On Lane-Emden equation and some variations

by

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Abstract

In this thesis some Lane-Emden problems of different order are studied. Tackling the issue of existence of a positive solution and regularity of the solutions are of paramount importance for each instance.

In addition to discussing the general Lane-Emden equation, the cases of having an advection term to the original problem and investigating some fourth order nonlinear Dirichlet and Navier problems are of considerable interest.

While the well-studied general equation points out that for $p \geq \frac{N+2}{N-2}$ and Ω a star-shaped domain in $\mathbb{R}^N$ there would be no non-trivial solution, some advantageous results regarding the existence of a positive solution and regularity of the solutions on a general bounded domain in $\mathbb{R}^N$ are addressed for the equations where an advection is involved, as well as some nonlinear fourth order problem with given Dirichlet and Navier boundary conditions.

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Introduction

Partial Differential Equations is one of the most interesting research topics amongst researchers in mathematical sciences since Leonhard Euler's momentous contribution to this subject in eighteenth century, as mentioned in a 1928 paper in [10], and it has always been of paramount importance with a multitude of applications in diverse branches of engineering, natural sciences, statistics and economics. This research field, in fact, could be adopted to model a broad range of phenomena where heat transfer, diffusion, sound, electrodynamics, electrostatics, quantum mechanics and gravitation are a number of them. Elliptic PDEs have numerous applications in many areas of mathematics and physics.

As for one of the very famous elliptic equations we can refer to the Lane-Emden equation which is a nonlinear Poisson equation as below

$$\begin{cases} -\Delta w = w^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < \infty$ and Ω is some bounded domain in $\mathbb{R}^N$ where $N \geq 3$.

This equation has a multitude of application in astrophysics as it models a stellar object as a gaseous ball and determines its gravitational potential, see [12, 7, 15].

This equation, in fact, is advantageous in proposing a model for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid, see [12, 7, 15]. As we take a look into the history of this equation we observe that it has been first introduced by Jonathan Homer Lane who was an American astrophysicist in [40] in 1870; and later, by Jacob Robert Emden who was a Swiss astrophysicist and meteorologist, in [28] in 1907.

There have been many interesting research work conducted on this problem from different perspectives that we would like to bring up a few of them such as [34, 14, 22, 30, 47, 1, 4, 24, 7, 9, 11].

The current thesis is a survey on Lane-Emden equation and some of its variations and consists of six chapters. We primarily discussed about nonexistence and existence of solutions, regularity results and some estimates for different types of Lane-Emden equations.

In addition to bringing the required essential mathematical background in the second chapter of our research, we have introduced the general Lane-Emden equation in chapter three and we have shown the nonexistence of a nontrivial solution for $p > \frac{N+2}{N-2}$ and Ω a star-shaped domain. This was first introduced by Pohozaev in a 1965 paper [48] and was shown by Gidas and Spruck in a 1981 paper [35]. It is also mentioned in several papers like a 1993 paper by Passaseo in [47] and is covered in the textbook [30] by Lawrence C. Evans on 1998, as well.

Moreover, the existence of a positive solution for the general Lane-Emden equation has been proposed for Ω a bounded domain with smooth boundary and a subset of $\mathbb{R}^N$, and $1 < p < \frac{N+2}{N-2}$ by Gidas and Spruck in the above-mentioned paper [35] in 1981 and some similar work for the critical Sobolev exponent where $p = \frac{N+2}{N-2}$ proposed by Brezis and Nirenberg in their paper [8] in 1983. Also, there have been numerous similar research performed on this equation for some different domains in $\mathbb{R}^N$ or slightly different cases with very useful results by many scholars that we can list some of them as follows, [9, 11, 45].

Also, by examining the appropriate values for p and N for the Lane-Emden equation we could address a regularity result for this well-studied problem that guarantees the smoothness of solutions for the right values of p and N, for $1 < p < \frac{N+2}{N-2}$, as well as $1 < p < \frac{N}{N-2}$. We have provided two approaches to prove it. The first one is a method based on a bootstrapping process and some iterations; besides, the second way is that we take the nonlinear problem to be a linear problem with some appropriate change of variable.

Furthermore, a more general version of the Lane-Emden equation could be considered as below where it would be possible to have negative solutions, as well.

$$\begin{cases} -\Delta w = |w|^{p-1}w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Along with introducing this problem we could find some similar theorem for this problem from some 2007 references [21, 31] stating that the solutions of this equation should be of one sign which means there would not be a sign-changing solution.

Furthermore, in the fourth chapter of our study on Lane-Emden equation we would like to discuss about the case where an advection term was added to the original problem (1.1) so that we will have the following equation

$$\begin{cases} -\Delta w + a(x)\cdot\nabla w = w^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In the case where $a(x) \equiv 0$ the problem is reduced to the well-studied Lane-Emden equation (1.1). For the case where we have $a(x) \neq 0$ some possible cases would be worthwhile to be considered. More specifically, if we have $a(x) = \nabla\gamma(x)$ where γ is a scalar function, the equation above would be considered as the following variational problem

$$-\Delta w + \nabla\gamma(x) \cdot \nabla w = w^p, \tag{1.4}$$

and so, we can assign an energy functional to this problem. Several existence and nonexistence results proposed for the general Lane-Emden equation (1.1) could be validated for the above problem involving gradient by considering some adjustments in the offered proofs for the general case; see [16]. We have tried to address the existence of a positive solution as well as some regularity results for the equation (1.3).

A multitude of studies with interesting results have been done on this problem which we can refer to a frequently cited 2012 paper by Dr. Craig Cowan and Dr. Mostafa Fazly that a much more general case is addressed in [18] where the following Dirichlet problem

$$-div(a_1 \nabla w) = a_2 w^p,$$

is discussed and some useful results are proposed. As remarked in this paper and a 2014 paper by Dr. Craig Cowan, provided we have $a_1 = a_2$, this problem is reduced to the case where we will obtain the above-mentioned variational problem for some scalar function γ. More details could be found in [16, 18, 26].

Along the same lines, in the fifth chapter of the current thesis we are interested in studying some clearly important fourth order nonlinear Dirichlet and Navier problems.

$$\text{(Dirichlet problem)} \quad \begin{cases} \Delta^2 w = |w|^{p-1}w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

as well as

$$
\text{(Navier problem)} \quad \begin{cases} \Delta^2 w = |w|^{p-1} w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ \Delta w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $N \geq 3$.

For $N = 2$ the problem would be defined for a two dimensional domain which represents a plate. In this case, the nonlinear Dirichlet problem (1.5) is called clamped plate equation and the Navier problem (1.6) is named buckled plate equation and some identities and results are proposed for these cases with various boundary conditions or in [33]; however, we will take some fourth order problems for $N \geq 3$ into consideration in this research.

First, we begin our discussion with more general cases where we have the following linear fourth order problems

$$
\begin{cases} \Delta^2 w = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}
$$

and

$$
\begin{cases} \Delta^2 w = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ \Delta w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}
$$

As we concentrate on our study on the nonlinear problems (1.6) and (1.5), we observe that Enzo Mitidieri in a 1990 paper [44] and Robert Van Der Vorst in his paper [53] in 1991, have addressed the nonexistence of a positive solution where Ω is a star-shaped domain for the nonlinear biharmonic Navier problem (1.6) and they have proposed similar advantageous results for higher order nonlinear polyharmonic problems, as well.

Furthermore, we propose some estimates for $w \in W^{4,p}(\Omega)$ as the unique solution of the linear problem (1.7) and $w \in H_0^2(\Omega)$, the unique solution of (1.8), taking advantage of some higher order L^p regularity theory precisely studied by Gazzola, Grunau and Sweers in the reference book [33] for higher order polyharmonic problems and it was initially introduced by Agmon, Douglis and Nirenberg in their 1959 paper [2] and 1964 paper [3].

Moreover, Boggio in 1905 presented a weak maximum principle for the equation (1.8) in a unit ball in his book [6]; however, it has not proven yet whether or not a maximum principle holds for this equation in general domains. Furthermore, a maximum principle holds for the linear biharmonic Navier problem (1.7) which has been demonstrated in chapter five of this thesis. More details on this problem and similar higher order polyharmonic problems could be found in [33].

Having discussed about the above-mentioned linear fourth order problems, we would like to investigate the nonlinear equations (1.5) and (1.6) which we offer an advantageous theorem for existence of a positive minimal solution for each problem. These problems, in fact, are a special case of some higher order polyharmonic semilinear problems of subcritical growth that are discussed in a multitude number of papers. For instance, we can refer to the a 1978 paper by von Wahl in [55], a 1985 paper by Oswald [46], Dalmasso's 1990 paper in French [20], the paper published in 1992 by Clement, de Figueiredo and Mitidieri [13], as well as Soranzo's 1994 paper [50].

Last but not least, the sixth chapter of the current thesis is dedicated to draw a brief conclusion on our research about various types of Lane-Emden equations. Also, we have recommended some opportunities for further studies on similar noteworthy topics.

In order to facilitate the readers' experience through reading and following the current thesis and finding the appropriate information in a timely manner, we have represented a table of norms including different spaces with their corresponding norm as an Appendix.

2

Mathematical Background

In order that we might be able to conduct research on analysis of partial differential equations, and more specifically on elliptic nonlinear problems such as Lane-Emden type equations, we will require a multitude of fundamental concepts, mathematical tools, function spaces, and useful theorems which are generally derived from functional analysis.

2.1 Some useful function spaces

In this section we introduce the essential function spaces of this study. Consider $\Omega \subset \mathbb{R}^N$ be the open and connected set, denote a bounded domain in $\mathbb{R}^N$ with smooth boundary.

2.1.1 Continuous functions.

We let $C(\Omega)$ denote the set of continuous functions on Ω.

2.1.2 Uniformly continuous functions.

Let $C(\overline{\Omega})$ stand for the set of uniformly continuous functions on Ω. It can be shown that $(C(\overline{\Omega}), \|\cdot\|)$ is a Banach space where

$$\|u\| := \sup_{x\in\Omega} |u(x)| = \max_{x\in\overline{\Omega}} |u(x)|.$$

For more information see [52].

2.1.3 The space of locally integrable functions

The space of locally integrable functions on Ω is shown by $L_{loc}(\Omega)$, which includes all measurable function u such that $\int_X |u|$ is finite for every compact subset X of Ω; see [41].

2.2 Hilbert Spaces

We have derived this section from [30]. Consider H is a real linear space:

Definition 1. *A mapping $(,) : H \times H \to \mathbb{R}$ is called an inner product if:*

1. $(u, v) = (v, u)$ *for all* $u, v \in H$*,*
2. *the mapping* $u \mapsto (u, v)$ *is linear for each* $v \in H$*,*
3. $(u, u) \geq 0$ *for all* $u \in H$*,*
4. $(u, u) = 0$ *if and only if* $u = 0$*.*

If $(,)$ is an inner product, the associated norm is for any $u \in H$

$$\|u\| := (u, u)^{\frac{1}{2}},$$

and from the $Cauchy - Schwarz$ inequality for any $u, v \in H$ we have:

$$|(u,v)| \leq \|u\| \, \|v\| \, .$$

Definition 2. *[30] A Hilbert space H is a Banach space equipped with an inner product generating the mentioned norm.*

2.2.1 Dual Space

For a normed vector space X, its dual space, which is denoted by X', is the space of bounded linear functionals on X. The norm on X' is determined by:

$$\|f\|' = \|f\|_{X'} := \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_X},$$

for $f \in X'$, see [41].

2.2.2 Some important theorems

Theorem 1. *(Riesz Representation Theorem) [41]*

Let H be a Hilbert space and f be a bounded linear functional on H'. Then there exists a unique $u \in H$ such that $f(v) = (v, u)$ for all $v \in H$ and $\|u\| = \|f\|'$, where $\|.\|'$ denotes the norm on H'.

Theorem 2. *(Lax-Milgram Theorem) [41, 30] Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear form which there are constants $c_1, c_2 > 0$ such that for every $u, v \in H$, both conditions below are satisfied:*

- *B is bounded*

$$|B(u,v)| \leq c_1 \, \|u\| \, \|v\| \, ,$$

- *B is coercive*

$$|B(u,u)| \geq c_2 \|u\|^2 .$$

Let $f \in H'$. Then there is a unique $u \in H$ such that $B(u,v) = f(v)$ for all $v \in H$ and $\|u\| \leq c \|f\|'$ where $\|.\|'$ denotes the norm on H'.

Furthermore, it is noticeable to mention that Lax-Milgram theorem reduces to Riesz Representation theorem when the bilinear form B is symmetric. It means $B(v,w) = B(w,v)$ for all $v, w \in H$. This is because of the fact that a symmetric coercive bilinear form is an inner product, see [41].

2.2.3 L^p Spaces

For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the Banach space of measurable functions f on Ω such that their L^p norm is finite , see [52]. as below:

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we have the $L^\infty(\Omega)$ space which is a Banach space and is the spaces of bounded functions defined on Ω whose essential sup is finite.

It is remarkable to note that the dual space of $L^p(\Omega)$ is $L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$, and Ω is a bounded domain in $\mathbb{R}^N$ for $1 \leq p, q < \infty$, see [41].

2.2.4 Smooth compactly supported functions.

We let $C_c^\infty(\Omega)$ denote the set of infinitely differentiable functions ϕ, where $\phi : \Omega \to \mathbb{R}$ and is compactly supported in Ω, see [30].

2.2.5 Convergence in $C_c^\infty(\Omega)$

We say that $\phi_m \to \phi$ in $C_c^\infty(\Omega)$ provided: $\phi_m, \phi \in C_c^\infty(\Omega)$ and there is some fixed compact set $K \subset \Omega$ such that for all m we have $supp(\phi_m), supp(\phi) \subset K$ and $\phi_m \to \phi$ in $C^k(K)$ for all $k = 0, 1, 2, \ldots$. Note the above notion of convergence is extremely strong.

2.2.6 Distributions

Let $T : C_c^\infty(\Omega) \to \mathbb{R}$ denote a linear function (ie. $T(au + bv) = aT(u) + bT(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in C_c^\infty(\Omega)$). We say $T \in \mathcal{D}'(\Omega)$, read T is a *distribution* on Ω, provided T is continuous with respect the above notion of convergence, ie.

$$\phi_m \to \phi \quad \text{in } C_c^\infty(\Omega) \qquad \Longrightarrow \qquad T(\phi_m) \to T(\phi).$$

Such function $\phi \in C_c^\infty(\Omega)$ that $\phi : \Omega \to \mathbb{R}$ is called a test function; see [30].

2.2.7 Sobolev Spaces

This section is mainly derived from references [30, 41]. For fixed $1 \leq p \leq \infty$ and a positive integer k, we need to consider some specific function spaces that functions have weak derivatives of different orders which belong to several L^p spaces.

Definition 3. *The Sobolev space $W^{k,p}(\Omega)$ is defined as below*

$$W^{k,p}(\Omega) := \{u \in L_{loc}(\Omega), D^\alpha u \in L^p(\Omega), |\alpha| \leq k\}, \tag{2.1}$$

and consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ which $D^\alpha u$ exists in the weak sense and lies in $L^p(\Omega)$ for any multi-index $|\alpha| \leq k$.

The appropriate norm for Sobolev spaces is the following for $1 \leq p < \infty$

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^{\alpha} u|^p \, dx \right)^{\frac{1}{p}}, \tag{2.2}$$

and when $p = \infty$ the norm is defined by:

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \|D^{\alpha} u\|_{\infty}.$$

It deserves to be noticed that for $p = 2$ we can write

$$H^k(\Omega) = W^{k,2}(\Omega),$$

where $H^k(\Omega)$ is a Hilbert space. Additionally, for $k = 0$ we would have $H^0(\Omega) = L^2(\Omega)$.

Definition 4. *[30]*

The closure of $C_c^{\infty}(\Omega)$ in $W^{k,p}(\Omega)$ is shown by

$$W_0^{k,p}(\Omega).$$

Therefore, $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in C_c^{\infty}(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$, see [30].

In other words, $W_0^{k,p}(\Omega)$ consists of the functions $u \in W^{k,p}(\Omega)$ such that for all $|\alpha| \leq k-1$ on on $\partial\Omega$ we have:

$$D^{\alpha} u = 0.$$

Furthermore, we can write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega). \tag{2.3}$$

For example, as indicated in the book [30], it is worthwhile to mention that the space $L^2(\Omega)$ is a Hilbert space with

$$(u, v) = \int_\Omega fg dx.$$

Moreover, the Sobolev space $H^1(\Omega)$ is a Hilbert space with:

$$(f, g) = \int_\Omega (fg + Df \cdot Dg) dx.$$

2.2.8 The space H^{-1}

We have taken this part of our research from the book [30]. We try to represent the dual space of H_0^1 as is outlined in the following definitions.

Definition 5. *We denote by $H^{-1}(\Omega)$ as the dual space to $H_0^1(\Omega)$.*

In other words, $f \in H^{-1}(\Omega)$ provided f is a bounded linear functional on $H_0^1(\Omega)$.

Definition 6. *If $f \in H^{-1}(\Omega)$ we define the following norm*

$$\|f\|_{H^{-1}(\Omega)} = \sup \left\{ \langle f, u \rangle \mid u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1 \right\},$$

where $\langle , \rangle$ stands for the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

In addition, we would like to suggest the following theorem for characterization of $H^{-1}(\Omega)$.

Theorem 3. *(Characterization of $H^{-1}(\Omega)$)[30]*

Assume $f \in H^{-1}(\Omega)$. Then, there exist functions $f^0, f^1, ..., f^N$ in $L^2(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\langle f, v \rangle = \int_\Omega (f^0 v + \sum_{i=1}^{N} f^i v_{x_i}) dx.$$

Moreover,

$$\|f\|_{H^{-1}(\Omega)} = \inf\left\{ (\int_\Omega \sum_{i=0}^{N} \left|f^i\right|^2 dx)^{\frac{1}{2}} \mid \langle f, v\rangle = \int_\Omega (f^0 v + \sum_{i=1}^{N} f^i v_{x_i}) dx \right\}.$$

Additionally, in a more general version, the dual space of $H_0^m(\Omega)$ is shown by $H^{-m}(\Omega)$ where

$$H^{-m}(\Omega) := (H_0^m(\Omega))',$$

for some positive integer m, see [41].

2.2.9 Hölder Spaces

We have taken this part of our research from the book [30]. We will require a very useful space, called Hölder spaces.

Assume $\Omega \subset \mathbb{R}^N$ is open and $0 < \gamma \leq 1$. Recall that for the class of Lipschitz continuous functions for some function $f : \Omega \to \mathbb{R}$ we had the following estimate for any $x, y \in \Omega$ and some constant C :

$$|f(x) - f(y)| \leq C\,|x - y|\,,$$

Now this implies that f is continuous. Having mentioned this idea, we can consider functions $u : \Omega \to \mathbb{R}$ satisfying an alternative of the above estimate as below:

$$|u(x) - u(y)| \leq C\,|x - y|^\gamma\,,$$

for some constant C. Such function is said to be Hölder continuous with exponent γ.

Definition 7. *[30]*

1. *If $u : \Omega \to \mathbb{R}$ is bounded and continuous, we write*

$$\|u\|_{C(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)| \, .$$

2. *The γ^{th}-Hölder semi-norm of $u : \Omega \to \mathbb{R}$ is*

$$[u]_{C^{o,\gamma}(\overline{\Omega})} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^{\gamma}} \right\}.$$

3. *The mentioned function's γ^{th}-Hölder norm is:*

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{o,\gamma}(\overline{\Omega})} \, .$$

Definition 8. *The Hölder space $C^{k,\gamma}(\overline{\Omega})$ consists of all functions $u \in C^k(\overline{\Omega})$ for which the following norm is finite:*

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha| \leq k} \|D^{\alpha} u\|_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [D^{\alpha} u]_{C^{0,\gamma}(\overline{\Omega})} \, .$$

Consequently, the space $C^{k,\gamma}(\overline{\Omega})$ consists of those functions u that are k-times continuously differentiable and whose k^{th}-partial derivatives are Hölder continuous with exponent γ. Such functions are well-behaved and the space $C^{k,\gamma}(\overline{\Omega})$, also, possesses a decent mathematical structure. As could be stated in the following theorem:

Theorem 4. *(Hölder spaces as function spaces) The space of functions $C^{k,\gamma}(\overline{\Omega})$ is a Banach space.*

2.2.10 Weak Derivatives

In this section we begin with weakening the idea of partial derivatives and we will dig into the definition of Sobolev spaces in another section. We have taken the majority of content in this section from [30].

Consider the function u continuously differentiable over the region Ω (which means $u \in C^1(\Omega)$) and the test function $\phi \in C_c^\infty(\Omega)$. By integration by parts formula we would have:

$$\int_\Omega u\phi_{x_i}dx = -\int_\Omega u_{x_i}\phi dx. \tag{2.4}$$

Because of the fact that ϕ has compact support in Ω and so vanishes close to the boundary $\partial\Omega$, there would be no boundary terms in (2.4).In addition,we will have the following for a multi-index α :

$$\int_\Omega uD^\alpha\phi dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u\phi dx,$$

Definition 9. *[30] For $u, v \in L^1_{loc}(\Omega)$, v is the α^{th}-weak derivative of u if*

$$\int_\Omega uD^\alpha\phi dx = (-1)^{|\alpha|}\int_\Omega v\phi dx,$$

holds for all test functions $\phi \in C_c^\infty(\Omega)$ and we will consider $D^\alpha u = v$. So v is the α^{th}-weak partial derivative of u.

2.2.11 Weighted L^p Spaces

In this section of our research we would like to represent the "weighted L^p spaces" which would be an advantageous tool in analysis of certain cases of Partial Differential Equations. The following definition could be found in the reference book [37] as it has covered a wide variety of the applications of such spaces.

Definition 10. *For $1 \le p < \infty$ and w be a real valued and positive almost everywhere function, $w : \mathbb{R} \to [0, \infty)$,the w-weighted Lebesgue space $L^p(\Omega, w(x))$ is defined as the set of measurable functions f on Ω such that the following norm is finite:*

$$\|f\|_{L^p(\Omega,w)} = (\int_\Omega |f|^p \, w(x)dx)^{\frac{1}{p}} < \infty. \tag{2.5}$$

It might be trivial to check but should be considered that by taking the weight function $w \equiv 1$ the above weighted L^p spaces become equivalent to L^p spaces that we introduced earlier and they share many similarities.

2.3 Sobolev embedding theorem

Theorem 5. *(Sobolev embedding theorem) [41] Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary. Also, consider $0 \le m < l$ an integer and $1 \le p, q < \infty$. Then*

1. *If $\frac{1}{p} \le \frac{1}{q} + \frac{l-m}{N}$, then $W^{l,p}(\Omega) \hookrightarrow W^{m,p}(\Omega)$. The embedding is compact if: $\frac{1}{p} < \frac{1}{q} + \frac{l-m}{N}$.*
2. *If $\frac{1}{p} < \frac{l-m}{N}$ then $W^{l,p}(\Omega) \hookrightarrow C^m(\overline{\Omega})$ and the embedding is compact.*

2.4 Green's identities

We would like to start off by introducing Green's identities which are very useful through our work. We have derived the majority of content in this section from the references [41, 30].

Theorem 6. *(Green's Identities)[41]*

$$\int_\Omega u_{x_i} v = -\int_\Omega u v_{x_i} + \int_{\partial\Omega} u v \nu^i, \tag{2.6}$$

where $u, v \in C^1(\overline{\Omega})$ and $\nu = (\nu^1, \nu^2, ..., \nu^N)$ is the outward pointing normal on $\partial\Omega$.

The equation (2.6) is called Green's first identity.

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} (-\Delta u) v + \int_{\partial\Omega} v \partial_\nu u, \tag{2.7}$$

where $\partial_\nu u := \nabla u \cdot \nu$. Here $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$.

The equation (2.7) is called Green's second identity.

A solution that fits in an elliptic partial differential equation but does not necessarily need to be differentiable for all points in the given domain is called a "weak solution".In fact, it satisfies the PDE in a weak sense.

We shall use integration by parts in our effort to find the weak form for a given PDE. Therefore, we will need to use Green's first and second identities many times.

2.5 Existence and Regularity theory for the linear Dirichlet problem

We would like to focus on the linear Dirichlet problem that are very useful within our study. More details on this could be found in references [30, 41]. Consider the following Dirichlet problem in its most general form:

$$\begin{cases} -\Delta u &= f(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $f \in H^{-1}$ and Ω is a smooth bounded domain in $\mathbb{R}^N$.

2.5.1 Motivation for an H_0^1 solution

Suppose $u \in C_c^\infty(\Omega)$ be a smooth solution of (2.8) and multiplying this equation by some smooth test function $\phi \in C_c^\infty(\Omega)$ and $\phi = 0$ on $\partial\Omega$, as well as considering

Green's second identity described in (2.7) we would get

$$\int_{\Omega} f\phi = \int_{\Omega} \nabla u \cdot \nabla \phi - \int_{\partial\Omega} \phi(\partial_{\nu} u) = \int_{\Omega} \nabla u \cdot \nabla \phi,$$

because of the fact that $\phi = 0$ on $\partial\Omega$.

The same identity holds when the smooth test function ϕ is substituted by any test function $\phi \in H_0^1(\Omega)$, and so, the followed identity would be valid provided only $u \in H_0^1(\Omega)$. In fact, the space $H_0^1(\Omega)$ is chosen to include the boundary condition $u = 0$ on $\partial\Omega$ from the original equation (2.8), see [30].

Definition 11. *[30, 41] The bilinear form associated with the equation (2.8) is*

$$B(u, v) := \int_{\Omega} \nabla u \cdot \nabla v,$$

for $u, v \in H_0^1(\Omega)$.

Definition 12. *[30, 41] We say $u \in H_0^1(\Omega)$ is a "weak solution" of the Dirichlet problem (2.8) provided*

$$B(u, v) := \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv,$$

for all $v \in H_0^1(\Omega)$.

Theorem 7. *(Existence of weak solutions)[30] For $f \in H^{-1}$ there exists a unique solution $u \in H_0^1$. which satisfies the PDE in (2.8).*

Proof. Because of $f \in H^{-1}$ and Riesz representation theorem there exists a unique $u \in H_0^1$ such that

$$(u, \phi) = \langle f, \phi \rangle,$$

for all $\phi \in H_0^1$ where (u, ϕ) is the following inner product on H_0^1

$$(u, \phi) = \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f \phi = \langle f, \phi \rangle .$$

Hence, we see that we obtained the definition of a weak solution.

$\square$

Furthermore, we will require the following regularity results for the linear Dirichlet problem which are significantly important within our discussion about different cases of such equations.

2.5.2 Regularity theory

We would like to introduce an essential part of our research where its significance is shown in answering the smoothness of weak solution for a PDE. In fact, for the Dirichlet problem presented in (2.8) where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $f \in H^{-1}$. Then, we know that there exists a unique solution $u \in H_0^1(\Omega)$ and the following regularity results hold. More details could be found in [30, 33].

Theorem 8. *[30, 33] (Regularity theory)*

1. ***(L^2 regularity theory)***

   For some integer $k \geq 0$, provided $f \in H^k(\Omega)$ we will have $u \in H^{k+2}(\Omega)$. In the special case we have $H^0 := L^2$. Therefore, if $f \in L^2$ then $u \in H^2$ and we will have the following estimate:

   $$\|u\|_{H^{k+2}} \leq C(\Omega, k) \|f\|_{H^k} .$$

2. ***(L^p regularity theory)***

For $1 < p < \infty$ provided $f \in L^p(\Omega)$ then $u \in W^{2,p}(\Omega)$. Furthermore, there will be some constant $C = C(p, \Omega)$ such that

$$\|u\|_{W^{2,p}} \leq C(p, \Omega) \|f\|_{L^p}.$$

3. ***Hölder space regularity; Schauder theory***

If $f \in C^{0,\alpha}(\Omega)$ where $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\Omega)$. Also, there will be some $C(\alpha, \Omega)$ such that

$$\|u\|_{C^{2,\alpha}} \leq C(\alpha, \Omega) \|f\|_{C^{0,\alpha}}.$$

4. ***Higher order Schauder theory***

If $f \in C^{k,\alpha}(\Omega)$, then $u \in C^{k+2,\alpha}(\Omega)$.

Furthermore, for the linear Dirichlet problem (2.8) we have the following higher order L^p regularity result, as well.

Theorem 9. *(see page 45 [33]) Assume for $1 < p < \infty$ and $\partial\Omega \in C^4$ the equation (2.8) holds. Then, for all $f \in W^{2,p}(\Omega)$ the equation (2.8) has a strong solution $v \in W^{4,p}(\Omega) \cap W_0^{1,p}(\Omega)$; furthermore, there is a constant $C = C(\Omega) > 0$ such that*

$$\|u\|_{W^{4,p}} \leq C\|f\|_{W^{2,p}}.$$

It is important to be considered that the above-mentioned L^p regularity result is still valid when L^p is not a subset of H^{-1}, which means p is too close to 1.

Definition 13. *(Very weak solutions approach)*

Set $X := \left\{\phi \in C^\infty(\overline{\Omega}) : \phi|_{\partial\Omega} = 0\right\}$. $u \in L^1(\Omega)$ is called a "very weak" solution of (2.8) if for all $\phi \in X$ we have:

$$\int_\Omega (-\Delta\phi)u = \int_\Omega f\phi.$$

It is significant to point out that the suggested definition of a very weak solution is similar to the distributional definition, except for the test functions that are not compactly supported. In fact, the given definition effectively conveys $u = 0$ on $\partial\Omega$.

By way of illustration, assume $u \in C^2(\overline{\Omega})$ and meets the above-mentioned criteria of a very weak solution while there is no assumption about the value of u on the $\partial\Omega$. Hence, we would have $u = 0$ on $\partial\Omega$.

Along the same line, we provide the following theorem to demonstrate the existence and regularity of the very weak solutions.

Theorem 10. *(Duality proofs; $f \in L^1(\Omega)$, $\delta f \in L^1(\Omega)$)*

1. *For $f \in L^1(\Omega)$ there exists a "very weak" solution of (2.8) and $u \in L^p(\Omega)$ for all $p < \frac{N}{N-2}$, and $u \in W^{1,p}$ for all $p < \frac{N}{N-1}$.*

2. *For $\delta f \in L^1(\Omega)$ where $\delta(x) = dist(x, \partial\Omega)$ there exists a "very weak" solution of (2.8) and $u \in L^p$ for all $p < \frac{N}{N-1}$.*

2.6 Fredholm Alternative

In order to be able to analyze the required regularity results for some given PDE we will need to consider Fredholm alternative for elliptic partial differential equations which is introduced as the following theorem with the appropriate definition for compact operators. We have taken this part of our research from the reference books [30, 49].

Definition 14. *[30] For A and B real Banach spaces, a bounded linear operator*

$$K : A \to B$$

is called "compact" if for each bounded sequence $\{u_k\}_{k=1}^{\infty} \subset A$, the sequence $\{Ku_k\}_{k=1}^{\infty}$ is precompact in B which means there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ such that $\left\{Ku_{k_j}\right\}_{j=1}^{\infty}$

converges in B.

Theorem 11. *[30] (Fredholm alternative) Let H denote a real Hilbert space and $K : H \to H$ be a compact linear operator. Then*

1. *$N(I-K)$ is finite dimensional,*

2. *$R(I-K)$ is closed,*

3. *$R(I-K) = N(I-K^*)^{\perp}$,*

4. *$N(I-K) = 0$ if and only if $R(I-K) = H$,*

   and

5. *$dim(N(I-K)) = dim(N(I-K^*))$.*

2.7 Weak sequential convergence

Weak sequential convergence is a very useful notion that we will exploit in our research and it could be presented for sequences in Banach spaces L^p spaces as well as Hilbert spaces. We have taken this part from the references [29, 30, 52].

2.7.1 Weak sequential convergence in Banach spaces

Definition 15. *(Weak convergence for Banach spaces)[30] Let X denote a Banach space. We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges weakly to $u \in X$ and shown as*

$$u_k \rightharpoonup u,$$

provided

$$\langle u^*, u_k \rangle \to \langle u^*, u \rangle$$

for each bounded linear functional $u^* \in X'$.

Theorem 12. *(Weak compactness) [30] Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is bounded. Then there exists a subsequence $\left\{u_{k_j}\right\}_{k=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that*

$$u_{k_j} \rightharpoonup u.$$

Now that we have defined weak convergence and weak compactness for Banach spaces, we would like to extend our definition to Hilbert spaces. In fact, a bounded sequence in a Hilbert space contains a weakly convergent subsequence, see [30].

Definition 16. *[29, 52] Let Ω be a bounded domain in $\mathbb{R}^N$ and $L^p(\Omega)$ and $L^q(\Omega)$ be dual spaces for some $1 \leq p, q < \infty$ which means we have $\frac{1}{p} + \frac{1}{q} = 1$. A sequence of functions u_m converges weakly in $L^p(\Omega)$ to a function u provided*

$$\int_{\Omega} u_m v \to \int_{\Omega} uv,$$

for all $v \in L^q(\Omega)$.

Moreover, if the sequence is bounded and we have the weak convergence, there would be a convergent subsequence available as stated in the following theorem.

Theorem 13. *(Weak sequential compactness in L^p spaces)[29, 52] Assume Ω be a bounded domain in $\mathbb{R}^N$. Let $1 < p < \infty$ and u_m be a sequence of functions such that*

$$\|u_m\|_p \leq C,$$

for some C. Then, there exists some function $u \in L^p(\Omega)$ such that

$$\|u\|_p \leq C,$$

and a subsequence u_{m_k} which is weakly convergent to u in $L^p(\Omega)$.

However, we should point out that for the case $p = 1$ the above-mentioned weak convergence fails in $L^1(\Omega)$. The following example would make it clear that there is no f in L^1 and subsequence f_{n_k} such that f_{n_k} converges weakly to f in $L^1(\Omega)$.

Example 1. *For $p = 1$ assume the weak convergence holds and so from Definition 16 we will have a sequence of functions $\{f_{n_k}\} \subset \{f_n\}$ that is weakly convergent to a function f in $L^1(\Omega)$ such that*

$$\int_\Omega f_{n_k} g \to \int_\Omega f g,$$

for all functions $g \in L^\infty(\Omega)$.

As an example of this case, let $\Omega := [0, 1]$ and $n \in \mathbb{N}$ be a positive integer. Define the sequence $\{f_n\}$ as below

$$f_n := n\chi_{[0,\frac{1}{n}]},$$

where $\chi_{[0,\frac{1}{n}]}$ is the characteristic function on $[0, \frac{1}{n}]$. Then, $\{f_n\}$ is bounded since $\|f_n\|_{L_1} = 1$ for all $n \in \mathbb{N}$. Assume the subsequence $\{f_{n_k}\}$ converges weakly in $L^1(\Omega)$ to $f \in L^1(\Omega)$. For each interval $[a, b] \subset [0, 1]$ integrating $\chi_{[a,b]}$ is a bounded linear functional on $L^1(\Omega)$ since $\chi_{[a,b]} \in L^\infty(\Omega)$. Therefore,

$$\int_a^b f = \int_0^1 f\chi_{[a,b]},$$

and by weak convergence we have

$$\int_0^1 f\chi_{[a,b]} = \lim_{k\to\infty} \int_0^1 f_{n_k}\chi_{[a,b]},$$

and so

$$\lim_{k\to\infty} \int_0^1 f_{n_k}\chi_{[a,b]} = \lim_{k\to\infty} \int_a^b f_{n_k}.$$

Now, as $f_{n_k} := n_k\chi_{[0,\frac{1}{n_k}]}$, for k sufficiently large and $a > 0$ we have $\int_a^b f_{n_k} = 0$.

Therefore, for all $0 < a < b \leq 1$ *we get*

$$\int_a^b f = \lim_{k\to\infty} \int_a^b f_{n_k} = 0,$$

and so, $f = 0$ *almost everywhere in* Ω*. Thus, by weak convergence and* $g \equiv 1 \in L^\infty(\Omega)$ *we have*

$$0 = \int_0^1 f = \lim_{k\to\infty} \int_0^1 f_{n_k} = 1,$$

which is a contradiction. So, there is no subsequence $\{f_{n_k}\}$ *that is weakly convergent in* $L^1(\Omega)$*.*

$\square$

2.7.2 Weak sequential convergence in Hilbert spaces

Along the same line, we can consider the approach of weak sequential convergence for Hilbert spaces in terms of the following definition.

Definition 17. *Consider* u_m *be a sequence in a Hilbert space* H*. We say* u_m *is weakly convergent to* u *if*

$$(u_m, v) \to (u, v),$$

for any $v \in H$*.*

Also, we will have a similar weak sequential compactness result for Hilbert spaces as the following theorem.

Theorem 14. *There exists a weakly convergent subsequence for a bounded sequence* u_m *in some Hilbert space* H*.*

As we introduce the weak sequential convergence, it would be required to take a look at compact operators within our research work. It should be noted that every compact operator is continuous. Conversely, not we can not conclude that all continuous operators would be compact. In fact, we know that continuous operators map convergent sequences convergent sequences but compact operators map weakly convergent sequences to convergent sequences. The following theorem provides us the information that we would need further in our proof of regularity for the Lane-Emden PDE containing some advection term.

Theorem 15. *[52] A continuous linear operator on a Hilbert space maps weakly convergent sequences to weakly convergent sequences.*

2.8 Existence of minimizers

The majority of content of this section of our work is extracted from the reference books [29, 30, 19].

In order to make sure that the functional $I[.]$ has a minimizer at least in a suitable Sobolev space, we would like to determine some conditions on some Lagrangian L.

Along the same line, we discuss the coercivity condition on the functional $I[.]$ and its lower semi-continuity, as well as convexity for the Lagrangian L. The functional $I[w]$ should have a minimizer and is defined by

$$I[w] := \int_{\Omega} L(Dw(x), w(x), x)dx,$$

where $w : \Omega \to \mathbb{R}$ and $w = g$ on the $\partial\Omega$.

2.8.1 Weak lower semi-continuity

First, let us define the class of admissible functions w as the following set

$$A := \{w \in W^{1,p}(\Omega) : w|_{\partial\Omega} = g\}.$$

Also, we will need the following definition for a weakly lower continuous function.

Definition 18. *[29, 30, 19] We say that a function $I[.]$ is (sequentially) weakly lower semi-continuous on $W^{1,q}(\Omega)$, provided*

$$I[u] \leq \liminf_{k\to\infty} I[u_k],$$

whenever

$$u_k \rightharpoonup u,$$

weakly in $W^{1,q}(\Omega)$.

We think of a function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be convex provided

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for every $x, y \in \mathbb{R}^N$ and $t \in [0, 1]$, see the reference book [19].

Theorem 16. *(Weak lower semi-continuity)[30] Assume that L is bounded below, and the mapping*

$$p \mapsto L(p, z, x),$$

is convex, for $z \in \mathbb{R}$ and $x \in \Omega$. Then, $I[.]$ is weakly lower semi-continuous on $W^{1,q}(\Omega)$.

Theorem 17. *(Existence of minimizers)[29, 30, 19] Suppose that L satisfies the the coercivity condition below which says there exist constants α_1 and α_2 and fixed $1 < q < \infty$ such that*

$$L(p, z, x) \geq \alpha_1 |p|^q - \alpha_2,$$

for all $p \in \mathbb{R}^N$, $z \in \mathbb{R}$ and $x \in \Omega$. Also, L is convex in the variable p and assume that the admissible set A is nonempty. Then, there exists at least one function $u \in A$ that solves

$$I[u] = \min_{w\in A} I[w].$$

2.9 A useful inequality and a norm

Here we discuss an advantageous theorem called *Poincare-Friedrichs* inequality that gives us an estimate and enables us to define a very useful norm, as well. We have taken the content of this section from the reference book [41].

Theorem 18. *(Poincare-Friedrichs inequality)[41] Let Ω be a bounded domain with Lipschitz boundary. Then, for every $u \in H_0^m(\Omega)$ and $m \geq 1$, there exists a constant c such that for all $u \in H_0^m(\Omega)$,*

$$\|u\|_{H_0^m}^2 \leq c \sum_{|\alpha|=m} \|D^\alpha u\|^2 . \tag{2.9}$$

Furthermore, from the above-mentioned inequality we would like to define the corresponding norm on $H_0^m(\Omega)$ as

$$\|u\|_m^* := (\sum_{|\alpha|=m} \|D^\alpha u\|^2)^{\frac{1}{2}}, \tag{2.10}$$

where $m \geq 1$ and $u \in H_0^m(\Omega)$, see [41].

We should remark that the norm defined in (2.10) is actually a semi-norm and is not a norm on $H^m(\Omega)$. The reason for emphasizing the difference of it with a norm is that we might have $u \equiv 1$, however, we get $\|u\|_m^* = 0$, for more details see the following [41].

2.10 Fredholm Operators

Within our analysis of regularity for different Lane-Emden problems we will need to introduce Fredholm operators and some of their useful properties.

Definition 19. *[36, 39] Let X and Y be Banach spaces and the operator $T : X \to Y$ be a bounded linear operator is called a Fredholm operator if the following hold:*

1. *$ker(T)$ is finite dimensional.*

2. *$ran(T)$ is closed.*

3. *$coker(T)$ is finite dimensional.*

Moreover, the index of the Fredholm operator T is denoted by $ind(T)$ and is defined as the following integer:

$$ind(T) := dim(ker(T)) - dim(coker(T)).$$

Definition 20. *[43] A bounded operator T is of finite rank if its range has finite dimension and that dimension is called the rank of T.*

Theorem 19. *[43] Every finite rank operator on a Banach space is a compact operator.*

Theorem 20. *[39] If T is a Fredholm operator and K is a compact operator, then $T + K$ is a Fredholm operator and*

$$ind(T) = ind(T + K).$$

Theorem 21. *[25, 39](Atkinson's Theorem) If H is a Hilbert space, then a Hilbert space operator T in H is a Fredholm operator if and only if the range of T is closed and $dim(ker(T))$ and $dim(coker(T))$ are finite.*

Theorem 22. *Let $T : X \to Y$ be a bounded linear operator on and X, Y be Banach spaces. Then, T is one to one and has closed range if and only if there exists a $C > 0$ such that for all $x \in X$*

$$\|x\| \leq \|Tx\|.$$

3

Solutions of the Lane-Emden Equation

In this chapter we would like to begin with introducing the celebrated Lane-Emden equation on some general domain in $\mathbb{R}^N$ and we are interested in discussing about non-existence, existence and regularity of solutions for this problem as well as proposing some ground state solution. A paramount number of research works have been conducted on this problem in the last four decades that we can refer to a few of them such as [1, 4, 24, 47, 7, 5, 26].

3.1 Derrick-Pohozaev Identity

The majority of the content of this part of our research could be found in the text book [30]. We are interested in exploring the solutions for a nonlinear elliptic PDE, general Lane-Emden equation, which is described as the following boundary value problem:

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We would like to show that the only nonnegative solution of (3.1), on some particular type of domains, should be $u = 0$. By this, we will show that $p > \frac{N+2}{N-2}$ in (3.1) implies $u \equiv 0$. It was first addressed by Pohozaev [48] and a plethora of details could be found in the following references [30, 47, 1, 4, 24, 22].

For the first step, let us define a star-shaped domain in terms of the following definition.

Definition 21. *[30] An open domain Ω is called "star-shaped" with respect to 0 if the line segment below lies inside $\bar{\Omega}$ for all $x \in \bar{\Omega}$*

$$\left\{\forall x \in \bar{\Omega} : \lambda x \in \bar{\Omega}, 0 \leq \lambda \leq 1\right\}.$$

In the general case, Ω does not necessarily need to be a convex domain. However, if Ω is convex and $0 \in \Omega$, then Ω is a star-shaped domain with respect to 0.

Lemma 1. *(Normals to a star-shaped region) [30] Assume $\partial\Omega \in C^1$ and Ω is a star-shaped domain with respect to 0. Then:*

$$x \cdot \nu(x) \geq 0,$$

$$\forall x \in \partial\Omega,$$

where ν stands for the unit outward normal.

Proof. As $\partial\Omega \in C^1$ if $x \in \partial\Omega$ then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|y - x| < \delta$ and $y \in \bar{\Omega}$ imply

$$\nu(x) \cdot \frac{(y - x)}{|y - x|} \leq \varepsilon.$$

In particular,

$$\limsup_{\substack{y \to x \\ y \in \bar{\Omega}}} \nu(x) \cdot \frac{(y - x)}{|y - x|} \leq 0.$$

For $0 < \lambda < 1$ let $y = \lambda x$. Then $y \in \bar{\Omega}$ because Ω is star-shaped. Therefore we will have

$$\nu(x) \cdot \frac{x}{|x|} = - \lim_{\lambda \to 1^-} \nu(x) \cdot \frac{(\lambda x - x)}{|\lambda x - x|} \geq 0.$$

$\square$

In order to show that there is no nontrivial solution to the problem (3.1) in the case where Ω is a star-shaped domain we would like to consider the following important theorem.

Theorem 23. *(Non-existence of the non-trivial solution)[30, 48, 35, 47] Suppose $u \in C^2(\bar{\Omega})$ be a smooth solution of the PDE in (3.1) and the exponent $p > \frac{N+2}{N-2}$. Moreover, assume Ω be a star-shaped domain with respect to 0 , and $\partial\Omega \in C^1$. Then:*

$$u \equiv 0$$

inside Ω .

Proof. For a chosen point x we multiply the PDE by $x.\nabla u$ and integrate over Ω so we will have:

$$\int_\Omega (-\Delta u)(x \cdot \nabla u) dx = \int_\Omega u^p (x \cdot \nabla u) dx \tag{3.2}$$

We can write the equation (3.2) as below:

$$A = B$$

On the left hand side of we have:

$$A =: - \sum_{i,j=1}^{N} \int_\Omega u_{x_i x_i} x_j u_{x_j} dx$$

And by integeration by parts we will get:

$$A=\sum_{i,j=1}^{N}\int_{\Omega}u_{x_i}(x_ju_{x_j})_{x_i}dx-\sum_{i,j=1}^{N}\int_{\partial\Omega}u_{x_i}\nu_ix_ju_{x_j}dS \tag{3.3}$$

Let:

$$A=:A_1+A_2.$$

Then, from (3.3) we will have:

$$A_1=\sum_{i,j=1}^{N}\int_{\Omega}(u_{x_i}\delta_{ij}u_{x_j}+u_{x_i}x_ju_{x_jx_i})dx,$$

where δ_{ij} denotes the Kronecker delta as below

$$\delta_{ij}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

Then,

$$A_1=\int_{\Omega}(|\nabla u|^2+\sum_{j=1}^{N}(\frac{|\nabla u|^2}{2})_{x_j}x_j)dx.$$

Also, by expanding the sigma and integration by parts we obtain:

$$=\int_{\Omega}|\nabla u|^2\,dx-\frac{N}{2}\int_{\Omega}|\nabla u|^2+\int_{\partial\Omega}\frac{|\nabla u|^2}{2}(\nu\cdot x)dS$$

$$=(1-\frac{N}{2})\int_{\Omega}|\nabla u|^2\,dx+\int_{\partial\Omega}\frac{|\nabla u|^2}{2}(\nu\cdot x)dS. \tag{3.4}$$

Now we try to get an expression for A_2.

As we mentioned:

$$A_2 = -\sum_{i,j=1}^{N} \int_{\partial\Omega} u_{x_i} \nu_i x_j u_{x_j} dS.$$

We, also, know that $u = 0$ on $\partial\Omega$ which implies ∇u be parallel to the normal $\nu(x)$ for all $x \in \partial\Omega$. So we would have:

$$\nabla u = \pm \left| \nabla u \right| \nu(x).$$

Now since $u = 0$ on $\partial\Omega$, ∇u points in the direction of $-\nu(x)$ at each point $x \in \partial\Omega$ and so we have

$$\nabla u = - \left| \nabla u \right| \nu(x),$$

and so for A_2, we obtain:

$$A_2 = -\int_{\partial\Omega} \left| \nabla u \right|^2 (\nu \cdot x) dS. \tag{3.5}$$

Now by combining (3.4) and (3.5) together we would obtain:

$$A = \frac{2-N}{2} \int_{\Omega} \left| \nabla u \right|^2 dx - \frac{1}{2} \int_{\partial\Omega} \left| \nabla u \right|^2 (\nu \cdot x) dS. \tag{3.6}$$

On the other hand, we have:

$$B := \sum_{j=1}^{N} \int_{\Omega} u^p x_j u_{x_j} dx$$

$$= \sum_{j=1}^{N} \int_{\Omega} \left(\frac{u^{p+1}}{p+1} \right)_{x_j} x_j dx,$$

which we will obtain:

$$B = -\frac{N}{p+1} \int_{\Omega} u^{p+1} dx. \tag{3.7}$$

Now by putting the equations (3.6) and (3.7) into (3.2) we will have:

$$\frac{N-2}{2}\int_{\Omega}|\nabla u|^{2}\,dx+\frac{1}{2}\int_{\partial\Omega}|\nabla u|^{2}\,(\nu\cdot x)dS=\frac{N}{p+1}\int_{\Omega}u^{p+1}dx. \tag{3.8}$$

It is noticeable to mention that the above equality is called *Derrick - Pohozaev Identity*. Furthermore, according to the Lemma 1 we have $x.\nu(x)\geq 0$. Thus, we would obtain:

$$\frac{N-2}{2}\int_{\Omega}|\nabla u|^{2}\,dx\leq\frac{N}{p+1}\int_{\Omega}u^{p+1}dx. \tag{3.9}$$

As we multiply the equation (3.1) by u and use integration by parts we also get:

$$\int_{\Omega}|\nabla u|^{2}\,dx=\int_{\Omega}u^{p+1}dx.$$

Plugging this in (3.9) would result as below:

$$(\frac{N-2}{2}-\frac{N}{p+1})\int_{\Omega}u^{p+1}dx\leq 0.$$

Therefore if $u\not\equiv 0$, we would have:

$$(\frac{N-2}{2}-\frac{N}{p+1})\leq 0,$$

which implies:

$$p\leq\frac{N+2}{N-2}.$$

□

3.2 Existence of a positive solution

In this part of our study we would like to discuss about the existence of a positive solution to Lane-Emden equation stated in (3.1) and we have found the following theorem that was proposed by Gidas and Spruck. More details could be found in the following [35, 23, 51, 33, 4, 8].

Theorem 24. *[35] Suppose $N \geq 3$ and Ω a bounded domain with smooth boundary in $\mathbb{R}^N$. Then for $1 < p < \frac{N+2}{N-2}$ there exists a positive $u \in H_0^1(\Omega)$ solution of (3.1).*

Proof. We try to take advantage of an analogous proof provided in [8] and adjusting it for our problem. For more details you can see [51, 4, 33, 38, 24]. So, we begin with minimizing some energy functional associated with the aforementioned partial differential equation (3.1) in order to demonstrate that it has a positive solution as below,

$$E(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2}, \tag{3.10}$$

where

$$u \in X := H_0^1 \setminus \{0\},$$

and:

$$E(u) = \min_{v \in X} E(v). \tag{3.11}$$

It is significant to indicate that such positive minimizer is, also, a positive solution of the PDE in (3.1). [51, 4, 33, 24, 17]

First, it should be noted that for $u \in X$ and all $t \neq 0$ we have :

$$E(tu) = E(u).$$

Therefore, in order to minimize E over X it would be sufficient minimizing it over some X_0 where we define:

$$X_0 := \{u \in X : \|u\|_{L^{p+1}} = 1\} .$$

So, let the sequence $\{u_m\} \in X_0$ such that:

$$E(u_m) \to \inf_{v \in X_0} E(v), \tag{3.12}$$

where we get:

$$E(u_m) = \|\nabla u_m\|_{L^2}^2 .$$

More specifically, as u_m is bounded in $H_0^1(\Omega)$, hence, by passing to a sub-sequence we can assume there is some $u \in H_0^1$ such that $u_m \rightharpoonup u$.

Because of the fact that H_0^1 is compactly embedded in $L^{p+1}(\Omega)$ we have $u_m \to u$ in L^{p+1} where would obtain:

$$\|u\|_{L^{p+1}} = 1.$$

On the other hand, since norms are weakly lower semi-continuous we would conclude that:

$$\|\nabla u\|_{L^2}^2 \leq \liminf \|\nabla u_m\|_{L^2}^2 ,$$

that we, accordingly, would have:

$$E(u) \leq \liminf \|\nabla u_m\|_{L^2}^2 .$$

Hence, u minimizes E over X_0.

The other point to mention is that:

$$E(|u|) = E(u).$$

Since the both L^{p+1} norm and the gradient norm did not change, we can assume $u \geq 0$ almost everywhere in Ω. Furthermore, the Sobolev embedding theorem demonstrates that E is bounded away from zero and so $E(u) > 0$.

Last but not least, let for a fixed ϕ where $\phi \in C_c^\infty(\Omega)$ and define:

$$g(t) := E(u + t\phi),$$

for $|t|$ small.

We need to illustrate that provided $E(u)$ is a minimizer of energy, g will have a minimum at $t = 0$ and if g is differentiable, then we would have $g'(0) = 0$ where:

$$g(t) = \frac{\|\nabla u + t\nabla\phi\|_{L^2}^2}{\|u + t\phi\|_{L^{p+1}}^2}.$$

We can rewrite $g(t)$ as below:

$$g(t) = \frac{f(t)}{h(t)^{\frac{2}{p+1}}}, \tag{3.13}$$

or equivalently

$$f(t) = g(t) \cdot h(t)^{\frac{2}{p+1}},$$

where by differentiating both sides and setting $g'(0) = 0$ we get:

$$f'(0) = \frac{2}{p+1} h(0)^{\frac{2}{p+1}-1} h'(0) \frac{f(0)}{h(0)^{\frac{2}{p+1}}},$$

and by substituting $t = 0$ we can see that:

$$h(0) = \|u\|_{L^{p+1}} = 1,$$

$$f(0) = \|\nabla u\|_{L^2}^2 = E(u).$$

It should be taken into account that we considered $f(t) = \|\nabla u + t\nabla\phi\|_{L^2}^2$ that gives us the following formulation:

$$f(t) = \int (\nabla u + t\nabla\phi)^2 dx$$

$$f(t) = \int (\nabla u)^2 dx + t^2 \int (\nabla\phi)^2 dx + 2t \int \nabla u \cdot \nabla\phi dx.$$

and some simple calculation for the derivative of $f(t)$ shows that

$$f'(t) = 2t \int (\nabla\phi)^2 dx + 2 \int \nabla u \cdot \nabla\phi dx$$

where

$$f'(0) = 2 \int \nabla u \cdot \nabla\phi dx.$$

So, we will have:

$$\int \nabla u \cdot \nabla\phi dx = \frac{E(u)}{p+1} h'(0) \tag{3.14}$$

Also, we can calculate the differentiation below and set $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u + t\phi|^{p+1}) \mid_{t=0} = (p+1)\,|u|^{p-1}\, u\phi = (p+1)u^p\phi.$$

Therefore, we will have:

$$h'(0) = (p+1) \int_\Omega |u|^p \,\phi dx.$$

Hence, by substituting the resulting term for $h'(0)$ into (3.14) it appears that:

$$\int \nabla u \cdot \nabla\phi dx = E(u) \int_\Omega |u|^p \,\phi dx.$$

Because of the fact that $E(u)$ is a positive number we can consider the following form for the integral above:

$$\int \nabla u \cdot \nabla \phi dx = \int_\Omega E(u) \left| u \right|^p \phi dx.$$

Consequently, using an integration by parts formula and considering the fact that $u = 0$ on the boundary $\partial\Omega$ the following could be presented:

$$\int -\Delta u \phi dx = \int_\Omega E(u) \left| u \right|^p \phi dx.$$

Thus, we will obtain this equation inside Ω:

$$-\Delta u = E(u) \left| u \right|^p$$

It should be pointed out that we had $u \geq 0$ as well as $E(u) > 0$ where we can establish the following equation:

$$-\Delta u = \lambda u^p$$

inside Ω where $\lambda := E(u)$ and $\lambda > 0$.

Now set $v := tu$ for $t > 0$. Then inside Ω we have:

$$-\Delta v = t(-\Delta u) = t\lambda u^p = t^{1-p} \lambda v^p$$

pick t such that

$$t^{1-p} \lambda = 1$$

so in Ω, we would obtain:

$$-\Delta v = v^p$$

and on the boundary $\partial\Omega$ it is gained $v = 0$.

Therefore, we have shown that a multiple of u solves the PDE in (3.1).

□

3.3 Regularity result for Lane-Emden equation

In this section of our research we would like to provide a proof for the regularity results for the given partial differential equation in (3.1) which our main goal is to determine for what p and N we will obtain a positive solution. Along the same line, we present the following theorem where the provided proof is based on a "bootstrapping method" and considering some iterations and an additional proof is, also, suggested based on a different approach that we consider the non-linear PDE in (3.1) to be viewed as a linear PDE. More details on similar works and results could be found in the following [35, 42, 47].

Theorem 25. *[35, 42]*

1. *Let $0 < u \in H_0^1$ denote a weak solution of (3.1) and assume that $1 < p < \frac{N+2}{N-2}$. Then $u \in C^\infty(\Omega)$.*

2. *Let $u \geq 0$ denote a very weak solution of (3.1) and assume $1 < p < \frac{N}{N-2}$. Then $u \in C^\infty(\overline{\Omega})$.*

Proof. In order to prove the theorem we would like to consider two steps where the first step consists of showing that u is bounded and the second step consists of showing that u is smooth.

Step 1. From Sobolev embedding, as $u \in H_0^1$ we get $u \in L^{2^*}$ where $2^* = \frac{2N}{N-2}$. Also, since $u \in L^{2^*}$ we can use Hölder to see that $u^p \in L^T$ for some T. Therefore, we will get:

$$-\Delta u = u^p \in L^T$$

We, accordingly, consider the original PDE in (3.1) and use the L^p regularity theory to see that $u \in W^{2,T}$. Then, since $u \in W^{2,T}$ we are able to use the Sobolev embedding again. Now, two possible cases could be considered:

1. $W^{2,T} \subset L^\infty$ and so we have finished step 1.
2. $W^{2,T}$ is not in L^∞ where we use the Sobolev embedding to see that $W^{2,T} \subset L^{\widetilde{T}}$ for some $\widetilde{T}$.

In fact, $\widetilde{T} > 2^*$ and hence we obtained a useful part of it. So we will iterate this process and advance at each step.

Along the same line, we start off by the first step showing that u is bounded. Assume $u \in L^{T_k}$ where $T_k \geq 2^*$, for $k \geq 1$ some integer. Then, we get:

$$u^p \in L^{\frac{T_k}{p}}$$

and so, by elliptic regularity theory, we obtain:

$$u \in W^{2,\frac{T_k}{p}}$$

If $2 > \frac{Np}{T_k}$ then $u \in L^\infty$ and we are done with step 1. So, assume $2 \leq \frac{Np}{T_k}$. Then, by Sobolev embedding we have:

$$u \in W^{\frac{T_k}{p}} \subset L^{\frac{NT_k}{Np-2T_k}}$$

So we have proven the following result:
If $u \in L^{T_k}$ then $u \in L^\infty$ or $u \in L^{\frac{NT_k}{Np-2T_k}}$.

Now define:

$$T_{k+1} := \frac{NT_k}{Np - 2T_k}$$

So we have proven that: If $u \in L^{T_k}$ then $u \in L^\infty$ or $u \in L^{T_{k+1}}$.

Also, recall that $u \in L^{2^*}$ and so we take $T_1 := 2^*$.

Considering the above iterative formula we can see where the iterates T_k go. Also, the condition on p that we had $p < \frac{N+2}{N-2}$ is what we need for $T_{k+1} > \frac{N_p}{2}$ in a finite number of steps. In this way, after a limited number of iterations we will have:

$$-\Delta u = u^p \in L^{\frac{T_{k+1}}{p}} \subset L^{\frac{N}{2}+\varepsilon},$$

for some $\varepsilon > 0$ small.

Also, another application of L^p regularity theory shows that $u \in W^{2,\frac{N}{2}+\varepsilon}$ but Sobolev embedding shows this in L^∞. Also, in some Hölder space like $u \in C^{0,\gamma}$.

Step 2. In this step we try to show that u is smooth. In fact, we have $-\Delta u = u^p$ and $u \in C^{0,\gamma}$ for some $\gamma > 0$.

As u is in the previously mentioned Hölder space, we will obtain $u^p \in C^{0,\alpha}$ for some $\alpha > 0$ since the composition of a function which is in Hölder space is in Hölder space, as well. Afterwards, by applying Schauder regularity theory it should be witnessed that $u \in C^{2,\alpha}$. Now since $u \in C^{2,\alpha}$ we would be able to see that $u^p \in C^{1,\alpha}$ and so we have $-\Delta u = u^p \in C^{1,\alpha}$.

By applying Schauder regularity theory once again we will observe that $u \in C^{3,\alpha}$. following this procedure we will obtain $u \in C^{k,\alpha}$ for all $k \geq 1$ and so, u is smooth.

$\square$

Furthermore, we can offer the following alternate proof for Theorem 25 where we approach the nonlinear problem in (3.1) as a linear equation.

Proof. Consider $-\Delta u = u^p$ and write it as $-\Delta u = c(x)u$ where $c(x) := u^{p-1}$. Then, we try to prove the following linear regularity result:

If $u \in H_0^1$ is a weak solution of $-\Delta u = c(x)u$ in Ω where $c \in L^{\frac{N}{2}+\varepsilon}$ for some $\varepsilon > 0$ then $u \in L^{\infty}$.

Having this linear result means we should have:

$$\int_{\Omega} c(x)^{\frac{N}{2}+\varepsilon} = \int_{\Omega} u^{(p-1)(\frac{N}{2}+\varepsilon)} < \infty$$

provided we have:

$$(p-1)(\frac{N}{2}+\varepsilon) < \frac{2N}{N-2} \tag{3.15}$$

Since we know that $u \in L^{2^*} = L^{\frac{2N}{N-2}}$. But the inequality (3.15) holds after we recall that $p < \frac{N+2}{N-2}$. Now, we would have $u \in L^{\infty}$ and we will continue as above. In fact, we need an iteration argument to prove the linear result.

□

3.4 Ground state solutions

So far through this chapter we were investigating the problem stated in (3.1) where we had $-\Delta u = u^p$ and the solutions for this equation are well defined only for $u \geq 0$ for certain values of p that we have discussed earlier. Now we try to generalize the equation (3.1) to the following problem [31, 32]

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

which it turns out that there is a possibility of having negative solutions as well. Accordingly, a good question could be whether the sign of solutions of the equation (3.16) could change or not and we try to provide an answer for it. However, a positive solution of (3.16) would be a positive solution of (3.1) too. For more details you can see the following [21, 31, 32].

Definition 22. *[24, 32, 30] A ground state solution for the generalized PDE in (3.16) is one which is a minimizer of the energy functional below*

$$E(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2}. \tag{3.17}$$

This energy is analogous to (3.10) but here the difference is that we have no restriction on the sign of u in the ground state problem (3.16). As we follow the provided proof of existence for (3.1), we have proven that there exists a positive ground state solution for that equation. The next theorem that we have brought from a similar case in [31] shows that all ground state solutions for the above-mentioned generalized equation should be of one sign.

Theorem 26. *[21, 31] A ground state solution of the problem (3.16) is of one sign.*

Proof. Suppose u is a sign changing solution for the PDE (3.16).Therefore, we can write u as below

$$u = u_+ - u_-$$

where

$$u_+, u_- \in H_0^1(\Omega)$$

and both are nonzero.

Therefore, for the energy functional we would obtain

$$E(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2} = \frac{\|\nabla u_+\|_{L^2}^2 + \|\nabla u_-\|_{L^2}^2}{(\|u\|_{L^{p+1}}^{p+1})^{\frac{2}{p+1}}},$$

and equivalently,

$$E(u) = \frac{\|\nabla u_+\|_{L^2}^2 + \|\nabla u_-\|_{L^2}^2}{(\|u_+\|_{L^{p+1}}^{p+1} + \|u_-\|_{L^{p+1}}^{p+1})^{\frac{2}{p+1}}}.$$

We can put some labels on each term in $E(u)$ for ease of calculations and we would have

$$E(u) = \frac{A + B}{(C + D)^{\frac{2}{p+1}}},$$

where all $A, B, C, D > 0$. From this we get

$$A + B = E(u)(C + D)^{\frac{2}{p+1}}.$$

Also, since $0 < \frac{2}{p+1} < 1$, we have

$$(C + D)^{\frac{2}{p+1}} < C^{\frac{2}{p+1}} + D^{\frac{2}{p+1}}.$$

Therefore, we get

$$A + B < E(u)(C^{\frac{2}{p+1}} + D^{\frac{2}{p+1}}) \tag{3.18}$$

Moreover, because of the fact that $E(u)$ is an energy minimizer for the equation (3.16), then for the energy functional of u_+ and u_- we should have

$$E(u) \leq \frac{\|\nabla u_+\|_{L^2}^2}{\|u_+\|_{L^{p+1}}^2} = \frac{\|\nabla u_+\|_{L^2}^2}{(\|u_+\|_{L^{p+1}}^{p+1})^{\frac{2}{p+1}}} = \frac{A}{C^{\frac{2}{p+1}}}$$

and similarly,

$$E(u) \leq \frac{\|\nabla u_-\|_{L^2}^2}{\|u_-\|_{L^{p+1}}^2} = \frac{\|\nabla u_-\|_{L^2}^2}{(\|u_-\|_{L^{p+1}}^{p+1})^{\frac{2}{p+1}}} = \frac{B}{D^{\frac{2}{p+1}}}.$$

From these we would have the following inequalities

$$E(u)C^{\frac{2}{p+1}} \leq A$$

and

$$E(u)D^{\frac{2}{p+1}} \leq B,$$

where by combining the above inequalities together we would obtain

$$E(u)(C^{\frac{2}{p+1}} + D^{\frac{2}{p+1}}) \leq A + B \tag{3.19}$$

Now considering the both inequalities (3.18) and (3.19) together at the same time and would give us to the following

$$A + B < E(u)(C^{\frac{2}{p+1}} + D^{\frac{2}{p+1}}) \leq A + B,$$

which is a contradiction.

Thus, it concludes that a solution for the ground state problem (3.16) should be of one sign and it points out that both u_+ and u_- minimize the energy and so they are the desired solutions.

$\square$

4

An advection in the PDE

In this part of our research, we are interested in evaluating the positive classical solutions of a Lane-Emden type equation where there is an advection term in the equation (3.1) so that it would form as below:

$$\begin{cases} -\Delta u + a(x) \cdot \nabla u &= u^p \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $1 < p < \frac{N+2}{N-2}$ and a is a smooth vector field. We try to investigate the positive solutions of this equation. Some of similar well-studied cases could be found in the following [16, 18, 4, 26].

In the situation where $a = 0$ we would have the same celebrated Lane-Emden equation mentioned in (3.1), where in the previous chapter the existence of positive solutions was analyzed and the regularity of solutions were discussed, as well; see [16, 18, 26].

In the case that $a \neq 0$ the given PDE in (4.1) could not be classified as a variational equation and it would not be possible for us to find the critical points of the corresponding energy associated with (4.1) so that we would be able to identify its positive solutions, see [16, 18, 4, 26].

Along the same line, we would like to consider a certain circumstance where $a(x) = \nabla\gamma(x)$ and γ is a scalar function. Now, we would have a variational equation in (4.1) and for the critical points of the energy related to this equation we would like to present the following minimizer

$$E_\gamma(u) := \frac{\int_\Omega e^{-\gamma} |\nabla u|^2 \, dx}{(\int_\Omega e^{-\gamma} |u|^{p+1} \, dx)^{\frac{2}{p+1}}}$$

Furthermore, it would be shown that the above-mentioned energy is a positive solution to the PDE in (4.1), for more details see the following [16, 4, 26].

4.1 Existence of a positive solution

In order to analyze the existence of a positive solution for the equation (4.1) we would like to present the following theorem. In fact, we have taken advantage of a very useful and well-known approach for this case within our proof of existence namely as weighted L^p spaces, along with their corresponding norm introduced in the equation (2.5) as well as the appropriate weight function for the above-mentioned equation. For more information on some similar problems you can see the following [16, 18, 26].

Theorem 27. *Assume $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ and Ω to be a bounded domain with smooth boundary in $\mathbb{R}^N$. Let $a(x)$ be a smooth bounded function where $a(x) = \nabla\gamma(x)$ and $e^{-\gamma}$ be the weight function defined on Ω. Then, there exists some positive solution $u \in H_0^1(\Omega, e^{-\gamma})$ for the PDE described in (4.1).*

Proof. Straight forward from the previously mentioned in Definition 10 we can consider the weighted L^p norm for the gradient of u to be defined as below

$$\|\nabla u\|_{L^2(\Omega, e^{-\gamma})} = (\int_\Omega e^{-\gamma} |\nabla u|^2 \, dx)^{\frac{1}{2}},$$

where $e^{-\gamma}$ is the weight function.

Consequently, the aforementioned energy functional would be viewed as the following form

$$E_\gamma(u) = \frac{\|\nabla u\|^2_{L^2(\Omega, e^{-\gamma})}}{\|u\|^2_{L^{p+1}(\Omega, e^{-\gamma})}}. \tag{4.2}$$

Now, consider the weighted Hilbert spaces and define X such that:

$$u \in X := H_0^1(\Omega, e^{-\gamma}) \setminus \{0\},$$

and since $E_\gamma(u)$ is a minimizer of energy over the space X we can re-write

$$E_\gamma(u) = \min_{v \in X} E_\gamma(v).$$

Also, for any $u \in X$ and all $t \neq 0$ it is important to notice that we have:

$$E_\gamma(tu) = E_\gamma(u).$$

Hence, for minimizing E_γ over X it is just required to minimize it over the space of functions that their weighted L^{p+1} norm is equal to 1 that could be represented below as X_1:

$$X_1 := \left\{ u \in X : \|u\|_{L^{p+1}(\Omega, e^{-\gamma})} = 1 \right\}.$$

Now consider the sequence $\{u_m\} \in X_1$ so that we have:

$$E_\gamma(u_m) \to \inf_{v \in X_1} E(v), \tag{4.3}$$

which would give us

$$E(u_m) = \|\nabla u_m\|^2_{L^2}.$$

As the sequence $\{u_m\}$ is bounded in $H_0^1(\Omega, e^{-\gamma})$, by passing to a sub-sequence there exists some $u \in H_0^1$ that $u_m \rightharpoonup u$.

Since we have the compact embedding of $H_0^1(\Omega, e^{-\gamma})$ in $L^{p+1}(\Omega, e^{-\gamma})$ we obtain $u_m \to u$ in $L^{p+1}(\Omega, e^{-\gamma})$ that we get:

$$\|u\|_{L^{p+1}(\Omega, e^{-\gamma})} = 1.$$

Because of the fact that norms are weakly lower semi-continuous we may consider the following for the weighted-L^2 norm on ∇u where:

$$\|\nabla u\|^2_{L^2(\Omega, e^{-\gamma})} \leq \liminf \|\nabla u_m\|^2_{L^2(\Omega, e^{-\gamma})}.$$

Or equivalently,

$$E_\gamma(u) \leq \liminf \|\nabla u_m\|^2_{L^2(\Omega, e^{-\gamma})},$$

and so, u minimizes E_γ over X_1.

It should be noted that we can assume $u \geq 0$ almost everywhere in Ω and we have $E_\gamma(u) > 0$ because of Sobolev embedding theorem for weighted L^{p+1} spaces that E_γ is bounded away from zero which is the similar case to the proof of existence that we demonstrated in the previous chapter with $a = 0$.

Along the same line, for a fixed ϕ where $\phi \in C_c^\infty(\Omega)$ consider:

$$g(t) := E_\gamma(u + t\phi)$$

for $|t|$ small.

Furthermore, in order to show that $E_\gamma(u)$ is an energy minimizer we should have $g'(0) = 0$ provided g is differentiable and has a minimum at $t = 0$.

That being said, $g(t)$ could be displayed in the following form in (4.2):

$$g(t) = \frac{\|\nabla u + t\nabla\phi\|^2_{L^2(\Omega, e^{-\gamma})}}{\|u + t\phi\|^2_{L^{p+1}(\Omega, e^{-\gamma})}}.$$

Considering the appropriate $f(t)$ and $h(t)$ we would be able to represent $g(t)$ as below which is remarkably in the similar form of (3.13) but with different functions for $f(t)$ and $h(t)$:

$$g(t) = \frac{f(t)}{h(t)^{\frac{2}{p+1}}} \tag{4.4}$$

or equivalently:

$$f(t) = g(t) \cdot h(t)^{\frac{2}{p+1}}$$

taking derivative from the above equation and since $g'(0) = 0$ we get:

$$f'(0) = \frac{2}{p+1} h(0)^{\frac{2}{p+1}-1} h'(0) \frac{f(0)}{h(0)^{\frac{2}{p+1}}}. \tag{4.5}$$

Also, for $f(0)$ and $h(0)$ we would have:

$$h(0) = \|u\|_{L^{p+1}(\Omega, e^{-\gamma})} = 1,$$

$$f(0) = \|\nabla u\|^2_{L^2(\Omega, e^{-\gamma})} = E_\gamma(u).$$

Since we decided $f(t)$ to be the following norm

$$f(t) = \|\nabla u + t\nabla\phi\|^2_{L^2(\Omega, e^{-\gamma})}$$

which is equal to the integral below

$$f(t) = \int_\Omega e^{-\gamma}(\nabla u + t\nabla\phi)^2 dx$$

and could be written in terms of the sum of following integrals:

$$f(t) = \int_{\Omega} e^{-\gamma}(\nabla u)^2 dx + t^2 \int_{\Omega} e^{-\gamma}(\nabla \phi)^2 dx + 2t \int_{\Omega} e^{-\gamma} \nabla u \cdot \nabla \phi dx.$$

So, for the derivative of $f(t)$ we would get:

$$f'(t) = 2t \int_{\Omega} e^{-\gamma}(\nabla \phi)^2 dx + 2 \int_{\Omega} e^{-\gamma} \nabla u \cdot \nabla \phi dx$$

and by setting $t = 0$ we see that the first term equals to zero and we get

$$f'(0) = \int_{\Omega} e^{-\gamma} \nabla u \cdot \nabla \phi dx.$$

Now, from (4.5) and the appropriate values for $f(0)$ and $h(0)$ as well as the above equation we can confirm that:

$$\int_{\Omega} e^{-\gamma} \nabla u \cdot \nabla \phi dx = \frac{E_{\gamma}(u)}{p+1} h'(0). \tag{4.6}$$

Moreover, we would require the following differentiation at some point further in our calculations for the case when $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u + t\phi|^{p+1} e^{-\gamma}) \mid_{t=0} = e^{-\gamma}(p+1) |u|^{p-1} u\phi = e^{-\gamma}(p+1) u^p \phi.$$

Also, for $h'(0)$ we could have the following equation to put in (4.6)

$$h'(0) = (p+1) \int_{\Omega} e^{-\gamma} |u|^p \phi dx.$$

Therefore, by plugging the right hand side of the above equality for $h'(0)$ in the equation (4.6) we can conclude that:

$$\int_\Omega e^{-\gamma}\nabla u\cdot\nabla\phi dx = \frac{E_\gamma(u)}{p+1}(p+1)\int_\Omega e^{-\gamma}\left|u\right|^p\phi dx$$

which is equivalent to

$$\int_\Omega e^{-\gamma}\nabla u\cdot\nabla\phi dx = E_\gamma(u)\int_\Omega e^{-\gamma}\left|u\right|^p\phi dx. \tag{4.7}$$

Furthermore, it is significant to mention that from the original PDE in (4.1) and the substitution of $a(x) = \nabla\gamma(x)$ we know that

$$-\Delta u + \nabla\gamma(x)\cdot\nabla u = u^p.$$

We, accordingly, just need to show that

$$\int_\Omega e^{-\gamma}\nabla u\cdot\nabla\phi dx = E_\gamma(u)\int_\Omega e^{-\gamma}(-\Delta u + \nabla\gamma(x)\cdot\nabla u)\phi dx.$$

Thus, considering an integration by parts for the left hand side of the above equation the following would be attained:

$$\int_\Omega e^{-\gamma}\nabla u\cdot\nabla\phi dx = \int_{\partial\Omega} e^{-\gamma}\nabla u\phi\cdot\hat{n}d\Gamma - \int_\Omega\left[e^{-\gamma}\cdot\nabla u\right]'\phi.$$

and since the term on the boundary is equal to zero, as $\phi = 0$ on the boundary, we would have

$$\int_\Omega e^{-\gamma}\nabla u\cdot\nabla\phi dx = -\int_\Omega(-e^{-\gamma}\nabla\gamma\cdot\nabla u + e^{-\gamma}\Delta u)\phi.$$

Therefore, considering the equation (4.7) we would obtain

$$\int_\Omega(e^{-\gamma}\nabla\gamma\cdot\nabla u - e^{-\gamma}\Delta u)\phi = E_\gamma(u)\int_\Omega e^{-\gamma}\left|u\right|^p\phi dx.$$

So, it is equivalent to

$$\int_\Omega e^{-\gamma}(\Delta u + \nabla\gamma \cdot \nabla u)\phi = E_\gamma(u) \int_\Omega e^{-\gamma} |u|^p \phi dx.$$

Thus, the following equation could be obtained for inside Ω:

$$-\Delta u + \nabla\gamma(x) \cdot \nabla u = E_\gamma(u) |u^p| .$$

We should note that we knew $u \geq 0$ and $E_\gamma(u) > 0$ where the following equation could be confirmed:

$$-\Delta u + a \cdot \nabla u = \lambda u^p$$

in Ω that $\lambda := E_\gamma(u)$ and $\lambda > 0$.

Now set $w := tu$ for $t > 0$. Then inside Ω we have:

$$-\Delta w + a \cdot \nabla w = t(-\Delta u + a \cdot \nabla u) = t\lambda u^p = t^{1-p}\lambda w^p$$

By choosing the appropriate t such that the coefficient of w^p becomes equal to one we can see that

$$t^{1-p}\lambda = 1$$

and for inside Ω we would obtain:

$$-\Delta w + a \cdot \nabla w = w^p$$

where on the boundary $\partial\Omega$ we have $w = 0$.

Therefore, it is proven that a multiple of u solves the PDE in (4.1).

$\square$

4.2 Regularity result for the PDE containing advection

In this part of our research we would like to discuss about the regularity of the solutions of the PDE containing some advection term in (4.1). We first begin with a linear case of this equation where we have $a(x) = \nabla\gamma(x)$ and $f \in L^p(\Omega)$ and the following equation holds

$$\begin{cases} -\Delta u + \nabla\gamma \cdot \nabla u &= f(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Theorem 28. *Let X_p be defined as below*

$$X_p = \left\{ u \in W^{2,p}(\Omega) : u \mid_{\partial\Omega} = 0 \right\},$$

Also, consider the operators T and K be the following

$$T(u) := -\Delta u$$

$$T : X_p \to L^p(\Omega),$$

and

$$K := \nabla\gamma \cdot \nabla u,$$

$$K : X_p \to L^p(\Omega).$$

Then, the operator $T + K : X_p \to L^p(\Omega)$ is a Fredholm operator with the same index of T and for all $f \in L^p(\Omega)$ and there exists a unique solution of the linear equation (4.8) and we will have the estimate below that states there exists some

$C > 0$ *such that*

$$\|u\|_{W^{2,p}} \leq C\|f\|_{L^p}. \tag{4.9}$$

Proof. Straight forward from Definition 19, T is a Fredholm operator of index zero.

Also, let

$$K := \nabla\gamma \cdot \nabla u,$$

$$K : X_p \to L^p(\Omega)$$

where γ is a scalar function defined earlier and $u \in W^{2,p}(\Omega)$.

We claim that K is a compact operator and this is proven by considering Definition 20 and Theorem 19.

Now that we have a Fredholm operator T and a compact operator K we take advantage of Theorem 20 which implies the operator $T + K$ is also a Fredholm operator with the same index of T. In other words,

$$ind(T + K) = ind(T) = 0.$$

For showing $T + K$ is onto, meaning $Im(T + K) = L^p(\Omega)$, we know that the kernel is trivial and the index for $T + K$ is zero. So we have

$$ker(T + K) = \{0\},$$

which means

$$dim(ker(T + K)) = 0;$$

and from the index being zero we know

$$dim(ker(T + K)) = dim(coker(T + K)).$$

So,

$$dim(coker(T+K)) = 0.$$

The cokernel is the quotient space $L^p(\Omega)/Im(T+K)$ and we have shown that its dimension is zero and since $T+K$ is a Fredholm operator it has closed range which we would have

$$L^p(\Omega) = Im(T+K),$$

and so, $T+K$ is onto and so we can apply the bounded inverse theorem to claim that it has a bounded inverse.

In order to prove the desired estimate let us give a proof by contradiction. Assume the inequality is false; so, there exists some sequence $u_m \in X_p$ and $f_m \in L^p(\Omega)$ such that

$$(T+K)u_m = f_m$$

and

$$\|u_m\|_{W^{2,p}} > m\|f_m\|_{L^p}, \tag{4.10}$$

and normalize

$$T(\frac{u_m}{\|u_m\|_{W^{2,p}}}) + K(\frac{u_m}{\|u_m\|_{W^{2,p}}}) = \frac{f_m}{\|u_m\|_{W^{2,p}}},$$

which we define

$$\frac{f_m}{\|u_m\|_{W^{2,p}}} := \hat{f}_m,$$

and

$$\hat{u}_m = \frac{u_m}{\|u_m\|_{W^{2,p}}}.$$

Therefore, we get

$$T(\hat{u}_m) + K(\hat{u}_m) = \hat{f}_m, \tag{4.11}$$

and

$$\|\hat{u}_m\|_{W^{2,p}} = 1,$$

and from (4.10) we will get

$$\|\hat{f}_m\|_{L^p} \leq \frac{1}{m}.$$

Now as $m \to \infty$ we get $\hat{f}_m \to 0$ in $L^p(\Omega)$ and from (4.11) we would have

$$T(\hat{u}_m) = 0 - K(\hat{u}_m),$$

As K is compact, it has a convergent subsequence in $L^p(\Omega)$ and so by passing to a subsequence $T(\hat{u}_m)$ converges in $L^p(\Omega)$; that is, $\hat{u}_m$ has a convergent subsequence in $W^{2,p}(\Omega)$ as

$$\hat{u}_m \to \hat{u},$$

and $\|\hat{u}\|_{W^{2,p}} = 1$. Therefore,

$$T(\hat{u}) + K(\hat{u}) = 0,$$

which is a contradiction since we know that $T + K$ has trivial kernel.

$\square$

Now we will go back to the nonlinear problem which we had

$$\begin{cases} -\Delta u + \nabla\gamma \cdot \nabla u &= u^p \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

It should be noted that the regularity of the solutions could be available through recalling the advection free case in the previous chapter that we will bring the following theorem.

Theorem 29. *For $u \in H_0^1(\Omega)$, which is a weak solution of the equation (4.1), and $1 < p < \frac{N+2}{N-2}$. Then, we have $u \in C^\infty(\Omega)$.*

Proof. We would need to recall the bootstrapping proof provided for the previous case where there is no advection.

First we should note that $u \in H_0^1$ implies $u \in L^{2^*}$ from Sobolev embedding and $2^* = \frac{2N}{N-2}$ and so we can use Hölder to see that $u^p \in L^t$ for some t which means

$$\int_\Omega u^{pt} < \infty,$$

and so $pt = 2^*$ or

$$t = \frac{2^*}{p}.$$

So,

$$u^p \in L^{\frac{2^*}{p}},$$

where we should note that $\frac{2^*}{p} > 1$.

Moreover we have

$$-\Delta u = u^p - \nabla\gamma \cdot \nabla u, \tag{4.12}$$

where $\nabla u \in L^2(\Omega)$ and $u^p \in L^{\frac{2^*}{p}}$.

Therefore, we obtain

$$\Delta u \in L^{\min\left\{\frac{2^*}{p},2\right\}},$$

that we would like to set

$$\min\left\{\frac{2^*}{p}, 2\right\} = \hat{t},$$

and so,

$$\Delta u \in L^{\hat{t}}.$$

Then, we use the L^p regularity theory to see that $u \in W^{2,\hat{t}}$. Now, we will see that $\nabla u \in W^{1,\hat{t}}$ and we should note that from Sobolev embedding $W^{1,\hat{t}} \subset L^t$ and $u^p \in L^{\hat{t}}$. Therefore, from the right hand side of the equation (4.12) we will get

$$\Delta u \in L^{\min\left\{\hat{t},t\right\}},$$

that we set the exponent to be

$$\min\left\{\hat{t}, t\right\} = \hat{t_1},$$

and similar to the base step, we will obtain $u \in W^{2,\hat{t_1}}$.

Following this iterative bootstrapping process and using L^p regularity theory we will get $u \in W^{2,\hat{t_k}}$ and $p < \frac{N+2}{N-2}$ is the condition we need to conclude that $\frac{2N}{N+2} < \hat{t_k}$ which, in fact, gives $\frac{N}{2} < \hat{t_k}$.

Therefore, we will get $u \in L^\infty$ from Sobolev embedding, and is similar to the case we studied in the previous chapter.

$\square$

5

A fourth order Lane-Emden equation

In this part of our research we are interested in examining the following nonlinear problems given by

$$\text{(Dirichlet problem)} \quad \begin{cases} \Delta^2 u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

and

$$\text{(Navier problem)} \quad \begin{cases} \Delta^2 u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

In order that we might be able to argue about these PDEs, we would like to introduce some similar linear problems and some useful function spaces. There have been a number of research works done on these fourth order equations that we can attract the readers attention to see the following for more details [33, 27]

5.1 Some linear problems and the required function spaces

We consider the following linear problems given by

$$\begin{cases} \Delta^2 u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \Delta^2 u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where we assume Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, see [33, 53]. It deserves attention that the equation $\Delta^2 u = f$ is widely known as the biharmonic equation. Also, the linear equation in (5.3) is called clamped plate equation and (5.4) is known as hinged plate equation in two dimensional cases in $\mathbb{R}^2$, for more details see the following [33].

5.1.1 Linear fourth order Navier problem

We begin with Navier linear problem stated in (5.4) and we are interested in investigating the existence and regularity results for this equation and we have the following theorem.

Theorem 30. *(Navier) [33] Let $1 < p < \infty$ and $f \in L^p(\Omega)$. Then there is some $C_p > 0$ and some unique $u \in W^{4,p}(\Omega)$ which solves (5.4). Moreover, one has $\|u\|_{W^{4,p}} \leq C_p \|f\|_{L^p}$.*

Proof. Because of the fact that

$$\Delta^2 u = -\Delta(-\Delta u)$$

we can consider the following change of variable

$$-\Delta u = v$$

where $v \in W^{2,p}(\Omega)$.

Therefore, we can decompose the PDE in (5.4) and treat it as a system so that we obtain the following equations

$$\begin{cases} -\Delta u &= & v & \text{in } \Omega, \\ u &= & 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

and

$$\begin{cases} -\Delta v &= & f(x) & \text{in } \Omega, \\ v &= & 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

For more details see the following [33, 38, 54].

Hence, we have reduced the problem to some second order linear problem with Dirichlet boundary condition and there are some well-studied existence results that we have discussed earlier in Theorem 7 and the corresponding regularity results. In fact, we are applying the second order theory twice to give a proof for this theorem.

Therefore, the existence of a unique $v \in W^{2,p}(\Omega)$ for the PDE (5.6) is obtained. For the regularity of the solutions for this equation we have

$$\|v\|_{W^{2,p}} \leq C_1 \|f\|_{L^p} \tag{5.7}$$

for some $C_1 > 0$.

Similarly, in the other linear Dirichlet problem stated in (5.5) we can consider the function $v \in W^{2,p}(\Omega)$ as a new f for the PDE and so the existence of a unique $u \in W^{4,p}(\Omega)$ would be guaranteed. Also, for this equation we will need the some higher order estimate for the L^p regularity result related to the first order Dirichlet problem.

Therefore, for the problem (5.5), by recalling Theorem 9, we will obtain

$$\|u\|_{W^{4,p}} \leq C_2 \|v\|_{W^{2,p}}, \tag{5.8}$$

for some $C_2 > 0$.

Thus, combining inequalities (5.7) and (5.8) we would have the following

$$\|u\|_{W^{4,p}} \leq C_2 \|v\|_{W^{2,p}} \leq C_2 C_1 \|f\|_{L^p}.$$

Hence, by interchanging $C_1 C_2 > 0$ with $C_p > 0$ we would obtain

$$\|u\|_{W^{4,p}} \leq C_p \|f\|_{L^p}.$$

$\square$

5.1.2 Linear fourth order Dirichlet problem

Theorem 31. *(Dirichlet)[33] Let $f \in H^{-2}(\Omega)$. Then there is some $C > 0$ and some unique $u \in H_0^2(\Omega)$ which solves (5.3) and we have $\|u\|_2^* \leq C\|f\|_{-2}$.*

Proof. We may recall Lax-Milgram theorem stated in Theorem 2 and consider the following bilinear form on $H_0^2(\Omega)$ for the equation (5.3)

$$B(u, v) = \int_\Omega \Delta u \Delta v, \tag{5.9}$$

where we would show later that the bilinear form B in (5.9) is bounded and coercive and we will be able to apply Lax-Milgram theorem to obtain the desired regularity result.

For the first step, by applying Green's identities we can see that the weak form for the equation (5.3) would be

$$B(u, v) = \langle f, v \rangle , \tag{5.10}$$

for all $u, v \in H_0^2(\Omega)$ and $f \in H^{-2}(\Omega)$.

Also, as we are working on $H_0^2(\Omega)$ we would like to bring the appropriate norm for this space. Recall for $m \geq 1$ we had

$$\|u\|_{H_0^m} = \|u\|_m^* := (\sum_{|\alpha|=m} \|D^\alpha u\|^2)^{\frac{1}{2}}.$$

In order to show that the weak form B is bounded it would be sufficient to check the following

$$|B(u, v)| \leq \alpha \|u\|_{H_0^2} \|v\|_{H_0^2},$$

for $u, v \in H_0^2(\Omega)$ and $\alpha > 0$.

Recalling Riesz representation theorem expressed in Theorem 1 we will have

$$B(u, v) = \langle f, v \rangle = (u, v),$$

where $(,)$ denotes the inner product on $H_0^2(\Omega)$. So we need to check if the following holds

$$|B(u, v)| = |(u, v)| \leq \alpha \|u\|_{H_0^2} \|v\|_{H_0^2},$$

which is directly obtained from Cauchy-Schwarz inequality when considering $\alpha = 1$. Therefore, B is bounded.

In order to show that B is coercive we should check the following

$$|B(u,u)| \geq \beta \|u\|_{H_0^2},$$

for $u \in H_0^2(\Omega)$ and $\beta > 0$.

It should be noted that

$$B(u,u) = \int_\Omega (\Delta u)^2 = \|\Delta u\|_{L^2}^2.$$

Plus, from Poincare-Friedrichs inequality stated in Theorem 18

$$\|u\|_{H_0^2}^2 \leq c \int_\Omega |\Delta u|^2 = c\|\Delta u\|_{L^2}^2,$$

for some $c > 0$.

Therefore, by setting $\beta = \frac{1}{c} > 0$ we have

$$\beta \|u\|_{H_0^2}^2 \leq \|\Delta u\|_{L^2}^2 = B(u,u),$$

which implies B is coercive.

Consequently, directly from Lax-Milgram theorem we can conclude that there will be a unique solution $u \in H_0^2(\Omega)$ for the PDE (5.3) and we have

$$\|u\|_2^* \leq C\|f\|_{-2}.$$

□

5.1.3 Energy minimizer for the both equations

In this section we are interested in finding the associated energy for the above-mentioned equations. A detailed work for a polyharmonic boundary value problem with different boundary conditions has been discussed in the reference book [33] and the authors have pointed out that the associated energy for the PDEs (5.4) and (5.3) would be as below

$$E(u) := \int_\Omega (\frac{1}{2}(\Delta u)^2 - fu)dx. \tag{5.11}$$

Also, some more details could be found in the following [27, 33]

As mentioned in the same reference, for the hinged plate equation (i.e. the equation with Navier boundary condition) we should minimize this energy on $H_0^2(\Omega)$; however, we should try minimizing it over $H^2 \cap H_0^1(\Omega)$ for the clamped plate equation which we refer to the equation with Dirichlet boundary condition. [33] In fact, we would like to take advantage of the energy functional (5.13) in showing the existence of a solution for the nonlinear equations stated in (5.1) and (5.2), so we will continue our discussion in the next few sections.

5.1.4 Maximum principle results for the linear problems

Here we would like to begin with the linear Navier problem (5.4) and try to inspect whether $f(x) \geq 0$ leads to have $u \geq 0$ or not. Very similar approaches for related PDEs could be found in references [33, 14].

We begin with the following definition for a weak solution for the equation (5.4).

Definition 23. *[33] For $f \in H^{-2}(\Omega)$ we say u is a weak solution of (5.4) for $u \in H^2 \cap H_0^1(\Omega)$ and*

$$\int_\Omega \Delta u \Delta v = \int_\Omega f v dx,$$

for all $v \in H^2 \cap H_0^1(\Omega)$.

Also, we continue by defining the following energy functional for the equation (5.4) over the space $H^2 \cap H_0^1(\Omega)$

$$I(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 - \int_\Omega f u dx, \tag{5.12}$$

for all $u \in H^2 \cap H_0^1(\Omega)$.

Theorem 32. *(Maximum principle for linear fourth order Navier problem) [33]*

For $N \geq 2$ let Ω be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^2$. If $f \geq 0$ then $u \geq 0$.

Proof. Recall we can seperate the linear fourth order Navier problem into two second order Dirichlet problems and so we would have the equations (5.5) and (5.6) at the same time. So for the equation (5.6) we have

$$\begin{cases} -\Delta v &= f(x) \qquad \text{in } \Omega, \\ v &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

where by weak maximum principle $f \geq 0$ implies that

$$\min_{v \in \bar{\Omega}} v = \min_{v \in \partial\Omega} v,$$

and so $v \geq 0$.

Now consider the equation (5.5) as below

$$\begin{cases} -\Delta u &= v \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

and apply weak maximum principle for the second order Dirichlet problem once again to see that $v \geq 0$ implies $u \geq 0$ and we are done.

□

On the other hand, it is very significant to mention that for Ω a bounded domain in $\mathbb{R}^N$, it has not yet proven whether or not the Dirichlet problem (5.3) could necessarily satisfy some maximum principle (in other words, a positivity) where for some $f(x) \geq 0$ we get $u \geq 0$. In fact, the case for general domains could be still an open problem, see the following for more details [33, 14]. However, there have been a multitude of research on different types of certain domains showing the linear Dirichlet problem (5.3) could hold some positivity feature in those domains. For instance we can refer to [6] where a weak maximum principle on the unit ball is achieved by Boggio in 1905.

5.2 The nonlinear equations

Now that we have brought adequate essential information about the more general linear cases, we would like to discuss about some similar nonlinear equations with Navier boundary condition as well as Dirichlet boundary conditions which are introduced in (5.2) and (5.1), respectively. It should be noted that in order to talk about the both problems we should take advantage of Sobolev embedding theorem stated in Theorem 5.

5.2.1 Existence of a solution for the Dirichlet problem

Here we are interested in finding the existence result for the nonlinear Dirichlet problem (5.1) and so we can consider the following theorem.

Theorem 33. *([33] pages 237-238) Assume Ω be a smooth bounded domain in $\mathbb{R}^N$ and $1 < p < \frac{N+4}{N-4}$. Then, the Dirichlet problem (5.1) has a nontrivial solution $u \in H_0^2(\Omega)$.*

Proof. [33, 38] In order to show the existence for the PDE (5.1) we minimize the following energy functional

$$E(u) := \int_\Omega (\Delta u)^2 dx = \|\Delta u\|_{L^2}^2 \tag{5.13}$$

over $H_0^2(\Omega)$ for all $u \in X_0$ such that

$$X_0 := \{u \in H_0^2(\Omega) : \int_\Omega |u|^{p+1} dx = 1\}.$$

So it would be sufficient to minimize $E(u)$ over X_0. Then, consider the sequence $\{u_m\} \in X_0$ such that

$$E(u_m) \to \inf_{v \in X_0} E(v),$$

and so we obtain

$$E(u_m) = \|\Delta u_m\|_{L^2}^2$$

Also, a very similar problem was proven in page 238 [33] that since the functional E is coercive, any minimizing sequence in X_0 like $\{u_m\}$ is bounded in $H_0^2(\Omega)$. So we would be able to pick a weakly convergent subsequence in X_0 such that

$$u_m \rightharpoonup \hat{u},$$

for some $\hat{u} \in H_0^2(\Omega)$.

Also, as u_m is bounded in $H_0^2(\Omega)$, hence, by passing to a subsequence we can assume there is some $\hat{u} \in H_0^2(\Omega)$ such that $u_m \rightharpoonup u$.

Therefore, from Sobolev embedding theorem, Theorem 5, $H_0^2(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$ and $H_0^2(\Omega)$ is compactly embedded in $L^2(\Omega)$. Therefore, we have $u_m \to \hat{u}$ in L^{p+1}. Now from this and because of lower semi-continuity of E we can conclude that $\hat{u} \in X_0$ is a minimizer for E in X_0 and let us label this as $E(u) = K$. By substituting the following in the equation (5.1)

$$u = \hat{u} K^{\frac{1}{p-1}},$$

we have obtained a solution for this problem.

□

5.2.2 Existence of a solution for the Navier equation

Now consider the nonlinear Navier problem stated in (5.2) and we try to prove the existence of some solution for this problem as the following theorem.

Theorem 34. *([33] pages 237-238) Assume Ω be a smooth bounded domain in $\mathbb{R}^N$ and $1 < p < \frac{N+4}{N-4}$. Then, the Navier problem (5.2) has a nontrivial solution $u \in H^2 \cap H_0^1(\Omega)$.*

Proof. [33, 38] Similar to the provided proof in the previous existence theorem for the Dirichlet problem, Theorem 33, we minimise exactly the same energy functional (5.13),

$$E(u) := \int_\Omega (\Delta u)^2 dx = \|\Delta u\|_{L^2}^2,$$

over the space $H^2 \cap H_0^1(\Omega)$ which is the suitable space for this problem for all $u \in X_1$ where

$$X_1 := \{u \in H^2 \cap H_0^1(\Omega) : \int_\Omega |u|^{p+1} dx = 1\}.$$

Now we just need to minimize $E(u)$ over X_1. Assuming a sequence $\{u_m\} \in X_1$ such that

$$E(u_m) \to \inf_{v \in X_1} E(v),$$

where

$$E(u_m) = \|\Delta u_m\|_{L^2}^2$$

And so, similar to Theorem 33, any minimizing sequence in $\{u_m\} \subset X_1$ is bounded in $H^2 \cap H_0^1(\Omega)$. We can take a subsequence in X_1 which is weakly convergent to some $\tilde{u}$ in $\in H^2 \cap H_0^1(\Omega)$ such that

$$u_m \rightharpoonup \tilde{u}.$$

Therefore, from Sobolev embedding theorem, Theorem 5, $H^2 \cap H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$ and $H^2 \cap H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. Thus, we have $u_m \to \tilde{u}$ in L^{p+1}. Now from lower semi-continuity of E and the above-mentioned proof, we can conclude that $\tilde{u} \in X_1$ is a minimizer for E in X_1 and set $E(u) = M$. By plugging the following in the equation (5.2)

$$u = \tilde{u} M^{\frac{1}{p-1}},$$

we have a solution for this problem.

$\square$

6

Conclusion

In this thesis, we studied the celebrated Lane-Emden equation and some of its remarkable variations in a bounded domain in $\mathbb{R}^N$. More specifically, we are interested in answering the question whether the given problem of Lane-Emden type admits the existence of a positive solution or not; moreover, we would like to examine if the solutions are satisfying certain regularity conditions under the appropriate constraints.

Firstly, we have shown that there would be no solution for the general Lane-Emden equation if Ω is star-shaped and $p > \frac{N+2}{N-2}$. Also, we have shown that for $1 < p < \frac{N+2}{N-2}$ there would be a positive solution for this problem in general domains; also, some regularity results are obtained for $1 < p < \frac{N+2}{N-2}$ and $1 < p < \frac{N}{N-2}$ through a bootstrapping iterative proof and we noticed that we can look in a different way so that we may be able to provide another proof for this issue. We note that for a more general case we can admit existence of a negative solution, but not a sign changing solution.

Moreover, in the case of having an advection term in the original equations we discussed the problem where $a = \nabla\gamma$ for some scalar function $\gamma(x)$ and we proposed the existence result using the notion of weighted L^p spaces. The regularity result for this problem is attained through Fredholm alternative and we have provided two proofs for this problem while they seem to share many similarities.

Furthermore, for studying fourth order nonlinear Navier and Dirichlet problems we begin with linear versions of each equation and propose some regularity results for each case. In addition, we have proceeded to the nonlinear problems and investigate the existence of a solutions which, in fact, is a minimizer of energy like above-mentioned second order problems. It should be remarked that we have offered a proof of a maximum principle for the fourth order linear problem, in spite of the fact that a maximum principle for the linear Dirichlet case does not necessarily hold in general bounded domains in $\mathbb{R}^N$ and could be an open problem.

We would like to suggest a number of possible interesting directions for further research on this topic listed as below.

1. Having different assumptions on $a(x) \neq 0$ for the equation containing advection; like letting it to be a divergence free vector field.

2. Considering more generalized higher order problems with multiple boundary conditions and eigenvalues.

3. Various conditions could be imposed on the geometry of the domain which many different results might be obtained.

4. Attempting to find some maximum principle for the fourth order Dirichlet problem on either bounded or unbounded domains.

Appendix A

Table of norms

Within our research we remark that this would be particularly useful to illustrate the following table including the norms defined for different spaces with their corresponding notation along with a brief definition.

Space	Norm notation	Definition	Conditions
Banach space X	$\|u\|_X$	$\sup_{x\in\Omega}\|u(x)\| = \max_{x\in\overline{\Omega}}\|u(x)\|$	
Dual space	$\|f\|' = \|f\|_{X'}$	$\sup_{x\neq 0}\frac{\|f(x)\|}{\|x\|_X}$	
$L^2(\Omega)$	$\|f\|_{L^2}$	$(\int_\Omega \|f(x)\|^2 dx)^{\frac{1}{2}} < \infty$	
$L^p(\Omega)$	$\|f\|_{L^p}$	$(\int_\Omega \|f(x)\|^p dx)^{\frac{1}{p}} < \infty$	$1 \leq p < \infty$
$W^{k,p}(\Omega)$	$\|u\|_{W^{k,p}(\Omega)}$	$\left(\sum_{\|\alpha\|\leq k}\int_\Omega \|D^\alpha u\|^p\, dx\right)^{\frac{1}{p}}$	$1 \leq p < \infty$
$W^{k,\infty}(\Omega)$	$\|u\|_{W^{k,\infty}(\Omega)}$	$\sum_{\|\alpha\|\leq k}\|D^\alpha u\|_\infty$	$p = \infty$
$H^{-1}(\Omega)$	$\|f\|_{H^{-1}(\Omega)}$	$\sup\left\{\langle f, u\rangle \mid u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1\right\}$	
$C^{k,\gamma}(\overline{\Omega})$	$\|u\|_{C^{k,\gamma}(\overline{\Omega})}$	$\sum_{\|\alpha\|\leq k}\|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{\|\alpha\|=k}[D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})}$	
$L^p(\Omega, w(x))$	$\|f\|_{L^p(\Omega,w)}$	$(\int_\Omega \|f\|^p\, w(x)dx)^{\frac{1}{p}} < \infty$	$1 \leq p < \infty$
$H_0^m(\Omega)$	$\|u\|_m^*$	$(\sum_{\|\alpha\|=m}\|D^\alpha u\|^2)^{\frac{1}{2}}$	$m \geq 1$

Table A.1: Table of norms for various spaces

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