to show $m(r) \leq r^2 \cdot 2^{r+1}$. In [304], Erdős considered the added restriction on the size of the ground set.

In 1969, Abbott and Hanson [1] showed that $m(4) \leq 24$, $m(5) \leq 51$, and $m(7) \leq 421$. In 1974, Seymour [852] slightly improved the first bound to $m(4) \leq 23$ by giving an example on 11 vertices (this bound was also found by Toft [925] a year later). In 1995, Manning [651] showed that $m(4) \geq 21$. Finally, in 2014, after an exhaustive search, Östergård [732] showed $m(4) = 23$.

The best known bounds for $m(r)$ are of the form (where c and c' are universal constants)

$$c(r/lnr)^{1/2} < m(r) < c'r^2 2^r,$$

where the lower bound is due to Radhakrishman and Srinivasan [767], and the upper bound is due to Erdős [301].

See also [753] on greedy colourings of hypergraphs.

Chapter 14

The reconstruction conjecture

14.1 Vertex-deletion version

Let $G = (V, E)$ be a graph (unlabelled). For any vertex $v \in V$, the *vertex-deleted subgraph* $G - x$ is the subgraph of G induced by $V \backslash \{x\}$. The *deck* of G is the set of all vertex-deleted subgraphs, denoted by

$$\text{deck}(G) = \{G - v : v \in V\}.$$

If $\{H_i; i = 1, \ldots, n\}$ is the deck of some graph G, can the original graph G be constructed from the H_is by a sequence of logical steps? If no two graphs (on n vertices, say) have the same deck, then any deck "determines" precisely one graph.

If a deck determines precisely one graph G, then G is said to be *reconstructible*. For example, both K_2 and $2K_1$ (the graph with two isolated vertices) are not reconstructible, since they both have the same deck, each consisting of two non-adjacent vertices.

The following is sometimes called "Ulam's conjecture", or the "Kelly–Ulam conjecture" first posed by P. J. Kelly [560] in 1942 and by Ulam in [946] in 1960.

Conjecture 14.1.1 (Reconstruction conjecture)**.** *If G and H are two finite graphs on at least three vertices and have the same deck, then G and H are isomorphic.*

Exercise 477. *For any $n \geq 1$, show that the complete graph K_n is reconstructible.*

The reconstruction conjecture has been shown to be true for all graphs on at most 11 vertices (see [667]). The reconstruction conjecture is (as of 2021) still open, but graphs of certain types have been found to be reconstructible (from their decks); see, *e.g.*, the surveys [141], [138], [488], or [713]. The literature on the reconstruction conjecture is quite extensive; only some highlights are given here.

A property (or parameter) of a graph G is called *reconstructible* (or *recognizable*) if and only if this property can be derived from the deck of G. For example, the number

of vertices in G is (trivially) reconstructible by adding 1 to the number of vertices to any vertex-deleted subgraph of G. The number of edges and the degree sequence are also reconstructible by the next two lemmas.

Lemma 14.1.2. *Let $G = (V, E)$ be a graph on $n = |V| \geq 3$ vertices with deck $\{G_i : i = 1, \ldots, n\}$. Then*

$$|E(G)| = \frac{1}{n-2} \sum_{i=1}^{n} |E(G_i)|. \tag{14.1}$$

Proof: Each edge of G appears in $n - 2$ of the G_is (since any one edge is deleted in precisely two vertex-deleted subgraphs). □

Lemma 14.1.3. *For a graph G on at least 3 vertices, the degree sequence of G can be derived from the deck of G.*

Proof: For some $n \geq 3$, let G be a graph on vertices $V(G) = \{v_1, \ldots, v_n\}$. For each v_i, $\deg_G(v_i) = |E(G)| - |E(G - v_i)|$. □

For any graphs G and H with $H \subseteq G$, let $s(H, G)$ denote the number of copies of H in G, i.e., the number of weak subgraphs of G isomorphic to H.

Exercise 478. *If P_2 denotes the path of length 2, verify that $s(P_2, K_4) = 12$.*

Lemma 14.1.4 (Kelly's lemma, 1957 [561])**.** *If deck$(G) = \{H_1, \ldots, H_n\}$, then for any subgraph F of G with $|V(F)| < |V(G)|$,*

$$s(F, G) = \frac{\sum_{i=1}^{n} s(H_i, G)}{|V(G)| - |V(F)|}.$$

Exercise 479. *Prove Kelly's lemma.*

The next exercise is a classic example given by Harary.

Exercise 480 (Harary, 1969 [486, Prob 2.2])**.** *Suppose that G has five vertices and its deck consists of the following 4-vertex induced subgraphs:*

- *$G_1 = K_4 - e$ (that is, K_4 minus an edge, giving 5 edges);*
- *$G_2 = P_2 \cup K_1$ (the disjoint union of a path of length two and an isolated vertex);*
- *$G_3 = K_{1,3}$;*
- *$G_4 = G_5 = K_{1,3} + e$ (an edge added to $K_{1,3}$ (giving a copy of K_3 with pendant edge attached).*

Find G. Hint: Kelly's lemma can be used, but none of the above lemmas are necessary—a direct proof is also available. First find the number of edges in G, and then search among all graphs with five vertices with the proper number edges (there are six such).

Theorem 14.1.5 (Kelly, 1957 [561]). *Regular graphs are recognizable.*

Proof: For some $k \in \mathbb{Z}^+$, let G be a k-regular graph on at least 3 vertices. By Lemma 14.1.3, the degrees of G are reconstructible. and so the property that G is k-regular is reconstructible from the deck of G. □

Kelly also showed that regular graphs are reconstructible.

Exercise 481. *Show that regular graphs are reconstructible.*

Theorem 14.1.6 (Kelly, 1957 [561]). *Disconnected graphs are reconstructible.*

Theorem 14.1.7 (Kelly, 1957 [561]). *Trees are reconstructible.*

The next exercise has a simple solution; see [586] for more related details.

Exercise 482. *Show that a graph is reconstructible if and only if its complement is reconstructible.*

Recall from Definition 4.1.11 that a graph F is *nonseparable* if and only if $\kappa(F) > 1$.

Theorem 14.1.8 (Tutte, 1976 [941]). *For any graph G, the number of nonseparable subgraphs of G is reconstructible.*

Theorem 14.1.9 (Bondy, 1969 [134]). *Separable graphs with no endpoints are reconstructible.*

Recall (see Definition 4.1.10) that for a graph G, the number $\kappa(G)$ is the connectivity of G.

Theorem 14.1.10 (see [141]). *For a graph G, the value $\kappa(G)$ is reconstructible.*

Using probability, a somewhat surprising result was found (that also might give strong evidence for the reconstruction conjecture):

Theorem 14.1.11 (Bollobás, 1990 [123]). *Almost all graphs can be reconstructed from any three members of its deck.*

The proof of Theorem 14.1.11 relies on a lemma using probability, similar to a lemma proved a few years earlier (but apparently unknown to Bollobás):

Lemma 14.1.12 (Müller, 1976 [704]). *Let $\varepsilon \in (0,1)$. For almost every graph, the induced subgraphs containing $\frac{(1+\varepsilon)}{2}|V(G)|$ vertices have no nontrivial isomorphisms and are pairwise non-isomorphic.*

For more on the minimum number of vertex-deleted subgraphs required to reconstruct certain graphs, see Section 14.2.

A graph is called a *cactus graph* if and only if every pair of cycles intersect in at most one vertex. (So a cactus graph can be constructed by replacing some edges of a tree by cycles.)

Theorem 14.1.13 (Geller–Manvel, 1969 [413]). *Cactus graphs are reconstructible.*

So, for example, graphs with only one cycle are reconstructible.

It is not known (see, *e.g.*, [141]) if bipartite graphs are reconstructible (they are recognizable, though).

In 1988, Yang [1003] showed that to prove the reconstruction conjecture, it suffices to prove the conjecture for 2-connected graphs.

It is not known if planar graphs are recognizable or reconstructible. Digraphs are not reconstructible (see [890, 891]); however, Harary and Palmer [494] showed that if a tournament T is not strongly connected then T is reconstructible. (Recall that byTheorem 12.7.11, almost all tournaments are strongly connected.) In general, hypergraphs are not reconstructible—in 1987, Kocay [583] (from U. of Manitoba, Computer Science, now retired) gave infinitely many examples of 3-uniform hypergraphs that are not reconstructible.

The reconstruction conjecture is not true for infinite graphs (even for infinite forests); see, *e.g.*, [915] and [54].

14.2 Graph reconstruction numbers

The *reconstruction number* of a graph G is the minimum number of cards in deck(G) required to reconstruct G (see [495]). This section contains only a few facts about reconstruction numbers and a few references for further research.

In 1985, Harary and Plantholt [495] conjectured that the reconstruction number of almost all graphs is 3; as in Theorem 14.1.11, Bollobás proved this conjecture (the reader is reminded that the Bollobás result is much stronger, in that for almost all graphs, *any three* members of its deck will do).

Exercise 483. *Let $G = 3K_2$, the graph on 6 vertices consisting of three disjoint edges. Show that the reconstruction number of G is 4.*

Exercise 484. *Prove that any graph (on at least 3 vertices) cannot have reconstruction number 2.*

Conjecture 14.2.1 (Harary–Plantholt, 1985 [495]). *The reconstruction number of a graph on n vertices is at most $\frac{n}{2}+1$.*

Only two more results are given in this section.

Theorem 14.2.2 (Myrvold, 1989 [708]). *If G is a disconnected with graph made up of c isomorphic components, then the reconstruction number of G is $c+2$.*

Theorem 14.2.3 (Myrvold, 1989 [708]). *The reconstruction number of a disconnected graph with at least two non-isomorphic components is 3.*

The reconstruction numbers defined above are called *existential* or *ally* reconstruction numbers. The minimum number k so that *any* k cards can be used to reconstruct G is called a *universal* reconstruction number. . For example, the Bollobás result is about universal reconstruction numbers. Much work has been done on both; for more on reconstruction numbers, one might look at [68], [671], or [771],

14.3 Edge-deletion

Instead of vertex deletion, consider the *edge-deck*, the multiset of graphs formed by deleting each edge of a graph. A graph G is called *edge-reconstructible* if and only if the edge-deck of G determines a unique graph.

Conjecture 14.3.1 (Harary, 1964 [485]). *Every (simple) graph with at least four edges is edge-reconstructible.*

In 1972, Lovász [636] showed that the edge-reconstruction conjecture is true for graphs with more than half of the possible edges (i.e., for graphs on n vertices and more than $\frac{1}{2}\binom{n}{2}$ edges). In 1977, Müller [705] improved this minimum number of edges to $n(\log_2 n - 1)$.

Brendan McKay [667] confirmed that the edge reconstruction is true for graphs G on n vertices falling into one of the following categories:

- $n \leq 11$;
- $n \leq 12$ and $\Delta(G) \leq 5$;
- $n \leq 14$ and G is triangle-free;
- $n \leq 15$ and G contains no C_4.

Theorem 14.3.2 (Greenwell, 1971 [443]). *Let G be a graph with no isolated points and more than 3 edges. If G is vertex-reconstructible, then G is edge-reconstructible.*

Recall that a graph G is called "claw-free" if G contains no induced copy of $K_{1,3}$.

Theorem 14.3.3 (Ellingham–Pyber–Yu, 1988 [283]). *Claw-free graphs are edge-reconstructible.*

For more on the edge reconstruction conjecture, see, *e.g.*, [138].

14.4 Deleting more than one vertex

The standard deck of a graph is formed by deleting a single vertex. For each integer $k > 1$, a similar deck is formed by deleting k vertices.

Definition 14.4.1. *For a positive integer k, and a graph G on n vertices, let $D_{n-k}(G)$ be the family (perhaps multiset) of the $\binom{n}{k}$ graphs formed by deleting each set of k vertices from $V(G)$. Say that $D_{n-k}(G)$ is the $(n-k)$-deck of G.*

So the standard vertex-deletion deck of a graph G on n vertices is denoted by $D_{n-1}(G)$.

Chapter 15

Algebraic graph theory

The area of research called "algebraic graph theory" is primarily the study of graph properties in terms of a graph's adjacency (or incidence) matrix. There are other "algebraic" aspects of graph theory, like graphs from finite geometries, group actions, and automorphism groups, however this chapter (and in general, algebraic graph theory *per se*) is more about the adjacency matrix and its eigenvalues or eigenvectors.

Some basic results in algebraic graph theory have already been looked at in this text. For example, in Section 1.12.1 it was shown that raising an adjacency matrix to some power gives information about the number of walks between vertices (and so whether or not a graph is connected). In Chapter 5, adjacency matrices were used in proving theorems about perfect matchings. Trivially, counting degrees of vertices is just counting the number of 1s in a row or column, which can be seen as using a dot product of a row/column with an all 1s vector, and so can be done by a product with the all 1s matrix J (see Exercise 84).

It is somewhat remarkable that also knowing the eigenvalues and eigenvectors of an adjacency matrix reveals so much more about a graph. This chapter is only a very brief survey of some selected graph properties that can be revealed by looking at eigenvalues and eigenvectors.

For more thorough examinations of algebraic graph theory, see [74], [103], [165], [237], [422], and [882]. Modern algebraic graph theory also includes the study of the "Laplacian" (the matrix $D - A$, where D is the diagonal matrix with degrees and A is the adjacency matrix). The Laplacian is not stressed in these notes, but the interested reader might consult [212] or many of the above references to see the power of this branch of algebraic graph theory.

For reference, well-known facts about matrices used in this chapter are given in the appendix Chapter 17. Calculating by hand the eigenvalues and eigenvectors for larger graphs can often be very labour-intensive (and prone to small but critical errors), the reader might learn how to use some of the online matrix calculators or the more powerful programs like the open-source program *Sage* (available at `https://www.sagemath.`

org/). [Note to student: If some question is assigned by an instructor, please be forthright about any programs you used to produce or check a result.]

15.1 Eiigenvalues and graphs

The eigenvalues of an adjacency matrix for a graph do not depend on the ordering of vertices used in the adjacency matrix. Hence, one can talk about "an eigenvalue" of a graph G without confusion, meaning that the eigenvalue is one for an adjacency matrix for G. An adjacency matrix for any undirected multigraph is symmetric. If loops are not allowed (in either a directed or undirected (multi)graph), the main diagonal of any adjacency matrix has 0s down the main diagonal. The adjacency matrix for a simple graph has entries that are 0 or 1.

A remarkable result in matrix theory (Theorem 17.3.12) and Corollary 17.3.13) says that any symmetric real matrix can be found by only knowing its eigenvalues and associated eigenvalues. Adjacency matrices of (undirected) graphs or multigraphs are symmetric, and so:

Theorem 15.1.1. *Any multigraph is completely determined by the eigenvalues and eigenvectors of its adjacency matrix.*

Theorem 15.1.1 is a rather strong statement. It may then be no surprise that general graph properties can be derived by only looking at on the eigenvalues of the adjacency matrix. A large part of algebraic graph theory is called "spectral graph theory", since the set of eigenvalues for a matrix is often called its *spectrum*.

Some notation might be useful for listing eigenvalues of large matrices. If A is an $n \times n$ matrix and $\lambda_1, \ldots, \lambda_s$ are its distinct eigenvalues, where for each $i = 1, \ldots s$, the eigenvalue λ_i is repeated r_i times (so $r_1 + \cdots + r_s = n$), then the set of eigenvalues (with repetition now) is sometimes denoted

$$(\lambda_1)^{r_1}, (\lambda_2)^{r_1}, \ldots, (\lambda_s)^{r_s}.$$

Example 15.1.2. *Let G be the path on 3 vertices (called P_2 here), which is $K_{1,2}$. An adjacency matrix is (putting the central vertex first)*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then the determinant of $\lambda I - A$ is $\lambda^3 - 2\lambda$. Hence the spectrum of G is $-\sqrt{2}, 0, \sqrt{2}$.

The exercises in this section are intended to be done without the use of mathematical software.

Exercise 485. *Find the eigenvalues and eigenvectors for P_3, the path of length 3 (on 4 vertices).*

Exercise 486. *Find the spectra of*
(a) the bowtie graph;
(b) $K_4 - e$;
(c) the bull graph.

Exercise 487. *Find the eigenvalues and eigenvectors of K_n. Hint: Look at Example 17.2.2 for K_3.*

Exercise 488. *Find the eigenvalues and eigenvectors for $S_n = K_{1,n-1}$ (a star with one central vertex and $n-1$ leaves). Hint: Rather than do the algebra behind finding the characteristic polynomial, look for obvious eigenvectors first.*

A generalization of Exercise 488 is:

Exercise 489. *Find the eigenvalues and eigenvectors for $K_{m,n}$.*

Exercise 490. *Find the eigenvalues and eigenvectors for the cycle C_n.*

Exercise 491. *Let d be a positive integer and let $G = Q_d$ be the d-dimensional cube graph, and let A be an adjacency matrix for G. Prove that the set of eigenvalues for A are $\{d - 2k : k = 0, 1, \ldots, d\}$, where the eigenvalue $d - 2k$ has multiplicity $\binom{d}{k}$. For each k, find an eigenvector for $d - 2k$.*

Exercise 492. *Show that the spectrum of $L(K_n)$ is $(2n-4)^1, (n-4)^{n-1}, (-2)^{-\frac{1}{2}n(n-3)}$.*

Exercise 493. *Let q be a prime power with $q \equiv 1 \pmod 4$, and let P_q denote the Paley graph whose vertex set is the finite field $\mathbb{F}_q$. Writing $q = 4t + 1$, show that the spectrum of P_q is*

$$(2t)^1, (-\frac{1}{2}(1 - \sqrt{4t+1}))^{2t}, (-\frac{1}{2}(1 + \sqrt{4t+1}))^{2t}.$$

For an extensive list (hundreds) of graphs and their characteristic polynomials or spectra see the appendices in [237]. Here is just a sample of a few graphs given there:

- The barbell graph (on 6 vertices—see Figure 1.31) has characteristic polynomial $x^6 - 7x^4 - 4x^3 = 11x^2 + 12x + 3$.
- The pentagonal wheel graph W_5 has characteristic polynomial $x^6 - 9x^4 - 8x^3 + 9x^2 + 8x - 1$.
- The hexagonal prism graph has spectrum $-3, (-2)^2, -1, (0)^4, 1, (2)^2, 3$.
- The graph of the icosahedron has characteristic polynomial $(x+1)^5(x^2-5)^3(x-5)$.

The interested reader is invited to verify one or two of the above. Also in [237] are a number of small multigraphs and their spectra.

Exercise 494. *Let G be the cubic multigraph with two vertices connected by three edges. Show that the spectrum of G is -3, 3.*

Figure 15.1: cospectral trees on 8 verices

15.2 Co-spectral graphs

A pair of graphs with the same spectrum are called *co-spectral* or *isospectral* graphs. ("co-spectral" is often spelled without the hyphen). It was conjectured that there are no co-spectral pairs, but in 1957, Collatz and Sinogowitz [225, p. 72] found the pair of co-spectral trees on 8 vertices given in Figure 15.1.

Without going into detail, work on co-spectral graphs was also motivated by problems in physics. For example, the in 1966, Baker published "Drum shapes and isospectral graphs" [66]. (For more on applications of spectral theory to quantum theory, chemistry, and physics, see [237, Ch. 8].)

Exercise 495. *Show that the following two graphs, given in 1966 by Baker, [67] are cospectral.*

Thee energetic reader can confirm that are no co-spectral pairs of graphs on 4 or fewer vertices. However, there are two co-spectral graphs on 5 vertices, as shown in Exercise 496, given by Cvetković in 1971 [236]:

Exercise 496. *Let G be the graph on 5 vertices consisting of a C_4 together with an isolated vertex, and let $H = K_{1,4}$. Show that G and H both have the same characteristic polynomial.*

The pair of graphs in Exercise 496 is the only co-spectral pair on 5 vertices; there are 6 co-spectral pairs on 6 vertices. The list of numbers of graphs on n vertices that have at least one co-spectral mate, for $n = 5, 6, 7, 8, 9, \ldots$, begins $2, 10, 110, 1722, 51039, \ldots$. For more details, see [950], [951], [420], [470], and the website (with unknown author(s), accessed April 2019) `https://www.win.tue.nl/~aeb/graphs/cospectral/cospectralA.html`. One might also compare the above sequence with the *OEIS* sequence A099883 [873, OEIS A099883]; that page also has references to related sequences.

The Hoffman graph [519] in Figure 15.2 on 16 vertices and the graph Q_4 are co-spectral mates.

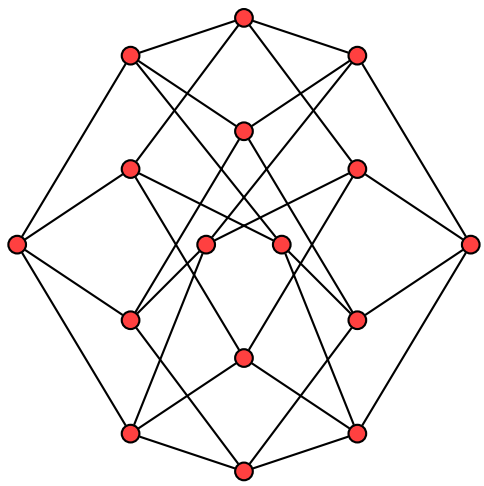


Figure 15.2: The Hoffman graph on 16 vertices

Exercise 497. *Compute the spectrum for the Hoffman graph (see Figure 15.2) and the cube graph Q_4.*

In 1965 Hoffman and Ray-Chaudhuri [521] found two cospectral graphs on 12 vertices, one of which is planar and the other not. (These also appear in [237, p. 157].)

In 1973, Schwenk [838] showed that almost all trees have a cospectral mate.

There is an extensive list of graphs that have no co-spectral mates, that is, they are determined uniquely by their spectrum. Examples include complete graphs, empty graphs, paths [950, Prop. 1], and regular multipartite graphs [361]; see [237] for more details. However, it has been conjectured (see [950]) that almost all graphs have no co-spectral mates.

15.3 The spectrum and related graph properties

Lemma 15.3.1. *Let A be an adjacency matrix for any (multi)graph G without loops, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0.$$

Proof: Apply Corollary 17.3.8 and the observation that the trace of any matrix with 0s along its main diagonal is 0. □

Although the following exercise does not use eigenvalues, it might be considered as a remarkable property of the adjacency matrix nevertheless:

Exercise 498. *Show that for a simple graph G with adjacency matrix A, the number of triangles in G is $\frac{1}{6}tr(A^3)$.*

Lemma 15.3.2. *If G is connected, then the largest eigenvalue is positive, has algebraic multiplicity 1, and there is a strictly positive vector associated with the largest eigenvalue.*

Proof: Let A be an adjacency matrix for a connected graph G. Since G is connected, by Corollary 12.6.2, $(I_n + A)^{n-1}$ is strictly positive. Hence, by Theorem 17.4.2, A is irreducible. So Theorem 17.5.16 applies. □

For the next useful result, some notation is reviewed: For a (multi)graph G, recall that the maximum degree is denoted $\Delta(G)$ and the minimum degree is denoted $\delta(G)$. Also, for matrices $X = (x_{ij})$ and $Y = (y_{ij})$ of the same size, write $X \leq Y$ if and only if for each i, j, $x_{ij} \leq y_{ij}$.

Remark 15.3.3. *In much (perhaps most) of the literature, eigenvalues are written in* decreasing *order, as in: $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ However, I prefer to write eigenvalues in natural order, so please be aware when comparing statements here with those in the literature.*

Theorem 15.3.4. *Let G be a connected graph on n vertices with adjacency matrix A and eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then*

$$\delta(G) \leq \lambda_n \leq \Delta(G),$$

and equality holds if and only if G is regular.

Notes on proof: One could apply Lemma 17.5.8 to the case when G is regular, and then Theorem 17.5.11 directly. For an even stronger theorem (that includes $\lambda_n \geq \sqrt{\Delta}$), and proof, see [640, 11.14(a)].

The following uses the incidence graph (or Levi graph) for a finite projective plane (the bipartite graph on points and lines where a point is adjacent to a line if and only if the point is on the line).

Theorem 15.3.5 (Hoffman, 1963 [518])**.** *For a positive integer $n \geq 2$, let G be a connected graph, regular of degree $n+1$, on $2(n^2+n+1)$ vertices. Then the adjacency matrix of G has eigenvalues $n+1$, $-(n+1)$, $\sqrt{n}$, $-\sqrt{n}$ if and only if G is the (incidence) graph of a finite projective plane of order n.*

In the same research note, Hoffman [518] gave a result on line graphs for FPPs. For more details, see [51].

By applying the interlacing theorem (Theorem 17.5.17) for principal submatrices, the following has a direct proof:

Theorem 15.3.6. *If G is a graph and H is an induced subgraph of G, then the largest eigenvalue for H is at most the largest eigenvalue for G.*

A set S of real (or complex) numbers is said to be *symmetric with respect to 0* if and only if for any $x \in S$, also $-x \in S$.

Theorem 15.3.7. *Let G be a bipartite (multi)graph with at least one edge. Then the spectrum of G is symmetric with respect to 0.*

Proof: Let $V(G) = X \cup Y$ be the bipartition, and let $|V(G)| = n$. Suppose that $|X| = k$ and $|Y| = \ell = n - k$. Let A be an adjacency matrix for G where all vertices in X are previous to all vertices in Y. Then for some $k \times \ell$ matrix B and $\ell \times k$ matrix C,

$$A = \begin{bmatrix} O_{k\times k} & B \\ C & O_{\ell\times\ell} \end{bmatrix}.$$

[Note: if G is undirected, then A is symmetric and so $C = B^T$, however this condition is not required for the proof.]

Let λ be an eigenvalue for A, and let $\mathbf{x} = [x_1, \ldots, x_n]^T$ be an eigenvector associated with λ. Put $\mathbf{u} = [x_1, \ldots, x_k]^T$ and $\mathbf{v} = [x_{k+1}, \ldots, x_n]^T$; then $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$. Working out the matrix multiplication,

$$A\mathbf{x} = \begin{bmatrix} O_{k\times k} & B \\ C & O_{\ell\times\ell} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} B\mathbf{v} \\ C\mathbf{u} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

and so $B\mathbf{v} = \lambda\mathbf{u}$ and $C\mathbf{u} = \lambda\mathbf{v}$. Now consider the column vector $\mathbf{y} = \begin{bmatrix} \mathbf{u} \\ -\mathbf{v} \end{bmatrix}$. Then

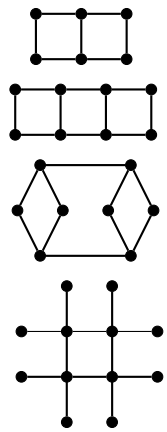
$$A\mathbf{y} = A \begin{bmatrix} \mathbf{u} \\ -\mathbf{v} \end{bmatrix} = \begin{bmatrix} -B\mathbf{v} \\ C\mathbf{u} \end{bmatrix} = \begin{bmatrix} -\lambda\mathbf{u} \\ \lambda\mathbf{v} \end{bmatrix} = -\lambda \begin{bmatrix} \mathbf{u} \\ -\mathbf{v} \end{bmatrix} = -\lambda\mathbf{y},$$

and so $-\lambda$ is an eigenvalue for A. □

Note: It appears that the above proof also works for digraphs.

To confirm Theorem 15.3.7, recall from Example 15.1.2 that the path on 3 vertices is bipartite and has spectrum $-\sqrt{2}, 0, \sqrt{2}$, which is symmetric about 0.

Exercise 499. *Show that each of the four small bipartite graphs given in Figure 15.3 indeed have symmetric spectra (the graphs and the characteristic polynomials are from [237]).*



$x^6 - 7x^4 + 7x^2 - 1$

$x^8 - 10x^6 + 23x^4 - 10x^2 + 1$

$x^8 - 10x^6 + 25x^4 - 16x^2$

$x^{12} - 12x^{10} + 40x^8 - 40x^6 + 16x^4$

Figure 15.3: Four bipartite graphs and their characteristic polynomials

In fact, if only the largest eigenvalue has a symmetric "mate", then the graph is bipartite (and so the entire spectrum is symmetric about 0!):

Theorem 15.3.8. *Let G be a connected loopless n-vertex (multi or di)graph with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. If both λ_n and $-\lambda_n$ are eigenvalues, then G is bipartite.*

Proof: Let A be an adjacency matrix for G, and suppose that $-\lambda_n$ is also an eigenvalue for A with $\mathbf{u} = [u_1, u_2, \ldots, u_n]^T$ being an eigenvector associated with $-\lambda_n$. Putting $\hat{\mathbf{u}} = [|u_1|, |u_2|, \ldots, |u_n|]^T$,

$$\begin{aligned}
\lambda_n \mathbf{u}^T\mathbf{u} =& |-\lambda_n \mathbf{u}^T\mathbf{u}| \\
=& |\mathbf{u}^T A\mathbf{u}| \\
=& \left|\sum_i \sum_j a_{ij} u_i u_j\right| \\
\leq& \sum_i \sum_j a_{ij} |u_i| \cdot |u_j| \\
=& \hat{\mathbf{u}}^T A \hat{\mathbf{u}} \\
\leq& \lambda_n \hat{\mathbf{u}}^T \hat{\mathbf{u}} && \text{(by Rayleigh–Ritz)} \\
=& \lambda_n \mathbf{u}^T\mathbf{u},
\end{aligned}$$

and so equality holds throughout. Hence, $\hat{\mathbf{u}}$ is an eigenvector associated with λ_n. Since G is connected, A is irreducible, and so by Theorem 17.5.16 assume that $\lambda_n > 0$ and $\mathbf{u}$ is strictly positive (written $\mathbf{u} > \mathbf{0}$).

Since $\sum_i \sum_j a_{ij} u_i u_j = \mathbf{u}^T A\mathbf{u} = -\lambda_n \mathbf{u}^T\mathbf{u}$ is negative, and since the first inequality above is now equality, each of the non-zero terms $a_{ij} u_i u_j$ is negative. Thus, for each

non-zero a_{ij}, u_i and u_j are of opposite sign. (Note: here it has been assumed that G has no loops, since if $a_{ii} \neq 0$, u_i cannot be both positive and negative.) Set $X = \{v_i \in V(G) : u_i > 0\}$ and $Y = \{v_i \in V(G) : u_i < 0\}$. □

Exercise 500. *Show that if T is a tree with adjacency matrix A, then A is invertible if and only if T contains a perfect matching.*

A square matrix A with integer entries is called *unimodular* if and only if $\det(A) = \pm 1$. An integer valued matrix B (not necessarily square) is *totally unimodular* if and only if every square principal submatrix is unimodular.

Exercise 501. *Show that the adjacency matrix of any tree (with at least one edge) is totally unimodular.*

For the next exercise, recall that for a graph G, the value $\alpha(G)$ is the order of a largest independent set of vertices.

Exercise 502. *Suppose that a tree T has k distinct non-negative eigenvalues. Show that $\alpha(T) = k$.*

Lemma 15.3.9. *Let A be an adjacency matrix for a graph G on n vertices with (real) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 = 2|E(G)|. \tag{15.1}$$

Proof: Each side of (15.1) is equal to $\mathrm{tr}(A^2)$. By Lemma 17.2.5, each λ_i^2 is an eigenvalue for A^2, and so by Corollary 17.3.8, the trace of A^2 is the sum of the λ_i^2s; also, by Lemma 1.12.1, elements on the main diagonal of A^2 are the degrees of the vertices (actually, the number of walks of length 2 from x_i to x_i, which is the number of edges from x_i), and the sum of the degrees is (by the handshaking lemma—see Exercise 43) is $2|E(G)|$. □

A statement very similar to that of Lemma 15.3.9 is found in [237, p. 94]:

Theorem 15.3.10. *Let A be an adjacency matrix for a graph G on n vertices with (real) eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots, \leq \lambda_n$. For $k \in \mathbb{Z}^+$, then G is k-regular if and only if*

$$n\lambda_n = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2, \tag{15.2}$$

in which case $\lambda_n = k$.

Exercise 503. *For positive integers n, k with $2k \leq n$, let $KG(n,k)$ denote a Kneser graph (see Definition 6.2.15). Verify that eigenvalues for $KG(n,k)$ are, for $i = 0, \ldots, k$,*

$$\lambda_i = (-1)^i \binom{n-j-i}{k-i},$$

where the multiplicity of λ_0 is 1, and for each $i = 1, \ldots, k$, the multiplicity of λ_i is $\binom{n}{i} - \binom{n}{i-1}$.

Another statement for regular graphs is found in [237, p. 94]:

Theorem 15.3.11. *Let A be an adjacency matrix for a regular graph G on n vertices with (real) eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then the number of components of G is the multiplicity of λ_n.*

Exercise 504. *Suppose that G is a k-regular graph with known eigenvalues. Find the spectrum of its complement $\overline{G}$.*

Exercise 505. *If G is a k-regular graph with known eigenvalues, find the spectrum of the line graph $L(G)$.*

Exercise 506. *Let A be an adjacency matrix for the Petersen graph, and let J_{10} be the 10×10 all 1s matrix. Show that $A^2 + A = 2I_{10} + J_{10}$.*

Exercise 507. *Find the eigenvalues and eigenvectors for the Petersen graph.*

Exercise 508. *Let A be the adjacency matrix of a graph G. Show that G is bipartite if and only if for all odd k, $tr(A^k) = 0$.*

Exercise 509. *Suppose that a bipartite graph G has no 1-factor. Prove that 0 is an eigenvalue for G.*

There are many results on the spectral radius of a tree (see [882] for a detailed examination of such bounds).

Theorem 15.3.12 (Lovász–Pelikán, 1973, [642])**.** *For any tree T on n vertices, $\lambda_{\max}(T) \leq \sqrt{n-1}$, and equality holds if and only if $T = K_{1,n-1}$.*

Exercise 510. *Let T be a tree with maximum degree Δ. Show that the largest eigenvalue of T is at most $2\sqrt{\Delta - 1}$.*

The following application of the interlacing theorem (Theorem 17.5.17), which can be found in [422, p.195], is apparently based on work by Mohar [689]. See also [165, p. 72].

Theorem 15.3.13. *The Petersen graph is not Hamiltonian.*

Proof sketch: Let P denote the Petersen graph. If P has a Hamiltonian cycle, then its line graph $L(P)$ contains an induced C_{10}. The spectrum of $L(P)$ is $(-2)^5, (-1)^4, 2^5, 4$. By the interlacing theorem, the seventh eigenvalue $2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2} \sim -.618$ of C_{10} needs to be at most -1, a contradiction.

Theorem 15.3.14 (Collatz–Sinogowitz, 1957 [225])**.** *The average degree in a graph G is at most $\lambda_{\max}(G)$, with equality if and only if G is regular.*

A proof of Theorem 15.3.14 can also be found in, *e.g.*, [237, 3.8].

The following can be found in, *e.g.*, [237, 3.13], where multiple sources are given.

Theorem 15.3.15. *If G is connected, and has m distinct eigenvalues, then the diameter of G is at most $m-1$.*

Theorem 15.3.16. *If $\lambda_{\max} > \sqrt{|E(G)|}$, then G contains at least one triangle. Also, if $\lambda_{\max}^2$ is larger than the sum of all other eigenvalues squared, then G contains a triangle.*

The following can be found in, *e.g.*, [237, 3.18, p. 91]. (Its proof requires a technical lemma, omitted here.)

Theorem 15.3.17 (Hoffman, 1970 [520])**.** *Let G be a graph with largest and smallest eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$, respectively. If $\lambda_{\max} \neq 0$, then*

$$\chi(G) \geq \frac{\lambda_{\max}}{-\lambda_{\min}} + 1.$$

Theorem 15.3.18 (Wilf, 1967 [984])**.** *For any connected graph G,*

$$\chi(G) \leq \lambda_{\max}(G) + 1. \tag{15.3}$$

Equality holds if and only if G is a complete graph or an odd cycle.

Proof: Let G be a graph and let $k = \chi(G)$. If $k = 1$, the result is trivial, so assume that $k \geq 2$. Let H be a subgraph of G with the smallest number of vertices possible so that $\chi(H) = k$.

Claim: $\delta(H) \geq k-1$.
Proof of claim: Argue by contradiction. Suppose that $x \in V(H)$ has $\deg(x) < k-1$. Since H is minimal, $\chi(H-x) < k$, and so there exists a proper $(k-1)$ colouring of $H-x$. Since $\deg(x) < k-1$, this colouring can be extended to x since x is not adjacent to vertices of all colours in $H-x$. This contradicts that H is k-colourable, proving the claim.

By Theorems 15.3.6 and 15.3.4,

$$\lambda_{\max}(G) \geq \lambda_{\max}(H) \geq \delta(H) \geq k-1, \tag{15.4}$$

concluding the proof of the first statement.

Now suppose that equality in (15.3) holds. Then equality in (15.4) also holds, and so $\lambda_{\max}(G) = \lambda_{max}(H)$. Since G is connected, $G = H$. Also, $\lambda_{max}(G) = \delta(G)$, and so by Theorem 15.3.14, G is regular. Thus, $k = 1 + \lambda_{max}(G) = 1 + \Delta(G)$. By Brooks' theorem (Theorem 6.2.10), G is either a complete graph or an odd cycle. □

The proof of the following theorem is left as an exercise. (The product $G \times H$ is explained in Definition 1.7.4.)

Theorem 15.3.19. *Let G and H be graphs whose adjacency matrices have eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ respectively. Then the eigenvalues for $G \times H$ are*

$$\{\lambda_i + \mu_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Exercise 511. *Prove Theorem 15.3.19.*

Exercise 512. *Let G be a planar graph on n vertices. If A is an adjacency matrix for G with maximum eigenvalue $\lambda_{\max}$ and minimum eigenvalue $\lambda_{\min}$, show that*

$$\lambda_{\max} \leq -3 \cdot \lambda_{\min}.$$

15.4 Strongly regular graphs

Definition 15.4.1. *A graph G that is neither complete nor empty is called* strongly regular *if and only if for some parameters (k, a, b), G is k-regular, every pair of adjacent vertices has a common neighbours, and every pair of non-adjacent neighbours has b common neighbours.*

A strongly regular graph on n vertices as defined above is also said to have parameters (n, k, a, b). In many texts, the parameters for a strongly regular graph are given as (v, k, λ, μ), but since v is occasionally used to indicate a vertex and λ is often used to indicate an eigenvalue, such notations can be confusing; hence the notation used here follows a combination of [125] and [422].

Many graphs with a high degree of symmetry are strongly regular (however, on average, strongly regular graphs do not have many symmetries; see [422, p. 217]). For example, C_5 is strongly regular with parameters (5,2,0,1). The graph $K_{4,4}$ is strongly regular with parameters $(8, 4, 4, 0)$. The Petersen graph is also strongly regular with parameters $(10, 3, 0, 1)$.

Exercise 513. *Show that $L(K_6)$, the line graph of K_6, is strongly regular with parameters $(15, 8, 4, 4)$.*

It is known (see Peter Cameron's notes[178, Ex. 6.4b]) that any strongly regular graph with parameters $(15, 8, 4, 4)$ is isomorphic to $L(K_6)$; the complete proof provided by Cameron takes a few pages, using some results not given in this text.

Exercise 514. *Show that for each prime power q congruent to 1 modulo 4, the Paley graph of order q is strongly regular with parameters*

$$\left(q, \frac{q-1}{2}, \frac{q-5}{2}, \frac{q-1}{4}\right).$$

Exercise 515. *Show that if G is a strongly regular graph with parameters (n, k, a, b), then $k(k - a - 1) = b(n - k - 1)$.*

Exercise 516. *Prove that if G is a strongly regular graph with parameters (n, k, a, b), then the complement $\overline{G}$ is also strongly regular. Find the parameters for $\overline{G}$.*

If G is a strongly regular graph with parameters (n, k, a, b), then G is triangle-free if and only if $a = 0$. At present, only seven triangle-free strongly regular graphs are known to exist. The seven such graphs include the three known Moore graphs of diameter 2:

- C_5.
- The Petersen graph.
- The Hoffman–Singleton graph (see Figure 11.6).
- The Clebsch graph (see Figure 9.6).
- The Gewirtz graph (also called the Sims–Gewirtz graph), a 4-chromatic graph with parameters $(56, 10, 0, 2)$ and independence number $\alpha = 16$.
- The Higman–Sims graph, with parameters $(100, 22, 0, 6)$. This graph was first constructed by Mesner [679] in 1956 and proved to be unique in 1964 [680]. It was rediscovered by Higman and Sims [513] in 1968 (see also [512]) and again proved to be unique by Gewirtz [417] in 1969.
- The unique $(77, 16, 0, 4)$ strongly regular graph, sometimes called the M_{22} graph. This graph is a subgraph of the Higman–Sims graph..

An eighth possible example, is, if it exists, the 57-regular "monster" diameter 2 Moore graph on 3250 vertices.

The following problem has attracted some attention (see, *e.g.*, [419]):

Problem 15.4.2. *Is there an eighth triangle-free strongly regular graph?*

Another class of strongly regular graphs are called "rook's graphs" or "lattice graphs". The lattice graph L_m is the line graph of $K_{m,m}$, which is isomorphic to $K_m \square K_m$ (this cartesian product is sometimes denoted by simply $K_m \times K_m$—see Definition 1.7.5 and Figure 1.37 for $K_5 \square K_4$). Each L_m is strongly regular with parameters $(m^2, 2(m-2), m-2, 2)$ (see [171, p. 153]).

The smallest number of vertices for which there exist two (non-isomorphic) strongly regular graphs with the same parameters is 16: both $L(K_{4,4})$ (for $L(K_{3,3})$, see Exercise 34, or see Figure 1.37 for $L(K_{4,5})$) and the Shrikhande graph (see Figure 15.4) have parameters $(16, 6, 2, 2)$ (see [861] or [486, p. 79]).

Lemma 15.4.3. *Let G be a strongly regular graph with parameters (n, k, a, b). If A is an $n \times n$ adjacency matrix for G and $J = J_n$ is the $n \times n$ all 1s matrix, then*

$$A^2 = (a - b)A + (k - b)I + bJ.$$

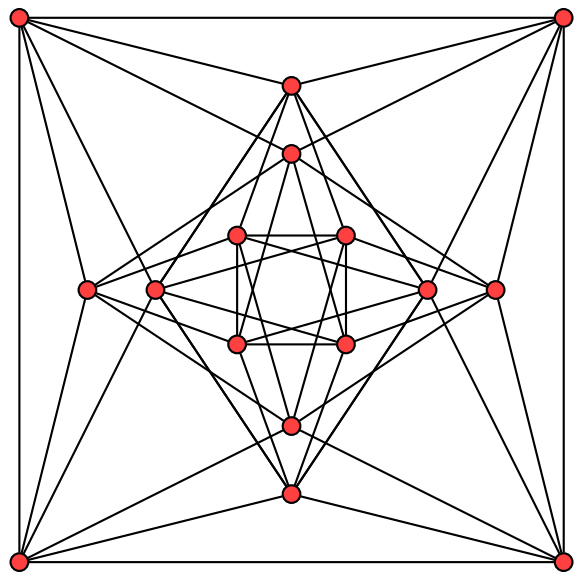


Figure 15.4: The Shrikhande graph

Proof: Let $V(G) = \{v_1, \ldots, v_n\}$ and let $A = (a_{i,j})$ be the $n \times n$ adjacency matrix for G with the given vertex ordering. With some simple checking, the (i, j) entry of A^2 is

$$(A^2)_{i,j} = \begin{cases} k & \text{if } i = j \\ a & \text{if } \{v_i, v_j\} \in E(G) \\ b & \text{if } i \neq j \text{ and } \{v_i, v_j\} \notin E(G), \end{cases}$$

which gives the equation in the statement of the lemma. □

Theorem 15.4.4 (Bhagwandas–Shrikhande, 1965 [99])**.** *Let G be a connected regular graph on at least 3 vertices. Then G is strongly regular if and only if G has exactly three eigenvalues.*

Proof: For one version of the proof, see [237, p. 104], which is based on some other sophisticated theorems. For another version using collapsed matrices, see [125]. Here is a direct version (which is based on some old notes of mine, but I failed to record my source):

Throughout the proof, G is a connected k-regular graph on $n \geq 3$ vertices $\{v_1, \ldots, v_n\}$ with adjacency matrix A. The matrix $J = J_n$ is the all 1s $n \times n$ matrix, and $O_{n\times n}$ is the all zeroes matrix. Let $\mathbf{j} = (1, 1, \ldots, 1)^T$ denote the $n \times 1$ all 1s column vector.

For one direction of the proof, let G be strongly regular with parameters (n, k, a, b). Since G is k-regular, $A\mathbf{j} = k\mathbf{y}$, and so k is an eigenvalue (and $\mathbf{j}$ is an eigenvector for k). By Theorem 15.3.4, k is the largest eigenvalue, and by Lemma 15.3.2, k has multiplicity 1.

By Theorem 17.2.8, extend $\{\mathbf{j}\}$ to a basis for $\mathbb{R}^n$ consisting of orthogonal eigenvectors. Thus, for an eigenvalue of $\lambda \neq k$ with eigenvector $\mathbf{z}$, assume that $\mathbf{j} \bullet \mathbf{z} = 0$. For such a $\mathbf{z}$, $J\mathbf{z} = \mathbf{0} = (0, 0, \ldots, 0)^T$. From Lemma 15.4.3, $A^2 = (a-b)A + (k-b)I + bJ$, and so

$$O_{n\times n} = A^2 + (b-a)A + (b-k)I - bJ. \tag{15.5}$$

Multiplying (15.5) on the right by one of the eigenvectors $\mathbf{z}$ associated with some eigenvalue $\lambda \neq k$, and applying $A\mathbf{z} = \lambda\mathbf{z}$ and $J\mathbf{z} = \mathbf{0}$ then gives

$$\mathbf{0} = \lambda^2\mathbf{z} + (b-a)\lambda\mathbf{z} + (b-k)\mathbf{z},$$

and so

$$\lambda^2 + (b-a)\lambda + (b-k) = 0.$$

Thus λ is a root of a quadratic, and so can take at most two values λ_1 and λ_2, where

$$\lambda_1 = \frac{(b-a) - \sqrt{(b-a)^2 - 4(b-k)}}{2}$$

and

$$\lambda_2 = \frac{(b-a) + \sqrt{(b-a)^2 - 4(b-k)}}{2}.$$

So G has at most three eigenvalues. To see that two are not possible, if $(b-a)^2 - 4(b-k) = 0$, then $b > k$, violating the fact that $b \leq k$.

For the other direction of the proof, assume that G has exactly three eigenvalues. Not only does the following show that G is strongly regular, but it surrenders the values for (k, a, b). As mentioned in the first part of the proof, k is the largest eigenvalue, so let $\lambda_1 < \lambda_2 < k$ be the eigenvalues. Consider the following polynomial:

$$f(x) = (x-k)(x-\lambda_1)(x-\lambda_2) = (x-k)(x^2 - (\lambda_1+\lambda_2)x + \lambda_1\lambda_2).$$

Then for any eigenvalue λ, $f(\lambda) = 0$. Since the degree of the minimal polynomial is the number of distinct eigenvalues, $f(x)$ is the minimal polynomial for A (this fact follows from a corollary to the Cayley–Hamilton theorem, Theorem 17.2.6; see, *e.g.*, [381]). Hence,

$$f(A) = (A - kI)(A^2 - (\lambda_1+\lambda_2)A + \lambda_1\lambda_2 I) = O_{n\times n}.$$

Thus, for any vector $\mathbf{v}$, the vector $(A^2 - (\lambda_1+\lambda_2)A + \lambda_1\lambda_2 I)\mathbf{v}$ is in the null space of $A - kI$, which is the eigenspace of the eigenvalue k, (which has dimension 1, and contains $\mathbf{j}$). Thus the null space of $A - kI$ is equal to $\{c\mathbf{j} : c \in \mathbb{R}\}$. In particular, for each unit basis vector $\mathbf{e}_i$, there exists $c_i \in \mathbb{R}$ so that

$$(A^2 - (\lambda_1+\lambda_2)A + \lambda_1\lambda_2 I)\,\mathbf{e}_i = c_i\mathbf{j}.$$

Then by block multiplication,

$$(A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I)I = [c_1\mathbf{j} \mid \cdots \mid c_n\mathbf{j}].$$

Since A is symmetric, the left side of the last equation is also symmetric, and hence so is the right side. By a simple inductive argument there exists a non-zero $c \in \mathbb{R}$ so that $c_1 = c_2 = \cdots = c_n = c$. Summarizing so far, there is $c \neq 0$ so that

$$(A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I)I = cJ,$$

and so

$$A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I = cJ.$$

Hence every entry of $A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I$ is equal to c. Then

$$\begin{aligned} A^2 &= (\lambda_1 + \lambda_2)A - \lambda_1\lambda_2 I + cJ \\ &= (c + \lambda_1 + \lambda_2)A + (c - \lambda_1\lambda_2)I + c(J - A - I). \end{aligned} \tag{15.6}$$

Since the (i, i) entry of A^2 is the degree of v_i, comparing entries above, and observing that $a_{i,i} = 0$,

$$\begin{aligned} \deg(v_i) &= (c + \lambda_1 + \lambda)a_{i,i} + (c - \lambda_1\lambda_2) + c(1 - a_{i,i} - 1) \\ &= c - \lambda_1\lambda_2, \end{aligned}$$

and so G is regular of degree $k = c - \lambda_1\lambda_2$. Examining (15.6) further, if two vertices are adjacent in G, they have $c + \lambda_1 + \lambda_2$ common neighbours, and if two (distinct) vertices in G are not adjacent, they have $c = k + \lambda_1\lambda_2$ neighbours in common. Thus, G is strongly regular with parameters $(n, k, a, b) = (n, c - \lambda_1\lambda_2, c + \lambda_1 + \lambda_2, c)$. □

Since the multiplicities of eigenvalues are non-negative integers, Theorem 15.4.4 can be used to impose conditions on the variables n, k, a, b for strongly regular graphs. The following theorem identifies the multiplicities of the two remaining eigenvalues of a strongly regular graph as found in Theorem 15.4.4, and thereby defines two expressions that are integers. For this reason, the following is sometimes referred to by its "integrality conditions" thereby imposed.

Theorem 15.4.5 (Integrality condition for strongly regular graphs)**.** *Let G be a strongly graph with parameters (n, k, a, b). If r and s are the multiplicities of the eigenvalues λ_1 and λ_2 given in Theorem 15.4.4 then*

$$r = \frac{1}{2}\left(n - 1 + \frac{(n-1)(b-a) - 2k}{\sqrt{(b-a)^2 + 4(k-b)}}\right)$$

and

$$s = \frac{1}{2}\left(n - 1 - \frac{(n-1)(b-a) - 2k}{\sqrt{(b-a)^2 + 4(k-b)}}\right).$$

Partial proof: Since the only application of this theorem used in these notes is (in Section 15.6) for Moore graphs of diameter 2, the theorem is proved only for $a = 0$ and $b = 1$. (The general case follows the same method and is only a bit more complicated looking.) In this case, by the proof of Theorem 15.4.4, the eigenvalues for A are k, $\lambda_1 = \frac{1-\sqrt{4k-3}}{2}$ and $\lambda_2 = \frac{1+\sqrt{4k-3}}{2}$.

To be shown is that the multiplicity of λ_1 is

$$r = \frac{1}{2}\left(n - 1 + \frac{n-1-2k}{\sqrt{1+4(k-1)}}\right)$$

and the multiplicity of λ_2 is

$$s = \frac{1}{2}\left(n - 1 - \frac{n-1-2k}{\sqrt{1+4(k-1)}}\right)$$

Since there are n eigenvalues, $n = 1 + r + s$. Since all eigenvalues sum to 0, $r\lambda_1 + k + s\lambda_2 = 0$. Replacing $r = n-1-s$ in the second equation, $(n-1-s)\lambda_1 + k + s\lambda_2 = 0$, solving for s, and replacing the expressions for λ_1 and λ_2 gives

$$\begin{aligned} s &= \frac{(n-1)\lambda_1 + k}{\lambda_1 - \lambda_2} \\ &= \frac{1}{2}\left(n - 1 - \frac{n-1-2k}{\sqrt{4k-3}}\right), \end{aligned}$$

as desired. It then follows that

$$\begin{aligned} r &= n - 1 - s \\ &= n - 1 - \frac{1}{2}\left(n - 1 - \frac{n-1-2k}{\sqrt{4k-3}}\right) \\ &= \frac{1}{2}\left(n - 1 + \frac{n-1-2k}{\sqrt{4k-3}}\right), \end{aligned}$$

also as desired. □

Exercise 517. *Let G be a strongly regular graph with parameters (n, k, a, b). Show that $b \leq k$, with equality holding only if G is a complete multi-partite graph.*

Exercise 518. *Let G be a strongly regular graph with parameters (n, k, a, b). Show that $a + 1 \leq k$, and that the following four conditions are equivalent:*

(i) G is a disjoint union of complete graphs.

(ii) $k = a + 1$.

(iii) $b = 0$.

(iv) G *is disconnected.*

Exercise 519. *Show that the line graph* $L(K_n)$ *is strongly regular with parameters* $(\binom{n}{2}, 2n-4, n-2, 4)$.

Exercise 520. *Show that the line graph* $L(K_{n,n})$ *is strongly regular with parameters* $(n^2, 2n-2, n-2, 2)$.

Exercise 521. *Let* $q \equiv 1 \pmod 4$ *be a prime power, and let* P_q *denote the Paley graph on* q *vertices. Show that if* $q = 4t+1$*, then* P_q *is strongly regular with parameters* $(4t+1, 2t, t-1, t)$.

15.5 The friendship theorem

In 1966, Erdős, Rényi, and Sós published a paper that showed how a finite projective plane can be used to give a bound on the number of edges in a C_4-free graph. In the last two pages of that paper, they used a theorem about finite projective planes to prove the following:

Theorem 15.5.1 (Erdős–Rényi–Sós, 1966 [339])**.** *Let* G *be a graph on* $n \geq 3$ *vertices with the property that between any pair of vertices there is a unique path of length 2. Then* n *is odd, say* $n = 2k+1$*, and* G *is the graph* F_k *consisting of* k *triangles sharing a common vertex.*

Theorem 15.5.1 is often called "the friendship theorem", because it implies that if there is a party where every pair of people have precisely one common friend, then there is someone who knows everyone at the party. [Perhaps I should say "...knows everyone else at the party", since it is arguable that nobody can know themself and not even a theorem can not guarantee that!] The resulting graph is called the *friendship graph* (an example was given in Figure 10.1).

Including the original, there are now (at least) three different types of proofs of Theorem 15.5.1. All three proof ideas begin in the same way—by showing that if there is no vertex adjacent to all others, then the graph is regular.

In the original proof [339], a result by Baer on finite projective planes shows that no such regular graphs exist (except trivial cases).

The second proof (based on ideas of Wilf [985]) uses spectral theory (algebraic graph theory) to show that only one such graph exists, namely K_3. This proof is presented below. (There are many variants of this algebraic proof, but perhaps differing only in the way certain quantities are calculated.)

The third proof (by Longyear and Parsons [631]) uses what one might call more straightforward counting. (I think that this third proof is slightly longer than either of the first two, but requires less "machinery".)

First part of proof of Theorem 15.5.1: Let G be a graph on $n \geq 3$ vertices so that between any two vertices, there exists a unique path of length 2. Then G contains no C_4 and each edge is contained in at most one triangle. Also, G is connected and has minimum degree $\delta(G) \geq 2$ (for if there is a vertex of degree one, there is no path of length 2 between that vertex and its unique neighbour).

Case 1: Suppose that there exists a vertex $v \in V(G)$ with $\deg(v) = n - 1$. In this case, since each edge is in at most one triangle, each vertex in $V(G)\backslash\{v\}$ has degree at most one; since between v and any other vertex there is a path of length 2, all vertices in $V(G)\backslash\{v\}$ have degree at least one. So $V(G)\backslash\{v\}$ induces a matching, and so $G = F_{\frac{n-1}{2}}$ as desired.

Case 2: Suppose that there is no vertex with degree $n - 1$.

Claim: G is regular.

Let $x \in V(G)$ (in the proof given in [339], the vertex x is chosen to be of maximum degree, but this is not necessary to make the rest of the proof to work). Since no vertex has degree $n - 1$, let y be a vertex not adjacent to x. Let $N(x) = \{v_1, \ldots, v_k\}$ and $N(y) = \{w_1, \ldots, w_\ell\}$. Since $\delta(G) \geq 2$. both $k \geq 2$ and $\ell \geq 2$.

Since there exists a unique path of length 2 between x and y, $|N(x) \cap N(y)| = 1$ (if the intersection of neighbourhoods is larger, there is more than one x-y path of length 2). Suppose that $v_1 = w_1$ is the common neighbour of x and y. Since there is a unique 2-path from x to v_1, there exists a unique vertex, say v_2 adjacent to both. Similarly, there is a unique vertex, say w_2 adjacent to both $v_1 = w_1$ and y. (So far, $\deg(v_1) \geq 4$.)

For each $i \geq 3$, there exists a unique 2-path from v_i to y, and so there is a unique $w_j \in \{w_3, \ldots, w_\ell\}$ so that $\{v_i, w_j\} \in E(G)$. (If v_i is adjacent to two neighbours of y, say w_j and w_m, then a 4-cycle is formed, contrary to the property mentioned above.) So $k \leq \ell$. By the same argument, but replacing v_i with some w_j, $\ell \leq k$. So $k = \ell$ and thus $\deg(x) = \deg(y)$. Hence, any two non-adjacent vertices have the same degree (in this case, the degree is k).

For each $i = 2, \ldots, k$, v_i is not adjacent to y, so $deg(v_i) = \deg(y)$ as well, and since $\deg(x) = \deg(y)$, it follows that $\deg(v_i) = \deg(x) = k$. The same argument applied to $w_2, \ldots, w_k$ shows that for each $i = 2, \ldots, k$, $\deg(w_i) = k$. Observe that v_k is not adjacent to v_1 (for otherwise, $v_k v_1 v_2 x$ forms a 4-cycle), so $4 \leq \deg(v_1) = \deg(v_k) = k$. So any two adjacent vertices also have the same degree.

Then G is regular (of degree $k \geq 4$) proving the claim.

Second part of proof of Theorem 15.5.1 (based on eigenvalues): Let $x \in V(G)$ have neighbours $v_1, \ldots, v_k$. Since all vertices are reachable from x by a path of length 2, each v_i is adjacent to just one other v_j. Thus, each of the v_is have $k - 2$ other neighbours, and since no C_4 exists, these neighbour sets are disjoint. In all, there are

$$n = 1 + k + k(k - 2) = k^2 - k + 1 \tag{15.7}$$

vertices.

Let A be an adjacency matrix for G, and examine A^2. Since G is k-regular, A^2 has ks down the main diagonal; since between any two vertices there exists a unique walk of length 2, there are 1s in all remaining positions. In other words, letting I_n be the identity matrix and J_n be the all 1s $n \times n$ matrix,

$$A^2 = (k-1)I_n + J_n.$$

Since the $n \times 1$ column vector $\mathbf{j} = (1, 1, \ldots, 1)^T$ is an eigenvector of A associated with k, the vector $\mathbf{j}$ is also an eigenvector for A^2 associated with k^2. By inspection, each of $\mathbf{v}_2 = (1, -1, 0, \ldots)^T$, $\mathbf{v}_3 = (0, 1, -1, 0, \ldots, 0)$, $\ldots$, $\mathbf{v}_n = (0, \ldots, 0, 1, -1)$ is an eigenvector for A^2 with eigenvalue $k-1$.

Hence the eigenvalues for A^2 are k^2 and $k-1$ (with multiplicity $n-1$). Since the square of any eigenvalue for A is an eigenvalue for A^2, eigenvalues for A are either $\pm k$ or $\pm\sqrt{k-1}$. For the first, since G is regular of degree k, the all 1s vector $\mathbf{j}$ shows that k is an eigenvalue of A. Suppose that the remaining eigenvalues $\sqrt{k-1}$ and $-\sqrt{k-1}$ have multiplicities r and s respectively (where $r + s = n - 1$).

By Lemma 15.3.1 the sum of eigenvalues of A is zero, and so

$$0 = k + r\sqrt{k-1} - s\sqrt{k-1} = k + (r-s)\sqrt{k-1}.$$

Rearranging and squaring gives

$$k^2 = (s-r)^2(k-1).$$

The above equation requires that $k-1$ divides k^2, which is impossible since $k-1$ divides $k^2 - 1$ and $k \geq 4$. Hence, it is not possible that every vertex has degree less than $n-1$, and so the first case holds. □

15.6 At most four diameter 2 Moore graphs

Recall from Section 11.4 that a Moore graph of diameter 2 is a k-regular graph on $n = k^2 + 1$ vertices with girth 5. By its definition, it is not difficult to see that a k-regular Moore graph of diameter 2 is also a strongly regular graph with parameters $(k^2 + 1, k, 0, 1)$. So far, three diameter 2 Moore graphs have been found: C_5 is 2-regular, the Petersen graph is 3-regular, and the Hoffman–Singleton (7,5)-cage graph (see Figure 11.6) is 7-regular. For each of $k = 2, 3, 7$, these graphs are known to be unique (e.g, for $k = 7$, see [540]). The following theorem shows that there is at most one more value of k possible.

Theorem 15.6.1 (Hoffman–Singleton, 1960 [522]). *If G is a k-regular Moore graph of diameter 2 and girth 5, then $k \in \{2, 3, 7, 57\}$.*

Proof: Let G be a k-regular Moore graph with diameter 2 and girth 5. Then G is strongly regular with parameters $(k^2+1, k, 0, 1)$. By Theorem 15.4.4, the eigenvalues are k, $\lambda_1 = \frac{1-\sqrt{4k-3}}{2}$, and $\lambda_2 = \frac{1+\sqrt{4k-3}}{2}$ with multiplicities, respectively 1,

$$\begin{aligned} r &= \frac{1}{2}\left(n-1+\frac{n-1-2k}{\sqrt{4k-3)}}\right) \\ &= \frac{1}{2}\left(k^2+\frac{k^2-2k}{\sqrt{4k-3}}\right), \end{aligned}$$

and

$$\begin{aligned} s &= \frac{1}{2}\left(n-1-\frac{n-1-2k}{\sqrt{4k-3}}\right) \\ &= \frac{1}{2}\left(k^2+\frac{k^2-2k}{\sqrt{4k-3}}\right). \end{aligned}$$

Since

$$r-s = \frac{k^2-2k}{\sqrt{4k-3}}$$

is an integer, either $k^2-2k=0$ or $4k-3$ is a perfect square, say $4k-3=t^2$ and t divides $k^2-2k>0$. In the first case, $k=2$ (which was already dealt with). In the second case, $k=\frac{t^2+3}{4}$, and so substituting this value in the expression above for r gives

$$r = \frac{1}{2}\left(\frac{(t^2+3)^2}{16}+\frac{\frac{(t^2+3)^2}{16}-2(\frac{t^2+3}{4})}{t}\right).$$

Rewriting the above equation, and simplifying gives

$$(t^4+t^3+6t^2-2t+9-32r)t = 15.$$

Thus t divides 15, and so $t \in \{1,3,5,15\}$.

If $t=1$, then $4k-3=1^2$, giving $k=1$, which is impossible. If $t=3$, then $4k-3=9$, giving $k=3$. If $t=5$, then $4k-3=25$, and so $k=7$. If $t=15$, then $4k-3=225$, giving $k=57$. In conclusion, $k \in \{2,3,7,57\}$. □

As mentioned previously, it is not known if a 57-regular Moore graph of diameter 2 (and girth 5) exists, and if it does exist, such a graph has 3250 vertices. Such a graph, if it exists, is called the "monster".

Exercise 522. *If the monster graph exists with parameters* $(3250, 57, 0, b)$*, what is* b*?*

15.7 The Petersen graph and decomposing K_{10}

In Exercise 507, it was asked to find the eigenvalues of the Petersen graph. They are -2 (with multiplicity 4), 1 (with multiplicity 5), and 3.

The following result was posed by Schwenk (then at University of Waterloo) as an unstarred problem in *The Monthly* in 1983 (if a problem has no star, a solution is known to the poser). Schwenk's solution was given a few years later (Lossers also provided the same solution) in [632].

Theorem 15.7.1 (Schwenk, 1983 [839])**.** *The edges of K_{10} can not be decomposed into three copies of the Petersen graph.*

Proof: For any ordering of the vertices of a copy of K_{10}, the adjacency matrix for K_{10} is $A = J_{10} - I_{10}$. Suppose that A is decomposed into three 0-1 matrices, $A = A_1 + A_2 + A_3$, where A_1 and A_2 are adjacency matrices for the Petersen graph. The eigenvalues for each of A_1 and A_2 are -2, 1, 3, where 1 has multiplicity 5 (see Exercise 507). Let E be the eigenspace of A_1 associated with eigenvalue 1, and let F be the eigenspace of A_2 associated with eigenvalue 1. The all 1s vector $\mathbf{j} = (1, 1, \ldots, 1)^T$ is an eigenvector for the eigenvalue 3 (in either A_1 or A_2), and so both E and F are 5-dimensional subspaces of the 9-dimensional subspace $\mathbf{j}^\perp$ of $\mathbb{R}^{10}$. So $\dim(E \cap F) \geq 1$ and so let $\mathbf{y}$ be a non-zero vector in $E \cap F$. Then $A_1\mathbf{y} = \mathbf{y}$ and $A_2\mathbf{y} = \mathbf{y}$. .

Since $\mathbf{y}$ is orthogonal to $(1, 1, \ldots, 1)^T$ it follows that $J\mathbf{y} = \mathbf{0} \in \mathbb{R}^{10}$. Hence

$$A\mathbf{y} = (J_{10} - I_{10})\mathbf{y} = \mathbf{0} - I_{10}\mathbf{y} = -\mathbf{y}.$$

Thus the equation $A\mathbf{y} = (A_1 + A_2 + A_3)\mathbf{y}$ becomes $-\mathbf{y} = \mathbf{y} + \mathbf{y} + A_3\mathbf{y}$, and so $A_3\mathbf{y} = -3\mathbf{y}$. However, since -3 is not an eigenvalue for the Petersen graph, the matrix A_3 is not an adjacency matrix for the Petersen graph. □

See [422, p. 195] for a proof of Theorem 15.7.1 based on the interlacing theorem (Theorem 17.5.17). In [632], it was noted that Theorem 15.7.1 also follows from an older theorem by Bosák, Rosa, and Znám [153], which says that for $n \leq 11$, the graph K_n does not have a decomposition into three spanning graphs of diameter 2.

15.8 The Matrix-Tree Theorem and its proof

Let $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ be an $n \times n$ matrix. For any $1 \leq i, j \leq n$, let $M_{i,j}$ denote the $(n-1) \times (n-1)$ principal submatrix of M formed by deleting row i and column j; matrix $M_{i,j}$ is called the (i, j) minor of M. The (i, j) cofactor of M is

$$C_{i,j} = (-1)^{i+j} \det(M_{i,j}).$$

The next theorem, now called "The Matrix-Tree theorem" counts the number of spanning trees in a graph. To prove the Matrix-Tree Theorem, some results from matrix theory are used, the first of which is elementary.

Lemma 15.8.1. *Let M be a square matrix with linearly dependent column vectors. Then det$(M) = 0$.*

Lemma 15.8.2. *Let M be a square matrix with the property that all row-sums and column-sums are zero. Then all cofactors of M have the same value.*

Exercise 523. *Prove Lemma 15.8.2. Hint: Letting J be the square all 1s matrix (with same size as M), show that* $\det(M + J)$ *is n^2 times any cofactor of M.*

The Binet–Cauchy formula for the determinant of a product of non-square matrices is also used (see Theorem 17.1.2), repeated here for convenience:
Theorem 17.1.2. Let $A \in \mathcal{M}_{n\times k}(\mathbb{R})$ and $B \in \mathcal{M}_{k\times n}(\mathbb{R})$. For any set $S \subset [k]$, let A_S be the (square) submatrix induced by rows and columns whose indices are S. Similarly, define B_S. Then

$$\det(AB) = \sum_{S\in[n]^k} \det(A_S)\det(B_S).$$

The Matrix-Tree Theorem is often credited to Gustav Kirchhoff, although it has been attributed to other authors before Kirchhoff.

Theorem 15.8.3 (Matrix-Tree Theorem, Kirchhoff, 1847 [571])**.** *Let G be a graph on $n \geq 3$ vertices $v_1, \ldots, v_n$ and let A be the adjacency matrix for this ordering of $V(G)$. Let $D = (d_{i,i})$ be the diagonal matrix whose entries are $d_{i,i} = \deg(v_i)$. Then the number of spanning trees of G is the value of any cofactor of $D - A$.*

Proof: (The proof given here follows the form used in, *e.g.*, [190]; there are other proofs that do not rely on the Binet-Cauchy formula.)

Since the number of 1s in the ith row (or column) of A is $\deg(v_i)$, the row sums and column sums of $D - A$ are all zero, and so by Lemma 15.8.2, all cofactors of $D - A$ are the same.

Let G be disconnected (in which case there are 0 spanning trees0, and without loss of generality, let G_1 be a component on k vertices $v_1, \ldots, v_k$. Let B the $(n-1)\times(n-1)$ submatrix of $D - A$ formed by deleting the last row and column of $D - A$. Then the sum of the first k rows of B is the zero vector, and because each of these rows ends in a 0 (since G_1 is disconnected from v_n), the sum of the first k rows in $D-A$ is also the zero vector, which then shows $D - A$ has linearly dependent rows and so $\det(D - A) = 0$, as desired.

So assume that G is connected and let $E(G) = \{e_1, e_2, \ldots, e_m\}$. Since G is connected, $m \geq n - 1$. Let M be the $n \times m$ matrix derived from the incidence matrix (columns correspond to edges) but for each e_i, the two entries in the ith column are 1 and -1 (in any order). (In other words, M is an incidence matrix for a digraph obtained by randomly directing each edge).

Writing $M = (m_{ij})$, for each i, j, the (i, j) entry of the MM^T is

$$\sum_{k=1}^{m} m_{ik}m_{jk} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } \{v_i, v_j\} \in E(G) \\ 0 & \text{if } i \neq j \text{ and } \{v_i, v_j\} \notin E(G), \end{cases}$$

which shows $MM^T = D - A$.

Let H be a spanning subgraph of G containing $n-1$ edges of G, For some k, let M_k be the $(n-1) \times (n-1)$ submatrix of M formed by columns corresponding to edges in H and then deleting the kth row.

Suppose that H is disconnected, and let H_1 be a component not containing v_k. The sum of the row vectors in M_k corresponding to vertices of H_1 is zero, and so rows of M_k are linearly dependent, and hence $\det(M_k) = 0$.

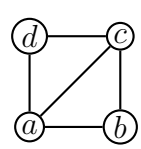
Now suppose that H is connected; then H is a spanning tree of G. After perhaps relabelling, let $v_1 \neq v_k$ be an end vertex of H, and let e_1 be the pendant edge incident with v_1. In the tree $H - v_1$, let v_2 be an endpoint, and let e_2 be the edge incident with v_2. Continue until v_k is the only remaining vertex. Then $M_k^* = (m_{ij}^*)$ is so that for each i, j, $|m_{ij}^*| = 1$ if and only if v_i and e_j are incident. Each v_i can only be incident with members of $\{e_1, \ldots, e_i\}$, and so after the relabelling, the new M_k^* is lower triangular, and since each $|m_i i| = 1$, $\det(M_k^*) = 1$. Since permuting rows or columns does only changes the determinant by -1s, $|\det(M_k)| = 1$.

So $|\det(M_k)|$ is 1 if and only if the spanning subgraph H is connected, i.e., when H is a tree (and 0 otherwise). Any cofactor of $D - A$ is a cofactor of MM^T, which, by the Binet-Cauchy formula, is the sum of the products of form $\det(M_k)\det(M_k^T) = \det(M_k)^2$, each of which is 1 if and only if the associated subset of columns correspond to a spanning tree. □

15.9 Matrix-Tree Theorem examples

Example 15.9.1. *Find the number of spanning trees for $K_4 - e$.*

One can find the answer either by the Matrix-Tree Theorem or directly. First, suppose that a copy of $K_4 - e$ is drawn on vertices a, b, c, d as follows:



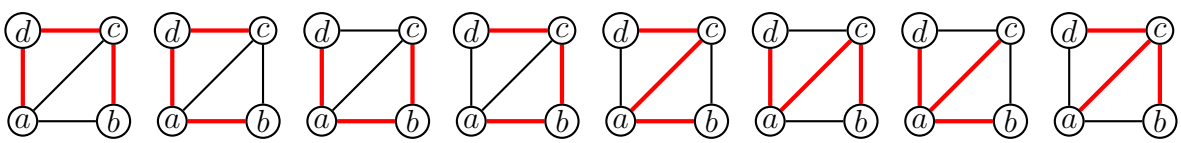
Using the natural order a, b, c, d, the adjacency matrix A and degree matrix D are

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The $(1,1)$-cofactor of $D - A$ is

$$\begin{vmatrix} -2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 12 - 2 - 2 = 8.$$

To confirm this answer, the eight spanning trees of $K_4 - e$ are given in red:



Exercise 524. *Let $G =$*  *; deleting any of the three edges of the triangle in G produces a spanning tree. Use the Matrix-Tree Theorem to confirm that G has exactly 3 spanning trees.*

Exercise 525. *Let $B =$*  *(the "bull graph"). Use the Matrix-Tree Theorem to find the number of spanning trees of B.*

Exercise 526. *Let $H =$*  *(the "house graph"). Use the Matrix-Tree Theorem to find the number of spanning trees of H.*

Exercise 527. *Show that the Matrix-Tree Theorem implies Cayley's tree formula (see Theorem 3.2.1).*

15.10 The automorphism group of a graph

Only a few results are given in this section. For a more thorough examination of automorphism groups, see, *e.g.*, Chapter 14 of [486].

Lemma 15.10.1. *Let H be a graph. If $\overline{H}$ denotes the complement of H, then $aut(\overline{H}) = aut(H)$.*

Lemma 15.10.2. *Let G be a graph and let $g \in aut(G)$. For any and $v \in V(G)$,* $\deg(v) = \deg(g(v))$.

Recall that in a graph H and $x, y \in V(H)$, the distance $d(x, y)$ between vertices x and y is the length of a shortest x–y path.

Lemma 15.10.3. *Let H be a graph and let $G = aut(H)$. For any $g \in G$ and vertices $x, y \in V(H)$, then $d(x, y) = d(g(x), g(y))$.*

Theorem 15.10.4 (Frucht, 1939 [385])**.** *For any group G, there is a graph H with $aut(H) = G$.*

A decade later, Frucht extended Theorem 15.10.4 in a way that might seem somewhat unexpected.

Theorem 15.10.5 (Frucht, 1949 [386])**.** *For any group G, there exists a 3-regular graph H with $aut(H) = G$.*

The Frucht graph (given by Frucht in 1939, see Figure 15.5) is an example of a 3-regular graph with no non-trivial automorphisms, which verifies Theorem 15.10.5 for the trivial group (consisting of only the identity map).

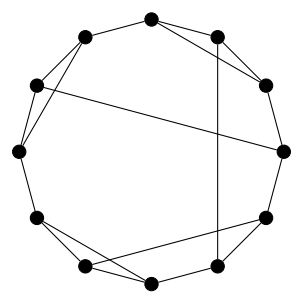


Figure 15.5: The Frucht graph, a 3-regular graph with no non-trivial automorphism

Definition 15.10.6. *If H is a graph with $aut(G) = \{id\}$, then H is called* asymmetric*.*
.

For example, the single vertex graph is asymmetric. The next smallest asymmetric graphs have 6 vertices (see Exercise 528), and the smallest non-trivial tree has 7 vertices (see Exercise 529). In 1946, Kagno [548] showed that an asymmetric graph with at one edge has at least 6 edges.

Exercise 528. *Find four 6-vertex asymmetric graphs.*

Theorem 15.10.7 (Quintas, 1967 [766], et al.)**.** *The smallest non-trivial asymmetric tree has seven vertices.*

Exercise 529. *Find an asymmetric tree on seven vertices.*

Quintas [766] found, for each n, the minimum number m_n of edges in an asymmetric graph on n vertices, and for each $n \geq 7$, produced an acyclic asymmetric graph (forest) on n vertices and m_n edges.

The opposite of asymmetric is "symmetric". After all, in geometry, any plane figure with at least one non-trivial "symmetry" (distance preserving isomorphism) is called "symmetric". However, there are various levels of "symmetric". The automorphism grouip of the complete graph K_n is the symmetric group S_n, which has $n!$ elements.

Exercise 530. *If $n \geq 3$, what is the automorphism group of the cycle C_n? How many automorphisms does C_6 have?*

In 1963, Erdős and Rényi [338] called a graph G "symmetric" if and only if G is not asymmetric, i.e., when G has at least one non-trivial automorphism. As hinted at above, there are other (stronger) definitions of "symmetric" used later; to avoid any confusion, meanings are given in parentheses.

The following lemma, observed in [338], has an easy proof, which is left to the reader.

Lemma 15.10.8. *A graph G is symmetric (has a non-trivial automorphism) if and only if its complement $\overline{G}$ is symmetric (has a non-trivial automorphism).*

Exercise 531. *Prove Lemma 15.10.8.*

Theorem 15.10.9 (Erdős–Rényi, 1963 [338])**.** *Almost all graphs are asymmetric.*

Theorem 15.10.10 (Erdős–Rényi, 1963 [338])**.** *If H is an asymmetric graph on n vertices, then H can be made symmetric (has a non-trivial automorphism) by the addition or removal of at most $n/2 + o(n)$ edges.*

Theorem 15.10.11 (Erdős–Renyi, 1963 [338])**.** *Almost all trees are symmetric (have a non-trivial automorphism).*

The idea behind the proof of Theorem 15.10.11 is simple: with high probability, there is a vertex adjacent to two or more leaves, and a map that fixes all other vertices but interchanges these leaves is an automorphism.

Theorem 15.10.12 (Babai, 1974 [62])**.** *Let G be a finite group that is not one of the cyclic groups $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_5$. Then there exists a graph H with automorphism group G where H has at most $2 \cdot |G|$ vertices.*

15.11 Highly symmetric graphs

Notation and terminology for graphs is still evolving, and there seems to be no one definition of what it means for a graph to be "symmetric"; this expression often no longer means only that at least one non-trivial automorphism exists, as Erdős and Rényi [338] first defined it.

Some terminology in this section is adapted from group theory. A group $\mathcal{G}$ is said to act transitively on a set S if and only if for any two elements $x, y \in S$, there exists $g \in \mathcal{G}$ so that $g(x) = y$. (Naturally, in this section, $\mathcal{G} = \mathrm{aut}(G)$.)

Definition 15.11.1. *A graph G is called* vertex-transitive *if and only if for every two vertices x, y there exists $\theta \in aut(G)$ so that $\theta(x) = y$.*

Other terminogy for "vertex-transitive" has been used; for example, Folkman [372] and Harary used the term "point-symmetric", and Tutte [940, p. 55] called such a graph "transitive on its vertices".

Exercise 532. *(Easy) Show that a vertex-transitive graph is regular.*

For example, the graphs K_n, C_n and the Petersen graph are vertex-transitive.

Exercise 533. *Do there exist regular graphs that are not vertex-transitive? Are there* connected *regular graphs that are not vertex-transitive?*

Exercise 534. *Find the three cubic (3-regular) vertex-transitive graphs on 10 vertices.*

An example of a vertex-transitive graph with no Hamiltonian cycle is the Coxeter graph (see Figure 15.6), which Coxeter calls "my graph" [233].

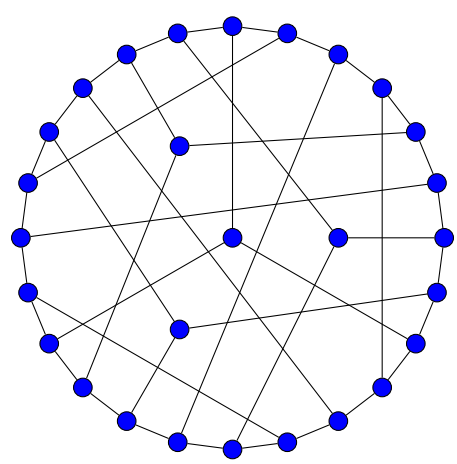


Figure 15.6: The Coxeter graph, a 3-regular non-Hamiltonian graph on 28 vertices

The Coxeter graph can be defined in a number of ways. For example, let the vertex set consist of the 28 anti-flags (pairs (P, ℓ) so that P is not on ℓ) of the Fano plane. Two

vertices (P, ℓ) and (Q, m) are adjacent if and only if $\{P, Q\} \cup \ell \cup m$ is all of the vertices in the Fano plane. (Some parameters of the Coxeter graph are given in Appendix 16.) For more information on the Coxeter graph, see [164] or [102].

Another feature of the Coxeter graph is that it is "distance-regular," a topic not considered here but for those interested, a definition and references are given in the following comment.

Remark 15.11.2. *A connected regular graph is called distance regular if for every i, j, k, and any two vertices x, y at distance k the number of vertices at distance i from x and at distance j from y depends only upon i, j, k. This definition is due to Biggs in the 1970s (see [103]), and is used in many areas of science and mathematics. For a rich survey of distance regular graphs and their applications, see [952].*

In 1972, Bondy [136] showed that the Coxeter graph is hypo-Hamiltonian (the deletion of any vertex leaves a Hamiltonian graph). (It might be of interest to see that the Coxeter graph is a subgraph of the Hoffman–Singleton graph.) By Brooks' theorem, the Coxeter graph is 3-colourable.

There are only five known (see [422]) vertex-transitive graphs with no Hamiltonian cycle, namely K_2, the Petersen graph, the Coxeter graph, and the two "triangle-replaced" versions of the Petersen graph and the Coxeter graph (where "triangle-replaced" means that each vertex x is replaced by a triangle, and at each vertex of the triangle, one of the three edges incident with x is joined—only cubic graphs have triangle-replaced versions).

Definition 15.11.3. *A graph G is called* edge-transitive *if for any two edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ there is $\theta \in aut(G)$ so that $\{\theta(x_1), \theta(y_1)\} = \{x_2, y_2\}$.*

An edge-transitive graph need not be vertex-transitive. For example, a path of length 2 is edge-transitive but not vertex transitive. In fact, it is simple to verify that if $m \neq n$, then the graph $K_{m,n}$ is edge-transitive but not vertex transitive (the case $m = 1$, $n = 2$ is precisely the path of length 2).

Exercise 535. *Let G be the graph of the pentagonal prism. Show that G is vertex-transitive but not edge-transitive.*

Both Harary [486, p. 171], and Holton and Sheehan [525, p. 209] used the word "symmetric" to mean both vertex-transitive and edge-transitive.

Exercise 536. *Show that if G is vertex-transitive and edge-transitive, then G is regular of even degree.*

In Harary's 1969 text [486, p. 172], the following is credited to Elayne Dauber (this also appeared three years earlier in Tutte's book *Connectivity in graphs* [940, pp. 55–56], where an unpublished paper by Dauber and Harary is cited):

Theorem 15.11.4 (Dauber, circa 1966)**.** *Let G be a graph with no isolated vertices. If G is edge-transitive but not vertex-transitive, then G is bipartite.*

Proof: Let G be edge-transitive, but not vertex-transitive. Let $e = \{x, y\} \in E(G)$, and let S and T be the sets of images of x and y respectively under elements of $\mathrm{aut}(G)$. Since G is edge-transitive and has no isolated points, each vertex in G is in an edge and so is in an image of either x or y, so $V(G) = S \cup T$. The identity automorphism shows that $x \in S$ and $y \in T$.

Case 1: Suppose that S and T are disjoint. If some two vertices s_1 and s_2 in S are adjacent, say with edge $f = \{s_1, s_2\}$, then since G is edge-transitive, there is an automorphism α so that $\alpha(e) = f$, and so $\alpha(y) \in \{s_1, s_2\}$, contradicting that $\alpha(y) \in T$. Thus, S is an independent set; similarly, T is an independent set. Hence, when S and T are disjoint, G is bipartite.

Case 2: Suppose that S and T are not disjoint, with, say, vertex $w \in S \cap T$. Let u and v be any two vertices in G.

Claim: There is an automorphism α with $\alpha(u) = v$.

Proof claim: If both u and v are in S, then automorphisms take $u \to x \to v$, so their composition takes u to v. (Note that if either u or v is equal to x, the desired automorphism trivially exists.) Similarly if both are in T, the desired automorphism exists. So suppose that u and v are contained in different sets, say $u \in S$ and $v \in T$. Then there exist automorphisms taking $u \to x \to w \to y \to v$, and so the composition of these proves the claim. □

Note that if G is edge-transitive, then G is "edge-regular", i.e., the same two degrees appear on endpoints of every edge. As in [486], Theorem 15.11.4 has three corollaries (whose proofs are left as exercises).

Corollary 15.11.5. *If G is edge-transitive and for every edge $\{x, y\} \in E(G)$, $\deg(x) \neq \deg(y)$, then G is bipartite.*

Corollary 15.11.6. *If G is edge-transitive and has an odd number of vertices, and if every edge $\{x, y\}$ has $\deg(x) = \deg(y)$, then G is vertex-transitive.*

Corollary 15.11.7. *If G is edge-transitive with $2k$ vertices, and is d-regular with $d \geq k$, then G is vertex-transitive.*

Exercise 537. *Prove Corollaries 15.11.5, 15.11.6, and 15.11.7.*

The only edge-transitive graphs not covered by the above are the graphs on an even number n of vertices and regular of degree $d < n/2$. Observe that even cycles are edge-transitive regular graphs that are bipartite. Also, the icosahedron, the dodecahedron, and the Petersen graph are edge-transitive, regular, vertex-transitive, but not bipartite.

Theorem 15.11.8 (Folkman, 1967 [372]). *If $n \geq 20$ is divisible by 4, there exists a regular edge-transitive graph on n vertices that is not vertex-transitive.*

An example of such a graph (with $n = 20$) found by Folkman is given in Figure 15.7; this graph is bipartite where the partite sets are the two diagonals.

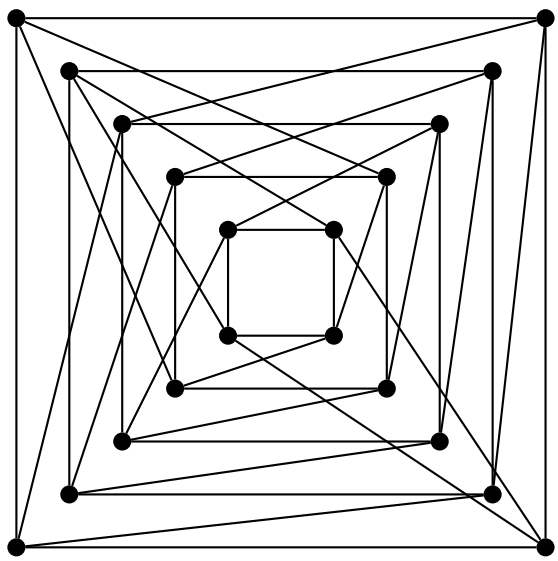


Figure 15.7: An example by Folkman of a graph that is regular, bipartite, and edge-transitive, but not vertex-transitive

In an edge-transitive graph, there is an automorphism that takes any edge to any other, but the order of the vertices in an edge is arbitrary. A stronger form of transitivity ensures that an automorphism exists that respects a given order of vertices in such edges:

Definition 15.11.9. *A graph G is called* arc-transitive *if and only if for any two edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ there is $\theta \in aut(G)$ so that $\theta(x_1) = x_2$ and $\theta(y_1) = y_2$.*

Arc-transitive graphs are also called *flag-transitive*, *1-arc-transitive*, or simply *1-transitive* [125, p. 291]. (The expression "1-arc-transitive" can be extended to s-arc-transitive, which means that for any two *ordered* paths of length s, there is an automorphism from one ordered s-path to another. See Tutte's text [940] for more on s-arc-transitivity, where an ordered path with n edges is called an "n-route".)

Note that arc-transitivity is stronger than edge-transitivity since the order of the vertices in each edge is important (hence "arc", as in a digraph).

Lemma 15.11.10. *An arc-transitive graph is also edge-transitive. If a graph G has no isolated vertices and is arc-transitive, then G is also vertex-transitive.*

Arc-transitive graphs are sometimes (*e.g.*, see [422, p. 59]) called "symmetric".

Theorem 15.11.11. *A Paley graph is arc-transitive.*

Paley graphs have no isolated vertices and so are also vertex-transitive. Thus,

Corollary 15.11.12. *Paley graphs are vertex-transitive, edge-transitive, and arc-transitive.*

.

Exercise 538. *Let G be a connected regular graph of odd degree that is both vertex-transitive and edge-transitive. Prove that G is arc-transitive.*

15.12 Graph homomorphisms and rigidity

For the definition of homomorphism, see Definition 6.4.1. A graph G is called *rigid* if and only if $|V(G)| \geq 2$ and the only homomorphism of G into itself (also called an endomorphism) is the identity.

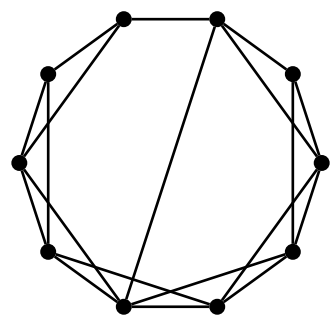


Figure 15.8: A rigid graph found by Hell

It is known [507] that if G is rigid, then G contains at least 8 vertices; it is also known [509] that rigid graphs have at least 14 edges. A minimal rigid graph (with 8 vertices and 14 edges, see Figure 15.9) was discovered by Hell and proved to be unique by Hell and Nešetřil [510].

For an extensive article that covers much of the history in studying rigid graphs, see the dedication for Hell's 60th birthday by Nešetřil [714].

Exercise 539. *Let G be a rigid graph. Show that each vertex in G has degree at least 2.*

Note: A geometric graph G (a graph embedded in some Euclidean space) is called *rigid* if the points of G cannot be perturbed while preserving edge-lengths in G (perturbations do not include simple translations, rotations, or reflections). For example, a square (on four vertices) is not rigid since it can be deformed into a rhombus, say, but any geometric triangle is rigid.

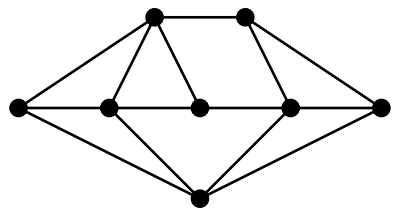


Figure 15.9: The Hell–Nešetřil graph, the unique minimal rigid graph on 8 vertices with 14 edges

15.13 Cayley graphs

A Cayley graph is a graph derived from a group; such a graph is actually the underlying graph of a digraph with labelled (coloured) edges, often called a Cayley diagram. If one ignores the direction of edges and the edge-labelling, the result is just a simple graph, also called a Cayley graph.

Definition 15.13.1. *Let G be a group and let S be a set of generators for G. The* Cayley diagram *$C(G, S)$ is defined to be a digraph on vertex set G, where (g, h) is a directed edge if and only if there exists $s \in S$ so that $sg = h$. In this case, say that the arc (g, h) is coloured s.*

In order to prevent loops in a Cayley diagram $C(G, S)$, it is usually assumed that the identity in the group G is not an element of S.

A simple undirected graph H is called a *Cayley graph* for a group G if some $C(G, S)$ has H as its underlying undirected graph. If the set of generators is symmetric (that is, if $s \in S$, then also $s^{-1} \in S$), then in $C(G, S)$, if (x, y) is a directed edge, then also (y, x) is an edge; in such a case, the Cayley diagram can be considered as an undirected simple graph. If the generating set S is the set of all non-identity elements of the group, then the Cayley graph is a complete graph.

Exercise 540. *Show that any Cayley graph is "vertex-transitive", that is, for any two vertices, there is an automorphism of the graph taking the first to the second. (In other words, show that the automorphism group of a Cayley graph is 1-transitive.)*

The smallest connected regular graph that is not a Cayley graph is the 8-vertex graph in Figure 15.10. This graph is not vertex-transitive since no automorphism takes 0 to 1.

Exercise 541. *Let $n \geq 3$ and let $G = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, the cyclic group of order n (with addition modulo n as the operation). Find a generating set S so the Cayley graph $C(G, S)$ is the cycle C_n. If n is prime, are there more generating sets S for which this true?*

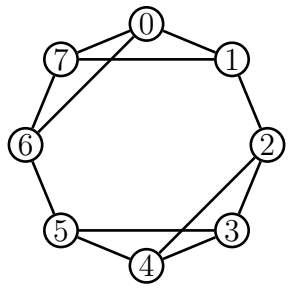


Figure 15.10: The smallest connected regular graph that is not a Cayley graph

Exercise 542. *Show that the graph of a prism is a Cayley graph for a dihedral group (with two generators).*

Exercise 543. *Show that the Petersen graph is not a Cayley graph.*

Exercise 544. *Show that for $k \geq 1$, the cube graph Q_k is a Cayley graph for abelian group $(\mathbb{Z}_2)^k$.*

Exercise 545. *Let G be a group and S be a generating set for G. Show that the Cayley graph $C(G, S)$ is connected if and only if for every $s \in S$, also $s^{-1} \in S$.*

It is known that the graphs of the icosahedron and dodecahedron are not Cayley graphs for any group.

Exercise 546. *Show that Cayley graphs for the alternating permutation group A_5 can be, depending upon the generators used, either the graph of the truncated icosahedron or the truncated dodecahedron (each has 60 vertices).*

[The result in Exercise 546 was first told to me by H. K. Farahat in 2000.]

It is known [990] that Cayley digraphs for p-groups have Hamiltonian cycles. For a survey of Cayley graphs that have Hamiltonian cycles, see [404].

15.14 Expander graphs

This section is only a very brief introduction to expander graphs; for more information and further references, see, *e.g.*, [30], [31], [34], [36], [424], [607], or [645].

Before giving a formal definition, one might think of an expander graph as one with a large number of vertices of bounded degree (and so is "sparse", i.e., has few edges), but yet any (not too big) set of vertices has "many" neighbours outside of the set.

Expander graphs have "connectivity" properties that can model numerous applications in networks. See, *e.g.*, [607] for an introduction to such applications. One such notable application [20] concerns sorting in parallel.

To check whether or not a graph is an expander graph can be done by checking each set and its neighbourhood, but such a method for large graphs is computationally intensive. However, by the mid-1980s, it was known (see, *e.g.*, [30], [34], [35], [266], and references therein) that a graph's eigenvalues alone can determine the "expansion" properties of a graph, thereby making it much simpler to determine whether or not a graph has the desired expansion properties.

For a graph $G = (V, E)$ and $S \subset V$, let

$$N(S) = \{v \in V \backslash S : \exists s \in S \text{ so that } \{s, v\} \in E\}.$$

(Some call $N(S)$ the border of S, but for present purposes, one might think of it as an "exclusive" neighbourhood.)

Definitions for expander graphs vary.

Definition 15.14.1. *For positive integers $\Delta < n$ and a real number $c \geq 1$, an (n, Δ, c) expander graph is a bipartite graph $G = (X \cup Y, E)$ with $|X| = |Y| = n$ and $\Delta = \Delta(G)$ (the maximum degree in G) so that for every $S \subseteq X$ with $|S| \leq \frac{n}{2}$,*

$$|N(S)| \geq (1 + c\left(1 - \frac{|S|}{n}\right)|S|.$$

The value c is sometimes called the expansion factor. The non-bipartite version of an expander graph is sometimes (*e.g.*, [30]) called a "magnifier" (although in [36] such is also called an expander).

Definition 15.14.2. *For positive integers $\Delta < n$ and a real number $c \geq 1$, a graph $G = (V, E)$ is an (n, Δ, c)-magnifier if and only if $|V| = n$, the maximum degree $\Delta(G)$ is at most Δ, and for any subset $S \subset V$ with $|S| \leq n/2$,*

$$|N(S)| \geq c|S|.$$

Exercise 547. *For some n, Δ and c, let $G = (V, E)$ be a (n, Δ, c)-magnifier on vertex set $V = \{v_1, \ldots, v_n\}$. Create an equibipartite graph $H = (X \cup Y, F)$ with partite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, and for $1 \leq i, j \leq n$, let $\{x_i, y_j\} \in F$ if and only if $i = j$ or $\{x_i, x_j\} \in F$. Show that H is, under the first definition, a $(n, \Delta + 1, c)$-expander.*

Only regular expander graphs (or magnifiers) are considered in this introduction. In 1973, Pinsker [751] observed that with high probability a random d-regular graph is a "good" expander.

The remainder of this introduction to (regular) expander graphs is largely based on [36, pp. 143–145]. Some of the details or generalizations thereof can be found in [34], [35], and [30] (but some details seem to be only available in [36]).

For a graph $G = (V, E)$ and disjoint subsets $A \subset V$ and $B \subset V$ let $e(A, B)$ denote the number of edges from A to B.

Theorem 15.14.3 (see [36]). *For any d-regular graph $G = (V, E)$ with second-largest eigenvalue λ and any partition $V = B \cup \overline{B}$,*

$$e(B, \overline{B}) \geq \frac{(d-\lambda)|B| \cdot |\overline{B}|}{|V|}.$$

Proof: Let $G = (V, E)$ be d-regular graph with $V = \{v_1, \ldots, v_n\}$, and with this ordering, let A be the adjacency matrix for G. Let $V = B \cup \overline{B}$ be a partition and let $b = |B|$. Let $D = dI_n$ (the diagonal matrix with ds along the main diagonal), and consider the matrix $D - A$ (sometimes called the "Laplacian").

Write the eigenvalues of A in increasing order (which is rarely done in the real world),

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda \leq \lambda_n = \lambda_{\max} = d,$$

where the second largest eigenvalue is $\lambda = \lambda_{n-1}$, and $d = \lambda_n$ by Theorem 15.3.4. The $n \times 1$ all 1s (column) vector $\mathbf{1}_n$ is an eigenvector for λ_n.

For any eigenvalue λ_i of A and associated eigenvector, $\mathbf{x}$ with $A\mathbf{x} = \lambda_i \mathbf{x}$,

$$(D - A)\mathbf{x} = D\mathbf{x} - A\mathbf{x} = d\mathbf{x} - \lambda_i \mathbf{x} = (d - \lambda_i)\mathbf{x}$$

shows that $d - \lambda_i$ is an eigenvalue for $D - A$ (with the same eigenvector). Hence the eigenvalues of $D - A$ are (in increasing order)

$$0 = d - \lambda_n \leq d - \lambda_{n-1} \leq d - \lambda_{n-2} \leq \cdots \leq d - \lambda_1.$$

For any real-valued $n \times 1$ vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, calculate the inner product

$$\begin{aligned}
\langle (D - A)\mathbf{x}, \mathbf{x} \rangle &= \langle D\mathbf{x}, \mathbf{x} \rangle - \langle A\mathbf{x}, \mathbf{x} \rangle \\
&= \langle d\mathbf{x}, \mathbf{x} \rangle - \langle A\mathbf{x}, \mathbf{x} \rangle \\
&= d \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j \\
&= d \sum_{i=1}^{n} x_i^2 - 2 \sum_{\{x_i, x_j\} \in E} x_i x_j \\
&= \sum_{\{x_i, x_j\} \in E} (x_i - x_j)^2.
\end{aligned}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ be defined by

$$x_i = \begin{cases} b - n & \text{if } v_i \in B \\ b & \text{if } v_i \in \overline{B} \end{cases}.$$

Then $\mathbf{x}$ is orthogonal to $\mathbf{1}_n$, and so $\mathbf{x}$ is not in the eigenspace corresponding to the smallest eigenvalue of $D-A$. It follows (with some work) that $\langle (D-A)\mathbf{x}, \mathbf{x}\rangle$ is at least the second smallest eigenvalue of $D-A$, giving

$$\begin{aligned}\langle (D-A)\mathbf{x}, \mathbf{x}\rangle &\geq (d-\lambda)\langle \mathbf{x}, \mathbf{x}\rangle \\ &= (d-\lambda)b(n-b)^2 + (n-b)b^2 \\ &= (d-\lambda)b(n-b)n.\end{aligned}$$

Also, using this particular value of $\mathbf{x}$, observe that if $\{x_i, x_j\} \in E$, then $x_i - x_j = \pm n$, and so

$$\sum_{\{x_i,x_j\}\in E} (x_i - x_j)^2 = e(B, \overline{G})n^2.$$

Putting together the two inequalities derived above, $e(B, \overline{B}) \geq \frac{d-\lambda)b(n-b)}{n}$. □

Corollary 15.14.4 (see [36]). *Let G be a d-regular graph on n vertices with second-largest eigenvalue λ. Then for $c = \frac{d-\lambda}{2d}$, G is an (n, d, c)-magnifier.*

Proof: Let $S \subset V(G)$ with $|S| \leq n/2$. By Theorem 15.14.3, there are at least

$$(d-\lambda)|S|(n-|S|)/n \leq \frac{d-\lambda}{2}|S|$$

edges from S to $\overline{S}$. Since no vertex in $\overline{S}$ is incident with more than d of these "crossing" edges, it follows that $|N(S)| \geq \frac{d-\lambda}{2d}|S|$. □

It is known (see [36], [723]) that in any infinite family of d-regular graphs the limsup of second largest eigenvalue is at least $2\sqrt{d-1}$. Any graph that achieves this bound is called a Ramanujan graph. (So Ramanujan graphs have the largest possible "spectral gap").

In 2003, Friedman [382] showed that a random d-regular graph is almost always "nearly" a Ramanujan graph, but explicitly constructing an infinite family of Ramanujan graphs is complicated.

In 1986, Lubotzky, Phillips, and Sarnak [646], and (independently) in 1988, Margulis [654] gave celebrated constructions of infinite families of d-regular Ramanujan graphs when d is a prime congruent to 1 modulo 4. Their graphs are Cayley graphs (see Section 15.13), but details are omitted here. For more on Ramanujan graphs, one might look at a thesis [722] by Timothy Nikkel while a U. of M. graduate student supervised by Michael Doob, a prof at U. of M. with over 50 years in the department—and an expert in spectral graph theory (*e.g.*, he is one of the authors of [237], one of the most comprehensive studies of algebraic graph theory).

Chapter 16

Appendix: Some parameters for a few named graphs

This list can be expanded greatly; some other values are mentioned earlier, with some in the exercises.

- The eigenvalues for K_n are $n-1$ and -1, where -1 has multiplicity $n-1$. (See Exercise 487.)
- The eigenvalues for the cycle C_n are 2, $2\cos(2\pi/n)$, $2\cos(4\pi/n)$,..., $2\cos(2(n-1)\pi/n)$. (See Exercise 490.)
- The eigenvalues for the cube graph Q_n are $\{n-2k : k = 0, 1, \ldots, n\}$, where $n-2k$ has multiplicity $\binom{n}{k}$. (See Exercise 491.)
- The eigenvalues for $K_{m,n}$ are $\pm\sqrt{mn}$ and 0. (See Exercise 489.)
- The eigenvalues for the Kneser graph $KG(n,k)$ are

  $$(-1)^i \binom{n-k-i}{k-i} : \quad i = 0, 1, \ldots, k.$$

  (For a proof, see [422, pp. 200–201].)
- The Petersen graph (call it P here) has 10 vertices, 15 edges, and characteristic polynomial $(x-3)(x-1)^5(x+2)^4$. (See Exercise 507.) Also, (see Exercise 204), P is cubic (3-regular), has radius 2, and is the largest cubic graph with diameter 2. The Petersen graph is the smallest example of a cubic bridgeless graph that is not 3-edge-colourable.

  It is the smallest cubic graph of girth 5; it is the unique $(3,5)$-cage and is a Moore graph. It is 3-connected, 3-edge-connected (and so bridgeless), $\alpha(P) = 4$, (see Exercise 204), $\chi(P) = 3$ (see Exercise 231), its domination number is 3, P has

6 perfect matchings and a 2-factor, it has 2000 spanning trees (the most of any 10-vertex cubic graph). The chromatic polynomial is $x(x-1)(x-2)(x^7-12x^6+67x^5-230x^4+529x^3-814x^2+775x-352$. The Petersen graph is also a unit-distance graph. The Petersen graph is vertex-transitive and its automorphism group is S_5 (and so has 120 automorphisms). Although P is not Hamiltonian, deleting any vertex produces a Hamiltonian subgraph (and so is called "hypo-Hamiltonian").

- If P is the Petersen graph, its line graph $L(P)$ has eigenvalues -2 (multiplicity 5), -1 (multiplicity 4), 2 (multiplicity 5), and 4 (multiplicity 1).

- For a prime power $q \equiv 1 \pmod 4$, the Paley graph (see Definition 9.6.2) of order q has eigenvalues q, $\frac{-1+\sqrt{q}}{2}$, $\frac{-1-\sqrt{q}}{2}$ with multiplicities 1, $\frac{q-1}{2}$, and $\frac{q-1}{2}$ respectively.

- The graph of the octahedron, denoted O_6, is isomorphic to $K_{2,2,2}$—see Figure 1.20. Then O_6 is 4-regular, has characteristic polynomial $(x-4)x^3(x-2)^2$, and has chromatic number 3.

- The graph D_{20} of the dodecahedron (see Figure 1.20) on 20 vertices and 30 edges is Hamiltonian (with 60 Hamiltonian cycles), has chromatic number 3, both vertex- and edge-connectivity 3, diameter 5, and has eigenvalues $-\sqrt{5}$ (multiplicity 3), -2 (multiplicity 4), 1 (multiplicity 5), 0 (multiplicity 4), $\sqrt{5}$ (mulitplicity 3) and 3 (multiplicity 1). The automorphism group of D_{20} is $A_5 \times \mathcal{Z}_2$, which has cardinality 120.

- The Herschel graph (see Figure 2.11) on 11 vertices and 18 edges is traceable (has a Hamiltonian path) but is not Hamiltonian (see Exercise 116). It is the smallest non-Hamiltonian 3-connected planar graph.

- The Shrikhande graph (see Figure 15.4) on 16 vertices and 48 edges has characteristic polynomial $(x-6)(x-2)^6(x+2)^9$, is strongly regular with parameters $(16, 6, 2, 2)$, is Hamiltonian and Eulerian, has chromatic number 4, girth 3, both vertex- and edge-connectivity 6, independence number 4, and has 192 automorphisms.

- The Hoffman–Singleton graph (see Figure 11.6) is strongly regular with parameters $(50, 7, 0, 1)$, and has eigenvalues -3 (multiplicity 21), 2 (multiplicity 28), and 7 (multiplicity 1). The edge-chromatic number is 7, the independence number is 15, and the girth is 5.

- The Higman–Sims graph (see Section 15.4 for details) is strongly regular with parameters $(100, 22, 0, 6)$. It has characteristic polynomial $(x-22)(x-2)^{77}(x+8)^{22}$.

- The Coxeter graph (see Figure 15.6) has 28 vertices, 42 edges, is connected, 3-regular, vertex-transitive, has girth 7, has no Hamiltonian cycle, and has characteristic polynomial $(x-3)(x-2)^8(x+1)^7(x^2+2x-1)^6$ (proof by van Dam and Haemers [949]). The Coxeter graph is the only graph with its spectrum, and so is uniquely identified by its eigenvalues. Its automorphism group has order 336. Both the chromatic number and chromatic index are 3.

- The Frucht graph (see Figure 15.5) has characteristic polynomial

  $$(x-3)(x-2)x(x+1)(x+1)(x+2)(x^3+x^2-2x-1)(x^4+x^3-6x^2-5x+4).$$

  The Frucht graph has 12 vertices, is 3-regular, has chromatic number 3, chromatic index 3, radius 3, diameter 4, girth 3, only one automorphism (the trivial one), and is pancyclic.

- The Tutte graph (see Figure 7.13) is planar, 3-connected, 3-regular, has 46 vertices, 69 edges, 25 faces, chromatic number 3, chromatic index 3, girth 4, and diameter 8, and has only three automorphisms.

Chapter 17

Appendix: Matrix theory

17.1 Some basic matrix theory

This section contains a review of some facts about matrices and eigenvalues, primarily for use in Chapter 15. Some of these facts are regularly taught in a first course in linear algebra, whereas many are taught in more advanced matrix theory courses.

The set of all $m \times n$ matrices with real entries is often denoted here by $M_{m\times n}(\mathbb{R})$, and when $m = n$, the set $M_{n\times n}(\mathbb{R})$ is occasionally denoted by simply $M_n(\mathbb{R})$. If entries are chosen from another field $\mathbb{F}$, (*e.g.*, $\mathbb{C}$, the complex numbers) the notation $M_{m\times n}(\mathbb{F})$ is used. Often, a matrix $A \in M_{m\times n}(\mathbb{R})$ is denoted by $A = (a_{ij})$, where lower case letters are used for entries in the matrix. The *transpose* of an $m \times n$ matrix A, denoted by A^T, is the $n \times m$ matrix $B = (b_{ij})$ defined by $b_{ij} = a_{ji}$. A matrix B is *symmetric* if and only if $B = B^T$. A diagonal matrix is a square matrix whose off-diagonal entries are 0s; the notation $\mathrm{diag}(d_1, \ldots, d_n)$ can be used to denote such a matrix $D = (d_{ij})$ where $d_{ii} = d_i$ (and for $i \neq j$, $d_{ij} = 0$).

For any positive integer n, the identity matrix I_n is the $n \times n$ matrix with 1s on its main diagonal and 0s elsewhere. Another commonly used matrix is the $n \times n$ matrix with every entry equal to 1; this matrix is denoted by J_n.

Let A be an $m \times p$ matrix and B be a $p \times n$ matrix. The product AB is the $m \times n$ matrix $C = (c_{ij})$ defined by, for each $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$,

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

An $n \times n$ matrix A is said to be *invertible* if there exists an $n \times n$ matrix B so that $AB = BA = I_n$. If such a B exists, B is unique, and is called the *inverse* of A, denoted A^{-1}.

There are many conditions on a matrix equivalent to being invertible. The first few of these conditions are given in the following (whose proof is a fundamental result often given in the first few days of a first linear algebra course).

Theorem 17.1.1. *Let A be an $n \times n$ matrix. The following conditions are equivalent.*

(a) A is invertible.

(b) The only solution to $A\mathbf{x} = \mathbf{0}$ is $n \times 1$ vector $\mathbf{x} = \mathbf{0}$.

(c) For any $n \times 1$ vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(d) The row-reduced-echelon form of A is I_n.

To each square matrix A, there is a number, called the *determinant* of A, denoted by either $\det(A)$ or by $|A|$, which can be defined in various ways. One way is by elementary products, another is by cofactor expansion. See any textbook on linear algebra for these definitions. Some properties of the determinant (for real valued matrices) are:

- A matrix A is invertible if and only if $\det(A) \neq 0$.
- If B is formed from A by multiplying a row (or column) of A by a constant k, then $\det(B) = k \cdot \det(A)$. Hence, if A is $n \times n$, $\det(kA) = k^n \cdot \det(A)$.
- If B is formed from A by interchanging two rows (or columns) of A, then $\det(B) = -\det(A)$.
- If B is formed from A by adding any multiple of one row (or column) to another row (or column) of A, then $\det(B) = \det(A)$.

If $x \in \mathbb{C}$, $x = a + bi$, where a and b are real, then the complex conjugate of x is $\overline{x} = a - bi$. If $B \in M_{m\times n}(\mathbb{C})$, the *complex transpose* is $B^* = (\overline{b}_{ij})^T$. A matrix A is called *Hermitian* if and only if $A^* = A$.

If $A \in \mathcal{M}_{m\times k}(\mathbb{R})$ and $B \in \mathcal{M}_{k\times n}(\mathbb{R})$, then the product AB is square, namely, $n \times n$. Binet and Cauchy independently noted a formula for $\det(AB)$ in terms of products of submatrices in A and B. [This result is now called either the Binet-Cauchy theorem/formula, but perhaps depending on which version is used, the Cauchy-Binet theorem; see [701] for details on Cauchy's role. The first version at least has the authors in alphabetical order.]

To state the Benet-Cauchy theorem, some notation for sets is convenient.

Recall (see Section 19.1.1) that for some set X and some $d \in [1, \ldots, |X|$, the notation $[X]^d$ denotes the collection of all d-element subsets of X. Also, let $[1, k] = \{1, \ldots, k\}$, and write $[k]^d$ to denote the d-subsets of $[k]$.

Theorem 17.1.2 (Binet, Cauchy, 1812)**.** *Let $A \in \mathcal{M}_{n\times k}(\mathbb{R})$ and $B \in \mathcal{M}_{k\times n}(\mathbb{R})$. For any set $S \subset [k]$, let A_S be the (square) submatrix induced by rows and columns whose indices are S. Similarly, define B_S. Then*

$$\det(AB) = \sum_{S\in[n]^k} \det(A_S)\det(B_S).$$

There is a slightly more general form of the Binet–Cauchy formula for determinants of minors of AB, even when AB is not square (see, *e.g.*, [528, p. 22]). Note that when $n = k$, this just says $\det(AB) = \det(A)\det(B)$.

17.2 Eigenvalues, eigenvectors, and characteristic polynomials

For a square matrix A, a number λ (either real or complex) is called an *eigenvalue* of A if and only if there exists a non-zero vector $\mathbf{x}$ so that $A\mathbf{x} = \lambda\mathbf{x}$; such a vector $\mathbf{x}$ is called an *eigenvector* associated with λ.

Exercise 548. *Show that if* $\mathbf{x}$ *is an eigenvector for* A*, then so is any scalar multiple of* $\mathbf{x}$.

For any $A \in M_{n\times n}(\mathbb{C})$ and any scalar $\lambda \in \mathbb{C}$, the system $A\mathbf{x} = \lambda\mathbf{x}$ has non-zero solutions precisely when $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$. Such non-zero solutions exist if and only if $\det(\lambda I_n - A) = 0$.

Definition 17.2.1. *If* A *is an* $n\times n$ *matrix, the* characteristic polynomial *of* A*, denoted* c_A*, is defined by* $c_A(\lambda) = \det(\lambda I_n - A)$*, a polynomial of degree* n *in the variable* λ*. The equation* $c_A(\lambda) = 0$ *is called the* characteristic equation *for* A.

It is not uncommon to use the variable x in the characteristic polynomial, in which case $c_A(x) = \det(xI_n - A)$.
Note: Many texts define c_A as the determinant of $A - \lambda I$, rather than $\lambda I - A$; when n is even, these are the same, however for odd n, one is the negative of the other. No matter which way the characteristic polynomial is defined, each have precisely the same set of roots.

The eigenvalues of A are precisely the roots of c_A. By the fundamental theorem of algebra, if A is $n \times n$, then there are n eigenvalues (possibly complex, counted with multiplicity). The multiplicity of any root λ of c_A is also called the *algebraic multiplicity* of the eigenvalue λ.

Example 17.2.2. *Let* $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$*. Expanding the determinant of* $xI - A$ *along the first row,*

$$c_A(x) = det(xI - A) = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix}$$

$$
\begin{aligned}
&= x \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & x \end{vmatrix} - \begin{vmatrix} -1 & x \\ -1 & -1 \end{vmatrix} \\
&= x(x^2-1) + (-x-1) - (1+x) \\
&= (x+1)[x(x-1)-2] \\
&= (x+1)(x+1)(x-2).
\end{aligned}
$$

So $c_A(x) = (x+1)^2(x-2)$, and thus $\lambda_1 = -1$ and $\lambda_2 = 2$ are the eigenvalues of A, where λ_1 has multiplicity 2.

To find eigenvalues and eigenvectors, one way is to find the characteristic polynomial, factor it, find its roots, and then for each root λ, solve $(\lambda I - A)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. Another common way to find eigenvalues is to first "spot" eigenvectors for A by carefully examining patterns in A; eigenvalues are then evident.

Continuing Example 17.2.2, for $\lambda_1 = -1$, the system $(-1(I) - A)\mathbf{v} = \mathbf{0}$ becomes

$$
\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right],
$$

which has general solution of the form

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},
$$

giving two linearly independent eigenvectors $[-1, 1, 0]^T$ and $[-1, 0, 1]^T$.

For $\lambda_2 = 2$, the system $(2I - A)\mathbf{v} = \mathbf{0}$ becomes

$$
\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right],
$$

which has general solution

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
$$

giving an eigenvector $[1, 1, 1]^T$. □

If A is an $n \times n$ matrix with characteristic polynomial

$$
c_A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \prod_{i=1}^{n} (x - \lambda_i),
$$

then $-a_{n-1} = \sum_{i=1}^{n} \lambda_i$. By setting $x = 0$, $a_0 = (-1)^n \det(A)$. (In Corollary 17.3.9, it is shown also that $-a_{n-1} = \mathrm{tr}(A)$.)

Note that the eigenvalues of a triangular (or diagonal) matrix are the main diagonal entries.

Exercise 549 (see [528, p. 56]). *Let $D = diag(d_1, \ldots, d_n)$ be a diagonal matrix where all d_i's are real and distinct. Show that for $A \in M_n(\mathbb{R})$, $AD = DA$ if and only if A is diagonal.*

A matrix that is not invertible is sometimes called *singular*.

Lemma 17.2.3. *A square matrix B is not invertible if and only if 0 is an eigenvalue for B.*

Proof: Let B be $n \times n$, and let I_n denote the $n \times n$ identity matrix.

First, suppose that $\lambda = 0$ is an eigenvalue for B. Then setting $\det(B - \lambda I_n) = 0$ gives $\det(B) = 0$, and so B is not invertible.

Next, assume that B is not invertible. Then there exist non-zero solutions to $B\mathbf{x} = \mathbf{0}$, and so there are non-zero vectors $\mathbf{x}$ such that $B\mathbf{x} = 0 \cdot \mathbf{x}$, that is, 0 is an eigenvalue for B. □

Definition 17.2.4. *A matrix $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ is called a* circulant matrix *if and only if for each i, j, $a_{i+1,j+1} = a_{i,j}$ (where all addition is modulo n).*

So, a circulant matrix is one whose rows are formed by consecutively shifting the next row one element to the right. Letting ω be an nth root of unity,

$$\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ b_n & b_1 & b_2 & \cdots & b_{n-1} \\ b_{n-1} & b_n & b_1 & \cdots & b_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_2 & b_3 & \cdots & b_n & b_1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \vdots \\ \omega^{n-1} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} b_i\omega^{i-1} \\ \sum_{i=1}^{n} b_i\omega^{i} \\ \vdots \\ \sum_{i=1}^{n} b_i\omega^{i-2} \end{bmatrix}$$

$$= \sum_{i=1}^{n} b_i\omega^{i-1} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \vdots \\ \omega^{n-1} \end{bmatrix},$$

so the corresponding eigenvalue is $\lambda_\omega = \sum_{i=1}^{n} b_i\omega^{i-1}$. Since there are n roots of unity, all eigenvalues (and associated eigenvectors) are found.

Lemma 17.2.5. *If λ is an eigenvalue for A, then for each $k \in \mathbb{Z}^+$, λ^k is an eigenvalue for A^k; furthermore, if $\mathbf{x}$ is an eigenvector of A associated with λ, then also $\mathbf{x}$ is an eigenvector of A^k associated with λ^k.*

Proof: Let $\mathbf{x}$ be an eigenvector associated with λ, that is, $A\mathbf{x} = \lambda\mathbf{x}$. Then, inductively,

$$A^k\mathbf{x} = A^{k-1}(A\mathbf{x}) = A^{k-1}\lambda\mathbf{x} = \lambda A^{k-2}A\mathbf{x} = \lambda A^{k-2}\lambda\mathbf{x} = \cdots = \lambda^k\mathbf{x}.$$

□

Theorem 17.2.6 (Cayley–Hamilton)**.** *Let A be an $n \times n$ matrix with characteristic polynomial $c_A(x)$. Then $c_A(A) = O_{n\times n}$, the $n \times n$ zero matrix.*

For a remarkable proof of the Cayley–Hamilton theorem that uses digraphs, see [166, pp. 34–38].

It is known (see, *e.g.*, [381, p. 516]) that if $f(x)$ is a polynomial for which $f(A) = O$, then $f(x)$ divides $c_A(x)$. The polynomial of the smallest degree that vanishes on A is called the *minimal polynomial* of A. The minimal polynomial has degree equal to the number of distinct eigenvalues. The minimal polynomial and the characteristic polynomial have the same roots except for multiplicities.

The following property of the trace has a direct proof.

Lemma 17.2.7. *For any $p, q \geq 1$ and matrices $A \in M_{p\times q}(\mathbb{C})$ and $B \in M_{q\times p}(\mathbb{C})$,*

$$tr(AB) = tr(BA).$$

Proof: The (i, i) entry of AB is $\sum_{k=1}^{q} a_{ik}b_{ki}$. So

$$\mathrm{tr}(AB) = \sum_{i=1}^{p}\sum_{k=1}^{q} a_{ik}b_{ki}.$$

Similarly, the (i, i) entry of BA is $\sum_{k=1}^{p} b_{ik}a_{ki}$, so

$$\mathrm{tr}(BA) = \sum_{i=1}^{q}\sum_{k=1}^{p} b_{ik}a_{ki}.$$

The two sums are equal by interchanging the roles of k and i, that is, by reversing the summation in the second expression. □

Theorem 17.2.8. *Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues for an $n \times n$ matrix A, and for each i, let $\mathbf{x}_i$ be an eigenvector associated with λ_i. Then the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent.*

Proof: For each $k \geq 1$, let $S(k)$ be the statement of the theorem.
BASE STEP: Since (by definition of eigenvector) each eigenvector is non-zero, $\{\mathbf{x}_1\}$ is an independent set, so $S(1)$ holds.

INDUCTION STEP: Fix some $m \geq 1$, suppose that $S(m)$ is true, and let $\lambda_1, \ldots, \lambda_{m+1}$ be distinct eigenvalues with associated eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_{m+1}$. Let $a_1, \ldots, a_{m+1}$ denote scalars (complex numbers) and set

$$a_1\mathbf{x}_1 + \cdots + a_{m+1}\mathbf{x}_{m+1} = \mathbf{0}. \tag{17.1}$$

(It remains to show that $a_1 = \cdots = a_{m+1} = 0$.) Multiply both sides of (17.1) by $A - \lambda_{m+1}I_n$ and obtain

$$a_1(\lambda_1 - \lambda_{m+1})\mathbf{x}_1 + \cdots + a_m(\lambda_m - \lambda_{m+1})\mathbf{x}_m = \mathbf{0}.$$

By $S(m)$, for each $i = 1, \ldots, m$, $a_i(\lambda_i - \lambda_{m+1}) = 0$, and since the λ_js are distinct, one concludes that $a_1 = \cdots = a_m = 0$. Replacing these values in (17.1), since $\mathbf{x}_{m+1} \neq \mathbf{0}$, then $a_{m+1} = 0$ as well, completing the proof of linear independence, thereby completing the inductive step.

By MI, for each $k = 1, \ldots, n$, the statement $S(k)$ holds. □

Lemma 17.2.9. *Let $A \in M_{n\times n}(\mathbb{C})$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be linearly independent eigenvectors and for each $i = 1, \ldots, k$, let λ_i be the eigenvalue associated with $\mathbf{v}_i$. Then there exists an invertible P so that $P^{-1}AP$ is of the form*

$$P^{-1}AP = \begin{bmatrix} D & B \\ O & C \end{bmatrix},$$

where $D = diag(\lambda_1, \ldots, \lambda_k)$, $B \in M_{k\times(n-k)}(\mathbb{C})$, and $C \in M_{(n-k)\times(n-k)}(\mathbb{C})$.

Proof: Extend $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $\mathbb{C}^n$. Put $P = [\mathbf{v}_1|\ldots|\mathbf{v_n}]$, the matrix with columns $\mathbf{v}_i$s; note that P is invertible. Since

$$I_n = P^{-1}P = P^{-1}[\mathbf{v}_1|\ldots|\mathbf{v}_n],$$

it follows that for each $i = 1, \ldots, n$, $P^{-1}\mathbf{v}_i = \mathbf{e}_i$, the standard basis vector. Hence

$$\begin{aligned} P^{-1}AP =& P^{-1}A[\mathbf{v}_1|\ldots|\mathbf{v}_n] \\ =& [P^{-1}A\mathbf{v}_1|\ldots|P^{-1}A\mathbf{v}_n] \\ =& [P^{-1}\lambda_1\mathbf{v}_1|\ldots|P^{-1}\lambda_k\mathbf{v}_k|P^{-1}A\mathbf{v}_{k+1}|\ldots P^{-1}A\mathbf{v}_n] \\ =& [\lambda_1\mathbf{e}_1|\ldots\lambda_k\mathbf{e}_k|P^{-1}A\mathbf{v}_{k+1}|\ldots P^{-1}A\mathbf{v}_n], \end{aligned}$$

which is of the desired form. □

Corollary 17.2.10. *If $A \in M_{n\times n}(\mathbb{C})$ has n linearly independent eigenvectors, then A is diagonalizable.*

For each eigenvalue λ of A, define

$$E_\lambda = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \lambda\mathbf{x}\},$$

called the *eigenspace* corresponding to λ. Note that $\mathbf{0} \in E_\lambda$, but by definition, $\mathbf{0}$ is not an eigenvector. Each eigenspace is indeed a subspace of $\mathbb{C}^n$. The dimension of E_λ is called the *geometric multiplicity* of λ.

Lemma 17.2.11. *The geometric multiplicity of an eigenvector is at most its algebraic multiplicity.*

Proof: Let A be an $n \times n$ matrix, and let λ be an eigenvalue with eigenspace E_λ. Let $\dim(E_\lambda) = k$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for E_λ. By Lemma 17.2.9, there exists P, B and C so that

$$P^{-1}AP = \begin{bmatrix} \lambda I_k & B \\ O & C \end{bmatrix}.$$

Then

$$\begin{aligned} \det(xI_n - A) &= \det(P^{-1}(xI_n - A)P) \\ &= \det(xI_n - P^{-1}AP) \\ &= \det \begin{bmatrix} (x-\lambda)I_k & -B \\ O & xI - C \end{bmatrix} = (x-\lambda)^k \det(xI - C). \end{aligned}$$

Hence the algebraic multiplicity of λ is at least k. □

17.3 Symmetric matrices and similar matrices

Since the adjacency matrix of a graph is symmetric, some basics regarding symmetric matrices are included here. Adjacency matrices also fall into another well-studied class of matrices, namely, non-negative matrices, and some general theorems for such are briefly reviewed in Section 17.4.

Theorem 17.3.1. *Let $B \in M_n(\mathbb{R})$ be a symmetric matrix. Then all eigenvalues of B are real.*

Proof: Let $\mathbf{x} \in M_{n\times 1}(\mathbb{C})$ be a column matrix with complex entries. Then

$$(\mathbf{x}^* B\mathbf{x})^* = \mathbf{x}^* B^* \mathbf{x}^{**} = \mathbf{x}^* B\mathbf{x},$$

hence $\mathbf{x}^* B\mathbf{x}$ is real. Let λ be an eigenvalue for B, and let $\mathbf{x}_1$ be an eigenvector associated with λ. Then $\mathbf{x}_1^* B\mathbf{x}_1 = \mathbf{x}_1^* \lambda\mathbf{x}_1 = \lambda\|\mathbf{x}_1\|^2$ is real, and so λ is real. [Note: one could take an eigenvector with norm 1, in which case $\lambda = \mathbf{x}_1^* B\mathbf{x}_1$, which is real.] □

Lemma 17.3.2. *Let $\lambda \in \mathbb{R}$ be a real eigenvalue for a matrix $B \in M_n(\mathbb{R})$. Then associated with λ, there exists an eigenvector with all real entries.*

Proof: Let $\mathbf{x} \in M_{n\times 1}(\mathbb{C})$ be a non-zero complex valued eigenvector associated with λ. Write $\mathbf{x} = \mathbf{a} + \mathbf{b}i$, where $\mathbf{a}$ and $\mathbf{b}$ are real-valued vectors. Then

$$B\mathbf{a} + iB\mathbf{b} = B(\mathbf{a} + \mathbf{b}i) = \lambda(\mathbf{a} + \mathbf{b}i) = \lambda\mathbf{a} + i\lambda\mathbf{b}.$$

Comparing real and imaginary parts, both $B\mathbf{a} = \lambda\mathbf{a}$ and $B\mathbf{b} = \lambda\mathbf{b}$. Since $\mathbf{x}$ is not zero, at least one of $\mathbf{a}$ or $\mathbf{b}$ is non-zero, and such is a real-valued eigenvector. □

So, from Theorem 17.3.1 and Lemma 17.3.2, one concludes that symmetric matrices have all real eigenvalues, and to each, one can associate a real eigenvector.

Two $n \times n$ matrices A and B are said to be *similar* if and only if there exists an invertible matrix Q so that $A = Q^{-1}BQ$.

Lemma 17.3.3. *Similar matrices have the same trace. Similar matrices also have the same determinant.*

Proof: Let A and B be similar matrices in $M_{n\times n}(\mathbb{C})$, and let $Q \in M_{n\times n}(\mathbb{C})$ be so that $A = Q^{-1}BQ$. By Lemma 17.2.7, $\operatorname{tr}(A) = \operatorname{tr}(Q^{-1}BQ) = \operatorname{tr}(Q^{-1}QB) = \operatorname{tr}(I_nB) = \operatorname{tr}(B)$. The second statement holds because

$$\begin{aligned}\det(A) &= \det(Q^{-1}BQ)\\ &= \det(Q^{-1})\det(B)\det(Q)\\ &= \frac{1}{\det(Q)}\det(B)\det(Q)\\ &= \det(B).\end{aligned}$$

□

Lemma 17.3.4. *Similar matrices have the same characteristic polynomial.*

Proof: Let A and B be similar matrices with Q so that $A = Q^{-1}BQ$. Then $c_A(x) = |xI - A| = |xI - Q^{-1}BQ| = |Q^{-1}(xI - B)Q| = |xI - B| = c_B(x)$. □

Hence similar matrices have the same eigenvalues (but not necessarily the same eigenvectors). Since permutation matrices are invertible (in fact, if P is a permutation matrix, that is, there is exactly one 1 in each row and in each column, then $P^{-1} = P^T$), relabelling vertices in a graph gives another adjacency matrix with precisely the same eigenvalues. Hence, the spectrum of one adjacency matrix for a graph is the same as the spectrum of another.

For appropriately sized matrices A and B, the next theorem relates the eigenvalues of AB and BA. Lovász [640, 11.2b] calls this result "well-known"; indeed, this result appears in [528, pp. 53–54]. (This theorem helps to find eigenvalues of line graphs—see Definition 1.7.6 for the definition of a line graph).

Theorem 17.3.5. *Let $A \in M_{m\times n}(\mathbb{C})$ and $B \in M_{n\times m}(\mathbb{C})$. Then the characteristic polynomials c_{BA} and c_{AB} are related by*

$$c_{BA}(x) = x^{n-m}c_{AB}(x).$$

So BA has the same eigenvalues as AB (with the same multiplicities) together with $n-m$ additional eigenvalues equal to 0. Furthermore, if $m = n$ and one of A or B is invertible, then BA is similar to AB.

Proof: It is nearly trivial to check that any non-zero eigenvalue of AB is also an eigenvalue of BA; however, more is needed. By block multiplication of matrices in $M_{m+n}(\mathbb{C})$,

$$\begin{aligned}\begin{bmatrix} AB & O_{m\times n} \\ B & O_{n\times n} \end{bmatrix}\begin{bmatrix} I_m & A \\ O_{n\times m} & I_n \end{bmatrix} &= \begin{bmatrix} AB & ABA \\ B & BA \end{bmatrix} \\ &= \begin{bmatrix} I_m & A \\ O_{n\times m} & I_n \end{bmatrix}\begin{bmatrix} O_{m\times m} & O_{m\times n} \\ B & BA \end{bmatrix}.\end{aligned}$$

Since $\begin{bmatrix} I_m & A \\ O_{n\times m} & I_n \end{bmatrix}$ is invertible,

$$\begin{bmatrix} I_m & A \\ O_{n\times m} & I_n \end{bmatrix}^{-1}\begin{bmatrix} AB & O_{m\times n} \\ B & O_{n\times n} \end{bmatrix}\begin{bmatrix} I_m & A \\ O_{n\times m} & I_n \end{bmatrix} = \begin{bmatrix} O_{m\times m} & O_{m\times n} \\ B & BA \end{bmatrix},$$

which shows that $C_1 = \begin{bmatrix} AB & O_{m\times n} \\ B & O_{n\times n} \end{bmatrix}$ and $C_2 = \begin{bmatrix} O_{m\times m} & O_{m\times n} \\ B & BA \end{bmatrix}$ are similar. The eigenvalues of C_1 are the eigenvalues of AB together with n zeroes, and the eigenvalues of C_2 are the eigenvalues of BA together with m zeroes. By Lemma 17.3.4, C_1 and C_2 have the same eigenvalues (with the same multiplicities), so $c_{BA}(x) = x^{n-m}c_{AB}(x)$, as desired. If $m = n$ and A is invertible (say), then $AB = A(BA)A^{-1}$ shows that AB and BA are similar. □

Definition 17.3.6. *A real matrix P is* orthogonal *if and only if P is invertible and $P^{-1} = P^T$. A complex matrix Q is called* unitary *if and only Q is invertible and $Q^* = Q^{-1}$.*

Theorem 17.3.7 (Schur's unitary triangulization)**.** *Let $B \in M_{n\times n}(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of B (in no prescribed order). Then there exists a unitary matrix $U \in M_{n\times n}(\mathbb{C})$ such that $U^*BU = T = (t_{ij})$ is upper triangular with, for each i, $t_{ii} = \lambda_i$. If $B \in M_{n\times n}(\mathbb{R})$ is symmetric, then there exists such a $U \in M_{n\times n}(\mathbb{R})$ satisfying $U^T = U^{-1}$.*

Proof: Without giving a formal proof by induction, the ideas required for such a proof are presented.

For each $i = 1, \ldots, n$, let $\mathbf{x}_i$ be a normalized eigenvector associated with λ_i, that is, $\|\mathbf{x}_i\| = 1$ and $B\mathbf{x}_i = \lambda_i \mathbf{x}_i$. Extend $\{\mathbf{x}_1\}$ to a basis $\mathcal{B} = \{\mathbf{x}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$ for $\mathbb{C}^n$, and by the Gram–Schmidt process, $\mathcal{B}$ can be chosen to be an orthonormal basis. Put U_1 to be the matrix $[\mathbf{x}_1 \ \mathbf{z}_2 \ \cdots \ \mathbf{z}_n]$ formed by using the basis vectors as columns. By checking that $U_1^* U_1 = I_n$, it is verified that U_1 is unitary.

Since the first column of BU_1 is $B\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$, write

$$U_1^* B U_1 = \begin{bmatrix} \mathbf{x}_1^* \\ \mathbf{z}_2^* \\ \vdots \\ \mathbf{z}_n^* \end{bmatrix} [\lambda_1 \mathbf{x}_1 \ B\mathbf{z}_2 \ \cdots \ B\mathbf{z}_n],$$

which is of the form

$$\left[\begin{array}{c|c} \lambda_1 \|\mathbf{x}_1\|^2 & \cdots\cdots\cdots \\ \hline \mathbf{z}_2^*(\lambda_1 \mathbf{x}_1) & \cdots \\ \vdots & \\ \mathbf{z}_n^*(\lambda_1 \mathbf{x}_1) & \cdots \end{array}\right].$$

Since $\|\mathbf{x}_1\| = 1$ and $\mathbf{z}_i \bullet \mathbf{x}_1 = 0$, then $U_1^* B U_1$ is of the form

$$U_1^* B U_1 = \left[\begin{array}{c|c} \lambda_1 & \cdots\cdots\cdots \\ \hline 0 & \\ \vdots & B_1 \\ 0 & \end{array}\right].$$

Since U_1 is unitary, B and $U_1^* B U_1$ are similar, and so (by Lemma 17.3.4) have the same eigenvalues, that is, B_1 has eigenvalues $\lambda_2, \ldots, \lambda_n$.

Now let $\mathbf{x}_2$ be chosen with $\|\mathbf{x}_2\| = 1$ and $B\mathbf{x}_2 = \lambda_2 \mathbf{x}_2$. Repeat the above procedure producing a unitary matrix $U_2 \in M_{n-1 \times n-1}(\mathbb{C})$ and B_2 satisfying

$$U_2^* B_1 U_2 = \left[\begin{array}{c|c} \lambda_2 & \cdots\cdots\cdots \\ \hline 0 & \\ \vdots & B_2 \\ 0 & \end{array}\right],$$

and set

$$V_2 = \left[\begin{array}{c|c} 1 & 0 \cdots\cdots 0 \\ \hline 0 & \\ \vdots & U_2 \\ 0 & \end{array}\right].$$

Then both V_2 and U_1V_2 are unitary (check that $V_2^* = V_2^{-1}$, and so $(U_1V_2)^* = (U_1V_2)^{-1}$), and

$$V_2^*U_1^*BU_1V_2 = \left[\begin{array}{c|c} 1 & 0\cdots\cdots 0 \\ \hline 0 & \\ \vdots & U_2^* \\ 0 & \end{array}\right] \left[\begin{array}{c|c} \lambda_1 & \cdots\cdots\cdots \\ \hline 0 & \\ \vdots & B_1 \\ 0 & \end{array}\right] \left[\begin{array}{c|c} 1 & 0\cdots\cdots 0 \\ \hline 0 & \\ \vdots & U_2 \\ 0 & \end{array}\right],$$

which is (using block multiplication) of the form

$$V_2^*U_1^*BU_1V_2 = \left[\begin{array}{c|c} \lambda_1 & \cdots\cdots\cdots \\ \hline 0 & \\ \vdots & U_2^*B_1U_2 \\ 0 & \end{array}\right] = \left[\begin{array}{cc|c} \lambda_1 & \cdot & \cdots\cdots\cdots \\ 0 & \lambda_2 & \cdots\cdots\cdots \\ \hline 0 & 0 & \\ \vdots & \vdots & B_2 \\ 0 & 0 & \end{array}\right].$$

Continue in a similar manner so that for each $i = 1, \ldots, n-1$, produce unitary matrices $U_i \in M_{(n-i+1)\times(n-i+1)}(\mathbb{C})$ and for each $i = 2, \ldots, n-1$, produce unitary matrices $V_i \in M_{n\times n}(\mathbb{C})$. Then the matrix $U = U_1V_2V_3\cdots V_{n-1}$ is unitary and U^*BU is of the desired form.

If $B \in M_{n\times n}(\mathbb{R})$ is symmetric, then by Theorem 17.3.1 and Lemma 17.3.2, B has real eigenvalues and real eigenvectors; so the above proof can be carried out entirely in $\mathbb{R}$, giving the last statement of the theorem. □

Corollary 17.3.8. *Let $A \in M_{n\times n}(\mathbb{C})$ have eigenvalues $\lambda_1, \ldots, \lambda_n$. Then*

$$\det(A) = \lambda_1\lambda_2\cdots\lambda_n$$

and

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Proof: By Theorem 17.3.7 A is similar to a triangular matrix T whose trace and determinant are as desired; by Lemma 17.3.3, $\mathrm{tr}(A) = \mathrm{tr}(T)$ and $\det(A) = \det(T)$. □

Corollary 17.3.9. *Let $A \in M_{n\times n}(\mathbb{C})$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, and let $c_A(x) = a_nx^n + \cdots + a_2x^2 + a_1x + a_0$ be the characteristic polynomial of A. Then $a_n = 1$, $a_{n-1} = -tr(A)$ and*

$$a_0 = (-1)^n\det(A) = (-1)^n\lambda_1\lambda_2\cdots\lambda_n.$$

Recall that an orthogonal matrix P satisfies $P^{-1} = P^T$, and has columns that are mutually orthogonal normalized vectors.

Definition 17.3.10. *Say that a square matrix A is* orthogonally diagonalizable *if and only if there exists an orthogonal matrix P so that $P^{-}1AP$ is diagonal.*

(Perhaps a more appropriate term is "orthonormally diagonalizable", but the above definition is the tradition.)

Theorem 17.3.11. *Let $B \in M_{n\times n}(\mathbb{R})$ be symmetric. Then B is orthogonally diagonalizable. Furthermore, the diagonal entries of the associated diagonal matrix are the eigenvalues of B.*

Proof: Two proofs of the first statement are given.

By the second statement in Schur's theorem (Theorem 17.3.7), let $U \in M_{n\times n}(\mathbb{R})$ satisfy $U^T = U^{-1}$ with $T = U^TBU$ triangular. But $T^T = (U^TBU)^T = U^TBU = T$, so T is symmetric. Since T is both triangular and symmetric, T is a diagonal matrix. The eigenvalues of any diagonal matrix are the entries on the main diagonal, and so the eigenvalues of T are its main diagonal entries. Since T is similar to B, and similar matrices have the same spectrum (Lemma 17.3.4), the proof is complete.

A proof by induction (see, *e.g.*, [719, p. 329]) of the first statement avoids the use of Schur's theorem. For each $n \geq 1$, let $S(n)$ be the statement that an $n \times n$ real symmetric matrix is orthogonally diagonalizable.
Base step: Since 1×1 matrices are diagonal already, the identity matrix [1] serves as P. So $S(1)$ is true.
: Inductive step. Let $k > 1$ and suppose that $S(k-1)$ is true. Let A be a real symmetric $k\times k$ matrix. By Lemma 17.3.2, let λ_1 be real eigenvalue with associated real eigenvector $\mathbf{x}_1$. Without loss of generality, let $\|\mathbf{x}_1\| = 1$. Extend $\{\mathbf{x}_1\}$ to a orthonormal basis (by, *e.g.*, the Gram-Schmidt algorithm) $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $\mathbb{R}^k$. Let $P_1 = [\mathbf{x}_1|\cdots|\mathbf{x}_k]$ be the $k \times k$ matrix whose columns are the $\mathbf{x}_i$s. Then by checking the definitions, P_1 is orthogonal. By block multiplication, for some matrices M and N,

$$P_1^TAP_1 = \begin{bmatrix} \lambda_1 & M \\ \mathbf{0} & N \end{bmatrix}.$$

Since A is symmetric, $P_1^TAP_1$, is also symmetric. Thus $M = \mathbf{0}$ and N is a real $(k-1) \times (k-1)$ symmetric matrix. By the induction hypothesis $S(k-1)$, N is orthogonally diagonalizable with, say, orthogonal Q and diagonal D_1 satisfying $D_1 = Q^TNQ$. Putting $P_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}$, it is easy to verify that P_2 is orthogonal. Observe that P_1P_2 is also orthogonal and

$$\begin{aligned}(P_1P_2)^TA(P_1P_2) &= P_2^T(P_1^TAP_1)P_2 \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}\begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & D_1 \end{bmatrix}.$$

. Since $\begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & D_1 \end{bmatrix}$ is diagonal, A is orthogonally diagonalizable, completing the inductive step.

By MI, for all $n \geq 1$, the statement $S(n)$ is true. □

Theorem 17.3.12 (Principal axis theorem)**.** *For an $n \times n$ matrix A, the following are equivalent.*

(i) A is symmetric.

(ii) A is orthogonally diagonalizable.

(iii) A has an orthonormal set of eigenvectors.

Proof:

(i) → (ii): This implication is contained in Theorem 17.3.11.

(ii) → (i): Suppose that A is orthogonally diagonalizable with orthogonal matrix P so that $D = P^{-1}AP$ is diagonal. Then $A = PDP^{-1}$ and so $A^T = (PDP^{-1})^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$; thus A is symmetric.

(ii) ↔ (iii): If P is an $n \times n$ matrix with column column vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, then P is orthogonal if and only if $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an orthonormal set in $\mathbb{R}^n$ and $P^{-1}AP = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal if and only if $AP = PD$, which says (by block multiplication)

$$[A\mathbf{x}_1| \cdots |A\mathbf{x}_n] = [\lambda_1\mathbf{x}_1 \cdots \lambda_n\mathbf{x}_n];$$

in other words, the $\mathbf{x}_i$s are eigenvectors for A. □

Corollary 17.3.13. *A symmetric matrix is uniquely determined by its eigenvalues and eigenvectors.*

The next theorem was published by Rayleigh and Ritz, two British physicists (see [528, pp. 176–7]). The theorem also applies to Hermitian matrices; however only the case for real symmetric matrices is needed here.

Theorem 17.3.14 (Rayleigh–Ritz)**.** *Let*

$$\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max}$$

be the (real) eigenvalues of a symmetric matrix $B \in M_{n\times n}(\mathbb{R})$. Then for all column vectors $\mathbf{x} \in \mathbb{C}^n$,

$$\lambda_1\mathbf{x}^*\mathbf{x} \leq \mathbf{x}^*B\mathbf{x} \leq \lambda_n\mathbf{x}^*\mathbf{x}. \tag{17.2}$$

Furthermore,

$$\lambda_n = \max_{\mathbf{x}\neq\mathbf{0}} \left\{ \frac{\mathbf{x}^* B\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \right\} = \max_{\mathbf{x}^*\mathbf{x}=1} \{\mathbf{x}^* B\mathbf{x}\}, \tag{17.3}$$

and

$$\lambda_1 = \min_{\mathbf{x}\neq\mathbf{0}} \left\{ \frac{\mathbf{x}^* B\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \right\} = \min_{\mathbf{x}^*\mathbf{x}=1} \{\mathbf{x}^* B\mathbf{x}\}. \tag{17.4}$$

Proof: (Only equations (17.2) and (17.4) are proved here; the proof of (17.3) is similar to that of (17.4) anyway.)

By the diagonalization theorem for symmetric matrices (Theorem 17.3.11), let $U, D \in M_{n\times n}(\mathbb{R})$ be with $U^T = U^{-1}$, $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ so that $D = U^T BU$ (and so $B = UDU^T$). For each column vector $\mathbf{x} \in \mathbb{C}^n$, using the notation $(\mathbf{v})_i$ to denote the ith element in a vector $\mathbf{v}$,

$$\mathbf{x}^* B\mathbf{x} = \mathbf{x}^* UDU^T\mathbf{x} = (U^T\mathbf{x})^* D(U^T\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \|(U^T\mathbf{x})_i\|^2.$$

Since each $\lambda_i\|(U^T\mathbf{x})_i\|^2$ is positive,

$$\lambda_{\min} \sum_{i=1}^{n} \|(U^T\mathbf{x})_i\|^2 \leq \sum_{i=1}^{n} \lambda_i \|(U^T\mathbf{x})_i\|^2 \leq \lambda_{\max} \sum_{i=1}^{n} \|(U^T\mathbf{x})_i\|^2.$$

The matrix U satisfies $U^T U = I_n$, so

$$\sum_{i=1}^{n} \|(U^T\mathbf{x})_i\|^2 = (U^T\mathbf{x})^*(U^T\mathbf{x}) = \mathbf{x}^* U^T U\mathbf{x} = \mathbf{x}^*\mathbf{x},$$

completing the proof of (17.2).

To prove (17.4), if $\mathbf{x}$ is a (non-zero) eigenvector corresponding to λ_1, then $\lambda_1\mathbf{x}^*\mathbf{x} = \mathbf{x}^*\lambda_1\mathbf{x} = \mathbf{x}^* B\mathbf{x}$, in which case the fraction in (17.4) is λ_1. For any other non-zero $\mathbf{x}$, equation (17.2) gives

$$\frac{\mathbf{x}^* B\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \geq \lambda_1,$$

so the minimum is λ_1, and equality is indeed attained for eigenvectors associated with λ_1. To see the second equality in (17.4) for any $\mathbf{x} \neq \mathbf{0}$, rewrite the quotient as

$$\frac{\mathbf{x}^* B\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \left(\frac{1}{\sqrt{\mathbf{x}^*\mathbf{x}}}\mathbf{x}\right)^* B \left(\frac{1}{\sqrt{\mathbf{x}^*\mathbf{x}}}\mathbf{x}\right),$$

where the vector $\mathbf{y} = \frac{1}{\sqrt{\mathbf{x}^*\mathbf{x}}}\mathbf{x}$ satisfies $\mathbf{y}^*\mathbf{y} = 1$. $\square$

17.4 Non-negative matrices

A real matrix is called *non-negative* if and only if each of its entries is non-negative; similarly, a real matrix is called *positive* if and only if each of its entries is positive. Slightly abusing notation, if A is non-negative, write $A \geq 0$ and if A is positive, write $A > 0$. A slightly more honest notation might be $A \geq O_{m\times n}$, the $m \times n$ zero matrix, however this can be clumsy.

Definition 17.4.1. *A matrix $A \in M_{n\times n}(\mathbb{C})$ is said to be* reducible *if and only if there exists a permutation matrix $P \in M_n(0,1)$ and $k \in [1, n-1]$, so that for some $B \in M_k(\mathbb{C})$, $C \in M_{k\times(n-k)}(\mathbb{C})$, $D \in M_{n-k}(\mathbb{C})$, and zero matrix $O = O_{(n-k)\times k}$,*

$$P^T AP = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

If no such P exists, then A is said to be irreducible.

Exercise 550. *Show that if all entries of A are positive, then A is irreducible. Also show that if A is reducible then A contains at least $n-1$ zero entries. Hint: minimize $f(x) = (n-x)x$.*

Theorem 17.4.2. *A non-negative matrix A is irreducible if and only if $(I_n + A)^{n-1}$ is positive (has all positive entries).*

Proof: First suppose that A is reducible with permutation matrix P so that $A' = P^T AP$, where

$$A' = \begin{bmatrix} B & C \\ O_{(n-k)\times k} & D \end{bmatrix}.$$

Since $P^T AP = A'$, and $P^T = P^{-1}$, it follows that $A = PA'P^T$. Also, for any positive integer i, $A^i = (PA'P^T)^i = P(A')^i P^T$, and so

$$\begin{aligned}(I_n + A)^{n-1} &= I_n + \sum_{i=1}^{n-1} \binom{n-1}{i} A^i \\ &= I_n + \sum_{i=1}^{n-1} \binom{n-1}{i} (PA'P^T)^i \\ &= I_n + \sum_{i=1}^{n-1} \binom{n-1}{i} P(A')^i P^T.\end{aligned}$$

Since each $(A')^i$ contains a $(n-k)\times k$ block of zeros in the same position, so does each $P(A')P^T$, and thus $(I_n + A)^{n-1}$ also contains zeros. In fact, since

$$I_n + \sum_{i=1}^{n-1} \binom{n-1}{i} P(A')^i P^T = P\left(I_n + \sum_{i=1}^{n-1} \binom{n-1}{i} (A')^i\right) P^T,$$

it follows that $(I_n + A)^{n-1}$ is also reducible.

For the other direction, suppose that for some $s \neq t$, the (s,t)-entry of $(I + A)^{n-1}$ is zero. Then for each $k = 1, \ldots, n-1$, the (s,t) entry of A^k is zero, and so also the (s,t) entry of

$$I_n + A + A^2 + \cdots + A^{n-1}$$

is zero. Let G be the (possibly directed, weighted) graph determined by A, with vertices $v_1, \ldots, v_n$ labelled to agree with A. By Theorem 12.6.1, the graph (or digraph with weighted edges) G is not connected since there is no (directed) path from v_s to v_t. Put $S = \{i : v_s \text{ is connected to } v_i\} \cup \{s\}$, and put $T = [1,n] \backslash S$. Then the rows and columns of A can be permuted so that rows corresponding to S come first, in which case the result is in the block form showing that A is reducible. □

Definition 17.4.3. *For any $n \times n$ matrix A, define the* digraph of A *(or simply, the graph of A) to be the graph on n vertices, say $v_1, \ldots, v_n$, defined by (v_i, v_j) is a directed edge if and only if $a_{ij} \neq 0$.*

If A is symmetric, then the graph of A is undirected. If A has any non-zero elements on its diagonal, its graph has loops.

Theorem 17.4.4. *A square matrix A is irreducible if and only if the graph of A is strongly connected.*

Proof: Apply Theorem 17.4.2 and analogues of Lemma 1.12.1 and Corollary 1.12.2. □

Definition 17.4.5. *A matrix A is called* positive regular *if and only if there exists some k so that A^k has only positive entries (which is sometimes written $A^k > 0$).*

Note that not all irreducible matrices are positive regular. For example, consider the graph consisting of a single edge; its adjacency matrix A is irreducible (either directly from the definition, or from Theorem 17.4.4 since the graph is connected), but every power of A contains two zeros. To prove some results about irreducible matrices, some facts about positive matrices are necessary, and the link between these two notions is sometimes positive regular matrices.

17.5 Matrix norms and spectral radius

The following definition could have been made immediately after the definition of eigenvalues; however it is finally needed here.

Definition 17.5.1. *For $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, define the* spectral radius *to be*

$$\rho(A) = \max\{|\lambda_i| : i = 1, \ldots, n\}.$$

Definition 17.5.2. *A matrix norm on $M_{n\times n}(\mathbb{C})$ is a function $|||\cdot||| : M_{n\times n}(\mathbb{C}) \to \mathbb{R}$ that satisfies, for all $A, B \in M_{n\times n}(\mathbb{C})$ and all scalars $z \in \mathbb{C}$,*

(a) $|||A||| \geq 0$.

(b) $|||A||| = 0$ *if and only if* $A = O_{n\times n}$, *the zero matrix.*

(c) $|||A + B||| \leq |||A||| + |||B|||$.

(d) $|||zA||| = |z|\,|||A|||$.

(e) $|||AB||| \leq |||A||| \cdot |||B|||$.

So a matrix norm is a vector norm with the added condition (e). A simple proof by induction shows that for any $k \in \mathbb{Z}^+$, and any matrix norm, $|||A^k||| \leq |||A|||^k$.

Some examples of matrix norms on $M_{n\times n}(\mathbb{C})$ are:

- The ℓ_1 norm: $\|A\|_1 = \sum_{i,j} |a_{ij}|$.
- The ℓ_2 norm (also called the Euclidean norm, the Frobenius norm, the Schur norm, or the Hilbert-Schmidt norm): $\|A\|_2 = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$.
- The ℓ_∞ norm: $\|A\|_\infty = n \cdot \max |a_{ij}|$.

Another matrix norm says something about how much multiplication by A affects the size of a vector.

Theorem 17.5.3. *For a vector (row or column) $\mathbf{v} \in \mathbb{C}^n$, let $\|\mathbf{v}\|$ be any vector norm of $\mathbf{v}$. Define*

$$|||A||| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Then $|||\cdot|||$ is a matrix norm with the properties that for any $\mathbf{x} \in \mathbb{C}^n$, $\|A\mathbf{x}\| \leq |||A||| \cdot \|\mathbf{x}\|$, and $|||I_n||| = 1$.

Proof: See, *e.g.* [528, Thm 5.6.2]. □

The matrix norm in Theorem 17.5.3 is sometimes called the *operator norm* associated with the particular vector norm.

Another kind of matrix norm is the *maximum-row-[column]sum norm*, which is the maximum of ℓ_1 norms of rows [columns].

One final example of a matrix norm is called the *spectral norm*, defined by

$$|||A||| = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\}.$$

Note that each eigenvalue of A^*A is a non-negative real since, if for some non-zero $\mathbf{x}$, $A^*A\mathbf{x} = \lambda\mathbf{x}$, then using the standard Euclidean norm and multiplying on the left by $\mathbf{x}^*$ yields

$$\|\mathbf{x}A\|^2 = (\mathbf{x}^*A^*)A\mathbf{x} = \lambda\mathbf{x}^*\mathbf{x} = \lambda\|\mathbf{x}\|^2,$$

and so $\lambda \geq 0$.

What may seem somewhat astonishing is that the spectral radius is bounded above by any matrix norm!

Theorem 17.5.4. *Let $|||\cdot|||$ be any matrix norm on $M_{n\times n}(\mathbb{C})$, then for any $A \in M_{n\times n}(\mathbb{C})$,*

$$\rho(A) \leq |||A|||.$$

Proof: Let $A \in M_{n\times n}(\mathbb{C})$, and let λ be an eigenvalue of A satisfying $|\lambda| = \rho(A)$. Let $\mathbf{x}$ be an eigenvector associated with λ and consider the matrix $X \in M_{n\times n}(\mathbb{C})$, each of whose columns is equal to $\mathbf{x}$. Then $AX = \lambda X$. Using any matrix norm,

$$|\lambda| \cdot |||X||| = |||\lambda X||| = |||AX||| \leq |||A||| \cdot |||X|||,$$

and so $\rho(A) = |\lambda| \leq |||A|||$. □

If one uses only a vector norm, the result in Theorem 17.5.4 need not hold; for example, consider the maximum (or l_∞) vector norm $\|A\| = \max_{i,j} |a_{ij}|$, and the matrix $J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then it is not difficult to check that $\rho(J_2) = 2$, yet $\|J_2\| = 1$.

With a little work (see [528, pp. 297–8]), one can show that $\rho(A)$ is actually the infimum over all matrix norms.

A useful fact (also given here without proof) when studying Markov processes is:

Theorem 17.5.5. *Let $A \in M_{n\times n}(\mathbb{C})$. Then $\lim_{k\to\infty} A^k = O_{n\times n}$ if and only if $\rho(A) < 1$.*

Perhaps one more fact about the spectrum can be mentioned (see [528, p. 299] for proof):

Theorem 17.5.6. *If $A \in M_{n\times n}(\mathbb{C})$, then for any matrix norm $\|\cdot\|$,*

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}.$$

Some properties of the spectral radius are easily verified for non-negative matrices:

Lemma 17.5.7. *Let $A, B \in M_{n\times n}(\mathbb{C})$ with $0 \leq A \leq B$. Then $\rho(A) \leq \rho(B)$.*

Proof: By properties of a matrix norm, for each $m \in \mathbb{Z}^+$, $\|A^m\|_2 \leq \|B^m\|_2$ and so $\|A^m\|_2^{1/m} \leq \|B^m\|_2^{1/m}$. Taking limits and applying Theorem 17.5.6 finishes the proof. □

Lemma 17.5.8. *Let $A \in M_{n\times n}(\mathbb{C})$ be non-negative. If the row sums of A are constant, then $\rho(A)$ is the maximum row sum norm of A; similarly, if the column sums are constant, then $\rho(A)$ is the maximum column sum norm of A.*

Proof: Let A have constant row sums α. Then using the vector $\mathbf{b} = [1, 1, \ldots, 1]^T$, $A\mathbf{b} = \alpha\mathbf{b}$, so α is an eigenvalue for A. Since the maximum row sum norm is indeed a matrix norm (sometimes denoted $|||A|||_\infty$) and for A, this norm is α. Since for any matrix norm $|||\cdot|||$, $\rho(A) \leq |||A|||$, conclude that $\rho(A) = \alpha$. The column sum case follows by applying this reasoning to A^T. □

An early simple bound on the spectral radius of a matrix is surprisingly easy to prove. In the following, for any $a \in \mathbb{C}$, use the notation $B_r(a) = \{z \in \mathbb{C} : |z - a| \leq r\}$ for the closed disk of radius r centred at a.

Theorem 17.5.9 (Gerschgorin, 1931 [416])**.** *Let $A \in M_{n\times n}(\mathbb{C})$, $A = (a_{i,j})$, and for each $i = 1, \ldots, n$, define*

$$\Lambda_i = \sum_{j=1,\ldots,n; j\neq i} |a_{i,j}|.$$

Then all eigenvalues of A lie in the union of disks

$$\bigcup_{i=1}^{n} B_{\Lambda_i}(a_{i,i}).$$

Proof: Let λ be an eigenvalue for A with associated eigenvector

$$\mathbf{x} = [x_1, \ldots, x_n]^T.$$

Suppose that the largest entry of $\mathbf{x}$ has modulus 1, say $|x_k| = 1$. Then for each $i = 1, \ldots, n$,

$$(\lambda - a_{i,i})x_i = \sum_{j\neq i} a_{i,j}x_j,$$

and so

$$|\lambda - a_{k,k}| = |(\lambda - a_{k,k})x_k| \leq \sum_{j\neq k} |a_{k,j}| \cdot |x_j| \leq \sum_{j\neq k} |a_{k,j}| = \Lambda_k.$$

Thus, $\lambda \in B_{\Lambda_k}(a_{k,k})$. □

Each disk $B_{\Lambda_i}(a_{i,i})$ in Theorem 17.5.9 is called a *Gerschgorin disk*.

Corollary 17.5.10. *If $A \in M_{n\times n}(\mathbb{C})$, $A = (a_{ij})$, then*

$$\rho(A) \le \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|.$$

Proof: For each i, $B_{\Lambda_i}(a_{i,i}) \subseteq B_{|a_{i,i}|+\Lambda_i}(0)$. □

Notice that Corollary 17.5.10 simply repeats the result for the maximum-row-sum norm in Theorem 17.5.4.

Theorem 17.5.11. *Let $A \in M_{n\times n}(\mathbb{R})$ be non-negative. Then*

$$\min_i \sum_{j=1}^{n} a_{ij} \le \rho(A) \le \max_i \sum_{j=1}^{n} a_{ij},$$

and

$$\min_j \sum_{i=1}^{n} a_{ij} \le \rho(A) \le \max_j \sum_{i=1}^{n} a_{ij}.$$

Proof: See [528, Thm 8.1.22, p. 492], for example. Only the first inequality is shown here; others are similar.

Let α be the smallest row sum, and construct a matrix B with $0 \le B \le A$, where each row sum in B is α. Then α is an eigenvalue (with eigenvector $[1, 1, \dots, 1]^T$). By Lemma 17.5.7, $\rho(B) \le \rho(A)$, and by Lemma 17.5.8, $\rho(B) = \alpha$, so the first inequality holds. The second inequality (upper bound) has a similar proof, and the second equation follows by examining A^T. □

Theorem 17.5.12. *Let $A \in M_{n\times n}(\mathbb{C})$ be a non-negative matrix. Then $\rho(A)$ is an eigenvalue of A and there exists a non-negative $\mathbf{x} \ne \mathbf{0}$ so that $A\mathbf{x} = \rho(A)\mathbf{x}$.*

Proof: See [528, Thm 8.3.1, pp. 503–504], for example. □

Lemma 17.5.13. *Let $A \in M_{n\times n}(\mathbb{C})$ have eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicities). Then $\lambda_1 + 1, \dots, \lambda_n + 1$ are eigenvalues for $I_n + A$, and $\rho(I_n + A) \le 1 + \rho(A)$. If A is non-negative, then $\rho(I_n + A) = 1 + \rho(A)$.*

Proof: Let λ be an eigenvalue for A with multiplicity k. Then λ is a root of the characteristic polynomial $c_A(t) = \det(tI_n - A)$ with multiplicity k. Using $s = t + 1$, $c_A(s) = \det(sI_n - (A + I_n))$, c_A is also the characteristic polynomial for $A + I_n$; so the eigenvalues of $A + I_n$ have the same multiplicities as those for A, and $s = \lambda + 1$ is an eigenvalue for $A + I_n$. Then $\rho(A + I_n) = \max_i\{|\lambda_i + 1|\} \le \max_i\{|\lambda_i| + 1\} = \rho(A) + 1$.

Hence, $\rho(A + I_n) \leq \rho(A) + 1$ and, since $A + I_n$ is non-negative, by Theorem 17.5.12, $\rho(A + I_n)$ is an eigenvalue for $A + I_n$. Thus, $\rho(A + I_n) = \rho(A) + 1$. □

The following theorem was proved by Oskar Perron [745] in 1907 and is central in the study of positive matrices; the details behind its proof are too many to give here (although most of the needed facts have been proved in this document). For a proof, see, *e.g.*, [528, Thm 8.2.11, p. 500].

Theorem 17.5.14 (Perron, 1907 [745])**.** *Let $A \in M_{n\times n}(\mathbb{C})$ be a positive matrix (that is, all entries are positive). Then*

(a) $\rho(A) > 0$.

(b) $\rho(A)$ *is an eigenvalue for* A.

(c) There is a positive eigenvector $\mathbf{x} \in (\mathbb{R}^+)^n$ *associated with the eigenvalue* $\rho(A)$.

(d) $\rho(A)$ *has (algebraic) multiplicity 1 (and so also geometric multiplicity 1).*

(e) For every eigenvalue $\lambda \neq \rho(A)$, $|\lambda| < \rho(A)$.

(f) If $\mathbf{x}$ *and* $\mathbf{y}$ *are positive vectors with* $\mathbf{x}^T\mathbf{y} = 1$ *satisfying* $A\mathbf{x} = \rho(A)\mathbf{x}$ *and* $A^T\mathbf{y} = \rho(A)\mathbf{y}$, *then* $\lim_{m\to\infty}(\rho(A)^{-1}A)^m = \mathbf{x}\mathbf{y}^T$.

The next result acts as a bridge between non-negative matrices and positive matrices—using positive regular matrices—which relate to irreducible matrices.

Lemma 17.5.15. *Let $A \in M_{n\times n}(\mathbb{C})$ be non-negative and positive regular, with, for some $k \geq 1$, A^k is strictly positive. Then $\rho(A)$ is a simple root of c_A, that is, the algebraic multiplicity of $\rho(A)$ is 1.*

Proof: Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues for A. Then for any $k \geq 1$, (by Lemma 17.2.5), $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues for A^k. If λ is a multiple root of c_A, then λ^k is a multiple root of A^k, which contradicts part (d) of Theorem 17.5.14) applied to A^k. □

Perron's theorem (Theorem 17.5.14) was extended to irreducible matrices by Georg Frobenius [384] in 1912.

Theorem 17.5.16 (Perron–Frobenius theorem)**.** *Let A be an irreducible non-negative matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then*

(a) $\rho(A) > 0$.

(b) $\lambda_n = \rho(A)$.

(c) There is a positive eigenvector (all entries positive) associated with λ_n.

(d) λ_n has multiplicity 1.

Proof: Part (a) follows from Theorem 17.5.11, since an irreducible non-negative matrix does not have a row of zeros and so all row sums are positive.

By Theorem 17.5.12, part (b) holds for any non-negative matrices.

To see (c), let $\mathbf{x} \neq \mathbf{0}$ be a non-negative vector for which $A\mathbf{x} = \rho(A)\mathbf{x}$; Then $(I_n+A)^{n-1}\mathbf{x} = (1+\rho(A))^{n-1}\mathbf{x}$, and since A is irreducible, by Theorem 17.4.2 $(I_n+A)^{n-1}$ has no zeros (and so is positive). Thus, $\mathbf{x}$ is a non-zero non-negative eigenvector for a positive matrix, and so $(I+A)^{n-1}\mathbf{x}$ is a strictly positive (the proof is trivial) vector. Hence

$$\mathbf{x} = \frac{1}{(1+\rho(A))^{n-1}}(1+A)^{n-1}\mathbf{x}$$

is also positive, proving (c).

[Note: One might try to start with, by Theorem 17.5.14, a positive eigenvector for the positive matrix $(I_n+A)^{n-1}$, if only one first shows that $(1+\rho(A))^{n-1}$ is the largest eigenvalue for $(I_n+A)^{n-1}$, which might follow from Lemma 17.5.13, but with a little extra work—instead, this approach works to settle (d) below.]

To see (d), by Theorem 17.4.2, $(I_n+A)^{n-1} > 0$, so Lemma 17.5.15 applies to I_n+A. Thus I_n+A has a simple eigenvalue, and so by Lemma 17.5.13, $\rho(A)+1$ is that simple eigenvalue—which means that $\rho(A)$ is a simple eigenvalue for A. □

Theorem 17.5.17 (Interlacing theorem, Cauchy, 1829 [183])**.** *Let $A \in M_n(\mathbb{R})$ be a real symmetric matrix and let $B \in M_{n-1}(\mathbb{R})$ be a principal submatrix formed by deleting one row and column, of A. If the eigenvalues of A are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and the eigenvalues of B are $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$, then*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

One proof of the interlacing theorem is by induction; see *e.g.*, [422, Ch. 9] for a more general version.

Chapter 18

Appendix: Inequalities and approximations

18.1 Landau notation

In his work on the distribution of the primes, Edmund Landau [615, p. 883] adopted some notation to describe asymptotic behaviour of some functions. Apparently, Bachman [64] [I have not verified this source] also developed some of this notation somewhat earlier, but today such notation is commonly called "Landau notation".

For the present discussion, consider only functions $f : \mathbb{Z}^+ \to \mathbb{R}$. For two such functions f and g, write $f = o(g)$ [read "f is little oh of g"] or $f(n) = o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. For example, $\ln(x) = o(x^2)$, and $\frac{1}{n} = o(1)$. Hence, the notation $f(n) = (1 + o(1))g(n)$ means that f and g are approximately equal for large n, that is, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, in which case one often writes $f \sim g$. (Since $o(g)$ actually describes a class of functions, some authors insist that the notation $f = o(g)$ be replaced by $f \in o(g)$—however, such cases are now rare, at least in combinatorics.)

When f is eventually bounded above by some fixed multiple of g, another notation, called "big oh" notation is used: write $f(n) = O(g(n))$ if and only if there exists a positive constant $C \in \mathbb{R}$ and an $n_0 \in \mathbb{Z}^+$ so that for all $n > n_0$, $f(n) \le Cg(n)$. [Note, instead of a capital O, some authors prefer $\mathcal{O}$.] So, for example, $x^2 + 1 = O(3x^2 + 14x)$ (*e.g.*, take $C = \frac{1}{5}$, and $n_0 = 10$). Turning the big oh notation inside out, define $f = \Omega(g)$ if and only if $g = O(f)$. If both $f = O(g)$ and $f = \Omega(g)$, write $f = \Theta(g)$; this essentially describes the situation where f and g satisfy $cg \le f \le Cg$ for some constants c and C.

It is often convenient to abbreviate an expression like $\lim_{n\to\infty} f(n) = L$ by $f(n) \to L$. The notation $\omega(n) \to \infty$ is reserved to describe functions whose values approach infinity, but arbitrarily slowly.

One more notation that is commonly used, particularly among number theorists, is the notation $f \ll g$, which sometimes means that there exists a constant $c < 1$ so that

for sufficiently large n, $f(n) \le cg(n)$. In other words, $f \ll g$ if $f = O(g)$ where the big oh constant C is less than one. If further, the constant depends on some parameter k, the notation $f \ll_k g$ is used.

18.2 Stirling's formula

Stirling's approximation formula says that

$$n! = (1 + o(1))\sqrt{2\pi n}(n/e)^n, \tag{18.1}$$

where e is the base of the natural logarithm. There are a number of proofs of Stirling's formula of varying degree of complexity (*e.g.*, see [881, p.292] or [360]). For a proof that uses the Wallis formula for $\pi/2$, see the book *The number* π [352].

An often-used application of Stirling's approximation is an asymptotic formula for binomial coefficients.

Lemma 18.2.1. *Let e be the base for the natural logarithm. For $k = o(n^{3/4})$,*

$$\binom{n}{k} \sim \frac{1}{\sqrt{2\pi k}} \left(\frac{ne}{k}\right)^k. \tag{18.2}$$

Proof: First consider the case when k is a constant and n grows arbitrarily large. Two facts are used: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ and because k is fixed, $\lim_{n\to\infty} \frac{n}{n-k} = 1$.

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&\sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \sqrt{2\pi(n-k)}\left(\frac{n-k}{e}\right)^{n-k}} && \text{(by Stirling)} \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{k(n-k)}} \cdot \frac{n^n}{k^k(n-k)^{n-k}} \\
&\sim \frac{1}{\sqrt{2\pi}}\sqrt{\frac{1}{k}} \cdot \frac{n^n}{k^k(n-k)^{n-k}} \\
&= \frac{1}{\sqrt{2\pi k}} \frac{n^n}{k^k(n-k)^{n-k}} \\
&= \frac{1}{\sqrt{2\pi k}} \frac{n^k n^{n-k}}{k^k(n-k)^{n-k}} \\
&= \frac{1}{\sqrt{2\pi k}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} \\
&= \frac{1}{\sqrt{2\pi k}} \left(\frac{n}{k}\right)^k \left(1 + \frac{k}{n-k}\right)^{n-k}
\end{aligned}$$

$$\begin{aligned}
&\sim \frac{1}{\sqrt{2\pi k}}\left(\frac{n}{k}\right)^k e^k \\
&= \frac{1}{\sqrt{2\pi k}}\left(\frac{ne}{k}\right)^k.
\end{aligned}$$

This completes the proof of (18.2) for a fixed k.

Now assume that k is not constant. The calculations above all hold true except when approximating $\left(\frac{n}{n-k}\right)^{n-k}$. Details are left to the reader. □

It might be worth noting that when n is even and $k = \frac{n}{2}$, the approximation (18.2) says

$$\binom{n}{n/2} \sim \frac{1}{\sqrt{2\pi\frac{n}{2}}}\left(\frac{ne}{\frac{n}{2}}\right)^{n/2} = \frac{1}{\sqrt{\pi n}}(2e)^{n/2}.$$

However, computing directly from Stirling's approximation,

$$\begin{aligned}
\binom{n}{n/2} &= \frac{n!}{((n/2)!)^2} \\
&\sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\left(\sqrt{\pi n}\left(\frac{n}{2e}\right)^{n/2}\right)^2} \\
&= \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\pi n\frac{n^n}{2^n e^n}} \\
&= 2^n \cdot \sqrt{2} \cdot \frac{1}{\sqrt{\pi n}} \\
&= c \cdot \frac{1}{\sqrt{n}} \cdot 2^n
\end{aligned}$$

for a constant c, giving an expression smaller than the value obtained by (18.2). This comparison confirms the need for a restriction on k in terms of n.

18.3 Basic inequalities

18.3.1 The Cauchy–Schwarz inequality

Let V be the n-dimensional vector space over $\mathbb{R}$, with the usual dot product (Euclidean inner product). Then the Cauchy–Schwarz inequality says that for any $\mathbf{x}, \mathbf{y} \in V$,

$$|\mathbf{x} \bullet \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|. \tag{18.3}$$

When $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, squaring each side of the Cauchy–Schwarz inequality (18.3) gives

$$(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

In particular, when $(x_1, \ldots, x_n) = (1, \ldots, 1)$, this last inequality becomes

$$(y_1 + \cdots + y_n)^2 \leq n(y_1^2 + \cdots + y_n^2).$$

18.3.2 The AM-GM inequality

The arithmetic mean (AM) of a set of n numbers is their sum divided by n, their average. The geometric mean (GM) of n numbers is the nth root of their product. The AM-GM inequality says that for $a_1, \ldots, a_n \in \mathbb{R}^+$,

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

In particular, when $n = 2$, this says that $ab \leq (a+b)^2/4$, which also follows directly by expanding $(a-b)^2 \geq 0$.

18.4 Convex functions, Jensen's inequality

For real numbers $a < b$, a function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* on $[a, b]$ if for every $x, y \in [a, b]$, and for any $t \in (0, 1)$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. Often this definition is stated only for $t = \frac{1}{2}$, in which case it says: A function f is convex if and only if for every $a < b$, $f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}$.

Note that a drawing of the graph of a convex function $f : \mathbb{R} \to \mathbb{R}$ is "concave up"!

For example, to check that f defined by $f(x) = x^2$ is convex on (on any finite closed interval), let $a < b$ be reals. Then $\frac{f(a)+f(b)}{2} = \frac{a^2+b^2}{2}$, and $f(\frac{a+b}{2}) = \frac{a^2+2ab+b^2}{4}$. It is an easy exercise to see that

$$\frac{a^2+b^2}{2} - \frac{a^2+2ab+b^2}{4} = \frac{(a-b)^2}{4} \geq 0,$$

so indeed $f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}$.

The function f defined by $f(x) = \binom{x}{2}$ is convex.

Another way to think of convex functions is by averages; a function is convex is if f(average of x and y) is at most the average of $f(x)$ and $f(y)$.

Exercise 551. *Prove that if f is convex on some closed interval $[a, b]$, then for any $x_1, \ldots, x_n \in [a, b]$,*

$$f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \leq \frac{\sum_{i=1}^n f(x_i)}{n}.$$

One can extend Exercise 551 to the following:

Exercise 552 (Jensen's inequality)**.** *Prove that for $p_1, p_2, \ldots, p_n \in \mathbb{R}^+$, if f is convex, then*

$$f\left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i}\right) \leq \frac{\sum_{i=1}^n p_i f(x_i)}{\sum_{i=1}^n p_i}.$$

Chapter 19

Appendix: Notation and definitions

Many of the definitions here are repeated in the body of the text. Some other universal definitions are given here for reference.

19.1 Universal definitions and notation

19.1.1 Standard notation

The following are standard throughout mathematics.

[Comments in square brackets are not part of the definitions; they are added only for understanding.]

- The integers, positive integers, rationals, and reals are denoted by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

  [So-called "natural numbers" have two definitions (depending on whether or not 0 is included). It seems that, roughly, European authors and logicians include 0, whereas North Amercian school texts start at 1—so to avoid confusion, the notation $\mathbb{N}$ is avoided here. If 0 is to be included, there are various forms in use, including $\mathbb{Z}^+ \cup \{0\}$; other less used forms are $\mathbb{Z}_0^+$ or $\mathbb{Z}_{\geq 0}$.]

- If A and B are statements, the notation $A \to B$ means "A implies B", or "If A then B", and the notation $A \Rightarrow B$ means "A logically implies B".

  [However, in mathematics, it is common to use the double arrow notation $\Rightarrow$ to indicate implication; perhaps this is because there are too many other arrows (like in a limit) already in use or that a double arrow looks cool? Even though most double arrows indicate simple implication, a logical implication is far stronger, one that follows only from the language of logic. Example: (I have a pencil) $\Rightarrow$ (I have either a pencil or a pen).]

- The notation $\forall$ means "for all"; the notation $\exists$ means "there exists". The notation $\wedge$ can be used in place of "and"; the notation $\vee$ can be used in place of "or". The expression "iff" is often used to abbreviate "if and only if".

- The expression "without loss of generality" is abbreviated "w.l.o.g.".

- The notations $\therefore$ and $\because$ (meaning "therefore" and "because") are quite commonly used by high school students, however are avoided in this text, since the author believes they are in poor style and are often overused.

  [In general, while learning how to write mathematics, adding reasons instead of simply "therefore" helps to clarify reasoning. A sequence of $\therefore$s might be correct, but is often not as helpful to the reader as explanations are.]

- A set S is a collection of items, and such items are called **elements** (of S). If x is an element of a set S, write $x \in S$, and if x is not an element of S, write $x \notin S$.

  [The definition of a "set" is complicated when dealing with infinitely many elements; for example, "the set of all sets" is a phrase that is troublesome, leading to a set that contains itself as an element—a consequence that one might not want!]

- The **empty set** is the set containing no elements, and is denoted $\emptyset$ or $\{\ \}$.

  [In the early 1900s, ϕ was instead denoted by either Γ or Λ, and was often called "the null set".]

- A set with m elements is sometimes called an m-set.

  [See below for definition of cardinality of a set.]

- If an element x is in a set S, write $x \in S$. If both $x \in S$ and $y \in S$, one may write $x, y \in S$ for brevity.

- For sets S and T, if every element of S is also in T, then S is a *subset* of T, written $S \subseteq T$. Also, S is a **proper subset** of T, denoted $S \subsetneq T$, if and only if $S \subseteq T$ and $S \neq T$.

  [Many texts say that the notation $S \subset T$ means that S is a proper subset of T, that is, that $S \subseteq T$ and $S \neq T$, however this usage is not standard and the notation $\subsetneq$ is clearer; many texts use $S \subset T$ to mean what is defined above to mean $S \subseteq T$, allowing equality.]

- Two sets are **equal** iff they have the same elements.

  [A standard proof of sets A and B being equal shows both $A \subseteq B$ and $B \subseteq A$.]

- For sets A and B, $A \backslash B = \{a \in A : a \notin B\}$.

  [Some call this operation "set subtraction", and denote it by $A - B$, but this can lead to confusion when A and B are sets of numbers, in which case $A - B$ can be the set $\{a - b : a \in A, b \in B\}$. When $B \subset A$ are graphs, the notation $A - B$ indicates what is left after the subgraph B of A is deleted.]

- For two sets A and B, a function $f : A \to B$ is a rule that assigns to each $a \in A$ a unique element $b \in B$. In this case, write $f(a) = b$.

  [Functions are defined further in the last section here as a type of relation.]

- A function $f : A \to B$ is *one-to-one* if and only if whenever $x, y \in A$ satisfy $x \neq y$, then $f(x) \neq f(y)$.

  [A one-to-one function is also called an *injection*. Another way to say that f is one-to-one is the contrapositive: if $f(x) = f(y)$, then $x = y$.]

- A function $f : A \to B$ is *onto* (or onto B) if for each $y \in B$, there exists $x \in A$ so that $f(x) = y$.

  [An onto function is sometimes called a *surjection*.]

- A *bijection* between sets A and B is a function from A to B (or from B to A) that is both one-to-one and onto.

- A *permutation* on a set S is a bijection from S to itself.

  [If S is listed in one accepted way, say, $\{s_1, s_2, \ldots, s_n\}$, then a permutation $\sigma : S \to S$ is often depicted by $(\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n))$, often without the commas. Other ways to depict a permutation includes writing two rows inside round parenthesis, the top row being the ordered list of elements in S, and the bottom row being the images of each under σ. Yet one more way to represent a permutation is what is called "cycle notation"; for example, the cycle notation $(145)(2)(36)$ is the permutation σ defined by $\sigma(1) = 4$, $\sigma(4) = 5$, $\sigma(5) = 1$, $\sigma(2) = 2$, $\sigma(3) = 6$ and $\sigma(6) = 3$. Another use of the word "permutation" is: "A *k-permutation* chosen from a set S" is an ordered list of k distinct elements chosen from S. This usage is not standard, so if need be, only say "an ordered list of k-distinct elements from S", or some such.]

- For a finite set A, the notation $|A| = m$ means that there is a bijection between A and the set $\{1, 2, \ldots, m\}$. For a finite set, $|A|$ is called the *cardinality* of A (the number of elements in A, or the "size" of A). In general (including infinite sets), two sets S and T have the *same cardinality* iff there is a bijection $f : S \to T$.

- A set is called infinite if is not finite. An infinite set S is called *countable* if there exists a bijection from S to $\mathbb{Z}^+$, in which case, write $|S| = \aleph_0$.

- For a set S, the *power set* of S is the collection of all subsets of S; the power set is denoted by
$$\mathcal{P}(S) = \{T : T \subseteq S\}.$$
[Note: The size of the power set is $|\mathcal{P}(S)| = 2^{|S|}$.]

- For a positive integer k, a *k-tuple* of elements chosen from a set S is a function f from the set $\{1, 2, \ldots, k\}$ to the set S.

  [Such a k-tuple is often written as $(f(1), f(2), \ldots, f(k))$. The round parentheses indicate that order is critical. Think of k-tuples as ordered lists, usually surrounded by round parenthesis, *e.g.*, (3,3,6), to indicate order. In some situations, a k-tuple might be denoted with angle brackets, or some other kind; the list inside is usually delineated by commas. When $k = 2$, a 2-tuple is called an ordered pair, usually written in the form (a, b), as opposed to $\{a, b\}$, an unordered pair.]

- For non-negative integers, k and n, the *binomial coefficient* $\binom{n}{k}$ denotes the number of different k-element sets contained in an n-element set.

  [With this definition, one can prove that for $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. From the definition, it follows that if $n < k$, then $\binom{n}{k} = 0$. Note that in graph theory, this definition is used most for $\binom{n}{2} = \frac{n(n-1)}{2}$, the number of edges in the complete graph K_n. Binomial coefficients can also be defined for fractions or negative numbers, but such forms are not used in this text.]

- For a positive integer n, write $[n] = \{1, 2, \ldots, n\}$.

- For a set X and a non-negative integer k, $[X]^k = \{S \subseteq X : |S| = k\}$. The notation $[n]^k$ is an abbreviation for $[[n]]^k$, the set of all k-element subsets from $\{1, 2, \ldots, n\}$. For any n-element set X, the collection $[X]^k$ is comprised of $\binom{n}{k}$ sets.

  [Note: elements of $[X]^k$ are all different unordered k-sets, and so $[X]^2$ is the collection of all unordered pairs of (distinct) elements of X, corresponding to all possible edges in a graph with vertex set X.]

19.1.2 Some more technical (but universal) definitions

Many of the following definitions are included only for background information only.

- The *cartesian product* of a set A with a set B is $A \times B = \{(a, b) : a \in A, b \in B\}$.

- For sets A and B, a (binary) *relation from A to B* is a subset of $A \times B$. [So, a relation is simply a set of ordered pairs.] The *domain* of a relation $\mathcal{R} \subset A \times B$ is the set
$$\{a \in A : a \text{ occurs as the first coordinate of some ordered pair in } \mathcal{R}\}.$$

The *range* of a relation $\mathcal{R} \subset A \times B$ is the set

$$\{b \in B : b \text{ occurs as the second coordinate of some ordered pair in } \mathcal{R}\}.$$

Any relation from A to A is called a *relation on* A.

- A *function* f from A to B is a relation from A to B so that for every $a \in A$, there exists one and only one $b \in B$ satisfying $(a, b) \in f$.

  [So a function is a kind of relation, one whose ordered pairs contain each element of A in the first coordinate once and only once. Note also that if $f : A \to B$ is a function from A to B, then A is the domain of f.]

- If $f : A \to B$ is a function and $(a, b) \in f$, write $f(a) = b$ and say that b is the *image* of a under the mapping f. The range of f is $\{f(a) : a \in A\}$.

  [Note: Although B contains the range, B is not the range; some say B is the "co-domain" of f.]

- If $\mathcal{R}$ is a relation from a set X to X, say that $\mathcal{R}$ is a relation on X. If $(a, b) \in \mathcal{R}$, write $a\mathcal{R}b$, and say that a is related to b (by the relation $\mathcal{R}$); write $a \not\mathcal{R} b$ to indicate that a is not related to b.

- Here are a few types of relations:

  A relation $\mathcal{R}$ on X is *reflexive* if and only if $\forall x \in X$, $x\mathcal{R}x$.

  [For example, equality is such a relation; another example is $\leq$.]

  A relation $\mathcal{R}$ on X is *symmetric* if and only if $\forall x, y \in X$, $x\mathcal{R}y$ implies $y\mathcal{R}x$.

  [For example, equality is symmetric; another example is "in the same family as".]

  A relation $\mathcal{R}$ on X is *transitive* if and only if $\forall x, y, z \in X$, $[x\mathcal{R}y \wedge y\mathcal{R}z]$ implies $x\mathcal{R}z$.

  [*E.g.*, the relation "is larger than" is transitive.]

  A relation $\mathcal{R}$ on X is *irreflexive* if and only if $\forall x \in X$, $x \not\mathcal{R} x$.

  A relation $\mathcal{R}$ on X is *antisymmetric* if and only if $\forall x, y \in X$, $[x\mathcal{R}y \wedge y\mathcal{R}x]$ implies that $x = y$.

  [A standard antisymmetric relation is $\leq$.]

- A relation $\mathcal{R}$ on a set X is said to be an *equivalence relation* (on X) if and only if $\mathcal{R}$ is reflexive, symmetric, and transitive.

  [Equality is an equivalence relation, and so is "has the same remainder upon division by 5".]

- A *group* is a set G together with a binary operation * on G so that

  (1) G is closed under *, that is, for any $a, b \in G$, also $a * b \in G$.

  (2) * is associative, that is, for any $a, b, c \in G$, $(a * b) * c = a * (b * c)$.

  (3) G contains an identity element e so that for each $g \in G$, $g * e = g = e * g$.

  (4) G contains inverses, that is, for any $g \in G$, there exists $h \in G$ so that $h * g = e = g * h$; in this case, write $h = g^{-1}$.

  [The set of all automorphisms on a given graph G is denoted $\mathrm{aut}(G)$, and it is not difficult to verify that $\mathrm{aut}(G)$ is a group where the operation is composition of functions. Note that each automorphism defines a permutation of the vertices.]

- A group $(G, *)$ is *abelian* if and only if for every $x, y \in G$, $x * y = y * x$.

19.2 Basic definitions for a first course in graph theory

Most definitions used in a first course in graph theory are listed in this section.

[Comments in square brackets are not part of a definition; they are added only for understanding.]

- A *graph* is an ordered pair $G = (V, E)$, where $V = V(G)$ is a non-empty set, and $E = E(G) \subseteq [V]^2$ is a set of unordered pairs from V; elements of V are called *vertices* and elements of E are called *edges*. Alternatively, [after you have learned the definition of a relation] a simple graph G is a set V together with an irreflexive symmetric binary relation on V. (This relation is usually denoted by E, considered as a set of unordered pairs from V.)

  [Note: many texts uses the word "graph" to describe a much larger class of structures called "multigraphs", or "pseudographs", however, the definition of graph as it is here is a "simple graph". In this text, *all graphs are simple unless otherwise specified.* Also note that the singular of "vertices" is "vertex".]

- An edge $e = \{x, y\} \in E(G)$ is said to *join* x and y; also, x and y are *endpoints* of e, x and y are *incident* with e, or x and y are *adjacent*.

- In a graph G, the *neighbourhood* of a vertex x is

  $$N_G(x) = \{y \in V(G) : \{x, y\} \in E(G)\},$$

  the set of those vertices adjacent to x.

  [When it is clear, the subscript G can be dropped, writing simply $N(x)$.]

- For each $n \in \mathbb{Z}^+$, the *complete graph* K_n is the graph on n vertices with all $\binom{n}{2}$ edges.

  [So if V is a set of n vertices, one can write $K_n = (V, [V]^2)$].

- If G and H are graphs, H is a (weak) *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subseteq E(G) \cap [V(H)]^2$, and H is an *induced subgraph* of G if further $E(H) = E(G) \cap [V(H)]^2$. If the word "induced" is not mentioned, a subgraph is, in general, considered to be a weak subgraph.

- If $G = (V, E)$ is a simple graph, the *complement* of G, denoted $\overline{G}$, is the graph with vertex set V and edge set $[V]^2 \backslash E$.

- For a simple graph G and $x \in V(G)$, the *degree* of x is $\deg_G(x) = |\{y \in V(G) : \{x, y\} \in E(G)\}| = |N(x)|$. If G is a multigraph, the degree of a vertex x is the number of edges incident with x, where loops are counted twice.

  [When clear, "$\deg(x)$" is used in place of "$\deg_G(x)$". Note that in a multigraph, the degree of a vertex and the size of its neighbourhood may differ.]

- The *degree sequence* of a graph G on n vertices is an ordered list $(d_1, d_2, \ldots, d_n)$ of the degrees of the vertices in G, usually written in either in non-decreasing order or non-increasing order.

- A sequence $(d_1, d_2, \ldots, d_n)$ is *graphic* (or *realizable*) if and only if there exists a graph on n vertices with these d_i's as the degrees.

- For a non-negative integer k, a graph G is *k-regular* if and only if every vertex has degree k. A *cubic* graph is a 3-regular graph.

- The maximum degree of vertices in a graph G is denoted $\Delta(G)$, and the minimum degree is denoted $\delta(G)$.

  [If G is k-regular, then $k = \Delta(G) = \delta(G)$.]

- In a graph G, a vertex x with $\deg_G(x) = 0$ is called *isolated*. A vertex x with $\deg(x) = 1$ is called a *leaf*, and the edge connecting a leaf is called a *pendant edge*.

- A *multigraph* is an ordered pair (V, E) where V is a non-empty set and E is a multiset of two-element multisets (called edges) chosen from V. An edge using the same vertex twice is called a *loop*.

  [If for some $x, y \in V(G)$, the set $\{x, y\}$ appears more than once as an edge in a multigraph G, say that there are multiple edges between x and y. The term "multigraph" includes graphs as defined above. For emphasis, the term "simple graph" is often used to denote "graph", that is, a multigraph with no multiple edges and no loops.]

- A *walk* in a multigraph G is an alternating sequence $v_1e_1v_2e_2\cdots e_{m-1}v_m$ of vertices and edges (not necessarily distinct) so that for each $i = 1, 2, \ldots, m-1$, $e_i = \{v_i, v_{i+1}\} \in E(G)$; such a walk has length $m-1$.

- A *trail* is a walk with no edge repeated.

- A *path* is a trail with no vertex repeated.

  [Since no vertices are repeated, it follows that no edges can be repeated, either. The length of a path is the number of edges in that path. In this text, P_n denotes a path of length n. Some texts use P_n to denote a path on n vertices, and so is a path of length $n-1$; this is not standard, so anytime the notation P_n is used, I will make clear what I mean.]

- If the first and last vertex of a walk are the same, the walk is *closed*. A closed trail is called a *circuit*. A walk is called *open* if and only if it is not closed.

- A *cycle* is a closed walk (or trail) where only the first and last vertex of the walk are the same, and no other vertices are repeated. A cycle on n vertices is denoted by C_n.

  [So a cycle is a circuit with no repeated vertices (other than the first/last).]

- If $G = (V, E)$ is a graph, its *line graph* $L(G)$ is the graph whose vertices are E and edges in $L(G)$ are pairs of edges in G that share a vertex.

- The *girth* of a graph G is the length of a shortest cycle in G, and the *circumference* of G is the length of a longest cycle in G.

- An *Eulerian trail* in a graph (or multigraph) G is a trail that includes all edges of G.

  [An Eulerian trail is still a trail, and so no edges are repeated; thus, an Eulerain trail is a trail that uses each edge precisely once.]

- An *Eulerian circuit* is a closed Eulerian trail.

- A graph (or multigraph) G is called *Eulerian* if and only if G has an Eulerian circuit.

- A graph G is *semi-Eulerian* if and only if G contains an open Eulerian trail but no Eulerian circuit.

  [Unfortunately, in some texts, an open Eulerian trail is called an Eulerian path.]

- A *Hamiltonian cycle* in a graph G is a cycle containing all vertices of G.

- A graph G is *Hamiltonian* if and only if G contains a Hamiltonian cycle.

- A graph G is *connected* if and only if for any two vertices x and y, there exists a path from x to y (called an x–y path). If a graph is not connected, it is *disconnected*.

- A maximal connected subgraph of a graph is called a *component*.

  [Above, H is a "maximal" connected subgraph of G if and only if H is connected, and any larger subgraph containing H is no longer connected.]

- For two vertices x, y in a connected graph G, the *distance* $d(x, y)$ between x and y is the length of a shortest x–y path in G.

- If G is a graph (or multigraph or digraph) with $V(G) = \{x_1, x_2, \ldots, x_n\}$ then an *adjacency matrix* for G is an $n \times n$ matrix $A = (a_{ij})$ whose (i, j) entry is the number of edges from x_i to x_j.

  [Note: An adjacency matrix depends upon the ordering of the vertices.]

- If G is a multigraph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$, then the *incidence matrix* (associated with these orderings) for G is the $n \times m$ 0-1 matrix I defined by $I(i, j) = 1$ if and only if x_i is incident with e_j.

  [If G is a digraph, its "oriented" incidence matrix has entries -1,0, or 1, where $I(i, j) = 1$ if and only if e_j leaves x_i and $I(i, j) = -1$ if e_j enters x_i.]

- The *diameter* of a connected graph G, denoted $\mathrm{diam}(G)$, is the maximum distance between vertices in G.

- The *radius* of a connected graph G is $\mathrm{rad}(G) = \min_{v \in V} \max_{y \neq v} d(v, y)$.

  [Note: $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.]

  [So the radius is the smallest distance d for which there exists a vertex v so that all other vertices are within distance d from v. For example, K_7 has radius 1.]

- For a connected graph G and a vertex $v \in V(G)$, the *eccentricity* of v, sometimes denoted $\varepsilon(v)$, is the maximum distance from v to any other vertex.

- For a graph G, a vertex $x \in V(G)$ is called *central* if x has minimum (among all vertices) eccentricity.

  [The maximum (over all $v \in V$) eccentricity is the diameter; the minimum eccentricity is the radius. A vertex x is central if the largest distance from x is smallest.]

- A graph G is called *acyclic* if and only if G contains no cycles.

- A *tree* is a connected acyclic (simple) graph.

- A *forest* is an acyclic (simple) graph.

  [So a forest is a graph whose components are trees.]

- A subgraph H of a graph G is called a *spanning subgraph* if and only if $V(H) = V(G)$.

  [So a *spanning tree* of a graph G is a spanning subgraph that is also a tree.]

- A *1-factor* of a graph G is a spanning subgraph that is regular of degree 1. In general, a *k-factor* of G is a k-regular spanning subgraph.

  [So a 1-factor is a collection of disjoint edges that use all vertices. A 2-factor is a collection of disjoint cycles that use all vertices.]

- For a graph G with edges labelled by real numbers, a *minimum weight spanning tree* is a spanning tree T of G so that the total of all the edge labels in T is minimum (over all spanning trees).

- A vertex x in a connected graph G is called a *cut-vertex* if the removal of x (and all edges incident with it) disconnects G. If G is already disconnected, a cut-vertex is one whose removal increases the number of components.

  [Note: see below for definition of cut-set.]

- A *bridge* in a connected graph G is an edge whose removal disconnects G.

  [Often, an edge in a disconnected graph G is called a bridge if its removal increases the number of components of G.]

- A *digraph* $D = (V, E)$ is an ordered pair $D = (V, E)$, where V is a non-empty set, and E is a set of ordered pairs of distinct vertices from V. Elements of V are called vertices and elements of E are called *directed edges* or *arcs*. If $(x, y) \in E$ is a directed edge, say that the edge is from x to y.

  [Alternatively, a directed graph is a set V together with an irreflexive binary relation on V].

- For a vertex x in a digraph D, the *outdegree* of x, denoted $\deg^+(x)$ is the number of arcs incident from x, that is, the number of arcs of the form (x, y), where $y \in V(D)$. Similarly, define the *indegree* $\deg^-(x)$ (or $d^-(x)$) to be the number of arcs to x.

- A digraph G is *strongly connected* if and only if for any ordered pair of vertices $x, y \in V(G)$, $x \neq y$, there exists a directed path from x to y.

- A graph is *orientable* if there is an orientation of its edges so that the resulting digraph is strongly connected.

  [An "orientation" of an edge $\{x, y\}$ is a replacing of the edge by one of the ordered pairs (x, y) or (y, x), so essentially, an edge is replaced by an arc, a directed edge.]

- A *tournament* is a directed graph D so that for every pair of vertices $x, y \in V(D)$, exactly one of $(x, y) \in E(D)$ or $(y, x) \in E(D)$. Alternatively, a tournament is an orientation of a complete graph.

- In a tournament T, a vertex $v \in V(T)$ is a *king* if and only if for every other vertex $x \in V(T)$, there is a directed path from v to x with length at most 2.

- A *ranking* of the vertices in a tournament T is an ordered list $v_1, v_2, v_3, \ldots, v_n$ of $V(T)$ so that for each $i = 1, \ldots, n-1$, $(v_i, v_{i+1}) \in E(T)$.

  [So a ranking in a tournament is a complete listing of the vertices of the form $v_1 \to v_2 \to v_3 \to \cdots \to v_n$.]

- A subset X of vertices in a graph G is called *independent* if and only if no two vertices in X are joined by an edge of G, that is, if and only if $[X]^2 \cap E(G) = \emptyset$. The maximum number of vertices in any independent set in G is called the *independence number* of G (or sometimes, the *stability number*), usually denoted by $\alpha(G)$.

- For any integer $n \geq 2$, a graph G is called *k-partite* if there is a partition of $V(G)$ into k sets, say $V = X_1 \cup \cdots \cup X_k$ (where for each $i \neq j$, $X_i \cap X_j = \emptyset$) so that for each $i = 1, \ldots, k$, $[X_i]^2 \cap E(G) = \emptyset$. The X_i's are called *partite sets*. A 2-partite graph is called *bipartite*. The notation $K_{m_1,\ldots,m_k}$ denotes the complete k-partite graph on partite sets with $m_1, \ldots, m_k$ vertices respectively, that is, $E(K_{m_1,\ldots,m_k}) = \{\{a, b\} : a \text{ and } b \text{ are in different partite sets}\}$.

  [Note: partite sets are independent sets—all edges go between partite sets. The complete bipartite graph $K_{a,b}$ has $a + b$ vertices and ab edges. If a graph G is bipartite on partite sets V_1 and V_2, it is common to denote this by writing $G = (V_1, V_2, E)$ or $G = (V_1 \cup V_2, E)$. Even cycles are bipartite, and the hypercube graph Q_n is bipartite.]

- A *Gray code* for n-bit words is an ordered list of all 2^n n-bit binary words so that consecutive words in the list differ in exactly one bit; a Gray code is *cyclic* if and only if the first and last word also differ in precisely one bit.

- For any $k \in \mathbb{Z}^+$, the k-dimensional *cube graph* Q_k is the graph defined on $V = \{0, 1\}^k$ where two vertices $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are adjacent if and only if there is exactly one i so that $a_i \neq b_i$.

[So the vertices of Q_k are k-bit binary words, and edges are between pairs of vertices that differ only in one place. Note that Q_k has 2^k vertices, and is regular of degree k. Thus by the HSL, there are $\frac{1}{2}2^k k = k \cdot 2^{k-1}$ edges in Q_k.]

- For a positive integer k and a graph G, a function $c : V(G) \to \{1, 2, \ldots, k\}$ is called a *proper (or good) vertex k-colouring* of G if and only if for every $\{x, y\} \in E(G)$, $c(x) \neq c(y)$. If G has a good vertex k-colouring, G is said to be k-colourable.

  [As the reader might already have noticed, if a graph is k-partite, then the graph is k-colourable.]

- If G is a graph, the *chromatic number* of G, denoted $\chi(G)$, is the least number k so that G has a proper vertex k-colouring.

- Two graphs G and H are *isomorphic* if and only if there exists a bijection (called an *isomorphism*) $f : V(G) \to V(H)$ so that for every pair of vertices $x, y \in V(G)$, $\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} \in E(H)$.

- A connected graph (or multigraph) G is *k-connected* if and only if the removal of fewer than k vertices does not disconnect the graph. Similarly, G is ℓ-edge-connected if and only if the removal of fewer than ℓ edges fails to disconnect G.

- For a connected graph (or multigraph) G, the *vertex-connectivity* $\kappa(G)$ is the least number (if it exists) of vertices whose removal disconnects the graph; $\kappa(K_n)$ is defined to be $n - 1$. The *edge-connectivity* $\lambda(G)$ is the least number of edges whose removal disconnects G.

  [So, if a graph G is k-connected, then $\kappa(G) \geq k$. Note that a connected graph is 1-connected. Also, the edge-connectivity of a graph (or multigraph) with only one vertex is not defined. Recall that $\kappa(G) \leq \lambda(G) \leq \delta(G)$.]

- A *planar drawing* of a graph (or multigraph) G is a drawing of G in the plane where two edges meet only at a vertex they are both incident with. Connected regions determined by a plane drawing are called *faces* or *regions*, where the (outer) unbounded face is called the *infinite face* (or simply, the outer face).

  [A planar drawing is sometimes called a *plane drawing*.]

- A graph G is called *planar* if and only if G has a planar drawing.

- For a connected planar graph G with a planar drawing, the *dual graph* G^* is the graph whose vertex set is the set of faces, and the edge set of G^* is the set of pairs of faces sharing a common border.

[The dual graphs arising from two different planar drawings of a polyhedral graph G can be different.
Also, some authors define the dual of a planar graph G to be a multigraph, where the number of edges in G^* between two regions F_i and F_j is the number of edges that border both F_i and F_j. Since such a multigraph has the same vertex-colouring properties as the simple dual graph described above, the simple dual graph definition given here is not restrictive.]

- For a graph G, a subset of edges $M \subset E(G)$ is called a *matching* if and only if no two edges in M share a vertex; a vertex x is said to be *matched* to y if $\{x, y\} \in M$. If $G = (V_1, V_2, E)$ is bipartite, a matching $M \subset E$ is called a *complete matching* of V_1 to V_2 if $|M| = |V_1|$.

- A *subdivision* of a graph G is another graph obtained by subdividing edges in G, that is, by inserting any number of vertices of degree 2 in edges of G. An *edge-contraction* of an edge e in a graph G is the process of creating a new graph obtained by identifying the endpoints of e and removing any multiple edges or loops thereby obtained. A graph H is a *contraction* of G if and only if H can be obtained by edge-contractions of edges in G.

  [Note: if H can be obtained from G by subdividing edges of G, then G can be obtained by contracting edges of H, but not necessarily the other way around.]

- A graph H is a *minor* of G if and only if H can be obtained from G by deleting vertices and/or edges and by contracting edges.

- In any graph $G = (V, E)$ and $S \subseteq V$, define the neighbourhood of S by $N(S) = \cup_{v \in S} N(v)$. In a bipartite graph $G = (X \cup Y, E)$, if $A \subseteq X$, then $N(A) \subseteq Y$. A bipartite graph $G = (X \cup Y, E)$ satisfies *Hall's condition* if and only if for every $A \subseteq X$, $|A| \leq |N(A)|$.

  [Note: In some instances, the neighbourhood of a set S is the set of those vertices in $V(G) \backslash S$ that are adjacent to at least one vertex in S. This definition is different from above in that it does not allow vertices of S to be in $N(S)$, and so might be called the "proper neighbourhood" of S.]

- For a set X and a family $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets of X, *a system of distinct representatives* (denoted "SDR") for $\mathcal{S}$ is a collection of distinct elements $x_1, x_2, \ldots, x_n \in X$ so that for each $i = 1, 2, \ldots, n$, $x_i \in S_i$.

Chapter 20

Solutions to selected exercises

Solution to Exercise 1: Let $G = (V, E) = (V(G), E(G))$ where V is a non-empty set and $E \subseteq [V]^2$. Define the binary relation $\mathcal{R} = \{(x, y) : \{x, y\} \in E\} \subseteq V \times V$. For any x, y, if $(x, y) \in \mathcal{R}$, then $\{x, y\} \in E$; but since $\{x, y\} = \{y, x\}$, it follows that $(y, x) \in \mathcal{R}$ as well. So, $\mathcal{R}$ is symmetric. For any $x \in V$, $\{x, x\} \notin E$, so $(x, x) \notin \mathcal{R}$, so $\mathcal{R}$ is irreflexive. $\square$

Solution to Exercise 2: Let G be a (simple) graph on n vertices. There are precisely $\binom{n}{2}$ pairs of vertices and since G is simple, each pair of vertices corresponds to at most one edge (and every edge is indeed such a pair), and the result follows. $\square$

Solution to Exercise 3: The number of vertices in K_{101} is 101, and there are $\binom{101}{2}$ pairs of vertices. Since each pair of vertices determines precisely one edge, there are

$$\binom{100}{2} = \frac{101 \cdot 100}{2} = 101 \cdot 50 = 5050$$

edges in K_{101}. $\square$

Solution to Exercise 4: In a graph on V, there are $\binom{n}{2}$ possible pairs, each of which can be an edge or a non-edge. Since three are two choices for each of the pairs, by the product principle, there are

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{\binom{n}{2} \text{ times}} = 2^{\binom{n}{2}}$$

different graphs on V. So when $n = 5$, there are

$$2^{\binom{5}{2}} = 2^{10} = 1024$$

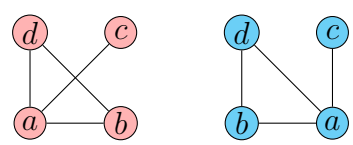
graphs on five labelled vertices. $\square$

Solution to Exercise 5: Let G be a connected graph and let $P = x_1, x_2, \ldots, x_m$ and $Q = y_1, y_2, \ldots, y_m$ be two longest paths.

Suppose, in hope of a contradiction, that P and Q share no common vertex. Since G is connected, there exists a path starting at some x_j and ending at some y_k. Then there is a path starting at one end of P, going to x_j, going to y_k, to an end of Q. At least one such path has more than m edges, contradicting that P and Q were maximal, the desired contradiction. □

Solution to Exercise 6: Even though both graphs have the same numbers of vertices and edges, they are not isomorphic, since in the first graph, vertex a is incident with just one edge, whereas there is no such vertex in the second graph. One might also show non-isomorphism by observing that the first graph has a triangle, whereas the second does not. □

Solution to Exercise 7: These two graphs seem to share features; they both have a triangle, and they both have a vertex incident with only one edge. In fact, they are isomorphic, and to prove this, one first finds a bijection between the two vertex sets. Label the vertices of the first graph with a, b, c, d and use these labels to identify the vertices of the second graph, thereby demonstrating the bijection between the vertices.



It is straightforward to check that all adjacent pairs of vertices in the left graph are also adjacent in the right, and all non-adjacent pairs in the left graph are also non-adjacent in the right. □

Solution to Exercise 8: The graph on the left is called Tietze's graph and the one on the right is called Franklin's graph. Tietze's graph contains a triangle (around the outside), but Franklin's graph has no odd cycles whatsoever. So they are not isomorphic. □

Solution to Exercise 9: One needs to show that $\cong$ is an equivalence relation, that is, that $\cong$ is (i) reflexive, (ii) symmetric, and (iii) transitive.

(i) For any graph G, $G \cong G$ because the identity map $f : V(G) \to V(G)$ is a bijection and preserves edges and non-edges. So $\cong$ is reflexive.

(ii) Suppose that $G \cong H$ with an isomorphism $f : V(G) \to V(H)$ as a witness. Then the inverse map $f^{-1} : V(H) \to V(G)$ is a bijection (since f is). It remains to show that f^{-1} preserves edges and non-edges. Let $\{a, b\} \in [V(H)]^2$. Since f is a bijection, there exists $x, y \in V(G)$ so that $f^{-1}(a) = x$ and $f^{-1}(b) = y$. Since

$\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} \in E(H)$, it follows that $\{a, b\} \in E(H)$ if and only if $\{f^{-1}(a), f^{-1}(b)\} \in E(G)$.

(iii) Let $F \cong G$ and $G \cong H$ with isomorphisms $f : V(F) \to V(G)$ and $g : V(G) \to V(H)$ as witnesses. Consider the composition h of f with g defined by $h(x) = g(f(x))$. Since both f and g are bijections, so is h. For any pair of vertices x, y in F, $\{x, y\} \in E(F)$ if and only if $\{f(x), f(y)\} \in E(G)$ and f or any pair of vertices w, z in G, $\{w, z\} \in E(G)$ if and only if $\{g(w), g(z)\} \in E(H)$. Using $w = f(x)$ and $z = f(y)$, $\{x, y\} \in E(F)$ if and only if $\{g(w), g(z)\} = \{h(x), h(y)\} \in E(H)$. □

Solution to Exercise 10: See Figure 20.1. Note that in the list of drawings given in Figure 20.1, most drawings have other isomorphic representations (for example, both the 6th and 7th graph can each be drawn in 4 ways). □

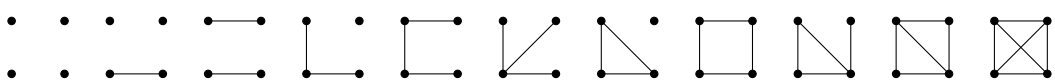
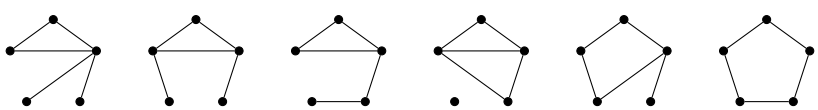


Figure 20.1: The 11 non-isomorphic graphs on 4 vertices

Solution to Exercise 11:



□

Hint for Exercise 12: There are 14. □

Solution to Exercise 13: Assume that G is bipartite on partite sets X and Y. If C is a cycle in G, consecutive vertices in C appear alternating between X and Y. Since C is a closed walk, C begins and ends in the same partite set. Thus, C has an even number of vertices.

Assume that $G = (V, E)$ has no odd cycles. Let $x \in V$, and define a colouring $f : V \to \{0, 1\}$ by $f(v) = 0$ if and only if the distance between x and v is an even number (and so $f(v) = 1$ if and only if the distance is odd). Since distances from x to v are well defined (such distance is the length of a shortest path to v), f is indeed a function. Define $X = \{v \in V : f(v) = 0\}$ and $Y = \{v \in V : f(v) = 1\}$. It remains to show that there are no edges in X and no edges in Y.

Suppose that $u, v \in X$, $u \neq v$, let $P = (x = u_0, u_1, \ldots, u_k = u)$ be a shortest x–u path, and let $Q = (x = v_0, v_1, \ldots, v_\ell = v)$ be a shortest x–v path. Let i be the largest index so that $u_i = v_i$ ($i = 0$ is possible). If $\{u, v\} \in E$, the vertices $u_i, u_{i+1}, \ldots, u_k = u, v = v_\ell, \ldots, v_{i+1}, v_i = u_i$ form an odd cycle, so $\{u, v\} \notin E$. The same argument shows that there are no edges in Y either. □

Solution to Exercise 14: Let the partite sets of $K_{a,b}$ be given by $V = V(K_{a,b} = A \cup B$, where $|A| = a$ and $|B| = b$. Since A and B are disjoint, $|V| = |A \cup B| = |A| + |B| = a + b$. Since the edge set of $K_{a,b}$ is $\{\{x, y\} : x \in A, y \in B\}$, which is equi-numerous with $A \times B$, and by the product rule, $|A \times B| = |A| \times |B| = ab$. □

Solution to Exercise 15: There are seven different (non-isomorphic) bipartite graphs on 4 vertices; these are listed in Figure 20.2. □

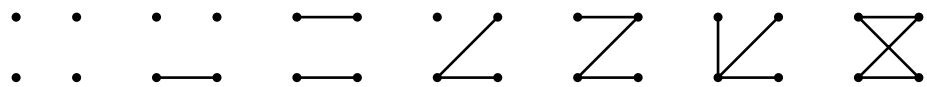


Figure 20.2: The non-isomorphic bipartite graphs on 4 vertices

Solution to Exercise 17: The labellings of the graphs in Figure 20.3 show one isomorphism.

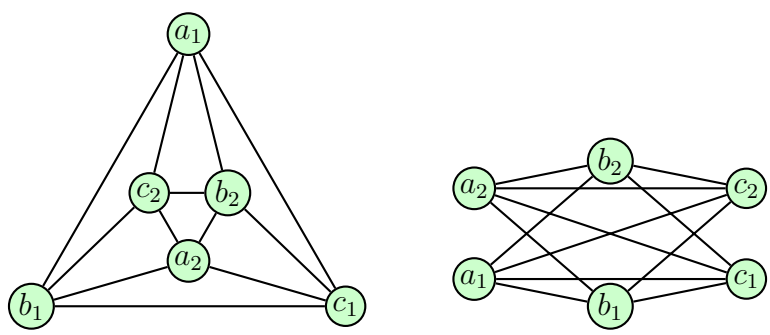


Figure 20.3: The octahedron graph is isomorphic to $K_{2,2,2}$

□

Solution to Exercise 18: Let $G = K_{2,3,7}$, wit partite sets A, B, C, where $|A| = 2$, $|B| = 3$, and $|C| = 4$. Then $|V(G)| = |A| + |B| + |C| = 2 + 3 + 7 - 12$. Between A and B there are $2 \cdot 3 = 6$ edges, between A and C there are $2 \cdot 7 = 14$ edges, and between B and C there are $3 \cdot 7 = 21$ edges. In all, there are $6 + 14 + 21 = 41$ edges. □

Solution to Exercise 19: Extending the solution given for Exercise 18, the number of vertices is $\sum_{i=1}^{k} t_i$ and the number of edges is $\sum_{i \neq j} t_i, t_j$. □

Solution to Exercise 20: There are many isomorphisms between the first two graphs. Here is one idea as to how to start. Label the vertices of the first drawing, where, say, the outside cycle is a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 the inner cycle, where for each

i, a_i is adjacent to b_i. In the second drawing, pick *any five cycle* (*e.g.*, the top five vertices form a 5-cycle) and label them cyclically with $a_1, \ldots, a_5$. It remains only to determine which other vertex a_1 is adjacent to in the second drawing and label it b_1. Continue the same with a_2, a_3, a_4, a_5. □

Solution to Exercise 21: Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let $X = \{x_1, \ldots, x_m\}$ be a set, and for each $i = 1, \ldots, n$, put $S_i = \{x_j : \{v_i, v_j\}\} = e_j\}$ and let $\mathcal{F} = \{S_1, \ldots, S_n\}$. Essentially, each vertex (set) is formed by adding an element unique to each incident edge. □

Solution to Exercise 23: There are $r + 1$ non-isomorphic connected subgraphs of $K_{1,r}$, one each for subgraphs with $0, 1, \ldots, r$ edges, respectively. □

Solution to Exercise 25: Let G be a self-complementary graph on n vertices. Then G has

$$|E(G)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$$

edges, which is an integer, so 4 divides $n(n-1)$. There are three cases to check: 4 divides n, 4 divides $n-1$, or 2 divides n and 2 divides $n-1$. The first two cases say that $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. The third case can never happen because two consecutive integers are not both even. □

Solution to Exercise 26: There is only one graph on $n = 1$ vertices, namely K_1, a single vertex, which is self-complementary. There are only two graphs on $n = 2$ vertices, neither of which are self-complementary. Similarly, no graph on $n = 3$ vertices is self-complementary (since there are only 3 possible edges, an odd number, so the equation $|E(G)| = 3/2$ never holds. When $n = 4$, there are $\binom{4}{2} = 6$ possible edges, so look for a graph with 3 edges. There are three such graphs: P_3 (a path of length 3, sometimes denoted P_4), a star $K_{1,3}$ (also called a claw), and a triangle with an isolated vertex. Of these three, only P_3 is self-complementary.

For $n = 5$, look for a graph with $\frac{1}{2}\binom{5}{2} = 5$ edges. After eliminating a few obvious cases, find that C_5 is self-complementary. (If C_5 is drawn as a cycle in a circular way, its complement is a pentagram (a star drawn in one stroke), and following the edges of a pentagram shows that it, too, is a 5-cycle. Also find that the so-called "bull graph" (a triangle with a pendant edge drawn from each of two of the triangle's vertices–see Exercise 525 for a drawing) is also self-complementary. There are no others. □

Hint to Exercise 27: One has degree sequence $(4, 4, 4, 4, 3, 3, 3, 3)$. Such a graph can be found in [585, p 162], along with an algorithm to generate self-complementary graphs by way of automorphisms. As already noted, there are 10 self-complementary graphs on 8 vertices.

Solution outline to Exercise 29: Let the vertices of K_2 be labelled $\{x_1, x_2\}$ and the vertices of K_3 be labelled $\{y_1, y_2, y_3\}$. Name the vertices of $K_2 \square K_3$ by $a = (x_1, y_1)$, $b = (x_2, y_1)$, $c = (x_1, y_2)$, $d = (x_2, y_2)$, $e = (x_1, y_3)$, and $f = (x_2, y_3)$. Then the graph $K_2 \square K_3$ can be drawn as in Figure 20.4.

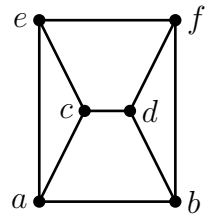


Figure 20.4: $K_2 \square K_3$

Then the complement is the graph formed by the cycle with the ordering of vertices given by a, d, e, b, c, f. □

Solution outline to Exercise 31: Let $V(G) = \{x_1, \ldots, x_k\}$ and $V(H) = \{y_1, \ldots, y_\ell\}$. Define $f : V(G \times H) \to V(H \times G)$ by, for each i, j, $f(x_i, y_j) = (y_j, x_i)$. It is routine to check that

$$\{(x, y), (x', y')\} \in E(G \square H) \text{ if and only if } \{f(x, y), f(x', y')\} \in E(H \square G).$$

□

Solution to Exercise 33: Since each of K_4 and $K_{2,3}$ have six edges, each line graph has six vertices.

The graph $L(K_4)$ is the graph of the octahedron (see Figure 1.20), which is also $K_{2,2,2}$.

The graph $L(K_{2,3})$ is formed by a cycle on six vertices, say x_1, x_2, x_3, x_4, x_5, x_6 (in cyclic order) and adding the three edges x_1x_5, x_2x_4, and x_3x_6. This graph is sometimes known by the cartesian product $K_3 \square K_2$; this same graph is also the graph of a triangular prism, which is $\overline{C_6}$ (see Exercise 29). □

Solution to Exercise 35: Both graphs have three edges, so both line graphs have three vertices. The line graphs are both equal to K_3. □

Solution to Exercise 37: Let G be a graph, and let $H = L(G)$ be its line graph. Suppose that H contains a copy of $K_{1,3}$ (not necessarily induced), where the singleton partite set is a vertex w and the remaining three vertices adjacent to w are x, y, z. Let e, f, g, h be edges in G corresponding to the respective vertices w, x, y, z in H, and let $e = \{v_1, v_2\}$.

Since e is incident with each of f, g, h, there are two possibilities: either all three of f, g, h contain the same v_i, or two of f, g, h are incident with one of v_1 or v_2 and the remaining edge is incident with the other. In either case, by the PHP, there is a vertex, say v_1 that is incident with two of f, g, h; without loss of generality, suppose that v_1 is adjacent with f and g. In this case, however, f and g are incident, so $\{f, g\} \in E(H)$, and so the copy of $K_{1,3}$ is not induced. □

Solution to Exercise 38: Let $G = (V(G), E(G))$ be a graph and let $H = L(G) = (X, F)$ be its line graph. By definition $X = E(G)$ so $|V(H)| = |X| = |E(G)|$. Two vertices in H form an edge if and only the corresponding edges from G are incident. Thus, the number of edges in H is

$$\sum_{x \in V} \binom{\deg(x)}{2}.$$

□

Solution to Exercise 39: Using the first diagram in Figure 1.26, contract each of the five spokes of the Petersen graph. □

Solution to Exercise 40: Let G be a graph on n vertices. For each $v \in V(G)$, $\deg(v) \in \{0, 1, 2, \ldots, n-1\}$. However, both a vertex of degree 0 and a vertex of degree $n-1$ can not exist in the same graph, so use PHP with n pigeons (vertices) and possible degrees as $n-1$ holes (either $[0, n-2]$ or $[1, n-1]$). □

Solution to Exercise 41: The construction is done inductively. When $n = 2$, either both degrees are 0 or both are 1.

Let $n > 2$ and suppose that a graph G on $n-1$ vertices has been constructed with precisely two vertices of the same degree, say x and y.

Case 1: If G has no isolated vertices, add an isolated vertex. The resulting graph still has just one pair of vertices with the same degree.

Case 2: If G has an isolated vertex, then G has no vertices of degree $n-1$, so add one vertex adjacent to each vertex in G, producing H on n vertices with only one pair having the same degree. □

Solution to Exercise 43: Two inductive proofs are available, one by induction on $|V(G)|$, and the other by induction on $|E(G)|$. The first natural choice might be to induct on the number of vertices, so this proof is presented first. Inducting on the number of edges follows.

For $n \geq 1$, let $S(n)$ be the statement that for any graph G on n vertices,

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|.$$

Base step: For $n = 1$, the only graph is a single vertex x; both $\deg(x) = 0$ and $2|E(G)| = 0$, so $S(1)$ holds.

Inductive step: Let $k \geq 1$ and suppose that $S(k)$ is true. Let G be a graph on $k+1$ vertices, and let $v \in V(G)$. Suppose that $\deg(v) = d$, and let $y_1, \ldots, y_d$ be the neighbours of x. Form G' by deleting v (and all incident edges). Then G' has k vertices, and so by induction hypothesis $S(k)$, $\sum_{x \in V(G')} \deg_{G'}(x) = 2|E(G')|$. Since the degree of each y_i is precisely one less in G' than in G,

$$\begin{aligned}
\sum_{x \in V(G)} \deg_G(x) &= d + \left[\sum_{x \in V(G')} \deg_{G'}(x) \right] + \deg_G(v) \\
&= d + 2|E(G')| + d && \text{(by } S(k)\text{)} \\
&= 2(d + |E(G')|) \\
&= 2|E(G)| && (d \text{ edges in } G \text{ contain } v).
\end{aligned}$$

This shows that $S(k+1)$ is true, completing the inductive step.

By MI, for any $n \geq 1$, $S(n)$ holds.

Next, induct on the number of edges (this method is nearly equivalent to the direct proof already given). For $m \geq 0$, let $A(m)$ be the statement that for any graph G with m edges, $\sum_{x \in V(G)} \deg(x) = 2|E(G)|$.

Base step: The only graphs with 0 edges are the empty graphs, where each vertex has degree 0, in which case the sum of degrees is 0, which is equal to $2 \cdot 0$, so $A(0)$ holds.

Inductive step: Let $k \geq 1$ and suppose that $A(k-1)$ is true, that is, for every graph G with $k-1$ edges, $\sum_{x \in V(G)} \deg(x) = 2(k-1)$. Let H be a graph with $k \geq 1$ edges, and delete one edge e, producing the graph $F = H - e$. Since removing e lowered the total degrees by 2,

$$\begin{aligned}
\sum_{x \in V(H)} \deg(x) &= 2 + \sum_{x \in V(F)} \deg(x) \\
&= 2 + 2|E(F)| && \text{(by } A(k-1) \text{ applied to } F\text{)} \\
&= 2 + 2(k-1) \\
&= 2k \\
&2|E(H)|.
\end{aligned}$$

Since H is an arbitrary graph with k edges, $A(k)$ holds, completing the inductive step.

By MI, for all $m \geq 0$, $A(m)$ is true. $\square$

Solution to Exercise 44: Suppose, in hopes of contradiction, that deleting some edge $e = \{x, y\}$ disconnects G, and let H be the component of $G - e$ containing x. Then $\deg_H(x) = 9$, and all other vertices in H have degree 10. But then the sum of degrees in H is odd, contrary to the HSL. $\square$

Solution to Exercise 45: Suppose, in hope of a contradiction, that there is a graph $G = (V, E)$ with an odd number of vertices with odd degree and examine the sum of all degrees split into the two types:

$$\sum_{v \in V} \deg(v) = \sum_{x:\ \deg x \text{ is odd}} \deg(x) + \sum_{y:\ \deg y \text{ is even}} \deg(y).$$

By the Handshaking lemma (Lemma 1.8.1), the sum on the left is even. However, an odd number of odd integers is again odd, and any sum of even numbers is again even, the expression above is of the form "even = odd + even", which is evidently false—the desired contradiction. $\square$

Answer to Exercise 46: Vertices, paths, and cycles are the only possibilities. $\square$

Solution to Exercise 47: Suppose that G is a k-regular graph on n vertices. By the Handshaking lemma,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg(v) = \frac{1}{2} \sum_{v \in V(G)} k = \frac{1}{2} nk.$$

$\square$

Solution to Exercise 48: For each positive integer k, the cube graph Q_k has 2^k vertices. For any one binary string of length k, there are precisely k other strings that differ in exactly one coordinate (namely one for each coordinate) and so each vertex of Q_k has degree k. Thus by the HSL, (or Lemma 1.8.4) there are $\frac{1}{2} \sum_{x \in V(Q_k)} k = \frac{1}{2} 2^k \cdot k = k 2^{k-1}$ edges in Q_k. $\square$

Solution to Exercise 49: (This exercise also appears in [190, q. 17a, p. 72] Supposing that one exists, let G be a 4-regular graph on 7 vertices. Then $\overline{G}$ is a 2-regular graph on 7 vertices. Any 2-regular graph is either a cycle or a union of cycles (as noted just before Lemma 1.8.4). Hence,'the only possibilities for $\overline{G}$ are C_7 or $C_3 \cup C_4$. $\square$

Solution to Exercise 51: Suppose that G is a graph that is not regular. Construct a new (larger) graph H by starting with two vertex disjoint copies of G, and for each vertex not of maximum degree, add an edge between that vertex and its twin. Then

$\delta(H) = \delta(G) + 1$, and $\Delta(H) = \Delta(G)$. If $\delta(H) < \Delta(H)$, repeat the above process applied to H creating another graph with larger minimum degree, and continue this process until a regular graph is obtained. □

First solution to Exercise 53: (Handshake problem) This problem is an old classic (see, *e.g.*, [409], [977, pp. 481–482], or [1009, pp. 84–85]). Of the $2n$ people, nobody shook more than $2n-2$ hands (everyone but the person and the spouse). Since everyone but Jack shook a different number of hands, the numbers of handshakes per person (not including Jack) are then, in some order, $0, 1, \ldots, 2n-2$. For everyone but Jack, for $i = 0, 1, \ldots, 2n-2$, let P_i be the person shaking i hands.

If P_{2n-2} is married to anyone other than P_0, then P_{2n-2} does not shake hands with their spouse, with P_0, or themselves, giving at most $2n-3$ possible shakes, so P_{2n-2} is married to P_0. Similarly, P_{2n-3} is married to P_1, P_{2n-4} is married to P_2. Continuing, P_{n+1} is married to P_{n-3}, and P_n is married to P_{n-2}. This leaves only P_{n-1}, who then is married to Jack. Thus, Jill shook $n-1$ hands. □

Second solution to Exercise 53: Use induction on n. Translating the problem to one of graph theory, for the moment, fix a party with the set of people $V = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$, where for each $i = 1, \ldots, n$, $\{x_i, y_i\}$ is the i-th couple. Let $G = (V, E)$ be the graph on V be defined by $\{v, w\} \in E(G)$ if and only if v and w shake hands.

For $n \geq 1$, let $H(n)$ be the statement that for any party of n couples, with host x_1 and hostess y_1, if in the graph for the party, the degrees of all vertices but the host are different, then $\deg(y_1) = n-1$.

Since no partners shake hands, then for any $v \in V$, $0 \leq \deg(v) \leq 2n-2$.

Base step: When $n = 1$, there is only one couple, no handshakes, and the hostess shakes $n-1 = 1-1 = 0$ hands as required.

Inductive step: Fix some $m \geq 1$ and suppose that $H(m)$ holds. Consider a party with $m+1$ couples $\{x_i, y_i\}$, G being its graph, no couple shaking hands, x_1 the host, y_1 the hostess, and where the degrees in G of all but the host are different. Since any vertex has degree at most $2(m+1) - 2 = 2m$, the degrees of all vertices but the host are $0, 1, 2, \ldots, 2m$.

The person other than the host with degree $2m$ shook hands with everyone else except the person of degree 0, so these two form a couple. Since the host is not part of this couple, neither is the hostess, so let x_{m+1}, y_{m+1} be the couple with degrees $0, 2m$ respectively.

Now consider the party where the couple x_{m+1}, y_{m+1} is deleted, with G' its graph on $2m$ vertices $V' = \{x_1, y_1, x_2, y_2, \ldots, x_m, y_m\}$, where x_1, y_1 are host, hostess, respectively. Because $\deg_G(x_{m+1}) = 2m$ and $\{x_{m+1}, y_{m+1}\} \notin E(G)$, x_{m+1} was connected in G to all vertices in V', so the degrees of vertices in $V' \backslash x_1$ are reduced by one in G'; thus, the degrees of vertices in $V' \backslash \{x_1\}$ are $0, 1, 2, \ldots, 2m-2$. So G' corresponds to a party

with m couples satisfying $H(m)$, and so $\deg_{G'} y_1 = m - 1$. Adjoining the two deleted vertices shows that $\deg_G(y_1) = m$, the required degree of the hostess. This concludes the inductive step $H(m) \to H(m+1)$.

By mathematical induction, for every $n \geq 1$, the statement $H(n)$ is true. □

Comment on Exercise 54: The result in this Exercise was noted by Hakimi [476]. Suppose that the sum of the d_is is even. One simple proof is to use up as many degrees as possible by adding loops (for each d_i, let vertex v_i have $\lfloor d_i/2 \rfloor$ loops). Since the number of odd d_is is even, it remains to add matchings between vertices of odd degree. □

Solution to Exercise 55: Try $(2, 2)$ or $(3, 2, 2, 1)$. □

Solution to Exercise 56: Since the sum of the elements in this sequence is odd, this sequence is not the degree sequence of any graph. □

Solution to Exercise 57: A first observation is that the sum of the degrees is even, so one can not say immediately that no such graph exists.

Deleting the 7 and subtracting 1s from the next seven positions gives a sequence $(5, 4, 3, 3, 3, 1, 0, 1)$, and after re-ordering, gives $(5, 4, 3, 3, 3, 1, 1, 0)$. Then deleting the 5 and subtracting 1 from the next five positions gives $(3, 2, 2, 2, 0, 1, 0)$, and after re-ordering, $(3, 2, 2, 2, 1, 0, 0)$. Deleting the 3 and subtracting 1 from the next three positions gives $(1, 1, 1, 1, 0, 0)$. This last sequence is realizable by two disjoint edges and two isolated vertices, so the original sequence is indeed realizable.

To find a graph with the original degree sequence, start with a graph realizing the last sequence (two disjoint edges, two isolated vertices, six vertices in all) and introduce a new vertex of degree 3 joined to the first three vertices. Then introduce another vertex of degree 5 joined to the vertices corresponding to the positions where 1 was subtracted. Repeat the same procedure with one last new vertex of degree 7. □

Solution to Exercise 58: Using Havel–Hakimi, $(6, 4, 4, 2, 2, 1, 1) \to (3, 3, 1, 1, 0, 0) \to (2, 0, 0, 0, 0)$, which is not realizable, so neither is the original sequence. □

Solution to Exercise 60: This classic result was observed by Erdős and Rado (see [325]) many decades ago. Suppose that G is not connected, and let x and y be vertices. If $\{x, y\} \notin E(G)$, then in $\overline{G}$, x and y are adjacent. So assume that $\{x, y\} \in E(G)$; then x and y are in the same component of G. For any other $z \in V(G)$ from a different component of G, both $\{x, z\} \in E(\overline{G})$ and $\{y, z\} \in E(\overline{G})$, and so x–z–y is a path connecting x and y in $\overline{G}$. So $\overline{G}$ is connected. □

Solution to Exercise 61: Indeed, the same proof works. □

Solution to Exercise 62: Let G be a graph on n vertices with components $G_1, \ldots, G_k$. Applying Lemma 1.9.2 to each component,

$$|E(G)| = \sum_{i=1}^{k} |E(G_i)| \geq \sum_{i=1}^{k} (|V(G_i)| - 1) = n - k.$$

With $m = |E(G)|$, this says $m \geq n - k$, from which the result follows. □

Solution to Exercise 63: Let G be a graph on n vertices that satisfies $\delta(G) \geq \frac{n-1}{2}$ and let $x, y \in V(G)$. If $\{x, y\} \notin E(G)$, then $N(x)$ contains at least $\frac{n-1}{2}$ vertices and does not contain y. Similarly, $N(y)$ contains at least $\frac{n-1}{2}$ vertices and does not contain y. Of the $n-2$ remaining vertices, more than half are neighbours of x and more than half are neighbours of y. By the PHP, any two vertices share a neighbour. □

Solution to Exercise 64: Assume that G is a graph on vertices $v_1, \ldots, v_n$ satisfying the degree conditions given. If all degrees are $n-1$, then $G = K_n$, which is connected. Suppose that G is not connected, and let H be a component not containing v_n. Set $X = V(H)$ and $Y = V(G) \backslash X$. Since Y contains v_n, $|Y| \geq \Delta + 1$, and so $|X| \leq n - \Delta - 1$. Hence, $\max_X \deg(x) \leq n - \Delta - n - 2$, and so all vertices of degree at least $n - 1 - \Delta$ are in Y. Hence, setting $k = n - 1 - \Delta$, $X \subseteq \{v_1, \ldots, v_{k-1}\}$. However, v_{k-1} has degree at least $k - 1$, contradicting the maximum degree in X already stated. Thus, G is connected. □

Solution to Exercise 65: Let G be a disconnected graph on n vertices with $c \geq 2$ components $G_1, \ldots, G_c$. For each i, let $|V(G_i)| = n_i \geq 1$ (where $\sum_{i=1}^{c} n_i = n$). For this choice of the n_is, the maximum number of edges in G is achieved when each component is complete, and so

$$|E(G)| \leq \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_c}{2}.$$

If $c \geq 3$, then the graph formed by adding all edges between G_1 and G_2 is still disconnected, but has more edges, so to maximize the number of edges, it suffices to consider only when $c = 2$. Thus it suffices to show that when $1 \leq k \leq n - 1$,

$$\binom{k}{2} + \binom{n-k}{2} \leq \binom{n-1}{2}. \tag{20.1}$$

Since the left side of (20.1) is a quadratic in k (whose graph is concave up), the left side is maximized at the endpoints of the interval $[1, n-1]$, the only values for which indeed equality is attained. □

Solution to Exercise 66: Let G be a graph with only two vertices u and v with odd degree. By Lemma 1.8.2, in any component of G, the number of odd degree vertices is

even. Thus, the number of odd degree vertices in any component is either 0 or 2. So if x is in one component, then y is also. $\square$

Solution to Exercise 68: This problem and solution appears in [192, Ex 3.13]. If f is distance preserving, then x is adjacent to y if and only if $d(x,y) = 1$ if and only if $d(f(x), f(y)) = 1$ if and only if $f(x)$ is adjacent to $f(y)$, so f preserves edges. Similarly, x is not adjacent to y if and only if $d(x,y) = \infty$ if and only if $d(f(x), f(y)) = \infty$ if and only if $f(x)$ is not adjacent to $f(y)$, so f preserves non-edges. Since f is a bijection that preserves edges and non-edges, f is an isomorphism. $\square$

Solution to Exercise 69: For $n \geq 1$, let $C(n)$ be the claim that for any connected graph G on n vertices, the sum of the distances from any fixed v to all other vertices of G, $\sum_{w \in V(G)} d(v,w)$, is at most $\binom{n}{2}$.

Base step: When $n = 1$, the sum of distances from the single vertex is 0, and $\binom{n}{2} = \binom{1}{2} = 0$ as well, so $C(1)$ holds. Also, when $n = 2$, the only connected graph is K_2, in which case the sum of the distances from one vertex is $1 = \binom{2}{2}$, so $C(2)$ holds as well.

Inductive step: For some $m \geq 2$, suppose that $C(m)$ holds, and let G be a graph on $m+1$ vertices with some vertex v fixed. Pick $x \in V(G) \backslash \{v\}$, and consider the graph G' on $m \geq 2$ vertices formed by deleting x (and all edges incident with x). Further suppose that x is not a cut-vertex (since the result of Exercise 177 says that there are always two such, one of which could be v). Since G is connected, $d(v,x) \leq m$. Since G' is connected, the induction hypothesis applies to G', and so

$$\begin{aligned}
\sum_{y \in V(G)} d(v,y) &= \left[\sum_{y \neq x} d(v,y)\right] + d(v,x) \\
&\leq \left[\sum_{y \neq x} d(v,y)\right] + m \\
&\leq \binom{m}{2} + m && \text{(by } C(m)\text{)} \\
&= \binom{m+1}{2},
\end{aligned}$$

showing that $C(m+1)$ follows, completing the inductive step.

By mathematical induction, for every $n \geq 1$, $C(n)$ holds. $\square$

Solution to Exercise 70: Let $a, b \geq 2$ and let $K_{a,b}$ have partite sets A and B, where $|A| = a$ and $|B| = b$. The distance between any two vertices $x \in A$ and $y \in B$ is 1. Two vertices in the same partite set are not adjacent, but they are both adjacent to a vertex in the other partite set, so their distance is 2. $\square$

Solution to Exercise 71: Let $m, n \geq 2$ and let $e = \{a, b\}$ be an in $K_{m,n}$. Let $G = K_{m,n} - e$ be the graph formed by deleting e (and not its end vertices). Any a–b path has length 3 (it is of the form a-y-x-b, where y is in the opposite partite of a and x is on the other side of b). Deletion of any further edges in $K_{m,n}$ does not reduce the distance between a and b. □

Solution to Exercise 72; After checking cases, the diameter of the Petersen graph is 2. □

Solution to Exercise 73: (This exercise with solution appears in [192].) Let G be the graph on six vertices formed by attaching a pendant edge to each vertex of a triangle. □

Comment on Exercise 74: See the old Bondy and Murty text, [142, p. 14, 1.6.12].

Comment on Exercise 75: This result follows from Exercise 74. This version of the exercise is from Harary [486], where references [790] and [817] are given.

Comment on Exercise 76: See the old Bondy and Murty text, [142, p. 14, 1.6.13].

Solution to Exercise 77: The diameter is 3 (since $d(u, y) = 3$) and the radius is 2 (since every vertex is within distance 2 of v). □

Comment on Exercise 75: This exercise appears in Harary's text [486, 2.18], where it is marked as difficult; credit for the result is given to Ringel [790] and Sachs [817].

Solution to Exercise 78: The Petersen graph has radius 2 (in fact, from any one vertex, all others are reachable by a path of length 2). □

Solution to Exercise 79: The inequality $\mathrm{rad}(G) \leq \mathrm{diam}(G)$ follows from the definitions. To see the second inequality, suppose that two vertices x, y have distance $d(x, y) = \mathrm{diam}(G)$ and let P be a shortest path between x and y (witnessing this distance). For any central vertex w, $d(x, w) \leq \mathrm{rad}(G)$ and $d(w, y) \leq \mathrm{rad}(G)$. Since any path x–w–y (which has length at most $2\mathrm{rad}(G)$) is at least as long as P (by def'n of distance), it follows that $\mathrm{diam}(G) \leq 2\mathrm{rad}(G)$. □

Solution to Exercise 80: This was proved in 1869 by Jordan [545] and is found in, *e.g.*, [977, p. 72].

One has to show that the center of a tree is either a vertex or a single edge. Recall that a vertex u is in the center of a graph G if and only if the eccentricity of u is minimal.

For $n \geq 1$, let $J(n)$ be the statement that the center of a tree on n vertices is either a vertex or a single edge. The theorem is proved by strong induction on n.

Base step: When $n = 1$ or $n = 2$, there is nothing to prove as the center of a tree on 1 or 2 vertices is the tree itself.

Inductive step: Let $n \geq 2$ and suppose that for each $i \leq n$, $J(i)$ holds. Let T be a tree on $n+1$ vertices. Form T' by deleting all leaves of T. Then T' is a tree with at least one vertex, and so by inductive hypothesis, the center of T' is either a vertex or a single edge. It suffices to show that the center of T is also the center of T'.

For any vertex $u \in V(T')$, another vertex at maximum distance from u is a leaf in T (otherwise one could extend the path from u farther). Since all leaves have been deleted and edges between other vertices remain, the length of any maximal path from a vertex $v \in V(T')$ is shortened by 1. Hence, for any $u \in V(T')$, $\epsilon_{T'}(u) = \epsilon_T(u) - 1$. The eccentricity of any leaf in T is greater than its neighbour, so the vertices minimizing eccentricity in T' also minimize eccentricity in T. Hence, the center of T' is the center of T, as required. This completes the inductive step.

By MI, for every $n \geq 1$, $J(n)$ is true. □

Solution to Exercise 81: One way to see this is to argue by contradiction. Suppose that G is a graph with diameter D and with girth $g \geq 2D + 2$. Let C be a cycle of length g. Let x and y be vertices on C such that their distance along C (from either side) is greater than D. Then $d(x, y) > D$, since there are no shorter x–y paths than a path along C (otherwise, a shorter cycle is created). This, of course, contradicts the definition of D. □

Solution for Exercise 86: Let G be a graph on n vertices with n edges. Form G by beginning with n isolated vertices and adding edges one at at time. Suppose that $e = \{x, y\} \in E(G)$ is being added. If both x and y are in the same component, there is already a x-y path, so adding e produces a cycle. If x and y are in different components, the number of components is reduced by 1. So at each stage, either a cycle is formed or the number of components is reduced by 1. Since the starting graph of isolated vertices has n components, and the fewest number of components possible is 1, adding edges between components can only be done at most $n - 1$ times. So at some point, a cycle is formed. □

Solution to Exercise 87: Yes; a loop is a cycle of length 1 and a pair of multi-edges form a cycle of length two. □

Solution to Exercise 88: (two cycles) To be written.

Solution to Exercise 89: A connected unicyclic graph G on n vertices has precisely n edges, so by the HSL, the average degree is $\frac{1}{n}\sum \deg(v) = \frac{1}{n}(2n) = 2$. □

Solution to Exercise 91: Let G be a connected graph on n vertices with $2n - 2$ edges, and let T be a spanning tree. In T, the longest path has length at most $n - 1$.

So adding any edge to T joins vertices with distance $d \in \{2, \ldots, n-1\}$ (of which there are $n-2$). However, to achieve $2n-2$ edges in all, $n-1$ edges are added to T, and so by the pigeonhole principle,, at least two of the edges join vertices at the same distance in T, and thus form cycles of the same length. (The restriction of $n \geq 4$ can be disposed with since if $n = 2$ then $2n - 2 = 2$ edges are impossible; similarly, if $n = 3$, no graph has $2n - 2 = 4$ edges.) □

Hint for Exercise 92: Join n triangles at a point, where $n = 2, 3, 4$. For example, see the bowtie in Figure 1.31 for the case $n = 2$ (and 5 vertices). These graphs are called "friendship graphs".

Hint for Exercise 93: Try induction on n. □

Solution to Exercise 94: The result of this exercise was likely known by Dirac since the proof very closely duplicates that of Dirac's theorem for Hamiltonian cycles (Theorem 2.3.2).

Let G be a graph on $n \geq 3$ vertices with minimum degree $\delta = \delta(G)$. Let P be a maximum length path on vertices (in order) $x_1, \ldots, x_m$. (Thus any neighbours of x_1 or x_m are on P.) If $m > 2\delta$, the desired path is found, so suppose that $m \leq 2\delta$.

Claim: There exists $i \in \{2, m\}$ so that $\{x_1, x_i\} \in E(G)$ and $\{x_{i-1}, x_m\} \in E(G)$.

Proof of claim: Let $A = \{i : \{x_1, x_i\} \in E(G)\}$ and let $B = \{i : \{x_{i-1}, x_m\} \in E(G)\}$. Since $\deg(x_1) \geq \delta$, $|A| \geq \delta$, and since $\deg(x_m) \geq \delta$, $|B| \geq \delta$ as well. Since both A and B are subsets of $[2, m]$, which has cardinality at most $m - 1 < 2\delta$, by the PHP, $A \cap B \neq \emptyset$, finishing the proof of the claim.

Let i be as in the claim. Then the walk

$$x_1, x_2, \ldots, x_{i-1}, x_m, x_{m-1}, \ldots, x_i, x_1$$

is a cycle, call it C, using all vertices of P.

If any vertices of G are not on P, since G is connected, there is a path from an outside vertex to a point on P, say, x_j. Then the cycle C contains a path of m vertices starting at x_j; adjoining x_j to C produces a path longer than P, which contradicts the assumption that P is a longest path. Hence, the assumption that $m \leq 2\delta$ leads to a contradiction. Thus, $m > 2\delta$. □

Solution to Exercise 95: Consider the infinite graph G on vertices x_0, x_1, x_2, ..., where $E(G)$ consists of $\{x_0, x_1\}$ and for each $i = 0, 1, 2, 3, \ldots$, the pair $\{x_i, x_{i+2}\}$. Then G is 2-regular but has no cycle (G consists of two infinite rays starting at x_0).

Solution to Exercise 96: Suppose that G is a graph where $E(G)$ is partitioned into cycles. Since each vertex of a cycle has degree 2, a simple proof by induction on the number of cycles shows that all vertices in G have even degree.

Now suppose that G is a graph with all degrees even. It suffices to show that each connected component can be partitioned into cycles, so let H be a component of G. Since H has no isolated vertices, $\delta(H) \geq 2$ and all degrees in H are even. By Lemma 2.1.2, H contains a cycle. Deleting the edges of that cycle in H produces another graph with all even degrees, and so by a simple inductive proof, the edges of H can be partitioned into cycles. □

Comment on Exercise 98: One easy way to see this is to draw the graph of a triangular prism, and show that the missing edges form a C_6.

Outline for Exercise 99: Let $v \in V(G)$. Consider all vertices that have distance at most 2 from v. For $k = 2$, the graph C_5 attains equality; are there others? □

Solution to Exercise 100: Let G be a graph on n vertices and $n + 1$ edges. By Exercise 88, G contains at least two cycles. If these two cycles are edge-disjoint, then the girth is at most $(n + 1)/2$. So, suppose that the two cycles share an edge. Then there are two vertices joined by three edge-disjoint paths, say of lengths n_1, n_2, n_3, respectively. Then $n_1 + n_2 + n_3 \leq n + 1$. The three cycles formed by taking two paths at a time have lengths $n_1 + n_2$, $n_1 + n_3$, and $n_2 + n_3$. Summing these lengths, $2(n_1+n_2+n_3) \leq 2(n+1)$. Hence one of these cycles has length at most $2(n+2)/3$. □

Solution to Exercise 102: For each graph, there are many such Eulerian circuits. Possibilities include:

Green graph: If one picks the outer cycle b–y–a–x–c–z–b to begin with, the remaining graph has four components, the three isolated vertices a, b, c, together with the triangle y–x–z–y. Splicing these together gives

$$b - y - \underbrace{x - z - y}_{\text{insert}} - a - x - c - z - b.$$

Blue graph: r–s–t–u–v–z–u–y–t–x–s–w–r (found by, *e.g.*, Fleury's algorithm).

Red graph: 1–5–2–6–3–9–4–8–3–7–4–6–1, where the first four vertices form a cycle. □

Solution to Exercise 104: Let G be a connected graph and form the multigraph H on $V(G)$ by doubling each edge in G. Then H is connected and has all degrees even, and so by Euler's theorem (Theorem 2.2.2), H has an Eulerian circuit. This circuit corresponds to the desired closed walk in G. □

Solution to Exercise 105: Since the underlying graph has only two vertices of odd degree (one at the top, the other at the bottom), this graph can be drawn in one stroke, beginning at either of the two ends of the vertical bar. □

Solution to Exercise 106: Since there are 28 vertices of odd degree (7 on each outer edge of the board), the figure can be drawn with 14 strokes. □

Solution to Exercise 107: For some $k, n \in \mathbb{Z}^+$, let G be k-regular graph on n vertices so that both G and $\overline{G}$ are connected. If k is even, then (by Theorem 2.2.2) G is Eulerian, so suppose k is odd.

Then $\overline{G}$ is $(n-1-k)$-regular. If n is even, then $n-1-k$ is even, and so (again by Theorem 2.2.2), $\overline{G}$ is Eulerian. It remains to observe that when k is odd, n is never odd (by the handshaking lemma, the odd sum of odd degrees equals an even number, which is impossible). □

Solution to Exercise 108: Let G be an Eulerian graph on $n \geq 3$ vertices. There may be isolated vertices, so first assume that G is connected (no isolated vertices).

If n is even, the possible degrees in G are $2, 4, \ldots, n-2$, of which there are $\frac{n-2}{2}$. If n is odd, the possible degrees are $2, 4, \ldots, n-1$, of which there are $\frac{n-1}{2}$. In either case, by the PHP, there are at least 3 degrees the same.

Now suppose that G contains an isolated vertex. When n is even, the possible degrees are $0, 2, 4, \ldots, n-2$, of which there are at most $\frac{n-2}{2}+1 = \frac{n}{2}$ values. When n is odd, the possible degrees are $0, 2, 4, \ldots, n-3$, of which there are at most $\frac{n-3}{2}+1 = \frac{n-1}{2}$ values. So when n is odd, again the PHP gives the desired result. The only troublesome case is when n is even, because the PHP alone does not give the result. However, when n is even, and each degree appears exactly twice, then there are two isolated vertices, which then implies that the maximum degree is $n-3$, and so the PHP argument works. □

Solution to Exercise 109: There are many solutions that close only three doors (or two for the open tour). One solution closes only two doors: closing the doors leading from room x to y and from v to z, produces a graph with no odd degree vertices (see Figure 20.5). Closing just one of these doors leaves two vertices of odd degree, and so gives an open tour (starting and stopping at different points).

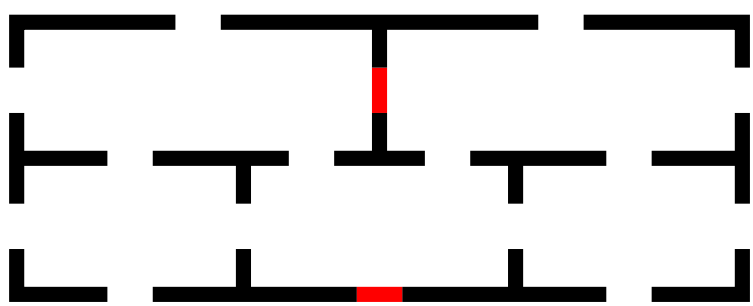


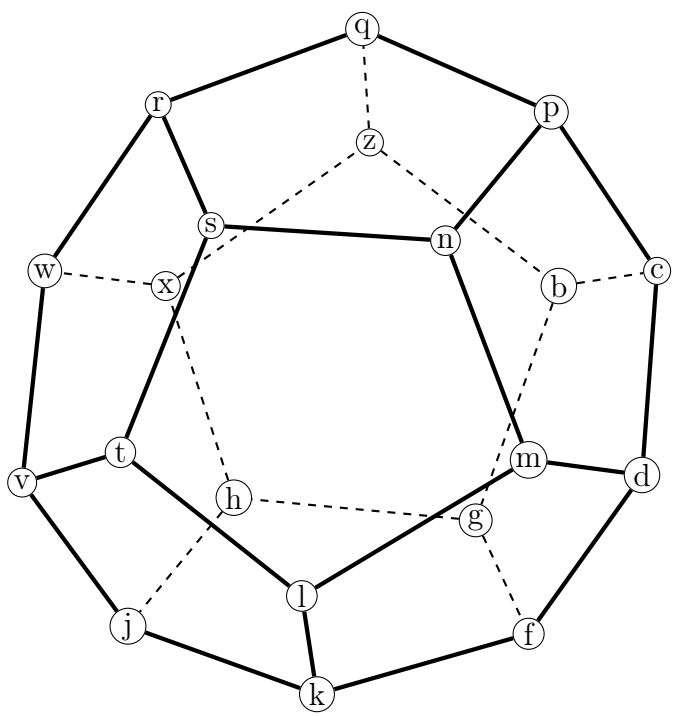
Figure 20.5: The five room puzzle with two doorways blocked

□

Solution to Exercise 110: Let G be the graph whose 27 vertices are the unit cubes and two vertices are adjacent if and only if the corresponding cubes share a face. Then the eight vertices on the eight corners each have degree 3. So by Theorem 2.2.2, G is not Eulerian, and by Theorem 2.2.3 (with $k \geq 4$), nor does G have an open Eulerian trail. □

Solution to Exercise 111: There are many possible solutions, one of which is (XWVTL)KJHGFDMNSRQPCBZ, then returning to X. □

Solution to Exercise 112: This problem (and its solution) was given by Donald Knuth as Exercise 8 in *The art of computer programming*, volume 4, pre-fascicle 8a, Hamiltonian paths and cycles, 25 Dec. 2020, p. 12, Q8; available at `www-cs-faculty.stanford.edu\T1\guilsinglright~knuth`. Knuth notes earlier that this result was observed by Hamilton. Consider the following labelling of the dodecahedron (which is different from Knuth's number labelling; lower case labels are used so not confuse L and R):



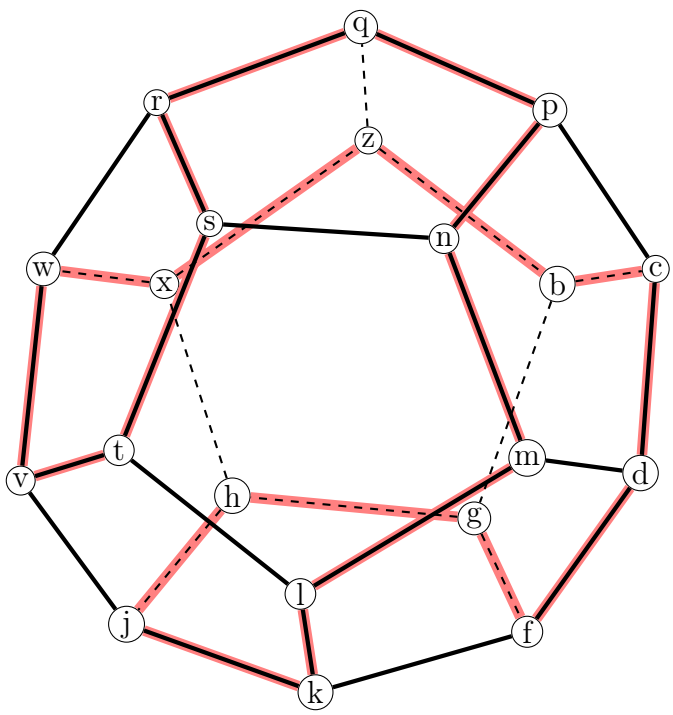
I analyze only the first six possibilities since a left-right reflection shows the others. Before proceeding, the reader is reminded that L and R need to be interpreted carefully when using edges from the "back side" of the pictorial representation.

- LLLL gives the cycle bcdfgb.
- LRRL gives bcdmns, which strands p.

- LRLRLRL gives bcdmltvwx, which forces the cycle qpnsrq.

- LLRLRLL, LLRLRR, and LLRLL: the opening moves give bcdfkj, which forces the path (in some direction) qzxhg, and so ... qzxhgb is forced as an ending. This then forces bcdfkjv...qzxhgb, which (after checking a few cases) cannot be completed.

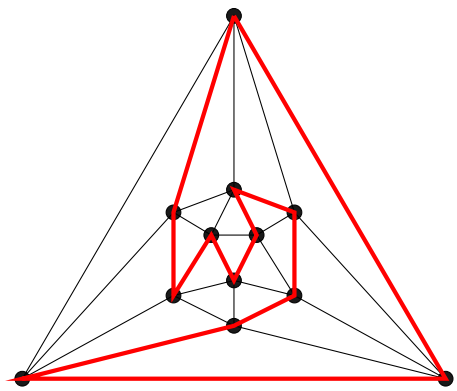
Since these cases are forbidden, the only sequences of turns allowed are LLLRRRLRLRLLLRRRLRLR and its "complement" RRRLLLRLRLRRRLLLRLRL. The Hamilton cycle for the first is given in red (which is, by reverse engineering, in alphabetical order!):



□

Solution to Exercise 113: Yes, the graph of the octahedron is Hamiltonian. This can be seen by viewing the graph as $K_{2,2,2}$, from which a Hamiltonian cycle is easy to find. □

Solution to Exercise 114: Yes, the graph of the icosahedron is Hamiltonian. One Hamiltonian cycle is:



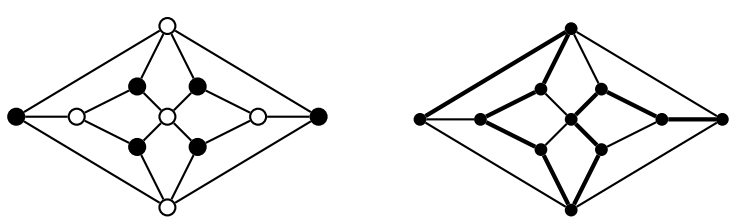
□

Solution to Exercise 115: Assume that $m = n \geq 2$. There are many ways to see that $K_{n,n}$ is Hamiltonian, but perhaps the easiest way is to exhibit such a cycle. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be the partite sets of a copy of $K_{n,n}$ (where $E = \{(x, y); x \in X, y \in Y\}$). One Hamiltonian cycle is given by the sequence of vertices (and the edges between consecutive vertices)

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1.$$

Now remove restrictions on m, and suppose that $K_{m,n} = (A \cup B, E)$ (where $|A| = m$, $|B| = n$, and $A \cap B = \emptyset$) is Hamiltonian. Suppose that $A \cup B = \{v_1, v_2, \ldots, v_{m+n}\}$, with the order in which the vertices occur in a Hamiltonian cycle. Since neither A nor B contains any edges, no two consecutive vertices in the Hamiltonian cycle are in the same partite set, and so each edge of the cycle is incident with one vertex from A and one from B. Thus $|A| = |B|$, and so $m = n$. □

Solution to Exercise 116: The Herschel graph is bipartite, where the partite sets are indicated by black or white in the first of the following diagrams; since there are six black vertices and only five white, by the argument in Exercise 115, the Herschel graph is not Hamiltonian. However, a Hamiltonian path is indicated in the second diagram below by thick edges:



Solution to Exercise 117: As mentioned, this problem has an instant solution given the existence of a cyclic Gray code. The inductive solution given below essentially

follows the recursive procedure used to create a reflected Gray code, (just using a different language) and so shows the existence of Gray codes as well.

Let Q_n denote the graph of the n-dimensional unit cube; to be specific,

$$V(Q_n) = \{0,1\} = \{(a_1, a_2, \ldots, a_n) : \forall i, a_i \in \{0,1\}\},$$

and $\{(a_1, \ldots, a_n), (b_1, \ldots, b_n)\} \in E(Q_n)$ if and only if there exists $i \in \{1, \ldots, n\}$ so that $a_i = b_i + 1 \pmod 2$ and for each $j \neq i$, $a_j = b_j$, that is, if the two vertices differ in precisely one coordinate.

For $n \geq 2$, let $A(n)$ be the assertion that Q_n is Hamiltonian.

Base step: For $n = 2$, the graph of Q_2 is simply a 4-cycle, which is itself Hamiltonian.

Inductive step: Let $k \geq 2$ and suppose that $A(k)$ holds, that is, suppose that Q_k is Hamiltonian. Let V_0 be the set of vertices with 0 in the last coordinate and V_1 be the set with 1 in the last coordinate. Then the vertices of Q_{k+1} are partitioned into two classes, $V_0 \cup V_1$. Each V_i induces a graph isomorphic to Q_k (as only the first k coordinates vary), call it H_i. By inductive hypothesis, each H_i has a Hamiltonian cycle, say C_i; these cycles can be taken to be the same (up to labelling of the last coordinate). Without loss of generality, suppose that in H_0, u_0 and v_0 are consecutive vertices in C_0 and let u_1 and v_1 be the corresponding consecutive vertices in C_2. Following Figure 20.6, which is for $k = 3$, generate a Hamiltonian cycle in Q_{k+1} by beginning at $u_0 \in C_0$, traverse C_0 the long way around to v_0, and instead of completing C_0 by going to u_0 (along red dotted edge), follow the edge $\{v_0, v_1\}$ to H_1, traverse C_1 the long way around to v_1 (in the opposite direction as in C_1), then finally close the cycle along the edge $\{u_1, u_0\}$ to v_0.

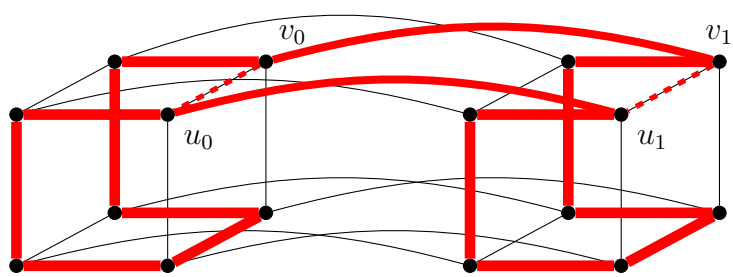


Figure 20.6: Splicing Hamiltonian circuits in H_0 and H_1 to give a Hamiltonian cycle in Q_{k+1}

By mathematical induction, for every $n \geq 2$, $A(n)$ is true. □

Solution to Exercise 118: The graph $K_4 - e$ (the complete graph K_4 minus any edge) is Hamiltonian but not Eulerian (it has two odd-degree vertices). Similarly, any cycle with a chord added is Hamiltonian but not Eulerian.

The bowtie graph (see Figure 1.31) is Eulerian but not Hamiltonian. Also, $K_{2,4}$ is Eulerian but not Hamiltonian. □

Solution to Exercise 119: Let G be a (connected) Eulerian graph. An Eulerian circuit in G corresponds to a Hamiltonian cycle in $L(G)$. Since all vertices of G have even degree, any edge is incident with an odd number of edges on one end, and an odd number of edges at the other end; thus, in $L(G)$, every vertex (which is an edge of G) has even degree. It only remains to check that $L(G)$ is connected to see that $L(G)$ is Eulerian. □

Solution to Exercise 120: (This is given as an exercise in [429, No. 17, p. 153].) One way to see this is to show that the conditions for Ore's theorem (Theorem 2.3.3) hold. Let G be a graph on n vertices and $\frac{n^2-3n+6}{2}$ edges. Let x and y be non-adjacent vertices in G. The maximum number of edges in $G\backslash\{x,y\}$ is $\binom{n-2}{2}$, and so the total number of edges incident with x or y is at least $|E(G)| - \binom{n-2}{2}$, and so

$$\deg(x) + \deg(y) \geq \frac{n^2-3n+6}{2} - \binom{n-2}{2} = n,$$

as desired. So by Ore's theorem, G is Hamiltonian. □

Solution to Exercise 121: Apply Chvátal's theorem (Theorem 2.3.4). □

Solution to Exercise 122: Let G be a k-regular graph on n vertices (where $n \geq 4$ is even). Since $\frac{n-1}{2}$ is not an integer, either $k \geq \frac{n}{2}$ or $k \leq \frac{n}{2} - 1$. If $k \geq \frac{n}{2}$, then Dirac's theorem gives that G is hamiltonian. If $k \leq \frac{n}{2} - 1$, then the degree in $\overline{G}$ is $n - 1 - k \geq n - 1 - (\frac{n}{2} - 1) = \frac{n}{2}$, so again Dirac's theorem applies to show that $\overline{G}$ is Hamiltonian. □

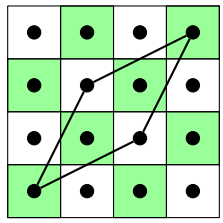
Comment: The case for odd n is more difficult, but can be found in [640]

Comment on Exercise 123: Theorem 2.3.8 was used (and proved) in a 2014 paper [624, Lemma 4] about Ramsey numbers for paths in random graphs, and was apparently published only slightly earlier in [267] and [755]. Hint: Begin with U and W, where $W = \emptyset$, and recursively move vertices from U either to W or to create a longer path.

Solution to Exercise 124: Let G be a cubic graph with Hamiltonian cycle C_1. By Theorem 2.3.16, for any edge in C, there exists at least two Hamiltonian cycles, so there is another Hamiltonian cycle C_2 (different from C_1). Let f be any edge of C_2 that is not on C_1. Again by Theorem 2.3.16, there exists yet another Hamiltonian cycle C_3 containing f. Since f is on C_3 and not on C_1, $C_3 \neq C_1$. □

Answer to Exercise 125: BRCG(4) is the (ordered) list 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100. 1100, 1101, 1111, 1110, 1101, 1011, 1001, 1000. □

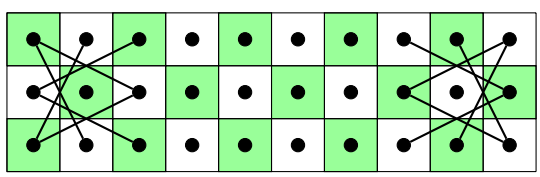
Solution to Exercise 126: Consider only two opposite corners, say positions $(1,1)$ and $(4,4)$ in the 4×4 knight's graph. The only neighbours of $(1,1)$ are $(2,3)$ and $(3,2)$ (and so any cycle through $(1,1)$ uses the edges to these neighbours); the neighbours of $(4,4)$ are also $(2,3)$ and $(3,2)$, and so a 4-cycle is forced.



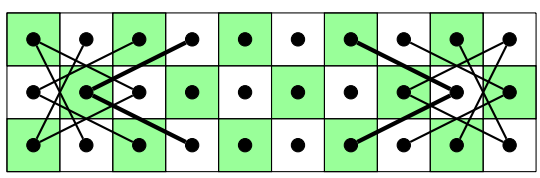
These four edges are required (since each corner vertex has degree 2), so no hamiltonian cycle exists. □

Solution to Exercise 127: No knight's tour exists; however, a Hamiltonian path exists. □

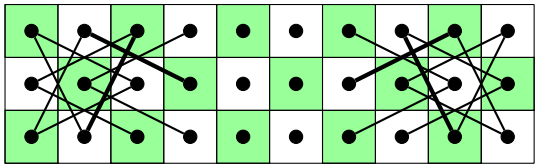
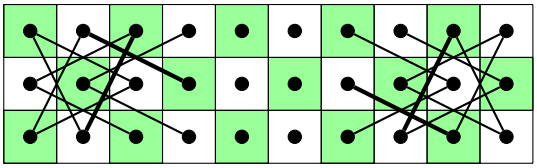
Solution to Exercise 128: There are 16 different solutions, however all of them begin in the same way. As in the solution to Exercise 126, certain edges are forced. For example, starting with the end vertices, the following configuration is forced:



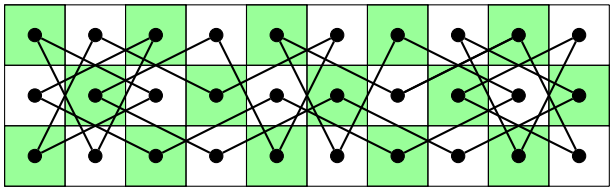
Then edges containing the squares in positions $(2,2)$ and $(9,2)$ are also forced (new edges are thick):



Looking at positions $(2,1)$ and $(2,3)$, if they are both adjacent to position $(4,2)$, then a cycle is formed, so either $(2,1)$ is adjacent to $(3,3)$ or $(2,3)$ is adjacent to $(3,1)$; if both occur, then another cycle is formed, so exactly one of these edges is present, and the other goes to $(3,2)$. Without loss of generality, suppose that $(2,1)$ is adjacent to $(3,1)$ and $(2,3)$ is adjacent to $(4,2)$. One can not yet make the symmetric choice on the other end, since one does not know if the symmetry will force a contradiction, so the other end might have corresponding edges the other way around. So far, any knight's tour begins in one of two possible ways (or the up-down reflection version of either):

 or 

is required. After many unsuccessful attempts, I managed to complete the first drawing to a knight's tour:



Since this example is not left-right symmetric nor up-down symmetric, this tour is but one of four equivalent ones. Finding the other 12 might take hours by hand. The purpose of this exercise was to show how difficult it might be to even construct an algorithm for a Hamilton cycle on 30 vertices in a graph with maximum degree 4. □

Comment on Exercise 129: (No knight's tour for a $4 \times n$ board) This exercise occurs in [977] without solution but with a vague hint. A solution can be found in [840].

Solution to Exercise 130: See Figure 20.7 for but one of many such tours.

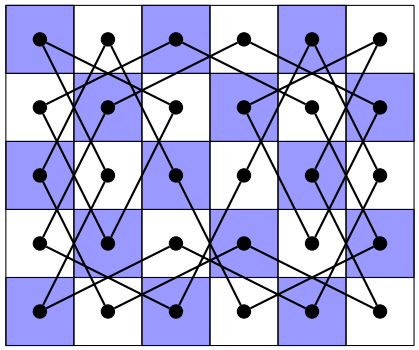


Figure 20.7: A knight's tour on a standard 5×6 chessboard

Solution to Exercise 131: Since there are 5! ways to order the vertices, 5 starting points all giving the same cycle, and two ways each cycle can be ordered, there are $5! \cdot \frac{1}{5 \cdot 2} = 12$ different cycles on 5 vertices.

For cycles on 4 vertices, there are $\binom{5}{4} = 5$ different sets of 4 vertices, and as above, there are $4! \frac{1}{4 \cdot 2}$ cycles for each 4-set, giving $5 \cdot 24 \cdot \frac{1}{8} = 15$ different cycles using 4 vertices.

Similarly, for 3 vertices, there are $\binom{5}{3} \cdot 3! \frac{1}{3 \cdot 2} = 10 \cdot 6 \frac{1}{6} = 10$ triangles.

In total, there are $12 + 15 + 10 = 37$ cycles on 5 labelled vertices. □

Solution to Exercise 132: Any cycle in $K_{3,4}$ either uses two vertices from each partite set or three vertices from each partite set. To choose two vertices from each partite set, there are $\binom{3}{2}\binom{4}{2} = 18$ ways to pick the vertices. For each such choice there is one four-cycle.

To count six-cycles, there is one way to pick 3 vertices from the first partite set, and 4 ways to pick 3 vertices from the second partite set. So in all, there are 4 possible sets of vertices for a six-cycle. Suppose that these six vertices are a_1, a_2, a_3 from the first partite set and b_1, b_2, b_3 from the second. In each six-cycle, a_1 is adjacent to two of the b_is, which can be done in three ways. Suppose that $i < j$ and a_1 is adjacent to b_i and b_j. Then in each case, b_j is adjacent to one of a_2 or a_3. In each case, there is only one way to complete a six-cycle. So there are $4 \cdot 3 \cdot 2 = 24$ possible six-cycles.

In total, there are $18 + 24 = 42$ cycles in $K_{3,4}$. □

Solution to Exercise 133: Let $G = K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$. Any cycle in G uses the same number of vertices in each of the partite sets. For each $k \in [2, \lfloor n/2 \rfloor]$ consider a cycle of length $2k$. There are $\binom{\lfloor n/2 \rfloor}{k}$ ways to pick k vertices from the small side and $\binom{\lceil n/2 \rceil}{k}$ ways to select k vertices from the large side. On each side there are $k!$ ways to permute the vertices. Each $2k$-cycle appears in $2k$ such permutation choices (k different starting points and 2 directions). So for a fixed k, there are

$$\frac{1}{2k} \cdot k!k! \binom{\lfloor n/2 \rfloor}{k} \cdot \binom{\lceil n/2 \rceil}{k}$$

cycles. Now simplify. □

Comment on Exercise 134: None of the loops in the associated graph can be used, so essentially the multigraph breaks down into two cycles, with at least two possible such pairs.

Comment on Exercise 135: This problem came from [973, Ex. 3.34] or [22, Ex. 2.17]. To start, if the YY edge from cube 3 is chosen in one subgraph, the rest of the subgraph is one of two 3-cycles; in each case, the other subgraph cannot be formed. Checking cases (details omitted) finishes the solution. □

Solution to Exercise 136: Let $T = (V, E)$ be a tree and let $v \in V$ be arbitrary. Colour a vertex in V red if its distance from v is even (so v gets red), and colour the remaining vertices blue (those that are at odd distance from v). This defines a partition of V. No two vertices at the same distance from v are adjacent (since T has no cycles), and no two vertices with distance from v differing by 2 or more also are not adjacent. So there are no edges among either the red vertices or among the blue vertices. Thus,

T is bipartite. In a forest with more than one tree, apply the same argument to each tree component.

To see that a tree (and hence a forest) is planar, one simple way is to apply induction starting at a fixed vertex and growing outward. □

Solution to Exercise 137: For $n \geq 1$ consider only graphs G on n vertices, and denote the three statements in the exercise by

$A(n)$: G is connected and acyclic (*i.e.*, G is a tree);
$B(n)$: G is connected and has $n-1$ edges;
$C(n)$: G is acyclic and has $n-1$ edges.

$A(n) \to B(n)$: The proof is by induction on n. For $n \geq 1$, let $S(n)$ be the statement that any tree on n vertices has $n-1$ edges.

Base step: For $n = 1$, the only tree has a single vertex and 0 edges, so $S(1)$ holds.

Inductive step: For some $k \geq 0$, suppose that $S(k)$ is true, and let T be a tree on $k+1$ vertices. There are two ways to go from here relying on different facts, Lemma 3.1.4 or by using the fact that a tree has at least one leaf. (Caution: One can not use Lemma 3.1.5 as its proof uses the result from this exercise!) Here is a proof using the latter.

By Lemma 3.1.2, let $x \in V(T)$ be a leaf, and form the tree T' by deleting x (and the edge incident with x). Then $|V(T')| = k$, and so by $S(k)$, T' has $k-1$ edges. Together with the edge removed, this shows that $|E(T)| = k = (k+1)-1 = |V(T)|-1$, showing that $S(k+1)$ follows, completing the inductive step.

By mathematical induction, for every $n \geq 1$, $S(n)$ holds. □

$B(n) \to C(n)$: (For a connected graph G on n vertices, if G has $n-1$ edges, then G is acyclic.)

Let G be connected on n vertices with $n-1$ edges. Let $G' \subseteq G$ be any acyclic connected subgraph of G (delete edges from cycles in G, which doesn't disconnect the graph) on $V(G) = V(G')$. By $A(n) \to B(n)$, G' has $n-1$ edges. Since $|E(G)| = n-1$ was assumed, $G' = G$, and so G is acyclic.

$C(n) \to A(n)$: Let G be an acyclic graph on n vertices, $n-1$ edges. One needs only to show that G is connected. Let $G_1, \ldots, G_k$ be the connected components of G. By $A(n) \to B(n)$, each component satisfies $|E(G_i)| = |V(G_i)| - 1$. Summing over all components, $|E(G)| = n-k$. But $|E(G)| = n-1$ was given, so $k = 1$, and thus G is connected. (So both $A(n)$ and $B(n)$ follow.) □

Solution to Exercise 138: Let x be the number of vertices with degree 1 and y be the number of vertices with degree 3. Then $x + y = n$ and so $y = n - x$. Also, by the handshaking lemma and Exercise 137, $x + 3y = 2|E(T)| = 2(n-1)$. Thus

$x = 2(n-1) - 3y = 2n - 2 - 3(n-x) = 2n - 2 - 3n + 3x = -n - 2 + 3x$, which gives $n + 2 = 2x$, and so $x = \frac{n+2}{2}$. □

Solution to Exercise 139: Let T be a tree on $n \geq 3$ vertices. Then by the HSL and Exercise 137, the average degree in T is

$$\frac{1}{n} \sum_{x \in V(T)} \deg(x) = \frac{2|E(T)|}{n} = \frac{2(n-1)}{n} = 2 - \frac{2}{n}.$$

□

Solution to Exercise 140: For any graph G on n vertices, $|E(G)| + |E(\overline{G}) = \binom{n}{2}$. If both T and $\overline{T}$ are trees on n vertices, then $|E(T)| = |E(\overline{T}| = n - 1$, and so $(n - 1) + (n - 1) = \binom{n}{2}$, which says $n^2 - 5n + 4 = 0$, which has solutions $n = 1, 4$. When $n = 1$, T is a single vertex (whose complement is again a single vertex, a trivial tree). When $n = 4$, there are only two trees on 4 vertices, namely $K_{1,3}$ and P_3, the path on 4 vertices. It is easy to check that $\overline{K_{1,3}}$ is a K_3 with an isolated vertex, which is not a tree. However, the complement of P_3 is again a copy of P_3, which is a tree. □

Solution to Exercise 141: By Exercise 140, there are only two trees whose complement is also a tree—either a single vertex or a path on 4 vertices. In both cases, the tree is isomorphic to its complement. □

Solution to Exercise 142: Let $G_1, G_2, \ldots, G_k$ be the components of G and for each $i \in [1, k]$, let $n_i = |V(G_i)|$. Each G_i is connected, and so $|E(G_i)| \geq n_i - 1$. If no component of G is a tree, then for each i, $|E(G_i)| \geq n_i$ and so

$$n - 1 = \sum_{i=1}^{k} |E(G_i)| \geq \sum_{i=1}^{k} n_i = n,$$

which is a contradiction, and so at least one component is a tree. □

Solution to Exercise 143: This problem is an old standard, and can be found in, *e.g.*, [977, p. 70, Prop. 2.1.8]. [I am sorry, I do not know the origin of this problem.] For $k \geq 0$, let $A(k)$ be the assertion that if G is a graph with $\delta(G) \geq k$, then G contains every tree with k edges as a (weak) subgraph.

Base step: The only tree with 0 edges is a single vertex, which is contained as a weak subgraph of any graph (with at least one vertex). Thus $A(0)$ is true.

Inductive step: Suppose that $t \geq 0$ and that $A(t)$ holds. Let G be a graph with $\delta(G) \geq t + 1$. It remains to show that G contains every tree on $t + 1$ vertices. Let T be a tree with $t + 1$ vertices. Because $t + 1 \geq 1$, T contains a leaf (vertex with degree 1) x, say, attached to $y \in V(T)$; form T' on t vertices by deleting x. By $A(t)$, since

$\delta(G) \geq t+1 > t$ G contains a copy of T'. Let $z \in V(G)$ be the vertex corresponding to y in T'. Since $\delta(G) \geq t+1$, y is adjacent to some vertex w not in the copy of T'; adjoining w to the copy of T' produces a copy of T. Thus, G contains a copy of T, ending the inductive step.

By the principle of mathematical induction, for any $k \geq 0$, $A(k)$ holds. □

Solution to Exercise 144: Let $n \geq 1$. Then $\delta(\overline{C_{n+2}}) = n-1$. Apply the result of Exercise 143. □

Hint for Exercise 145: First show that a graph with average degree $2k$ contains a graph with minimum degree k, then apply the result in Exercise 143. □

Solution to Exercise 146: Let T be a tree and let $v \in V(T)$ be a vertex of maximum degree, $\Delta(T) > 0$. Removing x produces $\Delta(T)$ subtrees, each of which has at least one vertex. If one of these subtrees is a single vertex, that vertex is a leaf in T; if one of these subtrees has at least two vertices, since each non-trivial tree has at least two leaves, one other than the neighbour of v, and so a leaf in T. □

Solution to Exercise 147: Let T be a tree on n vertices with no vertices of degree 2. Let ℓ be the number of leaves in T. Then,

$$2n-2 = \sum_{v \in V(T)} \geq 3(n-\ell) + \ell = 3n - 2\ell,$$

and so $2\ell \geq n+2$, which gives the desired result. □

Solution to Exercise 148: TBW.

Solution to Exercise 149: TBW that for each $i = 1, \ldots, n$, $d_i \leq \lceil \frac{n-1}{i} \rceil$.

Comment on Exercise 150: This exercise was found in [190, 2.2.20]. Solution TBW.

Solution to Exercise 151: In [190, 2.2.21], this exercise was given to find all such graphs G or to show that none exist, and the answer (without proof) of C_4 was given in that book's solutions. This answer follows essentially because the only tree on 3 vertices is a path of length 2. If every three vertices induce such a path, then a labelling of $V(G) = \{v_0, \ldots, v_{n-1}\}$ exists so that for each $i = 0, 2, \ldots, n$, $v_i v_{i+1} v_{i+2}$ (with addition of subscripts done modulo n) forms a path (and v_i, v_{i+2} is not an edge), so G contains a cycle C_n. However, if $n > 4$, then a triple of vertices of the form v_i, v_{i+2}, v_{i+4} induces at least two non-edges, and so do not form a path of length 2. □

Solution to Exercise 152: See Figure 20.8. □

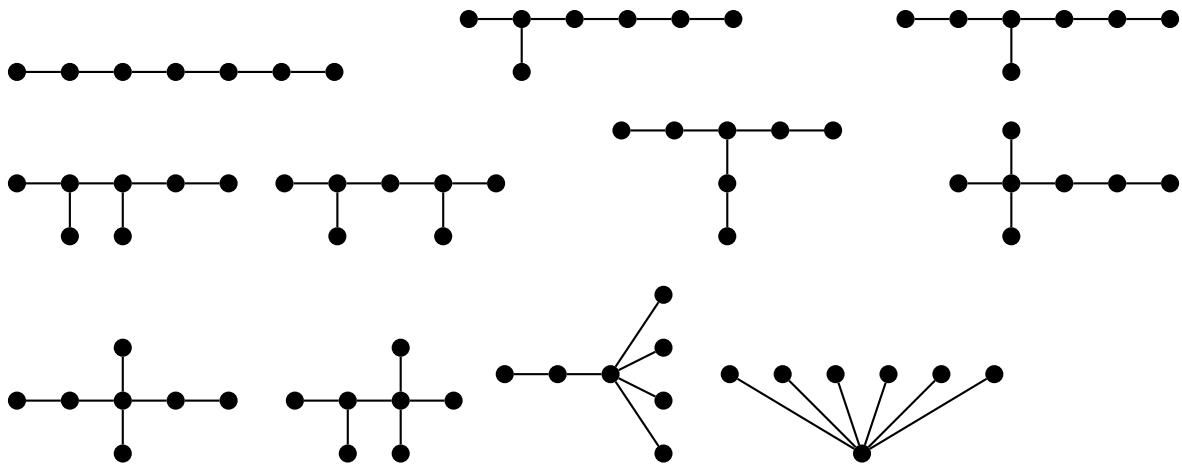


Figure 20.8: The 11 non-isomorphic trees on 7 vertices

Solution to Exercise 154: (Prüfer sequence with single digit) To be written.

Solution to Exercise 155: (Prüfer sequence with no repeats.) To be written.

Solution to Exercise 157: Let x, y be the vertices in the one partite set and let $z_1, \ldots, z_n$ be the vertices of the other partite set. To be connected, a spanning tree has at least one z_i so that both $\{x, z_i\}$ and $\{z_i, y\}$ are edges. Having chosen such a z_i, the remaining z_js are partitioned into two classes, those adjacent to x and those adjacent to y. Since a set of $n-1$ elements can be partitioned into two sets in 2^{n-1} ways (and there n choices for z_i), the result follows. □

Comment on Exercise 158: This result was proved in [841]. One idea is that the Prüfer sequence can be split into two subsequences of length $m-1$ and $n-1$ corresponding to vertices of each part; then show that the corresponding inverse bijection is defined properly. □

Comment on Exercise 160: See the proof of Prim's algorithm (the solution to Exercise 161) for a possible methodology.

Solution to Exercise 161: By an easy induction on i, each G_i is a tree, and so G_n is a spanning tree. It remains to prove that G_n is minimal.

For $k \geq 1$, let $S(k)$ be the statement that G_k is a subtree of a minimum weight spanning tree.

Base step: G_1 is a single vertex, which is a subtree of every spanning tree, so $S(1)$ is true.

Induction step: Fix $k \geq 1$, and suppose $S(k)$, that G_k is a subtree of a minimum weight spanning tree T. Consider G_{k+1}; if G_{k+1} is a subtree of T, there is nothing to show. So suppose that G_{k+1} is not a subtree of T.

Let $e = \{x, y\}$ be the edge chosen by the algorithm with $x \in V_k = V(G_k)$, $y \in V \backslash V_k$, $G_{k+1} = G_k \cup \{e\}$ and $e \notin E(T)$.

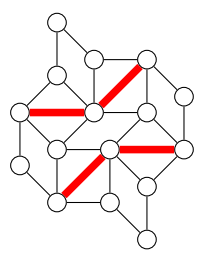
Consider the graph $T \cup \{e\}$; since T is a tree containing x and y, $T \cup \{e\}$ has a unique cycle containing e, and so there is another edge e' of T from some vertex in $V(G_k) = V_k$ to a vertex outside of V_k. Because e' was not chosen at step k, $w(e) \leq w(e')$. Examine the graph U formed by deleting e' from T and inserting e. Since e and e' are on the same cycle in $T \cup \{e\}$, U is a tree spanning $V_k \cup \{y\}$ with

$$w(U) = w(T) - w(e') + w(e) \leq w(T).$$

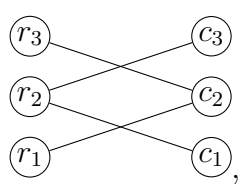
But T has minimum weight, so $w(U) = w(T)$, and $w(e) = w(e')$. Then U is a spanning tree with minimum weight. Hence, G_{k+1} is a subtree of the minimum spanning tree U, proving $S(k+1)$.

By mathematical induction, G_n produced by the algorithm is a tree contained in a minimum spanning tree—so G_n is a minimum spanning tree. □

Solution to Exercise 162: One deformation of the framework is given by



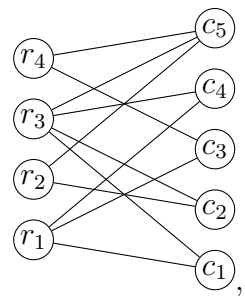
The associated bipartite graph is



which is not connected (and has only 4 of the required $3 + 3 - 1 = 5$ edges). The addition of any bipartite edge then connects the graph, and so adding any brace gives a rigid bracing. □

Solution to Exercise 163: If a rigid bracing uses more than $m + n - 1$ braces, then the associated graph contains a cycle, and the removal of any edge on that cycle does not disconnect the graph. □

Solution to Exercise 164: The associated bipartite graph is



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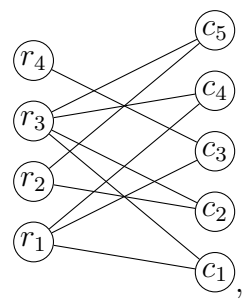
which is a connected graph on 9 vertices. Since the graph has 11 edges and only 8 edges occur in a minimum connected subgraph (a tree), look for three braces to be deleted. There are many ways to do this. For example, the edge $\{r_4, c_5\}$ lies on a cycle, so delete this edge, producing



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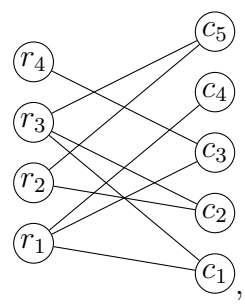
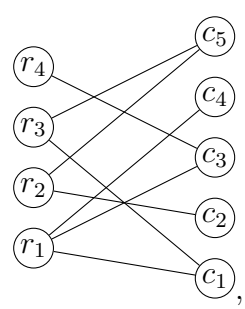
Since the edge $\{r_3, c_4\}$ lies on a cycle, delete this edge, giving



,

In this remaining graph, the vertices r_3, c_5, r_2, c_2 form a cycle, so delete, say, the edge $\{r_3, c_2\}$. It remains to observe that the remaining graph



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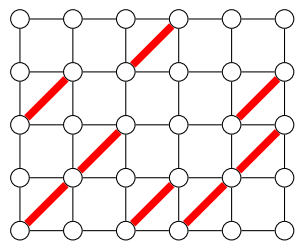
is a spanning tree, which is still connected. Hence, the corresponding subset of braces gives the rigid bracing



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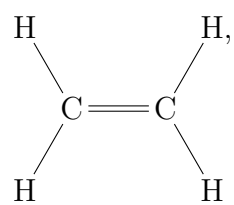
Note that this solution is but one of many. □

Solution to Exercise 165: The number of vertices in the graph of ethylene is 2+4=6. The number of edges in the graph is (by the handshaking lemma) one half of the sum of the degrees (valences), which is

$$\frac{1}{2}(2 \cdot 4 + 4 \cdot 1) = 6.$$

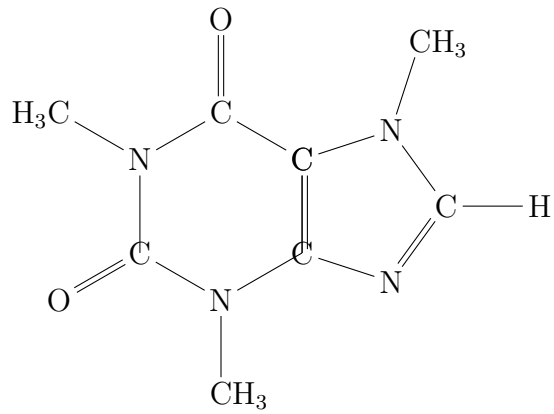
So the number of edges is not one less the number of vertices, so any graph for ethylene is not a tree. The molecule diagram is



ethylene

which has a double bond. □

Solution to Exercise 166: The graph of caffeine is not a tree; to witness this, here is one drawing of the molecule:



caffeine

The caffeine molecule has cycles and double bonds. To prove that the caffeine molecule does not form a tree, the number of vertices is 24, and the sum of all degrees is $8 \cdot 4 + 4 \cdot 3 + 2 \cdot 2 + 10 \cdot 1 = 58$, so (by the handshaking lemma) there are 29 edges, not 23, the number of edges in a tree on 24 vertices. □

Comment on Exercise 167: There are 14 (1 of height 5, 5 of height 4, 7 of height 3, 1 of height 2).

Solution to Exercise 169: For any non-negative integer h, let $S(h)$ be the statement that if T is any binary tree with height h, then $|V(T)| \leq 2^{h+1} - 1$.

Base step: When $h = 0$, there is only one binary tree of height 0, namely a single vertex, and $2^0 - 1 = 1$, so $S(0)$ is true.

Inductive step: Fix $\ell \geq 0$ and suppose $S(\ell)$ is true. Let T be a binary tree with height $\ell + 1$. Deleting the root of T produces two binary trees T_L and T_R, each of height at most ℓ, and so the number of vertices in T is

$$\begin{aligned} |V(T) &= 1 + |V(T_1)| + |V(T_2)| \\ &\leq 1 + (2^{\ell+1} - 1) + (2^{\ell+1} - 1) && \text{(by } S(\ell)\text{, twice)} \\ &= 1 + 2(2^{\ell+1} - 1) \\ &= 2^{\ell+2} - 1 \end{aligned}$$

and so $S(\ell + 1)$ is true.

By mathematical induction, for all $h \geq 0$, $S(h)$ is true. □

Solution outline for Exercise 170: When $n = 0$, there is only one increasing tree, and $0! = 1$, so the base case is true. To see the inductive step, first an observation is

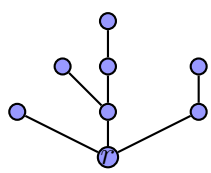
made. Let T be an increasing tree on $\{0, 1, \ldots, k\}$. Since T is increasing, k is a leaf and removal of k creates another increasing tree T'. Furthermore, if T' is increasing, the attaching of k by an edge to any $j \in V(T')$ creates an increasing tree on $\{0, 1, \ldots, k\}$. Since there are k vertices $0, 1, \ldots, k-1$ in T' where the new vertex k can be attached, the number of increasing trees on $\{0, 1, \ldots, k\}$ is k times the number of increasing trees on $\{0, 1, \ldots, k-1\}$. If one assumes the inductive hypothesis that for any $k \geq 1$, there are $(k-1)!$ increasing trees on $\{0, 1, \ldots, k-1\}$, then there are $k(k-1)! = k!$ increasing trees on $\{0, 1, \ldots, k\}$. □

Solution to Exercise 171; To be written.

Solution to Exercise 173:
(a) ((( ) ((( ))) (( ) (( )))))).
(b) ((( ) ( )) ((( ))) ((( )) (( )))).

□

Solution to Exercise 174:



□

Solution to Exercise 176: TBW.

Solution to Exercise 177: Let G be a connected graph. Let D be the maximum distance between vertices G, and pick two vertices u, v with $d(u, v) = D$.

Claim: Neither u or v is a cut-vertex.

Proof of claim: Suppose that u is a cut-vertex, and that $G - u$ contains at least two components. Let G_1 be the component containing v, and let w be any vertex in another component. Since any w–v path in G goes through u, $d(w, v) = d(w, u) + d(u, v) > d(u, v)$, contradicting the maximality of $d(u, v)$. So u is not a cut vertex. The identical argument shows also that v is not a cut-vertex. □

Comment on Exercise 178: See Jungnickel's book [547], Exercise 8.3.4, for a hint.

Solution to Exercise 179: Let G be a simple graph. Assume that G is not connected; it suffices to show that $\overline{G}$ is connected—in other words, it suffices to show that for any two vertices x and y, there exists an x–y path in $\overline{G}$.

Since G is not connected, there exist at least two components, say G_1 and G_2. If x and y are vertices in different components, then they are adjacent in $\overline{G}$, and so there is an x–y path in $\overline{G}$. If x and y are in the same component of G, say $x, y \in V(G_1)$,

then for any $z \in V(G_2)$, the edges $\{x, z\}$ and $\{y, z\}$ in $\overline{G}$ produce an x–y path in $\overline{G}$. Hence $\overline{G}$ is connected. □

Solution to Exercise 180: Assume that G is a graph on vertices $v_1, \ldots, v_n$ satisfying the degree conditions given. If all degrees are $n-1$, then $G = K_n$, which is connected. Suppose that G is not connected, and let H be a component not containing v_n. Set $X = V(H)$ and $Y = V(G) \backslash X$. Since Y contains v_n, $|Y| \geq \Delta+1$, and so $|X| \leq n-\Delta-1$. Hence, $\max_X \deg(x) \leq n - \Delta - n - 2$, and so all vertices of degree at least $n - 1 - \Delta$ are in Y. Hence, setting $k = n - 1 - \Delta$, $X \subseteq \{v_1, \ldots, v_{k-1}\}$. However, v_{k-1} has degree at least $k - 1$, contradicting the max degree in X already stated. Thus, G is connected. □

Solution to Exercise 181: If a graph contains a cycle, then the connected subgraph formed by removing an edge (namely, a path) is not induced. So any graph with the desired property is acyclic. An acyclic graph is also called a forest. It is not difficult to see that any tree satisfies the desired property, and so any forest also does. □

Solution to Exercise 182: This result is commonly proved in popular texts (*e.g.*, see [585, p. 209]). For every edge $\{u, v\}$ the gadget contains a directed u-v path and a directed v-u path, both of which use the directed edge (w, z) (the central arc of the gadget). For any two vertices $x, y \in V(G)$, edge-disjoint x-y paths in G define edge-disjoint directed paths in D, and edge-disjoint directed x-y paths in D define edge-disjoint x-y paths in G. Applying Theorem 4.1.9 finishes the proof. □

Solution to Exercise 183: By definition, $\kappa(K_n) = n-1$. Also, $\kappa(C_n) = 2$, $\kappa(Q_n) = n$, $\kappa(K_{a,b}) = \min\{a, b\}$. □

Solution to Exercise 184: This exercise occurred in [125, p. 93, q. 12] (without solution). For $d \geq 2$, let $G = (X \cup Y, E)$ be a d-regular bipartite graph. If G is disconnected, then $\kappa(G) = 0$, so assume that G is connected. To show that $\kappa(G) > 1$, it remains to show that deleting any vertex fails to disconnect the graph.

In hopes of a contradiction, suppose that $x \in V(G)$ is a cut-vertex, say $x \in X_1$, and let $C_1 = (X_1 \cup Y_1, E_1)$ be a component of $G - x$. Let ℓ be the number of neighbours of x that are in Y_1. Since x is a cut-vertex, $1 \leq \ell < d$.

Double counting edges in C_1 from either the X_1 side or the Y_1 side,

$$|X_1|d = (|Y_1| - \ell)d + \ell(d-1),$$

from which it follows that $|Y_1| - |X_1| = \frac{\ell}{d}$, which is impossible since $\frac{\ell}{d}$ is not an integer; this is a desired contradiction. □

Solution to Exercise 186: Consider first an example when $n = 3$. In this case, the vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ separate $u = (0, 0, 0)$ from $(1, 1, 1)$. By inspection,

this set of vertices is a minimal u–v separating set. On the other hand, the three paths

$$\begin{aligned}&(0,0,0)-(1,0,0)-(1,1,0)-(1,1,1),\\&(0,0,0)-(0,1,0)-(0,1,1)-(1,1,1),\ \text{and}\\&(0,0,0)-(0,0,1)-(1,0,1)-(1,1,1)\end{aligned}$$

are internally disjoint u–v paths. Note that these three paths show that the separating set chosen above is indeed minimum.

Essentially, these same patterns can be used for arbitrary dimension. □

Solution to Exercise 187: When $n = 2$, there is only one edge and deleting that edge disconnects the graph, so $\lambda(K_2) = 1$. So assume that $n \geq 3$. Removing all edges in K_n incident with one vertex disconnects the graph, so $\lambda(K_n) \leq n - 1$. It then remains to show that $\lambda(K_n) \geq n - 1$. For this, it suffices to show that the removal of any $n - 2$ edges fails to disconnect K_n.

Let G be a copy of K_n, and let W be a set of $n-2$ edges in G, and consider H formed by deleting the edges W from G. Then $\overline{H}$ has $n-2$ edges and so is not connected. By Exercise 179, H is connected. So removing $n - 2$ edges fails to disconnect K_n.

Another way to see $\lambda(K_n) \geq n-1$ is given in [190]. Let $F \subset E(K_n)$ be an edge cut, with deletion of edges in F giving two vertex disjoint graphs G_1 and G_2, say of orders $k \geq 1$ and $n - k \geq 1$ respectively. In K_n, there are $k(n - k)$ edges between $V(G_1)$ and $V(G_2)$, so $k(n - k) \leq |F|$. Then

$$0 \leq (k-1)(n-k-1) = kn - k^2 - k - n + k + 1 = k(n-k) - (n-1) \leq |F| - (n-1)$$

and so $|F| \geq n - 1$. □

Solution to Exercise 188: Let G be the bowtie graph (see Figure 1.31) formed by two triangles sharing a vertex. Then removing this shared vertex disconnects the graph, so $\kappa(G) = 1$. Since every edge lies on a cycle, removing a single edge does not disconnect the graph, but removing two edges from either triangle disconnects the graph, so $\lambda(G) = 2$.

To witness the second inequality, form H by joining two copies of K_3 by a bridge (so the result has 6 vertices). Since removing this bridge disconnects the graph, $\lambda(H) = 1$, but $\delta(H) = 2$.

For a graph that satisfies both strict inequalities, start with two copies of C_4 sharing only one vertex x; suppose that the vertices of each C_4 are $\{x, a, b, c\}$ and $\{x, w, y, z\}$. (So far, the graph has 7 vertices.) Add two new vertices, one adjacent to a, b, and c, and the other adjacent to w, y, and z. Call this resulting graph F. Then $\kappa(F) = 1$, $\lambda(F) = 2$, and $\delta(F) = 3$.

For another (more complicated) example that shows both inequalities, see [190, p. 91]. □

Solution to Exercise 189: For each $n \geq 1$, let $S(n)$ be the statement that the number of perfect matchings in $K_{n,n}$ is $n!$.

BASE STEP: When $n = 1$, there is only one matching, so $S(1)$ holds.

INDUCTION STEP: Let $k \geq 1$ and suppose that $S(k)$ is true. Let a copy of $K_{k+1,k+1}$ have partite sets $X = \{x_1, \ldots, x_k, x_{k+1}\}$ and $Y = \{y_1, \ldots, y_k, y_{k+1}\}$. Consider any edge of the form $\{x_{k+1}, y_i\}$ (of which there are $k+1$ many choices). Deleting these two vertices leaves a copy of $K_{k,k}$, which, by $S(k)$, has $k!$ perfect matchings. So in all, there are $(k+1)k!$ different perfect matchings. This proves $S(k+1)$.

By mathematical induction, for each $n \geq 1$, $S(n)$ is true. □

Solution to Exercise 190: The proof given here uses a probabilistic argument; a similar proof is available using just counting. (Here, $\mathbb{P}$ and $\mathbb{E}$ denote probability and expectation, respectively.) Let G be a subgraph of $K_{n,n}$ on partite sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ with $n^2 - n + 1$ edges.

For a permutation $\sigma : [n] \to [n]$, let $L_\sigma = \{(a_i, b_{\sigma(i)}) : i \in [n]\}$ be a set of pairs determined by σ. The goal is to show that some L_σ contains n edges and so is a perfect matching in G.

Let σ be a random permutation. For each edge $e \in E(G)$, define the random variable

$$X_e = X_e(\sigma) = \begin{cases} 1 & \text{if } e \in L_\sigma; \\ 0 & \text{otherwise.} \end{cases}$$

The number of permutations for which $e \in L_\sigma$ is $(n-1)!$, so $\mathbb{P}[X_e = 1] = \frac{(n-1)!}{n!} = \frac{1}{n}$. Let $X = \sum_{e \in E(G)} X_e$. Then

$$\mathbb{E}[X] = |E(G)|\frac{1}{n} = (n^2 - n + 1)\frac{1}{n} = n - 1 + \frac{1}{n} > n - 1,$$

counts the number of edges of G that are in L_σ. Thus there exists a permutation σ so that L_σ contains n edges. □

Solution to Exercise 191: The number of perfect matchings in K_{2n} is $\frac{(2n)!}{2^n(n!)}$. First, here is a proof by induction: For $n \geq 1$, let $P(n)$ be the statement that the number of perfect matchings in K_{2n} is $\frac{(2n)!}{2^n(n!)}$.

BASE STEP: K_2 has only one perfect matching, and $\frac{(2)!}{2^1(2!)} = 1$, so $P(1)$ is true.

INDUCTIVE STEP: For some $k \geq 1$, suppose that $P(k)$ is true. Examine a copy of K_{2k+2}, and fix one vertex, say x. If an edge containing x is fixed, there are $2k$ remaining vertices to be matched, and by $P(k)$, this can be done in $\frac{(2k)!}{2^k(k!)}$ ways. As there are $2k+1$ edges containing x, in all there are

$$(2k+1)\frac{(2k)!}{2^k(k!)} = \frac{(2k+1)(2k+2)}{2(k+1)}\frac{(2k)!}{2^k(k!)} = \frac{(2k+2)!}{2^{k+1}(k+1)!}$$

different perfect matchings in K_{2k+2}, proving $P(k+1)$, and completing the inductive step.

By MI, for every $n \geq 1$, the statement $P(n)$ is true. $\square$

Here is a direct proof:

To find a perfect matching, one can first pick $X = \{x_1, x_2, \ldots, x_n\} \subset V$ (think: left hand endpoints of edges in a matching) and put $Y = V \backslash X$, which also has n elements. For each labelling of $Y = \{y_1, y_2, \ldots, y_n\}$, the set

$$M = \{\{x_i, y_i\} : i \in [n]\}$$

is a perfect matching. There are $\binom{2n}{n}$ ways to choose X and then $n!$ ways to order Y, giving

$$\binom{2n}{n} n! = \frac{(2n)!}{n!}$$

ways to construct a perfect matching. However, if $M = \{\{x_i, y_i\} : i \in [n]\}$ is constructed by first picking X, for each i, one could have chosen y_i instead of x_i and the same matching is produced. Thus, there are 2^n options in choosing X that lead to the same construction. So divide the number of ways by 2^n. $\square$

Solution to Exercise 192: Let $n \geq 1$, and let Q_n have vertices of the form $(x_1, \ldots, x_n)$, where each $x_i \in \{0, 1\}$. Consider the set of edges

$$M = \{\{(0, x_2, x_3, \ldots, x_n), (1, x_2, x_3, \ldots, x_n)\} : x_2, \ldots, x_n \in \{0, 1\}\}.$$

For any particular choice of $x_2, \ldots, x_n$, if $(1, y_2, \ldots, y_n)$ is adjacent to $(0, x_2, \ldots, x_n)$, then since adjacent vertices can only differ in one coordinate, then $y_2 = x_2, \ldots, y_n = x_n$. This shows that in M, the vertex $(0, x_2, \ldots, x_n)$ is adjacent to one other vertex; it follows that M is a matching. Since all vertices have the first coordinate equal to either 0 or 1, all vertices of Q_n appear in some edge of M. Hence, M is a perfect matching. $\square$

In the above solution, there is nothing special about the first coordinate. One can get a perfect matching by varying any coordinate. For example, in Q_3, all edges parallel to the x-axis form a matching, as do all edges parallel to the y-axis, and all edges parallel to the z-axis.

If $n \geq 3$, there are perfect matchings in Q_n other those defined above. For example, in Q_3, consider set of edges (where commas and extra parentheses are dropped)

$$\{\{000, 100\}, \{010, 110\}, \{001, 011\}, \{101, 111\}\}.$$

In general, counting the number of perfect matchings in any bipartite graph can be done using permanents. See Theorem 5.2.2.

Solution to Exercise 196: For the family in Example 5.3.4, the triple (b, a, e) is also a SDR. □

Solution to Exercise 197: Let G be a subgraph of $K_{n,n}$ on disjoint n-sets X and Y, with $\delta(G) \geq \frac{n}{2}$.

By Hall's theorem (Theorem 5.3.1) it suffices to check that for each $S \subseteq X$, $|N(S)| \geq |S|$. For $S = X$, since there are no isolated vertices, $N(X) = Y$, and so Hall's condition holds.

Claim: If for some $S \subset X$ with $\frac{n}{2} < |S| < n$, Hall's condition fails, then Hall's condition also fails on the (smaller) set $T = Y \backslash N(S)$.
Proof of Claim: Since $\delta(G) \geq \frac{n}{2}$, $|N(S)| \geq \frac{n}{2}$, and so $|T| = |Y \backslash N(S)| = n - |N(S)| \leq \frac{n}{2} < |S|$. Also, $N(T) = X \backslash S$, and so if Hall's condition fails for S, $|N(T)| = n - |S| < n - |N(S)| = |T|$, and so T also violates Hall's condition, finishing the proof of the claim.

So, if Hall's condition fails, it fails for a small set. But this is impossible, since the neighbourhood of any non-empty subset of X has at least $n/2$ elements. □

Solution to Exercise 198: This result is a special case (after scaling) of a theorem due to Garett Birkhoff (1946) [106] and independently by John von Neumann (1953) [964]. A *doubly stochastic matrix* is a matrix with non-negative real entries so that entries in each row sums to 1 and each column sums to 1. They showed that an $n \times n$ doubly stochastic matrices is a non-negative linear combination (in fact, a convex combination) of permutation matrices. The proof for this exercise is found in various texts (*e.g.*, [954, Thm. 5.5]).

Let A be an $n \times n$ matrix with non-negative integer entries.

There are two directions to prove. First assume that A is the sum of k permutation matrices. Since each permutation matrix contributes precisely one 1 to each row or column sum, the row-sums and column-sums of A are all equal to k.

The other direction is proved by induction on k (where the size $n \times n$ is fixed). For each $k \geq 1$, let $S(k)$ be the statement that if A has non-negative integer entries with all column-sums and row-sums of A equal to k, then A is the sum of k permutation matrices.

Base step: When $k = 1$, the only matrices satisfying the hypothesis are permutation matrices, so $S(1)$ is true.

Inductive step: Let $\ell > 1$, and assume that $S(\ell - 1)$ is true. Let $A = (a_{ij})$ be an $n \times n$ matrix with all row-sums and column-sums equal to ℓ. Let R and C be disjoint copies of $[n]$ (used to index rows and columns, respectively). Define the bipartite graph $G = (R \cup C, E)$ by $\{i, j\} \in E$ if and only if $a_{ij} > 0$.

Claim: G satisfies Hall's condition; that is, for any collection $S \subset R$ of s vertices, $|N(S)| = |\cup_{v \in S} N(v)|$ is at least s.

Proof of Claim: Fix $S \subset R$ with $|S| = s$. Let B be the $s \times n$ matrix induced by rows from S. Since the sum of the elements in each row of A is ℓ, the sum of all elements in B is $s\ell$. Let $t = |N(S)|$ be the number of columns of B with non-zero entries. Since each column of A has sum ℓ, each column of B has sum at most ℓ, and so summing entries of B first by rows, then by columns, shows that $s\ell \leq t\ell$, and so $s \leq t$, proving the claim.

By Hall's theorem, there is a perfect matching of R to C; such a matching corresponds to a permutation matrix P. Form the new matrix $A_1 = A - P$. Then rows or columns in A_i sum to $\ell - 1$. By inductive hypothesis $S(\ell - 1)$, A_1 is the sum of $\ell - 1$ permutation matrices, and so A is the sum of ℓ permutation matrices, concluding the inductive step.

By mathematical induction, for all $k \geq 1$, the statement $S(k)$ is true. $\square$

Solution for Exercise 199: If in the first round every woman receives a proposal, the algorithm terminates. In this case, since each man proposed to exactly one woman, and all women received proposals, each man is married to his favorite woman. Such a marriage is stable just because no man would want to switch partners (though the women might not get their top picks).

Suppose that after round j some woman has still not received one proposal. Each man either proposed in round j, or was some woman's maybe at the end of round $j-1$. In either case, at the end of round j, each man is either freshly rejected, remains a "maybe", or freshly receives maybe status. Since not all women have a maybe on hold, there is at least one man that is freshly rejected in round j.

If some man receives a rejection and still has women he has not yet proposed to, the algorithm continues to the next round. Is it possible that the algorithm continues to the point where some man has been rejected by everyone and has no further proposals to make? No, because if some man has received n rejections, then he has proposed to all women (since he can only propose to each woman at most once), in which case no woman has been passed over, contrary to there existing a woman with no proposals. So the algorithm continues if and only if there are women with no proposals (or there are newly rejected men).

Why must the algorithm terminate? One easy way to see this is that since each man proposes to n women, so (by the pigeonhole principle) after at most $n^2 - n + 1$ rounds, some man will have exhausted all his choices, in which case each woman will have had at least one proposal. [In fact, with a closer analysis, at most $n^2 - 2n + 2$ rounds are necessary, and one can design a scenario requiring this many steps.]

To see that when the algorithm terminates a stable marriage ensues, suppose that the algorithm terminates (each woman has her "maybe"), and consider a man m and a woman w whom m prefers over the wife given him by the algorithm. It suffices to show that w does not prefer m over the husband given to her by the algorithm. Since m prefers w over his mate, m proposed to w at some earlier iteration and was either

outright rejected, or was w's maybe, but later rejected when w accepted the proposal from a another man she prefers to m. In either case, w prefers her mate to m. □

Comment on Exercise 200: For a solution, see [451, pp. 45–48], or for those who can read pseudocode, see [582, p. 50].

Solution outline to Exercise 202: This problem appears in [640, 7.39]. Since G is connected and has even degrees, G is Eulerian. Since G has an even number of edges, so does any Eulerian circuit. Let C be an Eulerian circuit and let H be the graph formed on $V(G)$ by every second edge of C. Then verify that H is a desired k-factor. □

Solution to Exercise 203: Let $n \geq 1$. Then $\alpha(K_n) = 1$ and $\nu(K_n) = \lfloor n/2 \rfloor$. To cover all edges of K_n, certainly $n-1$ vertices does the trick, so $\tau(K_n) \leq n-1$. Any smaller set of vertices misses two vertices, and hence an edge, so $\tau(K_n) = n-1$. To see what $\rho(L_n)$ is, if n is even, any perfect matching covers all vertices (and any fewer edges misses at least one vertex), so $\rho(K_n) = n/2$. If $n = 2k+1$ is odd, then in addition to the k vertices required to cover the vertices of K_2k, one more vertex covers all remaining edges, so $\rho(K_n) = k+1$. Putting the odd and even cases together, $\rho(K_n) = \lceil n/2 \rceil$.

For cycles, consider two cases, n even and n odd. Let $k \geq 2$ and consider C_{2k}. Then every second vertex forms an independent set (and such a set is largest, because any $k+1$ vertices contain an edge), so $\alpha(C_{2k} = k$. Similarly, $\nu(C_{2k}) = k$. Selecting every second vertex covers all edges (and fewer do not) so $\tau(C_{2k} = k$; similarly selecting every second edges gives a minimal set that covers all vertices, so $\rho(C_{2k} = k$. Using the same ideas, is not difficult to check that for $k \geq 1$, $\alpha(C_{2k+1}) = k$, $\nu(C_{2k+1}) = k$, $\tau(C_{2k+1}) = k+1$ and $\rho(C_{2k+1}) = k+1$. □

Solution to Exercise 204: $\alpha(P) = 4$. □

Solution to Exercise 205: Any independent set in H is also an independent set in G, so $\alpha(H) \leq \alpha(G)$. If G is the graph on two vertices consisting of a single edge, then $\alpha(G) = 1$; on the other hand, if H is the subgraph consisting of just the two vertices, then $\alpha(H) = 2$. □

Solution to Exercise 206: This result is due to, independently, in 1979 by Caro [180] and in 1981 by Wei [974]; this finally appeared in a journal article by Caro and Tuza [181] in 1991. Many proofs have been published. The inductive solution here can also be found in, *e.g.*, [126, p. 68, prob. 14]. Another completely different proof using probability and random orderings can be found in [36, p. 95].

For each positive integer n, let $S(n)$ denote that statement that for any graph G on n vertices, $\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$. The proof is by induction on $n = |V(G)|$.

Base step: When $n = 1$, the graph is an isolated vertex, with independence number 1. The sum on the right is then trivially equal to 1, so $S(1)$ holds (with equality).

Inductive step:. Essentially, this part of the proof recursively constructs a large independent set by removing vertices of minimum degree (and their neighbours) consecutively.

Let $k \geq 2$ and suppose that $S(1), \ldots, S(k-1)$ hold. Let $G = (V, E)$ be a graph on $|V(G)| = k$ vertices. Select any $x \in V(G)$ of minimum degree δ, and form a smaller graph H by removing x and all of its neighbours. Then since the degrees of x's neighbours are at least $\delta = \deg_G(x)$, and degrees of remaining vertices in H can only go down,

$$\begin{aligned}
\sum_{v \in V(G)} \frac{1}{\deg_G(v)+1} &= \frac{1}{\deg_G(x)+1} + \sum_{y \in N(x)} \frac{1}{\deg_G(v)+1} + \sum_{v \in V(H)} \frac{1}{\deg_G(y)+1} \\
&\leq \frac{1}{\delta+1} + \sum_{y \in N(x)} \frac{1}{\delta+1} + \sum_{v \in V(H)} \frac{1}{\deg_H(y)+1} \\
&= \frac{1}{\delta+1} + \delta \frac{1}{\delta+1} + \sum_{v \in V(H)} \frac{1}{\deg_H(y)+1} \\
&= 1 + \sum_{v \in V(H)} \frac{1}{\deg_H(y)+1}.
\end{aligned}$$

By the induction hypothesis applied to H, (the statement $S(|V(H)|)$),

$$\sum_{v \in V(G)} \frac{1}{\deg_G(v)+1} \leq 1 + \alpha(H).$$

Since x is independent of all vertices in H, x together with a largest independent set in H shows $\alpha(G) \geq 1 + \alpha(H)$. Tracing back the inequalities shows $S(k)$, completing the inductive step.

By mathematical induction, for every $n \geq 1$, $S(n)$ is true. □

Comment on Exercise 208: This exercise is from [643, Ex. 1.0.3]. See [724] for details.

Solution to Exercise 209: First direction: Let a tree T have a perfect matching $M \subseteq E(T)$. Let $x \in V(T)$, and let $\{x, y\} \in M$. Consider $T \backslash x$, with components, say $T_1, \ldots, T_c$. Without loss of generality, say $y \in T_1$. Since the deletion of x does not affect any edges in M outside of T_1, all other components have a perfect matching, and so are of even order. On the other hand, $T_1 \backslash y$ also has a perfect matching, so together with y, has odd order.

This first direction of the implication has another proof: If T has a perfect matching, then by Tutte's 1-factor theorem (Theorem 5.9.2) for any $x \in V(T)$, $o(T\backslash x) \leq |\{x\}| = 1$, and so $T\backslash x$ has at most one odd component. If T has a perfect matching, then T has an even number of vertices, so for any $x \in V(T)$, $T\backslash x$ has an odd number of vertices, and therefore has at least one odd component. Hence, if G has a perfect matching, $o(T\backslash x) = 1$.

Second direction: Suppose that T is a tree, and for every $x \in V(T)$, $T\backslash x$ has exactly one odd component. If x is a leaf, there is only one remaining component, and since this component is odd, the number of vertices in T is even, so it is "possible" to have a perfect matching. However, note that an even number of vertices is not sufficient to guarantee a perfect matching (for example, look at $K_{1,3}$, which has 4 vertices but no perfect matching).

Consider some $x \in V(T)$, and let H be any component (again, a tree) of $T\backslash x$. Since T is a tree, there is a unique vertex $x_H \in V(H)$ adjacent to x. In particular, let $y = y(x)$ be the unique neighbour of x in the (unique) odd component of $T\backslash x$. The pairs $M = \{\{x, y(x)\} : x \in V(T)\}$ cover all vertices. It remains to show that M is a matching. For this, one need only show that M contains no incident edges.

Suppose that $\{x, y(x)\} \in M$ and $\{y(x), z\} \in M$ (where $z = y(y(x))$). It remains to show that $x = z$. Let $H_1, \ldots, H_c$ be the components of $T\backslash x$, where, say H_1 is the odd component, and $y(x) \in V(H_1)$. Since $y(x)$ is not adjacent to any vertex in another component, the odd component C of $T\backslash y(x)$ is contained in $\{x\} \cup H_1$. Since x is adjacent to only one vertex in H_1, the singleton $\{x\}$ is an odd component of $T\backslash y(x)$, and by assumption, the only one. Thus, $y(y(x)) = x$. □

Solution outline to Exercise 210: This problem is from [143, p. 435, 16.4.14]. First, here is a construction of a graph with precisely one perfect matching and precisely n^2 edges. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. Define $G = (X \cup Y, E)$ by $E = \{\{x_i, y_j\} : i \leq j\}$. For each $i = 1, \ldots, n$, let y_i be adjacent to each vertex in $\{x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}\}$. Then the number of edges is $1 + 3 + \cdots + 2n + 1 = n^2$; the last equality has a simple proof by induction, or by observing that the sum is

$$n + 2(1 + 2 + \cdots n - 1) = n + 2\frac{(n-1)n}{2} = n^2,$$

and using the well-known formula for the sum of the first $n - 1$ integers.

To show that for a graph G with $2n$ vertices having exactly one perfect matching, then $|E(G)| \leq n^2$, use induction; to a graph on $2m$ vertices, add two new vertices, and show that at most $2m + 1$ edges can be added to retain the uniqueness of any perfect matching. □

Solution to Exercise 212: Any graph that is not bipartite contains odd cycles, so a natural first check is to see what happens for odd cycles. Let $k \geq 1$, and let $G = C_{2k+1}$.

Then the maximum number of edges in a matching is k, and the minimum number of vertices required to cover all edges is $k+1$. So $\nu(C_{2k+1}) < \tau(C_{k+1}$; in particular, K_3 is a an example that shows that "bipartite" is a necessary condition for $\nu(G) = \tau(G)$.

Is K_4 also an example (it has 4 triangles)? With checking a few possibilities, find that $\nu(K_4) = 2$, but $\tau(K_4) = 3$. □

Solution to Exercise 213: This exercise (with hint) is found in, *e.g.*, [977, Ex. 3.1.29]. First a claim is shown:

Claim: Every bipartite graph G has a matching of size at least $|E(G)|/\Delta(G)$.
Proof of Claim: Since any vertex is incident with at most $\Delta(G)$ edges, any minimum vertex cover (of all the edges) has at least $|E(G)|/\Delta(G)$ vertices. By the König–Egerváry theorem (Theorem 5.6.1), the largest matching in G is equal to the order of a minimum vertex cover (of all the edges), concluding the proof of the claim.

To conclude the solution, let $G = (X \cup Y, E)$ be an equibipartite graph on $2n$ vertices, and with more than kn edges. Since $\Delta(G) \leq n$, by the claim, the largest matching has at least

$$\frac{|E(G)|}{n} > \frac{kn}{n} = k$$

edges. □

Solution to Exercise 214: Let $G = (X \cup Y, E)$ be a bipartite graph that satisfies Hall's condition ($\forall S \subseteq X$, $|S| \leq |N(S)|$). Let M be a maximum matching (of size ν). By König–Egerváry, there exists a set $C \subseteq V(G)$ with $|C| = |M|$ vertices that covers all edges of G. Let $A = X \cap C$, $B = X \backslash A$, and $D = C \cap Y$. Then $|A| + |B| = |X|$, and $|A| + |D| = |C| = |M|$. Also, by Hall's condition, $|B| \leq |N(B)|$, but since D covers all edges from B, $N(B) \subseteq C \cap Y = D$, and so $|N(B)| \leq |D|$, which gives $|B| \leq |D|$.

So,

$$|X| = |A| + |B| \leq |A| + |D| = |C| = |M|,$$

giving $|X| \leq |M|$. However, each edge of M uses precisely one vertex from X, and so $|X| = |M|$, meaning that every vertex from X is matched into Y, which is the conclusion of Hall's theorem. □

Solution to Exercise 215: Let $|V(G)| = n$, and write simply α, ν, τ, and ρ in place of $\alpha(G)$, $\nu(G)$, $\tau(G)$, and $\rho(G)$, respectively.

By Lemma 5.5.3, $\alpha + \tau = n$, and so $\alpha = n - \tau$. By the König-Egerváry theorem (Theorem 5.6.1), $\nu = \tau$. By Gallai's theorem (Theorem 5.5.4), $\nu + \rho = n$, so $n - \nu = \rho$. Putting these together,

$$\alpha = n - \tau = n - \nu = \rho,$$

as desired. □

Solution to Exercise 217: There are many possible solutions; I give just one. The example given is 4-regular and connected, and so is Eulerian. Let

$$abcfdebfaecda$$

be one such Eulerian circuit. Then the edges of B are (abusing notation)

$$ab, ae, bc, bf, cd, cf, da, de, eb, ec, fa, fd.$$

A perfect matching is $\{ae, bc, cf, da, eb, fd\}$, and so the corresponding 2-factor in G is $aebcfda$. □

Solution to Exercise 218: See, for example, `http://math.stackexchange.com/questions/520203/` for one description and diagrams. (accessed Nov. 2014). Here is another that I learned from Juliana Felix [359]:

Let k be odd. The graph constructed here has $k(k+2)+(k-2)$ vertices. Let $G_1, G_2, \ldots, G_k$ be vertex disjoint copies of K_{k+1}. In each G_i, subdivide one edge, inserting a new vertex x_i, producing a graph H_i on $k+2$ vertices (where the degree of x_i is 2, and all other vertices have degree k. Note that $k+2$ is odd.

Finally, create G by adding a set $Y = \{y_1, \ldots, y_{k-2}\}$ of independent vertices, and connecting each y_j to every x_i. The resulting graph G is k-regular.

To see that G has no 1-factor, the graph $G \backslash Y$ has k odd components, which is larger than $|Y| = k-2$, so Tutte's theorem says there is no 1-factor. □

Here is yet another solution that I learned from Colin Desmarais [254]: First a construction is given for a graph H which is repeated k times in the final construction. Fix $k > 1$ odd, and let $m = \frac{k+1}{2}$. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$ be disjoint sets of vertices, and let s be a vertex not in X or Y. Define the graph H first by forming a K_n on each of X and Y, then for each $i \neq j$, let $\{x_i, y_j\}$ be an edge, and add one more edge $\{x_m, y_m\}$. Finally, attach s to the vertices $x_1, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}$, completing the construction of H.

Observe that for each of the vertices x_i and y_i have $m-1+m-1+1 = k+1-1 = k$ neighbours. The vertex s has $m-1+m-1 = k-1$ neighbours.

Form the graph G by attaching a new "central vertex" c to each of k disjoint copies of H at its s-vertex. Then G is k-regular. Deleting c gives k components, each a copy of H; each H has $m+m+1 = k+2$ vertices, which is odd. Thus, G fails Tutte's condition, and so has no 1-factor. □

It might be interesting to note that constructions above give a graph with k^2+3k-2 and k^2+2k+1 vertices respectively (and so when $k=3$, both have 16 vertices).

Solution to Exercise 223: (Inductive proof of Mirsky's theorem). To be written.

Comment on Exercise 224: This exercise appears in [547, Ex 6.1.12] along with a solution. □

Solution to Exercise 226: Let $V(C_5) = \{v_1, \ldots, v_5\}$. Since C_5 is not 2-colourable, every proper 3-colouring of $V(C_5)$ uses all three colours. Since any 3 vertices in C_5 induce an edge, each colour can be used at most twice. Thus any proper colouring of $V(C_5)$ uses one colour once and the other two colours used twice each. Suppose, for now, that the three colours are R, B, and Y, where R is used once, and each of B and Y are used twice. There are five places to put the R, and once R is placed, the remaining 4 vertices given clockwise from the R can be coloured either BYBY or YBYB.

There are 5 ways to pick the vertex that has the singleton colour, 3 choices for that colour, and for each, 2 ways to complete a proper colouring using the other two colours. Total: 30. □

Solution to Exercise 227: Let $n \geq 1$. It suffices to show that Q_n is bipartite. Let X be the set of all vertices in Q_n with an even number of 1s in their coordinates, and let Y be the set of all vertices with an odd number of 1s. Any two vertices in X are not adjacent, because they do not differ in just one place; similarly any pair of vertices in Y are non-adjacent. Therefore, X and Y are independent sets (and so can form the two colour classes that witness $\chi(Q_n) \leq 2$). □

Solutions to Exercise 228: Since each graph contains a triangle (a subgraph isomorphic to K_3), each chromatic number is at least 3. A proper 3-colouring of the vertices of each is not difficult to find. So, the answers are: (a) 3; (b) 3; (c) 3. □

Solution to Exercise 229: Let G and H be graphs with precisely one vertex x in common, and let $\chi(G) = k$ and $\chi(H) = \ell$, with $k \leq \ell$, say. Let $g : V(G) \to [k]$ and $h : V(H) \to [\ell]$ be proper colourings of G and H respectively. If $g(x) = h(x)$, then $g \cup h$ is a proper colouring of F. If $g(x) \neq h(x)$, then permute the colours on G so that x gets the colour $h(x)$. □

Solution to Exercise 230: (The Hajos graph has chromatic number 4.) TBW

Solution to Exercise 231: It is not difficult to generate a good vertex 3-colouring of the Petersen graph (see Figure 10.4 for one) so $\chi(P) \leq 3$. Since P contains a C_5, $\chi(P) \geq 3$. □

Comment on Exercise 232: This exercise came from from [486, p. 148, ex. 12.6].

Solution to Exercise 233: Observe that $\chi(C_5) = 3 > 2 = \omega(C_5)$. The Petersen graph also has the same parameters. □

Solution to Exercise 235: This problem occurs in [125, 56, p. 141]. When $m = 3$, the graph C_5 satisfies the requirements.

Solution to Exercise 237: Since each colour class is an independent set, $\chi(G)\alpha(G) \geq |V(G)|$. □

Solution to Exercise 238: Let $G = \overline{C_{2k+1}}$. First observe that $\alpha(G) = 2$. Hence by equation (6.1), $\chi(G) \geq \frac{2k+1}{2}$, and since $\chi(G)$ is an integer, $\chi(G) \geq k+1$. To finish the problem, it suffices to show a good $(k+1)$-colouring of G. Let $V(G) = \{0, 1, 2, \ldots, 2k\}$. Define $f : V(G) \to [0, k]$ by $f(v) = \lfloor \frac{v}{2} \rfloor$. Then vertices 0 and 1 receive colour 0, vertices 2 and 3 receive colour 1, and so on. One can verify that indeed f is a proper colouring. □

Solution to Exercise 239: This exercise appears in many places, *e.g.*, *Modern graph theory* by Bollobás [125]. Let $\chi(G) = k$, and let c be a good vertex k-colouring of G. If the number of edges is less, then some pair of different colours has no edge using both of these colours—in which case, one of these two colours can be eliminated.

To see the last part, let $e = |E(G)|$ and $k = \chi(G)$. Then equation (6.2) says $e \geq \frac{k(k-1)}{2}$, or equivalently, $k^2 - k - 2e \leq 0$. By the quadratic formula, the roots of this equation are $k = \frac{1 \pm \sqrt{1+8e}}{2}$, and so $k \leq \frac{1+\sqrt{1+8e}}{2} = \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$. □

Solution to Exercise 240: The solution to this exercise was noted in the review given by P. Ungar in MathSciNet for the paper [328].

Any subgraph that is an odd cycle fails to satisfy the property, so such a G has no odd cycles. Therefore, by Exercise 13, G is bipartite, and therefore is 2-colourable. □

Solution outline to Exercise 241: First show, by induction, that in any simple graph G, if G does not contain a K_{k+1}, then

$$|E(G)| \leq \left(1 - \frac{1}{k}\right) \frac{|V(G)|^2}{2}.$$

(For those familiar with Turán's theorem, count the number of edges in the Turán graph.) This gives a bound on the clique number (and hence the chromatic number).

Hint to Exercise 242: Delete an odd cycle.

Comment on Exercise 243: According to [143, 14.1.10, p. 363], this result is due to D. J. A. Welsh and M. B. Powell.

Solution to Exercise 244: Let G have chromatic number k, and let $c : V(G) \to [k]$ be a proper colouring of G. For each $i \in [k]$, let $X_i = \{v \in V(G) : c(v) = i\}$ be the ith colour class; note that each X_i is non-empty.

Claim: For each $i \in [k]$, there exists $v_i \in X_i$ with $\deg(v_i) \geq k - 1$.

Proof of claim: Let $i \in [k]$. If every vertex $v \in X_i$ has degree less than $k - 1$, then for each $v \in X_i$, there exists a $j \neq i$ so that v is not adjacent to any vertex in X_j, and so v can be recoloured with j (a different colour than i), thereby needing only $k - 1$ colours to properly colour G, contradicting that $\chi(G) = k$. Thus there is at least

one vertex in X_i with degree $k-1$, finishing the proof of the claim, and hence the result. $\square$

Solution to Exercise 245: This exercise can be found in many texts (*e.g.*, [977, 5.1.33]). Let G be a graph with $\chi(G) = k$, and let $f : V(G) \to [k]$ be a good colouring. For each $i = 1, \ldots, k$, let $V_i = f^{-1}(i)$ denote the ith colour class. Define an ordering $<^*$ of $V(G)$ by putting vertices of V_1 first (in any order), then those of V_2, and continuing until V_k (in notation, $V_1 <^* V_2 <^* \cdots <^* V_k$). Note that each V_i is an independent set.

With this ordering $<^*$ in mind, colour the vertices using the greedy algorithm; the first vertex gets colour 1. Since V_1 is an independent set, all vertices from V_1 are greedily coloured with colour 1. If $x \in V_2$ is adjacent to some vertex in V_1, the greedy algorithm colours x with 2; otherwise, x gets colour 1. In any case, all vertices of V_2 receive either colour 1 or 2. Suppose that for some $i < k$, vertices in $V_1 \cup \cdots \cup V_i$ have been properly coloured with colours $1, \ldots, i$. Any vertex in V_{i+1} is greedily coloured with either $i+1$ or some smaller colour. Inductively, all vertices get properly coloured with a colour which is at most k. (If the greedy algorithm properly colours V_k without using colour k, then such a colouring contradicts $\chi(G) = k$, so at least some vertex in V_k is coloured with some colour at least k.) $\square$

Solution to Exercise 246: Consider $K_{n,n}$ with a perfect matching deleted. In particular, if $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, delete all edges from $X \times Y$ of the form $\{x_i, y_i\}$. The ordering $x_1, y_1, x_2, y_2, \ldots$ forces the greedy algorithm to use n colours. $\square$

Solution to Exercise 247: The Petersen graph P is 3-regular and is neither a complete graph or an odd cycle, so by Brook's theorem, $\chi(P) \leq \Delta(P) = 3$. However, the Petersen graph contains a 5-cycle, so $\chi(P) > 2$. Thus, $\chi(P) = 3$. $\square$

Solution to Exercise 248: Induct on the number of vertices. For $n \geq 1$, let $S(n)$ be the statement that any triangle-free graph on n vertices is $(\lfloor 2\sqrt{n} \rfloor)$-colourable.

Base cases: Any triangle-free graph on at most 3 vertices is 2-colourable, and since $2 \leq 2\sqrt{3}$, then for $n = 1, 2, 3$, $S(n)$ is true.

Inductive step: Let $m \geq 4$ and suppose that for all $n < m$, $S(n)$ holds. Let G be a triangle-free graph with m vertices. If the maximum degree in G satisfies $\Delta(G) \leq 2\sqrt{m} - 1$, then by Lemma 6.2.7 (the greedy colouring bound) $\chi(G) \leq \Delta + 1 \leq 2\sqrt{m}$, verifying $S(m)$.

So suppose that there exists $v \in V(G)$ with $\deg(v) \geq 2\sqrt{m}$. Since G is triangle-free, the neighbourhood $N(v)$ is an independent set. Then $H = G \backslash N(v)$ is triangle-free and has at most $m - 2\sqrt{m}$ vertices. By $S(m - \lfloor 2\sqrt{m} \rfloor)$, there exists a good $(\lfloor 2\sqrt{m - \lfloor 2\sqrt{m} \rfloor} \rfloor)$-colouring of H. Put $k = \lfloor 2\sqrt{m - \lfloor 2\sqrt{m} \rfloor} \rfloor$ and let

$$c : V(H) \to \{1, \ldots, k\}$$

be a good k-colouring of H.

Extend the colouring c to a colouring c' of $V(G)$ defined by assigning the colour $k+1$ to each vertex $x \in N(v)$. Since $N(v)$ is an independent set, c' is a proper $(k+1)$-colouring of $V(G)$. To complete the proof of $S(m)$, it remains to show that

$$k+1 \leq 2\sqrt{m}. \tag{20.2}$$

In fact, (20.2) is true by a wide margin, and can be shown in a number of ways. To be pedantic, here is one that pays attention to the floor function:

First note that the floor functions can be removed by adding 2 on the inside term:

$$k = \lfloor 2\sqrt{m - \lfloor 2\sqrt{m} \rfloor} \rfloor \leq 2\sqrt{m - 2\sqrt{m} + 2}.$$

So to show the desired inequality (20.2) it suffices to show that

$$2\sqrt{m - 2\sqrt{m} + 2} \leq 2\sqrt{m} - 1,$$

which is easily shown by squaring each side. This completes the proof of $S(m)$, and so the inductive step.

By MI, for each $n \geq 1$, $S(n)$ is true. □

Solution to Exercise 249: The graph $KG(4,2)$ is a set of three independent edges (i.e., $3K_2$). □

Solution to Exercise 250: Each k-set is disjoint from $\binom{n-k}{k}$ other k-sets, and so each vertex has degree $\binom{n-k}{k}$. By the handshaking lemma, the number of edges is then

$$\frac{1}{2}\sum_v \deg(v) = \frac{1}{2}\binom{n}{k}\binom{n-k}{k} = \frac{n!(n-k)!}{2(n-k)!k!(n-2k)!k!} = \frac{n!}{2(n-2k)!k!k!}.$$

□

Solution to Exercise 251: Colour each set by its minimum element, as long is this minimum is at most $n-2k+1$. The sets that are left are k-sets in $[n-2k+2, n]$, of which there is $2k-1$. By the PHP, every pair of these k-sets from this intersect, and so the corresponding vertices are not adjacent; colour this independent set using one last colour. □

Hint to Exercise 252: Add isolated vertices.

Solution to Exercise 253: Let G be a bipartite graph on $n \geq 3$ vertices with at least one edge.

To see the upper bound, let $\{u, v\} \in E(G)$. Colour $V(G)$ by assigning the same colour to each of u and v, and colour the remaining vertices each with a new colour. In all, $n-1$ colours are used, and this colouring is a good $(n-1)$ colouring of $\overline{G}$.

To see the lower bound, let $V(G) = V(\overline{G}) = X \cup Y$ be a bipartition of the vertices in G. By the PHP, at least one of X or Y has at least $n/2$ vertices, say $|X| \geq n/2$. In $\overline{G}$, X induces a complete graph, and so requires $|X| \geq n/2$ colours to properly colour. Hence, $\chi(\overline{G}) \geq n/2$. □

Solution to Exercise 254: (See [640, 9.5].) Let $|V(G)| = n$. By Lemma 6.2.16 with $H = \overline{G}$, then $\chi(G)\chi(H) \geq \chi(G \cup \overline{G}) = \chi(K_n) = n$. □

Solution to Exercise 255: (This problem appears as an exercise, marked as easy, in [125, 6, p. 170].) Let $a = \chi(G)$ and $b = \chi(\overline{G})$. By the AM-GM inequality, $\frac{a+b}{2} \geq \sqrt{ab}$. By Gaddum–Nordhaus, $ab \geq n$, and so $\frac{a+b}{2} \geq \sqrt{n}$. Multiplying through by 2 gives the desired result. □

Solution to Exercise 256: In any proper colouring of $K_{1,n}$ the center is different colour from all others, so if the colouring is equitable, all other colour classes have at most 2 elements. Thus, $\chi_=(K_{1,n}) = 1 + \lceil n/2 \rceil$. □

Solution to Exercise 257: For some $k \geq 1$, let G be a graph on $n = 2k$ vertices with $\Delta(G) < k$. Then $\Delta(\overline{G}) > 2k - 1 - k = k - 1$, and so $\Delta(\overline{G}) \geq k = n/2$. By Dirac's theorem, $\overline{G}$ has a Hamiltonian cycle C, which is the union of two perfect matchings. Either of these perfect matchings in $\overline{G}$ corresponds to a proper k-colouring of G (each edge in the matching in $\overline{G}$ is an independent set in G, and so can form a colour class of 2 vertices). □

Solution to Exercise 258: For some $k \geq 1$, let G be a graph on $n = 3k$ vertices with $\Delta(G) < k$. Then $\Delta(\overline{G}) \geq 3k - 1 - (k-1) = 2k = \frac{2}{3}n$, and so by the Corrádi–Hajnal theorem, $V(G)$ can be covered by vertex disjoint copies of K_3. For any of these K_3s, the three vertices form an independent set in G, i.e., a colour class in G. □

Solution to Exercise 259: First suppose that n is even, where for some positive integer k, $n = 2k$. Colour the vertices of the first partite set using k colours, two vertices of each colour, and do the same for the second partite set using another k colours.

Next suppose that n is odd, with say, $n = 2k+1$. In hopes of contradiction, suppose that $K_{n,n}$ has an equitable n-colouring. Since there are $2n$ vertices, each colour class has two vertices (from the same partite set). Only k such pairs can occur in one partite set and another k pairs in the other. However, to properly colour the two remaining vertices, one from each partite set, two more colours are required, using $2k+2 = n+1$ colours, the required contradiction. □

Solution to Exercise 260: Let A and B be the partite sets in one copy of $K_{3,3}$ and let C and D be the partite sets of the second copy. Colour the vertices in A by all 1s and the vertices of B by 2,2,3. Colour the vertices of C by 2,2,1 and the vertices of D with all 3s. Each colour class has 4 vertices. □

Solution to Exercise 261: The K_3 requires 3 colours, but $K_{3,3}$ does not have an equitable 3-colouring, so at least 4 colours are necessary. □

Solution to Exercise 262: (4-colourable) TBW.

Solution to Exercise 263: TBW.

Solution to Exercise 264: (3-critical graphs are odd cycles). TBW

Solution to Exercise 265: The result in this exercise was noted (without proof) by Dirac [261] in 1952. Suppose that G is k-critical and let $x \in V(G)$. Since G is k-critical, $G - x$ is $(k-1)$-colourable; fix such a $(k-1)$-colouring of $V(G)\backslash\{x\}$. If $\deg_G(x) < k-1$, then x is adjacent to vertices receiving at most $k-2$ colours, leaving one colour available to colour x, thereby giving a proper $(k-1)$-colouring of G, contradicting that $\chi(G) = k$. □

Solution to Exercise 266: Let G be a k-critical graph, and in hopes of contradiction, suppose that $S \subset V(G)$ is a cut-set that induces a clique in G. Let $G_1, \ldots, G_m$ $(m \geq 2)$ be the components of $G - S$. For each i, let H_i be the graph induced by $S \cup V(G_i)$. since G is critically chromatic, for each $i = 1, \ldots, m$, $\chi(H_i) \leq k-1$, and so each H_i is $(k-1)$-colourable. However, in any proper $(k-1)$-colouring of some H_i, since S induces a clique, vertices of S receive all different colours. Thus, one can relabel the colouring of each H_i so that the colourings restricted to S agree. Together, these m colourings produce a proper $(k-1)$-colouring of G, the required contradiction. □

Solution to Exercise 267: To be written.

Solution to Exercise 268: This exercise (and solution outline) can be found in [142, p. 121], where it is stated that this result is due to P. C. Kainen.

Let $k \geq 1$ and let G be a graph with a partition $V(G) = X \cup Y$, where each of $G[X]$ and $G[Y]$ are k-colourable. Without loss of generality, suppose that both $X \neq \emptyset$ and $Y \neq \emptyset$ (for if one is empty, then G is trivially k-colourable). Let $X = X_1 \cup \cdots \cup X_k$ and $Y = Y_1 \cup \cdots \cup Y_k$ be proper colourings of $G[X]$ and $G[Y]$ respectively; so each X_i and Y_j is a (possibly empty) colour class.

Define the bipartite graph H with vertex set $\{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}$, where $\{X_i, Y_j\} \in E(H)$ if and only if there are no edges (in G) between X_i and Y_j.

Assume that G has at most $k-1$ edges between X and Y. Then H has at least $k^2 - (k-1)$ edges. By Exercise 190, H contains a perfect matching M. Let σ be a

permutation of $[k]$ so that $M = \{(X_i, Y_{\sigma(i)}) : i \in [k]\}$. Then for each i, the set $X_i \cup Y_{\sigma(i)}$ can be coloured with one colour, giving a proper k-colouring of G. □

Solution to Exercise 269: This exercise can be found in [142, p. 121], where it is stated that this result is due to G. A. Dirac; see [262] and [263].

Let G be a k-critical graph. If G is not $(k-1)$-connected, then there is a partition of $V(G) = X \cup Y$ (where X and Y are non-empty) so that the number of edges between X and Y is at most $k-2$. Since G is k-critical, both $G[X]$ and $G[Y]$ are $(k-1)$-colourable. By Exercise 268 (with $k-1$ replacing k), since there are at most $k-2$ edges between X and Y, then G is $(k-1)$-colourable, contradicting that G is k-critical. Hence, the assumption that G is not $(k-1)$-connected is false. □

Solution to Exercise 270: Let $G = (V, E)$ be a graph with $\chi(G) = k$, and for some edge $e = \{x, y\} \in E(G)$, let $H = G - e$ be the graph formed by deleting e from G. In hope of a contradiction, suppose that $\chi(H) \leq k-2$, and let $f : V \to [k-2]$ be a proper $(k-2)$-colouring of vertices in H. If $f(x) = f(y)$, alter the colouring f by colouring y a new colour, (one not used by f) to produce a proper colouring $(k-1)$-colouring of G, the desired contradiction. □

Solution to Exercise 271: Let G be a k-critical graph on n vertices. By Theorem 6.7.3, $\delta(G) \geq k-1$. By the handshaking lemma, $|E(G)| = \frac{1}{2}\sum_{v \in V(G)} \deg(v) \geq \frac{1}{2}n(k-1)$. □

Solution to Exercise 272: Let G be a (vertex) critically k-chromatic triangle-free graph, and let H be the $(k+1)$-chromatic Mycielski graph on vertices $X = V(G)$, Y, and z. Removing the vertex z drops the chromatic number (properly k-colour G, and colour vertices of Y correspondingly). Removing any vertex $y \in Y$ also drops the chromatic number, because in this case, z can be coloured with the same colour as was y. If a vertex from x is deleted, since G is k-critical, X can now be $(k-1)$-coloured, and with corresponding colours chosen for Y, z then can be coloured with a k-th colour. □

Solution to Exercise 273: By symmetry, there are only two cases to check—when an outside vertex is deleted, or when an inside vertex is deleted. If an outside vertex is deleted, colour the remaining 3 outside vertices 2,1,2 (in that order). Then it is not difficult to find a 3-colouring where only two (inner) verticesare coloured 3. Similarly, deleting an inner vertex allows a 3-colouring with the outside colours being, in order, 2,1,2,3 (and 6 of the remaining 7 inner vertices are all coloured 1 or 2). It only remains to show (by cases) that no proper 3-colouring of the graph exists—details are left to reader. □

Solution to Exercise 274: See [154] for this and other properties of the Mycielski construction. Let G be Hamiltonian with vertices $x_1, x_2, \ldots, x_n$ given in order forming a Hamiltonian cycle. Let H be the graph obtained by applying the Mycielski

construciton, where vertices $y_1, \ldots, y_n, z$ are added. If n is even, then the ordering

$$y_1, x_2, y_3, x_4, \ldots, x_n, x_1, y_2, x_3, y_4, \ldots, y_n, z$$

gives a Hamiltonian cycle. If n is odd, a Hamiltonian cycle is given by

$$y_1, x_2, y_3, x_4, \ldots, x_{n-1}, y_n, x_n, y_{n-1}, \ldots, y_4, x_3, y_2, z.$$

□

Solution to Exercise 275: Deletion of an edge of P_2 leaves a pair connected and an isolated point, which can be coloured in $k(k-1)k$ ways. Contracting an edge gives a single edge, which can be coloured in $k(k-1)$ ways. By Lemma 6.10.1, the number of good k-colourings of P_2 is $k(k-1)k - k(k-1) = k(k-1)^2$. □

Solution to Exercise 276: Let T be a tree on n vertices, and specify some $v \in V(T)$ as a root. The number of ways to colour v is k. The number of ways to colour each neighbour of v is $k-1$. Having coloured all vertices within distance d of v, the vertices at distance $d+1$ can be coloured with any colour except the neighbour at distance d from v, that is, these vertices can each be coloured with $k-1$ colours. Thus, colouring from the root out, each remaining vertex has $k-1$ colour choices. □

Solution to Exercise 277: There are many proofs; induction is perhaps the easiest. For $n \geq 3$, let $S(n)$ be the statement that $p_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$.

Base case ($n = 3$): There are $k(k-1)(k-2)$ proper k-colourings of C_3, and the RHS of $S(3)$ is

$$(k-1)^3 + (-1)^3(k-1) = (k-1)^3 - (k-1) = (k-1)[(k-1)^2 - 1] = (k-1)(k^2 - 2k),$$

which is $k(k-1)(k-2)$, so $S(3)$ is true.

Inductive step: Let $m \geq 4$ and suppose that $S(m-1)$ is true. Then

$$\begin{aligned} p_{C_m}(k) &= p_{C_m - e}(k) - p_{C_m/e}(k) && \text{(by (6.8))} \\ &= p_{P_{m-1}}(k) - p_{C_{m-1}}(k) \\ &= k(k-1)^{m-1} - \left[(k-1)^{m-1} + (-1)^{m-1}(k-1)\right] && \text{(Lemma 6.10.4 and } S(m-1)) \\ &= (k-1)^m + (-1)^m(k-1), \end{aligned}$$

which is the RHS of $S(m)$, so $S(m)$ is true.

By MI, for all $n \geq 3$, the statement $S(n)$ is true. □

Hint for Exercise 278: If the colours used are 1,2,3, there are 10 colourings that use 1 exactly once, 10 that use 2 exactly once, and 10 that use 3 exactly once. □

Solution to Exercise 279: Answer: $p_G(k) = k(k-1)^2(k-2)$. There are $k(k-1)(k-2)$ ways to colour the vertices of the triangle, and given such a colouring, there are $k-1$ ways to colour the pendant vertex. In all, there are $k(k-1)(k-2)(k-1)$ proper colourings of G. Hence, there are $p_G(3) = 12$ proper 3-colourings of $V(G)$, which are left to the interested reader to find.

To use the reduction formula, let e denote the pendant edge. Then $G - e$ is a triangle with an isolated vertex, and so $p_{G-e}(k) = k \cdot k(k-1)(k-2)$. Also, G/e is simply a triangle, and so $p_{G/e}(k) = k(k-1)(k-2)$. Using $p_{G-e}(k) = p_G(k) + p_{G/e}(k)$.

$$k \cdot k(k-1)(k-2) = p_G(k) + k(k-1)(k-2),$$

and thus $p_G(k) = k(k-1)(k-2)(k-1)$, as found above. □

Solution to Exercise 280: Apply equation 6.8 repeatedly to $G = K_{2,2,2}$. One such derivation gives

$$p_G(k) = k(k-1)(k-2)(k^3 - 9k^2 + 29k - 32),$$

and $p_G(3) = 6$. This makes sense since any three vertices taken one from each partite set form K_3, which has 6 colourings. Once a 3-colouring of such a triangle is given, the remaining vertices can only be 3-coloured in one way.

If, instead, equation (6.9) is applied recursively (see [190, p.539] for the diagrams of one such approach), arrive at $p_G = p_{K_6} + 3p_{K_5} + 3p_{K_4} + p_{K_3}$, from which the same answer is derived. □

Solution to Exercise 282: As was shown in lectures, $\chi'(K_4) = 3$. Also, $\chi'(K_5) = 5$.

It is also easy to see that an odd cycle can be 3-edge-coloured but is not 2-edge-colourable, so $\chi'(C_5) = 3$.

To see that $\chi'(K_{3,7}) = 7$, first observe that $\Delta(K_{3,7}) = 7$, and so $\chi'(K_{3,7}) \geq 7$. A good 7-edge-colouring is available by looking at the side with 3 vertices, say, x_1, x_2, x_3. The edges from x_1 can be coloured, in order, 0,1,2,3,4,5,6. the edges from x_2 can be coloured, in order, 1,2,3,4,5,6,0, and the edges from x_3 can be coloured, in order, 2,3,4,5,6,0,1.

Let G be the graph of the octahedron. Then $\Delta(G) = 4$, so $\chi'(G) \geq 4$. It is not difficult to find a proper 4-edge-colouring of G, so $\chi'(G) = 4$. □

Solution to Exercise 284: Both $\chi'(C_{2k+1}) = 3$ and $\Delta(C_{2k+1}) = 2$. □

Solution to Exercise 285: There is an edge colouring of K_4 using only 3 colours, and so the remaining pendant edge can be coloured with a fourth colour; thus $\chi'(H) = 4$, whereas $\Delta(H) = 4$, and so H is of class 1. □

Solution to Exercise 286: If P denotes the Petersen graph, since $\Delta(P) = 3$, $\chi'(P) \geq 3$ and by Vizing's theorem, $\chi(P) \in \{3, 4\}$.

To see that P is not 3-edge-colourable, suppose that a colouring using colours 1,2,3 is given. For such a colouring to be proper, the edges on the outer 5-cycle are coloured with (up to symmetry) with the pattern 1,2,1,2,3. This means that the spokes are then coloured 3,3,3,1,2. But then this forces two incident edges of the inner C_5 to be coloured the same. So no proper 3-edge colouring of P exists, and so by Vizing's theorem, $\chi(P) = 4 = \Delta(P) + 1$. Thus P is of class 2. To confirm this, (without Vizing's theorem) it is easy to find a proper edge colouring with only 4 colours (*e.g.*, colour spokes 1, and use colours 2,3,4 to colour the inner and outer copies of C_5). □

Solution to Exercise 287; Use two colours to colour the edges of a hamiltonian cycle, and since the remaining edges form a matching, use a third colour for the remaining edges. □

Solution to Exercise 288: Following the hint, let $X = \{x_1, \ldots, x_a\}$ and $Y = \{y_1, \ldots, y_b\}$ be the respective partite sets, and define $f : E(K_{a,b}) \to \{0, 1, \ldots, b-1\}$ by $f(\{x_i, y_j\}) = j - i + 1 \pmod b$. It remains to show that f is a proper edge b-colouring. If, for some i, k, ℓ, $f(\{x_i, y_k\}) = f(\{x_i, y_\ell\})$, then $k - i + 1 \equiv \ell - i + 1 \pmod b$, and so $k \equiv \ell \pmod b$, from which it follows that $k = \ell$. So for any $x_i \in X$, edges containing x_i are coloured differently. Similarly, if for some i, j, k, $f(\{x_i, y_k\}) = f(\{x_j, y_k\})$, then $k - i + 1 \equiv k - j + 1 \pmod b$ and so $i \equiv j \pmod b$, from which it follows that $i = j$. So for any vertex $y_k \in Y$, the edges containing y_k are coloured differently. □

Solution to Exercise 289: To be written.

Solution to Exercise 290: Since $\Delta(Q_k) = k$, $\chi'(Q_k) \geq k$. Colouring each edge by the coordinate in which the endpoints differ is a good edge k-colouring. □

Solution to Exercise 291: This exercise appears in [977, 7.1.26]. Let G be a k-regular graph. Vizing's theorem (Theorem 6.11.4) says that $k = \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 = k + 1$. So edges of G are $(k+1)$-colourable. It remains to show that no good edge k-colouring exists if G contains a cut-vertex and k is odd.

Let k be odd, and suppose that G is a k-regular graph. Suppose that $f : E(G) \to \{1, 2, ..., k\}$ is a good edge-colouring. Since each vertex is incident with k edges, at any given vertex, all k colours appear. Hence, each colour class is a perfect matching of G; write $E(G) = M_1 \cup \cdots \cup M_k$ be the partition of edges into the respective colour classes. (This yields that $|V(G)|$ is even—which also follows by the handshaking lemma.) In other words, G has a 1-factorization.

Let x be a cut-vertex, and let $G_1, \ldots, G_s$ be the components of $G \backslash x$ (where $s \geq 2$). Since k is odd, one of these components, say G_1, has an odd number a of neighbours of x, say $N(x) \cap V(G_1) = \{v_1, \ldots, v_a\}$, (where, in this case, $a \in \{1, 3, \ldots, k-2\}$). Without loss of generality, say that for each $i = 1, \ldots a$, $\{x, v_i\} \in M_i$. Looking at M_{a+1} restricted to G_1, since none of the edges $\{x, v_1\}, \ldots, \{x, v_a\}$ are coloured $a + 1$,

in G_1, the matching M_{a+1} covers all vertices of G_1, and so G_1 is even, which is a contradiction. $\square$

Comment on Exercise 292: This problem appears in [429, 8.19, p. 247].

Comment on Exercise 294: See [362].

Solution to Exercise 295: Let G be a class 1 graph with $\chi'(G) = \Delta(G) = k$. Suppose that G is "critical", i.e., the removal of any edge lowers the chromatic index. If G has no edges, then $G = K_1$ is a single vertex, which is vacuously critical. So assume that G has at least one edge, and since G is class 1, let $\chi'(G) = k \geq 1$. Since the removal of any edge lowers the chromatic index by at most 1, for any edge $e \in E(G)$, $\chi'(G-e) \geq k-1$. However, by Vizing's theorem, for any edge e, $\chi'(G-e) \geq \Delta(G-e) \geq k-1$ and so $\Delta(G-e) = k-1$. Hence the removal of any edge reduces the maximum degree. Thus there are at most two vertices in G of maximum degree k.

If there are two vertices of maximum degree, say x and y, there is only one edge, namely $e = \{x, y\}$, and so $G = K_2$. If there is only one vertex z of maximum degree, then every edge is incident with z, i.e., G is the star $K_{1,k}$. So, in any case, G is a star, and so for each $n \geq 1$, any critical class 1 graph on n vertices graph is unique. $\square$

Comment on Exercise 297: This result appears as Theorem 10.4, p. 69 of [363], and is given as Exercise 10b.

Solution to Exercise 298: See [363]. A proper edge-colouring of a graph G is equivalent to a proper vertex colouring of $L(G)$. $\square$

Comment on Exercise 299: This exercise appeared in [363, 10d, p. 70].

Solution to Exercise 300: This result was observed by Deretsky, Lee, and Mitchem [250].

Let P be a maximal path on consecutive vertices $x_0, x_1, \ldots, x_k$, and label the edges consecutively $1, 2, \ldots, k$. Vertex x_0 is incident with the edge coloured 1, so the gcd of the labels of edges incident with x_0 have labels with gcd $= 1$. For each $i = 1, \ldots, k-1$, vertex x_i is incident with (at least two) edges having consecutive labels, which are relatively prime. If x_k has only one edge incident, then the gcd requirement is not applicable. If x_k has another neighbour (which is on P since P is maximal), then label this edge $k+1$, which is relatively prime to the label k of the last edge in P.

Suppose that some vertex is not on P. Since G is connected, there is a maximal path Q with one endpoint on P (and the rest not on P). Colour consecutive edges of Q with the next available consecutive labels (beginning with either $k+1$ or $k+2$). In this case, the first vertex of Q (which lies on P) already has gcd of its edge labels 1, and arguing as above, all vertices on Q are labelled with gcd 1. If there are any vertices that remain, apply the same procedure as was done for Q, and continue until

all vertices satisfy the gcd condition. Lastly, label any unlabelled edges arbitrarily with the remaining labels. □

Solution to Exercise 301: Let G be bipartite. If G has no edges, then $\chi(G) = 1$ and $\omega(G) = 1$. If G has edges, then $\chi(G) = 2 = \omega(G)$. □

Comment on Exercise 302: See [429, pp. 245–246] for a proof by induction on the number of vertices. A more direct proof is available by observing that a P_3-graph is a disjoint union of stars, copies of K_3, edges, and vertices. If G is a P_3 graph containing a triangle, all other components have both smaller clique number (at most 2) and smaller chromatic number (also at most 2), so $\omega(G) = \chi(G) = 3$. The similar observation holds when there are no triangles. □

Comment on Exercise 303: This was given as an exercise in [429, Ex. 25, p. 247].

Solution to Exercise 304: Let $G = (X \cup Y, E)$ be a bipartite graph. Define a partial order $\leq$ on $V(G) = X \cup Y$ by $x \leq y$ if and only if $x \in X$, $y \in Y$, and $\{x, y\} \in E$. Then $(V(G), \leq)$ is a partially ordered set (the transitivity axiom is not needed). □

Solution outline for Exercise 306: Let $k \geq 2$, Let $G = (V, E)$ be a copy of C_{2k+1} on $V = \{v_1, \ldots, v_{2k+1}\}$, where (slightly abusing notation) $E = \{v_1v_2, v_2v_3, \ldots, v_{2k}v_{2k+1}, v_{2k+1}v_1\}$. Suppose, in hope of a contradiction, that G is a comparability graph for some poset $(P, <)$. Since $v_1v_2 \in E$, in the poset, either $v_1 < v_2$ or $v_2 < v_1$; without loss of generality, let $v_1 < v_2$. Since $v_2v_3 \in E$, either $v_2 < v_3$ or $v_3 < v_2$. If $v_2 < v_3$, then by transitivity, $v_1 < v_3$, which is impossible since $v_1v_3 \notin E$, so $v_3 < v_2$. (In general, $(P, <)$ does not contain any chain of length 3 or more.) In a similar manner, it follows that $v_3 < v_4$. So P looks like a zigzag (left to right) path, concluding with $v_{2k-1} < v_{2k}$ and $v_{2k+1} < v_{2k}$. Since $v_{2k+1}v_1 \in E$, either $v_{2k+1} < v_1$ or $v_1 < v2k + 1$; however, the first case leads to a $v_{2k+1} < v_2$, which is not possible, and the second case leads to $v_1 < v_{2k}$, again, which is impossible. In any case, a contradiction is reached. □

Solution to Exercise 308: Let G be a comparability graph with a partial order $\leq$ on $V(G)$ as a witness. A clique in G corresponds to a chain in $(V(G), \leq)$. Let m be the length of a longest chain in $(V(G), \leq)$. Then $\omega(G) = m$, and so $\chi(G) \geq m$. By Mirsky's theorem (Theorem 5.10.4), $(V(G), \leq)$, there exist disjoint antichains $A_1, \ldots, A_m$ so that $A_1 \cup \cdots \cup A_m = V(G)$. These antichains can be considered as colour classes in a proper m-colouring of $V(G)$, so $\chi(G) \leq m$. Thus, $\chi(G) = m = \omega(G)$ and so G is perfect. □

Solution to Exercise 312: Let A, B, C be the vertices of the triangle, and let AD and CE be the pendant edges. One set of intervals corresponding to the vertices is: $I_A = [0, 3]$, $I_B = [2, 3]$, $I_C = [3, 5]$, $I_D = [-1, 0]$ and $I_E = [5, 6]$. □

Solution to Exercise 313: Let G be an interval graph. Let C be a cycle in G with $V(C) = \{v_1, v_2, \ldots, v_k\}$, $k \geq 4$, with edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{k-1}, v_k\}$,

$\{v_k, v_1\}$, and for each $i = 1, \ldots, k$, let $I_i = (x_i, y_i) \subset \mathbb{R}$ be an interval corresponding to v_i. Without loss of generality, suppose that $x_1 \leq x_2$. If x_1 is adjacent to x_3, a chord of C is found so suppose that $\{x_1, x_3\} \notin E(G)$; then $I_1 \cap I_3 = \emptyset$. In this case, $x_2 < y_1 \leq x_3 < y_2$. If C does not have a chord, repeating the argument above, the x_is occur in a strictly increasing order; but then $y_1 \leq x_3 < x_k$ shows that $I_1 \cap I_k = \emptyset$, contradicting that $\{v_1, v_k\}$ is an edge. □

Solution to Exercise 314: Let G be an interval graph on vertices $V(G) = \{v_1, \ldots, v_n\}$ and for each i, let I_i be an interval corresponding to v_i. Define a partial order on $V(G)$ by $v_i \prec v_j$ if and only if I_i precedes I_j (that is, $I_i \cap I_j = \emptyset$ and every point in I_i precedes every point in I_j). Then in $\overline{G}$, x is adjacent to y if and only if the corresponding intervals intersect. □

Solution to Exercise 316: The condition "neither G nor $\overline{G}$ contains an odd hole" is self-complementary. So, if G is perfect, by the strong perfect graph theorem, G satisfies this condition, then so does $\overline{G}$, and so $\overline{G}$ is also perfect. □

Solution to Exercise 317: Consider the first drawing of the Petersen graph in Figure 1.26). Draw the inner five vertices with a star, each edge of length 1, and draw the outer ring of five vertices with a regular pentagon (of length 1). Rotate the outer ring until the five "spokes" (joining the inner ring and the outer ring) have length 1. □

Hint to Exercise 318: Use squares whose diagonal is just less than 1. □

Solution to Exercise 321: Pick a vertex x with $\deg(x) = \Delta(G)$. In any total colouring of G, all edges incident with x need different colours, and x needs yet one more colour. □

Solution to Exercise 322: Since $\chi(K_3) = \chi'(K_3) = 3$, $\chi''(K_3) \geq 3$. However, there is a total 3-colouring (where the colour of a vertex is the same as the opposite edge). □

Solution to Exercise 323: $\chi''(K_4) = 5$. □

Solution to Exercise 325: For example, $K_{2,4}$ is planar by the drawing in Figure 20.9.

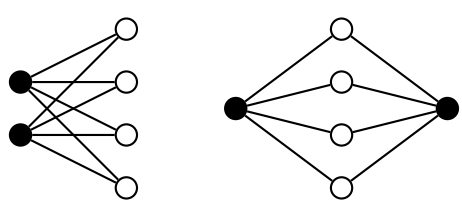


Figure 20.9: $K_{2,4}$ drawn in the standard way and in a planar way

Replacing 4 with n, the same idea for the plane drawing works. □

Solution to Exercise 326: The proof here is by induction on e. For any $e \geq 0$, let $S(e)$ be the statement that Euler's formula holds for all connected planar graphs with e edges.

BASE STEP: When $e = 0$, the only connected graph is a single vertex, in which case $v = 1$, and $f = 1$, and so $v + f = e + 2$.

INDUCTIVE STEP: For some $e \geq 0$, assume that $S(e)$ is true. Let G be a connected planar graph (with plane drawing fixed) with $e + 1$ edges, f faces, v vertices. To complete the inductive step, it remains to show that $S(e + 1)$ holds, that is, that $v + f = (e + 1) + 2$.

Case 1: If G is a tree, then $e + 1 = v - 1$, $f = 1$, and so $v + f = (e + 1) + 2$, as desired, showing that $S(e + 1)$ holds.

Case 2: Suppose that G contains a cycle C. Delete any edge $\{x, y\}$ from C to form a new graph G'. The number of edges in G' is $e' = e$, the number of vertices is $v' = v$, and since deleting an edge from a cycle joins two regions, the number of regions in G' is $f' = f - 1$. By induction hypothesis applied to G', $v' + f' = e' + 2$, that is, $v + f - 1 = e + 2$, or $v + f = (e + 1) + 2$, so $S(e + 1)$ holds. This completes the inductive step.

By mathematical induction, for all $e \geq 0$, Euler's formula holds. □

Comment on Exercise 327: Repeat virtually the same proof as the one for planar graphs (whereas now a 2-cycle can exist between two vertices). □

Solution to Exercise 328: (Euler's formula for planar graphs with k components) For each $k \geq 1$, the statement to be proved is "If a planar graph has v vertices, f faces, e edges, and k components, then $v + f = e + k + 1$." The proof is by induction on k.
BASE STEP: The base case $k = 1$ is true by Euler's formula for connected planar graphs (Exercise 326).
INDUCTIVE STEP: Let $\ell \geq 1$ and suppose that the formula holds for a planar graph with ℓ components. Consider a planar graph G with $\ell + 1$ connected components, say $C_0, C_1, \ldots, C_\ell$. Let G' be the graph obtained from G by removing C_0. If G' has v' vertices, e' edges and f' faces, then by the induction hypothesis,

$$v' - e' + f' = \ell + 1.$$

If the single component C_0 has v_0 vertices, e_0 edges and f_0 faces, then by Euler's formula (for a connected planar graph),

$$v_0 + f_0 = e_0 + 2.$$

Then G has $v = v' + v_0$ vertices, $e = e' + e_0$ edges, and $f = f' + f_0 - 1$ faces (since the infinite face of G' is a face of C_0), and hence

$$\begin{aligned} v - e + f &= (v_0 - e_0 + f_0) + (v' - e' + f') - 1 \\ &= 2 + (\ell + 1) - 1 \quad \text{(by Euler's formula and ind. hyp)} \\ &= (\ell + 1) + 1, \end{aligned}$$

proving that the formula holds for graphs with $\ell + 1$ components, completing the inductive step.

Therefore, by mathematical induction, for every $k \geq 1$, the result holds for all graphs with k components. □

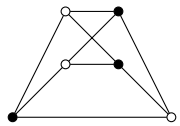
Solution to Exercise 329: Suppose that G is a graph with $k \geq 2$ components, say $G_1, \ldots, G_k$, where each G_i has at least 3 vertices. By Lemma 7.1.2, for each $i = 1, \ldots, k$, $|E(G_i)| \leq 3|V(G_i)| - 6$. Then

$$\begin{aligned} E(G)| &= \sum_{i=1}^{k} |E(G_i)| \\ &\leq \sum_{i=1}^{k} (3|V(G_i)| - 6) \\ &= 3(\sum_{i=1}^{k} |V(G_i)|) - 6k \\ &= 3|V(G)| - 6k \\ &< 3|V(G)| - 6. \end{aligned}$$

So not only does Lemma 7.1.2 hold, but something even stronger holds.

For the second part, one might look for a graph with at least one component on at most 2 vertices. When $v = 2$, and $G = K_2$, a single edge, then $e = 1$, in which case $3v - 6 = 0$, so the result fails. When G is a single vertex or a collection of isolated vertices, the result also fails. □

Solution to Exercise 330: If a simple graph has 5 vertices and 10 edges, then the graph is K_5, which is known not to be planar. So multiple edges are needed. When $v = 5$ and $e = 10$, solving $v + f = e + 2$ gives $f = 7$. Draw G by starting with a path P_4 (on five vertices), using up 4 edges. Put the remaining 6 edges between, say, the first and second vertices of the path (creating a multi-edge with 7 edges). Six regions are thereby formed, and with the outside region, there are 7. □

Solution to Exercise 331: The black vertices form one partite set (and white the other).  □

Solution to Exercise 332: Let G be a planar graph with girth g. Then in any planar drawing of G, each face has at least g edges. Let x be the number of pairs $(face, edge)$, where the edge is on the face. Counting x by looking at at each face, $x \geq fg$. Counting x by looking at each edge, since there are at most two faces incident with an edge, $x \leq 2e$. Putting these two together, $fg \leq 2e$. □

Hint to Exercise 333: Show $(g-2)e \leq g(v-2)$. □

Solution to Exercise 334: Suppose that a connected planar graph G on v vertices with e edges has at most 2 vertices of degree 5 or less; since G is connected each of these vertices has degree at least 1, and so

$$2e = \sum_{x \in V(G)} \deg(x) \geq 6(v-2) + 2 = 6v - 10,$$

and so $e \geq 3v - 5$, contradicting Lemma 7.1.2. □

Solution to Exercise 335: Let G be a connected planar graph (on $v \geq 4$ vertices). Suppose that there are at most 3 vertices of degree at most 5. If G has any vertex of degree 1, add an edge from that vertex to any other vertex in the face containing this vertex and call the new graph G'. Then G' is still planar and has minimum degree 2. By the handshaking lemma, and since at least 3 vertices have degree 2,

$$2|E(G')| = \sum_{x \in V(G')} \deg(x) \geq 6(v-3) + 3 \cdot 2,$$

and so $|E(G')| \geq 3v - 6$. However, in adding the new edges, at least one can be connected to a vertex of degree at least 6, thereby increasing the sum of degrees (of the high degree vertices) by 1, thereby increasing the right hand side of above by 1, producing the desired contradiction for G', and hence for G. □

Solution to Exercise 336: With $v = 8$, $f = 10$, and $e = 19$, Euler's formula does not hold, so there is no such planar graph that is connected. In fact, if one wanted a disconnected planar graph with these parameters, the formula $v + f = e + k + 1$ (from Exercise 328), then necessarily $k = 0$, which is nonsense, so no such planar graph exists. □

Solution to Exercise 337: For the moment, suppose that such a graph exists on v vertices, with f faces, and e edges. Count (face, vertex) pairs where the vertex is

incident with the face. Since each face is incident with 4 edges, each face is incident with 4 vertices. Therefore, the number of such pairs is $f \cdot 4$. Since each vertex is incident with 4 faces, the number of such pairs is $4v$. Hence $4f = 4v$, and thus $v = f$.

Since each vertex is incident with 4 faces, each vertex is incident with 4 edges. So the graph is 4-regular, and so by the handshaking lemma, $4v = 2e$. Thus $4f = 4v = 2e$, and so $e = 2v = 2f$.

If such a graph has k components, then $v + f = e + k + 1$ yields $v + v = 2v + k + 1$, which is impossible for $k \geq 1$. Therefore, no such graph exists. □

Solution to Exercise 342: This result was shown in [554]. For each $n \geq 2$, let $A(n)$ be the assertion that if the edges of a complete geometric graph on n vertices are 2-colored, then there exists a monochromatic plane spanning tree. The proof is by strong induction on n. The notation $\text{conv}(S)$ denotes the convex hull of a set S (the smallest convex polygon containing S together with its interior).
BASE STEP: $A(2)$ holds trivially as there is only one edge in a spanning tree.

INDUCTION STEP: Let $k \geq 3$ and suppose that $A(2), \ldots, A(k-1)$ are all true. Consider a set of points $P = \{p_1, \ldots, p_k\}$ in general position, let G denote the geometric graph on P, and let a red-blue colouring of $E(G)$ be given.

Case 1: At least two edges on the border of the convex hull of P receive different colors. Then there exist consecutive points, say m, p, q, on the border of $\text{conv}(P)$ so that the segments mp and pq are colored differently. By the induction hypothesis $A(k-1)$ on $P' = P \backslash \{p\}$, the geometric graph induced by P' has a monochromatic plane spanning tree, and together with one of mp or pq a monochromatic plane spanning tree for G is formed.

Case 2: All edges on the border of $\text{conv}(P)$ are coloured identically, say red. Without loss of generality, assume that the x-coordinates of all p_i's are strictly increasing (if not, rotate the picture a bit), and so the points are ordered $p_1, \ldots, p_k$ left to right. For each $1 < i < k$, let G_i^ℓ and G_i^r be the graphs induced by $\{p_1, p_2, \ldots, p_i\}$ and $\{p_i, \ldots, p_k\}$ respectively. For each such i, by the inductive hypotheses $A(i)$ and $A(k-i)$, each of G_i^ℓ and G_i^r has a monochromatic plane spanning tree, say T_i^ℓ and T_i^r, respectively. If for some i, both T_i^ℓ and T_i^r are coloured the same, then their union forms a monochromatic plane spanning tree for G, so assume that each such pair of trees has different colors. Furthermore, if either T_2^r or T_{k-1}^ℓ is red, a red edge on the border of $\text{conv}(P)$ joining either p_1 to T_2^r or T_{k-1}^ℓ to p_k produces a red plane spanning tree for G, so assume that both T_2^r and T_{k-1}^ℓ are blue. Then the sequence of left-right colour pairs begins red-blue, and ends in blue-red; hence there exists an $i \in \{2, \ldots, k-2\}$ so that $T_i\ell$ is red, T_i^r is blue, T_{i+1}^ℓ is blue, and T_{i+1}^r is red.

Adjoining any red edge from the border of $\text{conv}(P)$ that joins T_i^ℓ to T_{i+1}^r (which crosses a vertical line between p_i and p_{i+1}) yields a red planar spanning tree for G. In any case, $A(k)$ is true, completing the inductive step.

By strong mathematical induction, for each $n \geq 2$, the assertion $A(n)$ is true. □

Solution to Exercise 343: Although this result seems somewhat obvious when creating a drawing of a dual, a proof is needed. One idea is to use induction on the number of edges. Here is a sketch. For either 0 or 1 edges, the dual is a single vertex, which is obviously planar. Similarly for trees. So assume G is not a tree. In the inductive step, remove one edge e of a cycle in the planar drawing (that separates two regions). By induction hypothesis, $G - e$ has a planar dual. Now add back the original deleted edge, and examine what this does to the dual of $G - e$—it merely subdivides an edge, which is no obstacle to planarity. □

Solution to Exercise 344: The dual of the first graph has a vertex of degree 5' (corresponding to the outer region), whereas the dual of the second graph has no such vertex. □

Solution to Exercise 348: This exercise appears (without solution) in [22, 11.14, p. 276]. Let G be a connected planar graph with a given plane drawing. Then G is bipartite if and only if every cycle in G is even if and only if each face has even degree if and only if the dual has all even degrees. Since duals of planar graphs are always connected, G being bipartite is equivalent (by Euler's theorem) to the dual being Eulerian. □

Solution to Exercise 349: No. For example, consider a C_6 with three additional edges between the first, third, and fifth vertex. □

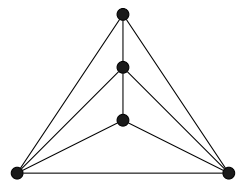
Hint for Exercise 350: Look at Bolivia.

Solution to Exercise 351: One approach to a solution is algorithmic: begin at any vertex, and alternately colour regions surrounding that vertex. Then do the same from any vertex on the border of regions already coloured. Another approach is to use mathematical induction on the number of edges, removing all edges surrounding one face (see [640, 5.26] for such a solution. This second approach works for multigraphs, where it can be assumed without loss of generality that G contains no loops.

Another approach to a solution is to look at the dual G^*. Since all degrees in G are even, all faces in G^* have an even number of edges, and so all cycles in G^* are even. Thus, G^* is bipartite, and so is 2-colourable. □

Solution to Exercise 352: Let $c : V(G) \to \{0, 1, 2\}$ be a proper colouring. Colour a face red if and only if the colours 0,1,2 appear in counterclockwise order, and blue if they appear in clockwise order. Adjacent triangular faces are coloured differently. □

Solution to Exercise 353: Consider the following graph:



This graph is the graph of the solid formed by attaching two tetrahedra by a common base—a double pyramid. □

Solution to Exercise 354: For each positive integer n, let $S(n)$ be the statement that every planar graph with n vertices is 6-colourable.

Base step: Every planar graph on six or fewer vertices is trivially 6-colourable (colour each vertex with a different colour) and so $S(n)$ is true for $n = 1, 2, 3, 4, 5, 6$.

Induction step: Let $k \geq 6$ and suppose that $S(k)$ is true. Let G be a planar graph with $k + 1$ vertices. By Lemma 7.1.5, let $x \in V(G)$ be a vertex of degree at most 5. Delete x (and all edges incident with x), forming a graph H. Since G was planar, so is H, and thus by $S(k)$, fix a good 6-colouring of H. Since x is of degree at most 5, there are at most 5 vertices in $V(H)$ connected to x in G, so colour x with a colour unused for these 5 (and colour remaining vertices of G as in H). This represents a good 6-colouring of G, and so shows that $S(k + 1)$ is true.

By mathematical induction, for every $n \geq 1$, $S(n)$ holds, and so the statement of the exercise is true. □

Hint to Exercise 356: The proof of this fact occurs in Tutte's paper [936].

Solution outline to Exercise 357: This problem is a standard one; *e.g.*, see [286, 8.9, p. 207].

For one direction, if all degrees are even, greedily colour starting with faces around one vertex. For the other direction, assume that the faces are properly 2-coloured. Then the number of faces surrounding any vertex is even (if odd, two adjacent faces are forced to have the same colour). Since the number of faces surrounding a vertex is equal to the number of edges leaving that vertex, each vertex has even degree. □

Solution to Exercise 358: Let G be Eulerian with a planar embedding so that all faces are triangles. Since G is Eulerian, G is connected and all degrees are even. Let G^* be the planar dual of G. Since all vertex degrees in G are even, all cycles in G^* are even, that is, $G*$ is bipartite. Properly colour the faces of G with two colours, say red and blue. This colouring determines a 3-colouring of the vertices according to: if the face is red, colour its vertices clockwise in increasing order (say 1,2,3). If the face is blue, colour its vertices counterclockwise. This colouring can be accomplished recursively. □

Solution to Exercise 359: Let G be a 3-regular Hamiltonian graph. Since all degrees are odd, $|V(G)|$ is even. Let C be a Hamiltonian cycle in G. Since C is an even cycle, its edges can be properly coloured with two (alternating) colours. All other edges can then be coloured with a third colour. □

Solution to Exercise 361: Let G be a planar graph containing a copy of K_4. The only way to draw K_4 as a planar graph has one vertex that is not adjacent to the outer region (see Figure 7.1), and so G is not outerplanar. □

Solution to Exercise 363: To be written.

Solution to Exercise 364: This result was stated (but not proved) in [626]. Let G be an outerplanar graph. If possible, add edges to G to make a graph H that is maximal outerplanar. If H has a vertex of degree at most 2, then so does G. It is not too difficult to see that vertices of H occur in a (Hamiltonian) cycle C (if G were to have a cut-vertex, then an edge can be added so that the cut-vertex is visible only from one side). Suppose C is given on vertices ordered by $v_1, v_2, \ldots, v_n, v_1$If $H = C$, then all vertices have degree 2. If H is not a cycle, then some chord $\{v_i, v_k\}$ of C is also in H. Since v_i is not adjacent to v_k, suppose that v_i and v_k are closest together, with say, v_j between v_i and v_k on the cycle. Then $\deg(v_j) = 2$. □

Solution to Exercise 365: This result is observed in, *e.g.*, [764]. One proof is by induction on the number of vertices. For 1, 2, or 3 vertices, the result is trivial. Let $n \geq 4$ and suppose that the result is true for all outerplanar graphs on less than n vertices, and let G be outerplanar with n vertices. By Exercise 364, let $x \in V(G)$ have degree at most 2. Delete x, and apply the inductive hypothesis to the remaining graph. Since x is adjacent to at most 2 vertices, a third colour remains to colour x, and so G is 3-colourable, finishing the inductive step. By MI, the result is true for all n.

Another proof is simpler: Let G be outerplanar. Since G is outerplanar, one can add another vertex in the infinite face that is adjacent to each vertex of G to produce a graph H, and this can be done with no new edges crossing, so that H is planar. By the 4CT, H is 4-colourable, and so G is 3-colourable. □

Solution to Exercise 366: See Figures 20.10 and 20.11. □

□

Solution to Exercise 368: A detailed solution for this exercise is given in [979, pp. 77–78]. First observe that $K_{3,2,2}$ contains a copy of $K_{3,3}$, and so is not planar. Hence, $\mathrm{cr}(K_{3,2,2}) \geq 1$. Put $x = \mathrm{cr}(K_{3,2,2})$, and consider an optimal drawing of $K_{3,2,2}$. This drawing has $v = 7 + x$ vertices, $e = 16 + 2x$ edges, and f faces. Not all faces are triangles (for if they were, a crossing point forces a K_4 around it, which is impossible in a tripartite graph), so by double counting, $2e \geq 3f+1$. By Euler's formula, $v+f = e+2$,

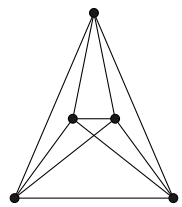


Figure 20.10: Drawing for $\mathrm{cr}(K_5) \leq 1$

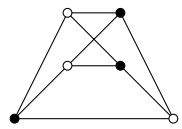


Figure 20.11: Drawing for $\mathrm{cr}(K_{3,3}) \leq 1$ with partite sets coloured differently

and so $7 + x + f = 16 + 2x + 2$, giving $f = 11 + x$. Then $2e = 32 + 4x \geq 3(11 + x) + 1$ shows $x \geq 2$.

A witness to $x = 2$ is the drawing (where vertices are shaded appropriately) is given in Figure 20.12. In that diagram, the second highest vertex can also be moved inside the small triangle below it, and there is still only two crossings. □

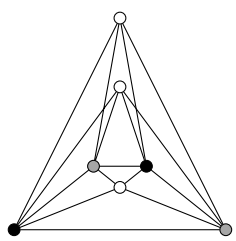


Figure 20.12: $\mathrm{cr}(K_{3,2,2}) \leq 2$

Solution to Exercise 369: See Figure 20.13. □

Solution to Exercise 370: See Figure 20.14 for a drawing of the Petersen graph with only two crossings.

Solution outline to Exercise 371: Let D be an optimal drawing with x crossings. If crossing points are considered vertices and D planar, then D has $v = 10 + x$ vertices, $e = 15 + 2x$ edges. Since the Petersen graph contains K_5 as a contraction, $x \geq 1$. Use the fact that the Petersen graph has no triangles and no 4-cycles to show that $x = 1$ is impossible. □

Solution to Exercise 372: There are many possible drawings; here is one:

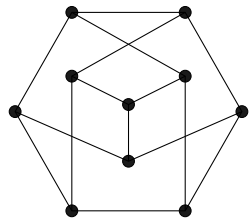


Figure 20.13: The Petersen graph with 3 crossings

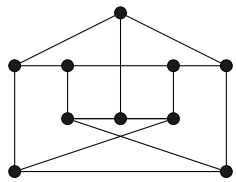
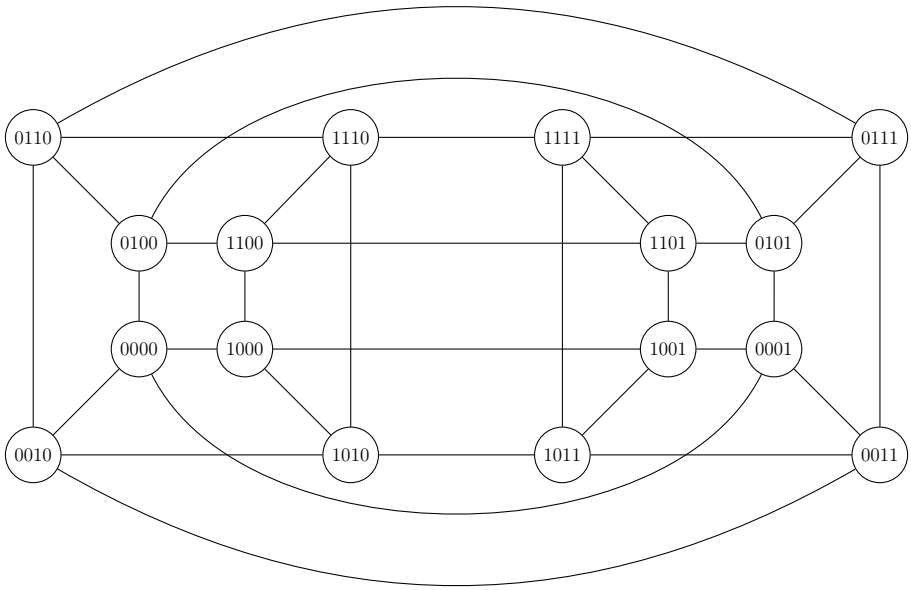


Figure 20.14: cr(Petersen graph) ≤ 2



Solution to Exercise 373: The following sketch can be found in [84], based on a drawing by Hill:

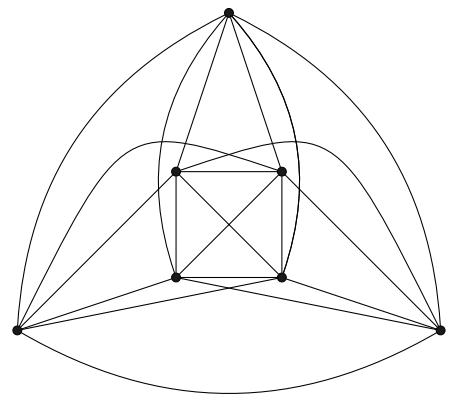


Figure 20.15: $\mathrm{cr}(K_7) \leq 9$

Solution to Exercise 375 At first glance, one might suspect that the degenerate crossing number of any K_n is at most one, since one can draw K_n by starting with a cycle C_n and then putting in all remaining arcs that pass through one common central point, as in Figure 20.16.

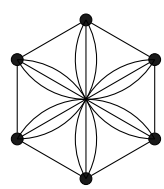


Figure 20.16: Is $\mathrm{cr}^*(K_6) = 1$?

However, the drawing in Figure 20.16 has pairs of arcs that touch at tangent points (they don't actually cross), which is forbidden under the definition of "drawing". So move three of the arcs to the outside, still giving only one crossing point. However, then the resulting drawing still violates the rule that two edges coming from the same vertex cannot cross. After checking other possible starts to a drawing, it appears that in each case, at least 3 crossings are required, and so $\mathrm{cr}^*(K_6) = 3$. $\square$

Solution to Exercise 376: Let $k \geq 1$, put $n = 2k$, and let $0, 1, \ldots, 2k-1$ be the vertices of K_n. Let the paths in the zig-zag pattern be labelled $G_0, G_1, \ldots, G_{k-1}$, where each G_i begins at vertex i. Each of the G_is is a Hamiltonian path, so it remains to show that no edge occurs in more than one G_i.

In the "starter" G_0, the vertices of the path are, in order, $0, 1, 2k-1, 2, 2k-2, 3, 2k-3, \ldots, k+1, k$, and for each $i = 1, \ldots, k-1$, the vertices of G_i are found by adding i to each of the vertices of G_0 (where addition is modulo $2k$). (So each G_i starts with i as one end of the path.)

There are various ways to show that the G_is are edge-disjoint. One way is to observe that for any G_i, if $\{a, b\} \in E(G_i)$, then $a+b \pmod{2k}$ is either $2i$ or $2i+1$. (So in G_0, the edge sums are either 0 or 1 (mod $2k$), in G_1, the edge sums are either 2 or 3, in G_2, the edge sums are either 4 or 5, ..., in G_{k-1}, the edge sums are either $2k-2$ or $2k-1$.) Hence, any $\{a, b\} \in E(K_n)$ is in precisely one G_i, and so the G_is are edge-disjoint. □

Solution to Exercise 377: Consider the complete balanced bipartite graph $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Then G is triangle-free with the most number of edges (see Mantel's theorem or Turán's theorem), in which case each edge requires one copy of K_2 to cover. □

Hint to Exercise 378: In the induction step, delete a vertex of minimum degree. □

Solution to Exercise 379: To be written.

Solution to Exercise 380: Stars centered at 100, 001, 010, and 011 cover all the edges. □

Solution to Exercise 381: The 1-factors from Exercise 192 are pairwise disjoint. □

Solution to Exercise 382: Let P be the Petersen graph. Removal of any 1-factor leaves a 2-regular graph H, and so is a union of cycles. Since the P is not hamiltonian, such cycles have length at most 9; however since the shortest cycle in P has length 5, the only choice is that $H = 2C_5$. However, neither copy of C_5 has a 1-factor, so neither does H. □

Solution to Exercise 383: As in the proof of 8.2.1 (using the second algebraic description), let $V(K_{2n}) = \{w, v_0, v_1, \ldots, v_{2n-2} = v_{p-1}\}$ and for $i = 0, 1, 2, \ldots, p-1$, let

$$M_i = \{w, v_i\} \cup \{\{v_{i-x}, v_{i+x}\} : x \in \{1, 2, \ldots, n-1\}\},$$

where arithmetic in the indices is done modulo p.

Let $k, \ell \in \{0, 1, \ldots, p-1\}$, where $k \neq \ell$. It remains to show that the graph $G_{k,\ell}$ induced by $M_k \cup M_\ell$ is a Hamiltonian cycle in K_{2n}. Since each of M_k and M_ℓ is a 2-factor, $G_{k,\ell}$ is 2-regular and so is a union of cycles. If $G_{k,\ell}$ is a single cycle, then $G_{k,\ell}$ is a Hamiltonian cycle. So suppose, in hopes of a contradiction, that $G_{k,\ell}$ is a union of at least two (vertex disjoint) cycles. Then at least one of these cycles does not contain the central vertex w. Let C be such a cycle, which necessarily is of even length (because two consecutive edges come from different matchings), and let the vertices of C be (in order of the cycle) $u_0, u_1, \ldots, u_{2q-1}$, where $2q < p$. Let $f : \{0, 1, \ldots, 2q-1\} \to \{0, 1, \ldots, 2p-2\}$ be (a 1:1 map) so that $u_i = x_{f(i)}$. Without loss of generality, let $\{u_0, u_1\} \in M_k$.

Then

$$\begin{array}{rlrl} f(0)+f(1) & \equiv 2k \pmod p; & f(1)+f(2) & \equiv 2\ell \pmod p; \\ f(2)+f(3) & \equiv 2k \pmod p; & f(3)+f(4) & \equiv 2\ell \pmod p; \\ & \vdots & & \vdots \\ f(2q-2)+f(2q-1) & \equiv 2k \pmod p; & f(2q-1)+f(0) & \equiv 2\ell \pmod p. \end{array}$$

Summing each of the two columns above,

$$\sum_{i=0}^{2q-1} f(i) \equiv q2k \pmod p, \quad \text{and} \quad \sum_{i=0}^{2q-1} f(i) \equiv q2\ell \pmod p.$$

Thus $q2k \equiv q2\ell \pmod p$, and since $2 < 2q < p$, it follows that since p is prime, $k = \ell$, which is the desired contradiction. □

Solution to Exercise 384: There are many such partitions. Since K_5 is Eulerian, let C be an Eulerian circuit (with 10 edges). Now partition C into 5 paths of length 2. For example, if $V(K_5) = \{1, 2, 3, 4, 5\}$, an Eulerian circuit is given by

$$12345135241,$$

and so 123, 345, 513, 352, and 241 give the desired five paths. □

Solution to Exercise 385: A tree on n vertices with diameter at most 2 is a star $K_{1,n-1}$. Colour the central vertex with 1, and colour the remaining vertices injectively with colours $2, \ldots, n$. (Colouring the central vertex with n also works.) □

Solution to Exercise 387: 1423, 4132, 2314, 3241. □

Solution to Exercise 388: Although an example is not a proof, consider a path on 9 vertices with the vertices coloured

$$1, 9, 2, 8, 3, 7, 4, 6, 5.$$

One can quickly check that the edges are then labelled 8, 7, 6, 5, 4, 3, 2, 1, and so the above colouring is graceful. Observe the similar colouring for a path on 10 vertices

$$1, 10, 2, 9, 3, 8, 4, 7, 5, 6$$

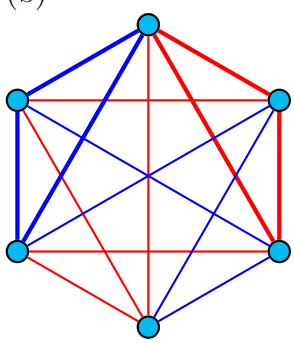
is also graceful. When n is odd or when n is even, the above type of colouring is graceful—to write out the details of the complete proof is straightforward. □

Comment on Exercise 389: See [577]. □

Solution for Exercise 390: (a) Let A, B, C, X, Y, Z be vertices, and assume that the pairs of vertices (*i.e.*, edges) are coloured either red or blue. By the proof of $R(3, 3) \leq 6$, assume without loss of generality that A, B, C form a red triangle.

If any of X, Y, Z have two red edges to A, B, C, then a second red triangle is formed. So suppose that each of X, Y, Z has at most one red edge to A, B, C, that is, each of X, Y, Z has at least two blue edges to A, B, C. If all of the edges XY, XZ, and YZ are red, a second triangle is found, so assume that at least one of these is blue, say XY. Since each of X and Y has two blue edges to A, B, C, one of A, B, C is adjacent to both of X and Y, in which case a blue triangle is formed.

(b)



□

Solution to Exercise 391: Let $\Delta : E(K_9) \to 2 = \{0, 1\}$ be given and fix a vertex $x \in V(K_9)$. Examine the edges containing x; if four of these edges are coloured 0, any colouring of the edges among the four endpoints produces either a 0-coloured K_3 or a 1-coloured K_4. So assume that at most 3 edges containing x are coloured 0, that is, at least 5 are coloured 1. If six are coloured 1, the proof is done by examining edges among the endpoints and employing the fact $R(3, 3) = 6$. Observe that not every vertex can have exactly three 0-edges and five 1-edges containing it, since if this were the case, one would have, by the degree sum formula, $9 \cdot 3/2$ 0-edges, a non-integer. so $R(3, 4) \leq 9$. □

Solution to Exercise 392: Examine a copy of K_8 on vertices $\{0, 1, 2, \ldots, 7\}$, and define a colouring $\Delta : \{0, 1, \ldots, 7\}^2 \to \{0, 1\}$ by $\Delta(\{x, y\}) = 0$ if and only if x and y differ by 1 or 4 (mod 8). See Figure 20.17, where 0-edges are red and 1-edges are blue.

It is straightforward to check that there is no 0-monochromatic K_3 and no 1-monochromatic K_4, showing $R(3, 4) > 8$. □

Solution to Exercise 395:

$$\begin{aligned}
\binom{2k-2}{k-1} &= \frac{(2k-2)!}{((k-1)!)^2} \\
&= (1+o(1))\frac{\sqrt{2\pi(2k-2)}(2k-2)^{2k-2}}{2\pi(k-1)(k-1)^{2k-2}} \qquad \text{(Stirling's approx)} \\
&= (1+o(1))\frac{2^{2k-2}}{\sqrt{\pi(k-1)}}
\end{aligned}$$

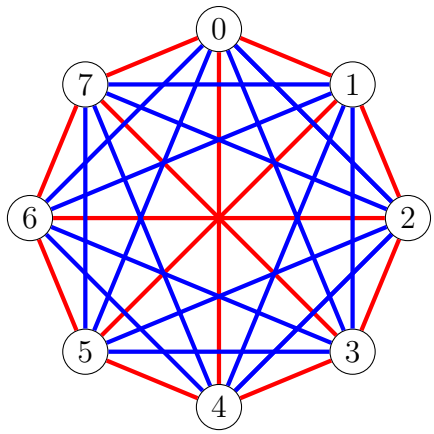


Figure 20.17: $R(3,4) > 8$.

$$< \frac{1}{6\sqrt{k}} 2^{2k}.$$

Solution to Exercise 396: Let $n = \left\lfloor \frac{k}{e\sqrt{2}} 2^{k/2} \right\rfloor$. To show that $n < R(k,k)$, by Theorem 9.5.1, it suffices to show that

$$2\binom{n}{k} 2^{-\binom{k}{2}} < 1. \tag{20.3}$$

Indeed, using the approximation for binomial coefficents (where k is fixed, and $n \to \infty$),

$$\begin{aligned}
2\binom{n}{k} 2^{-\binom{k}{2}} &= (1+o(1)) \frac{2}{\sqrt{2\pi k}} (en/k)^k 2^{-\binom{k}{2}} && \text{(by (18.2))} \\
&< \frac{1}{\sqrt{k}} (en/k)^k 2^{-\binom{k}{2}} \\
&= \frac{1}{\sqrt{k}} \left(\frac{ek2^{k/2}}{ek\sqrt{2}} \right)^k 2^{-\binom{k}{2}} \\
&= \frac{1}{\sqrt{k}} \\
&< 1,
\end{aligned}$$

which proves (20.3). □

Solution to Exercise 399: Let $w, x, y, z \in V(G)$. By adding (mod 17) some fixed number to labels of vertices, and relabelling, one can assume that $w = 0$. Recall that when p is a prime, if k is any non-zero element of $\mathbb{Z}_p$, then

$$\{ki : i \in \mathbb{Z}_q\} = \mathbb{Z}_p.$$

Since 17 is a prime, one can relabel vertices again my multiplying all labels by x^{-1}, thereby assuming that $x = 1$. Note that by multiplying by x^{-1}, one might have changed $\{w, x\}$ from a blue edge (or non-edge) to a red edge (edge), but if so, all other edges got reversed, too. This happens if x^{-1} is not a quadratic residue modulo 17, and, in fact, the proof that Paley graphs are self-complementary relies upon this fact—and that multiplying by a non-residue changes a residue to a non-residue and vice versa.

Since $\{0, 1\}$ is an edge in G (or red edge in a coloured K_{17}), and 1 is a quadratic residue, it remains to check that the remaining five edges $\{0.y\}, \{1, y\}\{0, z\}, \{1, z\}$ and $\{y, z\}$ do not all have distances that are quadratic residues. Hence, it suffices to check that no y, z exist ($y \neq z$ and not 0 or 1) so that $y, y-1, z, z-1$, and $z-y$ are all (non-zero) quadratic residues. In $\mathbb{Z}_{17}$, the quadratic residues are 1, 2, 4, 8, 9, 13, 15, 16. Only the pairs $(1, 2)$, $(8, 9)$, and $(15, 16)$ are consecutive, so assume that $\{y, z\} \subseteq \{2, 9, 16\}$; then $y - z \in \{7, -7, 3, -3\}$, none of which are quadratic residues. □

Solution to Exercise 401: One of the easiest ways to see this is to use a "graph product", as was shown by Lefmann [621]. For positive integers m, k, ℓ let $n_k = R(m; k) - 1$ and let $n_\ell = R(m; \ell) - 1$. Let $V = \{(x, y) : x \in [n_k], y \in [n_\ell]\}$.

By the definition of n_k, let $\Delta_1 : [n_k]^2 \to [k]$ be given so that there is no monochromatic K_m. Similarly let $\Delta_2 : [n_\ell]^2 \to [k+1, \ell]$ be so that no monochromatic K_ℓ exists. Define $\Delta : [V]^2 \to [k + \ell]$ by, for $(u, v) \neq (w, x)$,

$$\Delta(\{(u, v), (w, x)\}) = \begin{cases} \Delta_2(\{v, x\}) & \text{if } u = w \\ \Delta_1(\{u, w\}) & \text{if } u \neq w. \end{cases}$$

It remains to check that there is no monochromatic K_m. □

Note: Recall that there exists a constant c so that for every $m \geq 3$, $R(m; 2) > c2^{m/2}$, and in particular, $R(m; 2) > 2^{m/2}$; using this last inequality as a base step, an inductive proof (using the result above), Lefmann observes that for $m \geq 3$, $R(m; 2k) > 2^{mk/2}$.

Solution outline to Exercise 402: (This solution outline is from [415].) Let G be any graph on $n = k + \ell$ vertices and let $x \in V(G)$. Consider pairs of paths $Y \subseteq G$ and $Z \subseteq \overline{G}$ so that so that the only common vertex in Y and Z is x. The maximum number (taken over all x and all such pairs of paths) of vertices used in Y and Z is n. □

Solution to Exercise 403: This result was observed by Chvátal and Harary in [219] in 1972.

Case 1: Suppose that m is odd.
Since each vertex in K_{2m} has degree $2m-1$, any two-colouring of the edges in K_{2m} yields (by the pigeonhole principle) at least one vertex (in fact, every vertex) with at least m incident edges in the same colour, so $R(K_{1,m} \leq 2m$. To show that $R(K_{1,m}) > 2m - 1$,

it suffices to create an $(m-1)$-regular graph G on $2m-1$ vertices (since $\overline{G}$ is also $(m-1)$-regular, the colouring "edge vs non-edge" is a two-colouring of $E(K_{2m-1})$ with no monochromatic $K_{1,m}$). One way to construct G on vertices is given in Theorem 1.8.7.

Case 2: Suppose that m is even. Since $m-1$ and $2m-1$ are both odd, by Theorem 1.8.7, there is no $(m-1)$-regular graph on $2m-1$ vertices. Hence, for any subgraph G of K_{2m-1}, either G or $\overline{G}$ contains a vertex of degree at least m, and so $R(K_{1,m}) \leq 2m-1$. To see that $R(K_{1,m}) > 2m-2$, use Theorem 1.8.7 to find a $(m-1)$-regular graph on $2m-2$ vertices. □

Comment on Exercise 404: See [554].

Solution to Exercise 405: This result was given in 1972 by Chvátal and Harary [219].
Claim: $K_6 \to (C_4, C_4)$.
Proof of Claim: Let K_6 have vertices $\{u, v, w, x, y, z\}$, and let a colouring $c : E(K_6) \to \{\text{red}, \text{blue}\}$ be given. By Theorem 9.2.3, $R(3,3) = 6$, and so there exists a monochromatic triangle T; suppose that T has all red edges and is on vertices u, v, w. If any of x, y, z has two red edges to T, then a red C_4 is formed. So assume that each of x, y, z has at most one red edge to T, that is, each of x, y, z send at least two blue edges to T. If, say, x and y have two blue edges to the same two vertices of T, then a blue C_4 is found; the same is true for the pair x, z and y, z.

So the final case to consider is when all edges of T are red, and each of x, y, z has a pair of blue edges to each of a different pair of vertices in $\{u, v, w\}$ (and a red edge to the remaining one). Without loss of generality, suppose that the edges ux, vy, and wz are red, and uy, uz, vx, vz, wx, wy are blue. If any of xy, yz, xz are red, a red C_4 is formed (*e.g.*, if xy is red, then x, y, v, u is a red C_4). So assume that all edges between x, y, z are blue. In this case, for example, the vertices x, v, z, y is form a blue C_4. This ends the proof of the claim.

Now consider K_5; the colouring in Figure 9.2 that avoids a monochromatic triple also prevents a monochromatic C_4. So $K_5 \not\to (C_4.C_4)$. Thus, $R(C_4, C_4) = 6$. □

Solution to Exercise 406: ($R(sK_2, K_p) = 2s + p - 2$.) [I have not found the original source of this result, but it can be found in, *e.g.*, [125, Thm. 10, p. 192].]

The graph consisting of a K_{2s-1} with $p-2$ isolated vertices has no copy of sK_2 and its complement contains no K_p, so $R(sK_2, K_p) \geq 2s + p - 3$.

If G is a graph on $2s + p - 2$ with no copy of sK_2, then G contains at most $s-1$ independent edges using at most $2(s-1)$ vertices, and the remaining vertices, of which there are at least $2s + p - 2 - 2(s-1) = p$, are independent, and so in $\overline{G}$ form a K_p. Thus, $R(sK_2, K_p) \leq 2s + p - 3$. □

Solution to Exercise 407: Claim: $R(K_3, C_5) = 9$.

Let $\Delta : E(K_9) \to \{\text{red, blue}\}$ be given. Suppose that no red K_3 exists. Then since $R(3,4) = 9$, there exists a copy of K_4, all of whose edges are blue. Let H be the subgraph formed by the remaining five vertices. If any vertex in H has two blue edges to the blue K_4, then a blue C_5 is found, so assume that each vertex in H has at most one blue edge to the K_4. In this case, any two vertices in H share a common red-edged neighbour (actually, two) in the K_4. But H has no red K_3, so all edges in H are blue, in which case H trivially contains a blue C_5. Thus, $R(K_3, C_5) \le 9$.

To show that $R(K_3, C_5) > 8$, consider the graph $K_{4,4}$, which is K_3-free, and its complement is the disjoint union of two K_4s, and so does not contain a C_5. □

Solution to Exercise 408: (Find $\mathrm{ex}(n; 2K_2)$ and $\mathrm{EX}(n; 2K_2)$.) TBW

Solution to Exercise 409: The graph formed by adding one pendant edge to K_{n-1} has n vertices, $\binom{n-1}{2} + 1$ edges, and no Hamiltonian cycle, so $\mathrm{ex}(n; C_n) \ge \binom{n-1}{2} + 1$.

To show $\mathrm{ex}(n; C_n) \le \binom{n-1}{2} + 1$, it suffices to show that any graph on n vertices and $\binom{n-1}{2} + 2$ edges contains a C_n. This was shown in Exercise 120 by a direct application of Ore's theorem (Theorem 2.3.3). □

Solution to Exercise 410: Let G be a graph on n vertices and m edges. If $m > \lfloor n/2$, then $\sum_{v \in V(G)} \deg(v) = 2m > n$, so by the PHP, there exists a vertex of degree at least 2, and this vertex is then the center of a P_2. □

Solution to Exercise 411: (Erdős–Gallai extremal number for paths)—to be written.

Solution to Exercise 415: This exercise appeared in [286, 8.23, p. 208]. Let $S(n)$ be the statement that if G is a graph on n vertices with no tetrahedron, then G contains at most $n^2/3$ edges.

Base case: Since a tetrahedron contains 4 vertices, the statement is vacuously true for $n = 1, 2, 3$.

Inductive step $S(k) \to S(k+3)$: Suppose for some $k \ge 3$, $S(k)$ is true. Consider a graph G with $k+3$ vertices. If G does not contain a triangle, then it contains no tetrahedron, and by Mantel's theorem (Theorem 10.3.1), G contains at most $(k+3)^2/4 < (k+3)^2/3$ edges. So suppose that three vertices in G form a triangle. These three cannot be connected to another common neighbour, so there are at most $2k$ additional edges to the remaining vertices. Since the remainder of the graph contains no tetrahedron, by $S(k)$ that part of the graph contains at most $k^2/3$ edges. In all, the maximum number of edges is $k^2/3 + 2k + 3 = (k+3)^2/3$, completing the proof of $S(k+3)$.

By mathematical induction (actually, by three inductive proofs rolled into one), for all $n \ge 1$, the statement $S(n)$ is true. □

Solution outline to Exercise 416: The solution outlined here is a paraphrasing of that in [640, 10.1]. (See also [125, 6, p 136] for this exercise.) If the graph is disconnected or has a cut-vertex, apply induction. Otherwise, use the fact that in 2-connected graphs with more than n edges, there are at least two cycles and every edge is on a cycle. Essentially, do what is called an "ear-decomposition" on such graphs, showing that once one cycle is identified, another cycle uses at least one edge from the first cycle. Then optimize. □

Comment on Exercise 417: This exercise occurred in [125, 12, p. 136]. Hint: for the first part, use an idea from Exercise 416.

Solution to Exercise 418: The only published proof of Theorem 10.3.4 that I could find occurs in [291] (a paper written mostly in Hebrew), where at the end of the paper, Erdős gave a proof (in English) of Rademacher's theorem, and claimed a slightly stronger result. The proof is by induction (by removing a vertex of minimal degree).

For each $n \geq 3$, let $S(n)$ be the statement that if a graph G on n vertices has more than $\lfloor n^2/4 \rfloor$ edges, then G contains at least $\lfloor n/2 \rfloor$ triangles.

Base step: When $n = 3$, there is only one graph with more than $\lfloor 3^2/4 \rfloor = 2$ edges, namely K_3, which has $1 = \lfloor 3/2 \rfloor$ triangle, so $S(3)$ is true. Just to be sure, check also the next case: When $n = 4$, $n^2/4 = 4$, so $S(4)$ says that any graph on 4 vertices and 5 edges contains at least $\lfloor n/2 \rfloor = 2$ triangles. However, there is only one such graph, namely $K_4 - e$, which has two triangles, so $S(4)$ is true.

Inductive step: Fix $t \geq 3$ and suppose that $S(t)$ is true. Let G be a graph on $t+1$ vertices with more than $\lfloor (t+1)^2/4 \rfloor$ edges. It suffices to show that if G has exactly $\lfloor (t+1)^2/4 \rfloor + 1$ edges, then G contains at least $\lfloor \frac{t+1}{2} \rfloor$ triangles (because adding more edges does not destroy any triangles). Consider two cases: when t is even, and when t is odd. (In each case, the goal is to delete a vertex of smallest degree and apply the induction hypothesis to the remaining graph. Separating the cases of even and odd helps to avoid some floor functions.)

Let t be even. Then

$$|E(G)| = \left\lfloor \frac{(t+1)^2}{4} \right\rfloor + 1 = \frac{t^2+2t}{4} + 1.$$

The average degree of vertices in G is (by the handshaking lemma)

$$\begin{aligned} \frac{2|E(G)|}{t+1} &= \frac{2}{t+1}\left[\frac{t^2+2t}{4} + 1\right] \\ &= \frac{t}{2} + \frac{t+4}{2t+2} \\ &< \frac{t}{2} + 1 \qquad\qquad \text{(since } t \geq 4\text{)}. \end{aligned}$$

Hence there exists a vertex $v \in V(G)$ so that $\deg(v) \leq \frac{t}{2}$. Delete v (and all incident edges) to produce the graph H on t vertices. Then

$$|E(H)| = |E(G)| - \deg(v) \geq \frac{t^2+2t}{4} + 1 - \frac{t}{2} = \frac{t^2}{4} + 1,$$

and so by $S(t)$, H contains at least $\lfloor \frac{t}{2} \rfloor = \frac{t}{2}$ triangles. But $\frac{t}{2} = \lfloor \frac{t+1}{2} \rfloor$, so H contains the desired number of triangles, and thus so does G.

Let $t \geq 3$ be odd. Then

$$|E(G)| = \left\lfloor \frac{(t+1)^2}{4} \right\rfloor + 1 = \frac{(t+1)^2}{4} + 1.$$

The average degree is

$$\frac{2}{t+1}\left[\frac{(t+1)^2}{4} + 1\right] = \frac{t+1}{2} + \frac{2}{t+1}.$$

So there exists a vertex with degree at most $\frac{t+1}{2}$.

The proof now splits into two cases. First, suppose that there is a vertex w with $\deg(w) < \frac{t+1}{2}$. Delete w and let H be the remaining graph. Then

$$\begin{aligned} |E(H)| &= |E(G)| - \deg(w) \\ &> \frac{(t+1)^2}{4} + 1 - \frac{t+1}{2} \\ &= \frac{t^2-1}{4} + 1 \\ &= \left\lfloor \frac{t^2}{4} \right\rfloor + 1 \qquad\qquad \text{(since } t \text{ is odd)}, \end{aligned}$$

and so $|E(H)| \geq \left\lfloor \frac{t^2}{4} \right\rfloor + 2$. By Mantel's theorem, there exists at least one triangle; delete one edge of this triangle to form a graph H' on t vertices and at least $\lfloor \frac{t^2}{4} \rfloor + 1$ edges. By $S(t)$, H' contains at least $\lfloor t/2 \rfloor$ triangles. Together with the triangle whose edge was deleted, there are at least $\lfloor t/2 \rfloor + 1 = \lfloor (t+1)/2 \rfloor$ triangles, as required.

So now suppose that no vertex has degree smaller than $\frac{t+1}{2}$. Adding all the degrees gives at least $(t+1)\frac{t+1}{2}$, and since $2|E(G)| = \frac{(t+1)^2}{2} + 2$, there are at most two vertices of degree greater than $\frac{t+1}{2}$ and (by Mantel's theorem) there is at least one triangle, and on that triangle, at least one vertex, say y, has degree $\frac{t+1}{2}$. Delete y, giving leaving a graph with (as above) with $\frac{t^2}{4}+1$ edges. By $S(t)$, there are at least $\lfloor t/2 \rfloor$ triangles, and together with the one (or more) destroyed by removing y, gives $\lfloor t/2 \rfloor + 1 = \lfloor (t+1)/2 \rfloor$ triangles. This completes the inductive step when t is odd, and so the inductive step in general.

By mathematical induction, for each $n \geq 3$, the statement $S(n)$ is true, finishing the proof of the first part of Exercise 418.

For the second part, consider a copy of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Adding an edge to one side (the largest side if they differ by 1) creates exactly $\lfloor n/2 \rfloor$ triangles. With a bit of work, one can prove that this example is unique. □

Solution to Exercise 419:

An example with 9 vertices a,b,c,d,e,f,g,h,i with only 12 triangular cruises is given by edges ab, ac, ad, ae, af, be, bg, bi, cd, ce, ch, ci, dg, dh, ef, fh, fi, gh, gi. Triples abe, ace, bgi, cdh, dgh are triangles in G and triples afg, bcf, bcg, bfg, cfg, deg, dei, are triangles in $\overline{G}$. (This example can be found in [269].) □

Solution to Exercise 420: This exercise (without solution) occurs in [429, p. 320, Q. 15], and the result was generalized to an arbitrary number of triangles (the friendship graph) in [315].

Let $S(n)$ be the statement "if a simple graph G on $n \geq 5$ vertices has more than $\frac{n^2}{4} + 1$ edges, then G contains a bowtie (as a weak subgraph)".

Base step: For $n = 5$, $\frac{n^2}{4} + 1 = 7.25$, so it suffices to show that a graph on 5 vertices and 8 edges contains a bowtie. (If such a graph contains a bowtie, then any graph with 5 vertices and 9 or 10 edges does, too.) It is not difficult to verify that there are only two such graphs, each formed by removing two edges from K_5. One has degrees 4,4,3,3,2, and the other has degrees 4,3,3,3,3. Each has a bowtie, where the center of the bowtie is a vertex of degree 4.

Inductive step: Let $p \geq 5$ and suppose that $S(p)$ holds, that is, any graph with p vertices and more than $\frac{p^2}{4} + 1$ edges contains a bowtie. Now let G be a graph with $p+1$ vertices and q edges, where q is the smallest number larger than $\frac{(p+1)^2}{4} + 1$. Now divide the proof that G contains a bowtie into two cases. [Recall that $G \backslash x$ denotes a graph G with vertex x (and all edges incident with x) deleted.]

First consider the case when p is odd, say $p = 2m + 1$ (and so $m \geq 2$). Then

$$\frac{(p+1)^2}{4} + 1 = \frac{(2m+2)^2}{4} + 1 = m^2 + 2m + 2,$$

so $q = m^2 + 2m + 3$. The average degree of a vertex in G is

$$\frac{2(m^2 + 2m + 3)}{2m + 2} = m + 1 + \frac{4}{2m + 2} < m + 2,$$

and so there is a vertex $x \in V(G)$ with $\deg_G(x) \leq m + 1$. Deleting x leaves

$$\begin{aligned} |E(G \backslash x)| &\geq m^2 + 2m + 3 - (m + 1) \\ &= m^2 + m + 2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{4m^2+4m+8}{4} \\
&= \frac{(2m+1)^2+7}{4} \\
&> \frac{p^2}{4}+1
\end{aligned}
$$

edges, so by $S(p)$, $G\backslash x$ contains a bowtie, and hence so does G.

Next consider when p is even, say $p = 2m$. Since $p \geq 5$, it follows that $m \geq 3$. Then

$$\frac{(p+1)^2}{4}+1 = \frac{(2m+1)^2}{4}+1 = m^2+m+\frac{5}{4},$$

so $q = m^2+m+2$. The average degree of a vertex in G is

$$\frac{2(m^2+m+2)}{2m+1} = m + \frac{m+4}{2m+1},$$

and so there is a vertex $x \in V(G)$ with $\deg_G(x) \leq m + \frac{m+4}{2m+1}$. Now there is a bit of a problem. In general, $\frac{m+4}{2m+1}$ might be as large as 1 when $m = 3$, but deleting a vertex of degree $m+1$ leaves $m^2+1 = \frac{p^2}{4}+1$ edges, not quite enough to use the inductive hypothesis $S(p)$. It turns out, however, that when $m > 3$, this trick works, so take care of this first, then return to $m = 3$. Suppose for now that $m > 3$. Then $\frac{m+4}{2m+1} < 1$, making the average degree less than $m+1$, so there is a vertex x with degree at most m. Delete x, leaving $m^2+2 = \frac{p^2}{4}+2$ edges in $G\backslash x$, and so by $S(p)$, the graph $G\backslash x$ contains a bowtie, and hence so does G.

What remains is the annoying case $m = 3$, that is, $p = 7$. In this case $\frac{p^2}{4}+1 = 13.25$, so assume that G has 7 vertices and 14 edges. If some vertex x in G has $\deg_G(x) \leq 3$, then delete x, leaving a graph on 6 vertices and at least $11 = \frac{6^2}{4}+2$ edges, so by $S(6)$, (which follows from the above inductive step where $p = 5$ is odd) G contains a bowtie.

So suppose that every vertex has degree at least 4. Since the sum of the degrees ($\geq 7 \cdot 4$) is exactly twice the number of edges (14), every vertex has precisely degree 4 (that is, G is 4-regular). Thus, it remains to show that any 4-regular graph on 7 vertices contains a bowtie. There are many ways to show this. For example, by Exercise 49, there are only two 4-regular graphs on 7 vertices (the complement of C_7 and the complement $C_4 \cup C_3$) each of which evidently contain a bowtie. Another method that is only slightly better than an exhaustive analysis, is to delete a vertex, giving a graph on 6 vertices with 10 edges, and vertex degrees 4,4,3,3,3,3. Up to isomorphism, there are only three such graphs (whose complements are $C_4 \cup K_2$, P_5, and $K_3 \cup P_2$, where P_i denotes a path of length i), two of which are easily seen to contain a bowtie. For the remaining one, re-affix the deleted vertex and the bowtie is easily spotted.

Either argument concludes the inductive step where $p = 2m$, and hence the inductive step in general.

By mathematical induction, for each $n \geq 5$, the statement $S(n)$ is true. $\square$

Solution to Exercise 421: [This was proved by Erdős [294], as a consequence of his proof of Lemma 1.] Consider the following triangle-free graph on n (n odd) vertices. Divide the vertices into three groups A, B, C, of order 5, $(n-5)/2$, $(n-5)/2$, respectively. Let vertices of A be $x_1, \ldots, x_5$, and put a cycle of length five on A (with vertices in order). Let B and C induce a complete bipartite graph. Connect all vertices of B to x_1 and x_3; connect all vertices of C to x_2 and x_4. The total number of edges is

$$5 + 2(n-5) + \frac{(n-5)^2}{4} = \frac{(n-1)^2}{4} + 1.$$

$\square$

Note: The above solution is just a Turán graph with one edge subdivided. Is there anything special about 5?

Solution to Exercise 423: Let G be maximal triangle-free. If two non-adjacent vertices x and y do not have a common neighbour, adding the edge $\{x, y\}$ does not produce any triangles, and so G is not maximal. $\square$

Solution to Exercise 424: From Exercise 423, if G is maximal triangle-free, then any two vertices are connected by a path of length at most two, and since $|V(G)| \geq 3$, G is not a complete graph, so $\mathrm{diam}(G) = 2$. Now let G be triangle-free with $\mathrm{diam}(G) = 2$. Adding any missing edge produces a triangle, and so G is maximal. $\square$

Comment on Exercise 426: See [142, p. 111].

Solution outline to Exercise 427: To prove that for some constant $c = c(t)$, $\mathrm{ex}(n; K_{t,t}) < cn^{2-1/t}$, let G be a graph on n vertices and $cn^{2-1/t}$ edges (where c is yet to be determined). The average degree of a vertex in G is $2cn^{1-1/t}$. Counting pairs (x, T), where x is a vertex and T is a t-set in $N(x)$,

$$\begin{aligned}
|\{(x,T) : x \in V(G), T \in [N(x)]^t\}| &= \sum_{x \in V(G)} \binom{\deg(x)}{t} \\
&\geq n \binom{2cn^{1-1/t}}{t} && (\tbinom{*}{t} \text{ is convex}) \\
&= (1+o(1))n \frac{(2cn^{1-1/t})^t}{t!} \\
&= (1+o(1))n \frac{(2c)^t n^{t-1}}{t!} \\
&> (1+o(1)) \binom{n}{t} (2c)^t.
\end{aligned}$$

To complete the proof, $(2c)^t > t-1$ suffices, and so $c > \frac{1}{2}(t-1)^{1/t}$. $\square$

Solution to Exercise 429: By Turán's theorem, the maximum number of edges is $t(10;3)$, which is $\binom{10}{2} - \binom{3}{2} - \binom{3}{2} - \binom{4}{2} = 33$. $\square$

Solution to Exercise 430: I first found this exercise in [546]. Before discussing a (somewhat direct) solution, it might be interesting to know how the Erdős–Gallai paper [318] showed this result. Another reason I found this paper interesting is that it uses completely different notation for graph parameters. The parameters used in the paper are: $\pi(G)$ is the number of vertices in G; $\nu(G)$ is the number of edges; $\mu(G)$ is $\tau(G)$ here; $\overline{\mu}(G)$ is the independence number $\alpha(G)$ here; $\varepsilon(G)$ is the matching number ν. **In their notation**, Theorem 1.7 says that if G has at least one edge, then

$$\mu(G) \leq \frac{2\nu(G)\cdot\pi(G)}{2\nu(G)+\pi(G)},$$

and equality holds if and only if either G is complete or is the disjoint union of complete graphs all with the same number of vertices. Translated into present notation, this says that if G is a graph on n vertices with $m \geq 1$ edges, then

$$\tau \leq \frac{2mn}{2m+n}.$$

How does this show the desired result about α? Using Lemma 5.5.3, $\alpha + \tau = n$, and so the above becomes (with a bit of simplification)

$$\alpha \geq \frac{n^2}{2m+n}.$$

Thus, when $m \leq nk/2$, this says

$$\alpha \geq \frac{n}{k+1},$$

as desired.

Why is this result considered as a "complement" or "dual" version of Turán's theorem? By the calculations outlined in Lemma 10.5.1, if $k+1$ divides n, then

$$\begin{aligned} t\left(n;\frac{n}{k+1}\right) &= \binom{n}{2} - \frac{n}{k+1}\binom{k+1}{2} \\ &= \binom{n}{2} - \frac{n}{k+1}\frac{(k+1)k}{2} \\ &= \binom{n}{2} - \frac{nk}{2}. \end{aligned}$$

So if G has at most $nk/2$ edges, $\overline{G}$ has at least $\binom{n}{2} - \frac{nk}{2} = t(n;\frac{n}{k+1}) > t(n;\frac{n}{k+1}-1)$ edges, and so by Turán's theorem, $\overline{G}$ contains a $K_{\frac{n}{k+1}}$. Thus G contains an independent

set with more than $\frac{n}{k+1} - 1$ vertices, as desired. The calculations when $k+1$ does not divide n are similar (but with more details). Note: Another proof might be easier arguing by contradiction, in which case the conditions regarding divisibility might be avoided. (However, I have not yet worked out the details. I seem to remember finding a far easier proof, but it doesn't come to mind right now.) □

Outline of solution to Exercise 431: This exercise appears in [125, 57, p. 141], where it is marked as "easy". Let G satisfy the assumptions, and let $I \subset V(G)$ be a largest independent set. Put $X = V(G) \backslash I$. Since $|I| \leq n - k - 1$, then $|X| \geq k + 1$. From every vertex in X, there is at least one edge to I (otherwise, one could make a larger independent set), and so $|E(G)| \geq k+1$. If $|X| \geq k+2$, then $|E(G)| \geq k+2$ as desired, so assume that $|X| = k+1$. However, since $\nu(G) \leq k$, the edges from X to I can not form a matching. Thus, there exist two vertices $x, y \in X$ adjacent to the same point, say $z \in I$. If $|E(G)| = k+1$, then there are no additional edges in X from x or y, so in the graph induced by X, the points x and y are independent, and so replacing z with x and y gives a larger independent set, contrary to the assumption that I was maximal. Thus, G contains at least one more edge (inside of X), and so at least $k+2$ in all. □

Solution to Exercise 432: This result was observed by Erdős [289] in 1946; also see [142, p. 112]. Suppose that n points are in the plane. Form the unit distance graph G where the points are the vertices and two vertices are adjacent if and only if the two corresponding points are at distance 1.

Claim: G has no copy of $K_{2,3}$. Proof of claim: If there is a copy of $K_{2,3}$ with partite sets $\{a, b\}$ and $\{x, y, z\}$, each of x, y, z is at distance 1 from both a and b; but in the plane, there are at most two points at the same distance from any two points (both on the perpendicular bisector of a and b).

So by Exercise 426 (with $t = 3$), the solution follows. □

Solution to Exercise 433: See [142, p. 115]. ...to be written.

Solution to Exercise 434: This exercise was adapted from West's textbook [977, p. 217]. In that exercise, it mistakenly stated that the diameter, not the radius, is 4 miles. ... to be written.

Solution to Exercise 435: This standard exercise appeared in, *e.g.*, [429, Prob. 10.3.7, p. 314]. ...to be written. Hint: show that the ratio above decreases as $n \to \infty$.

Solution to Exercise 436: to be written

Comment on Exercise 437: See [342].

Solution to Exercise 439: Let $k \geq 2$ be given. Let $G \in G(n, p)$ be a random graph on n vertices, where the probability of any pair of vertices forming an edge is p. Let

$X = X(G)$ be the number of copies of C_{2k} in G. For any one set of $2k$ vertices, there are $(2k)!$ oriented cycles starting at any one the $2k$ vertices, so there are $\frac{(2k)!}{2\cdot 2k}$ undirected cycles possible on such a set. Then the expected number of copies of C_{2k} is

$$\begin{aligned}\mathbb{E}[X] &= \binom{n}{2k}\frac{(2k)!}{2\cdot 2k}!p^{2k}\\ &= \frac{n!}{(n-2k)!}\frac{1}{4k}p^{2k}\\ &\le \frac{1}{4k}n^{2k}p^{2k}.\end{aligned}$$

To create a C_{2k}-free graph, delete an edge from each cycle in G to create G^*. In order for there to be many edges left in G^*, it suffices to choose p so that $\mathbb{E}[X]$ is at most half of the expected number of edges in G, that is, so that

$$\frac{1}{4k}n^{2k}p^{2k} \le \frac{1}{2}\binom{n}{2}p.$$

A short calculation shows that $p \sim k^{\frac{1}{2k-1}}n^{-1+\frac{1}{2k-1}}$ satisfies the need, in which case G^* has at least $c_k n^{1+\frac{1}{2k-1}}$ edges remaining, as desired. It only remains to check that in $G(n,p)$ there is a graph G that satisfies both $X(G) \le \frac{1}{4k}n^{2k}p^{2k}$ and $|E(G)| \ge \binom{n}{2}p$ simultaneously; this can be shown explicitly using Markov's inequality. □

Solution to Exercise 442: Let $G = (V, E)$ is a graph and $I \subset V$ be a maximum independent set (with $|I| = \alpha(G)$). If $x \in V(G)$ is not dominated by any vertex in I, then $I \cup \{x\}$ is an independent set, contradicting that I is maximal, so all vertices in $V \backslash I$ are dominated, Thus $\mathrm{dom}(G) \le \alpha(G)$. □

Solution to Exercise 443: For any $n \ge 1$, let P_n be a path on $n+1$ vertices $v_0, v_1, \ldots, v_n$ (in order according to the path). Vertex v_1 dominates both v_0 and v_2, vertex v_4 dominates both v_3 and v_5, vertex v_7 dominates v_6 and v_8. Continuing in the manner, the vertices $D - \{v_1, v_4, v_7, \ldots\}$ form a dominating set. Since each vertex in P_n dominates at most two others, the dominating set D is minimal, and so $\mathrm{dom}(P_n) = |D| = \lceil \frac{n+1}{3} \rceil$. □

Solution to Exercise 444: Let $n \ge 3$, and let $x_0, \ldots, x_{n-1}$ be the vertices of C_n given in the cyclic order. For each i, the vertex x_i dominates both x_{i-1} and x_{i+1} (where indices are modulo n). So any three consecutive vertices have the two outside vertices dominated by the center vertex, and so $\mathrm{dom}(C_n) = \lceil n/3 \rceil$. □

Solution to Exercise 445: Consider a path of length 3 (on 4 vertices). No single vertex is a dominating set, but the two inner vertices (which are joined by an edge) form a dominating set. (This example is but one of many—for example, a "double-star"

formed by attaching edges to either end of a given edge works. Two disjoint complete graphs joined by an edge is another example.) □

Solution to Exercise 446: Let G be a connected graph with spanning tree T. Let S be the set of vertices of T that are not leaves. Then each leaf is dominated by some element of S (namely, by the leaf's neighbour in T), and since $V(G)$ is the union of S and the leaves of T, all vertices are dominated by S. □

Solution outline to Exercise 447: Suppose that G is an r-regular graph with girth g. In hope of contradiction, x and y be vertices with distance greater than g. Then the neighbourhoods $N(x)$ and $N(y)$ are disjoint r-sets. Create a new graph G' by removing x and y, and adding r disjoint edges between $N(x)$ and $N(y)$. Then G' has fewer vertices, and is still r-regular with girth g. Continue for every such pair of vertices with large distance, eventually producing an r-regular graph with girth g, but with smaller order. So the original G was not a cage. □

Solution to Exercise 449: to be written.

Solution to Exercise 450: The proof duplicates the undirected version in Lemma 2.1.2.

Let $D = (V, A)$ be a digraph where for each $x \in V$, the outdegree is $d^+(x) \geq 1$. Let $P = x_1x_2 \cdots x_p$ be a maximal directed path from x_1 to x_p. Since $d^+(x_p) \geq 1$, x_p has an outneighbour, but since P is maximal, such an outneighbour is on P, say x_j. Then $x_jx_{j+1} \cdots x_px_j$ is a cycle. □

Solution outline to Exercise 451: For each vertex x, let $f(x)$ be the maximum length of a path starting at x. Then f is a k-colouring. It remains to show that f is a proper colouring. □

Solution to Exercise 452: This problem can be found in [265, p. 345]; this problem was shortlisted for the 2009 IMO, attributed to Ross Atkins (Australia). The key to this problem is to rewrite the condition $x(y-1) \equiv 0$ as $xy \equiv y$. (All congruences here are modulo n.)

For some $k \geq 2$, suppose that $x_1, \ldots, x_k, x_1$ form a cycle. In the case of $k = 2$, $(x_1, x_2) \in E$ gives $x_1x_2 \equiv x_1$, and $(x_2, x_1) \in E$ gives $x_2x_1 \equiv x_2$. Hence $x_1 \equiv x_2$, and since $x_1, x_2 \in [n]$, $x_1 = x_2$, contradicting the definition of an edge. When $k \geq 2$ a similar idea works. Let $x_1, x_2, \ldots, x_k$ be a path in D. Then

$$\begin{aligned} x_1x_2 \cdots x_k &= x_1x_2 \cdots (x_{k-1}x_k) \\ &\equiv x_1x_2 \cdots x_{k-1} \\ &= x_1x_2 \cdots (x_{k-2}x_{k-1}) \\ &\equiv x_1x_2 \cdots x_{k-2} \end{aligned}$$

$$\begin{aligned} &\vdots && \text{(inductively)} \\ &= x_1. \end{aligned}$$

If $x_1, x_2, \ldots, x_k$ is a cycle, then $x_2, x_3, \cdots x_k, x_1$ is also a path, and so just as in the above sequence, $x_2 x_3 \cdots x_k x_1 \equiv x_2$, so $x_1 = x_2$, contrary to the definition of an edge. □

Hint for Exercise 455: If D is a digraph with minimum outdegree at least 1, then D contains a directed cycle (see Exercise 450); then use strong induction on the number of directed edges.

Solution to Exercise 456: Consider the Eulerian circuit in the de Bruijn graph for 2-bit words given by

$$00 - 00 - 01 - 11 - 11 - 10 - 01 - 10 - 00.$$

The corresponding labels give the sequence 01110100 (or if one starts with 00, get the sequence 00111010, which is the same up to circular ordering). □

Solution to Exercise 457: Let $T = (V, D)$ be a tournament. If T has only one or two vertices, then trivially T is both acyclic and transitive, so suppose that T has at least 3 vertices.

Let T be transitive. If T has a cycle, then by the result in Exercise 461, T has a 3-cycle, say $(a, b), (b, c), (c, a)$, and since T is a tournament, $(a, c) \notin D$, which violates transitivity. Hence T is acyclic.

On the other hand, suppose that T is acyclic, and let $(a, b), (b, c) \in D$. Since T is acyclic, $(c, a) \notin D$, and since T is a tournament, $(a, c) \in D$, showing that T is transitive. □

Solution to Exercise 458: The first of the tournaments is not strongly connected since there is no path from the upper right vertex to any other. (Also in the first tournament, the lower left vertex is not reachable from any other.) Similarly the third tournament is also not strongly connected since from the upper right vertex there is no path to any other vertex. The second tournament is also not strongly connected since the lower left vertex is not reachable from any other. The last tournament is indeed strongly connected, which can be checked with only a couple of cases. □

Solution to Exercise 459: No, it is not true. Consider the directed 3-cycle, with arcs (x, y), (y, z), (z, x). Removal of any one of the three vertices leaves a trivial tournament on two vertices (i.e., one directed edge), which is not strongly connected. Larger examples are also easily found: let T be a transitive tournament on vertices $x_1, x_2, \ldots, x_n$ so that for each $i < j$, (x_i, x_j) is an arc. T is not strongly connected. Now add one vertex, call it y, so that (x_n, y) and (y, x_1) are arcs. The new tournament is now strongly connected, but deletion of y produces T, which is not strongly connected. □

Solution to Exercise 460: (Rédei's theorem) For each integer $n > 1$, let $S(n)$ be the statement that if a tournament is held among n players, where every pair of players meets precisely once, then there is a listing of all the players $a, b, c, \ldots$, so that a beat b, b beat c, and so on, continuing until the last player. Call such a listing a *ranking*. Use strong induction on n.

Base step: When $n = 2$, there is only one match, and so simply list the winner first and the loser second.

Inductive step: Let $k \geq 2$, and suppose that for each $i = 2, \ldots, k$, $S(i)$ is true, and that a tournament T has been held with $k+1$ players. Choose a player p, and consider the two groups, A made of players who beat p, and B, those whom p beat. [Think: A=above, B=below.]

Since $|A| \leq k$, by induction hypothesis $S(|A|)$, a ranking of players in A exists (or A has at most one vertex); similarly a ranking for B exists. Since p was beaten by any member of A, in particular, by the last member of A, and p beat every member of B, in particular, the first member of B, the new listing formed by concatenating the ranking of A, followed by p, then followed by the ranking of B, is a ranking of the $k+1$ players as desired. So $S(k+1)$ is true, completing the inductive step.

By strong mathematical induction, for each $n \geq 2$, any tournament with n players has a ranking. □

Solution to Exercise 461: For $n \geq 3$, let $A(n)$ denote the assertion "if a tournament T has a directed cycle on n vertices, then T contains a directed cycle on three vertices."

Base step: When $n = 3$, there is nothing to prove.

Inductive step: Fix $k \geq 4$, and assume $A(k-1)$ holds, that is, if T is a tournament with a cycle on $k-1$ vertices, then T contains a directed cycle on 3 vertices.

Let $T = (V, D)$ be a tournament containing a cycle on k vertices, say x_0, x_1, $\ldots$, $x_{k-1} \in V$, where for each $i = 0, 1, \ldots, k-1$, $(x_i, x_{i+1}) \in D$ (addition in indices is done modulo k). If for any i, $(x_{i+2}, x_i) \in D$, then x_i, x_{i+1}, x_{i+2} is a directed triangle. So suppose that all such pairs (two apart on the cycle) are directed in the same direction as the cycle, *i.e.*, $(x_i, x_{i+2}) \in D$. Then, *e.g.*, the vertices $x_0, x_2, x_3, \ldots, x_{k-1}$ form a directed cycle with $k-1$ vertices. Thus, by $A(k-1)$, T contains a directed triangle, and so $A(k)$ is true.

By mathematical induction, for each $n \geq 3$, $A(n)$ is true. □

Solution to Exercise 462: If $\text{out}(a) = \{x_1, \ldots, x_s\}$ and $\text{out}(b) = \{y_1, \ldots, y_t\}$, where $s < t$, then there exists $y \in \{y_1, \ldots, y_t\} \backslash \{x_1, \ldots, x_s\}$. In this case, $y \in \text{in}(a)$ and $y \in \text{out}(b)$, so a, b, y forms a cycle. □

Solution to Exercise 463: There are at least two ways to prove this—directly, or by proving the contrapositive. To see the direct proof, let T be a tournament, and

let $x, y \in V(T)$ be vertices not on the same cycle. Then x and y are not in the same strongly connected component since, by Camion's theorem (Theorem 12.7.8), strongly connected components are Hamiltonian.

Between any two strongly connected components of T, all arcs go in one direction (since otherwise, the two components also form a strongly connected sub-tournament). Thus, the strongly connected components form the vertices of a tournament, and so by Redei's theorem, there is an ordering of the strongly connected tournaments, say $S_1, S_2, \ldots, S_k$, so that for $i < j$, all arcs go from S_i to S_j. Without loss of generality, let $x \in S_i$ and $y \in S_j$, where $i < j$. Then $N^+(x) \supseteq (\{y\} \cup N^+(y))$, and so $d^+(x) > d^+(y)$. □

Comment on Exercise 464: This exercise appeared in [977, 1.4.35, p. 65], marked as a harder problem. No solution is provided here.

Solution to Exercise 465: Let T be a tournament on n vertices, each with outdegree k. Counting arcs leaving vertices, there are then nk arcs. Since each vertex has indegree $n-1-k$, counting arcs entering vertices, there are $n(n-1-k)$ arcs in total. Thus the total number of arcs is counted in two ways, giving $nk = n(n-k-1)$ and so $k = n-k-1$. Hence, $n = 2k+1$. (Note: one can count only outdegrees giving nk arcs, but the total number of arcs is $\binom{n}{2}$, and setting these values equal gives the same result.) □

Solution to Exercise 466: When $n = 3$, and all outdegrees are 1, then the tournament is a 3-cycle.

When $n = 5$, and all outdegrees are 2, then all indegrees are also 2. Suppose that there exists a 2-regular tournament $T = (V, D)$ on vertices a, b, c, d, e. Starting at a, without loss of generality, suppose $(a, b), (a, c) \in D$ and $(d, a), (e, a) \in D$. Again without loss of generality, let $(b, c), (d, a) \in D$. Then $(c, d), (c, e) \in D$. It then follows that $(e, b), (a, d) \in D$. So up to isomorphism, there is only one regular tournament on 5 vertices. In this tournament, a 3-cycle is given by b, d, e, a 4-cycle by b, c, d, a, and a 5-cycle by a, b, c, d, e. □

Solution to Exercise 467: Let v be a vertex of maximum outdegree in a tournament T. Let $X = \{x \in V(T) : (v, x) \in E(T)$, and $Y = \{y \in V(T) : (y, v) \in E(T)\}$. (Then $X \cap Y = \emptyset$, $V(T) = \{v\} \cup X \cup Y$, and $d^+(v) = |X|$.) In hopes of contradiction, suppose that v is not a king, that is, there exists some $z \in V(T)$ so that there is no path of length of at most two from v to z. Then $z \in Y$, and for every $x \in X$, $(z, x) \in E(T)$. Since $(z, v) \in E(T)$ as well, $d^+(z) = |X| + 1$, contradicting that v has maximum outdegree. □

Solution to Exercise 468: For $n \geq 1$, let $C(n)$ be the claim that any tournament on n vertices contains a king.

Base step: If $n = 1$ or $n = 2$, the claim holds trivially.

Inductive step: Suppose that for some $k \geq 2$, $C(k)$ is true, and let $T = (V, E)$ be a tournament on $k + 1$ vertices, and fix some vertex $x \in V$. The remaining k vertices form a tournament $T' = T \backslash \{x\}$, and so by $C(k)$, T' contains a king w.

Now there are two cases. If $(w, x) \in E(T)$, then w is a king of T as well, as required. If there is a path of length two from w to x, again w is a king, as required.

So suppose that $(x, w) \in E(T)$ and that there is no path of length two from w to x. Thus for every y with $(w, y) \in E$, $(x, y) \in E$ (for if not, (w, y), (y, x) is a path of length two witnessing w as a king). Since all remaining vertices are reachable from w through such y's, so are they similarly reachable from x. Hence x is a king. This concludes the inductive step.

Therefore, by mathematical induction, for any $n \geq 1$, the statement $C(n)$ is true. □

Solution to Exercise 469: In the last tournament in Figure 12.4, the lower right vertex is a king, and this vertex has outdegree 1. □

Solution to Exercise 470: This was shown by Moon [695] in 1962. Virtually the same proof is used also in Exercises 471 and 472. The idea is fairly simple. Suppose that no element is an emperor; i.e., suppose that for each vertex x in T, $\deg^-(x) > 0$. Let $A = N^+(x)$ and $B = N^-(x)$. By Landau's theorem, in the tournament induced by B, there is a king $b \in B$. Then b dominates x and so all of A is reachable by a path of length 2, and so b is also a king in T. □

Solution to Exercise 471: Apply the result in Exercise 470 to a king. □

Solution to Exercise 472: This problem was posed and solved by Silverman [864] and was also solved by Moon [695] in 1962 (and repeated in [785]). Let $T = (V, D)$ be a tournament and suppose that vertices x and y are both kings and no other vertex is a king. Without loss of generality, assume that $(x, y) \in D$. In the tournament induced by $N^-(x)$, there exists (by Landau's theorem, see Exercise 467) a king v. Note that $v \neq y$. Then v is a king for all of T (since every vertex in $N^+(x)$ is reachable from v by a length 2 path through x), producing yet a third king. □

Solution to Exercise 473: By Landau's theorem (see Exercise 467), it suffices to find a tournament on 5 vertices so that each vertex has the same outdegree (which is then 2). Such an example on 5 vertices a, b, c, d, e is given by the two cycles $a \to b \to c \to d \to e \to a$ and $a \to c \to e \to b \to d \to a$. (Following these two cycles is a standard way to give an Eulerian circuit for K_5.) □

Solution to Exercise 474: (EKR with equality) To be written.

Solution to Exercise 475: (Double counting degrees) This lemma occurs in [546] . Let X be a set and let $\mathcal{F}$ be a finite family of subsets of X.

(i) Let N be the number of pairs of the form (x, F), where $x \in F$. For each $x \in X$, there are $\deg(x)$ such pairs, and so $N = \sum_{x \in X} \deg(x)$. On the other hand, since each $F \in \mathcal{F}$ contains $|F|$ elements from X, $N = \sum_{F \in \mathcal{F}} |F|$. Equating the two expressions for N finishes the proof. Note that this proof is, in principle, the same as that for Lemma 1.8.1, except that each F here is an edge with $|F| = 2$.

Another way to see (i) is to count the number of 1s in the incidence matrix for the family $\mathcal{F}$ in two different ways, by row and by column. This method is used for part (ii), of which (i) is a special case.

(ii) Let $m = |X|$ and $n = |\mathcal{F}|$. List $X = \{x_1, \ldots, x_m\}$ and $\mathcal{F} = \{F_1, \ldots, F_n\}$. For these orderings, let $M = (m_{ij})$ be the $n \times m$ incidence matrix for $\mathcal{F}$ defined by $m_{ij} = 1$ if and only if $x_i \in F_j$ (and $m_{ij} = 0$ otherwise). Let $Y \subseteq X$, have $b = |Y|$ elements, and without loss of generality, list $Y = \{x_1, \ldots, x_b\}$. Let N be the number of 1s in the first b rows of M, and count N in two ways. For each $i = 1, \ldots, b$, the number of 1s in row i is the degree of x_i, and so total number of ones in the first b rows is

$$N = \sum_{x_i \in Y} \deg(x).$$

On the other hand, the number of counted 1s in column j is $|F_j \cap Y|$ and so

$$N = \sum_{F_j \in \mathcal{F}} |F_j \cap Y|.$$

Equating the two expressions for N finishes the proof.

(iii) $\displaystyle\sum_{x \in X} (\deg(x))^2 = \sum_{F \in \mathcal{F}} \sum_{x \in F} \deg(x) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$ □

Solution to Exercise 476: Let X be an n-element set and let $\mathcal{F}$ be a Sperner family of subsets of X. By the unimodal property of binomial coefficients, for each $F \in \mathcal{F}$, $\binom{n}{|F|} \leq \binom{n}{\lfloor n/2 \rfloor}$, and so

$$\begin{aligned}
|\mathcal{F}| &= \sum_{F \in \mathcal{F}} 1 \\
&\leq \sum_{F \in \mathcal{F}} \frac{\binom{n}{\lfloor n/2 \rfloor}}{\binom{n}{|F|}} \\
&= \binom{n}{\lfloor n/2 \rfloor} \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}
\end{aligned}$$

$$\leq \binom{n}{\lfloor n/2 \rfloor} \qquad \text{(by LYM).}$$

□

Solution to Exercise 477: Let $n \geq 3$ and let $\text{deck}(K_n) = \{H_i : i = 1, \ldots, n\}$ be the deck of K_n. Then each H_i is a copy of K_{n-1}. Let G be any graph (on n vertices) with this deck. Then by Lemma 14.1.2, $|E(G)| = \frac{1}{n-2}\sum_{i=1}^{n} |E(H_i)| = \frac{1}{n-2} \cdot n\binom{n-1}{2} = \binom{n}{2}$, and so G is complete. □

Solution outline to Exercise 480: There are many possible solutions to this exercise, some of which are given here. Let $V(G) = \{v_1, \ldots, v_5\}$, where each $G_i = G - v_i$. At most 4 edges can be added to G_2, and so G has at most 6 edges. On the other hand, by G_1, G has at least 5 edges. To decide whether the answer is 5 or 6, the formula (14.1) gives

$$|E(G)| = \frac{1}{3}(5 + 2 + 3 + 4 + 4) = 6.$$

The same result is available by a graph theoretic argument as follows. If G were to have only 5 edges, then by G_1, v_1 is isolated (in G), giving $G = G_1 \cup \{v_1\}$. By G_3, the missing vertex is not isolated, a contradiction. So there are 6 edges. Thus v_1 is adjacent to exactly one vertex of $G_1 = K_4 - e$ and so G is connected.

The edge containing v_1 can attached to either a vertex of degree 2 (in G_1) or to a vertex of degree 3. After checking some cases, conclude that G is formed by making v_1 adjacent to precisely one degree 3 vertex in G_1. To verify this answer, one could look up (or find) all six graphs with 5 vertices and 6 edges and check the decks. Another way to verify G is that among these six graphs only one is both connected and contains $G_1 = K_4 - e$, namely G as found above. The energetic reader might also find a labelling of vertices in G so that for each $i = 12, 3, 4, 5$, $G_i = G - v_i$. □

Solution to Exercise 481: (Regular graphs are reconstructible.) Let G be a k-regular graph on n vertices and let $\text{deck}(G) = \{H_i : i = 1, \ldots, n\}$. Suppose that F is another graph on n vertices with $\text{deck}(F) = \text{deck}(G)$. By Theorem 14.1.5, F is k-regular. It remains to show that F is isomorphic to G. If some $H_i = F - x$, then adding a new vertex adjacent to the vertices of degree $k - 1$ in H_i gives G. □

Hints to Exercise 486:
(a) The characteristic polynomial is $x^5 - 6x^3 - 4x^2 + 5x + 4 = (x^2 - x - 4)(x - 1)(x + 1)^2$.
(b) The characteristic polynomial is $x^4 - 5x^2 - 4x = (x^2 - x - 4)(x + 1)x$.
(c) The characteristic polynomial is $x^5 - 5x^3 - 2x^2 + 3x = (x^2 - x - 3)(x^2 + x - 1)x$. □

Solution to Exercise 487: The vector $(1, 1, \ldots, 1)^T$ is an eigenvector associated with eigenvalue $n - 1$. Vectors of the form $(1, 0, 0, \ldots, 0, -1, 0, \ldots, 0)$ are eigenvectors associated with eigenvalue -1 (there are $n - 1$ such vectors). □

Solution to Exercise 488: Suppose that the vertices are ordered so that the center is first; then look for eigenvectors that have 1 in the first entry; let λ be the eigenvalue associated with such an eigenvector. Then other positions are $1/\lambda$; so $\frac{n-1}{\lambda} = \lambda$, so $\lambda = \pm\sqrt{n-1}$. This gives two eigenvalues and two eigenvectors. Now consider eigenvectors with first entry 0; the remaining elements of the vector sum to 0, and so there are $n-2$ linearly independent vectors associated with $\lambda = 0$. □

Hint to Exercise 489: Following a solution similar to that in Exercise 488, $\pm\sqrt{mn}$ are eigenvalues; all other eigenvalues are 0. See [640, 11.1] for a complete proof.

Hint to Exercise 490: Label vertices of C_n in a natural order. Use properties of a circulant matrix; see Definition 17.2.4. Eigenvalues are

$$2, 2\cos(2\pi/n), 2\cos(4\pi/n), \ldots, 2\cos(2(n-1)\pi/n).$$

□

Solution to Exercise 492: (The spectrum of $L(K_n)$.) TBW

Solution to Exercise 493: (The spectrum of a Paley graph.) TBW

Solution to Exercise 495: Let G be the first graph and H be the second, with vertices labelled for convenience by:

Since G has a vertex of degree 5 and H does not, these two graphs are not isomorphic. Using the ordering $v_1, v_2, v_3, v_4, v_5, v_6$, let A be the adjacency matrix for G and using the ordering u, v, w, x, y, z for $V(H)$, let B be the adjacency matrix for H. Then

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial for both is (see [237])

$$x^6 - 7x^3 - 4y^3 + 7x^2 + 4x - 1.$$

□

Hint for Exercise 496: Show that the characteristic polynomial for each is $(x+2)x^3(x-2)$. □

Solution to Exercise 497: The spectrum is: $(-4)^3, (-2)^3, (0)^6, (2)^3, 4$. □

Solution to Exercise 498: Since the diagonal entries of A^3 count the number of closed 3-walks from a given vertex, the trace of A^3 counts the number of directed triangles with a specific starting point; since each triangle has two orientations and 3 starting points, $\mathrm{tr}(A^3)$ is 6 times the number of triangles in G. □

Comment on Exercise 500: See [74, Lemma 3.32].

Comment on Exercise 501: This exercise appeared in [74, Ex. 9, p. 44].

Comment on Exercise 503: See [422, Ch. 7].

Solution to Exercise 504: Let G be k-regular on n vertices with adjacency matrix A.

Let B be the adjacency matrix for $\overline{G}$. Let $\mathbf{j} = [1, 1, \ldots, 1]^T \in M_{n\times 1}$; since G is regular, $A\mathbf{j} = k\mathbf{j}$ and so k is an eigenvalue for G; similarly, since $\overline{G}$ is $n-k-1$ regular, $B\mathbf{j} = (n-k-1)\mathbf{j}$, and so $n-k-1$ is an eigenvalue for $\overline{G}$.

For the other eigenvalues, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n = \mathbf{j}$ be a basis for $\mathbb{R}^n$ consisting of orthogonal eigenvectors for G and for each i, let λ_i be the eigenvalue for $\mathbf{v}_i$.

Letting J_n denote the all 1s $n \times n$ matrix, for $i \neq n$,

$$\begin{aligned} B\mathbf{v}_i &= (J_n - I_n - A)\mathbf{v}_i \\ &= J_n\mathbf{v}_i - \mathbf{v}_i - A\mathbf{v}_i \\ &= -\mathbf{v_i} - \lambda_i\mathbf{v}_i && (\mathbf{v}_i \text{ is orthogonal to } \mathbf{j}) \\ &= (-1-\lambda_i)\mathbf{v}_i. \end{aligned}$$

So, if the eigenvalues of A are $\lambda_1 \leq \cdots \leq \lambda_n = k$, with associated eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n = \mathbf{j}$, then B has eigenvalues $-1-\lambda_1$, $-1-\lambda_2$, ..., $n-1-k$, with associated eigenvectors the same as for A. □

Solution to Exercise 505: See [640, 11.2(b)] for full solution; here is only an outline. Apparently, this result is originally due to Sachs.

Let G be a k-regular graph on n vertices, and m edges. (Do the case $m \geq n$ first.)

Suppose that A is an adjacency matrix for G, and B is an incidence matrix for G (both based on the same vertex order, of course). Let A_L denote an adjacency matrix for $L(G)$. First show that

$$A = BB^T - kI, \text{ and } \ A_L = B^TB - 2I.$$

Then by Theorem 17.3.5 (with the m and n reversed),

$$\begin{aligned}\det(xI - A_L) &= \det((x+2)I - B^T B) \\ &= (x+2)^{m-n}\det((x+2)I - BB^T) \\ &= (x+2)^{m-n}\det((x+2-k)I - A).\end{aligned}$$

So -2 is an eigenvalue for A_L (with multiplicity $m-n$) and and for each eigenvalue of A, $k-2+\lambda$ is an eigenvalue for A_L.

The case when $m < n$ is similar. □

Comment for Exercise 507: If A is an adjacency matrix for the Petersen graph, then $A^2 + A = J_{10} + 2I_{10}$. The eigenvalues are -2, 1, and 3, with multiplicities 4, 5, and 1, respectively.

Solution to Exercise 508: (This problem was shown to me by Michael Doob, September 2013.) First assume that for each odd k, $\mathrm{tr}(A^k) = 0$. Then for each odd k, each element on the diagonal of A^k is zero, and so there are no walks of length k from any vertex back to itself. So for each odd k, there are no k-cycles in G. Hence (by Exercise 13), G is bipartite.

For the other direction, assume that G is bipartite. Every walk from a vertex back to itself contains an even number of edges, so there are no walks of odd length from a vertex back to itself. Thus, for odd k, the main diagonal of A^k is all zeroes, and so $\mathrm{tr}(A^k) = 0$. □

Hint to Exercise 509: A 1-factor corresponds to a non-zero elementary product (see Definition 5.2.1).

Hint to Exercise 510: See, *e.g.*, [640, 11.14c].

Solution to Exercise 511: TBW

Solution to Exercise 512: See [74, ex4, p. 43].

Solution to Exercise 513: [This exercise came from Peter Cameron's notes *Combinatorics 3: Finite geometry and strongly regular graphs*[178], exercise 6.4(a)., p. 62, with solution on p. 105.] The solution given here uses only slightly different notation.

Let $G = L(K_6)$, where K_6 has vertices labelled $1, 2, \ldots, 6$. Then $|V(G)| = |E(K_6)| = \binom{6}{2} = 15$. A typical vertex $\{i, j\}$ in G is adjacent to 8 others, namely four of the form $\{i, y\}$, $y \notin \{i, j\}$, and four of the form $\{j, z\}$, $z \notin \{i, j\}$. So G is regular of degree 8.

Two adjacent vertices in G are of the form $\{i, j\}$ and $\{j, k\}$, where i, j, k are distinct. There are four vertices adjacent to both, namely $\{i, k\}$ and three of the form $\{j, z\}$, $z \notin \{i, j, k\}$.

Two non-adjacent vertices in G are of the form $\{i,j\}$ and $\{k,\ell\}$, where i,j,k,ℓ are distinct. There are four vertices adjacent to both, namely $\{i,k\}$, $\{i,\ell\}$, $\{j,k\}$, and $\{j,\ell\}$.

In conclusion, G is strongly regular with parameters $(15,8,4,4)$. $\square$

Solution to Exercise 514: (Paley graphs are strongly regular.) TBW

Hint to Exercise 515: Pick any vertex v and count the number of induced paths of length 2 having v as an end vertex. $\square$

Solution to Exercise 516: Suppose that G is a strongly regular graph on n vertices with parameters (n,k,a,b). Then $\overline{G}$ is $(n-1-k)$-regular, also on n vertices.

Let $\{x,y\} \in E(\overline{G})$ be adjacent vertices in $\overline{G}$. Then $|N_G(x) \cap N_G(y)| = b$, and since G is k-regular, $|N_G(x)| = k = |N_G(y)|$. Common neighbours of x and y in $\overline{G}$ are those vertices not in $\{x,y\} \cup N_G(x) \cup N_G(y)$. Since G is k-regular, the number vertices adjacent to x but not to y is $k-b$; similarly, the number of vertices adjacent to y but not to x is $k-b$. Thus the number of vertices adjacent in G to either x or y is $2+(k-b)+b+(k-c) = 2+2k-b$, and so the number of common neighbours of x and y in $\overline{G}$ is $n-(2+2k-b) = n-2-2k+b$.

Now suppose that u,v are non-adjacent vertices in $\overline{G}$. Then $|N_G(u) \cap N_G(v)| = a$. Not counting v, the number of vertices adjacent to u is $k-1$, and not counting u, the number vertices adjacent to v is also $k-1$. As in the above case, the number of vertices in $V(G)\backslash\{u,v\}$ that are adjacent in G to at least one of u or v is $(k-1-a)+a+(k-1-a) = 2k-2-a$, and so in $\overline{G}$, the number of common neighbours to u and v is $n-2-(2k-2-a) = n-2k+a$.

Therefore, $\overline{G}$ is strongly regular with parameters $(n, n-2-2k+b, n-2k+a)$. $\square$

Solution to Exercise 517: Since each vertex has degree k, if x and y are not adjacent, then $b = |N(x) \cap N(y)| \leq |N(x)| = k$. To be finished...

Solution to Exercise 518: If x and y are adjacent, there are at most $k-1$ other neighbours of x, and hence at most $k-1$ neighbours of x in $N(x) \cap N(y)$. So $a \leq k-1$.

(i)$\rightarrow$ (ii): Suppose that G is a disjoint union of complete graphs. To be finished...

Solution to Exercise 519: When $n=1$, $L(K_1)$ has no vertices, so let $n \geq 2$ and let $G = L(K_n)$. Then $|V(G)| = |E(K_n)| = \binom{n}{2}$. Each edge in K_n is incident with $n-2$ edges at each end, so G is regular of degree $2n-4$. Let x and y be vertices in G, corresponding to edges e_1 and e_2 in K_n.

If x and y are adjacent in G, then e_1 and e_2 are incident in K_n, say at vertex $v \in V(K_n)$. The only other edges in K_n that are incident with both e_1 and e_2 are either incident with v, in which case there are $n-3$ choices, or the one edge joining the other endpoints of e_1 and e_2. In all, there are $n-4$ common neighbours to x and y.

If x and y are not adjacent in G, then e_1 and e_2 are disjoint in K_n, and so the only edges in K_n incident with both e_1 and e_2 are the four edges from e_1 to e_2. In this case, the joint neighbourhood of x and y has 4 elements.

In summary, G is strongly regular with parameters $(\binom{n}{2}, 2n-4, n-4, 4)$. □

Solution to Exercise 520: Let $n \geq 1$ and let $G = L(K_{n,n})$. Name the vertices of $K_{n,n}$ by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ so that $E(K_{n,n}) = \{\{x_i, y_i\} : i, j \in [n]\}$. Since $K_{n,n}$ has n^2 edges, $|V(G)| = n^2$. Each edge $\{x_i, y_j\}$ is incident to the remaining $n-1$ edges incident with x_i and the $n-1$ other edges incident with y_j, so G is regular of degree $2(n-1)$.

If two edges in $K_{n,n}$ are incident at, say, x_i, then the only other edges that are incident to both are the remaining $n-2$ edges incident with x_i. The same is true for a pair of edges incident at some y_i, and so any pair of adjacent vertices in G have $n-2$ neighbours.

If two edges in $K_{n,n}$ are not incident, say $\{x_i, y_j\}$ and $\{x_k, y_\ell\}$ then there are only two edges, namely $\{x_i, y_\ell\}$ and $\{x_k, y_j\}$ incident with both, so, in G, any two non-adjacent vertices have precisely 2 common neighbours.

In summary, G is strongly regular with parameters $(n^2, 2n-2, n-2, 2)$. □

Solution to Exercise 522: By the result in Exercise 515, $b = \frac{57 \cdot 56}{3192} = 1$. This fact also follows from the fact that in a Moore graph of diameter 2 and girth 5, two non-adjacent vertices have one common neighbour. □

Solution to Exercise 523: Let $M = (m_{i,j})$ be an $n \times n$ matrix with all row-sums and column-sums equal to 0. Let J_n denote that all 1s $n \times n$ matrix. Let $M_{i,j}$ denote the (i,j) minor (matrix) of M and $C_{i,j}$ denote the (i,j)-cofactor of M. The following calculations focus on $M_{1,1}$ and $C_{1,1}$:

$$\det(M + J_n)$$

$$= \begin{vmatrix} m_{1,1}+1 & m_{1,2}+1 & \ldots & m_{1,n}+1 \\ m_{2,1}+1 & m_{2,2}+1 & \ldots & m_{2,n}+1 \\ \vdots & & & \vdots \\ m_{n,1}+1 & m_{n,2}+1 & \ldots & m_{n,n}+1 \end{vmatrix}$$

$$= \begin{vmatrix} n & n & \ldots & n \\ m_{2,1}+1 & m_{2,2}+1 & \ldots & m_{2,n}+1 \\ \vdots & & & \vdots \\ m_{n,1}+1 & m_{n,2}+1 & \ldots & m_{n,n}+1 \end{vmatrix} \qquad \text{(add rows R2–Rn to R1)}$$

$$= n \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ m_{2,1}+1 & m_{2,2}+1 & \dots & m_{2,n}+1 \\ \vdots & & & \vdots \\ m_{n,1}+1 & m_{n,2}+1 & \dots & m_{n,n}+1 \end{vmatrix}$$

$$= n \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ m_{2,1} & m_{2,2} & \dots & m_{2,n} \\ \vdots & & & \vdots \\ m_{n,1} & m_{n,2} & \dots & m_{n,n} \end{vmatrix} \qquad \text{(subtract R1 from rows R2–Rn)}$$

$$= n \cdot \begin{vmatrix} n & 1 & \dots & 1 \\ 0 & m_{2,2} & \dots & m_{2,n} \\ \vdots & & & \vdots \\ 0 & m_{n,2} & \dots & m_{n,n} \end{vmatrix} \qquad \text{(add cols C2–Cn to C1)}$$

$$= n^2 \cdot \begin{vmatrix} m_{2,2} & \dots & m_{2,n} \\ \vdots & & \vdots \\ m_{n,2} & \dots & m_{n,n} \end{vmatrix} \qquad \text{(expanding along C1)}$$

$$= n^2 \cdot \det(M_{1,1}).$$
$$= n^2 C_{1,1}.$$

Similar calculations show that for each i, j, $\det(M + J) = n^2 C_{i,j}$. Hence, all cofactors of M are the same. □

Solution to Exercise 524: As in Figure 20.18, let $V(G) = \{v_1, v_2, v_3, v_4\}$, where v_1, v_2, v_3 form a triangle, and the pendant vertex v_4 is adjacent to v_3.

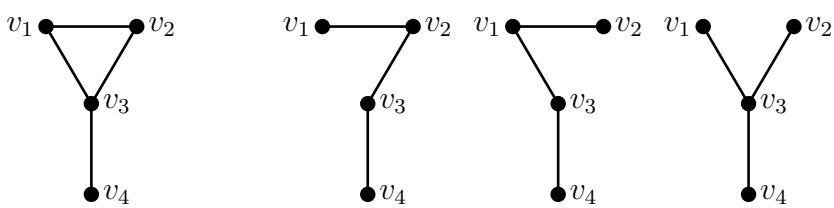


Figure 20.18: Example for Matrix-Tree Theorem, with three spanning trees

With this ordering the adjacency matrix A and degree matrix D are:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The $(3,3)$-cofactor of $D-A$ is

$$\begin{vmatrix} -2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3.$$

□

Solution to Exercise 525: (Use Matrix-Tree Theorem to find number of spanning trees of bull graph.) TBW.

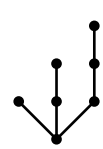
Solution to Exercise 526: (Use Matrix-Tree Theorem to find number of spanning trees of house graph.) TBW.

Solution to Exercise 527: Let $n \geq 2$ and let $A = J_n - I_n$ be the incidence matrix for the complete graph K_n, and let $D = \operatorname{diag}(n-1, n-1, \ldots, n-1)$. The $(1,1)$ cofactor of $D-A$ is

$$\begin{aligned}
&C_{1,1}(D-A) \\
&= (-1)^2 \begin{vmatrix} n-1 & -1 & -1 & \ldots & -1 \\ -1 & n-1 & -1 & \ldots & -1 \\ \vdots & \vdots & & & \vdots \\ -1 & -1 & \ldots & -1 & n-1 \end{vmatrix} && (n-1 \text{ rows/cols}) \\
&= \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ -1 & n-1 & -1 & \ldots & -1 \\ \vdots & \vdots & & & \vdots \\ -1 & -1 & \ldots & -1 & n-1 \end{vmatrix} && \text{(add R2, \ldots, R(n-1) to R1)} \\
&= \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & n & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & n \end{vmatrix} && \text{(add R1 to remaining rows)} \\
&= n^{n-2} && \text{(upper triangular determinant).}
\end{aligned}$$

So K_n has n^{n-2} spanning trees, as desired. □

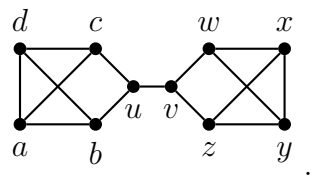
Solution to Exercise 529: Consider the tree formed by paths of length 1, 2, and 3, all starting at a common vertex:



□

Solution to Exercise 531: Any automorphism of a graph G is also an automorphism of $\overline{G}$. □

Solution to Exercise 533: The graph formed by the disjoint union of C_3 and C_4 is 2-regular but is not vertex-transitive. A connected 3-regular example on 10 vertices is



.

Vertices u and v are the only cut-vertices, so this graph is not vertex-transitive. □

Solution to Exercise 534: (See p. 161 of Read–Wilson [779].) The three cubic transitive graphs are the Petersen graph, the graph of the pentagonal prism (see Figure 1.21 for drawing of the hexagonal prism), and the 10 vertex Möbius graph (see Figure 1.33—start with C_{10} and add edges between opposite vertices). □

Solution to Exercise 535: The observation in this exercise was noted in [940]. Let x and y be two vertices in a pentagonal prism. If x and y are both on an end, a rotation along the central axis takes x to y. If x and y are on opposite end-faces, then a reflection along a central plane (parallel to the end-faces) and possibly a rotation gives an automorphism taking x to y.

There is no automorphism that takes a vertical edge to an edge along a base (top or bottom) since a vertical edge is an edge of two quadrilaterals whereas a base edge is an edge of both a quadrilateral and a pentagon. □

Solution to Exercise 536: Let $G = (V, E)$ be graph that is both vertex- and edge-transitive. TBW

Solution to Exercise 538: See [940, Thm. 7.53, p. 59]. □

Solution to Exercise 539: This result occurs as Lemma 1(d) in [507], without proof, and is used in [510, p. 169].

First show that in a rigid graph, there are no isolated vertices: If $G = (V, E)$ is a graph with an isolated vertex x, the for any other vertex y, the map that takes x to y and leaves all other vertices fixed is an endomorphism.

Similarly, if some vertex x has degree 1, a map that takes x to any non-neighbour of x and leaves all other vertices fixed is also an endomorphism. □

Solution to Exercise 540: (Cayley graphs are vertex-transitive.) TBW.

Solution to Exercise 541: TBW. Choose generator 1.

Solution to Exercise 542: (Graph of prism is a Cayley graph.) TBW.

Solution to Exercise 543: (Petersen graph not a Cayley graph.) TBW.

Solution to Exercise 544: (Q_k is a Cayley graph.)TBW.

Solution to Exercise 545: (Cayley graph connected.) TBW.

Solution to Exercise 546: (Cayley graph for A_5.) TBW.

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