Finite-Sample Properties and Applicability of Functional CLT Based Confidence Intervals for a Population Mean

by

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Abstract

We consider a Student process that is based on independent copies of a random variable X and has trajectories in the function space $D[0,1]$. If X is in the domain of attraction of the normal law, a weighted version of the Student process is known to follow a functional central limit theorem (FCLT). Accordingly, appropriate functionals of such a process converge in distribution to the same functionals of a weighted Wiener process. We use such a convergence for several functionals and derive asymptotic confidence intervals (CI) for the mean of X. Based on our investigation of the finite-sample coverage probabilities and expected lengths of the obtained CI's for different types of distributions of X, we suggest when these FCLT based CI's may be appealing alternatives to an asymptotic CI for the mean of X that is derived from the asymptotic normality of the Student t-statistic.

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Chapter 1

Introduction

Let $\{X_i, i \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) with an unknown mean μ. Consider the Student t-statistic

$$T_n(X_1, \ldots, X_n) := \frac{\sum_{i=1}^{n} X_i}{s_n \sqrt{n}}, \tag{1.1}$$

where

$$s_n := \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n-1}} \quad \text{and} \quad \overline{X}_n := \frac{\sum_{i=1}^{n} X_i}{n}.$$

For i.i.d. r.v.'s $\{X, X_i, i \geqslant 1\}$ and a constant μ, Giné et al. (1997) proved the following characterization of asymptotic normality of $T_n(X_1 - \mu, \ldots, X_n - \mu)$:

$$T_n(X_1 - \mu, ..., X_n - \mu) \xrightarrow[n \to \infty]{d} N(0,1) \quad \Longleftrightarrow$$

$$X \in \text{DAN} \quad \text{and} \quad E(X) = \mu, \tag{1.2}$$

where DAN denotes the domain of attraction of the normal law and $X \in$ DAN means that there exist constants a_n and $b_n > 0$ such that

$$\frac{\sum_{i=1}^{n} X_i - a_n}{b_n} \xrightarrow[n\to\infty]{d} N(0,1). \tag{1.3}$$

Remark 1. The distributions in DAN have finite moments of order $\nu \in (0,2)$, while their variances are positive, but may be infinite. Clearly, if $0 < Var(X) < \infty$, then (1.3) is satisfied on account of the Central Limit Theorem (CLT), with $a_n = nE(X)$ and $b_n = \sqrt{nVar(X)}$. If $Var(X) = \infty$, then a_n can also be taken as $nE(X)$, while $b_n = \sqrt{n} l_x(n)$, where $l_x(n)$ is a slowly varying function at infinity, meaning that $l_x(az)/l_x(z) \to 1$, as $z \to \infty$, for any $a > 0$. Moreover, in this case $l_x(n)$ is also nondecreasing and converges to ∞, as $n \to \infty$. Consider Pareto distribution of the first kind with the tail parameter α and the scale parameter β, denoted by $Pareto(\alpha, \beta)$ hereafter. $Pareto(1,2)$ distribution with the probability density function $2x^{-3}\mathbb{1}_{\{x \geqslant 1\}}$ is a well-known example of a distribution in DAN with an infinite variance (see, for example, Martsynyuk (2013)). Moreover, b_n in (1.3) can be taken as $\sqrt{n \log n}$ in this case. Clearly, if $\beta > 2$, then $Pareto(1, \beta)$ has a finite variance and hence is also in DAN.

The $\Leftarrow$ part of (1.2) can be used to construct an asymptotic confidence interval (CI) for an unknown population mean $\mu = E(X)$, provided that $X \in$ DAN. Accordingly, the $100(1-\alpha)\%$ asymptotic CI for mean μ is :

$$I_0 := \left[\overline{X}_n - \frac{z_{\alpha/2} s_n}{\sqrt{n}}, \overline{X}_n + \frac{z_{\alpha/2} s_n}{\sqrt{n}} \right], \tag{1.4}$$

where $\alpha \in (0,1)$ and $z_{\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution.

A Student process in $D[0,1]$, the space of real-valued functions on $[0,1]$ that are right-continuous and have left-hand limits, is defined as follows:

$$T_n^t(X_1,\ldots,X_n) := \frac{\sum_{i=1}^{[nt]} X_i}{s_n\sqrt{n}}, \quad 0 \leqslant t \leqslant 1, \tag{1.5}$$

where $\sum_{i=1}^{[0]} X_i := 0$. Thus, $T_n^t(X_1,\ldots,X_n)$ is a random step function on $[0,1]$:

$$T_n^t(X_1,\ldots,X_n) = \begin{cases} 0, & 0 \leqslant t < \frac{1}{n}, \\ \dfrac{X_1}{s_n\sqrt{n}}, & \frac{1}{n} \leqslant t < \frac{2}{n}, \\ \vdots & \\ \dfrac{X_1+X_2+\cdots+X_{n-1}}{s_n\sqrt{n}}, & \frac{n-1}{n} \leqslant t < 1, \\ \dfrac{X_1+X_2+\cdots+X_n}{s_n\sqrt{n}}, & t = 1. \end{cases} \tag{1.6}$$

When $t=1$, the Student process in (1.6) becomes the Student t-statistic of (1.1).

Let $\{W(t), 0 \leqslant t \leqslant 1\}$ denote a standard Wiener process. Csörgő et al. (2003) extended (1.2) as follows:

$$X \in \mathrm{DAN} \quad \text{and} \quad E(X) = \mu \quad \Longleftrightarrow$$
$$h(T_n^t(X_1 - \mu, \ldots.., X_n - \mu)) \xrightarrow[n\to\infty]{d} h(W(t))$$
$$\text{for all functionals } h\colon D[0,1] \to \mathbb{R} \text{ that are } \mathscr{D}\text{-measurable and } \rho\text{-continuous,} \tag{1.7}$$

where $\mathscr{D}$ is the σ-field of subsets of $D[0,1]$ that is generated by the finite-dimensional subsets of $D[0,1]$ and ρ is the sup-norm metric in $D[0,1]$.

The convergence in distribution in (1.7) is known as a Functional Central Limit Theorem (FCLT). The notion of $\mathscr{D}$-measurability for a functional is beyond the scope of the present work. The functionals considered in this thesis are $\mathscr{D}$-measurable, just like most functionals on $D[0,1]$. As for ρ-continuity, a functional $h\colon D[0,1] \to \mathbb{R}$ is called sup-norm continuous if for any $f_0(t) \in D[0,1]$ and $\epsilon > 0$, there exists $\delta > 0$, such that for any $f(t) \in D[0,1]$ satisfying $\sup_{0\leqslant t\leqslant 1} |f(t) - f_0(t)| < \delta$ we have $|h(f(t)) - h(f_0(t))| < \epsilon$.

In order to see that (1.7) generalizes (1.2), and thus that the CLT of (1.2) for the Student t-statistic is a special case of the FCLT of (1.7) for the Student process, one reads the convergence in distribution in (1.7) with the projection functional $h(\cdot)$ such that $h(f(t)) = f(1)$, for any $f(t) \in D[0,1]$.

Tuzov (2014) and Martsynyuk and Tuzov (2016) considered several concrete functionals as in (1.7) and derived respective asymptotic CI's, abbreviated as FACI's, for the mean of X based on convergence in distribution of these functionals. The focus of these works was to explore finite-sample properties of the obtained FACI's and compare them to those of I_0 of (1.4). Accordingly, from the comparison of the expected lengths and finite-sample coverage probabilities of these CI's, which was mostly performed numerically, it was concluded that some of the FACI's are shorter and/or have at least as high coverage probabilities as those of I_0, and thus can serve as reasonable alternatives to I_0.

Most of the FACI's for $\mu = E(X)$ obtained in Tuzov (2014) and Martsynyuk and Tuzov (2016) that may be more appealing than I_0 have higher finite-sample coverage probabilities than that of I_0 at an expense of being longer than I_0 on average. In these works, it was conjectured that there exists a FACI that outperforms I_0 both in terms of finite-sample coverage probability and an expected length. In partial support of this hypothesis, Tuzov (2014) and Martsynyuk and Tuzov (2016) constructed a FACI with a shorter expected length and approximately the same coverage probability as that of I_0. The latter example of a FACI and the fact that there are countless choices for the functional $h(\cdot)$ in (1.7) made this conjecture plausible.

The present work is an extension of Tuzov (2014) and Martsynyuk and Tuzov (2016). It deals with detailed analysis of finite-sample properties of FACI's for $\mu = E(X)$ based on the FCLT on (1.7) and further ones constructed from an FCLT for a weighted version of the Student process of (1.6) (cf. the upcoming (1.9)). This mostly numerical study is conducted for different types of distributions of X and new choices of functionals $h(\cdot)$. Moreover, based on the finite-sample coverage probabilities and expected lengths of the obtained FACI's and I_0, we discuss the scope of potential applicability of our FACI's, depending on a nature of the underlying distribution of X and the sample size.

Let Q be the class of positive functions $q(t)\colon (0,1) \to (0,\infty)$, i.e., $\inf_{\delta \leqslant t \leqslant 1-\delta} q(t) > 0$ for all $\delta \in (0,1/2)$, which are non-decreasing near 0 and non-increasing near 1. For $q(t) \in Q$, let

$$I_0^1(q,c) := \int_0^1 \frac{e^{-cq^2(t)/(t(1-t))}}{t(1-t)}dt, \quad c > 0. \tag{1.8}$$

Assuming that $I_0^1(q,c) < \infty$ for some, or possibly all, $c > 0$, it follows from Csörgő et al. (2008) combined with Csörgő and Horváth (1988) that the $\Rightarrow$ part of (1.7) for the weighted Student process $T_n^t/q([nt]/n)$ holds true:

$$X \in \text{DAN} \quad \text{and} \quad E(X) = \mu \quad \Rightarrow$$
$$h\left(\frac{T_n^t(X_1-\mu,\ldots..,X_n-\mu)}{q\left(\frac{[nt]}{n}\right)}\right) \xrightarrow[n\to\infty]{d} h\left(\frac{W(t)}{q(t)}\right)$$
for all functionals $h\colon D[0,1] \to \mathbb{R}$ that are $\mathscr{D}$-measurable and ρ-continuous.
(1.9)

A well-known weight function for the FCLT in (1.9) is

$$q_1(t) := \sqrt{t(1-t)\log\log(t(1-t))^{-1}}, \quad t \in (0,1). \tag{1.10}$$

In connection to studying finite-sample powers of tests for change in the mean via convergence in distribution of sup-functionals of weighted tied-down partial sums process, Orasch and Pouliot (2010) introduced a modification of $q_1(t)$:

$$q_2(t) := \begin{cases} q_1(t), & t \in (0, a] \bigcup [b, 1), \\ \sqrt{t(1-t)}, & t \in (a, b), \end{cases} \tag{1.11}$$

where

$$(a, b) = (0.071033..., 0.928966...). \tag{1.12}$$

The use of $q_2(t)$ led to a more powerful test than its counterpart with $q_1(t)$. Clearly, just like $q_1(t)$, the weight function $q_2(t)$ is also suitable for the FCLT in (1.9). Moreover, one can show that $I_0^1(\sqrt{q_1}, c) \leqslant I_0^1(q_1, c)$, that is $\sqrt{q_1(t)}$ and hence $\sqrt{q_2(t)}$ are additional choices for $q(t)$ in (1.9).

In this thesis, to derive our FACI's for μ, we use the FCLT in (1.7) for the Student process and the FCLT in (1.9) for its weighted version, using $q(t) = q_1(t)$ or $\sqrt{q_1(t)}$ and $q(t) = q_2(t)$ or $\sqrt{q_2(t)}$. The following functionals $h_1(\cdot), \ldots, h_4(\cdot)$ spelled out with a general function $f(t) \in D[0, 1]$ are used in this regard:

$$h_1(f(t)) = \int_0^1 f(t)\, dt, \tag{1.13}$$

$$h_2(f(t)) = \int_0^1 f^2(t)\, dt, \tag{1.14}$$

$$h_3(f(t)) = 0.457 f(0.97) + \int_0^1 f^2(t)\, dt, \tag{1.15}$$

and

$$h_4(f(t)) = -0.03f(0.99) + \int_0^1 f(t)\,dt. \tag{1.16}$$

The functionals $h_1(\cdot)$ and $h_2(\cdot)$ were used in Tuzov (2014) and Martsynyuk and Tuzov (2016) and led to obtaining two of their three best performing FACI's for μ based on (1.7) for the Student process. We study them here for more distributions along with applying them to the weighted Student process via (1.9), which results in FACI's for μ with better finite-sample properties. The functionals $h_3(\cdot)$ and $h_4(\cdot)$, which are linear combinations of the integral functional $h_1(\cdot)$ or $h_2(\cdot)$ and the projection functional $h_0(\cdot)$, where

$$h_0(f(t)) = f(t_0), \quad t_0 \in (0,1), \tag{1.17}$$

have not been studied before. Moreover, the FCLT in (1.9) with $h_3(\cdot)$ and $h_4(\cdot)$ yields the FACI's for μ that completely or partially outperform those that are based on $h_1(\cdot)$ or $h_2(\cdot)$ alone.

In order to derive our FACI's for μ, the distribution functions of the limiting r.v.'s in (1.7) and (1.9), namely $h_i(W(t))$, $h_i\left(W(t)/q(t)\right)$ and $h_i(W(t)/\sqrt{q(t)})$, must either have closed form expressions, or be tabulated. This is only the case for $h_1(W(t)) \stackrel{d}{=} N(0,1/3)$ and $h_2(W(t))$. The rest of the limiting distributions have to be tabulated by us, similarly to the way of Tuzov (2014) and Martsynyuk and Tuzov (2016) for their functionals, which is based on an invariance principle approach of Erdős and Kac (1946). Accordingly, in view of the fact that the limiting distributions in (1.7) and (1.9) do not depend on the underlying distribution structure of X, we estimate quantiles of our limiting distributions by those of the respective empirical distributions of the r.v.'s $h_i(T_n^t(X_1-\mu, X_2-\mu, ..., X_n-\mu))$ and $h_i(T_n^t(X_1-\mu, X_2-\mu, ..., X_n-\mu)/q([nt]/n))$, $j=1$ and 2, which are based on 500,000 simulated values of these r.v.'s, where each value is computed from an $N(0,1)$ random sample of size 10,000.

After deriving our FACI's, denoted by I_i (from (1.7) with $h(\cdot) = h_i(\cdot)$, $i = \overline{1,2}$) and I_{i,q_j} (from (1.9) with $h(\cdot) = h_i(\cdot)$, $i = \overline{1,4}$, and $q(t) = q_j(t)$ (for $h_1(\cdot)$ and $h_4(\cdot)$) or $\sqrt{q_j(t)}$ (for $h_2(\cdot)$ and $h_3(\cdot)$), $j = \overline{1,2}$), using tabulated limiting distributions of $h_i(W(t))$, $h_i\left(W(t)/q_j(t)\right)$ and $h_i\left(W(t)/\sqrt{q_j(t)}\right)$, we conduct a detailed, mostly numerical, study of the finite-sample coverage probabilities and expected lengths of our FACI's in comparison to the finite-sample properties of I_0 of (1.4). All the exact finite-sample coverage probabilities are approximated by their empirical counterparts $\widehat{CP}_i$ for I_i and $\widehat{CP}_{i,q_j}$ for I_{i,q_j}, which are based on 10,000 random samples of size n from a distribution of X:

$$\widehat{CP}_i := \frac{\sum_{k=1}^{10,000} \mathbb{1}_{\{\mu \in I_i, \text{ for sample } k\}}}{10,000} \quad \text{and}$$

$$\widehat{CP}_{i,q_j} := \frac{\sum_{k=1}^{10,000} \mathbb{1}_{\{\mu \in I_{i,q_j}, \text{ for sample } k\}}}{10,000}, \tag{1.18}$$

where $\mathbb{1}_A$ is the indicator function of set A. In order to compare the expected lengths of I_i and I_{i,q_j} to I_0, we set

$$r_i := \frac{E(\text{length of } I_i)}{E(\text{length of } I_0)} \quad \text{and} \quad r_{i,q_j} := \frac{E(\text{length of } I_{i,q_j})}{E(\text{length of } I_0)}. \tag{1.19}$$

With the exception of r_1, r_{1,q_j} and r_{4,q_j}, it is not feasible to obtain closed form expressions for r_i and r_{i,q_j} and hence, the latter are approximated from 10,000 random samples of size n using respectively

$$\widehat{r}_i := \frac{\sum_{k=1}^{10,000} (\text{length of } I_i \text{ for sample } k)/10,000}{\sum_{k=1}^{10,000} (\text{length of } I_0 \text{ for sample } k)/10,000} \quad \text{and}$$

$$\widehat{r}_{i,q_j} := \frac{\sum_{k=1}^{10,000} (\text{length of } I_{i,q_j} \text{ for sample } k)/10,000}{\sum_{k=1}^{10,000} (\text{length of } I_0 \text{ for sample } k)/10,000}. \tag{1.20}$$

In our simulation study, we use fourteen distributions that are listed in Table 1.1 with their number of moments and skewness, defined as the third moment $E[(X - E(X))/\sqrt{Var(X)}]^3$ of the standardized X, provided that $E|X|^3 < \infty$ and $Var(X) > 0$.

Distribution	Number of moments	Skewness	Probability density/mass function
$Pareto(1,2)$	<2	-	$2x^{-3}, \quad x \geqslant 1$
$Frechet(3)$	<3	-	$3x^{-4}, \quad x > 0$
$Burr(3,2)$	<6	1.603	$6x^2(1+x^3)^{-3}, \quad x > 0$
$Pareto(1,6)$	<6	3.81	$6x^{-7}, \quad x \geqslant 1$
$Burr(3,4)$	<7	0.684	$12x^2(1+x^3)^{-5}, \quad x > 0$
$t(8)$	<8	0	$\left[\Gamma\left(\frac{n+1}{2}\right)/\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)\right](1+x^2/n)^{-(n+1)/2}, x \in \mathbb{R}$
$Burr(8,7)$	<9	-0.354	$56x^7(1+x^8)^{-8}, \quad x \geqslant 0$
$Frechet(10)$	<10	1.91	$10x^{-11}, \quad x > 0$
$N(0,1)$	∞	0	$e^{-x^2/2}/\sqrt{2\pi}, \quad x \in \mathbb{R}$
$Pois(3)$	∞	0.577	$3^x e^{-3}/x!, \quad x = 0, 1, 2, ...$
$Exp(1)$	∞	2	$e^{-x}, \quad x \geqslant 0$
$Gamma(0.025,1)$	∞	12.65	$\Gamma^{-1}(0.025)x^{-0.975}e^{-x}, \quad x > 0$
$Gamma(0.001,1)$	∞	63.25	$\Gamma^{-1}(0.001)x^{-0.999}e^{-x}, \quad x > 0$
$Weibull(1,0.2)$	∞	190.11	$0.2x^{-0.8}e^{-0.2x}, \quad x \geqslant 0$

Table 1.1: Distributions, their number of moments and skewness

In Table 1.1, when the number of moments is listed as $< a$, it means that all the moments of order $< a$ are finite, while the a^{th} moment is infinite. On the other hand, the symbol ∞ means that the moments of all orders are finite.

According to our next Table 1.2, we distinguish the distributions of Table 1.1 as those having less than 7 moments, dubbed hereafter as heavy-tailed and those with at least 7 moments, called light-tailed hereafter. It may not exist (for the distributions with less than 3 moments), or be less than 2 in the absolute value, which we call small, or be at least 2 in the absolute value, which is dubbed as large hereafter. This results in the five distribution groups of Table 1.2 (our light-tailed distributions always have a finite skewness).

	No skewness	Small skewness	Large skewness
Heavy-tailed distributions	$Pareto(1, 2)$ $Frechet(3)$	$Burr(3, 2)$ $Burr(3, 4)$	$Pareto(1, 6)$
Light-tailed distributions	N/A	$t(8)$ $Burr(8, 7)$ $Frechet(10)$ $N(0, 1)$ $Pois(3)$	$Exp(1)$ $Gamma(0.025, 1)$ $Gamma(0.001, 1)$ $Weibull(1, 0.2)$

Table 1.2: Five distribution classes according to tails and skewness

Our detailed investigation of the finite-sample properties of the FACI's for μ that are derived in this thesis reveals that there is a trade-off between their finite-sample coverage probabilities and expected lengths. Thus, they have higher coverage probabilities than those of I_0, but are somewhat longer than I_0 on average. Our FACI's appear to provide reasonable alternatives to the classical CI I_0 when some improvement of the finite-sample coverage probability of I_0 is desirable and an expense of losing in the expected length is acceptable. Moreover, our FACI's may be particularly appealing for what we call here heavy-tailed and/or largely skewed distributions (cf. Table 1.2), since in this case there is a pressing need of having an asymptotic CI for μ with higher finite-sample coverage probabilities. Indeed, for such distributions, the empirical coverage probabilities of our FACI's and I_0 are lower, sometimes significantly, than the nominal confidence levels $1 - \alpha$, and the rates of convergence of these coverage probabilities to $1-\alpha$ are slow, which makes the FACI's good alternatives to I_0 even for very large sample sizes. At the same time, all the true and empirical ratios of the expected lengths of the FACI's and I_0, namely r_i, r_{i,q_j}, $\widehat{r}_i$, and $\widehat{r}_{i,q_j}$, are seen to decrease as the sample size n increases, at slow rates for $n > 100$. Some of them converge mathematically, as $n \to \infty$, while possible convergence of the rest is suggested numerically.

While each of our FACI's is derived and carefully studied separately in Chapter 2, we address their comparative finite-sample performance in the conclusions of this thesis in Chapter 3. We conclude that (1.9) based I_{1,q_j} and I_{2,q_j} outperform I_1 and I_2, respectively, while I_{4,q_2}, which is based on a linear combination of the integral functional $h_1(\cdot)$ and the projection functional $h_0(\cdot)$, may be preferred to I_{1,q_1} and I_{1,q_2}, which are based on $h_1(\cdot)$ alone, for our classes of heavy-tailed distributions with no skewness and light-tailed distributions with large skewness. The FACI I_{3,q_2} based on the functionals $h_2(\cdot)$ and $h_0(\cdot)$ has the best finite-sample properties overall. Moreover, it outperforms the best three FACI's of Tuzov (2014) and Martsynyuk and Tuzov (2016) (I_1, I_2 and I_5 of (3.1) with higher coverage probabilities than those of I_0).

Chapter 2

Main Results

In this chapter, we derive the FACI's I_i (from the FCLT of (1.7) with $h(\cdot) = h_i(\cdot)$, $i = \overline{1,2}$) and the FACI's I_{i,q_j} (from the FCLT of (1.9) with $h(\cdot) = h_i(\cdot)$ and the weight functions $q(\cdot) = q_j(\cdot)$ or $\sqrt{q_j(\cdot)}$, $i = \overline{1,4}$ and $j = \overline{1,2}$, with $q_1(\cdot)$ and $q_2(\cdot)$ of (1.10) and (1.11)). Most of the derivations rely on our simulations of the quantiles of the respective limiting r.v.'s $h_i(W(t))$, $h_i\left(W(t)/q_j(t)\right)$ and $h_i\left(W(t)/\sqrt{q_j(t)}\right)$ using the invariance principle approach of Erdős and Kac (1946). This is followed by a detailed investigation of the finite-sample properties of I_i and I_{i,q_j} in comparison to those of the CI I_0 of (1.4). While all the finite-sample coverage probabilities of our FACI's and I_0 are computed numerically via $\widehat{CP}_i$ and $\widehat{CP}_{i,q_j}$, the ratios r_i and r_{i,q_j} of the expected lengths of the FACI's and I_0 can sometimes be evaluated directly and seen to be functions of n, α and the quantiles of the limiting distributions.

For each of $h_i(\cdot)$, the simulations for the corresponding FACI's are done for the fourteen distributions of Table 1.1, three confidence levels of $1-\alpha =$ 0.9, 0.95 and 0.98, and for four sample sizes of $n = 50$, 100, 500, and 1,000, and, in case of some distributions, also for an additional sample size of $n =$ 5,000, when the coverage probabilities of those FACI's are lower than $1-\alpha$ and converge to $1-\alpha$ slowly. This results in large detailed tables that are abbreviated in additional summary tables with ranges of

$$\Delta_i := (\widehat{CP}_i - \widehat{CP}_0)100\% \quad \text{and}$$
$$\Delta_{i,q_j} := (\widehat{CP}_{i,q_j} - \widehat{CP}_0)100\%, \tag{2.1}$$

$i = \overline{1,5}$ and $j = \overline{1,2}$, and averages (in n) of r_i (or $\widehat{r}_i$) and r_{i,q_j} (or $\widehat{r}_{i,q_j}$), all computed for each of the three values of $1-\alpha$ and for each of the five distribution classes of Table 1.2.

2.1 FACI's based on $h_1(\cdot)$

The FCLT of (1.9) with $h(\cdot) = h_1(\cdot)$ of (1.13) reads as follows:

$$\int_0^1 \frac{T_n^t(X_1-\mu,\ldots,X_n-\mu)}{q\left(\frac{[nt]}{n}\right)}dt \xrightarrow[n\to\infty]{d} \int_0^1 \frac{W(t)}{q(t)}dt, \tag{2.2}$$

or, equivalently, in view of the definition of the Student process in (1.6),

$$\frac{1}{s_n n\sqrt{n}} \sum_{k=1}^{n-1} \left(\frac{\sum_{i=1}^k (X_i-\mu)}{q\left(\frac{k}{n}\right)} \right) \xrightarrow[n\to\infty]{d} \int_0^1 \frac{W(t)}{q(t)}dt, \tag{2.3}$$

where $q(\cdot)$ is later chosen as $q_1(\cdot)$ of (1.10) or $q_2(\cdot)$ of (1.11).

The distribution functions of $\int_0^1 (W(t)/q_1(t))dt$ and $\int_0^1 (W(t)/q_2(t))dt$ do not seem to be available in closed form expressions, or tabulated by others. We had to tabulate them ourselves. Some of the quantiles of these r.v.'s. are given in Tables 2.1 and 2.2.

γ	0.005	0.025	0.05	0.95	0.975	0.985
γ quantile	-5.41	-4.41	-3.58	3.48	4.04	4.59

Table 2.1: Some quantiles of $\int_0^1 \frac{W(t)}{q_1(t)} dt$

γ	0.01	0.015	0.04	0.94	0.965	0.99
γ quantile	-3.84	-3.46	-2.80	2.52	2.92	3.90

Table 2.2: Some quantiles of $\int_0^1 \frac{W(t)}{q_2(t)} dt$

Let $\alpha \in (0,1)$ and define $a, b \in \mathbb{R}$ such that $P\left(a \leqslant \int_0^1 (W(t)/q(t))dt \leqslant b\right) = 1-\alpha$. It follows from (2.3) that

$$P\left(a \leqslant \frac{1}{s_n n\sqrt{n}} \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k}(X_i - \mu)}{q\left(\frac{k}{n}\right)} \leqslant b\right) \xrightarrow[n\to\infty]{d} 1-\alpha. \tag{2.4}$$

Solving for μ under the probability sign in (2.4) yields:

$$a \leqslant \frac{1}{s_n n\sqrt{n}} \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k}(X_i - \mu)}{q\left(\frac{k}{n}\right)} \leqslant b$$

$$\iff \qquad a s_n n^{\frac{3}{2}} \leqslant \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)} - \mu \sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)} \leqslant b s_n n^{\frac{3}{2}}$$

$$\iff \qquad -b s_n n^{\frac{3}{2}} + \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)} \leqslant \mu \sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)} \leqslant -a s_n n^{\frac{3}{2}} + \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)}.$$

Therefore, the $1-\alpha$ FACI for μ from (2.4) is given by:

$$I_{1,q} := \left[\frac{-b s_n n^{\frac{3}{2}} + \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)}}{\sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)}}, \frac{-a s_n n^{\frac{3}{2}} + \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)}}{\sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)}} \right]. \tag{2.5}$$

The ratio of the expected lengths of $I_{1,q}$ and I_0 is

$$r_{1,q} = \frac{(b-a)n^2}{2 z_{\alpha/2} \sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)}}, \tag{2.6}$$

where the values of $n^2/\sum_{k=1}^{n-1}(k/q\left(\frac{k}{n}\right))$ are computed for $q(\cdot) = q_1(\cdot)$ and $q(\cdot) = q_2(\cdot)$ for different n in Table 2.3.

	50	100	500	1000	5,000	25,000	50,000	100,000	150,0000	200,000
$q = q_1$	0.572773	0.557482	0.539332	0.535407	0.530423	0.528310	0.527825	0.527486	0.527337	0.527249
$q = q_2$	0.747270	0.723086	0.693849	0.687385	0.679203	0.675743	0.674949	0.674395	0.674151	0.674008

Table 2.3: Values of $\frac{n^2}{\sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)}}$ for different n

For our three confidence levels of $1-\alpha$ =0.9, 0.95 and 0.98, the ratio r_{1,q_j} is always larger than one, regardless of the choice of a and b. While r_{1,q_j} is minimized whenever $b-a$ is the smallest, in view of a trade-off between the empirical coverage probability and expected length of I_{1,q_j}, we had to select a and b so that r_{1,q_j} would only be slightly higher than 1, while the difference $\widehat{CP}_{1,q_j} - \widehat{CP}_0 > 0$ would be reasonably significant. The respective choices of (a,b) are listed under the $1-\alpha$ values in the headers of Table 2.6 and 2.7.

The FACI from (1.7) with $h(\cdot) = h_1(\cdot)$ was obtained in Tuzov (2014) and Martsynyuk and Tuzov (2016) in a similar way, noting that $\int_0^1 W(t)dt \stackrel{d}{=} N(0,1/3)$, with $-a = b = z_{\alpha/2}/\sqrt{3}$ in the respective version of (2.4). Accordingly, the $1-\alpha$ FACI is

$$I_1 := \left[\frac{2}{n-1} \left(\sum_{k=1}^{n-1} \sum_{i=1}^{k} X_i - z_{\frac{\alpha}{2}} \frac{s_n\sqrt{n}}{\sqrt{3}} \right), \frac{2}{n-1} \left(\sum_{k=1}^{n-1} \sum_{i=1}^{k} X_i + z_{\frac{\alpha}{2}} \frac{s_n\sqrt{n}}{\sqrt{3}} \right) \right], \tag{2.7}$$

and the ratio of the expected lengths of I_1 and I_0 is

$$r_1 = \frac{2n}{\sqrt{3}(n-1)}. \tag{2.8}$$

Although r_1, $\widehat{CP}_1$ and $\widehat{CP}_0$ were computed before in these works, we resimulate $\widehat{CP}_1$ and $\widehat{CP}_0$ for a larger class of distributions in Table 2.5 and show that I_1 is outperformed by our I_{1,q_1} or I_{1,q_2} across all our distribution classes of Table 1.2.

Tables 2.5 - 2.7 contain the quadruplets $r_1\,(\widehat{CP}_1, \widehat{CP}_0)\,\Delta_1$ and $r_{1,q_j}\,(\widehat{CP}_{1,q_j}, \widehat{CP}_0)\,\Delta_{1,q_j}$ for each distribution of Table 1.1, n and $1-\alpha$. For a convenient quick reference, Tables 2.5 - 2.7 are abbreviated into summary Table 2.4.

We note that all three ratios r_1, r_{1,q_1} and r_{1,q_2} in Tables 2.5 - 2.7 are greater than one, but less than 1.243, indicating that the respective FACI's are longer on average than I_0 of (1.4). From Table 2.3, it appears that r_{1,q_1} and r_{1,q_2} converge, while $r_1 \to 2/\sqrt{3}$, as $n \to \infty$.

The empirical finite-sample coverage probabilities $\widehat{CP}_1$ of I_1, $\widehat{CP}_{1,q_1}$ of I_{1,q_1} and $\widehat{CP}_{1,q_2}$ of I_{1,q_2} are higher than $\widehat{CP}_0$ of I_0 for all the distributions and α in hand, except for a few instances. The estimated coverage probability differences Δ_1, Δ_{1,q_1} and Δ_{1,q_2} are generally higher for smaller n, as well as for the heavy-tailed distributions with no skewness and for the light-tailed distributions with large skewness as in Table 1.2 (cf. Table 2.4). For each of Δ_1,

Δ_{1,q_1} and Δ_{1,q_2}, their ranges are somewhat comparable across the other three distribution classes of the heavy-tailed distributions with small/large skewness and the light-tailed distributions with small skewness. For the confidence levels $1-\alpha = 0.9$ and 0.95, $\widehat{CP}_{1,q_2}$ is larger than the respective values of $\widehat{CP}_1$ and $\widehat{CP}_{1,q_1}$, while for $1-\alpha = 0.98$, I_{1,q_1} seems to have the best coverage probability among the three FACI's, except for a few cases. Hence, Δ_{1,q_1} and Δ_{1,q_2} are larger than Δ_1, as can also be quickly seen from Table 2.4.

Moreover, the empirical coverage probabilities of the three FACI's and that of I_0 are lower for the heavy-tailed distributions with fewer moments (e.g., $Pareto(1, 2)$ and $Frechet(3)$ with undefined skewness) and for the distributions with a larger absolute value of skewness (see $Burr(3, 2)$ and $Pareto(1, 6)$ for illustration, which have the same number of moments, while $Pareto(1, 6)$ is more skewed according to Table 1.1; see also the two very skewed gamma distributions with all moments). For some of our heavy-tailed and/or largely skewed distributions, the empirical coverage probabilities of the three FACI's and I_0 are lower than $1-\alpha$, more so for smaller n. In this regard, the three distributions $Gamma(0.025, 1)$, $Gamma(0.001, 1)$ and $Weibull(1, 0.2)$ particularly stand out, and the coverage probabilities appear to converge slowly to $1-\alpha$ in these cases, making the FACI's I_{1,q_1} and I_{1,q_2} with higher coverage probabilities clearly more appealing than I_0 at a little expense in terms of the expected lengths. On the other hand, for some of the distributions (such as $Burr(3, 4)$ and $Pois(3)$ for example), the coverage probabilities of I_{1,q_1} and

I_{1,q_2} are larger than $1-\alpha$, decreasing and approaching $1-\alpha$ as n increases.

In summary, from Tables 2.4 - 2.7, for all three FACI's I_1, I_{1,q_1} and I_{1,q_2}, we see a trade-off between their respective empirical coverage probabilities and expected lengths relative to that of I_0. Indeed, all three FACI's have somewhat higher coverage probabilities, while being slightly longer than I_0. When one prioritizes finite-sample coverage probability over expected length, one may view I_{1,q_2} (when $1-\alpha=0.9$ and 0.95) and I_{1,q_1} (when $1-\alpha=0.98$) as good alternatives to I_0 (and better ones than I_1, due to I_{1,q_1} and I_{1,q_2} having better coverage probabilities, with very close expected lengths). This is more true for smaller n and distributions with fewer moments and/or higher skewness.

	Heavy-tailed distributions with no skewness	Heavy-tailed distributions with small skewness	Heavy-tailed distributions with large skewness	Light-tailed distributions with small skewness	Light-tailed distributions with large skewness
$1-\alpha=0.9$					
Δ_1	0.29% - 1.84%	-0.08% - 0.91%	0.19% - 1.51%	-0.27% - 1.19%	0.01% - 2.12%
average r_1	1.164	1.164	1.164	1.164	1.164
Δ_{1,q_1}	1.09% - 3.73%	0.49% - 2.58%	0.67% - 3.14%	0.26% - 2.83%	0.46% - 3.16%
average r_{1,q_1}	1.178	1.183	1.183	1.183	1.178
Δ_{1,q_2}	0.71% - 4.38%	-0.10% - 2.72%	0.17% - 3.50%	-0.35% - 2.84%	0.42% - 3.46%
average r_{1,q_2}	1.147	1.153	1.153	1.153	1.147
$1-\alpha=0.95$					
Δ_1	0.30% - 2.64%	-0.06% - 0.56%	0.15% - 1.11%	-0.48% - 0.79%	-0.22% - 2.90%
average r_1	1.164	1.164	1.164	1.164	1.164
Δ_{1,q_1}	0.70% - 4.89%	0.39% - 2.01%	0.81% - 2.61%	0.16% - 2.10%	0.39% - 4.42%
average r_{1,q_1}	1.183	1.1885	1.1885	1.1885	1.183
Δ_{1,q_2}	0.54% - 5.57%	-0.06% - 2.25%	0.51% - 3.03%	-0.17% - 2.43%	0.35% - 5.03%
average r_{1,q_2}	1.154	1.161	1.161	1.161	1.154
$1-\alpha=0.98$					
Δ_1	0.19% - 2.50%	-0.12% - 0.74%	0.15% - 1.03%	-0.12% - 0.73%	0.17% - 3.11%
average r_1	1.164	1.164	1.164	1.164	1.164
Δ_{1,q_1}	0.86% - 5.17%	-0.12% - 1.43%	0.40% - 2.36%	-0.09% - 1.67%	0.28% - 5.47%
average r_{1,q_1}	1.179	1.184	1.184	1.184	1.179
Δ_{1,q_2}	-0.53% - 3.88%	0.37% - 1.53%	0.43% - 2.03%	0.29% - 1.52%	0.58% - 4.12%
average r_{1,q_2}	1.179	1.185	1.185	1.185	1.179

Table 2.4: Ranges of Δ_1, Δ_{1,q_1} and Δ_{1,q_2} and averages (in n) of r_1, r_{1,q_1} and r_{1,q_2}

Distribution	n	$1-\alpha=0.9$			$1-\alpha=0.95$			$1-\alpha=0.98$		
$Pareto(1,2)$	50	1.178	(0.7872,0.7688)	1.84	1.178	(0.8411,0.8147)	2.64	1.178	(0.8831,0.8581)	2.50
	100	1.166	(0.8086,0.7908)	1.78	1.166	(0.8592,0.8363)	2.29	1.166	(0.9054,0.8822)	2.32
	500	1.157	(0.8444,0.8379)	0.65	1.157	(0.9020,0.8855)	1.65	1.157	(0.9376,0.9235)	1.41
	1000	1.156	(0.8614,0.8516)	0.98	1.156	(0.9101,0.9006)	0.95	1.156	(0.9473,0.9357)	1.16
	5000	1.155	(0.8734,0.8664)	0.70	1.155	(0.9260,0.9173)	0.87	1.155	(0.9611,0.9518)	0.93
$Frechet(3)$	50	1.178	(0.8621,0.8472)	1.49	1.178	(0.9128,0.8968)	1.60	1.178	(0.9481,0.9346)	1.35
	100	1.166	(0.8752,0.8700)	0.52	1.166	(0.9264,0.9167)	0.97	1.166	(0.9611,0.9498)	1.13
	500	1.157	(0.8938,0.8909)	0.29	1.157	(0.9411,0.9381)	0.30	1.157	(0.9718,0.9669)	0.49
	1000	1.156	(0.9006,0.8958)	0.48	1.156	(0.9477,0.9445)	0.32	1.156	(0.9744,0.9725)	0.19
$Burr(3,2)$	50	1.178	(0.8988,0.8897)	0.91	1.178	(0.9451,0.9395)	0.56	1.178	(0.9774,0.9700)	0.74
	100	1.166	(0.8999,0.9005)	-0.06	1.166	(0.9481,0.9461)	0.20	1.166	(0.9796,0.9776)	0.20
	500	1.157	(0.9015,0.9015)	0.00	1.157	(0.9516,0.9487)	0.29	1.157	(0.9797,0.9783)	0.14
	1000	1.156	(0.9025,0.9016)	0.09	1.156	(0.9499,0.9505)	-0.06	1.156	(0.9783,0.9795)	-0.12
$Pareto(1,6)$	50	1.178	(0.8785,0.8634)	1.51	1.178	(0.9260,0.9149)	1.11	1.178	(0.9582,0.9479)	1.03
	100	1.166	(0.8908,0.8847)	0.61	1.166	(0.9366,0.9349)	0.17	1.166	(0.9704,0.9634)	0.70
	500	1.157	(0.9003,0.8984)	0.19	1.157	(0.9477,0.9462)	0.15	1.157	(0.9774,0.9746)	0.28
	1000	1.156	(0.9039,0.9013)	0.26	1.156	(0.9506,0.9469)	0.37	1.156	(0.9780,0.9765)	0.15
$Burr(3,4)$	50	1.178	(0.9027,0.8938)	0.89	1.178	(0.9508,0.9460)	0.48	1.178	(0.9790,0.9737)	0.53
	100	1.166	(0.9021,0.9008)	0.13	1.166	(0.9504,0.9486)	0.18	1.166	(0.9810,0.9786)	0.24
	500	1.157	(0.9008,0.9004)	0.04	1.157	(0.9501,0.9485)	0.16	1.157	(0.9795,0.9785)	0.10
	1000	1.156	(0.9025,0.9033)	-0.08	1.156	(0.9487,0.9493)	-0.06	1.156	(0.9792,0.9791)	0.01
$t(8)$	50	1.178	(0.9035,0.8928)	1.07	1.178	(0.9493,0.9466)	0.27	1.178	(0.9781,0.9781)	0.00
	100	1.166	(0.8993,0.8998)	-0.05	1.166	(0.9460,0.9508)	-0.48	1.166	(0.9796,0.9773)	0.23
	500	1.157	(0.8985,0.8951)	0.34	1.157	(0.9501,0.9493)	0.08	1.157	(0.9803,0.9798)	0.05
	1000	1.156	(0.8999,0.8999)	0.00	1.156	(0.9492,0.9488)	0.04	1.156	(0.9792,0.9791)	0.01
$Burr(8,7)$	50	1.178	(0.9053,0.8958)	0.95	1.178	(0.9505,0.9454)	0.51	1.178	(0.9810,0.9753)	0.57
	100	1.166	(0.9015,0.8994)	0.21	1.166	(0.9522,0.9495)	0.27	1.166	(0.9821,0.9796)	0.25
	500	1.157	(0.8990,0.9006)	-0.16	1.157	(0.9476,0.9496)	-0.20	1.157	(0.9792,0.9778)	0.14
	1000	1.156	(0.9015,0.9042)	-0.27	1.156	(0.9477,0.9495)	-0.18	1.156	(0.9783,0.9777)	0.06
$Frechet(10)$	50	1.178	(0.8965,0.8846)	1.19	1.178	(0.9426,0.9347)	0.79	1.178	(0.9733,0.9660)	0.73
	100	1.166	(0.9000,0.8999)	0.01	1.166	(0.9477,0.9431)	0.46	1.166	(0.9785,0.9760)	0.25
	500	1.157	(0.9028,0.9001)	0.27	1.157	(0.9501,0.9473)	0.28	1.157	(0.9795,0.9774)	0.21
	1000	1.156	(0.9033,0.9002)	0.31	1.156	(0.9496,0.9506)	-0.10	1.156	(0.9784,0.9791)	-0.07
$N(0,1)$	50	1.178	(0.8968,0.8955)	0.13	1.178	(0.9471,0.9460)	0.11	1.178	(0.9785,0.9737)	0.48
	100	1.166	(0.9023,0.8955)	0.68	1.166	(0.9526,0.9478)	0.48	1.166	(0.9812,0.9774)	0.38
	500	1.157	(0.8945,0.8956)	-0.11	1.157	(0.9471,0.9482)	-0.11	1.157	(0.9788,0.9791)	-0.03
	1000	1.156	(0.8981,0.8978)	0.03	1.156	(0.9489,0.9463)	0.26	1.156	(0.9789,0.9770)	0.19
$Pois(3)$	50	1.178	(0.9047,0.8951)	0.96	1.178	(0.9518,0.9464)	0.54	1.178	(0.9804,0.9740)	0.64
	100	1.166	(0.9022,0.9000)	0.22	1.166	(0.9512,0.9513)	-0.01	1.166	(0.9804,0.9792)	0.12
	500	1.157	(0.9014,0.9006)	0.08	1.157	(0.9527,0.9512)	0.15	1.157	(0.9803,0.9784)	0.19
	1000	1.156	(0.9052,0.9040)	0.12	1.156	(0.9505,0.9494)	0.11	1.156	(0.9779,0.9791)	-0.12
$Exp(1)$	50	1.178	(0.8900,0.8835)	0.65	1.178	(0.9373,0.9283)	0.90	1.178	(0.9693,0.9601)	0.92
	100	1.166	(0.8930,0.8899)	0.31	1.166	(0.9482,0.9368)	1.14	1.166	(0.9740,0.9693)	0.47
	500	1.157	(0.8998,0.8997)	0.01	1.157	(0.9479,0.9501)	-0.22	1.157	(0.9799,0.9782)	0.17
	1000	1.156	(0.9027,0.8978)	0.49	1.156	(0.9532,0.9471)	0.61	1.156	(0.9809,0.9789)	0.20
$Gamma(0.025,1)$	50	1.178	(0.6693,0.6585)	1.08	1.178	(0.7070,0.6882)	1.88	1.178	(0.7397,0.7153)	2.44
	100	1.166	(0.7483,0.7454)	0.29	1.166	(0.7904,0.7791)	1.33	1.166	(0.8241,0.8092)	1.49
	500	1.157	(0.8603,0.8541)	0.62	1.157	(0.9053,0.8966)	0.87	1.157	(0.9395,0.9308)	0.87
	1000	1.156	(0.8837,0.8728)	1.09	1.156	(0.9294,0.9221)	0.73	1.156	(0.9612,0.9544)	0.68
$Gamma(0.001,1)$	50	1.178	(0.1544,0.1521)	0.23	1.178	(0.1610,0.1574)	0.36	1.178	(0.1666,0.1622)	0.44
	100	1.166	(0.2491,0.2453)	0.38	1.166	(0.2595,0.2554)	0.41	1.166	(0.2696,0.2637)	0.59
	500	1.157	(0.5190,0.5116)	0.74	1.157	(0.5470,0.5319)	1.51	1.157	(0.5694,0.5552)	1.42
	1000	1.156	(0.6303,0.6243)	0.60	1.156	(0.6700,0.6527)	1.73	1.156	(0.6996,0.6787)	2.09
	5000	1.155	(0.8068,0.8005)	0.63	1.155	(0.8572,0.8385)	1.87	1.155	(0.8914,0.8732)	1.82
$Weibull(1,0.2)$	50	1.178	(0.5014,0.4827)	1.87	1.178	(0.5387,0.5135)	2.52	1.178	(0.5756,0.5445)	3.11
	100	1.166	(0.5835,0.5623)	2.12	1.166	(0.6274,0.5984)	2.90	1.166	(0.6639,0.6351)	2.88
	500	1.157	(0.7056,0.6857)	1.99	1.157	(0.7533,0.7303)	2.30	1.157	(0.7991,0.7712)	2.79
	1000	1.156	(0.7519,0.7309)	2.10	1.156	(0.8004,0.7796)	2.08	1.156	(0.8452,0.8220)	2.32
	5000	1.155	(0.8219,0.8119)	1.00	1.155	(0.8735,0.8602)	1.33	1.155	(0.9128,0.8979)	1.49

Table 2.5: Comparison table for I_1 vs I_0, values of $r_1(\widehat{CP}_1,\widehat{CP}_0)\Delta_1$

Distribution	n	$1-\alpha=0.9$ $(a,b)=(-3.58,3.48)$			$1-\alpha=0.95$ $(a,b)=(-4.41,4.04)$			$1-\alpha=0.98$ $(a,b)=(-5.41,4.59)$		
$Pareto(1,2)$	50	1.229	(0.8061,0.7688)	3.73	1.235	(0.8636,0.8147)	4.89	1.231	(0.9098,0.8581)	5.17
	100	1.196	(0.8214,0.7908)	3.06	1.202	(0.8796,0.8363)	4.33	1.198	(0.9284,0.8822)	4.62
	500	1.157	(0.8529,0.8379)	1.50	1.163	(0.9129,0.8855)	2.74	1.159	(0.9520,0.9235)	2.85
	1000	1.149	(0.8670,0.8516)	1.54	1.154	(0.9204,0.9006)	1.98	1.151	(0.9604,0.9357)	2.47
	5000	1.138	(0.8783,0.8664)	1.19	1.143	(0.9329,0.9173)	1.56	1.140	(0.9695,0.9518)	1.77
$Frechet(3)$	50	1.229	(0.8795,0.8472)	3.23	1.235	(0.9299,0.8968)	3.31	1.231	(0.9649,0.9346)	3.03
	100	1.196	(0.8893,0.8700)	1.93	1.202	(0.9403,0.9167)	2.36	1.198	(0.9729,0.9498)	2.31
	500	1.157	(0.9028,0.8909)	1.19	1.163	(0.9491,0.9381)	1.10	1.159	(0.9781,0.9669)	1.12
	1000	1.149	(0.9067,0.8958)	1.09	1.154	(0.9515,0.9445)	0.70	1.151	(0.9811,0.9725)	0.86
$Burr(3,2)$	50	1.229	(0.9155,0.8897)	2.58	1.235	(0.9596,0.9395)	2.01	1.231	(0.9843,0.9700)	1.43
	100	1.196	(0.9132,0.9005)	1.27	1.202	(0.9582,0.9461)	1.21	1.198	(0.9843,0.9776)	0.67
	500	1.157	(0.9102,0.9015)	0.87	1.163	(0.9555,0.9487)	0.68	1.159	(0.9809,0.9783)	0.26
	1000	1.149	(0.9076,0.9016)	0.60	1.154	(0.9544,0.9505)	0.39	1.151	(0.9805,0.9795)	0.10
$Pareto(1,6)$	50	1.229	(0.8948,0.8634)	3.14	1.235	(0.9410,0.9149)	2.61	1.231	(0.9715,0.9479)	2.36
	100	1.196	(0.9050,0.8847)	2.03	1.202	(0.9505,0.9349)	1.56	1.198	(0.9788,0.9634)	1.54
	500	1.157	(0.9070,0.8984)	0.86	1.163	(0.9545,0.9462)	0.83	1.159	(0.9805,0.9746)	0.59
	1000	1.149	(0.9080,0.9013)	0.67	1.154	(0.9550,0.9469)	0.81	1.151	(0.9805,0.9765)	0.40
$Burr(3,4)$	50	1.229	(0.9182,0.8938)	2.44	1.235	(0.9646,0.9460)	1.86	1.231	(0.9847,0.9737)	1.10
	100	1.196	(0.9162,0.9008)	1.54	1.202	(0.9597,0.9486)	1.11	1.198	(0.9856,0.9786)	0.70
	500	1.157	(0.9066,0.9004)	0.62	1.163	(0.9560,0.9485)	0.75	1.159	(0.9809,0.9785)	0.24
	1000	1.149	(0.9082,0.9033)	0.49	1.154	(0.9535,0.9493)	0.42	1.151	(0.9779,0.9791)	-0.12
$t(8)$	50	1.229	(0.9189,0.8928)	2.61	1.235	(0.9592,0.9466)	1.26	1.231	(0.9816,0.9781)	0.35
	100	1.196	(0.9120,0.8998)	1.22	1.202	(0.9568,0.9508)	0.60	1.198	(0.9820,0.9773)	0.47
	500	1.157	(0.9053,0.8951)	1.02	1.163	(0.9538,0.9493)	0.45	1.159	(0.9808,0.9798)	0.10
	1000	1.149	(0.9052,0.8999)	0.53	1.154	(0.9524,0.9488)	0.36	1.151	(0.9784,0.9791)	-0.07
$Burr(8,7)$	50	1.229	(0.9214,0.8958)	2.56	1.235	(0.9601,0.9454)	1.47	1.231	(0.9826,0.9753)	0.73
	100	1.196	(0.9167,0.8994)	1.73	1.202	(0.9609,0.9495)	1.14	1.198	(0.9841,0.9796)	0.45
	500	1.157	(0.9062,0.9006)	0.56	1.163	(0.9521,0.9496)	0.25	1.159	(0.9784,0.9778)	0.06
	1000	1.149	(0.9068,0.9042)	0.26	1.154	(0.9519,0.9495)	0.24	1.151	(0.9768,0.9777)	-0.09
$Frechet(10)$	50	1.229	(0.9129,0.8846)	2.83	1.235	(0.9557,0.9347)	2.10	1.231	(0.9827,0.9660)	1.67
	100	1.196	(0.9121,0.8999)	1.22	1.202	(0.9572,0.9431)	1.41	1.198	(0.9838,0.9760)	0.78
	500	1.157	(0.9097,0.9001)	0.96	1.163	(0.9555,0.9473)	0.82	1.159	(0.9803,0.9774)	0.29
	1000	1.149	(0.9078,0.9002)	0.76	1.154	(0.9544,0.9506)	0.38	1.151	(0.9790,0.9791)	0.04
$N(0,1)$	50	1.229	(0.9150,0.8955)	1.95	1.235	(0.9580,0.9460)	1.20	1.231	(0.9830,0.9737)	0.93
	100	1.196	(0.9170,0.8955)	2.15	1.202	(0.9610,0.9478)	1.32	1.198	(0.9851,0.9774)	0.77
	500	1.157	(0.9010,0.8956)	0.54	1.163	(0.9510,0.9482)	0.28	1.159	(0.9791,0.9791)	0.00
	1000	1.149	(0.9030,0.8978)	0.52	1.154	(0.9520,0.9463)	0.57	1.151	(0.9791,0.9770)	0.21
$Pois(3)$	50	1.229	(0.9210,0.8951)	2.59	1.235	(0.9650,0.9464)	1.86	1.231	(0.9853,0.9740)	1.13
	100	1.196	(0.9140,0.9000)	1.40	1.202	(0.9610,0.9513)	0.97	1.198	(0.9855,0.9792)	0.63
	500	1.157	(0.9070,0.9006)	0.64	1.163	(0.9570,0.9512)	0.58	1.159	(0.9803,0.9784)	0.19
	1000	1.149	(0.9100,0.9040)	0.60	1.154	(0.9510,0.9494)	0.16	1.151	(0.9786,0.9791)	-0.05
$Exp(1)$	50	1.229	(0.9050,0.8835)	2.15	1.235	(0.9500,0.9283)	2.17	1.231	(0.9788,0.9601)	1.87
	100	1.196	(0.9070,0.8899)	1.71	1.202	(0.9510,0.9368)	1.42	1.198	(0.9814,0.9693)	1.21
	500	1.157	(0.9060,0.8997)	0.63	1.163	(0.9540,0.9501)	0.39	1.159	(0.9821,0.9782)	0.39
	1000	1.149	(0.9080,0.8978)	1.02	1.154	(0.9570,0.9471)	0.99	1.151	(0.9817,0.9789)	0.28
$Gamma(0.025,1)$	50	1.229	(0.6831,0.6585)	2.46	1.235	(0.7233,0.6882)	3.51	1.231	(0.7615,0.7153)	4.62
	100	1.196	(0.7593,0.7454)	1.39	1.202	(0.8068,0.7791)	2.77	1.198	(0.8439,0.8092)	3.47
	500	1.157	(0.8678,0.8541)	1.37	1.163	(0.9153,0.8966)	1.87	1.159	(0.9507,0.9308)	1.99
	1000	1.149	(0.8898,0.8728)	1.70	1.154	(0.9365,0.9221)	1.44	1.151	(0.9686,0.9544)	1.42
$Gamma(0.001,1)$	50	1.229	(0.1567,0.1521)	0.46	1.235	(0.1640,0.1574)	0.66	1.231	(0.1705,0.1622)	0.83
	100	1.196	(0.2520,0.2453)	0.67	1.202	(0.2653,0.2554)	0.99	1.198	(0.2764,0.2637)	1.27
	500	1.157	(0.5220,0.5116)	1.04	1.163	(0.5538,0.5319)	2.19	1.159	(0.5810,0.5552)	2.58
	1000	1.149	(0.6355,0.6243)	1.12	1.154	(0.6777,0.6527)	2.50	1.151	(0.7109,0.6787)	3.22
	5000	1.138	(0.8103,0.8005)	0.98	1.143	(0.8663,0.8385)	2.78	1.140	(0.9031,0.8732)	2.99
$Weibull(1,0.2)$	50	1.229	(0.5135,0.4827)	3.08	1.235	(0.5577,0.5135)	4.42	1.231	(0.5992,0.5445)	5.47
	100	1.196	(0.5939,0.5623)	3.16	1.202	(0.6424,0.5984)	4.40	1.198	(0.6880,0.6351)	5.29
	500	1.157	(0.7121,0.6857)	2.64	1.163	(0.7691,0.7303)	3.38	1.159	(0.8199,0.7712)	4.87
	1000	1.149	(0.7571,0.7309)	2.62	1.154	(0.8132,0.7796)	3.36	1.151	(0.8635,0.8220)	4.15
	5000	1.138	(0.8239,0.8119)	1.20	1.143	(0.8808,0.8602)	2.06	1.140	(0.9266,0.8979)	2.87

Table 2.6: Comparison table for I_{1,q_1} vs I_0, values of $r_{1,q_1}(\widehat{CP}_{1,q_1},\widehat{CP}_0)\Delta_{1,q_1}$

Distribution	n	$1-\alpha=0.9$ $(a,b)=(-2.80,2.52)$			$1-\alpha=0.95$ $(a,b)=(-3.46,2.92)$			$1-\alpha=0.98$ $(a,b)=(-3.84,3.9)$		
$Pareto(1,2)$	50	1.209	(0.8126,0.7688)	4.38	1.217	(0.8704,0.8147)	5.57	1.243	(0.8969,0.8581)	3.88
	100	1.169	(0.8235,0.7908)	3.27	1.177	(0.8862,0.8363)	4.99	1.203	(0.9123,0.8822)	3.01
	500	1.122	(0.8547,0.8379)	1.68	1.129	(0.9153,0.8855)	2.98	1.154	(0.9411,0.9235)	1.76
	1000	1.112	(0.8676,0.8516)	1.60	1.119	(0.9210,0.9006)	2.04	1.140	(0.9488,0.9357)	1.31
	5000	1.098	(0.8774,0.8664)	1.10	1.105	(0.9330,0.9173)	1.57	1.119	(0.9610,0.9518)	0.92
$Frechet(3)$	50	1.209	(0.8843,0.8472)	3.71	1.217	(0.9341,0.8968)	3.73	1.243	(0.9588,0.9346)	2.42
	100	1.169	(0.8928,0.8700)	2.28	1.177	(0.9435,0.9167)	2.68	1.203	(0.9545,0.9498)	0.47
	500	1.122	(0.9020,0.8909)	1.11	1.129	(0.9494,0.9381)	1.13	1.154	(0.9632,0.9669)	-0.37
	1000	1.112	(0.9029,0.8958)	0.71	1.119	(0.9499,0.9445)	0.54	1.140	(0.9672,0.9725)	-0.53
$Burr(3,2)$	50	1.209	(0.9169,0.8897)	2.72	1.217	(0.9620,0.9395)	2.25	1.243	(0.9853,0.9700)	1.53
	100	1.169	(0.9137,0.9005)	1.32	1.177	(0.9594,0.9461)	1.33	1.203	(0.9873,0.9776)	0.97
	500	1.122	(0.9078,0.9015)	0.63	1.129	(0.9536,0.9487)	0.49	1.154	(0.9842,0.9783)	0.59
	1000	1.112	(0.9027,0.9016)	0.11	1.119	(0.9507,0.9505)	0.02	1.140	(0.9832,0.9795)	0.37
$Pareto(1,6)$	50	1.209	(0.8984,0.8634)	3.50	1.217	(0.9452,0.9149)	3.03	1.243	(0.9682,0.9479)	2.03
	100	1.169	(0.9054,0.8847)	2.07	1.177	(0.9541,0.9349)	1.92	1.203	(0.9772,0.9634)	1.38
	500	1.122	(0.9047,0.8984)	0.63	1.129	(0.9544,0.9462)	0.82	1.154	(0.9804,0.9746)	0.58
	1000	1.112	(0.9030,0.9013)	0.17	1.119	(0.9520,0.9469)	0.51	1.140	(0.9808,0.9765)	0.43
$Burr(3,4)$	50	1.209	(0.9204,0.8938)	2.66	1.217	(0.9634,0.9460)	1.69	1.243	(0.9875,0.9737)	1.38
	100	1.169	(0.9149,0.9008)	1.41	1.177	(0.9597,0.9486)	1.11	1.203	(0.9887,0.9786)	1.01
	500	1.122	(0.9053,0.9004)	0.49	1.129	(0.9533,0.9485)	0.48	1.154	(0.9842,0.9785)	0.57
	1000	1.112	(0.9053,0.9033)	-0.10	1.119	(0.9487,0.9493)	-0.06	1.140	(0.9829,0.9791)	0.38
$t(8)$	50	1.209	(0.9184,0.8928)	2.56	1.217	(0.9591,0.9466)	1.25	1.243	(0.9872,0.9781)	0.91
	100	1.169	(0.9111,0.8998)	1.13	1.177	(0.9562,0.9508)	0.54	1.203	(0.9863,0.9773)	0.90
	500	1.122	(0.9021,0.8951)	0.70	1.129	(0.9494,0.9493)	0.01	1.154	(0.9848,0.9798)	0.50
	1000	1.112	(0.9008,0.8999)	0.09	1.119	(0.9492,0.9488)	0.04	1.140	(0.9840,0.9791)	0.49
$Burr(8,7)$	50	1.209	(0.9216,0.8958)	2.58	1.217	(0.9598,0.9454)	1.44	1.243	(0.9884,0.9753)	1.31
	100	1.169	(0.9155,0.8994)	1.61	1.177	(0.9598,0.9495)	1.03	1.203	(0.9892,0.9796)	0.96
	500	1.122	(0.9020,0.9006)	0.14	1.129	(0.9498,0.9496)	0.02	1.154	(0.9841,0.9778)	0.63
	1000	1.112	(0.9007,0.9042)	-0.35	1.119	(0.9479,0.9495)	-0.16	1.140	(0.9821,0.9777)	0.44
$Frechet(10)$	50	1.209	(0.9130,0.8846)	2.84	1.217	(0.9590,0.9347)	2.43	1.243	(0.9812,0.9660)	1.52
	100	1.169	(0.9121,0.8999)	1.31	1.177	(0.9597,0.9431)	1.66	1.203	(0.9848,0.9760)	0.88
	500	1.122	(0.9078,0.9001)	0.77	1.129	(0.9540,0.9473)	0.67	1.154	(0.9839,0.9774)	0.65
	1000	1.112	(0.9044,0.9002)	0.42	1.119	(0.9502,0.9506)	-0.04	1.140	(0.9826,0.9791)	0.35
$N(0,1)$	50	1.209	(0.9154,0.8955)	1.99	1.217	(0.9573,0.9460)	1.13	1.243	(0.9880,0.9737)	1.43
	100	1.169	(0.9141,0.8955)	1.86	1.177	(0.9583,0.9478)	1.05	1.203	(0.9880,0.9774)	1.06
	500	1.122	(0.8985,0.8956)	0.29	1.129	(0.9465,0.9482)	-0.17	1.154	(0.9840,0.9791)	0.49
	1000	1.112	(0.8996,0.8978)	0.18	1.119	(0.9473,0.9463)	0.10	1.140	(0.9830,0.9770)	0.60
$Pois(3)$	50	1.209	(0.9211,0.8951)	2.60	1.217	(0.9636,0.9464)	1.72	1.243	(0.9880,0.9740)	1.40
	100	1.169	(0.9134,0.9000)	1.34	1.177	(0.9596,0.9513)	0.83	1.203	(0.9880,0.9792)	0.88
	500	1.122	(0.9044,0.9006)	0.38	1.129	(0.9537,0.9512)	0.25	1.154	(0.9840,0.9784)	0.56
	1000	1.112	(0.9059,0.9040)	0.19	1.119	(0.9480,0.9494)	-0.14	1.140	(0.9820,0.9791)	0.29
$Exp(1)$	50	1.209	(0.9093,0.8835)	2.58	1.217	(0.9554,0.9283)	2.71	1.243	(0.9770,0.9601)	1.69
	100	1.169	(0.9038,0.8899)	1.39	1.177	(0.9548,0.9368)	1.80	1.203	(0.9820,0.9693)	1.27
	500	1.122	(0.9040,0.8997)	0.43	1.129	(0.9536,0.9501)	0.35	1.154	(0.9840,0.9782)	0.58
	1000	1.112	(0.9028,0.8978)	0.50	1.119	(0.9527,0.9471)	0.56	1.140	(0.9850,0.9789)	0.61
$Gamma(0.025,1)$	50	1.209	(0.6870,0.6585)	2.85	1.217	(0.7287,0.6882)	4.05	1.243	(0.7479,0.7153)	3.26
	100	1.169	(0.7604,0.7454)	1.50	1.177	(0.8086,0.7791)	2.95	1.203	(0.8310,0.8092)	2.18
	500	1.122	(0.8662,0.8541)	1.21	1.129	(0.9159,0.8966)	1.93	1.154	(0.9421,0.9308)	1.13
	1000	1.112	(0.8851,0.8728)	1.23	1.119	(0.9384,0.9221)	1.63	1.140	(0.9635,0.9544)	0.91
$Gamma(0.001,1)$	50	1.209	(0.1563,0.1521)	0.42	1.217	(0.1654,0.1574)	0.80	1.243	(0.1684,0.1622)	0.62
	100	1.169	(0.2516,0.2453)	0.63	1.177	(0.2656,0.2554)	1.02	1.203	(0.2717,0.2637)	0.80
	500	1.122	(0.5221,0.5116)	1.05	1.129	(0.5560,0.5319)	2.41	1.154	(0.5691,0.5552)	1.39
	1000	1.112	(0.6348,0.6243)	1.05	1.119	(0.6789,0.6527)	2.62	1.140	(0.6972,0.6787)	1.85
	5000	1.098	(0.8062,0.8005)	0.57	1.105	(0.8664,0.8385)	2.79	1.119	(0.8885,0.8732)	1.53
$Weibull(1,0.2)$	50	1.209	(0.5173,0.4827)	3.46	1.217	(0.5638,0.5135)	5.03	1.243	(0.5857,0.5445)	4.12
	100	1.169	(0.5968,0.5623)	3.45	1.177	(0.6474,0.5984)	4.90	1.203	(0.6694,0.6351)	3.43
	500	1.122	(0.7145,0.6857)	2.88	1.129	(0.7722,0.7303)	4.19	1.154	(0.8003,0.7712)	2.91
	1000	1.112	(0.7580,0.7309)	2.71	1.119	(0.8155,0.7796)	3.59	1.140	(0.8446,0.8220)	2.26
	5000	1.098	(0.8228,0.8119)	1.09	1.204	(0.8815,0.8602)	2.13	1.119	(0.9115,0.8979)	1.36

Table 2.7: Comparison table for I_{1,q_2} vs I_0, values of $r_{1,q_2}(\widehat{CP}_{1,q_2}, \widehat{CP}_0)\Delta_{1,q_2}$

2.2 FACI's based on $h_2(\cdot)$

Substituting $h(\cdot) = h_2(\cdot)$ of (1.14) and $q(t) = \sqrt{q_j(t)}$, $j = \overline{1,2}$, into the FCLT of (1.9), we have:

$$\int_0^1 \frac{(T_n^t(X_1 - \mu, ..., X_n - \mu))^2}{q_j\left(\frac{[nt]}{n}\right)} dt \xrightarrow[n\to\infty]{d} \int_0^1 \frac{W^2(t)}{q_j(t)} dt, \tag{2.9}$$

which amounts to

$$\frac{1}{s_n^2 n^2} \sum_{k=1}^{n-1} \frac{\left(\sum_{i=1}^k (X_i - \mu)\right)^2}{q_j\left(\frac{k}{n}\right)} \xrightarrow[n\to\infty]{d} \int_0^1 \frac{W^2(t)}{q_j(t)} dt, \tag{2.10}$$

where the cumulative distribution function of $\int_0^1 (W^2(t)/q_j(t))dt$ has been unavailable, so that we have to tabulate the required quantiles ourselves in Table 2.8. Table 2.8 also lists the quantiles of $\int_0^1 W^2(t)dt$, needed for the upcoming FACI I_2 of (2.15), which we tabulated ourselves, though they are also available in Csörgő and Horváth. (1981).

Let $\alpha \in (0, 1)$ and define $b > 0$ such that $P\left(\int_0^1 (W^2(t)/q_j(t))dt \leqslant b\right) = 1-\alpha$. It follows from (2.10) that

$$P\left(\frac{1}{s_n^2 n^2} \sum_{k=1}^{n-1} \frac{\left(\sum_{i=1}^k (X_i - \mu)\right)^2}{q_j\left(\frac{k}{n}\right)} \leqslant b\right) \xrightarrow[n\to\infty]{d} 1 - \alpha. \tag{2.11}$$

	$\gamma = 0.9$	$\gamma = 0.95$	$\gamma = 0.98$
γ quantile of $\int_0^1 \frac{W^2(t)}{q_1(t)}dt$	4.41	6.27	8.7
γ quantile of $\int_0^1 \frac{W^2(t)}{q_2(t)}dt$	3.5	4.83	6.64
γ quantile of $\int_0^1 W^2(t)dt$	1.21	1.68	2.26

Table 2.8: Some quantiles of $\int_0^1 \frac{W^2(t)}{q_1(t)}dt$, $\int_0^1 \frac{W^2(t)}{q_2(t)}dt$ and $\int_0^1 W^2(t)dt$

The quadratic inequality in μ under the probability sign in (2.11) reads:

$$\mu^2 \sum_{k=1}^{n-1} \frac{k^2}{q_j\left(\frac{k}{n}\right)} - 2\mu \sum_{k=1}^{n-1} \frac{k\sum_{i=1}^{k} X_i}{q_j\left(\frac{k}{n}\right)} + \sum_{k=1}^{n-1} \frac{(\sum_{i=1}^{k} X_i)^2}{q_j\left(\frac{k}{n}\right)} - bn^2 s_n^2 \leqslant 0. \tag{2.12}$$

This leads to the following $1-\alpha$ FACI for μ :

$$I_{2,q_j} := \left[\frac{\sum_{k=1}^{n-1} \frac{k\sum_{i=1}^{k} X_i}{q_j\left(\frac{k}{n}\right)} - \sqrt{d_n}}{\sum_{k=1}^{n-1} \frac{k^2}{q_j\left(\frac{k}{n}\right)}}, \frac{\sum_{k=1}^{n-1} \frac{k\sum_{i=1}^{k} X_i}{q_j\left(\frac{k}{n}\right)} + \sqrt{d_n}}{\sum_{k=1}^{n-1} \frac{k^2}{q_j\left(\frac{k}{n}\right)}} \right], \tag{2.13}$$

where

$$d_n := 4\left(\sum_{k=1}^{n-1} \frac{k\sum_{i=1}^{k} X_i}{q_j\left(\frac{k}{n}\right)}\right)^2 - 4\sum_{k=1}^{n-1} \frac{k^2}{q_j\left(\frac{k}{n}\right)} \left(\sum_{k=1}^{n-1} \frac{(\sum_{i=1}^{k} X_i)^2}{q_j\left(\frac{k}{n}\right)} - bn^2 s_n^2\right). \tag{2.14}$$

We note that it is not feasible to obtain closed form expressions for the expected lengths of I_{2,q_1} and I_{2,q_2} and hence they are approximated via their empirical counterparts $\widehat{r}_{2,q_1}$ and $\widehat{r}_{2,q_2}$ as defined in (1.20).

Similarly to I_1 of (2.6), a version of I_{2,q_j} that is based on the FCLT in (1.7) with $h(\cdot) = h_2(\cdot)$ was studied in Tuzov (2014) and Martsynyuk and Tuzov (2016). The respective FACI was found to be

$$I_2 := \left[\frac{\sum_{k=1}^{n-1} k \sum_{i=1}^{k} X_i \mp \sqrt{(\sum_{k=1}^{n-1} k \sum_{i=1}^{k} X_i)^2 - c_n(\sum_{k=1}^{n-1}(\sum_{i=1}^{k} X_i)^2 - bn^2 s_n^2)}}{c_n} \right], \tag{2.15}$$

where

$$c_n := \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6} \tag{2.16}$$

and b is the 0.9, 0.95 or 0.98 quantile of $\int_0^1 W^2(t)dt$, as in Table 2.8. Similarly to I_{1,q_1} and I_{1,q_2}, evaluation of the expected length of I_2 is aggravated by the square root in (2.15). The finite-sample properties of I_2 are investigated here for more distributions and sample sizes as compared to Tuzov (2014) and Martsynyuk and Tuzov (2016).

A snapshot of finite-sample performance of I_2, I_{2,q_1} and I_{2,q_2} is provided in Table 2.9, with the ranges of the coverage probabilities differences Δ_2, Δ_{2,q_1} and Δ_{2,q_2} defined in (2.1), as well as with averages (in n) of $\widehat{r}_2$, $\widehat{r}_{2,q_1}$ and $\widehat{r}_{2,q_2}$, for each of the five classes of distributions as in Tables 1.2. Table 2.9 is followed by more detailed Tables 2.10 - 2.12 with the quadruplets $\widehat{r}_2(\widehat{CP}_2, \widehat{CP}_0)\Delta_2$, $\widehat{r}_{2,q_1}(\widehat{CP}_{2,q_1}, \widehat{CP}_0)\Delta_{2,q_1}$ and $\widehat{r}_{2,q_2}(\widehat{CP}_{2,q_2}, \widehat{CP}_0)\Delta_{2,q_2}$ calculated for each distribution of Table 1.1, sample sizes $n = 50, 100, 500, 1000$ and, for some distributions, also for $n = 5000$, as well as for the confidence levels $1 - \alpha = 0.9$, 0.95 and 0.98.

According to Tables 2.9 - 2.12, all three ratios $\widehat{r}_2$, $\widehat{r}_{2,q_1}$ and $\widehat{r}_{2,q_2}$ are above one and below 1.143, and they decrease as n increases. While $\widehat{r}_2$, $\widehat{r}_{2,q_1}$ and $\widehat{r}_{2,q_2}$ depend on the distribution of X_i's, their values are rounded up to three decimal places and hence appear as distribution free. Also, $\widehat{r}_{2,q_1}$ seems to increase in α, while $\widehat{r}_{2,q_2}$ appears to decrease in α.

As for the empirical finite-sample coverage probabilities, all $\widehat{CP}_2$, $\widehat{CP}_{2,q_1}$ and $\widehat{CP}_{2,q_2}$ are higher than $\widehat{CP}_0$, except for some cases (indicated by the negative signs of Δ_2, Δ_{2,q_1} and Δ_{2,q_2}). By in large, the difference in the probability coverage between I_2 and I_0, namely Δ_2, is smaller than one of Δ_{2,q_1} or Δ_{2,q_2}, depending on α, which can also be quickly concluded from summary Table 2.9. From the later table, for $1 - \alpha = 0.95$ and 0.98, all Δ_2, Δ_{2,q_1} and Δ_{2,q_2} appear to be somewhat larger for the heavy-tailed distributions with no skewness and light-tailed distributions with large skewness. All $\widehat{CP}_2$, $\widehat{CP}_{2,q_1}$, $\widehat{CP}_{2,q_2}$ and $\widehat{CP}_0$ are again seen to be lower for the heavy-tailed and/or largely skewed distributions. Moreover, all four are lower than, and slowly converging to, $1 - \alpha$ for distributions such as $Gamma(0.025, 1)$, $Gamma(0.001, 1)$ and $Weibull(1, 0.2)$. For such distributions, I_{2,q_1} and I_{2,q_2} are more appealing than I_0, at even smaller expense in terms of expected lengths than in cases of I_{1,q_1} and I_{1,q_2}.

All in all, compared to I_1, I_{1,q_1} and I_{1,q_2}, the FACI's I_2, I_{2,q_1} and I_{2,q_2} display similar, but less pronounced trade-offs between their respective expected lengths and finite-sample coverage probabilities. All three FACI's have a bit higher coverage probabilities than that of I_0 (less so compared to I_1, I_{1,q_1} and I_{1,q_2}), being also a bit longer than I_0 (less so compared to I_1, I_{1,q_1} and I_{1,q_2}). Just like for I_1, I_{1,q_1} and I_{1,q_2}, one may prefer to use I_2, I_{2,q_1} and I_{2,q_2} especially for smaller n and distributions with fewer moments and/or higher skewness. Taking into account that I_2 is mostly on average longer than (or very close to) I_{2,q_1} and I_{2,q_2}, while its coverage probability is by in large smaller than one of the I_{2,q_1} and I_{2,q_2}, we conclude that I_{2,q_1} and I_{2,q_2} perform overall better than I_2. Moreover, for $1-\alpha = 0.9$, I_{2,q_2} is better than I_{2,q_1}. While for $1-\alpha$ $= 0.95$ and 0.98, there is a bit of trade-off between I_{2,q_1} and I_{2,q_2}, the FACI I_{2,q_1} appears to be more appealing due to more gain in coverage probability at almost no expense in expected length.

	Heavy-tailed distributions with no skewness	Heavy-tailed distributions with small skewness	Heavy-tailed distributions with large skewness	Light-tailed distributions with small skewness	Light-tailed distributions with large skewness
$1-\alpha=0.9$					
Δ_2	0.63% - 1.64%	0.16% - 0.79%	0.17% - 1.41%	0.02% - 1.21%	0.21% - 1.82%
average r_2	1.115	1.116	1.116	1.116	1.115
Δ_{2,q_1}	-0.08% - 1.94%	-0.30% - 1.79%	-0.44% - 1.81%	-0.45% - 1.55%	-0.12% - 1.64%
average $\widehat{r}_{2,q_1}$	1.092	1.095	1.095	1.095	1.092
Δ_{2,q_2}	0.26% - 2.17%	0.10% - 1.88%	0.00% - 2.13%	-0.26% - 1.94%	-0.13% - 1.82%
average $\widehat{r}_{2,q_2}$	1.082	1.086	1.086	1.086	1.082
$1-\alpha=0.95$					
Δ_2	0.37% - 2.02%	0.12% - 0.61%	0.26% - 0.93%	-0.10% - 0.83%	-0.10% - 2.11%
average r_2	1.117	1.118	1.118	1.118	1.117
Δ_{2,q_1}	0.28% - 2.51%	0.04% - 1.07%	0.25% - 1.48%	-0.02% - 1.27%	0.09% - 2.09%
average $\widehat{r}_{2,q_1}$	1.111	1.111	1.111	1.111	1.111
Δ_{2,q_2}	0.09% - 2.26%	-0.20% - 1.15%	0.11% - 1.34%	-0.23% - 1.17%	-0.26% - 1.66%
average $\widehat{r}_{2,q_2}$	1.079	1.084	1.084	1.084	1.079
$1-\alpha=0.98$					
Δ_2	0.03% - 1.64%	-0.18% - 0.37%	0.01% - 0.47%	-0.29% - 0.79%	0.04% - 2.09%
average r_2	1.100	1.101	1.101	1.101	1.100
Δ_{2,q_1}	0.19% - 2.34%	0.06% - 0.99%	0.22% - 1.12%	0.04% - 0.97%	0.13% - 2.59%
average $\widehat{r}_{2,q_1}$	1.108	1.111	1.111	1.111	1.108
Δ_{2,q_2}	-0.03% - 1.84%	-0.33% - 0.80%	0.03% - 0.78%	-0.25% - 0.88%	-0.03% - 2.21%
average $\widehat{r}_{2,q_2}$	1.075	1.079	1.079	1.079	1.075

Table 2.9: Ranges of Δ_2, Δ_{2,q_1} and Δ_{2,q_2} and averages (in n) of $\widehat{r}_2$, $\widehat{r}_{2,q_1}$ and $\widehat{r}_{2,q_2}$

Distribution	n	$1-\alpha=0.9$			$1-\alpha=0.95$			$1-\alpha=0.98$		
$Pareto(1,2)$	50	1.127	(0.7852,0.7688)	1.64	1.128	(0.8349,0.8147)	2.02	1.111	(0.8745,0.8581)	1.64
	100	1.117	(0.8068,0.7908)	1.60	1.119	(0.8549,0.8363)	1.86	1.102	(0.8970,0.8822)	1.48
	500	1.110	(0.8475,0.8379)	0.96	1.112	(0.8976,0.8855)	1.21	1.096	(0.9328,0.9235)	0.93
	1000	1.110	(0.8677,0.8516)	1.61	1.112	(0.9097,0.9006)	0.91	1.095	(0.9419,0.9357)	0.62
	5000	1.109	(0.8760,0.8664)	0.96	1.111	(0.9247,0.9173)	0.74	1.094	(0.9578,0.9518)	0.60
$Frechet(3)$	50	1.127	(0.8605,0.8472)	1.33	1.128	(0.9108,0.8968)	1.40	1.111	(0.9415,0.9346)	0.69
	100	1.117	(0.8764,0.8700)	0.64	1.119	(0.9283,0.9167)	1.16	1.102	(0.9569,0.9498)	0.71
	500	1.110	(0.8976,0.8909)	0.67	1.112	(0.9418,0.9381)	0.37	1.096	(0.9692,0.9669)	0.23
	1000	1.110	(0.9021,0.8958)	0.63	1.112	(0.9484,0.9445)	0.39	1.095	(0.9728,0.9725)	0.03
$Burr(3,2)$	50	1.127	(0.8976,0.8897)	0.79	1.128	(0.9456,0.9395)	0.61	1.111	(0.9737,0.9700)	0.37
	100	1.117	(0.9021,0.9005)	0.16	1.119	(0.9507,0.9461)	0.46	1.102	(0.9784,0.9776)	0.08
	500	1.110	(0.9060,0.9015)	0.45	1.112	(0.9508,0.9487)	0.21	1.096	(0.9789,0.9783)	0.06
	1000	1.110	(0.9054,0.9016)	0.38	1.112	(0.9517,0.9505)	0.12	1.095	(0.9778,0.9795)	-0.17
$Pareto(1,6)$	50	1.127	(0.8775,0.8634)	1.41	1.128	(0.9242,0.9149)	0.93	1.111	(0.9524,0.9479)	0.45
	100	1.117	(0.8926,0.8847)	0.79	1.119	(0.9393,0.9349)	0.44	1.102	(0.9681,0.9634)	0.47
	500	1.110	(0.9013,0.8984)	0.29	1.112	(0.9488,0.9462)	0.26	1.096	(0.9747,0.9746)	0.01
	1000	1.110	(0.9030,0.9013)	0.17	1.112	(0.9515,0.9469)	0.46	1.095	(0.9780,0.9765)	0.15
$Burr(3,4)$	50	1.127	(0.9010,0.8938)	0.72	1.128	(0.9513,0.9460)	0.53	1.111	(0.9762,0.9737)	0.25
	100	1.117	(0.9044,0.9008)	0.36	1.119	(0.9510,0.9486)	0.24	1.102	(0.9804,0.9786)	0.18
	500	1.110	(0.9039,0.9004)	0.35	1.112	(0.9513,0.9485)	0.28	1.096	(0.9779,0.9785)	-0.06
	1000	1.101	(0.9056,0.9033)	0.23	1.112	(0.9505,0.9493)	0.12	1.095	(0.9773,0.9791)	-0.18
$t(8)$	50	1.127	(0.9049,0.8928)	1.21	1.128	(0.9487,0.9466)	0.21	1.111	(0.9767,0.9781)	-0.14
	100	1.117	(0.9007,0.8998)	0.09	1.119	(0.9498,0.9508)	-0.10	1.102	(0.9773,0.9773)	0.00
	500	1.110	(0.8995,0.8951)	0.44	1.112	(0.9519,0.9493)	0.26	1.096	(0.9786,0.9798)	-0.12
	1000	1.101	(0.9025,0.8999)	0.26	1.112	(0.9508,0.9488)	0.20	1.095	(0.9870,0.9791)	0.79
$Burr(8,7)$	50	1.127	(0.9057,0.8958)	0.99	1.128	(0.9508,0.9454)	0.54	1.111	(0.9782,0.9753)	0.29
	100	1.117	(0.9035,0.8994)	0.41	1.119	(0.9529,0.9495)	0.34	1.102	(0.9810,0.9796)	0.14
	500	1.110	(0.9008,0.9006)	0.02	1.112	(0.9510,0.9496)	0.14	1.096	(0.9783,0.9778)	0.05
	1000	1.110	(0.9049,0.9042)	0.07	1.112	(0.9499,0.9495)	0.04	1.095	(0.9762,0.9777)	-0.15
$Frechet(10)$	50	1.127	(0.8941,0.8846)	0.95	1.128	(0.9430,0.9347)	0.83	1.111	(0.9708,0.9660)	0.48
	100	1.117	(0.9017,0.8999)	0.18	1.119	(0.9487,0.9431)	0.56	1.102	(0.9767,0.9760)	0.07
	500	1.110	(0.9039,0.9001)	0.38	1.112	(0.9508,0.9473)	0.35	1.096	(0.9777,0.9774)	0.03
	1000	1.110	(0.9053,0.9002)	0.51	1.112	(0.9511,0.9506)	0.05	1.095	(0.9775,0.9791)	-0.16
$N(0,1)$	50	1.127	(0.8985,0.8955)	0.30	1.128	(0.9480,0.9460)	0.20	1.111	(0.9759,0.9737)	0.22
	100	1.117	(0.9057,0.8955)	1.02	1.119	(0.9539,0.9478)	0.61	1.102	(0.9792,0.9774)	0.18
	500	1.110	(0.8985,0.8956)	0.29	1.112	(0.9497,0.9482)	0.15	1.096	(0.9771,0.9791)	-0.20
	1000	1.110	(0.8999,0.8978)	0.21	1.112	(0.9510,0.9463)	0.47	1.095	(0.9771,0.9770)	0.01
$Pois(3)$	50	1.127	(0.9036,0.8951)	0.85	1.128	(0.9517,0.9464)	0.53	1.111	(0.9781,0.9740)	0.41
	100	1.117	(0.9036,0.9000)	0.36	1.119	(0.9509,0.9513)	-0.04	1.102	(0.9794,0.9792)	0.02
	500	1.110	(0.9034,0.9006)	0.28	1.112	(0.9524,0.9512)	0.12	1.096	(0.9787,0.9784)	0.03
	1000	1.110	(0.9087,0.9040)	0.47	1.112	(0.9516,0.9494)	0.22	1.095	(0.9762,0.9791)	-0.29
$Exp(1)$	50	1.127	(0.8892,0.8835)	0.57	1.128	(0.9370,0.9283)	0.87	1.111	(0.9671,0.9601)	0.70
	100	1.117	(0.8953,0.8899)	0.54	1.119	(0.9420,0.9368)	0.52	1.102	(0.9713,0.9693)	0.20
	500	1.110	(0.9020,0.8997)	0.23	1.112	(0.9491,0.9501)	-0.10	1.096	(0.9793,0.9782)	0.11
	1000	1.110	(0.9046,0.8978)	0.68	1.112	(0.9523,0.9471)	0.52	1.095	(0.9793,0.9789)	0.04
$Gamma(0.025,1)$	50	1.127	(0.6669,0.6585)	0.84	1.128	(0.7035,0.6882)	1.53	1.111	(0.7322,0.7153)	1.69
	100	1.117	(0.7477,0.7454)	0.23	1.119	(0.7874,0.7791)	0.83	1.102	(0.8179,0.8092)	0.87
	500	1.110	(0.8678,0.8541)	1.37	1.112	(0.9045,0.8966)	0.79	1.096	(0.9338,0.9308)	0.30
	1000	1.110	(0.8898,0.8728)	1.70	1.112	(0.9293,0.9221)	0.72	1.095	(0.9591,0.9544)	0.47
$Gamma(0.001,1)$	50	1.127	(0.1542,0.1521)	0.21	1.128	(0.1598,0.1574)	0.24	1.111	(0.1652,0.1622)	0.30
	100	1.117	(0.2496,0.2453)	0.43	1.119	(0.2594,0.2554)	0.40	1.102	(0.2672,0.2637)	0.35
	500	1.110	(0.5169,0.5116)	0.53	1.112	(0.5435,0.5319)	1.16	1.096	(0.5630,0.5552)	0.78
	1000	1.110	(0.6308,0.6243)	0.65	1.112	(0.6680,0.6527)	1.53	1.095	(0.6919,0.6787)	1.32
	5000	1.110	(0.8074,0.8005)	0.69	1.111	(0.8538,0.8385)	1.53	1.094	(0.8850,0.8732)	1.18
$Weibull(1,0.2)$	50	1.127	(0.4960,0.4827)	1.33	1.128	(0.5320,0.5135)	1.85	1.111	(0.5654,0.5445)	2.09
	100	1.117	(0.5802,0.5623)	1.79	1.119	(0.6195,0.5984)	2.11	1.102	(0.6548,0.6351)	1.97
	500	1.110	(0.7019,0.6857)	1.62	1.112	(0.7486,0.7303)	1.83	1.096	(0.7897,0.7712)	1.85
	1000	1.110	(0.7491,0.7309)	1.82	1.112	(0.7977,0.7796)	1.81	1.095	(0.8354,0.8220)	1.34
	5000	1.109	(0.8180,0.8119)	0.61	1.111	(0.8721,0.8602)	1.19	1.095	(0.9061,0.8979)	0.82

Table 2.10: Comparison table for I_2 vs I_0, values of $\widehat{r}_2(\widehat{CP}_2, \widehat{CP}_0)\Delta_2$

Distribution	n	$1-\alpha=0.9$	$1-\alpha=0.95$	$1-\alpha=0.98$
$Pareto(1,2)$	50	1.128 (0.7882,0.7688) 1.94	1.143 (0.8398,0.8147) 2.51	1.143 (0.8815,0.8581) 2.34
	100	1.104 (0.8060,0.7908) 1.52	1.120 (0.8569,0.8363) 2.06	1.121 (0.9015,0.8822) 1.93
	500	1.077 (0.8418,0.8379) 0.39	1.092 (0.8963,0.8855) 1.08	1.093 (0.9347,0.9235) 1.12
	1000	1.071 (0.8562,0.8516) 0.46	1.087 (0.9082,0.9006) 0.76	1.088 (0.9432,0.9357) 0.75
	5000	1.064 (0.8679,0.8664) 0.15	1.079 (0.9223,0.9173) 0.50	1.080 (0.9581,0.9518) 0.63
$Frechet(3)$	50	1.128 (0.8637,0.8472) 1.65	1.143 (0.9152,0.8968) 1.84	1.143 (0.9479,0.9346) 1.33
	100	1.104 (0.8779,0.8700) 0.79	1.120 (0.9265,0.9167) 0.98	1.121 (0.9609,0.9498) 1.11
	500	1.077 (0.8934,0.8909) 0.25	1.092 (0.9409,0.9381) 0.28	1.093 (0.9714,0.9669) 0.45
	1000	1.071 (0.8966,0.8958) -0.08	1.087 (0.9479,0.9445) 0.34	1.088 (0.9744,0.9725) 0.19
$Burr(3,2)$	50	1.128 (0.9076,0.8897) 1.79	1.143 (0.9502,0.9395) 1.07	1.143 (0.9799,0.9700) 0.99
	100	1.104 (0.9026,0.9005) 0.21	1.120 (0.9537,0.9461) 0.76	1.121 (0.9822,0.9776) 0.46
	500	1.077 (0.9017,0.9015) 0.20	1.092 (0.9504,0.9487) 0.17	1.093 (0.9802,0.9783) 0.19
	1000	1.071 (0.9000,0.9016) -0.16	1.087 (0.9509,0.9505) 0.04	1.088 (0.9801,0.9795) 0.06
$Pareto(1,6)$	50	1.128 (0.8815,0.8634) 1.81	1.143 (0.9297,0.9149) 1.48	1.143 (0.9591,0.9479) 1.12
	100	1.104 (0.8931,0.8847) 0.84	1.120 (0.9412,0.9349) 0.63	1.121 (0.9718,0.9634) 0.84
	500	1.077 (0.8987,0.8984) 0.03	1.092 (0.9487,0.9462) 0.25	1.093 (0.9768,0.9746) 0.22
	1000	1.071 (0.8969,0.9013) -0.44	1.087 (0.9507,0.9469) 0.38	1.088 (0.9795,0.9765) 0.30
$Burr(3,4)$	50	1.128 (0.9061,0.8938) 1.23	1.143 (0.9562,0.9465) 0.97	1.143 (0.9824,0.9737) 0.87
	100	1.104 (0.9051,0.9008) 0.43	1.120 (0.9549,0.9486) 0.63	1.121 (0.9851,0.9786) 0.65
	500	1.077 (0.9015,0.9004) 0.11	1.092 (0.9516,0.9485) 0.31	1.093 (0.9800,0.9785) 0.15
	1000	1.071 (0.9003,0.9033) -0.30	1.087 (0.9499,0.9493) 0.06	1.088 (0.9797,0.9791) 0.06
$t(8)$	50	1.128 (0.9083,0.8928) 1.55	1.143 (0.9535,0.9466) 0.69	1.143 (0.9813,0.9781) 0.32
	100	1.104 (0.9006,0.8998) 0.08	1.120 (0.9529,0.9508) 0.21	1.121 (0.9814,0.9773) 0.41
	500	1.077 (0.8957,0.8951) 0.06	1.092 (0.9516,0.9493) 0.23	1.093 (0.9808,0.9798) 0.10
	1000	1.071 (0.8980,0.8999) -0.19	1.087 (0.9498,0.9488) 0.10	1.088 (0.9804,0.9791) 0.13
$Burr(8,7)$	50	1.128 (0.9089,0.8958) 1.31	1.143 (0.9557,0.9454) 1.03	1.143 (0.9828,0.9753) 0.75
	100	1.104 (0.9052,0.8994) 0.58	1.120 (0.9559,0.9495) 0.64	1.121 (0.9846,0.9796) 0.50
	500	1.077 (0.8980,0.9006) -0.26	1.092 (0.9506,0.9496) 0.10	1.093 (0.9801,0.9778) 0.23
	1000	1.071 (0.8997,0.9042) -0.45	1.087 (0.9493,0.9495) -0.02	1.088 (0.9785,0.9777) 0.08
$Frechet(10)$	50	1.128 (0.8985,0.8846) 1.39	1.143 (0.9474,0.9347) 1.27	1.143 (0.9757,0.9660) 0.97
	100	1.104 (0.9023,0.8999) 0.24	1.120 (0.9524,0.9431) 0.93	1.121 (0.9801,0.9760) 0.41
	500	1.077 (0.9005,0.9001) 0.04	1.092 (0.9510,0.9473) 0.37	1.093 (0.9792,0.9774) 0.18
	1000	1.071 (0.9005,0.9002) 0.03	1.087 (0.9505,0.9506) -0.01	1.088 (0.9800,0.9791) 0.09
$N(0,1)$	50	1.128 (0.9022,0.8955) 0.67	1.143 (0.9538,0.9460) 0.78	1.143 (0.9811,0.9737) 0.74
	100	1.104 (0.9062,0.8955) 1.07	1.120 (0.9567,0.9478) 0.89	1.121 (0.9843,0.9774) 0.69
	500	1.077 (0.8924,0.8956) -0.32	1.092 (0.9511,0.9482) 0.29	1.093 (0.9795,0.9791) 0.04
	1000	1.071 (0.8960,0.8978) -0.18	1.087 (0.9491,0.9463) 0.28	1.088 (0.9789,0.9770) 0.19
$Pois(3)$	50	1.128 (0.9079,0.8951) 1.28	1.143 (0.9572,0.9464) 1.08	1.143 (0.9824,0.9740) 0.84
	100	1.104 (0.9055,0.9000) 0.55	1.120 (0.9544,0.9513) 0.31	1.121 (0.9845,0.9792) 0.53
	500	1.077 (0.9001,0.9006) -0.05	1.092 (0.9526,0.9512) 0.14	1.093 (0.9813,0.9784) 0.29
	1000	1.071 (0.9042,0.9040) 0.02	1.087 (0.9499,0.9494) 0.05	1.088 (0.9782,0.9791) 0.09
$Exp(1)$	50	1.128 (0.8925,0.8835) 0.90	1.143 (0.9417,0.9283) 1.34	1.143 (0.9715,0.9601) 1.14
	100	1.104 (0.8948,0.8899) 0.49	1.120 (0.9449,0.9368) 0.81	1.121 (0.9759,0.9693) 0.66
	500	1.077 (0.8991,0.8997) -0.06	1.092 (0.9510,0.9501) 0.09	1.093 (0.9815,0.9782) 0.33
	1000	1.071 (0.8988,0.8978) 0.10	1.087 (0.9515,0.9471) 0.44	1.088 (0.9802,0.9789) 0.13
$Gamma(0.025,1)$	50	1.128 (0.6697,0.6585) 1.12	1.143 (0.7060,0.6882) 1.78	1.143 (0.7381,0.7153) 2.28
	100	1.104 (0.7469,0.7454) 0.15	1.120 (0.7897,0.7791) 1.06	1.121 (0.8211,0.8092) 1.19
	500	1.077 (0.8561,0.8541) 0.20	1.092 (0.9042,0.8966) 0.76	1.093 (0.9356,0.9308) 0.48
	1000	1.071 (0.8774,0.8728) 0.46	1.087 (0.9277,0.9221) 0.56	1.088 (0.9606,0.9544) 0.62
$Gamma(0.001,1)$	50	1.128 (0.1544,0.1521) 0.23	1.143 (0.1602,0.1574) 0.28	1.143 (0.1662,0.1622) 0.40
	100	1.104 (0.2497,0.2453) 0.44	1.120 (0.2598,0.2554) 0.44	1.121 (0.2684,0.2637) 0.47
	500	1.077 (0.5146,0.5116) 0.30	1.092 (0.5423,0.5319) 1.04	1.093 (0.5638,0.5552) 0.86
	1000	1.071 (0.6270,0.6243) 0.27	1.087 (0.6664,0.6527) 1.37	1.088 (0.6910,0.6787) 1.23
	5000	1.064 (0.7993,0.8005) -0.12	1.079 (0.8504,0.8385) 1.19	1.080 (0.8834,0.8732) 1.02
$Weibull(1,0.2)$	50	1.128 (0.4967,0.4827) 1.40	1.143 (0.5344,0.5135) 2.09	1.143 (0.5704,0.5445) 2.59
	100	1.104 (0.5787,0.5623) 1.64	1.120 (0.6191,0.5984) 2.07	1.121 (0.6591,0.6351) 2.40
	500	1.077 (0.6972,0.6857) 1.15	1.092 (0.7448,0.7303) 1.45	1.093 (0.7911,0.7712) 1.99
	1000	1.071 (0.7430,0.7309) 1.21	1.087 (0.7939,0.7796) 1.43	1.088 (0.8359,0.8220) 1.39
	5000	1.064 (0.8110,0.8119) -0.09	1.079 (0.8676,0.8602) 0.74	1.081 (0.9064,0.8979) 0.85

Table 2.11: Comparison table for I_{2,q_1} vs I_0, values of $\widehat{r}_{2,q_1}(\widehat{CP}_{2,q_1}, \widehat{CP}_0)\Delta_{2,q_1}$

Distribution	n	$1-\alpha = 0.9$	$1-\alpha = 0.95$	$1-\alpha = 0.98$
$Pareto(1,2)$	50	1.124 (0.7905,0.7688) 2.17	1.121 (0.8373,0.8147) 2.26	1.115 (0.8765,0.8581) 1.84
	100	1.098 (0.8082,0.7908) 1.74	1.095 (0.8535,0.8363) 1.72	1.090 (0.8964,0.8822) 1.42
	500	1.065 (0.8430,0.8379) 0.51	1.062 (0.8921,0.8855) 0.66	1.058 (0.9308,0.9235) 0.73
	1000	1.058 (0.8564,0.8516) 0.48	1.056 (0.9042,0.9006) 0.36	1.052 (0.9390,0.9357) 0.33
	5000	1.049 (0.8698,0.8664) 0.34	1.047 (0.9187,0.9173) 0.14	1.043 (0.9535,0.9518) 0.17
$Frechet(3)$	50	1.124 (0.8670,0.8472) 1.98	1.121 (0.9130,0.8968) 1.62	1.115 (0.9448,0.9346) 1.02
	100	1.098 (0.8820,0.8700) 1.20	1.095 (0.9261,0.9167) 0.94	1.090 (0.9575,0.9498) 0.77
	500	1.065 (0.8944,0.8909) 0.35	1.062 (0.9399,0.9381) 0.18	1.058 (0.9695,0.9669) 0.26
	1000	1.058 (0.8984,0.8958) 0.26	1.056 (0.9454,0.9445) 0.09	1.052 (0.9722,0.9725) -0.03
$Burr(3,2)$	50	1.124 (0.9085,0.8897) 1.88	1.121 (0.9510,0.9395) 1.15	1.115 (0.9777,0.9700) 0.77
	100	1.098 (0.9078,0.9005) 0.73	1.095 (0.9525,0.9461) 0.64	1.090 (0.9811,0.9776) 0.35
	500	1.065 (0.9045,0.9015) 0.30	1.062 (0.9488,0.9487) 0.01	1.058 (0.9783,0.9783) 0.00
	1000	1.058 (0.9034,0.9016) 0.18	1.056 (0.9485,0.9505) -0.20	1.052 (0.9780,0.9795) -0.15
$Pareto(1,6)$	50	1.124 (0.8847,0.8634) 2.13	1.121 (0.9283,0.9149) 1.34	1.115 (0.9557,0.9479) 0.78
	100	1.098 (0.8966,0.8847) 1.19	1.095 (0.9407,0.9349) 0.58	1.090 (0.9711,0.9634) 0.77
	500	1.065 (0.9008,0.8984) 0.24	1.062 (0.9473,0.9462) 0.11	1.058 (0.9749,0.9746) 0.03
	1000	1.058 (0.9013,0.9013) 0.00	1.056 (0.9485,0.9469) 0.16	1.052 (0.9772,0.9765) 0.07
$Burr(3,4)$	50	1.124 (0.9108,0.8938) 1.70	1.121 (0.9557,0.9465) 0.92	1.115 (0.9817,0.9737) 0.80
	100	1.098 (0.9112,0.9008) 1.04	1.095 (0.9539,0.9486) 0.53	1.090 (0.9842,0.9786) 0.56
	500	1.065 (0.9049,0.9004) 0.45	1.062 (0.9485,0.9485) 0.00	1.058 (0.9786,0.9785) 0.01
	1000	1.058 (0.9043,0.9033) 0.10	1.056 (0.9489,0.9493) -0.04	1.052 (0.9758,0.9791) -0.33
$t(8)$	50	1.124 (0.9122,0.8928) 1.94	1.121 (0.9550,0.9466) 0.84	1.115 (0.9804,0.9781) 0.23
	100	1.098 (0.9059,0.8998) 0.61	1.095 (0.9524,0.9508) 0.16	1.090 (0.9799,0.9773) 0.26
	500	1.065 (0.8987,0.8951) 0.36	1.062 (0.9496,0.9493) 0.03	1.058 (0.9799,0.9798) 0.01
	1000	1.058 (0.9024,0.8999) 0.25	1.056 (0.9478,0.9488) -0.10	1.052 (0.9780,0.9791) -0.11
$Burr(8,7)$	50	1.124 (0.9137,0.8958) 1.79	1.121 (0.9565,0.9454) 1.11	1.115 (0.9817,0.9753) 0.64
	100	1.098 (0.9086,0.8994) 0.92	1.095 (0.9568,0.9495) 0.73	1.090 (0.9842,0.9796) 0.46
	500	1.065 (0.9005,0.9006) -0.01	1.062 (0.9481,0.9496) -0.15	1.058 (0.9786,0.9778) 0.08
	1000	1.058 (0.9016,0.9042) -0.26	1.056 (0.9486,0.9495) -0.09	1.052 (0.9758,0.9777) -0.19
$Frechet(10)$	50	1.124 (0.9034,0.8846) 1.88	1.121 (0.9464,0.9347) 1.17	1.115 (0.9748,0.9660) 0.88
	100	1.098 (0.9065,0.8999) 0.66	1.095 (0.9528,0.9431) 0.97	1.090 (0.9786,0.9760) 0.26
	500	1.065 (0.9037,0.9001) 0.36	1.062 (0.9497,0.9473) 0.24	1.058 (0.9779,0.9774) 0.05
	1000	1.058 (0.9036,0.9002) 0.34	1.056 (0.9483,0.9506) -0.23	1.052 (0.9781,0.9791) -0.10
$N(0,1)$	50	1.125 (0.9088,0.8955) 1.33	1.121 (0.9534,0.9460) 0.74	1.115 (0.9799,0.9737) 0.62
	100	1.098 (0.9102,0.8955) 1.47	1.095 (0.9556,0.9478) 0.78	1.090 (0.9824,0.9774) 0.50
	500	1.065 (0.8961,0.8956) 0.05	1.062 (0.9483,0.9482) 0.01	1.058 (0.9785,0.9791) -0.06
	1000	1.058 (0.8978,0.8978) 0.00	1.056 (0.9466,0.9463) 0.03	1.052 (0.9760,0.9770) -0.10
$Pois(3)$	50	1.124 (0.9123,0.8951) 1.72	1.121 (0.9568,0.9464) 1.04	1.115 (0.9814,0.9740) 0.74
	100	1.098 (0.9105,0.9000) 1.05	1.095 (0.9545,0.9513) 0.32	1.090 (0.9831,0.9792) 0.39
	500	1.065 (0.9039,0.9006) 0.33	1.062 (0.9513,0.9512) 0.01	1.058 (0.9797,0.9784) 0.13
	1000	1.058 (0.9050,0.9040) 0.10	1.056 (0.9491,0.9494) -0.03	1.052 (0.9766,0.9791) -0.25
$Exp(1)$	50	1.124 (0.8981,0.8835) 1.46	1.121 (0.9409,0.9283) 1.26	1.115 (0.9698,0.9601) 0.97
	100	1.098 (0.8992,0.8899) 0.93	1.095 (0.9439,0.9368) 0.71	1.090 (0.9742,0.9693) 0.49
	500	1.065 (0.9009,0.8997) 0.12	1.062 (0.9491,0.9501) -0.10	1.058 (0.9799,0.9782) 0.17
	1000	1.058 (0.8989,0.8978) 0.11	1.056 (0.9497,0.9471) -0.26	1.052 (0.9786,0.9789) -0.03
$Gamma(0.025,1)$	50	1.124 (0.6732,0.6585) 1.47	1.121 (0.7027,0.6882) 1.45	1.115 (0.7351,0.7153) 1.98
	100	1.098 (0.7505,0.7454) 0.51	1.095 (0.7877,0.7791) 0.86	1.090 (0.8178,0.8092) 0.86
	500	1.065 (0.8584,0.8541) 0.43	1.062 (0.9028,0.8966) 0.62	1.058 (0.9315,0.9308) 0.07
	1000	1.058 (0.8787,0.8728) 0.59	1.056 (0.9247,0.9221) 0.26	1.052 (0.9577,0.9544) 0.33
$Gamma(0.001,1)$	50	1.124 (0.1549,0.1521) 0.28	1.121 (0.1590,0.1574) 0.16	1.115 (0.1660,0.1622) 0.38
	100	1.098 (0.2489,0.2453) 0.36	1.095 (0.2590,0.2554) 0.36	1.090 (0.2673,0.2637) 0.36
	500	1.065 (0.5147,0.5116) 0.31	1.062 (0.5390,0.5319) 0.71	1.058 (0.5600,0.5552) 0.48
	1000	1.058 (0.6277,0.6243) 0.34	1.056 (0.6616,0.6527) 0.89	1.052 (0.6868,0.6787) 0.81
	5000	1.050 (0.7992,0.8005) -0.13	1.047 (0.8470,0.8385) 0.85	1.043 (0.8776,0.8732) 0.44
$Weibull(1,0.2)$	50	1.124 (0.4990,0.4827) 1.63	1.121 (0.5301,0.5135) 1.66	1.115 (0.5666,0.5445) 2.21
	100	1.098 (0.5805,0.5623) 1.82	1.095 (0.6147,0.5984) 1.63	1.090 (0.6540,0.6351) 1.89
	500	1.065 (0.6960,0.6857) 1.03	1.063 (0.7396,0.7303) 0.93	1.058 (0.7844,0.7712) 1.32
	1000	1.058 (0.7428,0.7309) 1.19	1.056 (0.7886,0.7796) 0.90	1.052 (0.8295,0.8220) 0.75
	5000	1.049 (0.8124,0.8119) 0.05	1.047 (0.8619,0.8602) 0.17	1.042 (0.9000,0.8979) 0.21

Table 2.12: Comparison table for I_{2,q_2} vs I_0, values of $\hat{r}_{2,q_2}(\widehat{CP}_{2,q_2}, \widehat{CP}_0)\Delta_{2,q_2}$

2.3 FACI's based on $h_3(\cdot)$

In this section, we consider FACI's based only on (1.9) with $h(\cdot) = h_3(\cdot)$ and $q(\cdot) = \sqrt{q_1(\cdot)}$ and $\sqrt{q_2(\cdot)}$. From the FCLT in (1.9) with $h(\cdot) = h_3(\cdot)$,

$$0.457 \frac{\sum_{i=1}^{[0.97n]}(X_i - \mu)}{s_n \sqrt{n q_j\left(\frac{[0.97n]}{n}\right)}} + \int_0^1 \frac{\left(T_n^t(X_1 - \mu, \ldots.., X_n - \mu)\right)^2}{q_j\left(\frac{[nt]}{n}\right)} dt$$

$$\xrightarrow[n\to\infty]{d} 0.457 \frac{W(0.97)}{\sqrt{q_j(0.97)}} + \int_0^1 \frac{W^2(t)}{q_j(t)} dt, \tag{2.17}$$

or equivalently,

$$0.457 \frac{\sum_{i=1}^{[0.97n]}(X_i - \mu)}{s_n \sqrt{n q_j\left(\frac{[0.97n]}{n}\right)}} + \frac{1}{s_n^2 n^2} \sum_{k=1}^{n-1} \frac{\left(\sum_{i=1}^{k}(X_i - \mu)\right)^2}{q_j\left(\frac{k}{n}\right)}$$

$$\xrightarrow[n\to\infty]{d} \frac{0.457}{\sqrt{q_j(0.97)}} W(0.97) + \int_0^1 \frac{W^2(t)}{q_j(t)} dt, \tag{2.18}$$

where $\frac{0.457}{\sqrt{q_1(0.97)}} = \frac{0.457}{\sqrt{q_2(0.97)}} \approx 1.043$.

Remark 2. The functional $h_3(\cdot)$ of (1.15) is a linear combination of $h_2(\cdot)$ of (1.14) and the projection functional h_0 of (1.17) with $t_0 = 0.97$. We selected t_0 to be close to one, so that $\sum_{i=1}^{[0.97n]}(X_i - \mu)/(s_n\sqrt{n})$ in (2.17) would practically be the Student t-statistic. By considering the linear combination of the latter and $h_2(T_n^t(X_1 - \mu, \ldots, X_n - \mu)/\sqrt{q_j([nt]/n)})$, we hoped that the FACI for μ

based on $h_3(\cdot)$ would inherit good finite-sample properties of the CLT based CI I_0 and the FACI I_{2,q_j} based on the integral functional $h_2(\cdot)$. Another important consideration for using $h_3(\cdot)$ was that it yields FACI's with closed form expressions (derived from a quadratic inequality). Moreover, considering a linear combination of the two functionals ($h_0(\cdot)$ and $h_2(\cdot)$) with different positive coefficients allowed for some flexibility and control over the contributions of each functional in the linear combination. After some careful analysis of many pairs of the coefficients and close to one options for t_0, we decided to settle on 0.457 and 1 in $h_3(\cdot)$. This choice leads to the FACI's with better finite-sample properties.

Let $P(0.457W(0.97)/\sqrt{q_j(0.97)} + \int_0^1 (W^2(t)/q_j(t))dt \leqslant b) = 1-\alpha$, where $\alpha \in (0,1)$. Using the invariance principle, we obtain $1-\alpha$ quantiles of the limiting distribution of (2.17) with $q = q_1(t)$ and $q_2(t)$ in Table 2.13.

	$1-\alpha = 0.9$	$1-\alpha = 0.95$	$1-\alpha = 0.98$
$1-\alpha$ quantile of $0.457\frac{W(0.97)}{\sqrt{q_1(0.97)}} + \int_0^1 \frac{W^2(t)}{q_1(t)}dt$	4.76	6.61	9.29
$1-\alpha$ quantile of $0.457\frac{W(0.97)}{\sqrt{q_2(0.97)}} + \int_0^1 \frac{W^2(t)}{q_2(t)}dt$	3.88	5.39	7.60

Table 2.13: Some quantiles of $0.457\frac{W(0.97)}{\sqrt{q_j(0.97)}} + \int_0^1 \frac{W^2(t)}{q_j(t)}dt,\ j = \overline{1,2}$

Now, in order to derive the FACI for μ based on

$$P\left(0.457\frac{\sum_{i=1}^{[0.97n]}(X_i-\mu)}{s_n\sqrt{nq_j\left(\frac{[0.97n]}{n}\right)}}+\frac{1}{s_n^2n^2}\sum_{k=1}^{n-1}\frac{\left(\sum_{i=1}^{k}(X_i-\mu)\right)^2}{q_j\left(\frac{k}{n}\right)}-b\leqslant 0\right)\xrightarrow[n\to\infty]{d}1-\alpha, \tag{2.19}$$

we first simplify the quadratic function in μ under the probability sign.

$$\begin{aligned}
&0.457\frac{\sum_{i=1}^{[0.97n]}(X_i-\mu)}{s_n\sqrt{nq_j\left(\frac{[0.97n]}{n}\right)}}+\frac{1}{s_n^2n^2}\sum_{k=1}^{n-1}\frac{\left(\sum_{i=1}^{k}(X_i-\mu)\right)^2}{q_j\left(\frac{k}{n}\right)}-b\\
&=\frac{0.457\sum_{i=1}^{[0.97n]}X_i-0.457\mu[0.97n]}{s_n\sqrt{nq_j\left(\frac{[0.97n]}{n}\right)}}\\
&\quad+\frac{1}{s_n^2n^2}\sum_{k=1}^{n-1}\frac{\left(\sum_{i=1}^{k}X_i\right)^2+k^2\mu^2-2\mu k\sum_{i=1}^{k}X_i}{q_j\left(\frac{k}{n}\right)}-b\\
&=a_{3,q}\mu^2+b_{3,q}\mu+c_{3,q},
\end{aligned}$$

where

$$a_{3,q}:=\frac{1}{s_n^2n^2}\sum_{k=1}^{n-1}\frac{k^2}{q_j\left(\frac{k}{n}\right)}, \tag{2.20}$$

$$b_{3,q} := -\left(\frac{0.457[0.97n]}{s_n\sqrt{nq_j\left(\frac{[0.97n]}{n}\right)}} + \frac{2}{s^2n^2}\sum_{k=1}^{n-1}\frac{k\sum_{i=1}^{k}X_i}{q_j\left(\frac{k}{n}\right)}\right) \tag{2.21}$$

and

$$c_{3,q} := \frac{0.457\sum_{i=1}^{[0.97n]}X_i}{s_n\sqrt{nq_j\left(\frac{[0.97n]}{n}\right)}} + \frac{1}{s_n^2n^2}\sum_{k=1}^{n-1}\frac{(\sum_{i=1}^{k}X_i)^2}{q_j\left(\frac{k}{n}\right)} - b. \tag{2.22}$$

Thus, (2.19) results in the following FACI for μ:

$$I_{3,q_j} := \left[\frac{-b_{3,q} - \sqrt{b_{3,q}^2 - 4a_{3,q}c_{3,q}}}{2a_{3,q}}, \frac{-b_{3,q} + \sqrt{b_{3,q}^2 - 4a_{3,q}c_{3,q}}}{2a_{3,q}}\right]. \tag{2.23}$$

Due to the presence of the square root in (2.23), it is not feasible to obtain a closed form expression for the ratio r_{3,q_j} of the expected length of I_{3,q_j} to that of I_0, and r_{3,q_j} is approximated by $\widehat{r}_{3,q_j}$, simulated in the quadruplets $\widehat{r}_{3,q_j}(\widehat{CP}_{3,q_j}, \widehat{CP}_0)\Delta_{3,q_j}$ in the upcoming Tables 2.15 and 2.16, which are proceeded by Table 2.14 with ranges of Δ_{3,q_j} and averages (in n) of $\widehat{r}_{3,q_j}$, computed for three confidence levels and each distribution class of Table 1.2.

Both I_{3,q_1} and I_{3,q_2} are seen to have mostly higher empirical finite-sample coverage probabilities than the respective one of I_0, but to be a bit longer than I_0 on average. The ratios $\widehat{r}_{3,q_1}$ and $\widehat{r}_{3,q_2}$ decrease as n increases. Perhaps, r_{3,q_1} and r_{3,q_2} converge, as $n \to \infty$. Moreover, for each $1-\alpha$, averages (in n) of $\widehat{r}_{3,q_2}$

are only insignificantly larger than those of $\widehat{r}_{3,q_1}$, which reduces the comparison between I_{3,q_1} and I_{3,q_2} to that between their respective empirical coverage probabilities $\widehat{CP}_{3,q_1}$ and $\widehat{CP}_{3,q_2}$. Accordingly, the coverage probabilities of I_{3,q_2} are generally better than those of I_{3,q_1}. For this reason, the FACI I_{3,q_2} appears to be more appealing than I_{3,q_1}.

Similarly to the FACI's based on $h_1(\cdot)$ and $h_2(\cdot)$, the coverage probabilities $\widehat{CP}_{3,q_1}$ and $\widehat{CP}_{3,q_2}$ are lower for the heavy-tailed and/or largely skewed distributions (cf. Tables 2.15 and 2.16 for $Pareto(1,2)$, $Gamma(0.025,1)$, $Gamma(0.001,1)$ and $Weibull(1,0.2)$). For the rest of the distributions, $\widehat{CP}_{3,q_1}$ and $\widehat{CP}_{3,q_2}$ deviate around or slightly above $1-\alpha$ (like for $Burr(3,4)$ and $N(0,1)$), or even slowly decrease in n, staying still above $1-\alpha$ even for $n = 1000$ (like for $Burr(3,2)$ and $Exp(1)$ for example).

All in all, I_{3,q_1} and I_{3,q_2} present reasonable alternatives to I_0, especially for distributions with fewer moments and/or larger absolute skewness.

	Heavy-tailed distributions with no skewness	Heavy-tailed distributions with small skewness	Heavy-tailed distributions with large skewness	Light-tailed distributions with small skewness	Light-tailed distributions with large skewness
$1-\alpha = 0.9$					
Δ_{3,q_1}	1.28% - 6.95%	-0.11% - 2.47%	0.38% - 4.18%	-0.50% - 2.87%	0.48% - 6.52%
average $\widehat{r}_{3,q_1}$	1.162	1.166	1.166	1.166	1.162
Δ_{3,q_2}	1.41% - 8.20%	0.03% - 3.20%	0.63% - 4.89%	-0.48% - 3.61%	0.66% - 7.97%
average $\widehat{r}_{3,q_2}$	1.182	1.187	1.187	1.187	1.182
$1-\alpha = 0.95$					
Δ_{3,q_1}	0.97% - 6.88%	-0.02% - 2.04%	0.50% - 3.25%	-0.34% - 2.36%	0.38% - 6.67%
average $\widehat{r}_{3,q_1}$	1.156	1.160	1.160	1.160	1.156
Δ_{3,q_2}	1.30% - 8.11%	-0.10% - 2.38%	0.61% - 4.16%	-0.40% - 2.86%	0.47% - 7.64%
average $\widehat{r}_{3,q_2}$	1.169	1.174	1.174	1.174	1.169
$1-\alpha = 0.98$					
Δ_{3,q_1}	1.10% - 5.82%	-0.04% - 1.45%	0.58% - 2.66%	-0.19% - 1.82%	0.35% - 6.41%
average $\widehat{r}_{3,q_1}$	1.159	1.162	1.162	1.162	1.159
Δ_{3,q_2}	1.32% - 6.84%	0.03% - 1.76%	0.76% - 3.21%	-0.09% - 2.10%	0.43% - 7.42%
average $\widehat{r}_{3,q_2}$	1.172	1.176	1.176	1.176	1.172

Table 2.14: Ranges of Δ_{3,q_1} and Δ_{3,q_2} and averages (in n) of $\widehat{r}_{3,q_1}$ and $\widehat{r}_{3,q_2}$

Distribution	n	$1-\alpha=0.9$	$1-\alpha=0.95$	$1-\alpha=0.98$
$Pareto(1,2)$	50	1.199 (0.8383,0.7688) 6.95	1.193 (0.8835,0.8147) 6.88	1.196 (0.9163,0.8581) 5.82
	100	1.178 (0.8552,0.7908) 6.44	1.171 (0.9033,0.8363) 6.70	1.173 (0.9376,0.8822) 5.54
	500	1.147 (0.8850,0.8379) 4.71	1.141 (0.9312,0.8855) 4.57	1.143 (0.9622,0.9235) 3.87
	1000	1.141 (0.8924,0.8516) 4.08	1.135 (0.9376,0.9006) 3.70	1.137 (0.9684,0.9357) 3.27
	5000	1.132 (0.8973,0.8664) 3.09	1.128 (0.9481,0.9173) 3.08	1.131 (0.9788,0.9518) 2.70
$Frechet(3)$	50	1.199 (0.8983,0.8472) 5.11	1.193 (0.9395,0.8968) 4.27	1.196 (0.9695,0.9346) 3.49
	100	1.178 (0.9044,0.8700) 3.44	1.171 (0.9541,0.9167) 3.74	1.173 (0.9786,0.9498) 2.88
	500	1.147 (0.9093,0.8909) 1.84	1.141 (0.9562,0.9381) 1.81	1.143 (0.9827,0.9669) 1.58
	1000	1.141 (0.9086,0.8958) 1.28	1.135 (0.9542,0.9445) 0.97	1.137 (0.9835,0.9725) 1.10
$Burr(3,2)$	50	1.199 (0.9144,0.8897) 2.47	1.193 (0.9599,0.9395) 2.04	1.196 (0.9845,0.9700) 1.45
	100	1.178 (0.9125,0.9005) 1.20	1.171 (0.9611,0.9461) 1.50	1.173 (0.9874,0.9776) 0.98
	500	1.147 (0.9067,0.9015) 0.52	1.141 (0.9550,0.9487) 0.63	1.143 (0.9816,0.9783) 0.33
	1000	1.141 (0.9034,0.9016) 0.18	1.135 (0.9521,0.9505) 0.16	1.137 (0.9803,0.9795) 0.08
$Pareto(1,6)$	50	1.199 (0.9052,0.8634) 4.18	1.193 (0.9474,0.9149) 3.25	1.196 (0.9745,0.9479) 2.66
	100	1.178 (0.9109,0.8847) 2.62	1.171 (0.9577,0.9349) 2.28	1.173 (0.9820,0.9634) 1.86
	500	1.147 (0.9092,0.8984) 1.08	1.141 (0.9560,0.9462) 0.98	1.143 (0.9832,0.9746) 0.86
	1000	1.141 (0.9051,0.9013) 0.38	1.135 (0.9519,0.9469) 0.50	1.137 (0.9823,0.9765) 0.58
$Burr(3,4)$	50	1.199 (0.9169,0.8938) 2.31	1.193 (0.9594,0.9465) 1.29	1.196 (0.9842,0.9737) 1.05
	100	1.178 (0.9104,0.9008) 0.96	1.171 (0.9591,0.9486) 1.05	1.173 (0.9858,0.9786) 0.72
	500	1.147 (0.9053,0.9004) 0.49	1.141 (0.9531,0.9485) 0.46	1.143 (0.9799,0.9785) 0.14
	1000	1.141 (0.9022,0.9033) -0.11	1.135 (0.9491,0.9493) -0.02	1.137 (0.9787,0.9791) -0.04
$t(8)$	50	1.199 (0.9072,0.8928) 1.44	1.193 (0.9526,0.9466) 0.60	1.196 (0.9809,0.9781) 0.28
	100	1.178 (0.9072,0.8998) 0.74	1.171 (0.9510,0.9508) 0.02	1.173 (0.9816,0.9773) 0.43
	500	1.147 (0.8959,0.8951) 0.08	1.141 (0.9486,0.9493) -0.07	1.143 (0.9803,0.9798) 0.05
	1000	1.141 (0.8972,0.8999) -0.27	1.135 (0.9494,0.9488) 0.06	1.137 (0.9792,0.9791) 0.01
$Burr(8,7)$	50	1.199 (0.9063,0.8958) 1.05	1.193 (0.9529,0.9454) 0.75	1.196 (0.9796,0.9753) 0.43
	100	1.178 (0.9053,0.8994) 0.59	1.171 (0.9481,0.9495) 0.46	1.173 (0.9837,0.9796) 0.41
	500	1.147 (0.8960,0.9006) -0.46	1.141 (0.9481,0.9496) -0.15	1.143 (0.9794,0.9778) 0.16
	1000	1.141 (0.8992,0.9042) -0.50	1.135 (0.9473,0.9495) -0.22	1.137 (0.9758,0.9777) -0.19
$Frechet(10)$	50	1.199 (0.9133,0.8846) 2.87	1.193 (0.9583,0.9347) 2.36	1.196 (0.9842,0.9660) 1.82
	100	1.178 (0.9126,0.8999) 1.36	1.171 (0.9604,0.9431) 1.73	1.173 (0.9866,0.9760) 1.06
	500	1.147 (0.9090,0.9001) 0.89	1.141 (0.9555,0.9473) 0.82	1.143 (0.9812,0.9774) 0.38
	1000	1.141 (0.9036,0.9002) 0.34	1.135 (0.9514,0.9506) 0.08	1.137 (0.9802,0.9791) 0.11
$N(0,1)$	50	1.199 (0.9047,0.8955) 0.92	1.193 (0.9505,0.9460) 0.45	1.196 (0.9824,0.9737) 0.87
	100	1.178 (0.9054,0.8955) 0.99	1.171 (0.9548,0.9478) 0.70	1.173 (0.9831,0.9774) 0.57
	500	1.147 (0.8935,0.8956) -0.21	1.141 (0.9448,0.9482) -0.34	1.143 (0.9795,0.9791) 0.04
	1000	1.141 (0.8990,0.8978) 0.12	1.135 (0.9487,0.9463) 0.27	1.137 (0.9794,0.9770) 0.24
$Pois(3)$	50	1.199 (0.9151,0.8951) 2.00	1.193 (0.9580,0.9464) 1.16	1.196 (0.9834,0.9740) 0.94
	100	1.178 (0.9095,0.9000) 0.95	1.171 (0.9596,0.9513) 0.83	1.173 (0.9857,0.9792) 0.65
	500	1.147 (0.9026,0.9006) 0.20	1.141 (0.9531,0.9512) 0.19	1.143 (0.9811,0.9784) 0.27
	1000	1.141 (0.9012,0.9040) 0.10	1.135 (0.9488,0.9494) -0.06	1.137 (0.9801,0.9791) 0.10
$Exp(1)$	50	1.199 (0.9117,0.8835) 2.82	1.193 (0.9553,0.9283) 2.70	1.196 (0.9809,0.9601) 2.08
	100	1.178 (0.9059,0.8899) 1.60	1.171 (0.9556,0.9368) 1.88	1.173 (0.9847,0.9693) 1.54
	500	1.147 (0.9045,0.8997) 0.48	1.141 (0.9539,0.9501) 0.38	1.143 (0.9839,0.9782) 0.57
	1000	1.141 (0.9030,0.8978) 0.52	1.135 (0.9530,0.9471) 0.59	1.137 (0.9824,0.9789) 0.35
$Gamma(0.025,1)$	50	1.199 (0.7049,0.6585) 4.64	1.193 (0.7396,0.6882) 5.14	1.196 (0.7654,0.7153) 5.01
	100	1.178 (0.7800,0.7454) 3.46	1.171 (0.8229,0.7791) 4.38	1.173 (0.8529,0.8092) 4.37
	500	1.147 (0.8796,0.8541) 2.55	1.141 (0.9275,0.8966) 3.09	1.143 (0.9589,0.9308) 2.81
	1000	1.141 (0.8987,0.8728) 2.59	1.135 (0.9476,0.9221) 2.55	1.137 (0.9741,0.9544) 1.97
$Gamma(0.001,1)$	50	1.199 (0.1615,0.1521) 0.94	1.193 (0.1672,0.1574) 0.98	1.196 (0.1710,0.1622) 0.88
	100	1.178 (0.2608,0.2453) 1.55	1.171 (0.2706,0.2554) 1.52	1.173 (0.2784,0.2637) 1.47
	500	1.147 (0.5415,0.5116) 2.99	1.141 (0.5637,0.5319) 3.54	1.143 (0.5886,0.5552) 3.34
	1000	1.141 (0.6613,0.6243) 3.70	1.135 (0.6939,0.6527) 4.12	1.137 (0.7184,0.6787) 3.97
	5000	1.132 (0.8362,0.8005) 3.57	1.128 (0.8827,0.8385) 4.42	1.131 (0.9104,0.8732) 3.72
$Weibull(1,0.2)$	50	1.199 (0.5431,0.4827) 6.04	1.193 (0.5762,0.5135) 6.27	1.196 (0.6051,0.5445) 6.06
	100	1.178 (0.6275,0.5623) 6.52	1.171 (0.6651,0.5984) 6.67	1.173 (0.6992,0.6351) 6.41
	500	1.147 (0.7499,0.6857) 6.42	1.173 (0.7953,0.7303) 6.50	1.143 (0.8314,0.7712) 6.02
	1000	1.141 (0.7938,0.7309) 6.29	1.135 (0.8367,0.7796) 5.71	1.137 (0.8749,0.8202) 5.29
	5000	1.132 (0.8530,0.8119) 4.11	1.128 (0.9032,0.8602) 4.30	1.131 (0.9391,0.8979) 4.12

Table 2.15: Comparison table for I_{3,q_1} vs I_0, values of $\widehat{r}_{3,q_1}(\widehat{CP}_{3,q_1},\widehat{CP}_0)\Delta_{3,q_1}$

Distribution	n	$1-\alpha=0.9$			$1-\alpha=0.95$			$1-\alpha=0.98$		
$Pareto(1,2)$	50	1.226	(0.8508,0.7688)	8.20	1.213	(0.8958,0.8147)	8.11	1.215	(0.9265,0.8581)	6.84
	100	1.201	(0.8694,0.7908)	7.86	1.188	(0.9144,0.8363)	7.81	1.189	(0.9457,0.8822)	6.35
	500	1.164	(0.8961,0.8379)	5.82	1.152	(0.9396,0.8855)	5.41	1.155	(0.9678,0.9235)	4.43
	1000	1.156	(0.8989,0.8516)	4.73	1.144	(0.9450,0.9006)	4.44	1.146	(0.9737,0.9357)	3.80
	5000	1.146	(0.9066,0.8664)	4.02	1.135	(0.9543,0.9173)	3.70	1.137	(0.9815,0.9518)	2.97
$Frechet(3)$	50	1.226	(0.9080,0.8472)	6.08	1.213	(0.9488,0.8968)	5.20	1.215	(0.9747,0.9346)	4.01
	100	1.201	(0.9143,0.8700)	4.43	1.188	(0.9594,0.9167)	4.27	1.189	(0.9834,0.9498)	3.36
	500	1.164	(0.9153,0.8909)	2.44	1.152	(0.9607,0.9381)	2.26	1.155	(0.9854,0.9669)	1.85
	1000	1.156	(0.9099,0.8958)	1.41	1.144	(0.9575,0.9445)	1.30	1.146	(0.9857,0.9725)	1.32
$Burr(3,2)$	50	1.226	(0.9217,0.8897)	3.20	1.213	(0.9633,0.9395)	2.38	1.215	(0.9867,0.9700)	1.76
	100	1.201	(0.9179,0.9005)	1.74	1.188	(0.9622,0.9461)	1.61	1.189	(0.9889,0.9776)	1.13
	500	1.164	(0.9081,0.9015)	0.66	1.152	(0.9551,0.9487)	0.64	1.155	(0.9820,0.9783)	0.37
	1000	1.156	(0.9044,0.9016)	0.28	1.144	(0.9517,0.9505)	0.11	1.146	(0.9806,0.9795)	0.11
$Pareto(1,6)$	50	1.226	(0.9123,0.8634)	4.89	1.213	(0.9565,0.9149)	4.16	1.215	(0.9800,0.9479)	3.21
	100	1.201	(0.9178,0.8847)	3.31	1.188	(0.9623,0.9349)	2.74	1.189	(0.9861,0.9634)	2.27
	500	1.164	(0.9102,0.8984)	1.18	1.152	(0.9582,0.9462)	1.20	1.155	(0.9846,0.9746)	1.00
	1000	1.156	(0.9076,0.9013)	0.63	1.144	(0.9530,0.9469)	0.61	1.146	(0.9841,0.9765)	0.76
$Burr(3,4)$	50	1.226	(0.9211,0.8938)	2.73	1.213	(0.9602,0.9465)	1.37	1.215	(0.9855,0.9737)	1.18
	100	1.201	(0.9136,0.9008)	1.28	1.188	(0.9607,0.9486)	1.21	1.189	(0.9869,0.9786)	0.83
	500	1.164	(0.9063,0.9004)	0.59	1.152	(0.9532,0.9485)	0.47	1.155	(0.9802,0.9785)	0.17
	1000	1.156	(0.9036,0.9033)	0.03	1.144	(0.9483,0.9493)	-0.10	1.146	(0.9794,0.9791)	0.03
$t(8)$	50	1.226	(0.9095,0.8928)	1.67	1.213	(0.9540,0.9466)	0.74	1.215	(0.9831,0.9781)	0.50
	100	1.201	(0.9077,0.8998)	0.79	1.188	(0.9525,0.9508)	0.17	1.189	(0.9819,0.9773)	0.46
	500	1.164	(0.8975,0.8951)	0.24	1.152	(0.9471,0.9493)	-0.22	1.155	(0.9799,0.9798)	0.01
	1000	1.156	(0.8994,0.8999)	-0.05	1.144	(0.9484,0.9488)	-0.04	1.146	(0.9802,0.9791)	0.11
$Burr(8,7)$	50	1.226	(0.9089,0.8958)	1.31	1.213	(0.9530,0.9454)	0.76	1.215	(0.9794,0.9753)	0.41
	100	1.201	(0.9077,0.8994)	0.83	1.188	(0.9530,0.9495)	0.35	1.189	(0.9834,0.9796)	0.38
	500	1.164	(0.8968,0.9006)	-0.38	1.152	(0.9468,0.9496)	-0.28	1.155	(0.9793,0.9778)	0.15
	1000	1.156	(0.8994,0.9042)	-0.48	1.144	(0.9470,0.9495)	-0.25	1.146	(0.9768,0.9777)	-0.09
$Frechet(10)$	50	1.226	(0.9207,0.8846)	3.61	1.213	(0.9633,0.9347)	2.86	1.215	(0.9870,0.9660)	2.10
	100	1.201	(0.9185,0.8999)	1.95	1.188	(0.9629,0.9431)	1.98	1.189	(0.9892,0.9760)	1.32
	500	1.164	(0.9078,0.9001)	0.77	1.152	(0.9562,0.9473)	0.89	1.155	(0.9827,0.9774)	0.53
	1000	1.156	(0.9037,0.9002)	0.35	1.144	(0.9505,0.9506)	-0.01	1.146	(0.9808,0.9791)	0.17
$N(0,1)$	50	1.226	(0.9057,0.8955)	1.02	1.213	(0.9527,0.9460)	0.67	1.215	(0.9826,0.9737)	0.89
	100	1.201	(0.9067,0.8955)	1.12	1.188	(0.9557,0.9478)	0.79	1.189	(0.9835,0.9774)	0.61
	500	1.164	(0.8974,0.8956)	0.18	1.152	(0.9442,0.9482)	-0.40	1.155	(0.9800,0.9791)	0.09
	1000	1.156	(0.9001,0.8978)	0.23	1.144	(0.9506,0.9463)	0.43	1.146	(0.9798,0.9770)	0.28
$Pois(3)$	50	1.226	(0.9189,0.8951)	2.38	1.213	(0.9592,0.9464)	1.28	1.215	(0.9851,0.9740)	1.11
	100	1.201	(0.9113,0.9000)	1.13	1.188	(0.9605,0.9513)	0.92	1.189	(0.9858,0.9792)	0.66
	500	1.164	(0.9021,0.9006)	0.15	1.152	(0.9530,0.9512)	0.18	1.155	(0.9812,0.9784)	0.28
	1000	1.156	(0.9040,0.9040)	0.00	1.144	(0.9494,0.9494)	0.00	1.146	(0.9797,0.9791)	0.06
$Exp(1)$	50	1.226	(0.9201,0.8835)	3.66	1.213	(0.9601,0.9283)	3.18	1.215	(0.9846,0.9601)	2.45
	100	1.201	(0.9114,0.8899)	2.15	1.188	(0.9588,0.9368)	2.20	1.189	(0.9862,0.9693)	1.69
	500	1.164	(0.9075,0.8997)	0.78	1.152	(0.9548,0.9501)	0.47	1.155	(0.9849,0.9782)	0.67
	1000	1.156	(0.9044,0.8978)	0.66	1.144	(0.9527,0.9471)	0.56	1.146	(0.9832,0.9789)	0.43
$Gamma(0.025,1)$	50	1.226	(0.7170,0.6585)	5.85	1.213	(0.7484,0.6882)	6.02	1.215	(0.7730,0.7153)	5.77
	100	1.201	(0.7923,0.7454)	4.69	1.188	(0.8323,0.7791)	5.32	1.189	(0.8605,0.8092)	5.13
	500	1.164	(0.8853,0.8541)	3.12	1.152	(0.9348,0.8966)	3.82	1.155	(0.9636,0.9308)	3.28
	1000	1.156	(0.9045,0.8728)	3.17	1.144	(0.9516,0.9221)	2.95	1.146	(0.9765,0.9544)	2.21
$Gamma(0.001,1)$	50	1.226	(0.1642,0.1521)	1.21	1.213	(0.1686,0.1574)	1.12	1.215	(0.1731,0.1622)	1.09
	100	1.201	(0.2632,0.2453)	1.79	1.188	(0.2730,0.2554)	1.76	1.189	(0.2817,0.2637)	1.80
	500	1.164	(0.5482,0.5116)	3.66	1.152	(0.5727,0.5319)	4.08	1.155	(0.5941,0.5552)	3.89
	1000	1.156	(0.6688,0.6243)	4.45	1.144	(0.7013,0.6527)	4.86	1.146	(0.7265,0.6787)	4.78
	5000	1.146	(0.8447,0.8005)	4.42	1.135	(0.8898,0.8385)	5.13	1.137	(0.9160,0.8732)	4.28
$Weibull(1,0.2)$	50	1.226	(0.5579,0.4827)	7.52	1.213	(0.5856,0.5135)	7.21	1.215	(0.6160,0.5445)	7.15
	100	1.201	(0.6420,0.5623)	7.97	1.188	(0.6748,0.5984)	7.64	1.189	(0.7093,0.6351)	7.42
	500	1.164	(0.7640,0.6857)	7.83	1.152	(0.8059,0.7303)	7.56	1.155	(0.8411,0.7712)	6.99
	1000	1.156	(0.8051,0.7309)	7.42	1.144	(0.8477,0.7796)	6.81	1.146	(0.8826,0.8220)	6.06
	5000	1.146	(0.8620,0.8119)	5.01	1.135	(0.9129,0.8602)	5.27	1.137	(0.9457,0.8979)	4.78

Table 2.16: Comparison table for I_{3,q_2} vs I_0, values of $\widehat{r}_{3,q_2}(\widehat{CP}_{3,q_2},\widehat{CP}_0)\Delta_{3,q_2}$

2.4 FACI's based on $h_4(\cdot)$

We now consider (1.9) with $h_4(\cdot)$ of (1.16):

$$-0.03\frac{\sum_{i=1}^{[0.99n]}(X_i-\mu)}{s_n\sqrt{n}q\left(\frac{[0.99n]}{n}\right)}+\int_0^1\frac{T_n^t(X_1-\mu,.....,X_n-\mu)}{q\left(\frac{[nt]}{n}\right)}dt$$

$$\xrightarrow[n\to\infty]{d}\frac{-0.03}{q(0.99)}W(0.99)+\int_0^1\frac{W(t)}{q(t)}dt. \tag{2.24}$$

Using the definition of the Student process in (1.6), (2.24) is seen to be equivalent to

$$-0.03\frac{\sum_{i=1}^{[0.99n]}(X_i-\mu)}{s_n\sqrt{n}q\left(\frac{[0.99n]}{n}\right)}+\frac{1}{s_nn\sqrt{n}}\sum_{k=1}^{n-1}\frac{\sum_{i=1}^{k}(X_i-\mu)}{q\left(\frac{k}{n}\right)}$$

$$\xrightarrow[n\to\infty]{d}\frac{-0.03}{q(0.99)}W(0.99)+\int_0^1\frac{W(t)}{q(t)}dt, \tag{2.25}$$

where $\frac{0.03}{q_1(0.99)}=\frac{0.03}{q_2(0.99)}\approx 0.25$. The idea of $h_4(\cdot)$ is similar to that of $h_3(\cdot)$: the FACI based on $h_4(\cdot)$ is not only available in closed form, but also its finite-sample properties are likely influenced by those of I_0 and $I_{1,q}$ of (2.4). The coefficients -0.03 and 1 (as well as $t_0=0.99$) in the linear combination of $h_4(f(t))=-0.03f(0.99)+h_1(f(t))$, $f(t)\in D[0,1]$, were selected experimentally so that I_{4,q_1} and I_{4,q_2}, the respective FACI's based on (1.9) with such $h_4(\cdot)$ and $q(t)=q_j(t)$, would have better finite-sample properties.

Let $P(a \leqslant -(0.03/q(0.99))W(0.99) + \int_0^1 (W(t)/q(t))dt \leqslant b) = 1 - \alpha$, where $\alpha \in (0,1)$. In order to solve for μ the inequality

$$a \leqslant \frac{-0.03 \sum_{i=1}^{[0.99n]}(X_i - \mu)}{s_n\sqrt{n}q\left(\frac{[0.99n]}{n}\right)} + \frac{1}{s_n n\sqrt{n}} \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k}(X_i - \mu)}{q\left(\frac{k}{n}\right)} \leqslant b, \tag{2.26}$$

we first simplify its middle part as follows:

$$-0.03n \sum_{i=1}^{[0.99n]} (X_i - \mu) + q\left(\frac{[0.99n]}{n}\right) \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k}(X_i-\mu)}{q\left(\frac{k}{n}\right)}$$

$$= -0.03n \sum_{i=1}^{[0.99n]} X_i + 0.03n[0.99n]\mu + q\left(\frac{[0.99n]}{n}\right)\left[\sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)} - \mu \sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)}\right]$$

$$= -0.03n \sum_{i=1}^{[0.99n]} X_i + q\left(\frac{[0.99n]}{n}\right) \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)}$$

$$-\mu\left(q\left(\frac{[0.99n]}{n}\right) \sum_{k=1}^{n-1} \frac{k}{q\left(\frac{k}{n}\right)} - 0.03n[0.99n]\right)$$

$$= k_{4,q} - \mu l_{4,q},$$

where

$$k_{4,q} := -0.03n \sum_{i=1}^{[0.99n]} X_i + q\left(\frac{[0.99n]}{n}\right) \sum_{k=1}^{n-1} \frac{\sum_{i=1}^{k} X_i}{q\left(\frac{k}{n}\right)} \tag{2.27}$$

and

$$l_{4,q} := q\left(\frac{[0.99n]}{n}\right)\sum_{k=1}^{n-1}\frac{k}{q\left(\frac{k}{n}\right)} - 0.03n[0.99n]. \tag{2.28}$$

Thus, the FACI for μ based on (2.25) is:

$$I_{4,q} := \left[\frac{-bs_n n^{3/2} q\left(\frac{[0.99n]}{n}\right) + k_{4,q}}{l_{4,q}}, \frac{-as_n n^{3/2} q\left(\frac{[0.99n]}{n}\right) + k_{4,q}}{l_{4,q}}\right], \tag{2.29}$$

where the quantiles a and b of the limiting distribution were selected for each $1-\alpha$ and $q(t)$ numerically, as per Table 2.17, so that the respective $I_{4,q}$ would have better finite-sample properties.

		$1-\alpha = 0.9$	$1-\alpha = 0.95$	$1-\alpha = 0.98$
$q(t) = q_1(t)$	a (γ quantile)	-3.52 ($\gamma = 0.03$)	-4.11 ($\gamma = 0.015$)	-4.84 ($\gamma = 0.005$)
	b ($1-\alpha+\gamma$ quantile)	2.83 (1-$\alpha+\gamma = 0.93$)	3.43 (1-$\alpha+\gamma = 0.965$)	4.09 (1-$\alpha+\gamma = 0.985$)
$q(t) = q_2(t)$	a (γ quantile)	-2.82 ($\gamma = 0.02$)	-3.25 ($\gamma = 0.01$)	-3.91 ($\gamma = 0.0025$)
	b ($1-\alpha+\gamma$ quantile)	1.94 (1-$\alpha+\gamma = 0.92$)	2.45 (1-$\alpha+\gamma = 0.96$)	2.93 (1-$\alpha+\gamma = 0.9825$)

Table 2.17: Some quantiles of $-0.03\frac{W(0.99)}{q(0.99)} + \int_0^1 \frac{W(t)}{q(t)}dt$ for $q(t) = q_1(t)$ and $q(t) = q_2(t)$

Tables 2.18 - 2.20 are the comparison tables for I_{4,q_1} vs I_0 and I_{4,q_2} vs I_0, with the quadruplets $r_{4,q_1}(\widehat{CP}_{4,q_1}, \widehat{CP}_0)\Delta_{4,q_1}$ and $r_{4,q_2}(\widehat{CP}_{4,q_2}, \widehat{CP}_0)\Delta_{4,q_2}$ provided in Tables 2.19 and 2.20, and with ranges of Δ_{4,q_i} and averages in n of r_{4,q_i} listed in summary Table 2.18, $i = \overline{1,2}$.

As seen from Tables 2.18 - 2.20, I_{4,q_1} and I_{4,q_2} exibit some trade-off between their respective empirical finite-sample coverage probabilities and expected lengths. Similarly to the FACI's for μ that are based on $h_1(\cdot) - h_3(\cdot)$, their coverage probabilities are mostly higher than those of I_0, but I_{4,q_1} and I_{4,q_2} are a bit longer than I_0 on average.

For $q(t) = q_1(t)$ and $q(t) = q_2(t)$, the ratio

$$r_{4,q} = \frac{(b-a)n^2 q\left(\frac{[0.99n]}{n}\right)}{2z_{\alpha/2} l_{4,q}} \tag{2.30}$$

of the expected lengths of $I_{4,q}$ to that of I_0 decreases in n for $n > 50$, is somewhat similar across the three confidence levels and perhaps r_{4,q_1} and r_{4,q_2} converge as $n \to \infty$. As for $\widehat{CP}_{4,q_1}$ and $\widehat{CP}_{4,q_2}$, it appears that I_{4,q_2} is more preferable than I_{4,q_1}, due to the coverage probabilities of I_{4,q_2} being a bit higher than those of I_{4,q_1}, regardless of $1-\alpha$, while r_{4,q_1} and r_{4,q_2} being similar. Moreover, for the heavy-tailed and/or largely skewed distributions, when the finite-sample coverage probabilities of I_{4,q_1}, I_{4,q_2} and I_0 are well below, and converging slowly to, $1-\alpha$, the FACI I_{4,q_2} may be a particularly appealing alternative to I_0.

	Heavy-tailed distributions with no skewness	Heavy-tailed distributions with small skewness	Heavy-tailed distributions with large skewness	Light-tailed distributions with small skewness	Light-tailed distributions with large skewness
$1-\alpha=0.90$					
Δ_{4,q_1}	0.84% - 4.81%	0.00% - 1.83%	0.38% - 2.85%	-0.11% - 2.02%	0.24% - 5.11%
average r_{4,q_1}	1.214	1.218	1.218	1.218	1.214
Δ_{4,q_2}	0.86% - 6.58%	-0.09% - 1.90%	0.12% - 3.53%	-0.47% - 2.38%	0.06% - 6.65%
average r_{4,q_2}	1.228	1.234	1.234	1.234	1.228
$1-\alpha=0.95$					
Δ_{4,q_1}	0.57% - 5.66%	0.01% - 1.54%	0.56% - 2.46%	-0.20% - 1.53%	0.28% - 5.76%
average r_{4,q_1}	1.210	1.214	1.214	1.214	1.210
Δ_{4,q_2}	1.12% - 7.17%	0.31% - 2.20%	0.90% - 3.29%	0.08% - 2.33%	0.64% - 7.42%
average r_{4,q_2}	1.234	1.240	1.240	1.240	1.234
$1-\alpha=0.98$					
Δ_{4,q_1}	0.73% - 5.01%	-0.11% - 1.17%	0.30% - 2.11%	-0.19% - 1.41%	0.30% - 5.89%
average r_{4,q_1}	1.207	1.211	1.211	1.211	1.207
Δ_{4,q_2}	1.13% - 6.72%	0.12% - 1.62%	0.62% - 2.95%	-0.07% - 1.91%	0.43% - 8.27%
average r_{4,q_2}	1.247	1.253	1.253	1.253	1.247

Table 2.18: Ranges of Δ_{4,q_i} and averages (in n) of r_{4,q_i}, $i=\overline{1,2}$

Distribution	n	$1-\alpha=0.9$			$1-\alpha=0.95$			$1-\alpha=0.98$		
$Pareto(1,2)$	50	1.235	(0.8169,0.7688)	4.81	1.231	(0.8679,0.8147)	5.32	1.228	(0.9082,0.8581)	5.01
	100	1.248	(0.8373,0.7908)	4.65	1.243	(0.8929,0.8363)	5.66	1.241	(0.9318,0.8822)	4.96
	500	1.201	(0.8685,0.8379)	3.06	1.197	(0.9194,0.8855)	3.39	1.194	(0.9538,0.9235)	3.03
	1000	1.191	(0.8752,0.8516)	2.36	1.187	(0.9266,0.9006)	2.60	1.184	(0.9622,0.9357)	2.65
	5000	1.178	(0.8836,0.8664)	1.72	1.174	(0.9359,0.9173)	1.86	1.171	(0.9705,0.9518)	1.87
$Frechet(3)$	50	1.235	(0.8811,0.8472)	3.39	1.231	(0.9297,0.8968)	3.29	1.228	(0.9630,0.9346)	2.84
	100	1.248	(0.8990,0.8700)	2.90	1.243	(0.9446,0.9167)	2.79	1.241	(0.9739,0.9498)	2.41
	500	1.201	(0.9039,0.8909)	1.30	1.197	(0.9501,0.9381)	1.20	1.194	(0.9787,0.9669)	1.18
	1000	1.191	(0.9042,0.8958)	0.84	1.187	(0.9502,0.9445)	0.57	1.184	(0.9798,0.9725)	0.73
$Burr(3,2)$	50	1.235	(0.9080,0.8897)	1.83	1.231	(0.9549,0.9395)	1.54	1.228	(0.9817,0.9700)	1.17
	100	1.248	(0.9127,0.9005)	1.22	1.243	(0.9579,0.9461)	1.18	1.241	(0.9836,0.9776)	0.60
	500	1.201	(0.9069,0.9015)	0.54	1.197	(0.9558,0.9487)	0.71	1.194	(0.9797,0.9783)	0.14
	1000	1.191	(0.9046,0.9016)	0.30	1.187	(0.9518,0.9505)	0.13	1.184	(0.9789,0.9795)	-0.06
$Pareto(1,6)$	50	1.235	(0.8919,0.8634)	2.85	1.231	(0.9395,0.9149)	2.46	1.228	(0.9798,0.9479)	2.11
	100	1.248	(0.9071,0.8847)	2.24	1.243	(0.9532,0.9349)	1.83	1.241	(0.9861,0.9634)	1.53
	500	1.201	(0.9071,0.8984)	0.87	1.197	(0.9535,0.9462)	0.73	1.194	(0.9846,0.9746)	0.61
	1000	1.191	(0.9051,0.9013)	0.38	1.187	(0.9525,0.9469)	0.56	1.184	(0.9841,0.9765)	0.30
$Burr(3,4)$	50	1.235	(0.9092,0.8938)	1.54	1.231	(0.9570,0.9465)	1.05	1.228	(0.9815,0.9737)	0.78
	100	1.248	(0.9125,0.9008)	1.17	1.243	(0.9581,0.9486)	0.95	1.241	(0.9854,0.9786)	0.68
	500	1.201	(0.9043,0.9004)	0.39	1.197	(0.9538,0.9485)	0.53	1.194	(0.9800,0.9785)	0.15
	1000	1.191	(0.9033,0.9033)	0.00	1.187	(0.9494,0.9493)	0.01	1.184	(0.9780,0.9791)	-0.11
$t(8)$	50	1.235	(0.9070,0.8928)	1.42	1.231	(0.9535,0.9466)	0.69	1.228	(0.9783,0.9781)	0.02
	100	1.248	(0.9105,0.8998)	1.07	1.243	(0.9541,0.9508)	0.33	1.241	(0.9811,0.9773)	0.38
	500	1.201	(0.8994,0.8951)	0.43	1.197	(0.9490,0.9493)	-0.03	1.194	(0.9807,0.9798)	0.09
	1000	1.191	(0.9020,0.8999)	0.21	1.187	(0.9492,0.9488)	0.04	1.184	(0.9786,0.9791)	-0.05
$Burr(8,7)$	50	1.235	(0.9077,0.8958)	1.19	1.231	(0.9541,0.9454)	0.87	1.228	(0.9799,0.9753)	0.46
	100	1.248	(0.9119,0.8994)	1.25	1.243	(0.9590,0.9495)	0.95	1.241	(0.9835,0.9796)	0.39
	500	1.201	(0.9002,0.9006)	-0.04	1.197	(0.9485,0.9496)	-0.11	1.194	(0.9781,0.9778)	0.03
	1000	1.191	(0.9031,0.9042)	-0.11	1.187	(0.9488,0.9495)	-0.07	1.184	(0.9767,0.9777)	-0.10
$Frechet(10)$	50	1.235	(0.9048,0.8846)	2.02	1.231	(0.9500,0.9347)	1.53	1.228	(0.9801,0.9660)	1.41
	100	1.248	(0.9137,0.8999)	1.47	1.243	(0.9567,0.9431)	1.36	1.241	(0.9841,0.9760)	0.81
	500	1.201	(0.9070,0.9001)	0.69	1.197	(0.9544,0.9473)	0.71	1.194	(0.9793,0.9774)	0.19
	1000	1.191	(0.9046,0.9002)	0.44	1.187	(0.9522,0.9506)	0.16	1.184	(0.9780,0.9791)	-0.11
$N(0,1)$	50	1.235	(0.9027,0.8955)	0.72	1.231	(0.9487,0.9460)	0.27	1.228	(0.9794,0.9737)	0.57
	100	1.248	(0.9128,0.8955)	1.73	1.243	(0.9587,0.9478)	1.09	1.241	(0.9851,0.9774)	0.77
	500	1.201	(0.8963,0.8956)	0.07	1.197	(0.9480,0.9482)	-0.20	1.194	(0.9789,0.9791)	-0.02
	1000	1.191	(0.8999,0.8978)	0.21	1.187	(0.9477,0.9463)	0.14	1.184	(0.9778,0.9770)	0.08
$Pois(3)$	50	1.235	(0.9075,0.8951)	1.24	1.231	(0.9563,0.9464)	0.99	1.228	(0.9833,0.9740)	0.93
	100	1.248	(0.9098,0.9000)	0.98	1.243	(0.9573,0.9513)	0.60	1.241	(0.9841,0.9792)	0.49
	500	1.201	(0.9037,0.9006)	0.31	1.197	(0.9519,0.9512)	0.07	1.194	(0.9794,0.9784)	0.10
	1000	1.191	(0.9079,0.9040)	0.39	1.187	(0.9493,0.9494)	-0.01	1.184	(0.9772,0.9791)	-0.19
$Exp(1)$	50	1.235	(0.9023,0.8835)	1.88	1.231	(0.9479,0.9283)	1.96	1.228	(0.9759,0.9601)	1.58
	100	1.248	(0.9075,0.8899)	1.76	1.243	(0.9528,0.9368)	1.60	1.241	(0.9822,0.9693)	1.29
	500	1.201	(0.9035,0.8997)	0.38	1.197	(0.9529,0.9501)	0.28	1.194	(0.9817,0.9782)	0.35
	1000	1.191	(0.9057,0.8978)	0.79	1.187	(0.9546,0.9471)	0.75	1.184	(0.9819,0.9789)	0.30
$Gamma(0.025,1)$	50	1.235	(0.6892,0.6585)	3.07	1.231	(0.7264,0.6882)	3.82	1.228	(0.7601,0.7153)	4.48
	100	1.248	(0.7992,0.7454)	2.38	1.243	(0.8134,0.7791)	3.43	1.241	(0.8495,0.8092)	4.03
	500	1.201	(0.8711,0.8541)	1.70	1.197	(0.9173,0.8966)	2.07	1.194	(0.9529,0.9308)	2.21
	1000	1.191	(0.8913,0.8728)	1.85	1.187	(0.9394,0.9221)	1.73	1.184	(0.9687,0.9544)	1.43
$Gamma(0.001,1)$	50	1.235	(0.1545,0.1521)	0.24	1.231	(0.1646,0.1574)	0.72	1.228	(0.1706,0.1622)	0.84
	100	1.248	(0.2516,0.2453)	0.63	1.243	(0.2674,0.2554)	1.20	1.241	(0.2782,0.2637)	1.45
	500	1.201	(0.5282,0.5116)	1.66	1.197	(0.5607,0.5319)	2.88	1.194	(0.5836,0.5552)	2.84
	1000	1.191	(0.6450,0.6243)	2.07	1.187	(0.6850,0.6527)	3.23	1.184	(0.7157,0.6787)	3.70
	5000	1.178	(0.8166,0.8005)	1.61	1.174	(0.8732,0.8385)	3.47	1.171	(0.9046,0.8732)	3.14
$Weibull(1,0.2)$	50	1.235	(0.5263,0.4827)	4.36	1.231	(0.5638,0.5135)	5.03	1.228	(0.5983,0.5445)	5.38
	100	1.248	(0.6134,0.5623)	5.11	1.243	(0.6565,0.5984)	5.76	1.241	(0.6940,0.6351)	5.89
	500	1.201	(0.7302,0.6857)	4.45	1.197	(0.7827,0.7303)	5.24	1.194	(0.8265,0.7712)	5.53
	1000	1.191	(0.7721,0.7309)	4.12	1.187	(0.8242,0.7796)	4.46	1.184	(0.8678,0.8220)	4.58
	5000	1.178	(0.8382,0.8119)	2.63	1.174	(0.8883,0.8602)	2.81	1.171	(0.9294,0.8979)	3.15

Table 2.19: Comparison table for I_{4,q_1} vs I_0, values of $r_{4,q_1}(\widehat{CP}_{4,q_1}, \widehat{CP}_0)\Delta_{4,q_1}$

Distribution	n	$1-\alpha=0.9$	$1-\alpha=0.95$	$1-\alpha=0.98$
$Pareto(1,2)$	50	1.253 (0.8300,0.7688) 6.12	1.259 (0.8843,0.8147) 6.96	1.273 (0.9253,0.8581) 6.72
	100	1.274 (0.8566,0.7908) 6.58	1.280 (0.9080,0.8363) 7.17	1.294 (0.9487,0.8822) 6.65
	500	1.212 (0.8808,0.8379) 4.29	1.217 (0.9315,0.8855) 4.60	1.231 (0.9672,0.9235) 4.37
	1000	1.198 (0.8842,0.8516) 3.26	1.204 (0.9394,0.9006) 3.88	1.217 (0.9719,0.9357) 3.62
	5000	1.181 (0.8879,0.8664) 2.15	1.186 (0.9442,0.9173) 2.69	1.200 (0.9798,0.9518) 2.80
$Frechet(3)$	50	1.253 (0.8908,0.8472) 4.36	1.259 (0.9412,0.8968) 4.44	1.273 (0.9727,0.9346) 3.81
	100	1.274 (0.9076,0.8700) 3.76	1.280 (0.9565,0.9167) 3.98	1.294 (0.9825,0.9498) 3.27
	500	1.212 (0.9050,0.8909) 1.41	1.217 (0.9573,0.9381) 1.92	1.231 (0.9837,0.9669) 1.68
	1000	1.198 (0.9044,0.8958) 0.86	1.204 (0.9557,0.9445) 1.12	1.217 (0.9838,0.9725) 1.13
$Burr(3,2)$	50	1.253 (0.9087,0.8897) 1.90	1.259 (0.9615,0.9395) 2.20	1.273 (0.9862,0.9700) 1.62
	100	1.274 (0.9158,0.9005) 1.53	1.280 (0.9661,0.9461) 2.00	1.294 (0.9883,0.9776) 1.07
	500	1.212 (0.9071,0.9015) 0.56	1.217 (0.9586,0.9487) 0.99	1.231 (0.9822,0.9783) 0.39
	1000	1.198 (0.9028,0.9016) 0.12	1.204 (0.9541,0.9505) 0.36	1.217 (0.9809,0.9795) 0.14
$Pareto(1,6)$	50	1.253 (0.8987,0.8634) 3.53	1.259 (0.9478,0.9149) 3.29	1.273 (0.9774,0.9479) 2.95
	100	1.274 (0.9143,0.8847) 2.96	1.280 (0.9626,0.9349) 2.77	1.294 (0.9849,0.9634) 2.15
	500	1.212 (0.9091,0.8984) 1.07	1.217 (0.9597,0.9462) 1.35	1.231 (0.9840,0.9746) 0.94
	1000	1.198 (0.9025,0.9013) 0.12	1.204 (0.9559,0.9469) 0.90	1.217 (0.9827,0.9765) 0.62
$Burr(3,4)$	50	1.253 (0.9110,0.8938) 1.72	1.259 (0.9613,0.9460) 1.48	1.273 (0.9851,0.9737) 1.14
	100	1.274 (0.9146,0.9008) 1.38	1.280 (0.9634,0.9486) 1.48	1.294 (0.9878,0.9786) 0.92
	500	1.212 (0.9054,0.9004) 0.50	1.217 (0.9561,0.9485) 0.76	1.231 (0.9809,0.9785) 0.24
	1000	1.198 (0.9024,0.9033) -0.09	1.204 (0.9524,0.9493) 0.31	1.217 (0.9803,0.9791) 0.12
$t(8)$	50	1.253 (0.9063,0.8928) 1.35	1.259 (0.9554,0.9466) 0.88	1.273 (0.9809,0.9781) 0.28
	100	1.274 (0.9150,0.8998) 1.52	1.280 (0.9583,0.9508) 0.75	1.294 (0.9840,0.9773) 0.67
	500	1.212 (0.8961,0.8951) 0.10	1.217 (0.9520,0.9493) 0.27	1.231 (0.9831,0.9798) 0.33
	1000	1.198 (0.8967,0.8999) -0.32	1.204 (0.9509,0.9488) 0.21	1.217 (0.9784,0.9791) -0.07
$Burr(8,7)$	50	1.253 (0.9048,0.8958) 0.90	1.259 (0.9560,0.9454) 1.06	1.273 (0.9808,0.9753) 0.55
	100	1.274 (0.9134,0.8994) 1.40	1.280 (0.9619,0.9495) 1.24	1.294 (0.9852,0.9796) 0.56
	500	1.212 (0.9017,0.9006) 0.11	1.217 (0.9506,0.9496) 0.10	1.231 (0.9795,0.9778) 0.17
	1000	1.198 (0.8995,0.9042) -0.47	1.204 (0.9503,0.9495) 0.08	1.217 (0.9780,0.9777) 0.03
$Frechet(10)$	50	1.253 (0.9084,0.8846) 2.38	1.259 (0.9580,0.9347) 2.33	1.273 (0.9851,0.9660) 1.91
	100	1.274 (0.9164,0.8999) 1.74	1.280 (0.9648,0.9431) 2.17	1.294 (0.9880,0.9760) 1.20
	500	1.212 (0.9068,0.9001) 0.67	1.217 (0.9587,0.9473) 1.14	1.231 (0.9823,0.9774) 0.49
	1000	1.198 (0.9020,0.9002) 0.18	1.204 (0.9543,0.9506) 0.37	1.217 (0.9813,0.9791) 0.22
$N(0,1)$	50	1.253 (0.9027,0.8955) 0.72	1.259 (0.9510,0.9460) 0.60	1.273 (0.9810,0.9737) 0.73
	100	1.274 (0.9131,0.8955) 1.76	1.280 (0.9627,0.9478) 1.49	1.294 (0.9868,0.9774) 0.94
	500	1.212 (0.8937,0.8956) -0.19	1.217 (0.9497,0.9482) 0.15	1.231 (0.9802,0.9791) 0.11
	1000	1.198 (0.8995,0.8978) 0.17	1.204 (0.9508,0.9463) 0.45	1.204 (0.9797,0.9770) 0.27
$Pois(3)$	50	1.253 (0.9102,0.8951) 1.51	1.259 (0.9604,0.9464) 1.40	1.273 (0.9859,0.9740) 1.19
	100	1.274 (0.9137,0.9000) 1.37	1.280 (0.9631,0.9513) 1.18	1.294 (0.9867,0.9792) 0.75
	500	1.212 (0.9034,0.9006) 0.28	1.217 (0.9544,0.9512) 0.32	1.231 (0.9818,0.9784) 0.34
	1000	1.198 (0.9025,0.9040) -0.15	1.204 (0.9519,0.9494) 0.25	1.217 (0.9803,0.9791) 0.12
$Exp(1)$	50	1.253 (0.9085,0.8835) 1.50	1.259 (0.9549,0.9283) 2.66	1.273 (0.9814,0.9601) 2.13
	100	1.274 (0.9115,0.8899) 2.16	1.280 (0.9604,0.9368) 2.36	1.294 (0.9863,0.9693) 1.70
	500	1.212 (0.9037,0.8997) 0.40	1.217 (0.9565,0.9501) 0.64	1.231 (0.9844,0.9782) 0.62
	1000	1.198 (0.9039,0.8978) 0.61	1.204 (0.9572,0.9471) 1.01	1.217 (0.9832,0.9789) 0.43
$Gamma(0.025,1)$	50	1.253 (0.6970,0.6585) 3.85	1.259 (0.7383,0.6882) 5.10	1.273 (0.7732,0.7153) 5.79
	100	1.274 (0.7811,0.7454) 3.57	1.280 (0.8269,0.7791) 4.78	1.294 (0.8657,0.8092) 5.65
	500	1.212 (0.8749,0.8541) 2.08	1.217 (0.9278,0.8966) 3.12	1.231 (0.9641,0.9308) 3.33
	1000	1.198 (0.8933,0.8728) 2.86	1.204 (0.9480,0.9221) 2.59	1.217 (0.9753,0.9544) 2.09
$Gamma(0.001,1)$	50	1.253 (0.1527,0.1521) 0.06	1.259 (0.1671,0.1574) 0.97	1.273 (0.1738,0.1622) 1.16
	100	1.274 (0.2555,0.2453) 1.02	1.280 (0.2725,0.2554) 1.71	1.294 (0.2841,0.2637) 2.04
	500	1.212 (0.5332,0.5116) 2.16	1.217 (0.5690,0.5319) 3.71	1.231 (0.5963,0.5552) 4.11
	1000	1.198 (0.6525,0.6243) 2.82	1.204 (0.6948,0.6527) 4.21	1.217 (0.7276,0.6787) 4.89
	5000	1.181 (0.8265,0.8005) 2.60	1.186 (0.8807,0.8385) 4.22	1.200 (0.9170,0.8732) 4.38
$Weibull(1,0.2)$	50	1.253 (0.5398,0.4827) 5.71	1.259 (0.5792,0.5135) 6.57	1.273 (0.6185,0.5445) 7.40
	100	1.274 (0.6288,0.5623) 6.65	1.280 (0.6726,0.5984) 7.42	1.294 (0.7178,0.6351) 8.27
	500	1.212 (0.7438,0.6857) 5.81	1.217 (0.7985,0.7303) 6.82	1.231 (0.8459,0.7712) 7.47
	1000	1.198 (0.7856,0.7309) 5.47	1.204 (0.8398,0.7796) 6.02	1.217 (0.8865,0.8220) 6.45
	5000	1.181 (0.8424,0.8119) 3.05	1.186 (0.8994,0.8602) 3.92	1.200 (0.9440,0.8979) 4.61

Table 2.20: Comparison table for I_{4,q_2} vs I_0, values of $r_{4,q_2}(\widehat{CP}_{4,q_2}, \widehat{CP}_0)\Delta_{4,q_2}$

Chapter 3

Conclusions

In this thesis, using the FCLT's in (1.7) and (1.9) respectively with the functionals $h_1(\cdot) - h_2(\cdot)$ and $h_1(\cdot) - h_4(\cdot)$ as in (1.13)-(1.16), as well as with the weight functions $q(t) = q_j(t)$ or $\sqrt{q_j(t)}$, $j = \overline{1,2}$, we derived several FACI's for the mean of a distribution in DAN. We compared the finite-sample coverage probabilities and expected lengths of the obtained FACI's to the respective ones of the CI I_0 of (1.4) obtained simply from the CLT in (1.2). Accordingly, we observed a trade-off between the coverage probabilities and expected lengths of our FACI's: they have higher coverage probabilities than those of I_0, but are somewhat longer than I_0 on average. Our FACI's provide reasonable alternatives to I_0 when some improvement of the finite-sample coverage probability of I_0 is desirable and the expense of losing in the expected length is acceptable. In particular, our FACI's may be particularly appealing for heavy-tailed and/or largely skewed distributions, especially for smaller sample sizes with $n > 50$. Moreover, for the FACI's based on $h_1(\cdot)$ and $h_2(\cdot)$, we concluded that I_1 and

I_2 derived from the FCLT of (1.7) for the Student process (which are among the top three FACI's in Tuzov (2014) and Martsynyuk and Tuzov (2016)) are overall outperformed by at least one of our respective FACI's I_{1,q_1} and I_{1,q_2}, and I_{2,q_1} and I_{2,q_2}, which are based on the FCLT of (1.9) for the weighted Student process.

For a convenient reference and comparison of all $h_1(\cdot) - h_4(\cdot)$ based FACI's, we now use the summary Tables 2.4, 2.9, 2.14, and 2.18 and compose Tables 3.1 - 3.3 for our three confidence levels $1 - \alpha = 0.9$, 0.95 and 0.98. In each of these tables, we list the ranges of the empirical finite-sample coverage probabilities of the better of I_{i,q_1} and I_{i,q_2} (determined already in Chapter 2) and averages (in n) of the ratio of its true/empirical expected length and that of I_0 for each of the five distribution classes of Table 1.2.

In Table 3.1 ($1 - \alpha = 0.9$), I_{3,q_2} clearly outperforms I_{1,q_2} and I_{4,q_2} (with the ratios r_{1,q_2} and r_{4,q_2} that are comparable to $\widehat{r}_{3,q_2}$), especially for the two classes of heavy-tailed distributions with no skewness and light-tailed distributions with large skewness. Moreover, although I_{2,q_2} is a bit shorter than I_{3,q_2} on average, the finite-sample coverage probabilities of I_{3,q_2} are significantly higher than those of I_{2,q_2}. Thus, I_{3,q_2} appears to be more appealing compared to I_{2,q_2} as well.

Arguing similarly, we conclude that I_{3,q_2} is the best FACI in Table 3.2 ($1 - \alpha = 0.95$) and Table 3.3 ($1 - \alpha = 0.98$). In the latter table, when comparing I_{3,q_2} and I_{4,q_2}, we notice that, except for the class of light-tailed distributions with large skewness, the coverage probabilities of the two FACI's are very close, but I_{3,q_2} is a bit shorter on average than I_{4,q_2} and hence is preferred to I_{4,q_2}.

		I_{1,q_1} or I_{1,q_2}	I_{2,q_1} or I_{2,q_2}	I_{3,q_1} or I_{3,q_2}	I_{4,q_1} or I_{4,q_2}
Heavy-tailed	Δ_{i,q_1}	-	-	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	-	-	-
with	Δ_{i,q_2}	0.71% - 4.38%	0.26% - 2.17%	1.41% - 8.20%	0.86% - 6.58%
no skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.147	1.082	1.182	1.228
Heavy-tailed	Δ_{i,q_1}	-	-	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	-	-	-
with	Δ_{i,q_2}	-0.10% - 2.72%	0.10% - 1.88%	0.03% - 3.34%	-0.09% - 1.90%
small skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.153	1.086	1.187	1.234
Heavy-tailed	Δ_{i,q_1}	-	-	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	-	-	-
with	Δ_{i,q_2}	0.17% - 3.50%	0.00% - 2.13%	0.63% - 4.89%	0.12% - 3.53%
large skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.153	1.086	1.187	1.234
Light-tailed	Δ_{i,q_1}	-	-	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	-	-	-
with	Δ_{i,q_2}	-0.35% - 2.84%	-0.26% - 1.94%	-0.48% - 3.61%	-0.47% - 2.38%
small skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.153	1.086	1.187	1.234
Light-tailed	Δ_{i,q_1}	-	-	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	-	-	-
with	Δ_{i,q_2}	0.42% - 3.46%	-0.13% - 1.82%	0.66% - 7.97%	0.06% - 6.65%
large skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.147	1.082	1.182	1.228

Table 3.1: Ranges of Δ_{i,q_j} and averages (in n) of r_{i,q_j} or $\widehat{r}_{i,q_j}$ for the better of the FACI's I_{i,q_1} and I_{i,q_2}, for $1-\alpha = 0.9$

We also note in passing that I_{4,q_2} based on the functional $h_4(\cdot)$, which is linear combination of the integral functional $h_1(\cdot)$ and the projection functional $h_0(\cdot)$, may be preferred to I_{1,q_1} or I_{1,q_2}, which are based on $h_1(\cdot)$ alone, for all classes of heavy-tailed distributions with no skewness and light-tailed distributions with large skewness Indeed, although I_{4,q_2} is a bit longer on average than I_{1,q_1} and I_{1,q_2}, the finite-sample coverage probabilities of I_{4,q_2} for these two classes are significantly larger than the respective those of I_{1,q_1} and I_{1,q_2}.

		I_{1,q_1} or I_{1,q_2}	I_{2,q_1} or I_{2,q_2}	I_{3,q_1} or I_{3,q_2}	I_{4,q_1} or I_{4,q_2}
Heavy-tailed	Δ_{i,q_1}	-	0.28% - 2.51%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	1.111	-	-
with	Δ_{i,q_2}	0.54% - 5.57%	-	1.30% - 8.11%	1.12% - 7.17%
no skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.154	-	1.169	1.234
Heavy-tailed	Δ_{i,q_1}	-	0.04% - 1.07%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	1.111	-	-
with	Δ_{i,q_2}	-0.06% - 2.25%	-	-0.10% - 2.38%	0.31% - 2.20%
small skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.161	-	1.174	1.240
Heavy-tailed	Δ_{i,q_1}	-	0.25% - 1.48%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	1.111	-	-
with	Δ_{i,q_2}	0.51% - 3.03%	-	0.61% - 4.16%	0.90% - 3.29%
large skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.161	-	1.174	1.240
Light-tailed	Δ_{i,q_1}	-	-0.02% - 1.27%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	1.111	-	-
with	Δ_{i,q_2}	-0.17% - 2.43%	-	-0.40% - 2.86%	0.08% - 2.33%
small skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.161	-	1.174	1.240
Light-tailed	Δ_{i,q_1}	-	0.09% - 2.09%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	-	1.111	-	-
with	Δ_{i,q_2}	0.35% - 5.03%	-	0.47% - 7.64%	0.64% - 7.42%
large skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	1.154	-	1.169	1.234

Table 3.2: Ranges of Δ_{i,q_j} and averages (in n) of r_{i,q_j} or $\widehat{r}_{i,q_j}$ for the better of the FACI's I_{i,q_1} and I_{i,q_2}, for $1-\alpha = 0.95$

		I_{1,q_1} or I_{1,q_2}	I_{2,q_1} or I_{2,q_2}	I_{3,q_1} or I_{3,q_2}	I_{4,q_1} or I_{4,q_2}
Heavy-tailed	Δ_{i,q_1}	0.86% - 5.17%	0.19% - 2.34%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	1.179	1.105	-	-
with	Δ_{i,q_2}	-	-	1.32% - 6.84%	1.13% - 6.72%
no skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	-	-	1.172	1.247
Heavy-tailed	Δ_{i,q_1}	-0.12% - 1.43%	0.06% - 0.99%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	1.184	1.111	-	-
with	Δ_{i,q_2}	-	-	0.03% - 1.76%	0.12% - 1.62%
small skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	-	-	1.176	1.253
Heavy-tailed	Δ_{i,q_1}	0.40% - 2.36%	0.22% - 1.12%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	1.184	1.111	-	-
with	Δ_{i,q_2}	-	-	0.76% - 3.21%	0.62% - 2.95%
large skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	-	-	1.176	1.253
Light-tailed	Δ_{i,q_1}	-0.09% - 1.67%	0.04% - 0.97%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	1.184	1.111	-	-
with	Δ_{i,q_2}	-	-	-0.09% - 2.10%	-0.07% - 1.91%
small skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	-	-	1.176	1.253
Light-tailed	Δ_{i,q_1}	0.28% - 5.47%	0.13% - 2.59%	-	-
distributions	aver. r_{i,q_1} or $\widehat{r}_{i,q_1}$	1.179	1.108	-	-
with	Δ_{i,q_2}	-	-	0.43% - 7.42%	0.43% - 8.27%
large skewness	aver. r_{i,q_2} or $\widehat{r}_{i,q_2}$	-	-	1.172	1.247

Table 3.3: Ranges of Δ_{i,q_j} and averages (in n) of r_{i,q_j} or $\widehat{r}_{i,q_j}$ for the better of the FACI's I_{i,q_1} and I_{i,q_2}, for $1-\alpha = 0.98$

Table 3.4 is a summary of the finite-sample performance of our best FACI, I_{3,q_2}, for the three $1-\alpha$ confidence levels.

$1-\alpha$	Range of Δ_{3,q_2} (in %) and average $\widehat{r}_{3,q_2}$ across all distributions
0.9	[-0.48%, 8.20%], 1.184
0.95	[-0.40%, 8.11%], 1.172
0.98	[-0.09%, 7.42%], 1.174

Table 3.4: Finite-sample performance of I_{3,q_2}, the best FACI

We also note in passing that in the works of Tuzov (2014) and Martsynyuk and Tuzov (2016), the FCLT in (1.7) with the sup-functional $h_5(f(t)) = \sup_{0\leqslant t\leqslant 1}|f(t)|$, $f(t) \in D[0,1]$, led to one of their best three FACI for μ with higher coverage probabilities and somewhat longer expected lengths compared to I_0 (the other two of their best three FACI's were I_1 and I_2, outperformed by our I_{3,q_2}). The formula for that FACI, called I_5 here, was

$$I_5 := \left[\max_{1\leqslant k\leqslant n}\frac{\sum_{k=1}^{n} X_i - bs_n\sqrt{n}}{k}\,,\,\min_{1\leqslant k\leqslant n}\frac{\sum_{k=1}^{n} X_i + bs_n\sqrt{n}}{k}\right]. \tag{3.1}$$

We decided to compare I_5 to our best FACI I_{3,q_2}. Accordingly, we generated a table similar to Table 2.16 (not included here) for all our distributions of Table 1.1, which was then reduced to Table 3.5, with ranges of the difference between the empirical finite-sample coverage probabilities of I_5 and those of I_0, denoted by Δ_5, and averages (in n) of the empirical ratio of the expected lengths of I_5 and I_0, denoted by $\widehat{r}_5$.

	Distributions				
	Heavy-tailed, no skewness	Heavy-tailed, small skewness	Heavy-tailed, large skewness	Light-tailed, small skewness	Light-tailed, large skewness
$1-\alpha=0.9$					
Δ_5	1.03% - 2.09%	0.33% - 1.98%	0.48% - 2.10%	0.36% - 2.21%	0.22% - 1.70%
average $\widehat{r}_5$	1.085	1.068	1.075	1.067	1.108
$1-\alpha=0.95$					
Δ_5	0.41% - 1.24%	0.16% - 0.93%	0.41% - 1.10%	0.07% - 1.08%	0.05% - 0.97%
average $\widehat{r}_5$	1.072	1.056	1.061	1.055	1.084
$1-\alpha=0.98$					
Δ_5	0.12% - 0.67%	-0.05% - 0.58%	0.26% - 0.54%	-0.06% - 0.64%	-0.01% - 0.85%
average $\widehat{r}_5$	1.056	1.044	1.049	1.044	1.066

Table 3.5: Ranges of Δ_5 and averages (in n) of $\widehat{r}_5$

Comparing the performance of I_5 to that of I_{3,q_2} from Tables 3.1 - 3.3, we see that, in general, I_5 exibits a more conservative trade-off between its coverage probabilities and expected length, with Δ_5 decreasing in α and being rather marginal for the confidence level $1-\alpha=0.98$. Overall, the performance of I_5 is very similar to that of the FACI's I_{2,q_1} or I_{2,q_2}, and the latter two were shown to be outperformed by I_{3,q_2}.

Bibliography

Csörgő, M. and L. Horváth (1988). Invariance principles for changepoint problems. *J. Multivariate Anal. 27*, 151-168. (Cited on page 6.)

Csörgő, M., B. Szyszkowicz, and Q. Wang (2003). Donsker's theorem for self-normalized partial sums process. *Ann. Probab. 31*(3), 1228-1240. (Cited on page 3.)

Csörgő, M., B. Szyszkowicz, and Q. Wang (2008). On weighted approximations in $D[0,1]$ with applications to self-normalized partial sum processes. *Acta Math. Hungar. 121*(4), 307-332. (Cited on page 6.)

Csörgő, S. and L. Horváth. (1981). On the Koziol-Green model for random censorship. *Biometrika 68*(2), 391-401. (Cited on page 27.)

Erdős, P. and M. Kac (1946). On certain limit theorems of the theory of probability. *Bull. Amer. Math. Soc. 52,* 292-302 (2nd ed.). (Cited on pages 8 and 15.)

Giné, E., F. Götze, and D. Mason (1997). When is the Student t-statistic asymptotically standard normal? *Ann. Probab. 25*(3), 1514-1531. (Cited on page 1.)

Martsynyuk, Y. V. (2013). On the generalized domain of attraction of the multivariate normal law and asymptotic normality of the multivariate Student t-statistic. *J. Multivariate Anal. 114,* 402-411. (Cited on page 2.)

Martsynyuk, Y. V. and E. Tuzov (2016). Exploring functional CLT confidence intervals for a population mean in the domain of attraction of the normal law. *Acta Math. Hungar. 148*(2) 493-508. (Cited on pages 4, 5, 8, 13, 19, 29, 52 and 56.)

Orasch, M. and W. Pouliot (2010). Tabulating weighted sup-norm functionals used in change-point analysis. *J. Stat. Comput. Simul. 74*(4), 249-276. (Cited on page 7.)

Tuzov, E. (2014). *Exploring Functional Asymptotic Confidence Intervals for a Population Mean.* M.Sc. Thesis, University of Manitoba, Winnipeg. (Cited on pages 4, 5, 8, 13, 19, 29, 52 and 56.)