Left-Orderable Groups and Amenability

by

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Abstract

This thesis provides a review of left-ordered groups and amenable groups. These are used to investigate a conjecture by Peter Linnell, which relates the existence of a nonabelian free group to a strengthening of left-orderability, by examining three research articles. Lastly, we propose a possible generalisation of a theorem by Dave Witte Morris and Peter Linnell.

Acknowledgment Page

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Chapter 1

Introduction to Orderable Groups

1.1 Introduction

Left-orderable groups have received a great deal of attention in the last century. A group G is left-orderable if we can give all its elements a strict total ordering $<$ which is left-invariant, i.e., $g < h$ implies $fg < fh$ for all $g, h, f \in G$. Some notable mathematicians who work in this field are O. Hölder, P. Dehornoy, and A. A. Vinogradov. They discovered that free groups, torsion-free abelian groups, fundamental groups of many important spaces, and braid groups are left-orderable. A group having an ordering implies powerful algebraic properties. For example, left-orderable groups satisfy Kaplansky's zero-divisor conjecture [31, Chapter 9] . In Chapter 1, we give an example of a fundamental group of a space which is torsion-free but not left-orderable, and in Chapter 5 we give an example of a torsion-free group which is not left-orderable.

This thesis begins with a review of left-orderable groups and amenable groups. The first three chapters of the thesis are dedicated to an introduction of some well-known results of left-ordered groups, the space of left-ordered groups, and amenable groups, respectively. The fourth chapter aims to look at what has been done in linking amenable groups to Conradian left-ordered groups by providing a review of three research papers which investigate the following conjecture by P. Linnell:

Conjecture 1.1. *[23, Conjecture 1.1] Let $(G, <)$ be a left-ordered group which does not contain a non-Abelian free subgroup. Then G is Conradian left-orderable.*

In the last chapter, we proposed a generalisation of a theorem by D. Witte Morris and P. Linnell by weakening a condition known as "local invariance".

In this chapter, we introduce orderable groups with an emphasis on left-orderable groups. Furthermore, we will give some algebraic properties of orderable groups and introduce Archimedean and Conradian orderings.

There are several useful references on orderable groups, for example, *Orderable Groups and Topology* by Clay and Rolfsen [9], *Groups, Orders, and Dynamics* by Deroin, Navas and Rivas [14], and *Orderable groups* by Mura and Rhemtulla [3]. Many results which will not be discussed here can be found in these books.

1.2 Definitions

Definition 1.1. *[9, Chapter 1] A group G is left-orderable if we can give all its elements a strict total ordering $<$ which is left invariant, i.e., $g < h$ implies $fg < fh$ for all $g, h, f \in G$.*

Remark 1.1. *We denote by $(G, <)$ a group G with an ordering $<$ and the identity element by* 1*.*

We can ask ourselves if there exists a notion of right-orderability. We say that a group G is right-orderable if we can give all the elements of G a strict total ordering $<$ which is right-invariant. We have that left-orderable groups are right-orderable as well; however, the orders might be different (see Proposition 1.1). Furthermore, if the ordering of a group is right and left invariant, we say that the group is *bi-orderable.*

1.3 Properties

What properties do left-orderable groups have? Let us investigate some of them.

Proposition 1.1. *[9, Chapter 1, Problem 1.2] Let $(G, <)$ be a left-ordered group, then $\prec$ defines a right ordering $g \prec h$ if and only if $h^{-1} < g^{-1}$.*

Proof. Let $g, h \in G$ such that $g \prec h$. We want to show that $\prec$ is right-invariant. Let $c \in G$ be given, then

$$\begin{aligned} g \prec h &\iff h^{-1} < g^{-1} \\ &\iff ch^{-1} < cg^{-1} \qquad \text{(multiplying by } c \in G \text{ on the left)} \\ &\iff (cg^{-1})^{-1} \prec (ch^{-1})^{-1} \\ &\iff gc^{-1} \prec hc^{-1}. \end{aligned}$$

Thus $\prec$ is a right-invariant ordering. □

Proposition 1.2. *[9, Chapter 1, Problem 1.1] Let $(G, <)$ be a left-ordered group. Then*

(a) For all $g \in G$, $1 < g$ if and only if $g^{-1} < 1$.

(b) If $1 < g$ and $1 < h$, then $1 < gh$.

Proof. (a) We can multiply $1 < g$ by g^{-1} on the left and we obtain

$$g^{-1} < g^{-1}g \implies g^{-1} < 1, \text{ since } g^{-1}g = 1.$$

Conversely, we can multiply $g^{-1} < 1$ by g on the left and we get

$$gg^{-1} < g \implies 1 < g \text{ since } gg^{-1} = 1.$$

(b) We can multiply $1 < h$ by g and we get that $g < gh$. From the hypothesis we have that $1 < g$, thus $1 < g < gh$. Therefore we have that $1 < gh$.

□

Proposition 1.3. *[9, Chapter 1, Proposition 1.3] Let $(G, <)$ be a left-ordered group. Then G is torsion-free.*

Proof. Let G be a left-orderable group. Let $g \in G$. We have two cases $1 < g$ and $g < 1$.

Let $g \in G$. If $1 < g$, then by multiplying by g on the left we get $g < g^2$. By repeating this process we obtain

$$1 < g < g^2 < g^3 < \cdots < g^n \qquad \forall n \in \mathbb{Z}^+.$$

Therefore $1 < g^n$ for all positive integers $n \in \mathbb{Z}^+$.

Similarly for the case $g < 1$. We get that

$$g^n < \cdots < g^2 < g < 1.$$

Then $g^n < 1$ for all positive integers n. $\square$

Example 1.1. *[9, Chapter 1, Example 1.5] Some examples of bi-orderable groups are* $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$, *and* $(\mathbb{Z}, +)$. *The group* $(\mathbb{R} \setminus \{0\}, \cdot)$ *is not bi-orderable, because* $(-1)^2 = 1$ *(where* $-1 \in \mathbb{R} \setminus \{0\}$*), i.e., it has an element which has finite order, and by Proposition 1.3 this cannot happen.*

Example 1.2. *[9, Chapter 1, Example 1.5] We have that left (or bi)-orderability is preserved under taking direct products. To see this let,* G *and* H *be left (or bi)-orderable groups, then the direct product* $G \times H$ *is left (or bi)-orderable using a lexicographic ordering* $(g, h) < (g', h')$ *if and only if* $g <_G g'$ *or* $g = g'$ *and* $h <_H h'$*, where* $<_G$ *and* $<_H$ *are left (or bi)-orderings of* G *and* H*, respectively.*

Proposition 1.4. *[9, Chapter 1, Problem 1.8] Let* G *be a group,* K *be a normal subgroup of* G *and* $H \cong G/K$ *be its quotient group. Assume that there exists a short exact sequence:*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{p} H \longrightarrow 1.$$

Then G *is left-orderable if* H *and* K *are left-orderable.*

Proof. Let $(H, <_H)$ and $(K, <_K)$ be two left-ordered groups. We want to show that G is left-orderable. Consider $g \in G$. We can define a left-ordering on G as follows: $1 < g$ if $p(g) \neq 1$ and $1 <_H p(g)$ or $p(g) = 1$ and $1 <_K g$. This defines a left ordering of G. $\square$

An example to illustrate Proposition 1.4 is the following:

Example 1.3. *[9, Chapter 1, Problem 1.9 and Problem 1.10] Let us consider the Klein bottle group. Recall that the Klein bottle is a non-orientable surface. Furthermore, we have that we can think of Klein bottle group as a square with sides identified as in Figure 1.1.*

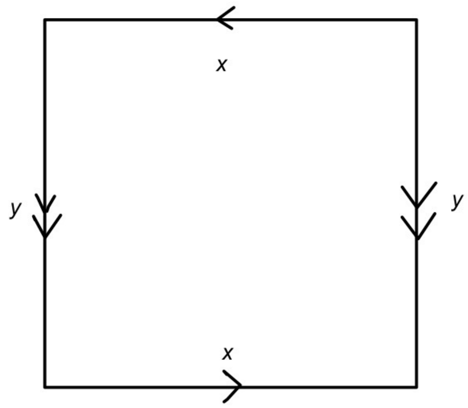

Figure 1.1: Klein Bottle seen as a square

The Klein bottle has the following fundamental group: [17, Chapter 1, Exercise 12]

$$K \cong \langle x,\, y \mid xyx^{-1} = y^{-1} \rangle.$$

We have that $\langle y \rangle$ is a normal subgroup of K and that $\langle y \rangle \cong \mathbb{Z}$. Furthermore, the quotient group $K/\langle y \rangle \cong \mathbb{Z}$. Thus we have the following short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow K \longrightarrow \mathbb{Z} \longrightarrow 1.$$

We know that $\mathbb{Z}$ is left-orderable, thus by Proposition 1.4 we have that K is left-orderable.

Moreover, the kernel and the quotient each admit two left-orderings. Thus K has at least four left-orderings. In Section 1.8, Example 1.6 we will see that these are the only left orderings of K.

Example 1.4. *[9, Chapter 1, Problem 1.7] How can we order $(\mathbb{Z}^2, +)$? There are two ways of doing this. One of them is by using Example 1.2 with $G = H = \mathbb{Z}$. The other one is by thinking of $\mathbb{Z}^2$ as sitting inside the xy-plane in the usual way. We choose a vector $\vec{v} = (v_1,\, v_2) \in \mathbb{R}^2$ which has irrational slope. Given two elements $\vec{a} = (a_1,\, a_2)$ and $\vec{b} = (b_1,\, b_2)$ of $\mathbb{Z}^2$, we declare the ordering to be*

$$\vec{a} < \vec{b} \iff a_1v_1 + a_2v_2 < b_1v_1 + b_2v_2.$$

We can also choose $\vec{v}$ to have rational slope. In this case, we pick another vector $\vec{t} = (t_1, t_2) \in \mathbb{R}^2$ which is orthogonal to $\vec{v}$. Then given $\vec{a}$ and $\vec{b}$, declare

$$\vec{a} < \vec{b} \iff a_1v_1 + a_2v_2 < b_1v_1 + b_2v_2,$$

unless $a_1v_1 + a_2v_2 = b_1v_1 + b_2v_2$, which implies that $\vec{a} - \vec{b}$ is orthogonal to $\vec{v}$. If this happens we declare

$$\vec{a} < \vec{b} \iff a_1t_1 + a_2t_2 < b_1t_1 + b_2t_2.$$

Proposition 1.5. *[9, Chapter 1, Problem 1.29] Let $(G, <)$ be a left-ordered group. Then the following are equivalent:*

(1) $(G, <)$ is right-ordered.

(2) If $g < h$, then $h^{-1} < g^{-1}$ for all $g, h \in G$.

(3) If $g < gh$, then $g < hg$, for all $g, h \in G$.

(4) If $g_1 < h_1$ and $g_2 < h_2$, then $g_1g_2 < h_1h_2$.

Proof. To prove this theorem we will show the following implications (1) $\iff$ (2) $\iff$ (3), and (4) $\iff$ (1).

Firstly, we will show (1) $\iff$ (2). Assume that $<$ is right-invariant. If we have that $g < h$, then we multiply by h^{-1} from the right and obtain

$$gh^{-1} < hh^{-1} \implies gh^{-1} < 1.$$

Now we multiply by g^{-1} from the left. We can do this because $<$ is left invariant, and we obtain

$$g^{-1}gh^{-1} < g^{-1} \implies h^{-1} < g^{-1}.$$

Conversely, assume that $g < h$ and (2) holds and let $c \in G$, then

$$\begin{aligned} g < h &\implies h^{-1} < g^{-1} \\ &\implies ch^{-1} < cg^{-1} \qquad \text{(multiplying by } c \in G \text{ on the left)} \\ &\implies gc^{-1} < hc^{-1} \qquad \text{(by part (2))}. \end{aligned}$$

Thus $<$ is right-invariant.

For (2) $\iff$ (3), suppose that $g < h \implies h^{-1} < g^{-1}$. If we have $g < gh$, then

$$\begin{aligned} g < gh &\implies g^{-1} < g^{-1}h \qquad \text{(multiplying by } g^{-2} \text{ on the left)} \\ &\implies h^{-1}g < g \qquad \text{(by part (2))} \\ &\implies g < hg \qquad \text{(multiplying by } h^{-1} \text{ on the left).} \end{aligned}$$

Conversely, assume that $g < h$ and (3) holds. Then

$$\begin{aligned} g < h &\implies g^{-1} < g^{-2}h = g^{-1}(g^{-1}h) \\ &\implies g^{-1} < g^{-1}hg^{-1} \qquad \text{(by part (3))} \\ &\implies 1 < hg^{-1} \qquad \text{(multiplying by } g \text{ on the left)} \\ &\implies h^{-1} < g^{-1} \qquad \text{(multiplying by } h^{-1} \text{ on the left).} \end{aligned}$$

For (4) $\implies$ (1), assume that if $g_1 < h_1$ and $g_2 < h_2$, then $g_1g_2 < h_1h_2$. We have that $g_1 < h_1$ implies that $h_1^{-1} < g_1^{-1}$. If not, then $g_1 < h_1$ and $g_1^{-1} < h_1^{-1}$ which gives $1 < 1$. Thus for all $g \in G$

$$\begin{aligned} g_1 < h_1 &\implies h_1^{-1} < g_1^{-1} \\ &\implies g^{-1}h_1^{-1} < g^{-1}g_1^{-1} \\ &\implies (g^{-1}g_1^{-1})^{-1} < (g^{-1}h_1^{-1})^{-1} \qquad (g^{-1} \in G) \\ &\implies g_1g < h_1g. \end{aligned}$$

For (1) $\implies$ (4), suppose that $g_1 < h_1$ and $g_2 < h_2$ and (1) holds. Then we have that

$$\left.\begin{aligned} g_1g_2 < h_1g_2 \\ h_1g_2 < h_1h_2 \end{aligned}\right\} \implies g_1g_2 < h_1g_2 < h_1h_2$$

$$\implies g_1g_2 < h_1h_2.$$

$\square$

Definition 1.2. *[9, Chapter 1, Theorem 1.24] Let $(G,<)$ be a left-ordered group. The positive cone P of G is defined as:*

$$P = \{g \in G \mid g > 1\}.$$

Furthermore, we call the elements of P positive.

Theorem 1.1. *[9, Chapter 1, Theorem 1.24] Let G be a group. Then G is left-orderable if and only if there exists P a subset of G such that*

(1) $P \cdot P \subset P$ and

(2) $G = P \sqcup P^{-1} \sqcup \{1\}$, where $\sqcup$ is the disjoint union.

Proof. Let $(G,<)$ be a left-ordered group and P its positive cone. Then we have

(1) $P \cdot P \subset P$ (by Proposition 1.2 part (b)) and

(2) $G = P \sqcup P^{-1} \sqcup \{1\}$ (by Proposition 1.2 part (a)).

Conversely, given such a P we can define a strict total ordering of G using the rule

$$g < h \iff g^{-1}h \in P.$$

□

Note that the proof of the above statement yields the stronger statement that the set of left-orders is in a bijection with the set of $P \subset G$ satisfying (1)-(2).

We know that a group is bi-orderable if there is a left-ordering on G which is right-invariant as well. How can we translate this to the language of positive cones? Well, the positive cone P of a bi-orderable group needs to be the positive cone of a left-ordering and we need to add the condition that $gPg^{-1} = P$ for all $g \in G$. Thus we have the following Theorem:

Theorem 1.2. *[9, Chapter 1, Problem 1.26] A group G is bi-orderable if and only if it there exists $P \subset G$ satisfying*

(1) $P \cdot P \subset P$,

(2) $G = P \sqcup P^{-1} \sqcup \{1\}$, *where* $\sqcup$ *is the disjoint union, and*

(3) $gPg^{-1} \subset P$ *for all* $g \in G$.

Note that the proof of the above statement yields the stronger statement that the set of bi-orders is in a bijection with the set of $P \subset G$ satisfying (1)-(3).

1.4 Bi-orderable Groups

Not all facts that hold for left-ordered groups hold for bi-ordered groups. In this section, we look at some facts about bi-orderable groups that do not hold for left-orderable groups.

Proposition 1.6. *[9, Chapter 1, Problem 1.20] Consider* $(G, <)$ *to be a bi-ordered group and let* $g, h \in G$. *If* $g^n = h^n$, *for some* $n > 0$, *then* $g = h$.

Proof. Using Proposition 1.5 we have that, if $g < h$ then $g^2 < h^2$ and then $g^3 < h^3$, and so on, thus distinct elements cannot have equal n-th powers. Thus, if $g^n = h^n$, then $g = h$. □

Proposition 1.7. *[9, Chapter 1, Problem 1.21] If* $(G, <)$ *is a bi-ordered group. Then* g^n *commutes with* h *if and only if* g *commutes with* h.

Proof. Assume that h and g^n commute, then we have that $(h^{-1}gh)^n = h^{-1}g^nh = g^n$. Thus, by Proposition 1.6 we have that $h^{-1}gh = g$, which implies that g and h commute.

The converse is trivial.

□

Proposition 1.8. *[9, Chapter 1, problem 1.22] If* $(G, <)$ *is a bi-ordered group, then* G *does not have generalised torsion.*

Proof. Let $(G, <)$ be a bi-ordered group and assume that $1 \neq g \in G$ is a generalised torsion element. Then there exists elements $a_1, \ldots, a_n \in G$ such that

$$(a_1^{-1} g a_1)(a_2^{-1} g a_2) \ldots (a_n^{-1} g a_n) = 1.$$

Since $g \neq 1$ we have that either $g < 1$ or $g > 1$. If $g < 1$, then $a_i^{-1} g a_i < 1$ for all i by bi-orderability. Thus

$$\underbrace{(a_1^{-1} g a_1)}_{<1} \underbrace{(a_2^{-1} g a_2)}_{<1} \ldots \underbrace{(a_n^{-1} g a_n)}_{<1} < 1,$$

which is a contradiction. The case $g > 1$ is similar. $\square$

The Klein bottle group shows that none of these propositions hold for a left-orderable group.

1.5 Example of a Torsion-free Group which is not Left-orderable

In this section we want to show that the fundamental group of the Weeks manifold is torsion-free but not left-orderable. The fundamental group is: [7]

$$\pi_1(W) \cong \langle a, b \mid bababa^{-1}b^2a^{-1}, ababab^{-1}a^2b^{-1} \rangle.$$

Assume that $\pi_1(W)$ is left-orderable. Then without loss of generality we can assume that $a \in P$. Furthermore, assume that $b \in P$. Then we have two possibilities, either: (i) $ab^{-1} \in P$ or (ii) $ba^{-1} \in P$.

If $ab^{-1} \in P$, then consider the relation $ababab^{-1}a^2b^{-1} = 1$. Thus

$$1 = \underbrace{a}_{\in P} \underbrace{b}_{\in P} \underbrace{a}_{\in P} \underbrace{b}_{\in P} \underbrace{ab^{-1}}_{\in P} \underbrace{a}_{\in P} \underbrace{ab^{-1}}_{\in P} \in P,$$

which is a contradiction.

If $ba^{-1} \in P$, then consider the relation $bababa^{-1}b^2a^{-1} = 1$. Thus

$$1 = \underbrace{b}_{\in P} \underbrace{a}_{\in P} \underbrace{b}_{\in P} \underbrace{a}_{\in P} \underbrace{ba^{-1}}_{\in P} \underbrace{b}_{\in P} \underbrace{ba^{-1}}_{\in P} \in P,$$

which is a contradiction.

Thus if $a \in P$, then $b \notin P$. However, this implies that $b^{-1} \in P$, and therefore,

$$\underbrace{b^{-1}}_{\in P}\underbrace{a}_{\in P}\underbrace{b^{-2}}_{\in P}\underbrace{a^2}_{\in P}\underbrace{b^{-1}}_{\in P}\underbrace{a^2}_{\in P}\underbrace{b^{-1}}_{\in P} \in P.$$

Let us show that $1 = b^{-1}ab^{-2}a^2b^{-1}a^2b^{-1}$, and so $1 \in P$ which is a contradiction.

We have

$$\begin{aligned}
b^{-1}ab^{-2}a^2(b^{-1}a^2b^{-1}) &= b^{-1}ab^{-2}a^2(a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}) \qquad \text{(using } ababab^{-1}a^2b^{-1} = 1\text{)}\\
&= b^{-1}(ab^{-2}a)b^{-1}a^{-1}b^{-1}a^{-1}\\
&= b^{-1}(babab)b^{-1}a^{-1}b^{-1}a^{-1} \qquad \text{(using } bababa^{-1}b^2a^{-1} = 1\text{)}\\
&= 1.
\end{aligned}$$

Hence $\pi_1(W)$ is not left-orderable.

Theorem 1.3. *[9, Chapter 5, Theorem 5.13] Let W be the Weeks manifold and $\pi(W)$ its fundamental group. Then $\pi_1(W)$ is a torsion-free group.*

The proof can be found in [7].

Note that the universal cover of W is $\mathbb{H}^3$ and recall that $\mathbb{H}^3$ is contactable; thus, $\pi_1(W)$ has finite cohomological dimension. Therefore we have that $\pi_1(W)$ is torsion-free [7, Theorem 9.2].

Hence, by Theorem 1.3, we have that $\pi_1(W)$ is a torsion-free group which is not left-orderable.

We will see another example of a torsion-free group which is not left-orderable in Chapter 5.

1.6 Archimedean Ordered Groups

Definition 1.3. *[9, Chapter 2] Let $(G, <)$ be a left-ordered group. We say that the left-ordering $<$ is an Archimedean ordering if for all elements $g, h \in G$ there exists an integer n such that $h^n \leq g < h^{n+1}$.*

Example 1.5. *The groups $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ with their usual ordering are Archimedean.*

Lemma 1.1. *[10, Section 3, Theorem 3.8] Let $(G, <)$ be an Archimedean left-ordering. Then $<$ is a bi-ordering.*

Proof. [9, Chapter 2] Let $(G, <)$ be an Archimedean left-ordered group. Let P be its positive cone. We want to show that G is bi-orderable, i.e., we want to show that $g^{-1}Pg \subset P$ for all $g \in G$.

Let $h \in P$ and $g \in G$. We have two cases: g is a positive element and g is not a positive element. Firstly, assume that g is positive, then because $<$ is an Archimedean ordering we have that there exists $n \in \mathbb{Z}^+$ such that $g < h^n$. Thus $1 < g^{-1}h^n$ and $1 < g^{-1}h^n g$ (because both $g^{-1}h^n$ and g are positive elements). This implies that $1 < g^{-1}hg$ (this is because its n-th power is positive), which implies that $g^{-1}hg \in P$.

Secondly, assume that g is a negative element of G and $h \in P$. Further assume that $g^{-1}hg \notin P$. Therefore, from $g^{-1}hg \notin P$, we have that

$$1 < (g^{-1}hg)^{-1} \implies 1 < g^{-1}h^{-1}g.$$

Now $g(g^{-1}h^{-1}g)g^{-1}$ is a positive element (by the above, conjugation of $g^{-1}hg$ with the positive element g^{-1} will result in a positive element). Therefore, we have that

$$1 < g(g^{-1}h^{-1}g)g^{-1} = h^{-1},$$

which is a contradiction. Thus G is bi-orderable. □

Lemma 1.2. *[9, Chapter 2, Lemma 2.5] Let G be a bi-ordered group which does not have a least positive element. Then given $g \in G$ with $g > 1$ there exists $h \in G$ with $h > 1$ such that $1 < h^2 < g$.*

Proof. Let $g > n > m > 1$, where $n, m \in G$, and consider $nm^{-1} > 1$. If $(nm^{-1})^2 \geq n$, then $m^{-1}nm^{-1} \geq 1$ and $n \geq m^2$, so we can choose $h = m$. Otherwise, choose $h = nm^{-1}$. $\square$

Lemma 1.3. *[9, Chapter 2, Lemma 2.4] Let $(G, <)$ be an Archimedean left-ordered group. Then G is abelian.*

Proof. [9, Chapter 2] We know that the ordering is bi-invariant. Let P be its cone and $p \in P$. Then we have two cases (i) P does not have a least element and (ii) P has a least element.

Consider (i). Let $g, h \in G$ and assume that g and h do not commute. Furthermore, without loss of generality we may assume that g,h, and $ghg^{-1}h^{-1}$ are positive elements of G. Then using Lemma 1.2 we have that there exists an element $f \in G$ such that $1 < f^2 < ghg^{-1}h^{-1}$. Using the fact that $<$ is Archimedian we have that there exists two integers m, n such that $f^n \leq g < f^{n+1}$ and $f^m \leq h < f^{m+1}$, this implies that $g^{-1} \leq f^{-n}$ and $h^{-1} \leq f^{-m}$. Now

$$\begin{aligned} ghg^{-1}h^{-1} &\leq ghg^{-1}f^{-m} \\ &\leq ghf^{-n}f^{-m} \\ &< f^{n+1}f^{m+1}f^{-n}f^{-m} \\ &= f^2, \end{aligned}$$

this is a contradiction. Thus G is abelian.

Consider (ii). We want to show that $\langle p \rangle \cong G$, where $p \in P$ is a least element. Let $g \in G \setminus \langle p \rangle$. Using the fact that $<$ is Archimedian we have that there exists an integer n such that $p^n \leq g < p^{n+1}$. This gives us that $1 \leq p^{-n}g < p$, which is a contradiction. Thus $G \cong \mathbb{Z}$, which implies that G is abelian. $\square$

Theorem 1.4 (Hölder, 1901)**.** *[9, Chapter 2, Theorem 2.6] If $(G, <)$ is an Archimedean left-ordered group, then G is isomorphic with a subgroup of $(\mathbb{R}, +)$.*

Proof. Let $(G, <)$ be a group with the Archimedean property. By Lemma 1.1 we have that $<$ is bi-ordering. Fix a positive element $f \in G$. Now for all $g \in G$ and $n_i \in \mathbb{Z}$ by the

Archimedean property we have an integer $a_{n_i} \in \mathbb{Z}$ such that

$$f^{a_{n_1}} \leq g^{n_1} < f^{a_{n_1}+1} \text{ and } f^{a_{n_2}} \leq g^{n_2} < f^{a_{n_2}+1},$$

then using bi-orderability we have that

$$f^{a_{n_1}+a_{n_2}} \leq g^{n_1+n_2} < f^{a_{n_1}+a_{n_2}+2}.$$

Hence $a_{n_1} + a_{n_2} \leq a_{n_1+n_2} \leq a_{n_1} + a_{n_2} + 1$; it follows that the sequence $\left\{\dfrac{a_n}{n}\right\}$ converges to a real number (this is due to Fekete's Lemma since the sequences a_n are subadditive). Then we can define a function $\phi : G \to \mathbb{R}$ by $\phi(g) = \dfrac{a_n}{n}$.

To see that ϕ is homomorphism let g, $h \in G$ and $n \in \mathbb{Z}$. By the Archimedean property we have integers a_n, $b_n \in \mathbb{Z}$ and $c_n \in \mathbb{Z}$ such that

$$f^{c_n} \leq (gh)^n < f^{c_n+1}$$

$$f^{a_n} \leq g^n < f^{a_n+1} \tag{1.1}$$

$$f^{b_n} \leq h^n < f^{b_n+1}. \tag{1.2}$$

We want to relate a_n and b_n to c_n. Thus by multiplying Equation 1.1 and Equation 1.2 we get

$$f^{a_n+b_n} \leq g^n h^n < f^{a_n+b_n+2} \implies f^{a_n+b_n} \leq (gh)^n < f^{a_n+b_n+2}. \tag{1.3}$$

Thus $a_n + b_n \leq c_n$ and $c_n + 1 \leq a_n + b_n + 2$. Applying ϕ to Equation 1.3 we get

$$\phi(f^{a_n+b_n}) \leq \phi\left((gh)^n\right) < \phi(f^{a_n+b_n+2})$$

$$a_n + b_n \leq \phi\left((gh)^n\right) < a_n + b_n + 2.$$

We have to find $\lim\limits_{n\to\infty} \dfrac{c_n}{n}$. Now

$$\lim_{n\to\infty} \frac{a_n + b_n}{n} \leq \lim_{n\to\infty} \frac{c_n}{n} = \lim_{n\to\infty} \frac{c_n + 1}{n} \leq \lim_{n\to\infty} \frac{a_n + b_n + 2}{n}.$$

Thus by the squeeze theorem we have that

$$\lim_{n\to\infty} \frac{c_n}{n} = \lim_{n\to\infty} \frac{a_n + b_n}{n}.$$

Then

$$\begin{aligned}\phi(gh) &= \lim_{n\to\infty} \frac{a_n + b_n}{n} \\ &= \lim_{n\to\infty} \frac{a_n}{n} + \frac{b_n}{b} \\ &= \lim_{n\to\infty} \frac{a_n}{n} + \lim_{n\to\infty} \frac{b_n}{n} \\ &= \phi(g) + \phi(h).\end{aligned}$$

Lastly, we need to show that ϕ is injective. Note that ϕ is an order preserving homomorphism, i.e., if $g < h$, then $\phi(g) < \phi(h)$, and that $\phi(f) = 1$. Assume that $h \in G$ is a nontrivial element. Then because the ordering is Archimedean we have that there exists an integer n such that $h^n \geq f$. Thus if $\phi(h) = 0$, then

$$0 = n\phi(h) = \phi(h^n) \geq \phi(f) = 1,$$

which is not possible. Therefore, $h = 1$ when $\phi(h) = 0$. □

1.7 Bi-orderable Groups are Locally Indicable

Definition 1.4. *[9, Chapter 2] Let $(G, <)$ be a left-ordered group and $C \subset G$ a subset of G. We say that C is convex relative to $<$ if for all $g, h \in C$ and $f \in G$ such that $g < f < h$, then $f \in C$. Furthermore, we say that a subset C of G is relatively convex if there exists an ordering of G relative to which C is convex. If C is a relatively convex set which is a subgroup of G, then we say that C is a relatively convex subgroup of G.*

One reason why we are interested in convex subgroups is because we can produce a left ordering by taking quotients by them. We have the following theorem:

Theorem 1.5. *[9, Chapter 2, Problem 2.13] Let $(G, <)$ be a left-ordered group and H a convex normal subgroup of G, then G/H is left-ordered.*

Sketch of Proof. Let $(G, <)$ be a left-ordered group and $H \subset G$ a normal convex subgroup of G. We have that on G/H there exists a natural left-ordering $\prec$ defined by

$$a < b \iff aH \prec bH.$$

Left-invariance follows from the definition and well-definedness follows from convexity. $\square$

Theorem 1.6. *[9, Chapter 2, Problem 2.9] Let* $(G, <)$ *be a left-ordered group and* C, $D \subset G$ *be convex subgroups of* G. *Then either* $C \subset D$ *or* $D \subset C$.

The conclusion of Theorem 1.6 says that convex subgroups are ordered by inclusion.

Proof. Let $(G, <)$ be a left-ordered group and C, D convex subgroups of G. Assume that the conclusion of the theorem is false. Then take elements $c \in C \setminus D$ and $d \in D \setminus C$. Without loss of generality, assume that $1 < c$ and $1 < d$ (if this is not the case, then take the inverses). Then we either have $1 < c < d$ or $1 < d < c$. This means that either $d \in C$ or $c \in D$ by convexity, both of which we assume is false. $\square$

Theorem 1.7. *[3, Chapter 1, Theorem 1.45] Let* $(G, <)$ *be a left-ordered group and* $\{C_i\}_{i\in I}$ *(*I *is an indexing set) a family of convex subgroups. Then* $\bigcap\limits_{i\in I} C_i$ *and* $\bigcup\limits_{i\in I} C_i$ *are convex subgroups.*

The proof can be found in [3] Chapter 1, Section 1.4.

Definition 1.5. *[9, Chapter 1] Let* G *be a group. We say that* G *is locally indicable if for every nontrivial finitely generated subgroup* H *of* G *we have a homomorphism* $H \to \mathbb{Z}$.

Theorem 1.8. *[9, Chapter 2, Corollary 2.20] Let* $(G, <)$ *be a bi-ordered group, then* G *is locally indicable.*

Corollary 1.1. *[9, Chapter 2, Theorem 2.19] Let* G *be a finitely generated group and* $<$ *a bi-ordering of* G. *Then then* G *surjects onto* $\mathbb{Z}$

Sketch of Proof of Theorem 1.8. [9, Chapter 2] Let $(G, <)$ be afinitely generated, bi-ordered group, and H a finitely generated subgroup of G. Let the generators of H be ordered as follows:

$$1 < g_1 < g_2 < \ldots < g_n.$$

Denote the union of all convex subgroups of H that do not contain g_n by K. The only convex subgroup which properly contains K is H. Recall that bi-orderings are conjugation invariant. Thus we have that K is conjugation invariant. Therefore we have that K is a normal convex subgroup in G by Theorem 1.7. Thus by Theorem 1.5, we have that H/K is left-orderable. In fact, H/K is Archimedean ordered; this is because there are no convex subgroups between K and H, thus by Hölder's Theorem 1.4 there exists $H \to (\mathbb{R}, +)$. Since H is finitely generated, we have that there exists a homomorphism

$$H \to H/K \to \mathbb{Z}.$$

Thus, G is locally indicable. □

1.8 Conradian left-ordering

In Chapter 4 we will be using Conradian left-orderings. Thus let us see what conditions we need for a left-ordering to be a Conradian left-ordering.

We can build a Conradian ordering from Archimedean orderings of abelian quotients. To do this, consider $(G, <)$ a left-ordered group and C, $D \subset G$ convex subgroups of G. We say that (C, D) is a Conradian convex jump if C is normal in D, and the natural ordering of D/C is an Archimedean ordering. A Conradian left-ordering is a left-ordering where all jumps are Conradian.

Example 1.6. *We have seen that the Klein bottle group, K, has at least four orderings (see Example 1.3). We want to show that K has exactly four orderings all of which are Conradian. To do this we will show that $\langle y \rangle$ is convex in any left-ordering of K.*
The steps are as follows:

- *We need to show that any element of K can be written as $y^n x^m$ with m, $n \in \mathbb{Z}$.*
- *We need to show that $\langle y \rangle$ is convex.*
- *We need to find the convex jumps and infer that all orderings are Conradian.*

Recall that

$$K \cong \langle x, y \mid xyx^{-1} = y^{-1} \rangle.$$

From the relation $xyx^{-1} = y^{-1}$ *we get four new relations*

- $xy = y^{-1}x$, *to show this multiply* $xyx^{-1} = y^{-1}$ *by* x *on the left;*
- $yx = xy^{-1}$, *this is because*
$$\begin{aligned} xyx^{-1} = y^{-1} &\implies yxyx^{-1} = 1 \quad \textit{multiply by } y \textit{ on the left} \\ &\implies yxy = x \quad \textit{multiply by } x \textit{ on the right} \\ &\implies yx = xy^{-1} \quad \textit{multiply by } y^{-1} \textit{ on the right;} \end{aligned}$$
- $x^{-1}y^{-1} = yx^{-1}$, *this is because*
$$xyx^{-1} = y^{-1} \implies yx^{-1} = x^{-1}y^{-1} \quad \textit{multiply by } x^{-1} \textit{ on the left}$$
- $x^{-1}y = y^{-1}x^{-1}$.

Thus by using these relationships we can write an element of K *with all powers of* x *on one side and all powers of* y *on the other side. Meaning that an element* g *of* K *can be written as* $y^n x^m$ $(m, n \in \mathbb{Z})$.

To show that $\langle y \rangle$ *is convex, we need to show that if* $g \in G$ *and there exists* $k, n \in \mathbb{Z}$ *such that* $y^{-k} < g < y^n$, *then this implies that* g *is a power of* y. *Assume that* $x > 1$, *and we will show that* $y^n < x^m$ *for all* $n \in \mathbb{Z}$ *and* m *and odd integer. Let us suppose that this does not happen to arrive at a contradiction. Let*

$$1 < x^m < y^n, \qquad (\textit{without loss of generality we can assume that } m \textit{ is an odd integer})$$
$$\implies x^{-m}y^n > 1$$
$$\implies x^{-m}y^n x^m > 1 \qquad (\textit{because } x \textit{ is positive})$$
$$\implies (x^{-m}yx^m)^n > 1$$
$$\implies (y^{-1})^n > 1 \qquad (\textit{because } xyx^{-1} = y^{-1} \textit{ and } m \textit{ is odd})$$
$$\implies y^{-n} > 1,$$

which is a contradiction. Thus $y^n < x^m$ for all $n \in \mathbb{Z}$ if $x > 1$.

Let $g \in K$ be an element where $g = y^n x^m$. Suppose that $y^{-k} < g < y^l$ for $k, l \in \mathbb{Z}$. Then

$$\begin{aligned} y^{-k} &< y^n x^m < y^l \\ \implies 1 &< y^{k+n} x^m < y^{l+k} \\ \implies 1 &< x^m < y^{l+k-k-n} = y^{l-n}, \end{aligned}$$

which is a contradiction if $m \neq 0$. Thus $g \in \langle y \rangle$.

Recall that $\langle y \rangle$ and $K/\langle y \rangle$ are infinite cyclic groups and also, that all the left-orderings of K arise from the following exact sequence:

$$1 \longrightarrow \langle y \rangle \longrightarrow K \longrightarrow K/\langle y \rangle \longrightarrow 1.$$

Thus we have that K has exactly four orderings, this is because the kernel and the quotient both have exactly two orderings. Furthermore, in every left-ordering of K the convex subgroups are $\{1\}$, $\langle y \rangle$, and K. Therefore the only convex jumps in a left-ordering of K are $(\{1\}, \langle y \rangle)$ and $(\langle y \rangle, K)$. Moreover, $\langle y \rangle$ and $Z/\langle y \rangle$ are Archimedean; thus every convex jump of a left-ordering of K is a Conradian jump. This implies that every left-ordering of K is Conradian

Theorem 1.9. *[10, Lemma 4.2][9, Chapter 9, Theorem 9.5] Let $(G, <)$ be a left-ordered group, then $<$ is a Conradian left-ordering if and only if for all g, $h \in G$ positive elements there exists $n > 0$ such that $g < hg^n$.*

From Theorem 1.9 we have that every bi-ordering is Conradian. Why is this? Well, we know that in a bi-ordering $g < hg^n$ is true for all positive elements g, $h \in G$ by taking $n = 1$.

In fact we have that requiring $n = 2$ is equivalent to Theorem 1.9. Thus we can think of Conradian groups as follows:

Proposition 1.9. *[29, Section 3, Proposition 3.7] Let $(G,<)$ be a left-ordered group. Then $<$ is Conradian if and only if $g < hg^2$ for all $g, h \in G$ with $g, h > 1$.*

Proof. Assume that $<$ is Conradian and that $hg^2 < g$. Then from $hg^2 < g$ we have that $(g^{-1}hg)g < 1$, this implies that $g^{-1}hg$ is a negative element. From $g^{-1}hg < 1$ we have that $hg < g$. Note that h and hg are two positive element of G. Then for all $n \geq 0$ we have:

$$\begin{aligned} h(hg)^n &= h(hg)^{n-2}(hg)(hg) \\ &< h(hg)^{n-2}(hg)g \\ &= h(hg)^{n-2}(hg^2) \\ &< h(hg)^{n-2}g \\ &\vdots \\ &< hg. \end{aligned}$$

Therefore, h and hg do not satisfy Theorem 1.9, which is a contradiction to our assumption.

The converse follows from Theorem 1.9. □

Theorem 1.10 (Conradian Burns-Hale)**.** *[9, Chapter 9, Theorem 9.17] A group G is Conradian left-orderable if and only if for every nontrivial finitely-generated subgroup $H \subset G$, there exists a Conradian left-orderable group K and a nontrivial homomorphism $H \to K$.*

A proof of Theorem 1.10 can be found in Chapter 9, Section 2 in [9].

Proposition 1.10. *[9, Chapter 9, Proposition 9.19] Let G be a nontrivial finitely-generated Conradian left-ordered group, then there exists a homomorphism $G \to \mathbb{Z}$.*

Proof. The proof is similar to Theorem 1.8 from Section 1.7. We choose a maximal convex subgroup C of G and conclude that G/C is Archimedean ordered, thus there exists a homomorphism $G \to \mathbb{Z}$ □

Corollary 1.2. *[5][9, Chapter 9, Corollary 9.21] Let $(G,<)$ be a left-ordered group. Then G is locally indicable if and only if G is Conradian left-ordered.*

Proof. Let G be a group with a Conradian ordering and let H be a non-trivial finitely generated subgroup of it. We have that H also has a Conradian left-ordering. Therefore, by Proposition 1.10 we have that there exists a map of H onto $\mathbb{Z}$.

Conversely, if for every nontrivial finitely-generated subgroup H of G there is a surjective map $H \to \mathbb{Z}$, then by Theorem 1.10 we have that G has a Conradian left-ordering . $\square$

Chapter 2

Spaces of Orderings

This chapter aims to see how we can topologise the set of left-orderings on a group G. This will be used in Chapter 4.

2.1 Introduction

Let us introduce some important definitions and facts about topologies, which we will be using in the coming chapters.

Let X be a set. A topology on X is a collection τ of subsets of X such that:

1. The empty set $\emptyset$ and X are in τ ;
2. If $\{X_i\}_{i\in I} \in \tau$, then $\bigcup\limits_{i\in I} X_i \in \tau$, where I is an indexing set;
3. If $\{X_i\}_{i=1}^{n} \in \tau$, then $\bigcap\limits_{i=1}^{n} X_i \in \tau$.

A pair (X, τ) is called a topological space. We say that a subset U of X is an open set of X if $U \in \tau$.

We say that a collection of open sets $\{B_i\}_{i\in I}$ of a topological space X is an open cover if $X = \bigcup\limits_{i\in I} B_i$. We say that a topological space X is compact if every open covering has a finite subcollection whose union is X. We say that a collection $\mathcal{B}$ of subsets of X is a basis of X if the open sets are exactly all unions of sets in $\mathcal{B}$.

Recall that if $\{X_i\}_{i\in I}$ is a family of topological spaces and I an indexing set, then the product topology on $\prod_{i\in I} X_i$ is the topology having as basis the collection

$$\mathcal{B} = \left\{ \prod_{i\in I} U_i \mid U_i \text{ is a open set in } X_i \text{ and } U_i \neq X_i \text{ for all but finitely many } i \in I \right\}.$$

Let us recall a theorem that will be used in the following subsection.

Theorem 2.1 (Tychonoff's Theorem)**.** *[28, Chapter 5] Let $\{X_i\}_{i\in I}$ be compact spaces, then $\prod_{i\in I} X_i$ is compact in the product topology.*

The proof can be found in [28] Chapter 5, Section 37.

2.1.1 Topology on the Power Set

We want to topologise the set of left-orderings on G. Before doing this, let us recall what is the topology of the power set, as this will be useful in understanding the topology of the set of left-orderings on G.

Let X be any set. Recall that the power set of X is the collection of all its subsets, denoted by $\mathcal{P}(X)$ or 2^X. The notation 2^X means that we can identify the power set with the set of all functions $f : X \to \{0, 1\}$ by using the characteristic function $\chi_A : X \to \{0, 1\}$ associated to a subset A of X. The characteristic function is defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Now, to give the set 2^X a topology we use the product topology. To do this, consider 2^X to be the product of $|X|$ copies of $\{0, 1\}$ such that for each $x \in X$ the sets

$$\{f \in 2^X \mid f(x) = 0\}$$

and

$$\{f \in 2^X \mid f(x) = 1\}$$

are a subbasis. Going back to the other notation, $\mathcal{P}(X)$, we have that the subbasic open sets are of the form:

$$V_x = \{A \subset X \mid x \in A\} \text{ and } V_x^c = \{A \subset X \mid x \notin A\}.$$

Note that these sets are closed as well, since the sets are complement of each other.

Theorem 2.2. *[9, Chapter 1] Let X be a set and $\mathcal{P}(X)$ be its power set. Then $\mathcal{P}(X)$ with the product topology is compact.*

2.2 The Space of Orderings

Now that the preliminaries are done, let us see how can we topologise the set of left-orderings of a group G.

Definition 2.1. *[9, Chapter 1, Definition 1.37] Let G be a group. We define the space of left-orderings of G, denoted by $LO(G)$, equipped with the subspace topology arising from $LO(G) \subset P(G)$ as*

$$LO(G) = \{P \subset G \mid P \textit{ is the positive cone}\}.$$

I.e., $LO(G)$ is the collection of all subsets $P \subset G$ such that:

1. *$P \cap P^{-1} = \emptyset$ and*

2. *$G = P \sqcup P^{-1} \sqcup \{1\}$, disjoint union.*

What are the open sets of $LO(G)$? To find them we need to relate the space of $LO(G)$ with $\mathcal{P}(G)$, the power set of G. To do this, we view $LO(G) \subset \mathcal{P}(G)$ as a subset of the power set of G. The subbasis of $LO(G)$ is sets of the form

$$U_g = V_g \cap LO(G) = \{P \in LO(G) \mid g \in P\}$$

and

$$U_g^c = V_g^c \cap LO(G) = \{P \in LO(G) \mid g^{-1} \in P\} = U_{g^{-1}}.$$

A basic open set of $LO(G)$ has the form $\bigcap\limits_{i=1}^{n} U_{g_i}$, for a finite subcollection $\{g_i\}_{i=1}^{n}$ of elements of G .

Is $LO(G)$ compact?

Proposition 2.1. *The space $LO(G)$ with the topology described above is compact.*

Proof. The sets which are not positive cones satisfy one of the following two conditions (i) $P \cdot P \not\subset P$ or (ii) $G \neq P \sqcup P^{-1} \sqcup \{1\}$ (disjoint union). In order for a set X to satisfy condition (i) we must have elements $g,\, h \in G$ such that $g,\, h \in X$ and $gh \notin X$, i.e., all the sets which satisfy (i) are

$$T = \bigcup_{g,\, h \in G} \{X \in 2^G \mid \exists g,\, h \in X \text{ such that } gh \notin X\} = \bigcup_{g,\, h \in G} \left(V_g \cap V_h \cap V_{gh}^c\right),$$

which is an open set.

In order for X to satisfy (ii) we must have an element $1 \neq g \in G$ such that $g,\, g^{-1} \in X$ or $g,\, g^{-1} \notin X$, or $1 \in X$. The collection of sets satisfying this is

$$S = \left(\bigcup_{g \in G} \{X \in 2^x \mid g,\, g^{-1} \in X\}\right) \cup \left(\bigcup_{g \in G \setminus \{1\}} \{X \in 2^x \mid g,\, g^{-1} \notin X\}\right)$$

$$= \left(\bigcup_{g \in G \setminus \{1\}} V_g \cap V_g^c\right) \cup \left(\bigcup_{g \in G \setminus \{1\}} V_g^c \cap V_g\right),$$

which is an open set as well.

This makes $LO(G)$ a closed subset of $\mathcal{P}(G)$, and thus a compact space. $\square$

Proposition 2.2. *[33, Theorem 1.4.] The space $LO(G)$ is totally disconnected .*

Proof. Let P_1 and P_2 be two distinct positive cones in a set U. such that for $g \in G$ we have that $g \in P_1$ and $g \notin P_2$. Then we have that $P_1 \in U_g$ and $P_2 \in U_{g^{-1}}$ and $U_g \cup U_{g^{-1}}$ is the entire space. Therefore U_g and $U_{g^{-1}}$ disconnect U. □

Proposition 2.3. *[33, Proposition 1.3] If G is countable infinite, then $LO(G)$ is metrizable.*

Proof. Let G be a countable infinite set and $g_0, g_1, \ldots$ its elements. Let $P_1, P_2 \in LO(G)$ be two positive cones such that the orderings determined by P_1 and P_2 agree on $g_0, \ldots, g_{r-1}$ and disagree on g_r. Define $d(P_1, P_2) = \dfrac{1}{2^r}$. This is a metric on $LO(G)$. □

It is important to know if our space is compact or not, totally disconnected or not, and if it has isolated points or not. This is because if we know this information we might be able to determine to what space it is homeomorphic to.

Theorem 2.3. *[6, Theorem 1] Let X be a nonempty, compact, totally disconnected, metrizable space without isolated points. Then X is homeomorphic to the Cantor space.*

Definition 2.2. *The Cantor set is defined as:*

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

A Cantor space is any space which is homeomorphic to the Cantor set C.

Note that $LO(G)$ is a totally disconnected since $\mathcal{P}(G)$ is totally disconnected. Furthermore, if G is countable infinite, we have that $LO(G)$ is metrizable, since $P(G)$ is metrizable.

Theorem 2.3 can give a characterisation of the homeomorphism type of $LO(G)$. This characterisation is that $LO(G)$ is homeomorphic with the Cantor set if and only if it contains no isolated points. By isolated points, we mean open one-point sets.
In this thesis, we have that an isolated ordering is one where the positive cone P satisfies

$$\{P\} = \bigcap_{i=1}^{n} U_{g_i},$$

where $g_1, \ldots, g_n$ are positive elements of G.

2.3 The Structure of $LO(G)$

Can we give $LO(G)$ a natural G-action? Let $(G, <)$ be a ordered group. Given $g \in G$ the action of g on $LO(G)$ is $P \mapsto g^{-1}Pg$. Moreover, G acts on $LO(G)$ by homeomorphisms, since the image of a basic open set $\bigcap\limits_{i=1}^{n} U_{g_i}$ under an action of g is the basic open set $\bigcap\limits_{i=1}^{n} U_{g^{-1}g_ig}$ [9, Chapter 10]. We will be using this in Chapter 4 Section 4.1.

More often than not, we do not know the structure or the action of $LO(G)$. An example when the structure and the action are known is the Klein bottle. In Example 1.6, we saw that the Klein bottle has exactly four orderings and that all orderings come from the short exact sequence

$$1 \to \langle y \rangle \to K \to K/\langle y \rangle \to 1,$$

note that $K/\langle y \rangle$ is generated by the cosets of x. The four orderings of K depend on if x and y are positive or negative. Now $LO(K) = \{P_{++}, P_{+-}, P_{-+}, P_{--}\}$, where

- P_{++} means that both x and y are positive;
- P_{+-} means that x is positive and y is negative;
- P_{-+} means that x is negative and y is positive;
- P_{--} means that x is negative, and y is negative.

Now, let us see what the action does to the above four orderings. Note that, from the relations, $xyx^{-1} = y^{-1}$ and $yxy^{-1} = x$ have that the sign of x does not change when we conjugate by y and that the sign of y changes when we conjugate by y. Thus when we conjugate the ordering P_{++} we get the ordering P_{+-} and when we conjugate the ordering P_{-+} we get P_{--}. Therefore the generator y acts trivially and the generator x acts by swapping pairs of left-orderings (Figure 2.1).

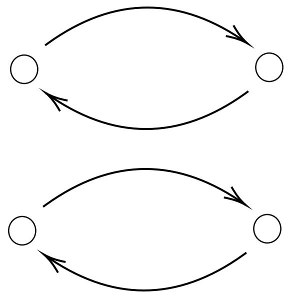

Figure 2.1: Action of the generators of K on $LO(K)$

Recall that K is not bi-orderable, thus every orbit of the action contains more than one point.

Let us give another example of a group where the structure of $LO(G)$ is known.

Example 2.1. *Consider the braid group with infinite many generators, B_∞. We will show that $LO(B_\infty)$ is homeomorphic to the Cantor set. This group has the following presentation [12, Chapter XIV, Proposition 2.10]*

$$B_\infty = \langle \sigma_1, \dots, \sigma_n, \dots \mid \sigma_i\sigma_j = \sigma_j\sigma_i, \ \textit{for } |i-j| \geq 2$$
$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \ \textit{for } |i-j| = 1\rangle.$$

Let $P \subset B_\infty$ be the positive cone of a left-ordering of B_∞. Let $H \subset P$ be a finite subset. Furthermore, let B_∞^n be the braid group with infinitely many generators where we shift the generators by n, i.e., the generator σ_i becomes σ_{i+n}. We want to show that there exists an i such that $\sigma_i P \sigma_i^{-1}$ is a positive cone of B_∞ with $P \neq \sigma_i P \sigma_i^{-1}$ and $H \subset \sigma_i P \sigma_i^{-1}$.

Fix an integer n such that $H \subset B_n$, where B_n is the braid group with $n-1$ generators. Note that B_n is the subgroup of B_∞ generated by the first $n-1$ generators. Furthermore, for $i > n$ we have that every element $h \in H$ commutes with σ_i. Therefore, $H = \sigma_i H \sigma_i^{-1} \subseteq \sigma_i P \sigma_i^{-1}$.

Now assume that there exists $i > n$ such that $P \neq \sigma_i P \sigma_i^{-1}$. Then we have that B_∞^n is isomorphic to B_∞. Thus both $P \cap B_\infty^n$ and $(\sigma_i P \sigma_i^{-1}) \cap B_\infty^n$ are positive cones of B_∞^n.

Therefore, if $P = \sigma_i P \sigma_i^{-1}$ *for some* $i > n$*, then* B_∞ *and* B_∞^n *are bi-orderable group, which is a contadiction since* B_∞ *is not bi-orderable (this is because the braid* $\sigma_1\sigma_2^{-1}$ *is conjugate to its inverse; see [12] Chapter II Proposition 1.2. for a proof that* B_∞ *is not bi-orderable).*

Chapter 3

Amenable Groups

Amenability is an important property for groups. There are many different definitions which characterise this class of groups. In this chapter, we will introduce the notion of amenable groups using some of these characterisations. We will restrict our discussion to discrete groups. We will be using amenable groups in Chapter 4.

3.1 Definitions and Examples

Let us look at an interesting example of a non-amenable groups. To do this we need to introduce Ponzi schemes:

Definition 3.1. *[26, Lecture 3, Definition 3.2] Let G be a discrete group. A Ponzi scheme on G is a function $m : G \to G$ such that*

1. *$\# m^{-1}(g) \geq 2$ for all $g \in G$,*
2. *there is a finite subset $S \subset G$ such that $m(g) \in gS$ for all $g \in G$.*

One of the reasons we are looking at Ponzi schemes first is because they are easier to understand than the following definition:

Definition 3.2. *[11, Chapter 9, Definition9.1.1] Let G be a group. If there exists a sequence of nonempty finite subsets of G, $\{F_n\}_{n \geq 1}$ such that*

$$\lim_{n \to \infty} \frac{| F_n \setminus gF_n |}{| F_n |} = 0,$$

for all $g \in G$, then we say that $\{F_n\}_{n\geq 1}$ is a Følner sequence.

Definition 3.3. *[11, Chapter 9, Proposition9.1.2] Let G be a group. If G has a Følner sequence, then G is amenable.*

Theorem 3.1 (Gromov)**.** *[26, Lecture 3, Theorem 3.4] Let G be a group. Then G is not amenable if and only if there exists a Ponzi scheme on G.*

Recall the following definition:

Definition 3.4. *[15] Let G be a finitely generated group and $H \subseteq G$ a set of generators for G. The Cayley graph, $Cay(G,\, H)$, corresponding to G has vertex set equal to G and two vertices $g_1,\, g_2 \in G$ are connected by a directed edge from g_1 to g_2 if and only if there exists $h \in H$ such that $g_1 = hg_2$.*

Example 3.1. *[26, Lecture 3, Example 3.1] Let us look at an example of a group that is not amenable. Consider the free group $\mathbb{F}_2 = \langle a,\, b\rangle$. Assume that each vertex of the Cayley graph of $\mathbb{F}_2$ represents a person holding one dollar. Let $f_t(g)$ denote the amount of money that a person, an element $g \in \mathbb{F}_2$, has at time t. At time $t = 0$ we have*

$$f_0(g) = \$1,\, \forall g \in \mathbb{F}_2.$$

At a given time every person passes its dollar to a neighbour, h, which is closer to the identity (by closer to the identity we mean that the path length between h and the identity, 1, in $Cay(\mathbb{F}_2, \{a,\, a^{-1},\, b,\, b^{-1}\})$ (see figure 3.1) is minimal). Furthermore, we have that the identity will keep its dollar as there is nowhere it can pass it to.

Now, at time $t = 1$ we have that all elements, except the identity element, have

$$f_1(g) = \$3 \text{ for all } g$$

and

$$f_1(e) = \$5.$$

The Ponzi scheme function is $m : \mathbb{F}_2 \to \mathbb{F}_2$ defined by $g \mapsto g'$. If $g = n_1n_2\dots n_m$ ($n_i \in \{a^{\pm1}, b^{\pm1}\}$) is a reduced word then g passes its dollar to $g' = n_1n_2\dots n_{m-1}$ and the *identity keeps its dollar. Hence we have that every element more than doubled their money, which satisfies condition (1) of the definition. By taking $S = \{a, b, a^{-1}, b^{-1}\}$ we satisfy condition (2) of Ponzi schemes.*

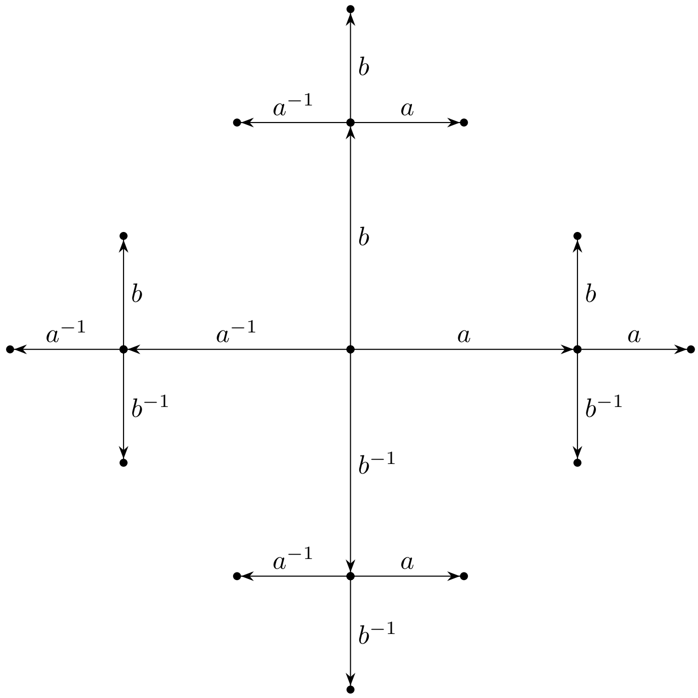

Figure 3.1: Cayley graph of $\mathbb{F}_2$ with generators a and b

Example 3.2. *[11, Chapter 9, Proposition 9.1.4] If G is a finite group, then G is amenable. To show this we need to find a Følner sequence for G. Let $\{F_n\}_{n\geq 1}$ be a nontrivial sequence of finite subsets of G such that $F_n = G$ for all $n \geq 1$. Then*

$$\lim_{n\to\infty} \frac{|F_n \setminus gF_n|}{|F_n|} = 0, \qquad \forall g \in G,$$

thus G is amenable.

Example 3.3. *[11, Chapter 9, Proposition 9.1.5] The set of integers $\mathbb{Z}$ is an amenable group. To show this we need to find a Følner sequence. Consider the following nontrivial finite sequence of subsets of $\mathbb{Z}$:*

$$F_n := \{0, \ldots, n-1\} \subset \mathbb{Z}.$$

Now, let $g \in \mathbb{Z}$ be a fixed point. Then for all n such that $|g| \leq n$ we have

$$F_n \setminus (g + F_n) = \begin{cases} \{0, \ldots, g-1\}, & g \geq 1 \\ \emptyset, & g = 0 \\ \{g+n, g+n+1, \ldots, n-1\}, & g \leq -1. \end{cases}$$

Now

$$\frac{|F_n \setminus (g + F_n)|}{|F_n|} = \begin{cases} \frac{|\{0, \ldots, g-1\}|}{|F_n|} = \frac{|g|}{n}, & g \geq 1 \\ \frac{|\emptyset|}{|F_n|} = 0, & g = 0 \\ \frac{|\{g+n, g+n+1, \ldots, n-1\}|}{|F_n|} = \frac{|g|}{n}, & g \leq -1. \end{cases}$$

Hence,

$$\lim_{n \to \infty} \frac{|F_n \setminus (g + F_n)|}{|F_n} = \begin{cases} \lim_{n \to \infty} \frac{|g|}{n} = 0, & g \geq 1 \\ \lim_{n \to \infty} 0 = 0, & g = 0 \\ \lim_{n \to \infty} \frac{|g|}{n} = 0, & g \leq -1. \end{cases}$$

Thus we have that $\{F_n\}_{n \geq 1}$ is a Følner sequence for $\mathbb{Z}$, which implies that $\mathbb{Z}$ is an amenable group.

3.2 Different Characterisations of Amenability

In this section, we want to prove that three different characterisations of amenability are equivalent.

Definition 3.5. *[24, Chapter 9] Let X be a set. The set of all bounded functions of the form $X \to \mathbb{R}$ is denoted by $\ell^\infty(X, \mathbb{R})$.*

Here, $\ell^\infty(X, \mathbb{R})$ is a real vector space where the binary operations are pointwise addition and scalar multiplication.

If we have a group G with a left G-action on X then we have a natural G-action on $\ell^\infty(X,\mathbb{R})$ as follows

$$\phi : G \times \ell^\infty(X,\, \mathbb{R}) \to \ell^\infty(X,\, \mathbb{R})$$

$$(g,\, f) \mapsto f \circ g^{-1}$$

Definition 3.6. *[24, Chapter 9, Definition 9.1.1] Let G be a group. A left-invariant mean on $\ell^\infty(G,\mathbb{R})$ is a linear map $\mu : \ell^\infty(G,\, \mathbb{R}) \to \mathbb{R}$ such that*

1. *(Normalisation.) If 1_G is the constant map: $g \mapsto 1$, then $\mu(1_G) = 1$;*
2. *(Positivity.) If $f \in \ell^\infty(G,\mathbb{R})$ is non-negative (i.e., $f(g) \geq 0$ for all $g \in G$), then $\mu(f) \geq 0$;*
3. *(Left-invariance.) For all $f \in \ell^\infty(G,\mathbb{R})$ and all $g \in G$ we have that $\mu(g \cdot f) = \mu(f)$, where $\cdot$ is the left-action of G on $\ell^\infty(G,\mathbb{R})$.*

Theorem 3.2. *[26, Lecture 3, Exercise 3.8.2 and Proposition 3.11][11, Chapter 9, Lemma9.2.1] If G is a countable discrete group, then the following are equivalent:*

1. *G is amenable;*
2. *For all finite subsets S of G and every $\epsilon > 0$, there is a nonempty finite subset F of G such that*

$$\frac{\mid F \setminus sF \mid}{\mid F \mid} \leq \epsilon \qquad \forall s \in S$$

   (this is also called almost-invariant);

3. *There is a left-invariant mean on $\ell^\infty(G,\mathbb{R})$.*

Proof. [26, Lecture 3][11, Chapter 9] (1) $\implies$ (2) Suppose G admits a Følner sequence, call it $\{F_n\}_{n\geq 1}$. Let S be a finite subset of G and $\epsilon > 0$. Then for each $s \in S$ we can find $t(s) \in \mathbb{Z}$ such that

$$\frac{\mid F_n \setminus sF_n \mid}{\mid F_n \mid} \leq \epsilon \qquad \forall n \geq t(s).$$

Take $m = \max_{s \in S} t(s)$, then we have that F_m satisfies

$$\frac{| F_m \setminus sF_m |}{| F_m |} \leq \epsilon \qquad \forall s \in S.$$

For (2) $\implies$ (1), assume that (2) holds. Let $S_1 \subset S_2 \subset \cdots \subset S_n \subset \ldots$ be finite subsets of G such that $G = \bigcup_n S_n$. For each $(n, m) \in \mathbb{Z} \times \mathbb{Z}$, let $F_{n,m}$ be the set corresponding to S_n from condition (2) using $\epsilon = \frac{1}{m}$. By condition (2) we have that $F_{n,m}$ satisfies

$$\frac{| F_{n,m} \setminus gF_{n,m} |}{| F_{n,m} |} \leq \frac{1}{m} \qquad \forall g \in S_n.$$

We want to show that $F_{n,n}$ is a Følner sequence. Let $g \in G$. There exists k such that $g \in S_n$ for all $n \geq k$. Thus

$$\frac{| F_{n,n} \setminus gF_{n,n} |}{| F_{n,n} |} \leq \frac{1}{n} \qquad \forall g \in S_n \text{ whenever } n \geq k.$$

Thus

$$\lim_{n \to \infty} \frac{| F_{n,n} \setminus gF_{n,n} |}{| F_{n,n} |} \leq \lim_{n \to \infty} \frac{1}{n} = 0$$

and

$$\lim_{n \to \infty} \frac{| F_{n,n} \setminus gF_{n,n} |}{| F_{n,n} |} \geq 0.$$

Therefore,

$$\lim_{n \to \infty} \frac{| F_{n,n} \setminus gF_{n,n} |}{| F_{n,n} |} = 0,$$

which implies that $F_{n,n}$ is a Følner sequence.

Now we will present a sketch of a proof that (1) implies (3). Let $\{F_n\}_{n \geq 1}$ be a Følner sequence and $f \in \ell^\infty(G, \mathbb{R})$. Let us consider

$$m_n(f) = \frac{1}{| F_n |} \sum_{x \in F_n} f(x).$$

We have that $m_n(f)$ is the average value of f on the set F_n, and thus $m_n(f)$ is a mean on $\ell^\infty(G, \mathbb{R})$. Since F_n satisfies condition (2) of the theorem, we have that $m_n(f)$ is "almost" left-invariant. This is because the linearity and left-invariant condition may not hold (i.e., $m_n(g \cdot f) = m_n(f)$ may not hold). To make $m_n(f)$ left-invariant we can choose a subsequence $\{n_k\}$ such that the limit $m(f) = \lim_{k\to\infty} m_{n_k}(f)$ exists. Such a choice can be made using either ultrafilters, Hahn - Banach Theorem, or a clever application of the Axiom of Choice (see [26, Lecture 3]).

To show that (3) $\Longrightarrow$ (1), (3) $\Longrightarrow$ (2), or (2) $\Longrightarrow$ (3), we need to use functional analysis, which is beyond the scope of this thesis. Proofs can be found in [27] Chapter 12 Section 12.3. □

One important characterisation of amenability which we will use in the coming chapters is that a group G is amenable if every space X with a G-action has an invariant probability measure (see Corollary 3.1). Let us recall the definition of probability measure:

Definition 3.7. *[16, Definition 1.12] Let G be a discrete group acting on a compact metric space X. We define a G-invariant finite additive probability measure on $\mathcal{P}(X)$ μ with the following properties*

1. $\mu(X) = 1$;
2. $\mu(A \sqcup B) = \mu(A) + \mu(B)$, $A, B \subset X$ *disjoint;*
3. $\mu(gA) = \mu(A)$, *for* $g \in G$ *and* $A \subset X$.

Corollary 3.1. *[26, Lecture 3, Corollary 3.12] Let G be an amenable group which acts on a compact metric space X by homeomorphisms. Then there exists a left-invariant probability measure μ on X.*

Proof. [16] Let G be an amenable group. Let $m \in \ell^\infty(G, \mathbb{R}) \to \mathbb{R}$ be a left-invariant mean on G. We want to show that there exists a probability measure on $\ell^\infty(G, \mathbb{R})$ which is invariant under multiplication from the left. Firstly, let $A \subset G$, then define $\mu(A) = m(1_A)$, where $1_A(g) = \begin{cases} 1, & g \in A \\ 0, & g \notin A. \end{cases}$

Let us show that μ is a measure:

- $\mu(G) = m(1_G) = m(1) = 1$;
- Let A, $B \subset G$ and note that $1_{A \sqcup B} = 1_A + 1_B$. Thus
$$\begin{aligned} \mu(A \sqcup B) &= m(1_{A \sqcup B}) \\ &= m(1_A + 1_B) \\ &= m(1_A) + m(1_B) \\ &= \mu(A) + \mu(B). \end{aligned}$$
- Let $A \subset G$ and $g \in G$ and note that
$$\begin{aligned} g \cdot 1_A(h) &= 1_A(gh) \\ &= \begin{cases} 1, & gh \in A \\ 0, & gh \notin A \end{cases} \\ &= \begin{cases} 1, & h \in g^{-1}A \\ 0, & h \notin g^{-1}A \end{cases} \\ &= 1_{g^{-1}A}(h). \end{aligned}$$
Thus
$$\begin{aligned} \mu(g^{-1}A) &= m(1_{g^{-1}A}) \\ &= m(g \cdot 1_A) \\ &= m(1_A) \\ &= \mu(A). \end{aligned}$$

Lastly, pick $x_0 \in X$ and define $t : G \to X$ by $t(g) = gx_0$. Then we can define a finitely additive G-invariant probability measure on X as follows:
$$\tau : \mathcal{P}(X) \to [0, 1]$$
$$\tau(A) = \mu(t^{-1}(A)), A \subset X.$$

□

3.3 Properties

In this section, let us introduce some important properties of amenable groups.

Proposition 3.1. *[24, Chapter 9, Proposition 9.1.6]*

(a) Subgroups of amenable groups are amenable.

(b) Consider the following short exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1.$$

Then G is amenable if and only if N and Q are amenable.

Proof. [24, Chapter 9, Proposition 9.1.6]

(a) Assume that G is an amenable group and let N be a subgroup of G. Since G is amenable we have that there exists a left-invariant mean μ_G. Consider the inclusion map $i : N \to G$. Then we can define a left-invariant mean of N as

$$\mu_N : \ell^\infty(N, \mathbb{R}) \to \mathbb{R}$$

$$f \mapsto \mu_G(i \circ f)$$

Thus subgroups of amenable groups are amenable, i.e., N is amenable.

(b) Assume that G is amenable, then by part (a) we have that N is amenable. We, now, have to show that Q is amenable. Consider the surjective homomorphism $\pi : G \to Q$. Then define a left-inavriant mean on Q as

$$\ell^\infty(Q, \mathbb{R}) \to \mathbb{R}$$

$$f \mapsto \mu_Q(\pi \circ f).$$

Therefore Q is amenable.

Conversely, assume that N and Q are amenable groups. Thus we have that there exists left-invariant means μ_N and μ_Q for N and Q, respectively. We want to build a left-invariant mean of μ_G for G.

Let $f \in \ell^\infty(G, \mathbb{R})$ and consider the map $f' : Q \to \mathbb{R}$ (i.e., $f' \in \ell^\infty(Q, \mathbb{R})$) defined by $f'(gN) = \mu_N(f_g)$, where $f_g \in \ell^\infty(N, \mathbb{R})$ is a the function defined by $f_g(n) = f(g \cdot n)$ for all the cosets of gN. Then we have that

$$\mu_G : \ell^\infty(G, \mathbb{R}) \to \mathbb{R}$$

$$\text{defined by } \mu_G(f) := \mu_Q(f')$$

is a left-invariant mean for G, which implies that G is amenable. To show it is indeed a left-invariant mean, we need to show that μ_G satisfies the conditions of Definition 3.6. Let us show the first condition. Because N and Q are amenable we have that $\mu_N(1) = 1$ and $\mu_Q(1) = 1$, where 1 is the constant map. Take $f = 1 \in \ell^\infty(G, \mathbb{R})$. Then we have that $f_g(n) = f(g \cdot n) = 1(g \cdot n) = 1$. Thus $f' \in \ell^\infty(Q, \mathbb{R})$ is $f'(gN) = \mu_N(f_g) = \mu_N(1) = 1$, which implies that

$$\mu_G(1) = \mu_Q(f') = \mu_Q(1) = 1.$$

The remaining conditions will follow similarly.

□

Proposition 3.2. *[35, Chapter 2, Proposition 2.8.8] Let $G_1 \subset G_2 \subset \cdots \subset G_n, \ldots$ be amenable groups. Then $G := \bigcup_n G_n$ is amenable.*

Proof. [35, Chapter 2, Section 2.8] We use Følner sequences. Let $S \subset G$ be a finite set. Given $\epsilon > 0$ we have that $S \subset G_n$ for some n. We know that G_n is amenable, thus we have that there exists $F_n \subset G_n$ such that

$$\lim_{n\to\infty} \frac{|F_n \setminus gF_n|}{|F_n|} = 0$$

for all $g \in S$. Thus G is amenable. □

Proposition 3.3. *[11, Chapter 9, Corollary 9.2.7] Let G be a countable abelian group. Then G is amenable.*

Proof. By Proposition 3.2 it is enough to show that finitely generated abelian groups are amenable. This follows from the fundamental theorem of finitely generated abelian groups, Example 3.2, Example 3.3, and Proposition 3.1.

□

Proposition 3.4. *If G is an amenable group, then G has no non-abelian free subgroups.*

Proof. The proof follows from Proposition 3.1 and Example 3.1 □

Chapter 4

A conjecture by Peter Linnell

We have seen that not all left-orderable groups are Conradian left-orderable groups. An example of such a group is the infinite braid group B_∞ [9, Chapter 10, Problem 10.11]. Thus a question we can ask ourselves is: What other condition must we include for a left-orderable group to always be a Conradian left-orderable group? We have seen (Corollary 1.2) that including locally indicability is one answer (in fact, local indicability is not just sufficient but also necessary). Another answer is including the condition of amenability. This chapter aims to look at what has been done in linking amenability to Conradian left-orderable groups. Throughout this chapter, we consider G a discrete group.

Recall that a group G is locally indicable if every nontrivial finitely generated subgroup H has an infinite cyclic quotient. In 1977 [3, Theorem 7.3.1] it was proved that a locally indicable group is left-orderable; however, the converse is not true [2]. Furthermore, in 1993, [8, Theorem A], it was shown that left-ordered elementary amenable groups are locally indicable. Moreover, in 1999, Peter Linnell [22] the following question: Are left-ordered amenable groups locally indicable? In 2006 Dave Witte Morris answer this question and showed that amenable left-ordered groups are locally indicable (see Section 4.1 for the proof of this theorem). In 2001 Peter Linnell [23] formulated the following stronger statement:

Conjecture 4.1. *[23, Conjecture 1.1] Let $(G, <)$ be a left-ordered group which does not contain a non-abelian free subgroup. Then G is Conradian left-orderable.*

Then in 2006, Dave Witte Morris [25] investigated Peter Linnell's conjecture and showed that amenable left-orderable groups are locally indicable (this is Theorem 4.1). Amenable groups are a natural class of groups to consider in light of Proposition 3.4

The structure of this chapter is as follows: We will be presenting three research articles and each article will have its own section. The articles are

- Dave Witte Morris: "Amenable groups that act on the line" [25]; this is Section 4.1,
- Peter Linnell, Dave Witte Morris: "Amenable groups with a locally invariant order are locally indicable" [21]; this is Section 4.2,
- Yago Antolín, Cristóbal Rivas: "The space of relative orders and a generalisation of Morris indicability theorem" [1]; this Section 4.3.

The proofs of the main theorems in all three articles follow the following four steps:

- Step 1: Define a topological space and find an action by homeomorphisms of the given group G;
- Step 2: we use the fact that the group G is amenable (co-amenable, in the case of the third article) to find an invariant probability measure on the topological space defined in Step 1;
- Step 3: we use the Poincaré Recurrence Theorem to find a recurrent point;
- Step 4: we use the previous steps to tie everything together with Conradian orders (local indicablility, a generalization of Conradian order, respectively).

4.1 Research Article 1

In 2006 Dave Witte Morris proved the following theorem:

Theorem 4.1. *[25, Theorem B] If G is an amenable left-orderable group, then G is locally indicable.*

See Chapter 1 Section 1.7 for the definition of locally indicable.

Because we want to show that G is locally indicable, we can assume without loss of generality that G is a non-trivial, finitely generated group [25]. We review the proof of Theorem 4.1 via the next 4 steps:

4.1.1 Step 1: Topology and an Action by Homeomorphisms

Let $LO(G)$ be the collection of all left-orderings of a group G. Recall that in Chapter 2 we saw how we can give $LO(G)$ a topology. Also, recall that $LO(G) \subset \mathcal{P}(G)$ and $LO(G)$ has subbasic sets of the form:

$$U_g = V_g \cap LO(G) = \{P \in LO(G) \mid g \in P\}$$

$$U_g^c = V_g^c \cap LO(G) = \{P \in LO(G) \mid g^{-1} \in P\} = U_{g^{-1}},$$

where V_g is the subset of sets containing g and V_g^c is its complement of $\mathcal{P}(G)$. Thus we have that U_g is a subbasic set of the subspace topology. A basic open set is

$$\bigcap_{i=1}^{n} U_{g_i}, \qquad g_i \in G.$$

Furthermore, from Proposition 2.1 we know that $LO(G)$ is a closed subset of $\mathcal{P}(G)$ and thus it is a compact topological space.

Let $<$ be an ordering on G. In Chapter 2 we saw that there is an action of $g \in G$ by homeomorphisms which sends P to $g^{-1}Pg$. It is an action by homeomorphisms because the image of a basic open set $\bigcap_{i=1}^{n} U_{g_i}$ under an element of g is the basic open set $\bigcap_{i=1}^{n} U_{g^{-1}g_ig}$.

4.1.2 Step 2: Applying Amenability

Let G be a group which acts on the compact metric space $LO(G)$. Since G is an amenable group, by Corollary 3.1, we have that there exists a left-invariant probability measure, μ, on $LO(G)$.

4.1.3 Step 3: Recurrent Ordering

Recall the Poincaré Recurrence Theorem:

Proposition 4.1 (Poincaré Recurrence Theorem)**.** *[25, Proposition 3.1] Let X be a measurable space with probability measure μ. Consider the homeomorphism $f : X \to X$ which preserves the measure μ and let $A \subset X$ be any measurable subset. Then there exists a subset B of X such that $\mu(B) = 0$ and for all $a \in A \setminus B$, there exists a sequence of positive integers $\{n_i\}_{n\geq 1}$ so that $f^{n_i}(a) \in A$ for every i.*

Note that as long as we avoid the "bad" points, which there are not that many, we have that the iterates $f^{n_i}(a)$ will return near to a infinitely often.

Definition 4.1. *[25, Definition 3.2] Let $(G, <)$ be a left-ordered group. We say that $<$ is recurrent for every cyclic subgroup if for all finite subsets $\{g_1, \dots, g_n\}$ such that $g_1 < g_2 < \cdots < g_n$ and for all $g \in G$ there is a sequence $\{n_i\}$ such that*

$$g_1 g^{n_i} < \cdots < g_n g^{n_i}$$

for every i.

Proposition 4.2. *[25, Corollary 3.4] Let G be a countable amenable left-orderable group. If there exists a left-invariant probability measure, μ, on $LO(G)$, then G has a left-invariant order that is recurrent for every cyclic subgroup.*

Proof. Let G be a countable amenable, left-orderable group, P its positive cone and $<$ its corresponding left-ordering. Let $g \in G$. Because G is amenable we have that there exists a probability measure on $LO(G)$. Consider a sequence of distinct elements of G, $g_1, \dots, g_r$. We can apply the Poincaré Recurrence Theorem with $X = LO(G)$ with the transformation $P \mapsto g^{-1}Pg$ in the role of f, and the basic open set $\bigcap_{i=1}^{r-1} U_{g_i^{-1}g_{i+1}}$ for A. Then we have that there exists a set $B_{g,g_1,\dots,g_r}$ of measure 0 such that for all $P \in \bigcap_{i=1}^{r-1} U_{g_i^{-1}g_{i+1}} \setminus B_{g,g_1,\dots,g_r}$ there exists a sequence of positive integers $n_1, n_2, \dots, n_i, \dots$ such that

$$g_1 g^{n_i} < \cdots < g_r g^{n_i} \qquad \forall i,$$

i.e., $g^{n_i} P g^{-n_i} \in \bigcap_{i=1}^{n-1} U_{g_i^{-1} g_{i+1}}$.

The union $\bigcup B_{g,g_1,\ldots,g_r}$ has measure 0 (since G is countable), thus there exists a positive cone P which is not in $\bigcup B_{g,g_1,\ldots,g_r}$. This order is recurrent for every cyclic subgroup. $\square$

Note that the basic open set $\bigcap_{i=1}^{r-1} U_{g_i^{-1} g_{i+1}}$ does not need to have positive measure. This is because when we take the union $\cup B_{g,g_1,\ldots,g_r}$, since G is countable, it will still have measure 0. Thus we have that there exist a point P which is outside of this union $\cup B_{g,g_1,\ldots,g_r}$, i.e., $P \notin \cup B_{g,g_1,\ldots,g_r}$. If the positive cone P satisfies the inequality $g_1 < g_2 < \cdots < g_r$, then we have that for this collection of elements $P \notin B_{g,g_1,\ldots,g_r}$; this means that P is a recurrent point.

4.1.4 Step 4: Tying Everything Together with Conradian Orderings

Lemma 4.1. *[25, Lemma 4.3] Let $(G, <)$ be a left-orderable group where $<$ is recurrent for every cyclic subgroup, then G is Conradian [1].*

Proof. Let g, h be positive elements of the group. Since we have that $g > 1$, then recurrence implies that there exists n_i such that $gh^{n_i} > 1h^{n_i} = h^{n_i}$ for all i. Since we also have $h > 1$, then $h^{n_i} > h$, thus $gh^{n_i} > h$ for all i, which implies that the order is Conradian. $\square$

Let G be a left-ordered group such that its left-ordering is recurrent for every cyclic subgroup. Then by Corollary 1.2 and Lemma 4.1 we have that G is locally indicable. This concludes the proof of Theorem 4.1.

Not all Conradian ordered groups are recurrent orderable, and there exist Conradian left-orderable groups which are not recurrent orderable.

[1]See Chapter 1 Section 1.8 for the definition of Conradian order.

4.2 Research Article 2

Definition 4.2. *[21, Definition 2.2] Let G be a group. We say that a partial order $<$ on G is locally invariant if for all $x, y \in G$ such that $y \neq 1$, then either $xy > x$ or $xy^{-1} > x$.*

In 2013 Peter Linnell and Dave Witte Morris [21] improved Dave Witte Morris' result, Theorem 4.1 as follows:

Theorem 4.2. *[21, Theorem 1.1] If G is an amenable group which has a locally invariant ordering, then G is locally indicable.*

We outline a proof when G is countable of Theorem 4.2 with the following 4 steps:

4.2.1 Step 1: Topology and Action by Homeomorphism

In Dave Witte Morris' paper [25], we have seen that the space of left-orderings on G was used. Unlike the first research article where we used left-invariant orderings on G, here we will be using binary relations on G.

How can we give the set of all relations which define a locally invariant ordering a topology? Using a similar method as we have shown in Chapter 2, let X be any set. To give $\mathcal{P}(X \times X)$ or $2^{X \times X}$ a topology we can identify $2^{X \times X}$ with the set of all functions $f : X \times X \to \{0, 1\}$. Furthermore, we can view $2^{X \times X}$ as the Cartesian product of $|X \times X|$ copies of the set $\{0, 1\}$. By Tychonoff's Theorem, we have that for any set X, $2^{X \times X}$ is a compact Hausdorff space when equipped with the product topology [2].

The subbasis of this topology are the following sets:

$$U_{(x,y)} = \{S \subset X \times X \mid (x, y) \in S\}$$

$$U^c_{(x,y)} = \{S \subset X \times X \mid (x, y) \notin S\}.$$

[2]We do not need X to be a compact topological space. Recall that $\{0, 1\}$ is a compact space and by the Tychonoff's theorem an arbitrary product of compact spaces is compact, i.e., $|X \times X|$ copies of $\{0, 1\}$ is compact, which implies that $2^{X \times X}$ is compact for any set X.

Note that $U_{(x,y)}$ is both open and closed.

Since the $U_{(x,y)}$'s are a subbasis, we have that the open basic sets are

$$\bigcap_{i=1}^{n} U_{(x_i,y_i)} \qquad x_i, y_i \in X, \forall i.$$

We denote by $LI(G) \subset \mathcal{P}(G \times G)$ the set of all relations which define a locally invariant ordering. In this embedding we have that the locally invariant ordering $<$ corresponds to $R \in G \times G$ by setting $x < y$ if and only if $(x, y) \in R$. It can be shown that $LI(G)$ using a similar method to the one we have presented for $LO(G)$.

The group G acts on $2^{G\times G}$ by both left and right translations. Thus we have that there exists an action of $G \times G$ on $2^{G\times G}$ defined as follows: Given $(g, h) \in G \times G$, set

$$\phi_{(g,h)} : \mathcal{P}(G \times G) \to \mathcal{P}(G \times G)$$

$$(x, y) \in \phi_{(g,h)}(R) \iff (gxh^{-1}, gyh^{-1}) \in R.$$

Let us check that this is an action. For this we need to show that

$$(x,\, y) \in \phi_{(g,\, h)} \circ \phi_{(s,\, t)}(R)$$

is the same as $(x,\, y) \in \phi_{(gs,\, ht)}(R)$. We have

$$(x,\, y) \in \phi_{(g,\, h)} \circ \phi_{(s,\, t)}(R) \iff (gsxt^{-1}h^{-1},\, gsyt^{-1}h^{-1}) \in R$$

and

$$\begin{aligned}(x,\, y) \in \phi_{(gs,\, ht)}(R) &\iff (gsx(ht)^{-1},\, gsy(ht)^{-1}) \in R \\ &\iff (gsxt^{-1}h^{-1},\, gsyt^{-1}h^{-1}) \in R.\end{aligned}$$

This is an action by homeomorphisms since the image of a basic open set $\bigcap_{i=1}^{n} U_{(x_i,y_i)}$, where $x_i,\, y_i \in G$ for all i, under the action of $(g,\, h) \in G \times G$ is the basic open set $\bigcap_{i=1}^{n} U_{(gx_ih^{-1},\, gyh^{-1})}$, where $x_i,\, y_i \in G$ for all i. It restricts to an action of homeomorphisms on $LI(G)$.

4.2.2 Step 2: Applying Amenability

Let G be a group which acts on the compact metric space $LI(G)$. Since G is an amenable group, by Proposition 3.2 we have that $G \times G$ is amenable. Now by Corollary 3.1 we have that there exists a left-invariant probability measure, μ, on $LI(G)$.

4.2.3 Step 3: Recurrent Locally Invariant Ordering

Definition 4.3. *[21, Definition 2.8] Let G be a group and R a relation on G, i.e., $R \in 2^{G\times G}$. We have that R is recurrent for $(g,\, h) \in G \times G$, if for every finite*

$$F = \{(x_1,\, y_1), \ldots, (x_k,\, y_k)\} \subset G \times G :$$

with $(x_i,\, y_i) \in R$ for all i, there exists a sequence of positive integers $\{n_i\}$ such that $\phi_{(g^{n_i},\, h^{n_i})}(R) \in \bigcap_{j=1}^{k} V_{(x_j,\, y_j)}$ for all i. We say that R is recurrent if it is recurrent for every element of $G \times G$.

This means that R is recurrent if:

$$(x, y) \in \phi_{(g^{n_i}, h^{n_i})}(R) \iff (g^{n_i} x h^{-n_i}, g^{n_i} y h^{-n_i}) \in R.$$

Then recurrent for (g, h) means that there exists n_i such that $(x_1, y_1), \ldots, (x_k, y_k) \in \phi_{(g^{n_i}, h^{n_i})}(R)$, i.e.,

$$x_1 \prec y_1, \ldots, x_k \prec y_k \iff g^{n_i} x_j h^{-n_i} \prec g^{n_i} y_j h^{-n_i} \qquad \forall j = 1, \ldots, k.$$

This means that if $\prec$ is a locally invariant ordering, then, by using Poincaré's Theorem, we have that the locally invariant order, $\prec$, is recurrent.

Corollary 4.1. *[21, Corollary 3.6] Let G be a countable, amenable group with a locally invariant order. Then the locally invariant order $<$ on G is recurrent.*

4.2.4 Step 4: Tying everything together with locally indicability

Proposition 4.3. *[21, Proposition 4.1] Let G be a group and $\prec$ be a recurrent locally invariant order. Then:*

1. *Let $H = \langle x \rangle$ be a cyclic subgroup of G. By restricting $\prec$ to a left coset of H we get either:*
$$\ldots \prec gx^{-2} \prec gx^{-1} \prec g \prec gx \prec gx^2 \prec \ldots, \tag{4.1}$$
or
$$\ldots \succ gx^{-2} \succ gx^{-1} \succ g \succ gx \succ gx^2 \succ \ldots. \tag{4.2}$$
for $g, x \in G$ with $x \neq 1$.

2. *The set $P = \{g \in G \mid g \succ 1\}$ is closed under multiplication.*

3. *The group $(G, <)$ is left-ordered.*

Proof. 1. Fix $1 \neq x \in G$ and $g \in G$. Then locally invariant implies that either $gx \succ g$ or $gx^{-1} \succ g$. Suppose that $gx \succ g$. Then by local invariance applied to gx and x either $gx^2 \succ gx$ or $gx(x^{-1}) \succ gx$ (which contradicts our assumption). Thus $g \prec gx \prec gx^2 \prec \ldots$. Similarly, by assuming $gx^{-1} \succ g$ we get that $\cdots \prec gx^{-2} \prec gx^{-1} \prec g$.

Now (4.1) happens when $gx^{-1} \prec g$ and $g \prec gx$ and (4.2) happens when $gx^{-1} \succ g$ and $g \succ gx$.

Let us assume neither (4.1) nor (4.2) hold. Then either $gx^{-1} \succ g$ and $g \prec gx$ or $gx^{-1} \prec g$ and $gx \prec g$. Using $gx^{-1} \succ g$ and $g \prec gx$ we get:

$$g \prec gx \prec gx^2 \prec gx^3 \prec \ldots$$

and

$$gx^{-1} \prec gx^{-2} \prec gx^{-3} \prec \ldots.$$

Because $\prec$ is recurrent for x and since $gx^{-1} \prec gx^{-2}$ we have that there exists a positive integer k such that $(gx^{-1})x^{k+2} \prec (gx^{-2})x^{k+2}$. This means $gx^{k+1} \prec gx^k$, which contradicts $g \prec gx \prec gx^2 \prec \dots$. Similarly we arrive at a contradiction if $gx^{-1} \prec g$ and $gx \prec g$.

2. Assume that there exists x and y, with $x \succ 1$ and $y \succ 1$, but $xy \not\succ 1$. From (1) we have that $\prec$ is a total order, and thus $1 \succ xy$. Then, because $x \succ 1 \succ xy$, we have that $x \succ xy$ and from (1) we obtain

$$x \succ xy \succ xy^2 \succ \dots,$$

thus

$$1 \succ xy \succeq xy^n, \text{ for all } n \in \mathbb{Z}^+.$$

Furthermore, $\prec$ is recurrent for y and $x \succ 1$. Using part (1) we have that for $x \succ 1$ there is some $n \in \mathbb{Z}^+$ such that $xy^n \succ 1y^n \succ 1$, which is a contradiction.

3. Let P be the positive cone of $\prec$. Given $x \in G$ a nontrivial element of G and using part (1) with $g = 1$ we get that either $x \in P$ or $x^{-1} \in P$. Now, using part (2) we have that P is closed under multiplication. Thus $(G, <)$ is a left-ordered group.

$\square$

To finish the proof of Theorem 4.2, assume that G is a countable group (because we want to show that G is a left-orderable group we may assume that G is finitely generated [20, Chapter 3]. Hence, we can assume that G is countable). Using Corollary 4.1, we have that G has a locally invariant order $<$. Now by part (3) of Proposition 4.3 we have that $(G, <)$ is left-ordered. Now using Theorem 4.1 and Step 4 of its proof, we have that G is a locally indicable group.

4.3 Research Article 3

In 2018 Yago Antolín and Cristóbal Rivas extended this result to groups acting on the line, which may or may not be amenable, but contain large point-stabilisers. A generalization of amenability is co-amenability, and a generalization of Conradian left-ordering is G surjects onto $\mathbb{Z}$ if G is finitely generated. Recall that in Corollary 1.2 we showed that G is Conradian left-orderable if and only if it is locally indicable.

Definition 4.4. *[1] Let G be a group and H a subgroup of G. We say that H is co-amenable in G if when G acts on a compact space by homeomorphisms such that H preserves a probability measure, then G preserves a probability measure.*

A generalization of Theorem 4.1 is:

Theorem 4.3. *[1, Theorem 1.2] Let G be a finitely generated left-orderable group. Suppose there is a proper, relatively convex (see below for a definition) co-amenable subgroup. Then G surjects onto $\mathbb{Z}$.*

We outline a proof of this theorem below:

4.3.1 Step 1: Topology and Action by Homeomorphism

Definition 4.5. *[1, Definition 1.3] Let G be a left-orderable group and C a proper subgroup of G. Recall that C is relatively convex in G if there exists a total order $<$ on G/C such that if $fC < gC$ then $hfC < hgC$ for all f, g, $h \in G$. We say that $<$ is a relative order of G with respect to C.*

Rivas and Antolín [1] noted that a subgroup C of G is relatively convex if and only if there exists $P \subset G$ a nonempty semigroup such that: (i) $CPC \subset P$ and (ii) $G = P \sqcup P^{-1} \sqcup C$. Such subsets P can be identified with the set of relative orders. Let $LO_R(G)$ be the set of $P \subset G$ such that P satisfies:

(i) $CPC \subset P$ and

(ii) $G = P \sqcup P^{-1} \sqcup C$, where C is a relatively convex subgroup.

To prove that the space $LO_R(G)$ is compact (using a similar method as the one presented in Chapter 2) we need a different characterisation of relative convexity.

Lemma 4.2. *Let G be a left-orderable group. A subgroup C is relatively convex in G if and only if there exists $P \subset G$ such that*

1. $P \cap P^{-1} = \emptyset$,
2. $P \cdot P \subset P$,
3. $C_P := G \setminus (P \cup P^{-1})$ *is a subgroup of* G,
4. $cPd \subset P$ *for all* c, $d \notin P \cup P^{-1}$.

Let $LO_{Rel}(G)$ to be the set of $P \subset G$ satisfying these properties.

We want to show that $LO_R(G)$ and $LO_{Rel}(G) \setminus \emptyset$ are the same, meaning our notion of relatively convex agrees with Antolin and Rivas'. Thus we need to show that if P is a set satisfying the conditions (i) and (ii) from above, then P satisfies the conditions from Lemma 4.2, and vice-versa.

Suppose that there exists C relatively convex in G and P satisfying (i) $CPC \subset P$ and (ii) $G = P \sqcup P^{-1} \sqcup C$. Then we have that P satisfies (1)-(4) of Lemma 4.2:

1. $P \cap P^{-1} = \emptyset$ is true, because $G = P \sqcup P^{-1} \sqcup C$ is a disjoint union.
2. $P \cdot P \subset P$ is true because P is a semigroup.
3. $C_P := G \setminus (P \cup P^{-1})$ is a subgroup because $C_P = C$ and $G = P \sqcup P^{-1} \sqcup C$.
4. $cPd \subset P$ is true for all $c, d \in C$ because $CPC \subset P$.

On the other hand, assume that P is nonempty and satisfies conditions (1)-(4) of Lemma 4.2. Then we need to show that C_P is relatively convex and P satisfies

(i) $C_P P C_P \subset P$

(ii) $G = P \sqcup P^{-1} \sqcup C_P$.

Now (i) $CPC \subset P$ and (ii) $G = P \sqcup P^{-1} \sqcup C$ follow from part (4) and part (3), respectively, of Lemma 4.2.

Lastly, we need to show that C_P is a relatively convex subgroup. Let $P' \subset C_P$ be a positive cone and set $Q = P' \cup P$. Let us show that Q is a positive cone. For this we need to show that $Q \cdot Q \subset Q$ and $Q \sqcup Q^{-1} \sqcup \{id\} = G$. Let $c \in P'$ and $d \in P$. Then $cd \in P \cup P'$, thus $Q \cdot Q \subset Q$ (property (4) is used). For the second one we have that $Q^{-1} = (P')^{-1} \cup P^{-1}$. Thus

$$\begin{aligned} Q \sqcup Q^{-1} \sqcup \{1\} &= (P' \cup P) \sqcup \left((P')^{-1} \cup P^{-1}\right) \sqcup \{1\} \\ &= G. \end{aligned}$$

Define $fC_p \prec gC_p$ if and only if $f^{-1}g \in Q$. Not that this definition is independent of the coset representatives. Furthermore, recall that by Theorem 1.5 the order $\prec$ is left-invariant. Thus we have that C_p is a relatively convex subgroup of G.

Thus $LO_{Rel}(G) \setminus \emptyset$ is the same set that Antolin and Rivas [1] have in their paper, but defined differently.

Since $LO_{Rel}(G) \subset \mathcal{P}(G)$, the subbasis of the topology of $LO_{Rel}(G)$ are the following sets:

$$U_g = V_g \cap LO_{Rel}(G)$$

$$U_g^c = V_g^c \cap LO_{Rel}(G),$$

where V_g, $V_g^c \subset \mathcal{P}(G)$. The basic open sets are $\bigcap_{i=1}^{n} U_{g_i} \cap \bigcap_{j=1}^{n} U_{g_j}^c$.

Theorem 4.4. *$LO_{Rel}(G)$ with the topology described above is compact.*

Proof. Note that $LO_{Rel}(G) \subset \mathcal{P}(G)$ and that $\mathcal{P}(G)$ is compact. Thus we need to show that $LO_{Rel}(G)$ is a closed subset of $\mathcal{P}(G)$. To do this let us see the ways a subset $X \subset G$ fails to be an element of $LO_{Rel}(G)$. This happens if X does not satisfy one of $(1)-(4)$.

Let us check that (1)-(4) are closed conditions by showing the complements are open.

(1) X such that $X \cap X^{-1} = \emptyset$ is an open set, by a proof similar to that of $LO(G)$ is compact (Theorem 2.1) .

(2) X such that $X \cdot X \not\subset X$ is an open set, by a proof similar to that of $LO(G)$ is compact (Theorem 2.1).

(3) Consider X such that $C_P := G \setminus (X \cup X^{-1})$ is not a subgroup. Thus either

(a) there exists $g \in C_P$ such that $g^{-1} \notin C_P$, or

(b) there exists $g,\, h \in C_P$ such that $gh \notin C_P$, or

(c) $id \notin C_P$.

If (a) holds, then there exists $g \notin X \cup X^{-1}$ such that $g^{-1} \in X \cup X^{-1}$, which is not possible. Note that (c) cannot happen, since $X \cap X^{-1} = \emptyset$.
Now, if (b) holds, then $g,\, h \in G \setminus (X \cup X^{-1})$ and $gh \in X \cup X^{-1}$ so that

$$X \in \left[U_g^c \cap U_{g^{-1}}^c \cap U_h^c \cap U_{h^{-1}}^c\right] \cap \left[U_{gh} \cup U_{(gh)^{-1}}\right] = W_{g,h}.$$

Therefore,

$$X \in \bigcup_{g,\, h \in G} W_{g,h}$$

if and only if X satisfies (3)(b). Note $\bigcup_{g,\, h \in G} W_{g,h}$ is open.

(4) Consider X such that $cXd \not\subset X$ for some $c,\, d \notin X \cup X^{-1}$. If $cXd \not\subset X$, then there exists $g \in X$ such that $cgd \notin X$, hence $X \in U_g \cup U_{cgd}^c$ for some g, which is open. Thus the collection of all such X is $\bigcup_{c,d \notin X \cup X^{-1}} \bigcup_{g \in G} \left(U_g \cup U_{cgd}^c\right)$, which is open.

Hence $LO_{Rel}(G)$ is a closed subset of $\mathcal{P}(G)$, which implies that $LO_{Rel}(G)$ is compact. $\square$

Note that $\emptyset \in LO_{Rel}(G)$ and this will pose a problem when we use the Poincaré Recurrence Theorem, because the empty set is a fixed point of the G-action and when we use the Poincaré Recurrence Theorem the empty set might always be the recurrent point.

However, if we add in Lemma 4.2 the condition that G is finitely generated, then $LO_{Rel}(G) \setminus \{\emptyset\}$ is also compact. This is because if $\{g_1, \ldots, g_n\}$ are the generators of G, then $\{\emptyset\} = \bigcap_{i=1}^{n} U_{g_i}^c \cap U_{g_i^{-1}}^c$ is an open set. In other words, the only $P \subset G$ satisfying (1)-(4) and containing no generators is the empty set.

Now when we use the Poincaré Recurrence Theorem, we will get a recurrent point which is guaranteed not to be the empty set.

Proposition 4.4. *Let G be a group. Then we have that G acts on $LO_{Rel}(G) \setminus \{\emptyset\}$ by conjugation. If P is a relative cone with respect to $C \subset G$, then for any $g \in G$, gPg^{-1} is a relative cone with respect to gCg^{-1}.*

Proof. We need to show that gPg^{-1} is a relative cone with respect to gCg^{-1}. To do this we need to show that gPg^{-1} satisfies the conditions in Lemma 4.2. I.e., we need

1. $gPg^{-1} \cap gP^{-1}g^{-1} = \emptyset$;

2. $(gPg^{-1})(gPg^{-1}) \subset gPg^{-1}$;

3. $gC_Pg^{-1} := G \setminus (gPg^{-1} \cup gP^{-1}g^{-1})$ is a subgroup;

4. $cgPg^{-1}d \subset gPg^{-1}$ for $c, d \notin gPd \cap gP^{-1}g^{-1}$.

Let us show that conditions (1)-(4) from above are true.

1. $gPg^{-1} \cap gPg^{-1} = \emptyset$ is true because P is a relative cone.

2. $(gPg^{-1})(gP^{-1}g^{-1}) \subset gPg^{-1}$ is true because P is a semigroup and P is a relative cone.

3. To show that $gC_Pg^{-1} := G \setminus (gPg^{-1} \cup gP^{-1}g^{-1})$ is a subgroup, we need to show that gC_Pg^{-1} is closed under multiplication and inverses. Let f, $h \in G \setminus (gPg^{-1} \cup gP^{-1}g^{-1})$.

Then consider $f,\, h \notin gP^{-1}g^{-1}$ and $fh \in gP^{-1}g^{-1}$, which implies

$$\begin{aligned}
&g^{-1}fg,\, g^{-1}hg \notin P \cup P^{-1}\\
\implies &g^{-1}fg,\, g^{-1}hg \in C_P = C \setminus (P \cup P^{-1})\\
\implies &g^{-1}fhg \in C_P\\
\implies &g^{-1}fhg \notin P \cup P^{-1}\\
\implies &fh \notin gPg^{-1} \cup gP^{-1}g^{-1}\\
\implies &fh \in gC_Pg^{-1}.
\end{aligned}$$

4. Assume $cPd \subset P$ for $c,\, d \notin P \cup P^{-1}$. We want to show that $cgPg^{-1}d \subset gPg^{-1}$. Note that

$$c, d \notin gPg^{-1} \cup gP^{-1}g^{-1} \implies g^{-1}cg, g^{-1}dg \notin P \cup P^{-1}.$$

Thus, $g^{-1}cgPg^{-1}d \subset P$ (since P satisfies condition (4)). Therefore $cgPg^{-1}d \subset gPg^{-1}$.

□

The action is by homeomorphisms, this is because the image of a basic set $\bigcap_{i=1}^{n} U_{g_i}$ under an element $g \in G$ is the basic open set $\bigcap_{i=1}^{n} U_{g^{-1}g_ig}$.

4.3.2 Step 2: Applying Co-amenability

Proposition 4.5. *[1, Corollary 2.6] If C is a relatively convex subgroup of G, then C fixes every point $P \in LO_{Rel}(G) \setminus \{\emptyset\}$ for which $C = C_P$.*

Proof. Let P be a relative cone with respect to C and $c \in C$. Then $G = cP^{-1}c^{-1} \sqcup C \sqcup cPc^{-1}$. By Lemma 4.2 we have that $cPc^{-1} \subset P$, which implies that $cPc^{-1} = P$. □

We have that the space $LO_{Rel}(G) \setminus \{\emptyset\}$ is compact and C acts with a fixed point and because C is co-amenable, we have that there exists a probability measure, μ, on $LO_{Rel}(G) \setminus \{\emptyset\}$. The probability measure is the Dirac measure.

4.3.3 Step 3: Recurrent Ordering

Definition 4.6. *Let P be a relative cone with respect to C with corresponding relative order $<$. We say that $<$ is recurrent for every cyclic subgroup if for every $g \in G$ and every finite sequence $g_1, \ldots, g_r$ in G such that $g_1 C \leq \cdots \leq g_r C$, there exists positive integers $n_i \to \infty$ such that*

$$g_1 g^{n_i} C \leq \cdots \leq g_r g^{n_i} C.$$

Corollary 4.2. *Suppose G is a finitely generated group and C is a relatively convex subgroup. Assume that there exists a G-invariant probability measure, μ, on $LO_{Rel}(G) \backslash \{\emptyset\}$. Then G admits a nontrivial relative order which is recurrent for every cyclic subgroup.*

Proof. Let G be a finitely generated group and $g_1, \ldots, g_r$ its generators. By using the Poincaré Recurrence Theorem we have that there exists a set $B_{g,g_1,\ldots,g_r}$ of measure 0 such that for all $P \notin \bigcap_{i=1}^{r} U_{g^{-1} g_i g} \setminus B_{g,g_1,\ldots,g_r}$ there exists a sequence of positive integers $n_1, n_2, \ldots, n_i, \ldots$ such that

$$g_1 g^{n_i} C \leq \cdots \leq g_r g^{n_i} C \qquad \forall i.$$

The union $\bigcup B_{g,g_1,\ldots,g_r}$ has measure 0, thus there exists a positive cone $P \notin \bigcup B_{g,g_1,\ldots,g_r}$. This order is recurrent for every cyclic subgroup.

□

Note that the basic open set $\bigcap_{i=1}^{r} U_{g^{-1} g_i g}$ does not need to have positive measure. This is because when we take the union $\cup B_{g,g_1,\ldots,g_r}$, since G is countable, it will still have measure 0. Thus we have that there exist a point P which is outside of this union $\cup B_{g,g_1,\ldots,g_r}$, i.e., $P \notin \cup B_{g,g_1,\ldots,g_r}$. If the relative cone P satisfies the inequality $g_1 g^{n_i} C \leq \cdots \leq g_r g^{n_i} C$, then we have that for this collection of elements $P \notin B_{g,g_1,\ldots,g_r}$; this means that P is a recurrent point.

4.3.4 Step 4: Tying everything together with a generalization of Conradian Orderings

In order to show that G surjects onto $\mathbb{Z}$ we need a generalisation of Conradian order. This generalisation has been developed by Andrés Navas in [29] and is called "crossings". Let us define what a crossing on an action is:

Definition 4.7. *[1, Definition 3.2.] Let $(\mathcal{T}, \leq)$ be a totally ordered space and G a group acting on $(\mathcal{T}, \leq)$ by order-preserving bijections. Let $f, g \in G$ and $u, v, w \in \mathcal{T}$. We say that $(f, g; u, v, w)$ is a crossing for the action if:*

1. *$u < w < v$*
2. *$g^n u < v$ and $u < f^n v$ for every $n \in \mathbb{N}$*
3. *there exists M, $N \in \mathbb{N}$ such that $f^N v < w < g^M u$.*

We can relate Conradian orderings with crossings as follows:

Theorem 4.5. *[30, Theorem 1.4] If $(G, <)$ is Conradian, then the left-action of G on itself has no crossings.*

Proof. Assume $\leq$ is not a Conradian ordering and let f, $g \in G$ be positive elements such that $fg^n < g$ for every $n \in \mathbb{Z}^+$. We want to show that there exists a crossing (i.e., we want to show that $(f,\, g :\, u,\, v,\, w)$ is a crossing) for $(G, \leq)$ where $u = 1$, $v = f^{-1}g$, and $w = g^2$.

For $n = 2$ we have that $fg^2 < g$ and thus we have that $g^2 < f^{-1}g$. Using $g > 1$ we get that $1 < g^2 < f^{-1}g$, i.e., $u < w < v$.

From $fg^n < g$ where $n \in \mathbb{Z}^+$ we have that $g^n < f^{-1}g$. Thus $g^n u < v$. Furthermore, since f and g are positive elements of G we have that $f^{n-1}g > 1$. Thus $f^n v > u$ for all $n \in \mathbb{Z}^+$. Lastly, observe that $f(f^{-1}g) = g < g^2$, thus $f^N v < v$ for $N = 1$ and $g^2 < g^3$ is $w < g^M u$ for $M = 3$. $\square$

Note that actions without crossings give maps onto the integers when the group is finitely generated [1].

Lemma 4.3. *[1, Lemma3.7] Let G be a group and $\leq$ be a relative order of G with respect to C that is recurrent for every cyclic subgroup. Then the left-action of G on $(G/C, \leq)$ is an action without crossings. If G is a finitely generated group, then G surjects onto $\mathbb{Z}$.*

Using Lemma 4.3 we have that: if G is a finitely generated group and $\leq$ a relative order of G which is recurrent for every cyclic subgroup, then G surjects onto $\mathbb{Z}$. This concludes the proof of Theorem 4.3.

Chapter 5

Unique Product Property for Groups

We want to give a generalisation of the theorems we saw in Chapter 4. We saw that in order to investigate Peter Linnell's Conjecture 4.1, we required our group to be amenable. Then we weakened left-orderablility to locally invariance, and lastly, we weakened amenability to co-amenability. What is a weakening of locally invariance? One weakening of locally invariance is the unique product property. What is a group with the unique product property?

Definition 5.1. *[32] Let G be a group. Let $A, B \subset G$ any two finite subsets of G. We say that G has the unique product property if for all $A, B \subset G$, there exists at least one element $x \in G$ such that $x = ab$ ($a \in A$ and $b \in B$) has a unique representation. The product ab will be called a unique product for A and B. We denote by $U(A, B)$ the set of all unique products from A and B.*

The unique product property fits with the classes of algebraic properties we have defined from Chapter 1 to Chapter 4 as follows: let

(A) bi-orderable;

(B) recurrent left-invariant orderable;

(C) locally indicable;

(D) Conradian orderable;

(E) left-orderable;

(F) locally-invariant orderable;

(G) unique product property (UPP);

(H) torsion-free.

We have that the class of bi-orderable groups is inside the class of recurrent left-invariant orderable groups which is a subset of locally indicable groups and so on, i.e.,

$$(A) \subset (B) \subset (C) \subset (D) \subset (E) \subset (F) \subset (G) \subset (H) \text{ and } (D) \subset (C).$$

The second and third inclusions were presented in this thesis (the second is Lemma 4 in Chapter 4, and the third one is Corollary 1.2 Chapter 1), the first inclusion can be found in [18], the fifth and sixth inclusions can be found in [18] and the seventh inclusion can be found in [13]. Note that it is an open problem whether or not $(G) \subset (F)$, and in the next section, we will give an example that torsion-free groups are not UPP. (I.e., that $(G) \subset (H)$ is proper.) Note that every let-orderedable group has the unique product property [14, Chapter 1, Section 1.4.3]. The converse is not true, i.e., groups with unique product property are not left-ordered; this was proved in [19, Appendix A, Theorem A.1]. Furthermore, in [2], it was proved that left-ordered groups are not locally indicable.

We can write all of this as a directed graph, Figure 5.1, where the edges of the graph denote the containments between the classes of groups. Note that the edge (C) to (D) is reversible, which means that (C) if and only if (D). As noted above, it is an open problem if (G) implies (F), all the other edges are not reversible.

Going back to Chapter 4, we saw that if we add the condition that the group is amenable, then some of the above implications are reversible (see Figure 5.2), the red edges correspond to implications whose converse hold under the additional assumption of amenability. We saw that amenable left-ordered groups are indicable (see Theorem 4.1) and that amenable groups with a locally invariant ordering are locally indicable (see

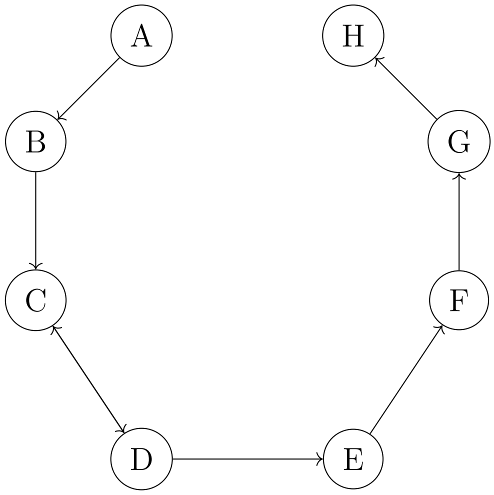

Figure 5.1: Directed graph which represents the implications for the defined class of groups

Theorem 4.2). Thus a question one can ask is what condition we must include for a group with the unique product property to make it left-ordered? In light of the theorems from Chapter 4, we have that amenability might be an answer. Theorem 4.2 makes natural the following question

Question 5.1. *[14, Chapter 4, Question 4.1.11]: Are amenable groups with the unique property product left-orderable?*

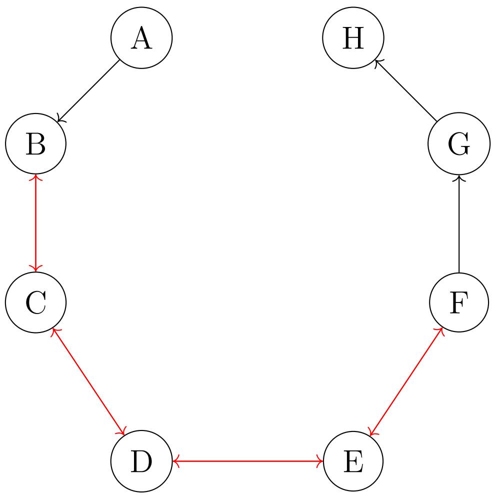

Figure 5.2: Directed graph under the additional assumption of amenability

5.1 Torsion-free groups are not UPP

Before we try to answer Question 5.1. Let us look at an example of a torsion-free groups which does not have the unique product property.

Consider the group [32]:

$$G = \langle x,\, y \mid x^{-1}y^2x = y^{-2},\, y^{-1}x^2y = x^{-2}\rangle.$$

To show that G is torsion-free and does not have the unique product property we will follow [31, Chapter 9, Proposition 37.1] and [14, Chapter 1].

Given

$$x(a,b,c) = (a+1, 1-b, -c)$$

$$y(a,b,c) = (-a, 1+b, 1-c),$$

consider $H = \langle x, y\rangle \subset \mathrm{Isom}(\mathbb{R}^3)$. It can be checked that x, y satisfy the same relations as G, and in fact $H \cong G$.

Note that every element of H is a map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(a,b,c) = (u_0a + u_1, v_0b + v_1, w_0 + c + w_1),$$

where $u_1, v_1, w_1 \in \mathbb{Z}$ and $u_0, v_0, w_0 \in \{\pm 1\}$.

In fact, there are only four kinds of elements :

(Type 1) $f_1(a,b,c) = (a+x, -b+y, -c+z)$, where x and y are odd and z is even;

(Type 2) $f_2(a,b,c) = (-a+x, b+y, -c+z)$, where x is even and y and z are odd;

(Type 3) $f_3(a,b,c) = (-a+x, -b+y, c+z)$, where x and z are odd and y is even;

(Type 4) $f_4(a,b,c) = (a+x, b+y, c+z)$, where x, y, and z are even.

Let T denote the set of all functions of Type 1, Type 2, Type 3, and Type 4. Then we can check that $(T, \circ)$ satisfies the conditions of a group. For the closure condition, let f_1 be a function of Type 1 and f_2 be a function of Type 2. Then

$$\begin{aligned} f_1 \circ f_2(a,b,c) &= f_1(-a+x, b+y, -c+z) \\ &= (-a+2x, -b, c) \end{aligned}$$

which is of Type 3. The other 15 cases are similar to verify. The other group conditions can be shown similarly.

Furthermore, the generators of H are in T, thus $H \subset T$. If we show that T is torsion-free, then we have that H is torsion-free and hence G is torsion-free.

To show that T is torsion-free, we need to show that $f_1^n, f_2^n, f_3^n, f_4^n \neq 1$, for $n \in \mathbb{Z} \setminus \{0\}$ and $f_1, f_2, f_3, f_4 \neq 1$, and that products of f_i's are different from the identity. Let us look at f_4:

$$\underbrace{f_4 \circ \cdots \circ f_4}_{n \text{ times}}(a,b,c) = (a+nx, b+ny, c+nz) \neq 1.$$

The other cases follow similarly.

To show that G is not UPP, consider the subset $S = Ax \cup By \cup C(xy)$ of G, where

$$\begin{aligned} A &= \{1,\, x^{-2},\, x^{-2}y^2,\, y^2,\, x^{-2}(xy)^{-2},\, (xy)^2\} \\ B &= \{1,\, x^2,\, y^{-2},\, y^{-2}(xy)^2,\, (xy)^2,\, x^2y^{-2}(xy)^2\} \\ C &= \{(xy)^2,\, (xy)^{-2}\}. \end{aligned}$$

Then $|S| = 14$ and $S \cdot S$ has no unique product. Unfortunately there is no simple way to check all products by hand, this was done in [32] using a computer.

5.2 Another characterisation of UPP

Let us look at another characterisation of the unique product property, called the two unique product property.

Definition 5.2. *[4] Let G be a group and A, $B \subset G$ such that $|A| \geq 2$ and $|B| \geq 2$. We say that G has the two unique property if there are at least two distinct elements, c_1, c_2, of AB each of which has a unique expression as a product, $c_i = a_i b_i$ where $a_i \in A$, $b_i \in B$ are unique products. Let $TU(A, B)$ be the set of all pairs of unique products for A and B where $|A|, |B| \geq 2$ are finite:*

$$TU(A, B) = \{(c_1,\, c_2) \mid c_1, c_2 \text{ are unique products for } AB\}.$$

In [34], it was shown that the unique product property is equivalent to the two unique product property.

5.3 Question

Now, going back to Question 5.1, we have that if the Question 5.1 were true, then the proof is likely follow the four steps we have seen in Chapter 4, i.e.,

- Step 1: Topology and action by homeomorphism
- Step 2: Using amenability to get a probability measure
- Step 3: Recurrence and Poincaré Recurrence Theorem
- Step 4: Tying the previous steps with left-orderability.

5.3.1 Step 1: Topology and Action by Homemorphism

Let us try to tackle the topology part of Step 1. Let

$$U(G) = \{U(A, B) \mid A,\, B \subset G,\, A,\, B \text{ finite and nonempty}\} \subset \mathcal{P}(G)$$

be the set of all sets of the form $U(A, B)$ where A, $B \subset G$ with $|A|$, $|B| \geq 1$, A and B finite. Recall that $\mathcal{P}(G)$ has a topology (see Chapter 2), thus $U(G)$ will inherit a topology from $\mathcal{P}(G)$.

We need $U(G)$ to be compact for us to apply Step 2, Step 3, and Step 4. This is equivalent to $U(G)$ being closed since $P(G)$ is a compact Hausdorff space.

Let us show by using a counterexample that $U(G)$ is not compact. Let $G = F_2 = \langle x, y \rangle$. Consider the subsets

$$A_1 = \{id\},\ A_2 = \{id, y\}, \ldots, A_k = \{id, y, \ldots, y^{k-1}\},\ B = \{x\}.$$

Then $U(A_k, B) = \{x,\ yx,\ y^2x,\ \ldots, y^{k-1}x\}$. Observe that $\mid U(A_k, B) \mid = k$ and that

$$U(A_1, B) \subset U(A_2, B) \subset \cdots \subset U(A_{k-1}, B) \subset U(A_k, B) \subset \ldots.$$

Now, taking the $\lim_{k\to\infty} U(A_k, B) = \bigcup_{k=1}^{\infty} U(A_k, B)$ we get a infinite set which is not in our space $U(F_2)$ (this is because of the way we have defined the space), thus $U(F_2)$ is not compact.

This is a bad definition of a space $U(G)$ in the sense that our space is not compact. If we can find a topology of $U(G)$, which gives compact spaces, it is reasonable to assume that Step 2, Step 3, and Step 4 will be similar to Step 2, Step 3, and Step 4 from Section 4.1.

Alternatively, because the two unique product property and the unique product property are equivalent we can try the set $TU(G)$ of all sets of the form $TU(A, B) \subset \mathcal{P}(G \times G)$ where A, $B \subset G$ with $|A|$, $|B| \geq 2$, A and B finite. Recall that $\mathcal{P}(G \times G)$ has a natural topology (see Chapter 4), thus $TU(A, B)$ will inherit a topology from $\mathcal{P}(G \times G)$.

We need $TU(G)$ to be compact for us to apply Step 2, Step 3, and Step 4. This is equivalent to $TU(G)$ being closed.

However, $TU(G)$ is not compact. To show this, let $G = F_2 = \langle x, y\rangle$. Consider the subsets

$$A_1 = \{id, y\},\ A_2 = \{id, y, y^2\}, \ldots, A_k = \{id, y, \ldots, y^{k-1}\},\ B = \{id, x\}.$$

Then $TU(A_k, B) = \{(id, id), (id, x), (y, id), (y, x)(y^2, id), (y^2, x), \ldots, (y^{k-1}, id), (y^{k-1}, x)\}$. Observe that $\mid TU(A_k, B) \mid= 2k$ and that

$$TU(A_{k-1}, B) \subset TU(A_k, B).$$

Now, taking the $\lim\limits_{k\to\infty} TU(A_k, B) = \bigcup\limits_{k=1}^{\infty} TU(A_k, B)$, we get a infinite set, thus $TU(F_2)$ is not compact.

This is a bad definition of topology of $TU(G)$ since our space is not compact if we can find a topology of $TU(G)$ which gives compact spaces it is reasonable to assume that Step 2, Step 3, and Step 4 will be similar to Step 2, Step 3, and Step 4 from Section 4.2.

Even though we do not have a compact topological space, let us try to see how the rest of the steps will proceed. Before we go to Step 2, we need to find subbasis of the topology, the basic open sets and an action by a homeomorphism.

Since $U(G) \subset \mathcal{P}(G)$, the subbasis of the topology of $LO_{Rel}(G)$ are the following sets:

$$U_g = V_g \cap U(G)$$

$$U_g^c = V_g^c \cap U(G),$$

where V_g, $V_g^c \subset \mathcal{P}(G)$. The basic open sets are $\bigcap\limits_{i=1}^{n} U_{g_i} \cap \bigcap\limits_{j=1}^{n} U_{g_j}^c$.

Similarly to Chapter 4, we have that there exists an action of G on $U(G)$. This action is an action by conjugation which sends $U(A, B)$ to $gU(A, B)g^{-1}$. Let us show that $U(A, B) \mapsto gU(A, B)g^{-1}$. For this we need to to show that $gU(A, B)g^{-1} =$

$U(gAg^{-1}, gBg^{-1})$, where

$$gU(A,B)g^{-1} = \{gabg^{-1} \mid gabg^{-1} \text{ is a unique expression}\}$$

$$U(gAg^{-1}, gBg^{-1})g = \{gag^{-1}gbg^{-1} \mid gag^{-1}gbg^{-1} \text{ is a unique expression}\}.$$

Let $gabg^{-1} \in gU(A,B)g^{-1}$, where $gabg^{-1}$ is a unique expression. Then

$$gabg^{-1} = gag^{-1}gbg^{-1}, \qquad g^{-1}g = 1.$$

We need to show that $gag^{-1}gbg^{-1}$ is a unique expression. Let $gcg^{-1} \in gAg^{-1}$ and $gdg^{-1} \in gBg^{-1}$ such that $gcg^{-1}gdg^{-1} = gag^{-1}gbg^{-1}$. Then

$$\begin{aligned} gcg^{-1}gdg^{-1} &= gag^{-1}gbg^{-1} \\ \implies cd &= ab \\ \implies c &= a \text{ and } d = b. \end{aligned}$$

Thus $gag^{-1}gba^{-1}$ is a unique product, and hence $gU(A,B)g^{-1} \subseteq U(gAg^{-1}, gBg^{-1})$. Similarly, we can show that $gU(A,B)g^{-1} \supseteq U(gAg^{-1}, gBg^{-1})$ This is an action by homeomorphism becasue the image of a basic open set $\bigcap_{i=1}^{n} U_{g_i}$ under an element $g \in G$ is the basic open set $\bigcap_{i=1}^{n} U_{gg_ig^{-1}}$.

5.3.2 Step 2: Left-invariant Probability Measure

As we have seen in Chapter 4, to move to Step 2 of the proof, we need our group to act on a compact space by a homeomorphism. However, the way we have defined $U(G)$ does not make it a compact topological space. Nevertheless, let us assume that we somehow have constructed a left-invariant probability μ on $LO(G)$.

5.3.3 Step 3: Recurrent Point (or ordering)

Because we want to show that G is a left-orderable group, we may assume that G is finitely generated [20, Chapter 3, Corollary 3.1.1] and hence we can assume that G is countable.

What we need for Step 3 of the proof is that G admits a recurrent point and also, we want to use the Poincaré Recurrence Theorem. In Chapter 4, we saw that in Step 3 in the three articles we used the fact that our group, G had an ordering (a left-ordering, a locally invariant ordering, and a relative ordering, respectively).

Definition 5.3. *Let G be a group with the unique product property. We say that a point $U(A,B) \in U(G)$ is a recurrent point if for every $g \in G$ and every finite sequence $g_1, \ldots, g_n$ there exists a positive integer $n_i \to \infty$ such that*

$$g^{n_i} U(A,B) g^{-n_i} \in \bigcap_{i=1}^{n} U_{g g_i g^{-1}}.$$

Proposition 5.1. *Let G be a countable amenable group which has the unique product property (or the two unique product property) and let τ be a left-invariant measure on $U(G)$ (or $TU(G)$). Then G admits a recurrent point.*

Proof. Let G be a countable amenable group and $U(A,B) \in U(G)$. Using Step 2, we have that there exists a probability measure, τ, on $U(G)$. Consider a sequence of distinct elements of G, $g_1, \ldots, g_n$. We can apply the Poincaré Recurrence Theorem with $X = U(G)$ with the transformation $U(A,B) \mapsto gU(A,B)g^{-1}$ in the role of f, and the basic open set $\bigcap\limits_{i=1}^{n} U_{g g_i g^{-1}}$ for A.Then, we have that there exist a set $C_{g,g_1,\ldots,g_n}$ of measure 0 such that for all $U(A,B) \in \bigcap\limits_{i=1}^{n} U_{g g_i g^{-1}} \setminus C_{g,g_1,\ldots,g_n}$ there exists a sequence of positive integers $\{n_i\}_{i \geq 1}$ such that

$$g^{n_i} U(A,B) g^{-n_i} \in \bigcap_{i=1}^{n} U_{g g_i g^{-1}}.$$

The union of $\cup C_{g,g_1,\dots,g_n}$ has measure 0 (since G is countable), thus there exists a point $U(A,B)$ which is not in $\cup C_{g,g_1,\dots,g_n}$. Thus G admits a recurrent point. $\square$

5.3.4 Step 4: Tying the previous steps together

From Step 3 we have that there exists a recurrent point $U(A,B)$. This means that for every $U(A,B) \in \bigcap_{i=1}^{n} U_{g_i}$ and $g \in G$, then there exists $n_i \in \infty$ such that

$$g^{n_i}U(A,B)g^{-n_i} = U(g^{n_i}Ag^{-n_i}, g^{n_i}Bg^{-n_i}).$$

Now, we have that $U(A,B)$ is a finite set such that for every element of the group there exists infinitely many n_i's such that $g^{n_i}U(A,B)g^{-n_i} = U(g^{n_i}Ag^{-n_i}, g^{n_i}Bg^{-n_i}) \in \bigcap_{i=1}^{n} U_{g_i}$. Note that this happens only for some groups.

For example, this does not happen for a free group. Let $\mathbb{F}_n(x_1,\dots,x_n)$ be the free group with n generators. Consider $S = \{a_1,\dots,a_r\} \subseteq \mathbb{F}_n$, where $a_1,\dots,a_r$ are reduced words where $a_i \neq a_j$ $(i \neq j)$ and where $S \in \bigcup_{i=1}^{r} U_{a_i}$. If S were a recurrent point, then for all $h \in \mathbb{F}_n$ there exists a positive integer $n_i \to \infty$ such that $h^{n_i}Sh^{-n_i} \in \bigcup_{i=1}^{r} U_{a_i}$. This happens if and only id $h^{n_i}a_jh^{-n_i} \in S$, i.e, $h^{n_i}Sh^{-n_i} = S$. However, this is not possible. Consider $a_j = x_{i_1}\dots x_{i_n} \in S$ be a word from S of biggest length and let $\ell = \ell(a_j)$, where $\ell(a_j)$ is the length of a_j. Now take $x_j \in \mathbb{F}_n$ an element that does not commute with a_j. When we conjugate a_j by x_j^m, where $m > \ell$ we know that at one point we will get that $x_j^m a_j x_j^{-m}$ is still longer than the longest a_j after we have reduced everything that can be reduced. Thus as we conjugate a_j with higher power of x_j we will get longer and longer words. Hence $a_j^m a_j x_j^{-m} \notin S$, which implies that for all $h \in \mathbb{F}_n$ there exists $n_i \to \infty$ such that $h^{n_i}Sh^{-n_i} \notin \bigcup_{i=1}^{r} U_{a_i}$.

Chapter 6

Conclusion

In Chapter 4 Sections 4.1, 4.2 we have seen that in order to investigate Peter Linnell's conjecture we needed to add the condition that the group is amenable and we needed to weaken left-orderability to locally invariance. In Chapter 4 Section 4.3 we have seen that Dave Witte Morris' result can be extended to groups acting on the line by giving a generalisation of amenability.

In Chapter 5 we proposed generalisations of the theorems presented in Chapter 4 by weakening the locally invariance condition and replacing it with the unique product property and the two unique product property, respectively. However, when we tried to tackle Step 1 of the proofs we saw that there is no obvious topological space to build on and mimic the proofs. If we able to somehow overcome this and construct a probability measure μ on $U(G)$, then our next hurdle will be to understand what

$$g^{n_i}U(A,B)g^{-n_i} = U(g^{n_i}Ag^{-n_i}, g^{n_i}Bg^{-n_i}) \in \bigcap_{i=1}^{n} U_{g_i}$$

means and how we can tie it with left-orderability.

Further works investigating Peter Linnell's conjecture will require to either:

1. define the set of all unique products property or the set of all two unique products in such a way that their respective topological spaces are compact, or

2. if we assume that we can construct a probability measure μ on $U(G)$ we need to analyse how to link the existence of a recurring point with left-orderability, or

3. weaken the amenability condition by replacing it with the non-existence of non-abelian free subgroups.

However, either of these things will require a new idea because there is no way to replicate the established proofs.

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