Schwartz Forms on Unitary Spaces & the Weil Representation

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Abstract

Stephen Kudla has conjectured a relationship between the Fourier coefficients of Eisenstein series, and the arithmetic heights of certain special cycles. Luis Garcia and Siddarth Sankaran confirmed the conjecture for certain Shimura varieties of type U(p, q), arising as the quotient of a symmetric space by a group action, when q=1. An essential step in their argument relies on establishing that a specific form is a highest weight vector of a particular weight, for the Weil representation. In an effort to extend the results of Garcia and Sankaran, we show that the aforementioned forms are highest weight vectors of the expected weight, under the action of the Weil representation, in various cases when q>1. In particular, we show that this result holds for all cases when q=2. We prove this result by using an inductive argument, which depends on a technical result about immersed submanifolds, and various results about splitting the action of the Weil representation on tensor products. The base cases are intractable to carry out by hand, and thus the final section of the thesis contains Sage code which was written to carry out the computations of the base cases.

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Contents

0.1 Introduction

In the paper [4], Garcia and Sankaran obtain an explicit formula for the local Archimedean height of a special cycle of a certain Shimura variety of type $U(p,1)$, in terms of a Fourier coefficient of a special derivative of an Eisenstein series. Their result supports a more general conjectured identity of Kudla, known as the *arithmetic Siegel-Weil formula*.
An important object in Kudla's investigations is the *Kudla-Millson form* $\varphi_{KM} \in \mathcal{S}(V) \otimes \mathcal{A}^{q,q}(\mathbb{D})$ where $\mathcal{S}(V)$ is the *Schwartz space* of (rapidly decreasing) functions on V for a $(p+q)$-dimensional $\mathbb{C}$-vector space V with Hermitian form of signature (p,q) (we call this a (p,q)-space for short), $\mathbb{D}(V)$ is diffeomorphic to the symmetric space G/K where $G = U(p,q)$, and $K \cong U(p) \times U(q)$ is a maximal compact subgroup of G, and $\mathcal{A}^{q,q}(\mathbb{D}(V))$ is the space of complex differential forms of type (q,q) on $\mathbb{D}(V)$.
A key step in the argument of [4] involves the use of Quillen's theory of *superconnections*, and the construction of forms $\varphi(\mathbf{v}), \nu_r(\mathbf{v}) \in \mathcal{S}(V^{\oplus r}) \otimes \Omega^\bullet(\mathbb{D}(V))$, generalizing the classical Chern-Weil theory, and when $r = 1$, the degree $2q$ component of $\varphi(v)_{[2q]}$ *is* φ_{KM}. The aforementioned Shimura variety of type $U(p,q)$ is realized as a quotient $\Gamma\backslash\mathbb{D}(V)$ by the action of a certain arithmetic group Γ.
The investigation of the forms ν_r is carried out by studying their behavior under the Weil representation for $U(r,r)$. In particular, in [4] it's proved that the form ν_r generates an irreducible subrepresentation of the Weil representation, when $q = 1$.
In this thesis, we will show that the form ν_r generates an irreducible subrepresentation for various cases when $q > 1$, including $(p,q) = (p,2)$ for *any* p. In fact, we also manage to prove an inductive step which shows that for any fixed q, and a p satisfying a particular bound in terms of q, if $\nu_r \in \mathcal{S}(V^r) \otimes \Omega^\bullet(\mathbb{D})$ is a *highest weight vector* for all $(p+q)$-dimensional vector spaces with a Hermitian form of signature (p,q), then the analogously constructed form $\nu_r' \in \mathcal{S}(V'^r) \otimes \Omega^\bullet(\mathbb{D}(V'))$ is highest weight vector for all $(p'+q)$-spaces where $p' \geq p$. Unfortunately the base cases remain un-established in general, but we include an algorithm written in Sage which determines any of these cases "by hand".

In chapter one, we recall some of the basic theory of complex manifolds, and Hermitian vector bundles. We also reconstruct much Quillen's seminal paper [8] on the theory of superconnections.
In the second chapter, we recall the basic theory of Lie groups, Lie algebras, and their representations. Important facts about the theory of highest weight vectors will be stated, and relevant explicit computations will be made along the way. This will culminate in explicit computations for the action of the Weil representation $\omega : \mathfrak{k}_{ss} \to \mathcal{S}(V^r)$, where $\mathfrak{k}_{ss}$ is the *semi-simplification* of the complexified Lie algebra $\mathrm{Lie}(U(p) \times U(q))$.
In the third chapter, we will describe the manifold $\mathbb{D}$, and reproduce the construction of the form $\nu_r \in \mathcal{S}(V^r) \otimes \Omega^\bullet(\mathbb{D})$ of [4], by exploiting Quillen's theory of superconnections for a particular vector bundle over $\mathbb{D}$.
The Weil representation can be made realized as $\omega : \mathfrak{k}_{ss} \to \mathcal{S}(V^r) \otimes \Omega^\bullet(\mathbb{D})$, just by acting on the $\mathcal{S}(V^r)$ factor. We then establish a series of inductive results for the form(s) ν_r, and together with the code laid out in the fourth chapter, establish the aforementioned results for ν_r.

Chapter 1

Geometric Background

1.1 Hermitian Geometry

As our main interest is involves constructing differential forms on the complex manifold $\mathbb{D}(V)$ through a generalization of connections, for this first chapter, we review the basic theory of real and complex manifolds, smooth and holomorphic vector bundles, connections, and introduce Quillen's theory of *superconnections* [8]. Much of this section closely follows the development as written in [10], with some minor adaptations to our specific interests.

Vector Bundles

Let K be $\mathbb{R}$ or $\mathbb{C}$, and let M be an $\mathcal{S}$-manifold, by which we mean either a smooth or holomorphic manifold with a sheaf $\mathcal{S} = \mathcal{A}$ of smooth or $\mathcal{S} = \mathcal{O}$ holomorphic functions.
We will speak of $\mathcal{S}$-morphisms, by which we either mean smooth or holomorphic maps, depending on the context.
Recall that an $\mathcal{S}$-vector bundle is a tuple (E, V, M, π) where E and M are $\mathcal{S}$-manifolds, V is an r-dimensional K-vector space and π is an $\mathcal{S}$-morphism such that

1. $\pi : E \to M$ is a surjective $\mathcal{S}$-map,

2. $\forall p \in M$, the *fiber* $E_p := \pi^{-1}(p)$ is a K-vector space, and

3. $\forall p \in M$ there exists an open set $U \subseteq M$ containing p and a homeomorphism $\psi : \pi^{-1}(U) \to U \times K^r$ where $\psi(E_p) \subseteq \{p\} \times K^r$, and for the projection

$$\begin{aligned} \mathrm{proj} : \{p\} \times K^r &\to K^r \\ (p, k) &\mapsto k \end{aligned}$$

   the map $\psi^p = \mathrm{proj} \circ \psi : E_p \to K^r$ is a K-linear isomorphism.

The pair (U, ψ) is called a *local trivialization*. Throughout this section we will adopt the convention of writing intersections of indexed open sets (say U_α and U_β) as a multi-indexed open set, i.e. $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Any two trivializations (U_α, ψ_α) and (U_β, ψ_β) determine a map

$$\psi_\alpha \circ \psi_\beta^{-1} : U_{\alpha\beta} \times K^r \to U_{\alpha\beta} \times K^r,$$

and thus a map

$$\begin{aligned} g_{\alpha\beta} : U_{\alpha\beta} &\to \mathrm{GL}(r, K) \\ p &\mapsto \psi_\alpha^p \circ (\psi_\beta^p)^{-1} \end{aligned}$$

called a *transition function*.

Given a pair of vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$, an $\mathcal{S}$-morphism of vector bundles is a map $f : E \to F$ such that $\pi_E = \pi_F \circ f$, and the map on each fiber $f : E_p \to F_{\pi_F \circ f(p)}$ is a K-linear morphism. We say that f is an $\mathcal{S}$-bundle *isomorphism* if f is an $\mathcal{S}$-isomorphism and the induced map on the fibers is a K-linear isomorphism.

Given an $\mathcal{S}$-morphism $f : N \to M$, we define the *pullback bundle*

$$f^*E := \{(p, e) \in N \times E : \pi(e) = f(p)\},$$

on N. We give f^*E the subspace topology of $N \times E$, with the projection map $f^*\pi(p, e) = p$, and the equip each fiber $(f^*E)_p$ with the linear structure induced from E_p. That is, for $u, v \in E_p$ and $\alpha, \beta \in K$

$$\alpha(p, u) + \beta(p, v) = (p, \alpha u + \beta v).$$

Thus, we see that $(p, v) \mapsto v$ gives us a K-linear isomorphism $(f^*E)_p \cong E_{f(p)}$.

For any $p \in N$, let (U, ψ) be a trivialization of $E \to M$ such that $f(p) \in U$. Thus $f^{-1}(U)$ is an open set of N containing p, and the map

$$\begin{aligned} f^{-1}(U) \times K^r &\to (f^*\pi)^{-1}(f^{-1}(U)) \\ (p, v) &\mapsto (p, \psi^{-1}(f(p), v)) \end{aligned}$$

is an $\mathcal{S}$-isomorphism, and thus it provides us with a local trivialization of $f^*E \to N$.

From an $\mathcal{S}$-bundle $E \to M$ we can also construct the bundle $\mathrm{End}(E) = \bigcup_{p\in M} \mathrm{End}(E_p)$ with the projection map $\pi(A) = p$ for $A \in \mathrm{End}(E_p)$. For a local trivialization (U, ψ) of $E \to M$, the isomorphisms $\psi^p : E_p \to K^r$ induce isomorphisms $\psi^p_{\mathrm{End}} : \mathrm{End}(E_p) \to \mathrm{End}(K^r) \cong K^{r^2}$. Then the map $\psi_{\mathrm{End}} : \pi^{-1}(U) \to U \times K^{r^2}$ sending $A \in \pi^{-1}(U)$ with $A \in \mathrm{End}(E_p)$ to $(p, \psi^p_{\mathrm{End}}(A))$ is an isomorpism. Thus $(U, \psi^p_{\mathrm{End}})$ gives us a local trivialization for $\mathrm{End}(E) \to M$.

Given an $\mathcal{S}$-morphism $f : N \to M$, we define a map $f^*(\mathrm{End}(E)) \to \mathrm{End}(f^*E)$ by sending $(p, A) \in f^*(\mathrm{End}(E))$ to the operator that acts on $(p, v) \in f^*E$ by $(p, A) \cdot (p, v) = (p, Av)$. (Noting that by definition $A \in \mathrm{End}(E_p)$ and $v \in E_p$, so this is in fact defined). This map determines an $\mathcal{S}$-morphism $f^*(\mathrm{End}(E)) \cong \mathrm{End}(f^*E)$.

We can extend the usual trace of a linear operator to a map $\mathrm{tr} : \mathrm{End}(E) \to K$ where for $A \in (\mathrm{End}(E))_p = \mathrm{End}(E_p)$ we define $\mathrm{tr}(A)$ to be the trace of A. Since the usual trace is preserved under K-linear isomorphisms of vector spaces, the isomorphisms of the fibers $(f^*\mathrm{End}(E))_p \cong \mathrm{End}(f^*)_p$ implies that the trace commutes with the pullback.

Any open set $U \subseteq M$ determines an $\mathcal{S}$-manifold, and thus we define the *restriction* $E|_U$ of E to U to be the vector bundle $\iota^* E \to U$, where $\iota : U \hookrightarrow M$ is the standard inclusion map.

For any open set $U \subseteq M$ an $\mathcal{S}$-*section* is an $\mathcal{S}$-map $s : U \to E$ such that $\pi \circ s = \mathrm{Id}_M$. We let $\mathcal{S}(M, E)$ be the space of all $\mathcal{S}$-sections on M. For each open set $U \subseteq M$ we define $\mathcal{S}(U, E) := \mathcal{S}(U, E|_U)$.

Suppose that for an open set $U \subseteq M$, there exists a collection $e = \{e_1, ..., e_r\} \subseteq \mathcal{S}(U, E)$ of $\mathcal{S}$-sections such that for each $p \in U$, the set $e = \{e_1(p), ..., e_r(p)\}$ is a basis for the fiber E_p. Then we call $e = \{e_1, ..., e_r\}$ a *local frame* for E.

Suppose $h = \{h_1, ..., h_r\}$ is another frame over U. For each $p \in U$ there exists a change-of-basis $e \to h$. That is, there is some (invertible) matrix $g(p)$ such that in terms of e, for each $1 \leq k \leq r$, we have $h_k(p) = \sum_{j=1}^r g_{j,k}(p)e_j(p)$. Note that $h_k(p)$ is just the kth entry of the matrix of columns $[e_1(p), ..., e_r(p)]$ multiplied by $g(p)$ on the right, and thus we write $h = eg$. When we want to specify the coordinates of a section ξ in a frame e, we'll write $\xi(e) = \sum_{j=1}^r \xi_j(e)e_j(p)$. Thus, writing $\xi(e)$ as a column vector

$$\xi(e) = \begin{pmatrix} \xi_1(e) \\ ... \\ \xi_r(e) \end{pmatrix},$$

we must have

$$\xi(h) = \xi(eg) = g^{-1}\xi(e). \tag{1.1}$$

That is, $g\xi(eg) = \xi(e)$. Conversely, if one defines a section in terms of frames for the open sets of some cover of M, such that the section obeys the transformation property above, then it defines a global section.

Given an $\mathcal{S}$-morphism $f : N \to M$, a vector bundle $\pi : E \to M$ and a frame $\{e_1, ..., e_r\}$ for some open set $U \subseteq M$, the isomorphism $(f^*E)_p \cong E_{f(p)}$ on the fibers implies that $\{f^*e_1, ..., f^*e_r\}$ is a local frame for $f^*E \to N$ over $f^{-1}(U)$.

Having an assignment for each of these opens sets, we define a sheaf $\mathcal{S}_M(E)$ with M complex or real, where for each open set $U \subseteq M$ we set $\mathcal{S}_M(E)(U) := \mathcal{S}(U, E)$. In fact, for the structure sheaf $\mathcal{S}_M$, the sheaf $\mathcal{S}(E)$ is a sheaf of $\mathcal{S}_M$-modules.

Recall that the *stalk* $\mathcal{S}_{M,p}$ is a K-algebra, defined to be the limit $\lim_{p \in U \subseteq M} \mathcal{S}_M(U)$ over all open sets containing $p \in M$.

A *derivation* is a K-linear map $D : \mathcal{S}_{M,p} \to K$ such that, for all $f, g \in \mathcal{S}_{M,p}$

$$D(fg) = D(f)g(p) + f(p)D(g).$$

We define the *tangent space* $T_p(M)$ to be the K-vector space of derivations on $\mathcal{S}_{M,p}$. For an $\mathcal{S}$-morphism $f : N \to M$ and an open set $U \subseteq M$, we define a K-algebra morphsim $f^* : \mathcal{S}_M(U) \to \mathcal{S}_N(f^{-1}(U))$ by $f^*(h) = h \circ f$, which descends to a K-algebra morphism of $\mathcal{S}_{N,f(p)} \to \mathcal{S}_{M,p}$. The map $df_p : T_p(N) \to T_p(M)$ defined by $df_p(D_p) = D_p \circ f^*$ is known as the *Jacobian*, *push-forward*, or *differential* of f.

For example, for the Euclidean $\mathcal{S}$-manifold K^n, for any $p \in K^n$, the vector space $T_p(K^n)$ has a basis provided by the coordinate derivatives $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$.

Define $T(M) := \bigcup_{p\in M} T_p(M)$, and $\pi : T(M) \to M$ where for $v \in T_p(M), \pi(v) = p$. We will show that $\pi : T(M) \to M$ is a vector bundle, which we call the *tangent bundle*.

Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M. The map $\phi_\alpha : U_\alpha \to K^n$ is an $\mathcal{S}$-isomorphism onto its image, and thus for any $p \in U_\alpha$, the induced map $(d\phi_\alpha)_p : T_p(M) \to T_{\phi_\alpha(p)}(K^n)$ is an isomorphism. Now, for any $v \in \pi^{-1}(U_\alpha)$, where $v \in T_p(M)$ and $p \in U_\alpha$, we have $d\phi_{\alpha,p}(\vec{v}) \in T_{\phi_\alpha(p)}(K^n)$, where we write the coefficients

$$d\phi_{\alpha,p}(v) = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial x_j}|_{\phi_\alpha(p)}.$$

Define the maps

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times K^n \tag{1.2}$$

$$v \mapsto (p, \xi_1(p), ..., \xi_n(p)) \tag{1.3}$$

and

$$\psi_\alpha^p : T_pM \xrightarrow{\psi_\alpha} \{p\} \times K^n \xrightarrow{\text{proj.}} K^n.$$

The fact that ψ_α^p is a K-linear isomorphism follows from the fact that $d\phi_{\alpha,p}$ is a K-linear isomorphism. Furthermore, for any α and β, we identify the vector spaces $T_{\phi_\alpha(p)}(K^n)$ and $T_{\phi_\beta(p)}(K^n)$ with K^n, and thus the K-linear isomorphsim $(d\phi_\alpha)\circ(d\phi_\beta)^{-1} : T_{\phi_\beta(p)}(K^n) \to T_{\phi_\alpha(p)}(K^n)$, determines an $\mathcal{S}$-isomorphism $K^n \to K^n$, which implies that $\psi_\alpha \circ \psi_\beta^{-1} : U_{\alpha\beta} \times K^n \to U_{\alpha\beta} \times K^n$ is an $\mathcal{S}$-isomorphism. Since the U_α cover M, if we can choose a topology on $T(M)$ such that the ψ_α are homeomorphisms, we could conclude that $T(M)$ is an $\mathcal{S}$-manifold, and thus $\pi : T(M) \to M$ is an $\mathcal{S}$-vector bundle. To this end, we decree that $U \subseteq T(M)$ is open iff $\psi_\alpha(U \cap \pi^{-1}(U_\alpha))$ is open in $U_\alpha \times K^r$ for all α. Thus, since the maps ψ_α are bijections, for each α, we have that $\psi_\alpha(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\alpha)) = \psi_\alpha(\pi^{-1}(U_\alpha)) = U_\alpha \times K^n$ is open in $U_\alpha \times K^n$. Therefore $\{(U_\alpha, \psi_\alpha)\}$ is an atlas for $T(M)$, as well as a collection of local trivializations for $\pi : T(M) \to M$.

Complex Structures

Just as the complex numbers have a richer structure than the reals, both complex manifolds and vector spaces have a richer structure than their real counterparts. Together, this translates into the tangent bundles (hence differential forms) of complex manifolds possessing extra structure. We will investigate such structures in this section.

Definition 1.1.1. A *complex structure* on an $\mathbb{R}$-vector space V, is a linear map $J : V \to V$ such that $J^2 = -I$.

For an arbitrary $\mathbb{C}$-vector space V, a vector $v \in V$ and $\alpha + \beta i = z \in \mathbb{C}$ we have $zv = (\alpha + \beta i)v = \alpha v + \beta(iv)$. That is, for a $\mathbb{C}$-basis $\{v_1, ..., v_n\}$ of V, the set $\{v_1, iv_1, ..., v_n, iv_n\}$ is an $\mathbb{R}$-basis for V. Furthermore, the map sending $v_j \mapsto iv_j$ and $iv_j \mapsto -v_j$ is a complex structure on V.

For the Euclidean vector space $\mathbb{C}^n$, and any vector $(z_1, ..., z_n) \in \mathbb{C}^n$, we write $x_j = \mathrm{Re}(z_j)$ and $y_j = \mathrm{Im}(z_j)$ for each $1 \le j \le n$. If we identify the point $(z_1, ..., z_n) \in \mathbb{C}^n$ with $(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}$, then multiplication by i in $\mathbb{C}^n$ induces the *standard complex structure* $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$J(x_1, y_1, ..., x_n, y_n) = (-y_1, x_1, ..., y_n, -x_n).$$

Definition 1.1.2. Let X be a complex manifold, with an atlas $\{(U_\alpha, \phi_\alpha)\}$. We let $X_0 = X$ as sets, and equip it with the same topology. For each chart (U_α, ϕ_α) we define

$$\begin{aligned}\tilde{\phi}_\alpha : U_\alpha &\to \mathbb{R}^{2n} \\ x &\mapsto \left(\operatorname{Re}(\phi_1(x)), \operatorname{Im}(\phi_1(x)), ..., \operatorname{Re}(\phi_n(x)), \operatorname{Im}(\phi_n(x))\right),\end{aligned}$$

Since ϕ_α is holomorphic, the map $\tilde{\phi}_\alpha$ is smooth. Thus $\{(U_\alpha, \tilde{\phi}_\alpha)\}$ is a smooth atlas for X_0, making X_0 into a smooth manifold which we refer to as the *underlying smooth manifold* of X.

The example of inducing a complex structure on $\mathbb{R}^{2n}$ from $\mathbb{C}^n$ can be applied in a consistent (and canonical) manner to tangent space of holomorphic manifolds. This is the content of *Example 3.2* on page 28 of [10], but we formalize it here as a proposition.

Proposition 1.1.3. *For a complex manifold X, the underlying real vector space of $T_x(X)$ is isomorphic to the vector space $T_x(X_0)$, and $T_x(X)$ induces a* canonical complex structure on $T_x(X_0)$.

Proof. For any complex manifold X and $x \in X$, we define the $\mathbb{R}$-linear map $\eta_{X,x} : T_x X_0 \to T_x X$ such that for all $f \in \mathcal{O}_{X,x}$ in real and complex components $f = u + iv$, and $D \in T_x X_0$, we define $\eta_{X,x}(D)f = D(u) + iD(v)$. Note that since $f \in \mathcal{O}_{X,x}$ is holomorphic, u and v are smooth, hence $u, v \in \mathcal{A}_{X_0,x}$, so the map is defined.
Our goal is to prove that $\eta_{X,x}$ is an isomorphism. First, we will demonstrate this for $\eta = \eta_{\mathbb{C}^n,p} : T_p(\mathbb{C}^n_0) \to T_p(\mathbb{C}^n)$. Then writing the coordinates of $(z_1, ..., z_n) \in \mathbb{C}^n$, as $z_j = x_j + iy_j$ the collections of partial coordinate derivatives

$$\left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p : 1 \le j \le n \right\}, \left\{ \left.\frac{\partial}{\partial z_j}\right|_p : 1 \le j \le n \right\}.$$

are respective $\mathbb{R}$ and $\mathbb{C}$ bases for $T_p\mathbb{C}^n_0$ and $T_p\mathbb{C}^n$, and thus $\left\{ \left.\frac{\partial}{\partial z_j}\right|_p, i\left.\frac{\partial}{\partial z_j}\right|_p : 1 \le j \le n \right\}$ is an $\mathbb{R}$-basis for $T_p\mathbb{C}^n$.
For any $f = u + iv \in \mathcal{O}_{\mathbb{C}^n,p}$, and recalling that the Cauchy-Riemann equations state that $\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}$ and $\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$ for each j, we find

$$\begin{aligned}\eta\left(\left.\frac{\partial}{\partial x_j}\right|_p \right) f = \left.\frac{\partial u}{\partial x_j}\right|_p + i\left.\frac{\partial v}{\partial x_j}\right|_p &= \frac{1}{2}\left(\left.\frac{\partial u}{\partial x_j}\right|_p + i\left.\frac{\partial v}{\partial x_j}\right|_p + \left.\frac{\partial u}{\partial x_j}\right|_p + i\left.\frac{\partial v}{\partial x_j}\right|_p \right) \\ &= \left(\left.\frac{\partial u}{\partial x_j}\right|_p + i\left.\frac{\partial v}{\partial x_j}\right|_p + \left.\frac{\partial v}{\partial y_j}\right|_p - i\left.\frac{\partial u}{\partial y_j}\right|_p \right) \\ &= \frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p - i\left.\frac{\partial}{\partial y_j}\right|_p \right)(u + iv) \\ &= \left.\frac{\partial f}{\partial z_j}\right|_p\end{aligned}$$

Similarly,

$$\begin{aligned}\eta\left(\left.\frac{\partial}{\partial y_j}\right|_p\right) f = \frac{1}{2}\left(\left.\frac{\partial u}{\partial y_j}\right|_p + i\left.\frac{\partial v}{\partial y_j}\right|_p + \left.\frac{\partial u}{\partial y_j}\right|_p + i\left.\frac{\partial v}{\partial y_j}\right|_p\right) &= \frac{1}{2}\left(\left.\frac{\partial u}{\partial y_j}\right|_p + i\left.\frac{\partial v}{\partial y_j}\right|_p - \left.\frac{\partial v}{\partial x_j}\right|_p + i\left.\frac{\partial u}{\partial x_j}\right|_p\right) \\ &= i\cdot\frac{1}{2}\left(-i\left.\frac{\partial u}{\partial y_j}\right|_p + \left.\frac{\partial v}{\partial y_j}\right|_p + i\left.\frac{\partial v}{\partial x_j}\right|_p + \left.\frac{\partial u}{\partial x}\right|_p\right) \\ &= i\cdot\frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p - i\left.\frac{\partial}{\partial y_j}\right|_p\right)(u+iv) \\ &= i\left.\frac{\partial f}{\partial z_j}\right|_p .\end{aligned}$$

Therefore η is surjective, and since $T_p\mathbb{C}^n_0$ and $T_p\mathbb{C}^n$ have the same *real* dimension, η is an isomorphism.

Returning to the general case, any holomorphic map $f : X \to Y$ of holomorphic manifolds defines a smooth map $\tilde{f} : X_0 \to Y_0$. For any $x \in X$, the $\mathbb{R}$-linear map between the underlying real vector spaces of $T_x(X)$and $T_{f(x)}Y$ induced by $(df)_x$ is simply the real Jacobian $(d\tilde{f})_x$. Thus, for any $g = u + iv \in \mathcal{O}_{X,x}$ and $D \in T_xX_0$, we compute

$$\eta_{Y,f(x)}\left((d\tilde{f})_x\right) g = (d\tilde{f})_x Du + i(d\tilde{f})_x Dv = (d\tilde{f})_x(Du + iDv) = (d\tilde{f})_x\eta_{X,x}(D)g,$$

and thus the diagram

$$\begin{tikzcd} T_xX_0 \arrow[r, "\eta_{X,x}"] \arrow[d, "(d\tilde{f})_x"] & T_xX \arrow[d, "(df)_x"] \\ T_{f(x)}Y_0 \arrow[r, "\eta_{Y,f(x)}"] & T_{f(x)}Y \end{tikzcd}$$

is commutative[1]. In particular, for any chart (U_α, ϕ_α) of X,

$$\begin{tikzcd} T_xX_0 \arrow[r, "\eta_{X,x}"] \arrow[d, "(d\tilde{\phi}_\alpha)_x"] & T_xX \arrow[d, "(d\phi_\alpha)_x"] \\ T_{\phi_\alpha(x)}\mathbb{C}^n_0 \arrow[r, "\eta_{Y,\phi_\alpha(x)}"] & T_{\phi_\alpha(x)}\mathbb{C}^n \end{tikzcd}$$

is commutative, and since $d\tilde{\phi}_\alpha$ and $\eta_{\mathbb{R}^{2n},\phi_\alpha(x)}$ are isomorphisms, so must be $\eta_{X,x}$.
Therefore, the complex structure on T_xX induces a *canonical* complex structure on T_xX_0 via $\eta_{X,x}$. $\square$

Definition 1.1.4. For a real manifold M of dimension $2n$, we call a vector bundle isomorphism

$$J : TM \to TM,$$

such that $J^2 = -\mathrm{Id}_{TM}$, an *almost complex structure*. In other words, for all $p \in M$, the map $J_p : T_pM \to T_pM$ is a complex structure. In this case, we call the pair (M, J) an *almost complex manifold*.

[1] The composition of maps in either direction agree

Theorem 1.1.5. *Every complex manifold X induces an almost complex structure on its underlying real manifold X_0.*

Proof. The previous proposition demonstrates that for each $x \in X$, the complex structure on $T_x X$ induces a a canonical complex structure on $T_x X_0$. Thus one need only check that these structures vary smoothly. The proof can be found in *Proposition 3.4* on page 30 of [10]. □

Recall that for an arbitrary K-vector space V, the tensor algebra $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, and the ideal $I = \langle v \otimes v \rangle$, we define $\bigwedge V := T(V)/I$. The image of $v_1 \otimes ... \otimes v_k$ in the quotient is denoted by $v_1 \wedge ... \wedge v_k$.
We set $\bigwedge^0 V := K$, and for $1 \leq k \leq \dim V$, we define $\bigwedge^k V := \operatorname{span}_K\{v_1 \wedge ... \wedge v_k | v_1, ..., v_k \in V\}$.

For a smooth (real) manifold M, we define the *cotangent bundle* $T^*M \to M$ to be the vector bundle whose fiber at $p \in M$ is the vector space dual $(T_pM)^*$ of T_pM. Writing $n = \dim_S M$, we define the *exterior algebra bundles* $\bigwedge^\bullet TM \to M$ and $\bigwedge^\bullet T^*M \to M$, where for $p \in M$ the respective fibers are

$$\bigwedge^\bullet T_pM := \bigoplus_{k=0}^{n} \bigwedge^k T_pM, \quad \text{and} \quad \bigwedge^\bullet T_p^*M := \bigoplus_{k=0}^{n} \bigwedge^k T_p^*M.$$

Definition 1.1.6. For any open set $U \subseteq M$, for $1 \leq k \leq n$ we define the C^∞ *differential forms of degree* k to be

$$\mathcal{A}^k(U) = \mathcal{A}\left(U, \bigwedge^k T^*M\right).$$

Thus the exterior derivative defines a map $d : \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$.

Complexification

In this section we combine some of the results of the previous section in order to define the *complex differential forms* on a holomorphic manifold X.

Definition 1.1.7. Given a (left) vector space V, we define the *complexification* to be the (right) $\mathbb{C}$-vector space $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.

Given a complex structure J on a vector space V, one can extend the action of J to $V_{\mathbb{C}}$ by $J(v \otimes z) = J(v) \otimes z$. As the relation $J^2 = -I$ still holds, we may decompose $V_{\mathbb{C}}$ into its respective $+i$ and $-i$ eigenspaces, $V^{(1,0)}$ and $V^{(0,1)}$. The subspaces $V^{(1,0)}, V^{(0,1)} \subseteq V_{\mathbb{C}}$ give rise to natural inclusions $\bigwedge^\bullet V^{(1,0)}, \bigwedge^\bullet V^{(0,1)} \subseteq \bigwedge^\bullet V_{\mathbb{C}}$.
We define

$$\bigwedge^{k,l} V := \operatorname{span}_{\mathbb{C}}\left\{u \wedge w : u \in \bigwedge^k V^{(1,0)}, w \in \bigwedge^l V^{(0,1)}\right\}.$$

Definition 1.1.8. For a real manifold M, we define the space of *complex valued differential forms of total degree r* to be

$$\mathcal{A}^r(M)_{\mathbb{C}} := \mathcal{A}\left(M, \bigwedge^r T^*M_{\mathbb{C}}\right).$$

Again, the exterior derivative can be extended to a map $d : \mathcal{A}^r(M)_{\mathbb{C}} \to \mathcal{A}^{r+1}(M)_{\mathbb{C}}$.
If (M, J) is an almost complex manifold, then complexifying the fibers, we obtain a complex-vector bundle $T(M)_{\mathbb{C}} \to M$, and the operator J extends to a complex-linear morphism of $T(M)_{\mathbb{C}}$ with eigenvalues $\pm i$, where we write the bundles of the respective eigenspaces $T(M)^{1,0}$ and $T(M)^{0,1}$.
By taking the wedges fiber-wise, we also obtain the vector bundle $\bigwedge^{k,l} T^*M$ where the fiber over $x \in M$ is $\bigwedge^{k,l} T_x^*M$.
We denote

$$\mathcal{A}^{k,l}(M) := \mathcal{A}\left(M, \bigwedge^{k,l} T^*M\right).$$

Thus we obtain a decomposition $\mathcal{A}^r(M)_{\mathbb{C}} = \bigoplus_{k+l=r} \mathcal{A}^{k,l}(M)$.
The exterior derivative can be extended again (see page 33 of [10]) to a map

$$d : \mathcal{A}^{k,l}(M) \to \mathcal{A}^{k+l+1}(M) = \bigoplus_{r+s=k+l+1} \mathcal{A}^{r,s}(M).$$

We define

$$(\partial = \pi_{k+1,l} \circ d) : \mathcal{A}^{k,l} \to \mathcal{A}^{k+1,l}(M)$$
$$(\bar{\partial} = \pi_{k,l+1} \circ d) : \mathcal{A}^{k,l} \to \mathcal{A}^{k,l+1}(M),$$

where the $\pi_{\bullet,\bullet}$ are natural projection maps. By (complex) linearity, these extend to all of $\mathcal{A}^\bullet(M)_{\mathbb{C}} = \bigoplus_{r=0}^{n} \mathcal{A}^r(M)_{\mathbb{C}}$.
We can decompose the above map d as

$$d = \sum_{r+s=k+l+1} \pi_{r,s} \circ d = ... + \pi_{k+1,l} \circ d + \pi_{k,l+1} \circ d + ... = ... + \partial + \bar{\partial} + ...$$

If it happens that the other terms in this sum cancel out/are zero, that is $d = \partial + \bar{\partial}$, we say that the almost complex structure is *integrable*.

Theorem 1.1.9. *Given a complex manifold X, the induced complex structure is integrable.*

Proof. See Page 34 of [10]. □

Hermitian Geometry

We have already seen that the notion of complex structures on vector spaces gives rise to a similar notion for vector bundles. Often, one is interested in complex vector spaces equipped with a kind of inner product known as a *Hermitian form* of *Hermitian inner product* (to be defined below). Hermitian inner products can also be defined for vector bundles, and we'll see that their existence in the case of holomorphic vector bundles, leads to a rich geometric theory.

Definition 1.1.10. A *connection* ∇ on a vector bundle $E \to M$ is a $\mathbb{C}$-linear map $\mathcal{A}(M, E) \to \mathcal{A}^1(M, E)$, such that $\forall \alpha \in \mathcal{A}(M), \forall \omega \in \mathcal{A}(M, E)$

$$\nabla(\alpha\omega) = (d\alpha) \wedge \omega + \alpha \wedge \nabla\omega.$$

Given a frame $\{e_1, ..., e_r\}$ over an open set $U \subseteq M$, there exist $\theta_{ij} \in \mathcal{A}^1(U)$ such that

$$\nabla e_j = \sum_{i=1}^{r} \theta_{ij} e_i,$$

and thus we obtain a matrix θ with values in $\mathcal{A}^1(U)$ whose i, j entry is $\theta_{i,j}$. Now given some section $\xi = \sum_{i=1}^{r} \xi_1 e_r \in \mathcal{A}(U, E)$,

$$\begin{aligned}
\nabla \xi = \nabla \left(\sum_{j=1}^{r} \xi_r e_j \right) = \sum_{j=1}^{r} \big((d\xi_j) \wedge e_j + \xi_j \wedge \nabla e_j \big) &= \sum_{j=1}^{r} \left((d\xi_j) \wedge e_j + \xi_j \wedge \sum_{i=1}^{r} \theta_{ij} e_i \right) \\
&= \sum_{j=1}^{r} \big((d\xi_j) \wedge e_j + \xi_j \wedge \theta e_j \big) \\
&= \left(\sum_{j=1}^{r} (d\xi_j) \wedge e_j \right) + \left(\theta \sum_{j=1}^{r} \xi_j \wedge e_j \right) \\
&= d\xi + \theta\xi \\
&= (d + \theta)\xi.
\end{aligned}$$

As θ is a matrix of 1-forms, it defines a map $\mathcal{A}^k(U, E) \to \mathcal{A}^{k+1}(U, E)$, and thus we can extend ∇ to a map

$$(d + \theta) : \mathcal{A}^k(U, E) \to \mathcal{A}^{k+1}(U, E).$$

In fact, as on page 74 of [10], these local descriptions glue in a consistent fashion so that we can view $\nabla = d + \theta$ as a map

$$\mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E),$$

known as the *covariant derivative*.

If M is an almost-complex manifold, then Since $\mathcal{A}^1(M, E) = \mathcal{A}^{1,0}(M, E) \oplus \mathcal{A}^{0,1}(M, E)$, any connection $\nabla : \mathcal{A}(M, E) \to \mathcal{A}^1(M, E)$ can be split into maps

$$\nabla' : \mathcal{A}(M, E) \to \mathcal{A}^{1,0}(M, E)$$
$$\nabla'' : \mathcal{A}(M, E) \to \mathcal{A}^{0,1}(M, E)$$

where $\nabla = \nabla' + \nabla''$.

Given a pair of $(n \times n)$ matrices A and B valued in $\mathcal{A}^\bullet(M)$, we define $A \wedge B$ (occasionally writing AB when the context is understood) such that the (j, k)th entry is given by $[AB]_{jk} = \sum_{l=1}^{n} A_{jl} \wedge B_{lk}$.

Definition 1.1.11. Given a connection ∇ for a vector bundle $E \to M$, with a local description $\nabla = d + \theta$, we define the *curvature* matrix Θ by $\Theta := d\theta + \theta \wedge \theta$.

For $\xi = (\xi_1, ..., \xi_r) \in \mathcal{A}(U, E)$, we recall that $d(\alpha \wedge \beta) = (-1)^{\deg \alpha}\alpha \wedge d\beta$, and writing $d\theta$ for the matrix whose i, jth entry is $d\theta_{i,j}$,

$$d(\theta\xi_i) = d\sum_{j=1}^{r} \theta_{i,j}\xi_j = \sum_{j=1}^{r} \left((d\theta_{ij})\xi_j - \theta_{ij}d\xi_j\right) = (d\theta)\xi_i - \theta(d\xi_i).$$

Using this result, we can compute the explicit form

$$\begin{aligned}\nabla^2\xi = (d+\theta)^2\xi = (d^2 + d\theta + \theta d + \theta^2)\xi &= d^2\xi + d(\theta\xi) + \theta d\xi + \theta^2\xi \\ &= 0 + ((d\theta)\xi - \theta d\xi) + \theta d\xi + \theta^2\xi \\ &= (d\theta)\xi + \theta^2\xi \\ &= (d\theta + \theta^2)\xi.\end{aligned}$$

Therefore, locally, the curvature can be expressed as the matrix $\Theta = (d+\theta)^2 = \nabla \circ \nabla = \nabla^2$. These local descriptions of Θ can also be glued together to obtain a global element $\Theta \in \mathcal{A}^2(M, \mathrm{End}E)$. That is, *Proposition 1.9* on page 74 of [10] states that $\nabla^2 = \Theta$ is an operator $\mathcal{A}^k(M, E) \to \mathcal{A}^{k+2}(M, E)$. Thus we can think of the operator Θ as a matrix of 2-forms.

Given an $\mathcal{S}$-morphism $f : N \to M$, we know that f induces a *pullback* map $f^* : \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(N)$. Furthermore, given a vector bundle $E \to M$, there exists a *pullback bundle* $f^*E \to N$, and a pair of maps which we also denote by f^*, where

$$\begin{aligned}&f^* : \mathcal{A}^0(M, E) \to \mathcal{A}^0(N, f^*E) \\ &f^* : \mathcal{A}^0(M, \mathrm{End}(E)) \to \mathcal{A}^0(N, \mathrm{End}(f^*E)).\end{aligned}$$

The first of these maps, together with the pullback of differential forms, induces another pullback map $f^* : \mathcal{A}^\bullet(M, E) \to \mathcal{A}^\bullet(N, f^*E)$.

Given that a connection ∇ on $E \to M$ is a map $\mathcal{A}^0(M, E) \to \mathcal{A}^1(M, E)$, one may hope there exists a connection $f^*\nabla$ on $f^*E \to N$, which is *naturally* induced by f and ∇. A prior, there are many possible choices of connections on $f^*E \to N$, however we would like $f^*\nabla$ to be compatible with the other notions of pullback. That is, letting $s : M \to E$ be a section of $E \to M$, a reasonable guess for dictating how $f^*\nabla$ should act on f^*s, is by simply having ∇ act on s, *and then* pullback. That is, $f^*\nabla(f^*s) = f^*(\nabla s)$. In fact, this request is enough to *uniquely* characterize $f^*\nabla$.

Theorem 1.1.12. *Let ∇ be a connection on an $\mathcal{S}$-bundle $E \to M$ and $f : N \to M$ and $\mathcal{S}$-morphism. Then*

1. *There exists a unique connection $f^*\nabla$ on $f^*E \to N$ such that, for all sections $s \in \mathcal{A}^0(M, E)$ we have $f^*\nabla(f^*s) = f^*(\nabla s)$,*

2. *If, given a frame $\{e_1, ..., e_r\}$ over an open set $U \subseteq M$, the connection has the local description $\nabla = d_M + \theta$, then for the frame $\{f^*e_1, ..., f^*e_r\}$ over $f^{-1}(U)$, the connection $f^*\nabla$ has local description $f^*\nabla = d_N + f^*\theta$, where f^* is the entry-wise pullback of the matrix of forms θ, and*

3. *If $\Theta : \mathcal{A}^k(M,E) \to \mathcal{A}^{k+2}(M,E)$ is the curvature operator associated to θ, then the curvature operator associated to $f^*\Theta$ is just the entry-wise pullback of Θ under f.*

Proof. For the proof of existence of 1), and the proofs of 2) and 3), see *Theorem 3.6, page 92* of [10]. We will prove uniqueness of the pullback connection here. Let ∇' be another connection satisfying $\nabla'(f^*s) = f^*(\nabla(s))$. Given an open set $U \subset M$ and a frame $\{e_1, ..., e_r\}$, the set $\{f^*e_1, ..., f^*e_r\}$ is a frame over $f^{-1}(U)$. By our assumption, for $1 \le j \le r$ we find

$$\nabla'(1 \otimes f^*e_j) = \nabla'\left(f^*(1 \otimes e_j)\right) = f^*\left(\nabla(1 \otimes e_j)\right) = f^*\nabla\left(f^*(1 \otimes e_j)\right) = f^*\nabla(1 \otimes f^*e_j).$$

Thus, writing an arbitrary $\xi \in \mathcal{A}^0(N, f^*E)$ as $\xi = \sum_{j=1}^r \xi_j \otimes f^*e_r$, by the definition of a connection

$$\begin{aligned}
\nabla'(\xi) &= \nabla'\left(\sum_{j=1}^r \xi_j \otimes f^*e_j\right) \\
&= \sum_{j=1}^r \nabla'(\xi_j \otimes f^*e_j) \\
&= \sum_{j=1}^r d\xi_j \otimes f^*e_j + \xi_j \wedge \nabla'(1 \otimes f^*e_j) \\
&= \sum_{j=1}^r d\xi_j \otimes f^*e_j + \xi_j \wedge f^*\nabla(1 \otimes f^*e_j) \\
&= \sum_{j=1}^r f^*\nabla(\xi_j \otimes f^*e_j) \\
&= f^*\nabla\left(\sum_{j=1}^r \xi \otimes e_j\right) \\
&= f^*\nabla(\xi).
\end{aligned}$$

As every point of N is contained in a set of the form $f^{-1}(U)$ for an open set $U \subseteq M$ possessing a frame, and the arbitrary choice of $\xi \in \mathcal{A}^0(N, f^*E)$, we conclude that $\nabla' = f^*\nabla$. $\square$

Definition 1.1.13. Given a $\mathbb{C}$-vector space W, a *Hermitian inner product* is a map $\langle , \rangle : W \times W \to \mathbb{C}$ such that, for all $u, v, w \in W$ and $\alpha \in \mathbb{C}$,

1. $\langle \alpha(u+v), w\rangle = \alpha\langle u, w\rangle + \alpha\langle v, w\rangle$,

2. $\langle u, \alpha(v+w)\rangle = \bar{\alpha}\langle u, v\rangle + \bar{\alpha}\langle u, w\rangle$,

3. $\langle v, u\rangle = \overline{\langle u, v\rangle}$,

4. $\langle u, u\rangle \ge 0$ with equality iff $u = 0$.

Definition 1.1.14. Given a smooth, complex vector bundle $E \to M$, a *Hermitian metric* on E is a choice of Hermitian inner product $\langle , \rangle_p$ on each fiber E_p, such that for any open set $U \subseteq M$ and $\xi, \eta \in \mathcal{A}(U, E)$, the map

$$U \to \mathbb{C}$$
$$p \mapsto \langle \xi(p), \eta(p) \rangle$$

is smooth. In this case we say that $E \to M$ is a Hermitian vector bundle.

Given a Hermitian vector bundle $E \to M$, we can define an extension of the Hermitian form $\langle , \rangle$ on $E \to X_0$ to a map $\langle , \rangle : \mathcal{A}^k(X, E) \otimes_{\mathcal{A}^0(M)} \mathcal{A}^l(X, E) \to \mathcal{A}^{k+l}(M)$. First, for all $\omega \in \wedge^k T_x^*(X_0), \eta \in \wedge^l T_x^*(X_0)$ and $\xi, \zeta \in E$ we define

$$\langle \omega \otimes \xi, \gamma \otimes \zeta \rangle_p := \omega \wedge \gamma \, \langle \xi, \zeta \rangle_p .$$

By linearity this extends to all of $\mathcal{A}^k(M, E) \otimes_{\mathcal{A}^0(M)} \mathcal{A}^l(M, E)$.

Definition 1.1.15. We say that a connection ∇ is *compatible* with a metric if for any open set $U \subseteq X$, and $\xi, \eta \in \mathcal{A}(U, E)$,

$$d \langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle .$$

Suppose now that we have a holomorphic vector bundle $E \to X$. This determines a smooth (complex) vector bundle on the underlying real manifold X_0. If $E \to X_0$ has a Hermitian metric, we call $E \to X$ a *Hermitian holomorphic* vector bundle.

Theorem 1.1.16. *Given a holomorphic Hermitian vector bundle $E \to X$, there exists a* unique *connection ∇ such that*

1. *∇ is compatible with the metric, and*
2. *for any open set $U \subseteq X$ and $\xi \in \mathcal{A}(U, E)$ we have $\nabla'' \xi = 0$.*

Proof. See *page 78* of [10]. □

The above connection is referred to as the *canonical connection* associated to the Hermitian holomorphic vector bundle. Both the canonical connection, and the associated curvature, have a particularly nice local description, which we make use of in the code at the end of this document.

Proposition 1.1.17. *Let $E \to X$ be a Hermitian holomorphic vector bundle, $\{e_1, ..., e_r\}$ be a local frame over an open set $U \subseteq X$, and write the canonical connection ∇ in its local description as $\nabla = d + \theta$.*
Then, for the Hermitian form $\langle , \rangle$, the matrix $H_{jk} = \left\langle e_j, e_k \right\rangle$, and the operator $\Theta \in \mathcal{A}^2(X, End E)$ determined by the curvature,

$$\theta = H^{-1} \partial H$$
$$\Theta = \bar{\partial} \theta$$

Proof. See *page 79* of [10]. □

1.2 Super Geometry

In [8], Quillen introduced the notion of *superconnections*, generalizing both the classical notion of connection, and Chern-Weil theory. Superconnections are essential to the constructions of [4] used in later chapters, and thus we recreate the relevant aspects of the theory in this section.

Super Vector Spaces

A K-vector space V with a $\mathbb{Z}_2$-grading, that is $V = V^0 \oplus V^1$, will be referred to as a *super vector space*. The $\mathbb{Z}_2$-grading on V induces a grading

$$(\mathrm{End}V)^0 = \mathrm{End}(V^0) \oplus \mathrm{End}(V^1)$$
$$(\mathrm{End}V)^1 = \mathrm{Hom}(V^0, V^1) \oplus \mathrm{Hom}(V^1, V^0),$$

making $\mathrm{End}V$ into a *superalgebra*. We say that an element $X \in (\mathrm{End}V)^i$ *homogeneous*, and call it *even* if $i = 0$, and *odd* if $i = 1$. We introduce a *supercommutator* on $\mathrm{End}V$, given by

$$[X, Y] := XY - (-1)^{\deg X \deg Y} YX.$$

Indeed, the supercommutator satisfies an augmented version of the definition of the Lie bracket.

Definition 1.2.1. For an arbitrary K, and a $\mathbb{Z}_2$-graded K-vector space $\mathfrak{g}$, a K-bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is said to be a *super Lie bracket* if for all $X \in \mathfrak{g}^i, Y \in \mathfrak{g}^j, Z \in \mathfrak{g}^k$,

1. $[Y, X] = -(-1)^{\deg X \deg Y}[X, Y]$,
2. $(-1)^{\deg X \deg Z}[X, [Y, Z]] + (-1)^{\deg Y \deg X}[Y, [Z, X]] + (-1)^{\deg Z \deg Y}[Z, [X, Y]]$ (the super Jacobi identity).

Any such pair $(\mathfrak{g}, [,])$ is referred to as a *super Lie algebra*.

Proposition 1.2.2. *For any $\mathbb{Z}_2$-graded (associative) K-algebra $\mathfrak{g}$, the bracket defined by*

$$[X, Y] := XY - (-1)^{\deg X \deg Y} YX,$$

is a super Lie bracket, and thus $(\mathfrak{g}, [,])$ is a Lie superalgebra.

We also make note of the fact that for a $\mathbb{Z}_2$-graded algebra $\mathfrak{g}$, and $X, \in \mathfrak{g}^i, Y \in \mathfrak{g}^j$ we have $XY, YX \in \mathfrak{g}^{i+j}$, and therefore $[X, Y] \in \mathfrak{g}^{i+j}$. Therefore $\deg[X, Y] := \deg X + \deg Y = i + j$. (Where both here and in the following, all arithmetic is done modulo 2.)
In particular, the super commutator on $\mathrm{End}V$ makes it into a Lie superalgebra.

Proposition 1.2.3. *The even and odd endomorphisms respectively commute and anti-commute with the involution $\epsilon(v) = (-1)^{\deg v} v$.*

Proof. Suppose $X \in (\mathrm{End}V)^0$, $Y \in (\mathrm{End}V)^1$, and let $v \in V$ be homogeneous. Since X is even $\deg v = \deg Xv$, and thus

$$X\epsilon(v) = X(-1)^{\deg v}v = (-1)^{\deg v}Xv = (-1)^{\deg(Xv)}Xv = \epsilon(Xv).$$

Since Y is odd, $\deg Yv = \deg v + 1 \Rightarrow \deg v = \deg(Yv) - 1$, hence

$$Y\epsilon v = Y(-1)^{\deg v}v = (-1)^{\deg v}Yv = (-1)^{\deg(Yv)-1}Yv = -(-1)^{\deg Yv}Yv = -\epsilon Yv.$$

By the arbitrary choice of $v \in V$, the even endomorphisms commute with ϵ, while the odd endomorphisms anti-commute. $\square$

We define the *supertrace* to be

$$\begin{aligned} \mathrm{tr}_s : \mathrm{End}V &\to \mathbb{C} \\ X &\mapsto \mathrm{tr}(\epsilon X) \end{aligned}$$

The additive property of the supertrace follows from that of the usual trace,

$$\mathrm{tr}_s(X+Y) = \mathrm{tr}(\epsilon(X+Y)) = \mathrm{tr}(\epsilon X + \epsilon Y) = \mathrm{tr}(\epsilon X) + \mathrm{tr}(\epsilon Y) = \mathrm{tr}_s(X) + \mathrm{tr}_s(Y).$$

Let $Y \in (\mathrm{End}V)^1$. By properties of tr we have $\mathrm{tr}(\epsilon Y) = \mathrm{tr}(Y\epsilon)$; however Y anti-commutes with ϵ and thus

$$\mathrm{tr}_s(Y) = \mathrm{tr}(\epsilon Y) = \mathrm{tr}(Y\epsilon) = \mathrm{tr}(-\epsilon Y) = -\mathrm{tr}(\epsilon Y) = -\mathrm{tr}_s(Y).$$

Therefore $\mathrm{tr}_s(Y) = 0$ for any odd operator Y.
If $X, Y \in (\mathrm{End}V)^1$, then by anti-commutativity we get

$$\mathrm{tr}_s(XY) = \mathrm{tr}(\epsilon XY) = \mathrm{tr}(-X(\epsilon Y)) = \mathrm{tr}\,(-(\epsilon Y)X) = -\mathrm{tr}(\epsilon YX) = -\mathrm{tr}_s(YX).$$

If $\deg X \neq \deg Y$, then XY and YX are both odd, and thus

$$\mathrm{tr}_s(XY) = 0 = \mathrm{tr}_s(YX).$$

Therefore we see that in general that

$$\mathrm{tr}_s XY = (-1)^{(\deg X)(\deg Y)}\mathrm{tr}_s YX.$$

This implies that for any $X, Y \in \mathfrak{g}$ we find

$$\begin{aligned} \mathrm{tr}_s[X,Y] &= \mathrm{tr}_s\,(XY) - (-1)^{(\deg X)(\deg Y)}\mathrm{tr}_s\,(YX) \\ &= (-1)^{\deg X \deg Y}\mathrm{tr}_s\,(YX) - (-1)^{\deg X \deg Y}\mathrm{tr}_s\,(YX) \\ &= 0. \end{aligned}$$

Super Vector Bundles

In this section we will extend the notion of a super vector space, to that of a super vector bundle. We will then upgrade our previous constructions, such as the curvature and connection, to account for the graded structure.
As in the previous section, we will write K for either $\mathbb{R}$ or $\mathbb{C}$, and we will let M be an $\mathcal{S}$-manifold.

Definition 1.2.4. For an $\mathcal{S}$-manifold M, a *super vector bundle* is a vector bundle $\pi : E \to M$ with a $\mathbb{Z}_2$-grading. Namely, it is the direct sum of vector bundles $E^0 \to M$ and $E^1 \to M$ called the *even* and *odd* components respectively.

Recall that for an $\mathcal{S}$-morphism $f : N \to M$, the pullback bundle $f^*E \to N$ is defined by setting $f^*E = \{(n, e) \in N \times E | f(n) = \pi(e)\}$. By the commutativity of direct sums and pullbacks of vector bundles

$$f^*E = f^*\left(E^0 \oplus E^1\right) \cong f^*\left(E^0\right) \oplus f^*\left(E^1\right),$$

and thus we obtain a natural $\mathbb{Z}_2$-grading by setting $(f^*E)^0 = f^*(E^0)$ and $(f^*E)^1 = f^*(E^1)$. Hence the usual pullback of vector bundles applied to a super vector bundle, produces a super vector bundle.
Recall that a section $s : M \to E$ defines a section $f^*s : N \to f^*E$ called the *pullback*, which is given by $n \mapsto (n, s \circ f(n))$. Throughout this section, the notation f^* and terminology "pullback" will be used for a variety of distinct notions, though each is compatible with the other in a natural way, and the meaning should be clear from context.
In the case of $\mathrm{End}E \to M$, we obtain an isomorphism $f^*(\mathrm{End}E) \cong \mathrm{End}(f^*E)$. In particular

$$f^*(\mathrm{End}E) \cong \mathrm{End}\left(f^*E\right) \cong \mathrm{End}\left((f^*E)^0\right) \oplus \mathrm{End}\left((f^*E)^1\right) \cong \mathrm{End}\left(f^*\left(E^0\right)\right) \oplus \mathrm{End}\left(f^*\left(E^1\right)\right) \cong f^*\left(\mathrm{End}E^0\right) \oplus f^*\left(\mathrm{End}E^1\right),$$

giving us the natural $\mathbb{Z}_2$-grading $(f^*(\mathrm{End}E))^0 = f^*\left(\mathrm{End}E^0\right), f^*(\mathrm{End}E)^1 = f^*\left(\mathrm{End}E^1\right)$. Therefore, we see that for any homogeneous $A \in \mathrm{End}E$, we have $\deg A = \deg f^*A$.
Furthermore, given the standard super Lie bracket $[,]$ on $\mathrm{End}E$, it follows that $[(n, A), (n, B)]_* := (n, [A, B])$, defines a super Lie bracket on $f^*(\mathrm{End}E)$, making it into a super vector bundle whose fibers are Lie superalgebras.

The vector space $\mathcal{A}^\bullet(M)$ of smooth differential forms is $\mathbb{Z}$-graded, and the space $\mathcal{A}^0(M, E)$ of $\mathcal{S}$-sections inherits a $\mathbb{Z}_2$-grading by $(\mathcal{A}^0(M, E))^i := \mathcal{A}^0(M, E^i)$.
As seen above, we have a map

$$f^*\left[\mathcal{A}^0(M, E)\right]^i = f^*\mathcal{A}^0(M, E^i) \to \mathcal{A}^0(N, f^*(E^i)) \cong \mathcal{A}^0(N, (f^*E)^i) = \left[\mathcal{A}^0(N, f^*E)\right]^i.$$

Recall that for any smooth map $f : N \to M$, we also denote the pullback on differential forms $\mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(N)$ by f^*. Since $\mathcal{A}^\bullet(M)$ is $\mathbb{Z}$-graded, we we obtain a natural $\mathbb{Z} \times \mathbb{Z}_2$-grading on the $\mathcal{A}^\bullet(M)$-module

$$\mathcal{A}(M, E) = \mathcal{A}^\bullet(M) \otimes_{\mathcal{A}^0(M)} \mathcal{A}^0(M, E).$$

However we'll only be interested in the total $\mathbb{Z}_2$-grading of $\mathcal{A}^\bullet(M, E)$ where for $A \in \mathcal{A}^i(M) \otimes_{\mathcal{A}^0(M)} \mathcal{A}^0(M, E^j)$ we set $\deg A = (i + j) \mod 2$.

We also define a map of algebras

$$f^*\left[\mathcal{A}^\bullet(M)\otimes_{\mathcal{A}^0(M)}\mathcal{A}^0(M,E)\right]\to\mathcal{A}^\bullet(N)\otimes_{\mathcal{A}^0(N)}f^*\mathcal{A}^0(M,E)$$
$$\omega\otimes s\mapsto f^*\omega\otimes f^*s,$$

which together with

$$\mathrm{id}\times f^*\,:\,\mathcal{A}^\bullet(N)\otimes_{\mathcal{A}^0(N)}f^*\mathcal{A}^0(M,E)\to\mathcal{A}^\bullet(N)\otimes_{\mathcal{A}^0(N)}\mathcal{A}^0(N,f^*E),$$

induces a map

$$f^*\mathcal{A}(M,E)=f^*\left[\mathcal{A}^\bullet(M)\otimes_{\mathcal{A}^0(M)}\mathcal{A}^0(M,E)\right]\to\mathcal{A}^\bullet(N)\otimes_{\mathcal{A}^0(N)}f^*\mathcal{A}^0(M,E)\to\mathcal{A}^\bullet(N)\otimes_{\mathcal{A}^0(N)}\mathcal{A}^0(N,f^*E)\cong\mathcal{A}(N,f^*E).$$

Similarly, we consider the algebra

$$\Omega=\mathcal{A}(M,\mathrm{End}E)=\mathcal{A}^\bullet(M)\widehat{\otimes}_{\mathcal{A}^0(M)}\mathcal{A}^0(M,\mathrm{End}E),$$

where $\widehat{\otimes}$ is the super tensor product given by

$$(\eta\otimes X)(\omega\otimes Y)=(-1)^{(\deg X)(\deg\omega)}(\eta\wedge\omega)\otimes XY.$$

The algebra Ω adopts a superalgebra structure from the total $\mathbb{Z}_2$-grading, of the $\mathbb{Z}\times\mathbb{Z}_2$-grading where

$$\mathcal{A}^i(M)\widehat{\otimes}_{\mathcal{A}^0(M)}\mathcal{A}^0(M,\mathrm{End}E^j)\subseteq\mathcal{A}^{(i+j)}(M,\mathrm{EndE}),$$

where arithmetic of indices is take modulo 2. More explicitly, for $\eta\otimes X\in\Omega$, where X is homogeneous, we compute $\deg(\eta\otimes X)=\deg\eta+\deg X$.

Therefore the map $f^*\,:\,f^*\mathcal{A}(M,\mathrm{End}E)\to\mathcal{A}(N,\mathrm{End}f^*E)$ is linear and preserves the grading by virtue of the notions of pullback comprising the map. Finally,

$$\begin{aligned} f^*\left[(\eta\otimes X)(\omega\otimes Y)\right]=f^*\left[(-1)^{(\deg X)\deg\omega}(\eta\wedge\omega)\otimes XY\right]&=(-1)^{(\deg X)\deg\omega f^*(\eta\wedge\omega)\otimes f^*(XY)} &(1.4)\\ &=(-1)^{(\deg f^*X)\deg f^*\omega}f^*\eta\wedge f^*\omega\otimes(f^*X)(f^*Y) &(1.5)\\ &=(f^*\eta\otimes f^*X)(f^*\omega\otimes f^*Y) &(1.6)\\ &=f^*(\eta\otimes X)f^*(\omega\otimes Y), &(1.7)\end{aligned}$$

and thus $f^*\,:\,\mathcal{A}(M,\mathrm{End}E)\to\mathcal{A}(N,f^*E)$ is a map of superalgebras.

We obtain a left action $\Omega\curvearrowright\mathcal{A}(M,E)$ given by

$$(\eta\otimes X)\cdot(\omega\otimes\alpha)=(-1)^{(\deg X)(\deg\omega)}(\eta\wedge\omega)\otimes X\alpha.$$

Definition 1.2.5. We say that an operator $\mathcal{A}(M,E)\to\mathcal{A}(M,E)$ is $\mathcal{A}^\bullet(M)$-*linear* if for each of its homogeneous components T, all $\omega\otimes\alpha\in\mathcal{A}(M,E)$ and $\eta\in\mathcal{A}^\bullet(M)$ we have

$$T(\eta\wedge(\omega\otimes\alpha))=(-1)^{\deg T\deg\omega}\eta\wedge T(\omega\otimes\alpha).$$

We now come to *Proposition 1* of [8].

Proposition 1.2.6. *The algebra Ω can be identified with the algebra of $\mathcal{A}^\bullet(M)$-linear operators.*

Proof. By the definitions of the actions above, we see that any homogeneous simple tensor in Ω is $\mathcal{A}^\bullet(M)$-linear, and by linearity we conclude that all of Ω is $\mathcal{A}^\bullet(M)$-linear.
Conversely, suppose T is an $\mathcal{A}^\bullet(M)$-linear operator on $\mathcal{A}(M, E)$. Let $e = \{e_1, ..., e_r\}$ be a local frame for some open set $U \subseteq M$. For all $1 \leq i, j \leq r$ there exists $\omega_{i,j}(e) \in \mathcal{A}^\bullet(U)$ such that

$$T(1 \otimes e_j) = \sum_{i=1}^{r} \omega_{i,j}(e) \otimes e_i = \sum_{i=1}^{r} \omega_{i,j}(e) \otimes E_{i,j}(1 \otimes e_j),$$

where E_{ij} is the matrix with 1 in the (i, j)th entry and 0 elsewhere, and we've written $\omega_{i,j}(e)$ to highlight the dependence on the frame e. For ease of reading we will drop the frame dependent notation.
Since T and $1 \otimes e_j$ are homogeneous $\sum_{i=1}^{r} \omega_{i,j} \otimes E_{i,j}(1 \otimes e_j)$ must be homogeneous, and thus every term of the sum must have the same degree. Thus, for any choice of i and j

$$\deg[T(1 \otimes e_j)] = \deg(\omega_{i,j} \otimes E_{i,j} \cdot 1 \otimes e_j) = \deg(\omega_{i,j} \otimes E_{i,j}) + \deg(1 \otimes e_j) = \deg \omega_{i,j} + \deg E_{i,j} + \deg e_j$$

and on the other hand

$$\deg[T(1 \otimes e_j)] = \deg T + \deg(1 \otimes e_j) = \deg T + \deg e_j,$$

therefore

$$\begin{aligned}\deg \omega_{i,j} + \deg E_{i,j} + \deg e_j &= \deg T(1 \otimes e_j) = \deg T + \deg e_j \\ &\Rightarrow \deg \omega_{i,j} + \deg E_{i,j} = \deg T.\end{aligned}$$

Now for any $\eta \in \mathcal{A}^\bullet(M)$,

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{r} \omega_{i,j} \otimes E_{i,j}(\eta \otimes e_k) &= \sum_{i=1}^{r} (-1)^{\deg E_{i,j} \deg \eta} \omega_{i,j} \wedge \eta \otimes E_{ik} e_k \\ &= \sum_{i=1}^{r} (-1)^{\deg E_{i,j} \deg \eta} (-1)^{\deg \omega_{i,j} \deg \eta} \eta \wedge \omega_{i,j} \otimes e_i \\ &= \eta \wedge \sum_{i=1}^{r} (-1)^{(\deg \omega_{i,j} + \deg E_{i,j}) \deg \eta} \omega_{i,j} \otimes e_i \\ &= \eta \wedge \sum_{i=1}^{r} (-1)^{\deg T \deg \eta} \omega_{i,j} \otimes e_i \\ &= (-1)^{\deg T \deg \eta} \sum_{i=1}^{r} \omega_{i,j} \otimes e_i \\ &= (-1)^{\deg T \deg \eta} T(1 \otimes e_j) \\ &= T(\eta \otimes e_j),\end{aligned}$$

where the last step follows from the $\mathcal{A}^\bullet(M)$-linearity of T. Since this holds for any $\eta \in \mathcal{A}^\bullet(M)$ and basis vector e_j, we conclude that for all $\xi \in \mathcal{A}(M, E)$, over U we have

$$T(\xi) = \sum_{i=1}^{r} \sum_{j=1}^{r} \omega_{i,j} \otimes E_{i,j}(\xi).$$

Thus there exists an element $\omega_U = \sum_{j=1}^{r} \omega_{i,j} \otimes E_{i,j} \in \mathcal{A}(U, \mathrm{End}(E))$ such that over U, for all global sections ξ, we have $T(\xi) = \omega_U(\xi)$.

We would like to construct an element $\omega \in \Omega$, such that $T(\xi) = \omega(\xi)$ for all global sections ξ. To this end, consider an open cover $\{U_\alpha\}$ and choice of ω_{U_α} for each α. (such a cover can always be obtained by starting from an arbitrary cover, by shrinking the open sets until they posses frames on which to apply the processes above.) We define ω such that for any open set U_α of the cover, $\omega|_{U_\alpha} = \omega_{U_\alpha}$. For any particular α and β, and $\xi \in \mathcal{A}(M, E)$, over the intersection $U_{\alpha\beta}$ we find

$$\omega|_{U_\alpha}(\xi) = \omega_{U_\alpha}(\xi) = T(\xi) = \omega_{U_\beta}(\xi) = \omega|_{U_\beta}(\xi).$$

Therefore our definition of ω is well-defined on any intersection of any two open sets of the cover, hence it is consistently defined on all of M. Therefore $\omega \in \Omega$. □

The supertrace on each fiber induces a $\mathcal{A}^0(M)$-module map $\mathrm{tr}_s : \mathcal{A}^0(M, \mathrm{End}E) \to \mathcal{A}^0(M)$, which extends to the $\mathcal{A}(M)$-module map

$$\begin{aligned} \mathrm{tr}_s : &\mathcal{A}(M, \mathrm{End}E) \to \mathcal{A}(M), \\ &\eta \otimes X \mapsto \eta \wedge \mathrm{tr}_s X. \end{aligned}$$

Thus writing tr_s for the supertrace on either bundle, for any $\eta \otimes X \in \mathcal{A}(M, \mathrm{End}E)$,

$$\mathrm{tr}_s\left(f^*(\eta \otimes X)\right) = \mathrm{tr}_s\left(f^*\eta \otimes f^*X\right) = f^*\eta \wedge \mathrm{tr}_s(f^*X) = f^*\eta \wedge f^*\mathrm{tr}_s(X) = f^*\left(\eta \wedge \mathrm{tr}_s X\right) = f^*\left(\mathrm{tr}_s(\eta \otimes X)\right). \tag{1.8}$$

Definition 1.2.7. A *superconnection* on a super vector bundle $E \to M$ is an operator $\nabla : \mathcal{A}(M, E) \to \mathcal{A}(M, E)$ of odd degree such that $\nabla(\eta \wedge (\omega \otimes \alpha)) = d\eta \wedge \omega \otimes \alpha + (-1)^{\deg \eta} \eta \wedge \nabla(\omega \otimes \alpha)$, for all $\eta \in \mathcal{A}(M)$, and $\omega \otimes \alpha \in \mathcal{A}(M, E)$.

Given two superconnections ∇_1, ∇_2 and any $\eta \in \mathcal{A}^\bullet(M), \omega \otimes \alpha \in \mathcal{A}^\bullet(M, E)$ we have

$$\begin{aligned} (\nabla_1 - \nabla_2)(\eta \wedge (\omega \otimes \alpha)) &= \nabla_1(\eta \wedge (\omega \otimes \alpha)) - \nabla_2(\eta \wedge (\omega \otimes \alpha)) \\ &= d\eta \wedge \omega \otimes \alpha + (-1)^{\deg \eta} \eta \wedge \nabla_1(\omega \otimes \alpha) - d\eta \wedge \omega \otimes \alpha + (-1)^{\deg \eta} \eta \wedge \nabla_2(\omega \otimes \alpha) \\ &= (-1)^{\deg \eta} \eta \wedge [\nabla_1 - \nabla_2](\omega \otimes \alpha). \end{aligned}$$

Since both ∇_1 and ∇_2 are odd, $\deg[\nabla_1 - \nabla_2]$ is defined, and by the identification of Ω with $\mathcal{A}^\bullet(M)$-linear operators on $\mathcal{A}(M, E)$,

$$(\nabla_1 - \nabla_2)(\eta \wedge (\omega \otimes \alpha)) = (-1)^{\deg(\nabla_1 - \nabla_2) \deg \eta} \eta \wedge [\nabla_1 - \nabla_2](\omega \otimes \alpha).$$

All other terms being equal, we're forced to conclude that $\deg\omega = \deg(\nabla_1 - \nabla_2)\deg\omega$ for any differential form ω, and thus $\deg(\nabla_1 - \nabla_2) = 1$. Therefore the difference of two superconnections is an odd endomorphism, hence the most general way to write a superconnection is $\nabla = d + \theta$, for an odd endomorphism $\theta \in \mathcal{A}$.
Since we usually work locally, it's useful to examine the case of the trivial bundle. Furthermore, the main vector bundles of interest later in this thesis are trivial bundles. Thus, supposing $E \to M$ is the trivial bundle $E = M \times V$ for some vector space V, we have $A = \mathcal{A}^\bullet \widehat{\otimes}_{\mathcal{A}^0(M)} \mathrm{End}V$.
Our main concern in later chapters is with connections $\nabla = d + \theta$ for trivial bundles where

$$\theta = A + L \in \left[\mathcal{A}^1(M)\widehat{\otimes}(\mathrm{End}V)^0\right] \oplus \left[\mathcal{A}^0(M)\widehat{\otimes}(\mathrm{End}V)^1\right].$$

Theorem 1.2.8. *Let ∇ be a superconnection on a super vector bundle $E \to M$, and let $f : N \to M$ be a smooth map. Then $f^*\nabla = d_N + f^*\theta$ is a superconnection on the pullback bundle $f^*E \to N$, which we call the* pullback connection*.*

Proof. Since pullbacks of both forms and operators preserve their respective degrees, given a superconnection $\nabla = d_M + \theta$ where $\theta = \eta \otimes X$ is odd, $f^*\eta \otimes f^*X = f^*(\eta \otimes X)$ is also odd, and thus we obtain a connection $f^*\nabla = d_N + f^*\theta$ on $f^*E \to N$. □

Following the definition of the curvature of a connection, we define the supercurvature of a superconnection ∇ to be ∇^2.

Lemma 1.2.9. *The supercurvature ∇^2 defines an element of Ω of even degree.*

Proof. For any simple tensor $\omega \otimes \alpha \in \mathcal{A}(M, E)$, and $\eta \in \mathcal{A}^\bullet(M)$ compute

$$\begin{aligned}
\nabla^2(\eta \wedge (\omega \otimes \alpha)) &= \nabla\left(d\eta \wedge \omega \otimes \alpha + (-1)^{\deg\eta}\eta \wedge \nabla(\omega \otimes \alpha)\right) \\
&= d^2\eta \wedge \omega \otimes \alpha + (-1)^{\deg d\eta} d\eta \wedge \nabla(\omega \otimes \alpha) + (-1)^{\deg\eta} d\eta \wedge \nabla(\omega \otimes \alpha) + (-1)^{\deg\eta}(-1)^{\deg\eta}\nabla^2(\omega \otimes \alpha) \\
&= 0 - (-1)^{\deg\eta} d\eta \wedge \nabla(\omega \otimes \alpha) + (-1)^{\deg\eta} d\eta \wedge \nabla(\omega \otimes \alpha) + \nabla^2(\omega \otimes \alpha) \\
&= \eta \wedge \nabla^2(\omega \otimes \alpha).
\end{aligned}$$

By linearity, this extends to all of $\mathcal{A}(M, E)$. More specifically, we see that ∇^2 is an even $\mathcal{A}^\bullet(M)$-linear operator, and thus by proposition 1.2.6 $\nabla^2 \in \Omega$. □

Note that as an element of $\mathcal{A}(M, \mathrm{End}E)$, by eq. (1.7), we have

$$f^*\left(\nabla^2\right) = \left(f^*\nabla\right)^2. \tag{1.9}$$

Chapter 2

Lie Theory Background

2.1 Lie Theory

A key component to the results of [4] mentioned in the introduction, is the behavior of special elements of $\mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D})$ under a certain Lie group representation $\tilde{\omega} : U(r,r) \to \mathrm{End}(\mathcal{S}(V^{\oplus r}) \otimes \mathcal{A}^\bullet(\mathbb{D}(V)))$ known as the *Weil representation*.

Here $\mathbb{D} = G/K$ where $G = U(p,q)$ is a *Lie group*, and $K = U(p) \times U(q)$ its maximal compact subgroup. The representation above induces a *Lie algebra representation* $\mathfrak{u}(r,r) \to \mathrm{End}(\mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}))$.

Thus, in order to extend the results of [4] it is imperative that in this chapter we develop the general theory of Lie groups, Lie algebras, their representations, and specifically the theory of the *highest weight vector*. We will then give a concrete description of the action of the representation of $\mathfrak{u}(r,r)$ induced by the Weil representation, and proof some technical lemmas which will be essential for the proof of the main theorem of the thesis.

Throughout this chapter we will closely follow [2], and most proofs not appearing in this chapter can be found there.

Lie Groups

Definition 2.1.1. A Lie group G is a both a group and smooth manifold, such that the group operation $G \times G \to G$ and the map sending an element to its inverse $G \to G$ are smooth.

Let F be $\mathbb{R}$ or $\mathbb{C}$, and let $M_n(F)$ be the space of $n \times n$ matrices, which we identify with F^{n^2}, giving it the structure of a F-manifold. As it is polynomial in the coordinates, the determinant $\det : M_n(F) \to F$ is continuous. The group $GL_n(F)$ of invertible matrices is $\det^{-1}(F - \{0\})$, and since $F - \{0\}$ is open, and det is continuous, $GL_n(F)$ is an open subset of $M_n(F)$, and thus it inherits the manifold structure of $M_n(F)$.

In terms of the matrix coordinates, by using cofactor expansion we can write the Jacobian of the determinant as

$$\left[\frac{\partial}{\partial x_{11}}\det X, \frac{\partial}{\partial x_{1,2}}\det X, ..., \frac{\partial}{\partial x_{n,n}}\det X\right] = \left[\frac{\partial}{\partial x_{11}}\sum_{i=1}^{n} x_{1i}(-1)^{i+1}M_{1i}, \frac{\partial}{\partial x_{12}}\sum_{i=1} x_{1i}(-1)^{i+1}M_{1,i}, ..., \frac{\partial}{\partial x_{n,n}}\sum_{i=1}^{n} x_{in}(-1)^{i+n}M_{n,i}\right]$$
$$= [M_{11}, -M_{1,2}, ..., M_{n,n}].$$

The Jacobian is zero iff each minor is zero, but then $\det X = 0$, which is impossible. Therefore the determinant has a constant rank of 1, hence by the level set theorem $SL_n(F) = \det^{-1}\{1\}$ is a closed embedded submanifold of codimension 1 in $GL_n(F)$. Since multiplication and inversion are also smooth, $SL_n(F)$ is also a Lie group.
As this is a subset of $GL_n(F)$, this leads us to, the concept of a Lie subgroup, but first we define *morphisms*.

Definition 2.1.2. A morphism $\rho : G \to H$ of Lie groups is a group homomorphism which is smooth.

Definition 2.1.3. Given a Lie group G, we define a (closed) *Lie subgroup* to be a subset which is a closed submanifold of G, and a subgroup, meanwhile an *immersed subgroup* is the image of an injective Lie group homomorphism $H \to G$.

Many important examples of Lie groups arise as subgroups of $GL(V)$ for some F-vector space V, by defining them as the subgroup of elements preserving some bilinear form $Q : V \times V \to F$.
If Q is symmetric and positive-definite, and V is real, the group preserving Q is the *orthogonal group*
$O_n(\mathbb{R}^n) = O(n)$.
If Q is skew-symmetric (that is, $Q(u, v) = -Q(v, u)$ for all $u, v \in V$), then the group $\mathrm{Sp}(Q)$ preserving Q is called the *symplectic group* and only occurs in even dimension if we demand the form is non-degenerate. If it is clear from context that V has a skew-symmetric form Q, we will write $\mathrm{Sp}(V)$ or $\mathrm{Sp}_{2n}(F)$ as $F^{2n} \cong V$.

Let V be a $\mathbb{C}$-vector space of dimension $p+q$, and a Hermitian form $\langle , \rangle$ (defined by 1.1.13) such that any maximal positive-definite subspace has dimension p, and any maximal negative-definite subspace has dimension q.
We are primarily interested in the *pseudo-unitary group* $U(p, q)$, of endomorphisms of V, preserving the form $\langle , \rangle$.
By Gram-schmidt orthonormalization, we may choose a basis $e_1, ..., e_{p+q}$ such that

$$\left\langle e_i, e_j \right\rangle = \begin{cases} \delta_{ij}, & 1 \leq i, j \leq p \\ -\delta_{ij}, & p+1 \leq i, j \leq p+q \\ 0, & \text{otherwise} \end{cases}$$

Define the matrix H such that $H_{ij} = \left\langle e_i, e_j \right\rangle$. Now for any two vectors $u, v \in V$, we write them in the above basis as

$$u = \sum_{i=1}^{p+q} \alpha_i e_i, \qquad v = \sum_{j=1}^{p+q} \beta_j e_j.$$

Now writing A^T for the transpose of the matrix A

$$\begin{pmatrix} \alpha_1 \\ ... \\ \alpha_{p+q} \end{pmatrix}^T H\overline{(\beta_1, ..., \beta_{p+q})} = \sum_{i=1}^{p+q} \alpha_i \sum_{j=1}^{p+q} \left\langle e_i, e_j \right\rangle \bar{\beta}_j = \left\langle \sum_{i=1}^{p+q} \alpha_i e_i, \sum_{j=1}^{p+q} \beta_j e_j \right\rangle = \langle u, v \rangle .$$

For $X \in U(p,q)$ we must have $\langle Xv, Xw \rangle = \langle v, w \rangle$, and writing $\bar{A}$ for the matrix whose (i,j)th entry is the complex-conjugate of the (i,j)th entry of A, we can write this as

$$v^T H\bar{w} = (Xv)^T H\overline{Xw} = v^T X^T H\bar{X}\bar{w}.$$

As this holds for every pair of vectors v, we must have $H = X^T H\bar{X}$. Thus

$$U(p,q) = \left\{X \in M_{p+q}(\mathbb{C}) | H = X^T H\bar{X}\right\}, \text{ with the special case,} \tag{2.1}$$

$$U(n) := U(n,0) = \left\{X \in M_n(\mathbb{C}) | X^{-1} = X^*\right\}, \tag{2.2}$$

where $X^* = \bar{X}^T$ is the *Hermitian conjugate*, and we simply refer to $U(n)$ as the *unitary group*. Our focus will mainly be on the Hermitian form defined by $H = \begin{pmatrix} I_{pp} & 0_{pq} \\ 0_{qp} & -I_{qq} \end{pmatrix}$, wherefore the condition defining $U(p,q)$ (in block form) becomes

$$\begin{aligned}
\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} &= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \\
&= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} \bar{A} & \bar{B} \\ -\bar{C} & -\bar{D} \end{pmatrix} \\
&= \begin{pmatrix} A^T\bar{A} - C^T\bar{C} & A^T\bar{B} - C^T\bar{D} \\ B^T\bar{A} - D^T\bar{C} & B^T\bar{B} - D^T\bar{D} \end{pmatrix},
\end{aligned}$$

providing us with the relations

$$\begin{aligned}
I = A^T\bar{A} - C^T\bar{C}, &\qquad A^T\bar{B} = C^T\bar{D}, \\
I = B^T\bar{B} - D^T\bar{D}, &\qquad B^T\bar{A} = D^T\bar{C}.
\end{aligned}$$

Proposition 2.1.4. *The group $U(p,q)$ is a Lie group.*

Proof. We will make use of the *regular level set theorem* (Corollary 5.24 of [7]) to prove this result. Equip the space of $(p+q)\times(p+q)$ Hermitian matrices $\mathrm{Herm}_q(\mathbb{C}) := \{X \in M_{p+q}(\mathbb{C}) : X^* = X\}$, with subspace topology of $M_{p+q}(\mathbb{C})$. Since the map

$$\begin{aligned}
F : M_{p+q}(\mathbb{C}) &\to \mathrm{Herm}_{p+q}(\mathbb{C}) \\
X &\mapsto X^T H\bar{X}
\end{aligned}$$

is polynomial in its coordinates, it's smooth. For any curve $\gamma : (-\epsilon, \epsilon) \to \mathrm{Herm}_{p+q}(\mathbb{C})$, differentiating $\gamma(t)^* = \gamma(t)$ simply yields $\gamma'(0)^* = \gamma'(0)$, and thus for any $Y \in \mathrm{Herm}_{p+q}(\mathbb{C})$ we have $T_Y \mathrm{Herm}_{p+q}(\mathbb{C}) = \mathrm{Herm}_{p+q}(\mathbb{C})$.
Given $X \in M_{p+q}(\mathbb{C})$ and $A \in T_Y \mathrm{Herm}_{p+q}(\mathbb{C}) = \mathrm{Herm}_{p+q}(\mathbb{C})$, we compute the push-forward

$$dF_X(A) = \frac{d}{dt}(X + tA)^T H \overline{(X + tA)}|_{t=0} = A^T H \bar{X} + X^T H \bar{A}.$$

We need to show that dF_X is surjective to apply the regular level set theorem. Let $B \in \mathrm{Herm}_{p+q}(\mathbb{C})$, and note that

$$H = X^T H \bar{X} \Rightarrow I = H X^T H \bar{X} \Rightarrow \bar{X}^{-1} = H X^T H.$$

Like-wise $\left(X^{-1}\right)^T = H \bar{X} H$, and thus taking $A = \frac{1}{2} X H B^T$,

$$\begin{aligned} F_X(A) &= \left(\frac{1}{2} X H B^T\right)^T H \bar{X} + X^T H \overline{\left(\frac{1}{2} X H B^T\right)} \\ &= \frac{1}{2} B (H X^T H) \bar{X} + \frac{1}{2} X^T (H \bar{X} H) B^* \\ &= \frac{1}{2}(B + B^*) \\ &= \frac{1}{2}(B + B) \\ &= B. \end{aligned}$$

Therefore dF_X is surjective. By the arbitrary choice of X, the map F is regular, and therefore $U(p,q) = F^{-1}\{H\}$ is a closed embedded submanifold of $M_{p+q}(\mathbb{C})$.
The fact that the group operation and inversion are smooth follows from the fact that these operations are rational functions in their coordinates (and defined everywhere). $\square$

Of particular import for us will be the subgroup

$$\begin{aligned} K := & \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}_{p+q}(\mathbb{C}) : \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in U(p,q) \right\} \\ = & \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}_{p+q}(\mathbb{C}) : H = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^T H \overline{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}} \right\}. \end{aligned}$$

By the defining relations of the Lie group,

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} &= \begin{pmatrix} A^T & 0 \\ 0 & D^T \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{D} \end{pmatrix} \\ &= \begin{pmatrix} A^T & 0 \\ 0 & D^T \end{pmatrix} \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{D} \end{pmatrix} \\ &= \begin{pmatrix} A^T \bar{A} & 0 \\ 0 & -D^T \bar{D} \end{pmatrix}, \end{aligned}$$

thus $A^{-1} = A^*$ and $D^{-1} = D^*$, so we find that

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in M_{p+q}(\mathbb{C}) : A \in U(p), D \in U(q) \right\} \cong U(p) \times U(q).$$

In fact, K is a *maximal compact subgroup* of $U(p,q)$, a fact which is significant in the general theory of Lie groups, though we will not discuss this here.

Representations of Lie Groups

As is the case with finite groups, we can learn a lot about Lie groups by studying their *representations*, which are homomorphisms $\rho : G \to GL(V)$ for some finite-dimensional F-vector space V. Note that if $\dim_F V = n$, then $GL(V) \cong GL_n(F)$, and so we require that ρ is a morphism in the sense of Lie groups, that is, it is both a group homomorphism and a smooth map.

In particular, since conjugation is a group morphism, by the smoothness assumptions

$$\begin{aligned} \Psi_g : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

is a morphism of Lie groups. Moreover, Ψ determines a morphism

$$\begin{aligned} \Psi : G &\to \mathrm{Aut}(G) \\ g &\mapsto \Psi_g \end{aligned}$$

For each $g \in G$ we will write $\mathrm{Ad}(g)$ for the differential $(d\Psi_g)_e : T_eG \to T_eG$. Therefore we also obtain a map

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(T_eG) \\ g &\mapsto \mathrm{Ad}(g) \end{aligned}$$

Now $\mathrm{Aut}(T_eG) = GL(T_eG)$ and thus itself a Lie group. Since Ad also happens to be smooth, we can also define ad to be the differential $d(\mathrm{Ad}) : T_eG \to T_I\mathrm{Aut}(T_eG)$, where $T_I\mathrm{Aut}(T_eG) = \mathrm{End}(T_eG)$.
For a morphism $\rho : G \to H$ of Lie groups, and any $g, h \in G$, we must have $\Psi_{\rho(g)} \circ \rho(h) = \rho \circ \Psi_g(h)$. Applying the differentials to this equation, one finds for the vector space morphism $(d\rho)_e : T_eG \to T_eH$ we have
$d\rho_e(\mathrm{ad}(X)(Y)) = \mathrm{ad}(d\rho_e(X))(d\rho_e(Y))$ for all $X, Y \in T_eG$.
Explicitly, for $GL_n(F)$ we define $[,] : T_eG \times T_eG \to T_eG$ by $[X,Y] = \mathrm{ad}(X)(Y)$. It follows that this map is bilinear.
By definition, for any $X \in T_eG$ there is a curve $\gamma : (-\epsilon, \epsilon) \to G$ such that $\gamma(0) = e$, and $\gamma'(0) = X$. We compute

$$\begin{aligned} [X,Y] = \mathrm{ad}(X)(Y) = \frac{d}{dt}(\mathrm{Ad}(\gamma(t))Y)\Big|_{t=0} &= \frac{d}{dt}(\gamma(t)Y\gamma(t)^{-1})\Big|_{t=0} \\ &= \gamma'(t)Y\gamma(t) + \gamma(t)Y(-\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1})\Big|_{t=0} \\ &= XY - YX. \end{aligned}$$

From this computation it also follows $[,]$ is skew-symmetric, and for $X, Y, Z \in T_eG$,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$$

which we call the *Jacobi identity*.
In the particular case of a representation $\rho : G \to GL_n(F)$, we have $T_eGL_n(F) = \mathrm{End}(F^n)$. Therefore, the general study of maps $\mathfrak{g} \to \mathrm{End}(F^n)$ for an F-vector space $\mathfrak{g}$, equipped with a skew-symmetric bilinear map $[,]$ satisfying the Jacobi identity, subsumes the study of differentials of Lie group representations, and thus Lie group representations themselves. This is the inspiration for the first definition of the following subsection.

Lie Algebras

Definition 2.1.5. A *Lie algebra* is a vector space $\mathfrak{g}$ with a skew-symmetric bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. That is, for all $X, Y, Z \in \mathfrak{g}$

1. $[X, X] = 0$
2. $[X, Y] = -[Y, X]$,
3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

If $\mathrm{char}(F) \neq 2$ then *1)* follows from *2)*.
Given a Lie group G, we've already seen that the tangent space $T_e(G)$ at the identity provides us with a Lie algebra. We will occasionally write $Lie(G)$ to refer to the Lie algebra $T_e(G)$ and occasionally we'll write $\mathfrak{g} := Lie(G)$.
The relevant operation for Lie algebras is the bracket, and thus we define a Lie algebra *morphism* of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ to be a vector space morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\phi\left([X,Y]_{\mathfrak{g}}\right) = [\phi(X), \phi(Y)]_{\mathfrak{h}},$$

for all $X, Y \in \mathfrak{g}$ where $[,]_{\mathfrak{g}}$ is the bracket of $\mathfrak{g}$, and $[,]_{\mathfrak{h}}$ is the bracket of $\mathfrak{h}$.

As a first example, we've already seen that $Lie(GL_n(F)) = \mathrm{End}(F^n)$, where the Lie bracket is given by
$[X, Y] = XY - YX$. We will often write this as $\mathfrak{gl}_n(F)$ for short. Just as many classical examples of Lie groups are given as subgroups of $GL_n(F)$, many classic examples of Lie algebras are realized as *subalgebras* of $\mathfrak{gl}_n$.

Definition 2.1.6. A subpsace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a Lie subalgebra if $[\mathfrak{h}, \mathfrak{h}] := \{[X, Y] : X, Y \in \mathfrak{h}\}$ is contained in $\mathfrak{h}$.

Proposition 2.1.7. *The Lie algebra $Lie(SL_n(F))$ is $\mathfrak{sl}_n(F) = \{X \in M_n(F) : tr(X) = 0\}$.*

Proof. Let $\gamma : (-\epsilon, \epsilon) \to SL_n(F)$, be a curve such that $\gamma(0) = I$, and write $X := \gamma'(0)$. Since $SL_n(F) \subseteq GL_n(F)$ we must have $\det \gamma(t) = 1$. Recall that the absolute value of the determinant of a matrix whose column vectors are the vectors determining a parallelogram, is the area of the parallelogram. Like-wise, the wedge product of vectors represents the volume spanned by those vectors. Thus, for the standard basis $\{e_1, .., e_n\}$ of F^n, the determinant condition becomes the requirement $\gamma(t)(e_1) \wedge ... \wedge \gamma(t)(e_n) = e_1 \wedge ... \wedge e_n$. Thus by the product rule, we have

$$\begin{aligned} 0 = \frac{d}{dt}\gamma(t)(e_1) \wedge ... \wedge \gamma(t)(e_n)\Big|_{t=0} &= \sum_{i=1}^{n} e_1 \wedge ... \wedge X(e_i) \wedge ... \wedge e_n \\ &= \mathrm{tr}(X) e_1 \wedge ... \wedge e_n, \end{aligned}$$

but this only holds iff $\mathrm{tr}(X) = 0$. Therefore the Lie algebra $\mathfrak{sl}_n(F)$ is the subspace of $\mathfrak{gl}_n(F)$ of traceless matrices. □

Proposition 2.1.8. *The Lie algebra $Lie(U(p, q))$ is*

$$\mathfrak{u}(p, q) = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} : A = -A^*, D = -D^* \right\}.$$

Proof. For some curve $\gamma : (-\epsilon, \epsilon) \to U(p, q)$, we must have $\gamma(t)^T H \overline{\gamma(t)} = H$, and thus we compute

$$\begin{aligned} 0 = \frac{\partial}{\partial t} H\Big|_{t=0} &= \frac{\partial}{\partial t}\left((\gamma(T))^T H \bar{A}_t\right)\Big|_{t=0} \\ &= \left(\frac{\partial}{\partial t}\gamma(t)\right)^T\Big|_{t=0} H\gamma(0) + \gamma(0)^T H \,\overline{\frac{\partial}{\partial t}\gamma(t)}\Big|_{t=0} \\ &= X^T H + H\bar{X}. \end{aligned}$$

Writing X in block-form as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, ($A$ is $p \times p$, and B is $q \times q$), and denoting conjugation by σ, the equation becomes

$$\begin{aligned} 0 &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \\ &= \begin{pmatrix} A^T & -C^T \\ B^T & -D^T \end{pmatrix} + \begin{pmatrix} \bar{A} & \bar{B} \\ -\bar{C} & -\bar{D} \end{pmatrix} \\ \Rightarrow \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} &= \begin{pmatrix} -\bar{A}^T & \bar{C}^T \\ -\bar{B}^T & \bar{D}^T \end{pmatrix}. \end{aligned}$$

Thus $-C = -B^* \Rightarrow C = B^*$. We conclude that

$$\mathfrak{u}(p, q) = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} : A = -A^*, D = -D^* \right\}.$$

□

In the special case $H = I$ from eq. (2.2), the Lie algebra

$$\mathfrak{u}(n) := \mathfrak{u}(n,0) = \left\{ X \in M_n\mathbb{C} | X = -X^* \right\}. \tag{2.3}$$

We will often write $\mathfrak{g}_0 = \mathfrak{u}(p,q)$ for short. To construct an explicit $\mathbb{R}$-basis for $\mathfrak{g}_0$, let $[A]_{jk} = a_{jk}, [B]_{jk} = b_{jk}, [D]_{jk} = d_{jk}$, and define E_{jk} be the elementary matrices with non-zero entry j,k. We decompose an arbitrary matrix as

$$\begin{aligned}
\begin{pmatrix} A & B \\ \bar{B}^T & D \end{pmatrix} &= \sum_{1\leq j,k\leq p} a_{jk}E_{jk} - \bar{a}_{jk}E_{kj} + \sum_{j\leq p, k>p} b_{jk}E_{jk} + \bar{b}_{jk}E_{kj} + \sum_{j,k>p} d_{jk}E_{jk} - \bar{d}_{jk}E_{kj} \\
&= \sum_{i=1}^{p} 2\mathrm{Im}(a_{jj})iE_{jj} + \sum_{k=p+1}^{p+q} 2\mathrm{Im}(d_{kk})iE_{kk} + \sum_{1\leq j<k\leq p} a_{jk}E_{jk} - \bar{a}_{jk}E_{kj} + \sum_{p+1\leq j<k\leq p+q} d_{jk}E_{jk} - \bar{d}_{jk}E_{kj} \\
&\quad + \sum_{j\leq p,k>p} b_{jk}E_{jk} + \bar{b}_{jk}E_{kj} \\
&= \sum_{i=1}^{p} 2\mathrm{Im}(a_{jj})iE_{jj} + \sum_{k=p+1}^{p+q} 2\mathrm{Im}(d_{kk})iE_{kk} + \sum_{j\leq p,k>p} \mathrm{Re}(b_{jk})(E_{jk} + E_{kj}) + \sum_{j\leq p,k>p} \mathrm{Im}(b_{jk})i(E_{jk} - E_{kj}) \\
&\quad + \sum_{1\leq j<k\leq p} \mathrm{Re}(a_{jk})(E_{jk} - E_{kj}) + \sum_{1\leq j<k\leq p} \mathrm{Im}(a_{jk})i(E_{jk} + E_{kj}) + \sum_{1\leq j<k\leq q} \mathrm{Re}(d_{jk})(E_{jk} - E_{kj}) \\
&\quad + \sum_{1\leq j<k\leq q} \mathrm{Im}(d_{jk})i(E_{jk} + E_{kj}).
\end{aligned}$$

Therefore, as an $\mathbb{R}$-vector space, $\mathfrak{u}(p,q)$ can be written

$$\mathrm{span}\{iE_{jj} | 1 \leq j \leq p+q\} \tag{2.4}$$

$$\oplus\, \mathrm{span}\{E_{jk} - E_{kj} | 1 \leq j < k \leq p\} \oplus \mathrm{span}\{i(E_{jk} + E_{kj}) | 1 \leq j < k \leq p\} \tag{2.5}$$

$$\oplus\, \mathrm{span}\{E_{jk} - E_{kj} | p+1 \leq j < k \leq p+q\} \oplus \mathrm{span}\{i(E_{jk} + E_{kj}) | p+1 \leq j < k \leq p+q\} \tag{2.6}$$

$$\oplus\, \mathrm{span}\{E_{jk} + E_{kj} | 1 \leq j \leq p, p+1 \leq k \leq p+q\} \oplus \mathrm{span}\{i(E_{jk} - E_{kj}) | 1 \leq j \leq p, p+1 \leq k \leq p+q\}. \tag{2.7}$$

Counting the dimensions of these sub-spaces we find $\dim_{\mathbb{R}} \mathfrak{u}(p,q) = p + q + 2\frac{p(p-1)}{2} + 2\frac{q(q-1)}{2} + 2pq = p^2 + q^2 + 2pq.$

Since

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} :\ A \in U(p), D \in U(q) \right\},$$

we have that

$$\mathfrak{k}_0 := \mathrm{Lie}(K) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} :\ A = -A^*, D = -D^* \right\}.$$

Thus by reading off the appropriate bases from the above, we have that

$$\begin{aligned}
\mathfrak{k}_0 &= \operatorname{span}\{iE_{jj} | 1 \le j \le p+q\} && (2.8)\\
&\oplus \operatorname{span}\{E_{jk} - E_{kj} | 1 \le j < k \le p\} \oplus \operatorname{span}\{i(E_{jk} + E_{kj}) | 1 \le j < k \le p\} && (2.9)\\
&\oplus \operatorname{span}\{E_{jk} - E_{kj} | p+1 \le j < k \le p+q\} \oplus \operatorname{span}\{i(E_{jk} + E_{kj}) | p+1 \le j < k \le p+q\}. && (2.10)
\end{aligned}$$

Now we define a few useful operations on Lie algebras.

Definition 2.1.9. Given two Lie algebra $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with respective Lie brackets $[,]_1$ and $[,]_2$, we define the Lie algebra direct sum by taking the vector space direct sum $\mathfrak{g} \oplus \mathfrak{h}$. For $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2, Y_2 \in \mathfrak{g}_2$ we define the Lie bracket $[,] : (\mathfrak{g} \oplus \mathfrak{h}) \times (\mathfrak{g} \oplus \mathfrak{h}) \to \mathfrak{g} \oplus \mathfrak{h}$

$$[X_1 + X_2, Y_1 + Y_2] = [X_1, Y_1]_1 + [X_2, Y_2]_2.$$

Ideals

In the last section we introduced the idea of Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, where we require that $\mathfrak{h}$ is a linear subspace and $[\mathfrak{h}, \mathfrak{h}] = \{[X, Y] : X, Y \in \mathfrak{h}\} \subseteq \mathfrak{h}$. If the stronger condition that $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$, is met, we say that $\mathfrak{h}$ is an *ideal*, and we write $\mathfrak{h} \lhd \mathfrak{g}$.

For any Lie algebra $\mathfrak{g}$, an ideal of central importance is the *commutator* $\mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. The next definition will provide a useful tool for understanding the structure of Lie algebras, and in particular, allow us to compute $\mathfrak{D}\mathfrak{gl}_{\mathfrak{n}}$.

Definition 2.1.10. For a Lie algebra $\mathfrak{g}$ with basis $\{e_i | 1 \le i \le n\}$, we refer to the coefficients

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k,$$

as the *structure constants* of $\mathfrak{g}$.

The set $\{E_{jk} : 1 \le j, k \le n\}$ forms a basis for $\mathfrak{gl}_n(F)$. Computing

$$[E_{jk}X]_{rs} = \sum_{t=1}^{n} (E_{jk})_{rt} x_{ts} = (E_{jk})_{rk} x_{ks} = \begin{cases} 0, & r \ne j \\ (E_{jk})_{jk} x_{ks}, & r = j \end{cases}$$

thus $E_{jk}X = \sum_{s=1}^{n} x_{ks} E_{js}$. Similarly

$$[XE_{jk}]_{rs} = \sum_{t=1}^{n} x_{rt} (E_{jk})_{ts} = x_{rj} (E_{jk})_{js} \begin{cases} 0, & s \ne k \\ x_{rj} (E_{jk})_{jk}, \end{cases}$$

so that $XE_{jk} = \sum_{r=1}^{n} x_{rj} E_{rk}$. Now we see that

$$E_{jk} E_{lm} = \sum_{s=1}^{n} (E_{lm})_{ks} E_{js} = (E_{lm}) E_{jm} = \delta_{kl} E_{jm}.$$

From this, we see that

$$[E_{jk}, E_{lm}] = \delta_{kl} E_{jm} - \delta_{mj} E_{lk}. \tag{2.11}$$

Thus, comparing this with

$$[E_{jk}, E_{lm}] = \sum_{r,s} C^{rs}_{jk,lm} E_{rs},$$

for $\mathfrak{gl}_n$, the structure constants are

$$C^{rs}_{jk,lm} = \begin{cases} \delta_{kl}, & rs = jm \\ -\delta_{jm}, & rs = lk \\ 0, & \text{otherwise} \end{cases} \tag{2.12}$$

Therefore

$$\mathfrak{D}\mathfrak{gl}_n = \operatorname{span}\left\{\delta_{kl} E_{jm} - \delta_{mj} E_{lk}\right\}.$$

We can be more specific by computing that

$$\delta_{kl} E_{jm} - \delta_{mj} E_{lk} = \begin{cases} E_{jj} - E_{kk}, & k = l, m = j \\ E_{jm}, & k = l, m \neq j \\ -E_{lk}, & k \neq l, m = j \\ 0, & \text{otherwise} \end{cases}$$

By the additivity of trace, we conclude that

$$\mathfrak{D}\mathfrak{gl}_n(F) = \left\{X \in \mathfrak{gl}_n : \operatorname{tr}(X) = 0\right\} = \mathfrak{sl}_n(F). \tag{2.13}$$

As $\mathfrak{D}\mathfrak{g}$ is a useful ideal for understanding the structure of $\mathfrak{g}$, and $\mathfrak{D}\mathfrak{g}$ is a Lie subalgebra in its own right, we can take the commutator of $\mathfrak{D}\mathfrak{g}$ itself.
Towards this end, we define two series $\mathfrak{D}_k\mathfrak{g}$ and $\mathfrak{D}^k\mathfrak{g}$ of Lie algebras, both with the initial terms $\mathfrak{D}_1\mathfrak{g} = \mathfrak{D}\mathfrak{g} = \mathfrak{D}^1\mathfrak{g}$. The *lower central series* is given by $\mathfrak{D}_k\mathfrak{g} = [\mathfrak{g}, \mathfrak{D}_{k-1}\mathfrak{g}]$, and the *derived series* $\mathfrak{D}^k\mathfrak{g} = [\mathfrak{D}^{k-1}\mathfrak{g}, \mathfrak{D}^{k-1}\mathfrak{g}]$. We will also classify Lie algebras according to their behavior with respect to these series by saying that a Lie algebra $\mathfrak{g}$ is

1. *Nilpotent*, if some k, we find $\mathfrak{D}_k\mathfrak{g} = 0$,

2. *Solvable*, if some k, we find $\mathfrak{D}^k\mathfrak{g} = 0$,

3. *Simple*, if $\dim \mathfrak{g} > 1$, and $\mathfrak{g}$ has no non-trivial ideals, and

4. *Semi-simple*, if $\mathfrak{g}$ has no non-zero solvable ideals.

The notion of semi-simplicity will be of particular importance to us. As we'll see in the next section, the representation theory of semi-simple Lie algebras enjoys some nice features, and information about general Lie algebras can be lifted from certain related semi-simple Lie algebras.
Given solvable ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, their sum $\mathfrak{a}+\mathfrak{b} := \{X+Y : X \in \mathfrak{a}, Y \in \mathfrak{b}\}$ is again a solvable ideal. Thus the sum $\sum_{\mathfrak{a}} \mathfrak{a}$ over all solvable ideals of $\mathfrak{g}$ is a maximal solvable ideal which we call the *radical* $\mathrm{Rad}(\mathfrak{g})$. As $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semi-simple, we write it as $\mathfrak{g}_{ss}$ and refer to it as the *semi-simplification* of $\mathfrak{g}$.

Representations of Lie algebras

We have already seen that given a pair of Lie groups G and H, with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, we can differentiate a morphism $\rho : G \to H$ to obtain a linear map $d\rho : \mathfrak{g} \to \mathfrak{h}$ which preserves the Lie bracket, and hence, is a morphism of Lie algebras. Thus, given a representation $\rho : G \to GL(V)$ for some vector space V, we obtain a morphism $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. In general, we refer to any map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$ as a *representation*.

As is the case with the representation theory of groups, much of the structure of a Lie algebra can be captured by understanding fundamental objects known as *irreducible representations*. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, and for some subspace $W \subset V$ we have that $\rho(X)W \subseteq W, \forall X \in \mathfrak{g}$, then we say that W is an *invariant subspace*. Notice that in this case ρ also determines a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$.

Definition 2.1.11. A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ is said to be *irreducible* if the only invariant subspaces of V and the 0-vector space, and V.

In the case of representations of groups, one way to build representations from irreducible ones is the use of tensor products. Given a pair of group representations $\rho_1 \ :\to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$, we can define the *tensor product representation* $\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ such that for all $g \in G, v_1 \in V_1$, and $v_2 \in V_2$, we have

$$\rho_1 \otimes \rho_2(g) \cdot \left(v_1 \otimes v_2\right) = \left(\rho_1(g) \cdot v_1\right) \otimes \left(\rho_2(g) \cdot v_2\right),$$

and then extend to the rest of $V_1 \otimes V_2$ by linearity.
If G above is a Lie group, then the representation $d(\rho_1 \otimes \rho_2)$ of $\mathfrak{g}$ is obtained taking the differential of $\rho_1 \otimes \rho_2$, and by *Proposition 4.19* on page 110 of [5], we may treat a tensor product of functions as a product of functions with respect to differentiation, and apply a kind of Leibniz formula. Thus, in general, we see that the correct definition for the tensor product of Lie algebras is the following.

Definition 2.1.12. Given a Lie algebra $\mathfrak{g}$, and two representations $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$, we define their *tensor product* $(\rho_1 \otimes \rho_2) : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$ such that for $X \in \mathfrak{g}$ and $v_1 \otimes v_2 \in V_1 \otimes V_2$

$$(\rho_1 \otimes \rho_2)(X)(v_1 \otimes v_2) = \rho_1(X)v_1 \otimes v_2 + v_1 \otimes \rho_2(X)v_2.$$

As we'll see below, the theory also becomes considerably easier for the case of semi-simple Lie algebras. While many common examples Lie algebra, such as $\mathfrak{gl}(\mathbb{C})$ (see below), fail to be semi-simple, this isn't always a great loss, as exemplified by*Proposition 9.17* of [2]

Theorem 2.1.13. *Let $\mathfrak{g}$ be a complex Lie algebra, and $\mathfrak{g}_{ss} = \mathfrak{g}/Rad(\mathfrak{g})$. Every irreducible representation of $\mathfrak{g}$ is of the form $\rho_0 \otimes \rho : \mathfrak{g} \to \mathfrak{gl}\left(V_0 \otimes L\right)$, where $\rho_0 : \mathfrak{g}_{ss} \to \mathfrak{gl}(V_0)$ is irreducible, and* $\dim_{\mathbb{C}} L = 1$.

For the remainder of this section, we will assume all our Lie algebras are semi-simple.
For an arbitrary Lie algebra $\mathfrak{g}$, one begins by finding a maximal Abelian subalgebra $\mathfrak{h}$ for which the restriction of the adjoint representation $\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ acts diagonally. We call any such subalgebra $\mathfrak{h}$ a *Cartan subalgebra*. Given this choice of $\mathfrak{h}$, the action decomposes $\mathfrak{g}$ into subspaces on which $\mathfrak{h}$ acts by a functional $\alpha \in \mathfrak{h}^*$. That is, there exists a set $\Lambda \subseteq \mathfrak{h}^*$ such that for each $\alpha \in \Lambda$, we have spaces

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \forall H \in \mathfrak{h}, \mathrm{ad}(H)(X) = \alpha(H) \cdot X\} \neq \{0\},$$

where $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha\right)$. The functionals are called *roots*, and we denote the set of all roots by Δ. The $\mathfrak{g}_\alpha$ are *root spaces*, and their elements are called *root vectors*.
Each of the root spaces $\mathfrak{g}_\alpha$ are 1-dimensional, and Δ generates a lattice $\Lambda_\Delta \subseteq \mathfrak{h}^*$ having rank equal to $\dim \mathfrak{h}$.
More generally, any representation $\mathfrak{g} \to \mathfrak{gl}(V)$ will decompose into spaces $V = \bigoplus_\alpha V_\alpha$ where $\mathfrak{h}$ acts diagonally, where $V_\alpha = \{v \in V | \forall H \in \mathfrak{h}, Hv = \alpha(H) \cdot \vec{v}\}$.
At this point, one must make a choice of hyperplane containing half the roots. This can by described more formally as choosing a linear functional $\ell : \Lambda_\Delta \otimes \mathbb{R} \to \mathbb{R}$, and extend it to a functional $\ell : \mathfrak{h}^* \to \mathbb{C}$ which is irrational with respect to the lattice Λ_Δ. We then define the positive and negative roots respectively

$$\Delta^+ = \{\alpha \in \Delta | \ell(\alpha) > 0\}, \qquad \Delta^- = \{\alpha \in \Delta | \ell(\alpha) < 0\}.$$

Definition 2.1.14. For a representation $\mathfrak{g} \to \mathfrak{gl}(V)$, with Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, a functional $\lambda \in \mathfrak{h}^*$ and positive roots Δ^+, a non-zero vector $v \in V$ is said to be of weight $\lambda \in \mathfrak{h}^*$ such that $Hv = \lambda(H)v$. We say that v is a *highest weight vector* if it has weight λ, and is in the Kernel of $\mathfrak{g}_\alpha$ for each $\alpha \in \Delta^+$.

The main point of the preceding definition is the following result.

Theorem 2.1.15. *For any representation $\mathfrak{g} \to \mathfrak{gl}(V)$ of a semisimple complex Lie algebra $\mathfrak{g}$, the subspace $W = \{Y \cdot v \in V | Y \in \mathfrak{g}_\beta, \beta \in \Delta^-\} \subseteq V$ is an irreducible subrepresentation.*

Computation of the Semi-simple part of $\mathfrak{gl}_n(\mathbb{C})$

For the main theorems of this thesis, we will be interested in a representation of a certain Lie subalgebra of $\mathfrak{sl}_n(\mathbb{C})$, and hence $\mathfrak{gl}_n(\mathbb{C})$. Thus, throughout the rest of this section we reproduce a number of well known results about the structure and representation theory of $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$, which will then restrict to results on the Lie subalgebra of our main interest, to be described later.

In this section, we define a symmetric bilinear form on arbitrary Lie algebras that has particular importance for representation theory, known as the *Killing form*.

Proposition 2.1.16. *For a Lie algebra $\mathfrak{g}$ over a field F, the map*

$$B : \mathfrak{g} \times \mathfrak{g} \to F$$
$$(X, Y) \mapsto tr(ad(X) \circ ad(Y))$$

is symmetric and bilinear.

Proof. The symmetry follows from the commutative property of trace, and the bilinearity follows from the linearty of trace, and the bilinearity of the Lie bracket. □

From the bilinearity of B, it's enough to determine its action on a basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$. Recalling the definition of the structure coefficients from the previous subsections, we can compute

$$\mathrm{ad}(e_i) \circ \mathrm{ad}(e_j)(e_k) = [e_i, [e_j, e_k]] = \left[e_i, \sum_{l=1}^n C_{jk}^l e_l \right] = \sum_{l=1}^n \sum_{m=1}^n C_{il}^m C_{jk}^l e_m.$$

Using this, we can compute a concrete description for the Killing form of $\mathfrak{gl}_n(\mathbb{C})$ here, from which the Killing form for $\mathfrak{sl}_n(\mathbb{C})$ will follow as an easy corollary.

Let $X, Y \in \mathfrak{gl}_n(\mathbb{C})$, then by the bi-linearity, and the computations of the Lie bracket eq. (2.11)

$$B(X, Y) = B\left(\sum_{j,k} x_{jk} E_{jk}, \sum_{l,m} y_{lm} E_{lm} \right) = \sum_{j,k} \sum_{l,m} x_{jk} y_{lm} B(E_{jk}, E_{lm}) = \sum_{j,k} \sum_{l,m} x_{jk} y_{lm} \sum_{r,s} \sum_{t,u} C_{jk,tu}^{rs} C_{lm,rs}^{tu}.$$

Now by eq. (2.12)

$$C_{lm,rs}^{tu} = \begin{cases} \delta_{mr}, & tu = ls \\ -\delta_{ls}, & tu = rm \\ 0, & \text{otherwise} \end{cases}$$

and therefore

$$B(X, Y) = \sum_{j,k} \sum_{l,m} x_{jk} y_{lm} \sum_{r,s} \left(C_{jk,ls}^{rs} \delta_{mr} - C_{jk,rm}^{rs} \delta_{ls} \right).$$

Again by eq. (2.12)

$$C_{jk,ls}^{rs} = \begin{cases} \delta_{kl}, & rs = js \\ -\delta_{js}, & rs = lk \\ 0, & \text{otherwise} \end{cases}$$

$$C_{jk,rm}^{rs} = \begin{cases} \delta_{kr}, & rs = jm \\ -\delta_{jm}, & rs = rk \\ 0, & \text{otherwise} \end{cases}$$

Thus the above becomes

$$
\begin{aligned}
B(X,Y) &= \sum_{j,k}\sum_{l,m} x_{jk}y_{lm}\left(\sum_{s=1}^{n}\delta_{kl}\delta_{mj} - \delta_{jk}\delta_{ml} - \delta_{kj}\delta_{lm} + \sum_{r=1}^{n}\delta_{jm}\delta_{lk}\right) \\
&= \sum_{j,k}\sum_{l,m}\left(x_{jk}y_{lm}n\delta_{kl}\delta_{mr} + x_{jk}y_{lm}n\delta_{jm}\delta_{lk} - x_{jk}y_{lm}\delta_{jk}\delta_{ml} - x_{jk}y_{lm}\delta_{kj}\delta_{lm}\right) \\
&= n\sum_{j=1}^{n}\sum_{k=1}^{n} x_{jk}y_{kj} + n\sum_{j=1}^{n}\sum_{k=1}^{n} x_{jk}y_{kj} - \sum_{j=1}^{n}\sum_{l=1}^{n} x_{jj}y_{ll} - \sum_{j=1}\sum_{l=1} x_{jj}y_{ll} \\
&= 2n\sum_{j=1}^{n}[XY]_{jj} - 2\sum_{j=1}^{n} x_{jj}\mathrm{tr}(Y) \\
&= 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y).
\end{aligned}
$$

Recall that $\mathfrak{sl}_n(\mathbb{C})$ is the subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ of traceless matrices, and therefore $B_{\mathfrak{sl}_n(F)}(X,Y) = 2n\mathrm{tr}(XY)$.

In order to compute $\mathrm{Rad}(\mathfrak{gl}_n(\mathbb{C}))$ we will make use of *Proposition C.22* of [2], which states that

Proposition 2.1.17. *For any complex Lie algebra Rad*$(\mathfrak{g}) = (\mathfrak{D}\mathfrak{g})^{\perp}$ *with respect to the Killing form.*

We computed the killing form for $\mathfrak{gl}_n(F)$ in the previous section to be $B(X,Y) = n\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)$. So we let $Y \in \mathfrak{gl}_n(\mathbb{C})$, and demand that it's Killing product with each basis vector of $\mathfrak{D}\mathfrak{gl}_n(\mathbb{C})$ be zero.
For $j \neq k$, we see that

$$
\begin{aligned}
0 = n\mathrm{tr}(E_{jk}Y) - \mathrm{tr}(E_{jk})\mathrm{tr}(Y) &= n\mathrm{tr}\left(\sum_{i=1}^{n}[E_{jk}Y]_{ii}\right) - 0 \\
&= n\sum_{i=1}^{n}\sum_{l=1}^{n}(E_{jk})_{il}y_{li} \\
&= n\sum_{i=1}^{n}(E_{jk})_{ik}y_{ki} \\
&= n(E_{jk})_{jk}y_{kj} \\
&= ny_{kj}.
\end{aligned}
$$

Therefore $y_{kj} = 0$, and by arbitrary selection of $j \neq k$, all off-diagonal terms of Y must be zero. Meanwhile

$$\begin{aligned} 0 = \operatorname{tr}((E_{jj} - E_{kk})Y) - \operatorname{tr}(E_{jj} - E_{jj})\operatorname{tr}(Y) &= \operatorname{tr}(E_{jj}Y) - \operatorname{tr}(E_{kk}Y) - 0 \\ &= n\sum_{i=1}^{n}(E_{jj}Y)_{ii} - n\sum_{i=1}^{n}(E_{kk}Y)_{ii} \\ &= n\sum_{i=1}^{n}\sum_{l=1}(E_{jj})_{il}y_{li} - n\sum_{i=1}^{n}\sum_{l=1}^{n}(E_{kk})_{il}y_{li} \\ &= n\sum_{i=1}^{n}(E_{jj})_{ij}y_{ji} - n\sum_{i=1}^{n}(E_{kk})_{ik}y_{ki} \\ &= n(E_{jj})_{jj}y_{jj} - n(E_{kk})_{kk}y_{kk} \\ &= n(y_{jj} - y_{kk}). \end{aligned}$$

Thus $y_{jj} = y_{kk}$, and by arbitrary selection of j and k, we conclude that $Y = \lambda I$ for some $\lambda \in \mathbb{C}$.
Altogether we conclude

$$\operatorname{Rad}(\mathfrak{gl}_n(\mathbb{C})) = (\mathfrak{D}\mathfrak{gl}_n(\mathbb{C})^{\perp} = \{\lambda I \, : \, \lambda \in \mathbb{C}\}\,.$$

Now, if we consider $\left(\mathfrak{gl}_n(\mathbb{C})\right)_{ss} = \mathfrak{gl}_n(\mathbb{C})/\operatorname{Rad}(\mathfrak{gl}_n(\mathbb{C}))$, we have that

$$X + \operatorname{Rad}(\mathfrak{gl}_n(\mathbb{C})) = Y + \operatorname{Rad}(\mathfrak{gl}_n(\mathbb{C})) \iff X - Y \in \operatorname{Rad}(\mathfrak{gl}_n(\mathbb{C})) \iff X - Y = \lambda I, \quad \text{for some } \lambda \in \mathbb{C}.$$

Thus two matrices represent the same equivalence class iff their diagonals differ by a constant. In particular
$X \sim X - \frac{\operatorname{tr}(X)}{n}I$, where

$$\operatorname{tr}\left(X - \frac{\operatorname{tr}(X)}{n}I\right) = t\operatorname{tr}X) - \frac{\operatorname{tr}(X)}{n}\operatorname{tr}(I) = \operatorname{tr}(X) - \operatorname{tr}(X) = 0.$$

Therefore each class of $\mathfrak{gl}_n\mathbb{C}/\operatorname{Rad}(\mathfrak{gl}_n\mathbb{C})$ can be represented by a matrix of zero trace.
Furthermore, we've seen that two matrices X and Y represent the same class iff $Y = X + \lambda I$. If we require that $\operatorname{tr}(Y) = 0$, then

$$0 = \operatorname{tr}(Y) = \operatorname{tr}(X + \lambda I) = \operatorname{tr}(X) - \lambda\operatorname{tr}(I) = \operatorname{tr}(X) - \frac{\lambda}{n} \Rightarrow \operatorname{tr}(X) = \frac{\lambda}{n}.$$

In other words, each equivalence class can *uniquely* be represented by a traceless matrix, and therefore we identify

$$\left(\mathfrak{gl}_n(\mathbb{C})\right)_{ss} = \mathfrak{gl}_n(\mathbb{C})/\operatorname{Rad}(\mathfrak{gl}_n(\mathbb{C})) \cong \{X \in \mathfrak{gl}_n(\mathbb{C}) \, : \, tr(X) = 0\}.$$

In this way, we see that $\left(\mathfrak{gl}_n\mathbb{C}\right)_{ss}$ is naturally identified with a Lie subalgebra of $\mathfrak{gl}_n$. In particular
$\left(\mathfrak{gl}_n(\mathbb{C})\right)_{ss} \cong \mathfrak{sl}_n(\mathbb{C})$. Thus as a consequence, we see that $\mathfrak{sl}_n(\mathbb{C})$ is a semi-simple Lie algebra.

The Cartan Subalgebra of $\mathfrak{sl}_n(\mathbb{C})$.

In this subsection, we will only work with the (semi-simple) Lie algebra $\mathfrak{g} := \mathfrak{sl}_n\mathbb{C} \cong \left(\mathfrak{gl}_n(\mathbb{C})\right)_{ss}$. First, we will need to choose a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$. In the case of $\mathfrak{g} = \left(\mathfrak{gl}_n(\mathbb{C})\right)_{ss}$, we define $\mathfrak{h}$ to be the subalgebra of diagonal matrices. First we observe that

$$[E_{ii}, E_{jj}] = \delta_{ij} E_{ij} - \delta_{ji} E_{ji} = \begin{cases} E_{ii} - E_{ii} = 0, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$[E_{ii} - E_{jj}, E_{kk} - E_{ll}] = [E_{ii}, E_{kk}] - [E_{jj}, E_{kk}] - [E_{ii}, E_{kk}] + [E_{jj}, E_{ll}] = 0,$$

and thus we see that $\mathfrak{h}$ is Abelian.

Now we examine the action $\mathrm{ad} : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{gl}(\mathfrak{g})$ restricted to $\mathfrak{h}$. First, since $H = \mathrm{span}\{E_{ii} - E_{jj}\}$, then letting $e_i \in \mathfrak{h}^*$ where

$$e_i(E_{jk}) = \begin{cases} 1, & i = j = k \\ 0, & \text{otherwise} \end{cases}$$

Fixing a particular j, we obtain a basis $\{e_i - e_j : 1 \leq i \leq n, i \neq j\}$ for $\mathfrak{h}^*$.

In the case of $\mathfrak{sl}_n = \mathfrak{sl}_2$, we have $\mathfrak{h} = \mathrm{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$. For $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{g}_{e_1 - e_2}$, and $H = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \in \mathfrak{h}$, we must have

$$[(e_1 - e_2)(H)](X) = \mathrm{ad}(H)(X) = \left[\begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} ha & hb \\ -hc & -hd \end{pmatrix} - \begin{pmatrix} ha & -hb \\ hc & -hc \end{pmatrix} = \begin{pmatrix} 0 & 2hb \\ -2hc & 0 \end{pmatrix}.$$

Meanwhile, $(e_1 - e_2)H(X) = (h - (-h))X = 2hX$ and thus

$$2h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 2hb \\ -2hc & 0 \end{pmatrix}.$$

Since $h \neq 0$, we know $2hc = -2hc \Rightarrow c = 0$. Therefore

$$\mathfrak{g}_{e_1 - e_2} = \mathrm{span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

Now we consider the case where $n \geq 3$.

Suppose $X \in \mathfrak{g}_{e_l - e_m}$. There must be at least one non-zero entry, say x_{ij}, and compute

$$[H, X]_{ij} = [HX]_{ij} - [XH]_{ij} = \sum_{k=1}^{n} h_{ik} x_{kj} - \sum_{k=1}^{n} x_{ik} h_{kj} = h_{ii} x_{ij} - x_{ij} h_{jj} = (h_{ii} - h_{jj}) x_{ij},$$

and $[H, X]_{ij} = [(e_l - e_m)(H)X]_{ij} = (h_{ll} - h_{mm}) x_{ij}$. Therefore $(h_{ii} - h_{jj}) x_{ij} = (h_{ll} - h_{mm}) x_{ij}$. Since $x_{ij} \neq 0$, we have $h_{ii} - h_{jj} = h_{ll} - h_{mm}$.

If $i = j$, we find $h_{ll} = h_{mm}$. However n is at least 3, and thus we can freely choose the values of both variables, and thus the equality won't hold in general. Therefore $x_{ii} = 0$ for all i, and so we now assume $i \neq j$.

We have that $0 = tr(H) = \sum_{k=1}^{n} h_k k = \sum_{k\in\{i,j,l,m\}} h_{kk} + \sum_{i\notin\{i,j,l,m\}} h_{kk}$. Since $i \neq j$, the first sum has at least 2 terms, thus we can freely set all the terms in the second sum to 0. Therefore we have a system of linear equations with some variables h_{ii}, h_{jj}, h_{ll}, and h_{mm}, which may not be distinct, and 2 equations $h_{ii} - h_{jj} = h_{ll} - h_{mm}$ and $\sum_{k\in\{i,j,k,l\}} h_{kk}$.
If there really are 3 or 4 distinct variables, that is 3 or 4 of the i, j, l, m are distinct, then there aren't enough constraints. That is, we can freely choose them to violate the prescribed equations. There can't be only one variable because $i \neq j$, thus there are exactly two variables.
It can't be that $j = l$ and $j = m$, because $m \neq l$.
If $j = l$, then $h_{ii} - h_{jj} = -h_{ii} + h_{jj}$ so $h_{ii} = h_{jj}$ and again, this won't hold for arbitrary H, therefore $j = m$ and thus $i = l$. In other words, all the x_{ij} are zero, except possibly x_{lm}, so $X = x_{lm}E_{lm}$.
We can also see that any such choice of x_{lm} will suffice, and therefore $\mathfrak{g}_{e_l-e_m} = \text{span}\{E_{lm}\}$.

By the argument above, we've determined that no other off-diagonal term be be added to $\mathfrak{h}$ while remaining Abelian, and therefore $\mathfrak{h}$ is a maximal Abelian subalgebra, making it a Cartan subalgebra.
Since the functionals $e_i - e_j$ are a basis for the dual $\mathfrak{h}^*$, we obtain the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{e_i-e_j} \mathfrak{g}_{e_i-e_j},$$

where $\mathfrak{g}_{e_i-e_j} = \text{span}\{E_{ij}\}$.

Usually, the set of *positive* roots is taken to be $\{e_i - e_j | 1 \leq i < j \leq n\}$. However, our main interest will be in a particular subalgebra $\mathfrak{k}$ of $\mathfrak{sl}_{2r}(\mathbb{C})$, to be defined in the next section. The subsets $\Delta^+ = \{e_i - e_j | 1 \leq i < j \leq r\}$ and $\Delta^- = \{-e_i + e_j | r+1 \leq i < j \leq 2r\}$, are positive and negative roots of $\mathfrak{k}$ respectively, and we'll write $\Delta := \Delta^+ \cup \Delta^-$.
Thus the root spaces are spanned by

$$\mathfrak{g}_{e_i-e_j} = \text{span}_{\mathbb{C}}\{E_{ij}\}, \quad 1 \leq i \leq r, i < j \leq 2r \tag{2.14}$$

$$\mathfrak{g}_{-e_i+e_j} = \text{span}_{\mathbb{C}}\{E_{ji}\}, \quad r+1 \leq i < j \leq 2r \tag{2.15}$$

2.2 The Weil Representation

Our ultimate goal is to investigate the behavior of certain vectors under the action of of the *Weil representation*. The Weil representation originated in physics, but was generalized to the realm locally compact Abelian groups by André Weil. This allowed for a representation theoretic interpretation of theta functions, and provided a powerful tool for the study of modular forms of half-integral weight.

We will only outline the theory here, the details for the general case can be found in [9]. Let G be a locally compact Abelian group, and define its *Pontryagin dual* $\hat{G} = Hom_{cts}(G, \mathbb{C}^1)$. The *symplectic group* $Sp(G \times \hat{G})$[1] is a certain special subgroup of $Aut(G \times \hat{G})$. Any finite dimensional $\mathbb{R}$-vector space $\mathbb{W}_1$ is isomorphic (as a vector space) to $\mathbb{R}^n$, and thus the underlying group G of $\mathbb{W}_1$ is isomorphic (as a group) to $\mathbb{R}^n$. As $\widehat{\mathbb{R}^n} \cong \mathbb{R}$ as a group, $\hat{G} \cong \widehat{\mathbb{R}^n}$ and can be realized as the underlying group of the linear dual space $(\mathbb{R}^n)^* \cong \mathbb{W}_1^*$. The natural pairing $G \times \hat{G} \to \mathbb{C}$ induces a form on $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_1^*$, which can be described by a matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

In this case $Sp(G \times \hat{G})$ is actually the Lie group $Sp(\mathbb{W})$ of matrices preserving the form J.

Recall that a *covering* of a topological space X is a space Y and a continuous surjection $p : Y \to X$ such that for all $x \in X$ there exists an open set $U \subseteq X$ containing U, where $p^{-1}(U)$ is a union of disjoint open sets of Y, each of which are homeomorphic to U under p. A cover $p : Y \to X$ is said to be a *double cover* if the fiber $p^{-1}(\{x\})$ contains 2 elements for all $x \in X$. A *covering group* of a topological group G is a covering $p : H \to G$ such that p is also a group homomorphism.

The *metaplectic group* $Mp(G \times \hat{G})$ is constructed as a double cover of $Sp(G \times \hat{G})$, and the Weil representation is a homomorphism $\omega_\psi : Mp(G \times \hat{G}) \to \mathrm{End}(\mathcal{S}(G))$, depending on a choice of $\psi \in \mathrm{Hom}(Z, \mathbb{C}^1)$, where Z is the center of $Sp(G \times \hat{G})$, and $\mathcal{S}(G)$ is the space of *Schwartz-Bruhat* functions. When G is the underlying group of $\mathbb{W}_1$ as above, the space $\mathcal{S}(G)$ is the *Schwartz space* $\mathcal{S}(\mathbb{W}_1)$. Given an identification $\mathbb{W}_1 \cong \mathbb{R}^n$, with coordinates $(x_1, ..., x_n) \in \mathbb{R}^n$, we can describe the Schwartz space as the collection of smooth $f : \mathbb{R}^n \to \mathbb{C}$ such that for all $\alpha = \{\alpha_1, ..., \alpha_n\}$,
$\beta = \{\beta_1, ..., \beta_n\} \in \mathbb{N}_0^n$,

$$\mathrm{sup}_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial f(x)}{\partial x^\beta} \right| < \infty,$$

in multi-index notation.

To describe the setting relevant to our purposes, consider the $(p+q)$-dimensional Hermitian space $(V, \langle , \rangle_V)$ of previous sections, and a $2r$-dimensional $\mathbb{C}$-vector space W, with a skew-Hermitian form $\langle , \rangle_W$ acting on a basis
$\{\vec{w}_s | 1 \leq s \leq 2r\}$ by

$$\langle w_s, w_t \rangle = \begin{cases} i, & 1 \leq s = t \leq r \\ -i, & r+1 \leq s = t \leq 2r \\ 0, & \text{otherwise} \end{cases} .$$

The vector space $\mathbb{W} := V \otimes_{\mathbb{C}} W$ considered as a $2(p+q)(2r)$-dimensional $\mathbb{R}$-vector space, has a symplectic form

$$\langle\langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle\rangle = \mathrm{Re}\left(\langle v, \tilde{v} \rangle_V \langle w, \tilde{w} \rangle_W \right).$$

We choose a maximal isotropic subsapce $\mathbb{W}_1$ of $\mathbb{W}$, and focus on the Weil representation $Mp(\mathbb{W}) = Mp(G \times \hat{G})$. In [6], a splitting $U(V) \times U(W) \to Mp(\mathbb{W})$ is provided, and thus by composition the Weil representation describes

[1] This is commonly written as $Sp(G)$, but this notation will become confusing in our context

and action of $U(V) \times U(W)$ on $\mathcal{S}(V^r)$. The $U(V)$ factor acts on $\Phi \in \mathcal{S}(W)$ by $g \cdot \Phi(x) = \Phi(g^{-1}x)$, but the action of $U(W)$ is more complicated.

Let $H_r(\mathbb{C})$ be the collection of $r \times r$ Hermitian matrices, and

$$m(a) := \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}, a \in GL_r(\mathbb{C}), \quad n(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in H_r(\mathbb{C}), \quad w_r = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$$

By an analogous argument to the proof of *Proposition 2.2* on page 8 of [1], $U(W)$ is generated by elements of the form $m(a), n(b)$, and w_r, and hence it suffices to describe the action of the Weil representation for these elements. Define $\psi : \mathbb{R} \to \mathbb{C}^1$ by $\psi(x) = e^{2\pi i x}$, let χ be a character of $\mathbb{C}^\times$ such that $\chi|_{\mathbb{R}^\times} = \text{sgn}(\cdot)^{p+q}$, and for any $\mathbf{v} = (v_1, ..., v_r) \in V^r$, set $T(\mathbf{v}) = \left[(1/2)\left\langle v_i, v_j \right\rangle_V\right]$. Then explicit formulas for the action of $\omega = \omega_{\psi,\chi}$ on $\Phi \in \mathcal{S}(W)$ are given by

$$\omega(m(a))\Phi(\mathbf{v}) = |\det a|^{\frac{p+q}{2}} \Phi(\mathbf{v} \cdot a)\chi(\det a)$$
$$\omega(n(b))\Phi(\mathbf{v}) = \psi(\text{tr}(bT(\mathbf{v})))\Phi(\mathbf{v})$$
$$\omega(w_r) = \gamma_{V^r} \hat{\Phi}(\mathbf{v})$$

where γ_{V^r} is a special constant associated to the representation known as the *Weil index*, and $\hat{\Phi}$ is the Fourier transform of Φ.

As mentioned in the section on Lie algebras, we can study representations of $U(W)$ by studying the induced representation $d\omega$ on its Lie algebra $\mathfrak{u}(W)$, which we will also just write as ω. Since the study of Lie algebra representations is easier for (and can be recovered from) the case of compact, complex, and semi-simple Lie groups, we will consider the Lie algebra $\mathfrak{k}_0$ of the maximal compact subgroup $U(r) \times U(r)$ of $U(r, r)$, and the complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, of $\mathfrak{g}_0 = Lie(U(r, r))$, where we set $\mathfrak{k} := \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

In the semi-simpification $\mathfrak{g}_{ss}$ of $\mathfrak{g}$, we define $\mathfrak{k}_{ss} = \mathfrak{k} \cap \mathfrak{g}_{ss} \cong \left(\mathfrak{k}_0\right)_{ss} \otimes_{\mathbb{R}} \mathbb{C}$. In fact, in this section we'll realize an explicit isomorphism $\mathfrak{g} \cong \mathfrak{gl}_{2r}(\mathbb{C})$, and thus $\mathfrak{g}_{ss} \cong \mathfrak{sl}_{2r}(\mathbb{C})$. Thus by realizing $\mathfrak{k}_{ss}$ as a Lie subalgebra of $\mathfrak{sl}_{2r}(\mathbb{C})$, we can apply the theory of the highest weight vector outlined in the Lie algebra section.

In [3], the action of the Lie algebra $\mathfrak{g}$ via the Weil representation is described by expressing $\mathfrak{g}$ as isomorphic to a quotient of $\left(\text{Sym}^2_{\mathbb{R}} W\right) \otimes_{\mathbb{R}} \mathbb{C}$, which we also make explicit in this section. Since our focus will be on $\mathfrak{k}_{ss}$, we will then restrict to the corresponding subspace.

The Skew-Hermitian space W

Let W be a complex vector space of dimension $2r$ with a basis $\{\vec{w}_a, \vec{w}_u | 1 \le a \le r, r+1 \le u \le 2r\}$, and a skew-Hermitian inner product defined by $\left\langle \vec{w}_a, \vec{w}_a \right\rangle = i$, $\left\langle \vec{w}_u, \vec{w}_u \right\rangle = -i$, and $\left\langle \vec{w}_s, \vec{w}_t \right\rangle = 0$ for $s \neq t$.

Letting $G = U(W)$ and $\mathfrak{g}_0 = \mathfrak{u}(W)$, the defining equation for the Lie group G becomes

$$\begin{pmatrix} iI_p & 0 \\ 0 & -iI_q \end{pmatrix} = X^T \begin{pmatrix} iI_p & 0 \\ 0 & -iI_q \end{pmatrix} \bar{X}.$$

Thus dividing both sides of the equation by i, we see $G = U(r,r)$ and $\mathfrak{g}_0 = \mathfrak{u}(r,r)$.
Recall that for a K-vector space U, the tensor algebra $T(U)$, and the ideal $I = \langle x \otimes y - y \otimes x \rangle$ of $T(U)$, we define $\mathrm{Sym}^\bullet := T(U)/I$, where we will write $x \circ y$ for the image of $x \otimes y \in T(U)$ in the quotient $T(U)/I$. We set $\mathrm{Sym}^0 U = K$, and for $1 \le k \le \dim U$ we define $\mathrm{Sym}^k(U) := \mathrm{span}_K\{u_1 \circ ... \circ u_k | u_1, ..., u_k \in U\}$.
We define a map

$$\begin{aligned} \phi : \mathrm{Sym}^2_{\mathbb{R}} W &\to \mathrm{End}(W) \\ \phi(\vec{u} \circ \vec{v})(\vec{w}) &= \left\langle \vec{w}, \vec{u} \right\rangle \vec{v} + \left\langle \vec{w}, \vec{v} \right\rangle \vec{u} \end{aligned}$$

Below we will derive the explicit action of the map, and as a result we will be able to see that the image is $\mathfrak{g}_0$.
Since $\{\vec{w}_s, i\vec{w}_t : 1 \le s, t \le 2r\}$ is an $\mathbb{R}$-basis for W, the set $\{\vec{w}_s \circ \vec{w}_t, (i\vec{w}_s) \circ \vec{w}_t, \vec{w}_s \circ (i\vec{w}_t), (i\vec{w}_s) \circ (i\vec{w}_t) : 1 \le s, t \le 2r\}$ is a basis for $\mathrm{Sym}^2_{\mathbb{R}} W$. Furthermore

$$\phi(\vec{u} \circ i\vec{v})\vec{w} = \left\langle \vec{w}, \vec{u} \right\rangle i\vec{v} + \left\langle \vec{w}, i\vec{v} \right\rangle \vec{u} = \left\langle \vec{w}, -i\vec{u} \right\rangle \vec{v} + \left\langle \vec{w}, \vec{v} \right\rangle (-i\vec{u}) = -\left(\left\langle \vec{w}, i\vec{u} \right\rangle \vec{v} + \left\langle \vec{w}, \vec{v} \right\rangle i\vec{u} \right) = -\phi(i\vec{u} \circ \vec{v})\vec{w}$$

$$\phi(i\vec{u} \circ i\vec{v})\vec{w} = \left\langle \vec{w}, i\vec{u} \right\rangle i\vec{v} + \left\langle \vec{w}, i\vec{v} \right\rangle \vec{u} = \left\langle \vec{w}, \vec{u} \right\rangle (-i^2)\vec{v} + \left\langle \vec{w}, \vec{v} \right\rangle (-i^2)\vec{u} = \left\langle \vec{w}, \vec{u} \right\rangle \vec{v} + \left\langle \vec{w}, \vec{v} \right\rangle \vec{u} = \phi(\vec{u} \circ \vec{v})\vec{w}.$$

Thus the image of ϕ is spanned by $\{\phi(\vec{w}_s \circ \vec{w}_t), \phi(i\vec{w}_s \circ \vec{w}_t)\}$, and thus it is sufficient to check the action of these elements on the basis vectors of W.
We compute

$$\phi(\vec{w}_s \circ \vec{w}_t)\vec{w}_m = \left\langle \vec{w}_m, \vec{w}_s \right\rangle \vec{w}_t + \left\langle \vec{w}_m, \vec{w}_t \right\rangle \vec{w}_s = \begin{cases} \left\langle \vec{w}_s, \vec{w}_s \right\rangle \vec{w}_t, & m = s \neq t \\ \left\langle \vec{w}_t, \vec{w}_t \right\rangle \vec{w}_s, & m = t \neq s \\ 2\left\langle \vec{w}_s, \vec{w}_s \right\rangle \vec{w}_s, & m = s = t, \\ 0, & \text{otherwise} \end{cases}$$

Letting $E_{st} \in M_{2r}(\mathbb{C})$ be the matrix with 1 in the s, t position,

$$\phi(\vec{w}_s \circ \vec{w}_t) = \left(\left\langle \vec{w}_s, \vec{w}_s \right\rangle E_{ts} + \left\langle \vec{w}_t, \vec{w}_t \right\rangle E_{st} \right) \in \mathfrak{g}_0, \tag{2.16}$$

which we can see by simply reading off the basis eqs. (2.4) to (2.7) from section 2.1.
Likewise, we compute

$$\begin{aligned} \phi(i\vec{w}_s \circ \vec{w}_t)\vec{w}_m &= \left\langle \vec{w}_m, i\vec{w}_s \right\rangle \vec{w}_t + \left\langle \vec{w}_m, \vec{w}_t \right\rangle i\vec{w}_s = -i\left\langle \vec{w}_m, \vec{w}_s \right\rangle \vec{w}_t + \left\langle \vec{w}_m, \vec{w}_t \right\rangle i\vec{w}_s \\ &= \begin{cases} -i\left\langle \vec{w}_s, \vec{w}_s \right\rangle \vec{w}_t, & m = s \neq t \\ i\left\langle \vec{w}_t, \vec{w}_t \right\rangle \vec{w}_s, & m = t \neq s \\ 0, & m = s = t \\ 0, & \text{otherwise} \end{cases} \\ &= i\left(-\left\langle \vec{w}_s, \vec{w}_s \right\rangle E_{ts} + \left\langle \vec{w}_t, \vec{w}_t \right\rangle E_{st} \right). \end{aligned}$$

Therefore, again by comparing with the basis eqs. (2.4) to (2.7),

$$\phi(i\vec{w}_s \circ \vec{w}_t) = i\left(-\left\langle \vec{w}_s, \vec{w}_s \right\rangle E_{ts} + \left\langle \vec{w}_t, \vec{w}_t \right\rangle E_{st}\right) \in \mathfrak{g}_0.$$

Theorem 2.2.1. *The map $\phi : Sym^2_{\mathbb{R}} W \to \mathfrak{g}_0$ is surjective.*

Proof. By comparing eq. (2.16) and eq. (3.1) to the basis eqs. (2.4) to (2.5) in section 2.1, we see that the map is surjective. □

Furthermore,

$$\{\phi(\vec{w}_s \circ \vec{w}_t) | 1 \leq s \leq t \leq 2r\} \cup \{\phi(i\vec{w}_s \circ \vec{w}_t) | 1 \leq s < t \leq 2r\}$$

is a basis for $\mathfrak{g}_0$.
Taking the complexifications $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, by the right-exactness of $\otimes$, the induced map $(\mathrm{Sym}^2_{\mathbb{R}} W) \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g}$ induced by ϕ is still surjective.
Now

$$\{\phi(\vec{w}_s \circ \vec{w}_t) \otimes 1 | 1 \leq s \leq t \leq 2r\} \cup \{\phi(i\vec{w}_s \circ \vec{w}_t) \otimes i | 1 \leq s < t \leq 2r\},$$

is a $\mathbb{C}$-basis for $\mathfrak{g}$, and thus so is

$$\left\{ \frac{1}{2}\left(\phi(\vec{w}_s \circ \vec{w}_t) \otimes 1 + \phi(i\vec{w}_s \circ \vec{w}_t) \otimes i\right) | 1 \leq s \leq t \leq 2r \right\} \cup \left\{ \frac{1}{2}\left(\phi(\vec{w}_s \circ \vec{w}_t) \otimes 1 - \phi(i\vec{w}_s \circ \vec{w}_t) \otimes i\right) | r+1 \leq s \leq t \leq 2r \right\}.$$

From the matrix expressions above, one can read off that $\phi(\vec{w}_t \circ \vec{w}_s) = \phi(\vec{w}_s \circ \vec{w}_t)$ and $\phi(i\vec{w}_t \circ \vec{w}_s) = -\phi(i\vec{w}_s \circ \vec{w}_t)$, and therefore

$$\left\{ \frac{1}{2}\left(\phi(\vec{w}_s \circ \vec{w}_t) \otimes 1 + \phi(i\vec{w}_s \circ \vec{w}_t) \otimes i\right) | 1 \leq s, t \leq 2r \right\},$$

is a $\mathbb{C}$-basis for $\mathfrak{g}$.

Theorem 2.2.2. *Let $H = \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix}$ and $\theta(X) = HX^*H$. Then the map*

$$\begin{aligned} \eta : \mathfrak{gl}_{2r}(\mathbb{C}) &\to \mathfrak{g} \\ X &\mapsto \frac{1}{2}\left[(X - \theta(X)) \otimes 1 - i(X + \theta(X)) \otimes i\right] \end{aligned}$$

is an isomorphism of (complex) Lie algebras.

Proof. Since θ is linear, so is η.
Suppose $\eta(X) = 0$, then $X = \pm\theta(X) \Rightarrow \theta(X) = 0 \Rightarrow X = 0$, so the map is injective.

$$\eta(E_{jk}) = \begin{cases} \frac{1}{2}\left((E_{jk} - E_{kj}) \otimes 1 - i(E_{jk} + E_{kj}) \otimes i\right), & 1 \leq j, k \leq r, \text{ or } r+1 \leq j, k \leq 2r \\ \frac{1}{2}\left((E_{jk} + E_{kj}) \otimes 1 - i(E_{jk} - E_{kj}) \otimes i\right), & 1 \leq j \leq r < k \leq 2r, \text{ or } 1 \leq k \leq r < j \leq 2r \end{cases}$$

and thus by comparison with the basis eqs. (2.4) to (2.7) from section 2.1, the map is surjective.
Finally, for any $X, Y \in \mathfrak{gl}_{2r}\mathbb{C}$

$$
\begin{aligned}
&[\eta(X), \eta(Y)]\\
=&\left[\frac{1}{2}\left((X-\theta(X))\otimes 1 - i(X+\theta(X))\otimes i\right), \frac{1}{2}\left((Y-\theta(Y))\otimes 1 - i(Y+\theta(Y))\otimes i\right)\right]\\
=&\frac{1}{4}\left([X-\theta(X), Y-\theta(Y)]\otimes 1 + [X-\theta(X), -i(Y+\theta(Y))]\otimes i\right)\\
&+\frac{1}{4}\left([-i(X+\theta(X)), Y-\theta(Y)]\otimes i + [-i(X+\theta(X)), -i(Y+\theta(Y))]\otimes(-1)\right)\\
=&\frac{1}{4}\left(([X,Y]-[X,\theta(Y)]-[\theta(X),Y]+[\theta(X),\theta(Y)]+[X,Y]+[X,\theta(Y)]+[\theta(X),Y]+[\theta(X),\theta(Y)])\otimes 1\right)\\
&+\frac{1}{4}\left(i\left(-[X,Y]-[X,\theta(Y)]+[\theta(X),Y]+[\theta(X),\theta(Y)]-[X,Y]+[X,\theta(Y)]-[\theta(X),Y]+[\theta(X),\theta(Y)]\right)\otimes i\right)\\
=&\frac{1}{4}\left(2\left([X,Y]+[\theta(X),\theta(Y)]\right)\otimes 1 + 2i\left(-[X,Y]+[\theta(X),\theta(Y)]\right)\otimes i\right)\\
=&\frac{1}{2}\left([X,Y]-\theta([X,Y])\right)\otimes 1 + i\left(-[X,Y]-\theta([X,Y])\right)\otimes i\\
=&\eta([X,Y]).
\end{aligned}
$$

Thus η is an isomorphism of Lie algebras. □

Now we will relate certain elements of $\mathfrak{sl}_{2r}(\mathbb{C})$, to the roots of $\mathfrak{k}_{ss}$. We write $w_{st} = \phi(\vec{w}_s \circ \vec{w}_t)\otimes 1 + \phi(i\vec{w}_s \circ \vec{w}_t)\otimes i$ as a shorthand, and then comparing with the previous theorem

$$
\begin{aligned}
w_{ba}\left(1\otimes\frac{-i}{2}\right) &= \left(i(E_{ab}+E_{ba})\otimes 1 + i(-iE_{ab}+iE_{ba})\otimes i\right)\frac{-i}{2}\\
&= \frac{1}{2}\left((E_{ab}-E_{ba})\otimes 1 - i(E_{ab}+E_{ba})\otimes i\right)\\
&= \eta(E_{ab}),\\
w_{uv}\left(1\otimes\frac{i}{2}\right) &= \left(-i(E_{vu}+E_{uv})\otimes 1 + i(iE_{vu}-iE_{uv})\otimes i\right)\frac{i}{2}\\
&= \frac{1}{2}\left((E_{vu}-E_{uv})\otimes 1 - i(E_{uv}+E_{vu})\otimes i\right)\\
&= \eta(E_{vu}).
\end{aligned}
$$

We now provide a concrete description of the Weil representation, as presented in [3]. For a standard basis $\{\vec{v}_1, ..., \vec{v}_{p+q}\}$ of V, we write the coordinates of each vector in a tuple $\mathbf{v} = (\vec{u}_1, ..., \vec{u}_r) \in V^r$ as $\vec{u}_t = \sum_{\gamma=1}^{p+q} z_{\gamma,t}\vec{v}_\gamma$. Let $\bar{t} = t \mod r$, and define

$$
D^+_{\gamma,t} := \left(z_{\gamma\bar{t}} + \pi^{-1}\frac{\partial}{\partial\bar{z}_{\gamma\bar{t}}}\right), \quad \text{and} \quad D^-_{\gamma,t} := \left(z_{\gamma\bar{t}} - \pi^{-1}\frac{\partial}{\partial\bar{z}_{\gamma\bar{t}}}\right).
$$

The actions of w_{ab} and w_{uv} under the Weil representation are described in *Section B.2* of [3] as

$$\omega(w_{ba}) = i\pi \left[\sum_{\alpha=1}^{p} \overline{D^-_{\alpha,a}} D^+_{\alpha,b} - \sum_{\mu=p+1}^{p+q} D^-_{\mu,b} \overline{D^+_{\mu,a}} \right] + (p-q)\delta_{ab},$$
$$\omega(w_{uv}) = i\pi \left[\sum_{\alpha=1}^{p} D^-_{\alpha,u} \overline{D^+_{\alpha,v}} - \sum_{\mu=p+1}^{p+q} \overline{D^-_{\mu,v}} D^+_{\mu,u} \right] - (p-q)\delta_{vu}.$$

Thus composing with η, we obtain a description of the Weil representation $\omega : \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V^r))$, for $1 \leq a, b \leq r < u, v \leq 2r$

$$\omega\circ\eta\left(E_{ab}\right) = \omega\left(w_{ba}\frac{-i}{2}\right) = \omega(w_{ba})\frac{-i}{2} = \frac{\pi}{2}\left[\sum_{\alpha=1}^{p} \overline{D^-_{\alpha,a}} D^+_{\alpha,b} - \sum_{\mu=p+1}^{p+q} D^-_{\mu,b} \overline{D^+_{\mu,a}} \right] + \frac{p-q}{2}\delta_{ab},$$
$$\omega\circ\eta\left(E_{vu}\right) = \omega\left(w_{uv}\frac{i}{2}\right) = \omega(w_{vu})\frac{i}{2} = -\frac{\pi}{2}\left[\sum_{\alpha=1}^{p} D^-_{\alpha,u} \overline{D^+_{\alpha,v}} - \sum_{\mu=p+1}^{p+q} \overline{D^-_{\mu,v}} D^+_{\mu,u} \right] - \frac{p-q}{2}\delta_{vu}.$$

We will mainly be working with the representation $\omega\circ\eta : \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V^r))$ and the restriction to $\left(\mathfrak{gl}_{2r}(\mathbb{C})\right)_{ss} \cong \mathfrak{sl}_{2r}(\mathbb{C})$, where we will identify $\mathfrak{k}$ with its pre-image in $\mathfrak{sl}_{2r}(\mathbb{C})$ under η. Thus for the remainder of the thesis, we will drop the η, and simply write $\omega : \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V^r))$.
Note that even though we began with the initial assumption that $p, q \geq 1$, the operators $\omega(E_{ab})$ and $\omega(E_{vu})$ are still defined for the degenerate case when $p = 0$ or $q = 0$. We will make use of this fact in the next section.

Propositions for Induction

In this section we will develop some relations between the action of the Weil representation on $\mathcal{S}(V)$, and the Schwartz space of the subspaces of V. We will also prove some propositions relating the Weil representation on $\mathcal{S}(V)$ to $\mathcal{S}(V^r)$ for $r > 1$. First we establish a result that is useful in the following chapter. Given $\mathbf{v} = (u_1, ..., u_r) \in V^r$, we define

$$\langle \mathbf{v}, \mathbf{v} \rangle := \langle u_1, u_1 \rangle + ... + \langle u_r, u_r \rangle .$$

Definition 2.2.3. If V is positive-definite with respect to $\langle , \rangle$, then we define the *Vacuum vector* $e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} \in \mathcal{S}(V^r)$.

This is often given as the prototypical example of an element of Schwartz space.

Lemma 2.2.4. *For a positive-definite vector space V with Hermitian form $\langle , \rangle$, and the Weil representation $\omega : \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathcal{S}(V^r)$, with $1 \leq a, b \leq r$ and $r+1 \leq u, v \leq 2r$*

$$\omega(E_{ab})e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} = \left(\frac{p}{2}\right)\delta_{ab}e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle}$$
$$\omega(E_{vu})e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} = -\left(\frac{p}{2}\right)\delta_{vu}e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle}$$

Proof. Choose a standard basis $\{v_1, ..., v_p\}$ for V, and write the coordinates of $\mathbf{v} = (u_1, ..., u_r) \in V^r$ as $\sum_{\alpha=1}^{p} z_{\alpha,s} v_\alpha$, for $1 \leq s \leq r$. Then for $1 \leq \alpha \leq p$ and $1 \leq s \leq r$

$$\begin{aligned} D^+_{\gamma,s} e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} &= \left(z_{\gamma,s} + \pi^{-1} \frac{\partial}{\partial \bar{z}_{\gamma,s}} \right) exp\left(-\pi \sum_{s=1} \sum_{\alpha=1}^{p} |z_{\alpha,s}|^2 \right) \\ &= z_{\gamma,s} e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} - z_{\gamma,s} e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} \\ &= 0. \end{aligned}$$

Similarly one finds $\bar{D}^+_{\gamma,s} e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} = 0$. Therefore (since $q = 0$)

$$\begin{aligned} \omega(E_{ab}) e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} &= \left(\frac{\pi}{2} \left[\sum_{\alpha=1}^{p} \overline{D^-_{\alpha,a}} D^+_{\alpha,b} \right] + \frac{p}{2} \delta_{ab} \right) e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle} \\ &= \frac{p}{2} \delta_{ab} e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle}. \end{aligned}$$

The proof that $\omega(E_{vu}) = -\left(\frac{p}{2}\right) \delta_{vu} e^{-\pi\langle \mathbf{v},\mathbf{v}\rangle}$ is entirely similar. $\square$

Given a decomposition $V = U \oplus U_\perp$, we will relate the Schwartz space $\mathcal{S}(U)$ and $\mathcal{S}(U_\perp)$ to $\mathcal{S}(V)$, which will be useful in the next chapter. Suppose $U \subseteq V$ is a subspace where the restriction $\langle , \rangle |_U$ is a Hermitian form of signature (m, n). Choose a standard basis $\{v_1, ..., v_m, v_{p+1}, ..., v_{p+n}\}$ for U, and extend it to a standard basis

$$\{v_1, ..., v_m, v_{m+1}, ..., v_p, v_{p+1}, ..., v_{p+n}, v_{p+n+1}, ..., v_{p+q}\},$$

of V. Then $\{v_{m+1}, ..., v_p, v_{p+m+1}, ..., v_{p+q}\}$ is a standard basis for $U_\perp$.
For $\mathbf{v} = (u_1, ..., u_r) \in V^r$, if for $1 \leq s \leq r$ we write the orthogonal decompositions $u_s = u'_s + u''_s$. Thus in the coordinates $u_s = \sum_{\gamma=1}^{p+q} z_{\gamma,s} v_\gamma$ the decompositions are

$$\begin{aligned} u'_s &= \sum_{\gamma=1}^{m} z_{\gamma,s} v_\gamma + \sum_{\gamma=p+1}^{p+m} z_{\gamma,s} v_\gamma \\ u''_s &= \sum_{\gamma=m+1}^{p} z_{\gamma,s} v_\gamma + \sum_{\gamma=p+m+1}^{p+q} z_{\gamma,s} v_\gamma \end{aligned}$$

Thus, in these coordinates, the action of the Weil representation

$$\begin{aligned} \omega &: \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V^r)) \\ \omega_U &: \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(U^r)) \\ \omega_\perp &: \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(U^r_\perp)) \end{aligned}$$

in terms of the operators

$$D^\pm_{\gamma,s} = \left(z_{\gamma,1} \pm \pi^{-1} \frac{\partial}{\partial \bar{z}_{\gamma,s}} \right),$$

is given for $1 \leq a, b \leq r$

$$\omega(E_{ab}) = \frac{\pi}{2}\left[\sum_{\alpha=1}^{p} \overline{D^-_{\alpha,a}} D^+_{\alpha,b} - \sum_{\mu=p+1}^{p+q} D^-_{\mu,b} \overline{D^+_{\mu,a}}\right] + \frac{p-q}{2}\delta_{ab},$$

$$\omega_U(E_{ab}) = \frac{\pi}{2}\left[\sum_{\alpha=1}^{m} \overline{D^-_{\alpha,a}} D^+_{\alpha,b} - \sum_{\mu=p+1}^{p+n} D^-_{\mu,b} \overline{D^+_{\mu,a}}\right] + \frac{m-n}{2}\delta_{ab},$$

$$\omega_\perp(E_{ab}) = \frac{\pi}{2}\left[\sum_{\alpha=m+1}^{p} \overline{D^-_{\alpha,a}} D^+_{\alpha,b} - \sum_{\mu=p+m+1}^{p+q} D^-_{\mu,b} \overline{D^+_{\mu,a}}\right] + \frac{(p-m)-(q-n)}{2}\delta_{ab},$$

and $r+1 \leq u, v \leq 2r$

$$\omega(E_{vu}) = -\frac{\pi}{2}\left[\sum_{\alpha=2}^{p} D^-_{\alpha,u} \overline{D^+_{\alpha,v}} - \sum_{\mu=p+1}^{p+q} \overline{D^-_{\mu,v}} D^+_{\mu,u}\right] - \frac{p-q}{2}\delta_{vu}$$

$$\omega_U(E_{vu}) = -\frac{\pi}{2}\left[\sum_{\alpha=1}^{m} D^-_{\alpha,u} \overline{D^+_{\alpha,v}} - \sum_{\mu=p+1}^{p+n} \overline{D^-_{\mu,v}} D^+_{\mu,u}\right] - \frac{m-n}{2}\delta_{vu}$$

$$\omega_\perp(E_{vu}) = -\frac{\pi}{2}\left[\sum_{\alpha=m+1}^{p} D^-_{\alpha,u} \overline{D^+_{\alpha,v}} - \sum_{\mu=p+m+1}^{p+q} \overline{D^-_{\mu,v}} D^+_{\mu,u}\right] - \frac{(p-m)-(q-n)}{2}\delta_{vu}.$$

The orthogonal projections $V \to U$ and $V \to U_\perp$ extend to component-wise projections $\pi_U : V^r \to U^r$ and $\pi_\perp : V^r \to U_\perp^r$, and thus we obtain pullbacks

$$\pi_U^* : \mathcal{S}(U^r) \to \mathcal{S}(V^r), \qquad \pi_\perp^* : \mathcal{S}(U_\perp^r) \to \mathcal{S}(V^r),$$

where for $\Phi \in \mathcal{S}(U^r)$ and $\Psi \in \mathcal{S}(U_\perp^r)$, we have $\pi_U^*(\Phi) = \Phi \circ \pi_U$ and $\pi_\perp^*(\Psi) = \Psi \circ \pi_\perp$.

Proposition 2.2.5. *Suppose $V = U \oplus U_\perp$, with $\Phi \in \mathcal{S}(U^r)$ and $\Psi \in \mathcal{S}(U_\perp^r)$. Then for $\left(\pi_U^*(\Phi)\pi_\perp^*(\Psi)\right) \in \mathcal{S}(V^r)$ and $E_{st} \in \mathfrak{gl}_{2r}(\mathbb{C})$ such that $1 \leq s, t \leq r$ or $r+1s, t \leq 2r$*

$$\omega(E_{st})\left(\pi_U^*(\Phi)\pi_\perp^*(\Psi)\right) = \pi_U^*\left(\omega_U(E_{st})\Phi\right) \cdot \pi_\perp^*(\Psi) + \pi_U^*(\Phi) \cdot \pi_\perp^*\left(\omega_\perp(E_{st})\Psi\right).$$

Proof. For the coordinates $\mathbf{v} = (u_1, ..., u_r) \in V^r$ described preceding the statement of the proposition, we write the orthogonal decompositions $u_s = u'_s + u''_s \in U \oplus U_\perp$ with $\mathbf{v}' = (u'_1, ..., u'_s)$ and $\mathbf{v}'' = (u''_1, ..., u''_r)$.
Observe that $\frac{\partial}{\partial z_{\gamma,s}}\pi_U^*\Phi = 0$ and $\frac{\partial}{\partial \bar{z}_{\gamma,s}}\pi_U^*\Phi = 0$ for all $1 \leq s \leq r$ and $m+1 \leq \gamma \leq p$ or $p+m+1 \leq \gamma \leq p+q$. Like-wise

$\frac{\partial}{\partial z_{\gamma,s}}\pi_\perp^*\Psi = 0$ and $\frac{\partial}{\partial \bar{z}_{\gamma,s}}\pi_\perp^*\Psi = 0$ for all $1 \leq s \leq r$ and $1 \leq \gamma \leq m$ or $p+1 \leq \gamma \leq p+m$. Thus, for $1 \leq a, b \leq r$

$$\begin{aligned}
\omega(E_{ab})\left(\pi_U^*(\Phi)\pi_\perp^*(\Psi)\right)(\mathbf{v}) &= \left(\frac{\pi}{2}\left[\sum_{\alpha=1}^{p}\overline{D^-_{\alpha,a}}D^+_{\alpha,b} - \sum_{\mu=p+1}^{p+q} D^-_{\mu,b}\overline{D^+_{\mu,a}}\right] + \frac{p-q}{2}\delta_{ab}\right)\left(\pi_U^*(\Phi)(\mathbf{v})\pi_\perp^*\Psi(\mathbf{v})\right)\\
&= \left(\left(\frac{\pi}{2}\left[\sum_{\alpha=1}^{m}\overline{D^-_{\alpha,a}}D^+_{\alpha,b} - \sum_{\mu=p+1}^{p+n} D^-_{\mu,b}\overline{D^+_{\mu,a}}\right] + \frac{m-n}{2}\delta_{ab}\right)\Phi(\mathbf{v}')\right)\pi_\perp^*(\Psi)(\mathbf{v})\\
&\quad + \pi_U^*(\Phi)(\mathbf{v})\left(\frac{\pi}{2}\left[\sum_{\alpha=m+1}^{p}\overline{D^-_{\alpha,a}}D^+_{\alpha,b} - \sum_{\mu=p+m+1}^{p+q} D^-_{\mu,b}\overline{D^+_{\mu,a}}\right] + \frac{(p-m)-(q-n)}{2}\delta_{ab}\right)\Psi(\mathbf{v}'')\\
&= \left(\omega_U(E_{ab})\Psi(\mathbf{v}')\right)\pi_\perp^*(\Psi)(\mathbf{v}) + \pi_U^*(\Phi)(\mathbf{v})\left(\omega_\perp(E_{ab})\Psi(\mathbf{v}'')\right)\\
&= \pi_U^*\left(\omega_U(E_{st})\Phi\right)(\mathbf{v})\cdot\pi_\perp^*\Psi(\mathbf{v}) + \pi_U^*\Phi(\mathbf{v})\cdot\pi_\perp^*\left(\omega_\perp(E_{st})\Psi\right)(\mathbf{v}).
\end{aligned}$$

The proof that for $r+1 \leq u, v \leq 2r$

$$\omega(E_{uv})\left(\pi_U^*(\Phi)\pi_\perp^*(\Psi)\right) = \pi_U^*\left(\omega_U(E_{st})\Phi\right)\cdot\pi_\perp^*(\Psi) + \pi_U^*(\Phi)\cdot\pi_\perp^*\left(\omega_\perp(E_{st})\Psi\right),$$

is entirely similar. □

We will also need to make use of some relations of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ to $\mathfrak{k}_r$ for $r > 2$. For an m-tuple $\mathbf{c} = (c_1, ..., c_m)$ with $1 \leq c_1 \leq ... \leq c_m \leq r$, we define $T_\mathbf{c} : V^r \to V^m$ by $T_\mathbf{c}(v_1, ..., v_r) = (v_{c_1}, ..., v_{c_m})$. We also obtain a pullback $T_\mathbf{c}^* : \mathcal{S}(V^m) \to \mathcal{S}(V^r)$ by $T_\mathbf{c}^*\Phi = \Phi\circ T_\mathbf{c}$.
Given a standard basis $v_1, ..., v_{p+q}$ for V, we write the coordinates of

$$(u_1, ..., u_{r-m}) \in V^{r-m}, \quad (u_1', ..., u_m') \in V^m,$$

as $u_s = \sum_{\gamma=1}^{p+q} z_{\gamma,s}v_\gamma$, and $u_t' = \sum_{\gamma=1}^{p+q} z_{\gamma,t}'v_\gamma$. Then the operators appearing in the Weil representation can be written as

$$D^\pm_{\gamma,s} = \left(z_{\gamma,s} \pm \pi^{-1}\frac{\partial}{\partial \bar{z}_{\gamma,s}}\right), \qquad D'^\pm_{\gamma,t} = \left(z'_{\gamma,t} \pm \pi^{-1}\frac{\partial}{\partial \bar{z}'_{\gamma,t}}\right).$$

The following lemma is a computation that will allow us to relate the action of the Weil representation on $\mathcal{S}(V^r)$ to $\mathcal{S}(V^m)$ with $m < r$.

Lemma 2.2.6. *Let $\Phi \in \mathcal{S}(V^m)$ and $\Psi \in \mathcal{S}(V^{r-m})$, with m-tuples $\mathbf{c}$ and $(r-m)$-tuple $\mathbf{d}$, with $s = c_j \in \mathbf{c}$. Then in the standard basis*

$$D^\pm_{\gamma,s}\left[(T_\mathbf{c}^*\Phi)(T_\mathbf{d}^*\Psi)\right] = \left(T_\mathbf{c}^*(D'^\pm_{\gamma,j}\Phi)\right)(T_\mathbf{d}^*\Psi).$$

Proof. This follows directly from the definitions, and the fact that $T_\mathbf{d}^*\Psi$ doesn't depend on the vectors in the coordinates in any of the positions in $\mathbf{c}$. □

The next proposition will allow us to determine that certain vectors of $\mathcal{S}(V^r)$ defined in the next chapter, will be eigenvectors for the action of the Cartan subalgebra of $\mathfrak{k}_r$. Furthermore, together with several base cases computed by the code presented at the end of thesis, we can determine their precise eigenvalues by induction.

Proposition 2.2.7. *Suppose $\Phi \in \mathcal{S}(V)$ and $\Psi \in \mathcal{S}(V^{r-1})$. For $1 \le a \le r$ and $r+1 \le u \le 2r$, $\mathbf{a} = \{1, ..., \hat{a}, ..., r\}$ with $E_{aa}, E_{uu} \in \mathfrak{k}_r$ and the Weil representation*

$$\omega : \mathfrak{k}_r \to End(\mathcal{S}(V^r)),$$
$$\omega_1 : \mathfrak{k}_1 \to End(\mathcal{S}(V)),$$
$$\omega_{r-1} : \mathfrak{k}_{r-1} \to End(\mathcal{S}(V^{r-1})),$$

we have

$$\omega(E_{aa})\left[(T_a^*\Phi)(T_{\mathbf{a}}^*\Psi)\right] = \left(T_a^*\omega_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\Phi\right)\left(T_{\mathbf{a}}\Psi\right)$$
$$\omega(E_{uu})\left[(T_s^*\Phi)(T_{\mathbf{a}}^*\Psi)\right] = \left(T_a^*\omega_1\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\Phi\right)\left(T_{\mathbf{a}}\Psi\right).$$

Proof. Given a standard basis $v_1, ..., v_{p+q}$ for V, we write $(u_1', u_2') \in V^2, (u_1, ..., u_r) \in V^r$ with coordinates $u_s' = \sum_{\gamma=1}^{p+q} z_{\gamma,s}' v_\gamma$, and $u_t = \sum_{\gamma=1}^{p+q} z_{\gamma,t} u_t$. The action of the Weil representation is expressed as

$$\omega(E_{aa}) = \frac{\pi}{2}\left[\sum_{\alpha=1}^{p}\overline{D_{\alpha,a}^-}D_{\alpha,a}^+ - \sum_{\mu=p+1}^{p+q} D_{\mu,a}^-\overline{D_{\mu,a}^+}\right] + \frac{(p-q)}{2},$$
$$\omega_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \frac{\pi}{2}\left[\sum_{\alpha=1}^{p}\overline{D_{\alpha,1}'^-}D_{\alpha,1}'^+ - \sum_{\mu=p+1}^{p+q} D_{\mu,1}'^-\overline{D_{\mu,1}'^+}\right] + \frac{(p-q)}{2}.$$

By linearity, it suffices to prove the relation for any of the operators, and by the previous lemma, for example,

$$\overline{D_{\alpha,a}^-}D_{\alpha,a}^+\left[(T_a^*\Phi)(T_{\mathbf{a}}^*\Psi)\right] = \overline{D_{\alpha,a}^-}\left[\left(T_a^*(D_{\alpha,1}'^+\Phi)\right)\left(T_{\mathbf{a}}\Psi\right)\right] = \left(T_a^*(\overline{D_{\alpha,1}'^-}D_{\alpha,1}'^+\Phi)\right)(T_{\mathbf{a}}\Psi).$$

The proof that $\omega(E_{uu})\left[(T_s^*\Phi)(T_{\mathbf{a}}^*\Psi)\right] = \left(T_a^*\omega_1\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\Phi\right)\left(T_{\mathbf{a}}\Psi\right)$, for $r+1 \le u \le 2r$ with $E_{uu} \in \mathfrak{k}_r$ is entirely similar. □

The following proposition will be used in the next chapter, to reduce the determination that certain vectors are killed by the positive roots of $\mathfrak{k}_r$, to the demonstration that related vectors are killed by the roots of $\mathfrak{k}_2$.

Proposition 2.2.8. *Whenever $1 \le a \le b \le r$ or $r+1 \le u \le v \le 2r$ for $\mathbf{c} = (1, ..., \hat{s}, ..., \hat{t}, ..., r)$, $\Phi \in \mathcal{S}(V^2)$, $\Psi \in \mathcal{S}(V^{r-2})$ and the Weil representation $\omega : \mathfrak{k}_r \to End(\mathcal{S}(V^r))$, $\omega_2 : \mathfrak{k}_2 \to End(\mathcal{S}(V^2))$, then for $E_{12}, E_{43} \in \mathfrak{k}_2$ and $E_{ab}, E_{vu} \in \mathfrak{k}_r$*

$$\omega(E_{ab})\left[\left(T_{ab}^*\Phi\right)\left(T_{\mathbf{c}}^*\Psi\right)\right] = \left(T_{ab}^*\omega_2(E_{12})\Phi\right)(T_{\mathbf{c}}^*\Psi)$$
$$\omega(E_{vu})\left[\left(T_{uv}^*\Phi\right)\left(T_{\mathbf{c}}^*\Psi\right)\right] = \left(T_{uv}^*\omega_2(E_{43})\Phi\right)(T_{\mathbf{c}}^*\Psi)$$

Proof. For $1 \leq a < b \leq r$ we can expand the operators

$$\omega(E_{ab}) = \frac{\pi}{2}\left[\sum_{\alpha=1}^{p} \overline{D^{-}_{\alpha,a}} D^{+}_{\alpha,b} - \sum_{\mu=p+1}^{p+q} D^{-}_{\mu,b} \overline{D^{+}_{\mu,a}}\right]$$
$$\omega(E_{vu}) = -\frac{\pi}{2}\left[\sum_{\alpha=1}^{p} D^{-}_{\alpha,u} \overline{D^{+}_{\alpha,v}} - \sum_{\mu=p+1}^{p+q} \overline{D^{-}_{\mu,v}} D^{+}_{\mu,u}\right]$$
$$\omega_2(E_{12}) = \frac{\pi}{2}\left[\sum_{\alpha=1}^{p} \overline{D'^{-}_{\alpha,1}} D'^{+}_{\alpha,2} - \sum_{\mu=p+1}^{p+q} D'^{-}_{\mu,2} \overline{D'^{+}_{\mu,1}}\right]$$
$$\omega_2(E_{43}) = -\frac{\pi}{2}\left[\sum_{\alpha=1}^{p} D'^{-}_{\alpha,3} \overline{D'^{+}_{\alpha,4}} - \sum_{\mu=p+1}^{p+q} \overline{D'^{-}_{\mu,4}} D'^{+}_{\mu,3}\right]$$

By linearity, and lemma 2.2.6 the result follows. □

Chapter 3

The Garcia-Sankaran Construction

In this chapter, we consider the manifold $\mathbb{D}$ of negative-definite q-dimensional subspace of a $(p+q)$-dimensional vector space V, with a Hermitian form $\langle , \rangle$ of signature (p, q).
Our goal is to construct a specific element $\nu_r \in \mathcal{S}(V^r) \otimes_{\mathbb{C}} \mathcal{A}^{\bullet}(\mathbb{D})$ where $\mathcal{S}(V^r)$ is the space of *Schwartz functions* on $V^r = V^{\oplus r}$, described in the previous chapter, and $\mathcal{A}^{\bullet}(\mathbb{D})$ is the algebra of differential forms on $\mathbb{D}$. We will then extend the Weil representation $\omega : \mathfrak{k}_r \to \mathrm{End}(\mathcal{S}(V^r))$ of the previous chapter, to act on $\mathrm{End}\left(\mathcal{S}(V^r) \otimes_{\mathbb{C}} \mathcal{A}^{\bullet}(\mathbb{D})\right)$, and show that the degree $2(qr-1)$ component (that is, the $(qr-1, qr-1)$-complex degree component) of ν_r generates an irreducible subrepresentation of the Weil representation.
In the first section, we recount the general set up as it appears in [4], by using Quillen's theory of superconnections as they appear in the first chapter.
We will prove the main results of this thesis in the second section, by relying on the theory of highest weight vectors, and a number of properties proved about the Weil representation from chapter 2.

3.1 The Manifold $\mathbb{D}$

Much of the material of this section comes from [4], including the definitions and basic properties of the forms φ and ν.

Definition 3.1.1. Let V be a $(p+q)$-dimensional $\mathbb{C}$-vector space, with Hermitian inner product $\langle , \rangle$ of signature (p, q). Define

$$\mathbb{D}(V) := \{\zeta \subseteq V | \zeta \text{ is a } q-\text{dimensional, negative definite subspace of } V\}.$$

Given $\zeta_0 \in \mathbb{D}$, by Gram-Schmidt orthonormalization, we may select a basis $\{v_{p+1}, ..., v_{p+q}\}$ for ζ_0 such that

$$\langle v_i, v_j \rangle = \begin{cases} -\delta_{ij}, & p+1 \leq i, j \leq p+q \\ 0, & \text{otherwise} \end{cases}$$

By applying Gram-Schmidt again, we can extend this to a *standard basis* $\{v_1, ..., v_{p+q}\}$ of V such that

$$\left\langle v_i, v_j \right\rangle = \begin{cases} \delta_{ij}, & 1 \le i, j \le p \\ -\delta_{ij}, & p+1 \le i, j \le p+q \\ 0, & \text{otherwise} \end{cases}$$

Now for any other $\zeta \in \mathbb{D}$, in the basis $\{v_i\}$,

$$\zeta = \operatorname{span}\left\{ f_1 = \begin{pmatrix} x_{1,1} \\ \dots \\ x_{p,1} \\ y_{1,1} \\ \dots \\ y_{q,1} \end{pmatrix}, ..., f_q = \begin{pmatrix} x_{1,q} \\ \dots \\ x_{p,q} \\ y_{1,q} \\ \dots \\ y_{q,q} \end{pmatrix} \right\}.$$

We must have $0 > \langle f_1, f_1 \rangle = \left(\sum_{i=1}^{p} |x_{i,1}|^2 \right) - \sum_{j=1}^{q} |y_{j,1}|^2$, and therefore not all $y_{j,1}$ can be zero. Without loss of generality, say $y_{1,1} \neq 0$. Thus we may replace the choice of basis by

$$\zeta = \operatorname{span}\left\{ \begin{pmatrix} \frac{x_{1,1}}{y_{1,1}} \\ \dots \\ \frac{x_{p,1}}{y_{1,1}} \\ 1 \\ \dots \\ \frac{y_{q,1}}{y_{1,1}} \end{pmatrix}, ..., \begin{pmatrix} x_{1,q} \\ \dots \\ x_{p,q} \\ y_{1,q} \\ \dots \\ y_{q,q} \end{pmatrix} \right\}.$$

For $i \neq 1$, we may replace f_i with $f_i - y_{1,i} f_1$ (note that the vectors will remain linearly independent) to obtain the basis of the form

$$\zeta = \operatorname{span}\left\{ f_1' = \begin{pmatrix} x'_{1,1} \\ \dots \\ x'_{p,1} \\ 1 \\ y'_{2,1} \\ \dots \\ y'_{q,1} \end{pmatrix}, f_2' = \begin{pmatrix} x'_{1,2} \\ \dots \\ x'_{p,2} \\ 0 \\ y'_{2,2} \\ \dots \\ y'_{q,2} \end{pmatrix} ..., f_q' = \begin{pmatrix} x'_{1,q} \\ \dots \\ x'_{p,q} \\ 0 \\ y'_{2,q} \\ \dots \\ y'_{q,q} \end{pmatrix} \right\}.$$

Now observe that we now require $\left\langle f_2', f_2' \right\rangle < 0$, and thus we may argue that the remaining $y'_{2,2}, ..., y'_{q,2}$ cannot all be zero. Thus we may take $y'_{2,2} \neq 0$ without loss of generality. After dividing f_2' by $y'_{2,2}$, we may repeat this process of eliminating the $y'_{\bullet,2}$ terms of the other vectors.

The above argument of dividing by a non-zero term, and eliminating entries may be repeated until one arrives at a basis of the form

$$\zeta = \operatorname{span}\left\{ \begin{pmatrix} f_{1,1} \\ \dots \\ f_{p,1} \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} f_{1,q} \\ \dots \\ f_{p,q} \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus we may identify each $\zeta \in \mathbb{D}$ with a matrix (in block from) $\begin{pmatrix} F_\zeta \\ I_q \end{pmatrix}$, where $F_\zeta = \begin{pmatrix} f_{1,1} & \dots & f_{1,q} \\ \dots & & \dots \\ f_{p,1} & \dots & f_{p,q} \end{pmatrix}$, and I_q is the $q \times q$ identity matrix.

Furthermore, this representation is unique, for given any $p \times q$ matrix F, the coefficients of any non-trivial linear combination of its columns, are the coefficients of a non-trivial linear combination of the columns of $\begin{pmatrix} F \\ I_q \end{pmatrix}$, and thus the resulting combination is a matrix $\begin{pmatrix} A \\ B \end{pmatrix}$, for which $B \neq I_q$. Therefore we have a bijection between $\mathbb{D}$ and

$$D := \left\{ A \in M_{pq}(\mathbb{C}) \,:\, \operatorname{Col}\begin{pmatrix} A \\ I_q \end{pmatrix} \text{ is negative-definite} \right\}.$$

For $F \in D$, writing the column vectors $\begin{pmatrix} F \\ I_q \end{pmatrix} = [v_1, \dots, v_q]$, we compute

$$[F^*F - I_q]_{ij} = \left(\sum_{k=1}^{p} [F^*]_{ik} E_{kj} \right) - \delta_{ij} = \left(\sum_{k=1}^{p} \bar{F}_{ki} F_{kj} \right) - \delta_{ij} = \left\langle v_i, v_j \right\rangle.$$

If the column vectors $[v_1, ..., v_q]$ have negative-definite span, then for every $z^T = (z_1, ..., z_q) \in (\mathbb{C}^q - \{\vec{0}\})$

$$\begin{aligned} & 0 > \left\langle \sum_{i=1}^{q} z_i v_i, \sum_{j=1}^{q} z_j v_j \right\rangle \\ \iff & 0 > \sum_{i=1}^{q} \sum_{j=1}^{q} z_i \bar{z}_j \left\langle v_i, v_j \right\rangle \\ \iff & 0 > \sum_{i=1}^{q} z_i \sum_{j=1}^{q} (F^*F - I_q)_{ij} \bar{z}_j \\ \iff & 0 > \sum_{i=1}^{q} (z^T)_{1i} [(F^*F - I_q)\bar{z}]_{i1} \\ \iff & 0 > z^T (F^*F - I_q)\bar{z}, \qquad \forall z^T \in (\mathbb{C}^q - \{\vec{0}\}) \end{aligned}$$

and therefore

$$D = \{F \in M_{pq}(\mathbb{C}) : z^T (F^*F - I_q)\bar{z} < 0, \forall z^T \in (\mathbb{C}^q - \{\vec{0}\})\}.$$

Note that, for *any* $F \in M_{pq}(\mathbb{C})$, we have

$$(F^*F - I_q)^* = (F^*F)^* - I_q = F^*F - I_q.$$

Thus $F^*F - I_q$ is Hermitian. We denote the space of $q \times q$ complex Hermitian matrices by $\mathrm{Herm}_q(\mathbb{C})$, given the subspace topology of $M_q(\mathbb{C})$. Since the map

$$\begin{aligned} h : M_{pq}(\mathbb{C}) &\to \mathrm{Herm}_q(\mathbb{C}) \\ A &\mapsto A^*A - I_q \end{aligned}$$

is polynomial in its coordinates, it's continuous.

For any $B \in \mathrm{Herm}_q(\mathbb{C})$ and $z \in \mathbb{C}^q$,

$$(z^T B\bar{z})^* = z^T B^* \bar{z} = z^T B\bar{z},$$

and since $z^T B\bar{z}$ is just a complex number, we conclude that $\overline{z^T B\bar{z}} = z^T B\bar{z}$, that is $z^T B\bar{z} \in \mathbb{R}$. Thus, we can define the set

$$X^- := \{B \in \mathrm{Herm}_q(\mathbb{C}) : z^T B\bar{z} < 0, \forall z \in (\mathbb{C}^q - \{\vec{0}\})\},$$

and note that $D = h^{-1}(X^-)$. Since h is continuous, if we can show that X^- is open in $\mathrm{Herm}_q(\mathbb{C})$ it will follow that D is open in $M_{pq}(\mathbb{C})$.

Theorem 3.1.2. *The set $\mathbb{D}$ is a complex manifold of dimension pq.*

Proof. First we will show that the set X^- is an open subset of $\mathrm{Herm}_q(\mathbb{C})$. The map

$$\begin{aligned} f : \mathrm{Herm}_q(\mathbb{C}) \times \mathbb{C}^q &\to \mathbb{R} \\ (B, v) &\mapsto v^T B\bar{v} \end{aligned}$$

is polynomial in its entries, hence continuous. Furthermore, since f is continuous, and $\mathbb{C}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is compact, $\hat{f} : \mathrm{Herm}_q(\mathbb{C}) \to \mathbb{R}$ defined by $\hat{f}(B) = \max_{|z|=1} f(B, z)$ is well-defined and continuous. Thus we conclude that $\hat{f}^{-1}(-\infty, 0) = X^-$ must be open.

Since $h : M_{pc}(\mathbb{C}) \to \mathrm{Herm}_q(\mathbb{C})$ is continuous, and $X^- \subseteq \mathrm{Herm}_q(\mathbb{C})$ is open, $h^{-1}(X^-) = D \subseteq M_{pq}(\mathbb{C})$ is open. Therefore D inherits the manifold structure of $M_{pq}(\mathbb{C})$. Thus we give $\mathbb{D}$ the structure of a complex manifold by demanding that the bijection $\mathbb{D} \to D$ is a biholomorphism.

Furthermore, choosing some $\zeta_0 \in \mathbb{D}$ gives us coordinates for $\mathbb{D}$ by representing $\zeta \in \mathbb{D}$ by E_ζ. □

3.2 The Super Connection and the Special Forms

In this section, we will apply Quillen's superconnections, and the generalized Chern-Weil theory, in order to construct the forms φ and ν of [4].

Definitions of φ and ν

On the manifold $\mathbb{D}$ of q-dimensional subspaces of a (p, q)-vector space V, we define the *tautological bundle* $\mathcal{E} \to \mathbb{D}$ by

$$\mathcal{E} = \{(\zeta, v) \in \mathbb{D} \times V : v \in \zeta\} \subseteq \mathbb{D} \times V$$

which we give the subspace topology induced by $\mathbb{D} \times V$, with the projection map

$$\pi : \mathcal{E} \to \mathbb{D}$$
$$(\zeta, v) \mapsto \zeta$$

The vector space structure on each fiber is simply given by

$$c(\zeta, u) + d(\zeta, v) = (\zeta, cu + dv).$$

We define a Hermitian form $\langle , \rangle_{\mathcal{E}}$ on $\mathcal{E}$, such that on the fiber over $\zeta \in \mathbb{D}(V)$, we set $\langle(\zeta, u), (\zeta, v)\rangle_{\mathcal{E}} = -\langle u, v\rangle$. We take the negative because $\langle , \rangle$ is negative-definite on $\zeta \cong \mathcal{E}_\zeta$, and Hermitian forms on bundles are required to be positive-definite. When the fiber is understood, we will just write u instead of (ζ, u).

The Hermitian form $\langle , \rangle_{\mathcal{E}}$ determines a canonical Hermitian form $\langle , \rangle_\vee$ on the dual bundle $\mathcal{E}^\vee$, and a Hermitian form $\langle , \rangle_\wedge$ on $\bigwedge^\bullet \mathcal{E} = \bigoplus_{k=0}^q \bigwedge^k \mathcal{E}$ by

$$\langle u_1 \wedge ... \wedge u_k, v_1 \wedge ... \wedge v_k\rangle_\wedge := \det\left[\left\langle u_i, v_j\right\rangle_{\mathcal{E}}\right],$$

and $\bigwedge^i \mathcal{E} \perp \bigwedge^j \mathcal{E}$ for $i \neq j$.

We also equip $\bigwedge^\bullet \mathcal{E}$ with a $\mathbb{Z}_2$-grading

$$\left(\bigwedge\nolimits^\bullet \mathcal{E}\right)^0 = \bigoplus_{k \text{ even}} \bigwedge\nolimits^k \mathcal{E}, \qquad \left(\bigwedge\nolimits^\bullet \mathcal{E}\right)^1 = \bigoplus_{l \text{ odd}} \bigwedge\nolimits^l \mathcal{E}.$$

For $v \in V$ we define a section $s_v : \mathbb{D} \to \mathcal{E}^\vee$ such that, for an open set $U \subseteq \mathbb{D}$, a section $\sigma : U \to \mathcal{E}$, and $\zeta \in \mathbb{D}$

$$s_v(\zeta)\sigma := \langle \sigma(\zeta), v \rangle .$$

Letting v_ζ and $v_\perp$ be the respective projections of v onto ζ and $\zeta^\perp$, we find that for a section $\sigma : \mathbb{D} \to \mathcal{E}$,

$$s_v(\zeta)\sigma = \langle \sigma(\zeta), v \rangle = \left\langle \sigma(\zeta), v_\zeta \right\rangle + \langle \sigma(\zeta), v_\perp \rangle = \left\langle \sigma(\zeta), v_\zeta \right\rangle = -\left\langle \sigma(\zeta), v_\zeta \right\rangle_{\mathcal{E}} = \left\langle \sigma(\zeta), -v_\zeta \right\rangle_{\mathcal{E}} .$$

Therefore $-v_\zeta$ is the vector in $\mathcal{E}_\zeta$ which represents s_v on $\mathcal{E}_\zeta^\vee$.
We define $\mathbb{D}_v$ to be the zero-locus of the section s_v. When $\langle v, v \rangle > 0$, the $\mathbb{D}_v$ are the special cycles relating to the arithmetic Siegel-Weil formula mentioned in the introduction.
More generally, for an r-tuple of vectors $\mathbf{v} = (v_1, ..., v_r) \in V^r$, we define a section $s_\mathbf{v} = (s_{v_1}, ..., s_{v_r})$ of $(\mathcal{E}^r)^\vee$ whose vanishing locus is $\mathbb{D}_\mathbf{v} := \cap_{i=1}^r \mathbb{D}_{v_i}$.
In order to define $s_\mathbf{v}$, we will need to define the *Koszul complex* $K(\mathbf{v})$. The Koszul complex is a general construction of homological algebra, originally introduced to define a cohomology theory for Lie algebras. We will only define it the particular case relevant to the discussion at hand. The Koszul complex $K(s_\mathbf{v})$ is the sequence $\bigwedge^{qr} \mathcal{E}^r \to ... \to \bigwedge^1 \mathcal{E}^r \to \mathcal{O}$ where $\mathcal{O}$ is the trivial line bundle, and the maps are given by extending the definition of $s_\mathbf{v}$ by setting $s_\mathbf{v}(c) = 0$ for all $c \in \mathbb{C}$, and for $\mathbf{u}_j = (u_{j,1}, ..., u_{j,r}) \in V^r$

$$s_\mathbf{v}(\mathbf{u}_1 \wedge ... \wedge \mathbf{u}_k) = \sum_{j=1}^{k} (-1)^{j+1} s_\mathbf{v}(\mathbf{u}_j)\mathbf{u}_1 \wedge ... \wedge \hat{\mathbf{u}}_j \wedge ... \wedge \mathbf{u}_k,$$

where the hat means that term is omitted.
We define $s_\mathbf{v}^*$ to be the adjoint of $s_\mathbf{v}$, and thus both $s_\mathbf{v}$ and $s_\mathbf{v}^*$ define odd endomorphisms of $\bigwedge^\bullet \mathcal{E}^r$ therefore $\sqrt{2\pi i}\left(s_\mathbf{v} + s_\mathbf{v}^*\right)$ is odd as well.
By theorem 1.1.16 of section 1.1, there exists a canonical connection ∇ on $\bigwedge^\bullet \mathcal{E}^r \to \mathbb{D}$ which is compatible with the metric. We define a superconnection $\nabla_\mathbf{v}$ on $\bigwedge^\bullet \mathcal{E}^r$ by $\nabla_\mathbf{v} = \nabla + \sqrt{2\pi i}\left(s_\mathbf{v} + s_\mathbf{v}^*\right)$.

Let $N \in \mathrm{End}\left(\bigwedge^\bullet \mathcal{E}^r\right)$ be the number operator which acts on $\bigwedge^k \mathcal{E}^r$ by multiplication by $-k$.
For a differential form α, we denote the component of degree m by $\alpha_{[m]}$, and set

$$\varphi^0(\mathbf{v}) := \sum_{k \geq 0} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}_s \left(e^{\nabla_\mathbf{v}^2}\right)_{[2k]}, \tag{3.1}$$

$$\nu^0(\mathbf{v}) := \sum_{k \geq 0} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}_s \left(N e^{\nabla_\mathbf{v}^2}\right)_{[2k]}. \tag{3.2}$$

Here tr_s is the supertrace defined in section 1.2. Finally, writing $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{j=1}^r \left\langle v_j, v_j \right\rangle$, we set

$$\varphi(\mathbf{v}) = e^{-\pi \langle \mathbf{v}, \mathbf{v} \rangle} \varphi^0(\mathbf{v}), \tag{3.3}$$

$$\nu(\mathbf{v}) = e^{-\pi \langle \mathbf{v}, \mathbf{v} \rangle} \nu^0(\mathbf{v}). \tag{3.4}$$

As proved in *Lemma 2.4.6* of [4], $\varphi, \nu \in \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D})$. By *Proposition 2.4.4* and *2.4.5* of [4], we have the following result

Proposition 3.2.1. *Given* $\mathbf{v} = (w_1, ..., w_r) \in V^r$,

1. $\varphi(v)_{[k]} = 0$ *for* $k < 2qr$,
2. $\varphi(\mathbf{v}) = \varphi(w_1) \wedge ... \wedge \varphi(w_r)$,
3. $\varphi(\mathbf{v})$ *is closed,*
4. $\forall g \in U(p,q)$, *we have* $g^*\varphi(gw_1, ..., gw_r) = \varphi(w_1, ..., w_r)$,
5. $\nu(v)_{[k]} = 0$, *for* $k < 2(q-1)$,
6. *for* $\nu_i(\mathbf{v}) = \nu(v_i) \wedge \varphi(v_1, ..., \widehat{v_i}, ..., v_r)$, *we have* $\nu(\mathbf{v}) = \sum_{i=1}^r \nu_i(\mathbf{v})$,
7. $\forall g \in U(p,q)$, *we find* $g^*\nu(gw_1, ..., gw_r) = \nu(w_1, ..., w_r)$.

Note that in particular, the $U(p,q)$-equivairance properties *4)* and *7)* imply that it's enough to understand the behaviour of φ and ν at any particular $\zeta \in \mathbb{D}$. From the definitions *2)* and *6)*, and applying *1)* and *5)*

$$\nu_r(\mathbf{v})_{[2(qr-1)]} = \nu(w_r)_{[2(q-1)]} \wedge \varphi(\mathbf{v})_{[2q(r-1)]} = \nu(w_r)_{[2(q-1)]} \wedge \varphi(w_1)_{[2q]} \wedge ... \wedge \varphi(w_{r-1})_{[2q]}.$$

Our goal for the next section is to prove that the degree $2(qr-1)$ component of ν_r generates an irreducible subrepresentation of the Weil representation. This has already been been established when $q = 1$ in [4]. We will obtain partial results for several small values of q, by establishing bases cases through computation, and applying an inductive argument. The general base case(s) for arbitrary q remain unknown.

3.3 Properties of the Forms

The Restriction Property

In this section we will establish some results relating the forms φ and ν defined on $\mathbb{D}(V)$ to the corresponding forms on $\mathbb{D}(W)$ for a certain proper subspace W of V. This will allow us to apply inductive arguments on the dimension of V, to obtain various properties of φ and ν.
Note that for $w \in V$, the special cycles $\mathbb{D}_w$ of the previous section are the sets

$$\mathbb{D}_w = \{\zeta \in \mathbb{D} : \zeta \perp w\}.$$

Since $\dim_{\mathbb{C}} \zeta = q$ for all $\zeta \in \mathbb{D}$, and q is the largest possible dimension of a negative-definite subspace of V, it must be the case that $\mathbb{D} = \emptyset$ when $\langle w, w\rangle < 0$. If $w = \vec{0}$, then $\mathbb{D}_w = \mathbb{D}$, and if $w \neq \vec{0}$ but $\langle w, w\rangle = 0$, then $\mathbb{D}_w = \emptyset$. Thus we will restrict to the case when $\langle w, w\rangle > 0$, whence $\mathbb{D}_w$ is precisely the set of q-dimensional negative-definite subspaces of the $(p-1+q)$-dimensional vector space $w^\perp = \left(\mathrm{span}_{\mathbb{C}} w\right)^\perp$, with respect to the Hermitian form $\langle,\rangle_w := \langle,\rangle|_{w^\perp}$, having signature $(p-1,q)$. Therefore the map

$$\begin{aligned} \iota : \mathbb{D}(w^\perp) &\to \mathbb{D}_w \\ \zeta &\mapsto \zeta \end{aligned}$$

is a bijection. The $\mathbb{D}_w$ are locally components of the special cycles relating to the conjectured arithmetic Siegel-Weil formula mentioned in the introduction.

In particular, given a standard basis $\{v_1,...,v_{p+q}\}$ of V, the vectors $\{v_1,...,\hat{v}_n,...,v_{p+q}\}$ are a standard basis for $v_n^\perp$. As for any $\zeta \in \mathbb{D}_{v_n}$ we have $\zeta \perp v_n$, the matrix corresponding to ζ in the standard basis has the form

$$\begin{pmatrix} \zeta_{1,1} & \cdots & \zeta_{1,q} \\ \cdots & & \cdots \\ \zeta_{n-1,q} & \cdots & \zeta_{n-1,q} \\ 0 & ... & 0 \\ \zeta_{n,1} & \cdots & \zeta_{n,q} \\ \cdots & & \cdots \\ \zeta_{p,1} & \cdots & \zeta_{p,q} \end{pmatrix}.$$

Using the coordinates for $\mathbb{D}(v_n^\perp)$ with respect to the standard basis $\{v_1,...,\hat{v}_n,...,v_{p+q}\}$ as described in the previous section, the inclusion map $\iota : \mathbb{D}(v_n^\perp) \hookrightarrow \mathbb{D}$ is given by

$$\iota \begin{pmatrix} \xi_{1,1} & \cdots & \xi_{p,q} \\ \cdots & & \cdots \\ \xi_{p-1,q} & \cdots & \xi_{p-1,q} \end{pmatrix} = \begin{pmatrix} \xi_{1,1} & \cdots & \xi_{1,q} \\ \cdots & & \cdots \\ \xi_{n-1,q} & \cdots & \xi_{n-1,q} \\ 0 & ... & 0 \\ \xi_{n,1} & \cdots & \xi_{n,q} \\ \cdots & & \cdots \\ \xi_{p-1,1} & \cdots & \xi_{p-1,q} \end{pmatrix}. \tag{3.5}$$

The above culminates in a useful lemma for the following section.

Lemma 3.3.1. *Let V be a $(p+1,q)$-vector space with standard basis $\{v_1,...,v_{p+1}\}$, take $m \in \mathbb{N}$ such that $p+1 \geq 2m+1$, and consider the coordinates for $\mathbb{D} = \mathbb{D}(V)$ described in section 3.1, with respect to the standard basis.*
Then for the basis $\left\{ \frac{\partial}{\partial \zeta_{i,j}} : 1 \leq i \leq p, 1 \leq j \leq q \right\}$ of $T_{\zeta_0}\mathbb{D}$, and for each

$$u = \frac{\partial}{\partial \zeta_{i_1,j_1}} \wedge ... \frac{\partial}{\partial \zeta_{i_m,j_m}} \wedge \frac{\partial}{\partial \bar{\zeta}_{k_1,l_1}} \wedge ... \wedge \frac{\partial}{\partial \bar{\zeta}_{k_m,l_m}} \in \bigwedge^{m,m} T_{\zeta_0}\mathbb{D},$$

there exists $w \in V$, and $u_w \in \bigwedge^{m,m} T_{\zeta_0}\mathbb{D}(w)$ such that $\iota_*(u_w) = u$.

Proof. Suppose that $p+1 \geq 2m+1 \iff p \geq 2m$. Since u is the wedge product of $2m$ vectors, and $(p+1) \geq 2m+1 > 2m$, there must exist some number
$n \in \{1, ..., p+1\}$ such that for all $1 \leq s \leq m$, we have $i_s, k_s \neq n$. Choosing the coordinates described in section 3.1 for $\mathbb{D}(v_n^{\perp})$ with respect to $\{v_1, ..., \hat{v}_n, ..., v_{p+q}\}$, by the description of $\iota : \mathbb{D}(v_n^{\perp}) \hookrightarrow \mathbb{D}$ in eq. (3.5), the push-forward $\iota_* : T_{\zeta_0}\mathbb{D}(v_n^{\perp}) \to T_{\zeta_0}\mathbb{D}$ acts by

$$\iota_*\left(\frac{\partial}{\partial \xi_{i,j}}\right) = \begin{cases} \frac{\partial}{\partial \zeta_{i,j}}, & i < n \\ \frac{\partial}{\partial \zeta_{i+1,j}}, & i \geq n \end{cases}$$

For $1 \leq p \leq m$, set

$$i'_k = \begin{cases} i_p, & i_p < n \\ i_p - 1, & i_p \geq n \end{cases}$$

$$k'_p = \begin{cases} k_p, & k_p < n \\ k_p - 1, & k_p \geq n \end{cases}$$

Then

$$u_w := \frac{\partial}{\partial \xi_{i'_1,j_i}} \wedge ... \wedge \frac{\partial}{\partial \xi_{i'_m,j_m}} \wedge \frac{\partial}{\partial \bar{\xi}_{k'_1,l_1}} \wedge ... \wedge \frac{\partial}{\partial \bar{\xi}_{k'_m,l_m}} \in \bigwedge^{m,m} T_{\zeta_0}\mathbb{D}(v_n^{\perp}),$$

where

$$\iota_*(u_w) = \frac{\partial}{\partial \zeta_{i_1,j_i}} \wedge ... \wedge \frac{\partial}{\partial \zeta_{i_m,j_m}} \wedge \frac{\partial}{\partial \bar{\zeta}_{k_1,l_1}} \wedge ... \wedge \frac{\partial}{\partial \bar{\zeta}_{k_m,l_m}} = u.$$

□

Returning to the general case of $w \in V$ with $\langle w, w\rangle > 0$, we will write $\pi : \mathcal{E} \to \mathbb{D}(V)$ and $\pi_w : \mathcal{E}_w \to \mathbb{D}(w^{\perp})$ for the tautological bundles. Then,

$$\begin{aligned} \iota^*\mathcal{E} = \{(\zeta, e) \in \mathbb{D}(w^{\perp}) \times \mathcal{E} : \iota(\zeta) = \pi(e)\} &= \left\{(\zeta, (\zeta', v)) \in \mathbb{D}(w^{\perp}) \times (\mathbb{D} \times V) : \zeta = \pi(\zeta', v), v \in \zeta'\right\} \\ &= \left\{(\zeta, (\zeta', v)) \in \mathbb{D}(w^{\perp}) \times (\mathbb{D} \times V) : \zeta = \zeta', v \in \zeta'\right\} \\ &= \left\{(\zeta, (\zeta, v)) \in \mathbb{D}(w^{\perp}) \times (\mathbb{D} \times V) : v \in \zeta\right\}. \end{aligned}$$

Thus the map $(\zeta, v) \mapsto (\zeta, (\zeta, v))$ is an isomorphism with the tautological bundle
$\mathcal{E}_w = \{(\zeta, v) \in \mathbb{D}(w^{\perp}) \times V : v \in \zeta\}$ on $\mathbb{D}(w^{\perp})$, and so we will take our model of the tautological bundle on $\mathbb{D}(w^{\perp})$ *to be* $\iota^*\mathcal{E}$. When the context is clear, we will just write u both in place of (ζ, u) and $(\zeta, (\zeta, u))$.
Note that for the Hermitian form $\langle , \rangle_{\mathcal{E}_w}$ on $\mathcal{E}_w$ we have

$$\langle u, v\rangle_{\mathcal{E}_w} := -\langle u, v\rangle_w = -\langle u, v\rangle,$$

since $u, v \in \zeta \subseteq w^{\perp}$. Therefore the inner product $\langle , \rangle_*$ on $\iota^*\mathcal{E}$ representing $\langle , \rangle_{\mathcal{E}_w}$ is

$$\langle u, v\rangle_* := -\langle u, v\rangle = \langle u, v\rangle_{\mathcal{E}_w}.$$

Given an open set $U \subseteq \mathbb{D}$ and frame $\{e_1, ..., e_q\}$ over U, the set $\{\iota^* e_1, ..., \iota^* e_q\}$ is a frame over $\iota^{-1}(U)$. The section $\zeta \mapsto \left\langle \iota^* e_j(\zeta), \iota^* e_k(\zeta) \right\rangle$ is just the pullback of the section $\zeta \mapsto \left\langle e_j(\zeta), e_k(\zeta) \right\rangle_{\mathcal{E}}$ thus the matrix H_* whose (j,k)th entry entry is $\left\langle \iota^* e_j, \iota^* e_k \right\rangle_*$ is just the entry-wise pullback of the matrix H whose (j,k)th entry is $\left\langle e_j, e_k \right\rangle$.
By 1.1.16, over U the canonical connection compatible with $\langle , \rangle$ is given by $\nabla = d_{\mathbb{D}} + \theta$ where $\theta = H^{-1} \partial H$, and over $\iota^{-1}(U)$, the canonical connection compatible with $\langle , \rangle_{\mathcal{E}_w}$ (that is, $\langle , \rangle_*$) can be written as $\nabla_* = d_* + \theta_*$, where d_* is the exterior derivative on $\mathbb{D}(w^\perp)$, and $\theta_* = H_*^{-1} \partial H_*$. Therefore

$$\nabla_* = d_* + \theta_* = d_* + H_*^{-1} \partial H_* = d_* + (\iota^* H)^{-1} \partial (\iota^* H) = \iota^* (d_{\mathbb{D}} + H^{-1} \partial H) = \iota^* \nabla.$$

Thus the pullback connection $\iota^* \nabla$, is the canonical connection compatible with $\langle , \rangle_*$. Furthermore the Hermitian form on $\bigwedge^\bullet \mathcal{E}_w \cong \bigwedge^\bullet \iota^* \mathcal{E} \cong \iota^* \bigwedge \mathcal{E}$ is the pullback of the Hermitian form on $\bigwedge^\bullet \mathcal{E}$ induced by $\langle , \rangle_{\mathcal{E}}$.
Given $v' \in w^\perp$, we define a section $s_{v'} : \mathbb{D}(w^\perp) \to (\iota^* \mathcal{E})^\vee$ in the same way we defined s_v on $\mathbb{D} \to \mathcal{E}^\vee$, that is for a section $\sigma : \mathbb{D}(w^\perp) \to (\iota^* \mathcal{E})^\vee$ and $\zeta \in \mathbb{D}(w^\perp)$,

$$\bar{s}_{v'}(\zeta)\sigma = \left\langle \sigma(\zeta), v' \right\rangle_* .$$

We define $\bar{s}^*_{v'}$ to be the adjoint of $\bar{s}_{v'}$. Now, given $v' \in w_\perp$, we define a superconnection $\bar{\nabla}_{v'}$ on $\iota^* \mathcal{E} \to \mathbb{D}(w^\perp)$ in the same way we defined ∇_v on $\mathcal{E}$, namely,

$$\bar{\nabla}_{v'} = \iota^* \nabla + \sqrt{2\pi i} \left(\bar{s}_{v'} + \bar{s}^*_{v'} \right).$$

For $v \in V$, we write $v_\perp$ for the projection of v onto $w^\perp$. Since for any $\zeta \in \mathbb{D}(w^\perp)$ the fiber $(\iota^* \mathcal{E})_\zeta$ is identified with $\zeta \subseteq w^\perp$, given any section $\sigma : \mathbb{D}(w^\perp) \to \iota^* \mathcal{E}$,

$$\bar{s}_{v_\perp}(\zeta)\sigma = - \langle \sigma(\zeta), v_\perp \rangle_* = - \langle \sigma(\zeta), v \rangle = s_v(\iota(\zeta))\sigma = (s_v \circ \iota)(\zeta)\sigma.$$

Therefore $\bar{s}_{v_\perp}$ is the pullback section $\iota^* s_v = s_v \circ \iota$. It follows that the adjoint section with respect to the Koszul complex $K(v_\perp)$ on $\mathbb{D}(w^\perp)$ is given by $\bar{s}^*_{v_\perp} = \iota^* s^*_v$. Therefore,

$$\bar{\nabla}_{v_\perp} = \iota^* \nabla + \sqrt{2\pi i} \left(\bar{s}_{v_\perp} + \bar{s}^*_{v_\perp} \right) = \iota^* \nabla + \sqrt{2\pi i} \left(\iota^* s_v + \iota^* s^*_v \right) = \iota^* \left(\nabla + \sqrt{2\pi i} \left(s_v + s^*_v \right) \right) = \iota^* \nabla_v.$$

By eq. (1.9) of section 1.2, we know $\left(\iota^* \nabla_v \right)^2 = \iota^* \left(\nabla_v \right)^2$, and thus by the above

$$\exp\left(\bar{\nabla}^2_{v_\perp} \right) = \exp\left(\left(\iota^* \nabla_v \right)^2 \right) = \exp\left(\iota^* \nabla^2_v \right) = \iota^* \exp\left(\nabla^2_v \right).$$

Writing φ^0_w, ν^0_w for the forms on $\mathbb{D}(w^\perp)$ corresponding to φ^0 and ν^0, we obtain

$$\varphi^0_w(v_\perp) = \sum_{k \geq 0} \left(\frac{i}{2\pi} \right)^k \mathrm{tr}_s \left(\exp\left(\bar{\nabla}^2_{v_\perp} \right) \right)_{[2k]} = \sum_{k \geq 0} \left(\frac{i}{2\pi} \right)^k \mathrm{tr}_s \left(\iota^* \exp\left(\nabla^2_v \right) \right)_{[2k]} = \iota^* \left(\sum_{k \geq 0} \left(\frac{i}{2\pi} \right)^k \mathrm{tr}_s \left(\exp\left(\nabla^2_v \right) \right)_{[2k]} \right)$$
$$= \iota^* \left(\varphi^0(v) \right).$$

Since the grading on $\bigwedge^\bullet \iota^*\mathcal{E} \cong \iota^* \bigwedge^\bullet \mathcal{E}$ is induced from the grading of $\bigwedge^\bullet \mathcal{E}$, the number operator of $\iota^* \bigwedge^\bullet \mathcal{E}$ is $\iota^* N$, where N is the number operator on $\mathcal{E}$, and thus

$$\begin{aligned} \nu_w^0(v_\perp) = \sum_{k\geq 0} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}_s \left((\iota^* N)\exp\left(\nabla^2_{v_\perp}\right)\right)_{[2k]} &= \sum_{k\geq 0} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}_s \left((\iota^* N)\iota^*\exp\left(\nabla^2_{v}\right)\right)_{[2k]} \\ &= \iota^* \left(\sum_{k\geq 0} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}_s \left(N\exp\left(\nabla^2_{v}\right)\right)_{[2k]}\right) \\ &= \iota^*\left(\nu^0(v)\right). \end{aligned}$$

Therefore, we obtain *Lemma 2.4.2* on Page 17 of [4],

Proposition 3.3.2. *Let $w \in V$ such that $\langle w, w\rangle > 0$, and φ_w, ν_w be the forms on $\mathbb{D}(w^\perp)$ defined analogously to φ and ν on $\mathbb{D}$. Writing V as the direct sum $V = span\{w\} \oplus w^\perp$, we write the respective components of $v \in V$ as $v = v_w + v_\perp$. Then, for the natural inclusion $\iota : \mathbb{D}(w^\perp) \hookrightarrow \mathbb{D}$*

$$\iota^*(\varphi(v)) = e^{-\pi\langle v_w, v_w\rangle}\varphi_w(v_\perp)$$
$$\iota^*(\nu(v)) = e^{-\pi\langle v_w, v_w\rangle}\nu_w(v_\perp).$$

Proof. This follow from the preceding discussion, and the eq. (3.4). □

3.4 Highest Weight Vectors

This section contains the main theorems of this thesis. We will use the properties of the immersed submanifolds $\mathbb{D}_w$ for $\langle w, w\rangle > 0$, the behaviour under the inclusions $\iota : \mathbb{D}(w^\perp) \hookrightarrow \mathbb{D}$, and the technical results of 2.2 to obtain inductive results regarding when the form $\nu_r(\mathbf{v})_{[2(q-1)]}$ is a highest weight vector.

Throughout, by a (p, q)-vector space we will mean a
$(p+q)$-dimensional $\mathbb{C}$-vector space with a Hermitian form of signature (p, q). Recall that for a differential form α, by $\alpha_{[k]}$ we mean the kth degree component.

Given such a (p, q)-vector space V, the action of the Weil representation $\omega : \mathfrak{gl}_{2r}(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}))$ can be extended to

$$\omega \otimes 1 : \mathfrak{k}_r \to \mathrm{End}\left(\mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D})\right)$$
$$\omega \otimes 1 : \mathfrak{k}_r \to \mathrm{End}\left(\mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}(w^\perp))\right).$$

Since the actions of these extensions are entirely determined by the action on the first factor, we will use the same notation for both as it should be clear from context. We will also drop the tensor notation, and simply use ω.

Our ultimate goal is to prove that, writing $\mathbf{v} = (w_1, ..., w_r)$, the form

$$\nu_r(w_1, ..., w_r)_{[2(qr-1)]} = \nu(w_r) \wedge \varphi(w_1) \wedge ... \wedge \varphi(w_{r-1})_{[2(qr-1)]} \in \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}(V)),$$

constructed in the previous section, generates an irreducible representation under the action of the Lie subalgebra $\mathfrak{k}_r = \mathfrak{k}_{ss}$ of $\mathfrak{gl}_{2r}(\mathbb{C}) \cong \mathfrak{u}(r,r) \otimes_{\mathbb{R}} \mathbb{C}$, via the Weil representation. We will apply the theory of highest weight vectors from section 2.1 in order to demonstrate this. Thus we need to demonstrate that $v_r(\mathbf{v})_{[2(qr-1)]}$ is an eigenvector for the action of the Cartan subalgebra $\mathfrak{h}_r$ of $\mathfrak{k}_r$. Recall from section 2.1 that $\mathfrak{h}_r$ has a basis $\{E_{ss} - E_{2r,2r} : 1 \leq s < 2r\}$. We define the weight $\mu_r \in \mathfrak{h}_r^*$ by

$$\mu_r(E_{ss} - E_{2r,2r}) = \begin{cases} (p+q-1), & 1 \leq s < r \\ (p+q-2), & s = r \\ -1, & r+1 \leq s < 2r \end{cases} \tag{3.6}$$

In particular, we will show that $v_r(\mathbf{v})_{[2(qr-1)]}$ has weight μ_r.

First, we will use induction on p in the $r = 1$ and $r = 2$ cases, extending some of the results from section 2.2. These results assume certain base cases that are not known in general, but for which we have established several cases by computation from the code displayed at the end of this document.

In order to lift these results to higher values of r, we will need to rely on the following result.

Theorem 3.4.1 (Kudla-Millson)**.** *For all (p,q)-vector spaces and $r \geq 1 \leq p$ (or $r = p+1$ when $q = 1$) the form $\varphi(w_1) \wedge ... \wedge \varphi(w_r)_{[2qr]} \in \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D})$ has weight $\lambda_r \in \mathfrak{h}_r^*$ given by*

$$\lambda_r(E_{ss} - E_{2r,2r}) = \begin{cases} p+q, & 1 \leq s \leq r \\ 0, & r+1 \leq s < 2r \end{cases}$$

for which it is a highest weight vector.

Proof. Page 364 of *Theorem 3.1* of [6]. □

When $r = 1$, for the form $\varphi_{[2q]} \in \mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})$, and the Weil representation $\omega : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D}))$

$$\omega\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(v)_{[2q]} = \left(\frac{p+q}{2}\right) \varphi(v)_{[2q]},$$
$$\omega\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \varphi(v)_{[2q]} = -\left(\frac{p+q}{2}\right) \varphi(v)_{[2q]}.$$

For the first result, we'll also need to extend the action of the pullback

$$1 \otimes \iota^* : \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}) \to \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}(w^\perp)),$$

where again we will drop the tensor notation for simplicity.

Lemma 3.4.2. *For any $X \in \mathfrak{k}_r$, we have $(1 \otimes \iota^*)\circ(\omega \otimes 1)(X) = (\omega \otimes 1)(X)\circ(1 \otimes \iota^*)$.*

Proof. Let $\Phi(\mathbf{v}) \otimes \gamma \in \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D})$ and $X \in \mathfrak{k}_r$, then

$$\begin{aligned}(1 \otimes \iota^*)\circ(\omega \otimes 1)(X)\Phi(\mathbf{v}) \otimes \gamma = (1 \otimes \iota^*)[(\omega(X)\Phi(\mathbf{v})) \otimes \gamma] = (\omega(X)\Phi(\mathbf{v})) \otimes \iota^*\gamma &= (\omega \otimes 1)(X)(\Phi(\mathbf{v}) \otimes \iota^*\gamma)\\ &= (\omega \otimes 1)(X)\circ(1 \otimes \iota^*)(\Phi(\mathbf{v}) \otimes \gamma).\end{aligned}$$

□

As a first step to showing $\nu_r(\mathbf{v})_{[2(qr-1)]} \in \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D})$ has weight μ_r in general, we will demonstrate (assuming certain base cases) by induction on p, that $\nu_{[2(q-1)]}$ has weight μ_1. More specifically, we will investigate the particular action of $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C})$ on $\nu(v)_{[2(q-1)]}$ under the Weil representation $\omega : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D}))$, which will be useful in lifting these results to the cases when $r > 1$.

Proposition 3.4.3. *Consider some fixed q such that $q \geq 1$, and let $p \geq 2(q-1)$. Suppose that for every (p,q)-vector space V, the Weil representation $\omega : \mathfrak{gl}_2(\mathbb{C}) \to End\,(\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D}))$, and $\nu(v)_{[2(q-1)]} \in \mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D}(V))$ we find*

$$\omega\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu(v)_{[2(q-1)]} = \left(\frac{p+q-2}{2}\right)\nu(v)_{[2(q-1)]},$$
$$\omega\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\nu(v)_{[2(q-1)]} = -\left(\frac{p+q-2}{2}\right)\nu(v)_{[2(q-1)]}.$$

Then, for all (p',q)-vector spaces V' with $p' \geq p$, for the form $\nu(v)'_{[2(q-1)]} \in \mathcal{S}(V') \otimes \mathcal{A}^\bullet(\mathbb{D}(V'))$ and the Weil representation $\omega' : \mathfrak{gl}_2(\mathbb{C}) \to End\left(\mathcal{S}(V') \otimes \mathcal{A}^\bullet(\mathbb{D}(V'))\right)$, we have

$$\omega'\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu(v)'_{[2(q-1)]} = \left(\frac{p'+q-2}{2}\right)\nu(v)'_{[2(q-1)]},$$
$$\omega'\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\nu(v)'_{[2(q-1)]} = -\left(\frac{p'+q-2}{2}\right)\nu(v)'_{[2(q-1)]}.$$

Proof. Supposing that the hypothesis holds, let V be a $(p+1,q)$-vector space, and writing $\mathbb{D} = \mathbb{D}(V)$, consider $\nu(v)_{[2(q-1)]} \in \mathcal{S}(V) \otimes \mathcal{A}^{q-1,q-1}(\mathbb{D})$. For $\zeta_0 \in \mathbb{D}$, if we can show that $\omega\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu(v)_{[2(q-1)]}$ and $((p+q-2)/2)\nu(v)_{[2(q-1)]}$ agree on any basis of $\bigwedge^{m,m} T_{\zeta_0}\mathbb{D}$, then by linearity they must be equal.
For a standard basis $\{v_1, ..., v_{p+q}\}$ of V, and the coordinates for $\mathbb{D}$ described in section 3.1,

$$\left\{\frac{\partial}{\partial\zeta_{i_1,j_1}} \wedge \dots \frac{\partial}{\partial\zeta_{i_m,j_m}} \wedge \frac{\partial}{\partial\bar{\zeta}_{k_1,l_1}} \wedge \dots \wedge \frac{\partial}{\partial\bar{\zeta}_{k_m,l_m}} \;:\; 1 \leq i_\bullet, k_\bullet \leq p, 1 \leq j_\bullet, l_\bullet \leq q\right\},$$

is a basis for $\bigwedge^{m,m} T_{\zeta_0}\mathbb{D}$. By lemma lemma 3.3.1, for any one of the above basis vectors u, there exists $w \in V$ where $\zeta_0 \in \mathbb{D}(w^\perp)$, and some $u_w \in \bigwedge^{m,m} T_{\zeta_0}\mathbb{D}(w^\perp)$, such that $\iota_*(u_w) = u$, for the inclusion map $\iota : \mathbb{D}(w^\perp) \hookrightarrow \mathbb{D}$.
Write $\omega : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}\,(\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D}))$, and $\omega_w : \mathfrak{k}_1 \to \mathrm{End}\left(\mathcal{S}(w^\perp) \otimes \mathcal{A}^\bullet(\mathbb{D}(w^\perp))\right)$, for the Weil representation.

Note that $w^{\perp}$ is a (p,q)-vector space with respect to the restricted Hermitian form $\langle , \rangle|_{w^{\perp}}$. Writing $\langle w \rangle := \mathrm{span}_{\mathbb{C}} w$, for the orthogonal direct sum $V = \langle w \rangle \oplus w^{\perp}$, we write the respective components of $v \in V$ as $v = v_w + v_{\perp}$, and the orthogonal projections $\pi_w : V \to \langle w \rangle$ and $\pi_{\perp} : V \to w^{\perp}$. We can extend the pullbacks
$\pi_w^* \mathcal{S}(\langle w \rangle) \to \mathcal{S}(V)$ and $\pi_{\perp}^* \otimes 1 : \mathcal{S}(w^{\perp}) \to \mathcal{S}(V)$ to

$$\pi_w^* \otimes 1 : \mathcal{S}(\langle w \rangle) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp})) \to \mathcal{S}(V) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp}))$$
$$\pi_{\perp}^* \otimes 1 : \mathcal{S}(w^{\perp}) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp})) \to \mathcal{S}(V) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp}))$$

where again, we will drop the tensor notation and simply write π_w^* and $\pi_{\perp}^*$ as there's no chance of confusion.
By proposition 3.3.2 $\iota^* \nu(v) = e^{-\pi\langle v_w, v_w \rangle} \nu_w(v_{\perp})$, and note that $e^{-\pi\langle v_w, v_w\rangle} \in \mathcal{S}(\langle w \rangle)$ and
$\nu_w(w^{\perp}) \in \mathcal{S}(w^{\perp}) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp}))$, where

$$e^{-\pi\langle v_w, v_w\rangle} \nu_w(v_{\perp}) = \pi_w^* \left(e^{-\pi\langle v_w, v_w\rangle} \right)(v) \pi_{\perp}^*(\nu_w)(v).$$

Therefore $\iota^* \nu(v) = \pi_w^* \left(e^{-\pi\langle v_w, v_w\rangle} \right) \pi_{\perp}^*(\nu_w)(v)$.
Writing the Weil representation

$$\omega_w : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(\langle w \rangle) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp})))$$
$$\omega_{\perp} : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(w^{\perp}) \otimes \mathcal{A}^{\bullet}(\mathbb{D}(w^{\perp})))$$

by the induction hypothesis, we have

$$\omega_{\perp} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu_w(v_{\perp})_{[2(q-1)]} = \left(\frac{p+q-2}{2} \right) \nu_w(v_{\perp})_{[2(q-1)]}.$$

The result of proposition 2.2.5 carries over to $\omega \otimes 1, \omega_w \otimes 1$, and $\omega_{\perp} \otimes 1$, and thus

$$\begin{aligned} \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu(v)_{[2(q-1)]} u &= \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu(v)_{[2(q-1)]} \iota_*(u_w) \\ &= \iota^* \left(\omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu(v)_{[2(q-1)]} \right) u_w \\ &= \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\iota^* \nu(v)_{[2(q-1)]} \right) u_w, \end{aligned}$$

by lemma 3.4.2,

$$\begin{aligned} \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu(v)_{[2(q-1)]} u &= \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\pi_w^* \left(e^{-\pi\langle v_w, v_w\rangle} \right) \pi_{\perp}^*(\nu_w) \right) u_w \\ &= \pi_w^* \left(\omega_w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-\pi\langle v_w, v_w\rangle} \right) \pi_{\perp}^* \nu_w(v_{\perp})_{[2(q-1)]} \\ &\quad + \pi_w^* \left(e^{-\pi\langle v_w, v_w\rangle} \right) \pi_{\perp}^* \left(\omega_{\perp} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu_w(v_{\perp})_{[2(q-1)]} \right) u_w, \end{aligned}$$

By proposition 2.2.5,

$$\omega\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu(v)_{[2(q-1)]}u = \frac{1}{2}\pi_w^*\left(e^{-\pi\langle v_w,v_w\rangle}\right)\pi_\perp^*\nu_w(v_\perp)_{[2(q-1)]}u_w$$
$$+\pi_w^*\left(e^{-\pi\langle v_w,v_w\rangle}\right)\pi_\perp^*\left(\left(\frac{p+q-2}{2}\right)\nu_w(v_\perp)_{[2(q-1)]}\right)u_w,$$

By lemma 2.2.4 and the induction hypothesis,

$$\begin{aligned}\omega\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu(v)_{[2(q-1)]}u &= \left(\frac{(p+1)+q-2}{2}\right)\pi_w^*\left(e^{-\pi\langle v_w,v_w\rangle}\right)\pi_\perp^*\nu_w(v_\perp)_{[2(q-1)]}u_w\\ &= \left(\frac{(p+1)+q-2}{2}\right)\iota^*\nu(v)_{[2(q-1)]}u_w\\ &= \left(\frac{(p+1)+q-2}{2}\right)\nu(v)_{[2(q-1)]}\iota_*(u_w)\\ &= \left(\frac{(p+1)+q-2}{2}\right)\nu(v)_{[2(q-1)]}u.\end{aligned}$$

By the arbitrary choice of $u\in T_{\zeta_0}\mathbb{D}$, we conclude that $\omega\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu_{[2(q-1)]} = \left(\frac{(p+1)+q-2)}{2}\right)\nu_{[2(q-1)]}$. By the arbitrary choice of $(p+1,q)$-vector space V, and the principle of induction, we conclude that the result holds for all (p',q)-vector spaces with $p'\geq p$.

The proof that $\omega\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu_{[2(q-1)]} = -\left(\frac{(p+1)+q-2}{2}\right)\nu_{[2(q-1)]}$ is entirely similar. ☐

We now use the results of the $r=1$ case, to extend to the case when $r>1$.

Corollary 3.4.4. *Let V be a (p,q)-vector space, and consider the form $\nu_{[2(q-1)]}\in\mathcal{S}(V)\otimes\mathcal{A}^\bullet(\mathbb{D})$ and the Weil representation $\omega_1:\mathfrak{gl}_2(\mathbb{C})\to End\left(\mathcal{S}(V)\otimes\mathcal{A}^\bullet(\mathbb{D})\right)$. If*

$$\omega_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\nu(v)_{[2(q-1)]} = \left(\frac{p+q-2}{2}\right)\nu(v)_{[2(q-1)]},$$
$$\omega_1\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\nu(v)_{[2(q-1)]} = -\left(\frac{p+q-2}{2}\right)\nu(v)_{[2(q-1)]},$$

then for all $r\geq 1$, the form $\nu_r(\mathbf{v})_{[2(qr-1)]}\mathcal{S}(V^r)\otimes\mathcal{A}^\bullet(\mathbb{D})$, has weight $\mu_r\in\mathfrak{h}_r^*$ (eq. (3.6)) for the Weil representation $\omega:\mathfrak{k}_r\to End\left(\mathcal{S}(V^r)\otimes\mathcal{A}^\bullet(\mathbb{D})\right)$.

Proof. Let $\mathbf{v}=(\vec{w}_1,...,\vec{w}_r)\in V^r$, and recall from proposition 3.2.1 that

$$\nu_r(\mathbf{v})_{[2(qr-1)]} = \nu(w_r)_{[2(q-1)]}\wedge\varphi(w_1,...,w_{r-1})_{[2q(r-1)]} = \nu(w_r)_{[2(q-1)]}\wedge\varphi(w_1)_{[2q]}\wedge...\varphi(w_{r-1})_{[2q]},$$

where $\nu(w_r)_{[2(q-1)]} \in \mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})$, for each $1 \leq s \leq r-1$ we have $\varphi(w_s)_{[2q]} \in \mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})$, and $\varphi(w_1, ..., w_{r-1})_{[2q(r-1)]} \in \mathcal{S}(V^{r-1}) \otimes \mathcal{A}^\bullet(\mathbb{D})$. Letting $\mathbf{d} = \{1, ..., r-1\}$, for the maps

$$\begin{aligned} T_{\{r\}} &: V^r \to V \\ T_{\mathbf{d}} &: V^r \to V^{r-1} \end{aligned}$$

described preceding proposition 2.2.7, we extend the pullbacks to

$$\begin{aligned} T^*_{\{r\}} \otimes 1 &: \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}) \to \mathcal{S}(V) \otimes \mathbb{D} \\ T^*_{\mathbf{d}} \otimes 1 &: \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}) \to \mathcal{S}(V^{r-1}) \otimes \mathcal{A}^\bullet(\mathbb{D}) \end{aligned}$$

As with previous extensions, we drop the tensor notation. Now we write

$$\begin{aligned} \nu_r(\mathbf{v})_{[2(qr-1)]} &= \nu(w_r)_{[2(q-1)]} \wedge \varphi(w_1, ..., w_{r-1})_{[2q(r-1)]} \\ &= \left(T^*_{\{r\}}\nu\right)(\mathbf{v})_{[2(q-1)]} \wedge \left(T^*_{\mathbf{d}}\varphi\right)(\mathbf{v})_{[2q(r-1)]} \end{aligned}$$

By proposition 2.2.7, for $E_{rr} \in \mathfrak{k}_r$

$$\begin{aligned} \omega(E_{rr})\nu_r(\mathbf{v})_{[2(qr-1)]} &= \omega(E_{rr})\left(T^*_{\{r\}}\nu\right)(\mathbf{v})_{[2(q-1)]} \wedge \left(T^*_{\mathbf{d}}\varphi\right)(\mathbf{v})_{[2q(r-1)]} \\ &= \left(T^*_{\{r\}}\omega_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\nu\right)(\mathbf{v})_{[2(q-1)]} \wedge \left(T_{\mathbf{d}}\varphi\right)(\mathbf{v})_{[2q(r-1)]} \\ &= \left(T^*_{\{r\}}\left(\frac{p+q-2}{2}\right)\nu\right)(\mathbf{v})_{[2(q-1)]} \wedge \left(T^*_{\mathbf{d}}\varphi\right)(\mathbf{v})_{[2q(r-1)]} \\ &= \left(\frac{p+q-2}{2}\right)\left(T^*_{\{r\}}\nu\right)(\mathbf{v})_{[2(q-1)]} \wedge \left(T^*_{\mathbf{d}}\varphi\right)(\mathbf{v})_{[2q(r-1)]} \\ &= \left(\frac{p+q-2}{2}\right)\nu_r(\mathbf{v})_{[2(qr-1)]}. \end{aligned}$$

The proof that for $E_{2r,2r} \in \mathfrak{k}_r$,

$$\omega(E_{2r,2r})\nu_r(\mathbf{v})_{[2(qr-1)]} = -\left(\frac{p+q-2}{2}\right)\nu_r(\mathbf{v})_{[2(qr-1)]},$$

is entirely similar to the above.

For $1 \leq s < r$,

$$\nu_r(\mathbf{v})_{[2(qr-1)]} = \nu(w_r) \wedge \varphi(w_1) \wedge ... \wedge \varphi(w_{r-1})_{[2(qr-1)]} = \varphi(w_s) \wedge \nu(w_r) \wedge \varphi(w_1) \wedge ... \wedge \widehat{\varphi(w_s)} \wedge ...\varphi(w_{r-1})_{[2(qr-1)]},$$

and since $\varphi(w_s)_{[2q]} \in \mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})$ and

$$\nu^{r-2}(w_1, ..., \hat{w}_s, ..., w_r) := \nu(w_r) \wedge \varphi(w_1) \wedge ... \wedge \widehat{\varphi(w_s)} \wedge ...\varphi(w_{r-1})_{[2(q(r-1)-1)]} \in \mathcal{S}(V^{r-1}) \otimes \mathcal{A}^\bullet(\mathbb{D}),$$

writing $\mathbf{d} = \{1, ..., \hat{s}, ..., r\}$ we have

$$\nu_r(\mathbf{v})_{[2(qr-1)]} = \left(T^*_{\{s\}}\varphi\right)(\mathbf{v})_{[2q]} \wedge \left(T^*_{\mathbf{d}}\nu^{r-1}\right)(\mathbf{v})_{[2(q(r-1)-1)]}.$$

Since $\omega_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi_{[2q]} = \left(\frac{p+q}{2}\right) \varphi_{[2q]}$, by Proposition 2.2.7

$$\begin{aligned}
\omega(E_{ss})\nu_r(\mathbf{v})_{[2(qr-1)]} &= \omega(E_{ss}) \left(T^*_{\{s\}}\varphi\right)(\mathbf{v})_{[2q]} \wedge \left(T^*_{\mathbf{d}}\nu^{r-1}\right)(\mathbf{v})_{[2(q(r-1)-1)]} \\
&= \left(T^*_{\{s\}}\omega_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi\right)(\mathbf{v})_{[2q]} \wedge \left(T^*_{\mathbf{d}}\nu^{r-1}\right)(\mathbf{v})_{[2(q(r-1)-1)]} \\
&= \left(T^*_{\{s\}}\frac{(p+q)}{2}\varphi\right)(\mathbf{v})_{[2q]} \wedge \left(T^*_{\mathbf{d}}\nu^{r-1}\right)(\mathbf{v})_{[2(q(r-1)-1)]} \\
&= \frac{(p+q)}{2}\left(T^*_{\{s\}}\varphi\right)(\mathbf{v})_{[2q]} \wedge \left(T^*_{\mathbf{d}}\nu^{r-1}\right)(\mathbf{v})_{[2(q(r-1)-1)]} \\
&= \frac{(p+q)}{2}\nu_r(\mathbf{v})_{[2(qr-1)]}.
\end{aligned}$$

The proof that for $r+1 \le s < 2r$

$$\omega(E_{ss})\nu_r(\mathbf{v})_{[2(qr-1)]} = -\frac{(p+q)}{2}\nu_r(\mathbf{v})_{[2(qr-1)]},$$

is entirely similar to the above.

As determined in Section 2.1, $\{E_{ss} - E_{2r,2r} : 1 \le s < 2r\}$ is a basis for the Cartan subalgebra of $\mathfrak{k}_r \subseteq \mathfrak{gl}_{2r}(\mathbb{C})$, and thus combining the computations above, the action of the Cartan subalgebra is given by

$$\begin{aligned}
\omega(E_{ss} - E_{2r,2r})\nu_r(\mathbf{v})_{[2(qr-1)]} &= \begin{cases} (p+q-1)\nu_r(\mathbf{v})_{[2(qr-1)]}, & 1 \le s \le r \\ (p+q-2)\nu_r(\mathbf{v})_{[2(qr-1)]}, & s = r \\ -\nu_r(\mathbf{v})_{[2(qr-1)]}, & r+1 \le s < 2r \end{cases} \\
&= \mu_r(E_{ss} - E_{2r,2r})\nu_r(\mathbf{v})_{[2(qr-1)]}.
\end{aligned}$$

□

Recall from Section 2.1, in order to show a certain vector is of highest weight, we must both demonstrate its weight for the action of the Cartan subalgebra, and show that it is killed by the *positive* root vectors. By eq. (2.14), the positive root vectors for $\mathfrak{gl}_{2r}(\mathbb{C})$ are

$$\{E_{st} : 1 \le s \le r, s < t \le 2r\} \cup \{E_{ts} : r+1 \le s < t \le 2r\}.$$

As we're restricting our attention to $\mathfrak{k}_r \subseteq \mathfrak{gl}_{2r}(\mathbb{C})$, we will only focus on *compact* positive roots,

$$\{E_{st} : 1 \le s < t \le r\} \cup \{E_{ts} : r+1 \le s < t \le 2r\}$$

i.e. those which belong to $\mathfrak{k}_r$.

When $r = 1$, the Lie algebra $\mathfrak{k}_1$ has no such roots. In the next theorem we will induct on p for (p, q)-vector spaces to show that $\nu_2(\mathbf{v})$ is killed by the positive roots.

Theorem 3.4.5. *Suppose $p \geq 4q-2$. If for every (p,q)-vector space V, the form $\nu_2(\mathbf{v})_{[4q-2]} \in \mathcal{S}(V^2) \otimes \mathcal{A}^\bullet(\mathbb{D}(V))$ is killed by the positive roots of $\mathfrak{k}_2$, then every (p',q)-vector space V' with $p' \geq p$, the corresponding form $\nu_2'(\mathbf{v}) = \nu'(\vec{w}_2) \wedge \varphi'(\vec{w}_1)$ is killed by the positive roots of $\mathfrak{k}_2$.*

Proof. Suppose that the hypothesis holds, and let V be a $(p+1,q)$-vector space. Let $\{v_{p+1},...,v_{p+q}\}$ be a standard basis for $\zeta_0 \in \mathbb{D}$, and extend to a standard basis $\{v_1,...,v_{p+1+q}\}$. Giving $\mathbb{D} = \mathbb{D}(V)$ the coordinates of section 3.1, let u be a basis vector of $\bigwedge^{2q-1,2q-1} T_{\zeta_0}\mathbb{D}(V)$ as appearing in the statement of lemma 3.3.1. Then by the result of lemma 3.3.1, there exists $w \in \{v_1,...,v_p\}$ where $\zeta_0 \in \mathbb{D}(w^\perp)$ and $u_w \in \bigwedge^{2q-1,2q-1} T_{\zeta_0}\mathbb{D}(w^\perp)$ such that for the inclusion $\iota : \mathbb{D}(w^\perp) \hookrightarrow \mathbb{D}$, we have $\iota_* u_w = u$. Note that $w^\perp$ is a (p,q)-vector space with respect to the restricted Hermitian form $\langle,\rangle|_{w^\perp}$. By our induction hypothesis, for the forms φ_w and ν_w on $\mathbb{D}(w^\perp)$, the Weil representation $\omega_w : \mathfrak{k}_2 \to \mathrm{End}(\mathcal{S}(V^2) \otimes \mathcal{A}^\bullet(\mathbb{D}(w^\perp)))$ and $E_{12}, E_{43} \in \mathfrak{k}_2$

$$\omega(E_{12})\nu_w(u_2)_{[2(q-1)]} \wedge \varphi_w(u_1)_{[2q]} = 0,$$
$$\omega(E_{43})\nu_w(u_2)_{[2(q-1)]} \wedge \varphi_w(u_1)_{[2q]} = 0.$$

Write $\langle w \rangle := \mathrm{span}_{\mathbb{C}} w$. For each entry of $\mathbf{v} = (u_1,u_2) \in V^2$ we'll write the components of orthogonal decomposition $V = \langle w \rangle \oplus w^\perp$ as $u_1 = u_1' + u_1''$ and $u_2 = u_2' + u_2''$.
By the proposition 3.3.2, for the inclusion $\iota : \mathbb{D}(w^\perp) \hookrightarrow \mathbb{D}$ we have

$$\iota^*\varphi(u_1)_{[2q]} = e^{-\pi\left\langle u_1',u_1'\right\rangle}\varphi_w(u_1'')_{[2q]}$$
$$\iota^*\nu(u_2)_{[2(q-1)]} = e^{-\pi\left\langle u_2',u_2'\right\rangle}\nu_w(u_2'')_{[2(q-1)]}.$$

We will write

$$\phi_0(u_1',u_2') = exp\left(-\pi\left(\left\langle u_1',u_1'\right\rangle + \left\langle u_2',u_2'\right\rangle\right)\right),$$

which is the vacuum vector for $\mathcal{S}(\langle w\rangle^2)$.
Thus, for the pullbacks $\pi_w^* : \mathcal{S}(\langle w\rangle^2) \otimes \mathcal{A}^\bullet(\mathbb{D}) \to \mathcal{S}(V^2) \otimes \mathcal{A}^\bullet(\mathbb{D})$ and $\pi_\perp^* : \mathcal{S}((w^\perp)^2) \otimes \mathcal{A}^\bullet(\mathbb{D}) \to \mathcal{S}(V^2) \otimes \mathcal{A}^\bullet(\mathbb{D})$ defined in the proof of proposition 3.4.3, we find

$$\begin{aligned}
\iota^*\nu_r(\mathbf{v})_{[4q-2]} &= \iota^*\left(\nu(u_2)_{[2(q-1)]} \wedge \varphi(u_1)_{[2q]}\right)\\
&= \iota^*\nu(u_2)_{[2(q-1)]} \wedge \iota^*\varphi(u_1)_{[2q]}\\
&= e^{-\pi\left\langle u_2',u_2'\right\rangle}\nu_w(u_2'')_{[2(q-1)]} \wedge e^{-\pi\left\langle u_1',u_1'\right\rangle}\varphi_w(u_1'')_{[2q]}\\
&= \phi_0\left(\nu_w(u_1'')_{[2(q-1)]} \wedge \varphi_w(u_1'')_{[2q]}\right)\\
&= \pi_w^*(\phi_0)\pi_\perp^*\left((\nu_w)_{[2(q-1)]} \wedge (\varphi_w)_{[2q]}\right)(u_1,u_2).
\end{aligned}$$

Thus, for the Weil representation

$$\omega : \mathfrak{k}_2 \to \mathrm{End}(\mathcal{S}(V^2) \otimes \mathcal{A}^\bullet(\mathbb{D}))$$
$$\omega_w : \mathfrak{k}_2 \to \mathrm{End}(\mathcal{S}(\langle w\rangle^2) \otimes \mathcal{A}^\bullet(\mathbb{D}(w^\perp)))$$
$$\omega_\perp : \mathfrak{k}_2 \to \mathrm{End}(\mathcal{S}((w^\perp)^2) \otimes \mathcal{A}^\bullet(\mathbb{D}(w^\perp))),$$

we have

$$
\begin{aligned}
&\omega(E_{12})\nu_r(\mathbf{v})_{[4q-2]}u \\
&= \omega(E_{12})\nu_r(\mathbf{v})_{[4q-2]}\iota_*(u) \\
&= \iota^*\left(\omega(E_{12})\nu_r(\mathbf{v})_{[4q-2]}\right)u_w \\
&= \omega(E_{12})\iota^*\left(\nu_r(\mathbf{v})_{[4q-2]}\right)u_w, \qquad \text{by lemma 3.4.2} \\
&= \omega(E_{12})\pi_w^*(\phi_0)\pi_\perp^*\left((\nu_w)_{[2(q-1)]}\wedge(\varphi_w)_{[2q]}\right)u_w \\
&= \pi_w^*\left(\omega_w(E_{12})\phi_0\right)\pi_\perp^*\left((\nu_w)_{[2(q-1)]}\wedge(\varphi_w)_{[2q]}\right)+\pi_w^*(\phi_0)\pi_\perp^*\left(\omega_\perp(E_{12})(\nu_w)_{[2(q-1)]}\wedge(\varphi_w)_{[2q]}\right), \qquad \text{by proposition 2.2.5} \\
&= \left(0\cdot\pi_\perp^*\left((\nu_w)_{[2(q-1)]}\wedge(\varphi_w)_{[2q]}\right)+\pi_w^*(\phi_0)\cdot 0\right)u_w, \qquad \text{by lemma 2.2.4 and the induction hypothesis} \\
&= 0.
\end{aligned}
$$

Therefore we've shown the equality of $\omega(E_{12})\nu_2(\mathbf{v})_{[4q-2]}$ and the 0-functional on any one of the basis vectors of $\bigwedge^{2q-1,2q-1}T_{\zeta_0}\mathbb{D}$ as described in lemma 3.3.1. By linearity $\omega(E_{12})\nu_2(\mathbf{v})_{[4q-2]}$ agrees with 0 on all of $\bigwedge^{2q-1,2q-1}T_{\zeta_0}\mathbb{D}$, and are therefore they are equal.
The proof that $\omega(E_{43})\nu_r(\mathbf{v})_{[4q-2]} = 0$, is entirely similar. □

Theorem 3.4.6. *If $\nu_2(\mathbf{v})_{[4q-2]}$ is killed by the positive roots of $\mathfrak{k}_2$, then $\nu_r(\mathbf{v})_{[2(qr-1)]}$ is killed by the positive roots of $\mathfrak{k}_r$ for any $r \geq 2$.*

Proof. Suppose that $\nu_2 \in \mathcal{S}(V^2)\otimes\mathcal{A}^\bullet(\mathbb{D})$ is killed by the positive roots of $\mathfrak{k}_2$. Let $\mathbf{v} = (w_1, ..., w_r) \in V^r$, and $E_{ar} \in \mathfrak{k}_r$ with $1 \leq a < r$. Observe that

$$\nu_r(\mathbf{v})_{[2(qr-1)]} = \nu(w_r)\wedge\varphi(w_1)\wedge...\wedge\varphi(w_{r-1})_{[2(qr-1)]} = \nu(w_r)\wedge\varphi(w_a)\wedge\varphi(w_1)\wedge...\wedge\widehat{\varphi(w_a)}\wedge...\wedge\varphi(w_{r-1})_{[2(qr-1)]}.$$

Now $\nu(w_r)\wedge\varphi(w_1)_{[4q-2]} = \nu_2(\mathbf{v})_{[4q-2]} \in \mathcal{S}(V^2)\otimes\mathcal{A}^\bullet(\mathbb{D})$ and
$\varphi^{r-2}(\mathbf{v}) := \varphi(w_1)\wedge...\wedge\widehat{\varphi(w_a)}\wedge...\wedge\varphi(w_{r-1})_{[2(q(r-1)-1)]} \in \mathcal{S}(V^{r-2})\otimes\mathcal{A}^\bullet(\mathbb{D})$, where if we let $\mathbf{d} = \{1, ..., \hat{a}, ..., \hat{b}, ..., r\}$

$$
\begin{aligned}
\nu_r(\mathbf{v})_{[2(qr-1)]} &= \nu(w_r)\wedge\varphi(w_a)\wedge\varphi(w_1)\wedge...\wedge\widehat{\varphi(w_a)}\wedge...\wedge\varphi(w_{r-1})_{[2(qr-1)]} \\
&= \left(T_{\{ab\}}\nu_2\right)(\mathbf{v})_{[4q-2]}\left(T_{\mathbf{d}}^*\varphi^{r-2}\right)(\mathbf{v})_{[2(q(r-1)-1)]}.
\end{aligned}
$$

Thus by proposition 2.2.8

$$
\begin{aligned}
\omega(E_{ar})\nu_r(\mathbf{v})_{[2(qr-1)]} &= \omega(E_{ar})\left[\left(T_{\{ab\}}\nu_2\right)_{[4q-2]}\left(T_{\mathbf{d}}^*\varphi^{r-1}\right)(\mathbf{v})_{[2(q(r-1)-1)]}\right] \\
&= \left[T_{\{ab\}}^*\omega_2(E_{12})\nu_2\right](\mathbf{v})_{[4q-2]}\left(T_{\mathbf{d}}^*\varphi^{r-2}\right)(\mathbf{v})_{[2(q(r-1)-1)]} \\
&= T_{\{ab\}}^*(0)\left(T_{\mathbf{d}}^*\varphi^{r-2}\right)(\mathbf{v})_{[2(q(r-1)-1)]} \qquad \text{by the hypothesis,} \\
&= 0.
\end{aligned}
$$

The proof that $\nu_r(\mathbf{v})_{[2(qr-1)]}$ is killed by the other positive root is entirely similar. □

Finally, we combine the previous results to yield our main theorem.

Theorem 3.4.7. *Let V be a (p,q)-vector space. The form $\nu_r(\mathbf{v})_{[2(qr-1)]}$ is a highest weight vector of weight $\mu_r \in \mathfrak{h}_r^*$ for the Weil representation $\omega : \mathfrak{k}_r \to \mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}(V))$ for (p,q,r) in the following cases*

1. *$(p,1,r)$ for any $r \leq p+1$.*

2. *$(p,2,r)$ for any $r \leq p$.*

3. *$(p,3,r)$ for $1 \leq p \leq 3$ and any $r \leq p$.*

4. *$(1,4,1), (2,4,1), (2,4,2)$, and $(3,4,r)$ for $r \leq 3$.*

Proof.

1. This is proven in [4].

2. Let V be a $(p,2)$-vector space, $\{v_{p+1}, ..., v_{p+q}\}$ a standard basis for some $\zeta_0 \in \mathbb{D}$. Extend to a standard basis $\{v_1, ..., v_{p+q}\}$ of V, and equip $\mathbb{D}$ with the coordinates of section 3.1. Let $\omega_1 : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}(\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D}))$ be the Weil representation. Using the code in the final chapter, we've established that for $p = 1$ or $p = 2$, evaluating the form $\nu(v)_{[2]}|_{\zeta_0}$ at ζ_0,
$$\omega_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu(v)_{[2]}|_{\zeta_0} = \left(\frac{p}{2}\right) \nu(v)_{[2]}|_{\zeta_0}$$
$$\omega_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \nu(v)_{[2]}|_{\zeta_0} = -\left(\frac{p}{2}\right) \nu(v)_{[2]}|_{\zeta_0}$$
As $p \geq 2$, by corollary 3.4.4, for every $(p,2)$-vector space V, $r \geq 1$, and the Weil representation $\omega_r : \mathfrak{k}_r \to \mathrm{End}(\mathcal{S}(V^r) \otimes \mathcal{A}^\bullet(\mathbb{D}))$ the form $\nu_r(\mathbf{v})_{[2(2r-1)]}|_{\zeta_0}$ has weight $\mu_r \in \mathfrak{h}_r^*$. By proposition 3.2.1 we conclude that $\nu_r(\mathbf{v})_{[2(2r-1)]}$ has weight μ_r.
We have also established by computation that for each $(p,2)$-vector space with $p \leq 4q-2 = 6$, the form $\nu_2(\mathbf{v})_{[6]}|_{\zeta_0}$ is killed by the positive roots of $\mathfrak{k}_2$ and thus by theorem 3.4.5, for any $(p,2)$-vector space the form $\nu_2(\mathbf{v})_{[6]}|_{\zeta_0}$ is killed by the positive roots of $\mathfrak{k}_2$. Again by proposition 3.2.1 we're able to conclude $\nu_2(\mathbf{v})_{[6]}$ is killed by the positive roots of $\mathfrak{k}_2$. Thus it follows from theorem 3.4.6 that for each $(p,2)$-vector space and r, the form $\nu_r(\mathbf{v})_{[2(2r-1)]}$ is killed by the positive roots of $\mathfrak{k}_r$.
In conclusion, for any $(p,2)$-vector space, the form $\nu_r(\mathbf{v})_{[2(qr-1)]}$ has weight $\mu_r \in \mathfrak{h}_r^*$ and is killed by the action of its root vectors. Therefore $\nu_r(\mathbf{v})_{[2(qr-1)]}$ is a *highest weight vector*, which implies that it generates an irreducible subrepresentation.

3. By direct computation.

4. By direct computation.

□

3.5 Computations

In this section we will walk through the computation of the $(p, q) = (1, 1)$ case "by hand". This can serve both as test that the code does what it promises to do, and also clarify what exactly needs to be computed. We will also display some of the output of the code, and highlight some interesting features there in.

The Case $(p, q) = (1, 1)$ by Hand

Let V be a $(1, 1)$-vector space. Given a standard basis $\{v_1, v_2\}$ for V, we equip $\mathbb{D} = \mathbb{D}(V)$ with the coordinates outlined in section 3.1. Under this description, the coordinates of any point $\zeta \in \mathbb{D} = \mathbb{D}(V)$ is just a 1×1 matrix, i.e. a complex number $\zeta_{1,1}$ such that $|\zeta_{1,1}|^2 - 1 < 0$, that is, $\mathbb{D}$ is (biholomorphic to) the open complex unit disc.

We have that $\zeta = \operatorname{span}\left\{\begin{pmatrix} \zeta_{1,1} \\ 1 \end{pmatrix}\right\}$. Letting $\mathcal{E} \to \mathbb{D}$ be the tautological bundle, for each $\zeta \in \mathbb{D}$ the fiber $\mathcal{E}_\zeta$ is ζ, and thus we take the *global* frame $\left\{e = \begin{pmatrix} \zeta_{1,1} \\ 1 \end{pmatrix}\right\}$, trivializing $\mathcal{E} \to \mathbb{D}$. Thus we will take $\{1, e\}$ as a global frame for $\mathcal{O} \oplus \mathbb{D}$, where $\mathcal{O} = \mathcal{O}_{\mathbb{D}}$ is the structure sheaf.

From this point on, we will simply write ζ for $\zeta_{1,1}$, as we will only make reference to the variable, not the space.

The Hermitian form $\langle , \rangle_{\mathcal{E}}$ on $\mathcal{E}$ is determined entirely by $h := \langle e, e \rangle_{\mathcal{E}} = -\langle e, e \rangle = 1 - |\zeta|^2$.

The Hermitian form $\langle , \rangle_{\wedge}$ of the total bundle $\mathcal{O} \oplus \mathcal{E}$ is thus

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |\zeta|^2 \end{pmatrix}.$$

For $v = (z_1, z_2)^T \in V$ the section $s_v : \mathbb{D} \to \mathcal{E}^\vee$ acts by

$$s_v(\zeta)(e) = \langle e, v \rangle = \bar{z}_1 \zeta - \bar{z}_2.$$

Thus the matrix S of the extension of s_v to the Koszul complex $\mathcal{E} \xrightarrow{s_v} \mathcal{O}$ is

$$S = \begin{pmatrix} 0 & \bar{z}_1 \zeta - \bar{z}_2 \\ 0 & 0 \end{pmatrix}.$$

To compute the adjoint s_v^* of s_v, for $u \in \mathcal{E}$ and $w \in \mathcal{O}$ we have

$$\begin{aligned}
\langle Su, w\rangle = \left\langle u, S^\star w\right\rangle \Rightarrow (Su)^T H\bar{w} &= u^T H\overline{S^\star w} \\
v^T S^T H\bar{w} &= u^T H\bar{S}^\star\bar{w}, \text{ since this holds for any pair of vectors,} \\
\Rightarrow S^T H &= H\bar{S}^\star \\
\overline{H^{-1}S^T H} &= S^\star \\
\bar{H}^{-1}\bar{S}^T\bar{H} &= S^\star \\
H^{-1}S^* H &= S^\star.
\end{aligned}$$

As shown in the previous section, the matrix of the adjoint $S^\star$ is computed as

$$\begin{aligned}
S^\star = H^{-1}S^*H &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-|\zeta|^2}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ z_1\bar{\zeta} - z_2 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-|\zeta|^2\end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-|\zeta|^2}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ z_1\bar{\zeta} - z_2 & 0\end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{z_1\bar{\zeta}-z_2}{1-|\zeta|^2} & 0\end{pmatrix}
\end{aligned}$$

Therefore

$$\begin{aligned}
(S+S^\star)^2 &= \left(\begin{pmatrix} 0 & \bar{z}_1\zeta - \bar{z}_2 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{z_1\bar{\zeta}-z_2}{1-|\zeta|^2} & 0\end{pmatrix}\right)^2 \\
&= \begin{pmatrix} 0 & \bar{z}_1\zeta - \bar{z}_2 \\ \frac{z_1\bar{\zeta}-z_2}{1-|\zeta|^2} & 0\end{pmatrix}\begin{pmatrix} 0 & \bar{z}_1\zeta - \bar{z}_2 \\ \frac{z_1\bar{\zeta}-z_2}{1-|\zeta|^2} & 0\end{pmatrix} \\
&= \frac{(\bar{z}_1\zeta - \bar{z}_2)(z_1\bar{\zeta} - z_2)}{1-|\zeta|^2}\cdot I
\end{aligned}$$

Recall that by proposition 1.1.17, the connection ∇ on $\mathcal{E}$ can be given a local description $\nabla = d + \theta$ where

$$\theta = \frac{\partial(1-|\zeta|^2)}{1-|\zeta|^2} = \frac{-\bar{\zeta}}{1-|\zeta|^2}d\zeta.$$

Again by proposition 1.1.17, the curvature can be expressed as the operator

$$\Theta = \bar{\partial}\frac{-\bar{\zeta}}{1-|\zeta|^2}dx = \frac{-(1-|\zeta|^2) - (-\bar{\zeta})(-\zeta)}{(1-|\zeta|^2)^2}d\bar{\zeta}\wedge d\zeta = \frac{d\zeta\wedge d\bar{\zeta}}{(1-|\zeta|^2)^2}.$$

For the superconnection $\nabla_v = \nabla + i\sqrt{2\pi}(S+S^\star)$, the super curvature is

$$\nabla_v^2 = -2\pi(S^2 + \left(S^\star\right)^2) + i\sqrt{2\pi}(\nabla(S+S^\star) + (S+S^\star)\nabla) + \nabla^2.$$

If we split $\nabla_v^2 = r_0 + r_1 + r_2$ into components where the form degree of r_i is i. Thus

$$-2\pi(S^2 + \left(S^\star\right)^2) + i\sqrt{2\pi}(\nabla(S+S^\star) + (S+S^\star)\nabla) + \nabla^2,$$

and therefore

$$r_0|_{\zeta=0} = \left(i\sqrt{2\pi}\left(S + S^\star\right)\right)^2|_{\zeta=0} = -2\pi\left(\frac{(\bar{z_1}\zeta - \bar{z_2})(z_1\bar{\zeta} - z_2)}{1-|\zeta|^2}\right)I_2|_{\zeta=0} = -2\pi|z_2|^2 I_2.$$

Note that the connection for $\mathcal{O}$ is just d, so the connection ∇ on the total bundle $\mathcal{E} \oplus \mathcal{O}$ is $\nabla(f \otimes 1) = df \otimes 1$, and $\nabla(f \otimes s_v^{-1}) = df \otimes s_v^{-1} + \theta f \otimes s_v^{-1}$.

$$\begin{aligned}
r_1(1 \otimes 1) &= i\sqrt{2\pi}\left[\nabla s_v(1 \otimes 1) + \nabla s_v^*(1 \otimes 1) + (s_v + s_v^*)\nabla(1 \otimes 1)\right] \\
&= i\sqrt{2\pi}\left[\nabla(0) + \nabla\left(\frac{z_1\bar{\zeta} - z_2}{1-|\zeta|^2} \otimes e\right) + (s_v + s_v^*)(0)\right] \\
&= i\sqrt{2\pi}\left[(d+\theta)\left(\frac{z_1\bar{\zeta} - z_2}{1-|\zeta|^2} \otimes e\right)\right] \\
&= i\sqrt{2\pi}\left[\partial\frac{z_1\bar{\zeta} - z_1}{1-|\zeta|^2} + \bar{\partial}\frac{z_1\bar{\zeta} - z_2}{1-|\zeta|^2} + \left(\frac{-\bar{\zeta}}{1-|\zeta|^2}d\zeta\right)\left(\frac{z_1\bar{\zeta} - z_2}{1-|\zeta|^2}\right)\right] \otimes e \\
&= i\sqrt{2\pi}\left[-\frac{(z_1\bar{\zeta} - z_2)(-\bar{\zeta})}{(1-|\zeta|^2)^2}d\zeta + \frac{z_1(1-|\zeta|^2) - (z_1\bar{\zeta} - z_2)(-\zeta)}{(1-|\zeta|^2)^2}d\bar{\zeta} + \frac{-\bar{\zeta}(z_1\bar{\zeta} - z_2)}{(1-|\zeta|^2)^2}d\zeta\right] \otimes e \\
\Rightarrow r_1(1 \otimes 1)|_{\zeta=0} &= \left(i\sqrt{2\pi}z_1\right)d\bar{\zeta} \otimes e.
\end{aligned}$$

$$\begin{aligned}
r_1(1 \otimes e) &= i\sqrt{2\pi}\left[\nabla s_v(1 \otimes e) + \nabla s_v^*(1 \otimes e) + s_v\nabla(1 \otimes e) + s_v^*\nabla(1 \otimes e)\right] \\
&= i\sqrt{2\pi}\left[\nabla(\bar{z_1}\zeta - \bar{z_2}) \otimes 1 + \nabla(0) + s_v\left(\frac{-\bar{\zeta}}{1-|\zeta|^2}d\zeta \otimes e\right) + 0\right] \\
&= i\sqrt{2\pi}\left[d(\bar{z_1}\zeta - \bar{z_2}) \otimes 1 + s_v\left(\frac{-\bar{\zeta}}{1-|\zeta|^2}d\zeta \otimes e\right)\right] \\
&= i\sqrt{2\pi}\left[\bar{z_1}d\zeta + \frac{-\bar{\zeta}}{1-|\zeta|^2}(\bar{z_1}\zeta - \bar{z_2})d\zeta\right] \otimes 1 \\
\Rightarrow r_1(1 \otimes e)|_{\zeta=0} &= \left(i\sqrt{2\pi}\bar{z_1}\right)d\zeta \otimes 1.
\end{aligned}$$

Now, by proposition 1.2.6, the definition of $\mathcal{A}^\bullet(\mathbb{D})$-linearity preceding it, and the fact that r_1 had even total degree, we compute

$$
\begin{aligned}
r_1^2(1\otimes 1)|_{\zeta=0} &= r_1\left(i\sqrt{2\pi}z_1 d\bar{\zeta}\otimes e\right)\Big|_{\zeta=0}\\
&= \left(i\sqrt{2\pi}z_1 d\bar{\zeta}\right)\wedge r_1\,(1\otimes e)\,|_{\zeta=0}\\
&= \left(i\sqrt{2\pi}z_1 d\bar{\zeta}\right)\wedge\left(i\sqrt{2\pi}\bar{z}_1 d\zeta\otimes 1\right)\\
&= 2\pi|z_1|^2 d\zeta\wedge d\bar{\zeta}\otimes 1,\\
r_1^2(1\otimes e)|_{\zeta=0} &= r_1\left(i\sqrt{2\pi}\bar{z}_1 d\zeta\otimes 1\right)\Big|_{\zeta=0}\\
&= \left(i\sqrt{2\pi}\bar{z}_1 d\zeta\right)\wedge r_1\,(1\otimes 1)\,|_{\zeta=0}\\
&= \left(i\sqrt{2\pi}\bar{z}_1 d\zeta\right)\wedge\left(i\sqrt{2\pi}z_1 d\bar{\zeta}\otimes e\right)\\
&= -2\pi|z_1|^2 d\zeta\wedge d\bar{\zeta}\otimes e.
\end{aligned}
$$

Therefore

$$
\frac{1}{2}\mathrm{tr}_s(r_1^2)|_{\zeta=0} = \frac{1}{2}\left(2\pi|z_1|^2 d\zeta\wedge d\bar{\zeta} - \left(-2\pi|z_1|^2 d\zeta\wedge d\bar{\zeta}\right)\right) = 2\pi|z_1|^2 d\zeta\wedge d\bar{\zeta}.
$$

Finally, we see that

$$
\mathrm{tr}_s(r_2)|_{\zeta=0} = \mathrm{tr}_s\begin{pmatrix}0 & 0\\ 0 & \frac{d\zeta\wedge d\bar{\zeta}}{(1-|\zeta|^2)^2}\end{pmatrix}|_{\zeta=0} = -d\zeta\wedge d\bar{\zeta}.
$$

We now compute $\varphi(v)_{[2]}|_{\zeta=0}$ and $\nu(v)_{[0]}|_{\zeta=0}$ according to their definitions eq. (3.3) and eq. (3.4). Since $r_0|_{\zeta=0} = -2\pi|z_2|^2 I_2$, it commutes with $(r_1+r_2)|_{\zeta=0}$ and thus

$$
e^{\nabla_v^2}|_{\zeta=0} = e^{r_0+r_1+r_2}|_{\zeta=0} = e^{r_0}e^{r_1+r_2}|_{\zeta=0}.
$$

Therefore

$$
\begin{aligned}
\varphi(v)_{[2]}|_{\zeta=0} &= e^{-\pi(|z_1|^2-|z_2|^2)}\left(\frac{i}{2\pi}\right)\mathrm{tr}_s\left(e^{\nabla_v^2}\right)_{[2]}|_{\zeta=0}\\
&= e^{-\pi(|z_1|^2-|z_2|^2)}\left(\frac{i}{2\pi}\right)\mathrm{tr}_s\left(e^{r_0}e^{r_1+r_2}\right)_{[2]}|_{\zeta=0}\\
&= \frac{ie^{-\pi(|z_1|^2-|z_2|^2)}}{2\pi}e^{-2\pi|z_2|^2}\mathrm{tr}_s\left(I+(r_1+r_2)+\frac{1}{2}(r_1+r_2)^2+\ldots\right)_{[2]}|_{\zeta=0}\\
&= \frac{ie^{-\pi(|z_1|^2+|z_2|^2)}}{2\pi}\mathrm{tr}_s\left(r_2+\frac{1}{2}r_1^2\right)|_{\zeta=0}\\
&= \frac{i}{2\pi}e^{-\pi(|z_1|^2+|z_2|^2)}\left(-d\zeta\wedge d\bar{\zeta}+2\pi|z_1|^2 dz\wedge d\bar{\zeta}\right)\\
&= \frac{ie^{-\pi(|z_1|^2+|z_2|^2)}}{2\pi}(2\pi|z_1|^2-1)d\zeta\wedge d\bar{\zeta}.
\end{aligned}
$$

The number operator N on $\mathcal{O} \oplus \mathcal{E}$ is given by $N = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, and thus

$$
\begin{aligned}
\nu(v)_{[0]}|_{\zeta=0} &= e^{-\pi(|z_1|^2-|z_2|^2)} \left(\frac{i}{2\pi}\right)^0 \mathrm{tr}_s \left(N e^{\nabla_v^2}\right)_{[0]} |_{\zeta=0} \\
&= e^{-\pi(|z_1|^2-|z_2|^2)} \mathrm{tr}_s \left(N e^{r_0} e^{r_1+r_2}\right)_{[0]} |_{\zeta=0} \\
&= e^{-\pi(|z_1|^2-|z_2|^2)} \mathrm{tr}_s \left(\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-2\pi|z_2|^2} & 0 \\ 0 & e^{-2\pi|z_2|^2} \end{pmatrix} (I + (r_1 + r_2) + \ldots) \right)_{[0]} |_{\zeta=0} \\
&= e^{-\pi(|z_1|^2-|z_2|^2)} \mathrm{tr}_s \begin{pmatrix} 0 & 0 \\ 0 & -e^{-2\pi|z_2|^2} \end{pmatrix} \\
&= e^{-\pi(|z_1|^2+|z_2|^2)}.
\end{aligned}
$$

thus $\nu(v)_{[0]}|_{\zeta=0}$ is just the vacuum vector, and thus by the computation of lemma 2.2.4, we know it's killed under the action of the Weil representation. That is, it has weight determined by $\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 = p + q - 2$, as expected. By the proof of corollary 3.4.4 we know that $\nu_2(\mathbf{v})_{[2]}|_{\zeta=0} = \nu(w_2)_{[0]} \wedge \varphi(w_1)|_{\zeta=0}$ has the weight μ_1 specified in eq. (3.6). Now we will determine the action of the positive compact roots. We write $\mathbf{v} = (w_1, w_2) \in V^2$ in the standard basis as $w_1 = z_{1,1}v_1 + z_{2,1}v_2$ and $w_2 = z_{1,2}v_1 + z_{2,2}v_2$. Again, from the proof of lemma 2.2.4, we know that

$$
\begin{aligned}
\overline{D^+_{1,2}} \exp\left(-\pi(|z_{1,1}|^2 + |z_{2,1}|^2 + |z_{1,2}|^2 + |z_{2,2}|^2\right) &= 0 \\
\overline{D^+_{2,1}} \exp\left(-\pi(|z_{1,1}|^2 + |z_{2,1}|^2 + |z_{1,2}|^2 + |z_{2,2}|^2\right) &= 0.
\end{aligned}
$$

Therefore, writing $\varphi^G = \exp\left(-\pi(|z_{1,1}|^2 + |z_{2,1}|^2 + |z_{1,2}|^2 + |z_{2,2}|^2\right)$ for short, we have

$$
\nu_2(w_1, w_2)_{[2]}|_{\zeta=0} = \nu(w_1)_{[0]} \wedge \varphi(v)_{[2]}|_{\zeta=0} = \varphi^G \cdot \left(\frac{i}{2\pi}\right) (2\pi|z_{1,1}|^2 - 1) d\zeta \wedge d\bar{\zeta},
$$

and thus

$$
\begin{aligned}
\omega(E_{12})\nu_2(w_1, w_2)_{[0]}|_{\zeta=0} &= i\pi \left[\overline{D^-_{1,1}} D^+_{1,2} - D^-_{2,2} \overline{D^+_{2,1}} \right] \varphi^G \cdot (2\pi|z_{1,1}|^2 - 1) d\zeta \wedge d\bar{\zeta} \\
&= i\pi \left[\overline{D^-_{1,1}} (2\pi|z_{1,1}|^2 - 1) D^+_{1,2} \varphi^G - (2\pi|z_{1,1}|^2 - 1) D^-_{2,2} \overline{D^+_{2,1}} \varphi^G \right] d\zeta \wedge d\bar{\zeta} \\
&= i\pi[0 - 0] d\zeta \wedge d\bar{\zeta} \\
&= 0.
\end{aligned}
$$

The computation that $\omega(E_{43})\nu_2(w_1, w_2)_{[2]}$ is entirely similar. Thus we've seen in this case that $\nu_2(\mathbf{v})_{[2q]}$ has weight μ_1, and is killed by the positive compact roots of $\mathfrak{k}_1$, implying that it is a highest weight vector.

The Case of (p, q) for $q > 1$ via Sage

As an example of how infeasible it is to carry these computations out by hand for other small q, for $(p, q) = (1, 2)$ the inverse matrix H^{-1} of the matrix H describing the Hermitian form on $\mathbb{D}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(|\zeta_{1,1}||\zeta_{1,2}|)^2}{\left(\frac{(|\zeta_{1,1}|\zeta_{1,2}|)^2}{|\zeta_{1,1}|^2-1}-|\zeta_{1,2}|^2+1\right)(|\zeta_{1,1}|^2-1)^2} - \frac{1}{|\zeta_{1,1}|^2-1} & -\frac{|\zeta_{1,2}|^2}{\left(\frac{(|\zeta_{1,1}||\zeta_{1,2}|)^2}{|\zeta_{1,1}|^2-1}-|\zeta_{1,2}|^2+1\right)(|\zeta_{1,1}|^2-1)} & 0 \\ 0 & -\frac{|\zeta_{1,2}|^2}{\left(\frac{(|\zeta_{1,1}||\zeta_{1,2}|)^2}{|\zeta_{1,1}|^2-1}-|\zeta_{1,2}|^2+1\right)(|\zeta_{1,1}|^2-1)} & \frac{1}{\frac{(|\zeta_{1,1}||\zeta_{1,2}|)^2}{|\zeta_{1,1}|^2-1}-|\zeta_{1,2}|^2+1} & 0 \\ 0 & 0 & 0 & -\frac{1}{|\zeta_{1,1}|^2|\zeta_{1,2}|^2-(|\zeta_{1,1}|^2-1)(|\zeta_{1,2}|^2-1)} \end{pmatrix}$$

In order to obtain the curvature, one must compute $\bar{\partial}\left(H^{-1}\partial H\right)$, which is already quite tedious, let alone the computations of the rest of $\nabla_v^2 = r_0 + r_1 + r_2$, and $\mathrm{tr}_s(Ne^{\nabla_v^2})$.

We now produce the forms $\nu(v)_{[2(q-1)]}|_{\zeta_0}$ in several different cases, as computed from Sage. For the ease of reading, we've set

$$\varphi^G = e^{-\pi\sum_{\gamma=1}^{p+q}|z_\gamma|^2}.$$

The $(p,q) = (1,2)$ case,

$$\nu(v)_{[2]}|_{\zeta_0} = \varphi^G \cdot \left(\frac{\left(2i\,\pi|z_1|^2 - i\right)}{2\,\pi}\mathrm{d}\zeta_{1,1}\wedge\mathrm{d}\bar{\zeta}_{1,1} + \frac{\left(2i\,\pi|z_1|^2 - i\right)}{2\,\pi}\mathrm{d}\zeta_{1,2}\wedge\mathrm{d}\bar{\zeta}_{1,2}\right).$$

Notice the similarity with the form $\varphi(v)_{[2]}|_{\zeta_0}$ from the $(1,1)$ case.

In the $(2,2)$ case,

$$\begin{aligned}\nu(v)_{[2]}|_{\zeta_0} =& \varphi^G \cdot \left(\frac{(2i\,\pi|z_1|^2 - i)}{2\,\pi} d\zeta_{1,1} \wedge d\bar{\zeta}_{1,1} + i\, z_2\bar{z}_1 d\zeta_{1,1} \wedge d\bar{\zeta}_{2,1} + i\, z_1\bar{z}_2 d\zeta_{1,2} \wedge d\bar{\zeta}_{1,1} + \frac{(2i\,\pi|z_2|^2 - i)}{2\,\pi} d\zeta_{1,2} \wedge d\bar{\zeta}_{2,1} \right) \\ &+ \varphi^G \cdot \left(\frac{(2i\,\pi|z_1|^2 - i)}{2\,\pi} d\zeta_{1,2} \wedge d\bar{\zeta}_{1,2} + i\, z_2\bar{z}_1 d\zeta_{1,2} \wedge d\bar{\zeta}_{2,2} + i\, z_1\bar{z}_2 d\zeta_{2,2} \wedge d\bar{\zeta}_{1,2} + \frac{(2i\,\pi|z_2|^2 - i)}{2\,\pi} d\zeta_{2,2} \wedge d\bar{\zeta}_{2,2} \right),\end{aligned}$$

and compare this with $\varphi_{[2]}$ of the $(2,1)$-case

$$\varphi(v)_{[2]}|_{\zeta_0} = \varphi^G \cdot \left(\frac{(2i\,\pi|z_1|^2 - i)}{2\,\pi} d\zeta_{1,1} \wedge d\bar{\zeta}_{1,1} + i\, z_2\bar{z}_1 d\zeta_{1,1} \wedge d\bar{\zeta}_{1,2} + i\, z_1\bar{z}_2 d\zeta_{1,1} \wedge d\bar{\zeta}_{1,1} + \frac{(2i\,\pi|z_2|^2 - i)}{2\,\pi} d\zeta_{1,2} \wedge d\bar{\zeta}_{1,2} \right).$$

The $(p,q) = (1,3)$ case,

$$\begin{aligned}\nu(v)_{[4]}|_{\zeta_0} =& \varphi^G \cdot \left(\frac{(2\,\pi^2|z_1|^4 - 4\,\pi|z_1|^2 + 1)}{2\,\pi^2} d\zeta_{1,1} \wedge d\zeta_{1,2} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,2} + \frac{(2\,\pi^2|z_1|^4 - 4\,\pi|z_1|^2 + 1)}{2\,\pi^2} d\zeta_{1,1} \wedge d\zeta_{1,3} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,3} \right) \\ &+ \varphi^G \cdot \frac{(2\,\pi^2|z_1|^4 - 4\,\pi|z_1|^2 + 1)}{2\,\pi^2} d\zeta_{1,2} \wedge d\zeta_{1,3} \wedge d\bar{\zeta}_{1,2} \wedge d\bar{\zeta}_{1,3},\end{aligned}$$

is also very similar to $\varphi_{[4]}|_{\zeta_0}$ of the $(1,2)$ case,

$$\varphi_{[4]}|_{\zeta_0} = \varphi^G \cdot \frac{(2\,\pi^2|z_1|^4 - 4\,\pi|z_1|^2 + 1)}{2\,\pi^2} d\zeta_{1,1} \wedge d\zeta_{1,2} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,2}.$$

Finally, for the $(p,q) = (1,4)$ case,

$$\begin{aligned}\nu(v)_{[6]}|_{\zeta_0} =& \varphi^G \cdot \frac{(4i\,\pi^3|z_1|^6 - 18i\,\pi^2|z_1|^4 + 18i\,\pi|z_1|^2 - 3i)}{4\,\pi^3} d\zeta_{1,1} \wedge d\zeta_{1,2} \wedge d\zeta_{1,3} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,2} \wedge d\bar{\zeta}_{1,3} \\ &+ \varphi^G \cdot \frac{(4i\,\pi^3|z_1|^6 - 18i\,\pi^2|z_1|^4 + 18i\,\pi|z_1|^2 - 3i)}{4\,\pi^3} d\zeta_{1,1} \wedge d\zeta_{1,2} \wedge d\zeta_{1,4} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,2} \wedge d\bar{\zeta}_{1,4} \\ &+ \varphi^G \cdot \frac{(4i\,\pi^3|z_1|^6 - 18i\,\pi^2|z_1|^4 + 18i\,\pi|z_1|^2 - 3i)}{4\,\pi^3} d\zeta_{1,1} \wedge d\zeta_{1,3} \wedge d\zeta_{1,4} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,3} \wedge d\bar{\zeta}_{1,4} \\ &+ \varphi^G \cdot \frac{(4i\,\pi^3|z_1|^6 - 18i\,\pi^2|z_1|^4 + 18i\,\pi|z_1|^2 - 3i)}{4\,\pi^3} d\zeta_{1,2} \wedge d\zeta_{1,3} \wedge d\zeta_{1,4} \wedge d\bar{\zeta}_{1,2} \wedge d\bar{\zeta}_{1,3} \wedge d\bar{\zeta}_{1,4},\end{aligned}$$

we compare $\varphi(v)_{[6]}$ of the $(1,3)$ case,

$$\varphi_{[6]}|_{\zeta_0} = \varphi^G \cdot \frac{(4i\,\pi^3|z_1|^3 - 18i\,\pi^2|z_1|^4 + 18i\,\pi|z_1|^2 - 3i)}{4\,\pi^3} d\zeta_{1,1} \wedge d\zeta_{1,2} \wedge d\zeta_{1,3} \wedge d\bar{\zeta}_{1,1} \wedge d\bar{\zeta}_{1,2} \wedge d\bar{\zeta}_{1,3}$$

Chapter 4

The Code

```
# Created April 23rd, 2021 this is not the official total code, but the best version yet
    that computes nu and mu in the specific
# 2q and 2(q-1), and checks the r = 2 case.  (No Funke - Hoffman)

# Setting the parameters of the manifold.

p = 2
q = 2

# Defining the manifold, with chart and frame.  The coordinates x are defined in such a way
    that for i<p*q, i+p*q represents
# the conjugate of x.  More specifically yet, x_1 through x_p are the entries of the column
    vector of the first basis vector of
# a given point of D.  To put it another way, the entries going down of the first column of
    the matrix Z representing the point.
# Thus x_{i+p} is the next colum, then x_{1+2p}, ..., to x_{i+(q-1)p}.  With conjugates x_{p
    *q+i}, ..., x_{p*q+i+(q-1)p}.
# That is to say, taking for coordinates 0<=i<=q-1, 0<=j<=p-1, z_{i,j} = x[j*p+i]

M = Manifold((2*p*q), 'M', field='complex')
U = M.open_subset('U')
x = U.chart(names=tuple('x_%d' % i for i in range(2*p*q)))
eU = x.frame()
# First coordinate controls conjuagtion (0 is normal, 1 is conjugated), second coordinate
    controls vector, 3rd controls
# vector coordinate.

e = {(i,j,k): var("e_{}{}{}".format(i,j,k), latex_name="e_{{{}{}{}}}".format(i,j,k)) for i
    in range(2) for j in range(q) for k in range((p+q))}

# Is the kth entry of the jth canonical frame vector is e[(0,j,k)], with conjugate e[(1,j,k)
```

```
    ].

# Here we set the p variables in the first p coordinates of each frame vector.
for i in range(q):
    for k in range(p):
        e[(0, i, k)] = x[((i*p)+k)]

# Here we set the conugate variables of the first p entries of the frame vectors.
for i in range(q):
    for k in range(p):
        e[(1,i,k)] = x[(((i*p)+k)+((p*q)))]

# Here we tell both the frame, and conjugate vectors where the 1 is.
for i in range(2):
    for j in range(q):
        e[(i,j,(p+j))] = 1

# Here we set the rest of the entries of the frame and conjugate vectors to 0.
for i in range(2):
    for j in range(q):
        for k in range(p+q):
            if k not in range(p) and k!=p+j:
                e[(i,j,k)] = 0

# Defining variables for an arbitrary vecotr v[(0,i)], and its conjugate v[(1,i)].

v = {(i,j): var("v_{}{}".format(i, j), latex_name="z_{{{}{}}}".format(i, j)) for i in range
    (2) for j in range(p+q)}

# Defining labels to call the entries of s_v(e_j).

sve = {(i): var("sve_{}".format(i)) for i in range(q)}

# Defining labels to call the entries of s_vbar(e_j).

svebar = {(i): var("svebar_{}".format(i)) for i in range(q)}

# Setting the entries of sve.

for i in range(q):
        prod = 0
        for j in range(p):
            prod = prod + (e[(0, i, j)]*v[(1, j)])
        for k in range(p, p+q):
            prod = prod - (e[(0, i, k)]*v[(1,k)])
```

```
        sve[i]  = prod

# Setting the entries of svebar.

for i in range(q):
        prod = 0
        for j in range(p):
            prod += (v[(0, j)]*e[(1, i, j)])
        for k in range(p, p+q):
            prod -= (v[(0,k)]*e[(1, i, k)])
        svebar[i]  = prod

# Making them into mixed forms. We want these to be zero forms, so the list P lets us set
    all other components to 0, with the
# appropriate size.

P = []
for i in range((2*p*q)):
    P.append(0)

Q = []
for i in range((2*p*q)-1):
    Q.append(0)

SveBar = [M.mixed_form(comp=([svebar[i]]+P)) for i in range(q)]

Sve = [M.mixed_form(comp=([sve[i]]+P)) for i in range(q)]

# Pre-computing the exterior derivatives for later use.

dSve = {(i): var("dSve_{}".format(i)) for i in range(q)}

for i in range(q):
    dSve[i] = Sve[i].exterior_derivative()

# Pre-computing the exterior derivative for later use.

dSveBar = {(i): var("dSveBar_{}".format(i)) for i in range(q)}

for i in range(q):
    dSveBar[i] = SveBar[i].exterior_derivative()
show(v[0,0])

# We want L[k] to be the list of all k-wedge basis vectors, so we set the o position to be 0
     by convention.
```

```

L = [[[]]]

# Here we set L[1] to be the q numbers from 0 to q-1.

LL = []
for i in range(q):
    LL.append([i])
L.append(LL)
# Now that we've set L[1], we can use this to determine L[k] for each k, and note that this
    gives the proper
# lexicographic order.

if q>=2:
    for k in range(2,q+1):
        LLL = []
        for i in range(binomial(q, k-1)):
            for j in range(q):
                if L[k-1][i][len(L[k-1][i])-1]<j:
                    mm = []
                    for t in L[k-1][i]:
                        mm.append(t)
                    mm.append(j)
                    LLL.append(mm)
        L.append(LLL)
# Now we simply copy this entire method, except we make lists with an eU so they can be
    called upon as coordinate frames later.
# First we create two lists of those numbers from 0 to pq-1, and pq to 2pq-1 respectively.

Lpq = [[[]]]
L2pq = [[[]]]

LLpq = []
for i in range(p*q):
    LLpq.append([i])
Lpq.append(LLpq)

LL2pq = []
for i in range(p*q, 2*p*q):
    LL2pq.append([i])
L2pq.append(LL2pq)

# Now we creat two lists Lpq and L2pq inductively, where Lpq[i] is the collection of all
    lists of length i, (i<=q) of numbers
# between 0 and pq-1, in strictly increasing order, and the same for L2pq but between pq and
```

```
2pq-1.

if q>=2 or p>=2:
    for i in range(2, 2*q):
        LLLpq = []
        for j in Lpq[i-1]:
            for k in range(p*q):
                if j[len(j)-1]<k:
                    mm = []
                    for t in j:
                        mm.append(t)
                    mm.append(k)
                    LLLpq.append(mm)
        Lpq.append(LLLpq)

elif (q == 1) and (p == 1):
    Lpq.append([[0,1]])

# And do the same for the lists of length q-1.

if q>=2 or p>=2:
    for i in range(2, 2*q):
        LLL2pq = []
        for j in L2pq[i-1]:
            for k in range(p*q, 2*p*q):
                if j[len(j)-1]<k:
                    mm = []
                    for t in j:
                        mm.append(t)
                    mm.append(k)
                    LLL2pq.append(mm)
        L2pq.append(LLL2pq)

elif (q == 1) and (p == 1):
    L2pq.append([[0,1]])

# Now we fuse these two lists in such a way that that qList simply contains those lists where the first q are from Lpq[q],
# and the second q are from L2pq[q].  The list q1List is similar, but draws on those entries of length q-1.

qList = []

for i in Lpq[q]:
    for j in L2pq[q]:
```

```
        temp = []
        for k in i:
            temp.append(k)
        for l in j:
            temp.append(l)
        qList.append(temp)

q1List = []

for i in Lpq[q-1]:
    for j in L2pq[q-1]:
        temp = []
        for k in i:
            temp.append(k)
        for l in j:
            temp.append(l)
        q1List.append(temp)

rList = []

for i in Lpq[2*q-1]:
    for j in L2pq[2*q-1]:
        temp = []
        for k in i:
            temp.append(k)
        for l in j:
            temp.append(l)
        rList.append(temp)
# Here we create a list so that the ith entry is a list of of all the ith wedge basis
    vectors.

WedgeList = []

for i in range(2*(2*q-1)+1):
    if i == 2*(q-1):
        W = []
        for j in q1List:
            WW = [eU]
            for k in j:
                WW.append(k)
            W.append(WW)
        WedgeList.append(W)
    elif i == 2*q:
        W = []
        for j in qList:
```

```
            WW = [eU]
            for k in j:
                WW.append(k)
            W.append(WW)
        WedgeList.append(W)
    elif i == 2*(2*q - 1):
        W = []
        for j in rList:
            WW = [eU]
            for k in j:
                WW.append(k)
            W.append(WW)
        WedgeList.append(W)
    else:
        WedgeList.append([])


# Defining a function which will help combine separate peices of hemritian matrices, SVe, and r_1 and r_2.

def F(i):
    Sum = 0
    for j in range(i):
        Sum += binomial(q,j)
    return Sum

# Computing the matrix for the conjugate of the operator s_v.

SVEbar = matrix(SR, 2^q, 2^q)

for i in range(q+1):
    if i == 0:
        for j in range(2^q):
            SVEbar[j,0] = 0
    else:
        for j in L[i]:
            for k in j:
                temp = []
                for l in j:
                    if l != k:
                        temp.append(l)
                SVEbar[L[i-1].index(temp)+F(i-1), L[i].index(j)+F(i)] = (-1)^(j.index(k))*svebar[k]

# Defining the Hermitian matrix of the tautological bundle.
```

```

HE = matrix(SR, q, q)

for i in range(q):
    for j in range(q):
        prod = 0
        for k in range(p):
            prod -= (e[(0, i, k)]*e[(1, j, k)])
        for l in range(p+q):
            if l>=p:
                prod += (e[(0, i, l)]*e[(0, j, l)])
        HE[i,j] = prod

# Making variables for Hermitian forms h[(i,j,k)], the first variable should control the
    wedge degree, the other two
# variables control that matrix entry.

# h = {(i,j,k): var("h_{}{}{}".format(i,j,k)) for i in range(1,q+1) for j in range(binomial(
    q, floor(q/2))) for k in
# range(binomial(q, floor(q/2)))}.

h = {(i): var("h_{}".format(i)) for i in range(q+1)}

# creating a list by which to index Hermitian matrices.  We give i a range such that that
    the index matches the wedge power.

for i in range(q+1):
    h[i] = matrix(SR, binomial(q, i), binomial(q, i))

# Here we set the values for each entry of each Hermitian matrix by use of the minor
    definiton of the inner product
# for k-wedges.

h[0][0,0] = 1

for i in range(1, q+1):
    for z in L[i]:
        for w in L[i]:
            h[i][L[i].index(z), L[i].index(w)] = HE[z, w].det()
hinv = {(i): var("hinv_{}".format(i)) for i in range(q+1)}

for i in range(q+1):
    hinv[i] = h[i].inverse()

#Setting up HW to mean H-wedge, as in the hermiitan matrix of the total bundle.
```

```

HW = matrix(SR, 2^q, 2^q)

for i in range(q+1):
    for j in range(binomial(q, i)):
        for k in range(binomial(q, i)):
            HW[j+F(i), k+F(i)] = h[i][j,k]

#Computing SveStar as a matrix for the total bundle.

SVEstr = (HW*(SVEbar)*(HW.inverse())).transpose()

# We create purely symbol versions of the x coordinates to speed up the computation of the sve* coefficients.

X = {(i): var("X_{}".format(i), latex_name="X_{}") for i in range(2*p*q)}

# Now we turn the entries of SVEstr into mixed forms. The notation is setup so that Cform[i][j][k] means the kth coefficient
# of the action of svestar on the jth vector of the ith wedge.

Cform = [[[M.mixed_form(comp=([SVEstr[k+F(i+1),j+F(i)]]+P)) for k in range(binomial(q,i+1))] for j in range(binomial(q, i))] for i in range(q)]

Cform[0][0][0]

# Here we pre-compute the exterior derivatives as they will probably be called upon anumber of times, first some labels.
# dC is going to be the mixed form differential of the C coefficients, so first we creat diff_forms dCdiff.

dCdiff = [[[M.diff_form(1, name='dCdiff_{}{}'.format(i,j,k)) for k in range(binomial(q,i+1))] for j in range(binomial(q,i))] for i in range(q)]

dC = {(i,j,k): var("dC_{}{}{}".format(i,j,k), latex_name="dC_{{{}{}{}}}".format(i,j,k)) for i in range(q) for j in range(binomial(q,i)) for k in range(binomial(q,i+1))}

for i in range(q):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i+1)):
            for l in range(2*p*q):
                dCdiff[i][j][k][l] = (diff(Cform[i][j][k][0].expr().subs({x[m] : X[m] for m in range(2*p*q)}),X[l])).subs({X[n] : 0 for n in range(2*p*q)})

for i in range(q):
```

```
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i+1)):
            dC[i,j,k] = M.mixed_form(comp=([0]+[dCdiff[i][j][k]]+Q))

# Setting up a labelling for the entries of the inverse Hermitian matrices as mixed forms.

Hinv = {(i): var("H_{}".format(i)) for i in range(q+1)}

#Setting the components for the mixed-form version of the matrices.

for i in range(q+1):
    Hinv[i] = [[M.mixed_form(comp=([hinv[i][j,k]]+P)) for j in range(binomial(q, i))] for k
    in range(binomial(q, i))]

# Setting up a labelling for the inverse Hermitian matrices as matrices of mixed forms.

Hminv = {(i): var("H_{}".format(i)) for i in range(q+1)}

# Initializing the inverse Hermitian matrices as mixed forms.

for i in range(q+1):
    Hminv[i] = matrix(M.mixed_form_algebra(), binomial(q, i), binomial(q, i))

# Setting the components of the inverse Hermitian mixed form matrices.

for i in range(q+1):
    for j in range(binomial(q, i)):
        for k in range(binomial(q, i)):
            Hminv[i][j,k] = Hinv[i][j][k]

# Making a list to label the differentials of Hermitian matrices as mixed form matrices.

dH = {(i): var("dH_{}".format(i)) for i in range(q+1)}

# Initializing entries of derivative Hermitian matrices as 1-forms.

for i in range(q+1):
    dH[i] = [[M.diff_form(1, name='dh_{}{}'.format(j, k)) for j in range(binomial(q, i))]
    for k in range(binomial(q,i))]

# Setting the components of the 1-forms.

for i in range(q+1):
    for j in range(binomial(q, i)):
        for k in range(binomial(q, i)):
```

```
            for l in range((p*q)):
                dH[i][j][k][eU,l] = diff(h[i][j, k], x[l])

# Making labels for the mixed form Hermitian derivative matrices.

DH = {(i): var("DH_{}".format(i)) for i in range(q+1)}

# Setting the components of the mixed forms for the matrix entries of the derivative of the Hermitian matrices.

for i in range(q+1):
    DH[i] = [[M.mixed_form(comp=([0]+[dH[i][j][k]]+Q)) for k in range(binomial(q, i))] for j in range(binomial(q, i))]

# Making a list of variables to label the matrices of derivatives of Hermitin matrices as mixed matrices.

DHm = {(i): var("DHm_{}".format(i)) for i in range(q+1)}

# Initializing matrices to be the derivatives of the Hermitian form matrices.

for i in range(q+1):
    DHm[i] = matrix(M.mixed_form_algebra(), binomial(q, i), binomial(q, i))

# Setting the components of the derivatives of the Hermitian matrcies as mixed matrices.

for i in range(q+1):
    for j in range(binomial(q, i)):
        for k in range(binomial(q, i)):
            DHm[i][j,k] = DH[i][j][k]

# Setting labels for the connections matrices.

omega = {(i): var("DHm_{}".format(i)) for i in range(1,q+1)}

# Computing the connection matrices.

for i in range(q+1):
    omega[i]=Hminv[i]*DHm[i]

# Creating a list of variables to index curvature matrix entries.

o = {(i): var("o_{}".format(i)) for i in range(q+1)}

# Initializing 2-forms for the curvature matrices.
```

```
for i in range(q+1):
    o[i] = [[M.diff_form(2, name='o_{}{}'.format(j, k)) for k in range(binomial(q, i))] for
    j in range(binomial(q,i))]

# setting the components of the 2-forms to be the derivatives of the components of the
    connection matrices.

for i in range(q+1):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i)):
            for l in range(p*q):
                for m in range(p*q, 2*p*q):
                    o[i][j][k][eU, m, l] = diff(omega[i][j,k][1][l].expr(), x[m])

#Creating a list of zeros for indexing mixed forms.

R = []
for i in range((2*p*q)-2):
    R.append(0)

# Creating labels for the mixed-forms of the curvature matrix.

O = {(i): var("o_{}".format(i)) for i in range(q+1)}

# Defining the mixed forms.

for i in range(q+1):
    O[i] = [[M.mixed_form(comp=([0,0]+[o[i][j][k]]+R)) for j in range(binomial(q, i))] for k
     in range(binomial(q, i))]

# Defining labels for the mixed form curvature matrices.

Omega = {(i): var("Omega_{}".format(i)) for i in range(q+1)}

# Initializng the curvature matrices of mixed forms.

for i in range(q+1):
    Omega[i] = matrix(M.mixed_form_algebra(), binomial(q, i), binomial(q, i))

# Setting the components of the mixed form curvature matrices.

for i in range(q+1):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i)):
```

```
            Omega[i][j,k] = O[i][j][k]
# We will break the computation of r_1 up into several small parts letting Svs mean SvStar, O omega, and d for d,
# we will look at dSv, SvO, OSv, dSvs, SvsO, OSvs.  We will write these as matrices, and then add them altogether.
# Note that the index i indicates coming from the ith component, and moving to either the i-1 componet for Sv terms, or the
# i+1 component for Svs terms.

dSv = {(i): var("NabSvd_{}".format(i), latex_name="dSv_{{}}".format(i)) for i in range(1, q+1)}

OSv = {(i): var("NabSvd_{}".format(i), latex_name="dSv_{{}}".format(i)) for i in range(1, q+1)}

SvO = {(i): var("NabSvd_{}".format(i), latex_name="dSv_{{}}".format(i)) for i in range(1, q+1)}

dSvs = {(i): var("NabSvd_{}".format(i), latex_name="dSv_{{}}".format(i)) for i in range(q)}

OSvs = {(i): var("NabSvd_{}".format(i), latex_name="dSv_{{}}".format(i)) for i in range(q)}

SvsO = {(i): var("NabSvd_{}".format(i), latex_name="dSv_{{}}".format(i)) for i in range(q)}

# I initialize these as matrices.

for i in range(1,q+1):
    dSv[i] = matrix(M.mixed_form_algebra(), binomial(q, i-1), binomial(q,i))

for i in range(1,q+1):
    OSv[i] = matrix(M.mixed_form_algebra(), binomial(q, i-1), binomial(q,i))

for i in range(1,q+1):
    SvO[i] = matrix(M.mixed_form_algebra(), binomial(q, i-1), binomial(q,i))

for i in range(q):
    dSvs[i] = matrix(M.mixed_form_algebra(), binomial(q, i+1), binomial(q,i))

for i in range(q):
    OSvs[i] = matrix(M.mixed_form_algebra(), binomial(q, i+1), binomial(q,i))

for i in range(q):
    SvsO[i] = matrix(M.mixed_form_algebra(), binomial(q, i+1), binomial(q,i))

# Finally, I compute their components.
```

```
for i in range(1, q+1):
    for j in range(binomial(q, i)):
        for z in L[i][j]:
            templist = []
            for w in L[i][j]:
                if w != z:
                    templist.append(w)
            dSv[i][L[i-1].index(templist), j] = (-1)^(L[i][j].index(z))*dSve[z]

for i in range(1,q+1):
    for j in range(binomial(q,i)):
        for k in range(binomial(q, i-1)):
            Sum = 0
            for l in L[i][j]:
                templist = []
                for m in L[i][j]:
                    if m != l:
                        templist.append(m)
                Sum += (-1)^(L[i][j].index(l))*Sve[l]*omega[i-1][k, L[i-1].index(templist)]
            OSv[i][k,j] = Sum

# This one has not been checked by hand for p=2.

for i in range(1,q+1):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i)):
            for l in L[i][k]:
                templist = []
                for m in L[i][k]:
                    if m != l:
                        templist.append(m)
                SvO[i][L[i-1].index(templist), j] += (-1)^(L[i][k].index(l))*omega[i][k,j]*
    Sve[l]

for i in range(q):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i+1)):
            dSvs[i][k,j] = dC[(i,j,k)]

for i in range(q):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i+1)):
            Sum = 0
            for l in range(binomial(q,i+1)):
```

```
                Sum += Cform[i][j][l]*omega[i+1][k,l]
            OSvs[i][k,j] = Sum

for i in range(q):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i+1)):
            Sum = 0
            for l in range(binomial(q,i)):
                Sum += omega[i][l,j]*Cform[i][l][k]
            SvsO[i][k,j] = Sum

# Setting labels for the matrices describing when r_1 move up or down from wedge i.

r1down = {(i): var("r1down_{}".format(i), latex_name="r1down_{{}}".format(i)) for i in range(1,q+1)}

r1up = {(i): var("r1up_{}".format(i), latex_name="r1up_{{}}".format(i)) for i in range(q)}

for i in range(1,q+1):
    r1down[i] = matrix(M.mixed_form_algebra(), binomial(q,i-1), binomial(q,i))

for i in range(q):
    r1up[i] = matrix(M.mixed_form_algebra(), binomial(q,i+1), binomial(q,i))

for i in range(1,q+1):
    r1down[i] = dSv[i]+OSv[i]+SvO[i]

for i in range(q):
    r1up[i] = dSvs[i]+OSvs[i]+SvsO[i]

# Initializing r_1.

r_1 = matrix(M.mixed_form_algebra(), 2^q, 2^q)

# Setting the entries of r_1.  .

for i in range(1,q+1):
    for j in range(binomial(q, i-1)):
        for k in range(binomial(q,i)):
            r_1[j+F(i-1),k+F(i)] += r1down[i][j,k]

for i in range(q):
    for j in range(binomial(q, i+1)):
        for k in range(binomial(q, i)):
            r_1[j+F(i+1), k+F(i)] += r1up[i][j,k]
```

```

# Initializing the 2-component of the curvature.

r_2 = matrix(M.mixed_form_algebra(), 2^q, 2^q)

# Setting the entries of the 2-component of the curvature, from the entires of the separate curvature matrices.

for i in range(q+1):
    for j in range(binomial(q,i)):
        for k in range(binomial(q,i)):
            r_2[j+F(i),k+F(i)] = Omega[i][j,k]


r_1EvalDForm = [[M.diff_form(1, name='r_1EvalDForm_{}{}'.format(i,j)) for j in range(2^q)] for i in range(2^q)]

for i in range(2^q):
    for j in range(2^q):
        for k in range(2*p*q):
            r_1EvalDForm[i][j][k] = r_1[i,j][1][k].expr().subs({x[i] : 0 for i in range(2*p*q)})

r_1EvalMForm = [[M.mixed_form(comp=([0]+[r_1EvalDForm[i][j]]+Q)) for j in range(2^q)] for i in range(2^q)]

r_2EvalDForm = [[M.diff_form(2, name='r_2EvalDForm_{}{}'.format(i,j)) for j in range(2^q)] for i in range(2^q)]

for i in range(2^q):
    for j in range(2^q):
        for l in range(p*q):
            for m in range(p*q, 2*p*q):
                r_2EvalDForm[i][j][m, l] = r_2[i,j][2][m, l].expr().subs({x[z] : 0 for z in range(2*p*q)})

r_2EvalMForm = [[M.mixed_form(comp=([0, 0]+[r_2EvalDForm[i][j]]+R)) for j in range(2^q)] for i in range(2^q)]


# Here we evaluate the matrices at the origin which belongs to our manifold, thus reducing the runtimes.  Note that we could
# replace this with evaluation at any other point.

r_1Eval = matrix(M.mixed_form_algebra(), 2^q, 2^q)
```

```
r_2Eval = matrix(M.mixed_form_algebra(), 2^q, 2^q)

for i in range(2^q):
    for j in range(2^q):
        r_1Eval[i,j] = r_1EvalMForm[i][j]
        r_2Eval[i,j] = r_2EvalMForm[i][j]


abc = [[[]],[[1], [2]]]

for i in range(2, 2*p*q+1):
    abcd = [1, 2]
    add = []
    for j in Tuples(abcd, i).list():
        Sum = 0
        for k in j:
            Sum += k
        if Sum == 2*(q-1) or Sum == 2*q:
            add.append(j)
    abc.append(add)

def is_cyc_perm (list1, list2):
    if len (list1) == len (list2):
        for shift in range (len (list1)):
            for i in range (len (list1)):
                if list1 [i] != list2 [(i + shift) % len (list1)]:
                    break
            else:
                return True
        else:
            return False
    else:
        return False

# Now we sift through Rnew, realizing that pairs of elements which are cyclic permutations
    of each other obtained by swapping
# the order of multiplication of endomorphisms, are in fact the same. (This only holds for
    even endos, which r_1 and r_2 are).
# I let it run over j so many times because I found running it once or twice it
# might miss an equivalence.  I think I have it so that it runs an appropriate number of
    times so that everything will be
# accounted for.

for j in range(binomial((2*p*q+1), floor((2*p*q+1)/2))):
        for i in range(2, 2*p*q+1):
```

```
            for z in abc[i]:
                for w in abc[i]:
                    if is_cyc_perm(z,w):
                        abc[i].remove(w)
                        abc[i].append(z)

# We create UR to be the union of Rnew, and then we shuffle things to make sure it always
    begins with a 1.

UR = []
for thing in abc:
    for stuff in thing:
        if stuff != []:
            UR.append(stuff)

for item in UR:
    for i in range(len(item)):
        if item[0] == 2:
            item.remove(2)
            item.append(2)

# UR cond creates a list which is UR with duplicates removed.

URcond = []
for i in UR:
    add = True
    for (j,k) in URcond:
        if i == j:
            add = False
    if add:
        One_Counter = 0 # this counts the number of 1's, because 2mod 4 r_1's gives minus,
    while 0mod 4 give +.
        for l in i:
            if l == 1:
                One_Counter += 1
        URcond.append((i, One_Counter % 4))

# URmult is a list that keeps track of the multiplicities of the elements of UR.

URmult = []
for (i,j) in URcond:
    count = 0
    for k in UR:
        if i == k:
            count = count + 1
```

```
    URmult.append(((i,j), count))

# URlen organizes URmult into collections based on number of matrix factors.  (which affects
     the constant out in front)

URlen = []
for i in range(2*p*q+1):
    URlen.append([])
for ((j,k),l) in URmult:
    URlen[len(j)].append(((j,k),l))

for i in range(2^q):
    for j in range(2^q):
        r_1Eval[i, j] = (I*sqrt(2*pi))*r_1Eval[i, j]

# Labelling r_1 and r_2 so that they can be called upon and computed from Rnew's
    arrangements.

RListTracker = {(i): var("R_{}".format(i), latex_name="R_{{}}".format(i)) for i in range
    (1,3)}
RListTracker[1] = r_1Eval
RListTracker[2] = r_2Eval

# Intiailizing a mixed form identity matrix.

Idform = M.mixed_form(comp=([1]+P))

IdMat = matrix(M.mixed_form_algebra(), 2^q, 2^q)

for i in range(2^q):
    IdMat[i,i] = Idform

# Setting up the first two terms of the taylor expansion of exp, however we don't write r_1
    because it vanishes in the
# supertrace(s).

E = IdMat+r_2Eval

# This computes the actual product of the powers of r_1+r_2, but instead of using URlen in
    previous programs, we run this with
# UR2q2 making it faster to compute the 2q-2 degree part of nu.

for i in range(2, 2*p*q+1):
    fac = factorial(i)
    for ((j,k),l) in URlen[i]:
```

```
        prod = IdMat
        for m in j:
            prod = prod*RListTracker[m]
        if k == 0:
            E += (prod.apply_map(lambda s: 1*((1/fac))*s))
        else:
            E += (prod.apply_map(lambda s: -1*((1/fac))*s))
SuperTrace = 0
for i in range(q+1):
    if (i % 2) == 0:
        for j in range(binomial(q, i)):
            SuperTrace += E[j+F(i), j+F(i)]
    else:
        for j in range(binomial(q, i)):
            SuperTrace -= E[j+F(i), j+F(i)]

SuperTraceN = 0
for i in range(q+1):
    if (i % 2) == 0:
        for j in range(binomial(q, i)):
            SuperTraceN -= i*E[j+F(i), j+F(i)]
    else:
        for j in range(binomial(q, i)):
            SuperTraceN += i*E[j+F(i), j+F(i)]

# Setting up the scalar term.

r_00 = 0
for i in range(q):
    r_00 += (SVEstr[i+1,0]*sve[i])

r_0 = r_00.subs({x[j] : 0 for j in range(2*p*q)})
r_0 = (-2*pi)*r_0

QQ = 0
for i in range(p):
    QQ += v[(0, i)]*v[(1, i)]
for i in range(p, p+q):
    QQ -= v[(0, i)]*v[(1, i)]

# We give Phi and Nu only in the relevent degrees.

Phi = exp(-pi*QQ)*((I/(2*pi))^q)*exp(r_0)*SuperTrace[2*q]
Nu = exp(-pi*QQ)*((I/(2*pi))^(q-1))*exp(r_0)*SuperTraceN[2*(q-1)]
# Defining the pieces of the operators to act on Phi and Nu, which suffices for the
```

```
    eigenvalue conditions.


Phi.display(eU)
def Dp(i,j,k,f):
    return ((v[(i,k)]*f)+((1/pi)*diff(f,v[(j, k)])))

def Dm(i,j,k,f):
    return ((v[(i,k)]*f)-((1/pi)*diff(f,v[(j, k)])))

def Opr_Alpha1(f):
    Sum = 0
    for i in range(p):
        Sum += Dm(1,0,i,Dp(0,1,i,f))
    return ((pi)/2)*Sum

def Opr_Alpha2(f):
    Sum = 0
    for i in range(p):
        Sum += Dm(0,1,i,Dp(1,0,i,f))
    return ((pi)/2)*Sum

def Opr_Mu1(f):
    Sum = 0
    for i in range(p,p+q):
        Sum -= Dm(0,1,i,Dp(1,0,i,f))
    return ((pi)/2)*Sum

def Opr_Mu2(f):
    Sum = 0
    for i in range(p,p+q):
        Sum -= Dm(1,0,i,Dp(0,1,i,f))
    return ((pi)/2)*Sum

def OP(f):
    return Opr_Alpha1(f)+Opr_Alpha2(f)+Opr_Mu1(f)+Opr_Mu2(f)+(p-q)*f

# Here we break the very operators from above open into different pieces in order to analyze
     the separate actions, which in turn
# should help us prove the r>1 case for Phi.

PhiPart_Alpha1 = [M.diff_form(2*q, name='PhiPartAlpha1_{}'.format(i)) for i in range(len(
    WedgeList[2*q]))]
for j in range(len(WedgeList[2*q])):
    PhiPart_Alpha1[j][WedgeList[2*q][j]] = Opr_Alpha1(Phi[WedgeList[2*q][j]].expr())
```

```

OprPhi_Alpha1 = 0
for k in range(len(WedgeList[2*q])):
    OprPhi_Alpha1 += PhiPart_Alpha1[k]

PhiPart_Alpha2 = [M.diff_form(2*q, name='PhiPartAlpha2_{}'.format(i)) for i in range(len(
    WedgeList[2*q]))]
for j in range(len(WedgeList[2*q])):
    PhiPart_Alpha2[j][WedgeList[2*q][j]] = Opr_Alpha2(Phi[WedgeList[2*q][j]].expr())

OprPhi_Alpha2 = 0
for k in range(len(WedgeList[2*q])):
    OprPhi_Alpha2 += PhiPart_Alpha2[k]

PhiPart_Mu1 = [M.diff_form(2*q, name='PhiPartMu1_{}'.format(i)) for i in range(len(WedgeList
    [2*q]))]
for j in range(len(WedgeList[2*q])):
    PhiPart_Mu1[j][WedgeList[2*q][j]] = Opr_Mu1(Phi[WedgeList[2*q][j]].expr())
OprPhi_Mu1 = 0
for k in range(len(WedgeList[2*q])):
    OprPhi_Mu1 += PhiPart_Mu1[k]

PhiPart_Mu2 = [M.diff_form(2*q, name='PhiPartMu2_{}'.format(i)) for i in range(len(WedgeList
    [2*q]))]
for j in range(len(WedgeList[2*q])):
    PhiPart_Mu2[j][WedgeList[2*q][j]] = Opr_Mu2(Phi[WedgeList[2*q][j]].expr())

OprPhi_Mu2 = 0
for k in range(len(WedgeList[2*q])):
    OprPhi_Mu2 += PhiPart_Mu2[k]
if OprPhi_Alpha1 == q*Phi:
    print('Eigen')
else:
    print('no')
if OprPhi_Alpha2 == q*Phi:
    print('Eigen')
else:
    print('no')
if OprPhi_Mu1 == 0:
    print('killed')
else:
    print('no')
if OprPhi_Mu2 == 0:
    print('killed')
else:
```

```
    print('no')
# Here we break the very operators from above open into different pieces in order to analyze
     the separate actions, which in turn
# should help us prove the r>1 case for Nu.

if q == 1:
    OprNu_Alpha1 = Opr_Alpha1(Nu.expr())
else:
    NuPart_Alpha1 = [M.diff_form(2*(q-1), name='NuPart_{}'.format(i)) for i in range(len(
    WedgeList[2*(q-1)]))]
    for i in range(len(WedgeList[2*(q-1)])):
        NuPart_Alpha1[i][WedgeList[2*(q-1)][i]] = Opr_Alpha1(Nu[WedgeList[2*(q-1)][i]].expr
    ())
    OprNu_Alpha1 = 0
    for i in range(len(WedgeList[2*(q-1)])):
            OprNu_Alpha1 += NuPart_Alpha1[i]

if q == 1:
    OprNu_Alpha2 = Opr_Alpha2(Nu.expr())
else:
    NuPart_Alpha2 = [M.diff_form(2*(q-1), name='NuPart_{}'.format(i)) for i in range(len(
    WedgeList[2*(q-1)]))]
    for i in range(len(WedgeList[2*(q-1)])):
        NuPart_Alpha2[i][WedgeList[2*(q-1)][i]] = Opr_Alpha2(Nu[WedgeList[2*(q-1)][i]].expr
    ())
    OprNu_Alpha2 = 0
    for i in range(len(WedgeList[2*(q-1)])):
            OprNu_Alpha2 += NuPart_Alpha2[i]

if q == 1:
    OprNu_Mu1 = Opr_Mu1(Nu.expr())
else:
    NuPart_Mu1 = [M.diff_form(2*(q-1), name='NuPart_{}'.format(i)) for i in range(len(
    WedgeList[2*(q-1)]))]
    for j in range(len(WedgeList[2*(q-1)])):
        NuPart_Mu1[j][WedgeList[2*(q-1)][j]] = Opr_Mu1(Nu[WedgeList[2*(q-1)][j]].expr())
    OprNu_Mu1 = 0
    for k in range(len(WedgeList[2*(q-1)])):
            OprNu_Mu1 += NuPart_Mu1[k]

if q == 1:
    OprNu_Mu2 = Opr_Mu2(Nu.expr())
else:
    NuPart_Mu2 = [M.diff_form(2*(q-1), name='NuPart_{}'.format(i)) for i in range(len(
    WedgeList[2*(q-1)]))]
```

```
    for j in range(len(WedgeList[2*(q-1)])):
        NuPart_Mu2[j][WedgeList[2*(q-1)][j]] = Opr_Mu2(Nu[WedgeList[2*(q-1)][j]].expr())
    OprNu_Mu2 = 0
    for k in range(len(WedgeList[2*(q-1)])):
            OprNu_Mu2 += NuPart_Mu2[k]

if OprNu_Alpha1 == (q-1)*Nu:
    print('Eigen')
else:
    print('no')

if OprNu_Alpha2 == (q-1)*Nu:
    print('Eigen')
else:
    print('no')

if OprNu_Mu1 == 0:
    print('killed')
else:
    print('no')

if OprNu_Mu2 == 0:
    print('killed')
else:
    print('no')
# Here begins the r = 2 code.
# We want to deal with the case when r>1 now, so we set up a list of r vectors indexed by the first coordinate, the second
# controls conjuagtion, and the third is the component.  That is, vr[i,0,k] is the kth component of the ith vector, and v[i,1,k]
# is its conjugate.

vr = {(i,j,k): var("v_{}{}{}".format(i,j,k), latex_name="v_{{{}{}{}}}") for i in range(2) for j in range(2) for k in range(p+q)}
# Everything seems to work fine here, but in other versions it doesn't like that we index the scalar part of PhiForm and NuForm.
# This should get changed eventually, but for now I'll just focus on updating it in Copy5-Copy2.

# Now we set up a protocol to compute Phi(v_i), by simply replacing the instances of v[i,j] with the appropriate instance of
# vr[j,k,l].  So we want a list Phiv[i], each of which is a mixed form, being Phi with all the v[i,j] swapped out.  Thus we
# first need to build differential forms for each component.

```

```
# Then we set-up the differential forms.

PhivForm = M.diff_form(2*q)

NuvForm = M.diff_form(2*(q-1))

for k in WedgeList[2*q]:
    PhivForm[k] = ((Phi[k].expr()).subs({v[0,l] : vr[0,0,l] for l in range(p+q)})).subs({v
    [1,m] : vr[0,1,m] for m in range(p+q)})

if q == 1:
    NuvForm = ((Nu.expr()).subs({v[0,l] : vr[1,0,l] for l in range(p+q)})).subs({v[1,m] : vr
    [1,1,m] for m in range(p+q)})
else:
    for k in WedgeList[2*(q-1)]:
        NuvForm[k] = ((Nu[k].expr()).subs({v[0,l] : vr[1,0,l] for l in range(p+q)})).subs({v
    [1,m] : vr[1,1,m] for m in range(p+q)})

# In the original we compute Nu(v), but We'll set up a List NuV, where NuV[i] is the i'th
    term, that is
# NuV[i]=Nu(v_i)^Phi(v_1)^...^Phi(v_{i-1})^Phi(v_{i+1})^... Phi(v_r).
# However, we really only care about the piece with highest weight, being NuV[r-1]

if q == 1:
    NuV = NuvForm*PhivForm
else:
    NuV = NuvForm.wedge(PhivForm)
# Here we try to develop the operators for Appendix B Funke-Hoffman, that may may analyze r
    >1.  First we define delta functions.

delta = matrix(SR, 2, 2)

for i in range(2):
    delta[i,i] = 1
# Now we start to build the operators piece by piece.  Since it doesn't affect the form
    degree, we can first define the operator
# as acting on the scalar parts, and then worry about stitching it together into a mixed
    form.

# This first operator is the terms of the sum of for the first p terms.

def W0a(i,j,f):
    Sum = 0
    for k in range(p):
        Sum += vr[j,1,k]*(vr[i,0,k]*f+(1/pi)*diff(f,vr[i,1,k])) - (1/pi)*diff(vr[i,0,k]*f
```

```
    +(1/pi)*diff(f,vr[i,1,k]),vr[j,0,k])
    for l in range(p,p+q):
        Sum -= vr[i,0,l]*(vr[j,1,l]*f+(1/pi)*diff(f,vr[j,0,l])) - (1/pi)*diff(vr[j,1,l]*f
    +(1/pi)*diff(f,vr[j,0,l]),vr[i,1,l])
    return ((pi/2)*Sum + ((p-q)/2)*delta[i,j]*f)

def WOu(i,j,f):
    Sum = 0
    for k in range(p):
        Sum+= vr[i,0,k]*(vr[j,1,k]*f+(1/pi)*diff(f,vr[j,0,k])) - (1/pi)*diff(vr[j,1,k]*f+(1/
    pi)*diff(f,vr[j,0,k]), vr[i,1,k])
    for l in range(p,p+q):
        Sum -= vr[j,1,l]*(vr[i,0,l]*f+(1/pi)*diff(f,vr[i,1,l])) - (1/pi)*diff(vr[i,0,l]*f
    +(1/pi)*diff(f,vr[i,1,l]), vr[j,0,l])
    return -((pi/2)*Sum - ((p-q)/2)*delta[i,j]*f)
# Okay... Things are going weird here with the roots, trying to find out when they really
    get killed.

Kila = True

for i in WedgeList[2*(2*q-1)]:
    if (WOa(1,0,NuV[i].expr())).expand() == 0:
        pass
    else:
        print(i)
        Kila = False
print(Kila)
Kilu = True

# Note that there's an issue here with Sage sometimes the comparison (esp. for forms) !=0
    will not register correctly,
# HOWEVER for some reason == always works, or at least it seems to.  Therefore, in the below
     I have made a fairly silly looking
# if statement to basically use != via ==.

for i in WedgeList[2*(2*q-1)]:
    if (WOu(0, 1,NuV[i].expr())).expand() == 0:
        pass
    else:
        print(i)
        Kilu = False
print(Kilu)
```

Index

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