Hardy Inequalities

by

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Preface

The main theme of this thesis is further scrutinizing classic Hardy inequalities and expanding the study carried out by my advisor Dr. Craig Cowan on "Optimal Hardy Inequality for General Elliptic Operators with Improvement" [8]. We rediscovered explicit integral form of Hardy inequality with the main focus on its functional aspects, including density of Sobolev space. We investigated Hardy inequality for E as a positive function in Ω and $\mu = -div(A\nabla E)$ where $A(x)$ is a $n \times n$ symmetric, uniformly positive definite matrix in Ω. Based on two main directions of function E, we examined Hardy type inequalities:

- $E \in H_0^1(\Omega)$ which is called 'Boundary Weight' in Ω
- $E \in C^\infty(\overline{\Omega} \backslash K)$ where $K \subset \Omega$ is the compact support of μ and $E = \infty$ on K, so called 'Interior Weight'.

Moreover, it is shown that the inequalities obtained by two types of E are optimal and not-attained.

In Chapter 1, we establish mathematical tools and the results of density in $W_0^{1,p}(\Omega)$ provides a basic characterization of functions in this space. The proof relies on showing that $W_0^{1,p}(\Omega\backslash K) = W_0^{1,p}(\Omega)$.

In Chapter 2, we discuss our motivation for exploring the main Hardy inequality, with a focus on the quadratic case $(p = 2)$. The fundamental solution of Laplacian is applied for illustrating the proof of classical Hardy inequality. Furthermore, we conclude the classic Hardy inequalities for a bounded domain in $\Omega \in \mathbb{R}^N$, $(N \geq 3)$, containing the origin and similar results for smooth boundary of Ω and $\delta(x) := dist(x, \partial\Omega)$. We will refer this inequality as Hardy's boundary inequality

In Chapter 3, the inequalities are considered for operators more general than Laplacian. One case of this is the result obtained by Adimurthi and Sekar [3]. Various Hardy inequalities are explored in terms of different boundary weight and interior wight of function E.

In Chapter 4, we investigate optimal weighted versions of aforementioned inequalities for $(p \neq 2)$. The generalization of Cafferelli-Kohn-Nirenberg inequality are examined to find the optimal and not attained constant. Furthermore, the possibility of more general weighted inequalities is investigated.

In Chapter 5, we consider one common type of improvement for the above-mentioned inequalities. The method we use for two types of function E was firstly adopted by Ghoussoub and Moradifam [13].

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Introduction

The Hardy inequality was originally discovered by Hardy in the one-dimensional case and later extended to higher dimensions [17],[6],[7],[10],[11],[15],[21]. The development of the famous Hardy inequality has received plenty of attention since 1960s. These topics are mainly about scaling invariant inequalities including integral of functions and their derivatives in various powers, and are well established in nonlinear analysis and PDEs [16]. These inequalities have numerous applications in geometry and mathematical physics [17], [16]. By adding positive terms to the right hand side of the Hardy type inequality, improved versions of the aforementioned inequalities have been generated. The corresponding results play an essential role in critical phenomena in elliptic and polynomial PDEs [19], [9].

1.1 Mathematical Tools

In this part some notations and tools are provided which will be applied throughout the thesis.

Definition 1.1. *For function E in Ω we have:*

- $E_+ = \max\{E(x), 0\}$,
- $E_- = \max\{-E(x), 0\}$,

and we have $E = E^+ - E^-$.

Theorem 1.2. *Co-area Formula*

$$\int_\Omega f(x)|\nabla u(x)|dx = \int_\mathbb{R} \left(\int_{\{x\in\Omega:u(x)=t\}} f(x)dH^{N-1}(x) \right) dt,$$

where $u \in C^1(\bar{\Omega})$ and H^{N-1} is the $N-1$ dimensional surface measure.

Remark 1.3. *(Notation for Derivatives).*

Suppose $\phi : \Omega \to \mathbb{R}$ and $x \in \Omega$, we denote:

- $\phi_{x_i} = \frac{\partial \phi}{\partial x_i} = \lim_{h\to 0} \frac{\phi(x+he_i)-\phi(x)}{h}$ *(if the limit exists),*
- $\nabla\phi := (\phi_{x_1}, \cdots, \phi_{x_n}) =$ *gradient vector,*
- Δu *is the Laplacian of a real valued smooth function u.*
- *For $i = 1, \cdots, n$, $\nabla_i f(x)$ is the partial derivative of function f at x in the direction of the vector whose coordinates are 0 except in the ith, which equals 1.*

Definition 1.4. *Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multi-index. We say that V is the α^{th}-weak partial derivative of u, written*

$$\partial^\alpha u = v,$$

provided

$$\int_\Omega u\partial^\alpha \phi dx = (-1)^{|\alpha|}\int_\Omega v\phi dx,$$

for all test functions $\phi \in C_c^\infty(\Omega)$.

Definition 1.5. *(Interior Ball Condition). There exists an open ball $B \subset \Omega$ such that for $x_0 \in \partial\Omega$ we have $x_0 \in \partial B$.*

Note that the interior ball condition automatically holds if $\partial\Omega$ is C^2.

Lemma 1.6. *(Hopf's Lemma), Suppose Ω is a smooth, open and bounded domain in $\mathbb{R}^N$ and suppose $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies $-\Delta u \le 0$ in Ω and there exists $x_0 \in \partial\Omega$ such that*

$$u(x_0) > u(x) \quad \forall x \in \Omega,$$

and Ω satisfies the interior ball condition at x_0 [Definition 1.5], then

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the outer unit normal to $\partial\Omega$ at x_0. The importance is the strict inequality.

Definition 1.7. *If $G : \mathbb{R} \to \mathbb{R}$ is a smooth function, and for $g = G'$ we have $|g(z)| \le C(|z| + 1)$ and $|G(z)| \le C(|z|^2 + 1)$ for some constant C. It is defined*

$$I[w] := \int_\Omega |\nabla w|^2 dx,$$

for all function w such that $w = 0$ on $\partial\Omega$, and

$$J[w] := \int_\Omega G(w)dx = 0,$$

and the appropriate admissible class is

$$\mathscr{A} := \{w \in H_0^1(\Omega) \quad s.t \quad J[w] = 0\}.$$

Theorem 1.8. *(Existence of Constrained Minimizer). Assume the admissible set $\mathscr{A}$ is nonempty. Then there exists $u \in \mathscr{A}$ such that*

$$I[u] = \min_{\omega \in \mathscr{A}} I[\omega],$$

where the problem of minimizing, say, energy functional is defined as

$$I[u] := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx,$$

among all function u belonging to the set $\mathscr{A}$.

Theorem 1.9. *(Harnack Inequality). Let Ω be connected subset of $\mathbb{R}^N$ and suppose $u \in C^2(\Omega)$ is harmonic. If $u \geq 0$ then for all $\Omega_0 \subset\subset \Omega$ there is some $C = C(\Omega_0, \Omega)$ (independent of u) such that*

$$\sup_{\Omega_0} u \leq C \inf_{\Omega_0} u.$$

Theorem 1.10. *(Strong Maximum Principle). Assume Ω is connected bounded in $\mathbb{R}^N$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic. Then if u attains its max or min over $\overline{\Omega}$ at some point $x_0 \in \Omega$ then u is constant.*

Theorem 1.11. *(Global Approximation by Smooth Functions). Assume Ω is bounded and suppose as well that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that*

$$u_m \to u \quad in \quad W^{k,p}(\Omega).$$

Definition 1.12. *Let H be a vector space. A function $< .,. >: H \times H \to \mathbb{C}$ is called an inner product if*

- $< \lambda_1 x_1 + \lambda_2 x_2, y >= \lambda_1 < x_1, y > + \lambda_2 < x_2, y >,$

- $< x,y >= \overline{< y,x >}$,

- $< x,x >> 0$ *if* $x \neq 0$,

for every $x_1, x_2, x, y \in H$ *and* $\lambda_1, \lambda_2 \in \mathbb{C}$ *and in particular*

$$< x, \lambda_1 y_1 + \lambda_2 y_2 >= \overline{\lambda_1} < x, y_1 > + \overline{\lambda_2} < x, y_2 >,$$

for every $x, y_1, y_2 \in H$ *and* $\lambda_1, \lambda_2 \in \mathbb{C}$, *it means an inner product is not quite bilinear (it is linear in the first variable, and conjugate linear in the second), moreover, note that*

$$< x,0 >=< 0,x >= 0 \quad x \in H.$$

Theorem 1.13. *(Riesz Representation Theorem). Let H be a Hilbert space. Then, there is an isomorphic conjugate-linear isomorphism* $\Phi : H \to H^*$ *such that*

$$(\Phi(x))(y) =< y,x >,$$

for $x,y \in H$.

Theorem 1.14. *(Riesz Representation Theorem). For any Hilbert space H,* $H^* \simeq H$. *In particular, for every* $x \in H$,

$$\phi_x(y) \equiv < x,y >$$

is bounded and has norm $\|\phi_x\|_{H^*} = \|x\|$. *Furthermore, for every bounded linear functional* $\phi \in H^*$ *there exists a unique* $x_1 \in H$ *such that*

$$\phi(y) =< x_1, y >,$$

for all $y \in H$.

Remark 1.15. *In Banach spaces X a sequence* $\{f_n\}$ *converges weakly to* f *if*

$$\phi(f_n) \to \phi(f),$$

fo all $\phi \in X^*$, *where* X^* *is the dual of* X.

In the case of Hilbert space H, every element of the dual space is realized by an element of H according to Reisz Representation Theorem, $f_n \to f$ *weakly if and only if*

$$< f_n, \phi > \to < f, \phi >$$

for all $\phi \in H$.

Theorem 1.16. *Every bounded sequence in Hilbert space H, has a subsequence which converges in the weak topology of H.*

Definition 1.17. *If* $\Omega \subset \mathbb{R}^N$ *is open and* $\varepsilon > 0$, *we write*

$$\Omega_\varepsilon := \{x \in \Omega \quad s.t \quad dist(x, \partial\Omega) > \varepsilon\},$$

and for $x \in \mathbb{R}^N$ *and* $\varepsilon > 0$ *we let*

$$B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}.$$

Definition 1.18.

- (i) *Define* $\eta \in C^\infty(\mathbb{R}^N)$ *by*

$$\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & if \quad |x| < 1 \\ 0 & if \quad |x| > 1, \end{cases}$$

  the constant $C > 0$ *selected so that* $\int_{\mathbb{R}^N} \eta\, dx = 1$.

- *(ii) For each* $\varepsilon > 0$, *set*

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right),$$

η is the standard mollifier. The function η_ε are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1, \quad supp(\eta_\varepsilon) \subset B(0,\varepsilon).$$

Definition 1.19. *A function is called locally integrable if around every point in the domain, there is a neighborhood in which the function is integrable. The space of locally integrable functions is denoted by $L^1_{loc}(\Omega)$.*

Definition 1.20. *Suppose that $f \in L^1_{Loc}(\Omega)$ is a locally integrable function, for $\varepsilon > 0$, we define $f^\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by*

$$f^\varepsilon(x) := \int_\Omega \eta_\varepsilon(x-y) f(y) dy,$$

where η_ε is the mollifier and we define f^ε for $x \in \Omega_\varepsilon$ so that $B_\varepsilon(x) \subset \Omega$ and we can define the average of f and if $\Omega = \mathbb{R}^N$ we have $\Omega_\varepsilon = \mathbb{R}^N$.

Theorem 1.21. *(Properties of mollifiers). For function f^ε as a smooth approximation of f, we have:*

- *(i) $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$,*
- *(ii) $f^\varepsilon \to f$ a.e as $\varepsilon \to 0$,*
- *(iii) If $f \in C(\Omega)$, then $f^\varepsilon \to f$ uniformly on compact subsets of Ω,*
- *(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$ then $f^\varepsilon \to f$ in $L^p_{loc}(\Omega)$.*
- *(v) In particular, if $f \in W^{k,p}_{loc}(\Omega)$, for some $1 \leq p < \infty$ then*

$$f^\varepsilon \to f \quad in \quad W^{k,p}_{loc}(\Omega),$$

as $\varepsilon \to 0$.

Theorem 1.22. *Let Ω denote a smooth bounded domain in $\mathbb{R}^N$ and suppose $F \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and $\phi \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega F(x) \cdot \nabla \phi(x) dx = -\int_\Omega divF \phi dx + \int_{\partial\Omega} F\phi \cdot \nu dS.$$

Theorem 1.23. *If $u, v \in C^1(\overline{\Omega})$, then the application of the divergence theorem to the vector field $X = (0, 0, \cdots, uv, \cdots, 0)$ with i-th component uv, gives the integration by parts formula*

$$\int_\Omega u(v_{x_i}) dx = -\int_\Omega (u_{x_i}) v dx + \int_{\partial\Omega} uv\nu_i dS,$$

where $\nu = (\nu^1, \nu^2, \cdots, \nu^N)$ is the outward pointing normal on $\partial\Omega$.

Theorem 1.24. *If $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega (-\Delta u) v dx + \int_{\partial\Omega} v \partial_\nu u,$$

where $\partial_\nu u := \nabla u \cdot \nu$.

Lemma 1.25. *(Fatou's lemma) Let (f_n) be a sequence of functions in L^1 that satisfies*

- *for all n, $f_n \geq 0$ a.e,*
- $\sup_n \int f_n < \infty$ *for almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq +\infty$, then $f \in L^1$ and*

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

Theorem 1.26. *If $1 \leq p < \infty$ and if $\{f_n\}_{n=1}^\infty \subset L^p(X, \mathscr{A}, \mu)$ is such that $f_n \to f$ in $L^p(X, \mathscr{A}, \mu)$ then there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ μ-a.e.*

Definition 1.27. *Given $1 < p < \infty$, its conjugate is the number $1 < q < \infty$ satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

for $p = 1$ *we choose* $q = \infty$.

Lemma 1.28. *(Young's Inequality). Let* $a, b \geq 0$ *and* $p > 1, q < \infty$ *are conjugate, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 1.29. *(Hölder's inequality). Let* $1 \leq p \leq \infty$ *and* p *and* q *be conjugate. Let* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then the pointwise product, given by* $(fg)(x) = f(x)g(x)$ *is in* $L^1(\Omega)$ *and*

$$\left| \int_\Omega fg d\mu \right| \leq \int_\Omega |f||g| d\mu \leq \|f\|_p \|g\|_q.$$

Theorem 1.30. *(Minkowski's Inequality). Fix* $1 \leq p \leq \infty$ *and* $f, g : X \to \mathbb{C}$ *be measurable functions. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Theorem 1.31. *(Young's Convolution Inequality). Suppose that* $g \in L^p(\mathbb{R}^N)$ *and* $f \in L^q(\mathbb{R}^N)$*. Then,*

$$\|g * f\|_{L^r} \leq \|g\|_{L^p} \|f\|_{L^q},$$

for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Definition 1.32. *Let* X *be a topological space with a countable basis,* B *as a* σ*-algebra of subsets of* X *containing the open sets and* $\mu : B \to [0, \infty]$ *a* σ*-additive measure. The support of* μ *(denoted by* $supp(\mu)$*) is the complement of the union of all open sets which are* μ*-null sets, i.e. the smallest closed set* C *such that* $\mu(X \backslash C) = 0$*. The support can be defined also in case* μ *is a signed measure. In this case the support of* μ *is then defined as the support of its total variation measure.*

Equivalently, a signed measure μ *is supported on set* K *if the complement of* K *is null.*

Definition 1.33. *If* $K \subset \mathbb{R}^N$ *is a compact subset we define the box-counting dimension of*

K by

$$diam_{box}(K) := N - lim_{r\searrow 0}\frac{\log(H^N(K_r))}{log(r)},$$

where $K_r := \{x \in \Omega : dist(x,K) < r\}$ *and* H^N *is N-dimensional Lebesgue Measure.*

Example 1.34. *(Line segment) Let* e_1 *denote the unit vector in* $\mathbb{R}^N$ *and set* $K := \{te_1 : t \in [0,1]\}$, *then* $dim_{box}(K) = 1$, *since the topological dimension of K is* 1 *and*

$$diam_{box}(K) = N - \lim_{r\to 0}\frac{\log r^{N-1}}{logr} = 1.$$

$\triangle$

1.2 Density in $W_0^{1,p}$

Our next results prove a basic characterization of functions in $W_0^{1,p}(\Omega)$. Recall that $W_0^{1,p}(\Omega)$ as the completion of the $C_c^\infty(\Omega)$ with respect to norm $\|.\|_{1,p}$.

Theorem 1.35. *Let* Ω *denote a smooth bounded domain and let K denote a compact set in* Ω *with box counting dimension;* $dim_{box}(K) < N-p$ *where* $p \in (1,N)$. *Then* $W_0^{1,p}(\Omega\backslash K) = W_0^{1,p}(\Omega)$.

Before proving this theorem we look at the proof of special case of this.

Theorem 1.36. *Let* $1 \le p < N$ *and* Ω *denote a smooth bounded domain in* $\mathbb{R}^N$ *then* $C_c^\infty(\Omega\backslash\{0\})$ *is dense in* $W_0^{1,p}(\Omega)$.

Proof. Recall these spaces are defined to be the completion of $C_c^\infty(\Omega\backslash\{0\})$ and $C_c^\infty(\Omega)$ respectively, under the L^p gradient norm. Its clear that $W_0^{1,p}(\Omega\backslash\{0\}) \subset W_0^{1,p}(\Omega)$ and to show the other inclusion it is sufficient to show that if $\phi \in C_c^\infty(\Omega)$ that we can get close (in L^p gradient norm) by a sequence of functions in $C_c^\infty(\Omega\backslash\{0\})$. So towards this let $0 \le \gamma \le 1$ be smooth with $\gamma = 0$ in B_1 and $\gamma = 1$ in B_2^c (complement of B_2) and for $\varepsilon > 0$ small consider $\gamma_\varepsilon(x) := \gamma(\varepsilon^{-1}x)$. Then set $\phi_\varepsilon(x) = \gamma_\varepsilon(x)\phi(x)$ and note that $\phi_\varepsilon(x) \in C_c^\infty(\Omega\backslash\{0\})$ then we have:

$$\begin{aligned}\left(\int_\Omega |\nabla\phi_\varepsilon - \nabla\phi|^p\right)^{\frac{1}{p}} &= \left(\int_\Omega \left|\frac{1}{\varepsilon}\nabla(\gamma(x/\varepsilon))\phi + (\gamma_\varepsilon - 1)\nabla\phi\right|^p dx\right)^{\frac{1}{p}} \\ &\le \left(\int_{\varepsilon<|x|<2\varepsilon} \left|\varepsilon^{-1}\nabla\gamma(x/\varepsilon)\phi\right|^p dx\right)^{\frac{1}{p}} + \left(\int_{B_{2\varepsilon}} |(\gamma_\varepsilon - 1)\nabla\phi|^p dx\right)^{\frac{1}{p}},\end{aligned}$$

since

$$\nabla\phi_\varepsilon(x) = \gamma_\varepsilon(x)\nabla\phi(x) + \frac{1}{\varepsilon}\nabla\gamma(x/\varepsilon)\phi(x),$$

by the Lebesgue dominated convergence theorem, we have $\gamma_\varepsilon\nabla\phi \to \nabla\phi$ in L^p and

$$\int_{\varepsilon<|x|<2\varepsilon} \left|\frac{1}{\varepsilon}\nabla\gamma(x/\varepsilon)\phi(x)\right|^p dx \le \varepsilon^{N-p} \sup_{B_{2\varepsilon}} |\phi(x)|^p C_N \sup_{B_{2\varepsilon}} |\nabla\gamma(x)|^p \to 0,$$

as $\varepsilon \to 0$ since $p < N$. $\square$

Lemma 1.37. *[11] Suppose f is a real valued Lipschitzian function on an open subset K of $\mathbb{R}^N$, and g is a real valued continuous function on K such that*

$$f_{x_i}(x) = g(x) \quad \textit{whenever } f \textit{ is differentiable at } x,$$

then

$$f_{x_i}(x) = g(x) \quad \textit{for all } x \in K.$$

Here we gather some facts for the distance function, the results will play an essential role in the next chapters.

Definition 1.38. *If $\emptyset \neq K \subset \mathbb{R}^N$ is closed then $\delta_K : \mathbb{R}^N \to [0,\infty)$ is distance function to K and:*

$$\delta_K(x) := dist(x,K) = \inf_{y\in K} |x-y|,$$

where $x \in \mathbb{R}^N$. In addition if K_1 is the set of points in $\mathbb{R}^N$ which have a unique closest *point on K i.e.*

$$K_1 := \left\{x \in \mathbb{R}^N : \exists! y \in K \quad \textit{such that} \quad \delta_K(x) = |x-y|\right\},$$

and the function $\zeta_K : K_1 \to K$ associate each $x \in K_1$ with its unique $y \in K$ which $\delta_K(x) = |x-y|$ and we have

$$\nabla \delta_K(x) = \frac{x - \zeta_K(x)}{\delta_K(x)}. \tag{1.1}$$

Theorem 1.39. *[11] If $\emptyset \neq K \subset \mathbb{R}^N$ is closed then*

- *i) for all $x, y \in \mathbb{R}^N$ we have*

$$|\delta_K(x) - \delta_K(y)| \leq |x-y|,$$

- *ii) for $x \in \mathbb{R}^N$, δ_K is differentiable at x if and only if $x \in K_1$.*

Proof.

- i) Choosing $a \in K$ such that $\delta_K(x) = |x-a|$ we have

$$|\delta_K(x) - \delta_K(y)| \leq |x-a| - |y-a| \leq |x-y|,$$

- ii) First we prove ζ_K in (1.1) is continuous. By contradiction assume ζ_K is not continuous, so there exists an $\varepsilon > 0$ and a sequence $x_1, x_2, x_3, \cdots,$ of points of K_1 convergent to a point $x \in K_1$ such that

$$|\zeta_K(x_i) - \zeta_K(x)| \geq \varepsilon \quad i = 1, 2, 3, \cdots,$$

then

$$|\zeta_K(x_i) - x_i| = \delta_K(x_i)|\zeta_K(x_i) - x| \leq \delta_K(x) + 2|x_i - x|,$$

the points $\zeta_K(x_i)$ are in bounded subset of the closed set of K.

Consider the sequence $\zeta_K(x_1), \zeta_K(x_2), \zeta_K(x_3), \cdots$ converges to the point $a \in K$. We have

$$\delta_K(x) = \lim_{i\to\infty} \delta_K(x_i) = \lim_{i\to\infty} |\zeta_K(x_i) - x_i| = |a - x|,$$

in fact $a = \zeta_K(x)$ so

$$|a - \zeta_K(x)| = \lim_{i\to\infty} |\zeta_K(x_i) - \zeta_K(x)| \geq \varepsilon,$$

a contradiction so ζ_K is continuous. Now since $\delta_K(x)$ is Lipschitz and ζ_K is continuous the RHS of (1.1) are continuous maps from K_1 into $\mathbb{R}^N$ and the elements of LHS of (1.1) are $\nabla_1 \delta_K(x), \cdots, \nabla_n \delta_K(x)$ so according to lemma (1.37) δ_K has continuous partial derivative on K_1. $\square$

Proof of Theorem 1.35 For proving the inclusion $W_0^{1,p}(\Omega) \subset W_0^{1,p}(\Omega\backslash K)$, we use Lipschitz function $\delta_K(x) := dist(x,K)$. Let $\psi \in C_c^\infty(\Omega)$ and we can get close by a sequence of functions in $C_c^\infty(\Omega\backslash K)$. For sufficiently small $\varepsilon > 0$ consider cutoff function as

$$\gamma_\varepsilon(x) = \begin{cases} 0 & \text{if} \quad \delta_K(x) < \varepsilon \\ \frac{\delta_K(x)}{\varepsilon} - 1 & \text{if} \quad \varepsilon < \delta_K(x) < 2\varepsilon \\ 1 & \text{if} \quad \delta_K(x) \geq 2\varepsilon, \end{cases}$$

note that according to Theorem 1.39, $\delta_K(x)$ is Lipschitz and is differentiable in K_1. and we have:

$$|\nabla\gamma_\varepsilon(x)| \leq \begin{cases} \frac{1}{\varepsilon} & \text{if} \quad \varepsilon < \delta_K(x) < 2\varepsilon \\ 0 & \text{if} \quad \text{otherwise}, \end{cases}$$

we consider $\psi_\varepsilon(x) = \gamma_\varepsilon(x)\psi(x)$ then we have

$$\left(\int_\Omega |\nabla\psi_\varepsilon - \nabla\psi|^p \, dx\right)^{\frac{1}{p}} = \left(\int_\Omega |\nabla\gamma_\varepsilon\psi + (\gamma_\varepsilon - 1)\nabla\psi|^p \, dx\right)^{\frac{1}{p}},$$

by Lebesgue dominated convergence theorem,

$$\gamma_\varepsilon \nabla \psi \to \nabla \psi,$$

in L^p norm and since $\psi(x)$ is bounded and $|\nabla \delta_K(x)| \leq 1$ we need to show that

$$\int_{\{x:\varepsilon<\delta_K(x)<2\varepsilon\}} \frac{|\nabla \delta_K(x)|^p}{\varepsilon^p} dx \leq \int_{\{x:\varepsilon<\delta_K(x)<2\varepsilon\}} \frac{1}{\varepsilon^p} dx \to 0, \tag{1.2}$$

note that

$$\int_{\{x:\varepsilon<\delta_K(x)<2\varepsilon\}} \left| \frac{1}{\varepsilon} \nabla \delta_K(x) \right|^p dx \leq \frac{|K_{2\varepsilon}|}{\varepsilon^p}, \tag{1.3}$$

if $dim_{box}(K) < N - p$ so there is some $t > 0$ such that $dim_{box}(K) = N - p - t$ and this is some $\rho(\varepsilon) \to 0$ such that

$$N - p - t = dim_{box}(K) = N - \frac{\log(|K_{2\varepsilon}|)}{\log(2\varepsilon)} - \rho(\varepsilon),$$

so we have

$$p + t = \frac{\log |K_{2\varepsilon}|}{\log(2\varepsilon)} + \rho(\varepsilon),$$

by rearranging

$$t - \rho(\varepsilon) = \frac{\log |K_{2\varepsilon}|}{\log(2\varepsilon)} - p, \tag{1.4}$$

raising both sides of (1.4) to the power of 2ε we have

$$(2\varepsilon)^{t-\rho(\varepsilon)} = \frac{(2\varepsilon)^{\frac{\log |K_{2\varepsilon}|}{\log(2\varepsilon)}}}{(2\varepsilon)^p},$$

using logarithm property concludes to:

$$\frac{|K_{2\varepsilon}|}{(2\varepsilon)^p} = (2\varepsilon)^{t-\rho(\varepsilon)}, \tag{1.5}$$

the right hand side of (1.5) goes to zero since $t > 0$, so (1.2) goes to zero.

$\square$

Remark 1.40. *Let $1 \leq p < \infty$; $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ so $C_c^\infty(\Omega)$ could equally well have been used instead of $C_c^1(\Omega)$ in the definition of $W_0^{1,p}(\Omega)$.*

2

Motivation of Hardy Inequality

Before starting the main results, we illustrate the proof of classical Hardy inequality using the fundamental solution of Laplacian. This is the main philosophy we adopt to obtain our main results in the next chapters.

Lemma 2.1. *Let Ω denote bounded domain in $\mathbb{R}^N$ with $\partial\Omega$ smooth. Suppose $E > 0$ is a smooth function with $-\Delta E(x) = f(x) \geq 0$ in Ω with $E = 0$ on $\partial\Omega$. Then*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx \geq 0, \tag{2.1}$$

for all $u \in H_0^1(\Omega)$. Moreover the constant $\frac{1}{4}$ is optimal in the sense that

$$\inf\left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx} : u \in H_0^1(\Omega)\backslash\{0\} \right\} = \frac{1}{4},$$

additionally the best constant is not attained.

Proof. Let $u \in C_c^\infty(\Omega)$ and $v := E^{\frac{-1}{2}} u$ then we have $u := E^{\frac{1}{2}} v$ so we have:

$$\nabla u = \frac{1}{2}\nabla E E^{\frac{-1}{2}} v + E^{\frac{1}{2}} \nabla v,$$

so we have

$$|\nabla u|^2 = \frac{1}{4}|\nabla E|^2 E^{-1} v^2 + E|\nabla v|^2 + \nabla E \cdot \nabla v v,$$

since

$$\int_\Omega \nabla E \cdot \nabla v v dx = \frac{1}{2}\int_\Omega \nabla E \cdot \nabla (v^2) dx,$$

and since $v^2 := E^{-1}u^2$ by rearranging $|\nabla u|^2$ we have

$$|\nabla u|^2 - \frac{1}{4}\frac{|\nabla E|^2}{E^2}u^2 = E|\nabla v|^2 + \frac{\nabla E \cdot \nabla (v^2)}{2},$$

after integration this over Ω we obtain

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}u^2 = \int_\Omega E|\nabla v|^2 dx + \frac{1}{2}\int_\Omega \nabla E \cdot \nabla (v^2) dx,$$

since we have

$$\frac{1}{2}\int_\Omega \nabla E \cdot \nabla (v^2) dx = -\frac{1}{2}\left(\int_\Omega div(\nabla E) v^2 dx + \int_{\partial\Omega} v^2 \partial_\nu E\right),$$

where $\partial_\nu E := \nabla E \cdot \nu$ and since $E = 0$ on $\partial\Omega$ so we have

$$\frac{1}{2}\int_\Omega \nabla E \cdot \nabla (v^2) dx = -\frac{1}{2}\int_\Omega \Delta E v^2 dx,$$

so we have

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{4E^2}u^2 = \int_\Omega E|\nabla v|^2 dx - \frac{1}{2}\int_\Omega \Delta E v^2 dx, \tag{2.2}$$

and since $-\Delta E \geq 0$ so $-\frac{1}{2}\int_\Omega \Delta E v^2 dx$ and $\int_\Omega E|\nabla v|^2 dx$ are non-negative, we have

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{4E^2}u^2 \geq 0,$$

for $u \in C_c^\infty(\Omega)$.

Since H_0^1 is the closure of $C_c^\infty(\Omega)$ so the inequality holds for $u \in H_0^1$.

Claim. For $\frac{1}{2} < t \le 1$ we have $E^t \in H_0^1(\Omega)$. We now prove the claim. For $\varepsilon > 0$ small, set $\phi_\varepsilon := (E^t - \varepsilon^t)_+$, then it is clear $\phi_\varepsilon \in H_0^1(\Omega)$. Let $\varepsilon_m \searrow 0$ and set $\phi_m := \phi_{\varepsilon_m}$ and note that $|\nabla \phi_m|^2 = t^2 E^{2t-2} |\nabla E|^2 \chi_{\{E(x) > \varepsilon\}}(x)$. We first show that ϕ_m is bounded in $H_0^1(\Omega)$ independent of m. First note

$$\|\phi_m\|_{H_0^1}^2 = \int |\nabla \phi_m|^2 = t^2 \int_{\{x \in \Omega : E(x) > \varepsilon_m\}} E^{2t-2} |\nabla E|^2 dx.$$

We now show this quantity is bounded independent of ε; two ways:

(i) **Use the co-area formula**. In what follows let $\Gamma_\tau := \{x \in \Omega : E(x) = \tau\}$. Let $M := \max_\Omega E$ and so we have

$$\begin{aligned}
\int_{\{x \in \Omega : E(x) > \varepsilon_m\}} E^{2t-2} |\nabla E|^2 dx &\le \sup_\Omega |\nabla E| \int_{\{x \in \Omega : E(x) > \varepsilon_m\}} E^{2t-2} |\nabla E| dx \\
&= \sup_\Omega |\nabla E| \int_{\varepsilon_m}^M \left(\int_{\Gamma_\tau} \tau^{2t-2} dH^{N-1}(x) \right) d\tau \qquad \text{co-area} \\
&= C_E \int_{\varepsilon_m}^M |\Gamma_\tau| \tau^{2t-2} d\tau \\
&\le \hat{C}_E \int_{\varepsilon_m}^M \tau^{2t-2} d\tau \text{ the } H^{N-1} \text{ measure of level sets bounded above,}
\end{aligned}$$

now note that $2t - 2 > -1$ and hence this final integral is bounded no matter how small ε_m is which proves the desired result.

(ii) **Use the pde directly**. For $\varepsilon > 0$ small we set $\psi_\varepsilon := (E(x)^{2t-1} - \varepsilon^{2t-1})_+ \in H_0^1(\Omega)$ and hence we can use this as a test function on the pde. Writing out

$$\int_\Omega \nabla E \cdot \nabla \psi_\varepsilon dx = \int_\Omega f(x) \psi_\varepsilon dx,$$

gives

$$(2t-1) \int_{\{x \in \Omega : E(x) > \varepsilon\}} E^{2t-2} |\nabla E|^2 dx = \int_\Omega f \psi_\varepsilon dx,$$

and sending $\varepsilon \searrow 0$ gives

$$\int_\Omega (2t-1)E^{2t-2}|\nabla E|^2 dx = \int_\Omega f(x)E^{2t-1}dx,$$

(note we used fact that $2t-1>0$ when we took the limit in ε). By weak compactness there is some subsequence of ϕ_m (don't rename) and $\phi \in H_0^1(\Omega)$ such that $\phi_m \rightharpoonup \phi$ in $H_0^1(\Omega)$. By compactness results we have $\phi_m \to \phi$ a.e. in Ω but since $E>0$ in Ω we see that $\phi(x) = E(x)^t$ for all $x \in \Omega$ and hence $E^t \in H_0^1(\Omega)$.

Now we want to show $\frac{1}{4}$ is optimal. For $\frac{1}{2} < t < 1$ set $u_t := E^t \in H_0^1(\Omega)$. A computation shows

$$Q_t := \frac{\int_\Omega |\nabla u_t|^2 dx}{\int_\Omega \frac{|\nabla E|^2}{E^2} u_t^2 dx} = \frac{t^2 \int_\Omega E^{2t-2}|\nabla E|^2 dx}{\int_\Omega E^{2t-2}|\nabla E|^2 dx} = t^2,$$

and so

$$\inf\left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx} : u \in H_0^1(\Omega)\backslash\{0\} \right\} \leq \frac{1}{4},$$

but recalling the fact we know this is greater than or equal $\frac{1}{4}$ gives the desired result.

Now we show the constant is not attained. Since $-\Delta E \geq 0$ we can drop a term in (2.2) to arrive at

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{4E^2} u^2 = \int_\Omega E|\nabla v|^2 dx, \tag{2.3}$$

if we can show that for all $0 \neq u \in H_0^1(\Omega)$ that $\int_\Omega E|\nabla v|^2 dx > 0$ then the best constant in (2.1) is not attained. We assume Ω is connected (but if it is not connected we can do this argument on each connected component). Lets assume this integral is zero; then we have $|\nabla v| = 0$ a.e. in Ω and hence there is some $c \in \mathbb{R}$ with $v(x) = c$ (to fully prove this one can use the fact that $v \in H_0^1(\Omega_0)$ for all $\Omega_0 \subset\subset \Omega$ then to see v is constant on Ω_0 and so we must have $u = c\sqrt{E}$ and since u is nonzero we must have $c \neq 0$. So to arrive a contradiction we need to show that $\sqrt{E} \notin H_0^1(\Omega)$; we will use the co-area formula to do that. Define $\Omega_\varepsilon = \{x \in \Omega : E(x) < \varepsilon\}$ and take $\varepsilon > 0$ small, by Hopf's lemma we have

$|\nabla E(x)|$ bounded away from zero on Ω_ε and since we have $|\nabla\sqrt{E}|^2 = \frac{1}{4E}|\nabla E|^2$, we obtain

$$\begin{aligned}\int_\Omega \frac{1}{E}|\nabla E|^2 dx \geq C\int_{\Omega_\varepsilon} \frac{1}{E}|\nabla E|^2 dx \\ = C\int_0^\varepsilon \left(\int_{\Gamma_\tau:=\{x\in\Omega: E(x)=\tau\}} \frac{1}{\tau} dH^{N-1}(x)\right) d\tau \\ = C\int_0^\varepsilon |\Gamma_\tau|\frac{1}{\tau} d\tau,\end{aligned} \tag{2.4}$$

and note there exists some $C_0 > 0$ such that for small $\tau > 0$ we have $|\Gamma_\tau| \geq C_0$. Since the last integral is unbounded so it is contradiction and we have $\int_\Omega E|\nabla v|^2 dx > 0$.

□

2.1 Classical Hardy Inequalities

We begin by recalling the various Hardy inequalities. The well known family of scale invariant inequalities consists of Hardy inequalities involve the distance function usually taken from a point or from the boundary.

Definition 2.2. *(Euclidean distance function $\delta(x)$). If Ω is a bounded open set in $\mathbb{R}^N$ and $x \in \Omega$, we define:*

$$\delta(x) := dist(x, \partial\Omega) := \min_{y\in\partial\Omega} |y - x|,$$

where $|\cdot|$ is the normal Euclidean norm on $\mathbb{R}^N$ and $\nu(x)$ is the outward pointing normal, given by

$$\nu(x) = -\nabla\delta(x) \quad x \in \partial\Omega, \quad \text{where the gradient is defined.}$$

If $\partial\Omega$ is smooth then δ will be smooth except on the "ridge of Ω" (kind of the centre denoted by Σ) and the ridge will be small in the sense that its dimension will be at most $N-1$.

Theorem 2.3. *(Classical Hardy inequality). Let $N \geq 3$ and suppose Ω is a open domain* *in $\mathbb{R}^N$ (bounded or unbounded), then*

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\phi^2}{|x|^2}dx \leq \int_\Omega |\nabla\phi|^2 dx \quad \forall\phi \in C_c^\infty(\Omega). \tag{2.5}$$

We will give a vector field approach, but later on we can use our unified approach to prove this.

Proof. Let $\phi \in C_c^\infty(\Omega\backslash\{0\})$ we have

$$div(|x|^{-t}x) = \frac{N-t}{|x|^t}, \tag{2.6}$$

multiplying both sides of (2.6) by ϕ^2 and integrating both sides we have

$$\int_\Omega \frac{N-t}{|x|^t}\phi^2 dx = \int_\Omega div\left(\frac{x}{|x|^t}\right)\phi^2 dx, \tag{2.7}$$

according to integration by parts formula (Theorem 1.22) we have

$$\int_\Omega div\left(\frac{x}{|x|^t}\right)\phi^2 dx = -\int_\Omega \nabla(\phi^2)\cdot\frac{x}{|x|^t}dx + \int_{\partial\Omega}\phi^2\frac{x}{|x|^t}\cdot\nu dS, \tag{2.8}$$

but since $\phi \in C_c^\infty(\Omega\backslash\{0\})$ so the boundary term vanishes and we have

$$\begin{aligned}\int_\Omega div\left(\frac{x}{|x|^t}\right)\phi^2 dx =& -2\int_\Omega \phi\nabla\phi\cdot\frac{x}{|x|^t}dx\\ \leq& 2\int_\Omega \frac{|\phi|}{|x|^{t-1}}|\nabla\phi|dx,\end{aligned} \tag{2.9}$$

using holder inequality for RHS of (2.9) we have

$$\int_\Omega \frac{N-t}{|x|^t}\phi^2 dx \leq 2\left(\int_\Omega \frac{|\phi|^2}{|x|^{2t-2}}dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\phi|^2 dx\right)^{\frac{1}{2}}, \tag{2.10}$$

putting $t = 2$ in (2.10) and simplifying we get

$$\frac{N-2}{2}\int_\Omega \frac{\phi^2}{|x|^2}dx \leq \left(\int_\Omega \frac{\phi^2}{|x|^2}dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\phi|^2 dx\right)^{\frac{1}{2}}, \tag{2.11}$$

dividing both sides of (2.11) by $\left(\int_\Omega \frac{\phi^2}{|x|^2}dx\right)^{\frac{1}{2}}$ we get

$$\frac{N-2}{2}\left(\int_\Omega \frac{\phi^2}{|x|^2}dx\right)^{\frac{1}{2}} \leq \left(\int_\Omega |\nabla\phi|^2 dx\right)^{\frac{1}{2}}, \tag{2.12}$$

raising both sides of (2.12) we get the inequality we sought for

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\phi^2}{|x|^2}dx \leq \int_\Omega |\nabla\phi|^2 dx,$$

now using Theorem 1.36

$$\overline{C_c^\infty(\Omega\backslash\{0\})} = H_0^1(\Omega),$$

and remark (1.40)

$$\overline{C_c^\infty(\Omega)} = H_0^1(\Omega),$$

we conclude inequality (2.5) holds for $\phi \in C_c^\infty(\Omega)$.

□

A similar result for Euclidean distance from x to $\partial\Omega$ is obtained.

Theorem 2.4. *(Boundary Hardy Inequality). Let Ω denote a bounded smooth convex domain in $\mathbb{R}^N$ and set $\delta(x) := dist(x,\partial\Omega)$, then*

$$\frac{1}{4}\int_\Omega \frac{|\psi|^2}{\delta^2}dx \leq \int_\Omega |\nabla\psi|^2 dx \quad \forall\psi \in H_0^1(\Omega), \tag{2.13}$$

moreover, the constant is optimal and not attained.

Proof. Here we consider $\psi \in C_c^\infty(\Omega)$ and the proof consists merely of an integration by parts and Young's inequality.

Let $F(x) = \frac{\nabla\delta}{\delta^{t-1}}$ and taking $\phi = (\psi(x))^2$, by applying Theorem 1.22 we have

$$\int_\Omega \nabla(\psi(x))^2 \cdot \frac{\nabla\delta}{\delta^{t-1}} = -\int_\Omega (\psi(x))^2 div\left(\frac{\nabla\delta}{\delta^{t-1}}\right)dx + \int_{\partial\Omega}(\psi(x))^2\frac{\nabla\delta}{\delta^{t-1}}\cdot \nu dS,$$

since $div\left(\frac{\nabla\delta}{\delta^{t-1}}\right) = \frac{(1-t)}{\delta^t} - \frac{(-\Delta\delta)}{\delta^{t-1}}$, for $t>1$ we get

$$\int_\Omega \frac{|\nabla(\psi(x))^2|}{\delta^{t-1}}dx - \int_\Omega \frac{|(\psi(x))^2|}{\delta^{t-1}}(-\Delta\delta)dx - \int_{\partial\Omega}\frac{|(\psi(x))^2|}{\delta^{t-1}}\nabla\delta\cdot\nu dS \geq (t-1)\int_\Omega \frac{|(\psi(x))^2|}{\delta^t}dx,$$

where $|\nabla|u(x)|| \leq |\nabla u(x)|$ for $x\in\Omega$. So we have

$$\begin{aligned}\frac{2}{t-1}\int_\Omega \frac{|\nabla\psi(x)||\psi(x)|}{\delta^{t-1}}dx - \frac{1}{t-1}\left(\int_\Omega \frac{|(\psi(x))^2|}{\delta^{t-1}}(-\Delta\delta)dx + \int_{\partial\Omega}\frac{|(\psi(x))^2|}{\delta^{t-1}}\nabla\delta\cdot\nu dS\right)\\ \geq \int_\Omega \frac{|(\psi(x))^2|}{\delta^t}dx,\end{aligned} \tag{2.14}$$

using Young inequality for the LHS of (2.14) gives

$$\begin{aligned}\frac{2}{t-1}\int_\Omega \frac{|\nabla\psi(x)||\psi(x)|}{\delta^{t-1}}dx =& \int_\Omega \left\{\frac{2}{t-1}\frac{|\nabla\psi(x)|}{\delta^{\frac{t}{2}-1}}\right\}\left\{\frac{|\psi(x)|}{\delta^{t-\frac{t}{2}}}\right\}\\ \leq& \frac{1}{2}\left(\frac{2}{t-1}\right)^2\int_\Omega \frac{|\nabla\psi(x)|^2}{\delta^{t-2}} + \frac{1}{2}\int_\Omega \frac{|\psi(x)|^2}{\delta^t}dx,\end{aligned} \tag{2.15}$$

there is no boundary term in LHS of (2.14) since ψ as a test function has compact support in Ω and thus vanishes near $\partial\Omega$ and we deduce the following inequality

$$\frac{1}{2}\frac{4}{(t-1)^2}\int_\Omega \frac{|\nabla\psi(x)|^2}{\delta^{t-2}} + \frac{1}{2}\int_\Omega \frac{|\psi(x)|^2}{\delta^t}dx - \frac{1}{t-1}\int_\Omega \frac{|\psi(x)|^2}{\delta^{t-1}}(-\Delta\delta)dx \geq \int_\Omega \frac{|\psi(x)|^2}{\delta^t}dx, \tag{2.16}$$

since $\delta(x)$ is smooth away from the ridge and Ω has sufficiently smooth boundary, the term including $(-\Delta\delta)$ cancel and simplifying (2.16) deduce

$$\frac{1}{2}\int_{\Omega}\frac{|\nabla\psi(x)|^2}{\delta^{t-2}} \geq \frac{1}{2}\frac{(t-1)^2}{4}\int_{\Omega}\frac{|\psi(x)|^2}{\delta^t}dx, \tag{2.17}$$

simplifying and putting $t=2$ in (2.17) gives

$$\frac{1}{4}\int_{\Omega}\frac{|\psi(x)|^2}{\delta^2}dx \leq \int_{\Omega}|\nabla\psi(x)|^2dx, \tag{2.18}$$

Now we show inequality (2.18) is also valid for $\psi \in H_0^1(\Omega)$.

Given $\psi \in H_0^1(\Omega)$, there exists a sequence $\psi_k \in C_c^\infty(\Omega)$ converging to ψ in H^1. (We may assume ψ_k converging to ψ pointwise.) Fatou's lemma implies

$$\int_{\Omega}|\nabla\psi|^2dx = \liminf_{k\to\infty}\int_{\Omega}|\nabla\psi_k|^2dx \geq \liminf_{k\to\infty}\frac{1}{4}\int_{\Omega}\frac{|\psi_k|^2}{\delta^2}dx \geq \frac{1}{4}\int_{\Omega}\frac{\psi^2}{\delta^2}dx,$$

for the best constant since $-\Delta\delta \geq 0$ we can drop the second term in RHS of (2.16) and we have

$$\int_{\Omega}|\nabla\psi|^2dx \geq \frac{1}{4}\int_{\Omega}\frac{\psi^2}{\delta^2}dx, \tag{2.19}$$

and we choose $\psi(x) = (\delta(x))^t$ for $\frac{1}{2} < t < 1$. According to the proof of lemma 2.1, $E(x) = \delta^t \in H_0^1(\Omega)$ for $\frac{1}{2} < t < 1$, so we have

$$\frac{\int_{\Omega}|\nabla\psi|^2dx}{\int_{\Omega}\frac{|\psi|^2}{\delta^2}dx} = \frac{t^2\int_{\Omega}(\delta(x))^{2t-2}dx}{\int_{\Omega}(\delta(x))^{2t-2}} = t^2, \tag{2.20}$$

since $\nabla\psi = t\nabla\delta\delta^{t-1}$ and $|\nabla\delta| = 1$ so we have

$$|\nabla\psi|^2 = t^2\delta(x)^{2t-2},$$

and for the denominator of (2.20) we have

$$\frac{|\psi|^2}{\delta^2} = \frac{\delta^{2t}}{\delta^2} = \delta^{2t-2},$$

from this we see that the optimal constant

$$\inf \left\{ \frac{\int_\Omega |\nabla \delta^t|^2 dx}{\int_\Omega \frac{(\delta^t)^2}{\delta^2} dx} \quad : \quad \delta^t \in H_0^1(\Omega) \backslash \{0\} \right\} = \frac{1}{4},$$

so the constant in (2.19) is optimal.

We will show not attainment of best constant in the next chapters when we use our general inequality.

$\square$

Remark 2.5. *If* Ω *is convex then* $-\Delta\delta \geq 0$ *in* Ω*.*

A general class of L^p Hardy inequalities in $\mathbb{R}^N$ invloving distance from a surface of general co-dimension $1 \leq K \leq N$ have been studied in [6], [14].

Theorem 2.6. *(Boundary Hardy* L^p *inequality). Let* Ω *be a domain in* $\mathbb{R}^N$, $N \geq 2$ *with nonempty boundary, and let* $\delta(x) = dist(x, \partial\Omega)$ *denote the distance of a point* $x \in \mathbb{R}^N$ *to the boundary of* Ω*. Fix* $p \in [1, \infty)$*. We say that the* L^p *Hardy inequality is satisfied in* Ω *if there exists* $c > 0$ *such that*

$$\int_\Omega |\nabla u|^p dx \geq c \int_\Omega \frac{|u|^p}{\delta^p} dx \quad \textit{for all} \quad u \in C_c^\infty(\Omega).$$

Moreover, adding lower order terms with optimal weight to the right hand side of the inequality permitted more improvement of the above inequalities. Defining a potenatial function, so called $V(x) \geq 0$ in Ω, was the common approach for improving Hardy inequalities. While numerous approached have been provided to improve the inequalities, the results were mainly presented in the form of infinite series that include complicated functions that are defined inductively. The explicit example of potential V of radial symmetric type on bounded domain $\Omega \subset \mathbb{R}^N$ was originally considered by Ghoussoub and Moradifam [13], [12]

$$\int_\Omega |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \geq \int_\Omega V(x) u^2 dx,$$

where $u \in H_0^1(\Omega)$.

Through an entirely different approach, some researchers utilized operators of sub-Laplacian and p-sub-Laplacian for extending the Hardy inequalities [4], [11], [18]. A successful result of this method was presented by Adimurthi and Sekar [3]. Suppose Ω is a smooth domain in $\mathbb{R}^N$ which contains the origin, $A(x) = ((a_{ij}(x)))$ denotes a symmetric, uniformly positive definite matrix with suitably smooth coefficients and for $\zeta \in \mathbb{R}^N$ we define $|\zeta|_A^2 := |\zeta|_A(x)^2 := A(x)\zeta \cdot \zeta$. Now suppose E is a solution of $L_{A,p}(E) := -div(|\nabla E|_A^{p-2} A \nabla E) = \delta_0$ in Ω with $E = 0$ on $\partial\Omega$ where δ_0 is the Dirac mass at 0. Then for all $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|_A^p - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E} |u|^p dx \geq 0,$$

for $\Omega \subset \mathbb{R}^N$, provided domain with $0 \in \Omega$.

3

Various Hardy Inequalities

The major objective of this chapter is to organize some of the new approaches that are available for Hardy inequalities. In this thesis Ω is assumed as a bounded connected domain in $\mathbb{R}^N$ with smooth boundary.

In the direction that Hardy inequalities considered for operators more general than Laplacian, these results are obtained by Adimurthy and Sekar [3]:

Suppose Ω is a smooth domain in $\mathbb{R}^N$ which contains the origin and $A(x) = (a^{ij}(x))$ denotes a symmetric, uniformly positive definite matrix with suitably smooth coefficients and for $\zeta \in \mathbb{R}^N$ we define $|\zeta|_A^2 := |\zeta|_{A(x)}^2 := A(x)\zeta \cdot \zeta$. Now suppose E is a solution of $L_{A,p}(E) := -div\left(|\nabla E|_A^{p-2} A\nabla E\right) = \delta_0$ in Ω with $E = 0$ on $\partial\Omega$, where δ_0 is the Dirac mass at 0.

Our method is an analogous to their approach but we consider the quadratic case $(p = 2)$ and for this we define $L_A(E) := -div(A\nabla E)$.

Definition 3.1. *Suppose $E > 0$ in Ω and $L_A(E)$ is a nonegative nonzero finite measure in Ω denoted by μ.*

- (1) *If in addition $E \in H_0^1(\Omega)$ then we call E a boundary weight on Ω.*

- (2) *If in addition $E \in C^\infty(\overline{\Omega}\backslash K)$ where $K \subset \Omega$ denotes the support of μ, $E = \infty$ on K*

and $diam_{box}(K) < N-2$ [definition 1.33], then we call E an interior weight on Ω.

Note that μ will denote the measure $L_A(E)$ and in this case where E is an interior weight on Ω, K will denote the support of μ [Definition 1.32].

Lemma 3.2. • *i) Suppose E is an interior weight on Ω, then*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega |\nabla v|_A^2 E dx,$$

for all $u \in C_c^{0,1}(\Omega \backslash K)$ and where $v = E^{\frac{-1}{2}} u$.

- *ii) Suppose E is a boundary weight on Ω, then*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega |\nabla v|_A^2 E dx + \frac{1}{2}\int_\Omega \frac{u^2}{E} d\mu, \tag{3.1}$$

for all $u \in H_0^1(\Omega)$ and $v := E^{\frac{-1}{2}} u$.

Proof. • i) Since $u \in C_c^{0,1}(\Omega \backslash K)$, u has compact support in Ω and since $v := E^{\frac{-1}{2}} u$ and E is a positive function in Ω so v has compact support in Ω as well, so there is no boundary term thus u and v vanish near $\partial\Omega$ so the integration by parts used in obtaining (2.2) is valid here and we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx = \int_\Omega |\nabla v|_A^2 E dx + \frac{1}{2}\int_\Omega L_A(E)\frac{u^2}{E} dx,$$

but since E is an interior weight so K is the support of μ, $E \in C^\infty(\overline{\Omega} \backslash K)$ and $L_A(E) \neq 0$ on K and $L_A(E) = 0$ on $\overline{\Omega} \backslash K$, so we have

$$\int_\Omega |\nabla v|_A^2 E dx \leq \int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx.$$

- ii) Suppose E is a boundary weight ($E \in H_0^1(\Omega)$), we extend E to all of $\mathbb{R}^N$ by $E = 0$ outside of $\overline{\Omega}$ and we approximate E with its ε-mollification, E_ε, such that

$F_\varepsilon = L_A(E_\varepsilon)$ and $v_\varepsilon := E_\varepsilon^{-\frac{1}{2}} u$ for $u \in C_c^\infty(\Omega)$.

According to mollification property $E_\varepsilon \in C^\infty(\mathbb{R}^N)$ and $E_\varepsilon \to E$ as $\varepsilon \to 0$ in $H^1(\mathbb{R}^N)$. We now fix $u \in C_c^\infty(\Omega)$ and note we can argue as before that (we are replacing E with E_ε)

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx = \int_\Omega |\nabla v_\varepsilon|_A^2 E_\varepsilon + \frac{1}{2}\int_\Omega F_\varepsilon \frac{u^2}{E_\varepsilon} dx, \tag{3.2}$$

there exist $\sigma > 0$ such that if $\Omega_\sigma := \{x \in \Omega : \delta(x) > \sigma\}$ we have $supp(u) \subset \Omega_\sigma \subset\subset \Omega$. Also, as $\varepsilon \to 0$, $E_\varepsilon \to E$ a.e. in Ω.

Claim. There exist $C > 0$ and $\varepsilon_0 > 0$ (small) such that

$$\inf_{\Omega_\sigma}(E(x)) \geq C \quad \text{and} \quad \inf_{\Omega_\sigma} E_\varepsilon \geq C \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

also for the second LHS of (3.2), since $E \in H_0^1(\Omega)$ and according to property (v) of mollifier, $\nabla E_\varepsilon \to \nabla E$ in L^2 so regarding Theorem 1.26 we have:

$$\frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} \to \frac{|\nabla E|_A^2}{E^2}, \quad \text{a.e.} \tag{3.3}$$

Claim.

$$uE_\varepsilon^{-1} \to uE^{-1} \quad \text{in} \quad H_0^1(\Omega). \tag{3.4}$$

Since we observe that

$$\begin{aligned} \|uE_\varepsilon^{-1} - uE^{-1}\|_{H^1(\Omega)} &= \left(\|uE_\varepsilon^{-1} - uE\|_{L^2(\Omega)}^2 + \|\nabla(uE_\varepsilon^{-1} - uE^{-1})\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &\leq \|uE_\varepsilon^{-1} - uE^{-1}\|_{L^2(\Omega)} + \|\nabla(uE_\varepsilon^{-1} - uE^{-1})\|_{L^2(\Omega)}, \end{aligned} \tag{3.5}$$

since E is 0 on $\partial\Omega$ and $u \in C_c^\infty(\Omega)$ so we restrict the domain of integral so we have:

$$\begin{aligned}
\|(uE_\varepsilon^{-1} - uE^{-1})\|_{L^2(\Omega)} &= \left(\int_\Omega |uE_\varepsilon^{-1} - uE^{-1}|^2 dx\right)^{\frac{1}{2}} \\
&= \left(\int_\Omega |u|^2 |E_\varepsilon^{-1} - E^{-1}|^2 dx\right)^{\frac{1}{2}} \\
&\leq \|u\|_{L^\infty(\Omega_\sigma)} \left(\int_{\Omega_\sigma} |E_\varepsilon^{-1} - E^{-1}|^2 dx\right)^{\frac{1}{2}}, \qquad (3.6)
\end{aligned}$$

and according to Fatou's lemma and the claim we have:

$$\int_{\Omega_\sigma} \left|\frac{E_\varepsilon - E}{E_\varepsilon E}\right|^2 \to 0$$

also by applying dominated convergence theorem we see that

$$\int_{\Omega_\sigma} |\nabla u|^2 \frac{|E_\varepsilon - E|^2}{|E_\varepsilon E|^2} \to 0 \qquad (3.7)$$

and according to (3.3) and (3.7) we have

$$\begin{aligned}
\|\nabla(uE_\varepsilon^{-1} - uE^{-1})\|_{L^2(\Omega_\sigma)}^2 &= \int_{\Omega_\sigma} \left|\frac{\nabla u}{E_\varepsilon} - \frac{\nabla u}{E} + \frac{u}{E_\varepsilon^2}\nabla E - \frac{u}{E^2}\nabla E_\varepsilon\right|^2 dx \\
&\leq C\int_{\Omega_\sigma} |\nabla u|^2 \left(\frac{1}{E_\varepsilon} - \frac{1}{E}\right)^2 dx + C\int_{\Omega_\sigma} u^2 \left|\frac{\nabla E}{E^2} - \frac{\nabla E_\varepsilon}{E_\varepsilon^2}\right|^2 dx \to 0 \qquad (3.8)
\end{aligned}$$

a.e in Ω, and using ε-mollification for the first term on RHS of (3.2) and since $\nabla v_\varepsilon \to \nabla v$ and $E_\varepsilon \to E$ a.e in Ω so we have:

$$E_\varepsilon |\nabla v_\varepsilon|_A^2 \to E|\nabla v|_A^2 \quad \text{a.e} \quad \text{in} \quad \Omega, \qquad (3.9)$$

and finally for the last part of RHS of (3.2) we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega uL_A(E_\varepsilon)dx &= -\lim_{\varepsilon\to 0}\int_\Omega u \quad div(A\nabla E_\varepsilon)dx\\ &= -\int_\Omega u \quad div(A\nabla E)dx\\ &= \int_\Omega uL_A(E)dx\\ &= \int_\Omega u\mu dx,\end{aligned}$$

so we have $uF_\varepsilon \rightharpoonup u\mu$ in $H^{-1}(\Omega)$ as dual of $H_0^1(\Omega)$ [Remark 1.15]. Plugging (3.4), (3.3) and (3.9) in (3.2) and using Fatou's lemma, we have:

$$\begin{aligned}\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx &= \int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\liminf\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}u^2 dx\\ &= \liminf\int_\Omega |\nabla v_\varepsilon|_A^2 E_\varepsilon dx + \frac{1}{2}\liminf\int_\Omega F_\varepsilon\frac{u^2}{E_\varepsilon}dx\\ &\geq \int_\Omega |\nabla v|_A^2 E dx + \frac{1}{2}\int_\Omega \frac{u^2}{E}d\mu.\end{aligned}$$

□

For obtaining the best constant we need the following lemma for having test functions. Note that E is interior weight and we consider g as a solution of $L_A(g) = 0$ in Ω and $g = E$ on $\partial\Omega$.

Lemma 3.3. *Suppose E is an interior weight on Ω and $0 < \gamma := min_{\partial\Omega}E$, then*

- *i) For $0 < t < \frac{1}{2}$, $u_t = E^t - g^t \in H_0^1(\Omega)$,*

- *ii) Define $I(t) := \int_\Omega |\nabla E|_A^2 E^{2t-2}dx$ then*
  - *For $\frac{1}{2} < t \quad I(t) < \infty$,*
  - *For $t \nearrow \frac{1}{2} \quad I(t) \to \infty$,*

- *iii) Suppose $E = \gamma > 0$ on $\partial\Omega$, define $v_{t,\tau} := E^t log^\tau(\gamma^{-1}E)$ and*

$$J_t(\tau) := \int_\Omega E^{2t-2}|\nabla E|_A^2 \log^{2\tau-2}(\gamma^{-1}E)dx,$$

then $v_{t,\tau} \in H_0^1(\Omega)$ for $0 < t < \frac{1}{2}$ and $\tau > \frac{1}{2}$. Moreover for each $0 < t < \frac{1}{2}$ we have $J_t(\tau) \to \infty$ as $\tau \searrow \frac{1}{2}$.

Proof.

- i,ii) Fix $0 < t < \frac{1}{2}$ and since $u_t = E^t - g^t$ so $\nabla u_t = t\nabla E E^{t-1} - t\nabla g g^{t-1}$. For $0 < t < \frac{1}{2}$ we have

$$\begin{aligned}|\nabla u_t|_A^2 =& t^2|\nabla E|_A^2 E^{2t-2} + t^2|\nabla g|_A^2 g^{2t-2} - 2t^2\nabla E \cdot \nabla g E^{t-1}g^{t-1} \\ \leq& c|\nabla E|_A^2 E^{2t-2} + c|\nabla g|_A^2 g^{2t-2},\end{aligned}$$

where c is some uniform constant. Now we should prove $u_t \in H_0^1(\Omega)$, note that the term involving g is harmless. We multiply both sides of $L_A(E) = \mu$ by E^{2t-1} and integrate over Ω we have

$$\begin{aligned}\int_\Omega E^{2t-1}L_A(E)dx =& -\int_\Omega E^{2t-1}div(A\nabla E)dx \\ =& \int_\Omega \nabla E^{2t-1} \cdot (A\nabla E)dx - \int_{\partial\Omega} E^{2t-1}(A\nabla E) \cdot \nu dS,\end{aligned}$$

since on $\partial\Omega$ we have $g = E$ so we have

$$\begin{aligned}\int_\Omega E^{2t-1}L_A(E)dx =& -\int_\Omega E^{2t-1}div(A\nabla E)dx \\ =& +\int_\Omega \nabla E^{2t-1} \cdot (A\nabla E)dx - \int_{\partial\Omega} g^{2t-1}(A\nabla E) \cdot \nu dS \\ =& (2t-1)\int_\Omega E^{2t-2}\nabla E \cdot (A\nabla E)dx - \int_{\partial\Omega} g^{2t-1}(A\nabla E) \cdot \nu dS \\ =& (2t-1)\int_\Omega |\nabla E|_A^2 E^{2t-2}dx - \int_{\partial\Omega} g^{2t-1}(A\nabla E) \cdot \nu dS,\end{aligned} \tag{3.10}$$

since the term involving g is harmless and considering $\varepsilon(t) := \int_\Omega E^{2t-1}L_A(E)dx =$

$\int_\Omega E^{2t-1} d\mu$ we have

$$(1-2t)\int_\Omega |\nabla E|_A^2 E^{2t-2} dx = \varepsilon(t) - \int_{\partial\Omega} (A\nabla E)\cdot \nu dS, \tag{3.11}$$

on the other hand we have

$$\int_{\partial\Omega} (A\nabla E)\cdot \nu dS = \int_\Omega div(A\nabla E) dx, \tag{3.12}$$

therefore considering (3.12) in (3.11) we have

$$\begin{aligned}(1-2t)\int_\Omega |\nabla E|_A^2 E^{2t-2} dx =& \varepsilon(t) - \int_\Omega div(A\nabla E) dx,\\ =& \varepsilon(t) + \int_\Omega L_A(E) dx\\ =& \varepsilon(t) + \int_\Omega \mu dx\\ =& \varepsilon(t) + \mu(\Omega),\end{aligned} \tag{3.13}$$

also note that we multiplied both sides of equation $L_A(E) = \mu$ by E^{2t-1} which derives

$$\int_\Omega E^{2t-1} L_A(E) dx = \int_\Omega E^{2t-1} \mu dx = \int_\Omega E^{2t-1} d\mu, \tag{3.14}$$

since $t < \frac{1}{2}$ and E is an interior weight (i.e $E = \infty$ on K), therefore

$$\varepsilon(t) = \int_\Omega E^{2t-1} d\mu = 0,$$

now since μ is finite measure to show $u_t \in H_0^1(\Omega)$ regarding (3.13) we have

$$\int_\Omega |\nabla u_t|^2 \leq c\int_\Omega |\nabla E|_A^2 E^{2t-2} dx + c|\nabla g|_A^2 g^{2t-2} < \infty,$$

also

$$\lim_{t \nearrow \frac{1}{2}} \left(\int_{\Omega} |\nabla E|_A^2 E^{2t-2} dx \right) = \infty.$$

- iii) Fix $0 < t < \frac{1}{2}$ and $\tau > \frac{1}{2}$ and

$$v_{t,\tau} = E^t log^{\tau}(\gamma^{-1}E),$$

$log\gamma^{-1}E$ is continuous near $\partial\Omega$ since $E \neq \gamma$ and it vanishes on $\partial\Omega$ since $E = \gamma$. To show $v_{t,\tau} \in H_0^1(\Omega)$ we should show that its weak derivative [Definition 1.4] belongs to $L^2(\Omega)$ so equivalently we need to show

$$w_1 = E^{2t-2}|\nabla E|^2 log^{2\tau}(\gamma^{-1}E) \in L^1(\Omega),$$

and

$$w_2 = E^{2t-2}|\nabla E|^2 log^{2\tau-2}(\gamma^{-1}E) \in L^1(\Omega),$$

for proving $v_{t,\tau} \in H_0^1(\Omega)$, since $E = \infty$ on K and $E = \gamma$ on $\partial\Omega$, w_1 and w_2 are singular near K and $\partial\Omega$. Thus we consider two sets Ω_ε and K_ε as following

  - K_ε is small neighbourhood of K where $E \neq \infty$
  - $\Omega_\varepsilon := \{x \in \Omega : E < \gamma + \varepsilon\}$ where $E \neq \gamma$,

in fact we consider w_1 and w_2 on $L^1(K_\varepsilon)$ and $L^1(\Omega_\varepsilon)$.

First we consider w_1 **and** w_2 **on** $L^1(K_\varepsilon)$. Set $w_\tau = E^{2t-2}|\nabla E|^2 log^{2\tau-2}(\gamma^{-1}E)$ and $w_2 = w_\tau$ thus $w_1 = w_{\tau+1}$, and suppose $t' \in (t, \frac{1}{2})$, since near K, $E \neq \infty$ and $\gamma \neq E$ so

$$\frac{log^{2\tau}(\gamma^{-1}E)}{E^{2t'-2t}} \leq c,$$

thus

$$w_{\tau+1} = E^{2t'-2t}|\nabla E|^2 \frac{log^{2\tau}(\gamma^{-1}E)}{E^{2t'-2t}} \leq cE^{2t'-2t}|\nabla E|^2 < \infty \quad \text{near K},$$

and therefore $w_1 = w_{\tau+1} \in L^1(K_\varepsilon)$. thus $w_2 = E^{2t-2}|\nabla E|^2 log^{2\tau-2}(\gamma^{-1}E) \in L^1(K_\varepsilon)$. **Now we consider the behaviour of w_1 and w_2 on** $\Omega_\varepsilon = \{x \in \Omega : E < \gamma + \varepsilon\}$. We should consider ε small sufficiently such that $K \subset \Omega \backslash \Omega_\varepsilon$. Now our goal is to show $w_2 \in L^1(\Omega_\varepsilon)$. By co-area formula and considering $C = \sup_{\Omega_\varepsilon} E$ and since $E\gamma^{-1} < 1 + \varepsilon\gamma^{-1}$, for $\Gamma_s := \{x \in \Omega : E(x) = s\}$ we have

$$\begin{aligned} \int_{\Omega_\varepsilon} E^{2t-2}|\nabla E|^2 log^{2\tau-2}(\gamma^{-1}E)dx &\leq \sup_{\Omega_\varepsilon}|\nabla E| \int_{\Omega_\varepsilon} E^{2t-2}|\nabla E| log^{2\tau-2}(\gamma^{-1}E)dx \\ &= C\int_1^{1+\frac{\varepsilon}{\gamma}} \left(\int_{\Gamma_s} s^{2t-2} log^{2\tau-2}(s) dH^{N-1}(x)\right) ds \\ &= C\int_1^{1+\frac{\varepsilon}{\gamma}} |\Gamma_s| s^{2t-2} log^{2\tau-2}(s) ds \\ &\leq C\int_1^{1+\frac{\varepsilon}{\gamma}} s^{2t-2} log^{2\tau-2}(s) ds, \end{aligned}$$

where H^{N-1} measure of level sets bounded above, the last term is finite for $\tau > \frac{1}{2}$. Therefore $w_2 \in L^1(\Omega_\varepsilon)$. Since w_1 behaves better than w_2 near $\partial\Omega$ so $w_1 \in L^1(\Omega_\varepsilon)$. As we see $w_1, w_2 \in L^1(K_\varepsilon)$ and $w_1, w_2 \in L^1(\Omega_\varepsilon)$ so $v_{t,\tau} \in H_0^1(\Omega)$.
Fix $0 < t < \frac{1}{2}$ and $\tau > \frac{1}{2}$. Since μ is a nonegative nonzero finite measure so $-\Delta E \geq 0$ and $E < \gamma + \varepsilon$ on Ω_ε, so according to Hopf's Lemma 1.6 $|\nabla E(x)|$ bounded away from zero on Ω_ε for $\varepsilon > 0$ sufficiently small.

Fix $\varepsilon > 0$ sufficiently small, then

$$\begin{aligned}
J_t(\tau) = &\int_{\Omega} E^{2t-2}|\nabla E|_A^2 log^{2\tau-2}(\gamma^{-1}E)dx \\
&\geq \sup_{\Omega_\varepsilon}|\nabla E| \int_{\Omega_\varepsilon} E^{2t-2} log^{2\tau-2}(E)|\nabla E|dx \\
&= C\int_1^{1+\frac{\varepsilon}{\gamma}} \left(\int_{\Gamma_s:=\{x\in\Omega:E(x)=s\}} s^{2t-2}log^{2\tau-2}(s)dH^{N-1}(x) \right) ds \\
&= \int_1^{1+\frac{\varepsilon}{\gamma}} |\Gamma_s| s^{2t-2} log^{2\tau-2}(s)ds, \\
&\geq \tilde{C}\int_1^{1+\frac{\varepsilon}{\gamma}} s^{2t-2}log^{2\tau-2}(s)ds
\end{aligned}$$

and note there exists some $\hat{C} > 0$ such that for small s we have $|\Gamma_s| \geq \hat{C}$ and for $\tau \searrow \frac{1}{2}$ the lase integral becomes unbounded, so $J_t(\tau) \to \infty$ as $\tau \searrow \frac{1}{2}$.

□

The main inequality for function E as interior weight and boundary weight is considered in the following theorem.

Theorem 3.4.
- *i) Suppose E is either an interior weight or a boundary weight on Ω. Then*

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4}\int_{\Omega} \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq 0, \tag{3.15}$$

for all $u \in H_0^1(\Omega)$ and moreover, $\frac{1}{4}$ is optimal and not attained.

- *ii) Suppose E is boundary weight on Ω, then*

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4}\int_{\Omega} \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \frac{1}{2}\int_{\Omega} \frac{u^2}{E} d\mu, \tag{3.16}$$

for all $u \in H_0^1(\Omega)$ and moreover, $\frac{1}{2}$ is optimal and not attained.

Proof. • i) According to Lemma 3.2 since E is interior weight so

$$\int_{\Omega}|\nabla u|_A^2dx-\frac{1}{4}\int_{\Omega}\frac{|\nabla E|_A^2}{E^2}u^2dx\geq\int_{\Omega}|\nabla v|_A^2Edx,$$

according to the proof of Lemma 2.1, $\int_{\Omega}|\nabla v|_A^2Edx$ is positive so we have

$$\int_{\Omega}|\nabla u|_A^2dx-\frac{1}{4}\int_{\Omega}\frac{|\nabla E|_A^2}{E^2}u^2dx\geq 0, \tag{3.17}$$

for $u\in C_c^{0,1}(\Omega\backslash K)$, since $C_c^{0,1}(\Omega\backslash K)$ is dense in $H_0^1(\Omega)$ so (3.15) holds for $u\in H_0^1(\Omega)$.

Now we show that the constant is optimal. Suppose E is an interior weight on Ω and define $E_\varepsilon=E+\varepsilon$ and $g_\varepsilon=g+\varepsilon$ where $\varepsilon>0$. Define

$$I_\varepsilon(t)=\int_{\Omega}E_\varepsilon^{2t-2}|\nabla E_\varepsilon|_A^2dx,$$

from Lemma 3.3 we have $\lim_{t\nearrow\frac{1}{2}}I_\varepsilon(t)=\infty$. We use $u_{t,\varepsilon}:=E_\varepsilon^t-g_\varepsilon^t$ as a test function. Let $0<t<\frac{1}{2}$ and $\varepsilon>0$ then

$$Q_{t,\varepsilon}:=\frac{\int_{\Omega}|\nabla u_{t,\varepsilon}|_A^2dx}{\int_{\Omega}\frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}u_{t,\varepsilon}^2dx}\leq\frac{t^2I_\varepsilon(t)+c_0+c_1\sqrt{I_\varepsilon(t)}}{I_\varepsilon(t)-c_2I_\varepsilon(\frac{t}{2})-c_3I_\varepsilon(0)},$$

here we explain how this inequality is obtained. Consider $u_{t,\varepsilon}=E_\varepsilon^t-g_\varepsilon^t$ so

$$\nabla u_{t,\varepsilon}=\nabla E_\varepsilon^t-\nabla g_\varepsilon^t=tE_\varepsilon^{t-1}\nabla E_\varepsilon-tg_\varepsilon^{t-1}\nabla g_\varepsilon,$$

so

$$|\nabla u_{t,\varepsilon}|^2=t^2E_\varepsilon^{2t-2}|\nabla E_\varepsilon|^2+t^2g_\varepsilon^{2t-2}|\nabla g_\varepsilon|^2-2t^2E_\varepsilon^{t-1}g_\varepsilon^{t-1}\nabla E_\varepsilon\cdot\nabla g_\varepsilon,$$

since g is harmless we can ignore the terms involving g and we have:

$$\int_\Omega |\nabla u_{t,\varepsilon}|_A^2 dx \le t^2 I_\varepsilon(t) + c_0 + c_1\sqrt{I_\varepsilon(t)}, \tag{3.18}$$

also

$$\begin{aligned}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u_{t,\varepsilon}^2 dx &= \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}(E_\varepsilon^t - g_\varepsilon^t)^2 dx \\ &= \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}(E_\varepsilon^{2t} - 2E_\varepsilon^t g_\varepsilon^t + g_\varepsilon^{2t}) dx \\ &= \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} dx - 2\int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{t-2} g_\varepsilon^t dx + \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} g_\varepsilon^{2t} dx,\end{aligned} \tag{3.19}$$

since g is harmless so (3.19) is

$$\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u_{t,\varepsilon}^2 dx = \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} dx - 2\int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{t-2} dx + \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{-2} dx, \tag{3.20}$$

so we can consider (3.20) as

$$\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u_{t,\varepsilon}^2 dx = I_\varepsilon(t) - c_2 I_\varepsilon(\frac{t}{2}) - c_3 I_\varepsilon(0), \tag{3.21}$$

where the constants c_k, $k = 0,1,2,3$ possibly depends on ε. Now from Lemma 3.3 we have

$$\lim_{t\nearrow\frac{1}{2}} I_\varepsilon(t) = \lim_{t\nearrow\frac{1}{2}} \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} dx = \infty,$$

so we have

$$\lim_{t\nearrow\frac{1}{2}} Q_{t,\varepsilon} := \lim_{t\nearrow\frac{1}{2}} \frac{\int_\Omega |\nabla u_{t,\varepsilon}|_A^2 dx}{\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u_{t,\varepsilon}^2 dx} \le \lim_{t\nearrow\frac{1}{2}} \frac{t^2 I_\varepsilon(t) + c_0 + c_1\sqrt{I_\varepsilon(t)}}{I_\varepsilon(t) - c_2 I_\varepsilon(\frac{t}{2}) - c_3 I_\varepsilon(0)} = \frac{1}{4},$$

for u and E replaced with $u_{t,\varepsilon}$ and E_ε in inequality (3.17) we have

$$Q_{t,\varepsilon} := \frac{\int_\Omega |\nabla u_{t,\varepsilon}|_A^2 dx}{\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u_{t,\varepsilon}^2 dx} \geq \frac{1}{4}, \tag{3.22}$$

now fix $\varepsilon > 0$ and let $0 \neq u \in C_c^\infty(\Omega)$, since $E_\varepsilon = E + \varepsilon$ so

$$\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \leq \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx,$$

thus

$$\frac{\int_\Omega |\nabla u|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx} \leq \frac{\int_\Omega |\nabla u|_A^2 dx}{\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx},$$

since $\lim_{t \nearrow \frac{1}{2}} Q_{t,\varepsilon} = \frac{1}{4}$ and according (3.22), $Q_{t,\varepsilon} \geq \frac{1}{4}$ so the best constant for (3.17) is $\frac{1}{4}$.

As it is mentioned in proof of Lemma 2.1 we drop the positive term $\int_\Omega |\nabla v|_A^2 E dx$ and so we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq 0,$$

but in case of this inequality

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega |\nabla v|_A^2 E dx,$$

so $\frac{1}{4}$ is not attained.

Now we show the inequality holds for E as a boundary weight for $t > \frac{1}{2}$ and $\varepsilon > 0$. In the proof of Lemma 2.1 it is denoted that $E^t \in H_0^1(\Omega)$ for $\frac{1}{2} < t < 1$ so we get the inequality (3.15).

To verify $\frac{1}{4}$ is optimal in (3.15) for $\frac{1}{2} < t < 1$ we take $u_t = E^t \in H_0^1(\Omega)$ [Lemma 3.3] so we have

$$\frac{\int_\Omega |\nabla u_t|^2 dx}{\int_\Omega \frac{|\nabla E|^2}{E^2} u_t^2 dx} = \frac{t^2 \int_\Omega E^{2t-2} |\nabla E|^2 dx}{\int_\Omega E^{2t-2} |\nabla E|^2 dx} = t^2,$$

which shows $\frac{1}{4}$ is optimal constant.

- ii) According to lemma 3.2 part (ii), for E as a boundary weight we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega |\nabla v|_A^2 E dx + \frac{1}{2}\int_\Omega \frac{u^2}{E} d\mu, \tag{3.23}$$

by dropping the positive term $\int_\Omega |\nabla v|_A^2 E dx$ we have the desire inequality (3.16). To show that $\frac{1}{2}$ is optimal, suppose E is a boundary weight on Ω, let $\frac{1}{2} < t < 1$ and according to the last section of part (i), $E^t \in H_0^1(\Omega)$ so we have

$$\begin{aligned}
\int_\Omega E^{2t-1} d\mu =& -\int_\Omega E^{2t-1} div(A\nabla E) dx \\
=& \int_\Omega \nabla E^{2t-1} \cdot (A\nabla E) dx \\
=& (2t-1)\int_\Omega E^{2t-2} \nabla E \cdot (A\nabla E) dx \\
=& (2t-1)\int_\Omega E^{2t-2} |\nabla E|_A^2 dx,
\end{aligned} \tag{3.24}$$

so we have

$$\frac{\int_\Omega |\nabla E|_A^2 E^{2t-2} dx}{\int_\Omega E^{2t-1} d\mu} = \frac{1}{2t-1}, \tag{3.25}$$

also according to (3.25) and inequality (3.16) we have

$$\begin{aligned}
\frac{\int_\Omega |\nabla E^t|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}(E^t)^2}{\int_\Omega \frac{(E^t)^2}{E} d\mu} &= \frac{t^2\int_\Omega |\nabla E|_A^2 E^{2t-2} dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}(E^t)^2 dx}{\int_\Omega \frac{(E^t)^2}{E} d\mu} \\
&= \frac{t^2\int_\Omega |\nabla E|_A^2 E^{2t-2} dx - \frac{1}{4}\int_\Omega |\nabla E|_A^2 E^{2t-2} dx}{\int_\Omega E^{2t-1} d\mu} \\
&= \frac{(t^2-\frac{1}{4})\int_\Omega |\nabla E|_A^2 E^{2t-2} dx}{\int_\Omega E^{2t-1} d\mu} \\
&= \frac{(t^2-\frac{1}{4})\int_\Omega |\nabla E|_A^2 E^{2t-2} dx}{(2t-1)\int_\Omega |\nabla E|_A^2 E^{2t-2} d\mu} \\
&= \frac{t^2-\frac{1}{4}}{2t-1} = \frac{t}{2} + \frac{1}{4},
\end{aligned}$$

which shows $\frac{1}{2}$ is optimal.

Finally as it is showed in proof of Lemma 2.1, $\int_\Omega |\nabla v|_A^2 E dx > 0$ so considering the positive term as in the inequality (3.23), the best constant is not attained.

$\square$

Example 3.5. *Suppose Ω is a bounded, convex domain in $\mathbb{R}^N$. We set $\delta(x) := dis(x, \partial\Omega)$. For $\frac{1}{2} < t < 1$ we set $E = \delta^t \in H_0^1$ [Lemma 2.1], since $|\nabla\delta| \le 1$ for all x, then*

$$\frac{|\nabla E|^2}{E^2} = \frac{t^2\delta^{2t-2}}{\delta^{2t}} = t^2\delta^{2t-2}\delta^{-2t} = t^2\delta^{-2} = \frac{t^2}{\delta^2},$$

by considering $\mu = -div(\nabla E) = -\Delta E$, we have

$$\Delta E = t(t-1)\delta^{t-2} + t\delta^{t-1}\Delta\delta,$$

and

$$-\Delta E = t(1-t)\delta^{t-2} + t\delta^{t-1}(-\Delta\delta) \ge 0,$$

now according to (3.15) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq 0, \tag{3.26}$$

where $u \in H_0^1(\Omega)$. Since $E = \delta^t > 0$ and $\mu = -\Delta E \geq 0$ and $E \in H_0^1(\Omega)$ by considering E as a boundary weight and inserting in (3.17) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx = \int_\Omega |\nabla u|_A^2 dx - \frac{t^2}{4}\int_\Omega \frac{u^2}{\delta^2} dx \geq 0, \tag{3.27}$$

so

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{t^2}{4}\int_\Omega \frac{u^2}{\delta^2} dx, \tag{3.28}$$

for $\frac{1}{2} < t < 1$, $\frac{1}{4}$ is not attained and (3.17) is not optimal, because in this example we can not use Theorem 3.4. Although $E \in H_0^1(\Omega)$ but according to the definition of boundary weight for positive function E, $L_A(E) = \mu$ should be finite and in this example since we have $-\Delta E = \mu$ and

$$\Delta E = t(t-1)\delta^{t-2}\nabla\delta + t\delta^{t-1}\Delta\delta,$$

Claim. *For $\frac{1}{2} < t < 1$, $\delta^{t-2} \notin L^1(\Omega)$.*

We consider $M = \max\delta$ and let $0 < \varepsilon < M$ and take $\Gamma_s := \{x \in \Omega : \delta(x) = s\}$, by applying *co-area formula we have:*

$$
\begin{aligned}
\int_{\Omega} \delta^{t-2} dx =& \int_{\Omega} \frac{|\nabla \delta(x)|}{\delta(x)^{2-t}} dx \\
=& \int_0^M \left(\int_{\Gamma_s} \frac{1}{s^{2-t}} dH^{N-1} \right) ds \\
=& \int_0^M \frac{|\Gamma_s|}{s^{2-t}} ds \\
\geq& \int_0^{\varepsilon} \frac{|\Gamma_s|}{s^{2-t}} ds \\
\geq& C \int_0^{\varepsilon} \frac{1}{s^{2-t}} ds
\end{aligned}
$$

note there exist some $C > 0$ such that for small $0 < s < \varepsilon$ we have $|\Gamma_s| \geq C$ so the last integral is unbounded. Now since $\delta^{t-2} \notin L^1(\Omega)$ thus μ is not locally finite measure. This shows that (3.15) is not optimal and this apparent failure of Theorem 3.4.

$\triangle$

Since for $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ we get the best constants in inequality (3.16) now we want to consider the best constant in

$$
\int_{\Omega} |\nabla u|_A^2 dx \geq \alpha \int_{\Omega} \frac{|\nabla E|_A^2}{E^2} u^2 dx + \beta \int_{\Omega} \frac{u^2}{E} d\mu, \tag{3.29}
$$

for $u \in H_0^1(\Omega)$ and $(\beta, \alpha) \in \mathbb{R}^2$.

Theorem 3.6. *If $E \in L^{\infty}(\Omega)$ and E is a boundary weight in Ω (i.e. $E \in H_0^1$), then by defining*

$$
\Upsilon = \left\{ (\beta, \alpha) \quad \textit{such that} \quad \beta > \frac{1}{2}, \quad \alpha \leq \beta - \beta^2 \right\} \tag{3.30}
$$

and

$$
\Upsilon' := \left(-\infty, \frac{1}{2} \right] \cup \left(-\infty, \frac{1}{4} \right], \tag{3.31}
$$

(3.29) is attained for $(\beta,\alpha)\subset\Gamma$*, where*

$$\Gamma=\{(\tau,\tau-\tau^2):\tau>\frac{1}{2}\}\subset\partial\Upsilon.$$

Proof. As it is shown in the proof of Lemma 2.1, $E^t\in H_0^1(\Omega)$ for $\frac{1}{2}<t<1$. If $(\beta,\alpha)\in\Upsilon$ and $\beta>\frac{1}{2}$, then by inserting $u=E^\beta\in H_0^1(\Omega)$ in (3.29) we have

$$\beta^2\int_\Omega E^{2\beta-2}|\nabla E|_A^2dx\geq\alpha\int_\Omega|\nabla E|_A^2E^{2\beta-2}dx+(2\beta-1)\beta\int_\Omega|\nabla E|_A^2dx, \tag{3.32}$$

because

$$\int_\Omega\frac{u^2}{E}d\mu=\int_\Omega E^{2\beta-1}L_A(E)dx=-\int_\Omega E^{2\beta-1}div(A\nabla E)dx=\int_\Omega\nabla E^{2\beta-1}\cdot A\nabla Edx$$
$$=(2\beta-1)\int_\Omega|\nabla_E|_A^2E^{2\beta-2}dx,$$

so by factoring of $\int_\Omega|\nabla_E|_A^2E^{2\beta-2}dx$ of both sides of (3.32) we have $\beta^2-2\beta^2+\beta\geq\alpha$ which result in $\alpha\leq\beta-\beta^2$.

Now for $\beta\leq\frac{1}{2}$ and $u=E^t$ as $t\searrow\frac{1}{2}$ testing (3.29) indicates that

$$t^2\int_\Omega|\nabla E|_A^2E^{2t-2}dx\geq\alpha\int_\Omega|\nabla E|_A^2E^{2t-2}dx+\beta\int_\Omega E^{2t-1}L_A(E)dx$$
$$\geq\alpha\int_\Omega|\nabla E|_A^2E^{2t-2}dx+\beta(2t-1)\int_\Omega|\nabla E|_A^2E^{2t-2}dx,$$

so by factoring of $\int_\Omega|\nabla E|_A^2E^{2t-2}dx$ of both sides we have $t^2+\beta(1-2t)\geq\alpha$ which for $t\searrow\frac{1}{2}$ and $\beta\leq\frac{1}{2}$ we have $\alpha\leq\frac{1}{4}$.

Now we fix $t\geq1$ and set $E_2:=E^t$, then we have $\frac{|\nabla E_2|_A^2}{{E_2}^2}=\frac{|\nabla E^t|_A^2}{E^{2t}}$. Since $\nabla E^t=tE^{t-1}\nabla E$ and $|\nabla E|_A^2=t^2E^{2t-2}|\nabla E|_A^2$ we have:

$$\frac{|\nabla E_2|_A^2}{{E_2}^2}=t^2\frac{|\nabla E|_A^2E^{2t-2}}{E^{2t}}=t^2\frac{|\nabla E|_A^2}{E^2}, \tag{3.33}$$

and

$$\frac{L_A(E_2)}{E_2} = \frac{-div(A\nabla E^t)}{E^t}, \tag{3.34}$$

since we have

$$\begin{aligned} div(A\nabla E^t) =& div(A(tE^{t-1}\nabla E)) \\ =& \big(A(t(t-1))E^{t-2}\nabla E\big)\cdot \nabla E + div(A\nabla E)tE^{t-1} \\ =& (t(t-1))E^{t-2}|\nabla E|_A^2 - tE^{t-1}L_A(E), \end{aligned}$$

so (3.34) is

$$\begin{aligned} \frac{L_A(E_2)}{E_2} = \frac{-div(A\nabla E^t))}{E^t} :=& t(1-t)\frac{E^{t-2}|\nabla E|_A^2}{E^t} + t\frac{E^{t-1}L_A(E)}{E^t} \\ =& t(1-t)\frac{|\nabla E|_A^2}{E^2} + t\frac{L_A(E)}{E}, \end{aligned} \tag{3.35}$$

by (3.33) and (3.35) we consider the terms in (3.16),

$$\frac{1}{4}\int_\Omega \frac{|\nabla E_2|_A^2}{{E_2}^2}u^2dx = \frac{t^2}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2dx, \tag{3.36}$$

and

$$\begin{aligned} \frac{1}{2}\int_\Omega \frac{u^2}{E_2}d\mu =& \frac{1}{2}\int_\Omega \frac{u^2}{E_2}L_A(E_2)dx \\ =& \frac{1}{2}\int_\Omega \frac{L_A(E_2)}{E_2}u^2dx \\ =& \frac{t}{2}(1-t)\int_\Omega \frac{|\nabla E_2|_A^2}{E^2}u^2dx + \frac{t}{2}\int_\Omega \frac{L_A(E)}{E}u^2dx \\ =& (\frac{t}{2}-\frac{t^2}{2})\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2dx + \frac{t}{2}\int_\Omega \frac{L_A(E)}{E}u^2dx \\ =& (\frac{t-t^2}{2})\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu, \end{aligned} \tag{3.37}$$

by inserting (3.36) and (3.37) in (3.16) we have

$$\begin{aligned}\int_\Omega |\nabla u|_A^2 dx \geq & \frac{t^2}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t-t^2}{2}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu \\ =& (\frac{t^2}{4}+\frac{t-t^2}{2})\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu \\ =& \frac{t^2+2t-2t^2}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu \\ =& (\frac{t}{2}-\frac{t^2}{4})\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu,\end{aligned}$$

and so we see that $\left(\frac{t}{2}, \frac{t}{2}-\frac{t^2}{4}\right) \in \Upsilon$ for all $t \geq 1$. Since we have

$$\int_\Omega |\nabla u|_A^2 dx \geq (\frac{t}{2}-\frac{t^2}{4})\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu, \tag{3.38}$$

and since Υ is the set of (β, α) satisfying in

$$\int_\Omega |\nabla u|_A^2 dx \geq \alpha\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \beta\int_\Omega \frac{u^2}{E}d\mu, \tag{3.39}$$

so by corresponding (3.39) and (3.38) we can see $\beta = \frac{t}{2}$ and $\alpha = \frac{t}{2} - \frac{t^2}{4}$ so $\alpha = \beta - \beta^2$ and the curve $\beta - \beta^2$ for $\beta \geq \frac{1}{2}$ is in Υ. Also, for $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ which are as $\partial\Upsilon'$ we have :

$$\int_\Omega |\nabla u|_A^2 dx \geq (\frac{t}{2}-\frac{t^2}{4})\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \frac{t}{2}\int_\Omega \frac{u^2}{E}d\mu,$$

and by corresponding with

$$\int_\Omega |\nabla u|_A^2 dx \geq \alpha\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \beta\int_\Omega \frac{u^2}{E}d\mu,$$

we conclude that the remaining portion of $\partial\Upsilon'$ is contained in Υ.

$\square$

Corollary 3.7. *Suppose* $(E, \lambda_A(\Omega))$ *is the first eigenpair(with* $E > 0$*) of* L_A *in* $H_0^1(\Omega)$*. For* $B \subset \Omega$ *we suppose* $\lambda_A(B)$ *is the first eigenvalue of* L_A *in* $H_0^1(B)$*.*

We define

$$\underline{\alpha}(B) := \inf_B \frac{|\nabla E|_A^2}{E^2},$$

and

$$\overline{\alpha}(B) := \sup_B \frac{|\nabla E|_A^2}{E^2},$$

- *i) if* $\underline{\alpha}(B) > \lambda_A(\Omega)$ *then*

$$4\lambda_A(B) \geq \frac{(\underline{\alpha}(B) + \lambda_A(\Omega))^2}{\underline{\alpha}(B)},$$

- *ii) if* $\overline{\alpha}(B) < \lambda_A(\Omega)$ *then*

$$4\lambda_A(B) \geq \frac{(\overline{\alpha}(B) + \lambda_A(\Omega))^2}{\overline{\alpha}(B)}.$$

Proof. Suppose $B \subset \Omega$ and $u \in C_c^\infty(B)$ such that $\int_B u^2 = 1$. According to the inequality (3.38) for $0 < t < 2$ we have

$$2\int_B |\nabla u|_A^2 dx \geq (t - \frac{t^2}{2})\int_B \frac{|\nabla E|_A^2}{E^2} u^2 dx + t\int_B E^{-1} d\mu, \tag{3.40}$$

since $(E, \lambda_A(\Omega))$ is the first eigenpair of L_A so we have $(L_A - \lambda_A(\Omega))E = 0$ and

$$\frac{L_A(E)}{E} = \lambda_A(\Omega) \tag{3.41}$$

according to (3.41) we have

$$\int_B E^{-1} d\mu = \int_B E^{-1} L_A(E) dx = \lambda_A(\Omega),$$

so according to (3.40) we have

$$2\int_B |\nabla u|_A^2 dx \geq (t - \frac{t^2}{2}) \inf_B \frac{|\nabla E|_A^2}{E^2} + t\lambda_A(\Omega), \tag{3.42}$$

and for $t > 2$ we have

$$2\int_B |\nabla u|_A^2 dx \geq (t - \frac{t^2}{2}) \sup_B \frac{|\nabla E|_A^2}{E^2} + t\lambda_A(\Omega),$$

now suppose $t := 1 + \frac{\lambda_A(\Omega)}{\underline{\alpha}(B)} < 2$ and $\underline{\alpha}(B) > \lambda_A(\Omega)$ and $\underline{\alpha}(B) = \inf_B \frac{|\nabla E|_A^2}{E^2}$ by inserting in (3.42) we have

$$2\int_B |\nabla u|_A^2 dx \geq (t - \frac{t^2}{2})\underline{\alpha}(B) + t\lambda_A(\Omega), \tag{3.43}$$

for right-hand side of the inequality (3.43) we have

$$\begin{aligned}(t - \frac{t^2}{2})\underline{\alpha}(B) + t\lambda_A(\Omega) =& t\underline{\alpha}(B) + t\lambda_A(\Omega) - \frac{t^2}{2}\underline{\alpha}(B)\\ =& t(\underline{\alpha}(B) + \lambda_A(\Omega)) - \frac{t^2}{2}\underline{\alpha}(B),\end{aligned} \tag{3.44}$$

since

$$\begin{aligned}t(\underline{\alpha}(B) + \lambda_A(\Omega)) =& 1 + \frac{\lambda_A(\Omega)}{\underline{\alpha}(B)}(\underline{\alpha}(B) + \lambda_A(\Omega))\\ =& \frac{\lambda_A(\Omega) + \underline{\alpha}(B)}{\underline{\alpha}(B)}(\underline{\alpha}(B) + \lambda_A(\Omega))\\ =& \frac{(\lambda_A(\Omega) + \underline{\alpha}(B))^2}{\underline{\alpha}(B)},\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\frac{t^2}{2}\underline{\alpha}(B) =& \frac{1}{2}\left(1+\frac{\lambda_A(\Omega)}{\underline{\alpha}(B)}\right)^2 \underline{\alpha}(B)\\ =&\frac{1}{2}\left(\frac{\lambda_A(\Omega)+\underline{\alpha}(B)}{\underline{\alpha}(B)}\right)^2 \underline{\alpha}(B)\\ =&\frac{1}{2}\frac{(\lambda_A(\Omega)+\underline{\alpha}(B))^2}{\underline{\alpha}(B)},\end{aligned} \tag{3.46}$$

by plugging (3.45) and (3.46) in (3.44) we have

$$\begin{aligned}(t-\frac{t^2}{2})\underline{\alpha}(B))+t\lambda_A(\Omega) =& \frac{(\lambda_A(\Omega)+\underline{\alpha}(B))^2}{\underline{\alpha}(B)}-\frac{1}{2}\frac{(\lambda_A(\Omega)+\underline{\alpha}(B))^2}{\underline{\alpha}(B)}\\ =&\frac{1}{2}\frac{(\lambda_A(\Omega)+\underline{\alpha}(B))^2}{\underline{\alpha}(B)},\end{aligned} \tag{3.47}$$

since $u \in C_c^\infty(B)$ for the left-hand side of the inequality (3.43) we have

$$\begin{aligned}\int_B |\nabla u|_A^2 dx = \int_B A\nabla u\cdot\nabla u dx =& -\int_B div(A\nabla u)u dx\\ =& -\int_B -L_A(u)u dx\\ =& \int_B L_A(u)u dx,\end{aligned} \tag{3.48}$$

by taking infimum over u and since $\lambda_A(B)$ is the first eigenvalue of $L_A(u)$ in $H_0^1(B)$ so we have

$$\int_B |\nabla u|_A^2 dx = \lambda_A(B), \tag{3.49}$$

considering (3.47) and (3.49) in (3.43) we have

$$4\lambda_A(B) \geq \frac{(\underline{\alpha}(B)+\lambda_A(\Omega))^2}{\underline{\alpha}(B)},$$

now by setting $t := 1+\frac{\lambda_A(\Omega)}{\overline{\alpha}(B)} > 2$ and using $\overline{\alpha}(B) = \sup_B \frac{|\nabla E|_A^2}{E^2}$ we get the same result for (ii). $\square$

4

Weighted Versions of Hardy Inequality

We proved the inequalities (3.15) and (3.16) for E as interior weight and boundary weight. In this chapter the weighted versions of these inequalities are examined. The weighted versions which can be considered as generalization of Cafferelli-Kohn-Nirenberg inequality [20]

$$\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2a}} dx\right) \geq C_{a,b} \int_{\mathbb{R}^N} \left(\frac{|u|^p}{|x|^{bp}} dx\right)^{\frac{2}{p}},$$

where

- for $N \geq 3$: $a < \frac{N-2}{2}, \qquad a \leq b \leq a+1 \qquad p = \frac{2N}{N-2+2(b-a)}$;
- for $N = 2$: $a < 0, \qquad a < b \leq a+1 \qquad p = \frac{2}{b-a}$;
- for $N = 1$: $a < \frac{-1}{2}, \qquad a+\frac{1}{2} < b \leq a+1 \qquad p = \frac{2}{-1+2(b-a)}$,

for $u \in C_0^\infty(\mathbb{R}^N)$ contains particular cases of the classical Sobolev inequality and Hardy inequality as

$$C^2 \int_\Omega \left(\frac{u^2}{|x|^2} dx\right) \leq \int_\Omega |\nabla u|^2 dx,$$

so the weighted versions of above inequalities are proved in Theorems (4.3) and (4.4).

Definition 4.1. *If E is an interior weight so $E = \infty$ on K as the support of μ . In this case, for considering the weighted version, the completion of $C_c^{0,1}(\Omega \backslash K)$ is needed. We define X_t as a completion of $C_c^{0,1}(\Omega \backslash K)$ which is equipped with the norm*

$$\|u\|_t^2 := \int_\Omega E^{2t} |\nabla u|_A^2 dx.$$

Now we consider E as boundary weight and similar to the previous part we need to work with X_t as completion of $C_c^{0,1}(\Omega)$ with the same norm.

Remark 4.2. *Assume E is an interior weight it means (i.e $E = \infty$ on $K = supp(\mu)$) , we want to consider $C_c^\infty(\Omega)$ in X_t equipped the norm*

$$\|u\|_t^2 := \int_\Omega E^{2t} |\nabla u|_A^2 dx,$$

for $t > \frac{1}{2}$ and $t < \frac{1}{2}$.

- *If $t > \frac{1}{2}$*

  Claim *X_t does not contain $C_c^\infty(\Omega)$.*

  Let $u \in C_c^\infty(\Omega)$ and as contradiction let $C_c^\infty(\Omega) \subset X_t$ so $u \in X_t$. According to the inequality (4.2) in the next theorem for $u \in X_t$ where $t \neq \frac{1}{2}$ we have

  $$\int_\Omega E^{2t} |\nabla u|_A^2 dx \geq (t - \frac{1}{2})^2 \int_\Omega E^{2t-2} |\nabla E|_A^2 u^2 dx,$$

  for the left hand side of the above inequality we have

  $$\int_\Omega E^{2t} |\nabla u|_A^2 dx = \|u\|_t^2,$$

which is finite. So the right hand side of the above inequality

$$(t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx < \infty, \tag{4.1}$$

so it is required all terms in (4.1) be finite. So $E^{2t-2}|\nabla E|_A^2$ need to be finite for $t > \frac{1}{2}$. For this E^t should be in $H^1_{loc}(\Omega)$ but since $t > \frac{1}{2}$ and E is interior weight ($E = \infty$ on compact set K) so it is not correct. For

- *If $t < \frac{1}{2}$, according to Lemma 3.3, $E^t \in H^1_{loc}(\Omega)$ so $C_c^\infty(\Omega) \subset X_t$.*

Now we consider the weighted version of the inequality (3.15) for interior weight E in the next theorem.

Theorem 4.3. *If $u \in X_t$ and $t \neq \frac{1}{2}$ and E is an interior weight on Ω, then*

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx \geq (t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx, \tag{4.2}$$

and the constant is optimal and not attained.

Proof. Suppose $w := E^t u \in C_c^{0,1}(\Omega \backslash K)$ for $t \neq 0, \frac{1}{2}$ and $u \in C_c^{0,1}(\Omega \backslash K)$. Since $C_c^{0,1}(\Omega \backslash K)$ is dense in $H_0^1(\Omega)$ so plugging into inequality (3.15) gives us

$$\int_\Omega |\nabla w|_A^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} w^2 dx. \tag{4.3}$$

The left hand side of inequality (4.3) is

$$\int_\Omega |\nabla w|_A^2 dx = \int_\Omega E^{2t}|\nabla u|_A^2 dx + t^2\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx + 2t\int_\Omega E^{2t-1} A\nabla E \cdot \nabla u u dx, \tag{4.4}$$

and for last part of (4.4) we have

$$\begin{aligned}2t\int_\Omega E^{2t-1}\nabla E\cdot\nabla u u dx =&\frac{2t}{2}\int_\Omega E^{2t-1}A\nabla E\cdot\nabla u^2 dx\\=&-t\int_\Omega div(E^{2t-1})\cdot A\nabla E u^2 dx\\=&-t(2t-1)\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx\\=&(-2t^2+t)\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx,\end{aligned}$$

so we have

$$\int_\Omega |\nabla w|_A^2 dx = \int_\Omega E^{2t}|\nabla u|_A^2 dx + (-t^2+t)\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx, \tag{4.5}$$

the right hand side of inequality (4.3) is

$$\begin{aligned}\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}w^2 dx =&\frac{1}{4}\int_\Omega |\nabla E|_A^2 E^{-2}E^{2t}u^2 dx\\=&\frac{1}{4}\int_\Omega |\nabla E|_A^2 E^{2t-2}u^2 dx,\end{aligned} \tag{4.6}$$

so by plugging (4.5) and (4.6) in (4.3) we have

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx + (-t^2+t)\int_\Omega |\nabla E|_A^2 E^{2t-2}u^2 dx \geq \frac{1}{4}\int_\Omega |\nabla E|_A^2 E^{2t-2}u^2 dx,$$

so

$$\begin{aligned}\int_\Omega E^{2t}|\nabla u|_A^2 dx \geq&(\frac{1}{4}+t^2-t)\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx\\=&(t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx.\end{aligned}$$

Now we show the constant is optimal. Since $v_m \in C_c^{0,1}(\Omega\backslash K)$ so according to (3.15) we have

$$\int_\Omega |\nabla v_m|_A^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx, \tag{4.7}$$

by dividing both sides of (4.7) by $\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2$ we get

$$D_m := \frac{\int_\Omega |\nabla v_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4}, \tag{4.8}$$

now, we define $u_m := E^{-t} v_m$ in the completion space of $C_c^{0,1}(\Omega \backslash K)$ (i.e X_t equipped with the norm $\|u\|_t^2 := \int_\Omega E^{2t} |\nabla u|_A^2 dx$). For the sharp constant in (4.2) we compute

$$\frac{\int_\Omega E^{2t} |\nabla u_m|_A^2 dx}{\int_\Omega E^{2t-2} |\nabla E|_A^2 u_m^2 dx}, \tag{4.9}$$

so for the numerator of (4.9) we have

$$E^{2t}|\nabla u_m|_A^2 = |\nabla v_m|_A^2 + t^2 |\nabla E|_A^2 E^{-2} v_m^2 - 2tE^{-1}\nabla E \cdot (\nabla v_m) v_m, \tag{4.10}$$

by integration of both sides of (4.10) we have

$$\begin{aligned}
\int_\Omega E^{2t}|\nabla u_m|_A^2 dx &= \int_\Omega |\nabla v_m|_A^2 dx + t^2 \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - 2t \int_\Omega E^{-1} \nabla E \cdot \nabla v_m v_m dx \\
&= \int_\Omega |\nabla v_m|_A^2 dx + t^2 \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - t \int_\Omega E^{-1} \nabla E \cdot \nabla (v_m)^2 dx \\
&= \int_\Omega |\nabla v_m|_A^2 dx + t^2 \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx + t \int_\Omega div(E^{-1}) \cdot \nabla E v_m^2 dx \\
&= \int_\Omega |\nabla v_m|_A^2 dx + t^2 \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - t \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx, \qquad (4.11)
\end{aligned}$$

so (4.9) is

$$\frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2 u_m^2 dx} = \frac{\int_\Omega |\nabla v_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} + t^2 \frac{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} - t \frac{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}, \tag{4.12}$$

according to definition of D_m in (4.8) and simplifying (4.12) we have

$$\frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2} u_m^2 dx} = D_m + t^2 - t,$$

since $D_m \to \frac{1}{4}$ so the constant $\frac{1}{4} + t^2 - t = (t - \frac{1}{2})^2$ is optimal. Now we want to show that the constant is not attained. In case of $\gamma := \min_{\partial\Omega} E = 0$, to show the inequality does not attain we just not drop the positive term $\int_\Omega E|\nabla v|_A^2 dx$ (see Lemma 2.1). By considering $w := E^t u \in C_c^{0,1}(\Omega \backslash K)$ which with $v := E^{\frac{-1}{2}} w = E^{t-\frac{1}{2}} u$ we have

$$\int_\Omega E \left| \nabla \left(\frac{w}{\sqrt{E}} \right) \right|_A^2 = \int_\Omega E |\nabla (E^t u E^{\frac{-1}{2}})|_A^2 dx = \int_\Omega E |\nabla (E^{t-\frac{1}{2}} u)|_A^2 dx,$$

now by plugging w and v into

$$\int_\Omega |\nabla w|_A^2 dx - \int_\Omega \frac{|\nabla E|_A^2}{4E^2} u^2 dx = \int_\Omega E|\nabla v|_A^2 dx + \frac{1}{2} \int_\Omega \frac{u^2}{E} L_A(E) dx, \tag{4.13}$$

we compute the first term of the LHS of equality (4.13)

$$\int_\Omega |\nabla w|_A^2 dx = t^2 \int_\Omega E^{2t-2} |\nabla E|_A^2 u^2 + \int_\Omega E^{2t} |\nabla u|_A^2 + (-2t^2 + t) \int_\Omega E^{2t-2} |\nabla E|_A^2 u^2 dx, \tag{4.14}$$

and for the second term of the LHS of equality (4.13) we have

$$\frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} w^2 dx = \frac{1}{4} \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx, \tag{4.15}$$

now, we compute the first term of RHS of the equality (4.13)

$$\int_\Omega E|\nabla(E^{t-\frac{1}{2}}u)|_A^2dx = \int_\Omega E^{2t}|\nabla u|_A^2 + (t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2u^2$$
$$+2(t-\frac{1}{2})\int_\Omega E^{2t-1}A\nabla E\cdot u\nabla u dx$$
$$=\int_\Omega E^{2t}|\nabla u|_A^2 + (t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2u^2 + (t-\frac{1}{2})\int_\Omega E^{2t-1}A\nabla E\cdot\nabla u^2dx$$
$$=\int_\Omega E^{2t}|\nabla u|_A^2 + (t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2u^2 - (t-\frac{1}{2})\int_\Omega div(E^{2t-1})\cdot A\nabla E u^2$$
$$=\int_\Omega E^{2t}|\nabla u|_A^2 + (t-\frac{1}{2})^2\int_\Omega E^{2t-2}|\nabla E|_A^2u^2 - (t-\frac{1}{2})(2t-1)\int_\Omega E^{2t-2}|\nabla E|_A^2u^2$$
$$=\int_\Omega E^{2t}|\nabla u|_A^2 - \int_\Omega E^{2t-2}|\nabla E|_A^2u^2, \tag{4.16}$$

and for the second term of RHS of the equality (4.13)

$$\frac{1}{2}\int_\Omega \frac{u^2}{E}L_A(E)dx = (t-\frac{1}{2})\int_\Omega |\nabla E|_A^2E^{2t-2}u^2dx, \tag{4.17}$$

by plugging (4.14), (4.15), (4.16) and (4.17) into (4.13) and divided by $\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx$ we have:

$$t^2\frac{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx} + \frac{\int_\Omega E^{2t}|\nabla u|_A^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx} + (-2t^2+t)\frac{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}$$
$$-\frac{1}{4}\frac{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx} = \frac{\int_\Omega E^{2t}|\nabla u|_A^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx} - \frac{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}$$
$$+(t-\frac{1}{2})\frac{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2u^2dx}, \tag{4.18}$$

by simplifying the equality (4.18) we have

$$t^2+1-2t^2+t-\frac{1}{4} = 1-1+t-\frac{1}{2} = t^2-\frac{5}{4},$$

which shows the constant is not attained.

We will also show the constant is not attained when $w := \min_{\partial\Omega} E > 0$, but we need to use results that will be presented in a later section. □

Theorem 4.4. *If $t \neq 0$ and $t < \frac{1}{2}$ and E is boundary weight on Ω and $u \in X_t$ then*

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx - (t-\frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx \geq 0, \tag{4.19}$$

and

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx - (t-\frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx \geq (\frac{1}{2}-t) \int_\Omega E^{2t-1} u^2 d\mu, \tag{4.20}$$

the constant $(t-\frac{1}{2})^2$ in (4.19) and constant $(\frac{1}{2}-t)$ in (4.20) are optimal and not attained.

Proof. Suppose $0 \neq t < \frac{1}{2}$, first we want to prove the inequality (4.20) for $u \in X_t$. Since X_t is the completion of $C_c^{0,1}(\Omega)$ so we consider $u \in C_c^{0,1}(\Omega)$. We use E_ε as ε-mollification of E and $F_\varepsilon = L_A(E_\varepsilon)$. Note that $H^{-1}(\Omega) = (H_0^1(\Omega))^* = \left(\overline{C_c^{0,1}(\Omega)}\right)^*$ and standard argument shows that $uF_\varepsilon \to u\mu$ in $H^{-1}(\Omega)$ for $u \in C_c^{0,1}(\Omega)$ [Lemma 3.2].
In Theorem 3.4 it is proved for E as a boundary weight and for $u \in H_0^1(\Omega)$

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx + \frac{1}{2} \int_\Omega \frac{u^2}{E} L_A(E) dx, \tag{4.21}$$

now for (ε-mollification of E) and $v \in H_0^1(\Omega)$ we have

$$\int_\Omega |\nabla v|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} v^2 dx + \frac{1}{2} \int_\Omega \frac{v^2}{E_\varepsilon} F_\varepsilon dx. \tag{4.22}$$

For $u \in C_c^{0,1}(\Omega)$ set $v = E_\varepsilon^t u \in C_c^{0,1}(\Omega)$. For left hand side of (4.22) we have

$$|\nabla v|_A^2 = t^2 E_\varepsilon^{2t-2} |\nabla E_\varepsilon|_A^2 u^2 + E_\varepsilon^{2t} |\nabla u|_A^2 + 2t E_\varepsilon^{2t-1} A \nabla E_\varepsilon \cdot \nabla u u. \tag{4.23}$$

For the last part of (4.23) we have

$$\begin{aligned}
2t\int_\Omega E_\varepsilon^{2t-1}A\nabla E_\varepsilon\cdot\nabla uu =& t\int_\Omega E_\varepsilon^{2t-1}A\nabla E_\varepsilon\cdot\nabla u^2dx\\
=& -t(2t-1)\int_\Omega E_\varepsilon^{2t-2}A\nabla E_\varepsilon\cdot\nabla E_\varepsilon u^2dx\\
=&(-2t^2+t)\int_\Omega E_\varepsilon^{2t-2}|\nabla E_\varepsilon|_A^2u^2dx,
\end{aligned}\tag{4.24}$$

so according to (4.23) and (4.24) we have

$$\begin{aligned}
\int_\Omega|\nabla v|_A^2dx =& t^2\int_\Omega E_\varepsilon^{2t-2}|\nabla E_\varepsilon|_A^2u^2dx+\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2dx+(-2t^2+t)\int_\Omega E_\varepsilon^{2t-2}|\nabla E_\varepsilon|_A^2u^2dx\\
=&(-t^2+t)\int_\Omega E_\varepsilon^{2t-2}|\nabla E_\varepsilon|_A^2u^2dx+\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2dx,
\end{aligned}\tag{4.25}$$

so for the first term of the right hand side of (4.22) we have

$$\begin{aligned}
\frac{1}{4}\int_\Omega\frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}v^2dx =& \frac{1}{4}\int_\Omega\frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}E_\varepsilon^{2t}u^2dx\\
=&\frac{1}{4}\int_\Omega|\nabla E_\varepsilon|_A^2E_\varepsilon^{2t-2}u^2dx,
\end{aligned}\tag{4.26}$$

and for the second term part of (4.22) we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega\frac{v^2}{E_\varepsilon}F_\varepsilon dx =& \frac{1}{2}\int_\Omega\frac{E_\varepsilon^{2t}u^2}{E_\varepsilon}F_\varepsilon dx\\
=&\frac{1}{2}\int_\Omega E_\varepsilon^{2t-1}u^2F_\varepsilon dx\\
=&\frac{1}{2}\int_\Omega E_\varepsilon^{2t-1}u^2F_\varepsilon dx\\
=&-\frac{1}{2}\int_\Omega E_\varepsilon^{2t-1}u^2div(A\nabla E_\varepsilon)dx\\
=&\frac{1}{2}\int_\Omega\nabla E_\varepsilon^{2t-1}\cdot u^2(A\nabla E_\varepsilon)dx\\
=&-(t-\frac{1}{2})\int_\Omega E_\varepsilon E_\varepsilon^{2t-2}u^2div(A\nabla E_\varepsilon)dx\\
=&\left(\frac{1}{2}-t\right)\int_\Omega E_\varepsilon^{2t-1}u^2F_\varepsilon dx,
\end{aligned}\tag{4.27}$$

inserting (4.25),(4.26), (4.27) in (4.22) we have

$$(-t^2+t)\int_\Omega E_\varepsilon^{2t-2}|\nabla E_\varepsilon|_A^2 u^2 dx + \int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2 dx \geq \frac{1}{4}\int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} u^2 dx + \left(\frac{1}{2}-t\right)\int_\Omega E_\varepsilon^{2t-1} F_\varepsilon u^2 dx, \tag{4.28}$$

so we have

$$\begin{aligned}\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2 dx &\geq \left(t^2-t+\frac{1}{4}\right)\int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} u^2 dx + \left(\frac{1}{2}-t\right)\int_\Omega E_\varepsilon^{2t-1} F_\varepsilon u^2 dx \\ &= \left(t-\frac{1}{2}\right)^2 \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} u^2 dx + \left(\frac{1}{2}-t\right)\int_\Omega E_\varepsilon^{2t-1} F_\varepsilon u^2 dx.\end{aligned}$$

As we know a sequence of smooth function E_ε (as mollification of E) approximate E. Since $E \in H_0^1(\Omega)$ so $E^{2t} \in H_0^1(\Omega)$ for $t < \frac{1}{2}$ and there are functions $E_\varepsilon^{2t} \in C_c^\infty(\Omega)$ such that $E_\varepsilon^{2t} \to E^{2t}$, also since E is boundary weight so $E^{2t} \in L^1_{loc}(\Omega)$ and according the properties of mollifiers (Theorem property (iv)), $E_\varepsilon^{2t} \to E^{2t}$ in $L^1_{loc}(\Omega)$ so we have

$$\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2 dx \to \int_\Omega E^{2t}|\nabla u|_A^2 dx, \tag{4.29}$$

and since $uF_\varepsilon \to u\mu$ in $H^{-1}(\Omega)$ so we have

$$\begin{aligned}\int_\Omega E_\varepsilon^{2t-1} u^2 F_\varepsilon dx &\to \int_\Omega E^{2t-1} uu\mu dx \\ &= \int_\Omega E^{2t-1} u^2 \mu dx \\ &= \int_\Omega E^{2t-1} u^2 d\mu,\end{aligned} \tag{4.30}$$

using the results (4.29) and (4.30) and Fatou's lemma we have

$$\begin{aligned}
&\int_\Omega E^{2t}|\nabla u|_A^2 dx - \left(t-\frac{1}{2}\right)^2 \int_\Omega |\nabla E|_A^2 E^{2t-2}u^2 dx \\
=&\liminf\left(\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2 dx - \left(t-\frac{1}{2}\right)^2 \int_\Omega |\nabla E_\varepsilon|^2 E_\varepsilon^{2t-2}u^2 dx\right) \\
\geq&\liminf\left(\frac{1}{2}-t\right)\int_\Omega E_\varepsilon^{2t-1}F_\varepsilon u^2 dx \\
\geq&\left(\frac{1}{2}-t\right)\int_\Omega E^{2t-1}u^2 d\mu,
\end{aligned} \tag{4.31}$$

for $u \in C_c^{0,1}(\Omega)$.

Now we show the constants in (4.19) and (4.20) are optimal.

Since E is boundary weight, we suppose $u_m := E^{-t}v_m$ and use similar proof of Theorem 4.3 and proof of Theorem 3.4 by considering

$$D_m := \frac{\int_\Omega |\nabla v_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4}, \tag{4.32}$$

and since

$$\begin{aligned}
\int_\Omega \frac{v_m^2}{E} d\mu = \int_\Omega v_m^2 E^{-1} L_A(E) dx &= -\int_\Omega v_m^2 E^{-1} div(A\nabla E) dx \\
&= \int_\Omega v_m^2 \nabla(E^{-1}) \cdot A\nabla E dx \\
&= -\int_\Omega v_m^2 E^{-2} \nabla E \cdot A\nabla E dx \\
&= -\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx
\end{aligned} \tag{4.33}$$

so according to (4.32) and (4.33) we have

$$F_m ;= \frac{\int_\Omega |\nabla v_m|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{v_m^2}{E} d\mu} \to \frac{1}{2}, \tag{4.34}$$

since E is boundary weight and $v_m \in C_c^\infty(\Omega)$ so $u_m \in \overline{C_c^{0,1}(\Omega)} \cap X_t = X_t$.

For seeing the optimal constant in (4.19) we consider $u_m := E^{-t}v_m$.

Now we compute

$$\phi_m := \frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx}{\int_\Omega E^{2t-2}|\nabla E|_A^2 u_m^2 dx}, \tag{4.35}$$

we have:

$$E^{2t}|\nabla u_m|_A^2 = t^2|\nabla E|_A^2 E^{-2} v_m^2 + |\nabla v_m|_A^2 - 2tE^{-1}\nabla E \cdot \nabla v_m v_m, \tag{4.36}$$

by integration of both sides of (4.36) we have:

$$\begin{aligned}
\phi_m := \frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2} u_m^2} =& \frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2} \\
=& t^2 \frac{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} + \frac{\int_\Omega |\nabla v_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} - 2t \frac{\int_\Omega \frac{A\nabla E \cdot \nabla v_m v_m}{E} dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \\
=& D_m + t^2 - 2t \frac{\int_\Omega E^{-1} v_m \nabla v_m \cdot (A\nabla E) dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx},
\end{aligned} \tag{4.37}$$

so for third term in (4.37) we have:

$$\begin{aligned}
2\int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx =& \int_\Omega E^{-1} \nabla v_m^2 \cdot A\nabla E dx \\
=& -\int_\Omega div(E^{-1}(A\nabla E)) v_m^2 dx \\
=& \int_\Omega \nabla E E^{-2} \cdot (A\nabla E) v_m^2 dx - \int_\Omega E^{-1} div(A\nabla E) v_m^2 dx \\
=& \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx + \int_\Omega \frac{L_A(E)}{E} v_m^2 dx \\
=& \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx + \int_\Omega \frac{v_m^2}{E} \mu dx \\
=& \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx + \int_\Omega \frac{v_m^2}{E} d\mu,
\end{aligned} \tag{4.38}$$

by inserting (4.38) in (4.37) we have:

$$\begin{aligned}\phi_m =&D_m+t^2-2t\frac{\int_\Omega E^{-1}v_m\nabla v_m A\nabla E dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}\\ =&D_m+t^2-t\frac{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx+\int_\Omega \frac{v_m^2}{E}d\mu}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}\\ =&D_m+t^2-t-t\frac{\int_\Omega \frac{v_m^2}{E}d\mu}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx},\end{aligned} \tag{4.39}$$

for computing the fourth term in (4.39), according to (4.34) we have

$$F_m.\int_\Omega \frac{v_m^2}{E}d\mu=\int_\Omega |\nabla v_m|_A^2 dx-\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx, \tag{4.40}$$

so regarding (4.40), the fourth term in (4.39) is

$$\begin{aligned}\frac{\int_\Omega v_m^2E^{-1}d\mu}{\int_\Omega |\nabla E|_A^2E^{-2}v_m^2 dx}=&\frac{\frac{\int_\Omega |\nabla v_m|_2^A dx-\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}{F_m}}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}\\ =&\frac{\int_\Omega |\nabla v_m|_2^A}{F_m\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}-\frac{\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}{F_m\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx},\end{aligned} \tag{4.41}$$

regarding (4.32) we have

$$\frac{\int_\Omega v_m^2E^{-1}d\mu}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}=\frac{D_m}{F_m}-\frac{\frac{1}{4}}{F_m}=\frac{D_m-\frac{1}{4}}{F_m},$$

since $D_m\to\frac{1}{4}$ so we have

$$\frac{\int_\Omega v_m^2E^{-1}d\mu}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}v_m^2 dx}=\frac{D_m-\frac{1}{4}}{F_m}\to 0, \tag{4.42}$$

so regarding (4.39) and (4.42) we have

$$\phi_m := D_m + t^2 - t = \frac{1}{4} + t^2 - t = (t - \frac{1}{2})^2,$$

and (4.19) is optimal.

To see (4.20) is optimal, in terms of computation in (4.37) and $u_m := E^{-t} v_m$ we have

$$\begin{aligned}\psi_m :=& \frac{\int_\Omega E^{2t} |\nabla u_m|_A^2 dx - (t-\frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u_m^2 dx}{\int_\Omega E^{2t-1} u_m^2 d\mu} \\ =& \frac{t^2 \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx + \int_\Omega |\nabla v_m|_A^2 dx - 2t \int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx}{\int_\Omega E^{2t-1} u_m^2 d\mu} \\ & \frac{-t^2 \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx + t \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega E^{2t-1} u_m^2 d\mu},\end{aligned} \tag{4.43}$$

by simplifying (4.43) and since

$$\int_\Omega E^{2t-1} u_m^2 d\mu = \int_\Omega E^{2t-1} E^{-2t} v_m^2 d\mu = \int_\Omega \frac{v_m^2}{E} d\mu,$$

so we have

$$\begin{aligned}\psi_m :=& \frac{\int_\Omega |\nabla v_m|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{v_m^2}{E} d\mu} + \frac{t \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - 2t \int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx}{\int_\Omega \frac{v_m^2}{E} d\mu} \\ =& F_m + \frac{t \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - 2t \int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx}{\int_\Omega \frac{v_m^2}{E} d\mu},\end{aligned} \tag{4.44}$$

according to (4.38) we have

$$2 \int_\Omega E^{-1} \nabla v_m v_m \cdot A\nabla E dx = \int_\Omega \frac{v_m^2}{E} d\mu + \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx,$$

so (4.44) is

$$\psi_m := F_m + \frac{t\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - t\int_\Omega \frac{v_m^2}{E} d\mu - t\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{v_m^2}{E} d\mu}$$
$$= F_m + t\frac{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{v_m^2}{E} d\mu} - t\frac{\int_\Omega \frac{v_m^2}{E} d\mu}{\int_\Omega \frac{v_m^2}{E} d\mu} - t\frac{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{v_m^2}{E} d\mu} = F_m - t,$$

since $F_m \to \frac{1}{2}$ so $(\frac{1}{2} - t)$ is optimal in (4.20).

Now for showing the constants are not attained we hold on the extra term in (3.1) for $v := E^t u$

$$\int_\Omega E\,|\nabla v|_A^2\,dx = \int_\Omega E\,\left|\nabla(uE^t)\right|_A^2 dx, \tag{4.45}$$

since we have

$$\int_\Omega E|\nabla E^t u|_A^2 dx = t^2\int_\Omega E^{2t-1}|\nabla E|_A^2 u^2 dx + \int_\Omega E^{2t+1}|\nabla u|_A^2 dx + 2t\int_\Omega E^{2t} A\nabla E\cdot\nabla u u dx, \tag{4.46}$$

and

$$\int_\Omega E^{2t-1}|\nabla E|_A^2 u^2 dx = \int_\Omega E^{2t-1}|\nabla E|_A^2 E^{-2t} v^2 dx = \int_\Omega |\nabla E|_A^2 E^{-1} v^2 dx, \tag{4.47}$$

and

$$\int_\Omega E^{2t+1}|\nabla u|_A^2 dx = t^2\int_\Omega E^{-1}|\nabla E|_A^2 v^2 dx + \int_\Omega E^{-2t}|\nabla v|_A^2 dx - 2t\int_\Omega E^{-2t-1} A\nabla E\cdot\nabla v v dx, \tag{4.48}$$

and

$$\int_\Omega E^{2t} A\nabla E\cdot\nabla u u dx = \frac{1}{2}\int_\Omega E^{2t} A\nabla E\cdot\nabla|u^2| dx = -\frac{1}{2}\int_\Omega div E^{2t} A\nabla E u^2 dx$$
$$= -\frac{1}{2}(2t)\int_\Omega E^{2t-1}|\nabla E|_A^2 E^{-2t} v^2 dx = -t\int_\Omega E^{-1}|\nabla E|_A^2 v^2 dx, \tag{4.49}$$

From the computation in (4.47), (4.48) and (4.49) and according to (2.4) we have $\int_\Omega E^{-1}|\nabla E|_A^2 dx = \infty$ so (4.45) is positive for $u \in X_t \backslash \{0\}$.

□

4.1 More General Weighted Inequalities

Here we investigate the possibility of inequalities of the form

$$\int_\Omega W(x)|\nabla u|_A^2 dx \geq \int_\Omega U(x) u^2 dx,$$

for $u \in C_c^{0,1}(\Omega \backslash K)$.

Theorem 4.5. *If f is a positive function in $C^\infty(\gamma, \infty)$ where $\gamma := \min_{\partial\Omega} E$ and E is interior weight on Ω then:*

$$\int_\Omega f(E)^2 |\nabla u|_A^2 dx \geq \int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f(E) f''(E) \right) u^2 dx, \tag{4.50}$$

where $u \in C_c^{0,1}(\Omega \backslash K)$. If $\liminf_{z \to \infty} f''(z) > 0$ then the constant 1 is optimal.

Proof. We consider $w := f(E)u$, since $u \in C_c^{0,1}(\Omega \backslash K)$ then $w \in C_c^{0,1}(\Omega \backslash K)$. Plugging w in (3.15) we get

$$\int_\Omega |\nabla w|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} w^2 dx \geq 0. \tag{4.51}$$

For the first part of RHS of (4.51) we have

$$\begin{aligned}
\int_\Omega |\nabla w|_A^2 dx &= \int_\Omega |\nabla E|_A^2 (f'(E))^2 u^2 dx + \int_\Omega |\nabla u|_A^2 f(E)^2 dx + 2\int_\Omega A\nabla E \cdot \nabla u f'(E) f(E) u dx \\
&= \int_\Omega |\nabla E|_A^2 (f'(E))^2 u^2 dx + \int_\Omega |\nabla u|_A^2 f(E)^2 dx + \int_\Omega A\nabla E \cdot \nabla u^2 f'(E) f(E) dx \\
&= \int_\Omega |\nabla E|_A^2 (f'(E))^2 u^2 dx + \int_\Omega |\nabla u|_A^2 (f(E))^2 dx - \int_\Omega div(f'(E) f(E)) \cdot A\nabla E u^2 dx,
\end{aligned} \tag{4.52}$$

for the last term of (4.52) we have

$$\begin{aligned}\int_\Omega div(f'(E)f(E))A\nabla Eu^2dx &= \int_\Omega A\nabla E\cdot\nabla Ef''(E)f(E)u^2dx+\int_\Omega A\nabla E\cdot\nabla E(f'(E))^2u^2dx\\ &= \int_\Omega f''(E)f(E)|\nabla E|_A^2u^2dx+\int_\Omega (f'(E))^2|\nabla E|_A^2u^2dx,\end{aligned} \tag{4.53}$$

putting (4.53) in (4.52) we have

$$\begin{aligned}\int_\Omega |\nabla w|_A^2dx = \int_\Omega |\nabla E|_A^2(f'(E))^2u^2dx+\int_\Omega |\nabla u|_A^2f(E)^2dx\\ -\int_\Omega f''(E)f(E)|\nabla E|_A^2u^2dx-\int_\Omega (f'(E))^2|\nabla E|_A^2u^2dx,\end{aligned} \tag{4.54}$$

by simplifying (4.54) we have

$$\int_\Omega |\nabla w|_A^2dx = \int_\Omega |\nabla u|_A^2f(E)^2dx-\int_\Omega f''(E)f(E)|\nabla E|_A^2u^2dx, \tag{4.55}$$

for the second part of (4.51) we have

$$\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}w^2dx = \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}(f(E))^2u^2dx, \tag{4.56}$$

inserting (4.55) and (4.56) in (4.51) we have:

$$\begin{aligned}\int_\Omega |\nabla w|_A^2dx-\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}w^2dx = \int_\Omega |\nabla u|_A^2f(E)^2dx-\int_\Omega f''(E)f(E)|\nabla E|_A^2u^2dx\\ -\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}(f(E))^2u^2dx\geq 0,\end{aligned}$$

so

$$\int_\Omega |\nabla u|_A^2f(E)^2dx\geq \int_\Omega f''(E)f(E)|\nabla E|_A^2u^2dx+\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}(f(E))^2u^2dx, \tag{4.57}$$

finally (4.57) gives us the inequality (4.50)

$$\int_\Omega |\nabla u|_A^2 f(E)^2 dx \geq \int_\Omega |\nabla E|_A^2 \left(\frac{(f(E))^2}{4E^2} + f''(E)f(E) \right) u^2 dx. \tag{4.58}$$

Now we consider

$$D_m := \frac{\int_\Omega |\nabla v_m|_2^A dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4}, \tag{4.59}$$

where $v_m \in C_c^{0,1}(\Omega \backslash K)$. We can consider the support of v_m in K which is the support of μ. Suppose $u_m := \frac{v_m}{f(E)} \in C_c^{0,1}(\Omega \backslash K)$. Regarding (4.58) we consider

$$Q_m := \frac{\int_\Omega |\nabla u_m|_A^2 f(E)^2 dx}{\int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f''(E)f(E) \right) u_m^2 dx}, \tag{4.60}$$

now for the numerator of (4.60) we have

$$f(E)^2 |\nabla u_m|_A^2 = \frac{|\nabla v_m|_A^2 f(E)^4 + |\nabla E|_A^2 (f'(E))^2 v_m^2 (f(E))^2 - 2A\nabla E \cdot \nabla v_m v_m (f(E))^3 f'(E) v_m}{(f(E))^4}, \tag{4.61}$$

by simplifying and integration of both sides of (4.61) we have

$$\int_\Omega f(E)^2 |\nabla u_m|_A^2 dx = \int_\Omega |\nabla v_m|_A^2 dx + \int_\Omega \frac{|\nabla E|_A^2 (f'(E))^2 v_m^2}{(f(E))^2} dx - 2\int_\Omega \frac{\nabla v_m \cdot A\nabla E f'(E) v_m}{f(E)} dx, \tag{4.62}$$

now for the last term of (4.62) we have

$$
\begin{aligned}
2\int_\Omega \frac{\nabla v_m \cdot A\nabla E f'(E) v_m}{f(E)} dx &= \int_\Omega \frac{f'(E)}{f(E)} A\nabla E \cdot \nabla v_m^2 \\
&= -\int_\Omega div\left(\frac{f'(E)}{f(E)}\right) \cdot \nabla E v_m^2 \\
&= -\int_\Omega \frac{|\nabla E| f''(E) f(E) - |\nabla E| (f'(E))^2}{(f(E))^2} \cdot A\nabla E v_m^2 \\
&= -\int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)} v_m^2 dx + \int_\Omega \frac{|\nabla E|_A^2 (f'(E))^2}{(f(E))^2} v_m^2 dx, \quad (4.63)
\end{aligned}
$$

plugging (4.63) in (4.62) we have

$$
\begin{aligned}
\int_\Omega (f(E))^2 |\nabla u_m|_A^2 dx = \int_\Omega |\nabla v_m|_A^2 dx + \int_\Omega \frac{|\nabla E|_A^2 (f'(E))^2}{(f(E))^2} v_m^2 dx + \int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)} v_m^2 dx \\
- \int_\Omega \frac{|\nabla E|_A^2 (f'(E))^2}{(f(E))^2} v_m^2 dx,
\end{aligned}
\tag{4.64}
$$

simplifying (4.64) gives us

$$
\int_\Omega (f(E))^2 |\nabla u_m|_A^2 dx = \int_\Omega |\nabla v_m|_A^2 dx + \int_\Omega \frac{|\nabla E|_A^2 (f''(E))^2}{f(E)} v_m^2 dx. \tag{4.65}
$$

for denominator of (4.60) we have

$$
\begin{aligned}
\int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f(E) f''(E)\right) u_m^2 dx &= \int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f(E) f''(E)\right) \frac{v_m^2}{f(E)^2} dx \\
&= \int_\Omega |\nabla E|_A^2 \left(\frac{v_m^2}{4E^2} + \frac{f''(E)}{f(E)} v_m^2\right) dx,
\end{aligned}
\tag{4.66}
$$

so (4.66) is

$$
\int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f(E) f''(E)\right) u_m^2 dx = \int_\Omega |\nabla E|_A^2 \frac{v_m^2}{4E^2} dx + \int_\Omega |\nabla E|_A^2 \frac{f''(E)}{f(E)} v_m^2 dx, \tag{4.67}
$$

plugging (4.65) and (4.67) in (4.60) we have

$$Q_m := \frac{\int_\Omega |\nabla u_m|_A^2 f(E)^2 dx}{\int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f''(E)f(E)\right) u_m^2 dx} = \frac{\int_\Omega |\nabla v_m|_A^2 dx + \int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)} v_m^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{4E^2} v_m^2 + \int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)} v_m^2 dx}, \tag{4.68}$$

as we know

$$D_m := \frac{\int_\Omega |\nabla v_m|_2^A dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4},$$

and according to Q_m in (4.60), if we consider $\liminf_{z\to\infty} f''(z) > 0$ since in Q_m (4.68), we have: $x \to \frac{\alpha+x}{\beta+x}$ and α and β are positive constant(regarding D_m) so $Q_m \to 1$.

□

5

Improvement

Improvement in the direction of the Hardy inequality have been posed by many researchers [2], [1], [10]. In this chapter the improved versions of inequality (3.15) are investigated. For an improved Hardy inequality on radial domain, Ghoussoub and Moradifam [12] gave necessary and sufficient conditions for a nonegative function V as potential. The method applied here is the analogous approach. Now we define precisely what is meant by the potential.

Definition 5.1. *Suppose $0 \leq V \in C^{\infty}(\Omega \backslash K)$ for E as an interior weight on Ω, V is potential for E if*

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4} \int_{\Omega} \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_{\Omega} V(x) u^2 dx,$$

where $u \in H_0^1(\Omega)$.

Definition 5.2. *Suppose $0 \leq V \in C^{\infty}(\Omega)$ for E as boundary weight on Ω, V is potential for E if*

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4} \int_{\Omega} \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_{\Omega} V(x) u^2 dx,$$

where $u \in H_0^1(\Omega)$.

Remark 5.3. *If E is an interior weight, $E = \infty$ on support of μ so we consider $V \in C^\infty(\Omega \backslash K)$ but for boundary weight E we consider $V \in C^\infty(\Omega)$.*

Theorem 5.4. *(Interior Weight Improvement) If E is an interior weight on Ω and $0 \leq V \in C^\infty(\Omega \backslash K)$*

If there exists some $0 < \phi \in C^2(\Omega \backslash K)$ such that

$$-L_A(\phi) + \frac{A\nabla E \cdot \nabla \phi}{E} + V\phi \leq 0 \quad in \quad \Omega \backslash K, \tag{5.1}$$

then V is potential for E. After change of variables $\theta := E^{\frac{1}{2}}\phi$ we have $\phi := E^{-\frac{1}{2}}\theta$ for $0 < \theta \in C^2(\Omega \backslash K)$ we have

$$-\frac{L_A(\theta)}{\theta} + \frac{|\nabla E|_A^2}{4E^2} + V \leq 0 \quad in \quad \Omega \backslash K. \tag{5.2}$$

Proof. If $V \in C^\infty(\Omega \backslash K)$ is nonegative and there exists some $0 < \phi \in C^2(\Omega \backslash K)$ which satisfies $-L_A(\phi) + \frac{A\nabla E \cdot \nabla \phi}{E} + V\phi \leq 0 \quad \text{in} \quad \Omega \backslash K$.

Let $u \in C^{0,1}(\Omega \backslash K)$ and define $v := E^{-\frac{1}{2}}u$ so by Lemma 3.2 we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx = \int_\Omega |\nabla v|_A^2 E dx.$$

Now define $\psi \in C^{0,1}(\Omega \backslash K)$ and consider $v = \phi\psi$. We have

$$E|\nabla v|_A^2 = E|\nabla \phi|_A^2 \psi^2 + E|\nabla \psi|_A^2 \phi^2 + 2E\phi\psi A\nabla\phi \cdot \nabla\psi, \tag{5.3}$$

now we compute

$$\int_\Omega \psi^2 E|\nabla\phi|_A^2 dx + 2\int_\Omega E\phi\psi A\nabla\phi \cdot \nabla\psi dx, \tag{5.4}$$

for first part of (5.4) we have

$$\int_\Omega \psi^2 E|\nabla\phi|_A^2 dx = \int_\Omega A\psi^2 E\nabla\phi\cdot\nabla\phi dx = -\int_\Omega E\psi^2\phi div(A\nabla\phi)$$
$$= \int_\Omega E\psi^2\phi L_A(\phi)dx, \tag{5.5}$$

for second part of (5.4) we have

$$\begin{aligned} 2\int_\Omega \psi\phi EA\nabla\phi\cdot\nabla\psi dx =& 2\int_\Omega \phi\psi EA\nabla\phi\cdot\nabla\psi dx \\ =& \int_\Omega \phi A\nabla\phi\cdot\nabla(\psi)^2 E dx \\ =& -\int_\Omega \phi A\nabla E\cdot\nabla\phi(\psi)^2 dx, \end{aligned} \tag{5.6}$$

so by the results of (5.5) and (5.6) and inserting in (5.4) we have

$$\int_\Omega \psi^2 E|\nabla\phi|_A^2 dx + 2\int_\Omega \psi\phi EA\nabla\phi\cdot\nabla\psi dx = \int_\Omega \psi^2\left(L_A(\phi)\phi E - \phi A\nabla E\cdot\nabla\phi\right)dx, \tag{5.7}$$

now since $v = E^{-\frac{1}{2}}u$ and $u = E^{\frac{1}{2}}v$ and $v = \phi\psi$ so for the left hand side of (5.7) we have

$$\psi^2 L_A(\phi)\phi E = \psi^2\frac{L_A(\phi)}{\phi}\phi^2 E = (\phi\psi)^2\frac{L_A(\phi)}{\phi}E = v^2 E\frac{L_A(\phi)}{\phi} = u^2\frac{L_A(\phi)}{\phi}, \tag{5.8}$$

and we have

$$\begin{aligned} (\phi A\nabla E\cdot\nabla\phi)\psi^2 =& \frac{\phi^2\psi^2\frac{A\nabla E\cdot\nabla\phi E}{E}}{\phi} \\ =& \frac{(\phi\psi)^2 E\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi} \\ =& \frac{v^2 E\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi} \\ =& \frac{u^2\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi}, \end{aligned} \tag{5.9}$$

so by inserting (5.8) and (5.9) in (5.7) we have

$$\begin{aligned}
\int_\Omega \psi^2 E|\nabla\phi|_A^2 dx + \int_\Omega \phi\psi E A\nabla\phi\cdot\nabla\psi dx &= \int_\Omega \psi^2\left(L_A(\phi)\phi E - \phi A\nabla E\cdot\nabla\phi\right)dx \\
&= \int_\Omega u^2\frac{L_A(\phi)}{\phi}dx - \int_\Omega \frac{u^2\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi}dx \\
&= \int_\Omega u^2\left(\frac{L_A(\phi)-\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi}\right)dx \\
&:= Q,
\end{aligned} \tag{5.10}$$

since in Theorem 5.4 we have

$$-L_A(\phi)+\frac{A\nabla E\cdot\nabla\phi}{E}+V\phi\le 0 \quad \text{in} \quad \Omega\backslash K,$$

so

$$L_A(\phi)-\frac{A\nabla E\cdot\nabla\phi}{E}\ge V\phi \quad \text{in} \quad \Omega\backslash K, \tag{5.11}$$

and since

$$Q := \int_\Omega u^2\left(\frac{L_A(\phi)-\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi}\right)dx, \tag{5.12}$$

by multiplying $\frac{u^2}{\phi}$ to both sides of (5.11) we have

$$Q := \int_\Omega \frac{u^2}{\phi}\left(\frac{L_A(\phi)-\frac{A\nabla E\cdot\nabla\phi}{E}}{\phi}\right)dx \ge \int_\Omega \frac{u^2}{\phi}V\phi,$$

so $Q\ge\int_\Omega u^2V(x)dx$.

Now, since for $v=E^{\frac{1}{2}}u$ by Lemma 3.2 we have

$$\int_\Omega |\nabla u|_A^2dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2dx \ge \int_\Omega |\nabla v|_A^2 E dx, \tag{5.13}$$

and considering (5.3) and (5.10) and integrating of (5.3) we have

$$\int_\Omega |\nabla v|_A^2 E = Q + \int_\Omega E\phi^2 |\nabla \psi|_A^2,$$

so for (5.13) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx = Q + \int_\Omega E\phi^2 |\nabla \psi|_A^2 dx,$$

since $Q \geq \int_\Omega u^2 V(x)dx$ so for (5.13) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega u^2 V(x)dx + \int_\Omega E\phi^2 |\nabla \psi|_A^2 dx,$$

for $u \in C_c^{0,1}(\Omega \backslash K)$ and since $C_c^{0,1}(\Omega \backslash K)$ is dense in $H_0^1(\Omega)$

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega u^2 V(x)dx + \int_\Omega E\phi^2 |\nabla \psi|_A^2 dx,$$

holds for $u \in H_0^1(\Omega)$, then V is potential for E.

Now we prove (5.2), since we have

$$\begin{aligned}-L_A(\phi) &= -(-div(A\nabla\phi)) = div(A\nabla(E^{-\frac{1}{2}}\theta)) \\ &= \frac{3}{4}E^{-\frac{5}{2}}|\nabla E|_A^2\theta - \frac{1}{2}AE^{-\frac{3}{2}}\nabla E\cdot\nabla\theta - \frac{1}{2}AE^{-\frac{3}{2}}\nabla E\cdot\nabla\theta + E^{-\frac{1}{2}}div(A\nabla\theta), \quad (5.14)\end{aligned}$$

and we have:

$$\begin{aligned}\frac{A\nabla E\cdot\nabla\phi}{E} = \frac{A\nabla E\cdot\nabla(E^{-\frac{1}{2}}\theta)}{E} &= \frac{A\nabla E\cdot(-\frac{1}{2}|\nabla E|E^{-\frac{3}{2}}\theta + E^{-\frac{1}{2}}|\nabla\theta|)}{E} \\ &= -\frac{1}{2}|\nabla E|_A^2 E^{-\frac{5}{2}}\theta + AE^{-\frac{3}{2}}\nabla E\cdot\nabla\theta, \quad (5.15)\end{aligned}$$

inserting (5.14) and (5.15) in (5.1) we have

$$\begin{aligned}-L_A(\phi)+\frac{A\nabla E\cdot\nabla\phi}{E}+V\phi=&\frac{3}{4}E^{-\frac{5}{2}}|\nabla E|_A^2\theta-\frac{1}{2}AE^{-\frac{3}{2}}\nabla E\cdot\nabla\theta-\frac{1}{2}AE^{-\frac{3}{2}}\nabla E\cdot\nabla\theta+E^{-\frac{1}{2}}div(A\nabla\theta)\\&-\frac{1}{2}|\nabla E|_A^2E^{-\frac{5}{2}}\theta+AE^{-\frac{3}{2}}\nabla E\cdot\nabla\theta\\=&\frac{1}{4}|\nabla E|_A^2E^{-\frac{5}{2}}\theta+E^{-\frac{1}{2}}div(A\nabla\theta)+V(E^{-\frac{1}{2}}\theta)\le 0,\end{aligned}\tag{5.16}$$

after simplifying (5.16) we have

$$-L_A(\phi)+\frac{A\nabla E\cdot\nabla\phi}{E}+V\phi=E^{-\frac{1}{2}}\theta\left(\frac{|\nabla E|_A^2}{4E^2}+\frac{div(A\nabla\theta)}{\theta}+V\right)\le 0,$$

since $0<\theta\in C^2(\Omega\backslash K)$ and $E>0$ so

$$\frac{|\nabla E|_A^2}{4E^2}-\frac{L_A(\theta)}{\theta}+V\le 0\quad\text{in}\quad\Omega\backslash K.$$

□

Theorem 5.5. *(Boundary Weight Improvement) Suppose E is a boundary weight on Ω and $0\le V\in C^\infty(\Omega)$. Suppose there exists some $0<\phi\in C^2(\Omega)$ such that*

$$-\frac{L_A(\theta)}{\theta}+\frac{A\nabla E\cdot\nabla\phi}{E\phi}-\frac{\mu}{2E}+V\le 0\quad in\quad\Omega,\tag{5.17}$$

then V is potential for E.

Proof. If $V\in C^\infty(\Omega)$ is nonegative and there exists some $0<\phi\in C^2(\Omega)$ which solves

$$-L_A(\phi)-\frac{A\nabla E\cdot\nabla\phi}{E}-\frac{\mu}{2E}+V\le 0,$$

let $u\in H_0^1(\Omega)$ and define $v=E^{\frac{-1}{2}}u$ by Lemma 3.2 we have

$$\int_\Omega|\nabla u|_A^2dx-\frac{1}{4}\int_\Omega\frac{|\nabla E|_A^2}{E^2}u^2dx\ge\int_\Omega|\nabla v|_A^2Edx+\frac{1}{2}\int_\Omega\frac{u^2}{E}d\mu,$$

now we define $\psi \in C_c^{0,1}(\Omega)$ by $v = \phi\psi$ so we have

$$E|\nabla v|_A^2 = E|\nabla\phi|_A^2\psi^2 + E|\nabla\psi|_A^2\phi^2 + 2AE\phi\psi\nabla\phi\cdot\nabla\psi,$$

according to (5.10) in proof of Theorem (5.4) we have

$$\begin{aligned}
&\int_\Omega \psi^2 E|\nabla\phi|_A^2 dx + 2\int_\Omega E\phi\psi\nabla\phi\cdot\nabla\psi A dx + \frac{1}{2}\int_\Omega \frac{u^2}{E}d\mu \\
=&\int_\Omega \psi^2\left(L_A(\phi)\phi E - \phi A\nabla E\cdot\nabla\phi\right)dx - \frac{1}{2}\int_\Omega \frac{u^2}{E}d\mu \\
=&\int_\Omega u^2\left(\frac{L_A(\phi)}{\phi} - \frac{A\nabla E\cdot\nabla\phi}{\phi E} - \frac{\mu}{2E}\right)dx \\
:=&Q, \qquad (5.18)
\end{aligned}$$

since according to assumption (5.17) we have

$$-\frac{L_A(\phi)}{\phi} + \frac{A\nabla E\cdot\nabla\phi}{\phi E} - \frac{\mu}{2E} + V \le 0 \quad \text{in} \quad \Omega,$$

so

$$\frac{L_A(\phi)}{\phi} - \frac{A\nabla E\cdot\nabla\phi}{\phi E} + \frac{\mu}{2E} \ge V, \qquad (5.19)$$

by multiplying u^2 to the both sides of (5.19) we have

$$\int_\Omega u^2\left(\frac{L_A(\phi)}{\phi} - \frac{A\nabla E\cdot\nabla\phi}{\phi E} + \frac{\mu}{2E}\right)dx \ge \int_\Omega u^2 V(x)dx,$$

so

$$Q \ge \int_\Omega u^2 V(x)dx, \qquad (5.20)$$

since for $v := E^{-\frac{1}{2}}u$ and by Lemma 3.2 we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \int_\Omega |\nabla v|_A^2 E dx + \frac{1}{2}\int_\Omega \frac{u^2}{E} d\mu, \tag{5.21}$$

so for $\int_\Omega |\nabla v|_A^2 E dx$ we have

$$\int_\Omega u^2 \left(\frac{L_A(\phi)}{\phi} - \frac{A\nabla E \cdot \nabla \phi}{\phi E} + \frac{\mu}{2E} \right) dx = Q + \int_\Omega E\phi^2 |\nabla \psi|_A^2 dx,$$

so rewriting (5.21) gives us

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx = Q + \int_\Omega E\phi^2 |\nabla \psi|_A^2 dx,$$

and finally according (5.20) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \int_\Omega u^2 V(x) dx + \int_\Omega E\phi^2 |\nabla \psi|_A^2 dx,$$

for $u \in H_0^1(\Omega)$ so V is potential for boundary weight E.

$\square$

Theorem 5.6. *Suppose E is an interior weight on Ω, $0 < \gamma := \min_{\partial\Omega} E$ and $0 < f \in C^2(\gamma, \infty)$, then for all $u \in C_c^{0,1}(\Omega \backslash K)$ we have*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \int_\Omega \frac{|\nabla E|_A^2}{f(E)} \left(-f''(E) - \frac{f'(E)}{E} \right) u^2 dx,$$

in particular by $f(E) := \sqrt{\log(\gamma^{-1}E)}$ we obtain

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2 \log^2(\gamma^{-1}E)}u^2 dx,$$

for all $u \in H_0^1(\Omega)$. Now suppose $0 < \gamma = E$ on $\partial\Omega$, then $\frac{1}{4}$ is optimal.

Proof. Let E be an interior weight on Ω and $0 < \gamma := \min_{\partial\Omega} E$ and $0 < f \in C^2(\gamma, \infty)$.

According to Theorem 5.4, if we have $-L_A(\phi)+\frac{A\nabla E\cdot\nabla\phi}{E}+V\phi\le 0$, so V is potential for E. From this we have $\frac{L_A(\phi)}{\phi}-\frac{A\nabla E\cdot\nabla\phi}{E\phi}\ge V$, so for $u\in C_c^{0,1}(\Omega\backslash K)$ we have

$$\left(\frac{L_A(\phi)}{\phi}-\frac{A\nabla E\cdot\nabla\phi}{E\phi}\right)u^2\ge V(x)u^2, \tag{5.22}$$

by considering $\phi=f(E)$ in above inequality, so for the first part of (5.22) we have

$$\begin{aligned}\frac{L_A(\phi)}{\phi}=\frac{L_A(f(E))}{f(E)}=-\frac{div(A\nabla f(E))}{f(E)}&=-\frac{Adiv(\nabla f(E))}{f(E)}\\&=-\frac{Adiv(\nabla E\cdot f'(E))}{f(E)}\\&=-\frac{A(\nabla E\nabla E\cdot f''(E))}{f(E)}\\&=-\frac{|\nabla E|_A^2}{f(E)}f''(E),\end{aligned} \tag{5.23}$$

and for the second part of (5.22) we have

$$\frac{A\nabla E\cdot\nabla(f(E))}{Ef(E)}=\frac{A\nabla E\cdot\nabla Ef'(E)}{Ef(E)}=\frac{|\nabla E|_A^2}{f(E)}\frac{f'(E)}{E}, \tag{5.24}$$

plugging (5.23) and (5.24) in (5.22) we have

$$\frac{|\nabla E|_A^2}{f(E)}\left(-f''(E)-\frac{f'(E)}{E}\right)u^2\ge V(x)u^2, \tag{5.25}$$

now if $f(E)\in C^2(\gamma,\infty)$ satisfies in

$$-L_A(f(E))+\frac{A\nabla E\cdot\nabla f(E)}{E}+V\phi\le 0,$$

so according to Theorem 5.4, V is potential for E and according definition (5.2) we have

$$\int_\Omega|\nabla u|_A^2dx-\frac{1}{4}\int_\Omega\frac{|\nabla E|_A^2}{E^2}u^2dx\ge\int_\Omega u^2V(x)dx, \tag{5.26}$$

so in terms of (5.25) we have

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4}\int_{\Omega} \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_{\Omega} \frac{|\nabla E|_A^2}{f(E)} \left(-f''(E) - \frac{f'(E)}{E}\right) u^2 dx. \tag{5.27}$$

in particular since

$$\begin{aligned} -\frac{f''(E)}{f(E)} - \frac{f'(E)}{Ef(E)} =& -\frac{div(A\nabla f(E))}{f(E)} - \frac{A\nabla E\cdot\nabla f(E)}{Ef(E)} \\ =& \frac{L_A(f(E)}{f(E)} - \frac{A\nabla E\cdot\nabla f(E)}{Ef(E)}, \end{aligned} \tag{5.28}$$

so according to Theorem 5.4, V is potential for E. So plugging $f(E) := \sqrt{\log(\gamma^{-1}E)} \in C^2(\Omega\backslash K)$ in (5.28) gives us

$$-\frac{f''(E)}{f(E)} - \frac{f'(E)}{Ef(E)} = \frac{1}{4}\frac{|\nabla E|_A^2}{E^2}(\log(\gamma^{-1}E))^{-2} + \frac{1}{2}\frac{|\nabla E|_A^2}{E^2}(\log(\gamma^{-1}E))^{-1} - \frac{1}{2}\frac{|\nabla E|_A^2}{E^2}(\log(\gamma^{-1}E))^{-1}, \tag{5.29}$$

so

$$V = \frac{1}{4}\frac{|\nabla E|_A^2}{E^2}(\log(\gamma^{-1}E))^{-2},$$

so by plugging into (5.26)

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4}\int_{\Omega} \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \frac{1}{4}\int_{\Omega} \frac{|\nabla E|_A^2}{E^2 \log^2(\gamma^{-1}E)} u^2 dx, \tag{5.30}$$

for $u \in C_c^{0,1}(\Omega\backslash K)$ and extend it by Theorem 1.36, we have (5.30) for $u \in H_0^1(\Omega)$.

For showing $\frac{1}{4}$ is optimal constant we fix $0 < t < \frac{1}{2}$ and $\tau > \frac{1}{2}$ and define $u_\tau := E^t \log^\tau(\gamma^{-1}E)$. According to Lemma 3.3 $u_t \in H_0^1(\Omega)$ so we need to compute

$$\frac{\int_\Omega |\nabla u_\tau|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u_\tau^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2 \log^2(\gamma^{-1}E)} u_\tau^2 dx},$$

since we have

$$\int_\Omega \frac{|\nabla E|_A^2 dx}{E^2\log^2(\gamma^{-1}E)} u_\tau^2 dx = \int_\Omega \frac{|\nabla E|_A^2 dx}{E^2\log^2(\gamma^{-1}E)} E^{2t} log^{2\tau}(\gamma^{-1}E)dx = \int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx,$$

so we compute

$$\frac{\int_\Omega |\nabla u_\tau|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} - \frac{\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} E^{2t} log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}, \tag{5.31}$$

since we have $u_\tau := E^t \log^\tau(\gamma^{-1}E)$ so we have

$$\begin{aligned}|\nabla u_\tau|_A^2 =& t^2|\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E) + \tau^2|\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)\\ &+2t\tau|\nabla E|_A^2 E^{2t-2}\log^{2\tau-1}(\gamma^{-1}E),\end{aligned} \tag{5.32}$$

so after integrating of both sides of (5.32) for the first part in (5.31) we have

$$\begin{aligned}\frac{\int_\Omega |\nabla u_\tau|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} =& \frac{t^2\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}\\ &+\frac{\tau^2\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}\\ &+\frac{2t\tau\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-1}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx},\end{aligned} \tag{5.33}$$

and for the second part of (5.31) we have

$$\frac{\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} E^{2t}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} = \frac{\frac{1}{4}\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}, \tag{5.34}$$

so (5.31) is

$$
\begin{aligned}
&\frac{\int_\Omega |\nabla u_\tau|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} - \frac{\frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}E^{2t}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}\\
=&\frac{t^2\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} + \frac{\tau^2\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}\\
+&\frac{2t\tau\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-1}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} - \frac{\frac{1}{4}\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx},
\end{aligned}
\tag{5.35}
$$

now since in Lemma 3.3 we defined

$$
J_t(\tau) = \int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx,
$$

and $2\tau = 2(\tau+1)-2$ and $2\tau-1 = 2(\tau+\frac{1}{2})-2$ so

$$
\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E)dx = J_t(\tau+1),
\tag{5.36}
$$

and

$$
\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-1}(\gamma^{-1}E)dx = J_t(\tau+\frac{1}{2}),
\tag{5.37}
$$

so according (5.36) and (5.37) we rewrite (5.35) as

$$
\begin{aligned}
\frac{\int_\Omega |\nabla u_\tau|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u_\tau|^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2\log^2(\gamma^{-1}E)}u_\tau^2 dx} =&(t^2-\frac{1}{4})\frac{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx} + \tau^2\\
&+2t\tau\frac{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-1}(\gamma^{-1}E)dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2}\log^{2\tau-2}(\gamma^{-1}E)dx}\\
=&(t^2-\frac{1}{4})\frac{J_t(\tau+1))}{J_t(\tau)} + \tau^2 + 2t\tau\frac{J_t(\tau+\frac{1}{2}))}{J_t(\tau)},
\end{aligned}
\tag{5.38}
$$

according to Theorem 3.3 we have

$$J_t(\tau) = \int_\Omega |\nabla E|_A^2 E^{2t-2} \log^{2\tau-2}(\nu^{-1}E)dx \to \infty,$$

when $\tau \searrow \frac{1}{2}$ so for

$$(t^2 - \frac{1}{4})\frac{J_t(\tau+1))}{J_t(\tau)} + \tau^2 + 2t\tau\frac{J_t(\tau+\frac{1}{2}))}{J_t(\tau)},$$

so $\frac{1}{4}$ is optimal.

$\square$

Remark 5.7. *As it is mentioned $V \in C^\infty(\Omega)$ is potential for E (as interior weight) if there exists some $0 < \phi \in C^2(\Omega \backslash K)$ such that*

$$\frac{-L_A(\phi)}{\phi} + \frac{A\nabla E \cdot \nabla\phi}{E} + V\phi \leq 0,$$

now we are going to consider more useful condition for V to be potential for E.

Theorem 5.8. *(Interior improvement using ode method) Assume E is interior weight on Ω, $E := \gamma \geq 0$ on $\partial\Omega$ and $0 \leq f \in C^\infty(\gamma, \Omega)$. If there exists some $0 < h \in C^2(\gamma, \infty)$ such that*

$$h''(t) + \left(f(t) + \frac{1}{4t^2}\right)h(t) \leq 0,$$

in (γ, ∞) then for all $u \in H_0^1(\Omega)$

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \int_\Omega f(E)|\nabla E|_A^2 u^2 dx.$$

Proof. For E as interior weight and $0 < f \in C^\infty(\Omega \backslash K)$ and $0 < h \in C^2(\gamma, \infty)$ we have

$$h''(t) + \left(f(t) + \frac{1}{4t^2}\right)h(t) \leq 0,$$

so

$$\frac{h''(t)}{h(t)}+\frac{1}{4t^2}+f(t)\leq 0,$$

by considering E and scaling both sides we have

$$|\nabla E|_A^2\frac{h''(E)}{h(E)}+\frac{|\nabla E|_A^2}{4E^2}+f(E)|\nabla E|_A^2\leq 0, \tag{5.39}$$

taking $\theta := h(E)$ and using Theorem 5.4 so (5.39) corresponds to:

$$\frac{-L_A(\theta)}{\theta}+\frac{|\nabla E|_A^2}{4E^2}+V\leq 0, \tag{5.40}$$

because

$$-L_A(\theta)=-L_A(h(E))=div(A\nabla h(E))=|\nabla E|_A^2h''(E),$$

now based on Theorem 5.4, $f(E)|\nabla E|_A^2$ is potential for E and according to Definition 5.2 we have

$$\int_\Omega|\nabla u|_A^2dx-\frac{1}{4}\int_\Omega\frac{|\nabla E|_A^2}{E^2}u^2dx\geq\int_\Omega f(E)|\nabla E|_A^2u^2dx.$$

□

Now we generalize the results obtained by Avkhadiev and Wirths.

Definition 5.9. *Suppose Ω is domain in $\mathbb{R}^N$ it has finite inradius if $\delta(x):=dist(x,\partial\Omega)$ is bounded in Ω.*

Theorem 5.10. *(Avkhadiev,Wirths[5]) Suppose Ω is a convex domain in $\mathbb{R}^N$ with finite inradius. Then*

$$\int_\Omega|\nabla u|_A^2dx\geq\frac{1}{4}\int_\Omega\frac{u^2}{\delta^2}dx+\frac{\lambda_0^2}{\|\delta\|_{L_\infty}^2}\int_\Omega u^2dx,$$

for $u \in H_0^1(\Omega)$ is optimal, where λ_0 denotes the first positive zero of $J_0(t) - 2tJ_1(t)$ and J_n is the Bessel function of order n.

The above theorem extends the results of H. Brezis and M. Marcus:

Theorem 5.11. *(H.Brezis and M. Marcus[7]) Suppose Ω is a convex subset of $\mathbb{R}^N$ then*

$$\int_{\Omega} |\nabla u|_A^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{\delta^2} dx + \frac{1}{4diam^2(\Omega)} \int_{\Omega} u^2 dx,$$

for $u \in H_0^1(\Omega)$ where $diam(\Omega)$ denotes the diameter of Ω.

We establish a generalized version of this result.

Theorem 5.12. *Suppose μ is nonegative nonzero locally finite measure in Ω (possibly unbounded) and $0 < E \in L^\infty(\Omega)$ is a solution to*

- $L_A(E) = \mu \quad in \quad \Omega,$
- $|\nabla E|_A = 1 \quad a.e\ in \quad \Omega,$
- $E = 0 \quad on \quad \partial\Omega,$

then

$$\int_{\Omega} |\nabla u|_A^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{E^2} dx + \frac{\lambda_0^2}{\|E\|_{L^\infty}^2} \int_{\Omega} u^2 dx,$$

for all $u \in C_c^\infty(\Omega)$.

Proof. We extend E to all of $\mathbb{R}^N$ and $E = 0$ on $\mathbb{R}^N \backslash \overline{\Omega}$. According to the setting in proof of lemma 2.1, we consider ε-mollification of E. We set $F_\varepsilon := L_A(E_\varepsilon)$ and $0 < f \in C^2((0, \|E\|_{L^\infty}])$. Starting with $v = E_\varepsilon^{-\frac{1}{2}} u$ and $u = E_\varepsilon^{\frac{1}{2}} v$ so $u^2 = E_\varepsilon v^2$. We have

$$\int_{\Omega} |\nabla u|_A^2 dx - \frac{1}{4} \int_{\Omega} \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 \geq \int_{\Omega} |\nabla v|_A^2 E_\varepsilon dx + \int_{\Omega} \frac{u^2}{2E_\varepsilon} d\mu, \tag{5.41}$$

also considering $\psi \in C_c^{0,1}(\Omega)$ and $\phi \in C^2(\Omega)$ we let

$$v = \psi\phi,$$

so we have

$$E_\varepsilon |\nabla v|_A^2 = E_\varepsilon |\nabla \phi|_A^2 \psi^2 + E_\varepsilon |\nabla \psi|_A^2 \phi^2 + 2AE_\varepsilon \phi\psi\nabla\phi \cdot \nabla\psi, \tag{5.42}$$

so according to (5.18) we have

$$\begin{aligned}
&\int_\Omega \psi^2 E_\varepsilon |\nabla\phi|_A^2 dx + 2\int_\Omega E_\varepsilon \phi\psi\nabla\phi\cdot\nabla\psi A dx + \frac{1}{2}\int_\Omega \frac{u^2}{E_\varepsilon} d\mu \\
&= \int_\Omega \psi^2 \left(L_A(\phi)\phi E_\varepsilon - \phi A\nabla E_\varepsilon \cdot \nabla\phi\right) dx - \frac{1}{2}\int_\Omega \frac{u^2}{E_\varepsilon} d\mu \\
&= \int_\Omega u^2 \left(\frac{L_A(\phi)}{\phi} - \frac{A\nabla E_\varepsilon \cdot \nabla\phi}{\phi E_\varepsilon} - \frac{\mu}{2E_\varepsilon}\right) dx,
\end{aligned}$$

by considering

$$\frac{L_A(\phi)}{\phi} - \frac{A\nabla E_\varepsilon \cdot \nabla\phi}{\phi E_\varepsilon} + \frac{\mu}{2E_\varepsilon} \geq V,$$

and plugging into (5.41) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 \geq \int_\Omega V(x) u^2 dx + \int_\Omega E\phi^2 |\nabla\psi|_A^2 dx, \tag{5.43}$$

we consider the RHS of (5.41) but first of all according to the first part of the Theorem 5.6 we have

$$\frac{L_A(\phi)}{\phi} - \frac{A\nabla E_\varepsilon \cdot \nabla\phi}{E_\varepsilon \phi} + \frac{\mu}{2E_\varepsilon} = \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}\left(-f''(E_\varepsilon) - \frac{f'(E_\varepsilon)}{f(E_\varepsilon)}\right) + \frac{F_\varepsilon}{2E_\varepsilon}, \tag{5.44}$$

now for $\int_\Omega E\phi^2|\nabla\psi|_A^2 dx$ we have

$$\begin{aligned}
\int_\Omega E_\varepsilon \phi^2 \nabla\psi\cdot\nabla\psi A dx =& -\int_\Omega div(E_\varepsilon\phi^2)\psi\nabla\psi A dx \\
=& -\int_\Omega A\nabla E_\varepsilon\cdot\nabla\psi\phi^2\psi dx - 2\int_\Omega \phi'\phi E_\varepsilon\psi\nabla\psi A dx \\
=& -\frac{1}{2}\int_\Omega A\nabla E_\varepsilon\cdot\nabla(\psi)^2\phi^2 dx - \frac{2}{2}\int_\Omega \phi'\phi E_\varepsilon\nabla\psi^2 A dx \\
=& \frac{1}{2}\int_\Omega div(A\nabla E_\varepsilon)\phi^2\psi^2 dx + \int_\Omega \phi'\phi\nabla E_\varepsilon\psi^2 A dx \\
=& -\frac{1}{2}\int_\Omega L_A(E_\varepsilon)v^2 dx + \int_\Omega \frac{\phi'}{\phi}\phi^2\psi^2 A\nabla E_\varepsilon dx \\
=& -\frac{1}{2}\int_\Omega L_A(E_\varepsilon)\frac{u^2}{E_\varepsilon} dx + \int_\Omega \frac{\phi'}{\phi}v^2 A\nabla E_\varepsilon dx,
\end{aligned}$$

by considering $\phi = f(E)$ we have

$$\begin{aligned}
-\frac{1}{2}\int_\Omega L_A(E_\varepsilon)\frac{u^2}{E_\varepsilon}dx + \int_\Omega \frac{(f(E_\varepsilon))'}{f(E_\varepsilon)}A\nabla E_\varepsilon v^2 dx =& -\frac{1}{2}\int_\Omega L_A(E_\varepsilon)\frac{u^2}{E_\varepsilon}dx + \int_\Omega \frac{(f(E_\varepsilon))'}{f(E_\varepsilon)}A\nabla E_\varepsilon\cdot\nabla E_\varepsilon v^2 dx \\
=& -\frac{1}{2}\int_\Omega L_A(E_\varepsilon)\frac{u^2}{E_\varepsilon}dx - \int_\Omega E_\varepsilon v^2\frac{(f(E_\varepsilon))'}{f(E_\varepsilon)}div(A\nabla E_\varepsilon)dx \\
=& -\frac{1}{2}\int_\Omega L_A(E_\varepsilon)\frac{u^2}{E_\varepsilon}dx + \int_\Omega \frac{(f(E_\varepsilon))'}{f(E_\varepsilon)}u^2 L_A(E_\varepsilon)dx \\
=& -\frac{1}{2}\int_\Omega F_\varepsilon\frac{u^2}{E_\varepsilon}dx + \int_\Omega \frac{(f(E_\varepsilon))'}{f(E_\varepsilon)}u^2 F_\varepsilon dx,
\end{aligned} \tag{5.45}$$

plugging (5.45) and (5.44) in (5.43) we have

$$\begin{aligned}
\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}u^2 dx \geq& \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E^2}\left(-f''(E_\varepsilon) - \frac{f'(E_\varepsilon)}{f(E_\varepsilon)}\right)u^2 dx + \frac{1}{2}\int_\Omega \frac{F_\varepsilon}{E_\varepsilon}u^2 dx \\
& -\frac{1}{2}\int_\Omega F_\varepsilon\frac{u^2}{E_\varepsilon}dx + \frac{1}{2}\int_\Omega F_\varepsilon\frac{u^2}{E_\varepsilon}dx + \int_\Omega \frac{f'(E_\varepsilon)}{f(E_\varepsilon}u^2 F_\varepsilon dx,
\end{aligned} \tag{5.46}$$

and finally we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \geq \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{f(E_\varepsilon)} \left(-f''(E_\varepsilon) - \frac{f'(E_\varepsilon)}{E_\varepsilon}\right) u^2 dx + \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon}\right) u^2 F_\varepsilon dx, \tag{5.47}$$

for all $u \in C_c^\infty(\Omega)$.

We consider two terms of RHS of (5.47):

According to assumption for the first part of (5.47), $|\nabla E|_A = 1$ on Ω a.e so we consider:

$$\frac{1}{f(E_\varepsilon)} \left(-f''(E_\varepsilon) - \frac{f'(E_\varepsilon)}{E_\varepsilon}\right) u^2,$$

and $I_\varepsilon := \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon}\right) F_\varepsilon u^2 dx$ where $0 < f \in C^2((0, \|E\|_{L^\infty}])$.
By setting $\lambda := \frac{\lambda_0^2}{\|E\|_{L^\infty}^2}$ where λ_0 (Lambs constant) denote the first positive zero of $J_0(t) - 2tJ_1(t)$ where J_n is the Bessel function of order n. Define $f(t) := J_0(\sqrt{\lambda}t)$ so we have

$$f'(t) = -\sqrt{\lambda}(J_1(\sqrt{\lambda}t)), \tag{5.48}$$

and

$$f''(t) = -\lambda J_0(\sqrt{\lambda}t) + \frac{\sqrt{\lambda}}{t} J_1(\sqrt{\lambda}t), \tag{5.49}$$

according to (5.48) and (5.49) we compute

$$\begin{aligned}\frac{1}{f(t)}\left(-f''(t) - \frac{f'(t)}{t}\right) &= \frac{1}{J_0(\sqrt{\lambda}t)}\left(\lambda J_0(\sqrt{\lambda}t) - \frac{\sqrt{\lambda}}{t}J_1(\sqrt{\lambda}t) + \frac{\sqrt{\lambda}}{t}J_1(\sqrt{\lambda}t)\right)\\ &= \frac{\lambda J_0(\sqrt{\lambda}t)}{J_0(\sqrt{\lambda}t)} - \frac{\frac{\sqrt{\lambda}}{t}J_1(\sqrt{\lambda}t)}{J_0(\sqrt{\lambda}t)} + \frac{\frac{\sqrt{\lambda}}{t}J_1(\sqrt{\lambda}t)}{J_0(\sqrt{\lambda}t)} = \lambda,\end{aligned}$$

we set

$$l(t)=\frac{f'(t)}{f(t)}+\frac{1}{2t}=\frac{-\sqrt{\lambda}J_1(\sqrt{\lambda}t)}{J_0(\sqrt{\lambda}t)}+\frac{1}{2t}=\frac{-2\sqrt{\lambda}tJ_1(\sqrt{\lambda}t)+J_0(\sqrt{\lambda}t)}{2tJ_0(\sqrt{\lambda}t)}, \tag{5.50}$$

since

$$\sqrt{\lambda}=\sqrt{\left(\frac{\lambda_0^2}{\|E\|_{L^\infty}^2}\right)}=\frac{0.940....}{\|E\|_{L^\infty}},$$

and

$$2\sqrt{\lambda}=\frac{1.880....}{\|E\|_{L^\infty}},$$

so $l(t)\geq 0$ in $(0,\|E\|_L^\infty)$. Since

$$\frac{1}{f(E_\varepsilon)}\left(-f''(E_\varepsilon)-\frac{f'(E_\varepsilon)}{E_\varepsilon}\right)=\lambda=\frac{{\lambda_0}^2}{\|E_\varepsilon\|_{L^\infty}^2},$$

by substitution in

$$\int_\Omega|\nabla u|_A^2dx-\frac{1}{4}\int_\Omega\frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2}u^2dx\geq\int_\Omega\frac{|\nabla E_\varepsilon|_A^2}{f(E_\varepsilon)}\left(-f''(E_\varepsilon)-\frac{f'(E_\varepsilon)}{E_\varepsilon}\right)u^2dx,$$

we have

$$\begin{aligned}\int_\Omega|\nabla u|_A^2dx-\frac{1}{4}\int_\Omega\frac{|\nabla E|_A^2}{E^2}u^2dx\geq&\int_\Omega\lambda|\nabla E|_A^2u^2dx+I_\varepsilon\\=&\lambda\int_\Omega|\nabla E|_A^2u^2dx+I_\varepsilon\\=&\frac{\lambda_0^2}{\|E_\varepsilon\|_{L^\infty}^2}\int_\Omega|\nabla E_\varepsilon|_A^2u^2dx+I_\varepsilon,\end{aligned} \tag{5.51}$$

now considering Young's convolution inequality, Theorem 1.31, and $p=1$, $q=\infty$ and $r=1$ we have

$$\|E_\varepsilon\|_{L^\infty}=\|\eta_\varepsilon*E\|_{L^1}\leq\|\eta_\varepsilon\|_{L^1}\|E\|_{L^\infty}$$

so according to (5.51) we have

$$\int_\Omega|\nabla u|_A^2dx-\frac{1}{4}\int_\Omega\frac{|\nabla E|_A^2}{E^2}u^2dx\geq\frac{\lambda_0^2}{\|E\|_{L^\infty}^2}\int_\Omega|\nabla E|_A^2u^2dx+I_\varepsilon,$$

where $l(t) = \frac{f'(t)}{f(t)} + \frac{1}{2t} \geq 0$ and $I_\varepsilon = \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon} \right) u^2 F_\varepsilon dx$ so

$$I_\varepsilon = \int_\Omega l_A(E_\varepsilon) u^2 F_\varepsilon dx,$$

since

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j} = 1 - \frac{x^2}{2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots,$$

and

$$J_1(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} \left(\frac{x}{2}\right)^{2j+1} = \frac{x}{2} - \frac{x^3}{2^3 2!} + \frac{x^5}{2^5(2!)(3!)} - \frac{x^7}{2^7(3!)(4!)} + \cdots,$$

and both are in C^∞ and since

$$l(x) = \frac{-2\sqrt{\lambda} x J_1(\sqrt{\lambda} x) + J_0(\sqrt{\lambda} x)}{2t J_0(\sqrt{\lambda} x)},$$

so $l \in C^\infty((0, \|E\|_{L^\infty}])$.

Since for each member in $H_0^1(\Omega)$ there is a sequence in $C_c^\infty(\Omega)$ so when $\varepsilon \to 0$ we have

$$l(E_\varepsilon) = l(E + \varepsilon) = l(E),$$

so we have $ul(E_\varepsilon) \to ul(E) \quad \text{in} \quad H_0^1(\Omega)$, also $F_\varepsilon \notin C_c^\infty(\Omega)$ and $\mu \in H_0^1(\Omega)$ so by using of continuous linear function we have $uF_\varepsilon \rightharpoonup u\mu \quad \text{in} \quad (H_0^1(\Omega))^* = H^{-1}(\Omega)$, so we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} I_\varepsilon(t) &= \lim_{\varepsilon \to 0} \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon} \right) u^2 F_\varepsilon dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega l(E_\varepsilon) u^2 F_\varepsilon dx \\
&= \int_\Omega l(E) u^2 d\mu \\
&= \int_\Omega \left(\frac{f'(E)}{f(E)} + \frac{1}{2E} \right) u^2 \mu dx,
\end{aligned}$$

since $l(E) \geq 0$ and $u^2\mu \geq 0$ so $I_\varepsilon \geq 0$ as $\varepsilon \to 0$. Since $|\nabla E|_A = 1$ a.e on Ω and passing the limit of

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \geq \frac{\lambda_0^2}{\|E\|_{L^\infty}^2}\int_\Omega |\nabla E_\varepsilon|_A^2 u^2 dx + \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon}\right) u^2 F_\varepsilon dx,$$

so

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{E} dx \geq \frac{\lambda_0^2}{\|E\|_{L^\infty}^2}\int_\Omega u^2 dx,$$

finally

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{E} dx + \frac{\lambda_0^2}{\|E\|_{L^\infty}^2}\int_\Omega u^2 dx.$$

□

Bibliography

[1] ADIMURTHI, CHAUDHURI, N., AND RAMASWAMY, M. An improved Hardy-Sobolev inequality and its application. *Proc. Amer. Math. Soc. 130*, 2 (2002), 489–505.

[2] ADIMURTHI, AND ESTEBAN, M. J. An improved Hardy-Sobolev inequality in $W^{1,p}$ and its application to Schrödinger operators. *NoDEA Nonlinear Differential Equations Appl. 12*, 2 (2005), 243–263.

[3] ADIMURTHI, AND SEKAR, A. Role of the fundamental solution in Hardy-Sobolev-type inequalities. *Proc. Roy. Soc. Edinburgh Sect. A 136*, 6 (2006), 1111–1130.

[4] ALLEGRETTO, W., AND HUANG, Y. X. A Picone's identity for the p-Laplacian and applications. *Nonlinear Anal. 32*, 7 (1998), 819–830.

[5] AVKHADIEV, F. G., AND WIRTHS, K.-J. Unified poincare and hardy inequalities with sharp constants for convex domains. *ZAMM Z. Angew. Math. Mech. 87*, 8-9 (2007), 632–642.

[6] BARBATIS, G., FILIPPAS, S., AND TERTIKAS, A. Series expansion for l^p hardy inequalities. *Indiana Univ. Math. J 52*, 1 (2003), 171–190.

[7] BREZIS, AND MARCUS, M. Hardy's inequalities revisited. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25*, 1-2 (1997), 217–237 (1998). Dedicated to Ennio De Giorgi.

[8] COWAN, C. Optimal Hardy inequalities for general elliptic operators with improvements. *Commun. Pure Appl. Anal. 9*, 1 (2010), 109–140.

[9] CYCON, H. L., FROESE, R. G., KIRSCH, W., AND SIMON, B. *Schrödinger operators with application to quantum mechanics and global geometry*, study ed. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1987.

[10] FILIPPAS, S., AND TERTIKAS, A. Optimizing improved Hardy inequalities. *J. Funct. Anal. 192*, 1 (2002), 186–233.

[11] FLECKINGER, J., HARRELL, II, E. M., AND DE THÉLIN, F. Boundary behavior and estimates for solutions of equations containing the p-Laplacian. *Electron. J. Differential Equations* (1999), No. 38, 19 pp.

[12] GHOUSSOUB, N., AND MORADIFAM, A. Bessel pairs and optimal hardy and hardy-rellich inequalities. *Stud. Univ. Babecs-Bolyai Math. 36* (2007).

[13] GHOUSSOUB, N., AND MORADIFAM, A. On the best possible remaining term in the Hardy inequality. *Proc. Natl. Acad. Sci. USA 105*, 37 (2008), 13746–13751.

[14] LAMBERTI, P. D., AND PINCHOVER, Y. L^p Hardy inequality on $C^{1,\gamma}$ domains. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19*, 3 (2019), 1135–1159.

[15] MARCUS, M., MIZEL, V. J., AND PINCHOVER, Y. On the best constant for Hardy's inequality in $\mathbb{R}^N$. *Trans. Amer. Math. Soc. 350*, 8 (1998), 3237–3255.

[16] MAZ'JA, V. G. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[17] OPIC, B., AND KUFNER, A. *Hardy-type inequalities*, vol. 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.

[18] PINCHOVER, Y., AND TINTAREV, K. Ground state alternative for p-Laplacian with potential term. *Calc. Var. Partial Differential Equations 28*, 2 (2007), 179–201.

[19] Vazquez, J. L., and Zuazua, E. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal. 173*, 1 (2000), 103–153.

[20] Wang, Z.-Q., and Willem, M. Caffarelli-Kohn-Nirenberg inequalities with remainder terms. *J. Funct. Anal. 203*, 2 (2003), 550–568.

[21] Wang, Z.-Q., and Zhu, M. Hardy inequalities with boundary terms. *Electron. J. Differential Equations* (2003), No. 43, 8.