On the Hahn and Levi-Civita Fields: Topology, Analysis, and Applications

BY

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A Thesis Submitted to

the Faculty of Graduate Studies

in Partial Fulfillment of the Requirements of the Degree of

DOCTOR OF PHILOSOPHY

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Abstract

In this thesis, we present a number of developments regarding the Hahn and Levi-Civita fields ($\mathcal{F}$ and $\mathcal{R}$ respectively). After reviewing the algebraic and order structures of the field $\mathcal{F}$, we introduce different vector topologies on $\mathcal{F}$ that are induced by families of semi-norms and all of which are weaker than the order or valuation topology. We compare those vector topologies and we identify the weakest one which we denote by τ_w and whose properties are similar to those of the weak topology on the Levi-Civita field [1]. In particular, we state and prove a convergence criterion for power series in $(\mathcal{F}, \tau_w)$ that is similar to that for power series on the Levi-Civita field in its weak topology [2]. We also state three conjectures regarding so-called simple regions and prove a version of Weierstrass' Preparation Theorem in their support. Moreover we show how these conjectures can be used to extend the two-dimensional integration theory [3] to higher dimensions. We prove a version of Leibniz' Rule for integration on $\mathcal{F}$ and show how it determines the necessary boundary conditions for Green's Functions derived from the non-Archimedian delta function [3]. We also include corrected and extended examples of the use of Green's Functions for solving linear ordinary differential equations. Finally we investigate some of the computational applications of the Levi-Civita field. We replicate the results of [4] regarding the computation of derivatives of real-valued functions representable on a computer and we show how a similar method can be employed to compute real numerical sequences using their generating functions. We discuss a number of methods of numerical integration that are viable on the field $\mathcal{R}$ and we compare their performance to conventional methods as well as to commercial mathematical software.

Acknowledgements

Over the course of my academic career, I've had help from numerous sources, and it would be impossible to list them all here; nevertheless, there are a few key groups and individuals that I would like to draw attention to. Firstly, on a professional level I would like to begin by thanking Dr. Shamseddine for his incredible support throughout my academic journey. It is impossible to put into words how grateful I am to him for all of his work. I would like to acknowledge Dr. Gwinner, Dr. Shalchi, and Dr. Craigen for agreeing to be on my Internal Examination Committee. Their recommendations for this thesis have undoubtedly made it stronger. Dr. Comicheo also provided valuable input to my writing during the drafting stage. The External Examiner, Dr. Zuniga-Galindo, also has my gratitude for taking the time to review my work, his comments were appreciated. I would especially like to express my appreciation to Susan Beshta, the Office Assistant for Graduate Students in Physics, for helping me to navigate the administrative aspects of being a graduate student. On the institutional level, I'd like to thank the University of Manitoba as a whole, and in particular the Department of Physics and Astronomy, and the Department of Mathematics. I'm also grateful to the GETS Program for contributing to my stipend. On a personal note, I would like to, again, thank Khodr, for his unwavering support, and the other members of the research group, Will and Angel, who also have my sincere appreciation. I'd like to send my regards to the members of the PAGSA Volleyball Team, Net Force, for the many hours of enjoyment I spent with them. And finally, I would like to thank my friends and family for their ongoing support.

For my family, I could never have done this without you.

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Chapter 1

Preliminaries

1.1 Motivation

Traditionally, physicists have used three fields to describe the universe, the rational numbers (denoted by $\mathbb{Q}$), the real numbers (denoted by $\mathbb{R}$), and the complex numbers (denoted by $\mathbb{C}$). These fields are **Archimedean fields** because they satisfy the so-called **Archimedean property**. Simply put, the Archimedean property states that if we are given two distinct distances within our field then if the shorter distance is added to itself sufficiently many times the sum will eventually exceed the size of the greater distance. The dominance of the Archimedean fields within physics is easily understood since the Archimedean property agrees with our intuitive understanding of distance. Unfortunately, phenomena such as the Heisenberg uncertainty principle and the Planck scale seem to prevent us from probing the fine structure of our universe. A good discussion of this topic can be found in [5]. From a philosophical perspective it makes sense to investigate non-Archimedean models of physics simply because we have no *a priori* reason to conclude the universe is Archimedean. One might say the goal of the mathematical physicist is to locate among all possible mathematical structures that one which most closely resembles a given physical system; in this light, it makes little sense to limit the scope of the search to include only Archimedean mathematical structures. Another exciting opportunity allowed by non-Archimedean fields is the possibility of finding rigorous representations for functions that are improper over the real numbers; a good example of this is the Dirac delta function which is fundamental to quantum field theory but has no proper Archimedean representation. As we shall see in Chapter 4, there is a smooth, integrable non-Archimedean delta function which retains the useful properties of the real Dirac delta function. Of course a rigorous treatment of the Dirac delta function can be done in $\mathbb{R}$ using the theory of distributions but at the cost of the intuitive interpretation; this is particularly problematic for the purposes of teaching, where the delta function is often described as something like 'an infinitely thin, infinitely tall spike.' It would be beneficial from a pedagogical perspective to have a mathematical framework capable of matching that description. Taking for granted that non-Archimedean physics is worthy of study it remains to ask which non-Archimedean field should be employed in the project. The choice is a difficult one because there are infinitely many non-Archimedean fields and some have wildly different properties to others, an excellent review of the available fields and their respective properties can be found in [6]. To date, most work in non-Archimedean mathematical physics has focused on the so-called p-adic numbers and their complex analog (denoted by $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively). Such work has been ongoing since at least 1984 [7] and has resulted in a number of interesting applications since then [8]. In this dissertation our goal is to develop new mathematical tools on the Hahn field (denoted by $\mathcal{F}$) and its proper subfield the Levi-Civita field (denoted by $\mathcal{R}$) as viable alternatives to $\mathbb{Q}_p$ for use in physics. There are a number of important differences between $\mathbb{Q}_p$ and $\mathcal{F}$, perhaps the most immediate difference is that $\mathcal{F}$ contains both the real numbers and the rational numbers as proper subfields while $\mathbb{Q}_p$ only has the rational numbers as a subfield. Additionally $\mathcal{F}$ is totally ordered whereas $\mathbb{Q}_p$ is not and $\mathcal{R}$ can be implemented on a computer whereas $\mathbb{Q}_p$ is too large for that purpose. A good illustration of the effects these differences can have on physical theories is suggested by work done regarding a p-adic theory of quantum mechanics: in [9] there are proposed two potential formulations for state functions; in particular we can either consider functions from

$\mathbb{Q}_p$ (resp. $\mathbb{C}_p$) to $\mathbb{R}$ (resp. $\mathbb{C}$) or we can consider functions from $\mathbb{Q}_p$ (resp. $\mathbb{C}_p$) to itself. In the case of $\mathcal{F}$ and $\mathcal{R}$ this distinction is inconsequential because $\mathbb{R}$ (resp. $\mathbb{C}$) is a proper subfield of $\mathcal{F}$ (resp. $\mathcal{F} + i\mathcal{F}$), so there is no need to consider two separate formulations because the functions from $\mathcal{F}$ to $\mathbb{R}$ form a proper subset of the functions from $\mathcal{F}$ to $\mathcal{F}$. Finally an advantage $\mathcal{R}$ has over the other non-Archimedean fields is its suitability for implementation on a computer, this is especially useful for studying the asymptotic behaviour of real-valued functions. As we shall see in Chapter 5, the implementation of $\mathcal{R}$ numbers on a computer can be used in effectively calculating derivatives and integrals of real-valued functions. Although this document includes a number of advancements related to $\mathcal{F}$ and $\mathcal{R}$ it is by no means exhaustive. Progress on the topic of quantum mechanics in this context currently suffers from the lack of a theory of Fourier analysis and until a theory of manifolds and differential geometry is developed investigations into general relativity will remain impossible.

1.2 Outline

In the remainder of this chapter we will review the definition of the Hahn and Levi-Civita fields and we will summarize the preexisting work on measure theory and integration. In Chapter 2 we investigate a variety of topologies on the Hahn field, all weaker than the order topology, and we show that under the weakest of such topologies power-series on the Hahn field satisfy a similar convergence criterion as for those on the Levi-Civita field. The results from this chapter have been accepted for publication in *Indagationes Mathematicae* and at the time of writing are in print (see [10]). In Chapter 3 we begin by proposing three conjectures and we discuss certain related results from real analysis. We then prove a version of the Weierstrass Preparation Theorem for the Hahn field which is key to the inductive step required to prove the aforementioned conjectures. Finally, by accepting the conjectures, we show how integration can be defined in higher dimensions using induction on the dimension. The results regarding integration from this chapter are a generalization of the work published in [11]. In Chapter 4 we return to the rigorous treatment of the Dirac Delta function (see [3]) where we show what boundary conditions are necessary for a usable Green's function. We also include expanded examples and some corrections to previous results. The results from this chapter have been published in [12]. In Chapter 5 we discuss the Levi-Civita field as it relates to numerical computation and we present an implementation of it as a static library in the C++ programming language. We replicate the results of [4] regarding a method of numerical differentiation and we show how the same method can be used to compute the terms of real infinite sequences using their generating functions. Finally, we investigate related methods of numerical integration and compare them to more conventional methods of numerical integration. At the time of writing, these results are being prepared for publication. We include the code of the static library as an appendix for the reader's reference. Ideally this code will be made publicly available, however, it is not at this time clear how or where that might happen.

1.3 The Hahn and Levi-Civita Fields

As stated in the outline, the goal for the remainder of this chapter is to summarize the fundamental results regarding the Hahn and Levi-Civita fields. The most exhaustive study of these results is found in [13] and where not otherwise stated the reader may understand this to be the relevant reference throughout this chapter. We begin with a number of definitions.

Definition 1.3.1 (Well-ordered subset of $\mathbb{Q}$)**.** *Let $A \subset \mathbb{Q}$. Then we say that A is a* ***well-ordered subset of*** $\mathbb{Q}$ *if every non-empty subset of A has a minimum element.*

Definition 1.3.2 (Left-finite subset of $\mathbb{Q}$)**.** *Let $A \subset \mathbb{Q}$. Then we say that A is* ***left-finite*** *if, for any $q \in \mathbb{Q}$, the set*

$$A_{<q} := \{a \in A | a < q\}$$

is finite.

Since every finite set of the rational numbers has a minimum element it follows that every left-finite set is also a well-ordered set.

Definition 1.3.3 (The support of a function from $\mathbb{Q}$ to $\mathbb{R}$)**.** *Let $f : \mathbb{Q} \to \mathbb{R}$. Then the support of f is denoted by* $\operatorname{supp}(f)$ *and is defined by*

$$\operatorname{supp}(f) := \{q \in \mathbb{Q} | f(q) \neq 0\} .$$

Definition 1.3.4 (The sets $\mathcal{F}$ and $\mathcal{R}$)**.** *We define*

$$\mathcal{F} := \{f : \mathbb{Q} \to \mathbb{R} | \operatorname{supp}(f) \textit{ is well-ordered}\}$$

and

$$\mathcal{R} := \{f : \mathbb{Q} \to \mathbb{R} | \operatorname{supp}(f) \textit{ is left-finite}\} .$$

The close relationship between these two fields implies that many definitions and propositions hold equally for both. To avoid needless repetition we will let $\mathcal{K}$ denote either of $\mathcal{F}$ or $\mathcal{R}$. Elements of $\mathcal{F}$ and $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and in the course of this dissertation we will sometimes discuss these functions evaluated at specific points in their domain, and some other times, we will consider functions on $\mathcal{F}$ and $\mathcal{R}$. To avoid confusion we reintroduce the following notations.

Remark 1.3.5 (Notation regarding elements versus functions)**.** *We employ the convention that square brackets (i.e. '[' and ']') denote an element of either $\mathcal{F}$ or $\mathcal{R}$ evaluated at some point in $\mathbb{Q}$ whereas curved brackets (i.e. '(' and ')') denote a function on $\mathcal{F}$ or $\mathcal{R}$ evaluated at a point in one of those sets. So for example if we have $x \in \mathcal{K}$, $q \in \mathbb{Q}$, and $f : \mathcal{K} \to \mathcal{K}$, then*

- *$x[q] \in \mathbb{R}$ denotes an element x of $\mathcal{K}$ evaluated at a point q in $\mathbb{Q}$.*
- *$f(x) \in \mathcal{K}$ denotes a function f evaluated at an element x in $\mathcal{K}$.*

- $f(x)[q] \in \mathbb{R}$ *denotes a function evaluated at a point in* $\mathcal{K}$ *and the result of that evaluation (itself an element of* $\mathcal{K}$*) evaluated at a point in* $\mathbb{Q}$.

We also define the following notion to aid in our discussion of these sets.

Definition 1.3.6. *Let* $x \in \mathcal{K}$. *Then we define*

$$\lambda(x) \coloneqq \begin{cases} \min \operatorname{supp}(x) & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

Notice in the above definition the minimum is guaranteed to exist by the well-orderedness of the support of a non-zero element of $\mathcal{K}$. For $x \in \mathcal{K}$, $\lambda(x)$ corresponds to the "order of magnitude" of x, we make this notion more rigorous below after we have defined the order on $\mathcal{F}$ and $\mathcal{R}$ in Definition 1.3.8.

Definition 1.3.7 (Addition and multiplication on $\mathcal{F}$ and $\mathcal{R}$)**.** *Let* $x, y \in \mathcal{K}$ *be given. Then we define for every* $q \in \mathbb{Q}$

- $(x + y)[q] = x[q] + y[q]$

- $(x \cdot y)[q] = \displaystyle\sum_{\substack{q_1 \in \operatorname{supp}(x) \\ q_2 \in \operatorname{supp}(y) \\ q_1 + q_2 = q}} x[q_1] \cdot y[q_2]$

If A, B are well-ordered sets and $r \in A + B$ then by [14, Theorem 1.3] there are only finitely many pairs $(a, b) \in A \times B$ such that $a + b = r$. This fact ensures that multiplication on $\mathcal{F}$ and $\mathcal{R}$ is well defined since the sum in the definition will only ever have finitely many terms and thus will always converge. Under these definitions of addition and multiplication $(\mathcal{K}, +, \cdot)$ is a field [15], and in fact we can isomorphically embed the real numbers into $\mathcal{K}$ as a subfield using the map $\Pi : \mathbb{R} \to \mathcal{K}$ defined by

$$\Pi(x)[q] \coloneqq \begin{cases} x & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}.$$

Definition 1.3.8. *(Order on $\mathcal{F}$ and $\mathcal{R}$) Let $x, y \in \mathcal{K}$ be distinct. We say $x > y$ if and only if $(x-y)[\lambda(x-y)] > 0$. We say $x < y$ if $y > x$ and we say $x \geq y$ if either $x = y$ or $x > y$.*

Under this order relation, $(\mathcal{K}, \geq)$ is a totally ordered field. Moreover, the embedding of $\mathbb{R}$ into this field via the map Π defined above is order preserving [15]. The next definition introduces some convenient notations for comparing elements of $\mathcal{F}$ and $\mathcal{R}$; it will also allow us to say more regarding λ, the notion of infinite and infinitesimal elements, and their relation to each other. The map $|\cdot|_u : \mathcal{F} \to \mathbb{R}$ (resp. $\mathcal{R} \to \mathbb{R}$), given by

$$|x|_u = \begin{cases} e^{-\lambda(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is an "ultrametric" (or "non-Archimedean") valuation, which is to say that it satisfies the strong triangle inequality [6]. This ultrametric valuation induces on $\mathcal{K}$ the same topology as the order relation [4, 6]; we will denote this topology by τ_v hereafter and refer to it as either the **order topology**, the **valuation topology**, or the **strong topology**. The fields $\mathcal{F}$ and $\mathcal{R}$ are complete with respect to τ_v [6].

Definition 1.3.9 ($\ll$, $\gg$, $\sim$, $\approx$, and $=_q$)**.** *Let $x, y \in \mathcal{K}$ be non-negative. We say that x is infinitely larger than y and write $x \gg y$ if for every $n \in \mathbb{N}$, $x > ny$; we say x is infinitely smaller than y and write $x \ll y$ if for every $n \in \mathbb{N}$, $y > nx$. We say that x is infinitely large if $x \gg 1$ and we say it is infinitely small or infinitesimal if $x \ll 1$. Suppose that $\lambda(x) = \lambda(y) = \lambda_0$ then we write $x \sim y$. If in addition we have that $x[\lambda_0] = y[\lambda_0]$, we write $x \approx y$. We say $x =_q y$ if $x[q'] = y[q']$ for all $q' \leq q$.*

Notice in the above definition that $x \gg y$ if and only if $\lambda(x) < \lambda(y)$; moreover, since $\lambda(1) = 0$, x is infinitely large if and only if $\lambda(x) < 0$ and x is infinitesimal if and only if $\lambda(x) > 0$. The non-zero real numbers satisfy $\lambda(x) = 0$ as does the sum of a real number and an infinitesimal number. We define $\lambda(0) = \infty$ so that for every $x \in \mathcal{K}$ with $x \neq 0$ we have $\lambda(x) < \lambda(0)$.

Definition 1.3.10 (The number d). *We define the element $d \in \mathcal{K}$ as follows: for every $q \in \mathbb{Q}$,*

$$d[q] := \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases}.$$

Under this definition $d > 0$ and it is infinitesimal ($\lambda(d) = 1$); moreover, following from the definition of multiplication, we have that for any $r \in \mathbb{Q}$,

$$d^r[q] := \begin{cases} 1 & \text{if } q = r \\ 0 & \text{if } q \neq r \end{cases}.$$

In particular we have that

$$d^{-1}[q] := \begin{cases} 1 & \text{if } q = -1 \\ 0 & \text{if } q \neq -1 \end{cases}.$$

Since $\lambda(d^{-1}) = -1$, d^{-1} is infinitely large. This is consistent with d being infinitesimal and in fact it allows the statement of an interesting inequality, namely

$$0 < d < z < d^{-1},$$

for all $z \in \mathbb{R}^+$. So in the Hahn and Levi-Civita fields the set of positive real numbers is bounded both above and below.

1.4 Power-Series and Analytic Functions

Functions on non-Archimedean fields often display properties that appear very different from those of real-valued functions on the real field $\mathbb{R}$. In particular it is possible to construct continuous functions that are not bounded on a closed and bounded interval, continuous and bounded functions that attain neither a maximum nor a minimum value on closed intervals, and continuous and differentiable functions with a derivative equal to zero everywhere on an interval which are nevertheless non-constant on that interval [4]. These unusual properties are a result of the total disconnectedness of these structures in the order topology [16, 4]. Much work has been done in showing that power series and analytic functions on the Levi-Civita field have the same smoothness properties as real power series and real analytic functions [2]. The effort to extend these properties to as large a class of functions as possible has been aided considerably by the introduction of the so-called weak topology on the Levi-Civita field which is strictly weaker than the order topology and thus allows for more power series to converge than the order topology. In Chapter 2 we will show how a similar weak topology may be induced on the Hahn field and we derive convergence criteria for sequences and power series in this new topology. Here we briefly review the properties of power-series and analytic functions on the Levi-Civita field, a detailed discusion can be found in [17, 16].

Definition 1.4.1. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}$. Suppose that*

$$\bigcup_{n\in\mathbb{N}} \operatorname{supp}(s_n)$$

*is a left finite set. Then we say that $(s_n)_{n\in\mathbb{N}}$ is a **regular sequence**.*

Definition 1.4.2. *We say that a sequence in $\mathcal{R}$ **converges strongly** if it converges with respect to the order topology.*

The "**weak topology**" mentioned above is constructed using the family of semi-norms defined below:

Definition 1.4.3 (A Family of Semi-Norms on $\mathcal{R}$)**.** *For every $q \in \mathbb{Q}$ define the map $\|\cdot\|_{(w,q)} : \mathcal{R} \to \mathbb{R}$ by*

$$\|x\|_{(w,q)} := \sup\left\{|x[r]| | r \in \operatorname{supp}(x) \cap (-\infty, q]\right\}.$$

Since every $x \in \mathcal{R}$ has a left-finite support, the supremum in the above definition is a maximum, it being the supremum of a finite set. This is not the case for $x \in \mathcal{F}$ where the support of an element need only be well-ordered, which allows the support to have accumulation points and so for some $x \in \mathcal{F}$, $\{|x[r]|| r \in \operatorname{supp}(x) \cap (-\infty, q]\}$ may have a divergent subsequence and thus the supremum may be $+\infty$.

Definition 1.4.4. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}$. Then we say that this sequence* ***converges weakly*** *or is* ***weakly convergent*** *if there is an $s \in \mathcal{R}$ such that for every $\epsilon > 0$ in $\mathbb{R}$ there is a $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$\|s_n - s\|_{\left(w, \frac{1}{\epsilon}\right)} < \epsilon.$$

Proposition 1.4.5 (Weak convergence criterion for sequences in $\mathcal{R}$ [16])**.** *Let $(s_n)_{n=0}^{\infty}$ be a weakly convergent sequence in $\mathcal{R}$. Then, for every $q \in \mathbb{Q}$, the sequence $(s_n[q])$ converges in $\mathbb{R}$ in the standard topology. Conversely, if (s_n) is a regular sequence and if, for every $q \in \mathbb{Q}$, $(s_n[q])$ converges in $\mathbb{R}$ in the standard topology then (s_n) converges weakly in $\mathcal{R}$.*

Definition 1.4.6 (Power series on the Levi-Civita field)**.** *A* ***power series*** *is any formal expression of the form*

$$S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{R}$, x_0 is fixed in $\mathcal{R}$, and x is an independent variable. We say that a power series ***converges weakly*** *in $\mathcal{R}$ for some $x \in \mathcal{R}$ if the sequence of partial sums*

$$S_m(x) = \sum_{n=0}^{m} a_n (x - x_0)^n$$

converges weakly in $\mathcal{R}$.

We can now state a criteria that is sufficient for power series to converge both in the order topology and in the weak topology.

Proposition 1.4.7 (Strong convergence criterion for power series on the Levi-Civita field

[16]). *Let* $(a_n)_{n=0}^{\infty}$ *be a regular sequence in* $\mathcal{R}$ *such that*

$$\limsup_{n\to\infty}\left(-\frac{\lambda(a_n)}{n}\right) = \lambda_0 \in \mathbb{R}\cup\{-\infty,\infty\}\,.$$

Let $x_0 \in \mathcal{R}$ *be fixed and suppose* $x \in \mathcal{R}$ *is given. The power series*

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

converges strongly if $\lambda_0 < \lambda(x-x_0)$ *and diverges strongly if* $\lambda_0 > \lambda(x-x_0)$ *or if* $\lambda_0 = \lambda(x-x_0)$ *and* $-\frac{\lambda(a_n)}{n} > \lambda_0$ *for infinitely many* $n \in \mathbb{N}$.

As the example below illustrates many power series representations of common analytic functions fail to converge in the order topology outside an infinitesimally small neighbourhood. This difficulty motivates the introduction of the weak topology.

Example 1.4.8. *The power series representation of the exponential function about* $x = 0$ *is given by (see [17])*

$$exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We wish to determine for which values of $x \in \mathcal{R}$ *the above sum converges. Recall that in the order topology if* $(a_n)_{n\in\mathbb{N}} \in \mathcal{R}$ *is an infinite sequence then the sum* $\sum_{n=0}^{\infty} a_n$ *converges if and only if* $\lim_{n\to\infty} a_n = 0$*, in our case this means that the above representation of* $exp(x)$ *converges if and only if* $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ *[14]. This condition is met if and only if* $\lim_{n\to\infty} x^n = 0$ *or equivalently if and only if* $\lim_{n\to\infty} n\lambda(x) = \infty$ *and the last condition is met if and only if* $\lambda(x) > 0$*. Thus we see that the series representation of exp is convergent in the order topology only on the infinitesimal neighbourhood about 0.*

As the following proposition makes clear, the weak topology has a more convenient convergence criterion for power series which allows us to avoid the difficulty illustrated above.

Proposition 1.4.9 (Weak convergence criterion for power series on the Levi-Civita field

[16]). *Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathcal{R}$ and suppose that*

$$\limsup_{n\to\infty}\left(-\frac{\lambda(a_n)}{n}\right)=\lambda_0.$$

Let $x_0\in\mathcal{R}$ be fixed and suppose $x\in\mathcal{R}$ is such that $\lambda(x-x_0)=\lambda_0$. Finally, suppose that $(a_n d^{n\lambda_0})_{n=0}^{\infty}$ is a regular sequence. Then

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

converges weakly if $|(x-x_0)[\lambda_0]|<r$ and diverges weakly if $|(x-x_0)[\lambda_0]|>r$ where

$$r:=\frac{1}{\sup\left\{\limsup_{n\to\infty}|(a_n d^{n\lambda_0})[q]|^{\frac{1}{n}}:q\in\bigcup_{n\in\mathbb{N}}\operatorname{supp}(a_n d^{n\lambda_0})\right\}}.$$

Definition 1.4.10 (Analytic function on the Levi-Civita field)**.** *Let $a<b$ in $\mathcal{R}$ be given and let $f:[a,b]\to\mathcal{R}$. Then we say that f is* ***analytic on*** $[a,b]$ *if for all $x\in[a,b]$ there exists a positive $\eta(x)\sim b-a$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y)=\sum_{n=0}^{\infty}a_n(x)(y-x)^n$ for all $y\in(x-\eta(x),x+\eta(x))\cap[a,b]$.*

Fortunately, analytic functions on $\mathcal{R}$ behave similarly to their counterparts on $\mathbb{R}$. For example, if $f,g:I\subset\mathcal{R}\to\mathcal{R}$ are two analytic functions on an interval I and if $\alpha\in\mathcal{R}$ is a constant then $f\cdot g$ and $f+\alpha g$ are analytic on I [15]. Additionally it has been shown that the composition of analytic functions are analytic and if f is analytic on a closed interval $[a,b]\subset\mathcal{R}$, then f must be bounded on $[a,b]$ [17]. Finally, analytic functions on an interval $[a,b]$ satisfy a mean value theorem, an inverse function theorem, and an intermediate value theorem [18, 19]. We also note that in [17] it is shown that if $a<b$ in $\mathcal{R}$ and $f:[a,b]\to\mathcal{R}$ is analytic on $[a,b]$, then there exists a rational number called the **index** of f defined by

$$i(f)=\min\{\lambda(f(x)):x\in[a,b]\}.$$

Moreover, it is shown that $\lambda(f(x)) = i(f)$ almost everywhere on $\{x \in \mathcal{R} : x \in [a,b], supp(x-a) = \{\lambda(b-a)\}\}$ and for any such x the same is true for every element y in the neighbourhood of x satisfying $\lambda(y-x) > \lambda(b-a)$. One convenient consequence of the above is that we may assume without loss of generality that analytic functions have an index equal to zero, this is so since scaling the function by d^{-l} does not affect the property of being analytic.

1.5 Measure Theory and Integration

For a subset A of the real numbers the Lebesgue outer measure is defined as

$$\mu^*(A) := \inf \left\{ \sum_{n=0}^{\infty} (b_n - a_n) \middle| (a_n, b_n)_{n\in\mathbb{N}} \text{ are mutually disjoint, open intervals, and } A \subset \bigcup_{n=0}^{\infty} (a_n, b_n) \right\}.$$

In the Levi-Civita field this infimum need not exist [15] and so it is not possible to define the measure in exactly the same way we do the Lebesgue measure in real analysis. Instead we use the definition below to construct a Lebesgue-like measure (and eventually a Lebesgue-like integral). Note that for $a < b$ in $\mathcal{R}$ $I(a,b)$ denotes any one of the intervals (a,b), $(a,b]$, $[a,b)$, or $[a,b]$ and $l(I(a,b)) = b - a$.

Definition 1.5.1 (Measurable set). *Let $A \subset \mathcal{R}$ be given. We say that A is **measurable** if for every $\epsilon > 0$ in $\mathcal{R}$ there exist two countable sequences of mutually disjoint intervals $(I_n)_{n=1}^{\infty}$ and $(J_n)_{n=1}^{\infty}$ such that*

$$\bigcup_{n=1}^{\infty} I_n \subset A \subset \bigcup_{n=1}^{\infty} J_n,$$

$\sum\limits_{n=1}^{\infty} l(I_n)$ *and* $\sum\limits_{n=1}^{\infty} l(J_n)$ *both converge, and*

$$\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) < \epsilon.$$

If $A \subset \mathcal{R}$ is a measurable set then for every $k \in \mathbb{N}$ there exist two countable sequences of mutually disjoint intervals $(I_n^k)_{n=1}^{\infty}$ and $(J_n^k)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k,$$

$\sum_{n=1}^{\infty} l(I_n^k)$ and $\sum_{n=1}^{\infty} l(J_n^k)$ both converge, and $\sum_{n=1}^{\infty} l(J_n^k) - \sum_{n=1}^{\infty} l(I_n^k) < d^k$. It is shown in [20] that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l(I_n^k)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l(J_n^k)$ both exist and are equal. Furthermore, the measure of A is denoted by $m(A)$ and is given by

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l(I_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l(J_n^k).$$

Every analytic function defined on an open interval can be extended to the end points of that interval and every analytic function has a unique (up to a constant) analytic anti-derivative [20]. These facts allow for the following definition of the integral of an analytic function over an interval.

Definition 1.5.2 (Integral of an analytic function over an interval)**.** *Let $a < b$ in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be an analytic function on $I(a,b)$. Let F be an analytic anti-derivative of f on $I(a,b)$, then we define the* ***integral of** f **over*** *$I(a,b)$ as follows:*

$$\int_{x\in I(a,b)} f(x) = \lim_{x\to b} F(x) - \lim_{x\to a} F(x).$$

The next definition is motivated by the desire to extend the theory of integration to as large a class of functions as possible.

Definition 1.5.3 (Measurable function)**.** *Let $A \subset \mathcal{R}$ be a measurable set and let $f : A \to \mathcal{R}$ be a bounded function on A. Then we call f a* ***measurable function*** *if for every $\epsilon > 0$ in $\mathcal{R}$ there exists a sequence of mutually disjoint intervals $(I_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n \subset A$, $\sum_{n=1}^{\infty} l(I_n)$ converges, $m(A) - \sum_{n=1}^{\infty} l(I_n) < \epsilon$, and for every $n \in \mathbb{N}$ f is analytic on I_n.*

With these definitions in place it is possible to define the integral of a measurable function over a measurable set.

Definition 1.5.4 (Integral of a measurable function over a measurable set)**.** *Let $A \subset \mathcal{R}$ be a measurable set and let $f : A \to \mathcal{R}$ be a measurable function on A. Then, for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint intervals $(I_n^k)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n^k \subset A$, $\sum_{n=1}^{\infty} l(I_n^k)$ converges, $m(A) - \sum_{n=1}^{\infty} l(I_n^k) < d^k$, and for every $n \in \mathbb{N}$ f is analytic on I_n^k. It can be shown [20] that for every $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} \int_{x \in I_n^k} f$ converges. The resulting sequence $\left(\sum_{n=1}^{\infty} \int_{x \in I_n^k} f \right)_{k=1}^{\infty}$ is Cauchy and hence it converges [20]. The* ***integral of*** *f* ***over*** *A is then defined by*

$$\int_{x \in A} f = \lim_{k \to \infty} \sum_{n=1}^{\infty} \int_{x \in I_n^k} f$$

It is shown in [20] that the above integral behaves in much the same way as the Lebesgue integral of real analysis. For example integration is linear, the sum of the integrals of a function over two measurable sets is equal to the sum of the integrals over their union and intersection, and if $f : A \to \mathcal{R}$ and $M \in \mathcal{R}$ satisfy $|f| \leq M$ everywhere on A then $|\int_A f| \leq Mm(A)$. Another theorem that will be of particular importance to us is [13, Theorem 3.9] which we state below.

Theorem 1.5.5 (Uniform Convergence Theorem for the Levi-Civita Field)**.** *Let $A \subset \mathcal{R}$ be a measurable set and for every $k \in \mathbb{N}$ let $f_k : A \to \mathcal{R}$ be measurable on A. Suppose that the sequence $(f_k)_{k \in \mathbb{N}}$ converges uniformly to $f : A \to \mathcal{R}$. Then f is measurable on A, $\lim_{k \to \infty} \int_A f_k$ exists, and*

$$\lim_{k \to \infty} \int_A f_k = \int_A f.$$

The integration theory discussed above can be extended to higher dimensions. In fact this task is not as easy as it might seem due to the unique convergence properties in the order topology. The natural way to extend our measure theory to two-dimensions would be to follow the same steps we took in one dimension using rectangles in place of intervals and areas in place of lengths, unfortunately; we define our (one-dimensional) measure in terms of sums of the lengths of intervals that converge in the order topology. Infinite sums converge in the order topology only if the terms in the sum form a null sequence and this entails that only finitely many of those terms can contribute to the leading term in the sum. This is not an issue in one dimension but in two-dimensions (or higher) it severely restricts which sets are measurable. We illustrate this with an example:

Example 1.5.6. *Suppose $T \subset \mathcal{R} \times \mathcal{R}$ is a triangle with finite area and suppose we have extended our theory of integration to two dimensions using rectangles in place of intervals. Assume T is measurable. Then, there are sequences of mutually disjoint rectangles $(R_n)_{n\in\mathbb{N}}, (S_n)_{n\in\mathbb{N}} \in \mathcal{R} \times \mathcal{R}$ such that*

$$\cup_{n=0}^{\infty} R_n \subset T \subset \cup_{n=0}^{\infty} S_n, \tag{1.1}$$

$\sum\limits_{n=0}^{\infty} area(R_n)$ *and* $\sum\limits_{n=0}^{\infty} area(S_n)$ *converge, and*

$$\sum_{n=0}^{\infty} area(S_n) - \sum_{n=0}^{\infty} area(R_n) < d.$$

However, since the sums of the areas of the rectangles both converge in the order topology we have that $\lim_{n\to\infty} area(S_n) = \lim_{n\to\infty} area(R_n) = 0$ so there are only finitely many n such that $\lambda(area(S_n)) = 0$ and the same is true for $\lambda(area(R_n)) = 0$. Of course there is no way to arrange a finite number of rectangles into a triangle and so $\sum\limits_{n=0}^{\infty} area(R_n)$ must be less than the area of T by a finite amount and $\sum\limits_{n=0}^{\infty} area(S_n)$ must be greater by a finite amount too;

$$\sum_{n=0}^{\infty} area(S_n) - \sum_{n=0}^{\infty} area(R_n) > d.$$

This contradicts Equation 1.1 above; thus, triangles are not measurable.

Obviously we want all common geometric shapes to be measurable if we want our measure to be useful in physics so we must find a more general way of extending our one-dimensional measure theory into higher dimensions, this fact motivates the definition of simple regions below. A detailed discussion of the topic of integration in two-dimensions can be found in [3] and [11] from which we get these results.

Definition 1.5.7 (Simple Region). *Suppose that $G \subset \mathcal{R}^2$. Then, we call G a **simple region** if there exist constants $a, b \in \mathcal{R}, a \leq b$ and analytic functions $g_1, g_2 : I(a,b) \to \mathcal{R}, g_1 < g_2$ on $I(a,b)$ such that*

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$$

or

$$G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(a,b)\}.$$

In order to use simple regions to construct a two-dimensional integration theory we need to be able to find their area. Fortunately, this can be done inductively using the already existing one-dimensional theory of integration.

Definition 1.5.8 (Area of a Simple Region). *Suppose $G \subset \mathcal{R}^2$ is a simple region given by $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$. Then we denote the **area of** G with $a(G)$ and we define it as*

$$a(G) = \int\limits_{x \in I(a,b)} [g_2(x) - g_1(x)]$$

We now use the simple regions defined above to construct a measure and to define measurable sets on $\mathcal{R}^2$. The proofs of the following propositions can be found in [3, 11].

Definition 1.5.9 (Measurable Set). *Let $A \subset \mathcal{R}^2$. We say A is **measurable** if for every $\epsilon > 0$ there are a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and another sequence of simple regions $(H_n)_{n=1}^{\infty}$ such that*

$$\bigcup_{n=1}^{\infty} G_n \subset A \subset \bigcup_{n=1}^{\infty} H_n,$$

$\sum_{n=1}^{\infty} a(G_n)$ *and* $\sum_{n=1}^{\infty} a(H_n)$ *both converge, and*

$$\sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) < \epsilon.$$

Definition 1.5.10 (The Measure of a Measurable Set)**.** *Let $A \subset \mathcal{R}^2$ be a measurable set. Then, for every $k \in \mathbb{N}$ there are two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty} \in \mathcal{R}^2$ and $(H_n^k)_{n=1}^{\infty} \in \mathcal{R}^2$ such that*

$$\bigcup_{n=1}^{\infty} G_n^k \subset A \subset \bigcup_{n=1}^{\infty} H_n^k,$$

$\sum_{n=1}^{\infty} G_n^k$ *and* $\sum_{n=1}^{\infty} H_n^k$ *both converge, and*

$$\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k.$$

In fact $(\sum_{n=1}^{\infty} a(G_n^k))_{k=1}^{\infty}$ *and* $(\sum_{n=1}^{\infty} a(H_n^k))_{k=1}^{\infty}$ *are both Cauchy sequences. Since $\mathcal{R}$ is Cauchy complete [6] it follows that*

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k)$$

and

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$$

both exist. Moreover the above two limits must be equal [11] and we define

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$$

and we call this the ***measure of*** *A.*

The next proposition shows that measurable sets have the expected properties.

Proposition 1.5.11. *Let $A, B \subset \mathcal{R}^2$.*

- *If A and B are measurable with $A \subset B$, then $m(A) \leq m(B)$.*

- *If B is measurable with $m(B) = 0$ and if $A \subset B$, then A is measurable and $m(A) = 0$.*

- *If A is countable then A is measurable and $m(A) = 0$.*

- *If A and B are measurable, then $A \cup B$ and $A \cap B$ are measurable and $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.*

Definition 1.5.12 (Order of Magnitude of a Simple Region in $\mathcal{R}^2$)**.** *Let $A \subset \mathcal{R}^2$ be a simple region. Without loss of generality we assume*

$$A = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$$

where $a \leq b$, $h_1, h_2 : I(a, b) \to \mathcal{R}$ are analytic functions, and $h_1 < h_2$. We define

$$\lambda_x(A) = \lambda(b - a)$$

and

$$\lambda_y(A) = i(h_2(x) - h_1(x))$$

*where $i(h_2(x) - h_1(x))$ is the index of the analytic function on $I(a, b)$, we call these A's **orders of magnitude in** x **and** y respectively.*

With the above definition we are able to define analytic functions in $\mathcal{R}^2$.

Definition 1.5.13 (Analytic Functions on $\mathcal{R}^2$)**.** *Suppose $A \subset \mathcal{R}^2$ is a simple region. Then, $f : A \to \mathcal{R}$ is an **analytic function** if for every $(x_0, y_0) \in A$, there are a simple region A_0 containing (x_0, y_0) that satisfies $\lambda_x(A_0) = \lambda_x(A)$, $\lambda_y(A_0) = \lambda_y(A)$ and a regular sequence $(a_{ij})_{i,j=0}^{\infty}$ such that for every $s, t \in \mathcal{R}$, if $(x_0 + s, y_0 + t) \in A \cap A_0$, then*

$$f(x_0 + s, y_0 + t) = \sum_{i,j=0}^{\infty} a_{ij} s^i t^j = f(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j \neq 0}}^{\infty} a_{ij} s^i t^j,$$

where the power series converges in the weak topology [16]. In this case we say that f is given ***locally*** *by a power series.*

The family of analytic functions on $\mathcal{R}^2$ is closed under addition, multiplication, and composition; each member of the family is bounded on closed simple regions and has an index analogous to that of their one-dimensional counterparts. We are therefore able to define integration in two-dimensions using the already existing theory of one-dimensional integration.

Definition 1.5.14 (Integration of Analytic Functions on Simple Regions)**.** *Suppose $H \subset \mathcal{R}^2$ is a simple region and $f : H \to \mathcal{R}$ is an analytic function. Without loss of generality we assume $H = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$, where $a, b \in \mathcal{R}$, $a \leq b$, and $h_1, h_2 : I(a, b) \to \mathcal{R}$ are analytic with $h_1 < h_2$. We note that since the composition and anti-derivative of analytic functions are also analytic, the integral $\int\limits_{y \in I(h_1(x), h_2(x))} f(x, y)$ will always yield an analytic function $F(x)$ on $I(a, b)$ [3]. We define*

$$\iint\limits_{(x,y)\in H} f(x, y) = \int\limits_{x \in I(a,b)} \left[\int\limits_{y \in I(h_1(x), h_2(x))} f(x, y) \right] = \int\limits_{x \in I(a,b)} F(x)$$

and call this the ***integral of*** f ***over*** H*.*

Proposition 1.5.15 (Properties of two-dimensional integration of analytic functions over simple regions [11])**.** *Let $G \subset \mathcal{R}^2$ be a simple region, let $\alpha \in \mathcal{R}$ be an arbitrary constant, and let $f, g : G \to \mathcal{R}$ be analytic functions. Then:*

- $\iint\limits_{(x,y)\in G} \alpha = \alpha a(G)$.
- $\iint\limits_{(x,y)\in G} (f + \alpha g)(x, y) = \iint\limits_{(x,y)\in G} f(x, y) + \alpha \iint\limits_{(x,y)\in G} g(x, y)$.
- *Suppose $f \leq g$ on G. Then $\iint\limits_{(x,y)\in G} f(x, y) \leq \iint\limits_{(x,y)\in G} g(x, y)$.*
- *Suppose f is non-positive on G. Then $\iint\limits_{(x,y)\in G} f(x, y) \leq 0$.*

- *Suppose M is an upper bound of $|f|$ on G. Then* $|\iint\limits_{(x,y)\in G} f(x,y)| \leq Ma(G)$.

Definition 1.5.16 (Measurable function)**.** *Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $f : A \to \mathcal{R}$ be bounded. Then we say that f is **measurable on** A if for every $\epsilon > 0$ there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that*

$$\bigcup_{n=0}^{\infty} G_n \subset A,$$

$\sum\limits_{n=1}^{\infty} a(G_n)$ *converges,*

$$m(A) - \sum_{n=1}^{\infty} a(G_n) < \epsilon,$$

and for all $n \in \mathbb{N}$ f is analytic on G_n.

The following proposition collects together several results which are stated and proven independently in [3] and [11].

Proposition 1.5.17. *Suppose $A, B \subset \mathcal{R}^2$ are measurable sets, let $f : A \cup B \to \mathcal{R}$ be a measurable function on both A and B and let $g : A \to \mathcal{R}$ also be measurable. Then we have that*

- *g is given locally by a power series almost everywhere on A*
- *f is measurable on both $A \cap B$ and $A \cup B$*
- *For any $\alpha \in \mathcal{R}$, $f + \alpha g$ and $f \cdot g$ are measurable on A*
- *If f and g are differentiable (see [2]) on A with respect to both x and y and if $\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} g(x,y)$ and $\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} g(x,y)$ everywhere on A then f and g are different by at most a constant on A.*

Definition 1.5.18 (The Integral of a Measurable Function over a Measurable Set)**.** *Let $A \subset \mathcal{R}^2$ be a measurable set and let $f : A \to \mathcal{R}$ be a measurable function. Since f is* *measurable then, by Definition 1.5.16, for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^k,*

$$\bigcup_{n=1}^{\infty} G_n^k \subset A,$$

$\sum\limits_{n=1}^{\infty} a(G_n^k)$ *converges, and*

$$m(A) - \sum_{n=1}^{\infty} a(G_n^k) \leq d^k.$$

It is shown in [11] that for every $k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y)$$

converges and moreover

$$\left(\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y) \right)_{k=1}^{\infty}$$

*is a Cauchy sequence and thus converges in $\mathcal{R}$. We call the limit of this Cauchy sequence the **integral of** f **over** A and we denote it $\iint\limits_{(x,y)\in A} f(x,y)$.*

The results listed in the following proposition are a consequence of Propositions 1.5.15 and 1.5.17, their proofs can be found in [3] and [11].

Proposition 1.5.19 (Properties of two-dimensional integration of measurable functions over measurable sets [11])**.** *Suppose $H, G \subset \mathcal{R}^2$ are measurable sets, let $\alpha \in \mathcal{R}$ be an arbitrary constant and let $f, g : H \cup G \to \mathcal{R}$ be measurable functions.*

- *If $h : G \to \mathcal{R}$ is defined by $h(x,y) = \alpha$, then h is a measurable function and*

$$\iint\limits_{(x,y)\in G} h(x,y) = \iint\limits_{(x,y)\in G} \alpha = \alpha m(G).$$

- $$\iint\limits_{(x,y)\in G} (f+\alpha g)(x,y) = \iint\limits_{(x,y)\in G} f(x,y) + \alpha \iint\limits_{(x,y)\in G} g(x,y).$$

- *If* $f \leq g$ *everywhere on* G*, then* $\iint\limits_{(x,y)\in G} f(x,y) \leq \iint\limits_{(x,y)\in G} g(x,y)$*.*

- *If* f *is non-positive on* G*, then* $\iint\limits_{(x,y)\in A} f(x,y) \leq 0$*.*

- *If* M *is an upper bound for* $|f|$ *on* G*, then* $|\iint\limits_{(x,y)\in A} f(x,y)| \leqslant Mm(A)$*.*

- $$\iint\limits_{(x,y)\in H\cup G} f(x,y) = \iint\limits_{(x,y)\in H} f(x,y) + \iint\limits_{(x,y)\in G} f(x,y) - \iint\limits_{(x,y)\in H\cap G} f(x,y).$$

- *If there is a sequence of measurable functions* $h_k : G \to \mathcal{R}$ *such that the sequence* $(h_k)_{k=1}^{\infty}$ *converges uniformly to* h*, then*

  $$\lim_{k\to\infty} \iint\limits_{(x,y)\in G} h_k(x,y)$$

  exists; moreover if h *is measurable then*

  $$\lim_{k\to\infty} \iint\limits_{(x,y)\in A} h_k(x,y) = \iint\limits_{(x,y)\in A} h(x,y).$$

Chapter 2

Topologies on the Hahn Field

2.1 Semi-Norms

Since every $x \in \mathcal{R}$ has left-finite support, the supremum in the family of semi-norms that induce the weak topology is a maximum, it being the supremum of a finite set. This is not the case for $x \in \mathcal{F}$ where the support of an element need only be well-ordered, which allows the support to have accumulation points and so for some $x \in \mathcal{F}$, $\{|x[r]| : r \in \operatorname{supp} x \cap (-\infty, q]\}$ may have a divergent subsequence and thus the supremum may be $+\infty$. In this chapter we propose two different ways to overcome this difficulty and we show how they may be employed to induce a variety of topologies that are weaker than the order topology. Then we show how these topologies are related to each other, and finally we show what conditions must be satisfied to ensure that the induced topology has the same convergence criterion as the weak topology on the Levi-Civita field [1, 2]. One straightforward way to overcome the issue described above is simply to allow a semi-norm to be equal to $+\infty$ in addition to values in $\mathbb{R}$. This idea is made clear in the following definition which we believe to be novel.

Definition 2.1.1 (Semi-Norms on $\mathcal{F}$). *For every $r \in \mathbb{Q}$ define a map $\|\cdot\|_{(u,r)} : \mathcal{F} \to \mathbb{R} \cup \{\infty\}$ by*

$$\|x\|_{(u,r)} := \sup\{|x[q]| : q \leq r\}$$

The family of semi-norms from definition 2.1.1 has the advantage that it reduces to 1.4.3 when restricted to the Levi-Civita field. However, as we shall see, the topology induced on the Hahn field by this family is somewhat stronger than the weak topology on the Levi-Civita field. The following new definitions will allow us to construct a similar family of semi-norms which, as we will see, has more useful properties.

Definition 2.1.2 (Well-Bounded Sets)**.** *Let $(S, \leq)$ be a totally ordered set such that every nonempty subset $A \subset S$ has a maximum element. Then we say that S is* ***well-bounded****.*

Definition 2.1.3 (Well-Bounded Partition of $\mathbb{Q}$)**.** *Let $\Gamma = \{\gamma_1, \ldots, \gamma_i, \ldots\}$ be a countable collection of mutually disjoint well-bounded subsets of $\mathbb{Q}$ such that*

$$\bigcup_{i=1}^{\infty} \gamma_i = \mathbb{Q}.$$

Then we say that Γ is a ***well-bounded partition*** *of $\mathbb{Q}$. If in addition to the above we have that for every $i \in \mathbb{N}$, γ_i is finite then we call Γ a* ***finite well-bounded partition*** *of $\mathbb{Q}$. For convenience we will use the notation*

$$\Gamma_n := \bigcup_{i=1}^{n} \gamma_i.$$

Example 2.1.4 (A well-bounded partition of $\mathbb{Q}$)**.** *For every $n \in \mathbb{N}$ define*

$$\gamma_n := \left\{ \frac{x}{y} \in \mathbb{Q} : x \text{ and } y \text{ are relatively prime}, |x| + |y| = n \right\},$$

then $\Gamma = \{\gamma_i\}_{i \in \mathbb{N}}$ is a well-bounded partition of $\mathbb{Q}$. Clearly we have

$$\bigcup_{i=1}^{\infty} \gamma_i = \mathbb{Q}$$

since for any $q \in \mathbb{Q}$ it is possible to find unique $x, y \in \mathbb{Z}$ such that x and y are relatively prime and $\frac{x}{y} = q$. We then have by definition that $q \in \gamma_{|x|+|y|}$. Moreover, since every rational *number has a unique reduced form, it follows that for any $i \neq j$ in $\mathbb{N}$, we have that $\gamma_i \cap \gamma_j = \emptyset$. Finally, each γ_i has only finitely many elements since*

$$\gamma_i \subseteq \left\{ \frac{i-1}{1}, \frac{i-2}{2}, \ldots, \frac{1}{i-1}, -\frac{1}{i-1}, \ldots, -\frac{i-1}{1} \right\}.$$

Thus, the γ_i's must be well-bounded. Therefore there is at least one well-bounded partition of $\mathbb{Q}$.

Figure 2.1: The rational numbers

$$\begin{array}{ccccccccccccc}
\cdots & \frac{-5}{1} & \mathbf{\frac{-4}{1}} & \frac{-3}{1} & \frac{-2}{1} & \frac{-1}{1} & \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \mathbf{\frac{4}{1}} & \frac{5}{1} & \cdots \\
\cdots & \cdots & & \mathbf{\frac{-3}{2}} & & \frac{-1}{2} & & \frac{1}{2} & & \mathbf{\frac{3}{2}} & & \cdots & \cdots \\
\cdots & \cdots & \cdots & & \mathbf{\frac{-2}{3}} & \frac{-1}{3} & & \frac{1}{3} & \mathbf{\frac{2}{3}} & & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{\frac{-1}{4}} & & \mathbf{\frac{1}{4}} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \frac{-1}{5} & & \frac{1}{5} & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

The bold face numbers are the elements of γ_5.

Definition 2.1.5 (The semi-norms on $\mathcal{F}$ induced by a well-bounded partition of $\mathbb{Q}$)**.** *Let Γ be any well-bounded partition of $\mathbb{Q}$ and for every $n \in \mathbb{N}$ define the map $\|\cdot\|_{(\Gamma,n)} : \mathcal{F} \to \mathbb{R}$ by*

$$\|x\|_{(\Gamma,n)} := \max \{|x[q]| : q \in \Gamma_n\}.$$

Note that, because the support of x is well-ordered and Γ_n is well-bounded the set $\{\|x[q]\| : q \in \Gamma_n\}$ in the definition above contains only finitely many non-zero elements and hence the maximum does exist. We have not yet shown that either definition 2.1.1 or 2.1.5 actually define semi-norms. To avoid unnecessary repetition we include only the proof of this for the latter definition; the proof for the former follows similarly.

Proposition 2.1.6. *Let Γ be any well-bounded partition of $\mathbb{Q}$ and let $n \in \mathbb{N}$. Then, $\|\cdot\|_{(\Gamma,n)}$ is a semi-norm.*

Proof. We need to show that, for every $x, y \in \mathcal{F}$ and for every $a \in \mathbb{R}$, we have that

- $\|x\|_{(\Gamma,n)} \geq 0$
- $\|ax\|_{(\Gamma,n)} = |a| \, \|x\|_{(\Gamma,n)}$
- $\|x+y\|_{(\Gamma,n)} \leq \|x\|_{(\Gamma,n)} + \|y\|_{(\Gamma,n)}$.

The first property follows trivially from the definition. Now let $a \in \mathbb{R}$ and $x \in \mathcal{F}$ be given. Then

$$\begin{aligned} \|ax\|_{(\Gamma,n)} &= \max\{|ax[q]| : q \in \Gamma_n\} \\ &= \max\{|a|\,|x[q]| : q \in \Gamma_n\} \\ &= |a| \max\{|x[q]| : q \in \Gamma_n\} \\ &= |a| \, \|x\|_{(\Gamma,n)}, \end{aligned}$$

Finally, let $x, y \in \mathcal{F}$ be given. Then

$$\begin{aligned} \|x+y\|_{(\Gamma,n)} &= \max\{|(x+y)[q]| : q \in \Gamma_n\} \\ &\leq \max\{|x[q]| + |y[q]| : q \in \Gamma_n\} \\ &\leq \max\{|x[q]| : q \in \Gamma_n\} + \max\{|y[q]| : q \in \Gamma_n\} \\ &= \|x\|_{(\Gamma,n)} + \|y\|_{(\Gamma,n)}. \end{aligned}$$

□

2.2 Vector Topologies

Having defined the semi-norms we will be working with, we proceed to show that both families can be used to induce vector topologies on $\mathcal{F}$ that are consistent with a translation-invariant metric. Naturally the proofs are very similar in both cases so we will present them each once making notes where there are significant differences or modifications that must be accounted for. Note that, unless otherwise stated, Γ will denote any arbitrary well-bounded partition of $\mathbb{Q}$ and $\|\cdot\|_{(\Gamma,n\in\mathbb{N})} : \mathcal{F} \to \mathbb{R}$ will denote the corresponding family of semi-norms.

Remark 2.2.1. *Note that, if $r_1 < r_2$, then $\{|x[q]|\,|q \le r_1\} \subset \{|x[q]|\,|q \le r_2\}$ for all $x \in \mathcal{F}$. It follows immediately that*

$$\|x\|_{(u,r_1)} \le \|x\|_{(u,r_2)}.$$

Similarly if $n, m \in \mathbb{N}$ with $n < m$, then $\Gamma_n \subset \Gamma_m$ and hence for any $x \in \mathcal{F}$

$$\|x\|_{(\Gamma,n)} \le \|x\|_{(\Gamma,m)}.$$

The following definition is similar to that used in [1] but is modified to adapt to the newly defined semi-norms.

Definition 2.2.2 (Pseudo-Ball)**.** *Let $x \in \mathcal{F}$, and let $r > 0$ in $\mathbb{R}$ be given (resp. let $q > 0$ in $\mathbb{Q}$ be given). Then we define*

$$\mathrm{PB}_\Gamma(x, r) := \left\{y \in \mathcal{F} : \|x - y\|_{(\Gamma,\mu(r))} < r\right\}$$

where

$$\mu(r) := \left\lceil \frac{1}{r} \right\rceil$$

*is the smallest natural number n such that $\frac{1}{n} < r$. We say that $\mathrm{PB}_\Gamma(x, r)$ is the **"pseudo-ball"** at x with radius r. Respectively we define*

$$\mathrm{PB}_u(x, q) := \{y \in \mathcal{F} |\, \|y - x\|_{(u,1/q)} < q\}$$

*and we call this a **"pseudo-ball"** at x with radius q.*

Proposition 2.2.3. *Let $x \in \mathcal{F}$ and let $0 < r_1 < r_2 \in \mathbb{R}$ be given (resp. let $0 < r_1 < r_2 \in \mathbb{Q}$*

be given). Define $r := \min\{r_1, r_2 - r_1\}$*, then for all* $y \in \mathrm{PB}_u(x, r)$ *we have that*

$$\mathrm{PB}_\Gamma(y, r_1) \subset \mathrm{PB}_\Gamma(x, r_2);$$

in particular we have that

$$\mathrm{PB}_\Gamma(x, r_1) \subset \mathrm{PB}_\Gamma(x, r_2).$$

Respectively, we have that

$$\mathrm{PB}_u(y, r_1) \subset \mathrm{PB}_u(x, r_2);$$

and hence

$$\mathrm{PB}_u(x, r_1) \subset \mathrm{PB}_u(x, r_2).$$

Proof. Let $y \in \mathrm{PB}_\Gamma(x, r)$ be given and let $z \in \mathrm{PB}_\Gamma(y, r_1)$. Then, by definition, we have that

$$\|y - z\|_{(\Gamma, \mu(r_1))} < r_1.$$

Since $r_1 < r_2$, it follows that $\mu(r_1) \geq \mu(r_2)$, and hence

$$\|x - z\|_{(\Gamma, \mu(r_2))} \leq \|x - z\|_{(\Gamma, \mu(r_1))}.$$

It follows that

$$\begin{aligned} \|x - z\|_{(\Gamma, \mu(r_1))} &\leq \|y - z\|_{(\Gamma, \mu(r_1))} + \|x - y\|_{(\Gamma, \mu(r_1))} \\ &< r_1 + \|x - y\|_{(\Gamma, \mu(r_1))}. \end{aligned}$$

Recall that $r = \min\{r_1, r_2 - r_1\} \leq r_1$, and hence

$$\|x - y\|_{(\Gamma, \mu(r_1))} \leq \|x - y\|_{(\Gamma, \mu(r))}.$$

Since $y \in \mathrm{PB}_\Gamma(x, r)$, we have that

$$\|x - y\|_{(\Gamma, \mu(r))} < r \leq r_2 - r_1.$$

Altogether, it follows that

$$\begin{aligned}
\|x - z\|_{(\Gamma, \mu(r_2))} &< r_1 + \|x - y\|_{(\Gamma, \mu(r_1))} \\
&\leq r_1 + \|x - y\|_{(\Gamma, \mu(r))} \\
&< r_1 + (r_2 - r_1) \\
&= r_2;
\end{aligned}$$

and hence $z \in \mathrm{PB}_\Gamma(x, r_2)$. This argument holds for any $z \in \mathrm{PB}_\Gamma(y, r_1)$, and hence

$$\mathrm{PB}_\Gamma(y, r_1) \subset \mathrm{PB}_\Gamma(x, r_2).$$

In particular, letting $y = x$ in $\mathrm{PB}_\Gamma(x, r)$, we have that

$$\mathrm{PB}_\Gamma(x, r_1) \subset \mathrm{PB}_\Gamma(x, r_2).$$

□

We can now define the topologies induced by these families of semi-norms by letting a set S be open if every point in S is contained in a pseudo-ball which is itself contained in S. Again we include a definition that is substantially similar to one in [1] but which is modified to incorporate the new semi-norms.

Definition 2.2.4 (The topologies induced by families of semi-norms)**.** *We define*

$$\tau_\Gamma := \{O \subset \mathcal{F} : \forall x \in O, \exists r > 0 \textit{ in } \mathbb{R} \textit{ such that } \mathrm{PB}_\Gamma(x, r) \subset O\},$$

and

$$\tau_u := \{O \subset \mathcal{F} : \forall x \in O, \exists r > 0 \text{ in } \mathbb{R} \text{ such that } \mathrm{PB}_u(x,r) \subset O\}.$$

*We call these the **topology induced by** Γ and the **locally uniform support topology** respectively. The name of the latter topology will be justified when we discuss the convergence criterion in this topology.*

Proposition 2.2.5. *τ_Γ is a topology on $\mathcal{F}$ (resp. τ_u is a topology on $\mathcal{F}$).*

Proof. We need to show that τ_Γ is closed under arbitrary unions and finite intersections, and that $\emptyset, \mathcal{F} \in \tau_\Gamma$. Let $\{O_\alpha\}_{\alpha \in A}$ be an arbitrary collection of elements of τ_Γ; and let $x \in \bigcup_{\alpha \in A} O_\alpha$ be given. Then there is an $\alpha_0 \in A$ such that $x \in O_{\alpha_0}$. But $O_{\alpha_0} \in \tau_\Gamma$ so by definition there is a $r > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x,r) \subset O_{\alpha_0}$. It follows immediately that $\mathrm{PB}_\Gamma(x,r) \subset \bigcup_{\alpha \in A} O_\alpha$. Thus, $\bigcup_{\alpha \in A} O_\alpha$ is open, and hence τ_Γ is closed under arbitrary unions.

Now let $O_1, O_2 \in \tau_\Gamma$ and let $x \in O_1 \cap O_2$ be given. Since $x \in O_1$, there exists $r_1 > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x,r_1) \subset O_1$. Similarly there exists $r_2 > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x,r_2) \subset O_2$. Let $r = \min\{r_1, r_2\}$. Then we have that $\mathrm{PB}_\Gamma(x,r) \subset \mathrm{PB}_\Gamma(x,r_1) \subset O_1$ and $\mathrm{PB}_\Gamma(x,r) \subset \mathrm{PB}_\Gamma(x,r_2) \subset O_2$; and hence $\mathrm{PB}_\Gamma(x,r) \subset O_1 \cap O_2$. This shows that τ_Γ is closed under the intersection of two of its elements; and by induction it is therefore closed under finite intersections.

Finally, that $\emptyset$ and $\mathcal{F}$ are in τ_Γ follows from that the fact that they both trivially satisfy the defining property of τ_Γ. $\square$

Proposition 2.2.6. *$(\mathcal{F}, \tau_\Gamma)$ (resp. $(\mathcal{F}, \tau_u)$) is a topological vector space over $\mathbb{R}$.*

Proof. To show that $(\mathcal{F}, \tau_\Gamma)$ is a topological vector space, we will prove the following.

- Every singleton is closed with respect to τ_Γ.
- Vector addition is continuous with respect to τ_Γ.
- Scalar multiplication is continuous with respect to τ_Γ.

Let $x \in \mathcal{F}$ be given. We will show that $\{x\}$ is closed in τ_Γ by showing that its complement is open. So, let $y \in \mathcal{F} \setminus \{x\}$ be given, let $q = \lambda(x-y)$, and let $r_0 := |(x-y)[q]|$. Choose $N \in \mathbb{N}$ large enough so that $q \in \Gamma_N$, then let $r = \min\left\{\frac{1}{N}, \frac{r_0}{2}\right\}$. We will show that $\mathrm{PB}_\Gamma(y,r) \subset \mathcal{F} \setminus \{x\}$ by showing that $x \notin \mathrm{PB}_\Gamma(y,r)$. Note that

$$
\begin{aligned}
\|x-y\|_{(\Gamma,\mu(r))} &\geq \|x-y\|_{(\Gamma,N)} \\
&= \max\{|(x-y)[q]| : q \in \Gamma_N\} \\
&\geq r_0 \\
&> r.
\end{aligned}
$$

Thus, $x \notin \mathrm{PB}_\Gamma(y,r)$, and hence $\mathcal{F} \setminus \{x\}$ is open.

Next, we show that $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a continuous operation on $(\mathcal{F}, \tau_\Gamma) \times (\mathcal{F}, \tau_\Gamma)$. Let $O \subset \mathcal{F}$ be any open set with respect to τ_Γ, let $A \subset \mathcal{F} \times \mathcal{F}$ be the inverse image of O under addition. We will show that A is open in $(\mathcal{F}, \tau_\Gamma) \times (\mathcal{F}, \tau_\Gamma)$. Fix $(x_1, x_2) \in A$, then $x_1 + x_2 \in O$. O is open so there exists a $r > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x_1 + x_2, r) \subset O$. Let $y \in \mathrm{PB}_\Gamma\left(x_1, \frac{r}{2}\right)$, $z \in \mathrm{PB}_\Gamma\left(x_2, \frac{r}{2}\right)$. Then,

$$
\begin{aligned}
\|y+z-x_1-x_2\|_{(\Gamma,\mu(r))} &\leq \|y-x_1\|_{(\Gamma,\mu(r))} + \|z-x_2\|_{(\Gamma,\mu(r))} \\
&\leq \|y-x_1\|_{\left(\Gamma,\mu\left(\frac{r}{2}\right)\right)} + \|z-x_2\|_{\left(\Gamma,\mu\left(\frac{r}{2}\right)\right)} \\
&< \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}
$$

Thus, $y+z \in O$ and hence $(y,z) \in A$. Therefore, A is open in $(\mathcal{F}, \tau_\Gamma) \times (\mathcal{F}, \tau_\Gamma)$, and addition is continuous.

Finally we show that $\cdot : \mathbb{R} \times \mathcal{F} \to \mathcal{F}$ is continuous with respect to τ_Γ. So, let $O \subset \mathcal{F}$ be open with respect to τ_Γ and let $S \subset \mathbb{R} \times \mathcal{F}$ be the inverse image of O under scalar multiplication. Let $(\alpha, x) \in S$, then $\alpha x \in O$ and since O is open there is a $r > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(\alpha x, r) \subset O$. Now we have two cases; either $\alpha = 0$ or $\alpha \neq 0$. We deal with these cases separately.

Assume that $\alpha = 0$. Then we have two possibilities: either $\|x\|_{(\Gamma,\mu(r))} = 0$ or $\|x\|_{(\Gamma,\mu(r))} \neq 0$. Consider first the subcase $\|x\|_{(\Gamma,\mu(r))} = 0$; we show that $(-1,1) \times \mathrm{PB}_\Gamma(x,r) \subset S$. Let $\beta \in (-1,1)$ and $y \in \mathrm{PB}_\Gamma(x,r)$ be given, then

$$\begin{aligned}
\|\beta y\|_{(\Gamma,\mu(r))} &= |\beta| \, \|y\|_{(\Gamma,\mu(r))} \\
&< \|y\|_{(\Gamma,\mu(r))} \\
&\leq \|y - x\|_{(\Gamma,\mu(r))} + \|x\|_{(\Gamma,\mu(r))} \\
&= \|y - x\|_{(\Gamma,\mu(r))} < r.
\end{aligned}$$

Thus, $\beta y \in \mathrm{PB}_\Gamma(0,r) \subset O$ and hence $(\beta, y) \in S$. Next we consider $\|x\|_{(\Gamma,\mu(r))} \neq 0$; let

$$r_1 = \min\left\{\frac{1}{2}, \frac{r}{2\,\|x\|_{(\Gamma,\mu(r))}}\right\}.$$

Then $r_1 \in \mathbb{R}$ and $r_1 > 0$; and we will show that $(-r_1, r_1) \times \mathrm{PB}_\Gamma(x,r) \subset S$. So let $\beta \in (-r_1, r_1)$ and $y \in \mathrm{PB}_\Gamma(x,r)$ be given. Then

$$\begin{aligned}
\|\beta y\|_{(\Gamma,\mu(r))} &\leq \|\beta(y - x)\|_{(\Gamma,\mu(r))} + \|\beta x\|_{(\Gamma,\mu(r))} \\
&\leq |\beta| \, \|y - x\|_{(\Gamma,\mu(r))} + |\beta| \, \|x\|_{(\Gamma,\mu(r))} \\
&< r_1 r + r_1 \|x\|_{(\Gamma,\mu(r))} \\
&\leq \frac{r}{2} + \frac{r}{2\,\|x\|_{(\Gamma,\mu(r))}} \|x\|_{(\Gamma,\mu(r))} \\
&= r.
\end{aligned}$$

Thus, $\beta y \in O$ and hence $(\beta, y) \in S$.

Now we consider the case $\alpha \neq 0$. Let

$$r_1 = \min\left\{\frac{r}{2}, \frac{r}{2\,|\alpha|}\right\}$$

and

$$\nu := \begin{cases} \frac{1}{2} & \text{if } \|x\|_{(\Gamma,\mu(r))} = 0 \\ \min\left\{\frac{1}{2}, \frac{r}{4\|x\|_{(\Gamma,\mu(r))}}\right\} & \text{if } \|x\|_{(\Gamma,\mu(r))} \neq 0 \end{cases}.$$

We will show that $(\alpha-\nu, \alpha+\nu) \times \mathrm{PB}_\Gamma(x, r_1) \subset S$. So let $\beta \in (\alpha-\nu, \alpha+\nu)$ and $y \in \mathrm{PB}_\Gamma(x, r_1)$ be given. Then

$$\begin{aligned} \|\beta y - \alpha x\|_{(\Gamma,\mu(r))} &= \|(\beta-\alpha)(y-x) + (\beta-\alpha)x + \alpha(y-x)\|_{(\Gamma,\mu(r))} \\ &\leq |\beta-\alpha| \, \|y-x\|_{(\Gamma,\mu(r))} + |\beta-\alpha| \, \|x\|_{(\Gamma,\mu(r))} + |\alpha| \, \|y-x\|_{(\Gamma,\mu(r))}. \end{aligned}$$

However, $r_1 \leq \frac{r}{2} < r$ and hence

$$\|y-x\|_{(\Gamma,\mu(r))} \leq \|y-x\|_{(\Gamma,\mu(r_1))} < r_1 \leq \frac{r}{2|\alpha|}.$$

Thus,

$$|\alpha| \, \|y-x\|_{(\Gamma,\mu(r))} < \frac{r}{2}.$$

Moreover,

$$|\beta-\alpha| \, \|y-x\|_{(\Gamma,\mu(r))} < |\beta-\alpha| \, r_1 < \nu r_1 \leq \frac{r}{4}.$$

And finally,

$$|\beta-\alpha| \, \|x\|_{(\Gamma,\mu(r))} < \nu \, \|x\|_{(\Gamma,\mu(r))} \leq \frac{r}{4}.$$

Thus, altogether, we obtain that

$$\begin{aligned} \|\beta y - \alpha x\|_{(\Gamma,\mu(r))} &\leq |\beta-\alpha| \, \|y-x\|_{(\Gamma,\mu(r))} + |\beta-\alpha| \, \|x\|_{(\Gamma,\mu(r))} + |\alpha| \, \|y-x\|_{(\Gamma,\mu(r))} \\ &< \frac{r}{4} + \frac{r}{4} + \frac{r}{2} = r. \end{aligned}$$

So $\beta y \in O$ and hence $(\beta, y) \in S$. Therefore we conclude that $(\mathcal{F}, \tau_\Gamma)$ is a topological vector space. $\square$

Proposition 2.2.7. *The family of pseudo-balls* $\{\mathrm{PB}_u(0,q) : q \in \mathbb{Q}^+\}$ *(resp. the family of pseudo-balls* $\{\mathrm{PB}_\Gamma(0,q) : q \in \mathbb{Q}^+\}$*) is a countable local base for* τ_u *(resp.* τ_Γ*) at* 0.

Proof. Let $O \in \tau_u$ be any open set in $\mathcal{F}$ containing 0. Then there exists $r > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_u(0,r) \subset O$. Let $q \in \mathbb{Q}$ be such that $0 < q < r$; then $\mathrm{PB}_u(0,q) \subset \mathrm{PB}_\Gamma(0,r) \subset O$. Thus, for any open set containing 0, there is a $q \in \mathbb{Q}$ such that $0 \in \mathrm{PB}_u(0,q) \subset O$ and hence $\{\mathrm{PB}_u(0,q) : q \in \mathbb{Q}^+\}$ is a countable local base for τ_u at 0. $\square$

Corollary 2.2.8. *For any* $x \in \mathcal{F}$*, the family of pseudo-balls* $\{\mathrm{PB}_u(x,q) : q \in \mathbb{Q}^+\}$ *(resp.* $\{\mathrm{PB}_\Gamma(x,q) : q \in \mathbb{Q}^+\}$*) is a countable local base for* τ_u *(resp.* τ_Γ*) at* x.

2.3 Relations Between Topologies

Now that we have established that τ_Γ and τ_u are vector topologies, we will investigate their relationship to each other and to τ_v. We begin by recalling the definition of a compact set.

Definition 2.3.1. *Suppose* τ *is a topology on* $\mathcal{F}$ *and let* $A \subset \mathcal{F}$. *Then we say that* A *is compact in* $(\mathcal{F},\tau)$ *if every open cover of* A *in* $(\mathcal{F},\tau)$ *admits a finite subcover.*

Proposition 2.3.2. *Let* τ *be any topology on* $\mathcal{F}$ *satisfying* $\tau \subsetneq \tau_v$. *Suppose* $A \subset \mathcal{F}$ *is compact in* $(\mathcal{F},\tau_v)$*, then* A *is also compact in* $(\mathcal{F},\tau)$.

Proof. Let $A \subset \mathcal{F}$, a compact set in $(\mathcal{F},\tau_v)$, be given. Let $T \subset \tau$ be any open cover of A in τ. Then, since $\tau \subsetneq \tau_v$ and $T \subset \tau$, we have that $T \subset \tau_v$. Thus T is an open cover of A in τ_v, however by choice A is compact in $(\mathcal{F},\tau_v)$ so T must admit a finite subcover $T' \subset T$. But $T' \subset T \subset \tau$ so T' is also a finite subcover in $(\mathcal{F},\tau)$. This argument holds for any choice of $A \subset \mathcal{F}$ compact in $(\mathcal{F},\tau_v)$; thus, if A is compact in $(\mathcal{F},\tau_v)$ it is also compact in $(\mathcal{F},\tau)$. $\square$

Proposition 2.3.3. $\tau_\Gamma \subsetneq \tau_v$ *(resp.* $\tau_u \subsetneq \tau_v$*).*

Proof. Let $G \subset \mathcal{F}$ be open with respect to τ_Γ and fix $x \in G$. Then there exists $r > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x,r) \subset G$. Let $n > \max\{\Gamma_{\mu(r)}\}$ which is possible because $\Gamma_{\mu(r)}$ is the finite union of well-bounded sets and hence it is itself well-bounded. We will show that $B(x, d^n) \subset G$. Let $y \in B(x, d^n)$ be given. So $|y - x| < d^n$ and hence, for any $q < n$ in $\mathbb{Q}$, we have $(x - y)[q] = 0$. However, by our choice of n, we have that for every $q \in \Gamma_{\mu(r)}$, $q < n$. Therefore, for every $q \in \Gamma_{\mu(r)}$, $(y - x)[q] = 0 < r$. It follows that $y \in \mathrm{PB}_\Gamma(x, r)$. This holds for any $y \in B(x, d^n)$. It follows that

$$B(x, d^n) \subset \mathrm{PB}_\Gamma(x, r) \subset G.$$

We have just shown that $\tau_\Gamma \subset \tau_v$; so it remains to show that there exists an $O \in \tau_v$ such that $O \notin \tau_\Gamma$. Choose $n > \max\{\Gamma_1\} = q$, then

$$(-d^n, d^n) = B(0, d^n) \in \tau_v.$$

Now fix $r > 0$ in $\mathbb{R}$ and let $x = \frac{r}{2} d^q$. Then clearly $x \notin (-d^n, d^n)$ since by choice $d^q \gg d^n$. However

$$\|0 - x\|_{(\Gamma, \mu(r))} = \left\| \frac{r}{2} d^q \right\|_{(\Gamma, \mu(r))} \leq \frac{r}{2} < r$$

so $x \in \mathrm{PB}_\Gamma(0, r)$. Since our choice of r was arbitrary we conclude that for every $r > 0$ in $\mathbb{R}$, $\frac{r}{2} d^q \in \mathrm{PB}_\Gamma(0, r)$ but $\frac{r}{2} d^q \notin (-d^n, d^n)$. It follows that for every $r > 0$ in $\mathbb{R}$,

$$\mathrm{PB}_\Gamma(0, r) \not\subset (-d^n, d^n).$$

Thus, $(-d^n, d^n) \notin \tau_\Gamma$ and hence $\tau_v \not\subset \tau_\Gamma$. ☐

Corollary 2.3.4. *Suppose $A \subset \mathcal{F}$ is compact in $(\mathcal{F}, \tau_v)$. Then A is also compact in $(\mathcal{F}, \tau_u)$ and $(\mathcal{F}, \tau_\Gamma)$.*

Proposition 2.3.5. *There exist translation invariant metrics Δ_Γ and Δ_w that induce the topologies τ_Γ and τ_u, respectively, on $\mathcal{F}$.*

Proof. This follows from the fact that both $(\mathcal{F}, \tau_\Gamma)$ and $(\mathcal{F}, \tau_u)$ are topological vector spaces with countable local bases, for details see Theorem 1.24 in [21]. $\square$

Example 2.3.6. *Let*

$$\Delta_\Gamma(x,y) := \sum_{k=1}^{\infty} 2^{-k} \frac{\|x-y\|_{(\Gamma,k)}}{1+\|x-y\|_{(\Gamma,k)}}$$

and let

$$\Delta_u(x,y) := \sum_{k=1}^{\infty} 2^{-k} \frac{\|y-x\|_{(u,k)}}{1+\|y-x\|_{(u,k)}}.$$

Then Δ_Γ and Δ_u are translation invariant metrics on $\mathcal{F}$ that induce topologies equivalent to τ_Γ and τ_u, respectively; see the proofs of Theorem 3.32 and Theorem 3.33 in [1]. We note that, in both infinite sums above, the 2^{-k} factor could be replaced with c^{-k} where c is any real number greater than 1 and we would still obtain translation invariant metrics.

Proposition 2.3.7. $\tau_\Gamma \subset \tau_u$.

Proof. Let $O \in \tau_\Gamma$ be given and fix $x \in O$. Then there exists $r > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x,r) \subset O$. Let $q_0 = \max\{\Gamma_{\mu(r)}\}$, which exists by the well-boundedness of $\Gamma_{\mu(r)}$. Pick $q_1 > \max\{q_0, \frac{1}{r}\}$; we claim that

$$\mathrm{PB}_u\left(x, \frac{1}{q_1}\right) \subset \mathrm{PB}_\Gamma(x,r).$$

So fix $y \in \mathrm{PB}_u\left(x, \frac{1}{q_1}\right)$, then $\|y-x\|_{(u,q_1)} < \frac{1}{q_1} < r$. Thus for every $q \le q_1$, we have that $(y-x)[q] < r$. However, by selection, we have that $q_1 > q_0 = \max\{\Gamma_{\mu(r)}\}$; thus, for every $q \in \Gamma_{\mu(r)}$ we have that $q \le q_0 < q_1$ and hence, for every $q \in \Gamma_{\mu(r)}$, we have that $(y-x)[q] < r$. It follows that $\|y-x\|_{(\Gamma,\mu(r))} < r$ and hence $y \in \mathrm{PB}_\Gamma(x,r)$. Therefore, as claimed, $\mathrm{PB}_u\left(x, \frac{1}{q_1}\right) \subset \mathrm{PB}_\Gamma(x,r) \subset O$. Thus, $O \in \tau_u$ and hence $\tau_\Gamma \subseteq \tau_u$. $\square$

Proposition 2.3.8. *Let Γ and Ω be a finite well-bounded partition and an infinite well-bounded partition of $\mathbb{Q}$, respectively. Then $\tau_\Gamma \subsetneq \tau_\Omega$.*

Proof. First we show that $\tau_\Gamma \subset \tau_\Omega$. So let $O \in \tau_\Gamma$ and fix $x \in O$. Then there exists $\epsilon > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x,\epsilon) \subseteq O$. Since Γ is a finite partition, we know that $\Gamma_{\mu(\epsilon)}$ is a finite set and thus there exists $N_0 \in \mathbb{N}$ such that $\Gamma_{\mu(\epsilon)} \subset \Omega_{N_0}$. Let $N_1 \in \mathbb{N}$ be large enough so that $\frac{1}{N_1} < \epsilon$; and let $N = \max\{N_0, N_1\}$. We claim that $\mathrm{PB}_\Omega\left(x, \frac{1}{N}\right) \subset \mathrm{PB}_\Gamma(x, \epsilon)$. Let $y \in \mathrm{PB}_\Omega\left(x, \frac{1}{N}\right)$ be given. By definition, we have that

$$\|y - x\|_{(\Omega,N)} < \frac{1}{N} \leq \frac{1}{N_1} < \epsilon.$$

So for every $q \in \Omega_N$, we have that $|(y-x)[q]| < \epsilon$. Since $\Gamma_{\mu(\epsilon)} \subset \Omega_{N_0} \subset \Omega_N$, it follows that, for every $q \in \Gamma_{\mu(\epsilon)}$, we have that

$$|(y-x)[q]| < \epsilon.$$

Thus,

$$\|y - x\|_{(\Gamma,\mu(\epsilon))} < \epsilon$$

and hence $y \in \mathrm{PB}_\Gamma(x, \epsilon)$. Hence $\mathrm{PB}_\Omega\left(x, \frac{1}{N}\right) \subset \mathrm{PB}_\Gamma(x, \epsilon) \subseteq O$. Thus, we have shown that, for every $x \in O$, there exists $N \in \mathbb{N}$ such that

$$\mathrm{PB}_\Omega\left(x, \frac{1}{N}\right) \subseteq O.$$

This proves that $\tau_\Gamma \subset \tau_\Omega$. To prove that the two topologies are not equal, we observe that since Ω is non-finite there exists $N \in \mathbb{N}$ such that for all $n \geq N$, Ω_n has infinitely many elements. Consider the pseudo-ball $\mathrm{PB}_\Omega\left(0, \frac{1}{N}\right) \in \tau_\Omega$. We will show that $\mathrm{PB}_\Omega\left(0, \frac{1}{N}\right) \notin \tau_\Gamma$. Since we have already shown that the family of pseudo-balls $\{\mathrm{PB}_\Gamma(0, q) \mid |q \in \mathbb{Q}^+\}$ is a countable local basis for τ_Γ at 0, it is enough to prove that for every $q \in \mathbb{Q}^+$, $\mathrm{PB}_\Gamma(0, q) \not\subset \mathrm{PB}_\Omega\left(0, \frac{1}{N}\right)$. So fix a $q \in \mathbb{Q}^+$. Since $\Gamma_{\mu(q)}$ is a finite set and Ω_N is not, there must be $q_0 \in \Omega_N \setminus \Gamma_{\mu(q)}$; let $x_0 = \frac{2}{N} d^{q_0}$. We have that

$$\|x_0\|_{(\Gamma,\mu(q))} = \sup\left\{|x_0[q']| : q' \in \Gamma_{\mu(q)}\right\} = 0 < q$$

because $x_0[q'] = 0$ for all $q' \in \Gamma_{\mu(q)}$. However, we also have that

$$\|x_0\|_{(\Omega,N)} = \sup\{|x_0[q']| : q' \in \Omega_N\} = \frac{2}{N} > \frac{1}{N}.$$

Thus, for every $q \in \mathbb{Q}^+$, there is $x_0 \in \mathcal{F}$ such that $x_0 \in \mathrm{PB}_\Gamma(0,q)$ but $x_0 \notin \mathrm{PB}_\Omega\left(0, \frac{1}{N}\right)$. It follows that $\tau_\Omega \not\subset \tau_\Gamma$ and hence $\tau_\Gamma \subsetneq \tau_\Omega$. $\square$

Lemma 2.3.9. *For every $q \in \mathbb{Q}^+$ and for every $n \in \mathbb{N}$ there exists $q' \in (0,q] \cap \mathbb{Q}$ such that $q' \notin \Gamma_n$.*

Proof. Suppose otherwise. Then there exist $q \in \mathbb{Q}$ and $n \in \mathbb{N}$ such that for every $q' \in (0,q] \cap \mathbb{Q}$, $q' \in \Gamma_n$. As an immediate consequence we have that $(0,q] \cap \mathbb{Q} \subset \Gamma_n$, which contradicts the fact that intervals in $\mathbb{Q}$ are not well-bounded. Thus, no such q and n can exist. $\square$

Proposition 2.3.10. $\tau_u \not\subset \tau_\Gamma$.

Proof. Fix $q \in \mathbb{Q}^+$ and consider the pseudo-ball $\mathrm{PB}_u\left(0, \frac{1}{q}\right) \in \tau_u$. We claim that for every $r \in \mathbb{Q}^+$, $\mathrm{PB}_\Gamma(0,r) \not\subset \mathrm{PB}_u\left(0, \frac{1}{q}\right)$. Since $\{\mathrm{PB}_\Gamma(0,r) : r \in \mathbb{Q}^+\}$ forms a local base for τ_Γ at 0, proving our claim will be sufficient to establish that $\mathrm{PB}_u\left(0, \frac{1}{q}\right) \notin \tau_\Gamma$. To prove our claim, let $r \in \mathbb{Q}^+$. Then

$$\begin{aligned}\mathrm{PB}_\Gamma(0,r) &= \left\{x \in \mathcal{F} : \|x\|_{(\Gamma,\mu(r))} < r\right\} \\ &= \left\{x \in \mathcal{F} \mid |\sup\left\{|x[q]| : q \in \Gamma_{\mu(r)}\right\} < r\right\}.\end{aligned}$$

By lemma 2.3.9, we have that $(-\infty,q] \cap (\mathbb{Q} \setminus \Gamma_{\mu(r)}) \neq \emptyset$; so pick $q' \in (-\infty,q] \cap (\mathbb{Q} \setminus \Gamma_{\mu(r)})$ and let $x = \frac{2}{q} d^{q'}$. We see that

$$x \in \mathrm{PB}_\Gamma(0,r)$$

since $x[q] = 0$ for all $q \in \Gamma_{\mu(r)}$, but

$$x \notin \mathrm{PB}_u \left(0, \frac{1}{q}\right)$$

because $q' \in (-\infty, q] \cap \mathbb{Q}$ and $|x[q']| = \frac{2}{q} > \frac{1}{q}$. Thus the claim (and hence the proposition) is proved. $\square$

Combining the results of Proposition 2.3.7 and Proposition 2.3.10, we readily obtain the following corollary.

Corollary 2.3.11. $\tau_\Gamma \subsetneq \tau_u$.

Proposition 2.3.12. *Let Γ and Ω be distinct finite well-bounded partitions of $\mathbb{Q}$. Then $\tau_\Gamma = \tau_\Omega$.*

Proof. Let $O \in \tau_\Omega$ and fix $x \in O$. Then there exists $\epsilon > 0$ in $\mathbb{R}$ such that

$$\mathrm{PB}_\Omega (x, \epsilon) \subseteq O.$$

Ω is a finite partition so $\Omega_{\mu(\epsilon)}$ contains only a finite number of elements. Moreover, since Γ is also a partition of $\mathbb{Q}$, for each $q \in \Omega_{\mu(\epsilon)}$ there exists $N_q \in \mathbb{N}$ such that $q \in \Gamma_{N_q}$. Let

$$N = \max \left\{ N_q : q \in \Omega_{\mu(\epsilon)} \right\}.$$

Then $\Omega_{\mu(\epsilon)} \subset \Gamma_N$ because for every $q \in \Omega_{\mu(\epsilon)}$, we have that $q \in \Gamma_{N_q} \subseteq \Gamma_N$. Now let $\delta = \min \left\{ \frac{1}{N}, \epsilon \right\}$. We claim that

$$\mathrm{PB}_\Gamma (x, \delta) \subseteq \mathrm{PB}_\Omega (x, \epsilon).$$

Let $y \in \mathrm{PB}_\Gamma (x, \delta)$ be given. Then we have that $\|y - x\|_{(\Gamma, \mu(\delta))} < \delta$, and hence, for every $q \in \Gamma_{\mu(\delta)}$, we have that $|(y - x)[q]| < \delta$. But $\delta \leq \frac{1}{N}$ so $\mu(\delta) \geq N$, and hence $\Gamma_N \subseteq \Gamma_{\mu(\delta)}$.

We have already shown that $\Omega_{\mu(\epsilon)} \subseteq \Gamma_N \subseteq \Gamma_{\mu(\delta)}$. So for every $q \in \Omega_{\mu(\epsilon)}$, we have that $|(y-x)[q]| < \delta \leq \epsilon$. It follows that

$$\|y - x\|_{(\Omega,\mu(\epsilon))} < \epsilon,$$

and hence $y \in \mathrm{PB}_\Omega(x, \epsilon)$. The above argument holds for any $y \in \mathrm{PB}_\Gamma(x, \delta)$; it follows that $\mathrm{PB}_\Gamma(x, \delta) \subset \mathrm{PB}_\Omega(x, \epsilon) \subset O$. Thus, we have shown that for any $x \in O$ there exists $\delta > 0$ in $\mathbb{R}$ such that $\mathrm{PB}_\Gamma(x, \delta) \subset O$, and hence $O \in \tau_\Gamma$. Since O is an arbitrary element of τ_Ω, we infer that $\tau_\Omega \subseteq \tau_\Gamma$. A symmetric argument shows that $\tau_\Gamma \subseteq \tau_\Omega$, and hence $\tau_\Gamma = \tau_\Omega$. □

Corollary 2.3.13. *All finite well-bounded partitions of $\mathbb{Q}$ induce the same topology on $\mathcal{F}$.*

Definition 2.3.14. *We call the topology induced by finite well-bounded partitions of $\mathbb{Q}$ the **weak topology** and denote it by τ_w. In a later section we will justify this choice of name by showing that τ_w share the same convergence criterion as the weak topology on the Levi-Civita field. A detailed study of the weak topology on the Levi-Civita field $\mathcal{R}$ can be found in [16, 1].*

2.4 Convergence of Sequences

In this section we will study convergence in τ_u, τ_Γ, and in particular in τ_w. We start with the following definition.

Definition 2.4.1. *Let τ be any topology on $\mathcal{F}$ induced by a countable family of semi-norms (which we will denote by $\|\cdot\|_{(\tau,n)}$) and let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. We say that $(s_n)_{n\in\mathbb{N}}$ **converges in** τ if there exists $s \in \mathcal{F}$ such that for every $\epsilon > 0$ in $\mathbb{R}$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|s_n - s\|_{(\tau,\mu(\epsilon))} < \epsilon$.*

We would like to find necessary conditions for a sequence to converge as defined above but first we introduce the following useful notion of regularity which originates from [15].

Definition 2.4.2 (Regular sequence (Hahn field))**.** *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. If*

$$\bigcup_{n=1}^{\infty} \operatorname{supp}(s_n)$$

is well-ordered then we say that $(s_n)_{n\in\mathbb{N}}$ is a ***regular sequence****.*

For comparison, regularity of a sequence in the Levi-Civita field is defined in Definition 1.4.1. We are now ready to state necessary conditions for a sequence in $\mathcal{F}$ to converge in each of τ_w and τ_u.

Proposition 2.4.3. *Let $(s_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(\mathcal{F}, \tau_w)$. Then for every $q \in \mathbb{Q}$, the real sequence $(s_n[q])_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ in the standard topology. Conversely, if (s_n) is regular and if for every $q \in \mathbb{Q}$, $(s_n[q])$ converges in $\mathbb{R}$ in the standard topology then (s_n) converges in $(\mathcal{F}, \tau_w)$.*

Proof. First suppose that (s_n) converges in $(\mathcal{F}, \tau_w)$. We may assume without loss of generality that $s_n \to 0$ because if $s_n \to s \neq 0$ then we can define the new sequence $s'_n = s_n - s$ and for any $q \in \mathbb{Q}$, $(s'_n[q])$ will converge to 0 as a real sequence if and only if $(s_n[q])$ converges to $s[q]$ in $\mathbb{R}$. Thus, for any $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|s_n\|_{(\tau_w,\mu(\epsilon))} < \epsilon$.

Now let $q_0 \in \mathbb{Q}$ be given. We will show that $s_n[q_0] \to 0$ in $\mathbb{R}$. So let $\epsilon > 0$ in $\mathbb{R}$ be given and let $n_0 \in \mathbb{N}$ be large enough so that $q_0 \in \Gamma_{n_0}$ where Γ is any finite well-bounded partition of $\mathbb{Q}$. Let $\epsilon_0 = \min\left\{\epsilon, \frac{1}{n_0}\right\}$. Since $s_n \to 0$ in $(\mathcal{F}, \tau_w = \tau_\Gamma)$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|s_n\|_{(\Gamma,\mu(\epsilon_0))} < \epsilon_0$. Recall that $\|s_n\|_{(\Gamma,\mu(\epsilon_0))} = \max\left\{|s_n[q]| : q \in \Gamma_{\mu(\epsilon_0)}\right\}$ and $q_0 \in \Gamma_{n_0} \subseteq \Gamma_{\mu(\epsilon_0)}$. It follows that if $n \geq N$ then $|s_n[q_0]| < \epsilon_0 \leq \epsilon$. Thus, we have shown that for any $\epsilon > 0$ in $\mathbb{R}$ there is a $N \in \mathbb{N}$ such that if $n \geq N$ then $|s_n[q_0]| < \epsilon$, and hence $s_n[q_0] \to 0$ in $\mathbb{R}$. Since the argument holds for an arbitrary $q_0 \in \mathbb{Q}$, it follows that, for every $q \in \mathbb{Q}$, $(s_n[q])$ converges in $\mathbb{R}$.

Now suppose we have a regular sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that for every $q \in \mathbb{Q}$, $s_n[q] \to 0$ in $\mathbb{R}$. We will show that $s_n \to 0$ in $(\mathcal{F}, \tau_w)$. So let $\epsilon > 0$ in $\mathbb{R}$ and let Γ be any finite well-bounded partition of $\mathbb{Q}$ (recall that $\tau_\Gamma = \tau_w$). We know that, for every $q \in \Gamma_{\mu(\epsilon)} \subset \mathbb{Q}$, there exists $N_q \in \mathbb{N}$ such that if $n \geq N_q$ then $|s_n[q]| < \epsilon$. Let

$$N = \max\left\{N_q : q \in \Gamma_{\mu(\epsilon)}\right\}$$

which must exist because Γ is a finite partition. We now observe that if $n \geq N$ then, for every $q \in \Gamma_{\mu(\epsilon)}$, we have that $|s_n[q]| < \epsilon$ which implies that $\|s_n\|_{(\Gamma,\mu(\epsilon))} < \epsilon$. This holds for any $\epsilon > 0$ in $\mathbb{R}$ and hence $s_n \to 0$ in $(\mathcal{F}, \tau_\Gamma = \tau_w)$. □

The following proposition justifies our choice to refer to τ_u as the "locally uniform support topology". As we will see, for a sequence to converge in this topology it is not sufficient that the sequence of values at each support point converges as a real sequence but rather the convergence must be locally uniform. That is, how quickly the sequence of values at one support point converges tells us something about how quickly the sequence of values at other nearby support points converge.

Proposition 2.4.4. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ that converges weakly to s. Then for every $\epsilon > 0$ in $\mathbb{R}$ and for every $q \in \mathbb{Q}$ there exist $N_q \in \mathbb{N}$ and $\delta_q \in \mathbb{Q}$ such that if $n \geq N_q$ and $|q' - q| < \delta_q$ then $|(s_n - s)[q']| < \epsilon$. Conversely, if $(s_n)_{n\in\mathbb{N}}$ is a regular sequence in $\mathcal{F}$ and if there exists $s \in \mathcal{F}$ such that for every $\epsilon > 0$ in $\mathbb{R}$ and for every $q \in \mathbb{Q}$ there exist $N_q \in \mathbb{N}$ and $\delta_q \in \mathbb{Q}$ such that if $n \geq N_q$ and $|q' - q| < \delta_q$ then $|(s_n - s)[q']| < \epsilon$, then (s_n) converges to weakly s in $\mathcal{F}$.*

Proof. First we suppose that $(s_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{F}, \tau_u)$. As in previous proofs, we may assume without loss of generality that the sequence converges to 0. So let $q \in \mathbb{Q}$ and $\epsilon > 0$ in $\mathbb{R}$ be given. Let

$$\epsilon_0 = \min\left\{\epsilon, \frac{1}{|q|+1}\right\}.$$

Since (s_n) converges in $(\mathcal{F}, \tau_u)$, there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$, we have that

$$\|s_n\|_{\left(u, \frac{1}{\epsilon_0}\right)} < \epsilon_0.$$

By our choice of ϵ_0, it follows that

$$\{q' \in \mathbb{Q} : |q' - q| < 1\} \subseteq \left\{q' \in \mathbb{Q} : q' < \frac{1}{\epsilon_0}\right\}.$$

Thus if $|q' - q| < 1$ then for all $n \geq N$,

$$|s_n[q']| < \epsilon_0 \leq \epsilon.$$

Now suppose that for every $\epsilon > 0$ in $\mathbb{R}$ and for every $q \in \mathbb{Q}$ there exist $N_q \in \mathbb{N}$ and $\delta_q \in \mathbb{Q}$ such that if $n \geq N$ and $|q' - q| < \delta_q$ then $|(s_n)[q']| < \epsilon$. We wish to show that $(s_n)_{n \in \mathbb{N}}$ converges to 0 in $(\mathcal{F}, \tau_u)$. Thus, for every $q \in \mathbb{Q}$ let $I_q = (q - \delta_q, q + \delta_q)$. Since (s_n) is a regular sequence

$$q_{\min} := \min \left\{ \bigcup_{i=0}^{\infty} \operatorname{supp}(s_n) \right\}$$

exists. So for any $q \in \mathbb{Q} \setminus \left[q_{\min}, \frac{1}{\epsilon}\right]$ either $s_n[q] = 0$ for all $n \in \mathbb{N}$ (if $q < q_{\min}$) or $\|\cdot\|_{\left(u, \frac{1}{\epsilon}\right)}$ is not dependent on the value of $s_n[q]$ (if $q > \frac{1}{\epsilon}$). Clearly $\left\{I_q : q \in \left[q_{\min}, \frac{1}{\epsilon}\right]\right\}$ is an open cover of $\left[q_{\min}, \frac{1}{\epsilon}\right]$ and since every closed bounded subset of $\mathbb{Q}$ is compact there must be a finite collection of rational numbers $\{q_j\}_{j \in \{1, \ldots, J\}} \in [q_{\min}, 1/\epsilon]$ such that

$$\left[q_{\min}, \frac{1}{\epsilon}\right] \subset \bigcup_{j=1}^{J} I_{q_j}.$$

Let $N := \max\left\{N_{q_j} : j \in \{1, \ldots, J\}\right\}$. Then, for every $q \in \left[q_{\min}, \frac{1}{\epsilon}\right]$, if $n \geq N$ then $\|s_n[q]\| < \epsilon$. But

$$\|s_n\|_{\left(u, \frac{1}{\epsilon}\right)} = \sup\left\{|s_n[q]| : q \in \left[q_{\min}, \frac{1}{\epsilon}\right]\right\}.$$

It follows that if $n \geq N$ then $\|s_n\|_{\left(u,\frac{1}{\epsilon}\right)} < \epsilon$. This argument holds for any $\epsilon > 0$ in $\mathbb{R}$ and hence (s_n) converges to 0 in $(\mathcal{F}, \tau_u)$. $\square$

We conclude this section with the following proposition which offers some insight into the relation between weak topologies and convergence in the Hahn field and the analogous concepts on the Levi-Civita field.

Proposition 2.4.5. $\tau_w|_{\mathcal{R}} \subsetneq \tau_u|_{\mathcal{R}}$.

Proof. First we show that $\tau_w|_{\mathcal{R}} \subset \tau_u|_{\mathcal{R}}$. So let $O \in \tau_w|_{\mathcal{R}}$ be given and let Γ be any finite well-bounded partition of $\mathbb{Q}$. Let $x \in O$ be given. Then there exists $r \in \mathbb{Q}$ such that $\mathrm{PB}_\Gamma(x, r) \cap \mathcal{R} \subset O$. Since Γ is well-bounded $\Gamma_{\mu(r)}$ must have a maximal element which we will denote by $q_{\max}$. Let

$$\epsilon = \begin{cases} \min\left\{r, \frac{1}{|q_{\max}|}\right\} & \text{if } q_{\max} \neq 0 \\ r & \text{if } q_{\max} = 0. \end{cases} \tag{2.1}$$

Then $\epsilon > 0$ in $\mathbb{R}$. We claim that $\mathrm{PB}_u(x, \epsilon) \cap \mathcal{R} \subset \mathrm{PB}_\Gamma(x, r) \cap \mathcal{R}$.

To prove the claim, it is sufficient to show that $\mathrm{PB}_u(x, \epsilon) \subset \mathrm{PB}_\Gamma(x, r)$. So let $y \in \mathrm{PB}_u(x, \epsilon)$. Then we have that $\|y - x\|_{\left(u,\frac{1}{\epsilon}\right)} < \epsilon$, that is,

$$\sup\left\{|(y-x)[q]| : q \in \left(-\infty, \frac{1}{\epsilon}\right] \cap \mathbb{Q}\right\} < \epsilon.$$

Using Equation (2.1), we have that $\frac{1}{\epsilon} \geq |q_{\max}| \geq q_{\max}$, so $\Gamma_{\mu(r)} \subset (-\infty, \frac{1}{\epsilon}] \cap \mathbb{Q}$ and hence

$$\sup\left\{|(y-x)[q]| : q \in \Gamma_{\mu(r)}\right\} < \epsilon \leq r.$$

But the left hand side is, by definition, $\|y - x\|_{(\Gamma,\mu(r))}$. Thus, $\|y - x\|_{(\Gamma,\mu(r))} < r$, and hence $y \in \mathrm{PB}_\Gamma(x, r)$. This is true for any $y \in \mathrm{PB}_u(x, \epsilon)$. Thus, $\mathrm{PB}_u(x, \epsilon) \subset \mathrm{PB}_\Gamma(x, r)$ and hence $\mathrm{PB}_u(x, \epsilon) \cap \mathcal{R} \subset \mathrm{PB}_\Gamma(x, r) \cap \mathcal{R} \subset O$, as claimed. It follows that $O \in \tau_u|_{\mathcal{R}}$ and hence $\tau_w|_{\mathcal{R}} \subset \tau_u|_{\mathcal{R}}$.

It remains to show that $\tau_u|_{\mathcal{R}} \not\subset \tau_w|_{\mathcal{R}}$. We already know that $\{\mathrm{PB}_\Gamma(0,q) : q \in \mathbb{Q}^+\}$ is a local base of τ_w at 0 so it is enough to show that for every $q \in \mathbb{Q}^+$ there exists $x \in \mathrm{PB}_\Gamma(0,q) \cap \mathcal{R}$ such that $x \notin \mathrm{PB}_u(0,1)$. So let $q \in \mathbb{Q}^+$ be given. Since Γ is a finite partition of $\mathbb{Q}$, we have that $(-\infty,1] \cap (\mathbb{Q} \setminus \Gamma_{\mu(q)}) \neq \emptyset$ and hence we can select $s \in (-\infty,1] \cap (\mathbb{Q} \setminus \Gamma_{\mu(q)})$. Let $x \in \mathcal{R}$ be given by $x = 2d^s$. Then

$$\|x - 0\|_{(\Gamma,\mu(q))} = 0 < q$$

and hence $x \in \mathrm{PB}_\Gamma(0,q) \cap \mathcal{R}$; but

$$\|x - 0\|_{(u,1)} \geq 2 > 1$$

and hence $x \notin \mathrm{PB}_u(0,1) \cap \mathcal{R}$. Since our choice of $q \in \mathbb{Q}^+$ was arbitrary, it follows that $\mathrm{PB}_u(0,1)|_{\mathcal{R}} \notin \tau_w|_{\mathcal{R}}$ and hence $\tau_u|_{\mathcal{R}} \not\subset \tau_w|_{\mathcal{R}}$. $\square$

2.5 Convergence of Power-Series

In this final section of the chapter we will show that for power series over the Hahn field we have the same convergence criterion in the weak topology as for those over the Levi-Civita field [16, 1]. We begin by recalling what we mean by convergence of a power series.

Definition 2.5.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$, and let x_0 be a fixed point and x a given arbitrary point in $\mathcal{F}$. Then we say that the power series*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

***converges weakly in** $\mathcal{F}$ if the sequence of partial sums*

$$S_m(x) := \sum_{n=0}^{m} a_n(x - x_0)^n$$

converges in $(\mathcal{F}, \tau_w)$.

The following theorem provides a criterion for convergence of power series in $(\mathcal{F}, \tau_w)$.

Theorem 2.5.2. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a regular sequence in* $\mathcal{F}$*, let* $S = \bigcup_{n=0}^{\infty} \operatorname{supp}(a_n)$*, and assume that*

$$-\liminf_{n\to\infty}\left(\frac{\lambda(a_n)}{n}\right) = \limsup_{n\to\infty}\left(-\frac{\lambda(a_n)}{n}\right) = 0.$$

Let

$$r = \frac{1}{\sup\left\{\limsup_{n\to\infty} |a_n[q]|^{\frac{1}{n}} : q \in S\right\}}$$

and let $x \in \mathcal{F}$ *be such that* $\lambda(x) \geq 0$*. Then the power series*

$$\sum_{n=0}^{\infty} a_n x^n$$

converges absolutely in $(\mathcal{F}, \tau_w)$ *if* $|x[0]| < r$ *and diverges in* $(\mathcal{F}, \tau_w)$ *if* $|x[0]| > r$.

Proof. If $\lambda(x) > 0$ then $\sum_{n=0}^{\infty} |a_n x^n|$ converges in $(\mathcal{F}, \tau_v)$ and hence in $(\mathcal{F}, \tau_w)$ [14, Theorem 2.13]; so it remains to consider the case where $\lambda(x) = 0$. First assume that $|x[0]| < r$; we will show that, for every $q \in S$, the real power series

$$\sum_{n=0}^{\infty} |a_n[q](x[0])^n| \tag{2.2}$$

converges in $\mathbb{R}$. So let $q_0 \in S$ be given. Since $|x[0]| < r$ we have that

$$|x[0]| < \frac{1}{\sup\left\{\limsup_{n\to\infty} |a_n[q]|^{\frac{1}{n}} : q \in S\right\}} \leq \frac{1}{\limsup_{n\to\infty} |a_n[q_0]|^{\frac{1}{n}}}.$$

It follows that $\limsup_{n\to\infty} |a_n[q_0](x[0])^n|^{\frac{1}{n}} < 1$. Let $c \in \mathbb{R}$ be such that

$$\limsup_{n\to\infty} |a_n[q_0](x[0])^n|^{\frac{1}{n}} < c < 1.$$

Then $\sum\limits_{n=0}^{\infty} c^n$ converges to $1/(1-c)$ in $\mathbb{R}$. Since $\limsup\limits_{n\to\infty} |a_n[q_0](x[0])^n|^{\frac{1}{n}} < c$, there exists $N \in \mathbb{N}$ such that

$$|a_n[q_0](x[0])^n|^{\frac{1}{n}} < c$$

for all $n \geq N$. Thus, using the comparison test, it follows that $\sum\limits_{n=0}^{\infty} |a_n[q_0](x[0])^n|$ converges in $\mathbb{R}$. Since our choice of $q_0 \in S$ was arbitrary, we conclude that the power series in Equation (2.2) converges in $\mathbb{R}$ for every $q \in S$.

Next we claim that for all $q \in S$,

$$\sum_{n=0}^{\infty} |a_n[q]x^n|$$

converges in $(\mathcal{F}, \tau_w)$. So let $q_0 \in S$ be given and consider the sequence of partial sums $(S_m)_{m\in\mathbb{N}}$, where for each $m \in \mathbb{N}$, $S_m = \sum\limits_{n=0}^{m} |a_n[q_0]x^n|$. We know already that (S_m) is a regular sequence because $\lambda(x) = 0$ [14, Theorem 2.3, Corollary 2.12.1], so it remains to show that for any $t \in \mathbb{Q}$, the real sequence $(S_m[t])$ converges in $\mathbb{R}$. So let $t \in \mathbb{Q}$ and $\epsilon > 0$ in $\mathbb{R}$ be given. We will show that there exists $N \in \mathbb{N}$ such that if $m_2 > m_1 > N$ then

$$|S_{m_2}[t] - S_{m_1}[t]| = \sum_{n=m_1}^{m_2} |a_n[q_0]x^n[t]| < \epsilon,$$

thus showing that $(S_m[t])$ is a Cauchy sequence and hence convergent in $\mathbb{R}$.

Let $h = x - x[0]$ and let $N' \in \mathbb{N}$ be such that $N'\lambda(h) > t$. We have, for any $n \in \mathbb{N}$, that

$$\begin{aligned}
((x[0]+h)^n)[t] &= \left(\sum_{l=0}^{n} \frac{n!}{(n-l)!l!} h^l (x[0])^{n-l}\right)[t] \\
&= \sum_{l=0}^{\min\{N',n\}} \frac{n!}{(n-l)!l!} h^l[t] (x[0])^{n-l}.
\end{aligned}$$

It follows that, for $m_2 > m_1 > N'$, we have that

$$\begin{aligned}\sum_{n=m_1}^{m_2} |a_n[q_0]x^n[t]| &= \sum_{n=m_1}^{m_2} |a_n[q_0](x[0]+h)^n[t]| \\ &= \sum_{n=m_1}^{m_2} |a_n[q_0]| \left| \sum_{l=0}^{N'} \frac{n!}{(n-l)!l!} h^l[t]x[0]^{n-l} \right| \\ &\leq \sum_{n=m_1}^{m_2} \sum_{l=0}^{N'} |a_n[q_0]| \left|h[t]^l\right| |x[0]|^{n-l} \frac{n!}{(n-l)!l!} \\ &\leq \left(\sum_{l=0}^{N'} \frac{\left|h[t]^l\right| |x[0]|^{N'-l}}{l!} \right) \left(\sum_{n=m_1}^{m_2} |a_n[q_0]| \, n^{N'} |x[0]|^{n-N'} \right).\end{aligned}$$

The first term in the final expression above is independent of m_1 and m_2; moreover, since

$$|x[0]| < r \leq \frac{1}{\limsup\limits_{n\to\infty} \left\{ |a_n[q_0]|^{\frac{1}{n}} \right\}} = \frac{1}{\limsup\limits_{n\to\infty} \left\{ |a_n[q_0] n^{N'}|^{\frac{1}{n}} \right\}},$$

the real series

$$\sum_{n=0}^{\infty} |a_n[q_0]| \, n^{N'} |x[0]|^{n-N'}$$

must converge in $\mathbb{R}$. Thus, there exists $N'' \in \mathbb{N}$ such that if $m_2 > m_1 > N''$ then

$$\sum_{n=m_1}^{m_2} |a_n[q_0]| \, n^{N'} |x[0]|^{n-N'} < \frac{\epsilon}{\sum\limits_{l=0}^{N'} \frac{\left|h[t]^l\right| |x[0]|^{N'-l}}{l!}}.$$

It follows that if $m_2 > m_1 > \max\{N', N''\}$ then

$$\begin{aligned}\sum_{n=m_1}^{m_2} |a_n[q_0](x[0]+h)^n[t]| &\leq \left(\sum_{l=0}^{N'} \frac{\left|h[t]^l\right| |x[0]|^{N'-l}}{l!} \right) \left(\sum_{n=m_1}^{m_2} |a_n[q_0]| \, n^{N'} |x[0]|^{n-N'} \right) \\ &< \epsilon.\end{aligned}$$

Since $q_0 \in \mathbb{Q}$ was arbitrary, it follows that $\sum_{n=0}^{\infty} |a_n[q]x^n|$ converges in $(\mathcal{F}, \tau_w)$, for every $q \in \mathbb{Q}$.

Finally we show that $\sum_{n=0}^{\infty} |a_n x^n|$ converges in $(\mathcal{F}, \tau_w)$. We have already shown that, for every $q \in \mathbb{Q}$, $\sum\limits_{n=0}^{\infty} |a_n[q]x^n|$ converges in $(\mathcal{F}, \tau_w)$. Moreover, $\lambda(\sum\limits_{n=0}^{\infty} a_n[q]x^n) \geq 0$ and hence

$$\sum_{q \in S} d^q \sum_{n=0}^{\infty} |a_n[q]x^n|$$

has a well ordered support. Let $t \in \mathbb{Q}$ be given. Then

$$\begin{aligned}
\left(\sum_{q \in S} d^q \sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t] &= \sum_{q \in S} \left(d^q \sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t] \\
&= \sum_{q \in S} \left(\sum_{t_1+t_2=t} d^q[t_1] \left(\sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t_2]\right) \\
&= \sum_{q \in S} d^q[q] \left(\sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t-q] \\
&= \sum_{q \in S} \sum_{n=0}^{\infty} |a_n[q]|\,|x^n|\,[t-q].
\end{aligned}$$

Now, let $S_0 \subset S$ be the set of all $q \in S$ such that $t - q \in \bigcup\limits_{n=0}^{\infty} \operatorname{supp}(x^n)$. Since S and $\bigcup\limits_{n=0}^{\infty} \operatorname{supp}(x^n)$ are both well-ordered, it follows that S_0 is finite [14, Theorem 1.3]. It follows that

$$\begin{aligned}
\left(\sum_{q \in S} d^q \sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t] &= \sum_{q \in S} \sum_{n=0}^{\infty} |a_n[q]|\,|x^n|\,[t-q] \\
&= \sum_{q \in S_0} \left(\sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t-q]
\end{aligned}$$

is finite. Thus, $\sum\limits_{q \in S} d^q \sum\limits_{n=0}^{\infty} |a_n[q]x^n|$ converges in $(\mathcal{F}, \tau_w)$. Moreover, we have that

$$
\begin{aligned}
&\left(\sum_{q\in S} d^q \sum_{n=0}^{\infty} |a_n[q]x^n|\right)[t] = \sum_{q\in S_0} d^q[q] \sum_{n=0}^{\infty} |a_n[q]|\,|x^n|\,[t-q] \\
&= \sum_{n=0}^{\infty}\sum_{q\in S_0} d^q[q]\,|a_n[q]|\,|x^n|\,[t-q] = \sum_{n=0}^{\infty}\sum_{q\in S} d^q[q]\,|a_n[q]|\,|x^n|\,[t-q] \\
&= \sum_{n=0}^{\infty}\sum_{q\in S} |a_n[q]| \sum_{t_1+t_2=t} d^q[t_1]\,|x^n|\,[t_2] = \left(\sum_{n=0}^{\infty}\sum_{q\in S}|a_n[q]|d^q|x^n|\right)[t] \\
&= \left(\sum_{n=0}^{\infty}\left(\sum_{q\in S}|a_n[q]|d^q\right)|x^n|\right)[t] \geq \left(\sum_{n=0}^{\infty}\left|\sum_{q\in S}a_n[q]d^q\right||x^n|\right)[t] \\
&= \left(\sum_{n=0}^{\infty}|a_nx^n|\right)[t].
\end{aligned}
$$

So for every $t \in \mathbb{Q}$, $\left((\sum_{n=0}^{m}|a_nx^n|)[t]\right)_{m\in\mathbb{N}}$ converges as a real sequence and, moreover, $\left(\sum_{n=0}^{m}|a_nx^n|\right)_{m\in\mathbb{N}}$ is a regular sequence because $(a_n)_{n\in\mathbb{N}}$ is regular and $\lambda(x) \geq 0$. Hence $\sum_{n=0}^{m}|a_nx^n|$ converges in $(\mathcal{F}, \tau_w)$.

Now let $x \in \mathcal{F}$ be such $|x[0]| > r$; we will show that $\sum_{n=0}^{\infty}a_nx^n$ diverges in $(\mathcal{F}, \tau_w)$. Assume to the contrary that $\sum_{n=0}^{\infty}a_nx^n$ converges in $(\mathcal{F}, \tau_w)$. Let $h = x - x[0]$. Then, since $\sum_{n=0}^{\infty}a_nx^n$ converges in $(\mathcal{F}, \tau_w)$, we have that

$$
\begin{aligned}
\sum_{n=0}^{\infty}a_nx^n &= \sum_{n=0}^{\infty}a_n(x[0]+h)^n = \sum_{n=0}^{\infty}a_n\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x[0]^{n-k}h^k\right) \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}a_n\frac{n!}{k!(n-k)!}x[0]^{n-k}h^k = \sum_{k=0}^{\infty}\frac{\left(\sum_{n=k}^{\infty}\frac{a_nn!}{(n-k)!}x[0]^n\right)}{k!}h^k \\
&= \sum_{n=0}^{\infty}a_nx[0]^n + \sum_{k=1}^{\infty}\frac{\left(\sum_{n=k}^{\infty}\frac{a_nn!}{(n-k)!}x[0]^n\right)}{k!}h^k.
\end{aligned}
$$

Observe that, for every $k \in \mathbb{N}$,

$$\limsup_{n\to\infty} \left\{ \left| \frac{n!}{(n-k)!} \right|^{\frac{1}{n}} \right\} = 1.$$

Thus, for every $k \in \mathbb{N}$ and $q \in \mathbb{Q}$, we have that

$$\limsup_{n\to\infty} \left\{ \left| a_n[q] \frac{n!}{(n-k)!} \right|^{\frac{1}{n}} \right\} = \limsup_{n\to\infty} \left\{ |a_n[q]|^{\frac{1}{n}} \right\}.$$

It follows that, for every $q \in \mathbb{Q}$,

$$\sum_{n=k}^{\infty} \frac{a_n[q]n!}{(n-k)!} x[0]^n$$

diverges in $\mathbb{R}$ only when

$$\sum_{n=k}^{\infty} a_n[q]x[0]^n$$

diverges in $\mathbb{R}$. Since $|x[0]| > r$, we have by definition of r that

$$\frac{1}{|x[0]|} < \sup \left\{ \limsup_{n\to\infty} |a_n[q]|^{\frac{1}{n}} : q \in S \right\}.$$

Therefore there is at least one $q \in S$ such that

$$\frac{1}{|x[0]|} < \limsup_{n\to\infty} |a_n[q]|^{\frac{1}{n}}$$

and hence

$$|x[0]| > \frac{1}{\limsup_{n\to\infty} |a_n[q]|^{\frac{1}{n}}}.$$

Thus, by the root test, we have that $\sum_{n=0}^{\infty} a_n[q]x^n[0]$ diverges in $\mathbb{R}$. Let $q_0 \in S$ be the smallest such element (which exists since S is well-ordered) and let $q_1 = \lambda(h)$. Then, for any $k \geq 1$, the smallest $q \in \mathbb{Q}$ such that

$$\left(\frac{\left(\sum\limits_{n=k}^{\infty} \frac{a_n n!}{(n-k)!} x[0]^n \right)}{k!} h^k \right) [q]$$

diverges is $q = q_0 + kq_1$. Since $kq_1 > 0$ we therefore have that

$$\left(\sum_{k=1}^{\infty} \frac{\left(\sum\limits_{n=k}^{\infty} \frac{a_n n!}{(n-k)!} x[0]^n \right)}{k!} h^k \right) [q_0]$$

must converge. However, $\sum\limits_{n=0}^{\infty} a_n[q_0]x^n[0]$ diverges in $\mathbb{R}$ and

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) [q_0] = \sum_{n=0}^{\infty} a_n[q_0]x^n[0] + \left(\sum_{k=1}^{\infty} \frac{\left(\sum\limits_{n=k}^{\infty} \frac{a_n n!}{(n-k)!} x[0]^n \right)}{k!} h^k \right) [q_0].$$

It follows that $\left(\sum\limits_{n=0}^{\infty} a_n x^n \right) [q_0]$ diverges in $\mathbb{R}$. This contradicts the assumption that $\sum\limits_{n=0}^{\infty} a_n x^n$ converges in $(\mathcal{F}, \tau_w)$. Hence $\sum\limits_{n=0}^{\infty} a_n x^n$ diverges in $(\mathcal{F}, \tau_w)$. □

Corollary 2.5.3. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$ and assume that*

$$-\liminf_{n \to \infty} \left(\frac{\lambda(a_n)}{n} \right) = \limsup_{n \to \infty} \left(-\frac{\lambda(a_n)}{n} \right) = \lambda_0.$$

Let

$$r = \frac{1}{\sup \left\{ \limsup\limits_{n \to \infty} |(a_n d^{n\lambda_0})[q]|^{\frac{1}{n}} : q \in \bigcup\limits_{n \in \mathbb{N}} \operatorname{supp}(a_n d^{n\lambda_0}) \right\}},$$

let $x_0 \in \mathcal{F}$ be fixed, and let $x \in \mathcal{F}$ be such that $\lambda(x - x_0) \geq \lambda_0$. Finally, assume that $(a_n d^{n\lambda_0})_{n \in \mathbb{N}}$ is a regular sequence. Then $\sum\limits_{n=0}^{\infty} a_n (x - x_0)^n$ converges absolutely in $(\mathcal{F}, \tau_w)$ if $|(x - x_0)[\lambda_0]| < r$ and diverges in $(\mathcal{F}, \tau_w)$ if $|(x - x_0)[\lambda_0]| > r$.

Proof. For every $n \in \mathbb{N}$ let $b_n = a_n d^{n\lambda_0}$; and let $y = d^{-\lambda_0}(x - x_0)$. Then

$$\begin{aligned}
\limsup_{n\to\infty}\left(-\frac{\lambda(b_n)}{n}\right) &= \limsup_{n\to\infty}\left(-\frac{\lambda(a_n d^{n\lambda_0})}{n}\right) = \limsup_{n\to\infty}\left(-\frac{\lambda(a_n)}{n} - \frac{n\lambda_0}{n}\right) \\
&= \limsup_{n\to\infty}\left(-\frac{\lambda(a_n)}{n}\right) - \lambda_0 = \lambda_0 - \lambda_0 = 0.
\end{aligned}$$

Moreover, for every $n \in \mathbb{N}$, we have that

$$b_n y^n = a_n d^{n\lambda_0}\left(d^{-\lambda_0}(x - x_0)\right)^n = a_n (x - x_0)^n,$$

and hence $\sum\limits_{n=0}^{\infty} a_n (x - x_0)^n = \sum\limits_{n=0}^{\infty} b_n y^n$. Finally note that

$$\begin{aligned}
\frac{1}{r} &= \sup\left\{\limsup_{n\to\infty}\left|(a_n d^{n\lambda_0})[q]\right|^{\frac{1}{n}} : q \in \bigcup_{n\in\mathbb{N}} \operatorname{supp}(a_n d^{n\lambda_0})\right\} \\
&= \sup\left\{\limsup_{n\to\infty}|b_n[q]|^{\frac{1}{n}} : q \in \bigcup_{n\in\mathbb{N}} \operatorname{supp}(b_n)\right\}.
\end{aligned}$$

Since $(b_n)_{n\in\mathbb{N}}$ is a regular sequence in $\mathcal{F}$ with

$$-\liminf_{n\to\infty}\left(\frac{\lambda(b_n)}{n}\right) = \limsup_{n\to\infty}\left(-\frac{\lambda(b_n)}{n}\right) = 0$$

and since $\lambda(y) = -\lambda_0 + \lambda(x - x_0) \geq 0$, it follows immediately from Theorem 2.5.2 that $\sum\limits_{n=0}^{\infty} a_n (x - x_0)^n = \sum\limits_{n=0}^{\infty} b_n y^n$ converges absolutely in $(\mathcal{F}, \tau_w)$ if $|y[0]| < r$ and diverges in $(\mathcal{F}, \tau_w)$ if $|y[0]| > r$. However, $|y[0]| < r$ if and only if $|(x - x_0)[\lambda_0]| < r$ and $|y[0]| > r$ if and only if $|(x - x_0)[\lambda_0]| > r$. Thus, the corollary is proved. $\square$

Chapter 3

Measure Theory and Integration in ν-Dimensions

3.1 Simple Regions in ν-Dimensions

The notion of simple regions is critical to our success in constructing a theory of 2-dimensional integration and this remains true in higher dimensions; unfortunately in higher dimensions it is more difficult to prove that simple regions have the necessary properties with respect to unions, intersections, and set differences. We begin this chapter by outlining the difficulties involved and stating three conjectures that seem necessary for constructing a measure theory in an arbitrary number of dimensions. Then we will prove a version of Weierstrass' Preparation Theorem for the Hahn field making some progress towards resolving the issue. In the later chapters we will assume that the aforementioned conjectures hold and we will show how they can be used to inductively construct a theory of measures and integration on the Hahn fields in an arbitrary number of dimensions. We begin with the definition of a ν-dimensional simple region; notice that the definition relies on there already existing a definition for an $(\nu-1)$-dimensional simple region. There is, as we have seen in Chapter 1, a definition for 2-dimensional simple regions and so by induction the definition below holds for an arbitrary $\nu \in \mathbb{N}$.

Definition 3.1.1 (ν-Simple region)**.** *Let $\nu \in \mathbb{N}$ satisfy $\nu > 2$ and let $S \subset \mathcal{K}^\nu$. Then we say S is a ν-**dimensional simple region** or equivalently a ν-**simple region** in $\mathcal{K}^\nu$ if there exists a $(\nu - 1)$-simple region $A \subset \mathcal{K}^{\nu-1}$, two analytic functions $h_1, h_2 : A \to \mathcal{K}$ such that $h_1 < h_2$ everywhere on A, and a permutation $\sigma : \mathbb{Z}_\nu \to \mathbb{Z}_\nu$ such that*

$$S = \{(x_{\sigma(1)}, \ldots, x_{\sigma(\nu)}) \in \mathcal{K}^\nu : x_1 \in I(h_1(x_2, \ldots, x_\nu), h_2(x_2, \ldots, x_\nu)), (x_2, \ldots, x_\nu) \in A\}.$$

Note that the reason for the inclusion of σ in the definition is to account for different orientations (e.g. the possibility that h_1 and h_2 are functions of $(x_1, x_3, \ldots, x_\nu)$ rather than $(x_2, \ldots, x_\nu)$.

Now we will define the volume of an ν-simple region; as before, we proceed by induction using as our base case the definition of the area of a 2-simple region from Chapter 1.

Definition 3.1.2 (Volume of an ν-simple region)**.** *Let $\nu \in \mathbb{N}$ satisfy $\nu > 2$ and let $S \subset \mathcal{K}^\nu$ be a simple region with A, h_1 and h_2 as in Definition 3.1.1. Then, we denote the volume of S with $v(S)$ and define it to be*

$$v(S) = \int\limits_{(x_2,\ldots,x_\nu)\in A} [h_2(x_2, \ldots, x_\nu) - h_1(x_2, \ldots, x_\nu)].$$

As alluded to above, we will now state three conjectures regarding ν-simple regions which to date have only been proven for the case of $\nu = 2$ and $\mathcal{K} = \mathcal{R}$, the proofs in that case can be found in [3].

Conjecture 3.1.3. *Let $\nu \in \mathbb{N}$ satisfy $\nu > 2$ and let $A, B \subset \mathcal{K}^\nu$ be two ν-simple regions. Then there exists a finite collection of mutually disjoint n-simple regions $(F_k)_{k=0}^K$ such that*

$$\bigcup_{k=0}^{K} F_k = A \cap B$$

Conjecture 3.1.4. *Let $\nu \in \mathbb{N}$ satisfy $\nu > 2$ and let $A, B \subset \mathcal{K}^{\nu}$ be two n-simple regions. Then there exists a finite collection of mutually disjoint n-simple regions $(F_k)_{k=0}^{K}$ such that*

$$\bigcup_{k=0}^{K} F_k = A \backslash B$$

Conjecture 3.1.5. *Let $\nu \in \mathbb{N}$ satisfy $\nu > 2$ and let $A, B \subset \mathcal{K}^{\nu}$ be two n-simple regions. Then there exists a finite collection of mutually disjoint n-simple regions $(F_k)_{k=0}^{K}$ such that*

$$\bigcup_{k=0}^{K} F_k = A \cup B$$

The proofs of these conjectures in two dimensions rely on the fact that (as in the real numbers) a power series in one dimension may have only finitely many roots on a closed interval. Similar statements made in higher dimensions are significantly more complex and proving them (even in the real case) requires extensive use of the notions of semi- and sub-analytic sets (defined below) as well as differential manifolds. None of theses concepts have been developed yet on either the Hahn or Levi-Civita fields thus making completely rigorous proofs in these fields impossible at this time. Nevertheless we would like to give some justification for our use of these conjectures in the remainder of this chapter and to that end we now present some results from real analysis which are relevant to how simple regions might be described in that context. A summary of the relevant results from real analysis can be found in [22] and [23] contains a more detailed discussion which includes proofs.

Definition 3.1.6 ([22, Definition 5.4.7])**.** *Let M be a real analytic manifold.*

- *Let U be an open coordinate neighbourhood of M. An* ***analytic subset of*** *U is a set of the form*

$$U \bigcap \{(x_1, \ldots, x_n) : F(x_1, \ldots, x_n) = 0\}$$

  where F is a real analytic function on U.

- *Let U be an open coordinate neighbourhood of M. The family of* ***semianalytic subsets of*** *U is the smallest family containing the analytic subsets of U that is closed under finite intersection, finite union, and complement. [A subset of U is said to be semianalytic if it belongs to the family of semianalytic subsets of U.]*

- *A subset S of M is* ***semianalytic*** *if each point $p \in S$ has an open coordinate neighbourhood U such that $S \bigcap U$ is a semianalytic subset of U.*

- *A subset S of M is* ***subanalytic*** *if each point $p \in S$ has an open coordinate neighbourhood U such that $S \bigcap U$ is the projection of a relatively compact semianalytic subset of $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$.*

It should be clear that every ν-simple region is a semianalytic subset of $\mathbb{R}^\nu$; however, not every semianalytic subset of $\mathbb{R}^\nu$ is a ν-simple region. This is because simple regions have the additional requirement that when projected along a particular vector (which vector specifically depends on the simple region's orientation) into $\mathbb{R}^{\nu-1}$, its footprint there is a $(\nu-1)$-simple region. Although we do not attempt to give a complete description of simple regions in terms of semi- and subanalytic sets we note that the following theorem gives us the requirement that closed semianalytic subsets of real manifolds are locally the finite union of analytic sets.

Theorem 3.1.7 ([22, Theorem 5.4.12])**.** *Let S be a semianalytic subset of the real analytic manifold M. Then:*

- *Every connected component of S is semianalytic.*

- *The family of connected components of S is locally finite.*

- *S is locally connected.*

- *The closure and interior of S are semianalytic.*

- *Let U be a semianalytic subset of M with $U \subset S$ which is open relative to S. Then U is locally a finite union of sets of the form*

$$S \bigcap \{x \in M : f_1(x) > 0, \ldots, f_k(x) > 0\},$$

where $f_1, \ldots, f_k$ are real analytic functions.

- *If S is closed, then S is locally the finite union of sets of the form*

$$\{x \in M : f_1(x) \geq 0, \ldots, f_k(x) \geq 0\},$$

where $f_1, \ldots, f_k$ are real analytic functions.

The two following propositions make use of Conjectures 3.1.3 and 3.1.4 to establish crucial properties of ν-simple regions.

Proposition 3.1.8. *Suppose $(H_i)_{i=1}^{\infty}$ and $(G_j)_{j=1}^{\infty}$ are sequences of mutually disjoint ν-simple regions in $\mathcal{K}^\nu$ such that $\sum\limits_{i=1}^{\infty} v(H_i)$ and $\sum\limits_{j=1}^{\infty} v(G_j)$ both strongly converge. Then, there exists a sequence of mutually disjoint ν-simple regions $(T_k)_{k=1}^{\infty}$ such that $\bigcup\limits_{i=1}^{\infty} H_i \cap \bigcup\limits_{j=1}^{\infty} G_j = \bigcup\limits_{k=1}^{\infty} T_k$ and $\sum\limits_{k=1}^{\infty} v(T_k)$ strongly converges.*

Proof. From Conjecture 3.1.3 we know that for every $i, j \in \mathbb{N}$, there is a finite collection of mutually disjoint simple regions $(T_k^{i,j})_{k=1}^{l_{i,j}}$ such that $H_i \cap G_j = \bigcup\limits_{k=1}^{l_{i,j}} T_k^{i,j}$. We assert that the collection

$$(((T_k^{i,j})_{k=1}^{l_{i,j}})_{i=1}^{\infty})_{j=1}^{\infty}$$

is mutually disjoint; so consider $i_1, j_1, k_1 \in \mathbb{N}$ and $i_2, j_2, k_2 \in \mathbb{N}$ such that either $i_1 \neq i_2$, $j_1 \neq j_2$ or $k_1 \neq k_2$. Of course if $i_1 = i_2$ and $j_1 = j_2$ then $T_{k_1}^{i_1,j_1}$ and $T_{k_2}^{i_2,j_2}$ are both contained in $(T_k^{i_1,j_1})_{k=1}^{l_{i,j}}$ which is known to be mutually disjoint. If $j_1 \neq j_2$ then $T_{k_1}^{i_1,j_1} \subset H_{i_1} \cap G_{j_1} \subset G_{j_1}$ and $T_{k_2}^{i_2,j_2} \subset H_{i_2} \cap G_{j_2} \subset G_{j_2}$; since G_{j_1} and G_{j_2} are disjoint, it follows that $T_{k_1}^{i_1,j_1}$ and $T_{k_2}^{i_2,j_2}$ must be disjoint. The same argument can be made in the case that $i_1 \neq i_2$, so the assertion is correct. Since $(((T_k^{i,j})_{k=1}^{l_{i,j}})_{i=1}^{\infty})_{j=1}^{\infty}$ is a countable collection it may be rewritten as $(T_k)_{k=1}^{\infty}$. Thus, $(T_k)_{k=1}^{\infty}$ is a collection of mutually disjoint simple regions such that

$$\bigcup_{i=1}^{\infty} H_i \cap \bigcup_{j=1}^{\infty} G_j = \bigcup_{k=1}^{\infty} T_k.$$

Since

$$\bigcup_{k=1}^{\infty} T_k \subset \bigcup_{i=1}^{\infty} H_i,$$

$\sum\limits_{i=1}^{\infty} v(H_i)$ converges in the order topology, and $(T_k)_{k=1}^{\infty}$ is a collection of mutually disjoint simple regions, $\sum\limits_{k=1}^{\infty} v(T_k)$ must also converge in the order topology. □

Proposition 3.1.9. *Suppose that for every $i \in \mathbb{N}$, $(G_n^i)_{n=1}^{\infty}$ is a countable sequence of mutually disjoint simple regions such that*

$$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} v(G_n^i)$$

converges in the order topology. Then there exists a collection of mutually disjoint simple regions $(H_j)_{j=1}^{\infty}$ such that

$$\bigcup_{j=1}^{\infty} H_j = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} G_n^i$$

and $\sum\limits_{j=1}^{\infty} v(H_j)$ converges in the order topology.

Proof. First we note that $((G_n^i)_{n=1}^{\infty})_{i=1}^{\infty}$ is a countable collection of simple regions and so may be rewritten as $(G_n)_{n=1}^{\infty}$. To create the sequence $(H_j)_{j=1}^{\infty}$ we begin by defining $H_1 = G_1$. Next we observe that by Conjecture 3.1.4, for every $n_1, n_2 \in \mathbb{N}$, $G_{n_1} \backslash G_{n_2}$ is given by a finite number of mutually disjoint simple regions $(F_l^{n_1,n_2})_{l=1}^{t_{n_1,n_2}}$. So, for every $n_0 \in \mathbb{N}$,

$$G_{n_0} \backslash \bigcup_{n=1}^{n_0-1} G_n = \bigcap_{n=1}^{n_0-1} (G_{n_0} \backslash G_n) = \bigcap_{n=1}^{n_0-1} \bigcup_{l=1}^{t_{n_0,n}} F_l^{n_0,n}$$

However, using the same argument as in the proof of Proposition 3.1.8, we deduce that for every $n_0 \in \mathbb{N}$,

$$\bigcap_{n=1}^{n_0-1} \bigcup_{l=1}^{t_{n_0,n}} F_l^{n_0,n}$$

can be expressed as the union of a finite number of mutually disjoint simple regions $(F_l^{n_0})_{l=1}^{t_{n_0}}$. We define

$$
\begin{aligned}
H_2 &= F_1^2, \ldots, H_{t_2+1} = F_{t_2}^2 \\
H_{2+t_2} &= F_1^3, \ldots, H_{1+t_2+t_3} = F_{t_3}^3 \\
H_{2+t_2+t_3} &= F_1^4, \ldots, H_{1+t_2+t_3+t_4} = F_{t_4}^4 \\
&\vdots \\
H_{2+\sum\limits_{n=2}^{n_0} t_n} &= F_1^{n_0+1}, \ldots, H_{1+\sum\limits_{n=2}^{n_0+1} t_n} = F_{t_{n_0+1}}^{n_0+1} \\
&\vdots
\end{aligned}
$$

We see that by construction the H_j's are mutually disjoint and $\bigcup\limits_{j=1}^{\infty} H_j = \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{n=1}^{\infty} G_n^i$, so $0 \leq \sum\limits_{j=0}^{\infty} v(H_j) \leq \sum\limits_{i=1}^{\infty} \sum\limits_{n=1}^{\infty} v(G_n^i)$. By our premise, $\sum\limits_{i=1}^{\infty} \sum\limits_{n=1}^{\infty} v(G_n^i)$ converges in the order topology and so $\sum\limits_{j=0}^{\infty} v(H_j)$ must converge in the order topology as claimed. $\square$

3.2 Weierstrass' Preparation Theorem for the Hahn Field

Although the language from real analysis needed to prove the real counterparts of 3.1.3, 3.1.4, and 3.1.5 remains undeveloped on the Hahn field at this time, it is nevertheless possible to prove for the Hahn field the basic theorem required in the real case, namely Weierstrass' Preparation Theorem which "allows one to establish properties of analytic varieties by inducting on dimension" [22]. Below we include a proof of that theorem which has the additional benefit of illustrating how convenient the Hahn field can make arguments related to convergence.

Remark 3.2.1. *We introduce the following notation, borrowed from [24], to simplify our arguments*

- $\alpha = \alpha_0, \ldots, \alpha_{\nu-1}$
- $|\alpha| = |\alpha_0| + \cdots + |\alpha_{\nu-1}|$
- $x = x_0, \ldots, x_{\nu-1}$
- $x^\alpha = x_0^{\alpha_0} \cdots x_{\nu-1}^{\alpha_{\nu-1}}$

The definition below follows the example of [24] which "Let[s] $K\{x\} = k\{x_1, \ldots, x_\nu\}$ denote the ring of all power series $\sum c_\alpha x^\alpha$ which converge for $|x| < \delta$, where $\delta > 0$, but may depend on the series in question." There "K" is either the real or complex field.

Definition 3.2.2 ($\mathcal{K}\{x\}$)**.** *Let $\mathcal{K}\{x\} = \mathcal{K}\{x_1, \ldots, x_\nu\}$ denote the set of all power series for which there is a $\rho > 0$ in $\mathcal{K}$ such that the power series in question converges for $|x| < \rho$.*

For convenience we also define the following subsets of $\mathcal{K}\{x\}$.

Definition 3.2.3. *Let $r_1, \ldots, r_\nu \in \mathcal{K}$ be positive. We define*

$$\begin{aligned} L_r &= L_{r_1,\ldots,r_\nu} \\ &:= \left\{ f \in \mathcal{K}\{x\} \middle| f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha,\ \sum_{\alpha \in \mathbb{N}^\nu} |c_\alpha|\, r_1^{\alpha_1} \cdots r_\nu^{\alpha_\nu} \text{ converges in the order topology.} \right\}. \end{aligned}$$

Lemma 3.2.4. *Fix $\nu \in \mathbb{N}$ and let $r_1, \ldots, r_\nu \in \mathcal{K}$ be positive and satisfy $\lambda(r_i) > q$ for all $1 \le i \le \nu$. Every $f \in L_r$ can be given by*

$$f(x) = \sum_{\alpha \in \mathbb{N}^\nu} c_\alpha x^\alpha.$$

If we define $\|\cdot\|_r$ by

$$\|f\|_r = \sum_{\alpha \in \mathbb{N}^\nu} |c_\alpha|\, r^\alpha,$$

then $(L_r, \|\cdot\|)$ *is a Banach space.*

Proof. Clearly $\|\cdot\|_r$ is a norm. We show that that every Cauchy sequence in L_r must converge to a point also in L_r. So let $(f_i)_{i\in\mathbb{N}}$ be a Cauchy sequence; and for every $i \in \mathbb{N}$ let

$$f_i(x) = \sum_{\alpha\in\mathbb{N}^\nu} c_{i,\alpha} x^\alpha.$$

We claim that for every $\alpha \in \mathbb{N}^\nu$, $(|c_{i,\alpha}|)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}$. To see this, fix $\alpha_0 \in \mathbb{N}^\nu$, $\epsilon > 0$, and define $\epsilon_0 = \epsilon r^{\alpha_0}$. Since $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence there must exist a $N \in \mathbb{N}$ such that for all $i, j > N$

$$\|f_i - f_j\|_r < \epsilon_0.$$

But

$$\begin{aligned} \|f_i - f_j\|_r &= \sum_{\alpha\in\mathbb{N}^n} |c_{i,\alpha} - c_{j,\alpha}|\, r^\alpha \\ &\geq |c_{i,\alpha_0} - c_{j,\alpha_0}|\, r^{\alpha_0}. \end{aligned}$$

Thus,

$$|c_{i,\alpha_0} - c_{j,\alpha_0}|\, \delta^{\alpha_0} < \epsilon_0 = \epsilon r^{\alpha_0}$$

and hence $|c_{i,\alpha_0} - c_{j,\alpha_0}| < \epsilon$ which proves the claim. Since the Hahn field is complete we may define for every $\alpha \in \mathbb{N}^\nu$

$$c_\alpha \coloneqq \lim_{i\to\infty} c_{i,\alpha}$$

and we let

$$f(x) = \sum_{\alpha\in\mathbb{N}^\nu} c_\alpha x^\alpha.$$

Clearly our choice of c_α's ensures that

$$\lim_{i\to\infty} f_i = f$$

so it remains to show that $f \in L_r$. Suppose otherwise, then we must have that $\sum_{\alpha\in\mathbb{N}^\nu} |c_\alpha| r^\alpha$ diverges in the order topology. Notice that

$$\sum_{\alpha\in\mathbb{N}^\nu} |c_\alpha| r^\alpha = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} |c_\alpha| r^\alpha \tag{3.1}$$

and for every $i \in \mathbb{N}$

$$\sum_{\alpha\in\mathbb{N}^\nu} |c_{i,\alpha}| r^\alpha = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} |c_{i,\alpha}| r^\alpha. \tag{3.2}$$

In the order topology sums converge if and only if their terms form a null sequence so we must have that

$$\lim_{m\to\infty} \sum_{|\alpha|=m} |c_\alpha| r^\alpha \neq 0 \tag{3.3}$$

so that the sum in Equation 3.1 diverges and of course for every $i \in \mathbb{N}$,

$$\lim_{m\to\infty} \sum_{|\alpha|=m} |c_{i,\alpha}| r^\alpha = 0 \tag{3.4}$$

so that the sum in Equation 3.2 converges. By equation (3.3) there must exist a $h > 0$ in $\mathcal{K}$ such that for all $m \in \mathbb{N}$ there is a $n > m$ such that

$$\sum_{|\alpha|=n} |c_\alpha| r^\alpha > h.$$

Choose $N \in \mathbb{N}$ to be large enough that for any $i, j > N$

$$\|f_i - f_j\|_r < \frac{h}{3}.$$

Fix $i > N$. Then, since the series in equation (3.4) converges, $(\sum_{|\alpha|=m} |c_{i,\alpha}| r^\alpha)_{m\in\mathbb{N}})$ must be a null sequence. So we may choose $M \in \mathbb{N}$ large enough that for every $m > M$,

$$\sum_{|\alpha|=m} |c_{i,\alpha}| r^\alpha < \frac{h}{3}.$$

Thus we have that for every $j \in \mathbb{N}$ with $j > i$,

$$\begin{aligned}\sum_{|\alpha|=m} |c_{j,\alpha}| r^{\alpha} &= \sum_{|\alpha|=m} |c_{i,\alpha} + c_{j,\alpha} - c_{i,\alpha}| r^{\alpha} \\ &\leq \sum_{|\alpha|=m} |c_{i,\alpha}| r^{\alpha} + \sum_{|\alpha|=m} |c_{j,\alpha} - c_{i,\alpha}| r^{\alpha} \\ &\leq \sum_{|\alpha|=m} |c_{i,\alpha}| r^{\alpha} + \|f_j - f_i\|_r \\ &\leq \frac{h}{3} + \frac{h}{3} = \frac{2h}{3}.\end{aligned}$$

However this contradicts the fact that for every $\alpha \in \mathbb{N}^n$,

$$\lim_{i\to\infty} c_{i,\alpha} = c_{\alpha}$$

and so the lemma is proven. $\square$

Lemma 3.2.5. *Let $r_1, \ldots, r_\nu \in \mathcal{K}$ be positive and suppose that $A : L_{r_1,\ldots,r_\nu} \to L_{r_1,\ldots,r_\nu}$ is a linear operator satisfying $\|A\|_r \leq s$ where $\lambda(s) > 0$. Then, if I denotes the identity operator, $I - A$ is invertible.*

Proof. We have by the Triangle Inequality that

$$\left\| \sum_{n=0}^{\infty} A^n \right\|_r \leq \sum_{n=0}^{\infty} \|A^n\|_r .$$

However, $\|A^n\|_r \leq \|A\|_r^n \leq s^n$ and $\lim\limits_{n\to\infty} s^n = 0$ so the sum on the right above must converge in the order topology. Thus $\sum\limits_{n=0}^{\infty} A^n$ is a well-defined linear operator on L_r. Moreover, notice that

$$
\begin{aligned}
(I-A)\sum_{n=0}^{\infty} A^n &= \lim_{N\to\infty}(I-A)\sum_{n=0}^{N} A^n \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}(A^n - A^{n+1}) \\
&= \lim_{N\to\infty}(I - A^N).
\end{aligned}
$$

But $\lim_{N\to\infty} A^N \to 0$ because $\|A\|_r < s$ so we have that

$$
(I-A)\sum_{n=0}^{\infty} A^n = I.
$$

This proves that

$$
\sum_{n=0}^{\infty} A^n = (I-A)^{-1}
$$

and hence that $I - A$ is invertible. $\square$

Corollary 3.2.6. *Let $r_1, \ldots, r_\nu \in \mathcal{K}$ be positive and suppose that $A : L_{r_1,\ldots,r_\nu} \to L_{r_1,\ldots,r_\nu}$ is a linear operator satisfying $\|I - A\|_r \le s$ where I denotes the identity operator and $\lambda(s) \ge 0$. Then A is invertible.*

Proof. This follows immediately from the previous lemma by making the change, $A \to I - A'$. $\square$

Theorem 3.2.7 (Weierstrass' Preparation Theorem)**.** *Let $f \in \mathcal{K}\{x_1, \ldots, x_\nu\}$ and suppose $f(0, \ldots, 0, x_\nu) \in \mathcal{K}\{x_n\}$ is not identical to zero and has a root of multiplicity $p \ge 1$ at the origin. Then we have that for any $\phi \in \mathcal{K}\{x_1, \ldots, x_\nu\}$ there is an $a \in \mathcal{K}\{x_1, \ldots, x_\nu\}$ and $b_1, \ldots, b_p \in \mathcal{K}\{x_1, \ldots, x_{\nu-1}\}$ such that*

$$
\phi = a \cdot f + \sum_{i=1}^{p} b_i \cdot x_\nu^{p-i}.
$$

Proof. First we notice that we may write

$$f(x_1,\dots,x_\nu)=\sum_{i=0}^{\infty} f_i(x_1,\dots,x_{\nu-1})x_\nu^i$$

and by our premise we must have $f_i(0,\dots,0)=0$ for $i<p$ and $f_p(0)\neq 0$, moreover without loss of generality we may assume that $f_p=1$ since we can always change f to $\hat{f}=\frac{f}{f_p}$ and the only effect this will have on our theorem is to change a to $\hat{a}=af_p$. Since $f\in\mathcal{K}\{x\}$ there is a $\rho\in\mathcal{K}$ such that if $f(x)=\sum_{\alpha\in\mathbb{N}^\nu} a_\alpha x^\alpha$ then $\sum_{\alpha\in\mathbb{N}^n}|a_\alpha|\rho^{|\alpha|}$ converges. Let $q=\lambda(\rho)$ and let $r_1,\dots,r_\nu\in\mathcal{K}$ be positive with $\lambda(r_i)>q$ for all $i\in\{1,\dots,\nu\}$. Clearly we have that $f\in L_{r_1,\dots,r_\nu}$ and for any $\phi\in L_{r_1,\dots,r_\nu}$ we may write

$$\phi=a(\phi)x_\nu^p+b(\phi)$$

where $a,b\in L_{r_1,\dots,r_\nu}$ and b is a polynomial of degree less than p in x_ν. We define a linear operator $A:L_r\to L_r$ by

$$A\phi:=a(\phi)f+b(\phi).$$

Notice that

$$\|A\phi-\phi\|_r=\|a(\phi)(f-x_\nu^p)\|_r\leq\|a(\phi)\|_r\|f-x_\nu^p\|_r\,.$$

We may, by carefully choosing $r_1,\dots,r_\nu\in\mathcal{K}$, ensure that

$$\begin{aligned}\|f-x_\nu^p\|_r&=\left\|\sum_{i=0}^{\infty}f_i(x_1,\dots,x_{\nu-1})x_\nu^i-f_p(x_1,\dots,x_{\nu-1})x_\nu^p\right\|\\&\leq\left\|\sum_{i=0}^{p-1}f_i(x_1,\dots,x_{\nu-1})x_\nu^i\right\|+\left\|\sum_{i=p+1}^{\infty}f_i(x_1,\dots,x_{\nu-1})x_\nu^i\right\|\\&\leq dr_\nu^p.\end{aligned}$$

This will ensure that

$$\|A\phi-\phi\|_r\leq d\,\|a(\phi)\|_r\,r_\nu^p\leq d\,\|\phi\|_r\,.$$

So, if $I : L_r \to L_r$ is the identity operator, we have that $\|A - I\|_r \le d$ and so by Corollary 3.2.6 A is invertible and hence surjective. Thus for any $\phi \in L_r$ there must be a $\psi \in L_r$ such that

$$\phi = A\psi = a(\psi)f + b(\psi),$$

which finishes the proof of the theorem. $\square$

3.3 Measure Theory in ν-Dimensions

Having argued for Conjectures 3.1.3, 3.1.4, and 3.1.5 to the best of our current ability we now take them for granted and proceed to construct a theory of measures, functions, and integration on $\mathcal{K}^\nu$. We begin by defining measurable sets in ν dimensions and proving certain related propositions. The definitions and results in this and the remaining sections of this chapter are closely related to those in [3] and [11] but generalize them to an arbitrary number of dimensions.

Definition 3.3.1 (Measurable Set)**.** *Let $S \subset \mathcal{K}^\nu$. Then we say that S is a* ***measurable set*** *if for every $\epsilon > 0$ there exist two sequences of mutually disjoint simple regions, $(G_n)_{n=1}^\infty$ and $(H_n)_{n=1}^\infty$, such that*

$$\bigcup_{n=1}^\infty G_n \subset S \subset \bigcup_{n=1}^\infty H_n,$$

$\sum_{n=1}^\infty v(G_n)$ and $\sum_{n=1}^\infty v(H_n)$ converge, and

$$\sum_{n=1}^\infty v(H_n) - \sum_{n=1}^\infty v(G_n) < \epsilon.$$

Definition 3.3.2 (Measure of a Measurable Set)**.** *Suppose $S \subset \mathcal{K}^\nu$ is a measurable set. By definition we have that for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple* *regions,* $(G_n^k)_{n=1}^{\infty}$ *and* $(H_n^k)_{n=1}^{\infty}$*, such that*

$$\bigcup_{n=1}^{\infty} G_n^k \subset S \subset \bigcup_{n=1}^{\infty} H_n^k,$$

$\sum\limits_{n=1}^{\infty} v(G_n^k)$ *and* $\sum\limits_{n=1}^{\infty} v(H_n^k)$ *converge, and*

$$\sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k.$$

We note that that since for every $k \in \mathbb{N}$, $(G_n^k)_{n=1}^{\infty}$ *and* $(H_n^k)_{n=1}^{\infty}$ *are mutually disjoint we can arrange them so that*

$$\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} G_n^{k+1} \subset S \subset \bigcup_{n=1}^{\infty} H_n^{k+1} \subset \bigcup_{n=1}^{\infty} H_n^k.$$

We claim that $(\sum\limits_{n=1}^{\infty} v(G_n^k))_{k=1}^{\infty}$ *is a Cauchy sequence. To prove this we fix* $\epsilon > 0$ *in* $\mathcal{K}$ *and let* $k_0 \in \mathbb{N}$ *be large enough so that* $d^{k_0} < \epsilon$*. Now, for every* $l > k_0$,

$$\bigcup_{n=1}^{\infty} G_n^l \subset S \subset \bigcup_{n=1}^{\infty} H_n^{k_0},$$

so

$$\sum_{n=1}^{\infty} v(G_n^l) \leq \sum_{n=1}^{\infty} v(H_n^{k_0}).$$

Thus,

$$0 \leqslant \sum_{n=1}^{\infty} v(G_n^l) - \sum_{n=1}^{\infty} v(G_n^{k_0}) \leq \sum_{n=1}^{\infty} v(H_n^{k_0}) - \sum_{n=1}^{\infty} v(G_n^{k_0}) < d^{k_0} < \epsilon.$$

Therefore the claim is proven, and a similar argument shows that the sequence $(\sum\limits_{n=1}^{\infty} v(H_n^k))_{k=1}^{\infty}$ *is Cauchy. Since* $\mathcal{K}$ *is Cauchy complete*

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k)$$

and

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} v(H_n^k)$$

both exist, and hence

$$\lim_{k\to\infty}\sum_{n=1}^{\infty}(v(H_n^k)-v(G_n^k))$$

exists and

$$\lim_{k\to\infty}\sum_{n=1}^{\infty}(v(H_n^k)-v(G_n^k)) = \lim_{k\to\infty}\sum_{n=1}^{\infty} v(H_n^k) - \lim_{k\to\infty}\sum_{n=1}^{\infty} v(G_n^k).$$

Furthermore, for every $k \in \mathbb{N}$,

$$0 \le \sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k,$$

so

$$0 \le \lim_{k\to\infty}\left(\sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k)\right) \le 0.$$

We conclude that

$$\lim_{k\to\infty}\left(\sum_{n=1}^{\infty} v(G_n^k) - \sum_{n=1}^{\infty} v(H_n^k)\right) = 0$$

and hence

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} v(G_n^k) = \lim_{k\to\infty}\sum_{n=1}^{\infty} v(H_n^k).$$

We call this limit the ***measure of*** *S and we denote it by $m(S)$.*

Proposition 3.3.3. *Suppose $S \subset \mathcal{K}^\nu$ is a measurable set. Then*

$$\begin{aligned} m(S) = \inf \Bigg\{ &\sum_{n=1}^{\infty} v(H_n) : H_n\text{'s are mutually disjoint simple regions,} \\ &S \subset \bigcup_{n=1}^{\infty} H_n, \text{ and } \sum_{n=1}^{\infty} v(H_n) \text{ converges} \Bigg\} \\ &= \sup \Bigg\{ \sum_{n=1}^{\infty} v(G_n) : G_n\text{'s are mutually disjoint simple regions,} \\ &\bigcup_{n=1}^{\infty} G_n \subset S, \text{ and } \sum_{n=1}^{\infty} v(G_n) \text{ converges} \Bigg\}. \end{aligned}$$

Proof. First we show that the infimum exists and is equal to $m(S)$. Since S is a measurable set we know that for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^\infty$ and $(H_n^k)_{n=1}^\infty$ such that

$$\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} G_n^{k+1} \subset S \subset \bigcup_{n=1}^{\infty} H_n^{k+1} \subset \bigcup_{n=1}^{\infty} H_n^k,$$

$\sum_{n=1}^{\infty} v(G_n^k)$ and $\sum_{n=1}^{\infty} v(H_n^k)$ both converge, and

$$\sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k.$$

By definition

$$m(S) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k)$$

and, for every $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} v(G_n^k) \le m(S) \le \sum_{n=1}^{\infty} v(H_n^k).$$

It remains to be shown that if $(H_n)_{n=1}^\infty$ is a sequence of mutually disjoint simple regions such that $S \subset \bigcup\limits_{n=1}^{\infty} H_n$ and $\sum\limits_{n=1}^{\infty} v(H_n)$ converges, then

$$\sum_{n=1}^{\infty} v(H_n) \geq \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) = m(S).$$

Suppose not. Then, there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} v(H_n)$ converges, $S \subset \bigcup\limits_{n=1}^{\infty} H_n$, and

$$\sum_{n=1}^{\infty} v(H_n) < \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) = m(S).$$

Let

$$\eta = m(S) - \sum_{n=1}^{\infty} v(H_n),$$

then

$$\sum_{n=1}^{\infty} v(H_n) = m(S) - \eta.$$

However

$$m(S) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k),$$

thus there exists a $k_0 \in \mathbb{N}$ such that

$$m(S) - \sum_{n=1}^{\infty} v(G_n^{k_0}) < \eta.$$

Therefore,

$$\sum_{n=1}^{\infty} v(G_n^{k_0}) > m(S) - \eta = \sum_{n=1}^{\infty} v(H_n),$$

but this contradicts the fact that $\bigcup_{n=1}^{\infty} G_n^{k_0} \subset S \subset \bigcup_{n=1}^{\infty} H_n$. It follows that

$$\begin{aligned} m(S) &= \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) \\ &= \inf\left\{ \sum_{n=1}^{\infty} v(H_n) : H_n\text{'s are mutually disjoint, } S \subset \bigcup_{n=1}^{\infty} H_n, \sum_{n=1}^{\infty} v(H_n) \text{ converges} \right\}. \end{aligned}$$

A similar argument shows that

$$m(S) = \sup\left\{ \sum_{n=1}^{\infty} v(G_n) : G_n\text{'s are mutually disjoint, } \bigcup_{n=1}^{\infty} G_n \subset S, \sum_{n=1}^{\infty} v(G_n) \text{ converges} \right\}.$$

□

Proposition 3.3.4. *Fix $J \in \mathbb{N}$, let $S \subset \mathcal{K}^\nu$ be an arbitrary set and let $A, B, (A_i)_{i\in\mathbb{N}}, \{B_j\}_{j\in(1,\ldots,J)} \in \mathcal{K}^\nu$ be measurable sets. Then we have the following.*

1. *If $A \subset B$ then $m(A) \leq m(B)$.*

2. *If $S \subset A$ and $m(A) = 0$ then S is measurable and $m(S) = 0$*

3. *If S is a countable set then S is measurable and $m(S) = 0$*

4. *If $\lim_{i\to\infty} m(A_i) = 0$ then $\bigcup_{i=1}^{\infty} A_i$ is measurable and*

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i).$$

   Moreover, if the sets $(A_i)_{i=1}^{\infty}$ are mutually disjoint then

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i).$$

5. *$\bigcap_{j=1}^{J} B_j$ is measurable and $m\left(\bigcap_{j=1}^{J} B_j\right) \leq \min\{m(B_j) : j \in \{1, \ldots, J\}\}$*

6. $m(A \cup B) = m(A) + m(B) - m(A \cap B)$

Proof. 1. Suppose not, then $A \subset B$ but $m(A) > m(B)$. Let $m(A) - m(B) = \eta$. Since A is measurable there is a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} v(G_n)$ converges, and

$$m(A) - \sum_{n=1}^{\infty} v(G_n) < \frac{\eta}{4}.$$

Since B is measurable there is a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $B \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} v(H_n)$ converges, and

$$\sum_{n=1}^{\infty} v(H_n) - m(B) < \frac{\eta}{4}.$$

So we see that

$$\sum_{n=1}^{\infty} v(H_n) < m(B) + \frac{\eta}{4} < m(A) - \frac{\eta}{4} < \sum_{n=1}^{\infty} v(G_n),$$

thus

$$\sum_{n=1}^{\infty} v(H_n) < \sum_{n=1}^{\infty} v(G_n).$$

However,

$$\bigcup_{n=1}^{\infty} G_n \subset A \subset B \subset \bigcup_{n=1}^{\infty} H_n,$$

so

$$\bigcup_{n=1}^{\infty} G_n \subset \bigcup_{n=1}^{\infty} H_n$$

and thus we have reached a contradiction.

2. Fix $\epsilon > 0$ in $\mathbb{R}$. Since A is measurable and $m(A) = 0$, for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(H_n^k)_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1}^{\infty} H_n^k$, $\sum_{n=1}^{\infty} v(H_n^k)$ converges, and

$$\sum_{n=1}^{\infty} v(H_n^k) - m(A) = \sum_{n=1}^{\infty} v(H_n^k) < d^k.$$

For every $n \in \mathbb{N}$, let $G_n = \emptyset$ which is a simple region. Let $k_0 \in \mathbb{N}$ be large enough that $d^{k_0} < \epsilon$. Then,

$$\bigcup_{n=1}^{\infty} G_n \subset S \subset \bigcup_{n=1}^{\infty} H_n^{k_0}$$

and

$$\sum_{n=1}^{\infty} v(H_n^{k_0}) - \sum_{n=1}^{\infty} v(G_n) = \sum_{n=1}^{\infty} v(H_n^{k_0}) < d^{k_0} < \epsilon.$$

Hence S is measurable. Since for every $k \in \mathbb{N}$, $S \subset \bigcup_{n=1}^{\infty} H_n^k$ it follows that $m(S) \leq \sum_{n=1}^{\infty} v(H_n^k) < d^k$, by letting $k \to \infty$ we see that $m(S) = 0$.

3. Since S is a countable set there is a sequence of points $((x_{1,i}, \ldots, x_{n,i}))_{i=1}^{\infty}$ such that $S = \bigcup_{i=1}^{\infty} (x_{1,i}, \ldots, x_{n,i})$. Fix $\epsilon > 0$. For every $i \in \mathbb{N}$ define

$$H_i = \left\{ (x_1, \ldots, x_n) \in \mathcal{K}^n : \text{ For every } m \in \{1, \ldots, n\}\ x_m \in \left(x_{m,i} - \frac{1}{2}(d^i\epsilon)^{\frac{1}{n}}, x_{m,i} + \frac{1}{2}(d^i\epsilon)^{\frac{1}{n}} \right) \right\}.$$

Note that for every $i \in \mathbb{N}$, H_i is a simple region with $v(H_i) = d^i\epsilon$. So,

$$\lim_{i\to\infty} v(H_i) = \lim_{i\to\infty} d^i\epsilon = 0,$$

thus $\sum_{i=1}^{\infty} v(H_i)$ converges in the order topology. For every $j \in \mathbb{N}$, let $G_j = \emptyset$. Then,

$$\bigcup_{j=1}^{\infty} G_j \subset A \subset \bigcup_{i=1}^{\infty} H_i,$$

where $\sum_{j=1}^{\infty} v(G_j)$ and $\sum_{i=1}^{\infty} v(H_i)$ both strongly converge, and

$$\sum_{i=1}^{\infty} v(H_i) - \sum_{j=1}^{\infty} v(G_j) = \sum_{i=1}^{\infty} v(H_i) \leq \sum_{i=1}^{\infty} d^i\epsilon = \frac{d\epsilon}{1-d} < \epsilon,$$

which proves that A is measurable. Furthermore, since $A \subset \bigcup_{i=1}^{\infty} H_i$, $m(A) \leq \sum_{i=1}^{\infty} v(H_i) < \epsilon$.

Taking the limit as $\epsilon \to 0$ shows that $m(A) = 0$.

4. Note that since $\lim_{i\to\infty} m(A_i) = 0$, $\sum_{i=1}^{\infty} m(A_i)$ converges in the order topology. Fix $\epsilon > 0$. Since each A_i is measurable we see that for every $i \in \mathbb{N}$ there are two sequences of mutually disjoint simple regions $(G_n^i)_{n=1}^{\infty}$ and $(H_n^i)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} G_n^i \subset A_i \subset \bigcup_{n=1}^{\infty} H_n^i,$$

$\sum_{n=1}^{\infty} v(G_n^i)$ and $\sum_{n=1}^{\infty} v(H_n^i)$ both converge, and

$$\sum_{n=1}^{\infty} v(H_n^i) - \sum_{n=1}^{\infty} v(G_n^i) < d^i \epsilon.$$

Since $\lim_{i\to\infty} m(A_i) = 0$ and

$$\sum_{n=1}^{\infty} v(G_n^i) \leq \sum_{n=1}^{\infty} v(H_n^i) < m(A_i) + d^i \epsilon,$$

we arrive at the conclusion that

$$\lim_{i\to\infty} \sum_{n=1}^{\infty} v(G_n^i) = \lim_{i\to\infty} \sum_{n=1}^{\infty} v(H_n^i) = 0.$$

Thus, $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} v(G_n^i)$ and $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} v(H_n^i)$ both converge. From the proof of Proposition 3.1.9 we know there exist two sequences of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} G_n^i$$

and

$$\bigcup_{n=1}^{\infty} H_n = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} H_n^i.$$

Therefore,

$$\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} H_n^i \backslash \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} G_n^i$$

Now, for every $i \in \mathbb{N}$ we have that

$$\bigcup_{n=1}^{\infty} G_n^i \subset \bigcup_{n=1}^{\infty} H_n^i,$$

so

$$\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} H_n^i \backslash \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} G_n^i = \bigcup_{i=1}^{\infty} (\bigcup_{n=1}^{\infty} H_n^i \backslash \bigcup_{n=1}^{\infty} G_n^i)$$

Moreover, since for every $i \in \mathbb{N}$, the sequences $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ are both mutually disjoint we can arrange them in such a way that for every $n \in \mathbb{N}$, $G_n^i \subset H_n^i$. Thus, for every $i \in \mathbb{N}$,

$$\bigcup_{n=1}^{\infty} H_n^i \backslash \bigcup_{n=1}^{\infty} G_n^i = \bigcup_{n=1}^{\infty} (H_n^i \backslash G_n^i).$$

So we conclude that

$$\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} (H_n^i \backslash G_n^i).$$

Therefore,

$$\begin{aligned}
m\left(\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n\right) &= m\left(\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} (H_n^i \backslash G_n^i)\right) \\
&\leq \sum_{i=1}^{\infty} m\left(\bigcup_{n=1}^{\infty} (H_n^i \backslash G_n^i)\right) \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} m(H_n^i \backslash G_n^i) \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (v(H_n^k) - v(G_n^k)) \\
&= \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} v(H_n^i) - \sum_{n=1}^{\infty} v(G_n^i)\right)
\end{aligned}$$

But,

$$m\left(\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n\right) = \sum_{n=1}^{\infty} v(H_n) - \sum_{n=1}^{\infty} v(G_n)$$

and from above we know that for every $i \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} v(H_n^i) - \sum_{n=1}^{\infty} v(G_n^i) \leq d^i \epsilon.$$

Therefore,

$$\sum_{n=1}^{\infty} v(H_n) - \sum_{n=1}^{\infty} v(G_n) \leq \sum_{i=1}^{\infty} d^i \epsilon = \frac{d}{1-d}\epsilon < \epsilon$$

which proves that $\bigcup_{i=1}^{\infty} A_i$ is measurable. Since

$$\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} H_n^k$$

we have that

$$\begin{aligned}
m\left(\bigcup_{i=1}^{\infty} A_i\right) &\leq m\left(\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} H_n^i\right) \\
&\leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} v(H_n^i) \\
&\leq \sum_{i=1}^{\infty} (m(A) + d^i \epsilon) \\
&< \sum_{i=1}^{\infty} m(A) + \epsilon.
\end{aligned}$$

The above holds for any $\epsilon > 0$ so we obtain

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i).$$

Now, assume that the A_i's are mutually disjoint, and let $\epsilon > 0$ in $\mathcal{R}$ be given. There exists a $I \in \mathbb{N}$ such that $\sum_{i>I} m(A_i) < \frac{\epsilon}{2}$. Since $\bigcup_{i=1}^{\infty} A_i$ is measurable there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{n=1}^{\infty} H_n,$$

$\sum\limits_{n=1}^{\infty} v(H_n)$ converges, and

$$\sum_{n=1}^{\infty} v(H_n) - m\left(\bigcup_{i=1}^{\infty} A_i\right) < \frac{\epsilon}{2}.$$

Because the A_i's and the H_n's are mutually disjoint, and because

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{n=1}^{\infty} H_n,$$

we can find for every $i \in \{1, \ldots, I\}$ a sequence of mutually disjoint simple regions $(H_n^i)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} v(H_n^i)$ converges,

$$A_i \subset \bigcup_{n=1}^{\infty} H_n^i \subset \bigcup_{n=1}^{\infty} H_n,$$

and $\bigcup\limits_{n=1}^{\infty} H_n^1, \bigcup\limits_{n=1}^{\infty} H_n^2, \ldots, \bigcup\limits_{n=1}^{\infty} H_n^I$ are mutually disjoint. Thus,

$$\begin{aligned}
\sum_{i=1}^{I} m(A_i) &\leq \sum_{i=1}^{I} m\left(\bigcup_{n=1}^{\infty} H_n^i\right) \\
&= \sum_{i=1}^{I} \sum_{n=1}^{\infty} v(H_n^i) \\
&\leq \sum_{n=1}^{\infty} v(H_n) \\
&< m\left(\bigcup_{i=1}^{\infty} A_i\right) + \frac{\epsilon}{2}.
\end{aligned}$$

So,

$$\begin{aligned}\sum_{i=1}^{\infty} m(A_i) &= \sum_{i=1}^{I} m(A_i) + \sum_{i>I} m(A_i) \\ &< m(\bigcup_{i=1}^{\infty} A_i) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= m(\bigcup_{i=1}^{\infty} A_I) + \epsilon.\end{aligned}$$

Taking the limit as $\epsilon \to 0$ reveals that

$$\sum_{i=1}^{\infty} m(A_i) \leq m\left(\bigcup_{i=1}^{\infty} A_i\right).$$

This with the above result that

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i)$$

allows the conclusion that

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A).$$

5. It is sufficient to show that the statement holds for the case of two measurable sets A and B since the rest follows easily by induction. So suppose that A and B are measurable sets in $\mathcal{K}^n$ and fix $\epsilon > 0$ in $\mathcal{K}$. By the definition of measurability, there exist four sequences of mutually disjoint simple regions $(G_n^A)_{n=1}^{\infty}$, $(G_n^B)_{n=1}^{\infty}$, $(H_n^A)_{n=1}^{\infty}$, and $(H_n^B)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} G_n^A \subset A \subset \bigcup_{n=1}^{\infty} H_n^A,$$

$$\bigcup_{n=1}^{\infty} G_n^B \subset B \subset \bigcup_{n=1}^{\infty} H_n^B;$$

$\sum\limits_{n=1}^{\infty} v(G_n^A)$, $\sum\limits_{n=1}^{\infty} v(G_n^B)$, $\sum\limits_{n=1}^{\infty} v(H_n^A)$, and $\sum\limits_{n=1}^{\infty} v(H_n^B)$ all converge; and finally

$$\sum_{n=1}^{\infty} v(H_n^A) - \sum_{n=1}^{\infty} v(G_n^A) \leq \frac{\epsilon}{2},$$

$$\sum_{n=1}^{\infty} v(H_n^B) - \sum_{n=1}^{\infty} v(G_n^B) \leq \frac{\epsilon}{2}.$$

From Proposition 3.1.8 we know that there exist two sequences of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ and $(G_n)_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} H_n^A \cap \bigcup_{n=1}^{\infty} H_n^B,$$

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} G_n^A \cap \bigcup_{n=1}^{\infty} G_n^B;$$

and $\sum\limits_{n=1}^{\infty} v(H_n)$ and $\sum\limits_{n=1}^{\infty} v(G_n)$ both converge. Obviously

$$\bigcup_{n=1}^{\infty} G_n \subset A \cap B \subset \bigcup_{n=1}^{\infty} H_n.$$

Since

$$\bigcup_{n=1}^{\infty} G_n \subset \bigcup_{n=1}^{\infty} G_n^A$$

and

$$\bigcup_{n=1}^{\infty} H_n \subset \bigcup_{n=1}^{\infty} H_n^A,$$

and since all four sequences of simple regions are mutually disjoint simple regions, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} v(H_n) - \sum_{n=1}^{\infty} v(G_n) &\leq \left(\sum_{n=1}^{\infty} v(H_n^A) - \sum_{n=1}^{\infty} v(G_n^A)\right) + \left(\sum_{n=1}^{\infty} v(H_n^B) - \sum_{n=1}^{\infty} v(G_n^B)\right) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

which proves that $A \cap B$ is measurable. Since $A \cap B \subset A$, $m(A \cap B) \leq m(A)$ and since $A \cap B \subset B$, $m(A \cap B) \leq m(B)$. So, $m(A \cap B) \leq \min\{m(A), m(B)\}$.

6. Fix $\epsilon > 0$. First we note that by previous parts of this same proposition $A \cup B$ and $A \cap B$ are measurable. Since $A \cup B$ is measurable there exists a collection of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that

$$A \cup B \subset \bigcup_{n=1}^{\infty} H_n,$$

$\sum\limits_{n=1}^{\infty} v(H_n)$ converges and

$$\sum_{n=1}^{\infty} v(H_n) - m(A \cup B) < \frac{\epsilon}{2}.$$

Now, $A \backslash (A \cap B)$, $B \backslash (A \cap B)$, and $A \cap B$ are all mutually disjoint subsets of $A \cup B$, so there exist three subsequences of $(H_n)_{n=1}^{\infty}$ denoted by $(H_n^1)_{n=1}^{\infty}$, $(H_n^2)_{n=1}^{\infty}$, and $(H_n^3)_{n=1}^{\infty}$ such that

$$A \backslash (A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^1,$$

$$B \backslash (A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^2,$$

and

$$(A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^3.$$

Note that

$$\bigcup_{n=1}^{\infty} H_n^1 \cup \bigcup_{n=1}^{\infty} H_n^2 \cup \bigcup_{n=1}^{\infty} H_n^3 = \bigcup_{n=1}^{\infty} H_n.$$

Since $A = A \backslash (A \cap B) \cup (A \cap B)$ we see that

$$A \subset \bigcup_{n=1}^{\infty} H_n^1 \cup \bigcup_{n=1}^{\infty} H_n^3,$$

so

$$m(A) \leq \sum_{n=1}^{\infty} v(H_n^1) + \sum_{n=1}^{\infty} v(H_n^3).$$

Since $B = B \backslash (A \cap B) \cup (A \cap B)$ we see that

$$B \subset \bigcup_{n=1}^{\infty} H_n^2 \cup \bigcup_{n=1}^{\infty} H_n^3,$$

so

$$m(B) \leq \sum_{n=1}^{\infty} v(H_n^2) + \sum_{n=1}^{\infty} v(H_n^3).$$

Thus,

$$\begin{aligned} m(A) + m(B) &\leq \sum_{n=1}^{\infty} v(H_n^1) + \sum_{n=1}^{\infty} v(H_n^3) + \sum_{n=1}^{\infty} v(H_n^2) + \sum_{n=1}^{\infty} v(H_n^3) \\ &= \sum_{n=1}^{\infty} v(H_n) + \sum_{n=1}^{\infty} v(H_n^3) \end{aligned}$$

We have from above that

$$\sum_{n=1}^{\infty} v(H_n) \leq m(A \cup B) + \frac{\epsilon}{2},$$

we also know that

$$\sum_{n=1}^{\infty} v(H_n^3) \leq m(A \cap B) + \frac{\epsilon}{2}$$

since otherwise $\sum_{n=1}^{\infty} v(H_n)$ would be greater than $m(A \cup B) + \frac{\epsilon}{2}$. Therefore, $m(A) + m(B) \leq m(A \cup B) + m(A \cap B) + \epsilon$, this holds for any $\epsilon > 0$ so we obtain that $m(A) + m(B) \leq m(A \cup B) + m(A \cap B)$.

Now we prove the other inequality. By the measurability of A and B there exist two sequences of mutually disjoint simple regions $(H_n^A)_{n=1}^{\infty}$ and $(H_n^B)_{n=1}^{\infty}$ such that

$$A \subset \bigcup_{n=1}^{\infty} H_n^A,$$

$$B \subset \bigcup_{n=1}^{\infty} H_n^B;$$

$\sum\limits_{n=1}^{\infty} v(H_n^A)$ and $\sum\limits_{n=1}^{\infty} v(H_n^B)$ both converge; and

$$\sum_{n=1}^{\infty} v(H_n^A) < m(A) + \frac{\epsilon}{2}$$

and

$$\sum_{n=1}^{\infty} v(H_n^B) < m(B) + \frac{\epsilon}{2}.$$

B is the disjoint union of $B \backslash (A \cap B)$ and $A \cap B$. Thus $(H_n^B)_{n=1}^{\infty}$ can be split into two subsequences $(H_n^{B,1})_{n=1}^{\infty}$ and $(H_n^{B,2})_{n=1}^{\infty}$ where

$$B \backslash (A \cap B) \subset \bigcup_{n=1}^{\infty} H_n^{B,1}$$

and

$$A \cap B \subset \bigcup_{n=1}^{\infty} H_n^{B,2}.$$

So,

$$A \cup B = A \cup (B \backslash (A \cap B)) \subset \bigcup_{n=1}^{\infty} H_n^A \cup \bigcup_{n=1}^{\infty} H_n^{B,1},$$

and thus

$$m(A \cup B) \leq \sum_{n=1}^{\infty} v(H_n^A) + \sum_{n=1}^{\infty} v(H_n^{B,1}) \leq m(A) + \frac{\epsilon}{2} + \sum_{n=1}^{\infty} v(H_n^{B,1}).$$

Since $A \cap B \subset \bigcup\limits_{n=1}^{\infty} H_n^{B,2}$, it is clear that $m(A \cap B) \leq \sum\limits_{n=1}^{\infty} v(H_n^{B,2})$. So we obtain

$$\begin{aligned} m(A \cup B) + m(A \cap B) &\leq m(A) + \frac{\epsilon}{2} + \sum_{n=1}^{\infty} v(H_n^{B,1}) + \sum_{n=1}^{\infty} v(H_n^{B,2}) \\ &\leq m(A) + \frac{\epsilon}{2} + \sum_{n=1} v(H_n^{B}) \\ &\leq m(A) + m(B) + \epsilon \end{aligned}$$

The above inequality holds for any $\epsilon > 0$ so we find that $m(A \cup B) + m(A \cap B) \leq m(A) + m(B)$. This with our previous result gives the equality $m(A) + m(B) = m(A \cap B) + m(A \cup B)$ or $m(A \cup B) = m(A) + m(B) - m(A \cap B)$. □

3.4 Properties of Analytic Functions in ν-Dimensions

Continuing with our effort to develop a theory of integration in ν dimensions we now move on to a discussion of functions in ν dimensions. The primary goal of this section is to define analytic and measurable functions in ν dimensions as well as to prove a result related to the composition of analytic functions which will be used in the following section.

Definition 3.4.1 (Finite Simple Region and Order of Magnitude)**.** *Let S be a simple region given by*

$$S = \{(x_1, \ldots, x_\nu) \in \mathcal{K}^\nu : x_1 \in I(h_1(x_2, \ldots, x_\nu), h_2(x_2, \ldots, x_\nu)), (x_2, \ldots, x_\nu) \in A\}$$

*For $i \in \{2, \ldots, \nu\}$ we inductively define $\lambda_{x_i}(S) = \lambda_{x_i}(A)$ and we define $\lambda_{x_1}(S) = i(h_2(x_2, \ldots, x_\nu) - h_1(x_2, \ldots, x_\nu))$, the index of the analytic function $h_2 - h_1$ on A; we call these the **orders of magnitude of** S **in the** x_i **direction** and we call S a **finite region** if $\lambda_{x_1}(S) = \cdots = \lambda_{x_\nu}(S) = 0$.*

Definition 3.4.2 (Analytic Function in $\mathcal{K}^\nu$)**.** *Suppose $S \subset \mathcal{K}^\nu$ is a simple region and let f :*

*$S \to \mathcal{K}$. Then we call f an **analytic function on** S if for every point $p = (p_1, \dots, p_\nu) \in S$, there exists a simple region $A \subset \mathcal{K}^\nu$ containing p and a regular sequence*

$$(a_{i_1,\dots,i_\nu})_{i_1,\dots,i_\nu=0}^{\infty}$$

in $\mathcal{K}$ such that for every $i \in \{1, \dots, \nu\}$, $\lambda_{x_i}(A) = \lambda_{x_i}(S)$ and if $(x_1+\delta_1, \dots, x_\nu+\delta_\nu) \in S \cap A$ then

$$f(x_1+\delta_1, \dots, x_\nu+\delta_\nu) = \sum_{i_1,\dots,i_\nu=0}^{\infty} a_{i_1,\dots,i_\nu}\delta_1^{i_1}\dots\delta_\nu^{i_\nu},$$

*where the power series converges in the weak topology. As in two dimensions we say that f is given **locally** by a power series.*

Lemma 3.4.3. *Suppose $S \subset \mathcal{K}^\nu$ is a ν-simple region and let $f : S \to \mathcal{K}$ be an analytic function. Then there exist linear transformations $L_1, \dots, L_\nu : [0,1] \to \mathcal{K}$ and an analytic function $F : [0,1]^\nu \to \mathcal{K}$ such that*

$$F(x_1, \dots, x_\nu) = f(L_1(x_1), \dots, L_\nu(x_\nu)).$$

Proof. The lemma has already been proven for the cases of $\nu = 1, 2$ (see the proof of [17, Lemma 4.7] for the former and the proof of [3, Proposition 2.20] for the latter) so we assume that the lemma holds for the $(\nu-1)$ case and show that it must hold for the ν case as well. Since S is an ν-simple region there must exist an $(\nu-1)$-simple region A and two analytic functions $h_1, h_2 : A \to \mathcal{K}$ such that

$$S = \{(x_1, \dots, x_\nu) \in \mathcal{K}^\nu : x_\nu \in [h_1(x_1, \dots, x_{\nu-1}), h_2(x_1, \dots, x_{\nu-1})], (x_1, \dots, x_{\nu-1}) \in A\}.$$

By our inductive hypothesis there exist linear transformations $L_1^1, \dots, L_{\nu-1}^1 : [0,1] \to \mathcal{K}$ and $L_1^2, \dots, L_{\nu-1}^2 : [0,1] \to \mathcal{K}$ as well as analytic functions $H_1, H_2 : [0,1]^{\nu-1} \to \mathcal{K}$ such that

$$H_1(x_1, \dots, x_{\nu-1}) = h_1(L_1^1(x_1), \dots, L_{\nu-1}^1(x_{\nu-1})),$$

and

$$H_2(x_1, \ldots, x_{\nu-1}) = h_2(L_1^2(x_1), \ldots, L_{\nu-1}^2(x_{\nu-1})).$$

In fact since all these transformations do is map the set $[0,1]^{\nu-1}$ to a $(\nu-1)$-simple region we may assume without loss of generality that for every $i \in \{1, \ldots, \nu-1\}$, $L_i^1 = L_i^2 \coloneqq L_i$. We define

$$F(x_1, \ldots, x_\nu) = f(L_1(x_1), \ldots, L_{\nu-1}(x_{\nu-1}), (H_2(x_1, \ldots, x_{\nu-1}) - H_1(x_1, \ldots, x_{\nu-1}))x_\nu + H_1(x_1, \ldots, x_{\nu-1}))$$

and the lemma is proven. $\square$

Proposition 3.4.4. *Let*

$$S = \{(x_1, \ldots, x_\nu) \in \mathcal{K}^\nu : x_\nu \in [h_1(x_1, \ldots, x_{\nu-1}), h_2(x_1, \ldots, x_{\nu-1})], (x_1, \ldots, x_{\nu-1}) \in A\}$$

be an ν-simple region in $\mathcal{K}^\nu$, and let $f : S \to \mathcal{K}$ be an analytic function on S. Then, f is bounded on S.

Proof. By Lemma 3.4.3 there exist linear transformations $L_1, \ldots, L_\nu : [0,1] \to \mathcal{K}$ and an analytic function $F : [0,1]^\nu \to \mathcal{K}$ such that

$$F(x_1, \ldots, x_\nu) = f(L_1(x_1), \ldots, l_\nu(x_\nu)).$$

Clearly f is bounded on A if and only if F is bounded on $[0,1]^\nu$. For every $(y_1, \ldots, y_\nu) \in \mathcal{K}^\nu$ let

$$N((y_1, \ldots, y_\nu), \eta) = \{(x_1, \ldots, x_\nu) \in \mathbb{N} : \sqrt{(x_1 - y_1)^2 + \ldots + (x_\nu - y_\nu)^2} < \eta\}.$$

By the definition of analytic functions, for every $(y_1, \ldots, y_\nu) \in [0,1]^\nu \cap \mathbb{R}^\nu$, there exists a finite $\eta(y_1, \ldots, y_\nu) > 0$ and a regular sequence $(a_{i_1, \ldots, i_\nu}(y_1, \ldots, y_\nu))_{i_1, \ldots, i_\nu = 0}^\infty$ such that for every

$(x_1,\dots,x_\nu)\in N((y_1,\dots,y_\nu),\eta(y_1,\dots,y_\nu))\cap[0,1]^\nu$,

$$F(x_1,\dots,x_\nu)=\sum_{i_1,\dots,i_\nu=0}^{\infty}a_{i_1,\dots,i_\nu}(y_1,\dots,y_\nu)(x_1-y_1)^{i_1}\cdots(x_\nu-y_\nu)^{i_\nu}.$$

The set $\{N((y_1,\dots,y_\nu),\frac{\eta(y_1,\dots,y_\nu)}{2})\cap\mathbb{R}^\nu : (y_1,\dots,y_\nu)\in[0,1]^\nu\cap\mathbb{R}^\nu\}$ is an open cover of $[0,1]^\nu\cap\mathbb{R}^\nu$ which is a compact set of the Euclidean space $\mathbb{R}^\nu$, so there exists a finite set of points $\{(y_{k,1},\dots,y_{k,\nu})\}_{k=1}^m$ contained in $[0,1]^\nu\cap\mathbb{R}^\nu$ such that

$$[0,1]^\nu\cap\mathbb{R}^\nu\subset\bigcup_{k=1}^m N\left((y_{k,1},\dots,y_{k,\nu}),\frac{\eta(y_{k,1},\dots,y_{k,\nu})}{2}\right)\cap\mathbb{R}^\nu.$$

From this we have that $[0,1]^\nu\subset\bigcup\limits_{k=1}^m N((y_{k,1},\dots,y_{k,\nu}),\eta(y_{k,1},\dots,y_{k,\nu}))$. Let

$$l=\min_{1\le k\le m}\left\{\min\left\{\bigcup_{i_1,\dots,i_\nu=0}^{\infty}supp(a_{i_1,\dots,i_\nu}(y_{k,1},\dots,y_{k,\nu}))\right\}\right\},$$

which exists by the regularity of the sequence $(a_{i_1,\dots,i_\nu}(y_{k,1},\dots,y_{k,\nu}))$ for each k. It follows from the above that $|F(x_1,\dots,x_\nu)|<d^{l-1}$ for every $(x_1,\dots,x_\nu)\in[0,1]^\nu$. Thus F is bounded on $[0,1]^\nu$ and hence f is bounded on A. □

Proposition 3.4.5. *Let $\{(y_{k,1},\dots,y_{k,\nu})\}_{k=1}^m$, l, $F(x_1,\dots,x_\nu)$, and $\eta(y_1,\dots,y_\nu)$ be as in Proposition 3.4.4 and the proof thereof. Then, l is independent of our choice of $\{(y_{k,1},\dots,y_{k,\nu})\}_{k=1}^m$.*

Proof. Assume not, then there exists a set of points $\{(z_{k,1},\dots,z_{k,\nu})\}_{k=1}^{m_0}$ such that

$$\{N((z_{k,1},\dots,z_{k,\nu}),\eta(z_{k,1},\dots,z_{k,\nu})) : k\in\{1,\dots,m_0\}\}$$

is a finite open cover of $[0,1]^\nu$,

$$l_0=\min_{1\le k\le m_0}\left\{\min\left\{\bigcup_{i_1,\dots,i_\nu}supp(a_{i_1,\dots,i_\nu}(z_{k,1},\dots,z_{k,\nu}))\right\}\right\}$$

and

$$l_0 \neq l.$$

We assume without loss of generality that $l < l_0$, in particular, $l < \infty$. Define $F_{\mathbb{R}} : [0,1]^\nu \cap \mathbb{R}^\nu \to \mathbb{R}$, by

$$F_{\mathbb{R}}(x_1, \ldots, x_\nu) = (F(x_1, \ldots, x_\nu))\,[l].$$

For $(x_1, \ldots, x_\nu) \in N((z_{k,1}, \ldots, z_{k,n}), \eta((z_{k,1}, \ldots, z_{k,n}))) \cap [0,1]^\nu \cap \mathbb{R}^\nu$,

$$\begin{aligned} F_{\mathbb{R}}(x_1, \ldots, x_\nu) &= \left(\sum_{i_1,\ldots,i_\nu=0}^{\infty} a_{i_1,\ldots,i_\nu}(z_{k,1}, \ldots, z_{k,n})(x_1 - z_{k,1})^{i_1} \cdots (x_\nu - z_{k,n})^{i_\nu} \right)[l] \\ &= \sum_{i_1,\ldots,i_\nu=0}^{\infty} (a_{i_1,\ldots,i_\nu}(z_{k,1}, \ldots, z_{k,n}))\,[l](x_1 - z_{k,1})^{i_1} \cdots (x_\nu - z_{k,n})^{i_\nu} \end{aligned}$$

So, $F_{\mathbb{R}}(x_1, \ldots, x_\nu)$ is an $\mathbb{R}^\nu$-analytic function. Furthermore, $F_{\mathbb{R}}(x_1, \ldots, x_\nu) = 0$ everywhere in

$$N((z_{k,1}, \ldots, z_{k,n}), \eta((z_{k,1}, \ldots, z_{k,n}))) \cap [0,1]^\nu \cap \mathbb{R}^\nu,$$

so by the identity theorem for real analytic functions $F_{\mathbb{R}} = 0$ everywhere on $[0,1]^\nu \cap \mathbb{R}^\nu$. Then, for every $i_1, \ldots, i_\nu \in \mathbb{N} \cup \{0\}$ and for every $k \in \{1, \ldots, m\}$,

$$(a_{i_1,\ldots,i_\nu}(y_{k,1}, \ldots, y_{k,\nu}))\,[l] = 0,$$

which contradicts the definition of l. $\square$

Proposition 3.4.6. *Let A, f, F, $\{(y_{k,1}, \ldots, y_{k,n})\}_{k=1}^m$, $\eta(y_1, \ldots, y_\nu))$, and l be as in Proposition 3.4.4 and the subsequent proof. Then, $\lambda(f(x_1, \ldots, x_\nu)) = l$ almost everywhere on*

$$\{(x_1, \ldots, x_\nu) \in A : \textit{for every } i \in \{1, \ldots, n\}, \textit{supp}(x_i) = \{\lambda_{x_i}(A)\}\}$$

and for any point $(x_1, \ldots, x_\nu)$ in the above set where $\lambda(f(x_1, \ldots, x_\nu)) = l$, the same is true for points $(z_1, \ldots, z_\nu)$ satisfying $\lambda(x_i - z_i) > \lambda_{x_i}(A)$ for every $i \in \{1, \ldots, \nu\}$.

Proof. First note that $\lambda(f(x_1,\ldots,x_\nu)) = l$ almost everywhere on

$$\{(x_1,\ldots,x_\nu) \in A : \text{for every } i \in \{1,\ldots,\nu\}, supp(x_i) = \{\lambda_{x_i}(A)\}\}$$

if and only if $\lambda(F(x_1,\ldots,x_\nu)) = l$ almost everywhere on $[0,1]^\nu \cap \mathbb{R}^\nu$. Fix $(x_{0,1},\ldots,x_{0,\nu}) \in [0,1]^\nu \cap \mathbb{R}^\nu$. Then there is a $k \in \{1,\ldots,m\}$ such that

$$(x_{0,1},\ldots,x_{0,\nu}) \in N((y_{k,1},\ldots,y_{k,\nu}),\eta(y_{k,1},\ldots,y_{k,\nu})),$$

hence

$$F(x_{0,1},\ldots,x_{0,\nu}) = \sum_{i_1,\ldots,i_\nu=0}^{\infty} a_{i_1,\ldots,i_\nu}(y_{k,1},\ldots,y_{k,\nu})(x_{0,1}-y_{k,1})^{i_1}\cdots(x_{0,\nu}-y_{k,\nu})^{i_\nu}.$$

Since $\lambda(a_{i_1,\ldots,i_\nu}(y_{k,1},\ldots,y_{k,\nu})) \geq l$, and since for every $j \in \{1,\ldots,\nu\}$, $\lambda(x_{0,i}-y_{k,i}) \geq 0$, we must have that $\lambda(F(x_{0,1},\ldots,x_{0,\nu})) \geq l$. Moreover, the real non-zero analytic function

$$F_{\mathbb{R}}(x_{0,1},\ldots,x_{0,\nu}) = \left(\sum_{i_1,\ldots,i_\nu=0}^{\infty} a_{i_1,\ldots,i_\nu}(y_{k,1},\ldots,y_{k,\nu})(x_{0,1}-y_{k,1})^{i_1}\cdots(x_{0,\nu}-x_{k,\nu})^{i_\nu}\right)[l]$$

must be non-zero almost everywhere on $[0,1]^\nu \cap \mathbb{R}^\nu$, and if $F_{\mathbb{R}}(x_{0,1},\ldots,x_{0,\nu}) \neq 0$ then $\lambda(F(x_{0,1},\ldots,x_{0,\nu})) = l$ so $\lambda(F(x_{0,1},\ldots,x_{0,\nu})) = l$ almost everywhere on $[0,1]^\nu \cap \mathbb{R}^\nu$. Finally, suppose that the point $(x_1,\ldots,x_\nu) \in [0,1]^\nu \cap \mathbb{R}^\nu$ is such that $\lambda(F(x_1,\ldots,x_\nu)) = l$ and let $(z_1,\ldots,z_\nu) \in [0,1]^\nu$ satisfy $\sqrt{(x_1-z_1)^2+\cdots+(x_\nu-z_\nu)^2} \ll 1$. Then,

$$F(z_1,\ldots,z_\nu) = F(x_1,\ldots,x_\nu) + \sum_{\substack{i_1,\ldots,i_\nu=0 \\ i_1+\cdots+i_\nu\neq 0}}^{\infty} a_{i_1,\ldots,i_\nu}(x_1,\ldots,x_\nu)(x_1-z_1)^{i_1}\cdots(x_\nu-z_\nu)^{i_\nu}$$

but for every $j \in \{1, \ldots, \nu\}$, $\lambda((x_j - z_j)^{i_j}) > 0$ and $\lambda(a_{i_1,\ldots,i_\nu}(z_1, \ldots, z_\nu)) \geq l$. Thus,

$$\lambda\left(\sum_{\substack{i_1,\ldots,i_\nu=0\\ i_1+\cdots+i_\nu\neq 0}}^{\infty} a_{i_1,\ldots,i_\nu}(x_1, \ldots, x_\nu)(x_1 - z_1)^{i_1}\cdots(x_\nu - z_\nu)^{i_\nu}\right) > l$$

and hence

$$\lambda(F(z_1, \ldots, z_\nu)) = \lambda(F(x_1, \ldots, x_\nu)) = l.$$

□

Definition 3.4.7 (Index of an Analytic Function on $\mathcal{K}^\nu$)**.** *Let A and f be as in Proposition 3.4.4 and let l be as in the subsequent proof. Then we call l the **index of** f **on** A and we denote it by $i(f)$.*

Proposition 3.4.8. *Suppose $A \subset \mathcal{K}^\nu$ is a finite simple region and let $f, g : A \to \mathcal{K}$ be two analytic functions on A. Let $\alpha \in \mathcal{K}$ be an arbitrary constant, then $f + \alpha g$ and $f \cdot g$ are analytic functions on A.*

Proof. Fix $p = (p_1, \ldots, p_\nu) \in A$. Since f and g are analytic there exist two finite constants $\eta_1, \eta_2 > 0$ such that for every $x_1, \ldots, x_\nu \in \mathcal{K}$ satisfying $x_1^2 + \cdots + x_\nu^2 < \eta_1$, if $(p_1 + x_1, \ldots, p_\nu + x_\nu) \in A$ then

$$f(p_1 + x_1, \ldots, p_\nu + x_\nu) = \sum_{i_1,\ldots,i_\nu=0}^{\infty} a_{i_1,\ldots,i_\nu} x_1^{i_1} \ldots x_\nu^{i_\nu}$$

and for every $x_1, \ldots, x_\nu \in \mathcal{K}$ satisfying $x_1^2 + \cdots + x_\nu^2 < \eta_2$, if $(p_1 + x_1, \ldots, p_\nu + x_\nu) \in A$ then

$$g(p_1 + x_1, \ldots, p_\nu + x_\nu) = \sum_{i_1,\ldots,i_\nu=0}^{\infty} b_{i_1,\ldots,i_\nu} x_1^{i_1} \ldots x_\nu^{i_\nu}.$$

Let $\eta = \min\{\eta_1, \eta_2\}$. Thus, for every $x_1, \ldots, x_\nu \in \mathcal{K}$ satisfying $x_1^2 + \cdots + x_\nu^2 < \eta$, if $(p_1 +$

$x_1, \ldots, p_\nu + x_\nu) \in A$ then

$$
\begin{aligned}
(f + \alpha g)(p_1 + x_1, \ldots, p_n + x_\nu) &= \sum_{i_1,\ldots,i_\nu=0}^{\infty} a_{i_1,\ldots,i_\nu} x_1^{i_1} \ldots x_\nu^{i_\nu} + \alpha \sum_{i_1,\ldots,i_\nu=0}^{\infty} b_{i_1,\ldots,i_\nu} x_1^{i_1} \ldots x_\nu^{i_\nu} \\
&= \sum_{i_1,\ldots,i_\nu=0}^{\infty} (a_{i_1,\ldots,i_\nu} + \alpha b_{i_1,\ldots,i_\nu}) x_1^{i_\nu} \ldots x_\nu^{i_\nu},
\end{aligned}
$$

so $f + \alpha g$ is analytic on A. Furthermore, for every $x_1, \ldots, x_\nu \in \mathcal{K}$ satisfying $x_1^2 + \cdots + x_\nu^2 < \eta$, if $(p_1 + x_1, \ldots, p_\nu + x_\nu) \in A$ then

$$
\begin{aligned}
(f \cdot g))(p_1 + x_1, \ldots, p_\nu + x_\nu) &= (\sum_{i_1,\ldots,i_\nu=0}^{\infty} a_{i_1,\ldots,i_\nu} x_1^{i_1} \ldots x_\nu^{i_\nu})(\sum_{j_1,\ldots,j_\nu=0}^{\infty} b_{j_1,\ldots,j_\nu} x_1^{j_1} \ldots x_\nu^{j_\nu}) \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_\nu=0}^{\infty} \sum_{i_1+j_1=k_1} \cdots \sum_{i_\nu+j_\nu=k_\nu} a_{i_1,\ldots,i_\nu} b_{j_1,\ldots,j_\nu} x_1^{k_1} \ldots x_\nu^{k_\nu} \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} c_{k_1,\ldots,k_n} x_1^{k_1} \ldots x_\nu^{k_n}.
\end{aligned}
$$

We infer that the Cauchy product converges in $\mathcal{K}^\nu$ from the fact that Cauchy products converge in $\mathcal{K}$. So $f \cdot g$ is an analytic function. $\square$

Corollary 3.4.9. *Suppose $A \subset \mathcal{K}^\nu$ is a simple region and let $f, g : A \to \mathcal{K}$ be two analytic functions on A. Let $\alpha \in \mathcal{K}$ be an arbitrary constant. Then $f + \alpha g$ and $f \cdot g$ are analytic functions on A.*

Proposition 3.4.10. *Suppose $A \subset \mathcal{K}^\nu$ is a finite simple region and let $f : A \to \mathcal{K}$ be an analytic function on A. Let $B \subset \mathcal{K}^{\nu-1}$ be a finite simple region and let $g : B \to \mathcal{K}$ be an analytic function on B such that for every $(x_1, \ldots, x_{\nu-1}) \in B$, $(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1})) \in A$. Then, $f(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1}))$ is an analytic function on B.*

Proof. Since A and B are finite simple regions and since for every $(x_1, \ldots, x_{\nu-1}) \in B$ we have by our premise that $(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1})) \in A$, we may assume without loss of generality that the index of g is zero. Fix $(x_1, \ldots, x_{\nu-1}) \in B$, since f and g are analytic functions there exist $\eta_1, \eta_2 > 0$ such that $\eta_1, \eta_2 \sim 1$ and for every $\delta_1, \ldots, \delta_{\nu-1} \in \mathcal{K}$ satisfying

$\delta_1^2 + \cdots + \delta_{\nu-1}^2 < \eta_1^2$, if $(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1}) \in B$, then

$$g(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1}) = g(x_1, \ldots, x_{\nu-1}) + \sum_{\substack{i_1,\ldots,i_{\nu-1}=0 \\ i_1+\cdots+i_{\nu-1}\neq 0}}^{\infty} a_{i_1\ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}},$$

and, for every $\mu_1, \ldots, \mu_\nu \in \mathcal{K}$ satisfying $\mu_1^2 + \cdots + \mu_\nu^2 < \eta_2^2$, if

$$(x_1 + \mu_1, \ldots, x_{\nu-1} + \mu_{n-1}, g(x_1, \ldots, x_{\nu-1}) + \mu_\nu) \in A,$$

then

$$\begin{aligned} f(x_1 + \mu_1, \ldots, x_{\nu-1} + \mu_{n-1}, g(x_1, \ldots, x_{\nu-1}) + \mu_\nu) &= f(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1})) \\ &\quad + \sum_{\substack{j_1,\ldots,j_\nu=0 \\ j_1\cdots+j_\nu\neq 0}}^{\infty} b_{j_1\ldots j_\nu} \mu_1^{j_1} \cdots \mu_\nu^{j_\nu}. \end{aligned}$$

Define $H : \mathbb{R}^{\nu-1} \to \mathbb{R}$ by

$$H(y_1, \ldots, y_{\nu-1}) = \left(y_1^2 + \cdots y_{\nu-1}^2 + \left(\sum_{\substack{k_1\ldots k_{\nu-1}=0 \\ k_1+\cdots k_{\nu-1}\neq 0}} a_{k_1\ldots k_{\nu-1}} y_1^{k_1} \ldots y_{\nu-1}^{k_{\nu-1}} \right)^2 \right) [0],$$

clearly H is a continuous real function with $H(0, \ldots, 0) = 0$ so there exists a $\eta \in (0, \eta_1] \cap \mathbb{R}$ such that if $y_1^2 + \cdots + y_{\nu-1}^2 < \eta^2$ then $H(y_1, \ldots, y_{\nu-1}) < \eta_2^2$. Now, if $\delta_1, \ldots, \delta_{\nu-1} \in \mathcal{K}$ with

$\delta_1^2 + \cdots + \delta_{\nu-1}^2 < \eta^2$ and $(x_1 + \delta_1, \ldots x_{\nu-1} + \delta_{\nu-1}) \in B$ then

$$\left| \delta_1^2 + \cdots + \delta_{\nu-1}^2 + \left(\sum_{\substack{i_1,\ldots,i_{\nu-1}=0 \\ i_1+\cdots+i_{\nu-1}\neq 0}}^{\infty} a_{i_1\ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}} \right)^2 \right. \\ \left. - \left[(\delta_1[0])^2 + \ldots + (\delta_{\nu-1}[0])^2 + \left(\sum_{\substack{i_1,\ldots,i_{\nu-1}=0 \\ i_1+\cdots+i_{n-1+\neq 0}}}^{\infty} a_{i_1\ldots i_{\nu-1}} (\delta_1[0])^{i_1} \cdots (\delta_{\nu-1}[0])^{i_{\nu-1}} \right)^2 [0] \right] \right| \\ \ll 1$$

since $\delta_1 =_0 \delta_1[0], \ldots, \delta_{\nu-1} =_0 \delta_{\nu-1}[0]$, then

$$\sum_{\substack{i_1,\ldots,i_{\nu-1}=0 \\ i_1+\cdots+i_{\nu-1}\neq 0}}^{\infty} a_{i_1\ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}} =_0 \left(\sum_{\substack{i_1,\ldots,i_{\nu-1}=0 \\ i_1+\cdots+i_{\nu-1}\neq 0}}^{\infty} a_{i_1\ldots i_{\nu-1}} \delta_1[0]^{i_1} \cdots \delta_{\nu-1}[0]^{i_{\nu-1}} \right) [0].$$

So if $H(\delta_1[0], \ldots, \delta_{\nu-1}[0]) < \eta_2^2$ then

$$\delta_1^2 + \cdots + \delta_{\nu-1}^2 + \left(\sum_{\substack{i_1,\ldots,i_{\nu-1}=0 \\ i_1+\cdots+i_{\nu-1}\neq 0}} a_{i_1\ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}} \right)^2 < \eta_2^2.$$

So we have that for every $\delta_1, \ldots, \delta_{\nu-1} \in \mathcal{K}$ satisfying $\delta_1^2 + \cdots + \delta_{\nu-1}^2 < \eta^2$, if $(x_1 +$

$\delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1}) \in B$ then

$$
\begin{aligned}
& f(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1}, g(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1})) \\
& \quad = f\left(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1}, g(x_1, \ldots, x_{\nu-1}) + \sum_{\substack{i_1, \ldots, i_{\nu-1} = 0 \\ i_1 + \cdots i_{\nu-1} \neq 0}}^{\infty} a_{i_1 \ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}} \right) \\
& \quad = f(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1})) \\
& \quad + \sum_{\substack{j_1, \ldots, j_\nu = 0 \\ j_1 + \cdots + j_\nu \neq 0}}^{\infty} b_{j_1 \cdots j_\nu} \delta_1^{j_1} \cdots \delta_{\nu-1}^{j_{\nu-1}} \left(\sum_{\substack{i_1, \ldots, i_{\nu-1} = 0 \\ i_1 + \cdots + i_{\nu-1} \neq 0}}^{\infty} a_{i_1 \ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}} \right)^{j_\nu} .
\end{aligned}
$$

However, we can rewrite

$$
b_{j_1 \ldots j_\nu} \delta_1^{j_1} \cdots \delta_{\nu-1}^{j_{\nu-1}} \left(\sum_{\substack{i_1, \ldots, i_{\nu-1} = 0 \\ i_1 + \cdots + i_{\nu-1} \neq 0}}^{\infty} a_{i_1 \ldots i_{\nu-1}} \delta_1^{i_1} \cdots \delta_{\nu-1}^{i_{\nu-1}} \right)^{j_\nu}
$$

as the convergent series

$$
\sum_{\substack{l_1, \ldots, l_\nu \\ l_1 + \cdots + l_\nu \neq 0}}^{\infty} c_{j_1 \ldots j_\nu l_1 \ldots l_\nu} \delta_1^{l_1} \cdots \delta_{\nu-1}^{l_{\nu-1}} .
$$

We may also change the order of summation so that

$$
\begin{aligned}
& f(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1}, g(x_1 + \delta_1, \ldots, x_{\nu-1} + \delta_{\nu-1})) \\
& \quad = f(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1})) + \sum_{\substack{j_1, \ldots, j_\nu = 0 \\ j_1 \cdots + j_\nu \neq 0}}^{\infty} \sum_{\substack{l_1, \ldots, l_\nu \\ l_1 + \cdots + l_\nu \neq 0}}^{\infty} c_{j_1 \ldots j_\nu l_1 \ldots l_\nu} \delta_1^{l_1} \cdots \delta_{\nu-1}^{l_{\nu-1}} \\
& \quad = f(x_1, \ldots, x_{\nu-1}, g(x_1, \ldots, x_{\nu-1})) + \sum_{\substack{l_1, \ldots, l_\nu \\ l_1 + \cdots + l_\nu \neq 0}}^{\infty} \sum_{\substack{j_1, \ldots, j_\nu = 0 \\ j_1 + \cdots + j_\nu \neq 0}}^{\infty} c_{j_1 \ldots j_\nu l_1 \ldots l_\nu} \delta_1^{l_1} \cdots \delta_{\nu-1}^{l_{\nu-1}} .
\end{aligned}
$$

Finally, letting $e_{l_1,\ldots,l_\nu} = \sum\limits_{\substack{j_1,\ldots,j_\nu=0\\ j_1\cdots+j_\nu\neq 0}}^{\infty} c_{j_1\ldots j_\nu l_1\ldots l_\nu}$ we find that

$$\begin{aligned} & f(x_1+\delta_1,\ldots,x_{\nu-1}+\delta_{\nu-1},g(x_1+\delta_1,\ldots,x_{\nu-1}+\delta_{\nu-1})) \\ & \quad = f(x_1,\ldots,x_{\nu-1},g(x_1,\ldots,x_{\nu-1})) + \sum_{\substack{l_1,\ldots,l_\nu \\ l_1+\cdots+l_\nu\neq 0}}^{\infty} e_{l_1,\ldots,l_\nu}\delta_1^{i_1}\cdots\delta_{\nu-1}^{i_{\nu-1}} \end{aligned}$$

Thus $f(x_1,\ldots,x_{\nu-1},g(x_1,\ldots,x_{\nu-1}))$ is analytic on B. □

Corollary 3.4.11. *Suppose $A \subset \mathcal{K}^\nu$ is a simple region and let $f : A \to \mathcal{K}$ be an analytic function on A. Let $B \subset \mathcal{K}^{\nu-1}$ be a simple region and let $g : B \to \mathcal{K}$ be an analytic function on B such that for every $(x_1,\ldots,x_{\nu-1}) \in B$, $(x_1,\ldots,x_{\nu-1},g(x_1,\ldots,x_{\nu-1})) \in A$. Then, $f(x_1,\ldots,x_{\nu-1},g(x_1,\ldots,x_{\nu-1}))$ is an analytic function on B.*

Definition 3.4.12 (Measurable Function)**.** *Let $S \subset \mathcal{K}^\nu$ be a measurable set and let $f : S \to \mathcal{K}$ be bounded on S. Then we say that f is* ***measurable on*** *S if for every $\epsilon > 0$, there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^\infty$ such that*

$$\bigcup_{n=1}^{\infty} G_n \subset S,$$

$\sum\limits_{n=1}^{\infty} v(G_n)$ *converges,*

$$m(S) - \sum_{n=1}^{\infty} v(G_n) < \epsilon,$$

and f is analytic on G_n for every $n \in \mathbb{N}$.

Proposition 3.4.13. *Let $S \subset \mathcal{K}^\nu$ be a measurable set and let $f : S \to \mathcal{K}$ be a measurable function. Then, f is given locally by a power series almost everywhere on S.*

Proof. Let $S_0 = \{(x_1,\ldots,x_\nu) \in S : f \text{ is not given locally by a power series about } (x_1,\ldots,x_\nu)\}$. We will show that S_0 is measurable and $m(S_0) = 0$. So let $\epsilon > 0$ be given in $\mathcal{K}$. Since f is measurable on S there exists a sequence of mutually disjoint open simple regions $(G_n)_{n=1}^\infty$

such that $\bigcup_{n=1}^{\infty} G_n \subset S$, $\sum_{n=1}^{\infty} v(G_n)$ converges,

$$m(A) - \sum_{n=1}^{\infty} v(G_n) \leq \frac{\epsilon}{2},$$

and f is analytic on G_n for all n. By the measurability of S there must exist a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $S \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} v(H_n)$ converges, and

$$\sum_{n=1}^{\infty} v(H_n) - m(A) \leq \frac{\epsilon}{2}.$$

Since f is given locally by a power series about every point in $\bigcup_{n=1}^{\infty} G_n$ and since $S_0 \subset S$, we know that

$$S_0 \subset S \backslash \bigcup_{n=1}^{\infty} G_n.$$

Furthermore, since $S \subset \bigcup_{n=1}^{\infty} H_n$ we can conclude that

$$S_0 \subset \bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n.$$

Because both $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ are mutually disjoint sequences we can arrange them so that for every $n \in \mathbb{N}$, $G_n \subset H_n$. It follows that

$$\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (H_n \backslash G_n).$$

The $H_n \backslash G_n$'s are mutually disjoint; and by Conjecture 3.1.4, for every $n \in \mathbb{N}$, $H_n \backslash G_n$ may be expressed as the union of a finite number of mutually disjoint simple regions. So we see that

$$\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n$$

may be rewritten as the union of countably many mutually disjoint simple regions $(H_n^0)_{n=1}^{\infty}$.

For every $n \in \mathbb{N}$, let $G_n^0 = \emptyset$. Then, we have

$$\bigcup_{n=1}^{\infty} G_n^0 \subset S_0 \subset \bigcup_{n=1}^{\infty} H_n^0.$$

Furthermore we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} v(H_n^0) - \sum_{n=1}^{\infty} v(G_n^0) &= \sum_{n=1}^{\infty} v(H_n^0) \\
&= \sum_{n=1}^{\infty} v(H_n) - \sum_{n=1}^{\infty} v(G_n) \\
&= \left(\sum_{n=1}^{\infty} v(H_n) - m(S)\right) + \left(m(S) - \sum_{n=1}^{\infty} v(G_n)\right) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So S_0 is measurable. Since $S_0 \subset \bigcup_{n=1}^{\infty} H_n^0$ we have that

$$m(S_0) \leq \sum_{n=1}^{\infty} v(H_n^0) \leqslant \epsilon,$$

and hence by taking the limit as $\epsilon \to 0$, it follows that $m(S) = 0$. □

Proposition 3.4.14. *Let $S \subset \mathcal{K}^\nu$ be a measurable set and let $f : S \to \mathcal{K}$ be a function on S. Let there be a sequence of measurable functions $f_k : S \to \mathcal{K}$ such that the sequence $(f_k)_{k=1}^{\infty}$ converges uniformly to f.Then*

$$\lim_{k\to\infty} \int_{(x_1,\ldots,x_\nu)\in S} f_k(x_1,\ldots,x_\nu)$$

exists; moreover, if f is measurable on S then

$$\lim_{k\to\infty} \int\limits_{(x_1,\ldots,x_\nu)\in S} f_k(x_1,\ldots,x_\nu) = \int\limits_{(x_1,\ldots,x_\nu)\in S} f(x_1,\ldots,x_\nu).$$

Proof. Let $\epsilon > 0$ in $\mathcal{K}$ be given and let

$$\epsilon_0 = \begin{cases} \frac{\epsilon}{m(S)} & \text{if } m(S) \neq 0 \\ \epsilon & \text{if } m(S) = 0 \end{cases}.$$

Since the sequence $(f_k)_{k=1}^{\infty}$ converges uniformly, the sequence is uniformly Cauchy. Thus, there exists a $k_0 \in \mathbb{N}$ such that for every $i, j \geq k_0$,

$$|f_i(x_1,\ldots,x_\nu) - f_j(x_1,\ldots,x_\nu)| \leq \epsilon_0$$

for every $(x_1,\ldots,x_\nu) \in S$. Thus,

$$\begin{aligned} &\left| \int\limits_{(x_1,\ldots,x_\nu)\in S} f_i(x_1,\ldots,x_\nu) - \int\limits_{((x_1,\ldots,x_\nu)\in S} f_j((x_1,\ldots,x_\nu) \right| \\ &= \left| \int\limits_{(x_1,\ldots,x_\nu)\in S} (f_i(x_1,\ldots,x_\nu) - f_j(x_1,\ldots,x_\nu)) \right| \\ &\leq \int\limits_{(x_1,\ldots,x_\nu)\in S} |f_i(x_1,\ldots,x_\nu) - f_j(x_1,\ldots,x_\nu)| \\ &\leq \epsilon_0 m(S) \leq \epsilon. \end{aligned}$$

It follows that

$$\left(\int\limits_{(x_1,\ldots,x_\nu)\in S} f_k(x_1,\ldots,x_\nu) \right)_{k=1}^{\infty}$$

is a Cauchy sequence in $\mathcal{K}$; since $\mathcal{K}$ is Cauchy complete

$$\lim_{k\to\infty} \int\limits_{(x_1,\ldots,x_\nu)\in A} f_k(x_1,\ldots,x_\nu)$$

exists.

Now assume that f is measurable, let $\epsilon > 0$ in $\mathcal{K}$ be given and let ϵ_0 be as above. Then since $(f_k)_{k=1}^\infty$ converges uniformly to f, there exists a $k_0 \in \mathbb{N}$ such that for every $i \geq k_0$,

$$|f_i(x_1,\ldots,x_\nu) - f(x_1,\ldots,x_\nu)| \leq \epsilon_0$$

for every $(x_1,\ldots,x_\nu) \in S$. Therefore,

$$\begin{aligned}
&\left| \int\limits_{(x_1,\ldots,x_\nu)\in S} f_i(x_1,\ldots,x_\nu) - \int\limits_{(x_1,\ldots,x_\nu)\in S} f(x_1,\ldots,x_\nu) \right| \\
&= \left| \int\limits_{(x_1,\ldots,x_\nu)\in S} \left(f_i(x_1,\ldots,x_\nu) - f(x_1,\ldots,x_\nu)\right) \right| \\
&\leq \int\limits_{(x_1,\ldots,x_\nu)\in S} |f_i(x_1,\ldots,x_\nu) - f(x_1,\ldots,x_\nu)| \leq \epsilon_0 m(S) \leq \epsilon,
\end{aligned}$$

and hence

$$\lim_{k\to\infty} \int\limits_{(x_1,\ldots,x_\nu)\in S} f_k(x_1,\ldots,x_\nu) = \int\limits_{(x_1,\ldots,x_\nu)\in S} f(x_1,\ldots,x_\nu).$$

□

3.5 Integration in ν-Dimensions

We conclude this chapter with the definition of the integral of an analytic function over a ν dimensional simple region and the integral of a measurable function over a ν dimensional measurable set. Note the inductive nature of the first definition which assumes that integration has already been defined for an analytic function over a $\nu - 1$ dimensional simple region, this is possible because this definition exists for the case of 2 dimensions (see [3] and [11]).

Definition 3.5.1 (Integral of an Analytic Function Over a Simple Region in $\mathcal{K}^\nu$)**.** *Let $S \subset \mathcal{K}^\nu$ be a simple region, and let $f : S \to \mathcal{K}$ be an analytic function on S. Since S is a simple region, we may assume without loss of generality that*

$$S = \{(x_1, \ldots, x_n) \in \mathcal{K}^n : x_1 \in I(h_1(x_2, \ldots, x_n), h_2(x_2, \ldots, x_n)), (x_2, \ldots, x_n) \in A\}.$$

We define

$$\int\limits_{(x_1,\ldots,x_n)\in S} f(x_1, \ldots, x_n) = \int\limits_{(x_2,\ldots,x_n)\in A} \left[\int\limits_{x_1\in I(h_1(x_2,\ldots,x_n),h_2(x_2,\ldots,x_n))} f(x_1, \ldots, x_n) \right]$$

*and we call this the **integral of** f **over** S. Note that for fixed $(x_2, \ldots, x_n) \in A$, $f(x_1, \ldots, x_n)$ is an analytic function on the interval $I(h_1(x_2, \ldots, x_n), h_2(x_2, \ldots, x_n))$. It follows that*

$$F(x_2, \ldots, x_n) := \int\limits_{z\in I(h_1(x_2,\ldots,x_n),h_2(x_2,\ldots,x_n))} f(x_1, \ldots, x_n)$$

is well-defined and is an analytic function on A, and hence the integral is well-defined by induction.

We include the following four propositions and corollaries without proofs as those proofs are virtually identical to the one and two dimensional cases which may be found in [20], [3], and [11].

Proposition 3.5.2. *Let $S \subset \mathcal{K}^\nu$ be a simple region and let $M \in \mathcal{K}$ be constant. Then*

$$\int\limits_{(x_1,\ldots,x_n)\in S} M = Mv(S).$$

Proposition 3.5.3. *Let $S \subset \mathcal{K}^\nu$ be a simple region and let $f : S \to \mathcal{K}$ be a non-positive analytic function. Then*

$$\int\limits_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n) \le 0.$$

Corollary 3.5.4. *Let $S \subset \mathcal{K}^\nu$ be a simple region and let $f, g : S \to \mathcal{K}$ be analytic functions such that $f \le g$ everywhere on S. Then*

$$\int\limits_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n) \le \int\limits_{(x_1,\ldots,x_n))\in S} g(x_1,\ldots,x_n).$$

Corollary 3.5.5. *Let $S \subset \mathcal{K}^\nu$ be a simple region and let $f : S \to \mathcal{K}$ be an analytic function bounded by $M \in \mathcal{K}$ on S. Then*

$$\left| \int\limits_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n) \right| \le Mv(S).$$

Definition 3.5.6 (Integration of a Measurable Function Over a measurable Set)**.** *Let $S \subset \mathcal{K}^\nu$ be a measurable set and let $f : S \to \mathcal{K}$ be a measurable function such that $|f|$ is bounded by $M \in \mathcal{K}$ everywhere on S. We have that for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint simple regions $(G_l^k)_{l=1}^\infty$ such that*

$$\bigcup_{l=1}^\infty G_l^k \subset S,$$

$\sum\limits_{l=1}^\infty v(G_l^k)$ *converges,*

$$m(S) - \sum_{l=1}^\infty v(G_l^k) < d^k,$$

and for every $l \in \mathbb{N}$, f is analytic on G_l^k. It follows that for every $k, l \in \mathbb{N}$,

$$\left| \int\limits_{(x_1,\ldots,x_n)\in G_l^k} (x_1, \ldots, x_n) \right| \leq \int\limits_{(x_1,\ldots,x_n)\in G_l^k} M = Mv(G_l^k),$$

by Corollary 3.5.5. However, $\sum\limits_{l=1}^{\infty} v(G_l^k)$ converges in the order topology, and hence $\lim\limits_{l\to\infty} Mv(G_l^k) =$ 0. Therefore,

$$\lim_{l\to\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^k} f(x_1, \ldots, x_n) = 0$$

and hence

$$\sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^k} f(x_1, \ldots, x_n)$$

also converges in the order topology. We claim that

$$\left(\sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^k} f(x_1, \ldots, x_n) \right)_{k=1}^{\infty}$$

is a Cauchy sequence. So fix $\epsilon > 0$ and let $k_0 \in \mathbb{N}$ be such that $Md^{k_0} < \epsilon$. Fix $i \neq j \in \mathbb{N}$ so that $i > j \geq k_0$. Because for every $k \in \mathbb{N}$, $(G_l^k)_{l=1}^{\infty}$ is a sequence of mutually disjoint simple regions, we may arrange them so that for every $l \in \mathbb{N}$, $G_l^j \subset G_l^i$. Thus,

$$\begin{aligned}
&\left| \sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^i} f(x_1, \ldots, x_n) - \sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^j} f(x_1, \ldots, x_n) \right| \\
&= \left| \sum_{l=1}^{\infty} \left(\int\limits_{(x_1,\ldots,x_n)\in G_l^i} f(x_1, \ldots, x_n) - \int\limits_{(x_1,\ldots,x_n)\in G_l^j} f(x_1, \ldots, x_n) \right) \right| \\
&\leq \sum_{l=1}^{\infty} \left| \int\limits_{(x_1,\ldots,x_n)\in G_l^i} f(x_1, \ldots, x_n) - \int\limits_{(x_1,\ldots,x_n)\in G_l^j} f(x_1, \ldots, x_n) \right|.
\end{aligned}$$

Now, we have that for every $l \in \mathbb{N}$,

$$\begin{aligned}\left| \int\limits_{(x_1,\ldots,x_n)\in G_l^i} f(x_1,\ldots,x_n) - \int\limits_{(x_1,\ldots,x_n)\in G_l^j} f(x_1,\ldots,x_n) \right| &= \left| \int\limits_{(x_1,\ldots,x_n)\in G_l^i\setminus G_l^j} f(x_1,\ldots,x_n) \right| \\ &\leq \int\limits_{(x_1,\ldots,x_n)\in G_l^i\setminus G_l^j} |f(x_1,\ldots,x_n)| \\ &\leq M(v(G_l^i) - v(G_l^j)).\end{aligned}$$

Finally, we obtain that

$$\begin{aligned}&\left| \sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^i} f(x_1,\ldots,x_n) - \sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^j} f(x_1,\ldots,x_n) \right| \\ &\leq M \sum_{l=1}^{\infty} (v(G_l^i) - v(G_l^j)) \\ &= M \left(\sum_{l=1}^{\infty} v(G_l^i) - \sum_{l=1}^{\infty} v(G_l^j) \right) \\ &< M(m(S) - \sum_{l=1}^{\infty} v(G_l^j)) \\ &< Md^j \\ &\leq Md^{k_0} \\ &< \epsilon.\end{aligned}$$

Thus, the sequence $\left(\sum_{l=1}^{\infty} \int_{(x_1,\ldots,x_n)\in G_l^k} f(x_1,\ldots,x_n) \right)_{k=1}^{\infty}$ *is Cauchy. Since $\mathcal{K}$ is Cauchy complete we have that*

$$\lim_{k\to\infty} \sum_{l=1}^{\infty} \int\limits_{(x_1,\ldots,x_n)\in G_l^k} f(x_1,\ldots,x_n)$$

exists. We define

$$\int_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n) = \lim_{k\to\infty} \sum_{l=1}^{\infty} \int_{(x_1,\ldots,x_n)\in G_l^k} f(x_1,\ldots,x_n)$$

and we call this the ***integral of*** f ***over*** S.

Again we include three propositions and corollaries without proofs, as above the detailed proofs may be found in [20], [3], and [11].

Proposition 3.5.7. *Let* $S \subset \mathcal{K}^\nu$ *be a measurable set and let* $M \in \mathcal{K}$ *be given. Then*

$$\int_{(x_1,\ldots,x_n)\in S} M = Mm(S).$$

Proposition 3.5.8. *Let* $S \subset \mathcal{K}^\nu$ *be a measurable set and let* $f, g : S \to \mathcal{K}$ *be measurable functions. Furthermore, suppose that* $f \leq g$ *everywhere on S. Then*

$$\int_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n) \leq \int_{(x_1,\ldots,x_n)\in S} g(x_1,\ldots,x_n).$$

Corollary 3.5.9. *Let* $S \subset \mathcal{K}^\nu$ *be a measurable set and let* $f : S \to \mathcal{K}$ *be a measurable function bounded by* M *on* S. *Then*

$$\left| \int_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n) \right| \leq \int_{(x_1,\ldots,x_n)\in S} |f(x_1,\ldots,x_n)| \leq Mm(S).$$

Chapter 4

An Explicit Dirac Delta Function on the Hahn Field

4.1 The Delta Function in One Dimension

One of the most exciting aspects of the Levi-Civita and Hahn fields is the possibility of having representations of functions which are improper over the real numbers. In particular it has been shown that Dirac's delta function has a representation and that this non-Archimedean version of the delta function satisfies many of the properties of its conventional counterpart [12]. Below we give one possible representation of the function and state a number of propositions regarding it.

Definition 4.1.1. *Let $\delta : \mathcal{K} \to \mathcal{K}$ be given by*

$$\delta(x) = \begin{cases} \frac{3}{4} d^{-3}(d^2 - x^2) & \textit{if } |x| < d \\ 0 & \textit{if } |x| \geq d \end{cases}.$$

Proposition 4.1.2. *Let $I \subset \mathcal{K}$ be an interval. If $(-d, d) \subset I$ then*

$$\int_{x \in I} \delta(x) = 1.$$

Moreover, if $(-d, d) \cap I = \emptyset$ then

$$\int_{x \in I} \delta(x) = 0.$$

Proof. Note that $\delta(x)$ is measurable on I [20]. If $(-d, d) \subset I$ then

$$\begin{aligned}
\int\limits_{x \in I} \delta(x) &= \int\limits_{x \in (-d,d)} \delta(x) \\
&= \int\limits_{x \in (-d,d)} \frac{3}{4} d^{-3}(d^2 - x^2) \\
&= \frac{3}{4} d^{-3} \left(\left[d^2 x\right]_{-d}^{d} - \left[\frac{1}{3} x^3\right]_{-d}^{d} \right) \\
&= \frac{3}{4} d^{-3} \left(2d^3 - \frac{2}{3} d^3 \right) = 1.
\end{aligned}$$

If $(-d, d) \cap I = \emptyset$ then $\delta(x) = 0$ for all $x \in I$; and hence

$$\int_{x \in I} \delta(x) = \int_{x \in I} 0 = 0.$$

$\square$

Proposition 4.1.3. *Let $I \subset \mathcal{K}$ be an interval containing $(-d, d)$. Then $\delta(x)$ has a measurable anti-derivative on I that is equal to the Heaviside function on $I \cap \mathbb{R}$.*

Proof. Let $H : I \to \mathcal{K}$ be given by

$$H(x) = \begin{cases} 0 & \text{if } x \leq -d \\ \frac{3}{4} d^{-3}(d^2 x - \frac{1}{3} x^3) + \frac{1}{2} & \text{if } -d < x < d \\ 1 & \text{if } x \geq d \end{cases} \cdot$$

Then $H(x)$ is measurable and differentiable on I with $H'(x) = \delta(x)$ on I. Moreover,

$$H(x)|_{\mathbb{R}} = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases},$$

which is the so-called Heaviside function. $\square$

Proposition 4.1.4. *Let $\alpha \in \mathcal{K} \setminus \{0\}$ be given, and let $I \subset \mathcal{K}$ be any interval satisfying $\left(-\frac{d}{|\alpha|}, \frac{d}{|\alpha|}\right) \subset I$. Then*

$$\int_{x \in I} \delta(\alpha x) = \frac{1}{|\alpha|}.$$

Proof. Note that, by definition of the delta function, we have that

$$\delta(\alpha x) = \begin{cases} \frac{3}{4} d^{-3} \left(d^2 - (\alpha x)^2\right) & \text{if } |\alpha x| < d \\ 0 & \text{if } |\alpha x| \geq d \end{cases}$$
$$= \begin{cases} \frac{3}{4} d^{-3} \left(d^2 - (\alpha x)^2\right) & \text{if } |x| < \frac{d}{|\alpha|} \\ 0 & \text{if } |x| \geq \frac{d}{|\alpha|} \end{cases}$$

It follows that

$$\int_{x \in I} \delta(\alpha x) = \int_{x \in \left(-\frac{d}{|\alpha|}, \frac{d}{|\alpha|}\right)} \frac{3}{4} d^{-3}(d^2 - (\alpha x)^2) = \left[\frac{3}{4} d^{-1}\left(x - d^{-2}\frac{\alpha^2 x^3}{3}\right)\right]\Bigg|_{-\frac{d}{|\alpha|}}^{\frac{d}{|\alpha|}} = \frac{1}{|\alpha|}.$$

$\square$

The most useful property of the conventional delta function is how it behaves with other functions under integration. In particular, integrating the product of conventional delta function and some other function results in the value of the second function at the point where the delta function has its peak. Below we see that this property holds for the non-Archimedean delta function in the sense that when integrated with finite analytic or piecewise-analytic function the result is equal in its real part to the real part of the second function at the point where the delta function has its peak. To prove this statement however, we first prove the following two preliminary results.

Lemma 4.1.5. *Suppose $f : I(0,1) \to \mathcal{K}$ is an analytic function with $i(f) = 0$. Then for every $x \in I(0,1)$ and for every $n \in \mathbb{N}$, we have that $\lambda(f^{(n)}(x)) \geq 0$.*

Proof. Let $x \in I(0,1)$ and let $q < 0$ in $\mathbb{Q}$ be given. Since f is analytic over the finite interval $I(0,1)$, there must exist $\delta > 0$ in $\mathbb{R}$ such that for all $y \in (x-\delta, x+\delta) \cap I(0,1)$,

$$f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n.$$

Since $i(f) = 0$ it follows that for almost every $h \in (0,\delta) \cap \mathbb{R}$, we have $\lambda(f(x+h)) = 0$. In other words for almost every $h \in (0,1) \cap \mathbb{R}$ we have that for all $q < 0$, $f(x+h)[q] = 0$. But,

$$\begin{aligned} f(x+h)[q] &= \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n \right) [q] \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[q]}{n!} (h[0])^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[q]}{n!} h^n. \end{aligned}$$

So for almost every $h \in (0,\delta) \cap \mathbb{R}$ we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)[q]}{n!} h^n = 0.$$

Since the above is a real power series, this is possible only if $f^{(n)}(x)[q] = 0$ for all $n \in \mathbb{N}$. Therefore for any $n \in \mathbb{N}$ and $x \in (0,1)$, we have that $\lambda(f^{(n)}(x)) \geq 0$. □

Theorem 4.1.6. *Let $a < b$ in $\mathcal{K}$ be given and let $f : I(a,b) \to \mathcal{K}$ be analytic on $I(a,b)$ with*

$i(f) = 0$. *Then for any* $x \in I(a, b)$ *and for any* $n \in \mathbb{N}$ *we have that*

$$\lambda(f^{(n)}(x)) \geq -n\lambda(b - a).$$

Proof. Define $F : I(0, 1) \to \mathcal{K}$ by

$$F(x) \coloneqq f((b - a)x + a).$$

Then F is analytic on $I(0, 1)$ and $i(F) = i(f) = 0$ so by the Lemma 4.1.5, for all $x \in I(0, 1)$ and $n \in \mathbb{N}$,

$$\lambda\left(F^{(n)}(x)\right) \geq 0.$$

Notice that, for all $x \in I(0, 1)$ and $n \in \mathbb{N}$, we have that

$$F^{(n)}(x) = (b - a)^n f^{(n)}((b - a)x + a);$$

it follows that

$$\begin{aligned} 0 \leq \lambda(F^{(n)}(x)) &= \lambda((b - a)^n f^{(n)}((b - a)x + a)) \\ &= \lambda(((b - a)^n) + \lambda(f^{(n)}((b - a)x + a)) \\ &= n\lambda(b - a) + \lambda(f^{(n)}((b - a)x + a)); \end{aligned}$$

and hence

$$\lambda(f^{(n)}((b - a)x + a)) \geq -n\lambda(b - a).$$

Therefore, for all $x \in I(a, b)$, $\lambda(f^{(n)}(x) \geq -n\lambda(b - a)$. $\square$

Proposition 4.1.7. *Let* $a < b$ *in* $\mathcal{K}$ *be such that* $\lambda(b - a) < 1$ *and let* $f : [a, b] \to \mathcal{K}$ *be* *analytic on $[a, b]$ with $i(f) = 0$. Then for any $x_0 \in [a + d, b - d]$, we have that*

$$\int_{x \in [a,b]} f(x)\delta(x - x_0) =_0 f(x_0).$$

Proof. Fix $x_0 \in [a + d, b - d]$. Since f is a finite analytic function, there exists a $\eta > 0$ in $\mathcal{K}$ with $\lambda(\eta) = \lambda(b - a)$ such that, for any $x \in I(a, b)$ satisfying $|x - x_0| < \eta$, we have that $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$. Therefore,

$$\begin{aligned}
\int\limits_{x \in I(a,b)} f(x)\delta(x - x_0) &= \int\limits_{x \in (x_0 - d, x_0 + d)} f(x)\delta(x - x_0) \\
&= \int\limits_{x \in (x_0 - d, x_0 + d)} \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0) \\
&= \int\limits_{x \in (x_0 - d, x_0 + d)} f(x_0)\delta(x - x_0) \\
&+ \int\limits_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0) \\
&= f(x_0) + \int\limits_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0).
\end{aligned}$$

Now, for any $x \in (x_0 - d, x_0 + d)$, we have that $|x - x_0| < d$, and hence

$$\left| \int\limits_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0) \right| \leqslant \int\limits_{x \in [x_0 - d, x_0 + d]} \sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!} d^k \delta(x - x_0)$$

It follows that

$$\left| \int\limits_{x \in (x_0 - d, x_0 + d)} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0) \right| \leqslant \sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!} d^k$$

Thus

$$\lambda\left(\int\limits_{x\in(x_0-d,x_0+d)}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)\right)\geqslant\lambda\left(\sum_{k=1}^{\infty}\frac{|f^{(k)}(x_0)|}{k!}d^k\right).$$

However, since $i(f)=0$ we can apply Theorem 4.1.6 to establish that for all $k\in\mathbb{N}$, $\lambda(f^{(k)}(x_0))>-k$ and hence $\lambda\left(\sum\limits_{k=1}^{\infty}\frac{|f^{(k)}(x_0)|}{k!}d^k\right)>0$. Thus,

$$\lambda\left(\int\limits_{x\in(x_0-d,x_0+d)}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)\right)>0.$$

Therefore,

$$\int\limits_{x\in(x_0-d,x_0+d)}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)=_0 0.$$

It follows that

$$\int\limits_{x\in I(a,b)}f(x)\delta(x-x_0)=_0 f(x_0).$$

□

Proposition 4.1.8. *Let $a<b<c$ in $\mathcal{K}$ be such $\lambda(b-a)<1$ and $\lambda(c-b)<1$; let $g:[a,b]\to\mathcal{K}$ and $h:[b,c]\to\mathcal{K}$ be analytic functions satisfying $g(b)=h(b)$ and $i(h)=i(g)=0$; and let $f:[a,c]\to\mathcal{K}$ be given by*

$$f(x)=\begin{cases}g(x) & \text{if } x\in[a,b)\\ h(x) & \text{if } x\in[b,c]\end{cases}.$$

Then for any $x_0\in[a+d,c-d]$, we have that

$$\int\limits_{x\in[a,c]}f(x)\delta(x-x_0)=_0 f(x_0).$$

Proof. Without loss of generality, we may assume that $b=0$. Fix $x_0\in[a+d,c-d]$. If

$|x_0| \geq d$ then by Proposition 4.1.7 we are done; so without loss of generality we may assume that $|x_0| < d$. Thus, we have that

$$\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) = \int\limits_{x\in[x_0-d,0]} g(x)\delta(x-x_0) + \int\limits_{x\in[0,x_0+d]} h(x)\delta(x-x_0).$$

Both g and h are analytic functions defined on $[a,0]$ and $[0,c]$ respectively; and hence they both can be expanded as power series centered at 0. Thus,

$$g(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$

and

$$h(x) = \sum_{k=0}^{\infty} \beta_k x^k,$$

where

$$\alpha_k = \frac{g^{(k)}(0)}{k!} \text{ and } \beta_k = \frac{h^{(k)}(0)}{k!} \text{ for } k = 0, 1, 2, \ldots.$$

Since $\lambda(b-a) = \lambda(-a) = \lambda(a) < 1$ and $\lambda(c-b) = \lambda(c) < 1$ both power series will have radii of convergence infinitely larger than d, and hence they will converge everywhere on $[x_0-d, 0]$ and $[0, x_0+d]$, respectively. Thus,

$$\int\limits_{x\in[x_0-d,0]} g(x)\delta(x-x_0) = \int\limits_{x\in[x_0-d,0]} \sum_{k=0}^{\infty} \alpha_k x^k \delta(x-x_0)$$

and

$$\int\limits_{x\in[0,x_0+d]} h(x)\delta(x-x_0) = \int\limits_{x\in[0,x_0+d]} \sum_{k=0}^{\infty} \beta_k x^k \delta(x-x_0).$$

Therefore,

$$\begin{aligned}\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) &= \int\limits_{x\in[x_0-d,0]} \sum_{k=0}^{\infty} \alpha_k x^k \delta(x-x_0) + \int\limits_{x\in[0,x_0+d]} \sum_{k=0}^{\infty} \beta_k x^k \delta(x-x_0) \\ &= \alpha_0 \int\limits_{x\in[x_0-d,0]} \delta(x-x_0) + \beta_0 \int\limits_{x\in[0,x_0+d]} \delta(x-x_0) \\ &+ \int\limits_{x\in[x_0-d,0]} \sum_{k=1}^{\infty} \alpha_k x^k \delta(x-x_0) \\ &+ \int\limits_{x\in[0,x_0+d]} \sum_{k=1}^{\infty} \beta_k x^k \delta(x-x_0).\end{aligned}$$

However, $\alpha_0 = g(0) = f(0) = h(0) = \beta_0$, and hence

$$\alpha_0 \int\limits_{x\in[x_0-d,0]} \delta(x-x_0) + \beta_0 \int\limits_{x\in[0,x_0+d]} \delta(x-x_0) = f(0) \int\limits_{x\in[x_0-d,x_0+d]} \delta(x-x_0) = f(0).$$

Thus,

$$\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) = f(0) + \int\limits_{x\in[x_0-d,0]} \sum_{k=1}^{\infty} \alpha_k x^k \delta(x-x_0) + \int\limits_{x\in[0,x_0+d]} \sum_{k=1}^{\infty} \beta_k x^k \delta(x-x_0).$$

But

$$\begin{aligned}&\lambda\left(\int\limits_{x\in[x_0-d,0]} \sum_{k=1}^{\infty} \alpha_k x^k \delta(x-x_0) + \int\limits_{x\in[0,x_0+d]} \sum_{k=1}^{\infty} \beta_k x^k \delta(x-x_0)\right) \\ &\geqslant \lambda\left(\sum_{k=1}^{\infty} |\alpha_k|\,(2d)^k \int\limits_{x\in[x_0-d,0]} \delta(x-x_0) + \sum_{k=1}^{\infty} |\beta_k|\,(2d)^k \int\limits_{x\in[0,x_0+d]} \delta(x-x_0)\right) > 0\end{aligned}$$

as in the proof of Proposition 4.1.7. Thus,

$$\int\limits_{x\in[x_0-d,0]} \sum_{k=1}^{\infty} \alpha_k x^k \delta(x-x_0) + \int\limits_{x\in[0,x_0+d]} \sum_{k=1}^{\infty} \beta_k x^k \delta(x-x_0) =_0 0,$$

and hence $\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) =_0 f(0)$. However, $f(0) =_0 f(x_0)$ [15]; therefore

$$\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) =_0 f(x_0).$$

$\square$

4.2 A Version of Leibniz' Rule

We would like to use the delta function to solve differential equations via the method of Green's functions, this method involves passing a differential operator under an integral. To establish the legitimacy of this operation we now prove a version of Leibniz' Rule for the field $\mathcal{K}$. As we shall see in a later section, Leibniz' Rule gives us certain boundary conditions that the Green's functions must satisfy if they are to produce a valid solution. We begin by introducing the notion of **uniform differentiability** which is defined on an interval of $\mathcal{K}$ the same way it is defined on $\mathbb{R}$.

Definition 4.2.1 (Uniform differentiability on an interval of $\mathcal{K}$)**.** *Let $a, b \in \mathcal{K}$ be given with $a < b$ and let $f : I(a, b) \to \mathcal{K}$ be differentiable with derivative f' on $I(a, b)$. Then we say that f is **uniformly differentiable** on $I(a, b)$ if for every $\epsilon > 0$ in $\mathcal{K}$ there is a $\delta > 0$ in $\mathcal{K}$ such that for all $x, y \in I(a, b)$,*

$$0 < |y - x| < \delta \implies \left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| < \epsilon.$$

Lemma 4.2.2. *Suppose $f : I(0, 1) \to \mathcal{K}$ is analytic with $i(f) = 0$. Then f is uniformly differentiable on $I(a, b)$.*

Proof. First note that by Theorem 4.1.6, $\lambda(f^{(n)}(x)) \geq 0$ for all $n \in \mathbb{N}$ and $x \in I(a, b)$. Now let $\epsilon > 0$ in $\mathcal{K}$ and let

$$\delta := \min\left\{d^4\epsilon, d\right\}.$$

Then for any $x, y \in I(a,b)$ satisfying $0 < |y-x| < \delta$, we have that

$$f(y) = f(x) + f'(x)(y-x) + \sum_{n=2}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n,$$

where the power series converges in the order topology since $\lambda(f^{(n)}(x)) \geq 0$ for all $n \in \mathbb{N}$ and since $0 < |y-x| < \delta \ll 1$ so that

$$\lim_{n\to\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n = 0.$$

It follows that

$$|f(y) - f(x) - f'(x)(y-x)| = \left| \sum_{n=2}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n \right| < \sum_{n=2}^{\infty} \frac{d^{-1}}{n!} d^{n-2}(y-x)^2;$$

and hence

$$\begin{aligned} \left| \frac{f(y)-f(x)}{y-x} - f'(x) \right| &< \sum_{n=0}^{\infty} \frac{d^{n-3}}{n!} |y-x| \\ &< \left(\sum_{n=0}^{\infty} \frac{d^{n-3}}{n!} \right) d^4 \epsilon \\ &= \left(\sum_{n=0}^{\infty} \frac{d^n}{n!} \right) d\epsilon \\ &= \frac{d}{1-d}\epsilon < \epsilon. \end{aligned}$$

□

Theorem 4.2.3. *Let $a < b$ in $\mathcal{K}$ be given and let $f : I(a,b) \to \mathcal{K}$ be an analytic function. Then f is uniformly differentiable on $I(a,b)$.*

Proof. Let $F : I(0,1) \to \mathcal{K}$ be given by

$$F(x) = d^{-i(f)} f(a + (b-a)x).$$

Then F is analytic on $I(0,1)$ with $i(F)=0$; and hence, by Lemma 4.2.2, F is uniformly differentiable on $I(0,1)$. Now fix $\epsilon>0$ in $\mathcal{K}$. Since F is uniformly differentiable on $I(0,1)$, there is a $\delta>0$ in $\mathcal{K}$ such that if $x,y\in I(0,1)$ and $|x-y|<\frac{\delta}{b-a}$ then

$$|F(x)-F(y)-F'(x)(x-y)|<d^{-i(f)}\epsilon.$$

However,

$$\begin{aligned}&|F(x)-F(y)-F'(x)(x-y)|\\&=|d^{-i(f)}f(a+(b-a)x)-d^{-i(f)}f(a-(b-a)y)-d^{-i(f)}f'(a+(b-a)x)(b-a)(x-y)|\\&=d^{-i(f)}|f(a+(b-a)x)-f(a+(b-a)y)-f'(a+(b-a)x)(b-a)(x-y)|.\end{aligned}$$

Thus, if $x,y\in I(0,1)$ and $|x-y|<\frac{\delta}{b-a}$ then we have that

$$|f(a+(b-a)x)-f(a+(b-a)y)-f'(a+(b-a)x)(b-a)(x-y)|<\epsilon.$$

Now let $u,v\in I(a,b)$ be such that $|u-v|<\delta$; and let

$$x=\frac{u-a}{b-a}\text{ and }y=\frac{v-a}{b-a}.$$

Then $u=a+(b-a)x$, $v=a+(b-a)y$, $x,y\in I(0,1)$ and $|x-y|<\frac{\delta}{b-a}$. It follows that

$$\begin{aligned}&|f(u)-f(v)-f'(u)(u-v)|\\&=|f(a+(b-a)x)-f(a+(b-a)y)-f'(a+(b-a)x)(b-a)(x-y)|\\&<\epsilon.\end{aligned}$$

Thus, f is uniformly differentiable on (a,b). $\square$

Remark 4.2.4. *In the following, and to avoid confusion with the number d, we will use D_x*

to denote the differential operator $\frac{d}{dx}$*, moreover we will use* D_x^n *to denote* $\frac{d^n}{dx^n}$*.*

Proposition 4.2.5. *Let* x_0*,* $a < b$*, and* $\epsilon > 0$ *in* $\mathcal{K}$ *be given; and let* $f : [x_0-\epsilon, x_0+\epsilon]\times[a,b] \to \mathcal{K}$ *be a (2-variable) power series. Then*

$$D_x \int_{y\in[a,b]} f(x,y) = \int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y).$$

Proof. Let $N \in \mathbb{N}$ be such that $d^N < \epsilon$. By Theorem 4.2.3 above we have that f is uniformly differentiable with respect to x and hence

$$\lim_{k\to\infty} \frac{f(x+d^{N+k},y) - f(x,y)}{d^{N+k}} = \frac{\partial}{\partial x} f(x,y) \text{ (uniformly)}.$$

Moreover, by definition

$$D_x \int_{y\in[a,b]} f(x,y) = \lim_{k\to\infty} \int_{y\in[a,b]} \frac{f(x+d^{N+k},y) - f(x,y)}{d^{N+k}}.$$

However, by Theorem 3.9 in [13] we have that

$$\begin{aligned} \lim_{k\to\infty} \int_{y\in[a,b]} \frac{f(x+d^{N+k},y) - f(x,y)}{d^{N+k}} &= \int_{y\in[a,b]} \lim_{k\to\infty} \frac{f(x+d^{N+k},y) - f(x,y)}{d^{N+k}} \\ &= \int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y). \end{aligned}$$

This completes the proof of the proposition. □

Corollary 4.2.6. *Let* x_0*,* $a < b$*, and* $\epsilon > 0$ *in* $\mathcal{K}$ *be given; and let* $f : [x_0-\epsilon, x_0+\epsilon]\times[a,b] \to \mathcal{K}$ *be analytic on* : $[x_0-\epsilon, x_0+\epsilon]\times[a,b]$*. Then*

$$D_x \int_{y\in[a,b]} f(x,y) = \int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y).$$

Proof. This follows immediately from the fact that analytic functions are given locally by power series [11]. $\square$

Proposition 4.2.7 (Leibniz's Rule)**.** *Fix $x_0 \in \mathcal{K}$ and let $\epsilon > 0$ in $\mathcal{K}$ be given. Let $\alpha, \beta : [x_0 - \epsilon, x_0 + \epsilon] \to \mathcal{K}$ be analytic functions with $\alpha(x) \leq \beta(x)$ for all $x \in [x_0 - \epsilon, x_0 + \epsilon]$. Let S be the simple region given by*

$$S = \{(x, y) \in \mathcal{K}^2 | x \in [x_0 - \epsilon, x_0 + \epsilon], y \in [\alpha(x), \beta(x)]\}$$

and let $f : S \to \mathcal{K}$ be analytic. Then

$$D_x \int\limits_{y \in [\alpha(x), \beta(x)]} f(x, y) = f(x, \beta(x))\beta'(x) - f(x, \alpha(x))\alpha'(x) + \int\limits_{y \in [\alpha(x), \beta(x)]} \frac{\partial}{\partial x} f(x, y).$$

Proposition 4.2.8. *Let x_0, $a < b$, and $\epsilon > 0$ in $\mathcal{K}$ be given and let $\mu : [x_0 - \epsilon, x_0 + \epsilon] \to [a, b]$ be a non-constant analytic function. Let $g : [x_0 - \epsilon, x_0 + \epsilon] \times [a, \mu(x)] \to \mathcal{K}$ and $h : [x_0 - \epsilon, x_0 + \epsilon] \times [\mu(x), b] \to \mathcal{K}$ be analytic and let $f : [x_0 - \epsilon, x_0 + \epsilon] \times [a, b]$ be given by*

$$f(x, y) = \begin{cases} g(x, y) & \text{if } y \leq \mu(x) \\ h(x, y) & \text{if } y > \mu(x) \end{cases}.$$

Then

$$D_x \int\limits_{y \in [a, b]} f(x, y) = \int\limits_{y \in [a, b]} \frac{\partial}{\partial x} f(x, y)$$

if and only if $f(x, y)$ is continuous.

Proof. Observe that

$$D_x \int\limits_{y \in [a, b]} f(x, y) = D_x \int\limits_{y \in [a, \mu(x)]} g(x, y) + D_x \int\limits_{y \in [\mu(x), b]} h(x, y).$$

But by Proposition 4.2.7 we have that

$$D_x \int\limits_{y\in[a,\mu(x)]} g(x,y) = g(x,\mu(x))\mu'(x) + \int\limits_{y\in[a,\mu(x)]} \frac{\partial}{\partial x} g(x,y)$$

and

$$D_x \int_{y\in[\mu(x),b]} h(x,y) = \int\limits_{y\in[\mu(x),b]} \frac{\partial}{\partial x} h(x,y) - h(x,\mu(x))\mu'(x).$$

So

$$\begin{aligned} D_x \int\limits_{y\in[a,b]} f(x,y) = g(x,\mu(x))\mu'(x) &+ \int\limits_{y\in[a,\mu(x)]} \frac{\partial}{\partial x} g(x,y) \\ &+ \int\limits_{y\in[\mu(x),b]} \frac{\partial}{\partial x} h(x,y) - h(x,\mu(x))\mu'(x) \\ &= \int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y) + \left[g(x,\mu(x)) - h(x,\mu(x))\right]\mu'(x). \end{aligned}$$

Since μ is a non-constant analytic function we know that $\mu' \neq 0$; and it follows that the above expression equals $\int_{y\in[a,b]} \frac{\partial}{\partial x} f(x,y)$ if and only if $g(x,\mu(x)) = h(x,\mu(x))$ for all $x \in [x_0-\epsilon, x_0+\epsilon]$, that is if and only if f is continuous at $y = \mu(x)$ and hence everywhere (since g and h are analytic). □

4.3 Examples in One Dimension

In this section, we present two simple examples in which we illustrate the applications of the delta function defined on $\mathcal{K}$ above.

Example 4.3.1. *[Solving Poisson's Equation in One Dimension] Suppose that we wish to find the solution to the real differential equation $\ddot{x}(t) = f(t)$ on the interval $[0, +\infty)$ and subject to the initial conditions $x(0) = 0, \dot{x}(0) = 0$. To begin, we observe that the piecewise* *analytic solution to* $\frac{\partial^2}{\partial t^2}G(t,s)=\delta(t-s)$ *is*

$$G(t,s)=\begin{cases} A_1(t-s)+B_1 & s\le t-d \\ A_2(t-s)+B_2+\frac{3}{8}d^{-3}(d^2(t-s)^2-\frac{1}{6}(t-s)^4) & t-d<s<t+d \\ A_3(t-s)+B_3 & s\ge t+d \end{cases},$$

where A_1*,* A_2*,* A_3*,* B_1*,* B_2 *and* B_3 *are constants to be determined. To ensure that our solution satisfies the given initial conditions we must have that the real parts of* $G(0,s)$ *and* $\frac{\partial G}{\partial t}(0,s)$ *equal zero everywhere on the interval of interest; and to accomplish that, it is enough to set* $G(r,s)=0$ *and* $\frac{\partial G}{\partial t}(r,s)=0$ *where* $r\in\mathcal{K}$ *is any number that is infinitely small in absolute value. We will use* $r=-d$ *since that ensures that for every* $s\in[0,\infty)$*,* $s\ge r+d$ *and so this requirement only affects* A_3 *and* B_3*. If in contrast we chose* $r=0$ *that would dictate the values of* A_2 *and* B_2 *as well and it would not be possible to change them in order to ensure continuity of* G *and its derivative in* t*. In order to apply Proposition 4.2.8 we require that* G *be continuous (so that* $D_t\int_s G(t,s)=\int_s\frac{\partial}{\partial t}G(t,s)$*) and that* $\frac{\partial G}{\partial t}(t.s)$ *be continuous (so that* $D_t\int_s\frac{\partial}{\partial t}G(t,s)=\int_s\frac{\partial^2}{\partial t^2}G(t,s)$*). The requirement of continuity of* $G(t,s)$ *and its derivative at* $s=t\pm d$ *allows us to work backwards from the (now known) values of* A_3*, and* B_3 *to solve for* A_1*,* B_1*,* A_2*,* B_2*. The result is*

$$G(t,s)=\begin{cases} t-s & s\le t-d \\ \frac{1}{2}(t-s)+\frac{13}{16}d+\frac{3}{8}d^{-3}(d^2(t-s)^2-\frac{1}{6}(t-s)^4) & t-d<s<t+d \\ 0 & s\ge t+d \end{cases}.$$

Note that when restricted to real points, the real part of $G(t,s)$ *reduces to the classical Green's* *function for D_t^2. Applying Proposition 4.2.8, we obtain that*

$$\begin{aligned}
D_t^2 \int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s) &= \int\limits_{s\in[0,d^{-\frac{1}{2}}]} \frac{\partial^2}{\partial t^2} G(t,s)f(s) \\
&= \int\limits_{s\in[0,d^{-\frac{1}{2}}]} \delta(t-s)f(s) \\
&=_0 f(t).
\end{aligned}$$

It follows that $\left(\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s)\right)[0]$ *is a (real) solution to the equation*

$$\ddot{u}(t) = f(t)$$

with the initial conditions

$$\left(\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(0,s)f(s)\right)[0] = \left(\int\limits_{s\in[0,d^{-\frac{1}{2}}]} \frac{\partial G}{\partial t}(0,s)f(s)\right)[0] = 0;$$

and hence we must have that

$$x(t) = \left(\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s)\right)[0].$$

Now, if we set $f(t) = t$ then we see that

$$\begin{aligned}
&\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s) = \int\limits_{s\in[0,t+d]} G(t,s)f(s) \\
&= \int\limits_{s\in[0,t-d]} (t-s)s + \int\limits_{s\in[t-d,t+d]} \left(\frac{1}{2}(t-s) + \frac{3}{16}d + \frac{3}{8}d^{-3}\left(d^2(t-s)^2 - \frac{1}{6}(t-s)^4\right)\right) s.
\end{aligned}$$

But,

$$\int\limits_{s\in[0,t-d]} (t-s)s = \frac{t^3}{6} - \frac{t}{2}d^2 + \frac{1}{3}d^3$$

and

$$\int\limits_{s\in[t-d,t+d]} \left(\frac{1}{2}(t-s) + \frac{3}{16}d + \frac{3}{8}d^{-3}\left(d^2(t-s)^2 - \frac{1}{6}(t-s)^4\right)\right) s = \frac{9}{40}td^2 - \frac{1}{3}d^3$$

Thus,

$$\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s) = \frac{t^3}{6} - \frac{1}{2}td^2 + \frac{1}{3}d^3 + \frac{9}{40}td^2 - \frac{1}{3}d^3 = \frac{t^3}{6} - \frac{11}{40}td^2 =_0 \frac{1}{6}t^3$$

and hence the real solution is $x(t) = \frac{1}{6}t^3$. *One benefit of solving differential equations with the method of Green's functions is that the same Green's function will work for every analytic source function. Suppose we now wish to find the same equation as above but with* $f(t) = \sin(t)$. *Then we have that*

$$\begin{aligned}
&\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s) = \int\limits_{s\in[0,t+d]} G(t,s)f(s) \\
&= \int\limits_{s\in[0,t-d]} (t-s)\sin(s) + \int\limits_{s\in[t-d,t+d]} \left(\frac{1}{2}(t-s) + \frac{3}{16}d + \frac{3}{8}d^{-3}\left(d^2(t-s)^2 - \frac{1}{6}(t-s)^4\right)\right)\sin(s) \\
&= t + \sin(d-t) - \cos(d-t)d + \cos(d)\left(\cos(t)d + \sin(t) + 3\sin(t)d^{-2}\right) \\
&+ \frac{1}{8}\sin(d)\left(-8\cos(t) - 24\sin(t)d^{-3} + 5\sin(t)d\right).
\end{aligned}$$

Taylor expanding in powers of d and taking the first term (i.e. the real part) yields

$$x(t) = \left(\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)f(s)\right)[0] = t - \sin(t).$$

Example 4.3.2 (Damped Driven Harmonic Oscillator)**.** *Consider now an underdamped, driven harmonic oscillator with mass m, viscous damping constant c, spring constant k, and driving force $f(t)$. Let $x(t)$ be the position of the oscillator at time t with $x(0) = 0$ and $\dot{x}(0) = 0$. The oscillator's equation of motion is*

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{f(t)}{m}. \tag{4.1}$$

With the following change of variables

$$\gamma = \frac{c}{2\sqrt{mk}} \text{ and } \omega_0 = \sqrt{\frac{k}{m}},$$

Equation (4.1) takes the form

$$\ddot{x}(t) + 2\gamma\omega_0\dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}.$$

Since the oscillator is underdamped we have that $\gamma^2\omega_0^2 - \omega_0^2 < 0$ which is equivalent to $\gamma < 1$. To solve the equation of motion we first find the Green's function for the differential operator $(D_t^2 + 2\gamma\omega_0 D_t + \omega_0^2)$; that is, we find a solution for the differential equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2\right) G(t,s) = \delta(t-s).$$

First we observe that the analytic solution to the homogeneous partial differential equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2\right) G_{hom}(t,s) = 0$$

is

$$G_{hom}(t,s) = e^{-\gamma\omega_0(t-s)}\left(A\sin(\omega(t-s)) + B\cos(\omega(t-s))\right)$$

where $\omega = \sqrt{1-\gamma^2}\omega_0$ and where A and B are arbitrary constants. One particular solution *to the inhomogeneous partial differential equation*

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2\right) G_{inhom}(t,s) = \frac{3}{4}d^{-3}(d^2 - (t-s)^2)$$

is given by

$$G_{inhom}(t,s) = \frac{3}{\omega_0^2}d^{-3}\left(\frac{d^2-(t-s)^2}{4} + \frac{\gamma(t-s)}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2}\right).$$

Since

$$\delta(t) = \begin{cases} 0 & \text{if } t \le -d \\ \frac{3}{4}d^{-3}(d^2-t^2) & \text{if } -d < t < d \\ 0 & \text{if } d \le t \end{cases},$$

we must have

$$G(t,s) = \begin{cases} e^{-\gamma\omega_0(t-s)}\left(A_1\sin\left(\omega(t-s)\right) + B_1\cos\left(\omega(t-s)\right)\right) & \text{if } s \le t-d \\ \begin{array}{l} e^{-\gamma\omega_0(t-s)}\left(A_2\sin\left(\omega(t-s)\right) + B_2\cos\left(\omega(t-s)\right)\right) \\ +\frac{3}{\omega_0^2}d^{-3}\left(\frac{d^2-(t-s)^2}{4} + \frac{\gamma(t-s)}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2}\right) \end{array} & \text{if } t-d < s < t+d \\ e^{-\gamma\omega_0(t-s)}\left(A_3\sin\left(\omega(t-s)\right) + B_3\cos\left(\omega(t-s)\right)\right) & \text{if } s \ge t+d \end{cases}$$

where $A_1, A_2, A_3, B_1, B_2, B_3$ are constants to be determined by the initial conditions. As in the previous example the real part of our Green's function must satisfy the same initial conditions as the desired solution. To this end we require that for all $s \in [0,\infty)$

$$G(-d,s) = 0 \text{ and } \frac{\partial G}{\partial t}(-d,s) = 0.$$

Solving for the relevant constants yields

$$A_3 = 0 \text{ and } B_3 = 0.$$

Again, as in the previous example, requiring continuity of G and $\partial G/\partial t$ gives us the remaining 4 constants:

$$
\begin{aligned}
A_1 &= \frac{3}{2d^3\omega_0^4\omega}e^{-d\gamma\omega}\left(\left(4\gamma^3\omega_0+2d\gamma^2\omega_0^2-3\gamma\omega_0-d\omega_0^2\right)\cos(d\omega)-\left(4\gamma^2\omega+2d\gamma\omega_0\omega-\omega\right)\sin(d\omega)\right)\\
&\quad+\frac{3}{2d^3\omega_0^4\omega}e^{d\gamma\omega_0}\left(\left(-4\gamma^3\omega_0+2d\gamma^2\omega_0^2+3\gamma\omega_0-d\omega_0^2\right)\cos(d\omega)-\left(4\gamma^2\omega-2d\gamma\omega_0\omega-\omega\right)\sin(d\omega)\right)\\
B_1 &= \frac{3}{2d^3\omega_0^4\omega}e^{-d\gamma\omega_0}\left(\omega\left(2d\gamma\omega_0+4\gamma^2-1\right)+\omega_0\left(2d\gamma^2\omega_0-d\omega_0+4\gamma^3-3\gamma\right)\sin(d\omega)\right)\\
&\quad-\frac{3}{2d^3\omega_0^4\omega}e^{d\gamma\omega_0}\left(\omega\left(-2d\gamma\omega_0+4\gamma^2-1\right)+\omega_0\left(2d\gamma^2\omega_0-d\omega_0-4\gamma^3+3\gamma\right)\sin(d\omega)\right)\\
A_2 &= \frac{3}{2\omega_0^4\omega d^3}e^{-d\gamma\omega_0}\left(\omega\left(-2d\gamma\omega_0-4\gamma^2+1\right)\sin(d\omega)+\omega_0\left(2d\gamma^2\omega_0-d\omega_0+4\gamma^3-3\gamma\right)\cos(d\omega)\right)\\
B_2 &= \frac{3}{2\omega_0^4\omega d^3}e^{-d\gamma\omega_0}\cos(d\omega)\left(\omega\left(2d\gamma\omega_0+4\gamma^2-1\right)+\omega_0\left(2d\gamma^2\omega_0-d\omega_0+4\gamma^3-3\gamma\right)\tan(d\omega)\right)
\end{aligned}
$$

While at first glance these constants seem too cumbersome, we have that

$$A_1 =_0 \frac{1}{\omega} \textit{ and } B_1 =_0 0;$$

and hence

$$G(t,s)|_{\mathbb{R}} =_0 \begin{cases} \frac{1}{\omega}e^{-\gamma\omega_0(t-s)}\sin(\omega(t-s)) & \textit{if } s<t \\ 0 & \textit{if } s\geq t \end{cases}$$

which is the classical Green's function for this problem. Now, suppose that the driving force is given by

$$f(t) = me^{-\gamma\omega_0 t}.$$

Then the equation of motion becomes

$$\ddot{x}(t)+2\gamma\omega_0\dot{x}(t)+\omega_0^2x(t) = e^{-\gamma\omega_0 t}.$$

Thus, as in the previous example, we can obtain the real solution as the real part of

$$\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)\frac{f(s)}{m}.$$

Therefore,

$$x(t) =_0 \int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)\frac{f(s)}{m}.$$

But $G(t,s) = 0$ for $s > t+d$, and hence

$$\int\limits_{s\in[0,d^{-\frac{1}{2}}]} G(t,s)\frac{f(s)}{m} = \int\limits_{s\in[0,t+d]} G(t,s)e^{-\gamma\omega_0 s}.$$

Thus,

$$\begin{aligned}
x(t) &=_0 \int\limits_{s\in[0,t+d]} G(t,s)e^{-\gamma\omega_0 s} \\
&= e^{-\gamma\omega_0 t} \int\limits_{s\in[t-d,t+d]} (A_2\sin(\omega(t-s)) + B_2\cos(\omega(t-s))) \\
&+ e^{-\gamma\omega_0 t} \int\limits_{s\in[t-d,t+d]} \frac{3}{\omega_0^2 d^3}\left(\frac{d^2-(t-s)^2}{4} + \frac{\gamma(t-s)}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2}\right) \\
&+ e^{-\gamma\omega_0 t} \int\limits_{s\in[0,t-d]} (A_1\sin(\omega(t-s)) + B_1\cos(\omega(t-s))) \\
&= e^{-\gamma\omega_0 t}\left[\frac{2A_2\sin(\omega d)}{\omega} + A_1\frac{\sin(\omega t)-\sin(\omega d)}{\omega} + B_1\frac{\cos(\omega t)-\cos(\omega d)}{\omega}\right. \\
&\left. -\frac{3}{\omega_0^2}d^{-3}\left(\frac{2}{4\gamma^3\omega_0^3} + \frac{1+4\gamma^2}{2\gamma\omega_0^3}\right)(e^{\gamma\omega_0 d} - e^{-\gamma\omega_0 d}) + \frac{3}{\omega_0^2}\left(\frac{d}{2\gamma^2\omega_0^2} + \frac{d}{\omega_0^2}\right)(e^{\gamma\omega_0 d} + e^{-\gamma\omega_0 d})\right] \\
&=_0 e^{-\gamma\omega_0 t}\frac{\cos(\omega t)-1}{\omega^2},
\end{aligned}$$

which agrees with the classical solution. In the above example we have chosen the driving force carefully so as to simplify the subsequent calculations, this is useful for explanation and *necessary where the calculations are to be preformed by hand but with the use of mathematical software (Mathematica in our case) it is possible to solve more interesting problems. Define* $h : [0, \infty) \cap \mathcal{K} \to \mathcal{K}$ *by*

$$h(t) = \begin{cases} \frac{t}{1-d} & \text{if } t \leq 1-d \\ (t-1)^2 \left(\frac{1+2d(1-d)}{d^2(1-d)} t + 3 - \frac{2}{d} - \frac{1}{d^2} \right) & \text{if } 1-d < t \leq 1 \\ 0 & \text{if } t > 1 \end{cases},$$

in fact $h|_{\mathbb{R}}$ *is a single saw-tooth wave and the constants have been chosen so that* $h(t)$ *is continuous and has a continuous derivative; this would not be possible if* h *was a real analytic function. We may now solve the real differential equation*

$$\ddot{x}(t) + 2\gamma\omega_0 \dot{x}(t) + \omega_0^2 x(t) = h|_{\mathbb{R}}(t)$$

subject to the boundary conditions $x(0) = 0$*, and* $\dot{x}(0) = 0$*. We have that for* $t \in \mathbb{R}^+$*,*

$$x(t) = \left(\int_{s \in [0, d^{-\frac{1}{2}})} G(t,s) h(s) \right) [0].$$

To begin, suppose that $t<1$ then

$$
\begin{aligned}
\int_{s\in[0,d^{-\frac{1}{2}})} G(t,s)h(s) &= \int_{s\in[0,t-d]} e^{-\gamma\omega_0(t-s)}\left(A_1\sin\left(\omega(t-s)\right)+B_1\cos\left(\omega(t-s)\right)\right)\frac{s}{1-d}\\
&\int_{s\in[t-d,t+d]}\Bigg(e^{-\gamma\omega_0(t-s)}\left(A_2\sin\left(\omega(t-s)\right)+B_2\cos\left(\omega(t-s)\right)\right)\\
&+\frac{3}{\omega_0^2}d^{-3}\left(\frac{d^2-(t-s)^2}{4}+\frac{\gamma(t-s)}{\omega_0}+\frac{1-4\gamma^2}{2\omega_0^2}\right)\Bigg)\frac{s}{1-d}\\
&=_0 \frac{\frac{\left(2\gamma^2-1\right)e^{-\gamma\omega_0 t}\sin(\omega t)}{\sqrt{1-\gamma^2}}+2\gamma e^{-\gamma\omega_0 t}\cos\left(\omega t\right)-2\gamma+\omega_0 t}{\omega_0^3}\\
&-\frac{2\left(4\gamma^4\omega_0^2 t-5\gamma^2\omega_0^2 t+4\gamma^2\omega^2 t+\omega_0^2 t-\omega^2 t\right)}{\omega_0^4}\\
&=\frac{\frac{\left(2\gamma^2-1\right)e^{-\gamma\omega_0 t}\sin(\omega t)}{\sqrt{1-\gamma^2}}+2\gamma e^{-\gamma\omega_0 t}\cos\left(\omega t\right)-2\gamma+\omega_0 t}{\omega_0^3}
\end{aligned}
$$

Now let $t>1$. Then we have that

$$
\begin{aligned}
\int_{s\in[0,d^{-\frac{1}{2}})} G(t,s)h(s) &= \int_{s\in[0,1-d]} e^{-\gamma\omega_0(t-s)}\left(A_1\sin\left(\omega(t-s)\right)+B_1\cos\left(\omega(t-s)\right)\right)\frac{s}{1-d}\\
&+\int_{s\in[1-d,1]} e^{-\gamma\omega_0(t-s)}\left(A_1\sin\left(\omega(t-s)\right)+B_1\cos\left(\omega(t-s)\right)\right)\\
&\times\left((t-1)^2\left(\frac{1+2d(1-d)}{d^2(1-d)}t+3-\frac{2}{d}-\frac{1}{d^2}\right)\right)\\
&=_0 \frac{e^{-\gamma\omega_0 t}}{\omega\omega_0^2}\left(-2\gamma^2 e^{\gamma\omega_0}\sin\left(\omega(t-1)\right)+2\gamma^2\sin\left(\omega t\right)+\gamma\omega_0 e^{\gamma\omega_0}\sin\left(\omega(t-1)\right)\right)\\
&+\frac{e^{\gamma\omega_0}}{\omega\omega_0^2}\Big(\sin\left(\omega(t-1)\right)-\sin\left(\omega t\right)\\
&+2\sqrt{1-\gamma^2}\gamma\cos\left(\omega t\right)-\sqrt{1-\gamma^2}e^{\gamma\omega}(2\gamma-\omega_0)\cos\left(\omega(t-1)\right)\Big)
\end{aligned}
$$

Thus the real solution is given by

$$
\begin{aligned}
x(t) = {} & \frac{e^{-\gamma\omega_0 t}}{\omega\omega_0^2}\left(-2\gamma^2 e^{\gamma\omega_0}\sin\left(\omega(t-1)\right) + 2\gamma^2\sin\left(\omega t\right) + \gamma\omega_0 e^{\gamma\omega_0}\sin\left(\omega(t-1)\right)\right) \\
& + \frac{e^{\gamma\omega_0}}{\omega\omega_0^2}\Big(\sin\left(\omega(t-1)\right) - \sin\left(\omega t\right) \\
& + 2\sqrt{1-\gamma^2}\gamma\cos\left(\omega t\right) - \sqrt{1-\gamma^2}e^{\gamma\omega}(2\gamma-\omega_0)\cos\left(\omega(t-1)\right)\Big).
\end{aligned}
$$

4.4 The Delta Function in n-Dimensions

Capitalizing on the results of Chapter 3 we are able, in this section, to show that the non-Archimedean delta function behaves as expected in an arbitrary number of dimensions.

Definition 4.4.1. *Let $\delta_n : \mathcal{K}^n \to \mathcal{K}$ be given by*

$$\delta_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \delta(x_i).$$

Proposition 4.4.2. *Let $S \subset \mathcal{K}^n$ be measurable. If $\prod_{i=1}^{n}(-d, d) \subset S$ then*

$$\int_S \delta_n(x_1, \ldots, x_n) = 1.$$

If $\prod_{i=1}^{n}(-d, d) \cap S = \emptyset$ then

$$\int_S \delta_n(x_1, \ldots, x_n) = 0.$$

Proof. The proposition has already been proven for $n = 1$ and $n = 2$; so let $n > 2$ and suppose the proposition holds for the $n-1$ case. If $\prod_{i=1}^{n}(-d,d) \subset S$ then

$$\begin{aligned}
\int_S \delta_n(x_1,\ldots,x_n) &= \int_{(x_1,\ldots,x_n))\in\prod_{i=1}^{n}(-d,d)} \delta_n(x_1,\ldots,x_n) \\
&= \int_{(x_1,\ldots,x_{n-1})\in\prod_{i=1}^{n-1}(-d,d)} \left(\delta_{n-1}(x_1,\ldots,x_{n-1}) \int_{x_n\in(-d,d)} \delta(x_n)\right) \\
&= \int_{(x_1,\ldots,x_{n-1})\in\prod_{i=1}^{n-1}(-d,d)} \delta_{n-1}(x_1,\ldots,x_{n-1}) = 1
\end{aligned}$$

If $\prod_{i=1}^{n}(-d,d) \cap S = \emptyset$, then $\delta_n(x_1,\ldots,x_n) = 0$ everywhere on S; and hence

$$\int_S \delta_n(x_1,\ldots,x_n) = \int_S 0 = 0.$$

Therefore, using induction, it follows that the statement of the proposition holds for all $n \in \mathbb{N}$. □

Proposition 4.4.3. *Let $S \subset \mathcal{K}^n$ be a simple region with $\lambda_{x_i}(S) < 1$ for all $i \in \{1,\ldots,n\}$, and let $f : S \to \mathcal{K}$ be an analytic function on S with $i(f) = 0$ on S. Then, for any $(x_{1,0},\ldots,x_{n,0}) \in S$ that satisfies*

$$\prod_{i=1}^{n}(x_{i,0}-a, x_{i,0}+a) \subset S$$

for some positive $a \gg d$ in $\mathcal{K}$, we have that

$$\int_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n)\delta_n(x_1-x_{1,0},\ldots,x_n-x_{n,0}) =_0 f(x_{1,0},\ldots,x_{n,0}).$$

Proof. First we note this proposition has already been proven for the cases of $n=1$ and $n=$

2, so we let $n > 2$ and assume the proposition holds in the $n-1$ case. By definition we have that $\delta_n(x_1 - x_{1,0}, \ldots, x_n - x_{n,0}) = 0$ everywhere except on the simple region $\prod_{i=1}^{n}(x_{i,0}-d, x_{i,0}+d)$. Thus,

$$\int_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n)\delta_n(x_1 - x_{1,0},\ldots,x_n - x_{n,0})$$

$$= \int_{(x_1,\ldots,x_n)\in \prod_{i=1}^{n}(x_{i,0}-d,x_{i,0}+d)} f(x_1,\ldots,x_n)\delta_n(x_1 - x_{1,0},\ldots,x_n - x_{n,0})$$

$$= \int_{(x_1,\ldots,x_{n-1})\in \prod_{i=1}^{n-1}(x_{i,0}-d,x_{i,0}+d)} \Bigg(\delta_{n-1}(x_1 - x_{1,0},\ldots,x_{n-1} - x_{n-1,0})$$

$$\int_{x_n\in(x_{n,0}-d,x_{n,0}+d)} f(x_1,\ldots,x_n)\delta(x_n - x_{n,0}) \Bigg).$$

Now, for a fixed $(x_1, \ldots, x_{n-1}) \in \prod_{i=1}^{n-1}(x_{i,0} - d, x_{i,0} + d)$, $h(x_n) := f(x_1, \ldots, x_n)$ is an analytic function on the interval $(x_{n,0} - a, x_{n,0} + a)$ which contains $(x_{n,0} - d, x_{n,0} + d)$; and hence, by Proposition 4.1.7, we have that

$$\int_{x_n\in(x_{n,0}-d,x_{n,0}+d)} h(x_n)\delta(x_n - x_{n,0}) =_0 h(x_{n,0}) = f(x_1,\ldots,x_{n-1},x_{n,0}).$$

Furthermore, $g(x_1, \ldots, x_{n-1}) := f(x_1, \ldots, x_{n-1}, x_{n,0})$ is analytic on the simple region $S_{x_1,\ldots x_{n-1}} := \prod_{i=1}^{n-1}(x_{i,0} - a, x_{i,0} + a)$ containing $\prod_{i=1}^{n-1}(x_{i,0} - d, x_{i,0} + d)$; and hence, by our inductive hypothesis, we have that

$$\int_{(x_1,\ldots,x_{n-1})\in \prod_{i=1}^{n-1}(x_{i,0}-d,x_{i,0}+d)} g(x_1,\ldots,x_{n-1})\delta_{n-1}(x_1 - x_{1,0},\ldots,x_{n-1} - x_{n-1,0})$$

$$=_0 g(x_{1,0},\ldots,x_{n-1,0}) = f(x_{1,0},\ldots,x_{n,0}).$$

Thus,

$$\begin{aligned}
&\int\limits_{(x_1,\ldots,x_n)\in S} f(x_1,\ldots,x_n)\delta_n(x_1-x_{1,0},\ldots,x_n-x_{n,0}) \\
&= \int\limits_{(x_1,\ldots,x_{n-1})\in\prod\limits_{i=1}^{n-1}(x_{i,0}-d,x_{i,0}+d)} \Bigg(\delta_{n-1}(x_1-x_{1,0},\ldots,x_{n-1}-x_{n-1,0}) \\
&\int\limits_{x_n\in(x_{n,0}-d,x_{n,0}+d)} f(x_1,\ldots,x_n)\delta(x_n-x_{n,0})\Bigg) \\
&=_0 \int\limits_{(x_1,\ldots,x_{n-1})\in\prod\limits_{i=1}^{n-1}(x_{i,0}-d,x_{i,0}+d)} \delta_{n-1}(x_1-x_{1,0},\ldots,x_{n-1}-x_{n-1,0})f(x_1,\ldots,x_{n-1},x_{n,0}) \\
&=_0 f(x_{1,0},\ldots,x_{n,0}).
\end{aligned}$$

□

4.5 The Spherical Delta Function

In this section we investigate the possibility of employing the non-Archimedean delta function in spherical coordinates. The arguments used are somewhat less rigorous than we would like due to the lack of development on the subject of curvilinear coordinate systems. Fortunately, the mathematical theory needed here is closely related to that needed for the proof of the three conjectures from Chapter 3 and so research into either problem is research into both problems.

Remark 4.5.1. *Note that in the following discussion of the delta function in spherical coordinates we assume that the spherical volume element is the same as in the real case, that is to say*

$$\iiint\limits_{\mathbf{r}'\in\mathcal{K}^3} 1 = \int\limits_{r'\in\mathcal{K}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} r'^2\sin\theta'.$$

In fact proving this statement requires framework that has not yet been established and so is outside the scope of this discussion, for now we will take it for granted so that we may proceed.

As in the classical case it is also possible to define the delta function in spherical coordinates, in particular we have that

$$\delta_{sph}(\mathbf{r}-\mathbf{r}') = F(r,\phi,\theta)\delta(r-r',\phi-\phi',\theta-\theta')$$

where of course $\mathbf{r}$ is the point (r,ϕ,θ) and $\mathbf{r}'$ is the point (r',ϕ',θ') and F is some as yet unknown function. Naturally we must have that

$$\iiint\limits_{\mathbf{r}'\in\mathcal{K}^3} \delta_{sph}(\mathbf{r}-\mathbf{r}') = 1,$$

from which it follows that

$$\int\limits_{r'\in\mathcal{K}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} F(r',\phi',\theta')\delta(r-r',\phi-\phi',\theta-\theta')r'^2\sin\theta' = 1.$$

However, we already know that

$$\int\limits_{r'\in\mathcal{K}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} \delta(r-r',\phi-\phi',\theta-\theta') = 1$$

since $\delta(r-r',\phi-\phi',\theta-\theta')$ is normalized by definition, so we obtain

$$\int\limits_{r'\in\mathcal{K}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} \delta(r-r',\phi-\phi',\theta-\theta') = \int\limits_{r'\in\mathcal{K}^+}\int\limits_{\phi'\in[0,2\pi]}\int\limits_{\theta'\in[0,\pi]} F(r',\phi',\theta')\delta(r-r',\phi-\phi',\theta-\theta')r'^2\sin\theta'$$

and hence

$$F(r',\phi',\theta') = \frac{1}{r'^2\sin\theta'}.$$

Definition 4.5.2. *We define $\delta_{sph} : \mathcal{R}^3 \to \mathcal{R}$ by*

$$\delta_{sph}(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r', \phi - \phi', \theta - \theta')}{r^2 \sin\theta}.$$

Note that as in the classical case, if a problem has spherical symmetry then the delta function takes the form

$$\delta_{sph}(\mathbf{r} - \mathbf{r}') = F(r')\delta(r - r')$$

and since

$$\int_{\phi' \in [0,2\pi]} \int_{\theta' \in [0,\pi]} r'^2 \sin(\theta') = 4\pi r'^2$$

it follows that

$$F(r') = \frac{1}{4\pi r'^2}$$

and hence

$$\delta_{sph}(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r')}{4\pi r'^2}.$$

Example 4.5.3 (Electric Field of a thick spherical shell)**.** *Suppose we wish to find the electric field of a thick spherical shell centred at the origin with inner radius R_1, outer radius R_2 and a uniform charge density ρ_0. One way to accomplish this is to solve the differential equation implied by Gauss's law:*

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})$$

where $\mathbf{D}(\mathbf{r})$ is the electric displacement field at the point $\mathbf{r}$ and $\rho(\mathbf{r})$ is given by

$$\rho(\mathbf{r}) = \begin{cases} 0 & \text{if } r < R_1 \\ \rho_0 & \text{if } R_1 \leq r \leq R_2 \\ 0 & \text{if } r > R_2 \end{cases} \cdot$$

As in previous examples we can solve this differential equation by finding the Green's function

$\mathbf{G}(\mathbf{r},\mathbf{r'}) = G_r\mathbf{e_r} + G_\phi\mathbf{e}_\phi + G_\theta\mathbf{e}_\theta$ *corresponding to the operator* $\nabla\cdot$, *in particular* $\mathbf{G}(\mathbf{r},\mathbf{r'})$ *must satisfy*

$$\nabla \cdot \mathbf{G}(\mathbf{r},\mathbf{r'}) = \delta_{sph}(\mathbf{r}-\mathbf{r'}). \tag{4.2}$$

However our system is spherically symmetric about the origin so we may infer that

$$G_\phi = G_\theta = 0$$

and

$$\delta_{sph}(\mathbf{r}-\mathbf{r'}) = \frac{\delta(r-r')}{4\pi r^2}.$$

Thus, Equation (4.2) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 G_r(r,r')\right) = \frac{\delta(r-r')}{4\pi r^2}.$$

Solving this differential equation yields

$$G_r(r,r') = \begin{cases} \frac{c_1}{r^2} & \textit{if } r' \le r-d \\ \frac{1}{4\pi r^2}\frac{3}{4}d^{-3}\left(d^2(r-r') - \frac{1}{3}(r-r')^3\right) + \frac{c_2}{r^2} & \textit{if } r-d < r' < r+d \\ \frac{c_3}{r^2} & \textit{if } r' \ge r+d \end{cases}$$

where c_1, c_2, *and* c_3 *are constants of integration. We know that* $\mathbf{D}(\mathbf{r}) = 0$ *for* $r < R_1$ *since there is no charge inside the shell. To ensure our solution satisfies this initial condition we must have* $G_r(0,r') =_0 0$ *and as in previous examples we accomplish this by setting* $G_r(-d,r') = 0$. *Using this initial condition as well as the continuity of* G_r, *we are able to* *solve for the constants in $G_r(r,r')$; in fact we find that*

$$G_r(r,r') = \begin{cases} \frac{1}{4\pi r^2} & \text{if } r' \leqslant r-d \\ \frac{1}{4\pi r^2}\frac{3}{4}d^{-3}\left(d^2(r-r') - \frac{1}{3}(r-r')^3\right) + \frac{1}{8\pi r^2} & \text{if } r-d < r' < r+d \\ 0 & \text{if } r' \geq r+d \end{cases}$$

Now that we know the Green's function of the operator $\nabla\cdot$ and have made it satisfy the relevant boundary conditions we can solve for the (real) electric displacement field of the spherical shell by recalling that

$$\mathbf{D}(r) =_0 \int\limits_{r'\in\mathcal{K}^+}\int\limits_{\phi'\in[0,2\pi)}\int\limits_{\theta'\in[0,\pi]} G_r(r,r')\rho(r')r'^2\sin\theta'\mathbf{e_r} = 4\pi\int\limits_{r'\in\mathcal{K}^+} G_r(r,r')\rho(r')r'^2\mathbf{e_r} = D_r(r)\mathbf{e_r}.$$

If $r < R_1$ then we have

$$D_r(r) =_0 0,$$

if $R_1 \leq r \leq R_2$ then the integral reduces to

$$\begin{aligned} D_r(r) &=_0 4\pi\left(\int\limits_{r'\in[R_1,r-d]}\frac{\rho_0 r'^2}{4\pi r^2} + \int\limits_{r'\in[r-d,r+d]}\left(\frac{1}{4\pi r^2}\frac{3}{4}d^{-3}\left(d^2(r-r') - \frac{1}{3}(r-r')^3\right) + \frac{1}{8\pi r^2}\right)\rho_0 r'^2\right) \\ &=_0 \frac{\rho_0}{3r^2}\left(r^3 - R_1^3\right), \end{aligned}$$

and finally if $r > R_2$ then we get

$$D_r(r) =_0 4\pi\int\limits_{r'\in[R_1,R_2]}\frac{\rho_0 r'^2}{4\pi r^2} = \frac{\rho_0}{3r^2}\left(R_2^3 - R_1^3\right).$$

Hence the electric displacement field of a uniformly charged thick spherical shell is given by

$$D_r(r) = \begin{cases} 0 & \textit{if } r < R_1 \\ \frac{\rho_0}{3r^2}\left(r^3 - R_1^3\right) & \textit{if } R_1 \leq r \leq R_2 \\ \frac{\rho_0}{3r^2}\left(R_2^3 - R_1^3\right) & \textit{if } r > R_2 \end{cases} .$$

While the examples given in this chapter are admittedly simple they serve to illustrate how to use the newly defined delta functions. In the future we hope to engage in a more detailed study of the delta functions in two and three dimensions as well as in constructing more complex and challenging examples.

Chapter 5

Computational Applications of the Levi-Civita Field

5.1 Computation and the Levi-Civita Field

Non-Archimedean valued fields have applications in computing the limits and asymptotic behaviour of analytic functions; the seminal work seems to be that of Lightstone and Robinson who for example are able to compute the incomplete factorial function to a high degree of precision by summing a finite number of terms of a divergent series [25]. Another value of the Levi-Civita field from the perspective of computational applications is that it allows one to compute limits of real valued functions directly rather than by approximation. For example given a differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$, the derivative of f at some point $x_0 \in I$ is given by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h}.$$

To compute this conventionally one would either use symbolic manipulation to reduce the fraction on the right in a way that makes all terms with h in the denominator vanish or one would choose h to be small enough that the error caused by its inclusion is less than some predetermined tolerance. The first possibility, while it produces accurate results, is unsatisfactory firstly because symbolic manipulation is computationally slower than numerical calculations and secondly because it is not always clear what manipulations are necessary to produce the desired result. The second approach retains the speed of numerical computation but suffers from the issue that it is often difficult to determine how small h must be to ensure the result is sufficiently precise. Moreover the numerical method is highly susceptible to rounding errors. It has been shown [4] that by employing the Levi-Civita field all of these issues can be addressed; we will discuss this further in Section 5.3. Our purpose in this chapter is to investigate new computational applications of the Levi-Civita field; in the course of our investigation, we will also have the opportunity to compare our results with those from [4].

5.2 The Tulliotools Software

To be able to use the Levi-Civita field in computational applications we will first construct a code that will allow a computer to operate on these numbers. The code that has been used in previous papers on this topic (COSY ININITY) [4] is not easily accessible and so we construct our own software for this purpose. Our code forms a static library in the C++ programming language and we tentatively name it Tulliotools in honour of the Italian mathematician Tullio Levi-Civita who first discovered the field that bears his name [26]. Tulliotools was created using **Microsoft Visual Studio 2015 Community Edition** and was compiled using default setting. The Tulliotools library defines how a computer can store an element of the Levi-Civita field (up to some specific depth) and defines the operations of addition, multiplication, and inversion. Subtraction and division are defined by addition of the negative and multiplication with the inverse, respectively. Tulliotools also includes the basic trigonometric and inverse trigonometric functions, the hyperbolic trigonometric functions, the exponential function, the natural logarithm, and the n^{th} root for an arbitrary integer n. Addition and multiplication are easily defined in accordance with Definition 1.3.7;

however, calculating the inverse of an element is more difficult and we explain the process we use in the following remark.

Remark 5.2.1 (Process for calculating the inverse of an element)**.** *Suppose* $x \in \mathcal{R}\setminus\{0\}$ *and suppose we wish to find the element* x^{-1}*. Let* $\lambda_0 = \lambda(x)$ *and suppose that* $x[\lambda_0] = a$*, then we can write* $x = ad^{\lambda_0}(1+s)$ *where* $s \in \mathcal{R}$ *with* $|s| \ll 1$*. Clearly we have that*

$$\frac{1}{x} = \frac{1}{ad^{\lambda_0}(1+s)} = \frac{1}{a}d^{-\lambda_0}\frac{1}{1+s}.$$

The factor of $\frac{1}{a}d^{-\lambda_0}$ *can be computed directly and we have that*

$$\frac{1}{1+s} = \sum_{n=0}^{\infty}(-1)^n s^n$$

which converges in the order topology. Thus,

$$\frac{1}{x} = \frac{1}{a}d^{-\lambda_0}\sum_{n=0}^{\infty}(-1)^n s^n.$$

Of course we cannot compute all infinitely many terms of the sum; however, if we let $\lambda_1 = \lambda(s)$ *and if we store elements of* $\mathcal{R}$ *up to a depth* $q \in \mathbb{Q}$ *then, because we know that* $\lambda_1 > 0$*, we have that*

$$\frac{1}{1+s} =_q \sum_{n=0}^{\lceil \frac{q}{\lambda_1} \rceil}(-1)^n s^n.$$

This yields a finite expression

$$\frac{1}{x} =_q \frac{1}{a}d^{-\lambda_0}\sum_{n=0}^{\lceil \frac{q}{\lambda_1} \rceil}(-1)^n s^n$$

which defines the process of inversion used in Tulliotools.

When computing elementary functions we wish to employ the language's built-in functions as much as possible both for the sake of speed and accuracy. For clarity in our explanation below, we use the convention that for some real analytic function f, $f_{lc} : \mathcal{R} \to \mathcal{R}$ is the non-Archimedean continuation of the function, $f_r : \mathbb{R} \to \mathbb{R}$ is the real (built-in) function, and $f_t : \mathcal{R} \to \mathcal{R}$ is the Taylor expansion (up to whatever depth is required) of the function which we use to compute the infinitesimal part.

Remark 5.2.2 (Methods for computing elementary functions)**.** *Let* $x \in \mathcal{R}$ *be such that* $\lambda(x) \geq 0$ *and let* $x_r = x[0]$ *and* $x_i = x - x[0]$*. Then we have*

- $\sin_{lc}(x) = \sin_{lc}(x_r + x_i) = \sin_r(x_r)\cos_t(x_i) + \cos_r(x_r)\sin_t(x_i)$
- $\cos_{lc}(x) = \cos_r(x_r)\cos_t(x_i) - \sin_r(x_r)\sin_t(x_i)$
- $\tan_{lc}(x) = \frac{\sin_{lc}(x)}{\cos_{lc}(x)}$
- $\sinh_{lc}(x) = \sinh_r(x_r)\cosh_t(x_i) + \cosh_r(x_r)\sinh_t(x_i)$
- $\cosh_{lc}(x) = \cosh_r(x_r)\cosh_t(x_i) + \sinh_r(x_r)\sinh_t(x_i)$
- $\tanh_{lc}(x) = \frac{\sinh_{lc}(x)}{\cosh_{lc}(x)}$
- $\exp_{lc}(x) = \exp_r(x_r)\exp_t(x_i)$
- $\ln_{lc}(x) = \ln_r(x_r) + \ln_t(1 + \frac{x_i}{x_r})$

In the following, we will make use of the exponential function and natural logarithm to compute the n^{th} root of a positive element of $\mathcal{R}$.

Remark 5.2.3 (Process for computing arbitrary roots)**.** *Let* $x \in \mathcal{R}$ *with* $x > 0$*, let* $n \in \mathbb{N}$ *and suppose we wish to find* $x^{\frac{1}{n}}$*. To begin, let* $\lambda_0 = \lambda(x)$ *then we have that* $x = d^{\lambda_0}s$ *where* $s \in \mathcal{R}$ *with* $s > 0$ *and* $\lambda(s) = 0$*. It follows that* $x^{\frac{1}{n}} = d^{\frac{\lambda_0}{n}} s^{\frac{1}{n}}$*; thus we have reduced the problem to finding a* $y \in \mathcal{R}$ *such that* $y^n = s$*. Taking the natural logarithm of both sides yields*

$$\ln(y^n) = \ln(s);$$

from which we obtain that

$$\ln(y) = \frac{\ln(s)}{n}.$$

Thus, we have that

$$y = \exp(\ln(y)) = \exp\left(\frac{\ln(s)}{n}\right)$$

and hence we arrive at our desired equation

$$x^{\frac{1}{n}} = d^{\frac{\lambda_0}{n}} e^{\frac{\ln(s)}{n}}.$$

Computing the inverse trigonometric functions is more difficult than their trigonometric counterparts because they lack convenient additive angle formulas. Instead we make use of integration (which we discuss in a later section) and the fact that the derivatives of these functions are well known.

Remark 5.2.4 (Method for computing the inverse trigonometric functions)**.** *Let $x \in (-1, 1) \subset \mathcal{R}$ and let $y \in \mathcal{R}$ satisfy $\lambda(y) \geq 0$; let $x_r = x[0]$ and $y_r = y[0]$; and let $x_i = x - x_r$ and $y_i = y - y_r$. Then we have that*

- $\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ *and it follows that if $x_i \geq 0$ then*

$$\begin{aligned}
\arcsin_{lc}(x) &= \int\limits_{t\in(0,x)} \frac{1}{\sqrt{1-t^2}} \\
&= \int\limits_{t\in(0,x_r)} \frac{1}{\sqrt{1-t^2}} + \int\limits_{t\in(x_r,x_r+x_i)} \frac{1}{\sqrt{1-t^2}} \\
&= \arcsin_r(x_r) + \int\limits_{t\in(x_r,x_r+x_i)} \frac{1}{\sqrt{1-t^2}}
\end{aligned}$$

and if $x_i < 0$ *then*

$$\begin{aligned}
\arcsin_{lc}(x) &= \int\limits_{t\in(0,x)} \frac{1}{\sqrt{1-t^2}} \\
&= \int\limits_{t\in(0,x_r)} \frac{1}{\sqrt{1-t^2}} - \int\limits_{t\in(x_r+x_i,x_r)} \frac{1}{\sqrt{1-t^2}} \\
&= \arcsin_r(x_r) - \int\limits_{t\in(x_r+x_i,x_r)} \frac{1}{\sqrt{1-t^2}}
\end{aligned}$$

- $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$ *and it follows that if* $x_i \geq 0$ *then*

$$\begin{aligned}
\arccos_{lc}(x) &= \int\limits_{t\in(x,1)} \frac{1}{\sqrt{1-t^2}} \\
&= \int\limits_{t\in(x_r,1)} \frac{1}{\sqrt{1-t^2}} - \int\limits_{t\in(x_r,x_r+x_i)} \frac{1}{\sqrt{1-t^2}} \\
&= \arccos_r(x_r) - \int\limits_{t\in(x_r,x_r+x_i)} \frac{1}{\sqrt{1-t^2}}
\end{aligned}$$

and if $x_i < 0$ *then*

$$\begin{aligned}
\arccos_{lc}(x) &= \int\limits_{t\in(x,1)} \frac{1}{\sqrt{1-t^2}} \\
&= \int\limits_{t\in(x_r,1)} \frac{1}{\sqrt{1-t^2}} + \int\limits_{t\in(x_r+x_i,x_r)} \frac{1}{\sqrt{1-t^2}} \\
&= \arccos_r(x_r) + \int\limits_{t\in(x_r+x_i,x_r)} \frac{1}{\sqrt{1-t^2}}
\end{aligned}$$

- $\frac{d}{dy}\arctan(y) = \frac{1}{y^2+1}$ *and it follows that if* $y_i \geq 0$ *then*

$$\begin{aligned}\arctan_{lc}(y) &= \int\limits_{t\in(0,y)} \frac{1}{t^2+1} \\ &= \int\limits_{t\in(0,y_r)} \frac{1}{t^2+1} + \int\limits_{t\in(y_r,y_r+y_i)} \frac{1}{t^2+1} \\ &= \arctan_r(y_r) + \int\limits_{t\in(y_r,y_r+y_i)} \frac{1}{t^2+1}\end{aligned}$$

and if $y_i < 0$ *then*

$$\begin{aligned}\arctan_{lc}(y) &= \int\limits_{t\in(0,y)} \frac{1}{t^2+1} \\ &= \int\limits_{t\in(0,y_r)} \frac{1}{t^2+1} - \int\limits_{t\in(y_r+y_i,y_r)} \frac{1}{t^2+1} \\ &= \arctan_r(y_r) - \int\limits_{t\in(y_r+y_i,y_r)} \frac{1}{t^2+1}\end{aligned}$$

Notice that the only integrals that we actually need to compute are all over an infinitesimal interval, this allows us to compute them exactly (up to a given depth) by integrating the Taylor series of the integrand. We discuss this more thoroughly in a later section.

5.3 Numerical Computation of Derivatives

The first thing we would like to do with our newly developed library is to explore the applications to the numerical computation of derivatives developed in [4] and [27]. We begin by restating a number of definitions in our own notation and reviewing the underlying mathematical theory.

Definition 5.3.1 (Computer Function)**.** *Let* I *be the set of all functions intrinsic to the C++ programming language as well as their inverse functions and the step function* $s : \mathbb{R} \to \mathbb{R}$

defined by

$$s(x) := \begin{cases} 0 & \textit{if } x \leq 0 \\ 1 & \textit{if } x > 0 \end{cases}.$$

We define a ***computer function*** *to be any function that can be obtained preforming a finite number of arithmetic operations and compositions using functions in I.*

Because the functions intrinsic to C++ as well as the step function are real valued it follows that all computer functions must be real valued. It is possible to extend computer functions to $\mathcal{R}$ using the extensions of power series with purely real coefficients; the step function, $x^{\frac{1}{n}}$, and $\frac{1}{x}$ defined below [27].

Definition 5.3.2 (Continuation of certain real valued functions to $\mathcal{R}$)**.** *$x^{\frac{1}{n}}$ and $\frac{1}{x}$ can be continued to $\mathcal{R}$ by the existence of roots and multiplicative inverses of non-zero elements in $\mathcal{R}$. We define the continuation of s to $x \in \mathcal{R}$ by*

$$\bar{s}(x) := \begin{cases} 0 & \textit{if } x \leq 0 \\ 1 & \textit{if } x > 0 \end{cases}.$$

Let $\sum_{n=0}^{\infty} a_n X^n$ be a real power series (i.e. $X \in \mathbb{R}$ and for all $n \in \mathbb{N}$, $a_n \in \mathbb{R}$) with classical radius of convergence $\eta > 0$. Then for $x \in \mathcal{R}$ with $x < \eta$ and $x[0] \neq \eta$ the series

$$\sum_{n=0}^{\infty} a_n x^n$$

must converge in the weak topology ([27]) and we define this to be the ***continuation of the real power series to*** *$\mathcal{R}$.*

Definition 5.3.3 (Extendable computer function)**.** *Let $f \in I$, let $x_0 \in \mathbb{R}$ be in the domain of f, and let $x \in \mathcal{R}$. Then we say that f is extendable to $x_0 + x \in \mathcal{R}$ if $x_0 + x$ is in the domain of the extension of f to $\mathcal{R}$ obtained via the above definition. Let $f_1, f_2 \in I$, let $x_0 \in \mathbb{R}$*

be in the domain of both f_1 and f_2, and let $x \in \mathcal{R}$. Then we say that $f_1 + f_2$ and $f_1 \cdot f_2$ are extendable to $x_0 + x$ if $x_0 + x$ is in the domain of both the continuation of f_1 and the continuation of f_2, and we say that $f_2 \circ f_1$ is extendable to $x_0 + x$ if $x_0 + x$ is in the domain of the continuation of f_1 and $f_1(x_0 + x)$ is in the domain of the continuation of f_2. Since all computer functions are obtained through finite combinations of functions in I we may define extendability for general computer functions inductively.

Definition 5.3.4 (Continuation of computer functions to $\mathcal{R}$)**.** *It is shown in [27] that if f is a real computer function, x_0 is in the domain of f, and f is extendable to $x_0 \pm d$ then there is a $\eta > 0$ in $\mathbb{R}$ such that for $x \in \mathbb{R}$ with $0 < x < \eta$*

$$f(x_0 \pm x) = \sum_{i=0}^{\infty} a_i^{\pm} x^i + \sum_{j=1}^{j^{\pm}} x^{q_j^{\pm}} R_j^{\pm}(x) \tag{5.1}$$

where for all $j \in \{1, \ldots, j^{\pm}\}$, $R_j^{\pm}$ is a power series with $R_j^{\pm}(0) \neq 0$ and with a radius of convergence at least as large as η, and $q_j^{\pm}$ are nonzero rational numbers that are not positive integers. Since the right hand side of Equation 5.1 contains only roots, negative integer powers, and power series (for which we have already defined a continuation to $\mathcal{R}$) we may define the continuation of f to $x_0 + x \in \mathcal{R}$ such that $0 < x < \eta$ and $x[0] \neq \eta$ by

$$\bar{f}(x_0 \pm x) := \sum_{i=0}^{\infty} a_i^{\pm} x^i + \sum_{j=1}^{j^{\pm}} x^{q_j^{\pm}} \bar{R}_j^{\pm}(x)$$

where $\bar{R}_j^{\pm}$ is the continuation of $R_j^{\pm}$ to $\mathcal{R}$.

Now suppose that f is a Real computer function defined at $x_0 \in \mathbb{R}$ and extendable to $x_0 \pm d$. Then we have that

$$\bar{f}(x_0 \pm d) = \sum_{i=0}^{\infty} a_i^{\pm} d^i + \sum_{j=1}^{j^{\pm}} d^{q_j^{\pm}} \bar{R}_j^{\pm}(d).$$

The equation above entails that for any $n \in \mathbb{N}$

$$(\bar{f}(x_0 \pm d))[n] = \left(\sum_{i=0}^{\infty} a_i^{\pm} d^i + \sum_{j=1}^{i^{\pm}} d^{q_j^{\pm}} \bar{R}_j^{\pm}(d) \right) [n]$$

However if it happens that for some $m \in \mathbb{N}$, f is m times differentiable at x_0, then we must have that $q_j^{\pm} > m$ and for every $j \in \{1, \ldots, j^{\pm}\}$, $a_i^+ = (-1)^i a_i^- = \frac{f^{(i)}(x_0)}{i!}$ [27]. Hence we have that

$$\bar{f}(x_0 + d) =_m \sum_{i=0}^{m} a_i^+ d^i$$

and

$$\bar{f}(x_0 - d) =_m \sum_{i=0}^{m} a_i^- d^i$$

with $a_i^- = (-1)^i a_i^+$ in which case we have that for any $i \in \{1, \ldots, m\}$

$$i! a_i^+ = i! \bar{f}(x_0 + d)[i] = f^{(i)}(x_0) = (-1)^i i! \bar{f}(x_0 - d)[i] = (-1)^i i! a_i^-.$$

In fact it is possible to make an argument similar to the above but in the opposite direction which allows for the following theorem from [27].

Theorem 5.3.5. *Let f be a computer function that is continuous at x_0 and extendable to $x_0 \pm d$. Then f is m times differentiable at x_0 if and only if*

$$\bar{f}(x_0 + d) =_m \sum_{i=0}^{m} a_i^+ d^i,$$

and

$$\bar{f}(x_0 - d) =_m \sum_{i=0}^{m} a_i^- d^i$$

with $a_i^- = (-1)^i a_i^+$. Moreover, in this case

$$i! a_i^+ = i! \bar{f}(x_0 + d)[i] = f^{(i)}(x_0) = (-1)^i i! \bar{f}(x_0 - d)[i] = (-1)^i i! a_i^-.$$

for all $i \in \{1, \ldots, m\}$.

Theorem 5.3.5 gives us an easy way both to check the differentiability of and numerically compute the derivatives of real computer functions and it was used to great effect in [4]. Below we replicate the success of that paper using the Tulliotools library and we produce some additional examples. As in that paper, we compare our results against **Wolfram Mathematica 11.3**; we also use the **SymbolicC++3** library which is similar to Tulliotools in its implementation (both are C++ libraries). It is worth noting that, by the nature of the software, Tulliotools computes all (up to a given depth) derivatives simultaneously whereas Mathematica computes them each individually. SymbolicC++3 can be made to use either method and we make it similar to Tulliotools. This difference has no significant effect on our conclusions, however, because Mathematica takes significantly longer than Tulliotools to compute higher order derivatives. Even if we generously assume that Mathematica could, in the time it takes to compute the n^{th} derivative, compute the first $(n-1)$ derivatives as well the above method still easily out-performs it. Indeed for the sake of time we aborted Mathematica's calculations wherever they lasted for longer than an hour, these cases are denoted by the word "aborted" in the tables below. We use three functions to test this method of differentiation and we find the derivatives of each at three different points. The time to compute the results was found for Mathematica using the built-in **Absolutetiming** function and for Tulliotools and SymbolicC++3 using the **Chrono** libraries **high precision clock** function (which automatically uses the highest precision clock available on the given system). We use the following three functions to test the relative ability of our software to compute derivatives.

$$f(x) := e^{x^2 - x + 2}$$

$$g(x) := \frac{\sin(\sin(\sin(\sin(\sin(x)))))}{\cos(\cos(\cos(\cos(\cos(x)))))}$$

$$h(x) := \frac{\sin\left(x^3+2x+1\right) + \frac{3+\cos(\sin(\ln|1+x|))}{\exp\left(\tanh\left(\sinh\left(\cosh\left(\frac{\sin(\cos(\tan(\exp(x))))}{\cos(\sin(\exp(\tan(x+2))))}\right)\right)\right)\right)}}{2+\sin\left(\sinh\left(\cos\left(\arctan\left(\ln\left(\exp x + x^2 + 3\right)\right)\right)\right)\right)}$$

The function f was chosen to be simple enough that the derivatives could (at least in principle) be verified by hand. A careful examination of the following tables will show that for such simple functions Mathematica outperforms Tulliotools although both produce the correct values up to at least the 149^{th} derivative. SymbolicC++ failed to find anything beyond the 8^{th} derivative and took several orders of magnitude longer than the other two.

Table 5.1: Select derivatives of f as computed by Mathematica

n	$f^{(n)}(0)$	Time (sec.)	$f^{(n)}(1)$	Time (sec.)	$f^{(n)}(2)$	Time (sec.)
0	7.38905609893	0.001	7.38905609893	0.0001	54.5981500331	0.0003
1	-7.38905609893	0.001	7.38905609893	0.0002	163.794450099	0.0003
2	22.1671682968	0.0004	22.1671682968	0.0002	600.579650365	0.0002
3	-51.7233926925	0.0005	51.7233926925	0.0003	2456.91675149	0.0002
4	184.726402473	0.0005	184.726402473	0.0002	10974.2281567	0.0002
5	-598.513544013	0.0006	598.513544013	0.0002	52578.0184819	0.0002
6	2445.77756875	0.0007	2445.77756875	0.0002	267476.337012	0.0002
7	-9627.94009691	0.0008	9627.94009691	0.0002	1433365.23282	0.0002
8	43868.8260594	0.0009	43868.8260594	0.0003	8044764.41663	0.0003
9	-197915.867610	0.001	197915.867610	0.0003	47068136.9750	0.0004
10	987554.736678	0.004	987554.736678	0.0003	286010170.424	0.0004
11	-4945872.08888	0.001	4945872.08888	0.0003	1.799393250.77	0.0003
12	26672076.2958	0.006	26672076.2958	0.0003	11690403501.7	0.0003
13	-145373006.429	0.001	145373006.429	0.0003	78256648523.6	0.0003
14	838846990.119	0.001	838846990.119	0.0003	538720436614	0.0003
50	$9.74333610738\times10^{41}$	0.014	$9.74333610738\times10^{41}$	0.001	$6.44014243326\times10^{46}$	0.004
100	$1.20037473498\times10^{97}$	0.014	$1.20037473498\times10^{97}$	0.001	$4.85652586772\times10^{103}$	0.002
149	$-2.47164879131\times10^{156}$	0.034	$2.47164879131\times10^{156}$	0.003	$2.23779844382\times10^{164}$	0.005

Table 5.2: Select derivatives of f as computed by Tulliotools

n	$f^{(n)}(0)$	$f^{(n)}(1)$	$f^{(n)}(2)$
0	7.38905609893	7.38905609893	54.5981500331
1	-7.38905609893	7.38905609893	163.794450099
2	22.1671682968	22.1671682968	600.579650365
3	-51.7233926925	51.7233926925	2456.91675149
4	184.726402473	184.726402473	10974.2281567
5	-598.513544013	598.513544013	52578.0184819
6	2445.77756875	2445.77756875	267476.337012
7	-9627.94009691	9627.94009691	1433365.23282
8	43868.8260594	43868.8260594	8044764.41663
9	-197915.86761	197915.86761	47068136.975
10	987554.736678	987554.736678	286010170.424
11	-4945872.08888	4945872.08888	1799393250.77
12	26672076.2958	26672076.2958	11690403501.7
13	-145373006.429	45104788.8251	24340676103.1
14	838846990.119	12306318.9474	8332806653.96
50	$9.74333610738 \times 10^{41}$	$9.74333610738 \times 10^{41}$	$6.44014243326 \times 10^{46}$
100	$1.20037473498 \times 10^{97}$	$1.20037473498 \times 10^{97}$	$4.85652586772 \times 10^{103}$
149	$-2.47164879131 \times 10^{156}$	$2.47164879131 \times 10^{156}$	$2.23779844382 \times 10^{164}$
Time (sec.)	0.597	0.431	0.437

Table 5.3: Select derivatives of f as computed by SymbolicC++

n	$f^{(n)}(0)$)	$f^{(n)}(1)$	$f^{(n)}(2)$
0	7.38905609893	7.38905609893	54.5981500331
1	-7.38905609893	7.38905609893	163.794450099
2	22.1671682968	22.1671682968	600.579650365
3	-51.7233926925	51.7233926925	2456.91675149
4	184.726402473	184.726402473	10974.2281567
5	-598.513544013	598.513544013	52578.0184819
6	2445.77756875	2445.77756875	267476.337012
7	-9627.94009691	9627.94009691	1433365.23282
8	43868.8260594	43868.8260594	8044764.416633637
Time (sec.)	286.142	264.948	265.715

The function g provides us with an intermediate challenge and already we can see Mathematica falling behind Tulliotools for higher order derivatives.

Table 5.4: First 13 derivatives of g as computed by Mathematica

n	$g^{(n)}(0)$	Time (sec.)	$g^{(n)}(1)$	Time (sec.)	$g^{(n)}(2)$	Time (sec.)
0	0	0.0002	0.837192955627	0.308	0.888584820075	0.196
1	1.26027064058	0.002	0.407172848084	0.054	-0.317934898588	0.009
2	0	0.003	-0.618746127149	0.014	-0.651895577342	0.014
3	-5.35211351959	0.03	0.0122192107521	0.062	0.416693615024	0.109
4	0	0.112	-4.31613114141	0.171	-1.64786996410	0.178
5	121.167674235	0.329	15.652	0.542463	-19.6728802712	0.838
6	0	0.953	78.5779028176	1.747	-20.2596967220	2.106
7	-5627.09443507	3.0960	-685.282937503	5.835	615.708023511	6.997
8	0	10.691	-1285.70479011	19.589	2622.42370100	21.9916
9	429913.385688	32.896	16481.3309024	57.559	-30298.4169665	61.748
10	0	91.505	227724.788971	153.641	-114129.369772	161.231
11	-49831093.1255	238.66	-257502.130656	397.697	3525304.24927	399.671
12	0	583.886	aborted	>1 hour	aborted	>1 hour
13	8083947834.90	1623.43	aborted	>1 hour	aborted	>1 hour

Table 5.5: First 14 derivatives of g as computed by Tulliotools

n	$g^{(n)}(0)$	$g^{(n)}(1)$	$g^{(n)}(2)$
0	0	0.837192955627	0.888584820075
1	1.26027064058	0.407172848084	-0.317934898588
2	0	-0.618746127149	-0.651895577342
3	-5.35211351959	0.0122192107521	0.416693615024
4	0	-4.31613114141	-1.6478699641
5	121.167674235	15.6517446222	-19.6728802712
6	0	78.5779028176	-20.259696722
7	-5627.09443507	-685.282937503	615.708023511
8	0	-1285.70479011	2622.423701
9	429913.385688	16481.3309024	-30298.4169665
10	0	227724.788971	-114129.369772
11	-49831093.1255	-257502.130656	3525304.24927
12	0	-37912424.234	4688958.30662
13	8083947834.9	-13666350.8705	-495861347.515
14	0	4734886537.81	-152712264.273
Total (sec.)	0.207	0.193	0.218

Finally we have function h which was obtained from [4]. Here Mathematica is slower than Tulliotools even for relatively low order derivatives and it was unable to find any derivatives past the eighth in less than an hour. SymbolicC++ unfortunately is unable to work with this function at all because it does not have the inverse trigonometric functions implemented in it.

Table 5.6: First 8 derivatives of h as computed by Mathematica

n	$h^{(n)}(0)$	Time (sec.)	$h^{(n)}(1)$	Time (sec.)	$h^{(n)}(5)$	Time (sec.)
0	1.00484531901	0.16	0.268357844508	0.10	0.283393816437	0.30
1	0.460143808963	0.15	-1.44525348415	0.04	12.1382777290	0.14
2	-5.26609756823	0.30	7.31608659872	0.14	28594.4371105	0.13
3	-52.8216335199	0.70	40.8666551717	0.32	10161444.9755	0.31
4	-108.468284784	1.67	404.249076373	1.13	-32567374548.9	1.59
5	16451.4428641	4.73	-5092.63654924	4.53	$-1.29110802579\times10^{14}$	5.70
6	541334.997022	21.11	-19854.7155232	28.96	$-2.98281735849\times10^{17}$	35.47
7	794864118.936	124.35	1611673.41227	171.20	$-4.20384900033\times10^{20}$	184.15
8	-144969388.210	787.34	-86895133.1031	2426.91	aborted	>1 hour

Table 5.7: First 14 derivatives of h as computed by Tulliotools

n	$h^{(n)}(0)$	$h^{(n)}(1)$	$h^{(n)}(5)$
0	1.00484531901	0.268357844508	0.283393816437
1	0.460143808963	-1.44525348415	12.138277729
2	-5.26609756823	7.31608659872	28594.4371105
3	-52.8216335199	40.8666551717	10161444.9755
4	-108.468284784	404.249076373	-32567374548.9
5	16451.4428641	-5092.63654924	$-1.29110802579\times10^{14}$
6	541334.997022	-19854.7155232	$-2.98281735849\times10^{17}$
7	7948641.18936	1611673.41227	$-4.20384900033\times10^{20}$
8	-144969388.21	-86895133.1031	$2.78479886876\times10^{23}$
9	-15395959663	3193445289.11	$4.77510276588\times10^{27}$
10	-618406836695	-90967229524	$2.1329279112\times10^{31}$
11	$-1.17903146156\times10^{13}$	$1.74199571026\times10^{12}$	$6.24639715614\times10^{34}$
12	$4.03355397865\times10^{14}$	$1.49155784151\times10^{13}$	$9.55133940595\times10^{37}$
13	$5.51065265978\times10^{16}$	$-3.85982238753\times10^{15}$	$-2.68590144823\times10^{41}$
14	$3.27278740268\times10^{18}$	$2.59042564116\times10^{17}$	$-3.11629245228\times10^{45}$
Time (sec.)	0.826	0.505	0.580

5.4 Numerical Computation of Bernoulli Numbers

Evaluating analytic functions with real coefficients at infinitesimal points can do more than finding the derivatives of the function, it also allows us to calculate sequences of numbers defined by a generating function. Consider for example the following definition of the Bernoulli numbers.

Definition 5.4.1 (Bernoulli numbers)**.** *The* ***Bernoulli numbers*** *are precisely those numbers $B_n \in \mathbb{R}$ such that for any $t \in \mathbb{R}$*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

This is a perfectly rigorous definition but it does not (in the field of real numbers) provide an obvious way to calculate the value of each term. Typically the Bernoulli numbers are instead calculated using either the summation formula or the recursive formula.

$$B_n = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{j^n}{i+1}$$

$$B_n = \delta_{n,0} - \sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i}{n-i+1}$$

Using the Levi-Civita numbers however, we are able to calculate the Bernoulli numbers directly from the generating function. Notice that

$$\left(\frac{d}{e^d - 1}\right) = \sum_{n=0}^{\infty} \frac{B_n}{n!} d^n$$

from which we find that

$$B_n = \left(\frac{d}{e^d - 1}\right)[n]n!.$$

In fact there are a number of different ways we can calculate the Bernoulli numbers along the same lines. Recall that for $t \in \mathcal{R}$ with $|t| < \frac{\pi}{2}$ we have

$$\tan(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} t^{2n-1}$$

this entails that

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n}(2^{2n}-1)} (\tan(d)) [2n-1]$$

Additionally, for $t \in \mathcal{R}$ with $|t| < \frac{\pi}{2}$, we have that

$$\tanh(t) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} t^{2n-1}$$

and hence

$$B_{2n} = \frac{(2n)!}{2^{2n}(2^{2n}-1)} (\tanh(d)) [2n-1].$$

One might object that these equations only allow the calculation of every other Bernoulli number; however, aside from $B_1 = -\frac{1}{2}$, every odd Bernoulli number is zero.

Table 5.8: Bernoulli Numbers computed in various ways

n	Additive Formula	Recursive Formula	Generating Function	tan Formula	tanh Formula	Exact (to six decimals)
0	1	1	1	—	—	1
1	-0.5	-0.5	-0.5	—	—	-0.5
2	0.166667	0.166667	0.166667	0.166667	0.166667	0.166667
3	0	0	0	0	0	0
4	-0.0333333	-0.0333333	-0.0333333	-0.0333333	-0.0333333	-0.0333333
5	0	0	0	0	0	0
6	0.0238095	0.0238095	0.0238095	0.0238095	0.0238095	0.0238095
7	2.23517×10^{08}	0	-3.14748×10^{12}	0	0	0
8	0.0757571	0.0757576	0.0757576	0.0757576	0.0757576	0.0757576
9	3.8147×10^{06}	-6.10623×10^{16}	-1.73112×10^{11}	0	0	0
10	-0.252197	-0.253114	-0.253114	-0.253114	-0.253114	-0.253114
11	-0.0078125	1.11022×10^{15}	2.70054×10^{09}	0	0	0
12	1.125	1.16667	1.16667	1.16667	1.16667	1.16667
13	-32	-3.73035×10^{14}	1.84312×10^{06}	0	0	0
14	-256	-7.09216	-7.09216	-7.09216	-7.09216	-7.09216
15	98304	1.42109×10^{14}	-0.000173537	0	0	0
16	-3.93216×10^{06}	54.9712	54.966	54.9712	54.9712	54.9712
17	1.24151×10^{09}	-3.18323×10^{12}	0.0783084	0	0	0
18	7.62357×10^{10}	-529.124	-530.399	-529.124	-529.124	-529.124
19	3.43597×10^{11}	2.09184×10^{11}	-1.38482	0	0	0
20	-3.40849×10^{14}	6192.12	5976.59	6192.12	6192.12	6192.12
21	1.28071×10^{16}	-3.7835×10^{10}	98.5388	0	0	0
22	-1.97258×10^{18}	-86580.3	80473.3	-86580.3	-86580.3	-86580.3
23	-1.29704×10^{20}	4.42378×10^{09}	-3.88571×10^{06}	0	0	0
24	5.27577×10^{21}	1.42552×10^{06}	1.41551×10^{07}	1.42552×10^{06}	1.42552×10^{06}	1.42552×10^{06}
25	1.01531×10^{23}	-7.82311×10^{08}	5.59158×10^{08}	0	0	0
26	-7.20822×10^{25}	-2.72982×10^{07}	1.14962×10^{10}	-2.72982×10^{07}	-2.72982×10^{07}	-2.72982×10^{07}
27	6.50886×10^{27}	1.3113×10^{06}	-3.67793×10^{11}	0	0	0
28	-9.06172×10^{29}	6.01581×10^{08}	6.06192×10^{12}	6.01581×10^{08}	6.01581×10^{08}	6.01581×10^{08}
29	-4.69823×10^{31}	5.72205×10^{06}	-5.40877×10^{13}	0	0	0

Even a cursory glance at Table 5.8 will show that the first three methods displayed produce incorrect values for the odd Bernoulli numbers. This is caused by “rounding error” an artifact of the fact that decimal numbers are only stored in memory up to a specific number of significant digits. For example in our case each decimal number is stored with 52 significant binary digits (the maximum allowed by Microsoft Visual Studio) which translates to 15 to 17 significant decimal digits.

Example 5.4.2 (Rounding error)**.** *Suppose we wish to sum the numbers* 1010.5*,* -1010.0*,* -0.5*, clearly the sum is zero. Suppose however that we are using a program which only stores decimal numbers up to three digits of precision. Then in the computers memory*

- 1010.5 *is stored as* 1.01×10^3
- -1010.0 *is stored as* -1.01×10^3
- -0.5 *is stored as* -5.00×10^{-1}

so when the program sums these numbers it computes the result to be -0.5 *because the trailing digit in the first term was erased. This effect is called* ***rounding error****.*

Rounding error is most problematic when summing many different numbers of differing orders of magnitude, this explains why the summation formula experiences the worst error of all five methods discussed above. In Tulliotools the greatest source of rounding error is the Taylor expansion used in the inversion process, this explains the poor performance of the generating function method when compared with the tan and tanh methods. The tan and tanh methods have the greatest precision of the five methods showing no discernible rounding error for the first 29 Bernoulli numbers. The problem of rounding errors can be overcome entirely with the use of a so-called “arbitrary precision library” and it would be interesting to compare the computation time of these methods with the use of such a library; however, no such software is currently available for use with Microsoft Visual Studio and so further investigation will have to wait.

5.5 Methods of Numerical Integration

The ability to compute high-order derivatives of analytic functions allows for some interesting strategies for numerical integration, in particular we have **Darboux's Formula** which we shall see allows us to approximate an integral using our knowledge of the integrand and its derivatives at the end points of the interval of integration [28].

Proposition 5.5.1 (Darboux's Formula for the Hahn and Levi-Civita fields)**.** *Let $a, b \in \mathcal{K}$ satisfy $a < b$ and let $f : \mathcal{K} \to \mathcal{K}$ be an analytic function on the interval $[a, b]$. Suppose $\phi : \mathcal{K} \to \mathcal{K}$ is a polynomial of degree n, then we have that*

$$\begin{aligned}&\sum_{m=0}^{n}(-1)^m(b-a)^m\left[\phi^{(n-m)}(1)f^{(m)}(b)-\phi^{(n-m)}(0)f^{(m)}(a)\right]\\&=(-1)^n(b-a)^{n+1}\int\limits_{t\in[0,1]}\phi(t)f^{(n+1)}(a+t(b-a)).\end{aligned}$$

Proof. As in the conventional case, this identity can be proven by repeated integration by parts. □

In fact Darboux's formula is more general than the one stated above as it can be extended to include integration over complex numbers as well, the above however is sufficient for our purposes. Rearranging the terms in Darboux's formula yields the relation

$$\begin{aligned}\phi^{(n)}(1)f(b)-\phi^{(n)}(0)f(a)&=(-1)^n(b-a)^{n+1}\int\limits_{t\in[0,1]}\phi(t)f^{(n+1)}(a+t(b-a))\\&\quad-\sum_{m=1}^{n}(-1)^m(b-a)^m\left[\phi^{(n-m)}(1)f^{(m)}(b)-\phi^{(n-m)}(0)f^{(m)}(a)\right].\end{aligned}$$

However because ϕ is a polynomial of degree n we have that $\phi^{(n)}(0) = \phi^{(n)}(1) = \phi_0$ so we have that

$$\phi^{(n)}(1)f(b)-\phi^{(n)}(0)f(a)=\phi_0\int\limits_{t\in[a,b]}f'(t).$$

Thus, by relabelling our functions $f^{(n)} \to f^{(n-1)}$, we have that

$$\int\limits_{t\in[a,b]} f(t) = \frac{1}{\phi_0}\sum_{m=1}^{n}(-1)^{m+1}(b-a)^m\left[\phi^{(n-m)}(1)f^{(m-1)}(b) - \phi^{(n-m)}(0)f^{(m-1)}(a)\right] \tag{5.2}$$
$$+\frac{1}{\phi_0}(-1)^n(b-a)^{n+1}\int\limits_{t\in[0,1]}\phi(t)f^{(n)}(a+t(b-a)).$$

So we can integrate f by finding its derivatives as well as the derivatives of ϕ, the integral term on the right hand side of Equation 5.2 is in effect our error. Different choices of ϕ will reduce Equation 5.2 to different summation formula; for example, if ϕ is the n_{th} degree Bernoulli polynomial then Equation 5.2 is equivalent to the Euler-Maclaurin equation. Similarly if ϕ is $(t-1)^n$ or t^n then Equation 5.2 goes to the Taylor series of the integrand about the left or right endpoint as $n \to \infty$ [28]. Another polynomial we investigate is $\prod_{i=1}^{n}(t - \frac{i}{n+1})$ with the idea that even if the n^{th} derivative of the integrand is large the frequent sign changes in ϕ will cause the integral term of Equation 5.2 to be small. Although Darboux's formula can be made equivalent to the Taylor expansion of the integrand about an end point, the same is not possible for arbitrary points in the interval of integration. For this reason we also experiment with integrating directly by a Taylor series about the central point in the interval of integration. First we integrate a selection of high order polynomials only evaluating them in the infinitesimal neighbourhood about the end points (or centre point as the case may be). For comparison we integrate the same polynomials using the Trapezoidal Rule and Simpsons Rule in the normal way (i.e. without use of infinitesimals). We also compute the integrals symbolically using Mathematica and, where it is possible, SymbolicC++. In addition to its symbolic integration method Mathematica also provides a method of numerical integration; in fact, this method does not correspond to any single integration technique but instead selects from a number of different techniques depending on the specific integral in question. In principle, as long as the degree of the polynomial to be integrated is less than the depth to which we can find its derivatives (which is to say the depth of calculation minus 1), our methods should produce an exact answer. We consider both this case and the case where the degree of the integrand is greater than our depth of calculation (which happens to be 15 for this experiment). The following two families of polynomials provide us with an ample supply of integrands to test. Let $n \in \mathbb{N}$ be given. Then we define

- A polynomial of degree n
$$P_n(x) := \prod_{i=1}^{n}(x - \frac{i}{\pi})$$

- The Bernoulli polynomial of degree n, note that B_i is the i^{th} Bernoulli number, given by the recurrence relation
$$Q_n(x) := \sum_{i=0}^{n} \binom{n}{i} B_{n-i} x^i$$

Table 5.9: Integral of P_9 from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	167338556.792	0.0005
Darboux's Formula (Bernoulli polynomials)	167338556.792	0.0187
Darboux's Formula (Frequent sign change polynomial)	167338556.792	0.0025
Simpsons rule (1000 steps)	164684838.888	0.2523
Trapezoidal rule (1000 steps)	165350080.353	0.2744
Mathematica (symbolic)	167338556.792	0.2550
Mathematica (numeric)	167338556.792	0.1268
SymbolicC++	167338556.792	31.8075

Table 5.10: Integral of P_{14} from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	879984428861	0.0015
Darboux's Formula (Bernoulli polynomials)	879984428939	0.0198
Darboux's Formula (Frequent sign change polynomial)	879882373891	0.0037
Simpsons rule (1000 steps)	856565103943	0.3413
Trapezoidal rule (1000 steps)	862448189383	0.3586
Mathematica (symbolic)	879984428861	0.3345
Mathematica (numeric)	879984428870	0.1463

Table 5.11: Integral of P_{19} from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	$6.11917660042\times10^{14}$	0.0015
Darboux's Formula (Bernoulli polynomials)	$1.12283166334\times10^{15}$	0.0216
Darboux's Formula (Frequent sign change polynomial)	$-6.36011035051\times10^{15}$	0.0044
Simpsons rule (1000 steps)	$1.07683615415\times10^{15}$	0.5900
Trapezoidal rule (1000 steps)	$1.08842381018\times10^{15}$	0.5017
Mathematica (symbolic)	$1.12283199608\times10^{15}$	4.6772
Mathematica (numeric)	$1.12283199603\times10^{15}$	0.1906

As expected due to the low degree of the integrand polynomial our method of integration found the exact value of the integral of P_9 and did so an order of magnitude faster than either Mathematica's symbolic or numerical methods and several orders of magnitude faster than SymbolicC++. For the integrand P_{14} there is some small error in the result of the integration due to the degree of the polynomial being exactly equal to the number of its derivatives we can find; however, the time to calculate the integral remained an order of magnitude less than that for Mathematica and Mathematica's numerical method also produced a close but not exact result. In the case where the integrand was P_{14} the only one of our methods to produce a reasonable result was Darboux's formula with Bernoulli polynomials; the other two non-Archimedean methods were wildly inaccurate, this again is as expected due to the high degree of the integrand polynomial relative to our depth of calculation.

Table 5.12: Integral of Q_{10} from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	4914341925	0.0022
Darboux's Formula (Bernoulli polynomials)	4914341925	0.0231
Darboux's Formula (Frequent sign change polynomial)	4914341925	0.0042
Simpsons rule (1000 steps)	4838576835.6	0.7003
Trapezoidal rule (1000 steps)	4857568906.18	0.7625
Mathematica (symbolic)	4914341925.00	0.0005
Mathematica (numeric)	4914341925.00	0.0928
SymbolicC++	4914341925	2.9619

Table 5.13: Integral of Q_{15} from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	$2.4632485638\times10^{14}$	0.0060
Darboux's Formula (Bernoulli polynomials)	$2.46324856386\times10^{14}$	0.0259
Darboux's Formula (Frequent sign change polynomial)	$2.46317967669\times10^{14}$	0.0082
Simpsons rule (1000 steps)	$2.40806310322\times10^{14}$	1.6375
Trapezoidal rule (1000 steps)	$2.42191551765\times10^{14}$	1.3542
Mathematica (symbolic)	$2.46324856380\times10^{14}$	0.0005
Mathematica (numeric)	$2.46324856380\times10^{14}$	0.1304

Table 5.14: Integral of Q_{20} from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	$1.32124577735\times10^{19}$	0.0102
Darboux's Formula (Bernoulli polynomials)	$1.33941318348\times10^{19}$	0.0289
Darboux's Formula (Frequent sign change polynomial)	$1.20781435476\times10^{19}$	0.0126
Simpsons rule (1000 steps)	$1.3000785144\times10^{19}$	2.0887
Trapezoidal rule (1000 steps)	$1.30996596445\times10^{19}$	2.3008
Mathematica (symbolic)	$1.33941318588\times10^{19}$	0.0006
Mathematica (numeric)	$1.33941318589\times10^{19}$	0.1778

The results of integrating Bernoulli polynomials largely agree with the results from the previous family of polynomials; specifically our methods perform well when the degree is less than the depth of calculation, they perform reasonably well but imperfectly when the degree of the polynomial is close to the depth of calculation and they become inaccurate when the degree of the integrand polynomial is greater than the depth of calculation. One thing to notice is that Mathematica's symbolic integration method preformed significantly better for Bernoulli polynomials than for the previous family of test polynomials; we suspect this is because Mathematica is internally substituting some known identity in place of the integral thus significantly reducing the complexity of the computation. Since Tulliotools seems to compare favourably in the above tests we investigate further with the integrands P_6Q_4, $P_{10}Q_5$, and $P_{12}Q_8$, the depth of calculation throughout is 25.

Table 5.15: Integral of P_6Q_4 from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	3256892681.95	0.0011
Darboux's Formula (Bernoulli polynomials)	3256892681.95	0.0224
Darboux's Formula (Frequent sign change polynomial)	3256892681.95	0.0058
Simpsons rule (1000 steps)	3204897511.71	0.2974
Trapezoidal rule (1000 steps)	3217932446.13	0.3816
Mathematica (symbolic)	3256892681.95	4.0018
Mathematica (numeric)	3256892681.95	0.9128
SymbolicC++	64890102.8482	36.3266

Table 5.16: Integral of $P_{10}Q_5$ from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	$5.60845249807\times10^{13}$	0.0011
Darboux's Formula (Bernoulli polynomials)	$5.60845249842\times10^{13}$	0.0249
Darboux's Formula (Frequent sign change polynomial)	$5.60845249807\times10^{13}$	0.0060
Simpsons rule (1000 steps)	$5.47069797559\times10^{13}$	0.6372
Trapezoidal rule (1000 steps)	$5.50529044403\times10^{13}$	0.6114
Mathematica (symbolic)	$5.60845249807\times10^{13}$	4.3917
Mathematica (numeric)	$5.60845249807\times10^{13}$	0.1592

Table 5.17: Integral of $P_{12}Q_8$ from 0 to 10

Method of Computation	Result	Time (sec.)
Central point Taylor series	$1.41666070259\times10^{18}$	0.0021
Darboux's Formula (Bernoulli polynomials)	$1.41666067675\times10^{18}$	0.0224
Darboux's Formula (Frequent sign change polynomial)	$1.41666070259\times10^{18}$	0.0111
Simpsons rule (1000 steps)	$1.37030525479\times10^{18}$	0.8658
Trapezoidal rule (1000 steps)	$1.38196505555\times10^{18}$	1.0181
Mathematica (symbolic)	$1.41666070259\times10^{18}$	6.5037
Mathematica (numeric)	$1.41666070259\times10^{18}$	0.2105

As the tables above show, non-Archimedean methods of integration produced good numerical values for the given integrals and reliably did so an order of magnitude faster than Mathematica was able to; this would seem to suggest that non-Archimedean methods provide a real advantage at least when it comes to integrating polynomials. Next we would like to investigate the performance of non-Archimedean methods when integrating analytic functions; to that end, we obtained a selection of analytic functions from [29]. In this case we break the interval of integration into smaller steps so as to ensure that the error terms involved don't diverge, the depth of calculation remains 15 throughout. A number of these functions have integrable singularities at the endpoints of integration so the Trapezoidal

Rule and Simpsons Rule are inapplicable; in these cases we integrate over the piece of the interval that contains the singularity using the Taylor expansion of the integrand about some point where it is actually defined. For comparison, we compute these same integrals using Mathematica both symbolically and numerically, SymbolicC++ was unable to produce a result for any of the given integrals. The test functions we used are listed below:

- $$f_1(x) := \frac{x^2 \ln(x)}{(x^2-1)(x^4+1)}$$

  This function has singularities at both $x = 0$ and $x = 1$, its integral from 0 to 1 is conjectured to be $\frac{\pi^2(2-\sqrt{2})}{32}$ [29].

- $$f_2(x) := \frac{x^2}{\sin^2(x)}$$

  This function has a singularity at $x = 0$, its integral from 0 to $\frac{\pi}{4}$ is conjectured to be $\frac{\pi^2}{16} + \frac{\pi \ln(2)}{4} + G$ where G is Catalans constant [29].

- $$f_3(x) := \frac{x \sin(x)}{1+\cos^2(x)}$$

  This function has no singularities, its integral from 0 to π is conjectured to be $\frac{\pi^2}{4}$ [29].

- $$f_4(x) := x \ln(1+x)$$

  This function has no singularities for positive arguments, its integral from 0 to 1 is known to be $\frac{1}{4}$ [29].

- $$f_5(x) := e^x \cos(x)$$

  This function has no singularities, its integral from 0 to $\frac{\pi}{2}$ is known to be $\frac{e^{\frac{\pi}{2}}-1}{2}$ [29].

- $$f_6(x) \coloneqq \ln^2(x)$$

  This function has a singularity at $x = 0$, its integral from 0 to 1 is known to be 2. [29]

- $$f_7(x) \coloneqq \ln(\cos(x))$$

  This function has a singularity at $x = \frac{\pi}{2}$, its integral from 0 to $\frac{\pi}{2}$ is known to be $-\frac{\pi \ln(2)}{2}$ [29].

Table 5.18: Integral of f_1 from 0 to 1 with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	0.206802333557	0.055	0.185720449275	0.274	0.183183663033	0.593	0.181171761808	2.923	0.180921387493	7.300
Darboux's Formula (Bernoulli polynomials)	0.180671031514	0.123	0.180671260743	1.012	0.18067126236	1.862	0.180671262589	10.142	0.18067126259	17.610
Darboux's Formula (Frequent sign change polynomial)	0.180671018257	0.108	0.180671260637	0.858	0.180671262346	1.581	0.180671262589	6.636	0.18067126259	12.909
Method of Computation	Integral						Time (sec.)			
Mathematica (symbolic)	0.180671262591						10.389			
Mathematica (numeric)	0.180671262709						0.890			
$\frac{\pi^2(2-\sqrt{2})}{32}$	0.180671262591						n/a			

Table 5.19: Integral of f_2 from 0 to $\frac{\pi}{4}$ with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	0.938429184613	0.215	0.862808426244	0.709	0.853180614292	1.280	0.845448903068	5.241	0.844480579999	10.446
Darboux's Formula (Bernoulli polynomials)	0.843511841685	0.238	0.843511841685	1.706	0.843511841685	4.014	0.843511841685	13.269	0.843511841685	26.723
Darboux's Formula (Frequent sign change polynomial)	0.843511841685	0.208	0.843511841685	1.580	0.843511841685	2.317	0.843511841685	11.468	0.843511841685	22.712
Method of Computation	Integral						Time (sec.)			
Mathematica (symbolic)	0.843511841685						6.420			
Mathematica (numeric)	0.843511841685						0.069			
$\frac{\pi^2}{16} + \frac{\pi \ln(2)}{4} + G$	0.843511841685						n/a			

Table 5.20: Integral of f_3 from 0 to π with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	2.38333894223	0.212	2.46425705799	1.087	2.46662064711	1.376	2.46737005245	6.856	2.46739334352	13.503
Darboux's Formula (Bernoulli polynomials)	2.38333890545	0.499	2.46425705799	2.252	2.46662064711	3.323	2.46737005245	16.180	2.46739334352	32.000
Darboux's Formula (Frequent sign change polynomial)	2.38333894223	0.403	2.46425705799	1.658	2.46662064711	2.850	2.46737005245	14.118	2.46739334352	28.132

Method of Computation	Integral	Time (sec.)
Mathematica (symbolic)	2.46740110027	12.811
Mathematica (numeric)	2.46740110025	0.250
$\frac{\pi^2}{4}$	2.46740110027	n/a

Table 5.21: Integral of f_4 from 0 to 1 with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	0.325403421197	0.011	0.264102569731	0.084	0.256991253957	0.097	0.251388681655	0.535	0.250693743879	1.063
Darboux's Formula (Bernoulli polynomials)	0.325403421197	0.076	0.264102569731	0.479	0.256991253957	0.850	0.251388681655	4.067	0.250693743879	8.131
Darboux's Formula (Frequent sign change polynomial)	0.325403421197	0.037	0.264102569731	0.268	0.256991253957	0.439	0.251388681655	2.001	0.250693743879	4.066

Method of Computation	Integral	Time (sec.)
Mathematica (symbolic)	0.250000000000	5.703
Mathematica (numeric)	0.250000000000	0.2172
$\frac{1}{4}$	0.25	n/a

Table 5.22: Integral of f_5 from 0 to $\frac{\pi}{2}$ with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	1.83943337875	0.032	1.90281470625	0.200	1.90463898243	0.327	1.90521490197	1.846	1.90523274958	3.560
Darboux's Formula (Bernoulli polynomials)	1.83943337875	0.262	1.90281470625	0.725	1.90463898243	1.327	1.90521490197	6.581	1.90523274958	13.211
Darboux's Formula (Frequent sign change polynomial)	1.83943337875	0.229	1.90281470625	0.495	1.90463898243	0.907	1.90521490197	4.568	1.90523274958	9.106

Method of Computation	Integral	Time (sec.)
Mathematica (symbolic)	1.90523869048	0.556
Mathematica (numeric)	1.90523869048	0.214
$\frac{e^{\frac{\pi}{2}}-1}{2}$	1.90523869048	n/a

Table 5.23: Integral of f_6 from 0 to 1 with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	1.91297168502	0.017	1.97823314635	0.097	1.98819135915	0.149	1.99720902389	0.785	1.99851209132	1.630
Darboux's Formula (Bernoulli polynomials)	1.91347701776	0.069	1.97845551707	0.562	1.98831474868	0.863	1.99723896109	4.543	1.99852817956	9.016
Darboux's Formula (Frequent sign change polynomial)	1.91261136441	0.068	1.97823043825	0.249	1.98819102284	0.555	1.99720902111	2.504	1.99851209092	5.184

Method of Computation	Integral	Time (sec.)
Mathematica (symbolic)	2.00000000000	0.879
Mathematica (numeric)	2.00000000000	0.102
2	2	n/a

Table 5.24: Integral of f_7 from 0 to $\frac{\pi}{2}$ with various step sizes

Method of Computation	10 steps	Time (sec.)	50 steps	Time (sec.)	100 steps	Time (sec.)	500 steps	Time (sec.)	1000 steps	Time (sec.)
Central point Taylor series	-1.5213090659	0.207	-1.22581686958	0.728	-1.16819224184	1.321	-1.10972904147	6.797	-1.10034983571	13.390
Darboux's Formula (Bernoulli polynomials)	-1.07814289959	0.320	-1.08666301604	1.782	-1.0877280306	3.326	-1.08858004224	16.587	-1.0886865437	32.987
Darboux's Formula (Frequent sign change polynomial)	-1.07832106916	0.298	-1.08669864995	1.600	-1.08774584755	2.786	-1.08858360563	14.221	-1.08868832539	28.640

Method of Computation	Integral	Time (sec.)
Mathematica (symbolic)	-1.08879304515	12.381
Mathematica (numeric)	-1.08879304515	0.095
$-\frac{\pi \ln(2)}{2}$	-1.08879304515	n/a

It is not at all clear what conclusions can be drawn from the results of the above test. On the one hand, it is promising that non-Archimedean methods managed to produce recognizable solutions for the integrals in time periods that were of the same order of magnitude as Mathematica's symbolic method. On the other hand, Mathematica's numeric method of integration consistently produced more accurate solutions in times that were orders of magnitude shorter than for any of the other methods. We caution the reader against forming strong conclusions based off this data for a few reasons. Firstly, as we stated above, Mathematica's numerical integration method does not employ a single integration technique but rather chooses among several different ones; so there is no guarantee that the method used to integrate f_1 is the same as that used to integrate f_7. Secondly, even when using a single technique, Mathematica actively adjusts the parameters of integration (e.g. step size and number) based on information it obtains about the integrand; although mathematically it is possible to do this with non-Archimedean methods as well, the current state of the TullioTools software does not allow it. Another possibility that might improve the accuracy and speed of non-Archimedean methods would be to continue using Darboux's formula but to change the depth of calculation and the specific polynomial to be used based on the function to be integrated. This could be as simple as using the frequent sign change polynomial but changing the location of the roots to improve cancellation. A more sophisticated idea might be to employ the calculus of variations to make stationary the error term; either way, there seems to be ample room for improvement in this area. The depth of calculation used in the non-Archimedean methods had a pronounced effect on the accuracy of the integrals involving polynomials and so we would like to investigate the effect it has when the integrand is analytic. To do that we calculate the same integrals as above but this time we vary the depth of the calculation, the number of steps remains 100 throughout.

Table 5.25: Integral of f_1 from 0 to 1 with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	0.183183662351	0.250	0.18318366318	0.958	0.18318366325	2.803	0.183183663266	7.182
Darboux's Formula (Bernoulli polynomials)	0.180671261668	1.295	0.180671261529	3.605	0.177340121748	9.945	$-7.55504318522\times10^{47}$	19.897
Darboux's Formula (Frequent sign change polynomial)	0.180671261665	1.0309	0.180671262493	3.566	0.180671262563	5.554	0.180671262579	12.538

Table 5.26: Integral of f_2 from 0 to $\frac{\pi}{4}$ with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	0.853180614292	0.627	0.853180614292	3.331	0.853180615957	13.848	0.853181457083	49.891
Darboux's Formula (Bernoulli polynomials)	0.843511841685	1.356	0.84351184168	7.543	1.3421471305	30.701	$-7.05373532399\times10^{64}$	99.852
Darboux's Formula (Frequent sign change polynomial)	0.843511841685	1.215	0.843511841685	6.479	0.84351184335	27.641	0.843512684476	90.993

Table 5.27: Integral of f_3 from 0 to π with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	2.46662064711	0.992	2.46662064711	3.473	2.46662064711	14.003	2.46662064711	46.957
Darboux's Formula (Bernoulli polynomials)	2.46662064711	1.754	2.46662064711	8.292	2.46662064711	32.694	2.46662064711	102.442
Darboux's Formula (Frequent sign change polynomial)	2.46662064711	1.175	2.46662064711	7.295	2.46662064711	29.311	2.46662064711	95.222

Table 5.28: Integral of f_4 from 0 to 1 with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	0.256991253957	0.123	0.256991253957	0.145	0.256991253957	0.256	0.256991253957	0.373
Darboux's Formula (Bernoulli polynomials)	0.256991253957	0.431	0.256991253957	1.657	0.256991253957	4.025	0.256991253957	9.006
Darboux's Formula (Frequent sign change polynomial)	0.256991253957	0.240	0.256991253957	0.726	0.256991253957	1.173	0.256991253957	2.011

Table 5.29: Integral of f_5 from 0 to $\frac{\pi}{2}$ with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	1.90463898243	0.220	1.90463898243	0.586	1.90463898243	1.214	1.90463898243	2.359
Darboux's Formula (Bernoulli polynomials)	1.90463898243	0.696	1.90463898243	2.602	1.90463898243	5.928	1.90463898243	12.795
Darboux's Formula (Frequent sign change polynomial)	1.90463898243	0.594	1.90463898243	1.611	1.90463898243	3.053	1.90463898243	5.822

Table 5.30: Integral of f_6 from 0 to 1 with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	1.9831320353	0.096	1.99084742601	0.302	1.9936224532	0.696	1.99507097924	1.580
Darboux's Formula (Bernoulli polynomials)	1.98314867821	0.401	1.96897925712	1.867	-200559.732157	4.849	$1.81223968541\times10^{49}$	10.717
Darboux's Formula (Frequent sign change polynomial)	1.98313186078	0.276	1.99084709016	0.900	1.99362211734	2.096	1.99507064339	4.175

Table 5.31: Integral of f_7 from 0 to $\frac{\pi}{2}$ with various depths of calculation

Method of Computation	Depth of 10	Time (sec.)	Depth of 20	Time (sec.)	Depth of 30	Time (sec.)	Depth of 40	Time (sec.)
Central point Taylor series	-1.16735034886	0.515	-1.16855772081	3.760	-1.16895586425	15.367	-1.16915412109	51.043
Darboux's Formula (Bernoulli polynomials)	-1.08723860077	1.296	-1.09014091981	8.627	308135.281549	33.606	$6.65851559939\times10^{52}$	104.376
Darboux's Formula (Frequent sign change polynomial)	-1.08722226363	1.076	-1.08800764699	8.030	-1.08826944638	30.735	-1.08840034607	98.800

Although we find that increasing the depth of calculation did improve the accuracy of the result in most cases, in the case of f_3, f_4, and f_5 it had no effect on accuracy at all. Moreover, increasing the depth of calculation significantly increases the time needed to compute the result; this does not necessarily mean however that we should always use the smallest depth of calculation possible. One complicating factor is that increasing the depth of calculation might reduce the number of steps needed to achieve the same accuracy. For example, if increasing the depth of calculation doubled the time it took to evaluate the integrand at a given point but allowed us to achieve an equivalent level of accuracy using only a quarter as many steps it might still be a net gain in performance to use the larger depth of calculation. To conclude, we would like to determine if there is any class of functions which TullioTools is better at integrating than Mathematica, and in fact, it seems that there is: Mathematica has a well-known difficulty integrating highly oscillatory functions [30]. Since TullioTools has access to not only the value of the integrand but also its derivatives it should be substantially better at integrating highly oscillatory functions. To test this, we consider the function

$$g(x) \coloneqq P_5(cos(100x))Q_5(cos(100x)).$$

As may be seen in the table below, when Mathematica numerically integrates this function under default settings it produces an incorrect answer; TullioTools, on the other hand, is able to attain the first four digits correctly in approximately half the time Mathematica takes. When Mathematica attempts to integrate g under default settings it produces two warnings; the first states that the maximum allowed number of recursive steps was reached and the second states that the integrand may be highly oscillatory and the working precision may be too small. The first warning can be addressed by increasing the maximum allowed number of recursive steps, we set this to 100; note that this does not require Mathematica to use 100 recursive steps but it prevents it from using more than that. The second issue can be addressed by increasing the working precision; this was done in increments of 10 until the warning was no longer present, resulting in a working precision of 40. The depth of calculation is 5.

Table 5.32: Integral of g from 0 to 10

Method of Computation	1000 steps	Time (sec.)	5000 steps	Time (sec.)	10000 steps	Time (sec.)
Central point Taylor series	186.369934703	2.821	186.372340098	14.669	186.372340207	25.436123
Darboux's Formula (Bernoulli polynomials)	186.389214786	6.433	186.372340097	26.562	186.372340207	52.864
Darboux's Formula (Frequent sign change polynomial)	186.365945702	4.663	186.372340097	24.594	186.372340207	46.185

Method of Computation	Integral	Time (sec.)
Mathematica (symbolic)	186.372340189	4324.78
Mathematica (numeric with adjusted settings)	186.372340188	66.96
Mathematica (numeric with default settings)	182.322	5.964

As can be seen in the table above, TullioTools was able to integrate g with a good degree of precision in less time than Mathematica, the central point Taylor series method in particular was substantially faster than both Mathematica and the other methods implemented by TullioTools. The results in this section are hardly conclusive; in particular a fair comparison would require that TullioTools be able to actively adjust how the interval of integration is partitioned based on the specific integrand. Moreover, a more sophisticated method of comparison will be necessary to ensure that Mathematica and TullioTools are always competing to attain the same degree of precision. Nevertheless, we believe that these results are sufficient to establish that non-Archimedean methods of numerical integration are highly versatile and have the potential to improve upon conventional methods.

Chapter 6

Concluding Remarks

In this final chapter, we offer some remarks to conclude this thesis. We will summarize the primary results presented here and discuss their importance. As the work was highly collaborative we will clarify the origin of various contributions and endeavour to distinguish between those results that are entirely new and those that where preexisting. We will also describe some future possibilities for research. We outline certain mathematical topics that it will be necessary to investigate going forward. Moreover, we speculate as to how the existing mathematical framework might be applied to the theory of quantum mechanics. Finally, we outline a number of improvements that could be made to the Tulliotools software and we highlight some promising avenues of investigation into computational applications of the Levi-Civita field.

6.1 Summary of Results

The work presented in this thesis is a continuation of research done by Dr. Shamseddine as well as the author. We would like to take this opportunity to summarize the aspects of this work that constitute new results and to give some explanation as to their importance. In Chapter 2 we study the topological structure of the Hahn field. The key result is a description of one topology in particular which is similar to the weak topology on the

Levi-Civita field. Using this newly discovered topology we are able to derive convergence criteria both for sequences and for series. These convergence criteria are again similar to those for the weak topology on the Levi-Civita field [16]. This work is of key importance as it allows many results from the Levi-Civita field to be extended to the Hahn field. In Chapter 3 we address the problem of extending the theory of measures, analysis, and integration to dimensions greater than two. This begins with the proposal of three conjectures which appear necessary in order to allow this development. We also begin the effort to prove these conjectures by proving a version of the Weierstrauss Preparation Theorem for the Hahn field. The remainder of the chapter goes on to construct a theory of ν dimensional integration, by assuming the soundness of the three conjectures. These results are of particular importance in the ongoing effort to develop the mathematical theory necessary for applications in physics because much of physics involves more than two dimensions. Chapter 4 continues our investigations into the non-Archimedean delta function; this has been a motivating example from the beginning of our research, and much of the work in this chapter was in fact done for the author's Master's degree. The key result which merits this chapter's inclusion in the author's doctoral thesis is a derivation of a version of Leibniz rule for the Hahn field. Another significant development detailed in this chapter is how the aforementioned Leibniz rule explains what boundary conditions are sufficient for non-Archimedean Greens functions to be used to solve real differential equations. In Chapter 5 we turn our attention to the computational applications of the Levi-Civita field. This work entailed the development of a static library in the C++ programming language which defines a new class for Levi-Civita numbers and which implements numerous functions for those numbers including basic arithmetic and trigonometric operations. The entirety of this library is included as an Appendix. This newly developed software is used to verify the results of a previous paper in computing numerical derivatives using the Levi-Civita field [4]. We also show how a similar method can be used to compute the values of numerical sequences using generating functions. We conclude the chapter by investigating a variety of methods for performing numerical integration employing the Levi-Civita field and we compare the performance of these methods to other commercial software.

6.2 Statement of Contributions

The research done throughout my doctoral studies was completed in collaboration with my adviser, Dr. Shamseddine. Moreover, this research is a continuation of my work done for my Master's thesis as well as Dr. Shamseddine's work over the past couple of decades. For the sake of clarity, I will endeavour to highlight my own original contributions and distinguish those which are new results from ones contained in my Master's thesis. Regarding Chapter 2 the problem of inducing a weak topology on the Hahn field was originally brought to my attention by Dr. Shamseddine, however, the proposed solution is my own work. The subsequent propositions and theorems were done largely independently, although Dr. Shamseddine did provide key input in some areas to clarify arguments and fix minor mistakes. Naturally, many of the propositions are closely related to Dr. Shamseddine's work on the Levi-Civita field [1], however, substantial modification was often required to adapt to the newly identified weak topology. Chapter 3 addresses the problem of constructing a theory of measures, analysis, and integration in higher dimensions. This task began as part of my Master's thesis, where Dr. Shamseddine conceived of the notion of a simple region and I was able to develop that idea into a theory of integration in two and three dimensions. Notably in this work were a number of unfounded assumptions related to the behaviour of simple regions in three dimensions. This gap in my Master's thesis motivated my proposal of three conjectures and proof of the Weierstrauss Preparation Theorem. The developments related to this topic are my own work conducted during my Doctoral program and are meant to clarify the difficulty involved in generalizing the theory to higher dimensions. The definitions and propositions related to measure theory, analysis, and integration are based on the work done in three dimensions for my Master's thesis, however, the generalized versions of the arguments are new and my own work. The idea of a non- Archimedean delta function, which is discussed in Chapter 4 was brought to my attention by Dr. Shamseddine. Much of the related theory was conducted by myself under his supervision for my Master's degree. New and original work, however, include the proof of Leibniz's rule and the related arguments regarding the boundary conditions of Green's functions. The corrected and expanded examples are also my own original work and are relatively different to the original versions published in my Master's thesis. In chapter 5 we discuss the computational applications of the Levi-Civita field. The idea for how to compute numerical derivatives of functions originated with Dr. Shamseddine. The C++ static library used to recreate those results was created entirely by myself as was the specific implementation of this library in the context of the various tests performed. I was also responsible for the idea of computing numerical sequences using the generating functions, and for the work related to numerical integration.

6.3 Research Outlook

The results presented in this thesis allow for numerous possibilities for future research. They can be loosely divided into three branches, those being mathematical, physical, and computational. Mathematically there are a number of interesting ideas to pursue, perhaps the most obvious is the development of a theory of manifolds. Developing a theory of manifolds is important for two reasons, firstly, because it would constitute substantial progress towards proving the three conjectures put forward in Chapter 3. Secondly, it would also allow the possibility of investigating general relativity in the context of the Hahn and Levi-Civita fields. Another possibility for mathematical research is to further investigate the delta function. In particular, we would like to know if the non-Archimedean delta function allows us a way to easily compute the real Green's function of a given differential operator. It would also be interesting to extend this work to include partial differential equations, and non-linear differential equations. Additionally, it may be interesting to conduct a survey of differential operators with no known Green's function as the non-Archimedean context may allow us to find them in cases where conventional techniques fail. The field of p-adic (non-Archimedean) mathematical physics has largely focused on quantum mechanics and we believe there is room for progress on this topic. One possibility, briefly discussed in the motivation section, is to consider wave functions which go from the complex Hahn field to itself. This idea falls in line with the way that non-Archimedean mechanics has been approached in the past. One interesting possibility relates to perturbation theory. We suspect that it is possible to arrange things so that the real part of an equation corresponds to the first order of approximation, and every subsequent order of magnitude in the equation corresponds to another term in the approximation. Another approach to quantum mechanics is suggested by the success of the non-Archimedean delta function and its relation to the Dirac delta function which is commonly seen as describing the eigenstates of the position operator. In this approach one would endeavour to find a subset of functions from the complex Hahn field to itself, which is somehow isomorphic to the conventional family of wave functions. Ideally this new family of non–Archimedean wave functions could be used in place of conventional wave functions. It should be chosen to have a more convenient mathematical description, or more convenient computational properties, or possibly a clear intuitive meaning. Finally, pending some developments regarding manifolds on the Hahn field it would be interesting to investigate a non-Archimedean model of general relativity. There are also a number of ideas to investigate in terms of computational research. On the theoretical side of things, one might start by determining the computational complexity of various operations on the Levi-Civita field, and comparing them to their classical counterparts. More practically, the C++ implementation of the Levi-Civita field could be expanded to allow it to operate in conjunction with an arbitrary precision library. That would be particularly helpful in the study of generating functions and numeric sequences, as it would substantially reduce the effect of rounding errors. Finally, there are many ideas to pursue with regards to numerical integration. We would like to expand the C++ implementation, so that it is capable of actively adjusting parameters in response to the various numerical approximations produced. We are also interested in implementing Monte Carlo methods of integration, as these might make the best use of the ability to compute high order derivatives of the integrand. One key mathematical problem related to this topic is finding a way to easily approximate the radius of convergence of the power series representation of an integrand. Here too, we hope to benefit from our ability to compute high order derivatives of the integrand. We hope that the results put forward in this thesis can now be used to apply the Hahn and Levi-Civita fields to problems in physics in a mathematically rigorous manner. There is still more mathematical theory to be developed; however, we believe there is now enough in place to begin investigating descriptions of quantum mechanics that employ the Hahn and Levi-Civita fields. Now that there exists a C++ library implementing the Levi-Civita field it should be possible to explore a wide variety of computational applications with relative ease. One interesting project along these lines would be to incorporate the work done regarding the non-Archimedean delta function into the work done in computation for the purposes of solving differential equations.

Appendix A

Tulliotools

A.1 Documentation

A.1.1 ReadMe.txt

```
==============================================================================

                TullioTools v1.8
==============================================================================

Welcome to TullioTools v1.8. This static library (named for the Italian
    mathematician Tullio Levi-Civita) includes a number of header and
   source files intended to implement the so called Levi-Civita field
   in the C++ language. All code and comments were written by Darren M.
    Flynn-Primrose in pursuit of his Ph.D in Physics at the University
   of Manitoba. The code was written and compiled with Microsoft Visual
    Studios 2015 Community Edition.
==============================================================================

                Header Files
```

```
tulliotoolsv18.h: Header file to be included in projects that wish to
   use this library, this file automatically includes the other headers
    in the library

definitions.h: Defines a number of Macros such as the depth to which
   Taylor series are calculated and the number of significant digits to
    be used changing the definitions in this file will cause the values
    to change everywhere they appear in the library

rationals.h: Defines the "rat" class for rational numbers, elements are
    stored as two unsigned integers and a bool.

levicivita.h: Defines the "lcf" class for elements of the Levi-Civita
   field elements are stored as two vectors. The first vector (a vector
    of floats) contains the value of the element at each support point
   and the second vector (a vector of rats) contains the corresponding
   support points themselves.

functions.h: Includes a number of functions for use with the library.

difops.h: Includes operations related to differentiation.

intops.h Includes operations related to integration.
```

Source Files

In most cases each header file has a coresponding source file which contains the definitions of the classes/functions/etc. declared in the header; in **this** library there are three exeptions to **this**. In particular we have tulliotoolsv18.h **and** definitions.h neither of which has a corresponding source file. We also have functions.h which has three corresponding source files (primarily **for** ease of reading); namely, there are trigfuncs.cpp which contains the definitions **for** tig functions, specfuncs.cpp which provides the definitions **for** a miscellaneous assortment of special functions, **and** testfuncs.cpp which defines a number of functions used **for** the purpose of testing the library **and** which are **not** neccessary to its internal workings.

Issues & Bugs

1) The **operator** templates **for** the RAT **and** LCF classes all assume the first argument is of the **class** type **and** the second argument is of an arbitrary type. So **for** example **if** x is an LCF data type **and** **if** y is a **double** then to multiply them we need to write x*y. If we write y*x then the compiler states that there is no operation corresponding to those argument types the same is **true** **for** addition, subtraction, **and** division. The obvious thing to **do** is to include a second **template** which makes the oposite assumption (i.e. assume the first argument is arbitrary **and** the second is of the given **class** type) but

this fails in the **case** that both arguments are of the given **class** type. When **this** happens the compiler can't decide which template to use it can't see that they are equivalent. This could in principle be solved with yet a third **template** specifically **for this case** but **this** would nearly triple the size of both **class** definitions there must be a better way.

2) The two **class** definitions RAT **and** LCF use **unsigned** ints **and** doubles respectively. It would be better to change them so they are **class** templates they could then be changed to use any available data type. In particular the Boost Multiprecision library (which is **not** currently compatible with visual studios) has the float128 data type that it would be interesting to **try**. Alternatively an arbitrary precision library could be incorperated in **this** way.

3) The number of support points included in calculations is currently controled by the _SIG_SUPP_ macro **this** means that currently all computations have to be done to the same depth **and** there is no way to change that depth within the program itself. It would be better to make **this** more versatile so we could experiment with calculations that use varying numbers of support points as well as with programs which actively adjust the depth of calculation.

A.2 Headers

A.2.1 definitions.h

```
#pragma once
#ifndef _DEFINITIONS_H_
```

```
#define _DEFINITIONS_H_
#define _USE_MATH_DEFINES//Allows us to use the constants defined in
    the math.h library
#define _SIG_DIGS_ 12//Defines the number of significant digits to use
    when converting between types.
#define _SIG_SUPP_ 25//Defines the number of support points stored in
    the lcf class.
#define _SIG_TERMS_ 26//Defines how many terms of the taylor series are
     considered.
#define _STEP_NO_ 1000//Defines how many steps are used when employing
    simpsons rule. Must be even.

#define _CATS_CONST_  0.915965594177219015054603514932384110774//
    Catalan's Constant. We use the first 21 sig digs b/c that is the
    precision of the constants in math.h

//First 40 Bernoulli's numbers (from http://www.gutenberg.org/ebooks
    /2586 by Simon Plouffe, retrived 2019-01-23). Cast as double
#define _BERN_0_ double(1)

#define _BERN_1_ double(-1) / 2

#define _BERN_2_ double(1) / 6

#define _BERN_4_ double(-1) / 30

#define _BERN_6_ double(1) / 42

#define _BERN_8_ double(-1) / 30
```

```
#define _BERN_10_ double(5)/ 66

#define _BERN_12_ double(-691) / 2730

#define _BERN_14_ double(7) / 6

#define _BERN_16_  double(-3617) / 510

#define _BERN_18_ double(43867) / 798

#define _BERN_20_ double(-174611) / 330

#define _BERN_22_ double(854513) / 138

#define _BERN_24_ double(-236364091) / 2730

#define _BERN_26_ double(8553103) / 6

#define _BERN_28_ double(-23749461029) / 870

#define _BERN_30_ double(8615841276005) / 14322

#define _BERN_32_ double(-7709321041217) / 510

#define _BERN_34_ double(2577687858367) / 6

#define _BERN_36_ double(-26315271553053477373) / 1919190
```

```
#define _BERN_38_ double(2929993913841559) / 6

#define _BERN_40_ double(-261082718496449122051) / 13530
#endif
```

A.2.2 tulliotoolsv18.h

```
#pragma once
//This header is for inclusion in new projects, it includes all the
    others automatically
#ifndef _TULIOTOOLS_18_
#define _TULIOTOOLS_18_

#include "definitions.h"
#include "difops.h"
#include "intops.h"
#include "rational.h"
#include "levicivita.h"
#include "functions.h"
#endif
```

A.2.3 rational.h

```
#pragma once
#ifndef _RATIONAL_H_
#define _RATIONAL_H_
#include "definitions.h"

namespace tulliotools
{
```

```
class rat//A class for rational numbers
{
  bool sign;//TRUE=Negative, FALSE=Positive

  unsigned int numerator, denominator;

  void simplify();

public:
  // Constructors: These functions construct an instance of the rat
     class given some initial information
  explicit rat();

  explicit rat(bool, unsigned int, unsigned int);

  explicit rat(const unsigned int&);

  explicit rat(const int&);

  explicit rat(const double&);

  //Retrieval: these functions give us a way to retrieve the sign,
     numerator, and denominator from outside the class
  unsigned int get_num() const;

  unsigned int get_den() const;

  bool get_sign() const;
```

```
  //Comparison
  bool is_greater(const rat&) const;

  bool is_less(const rat&) const;

  bool is_equal(const rat&) const;

  //Arithmatic Operations
  rat rat_add(const rat&) const;

  rat rat_mult(const rat&) const;

  rat rat_invert() const;

  rat rat_dev(const rat&) const;

  rat exp(const unsigned int) const;//Exponent: this has nothing to
     do with Eulers number

  void operator= (const int&);

  void operator= (const float&);

  //Conversion
  explicit operator int();

  explicit operator float();
};
```

```
unsigned int gcd(unsigned int, unsigned int);//Find the greatest
    common denominator

unsigned int lcm(unsigned int, unsigned int);//Find the least common
    multiple

//Operators: These relate each symbol to their respective opperation,
     so the compiler knows for example that x+y means x.rat_add(y)
template <class T>
inline rat operator+ (const rat arg1, const T arg2)
{
  return arg1.rat_add(rat(arg2));
}

template <class T>
inline rat operator- (const rat arg1, const T arg2)
{
  return arg1.rat_add(rat(arg2)*-1);
}

template <class T>
inline rat operator* (const rat arg1, const T arg2)
{
  return arg1.rat_mult(rat(arg2));
}

template <class T>
inline rat operator/ (const rat arg1, const T arg2)
{
```

```
  return arg1.rat_dev(rat(arg2));
}

template <class T>
inline bool operator== (const rat arg1, const T arg2)
{
  if (arg1.is_equal(rat(arg2))) return true;
  else return false;
}

template <class T>
inline bool operator< (const rat arg1, const T arg2)
{
  if (arg1.is_less(rat(arg2))) return true;
  else return false;
}

template <class T>
inline bool operator> (const rat arg1, const T arg2)
{
  if (arg1.is_greater(rat(arg2))) return true;
  else return false;
}

template <class T>
inline bool operator<= (const rat arg1, const T arg2)
{
  return (arg1.is_less(rat(arg2))) || (arg1.is_equal(rat(arg2)));
}
```

```
  template <class T>
  inline bool operator>= (const rat arg1, const T arg2)
  {
    return (arg1.is_greater(rat(arg2))) || (arg1.is_equal(rat(arg2)));
  }

  //Some constants that use frequently
  const rat rat1 = rat(false, 1, 1);
  const rat rat0 = rat(false, 0, 1);
}
#endif
```

A.2.4 levicivita.h

```
#pragma once
#ifndef _LEVICIVITA_H_
#define _LEVICIVITA_H_


#include "rational.h"
#include "definitions.h"
#include <vector>


namespace tulliotools
{
  class lcf//Class for elements of the Levi-Civita field
  {
```

```
  std::vector<double> value;//Stores the values at each support point
  std::vector<rat> support;//Stores the support points
  void simplify();
public:
  //Constructors: These construct an instance of the lcf class givin
     some initial information
  explicit lcf();
  explicit lcf(std::vector<double>&, std::vector<rat>&);
  explicit lcf(const double&, const rat&);
  template <class T>
  explicit lcf(const T &x)
  {
    this->support.push_back(rat0);
    this->value.push_back(double(x));
  }

  //Comparison
  bool is_greater(const lcf&) const;
  bool is_less(const lcf&) const;
  bool is_equal(const lcf&) const;

  //Operations
  lcf lcf_add(const lcf&) const;
  lcf lcf_mult(const lcf&) const;
  lcf lcf_invert(unsigned int n = _SIG_SUPP_) const;
  lcf exp(const unsigned int) const;//Exponent
  lcf abs() const;//Absolute value

  //Retrieval
```

```
  rat get_order() const;//get minimum support point
  rat get_depth() const;//get maximum support point
  double get_real() const;//get the real part
  std::vector<double> get_value() const;//Returns a vector with all
     the values
  double get_value(const rat&) const;//Returns the value at a
     specific support point
  double get_value(const unsigned int&) const;//Returns the value at
     the given position in the value vector
  std::vector<rat> get_support() const;//Returns a vector with all
     support points
  rat get_support(const unsigned int&) const;//Returns the support
     point at the given position in the support vector

  template <class T>
  void operator= (const T &arg)
  {
    this->support.push_back(rat(0));
    this->value.push_back(double(arg));
  }

  template <class T>
  explicit operator T() { return T(this->get_value[rat(false, 0, 1)])
       };
};

//Operators: These associate a symbol with the appropriate operation
template <class T>
inline lcf operator+ (const lcf &arg1, const T &arg2)
```

```
{
  lcf temp = lcf(arg2);
  return arg1.lcf_add(temp);
}

template <class T>
inline lcf operator- (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
  return arg1.lcf_add(temp*-1);
}

template <class T>
inline lcf operator* (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
  return arg1.lcf_mult(temp);
}

template <class T>
inline lcf operator/ (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
  return arg1.lcf_mult(temp.lcf_invert());
}

template <class T>
inline bool operator< (const lcf &arg1, const T &arg2)
{
```

```
  lcf temp = lcf(arg2);
  return arg1.is_less(temp);
}

template <class T>
inline bool operator<= (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
  return (arg1.is_less(temp) || (arg1.is_equal(temp)));
}

template <class T>
inline bool operator> (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
  return arg1.is_greater(arg2);
}

template <class T>
inline bool operator>= (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
  return (arg1.is_greater(arg2)) || (arg1.is_equal(arg2));
}

template <class T>
inline bool operator== (const lcf &arg1, const T &arg2)
{
  lcf temp = lcf(arg2);
```

```
    return arg1.is_equal(arg2);
  }

  lcf NK_sum(std::vector<lcf> &);//Sums all elements in the vector
     using Kahan's summation algorithm.

  //Constants
  const lcf lcf0 = lcf(0);
  const lcf lcf1 = lcf(double(1), rat0);
  const lcf lcfd = lcf(double(1), rat1);
}
#endif
```

A.2.5 difops.h

```
//difops.h
#pragma once
#ifndef _DIFF_H_
#define _DIFF_H_

#include <functional>
#include "levicivita.h"
#include "definitions.h"




namespace tulliotools
{
  std::vector<double> fastdif(std::function<lcf(lcf)>, double);//
     Computes all derivatives up to _SIG_SUPP_-1 and returns them as a
```

```
    vector. Note that this operation only works for functions that
    take real values at real points

  double ndif(std::function<lcf(lcf)>, double, unsigned int);//Computes
     the real part of the nth derivative, this operation works even
    for functions that take non-real values at real points.
}
#endif
```

A.2.6 intops.h

```
#pragma once
#ifndef _int_H_
#define _int_H_

#include <functional>
#include "levicivita.h"
#include "definitions.h"



namespace tulliotools
{
  double real_trap_int(std::function<lcf(lcf)>, double, double);//
    Trapazoidal rule

  double real_simps_int(std::function<lcf(lcf)>, double, double);//
    Simpson's rule

  double real_r_int(std::function<lcf(lcf)>, double, double);//Taylor
    series of the integral to the right using a real length
```

```
lcf mixed_r_int(std::function<lcf(lcf)>, double, lcf&);//Taylor
    series of the integral to the right using a lcf length

double real_l_int(std::function<lcf(lcf)>, double, double);//Taylor
    series of the integral to the left using a real length

lcf mixed_l_int(std::function<lcf(lcf)>, double, lcf&);//Taylor
    series of the integral to the left using a lcf length

double real_c_int(std::function<lcf(lcf)>, double, double, double);//
    Taylor series of the integral from a cental point using real
    lengths

lcf mixed_c_int(std::function<lcf(lcf)>, double, lcf&, lcf&);//Taylor
     series of the integral from a cental point using lcf lengths

double real_darbint(std::function<lcf(lcf)>, std::function<lcf(lcf)>,
     double, double);//Darboux integration between real bounds

double real_tay_step(std::function<lcf(lcf)>, double, double);//
    Integrate using central point taylor series breaking the interval
    of integration into smaller steps

double real_tay_step_sing(std::function<lcf(lcf)>, double, double);//
    Same as above but with modifications to allow for singularities at
     the end points

double real_darb_step(std::function<lcf(lcf)>, std::function<lcf(lcf)
```

```
>, double, double);//Integrat using Darboux's formula breaking the
    interval of integration into smaller steps

  double real_darb_step_sing(std::function<lcf(lcf)>, std::function<lcf
      (lcf)>, double, double);//Modification of the above which uses
      Taylor series on the left and right most steps to allow for
      singularities at the end points
}
#endif
```

A.2.7 functions.h

```
#pragma once
#ifndef _FUNCS_H_
#define _FUNCS_H_
//This header defines a number of functions for inclusion in the
    library.
#include "tulliotoolsv18.h"
#include <vector>
#include <functional>

using namespace std::placeholders;

namespace tulliotools
{

  //Special Functions

  lcf bessel_arg(unsigned int, double, lcf&);
```

```
double bessel(unsigned int, double);

lcf erf_arg(lcf&);

lcf erf(lcf&);

double factorial(unsigned int);

double bico(unsigned int, unsigned int);

std::vector<double> bernoulli(unsigned int);

std::vector<double> bernoulli_rec(unsigned int);

std::vector<double> bernoulli_gf(unsigned int);

std::vector<double> bernoulli_tan(unsigned int);

std::vector<double> bernoulli_tanh(unsigned int);

std::vector<double> bernoulli_init();

lcf nroot(lcf&, unsigned int);

//Trig Functions

lcf sin_taylor(lcf&, unsigned int n = _SIG_TERMS_);

lcf sin(lcf&);
```

```
lcf cos_taylor(lcf&, unsigned int n = _SIG_TERMS_);

lcf cos(lcf&);

lcf tan(lcf&);

lcf exp_taylor(lcf&, unsigned int n = _SIG_TERMS_);

lcf exp(lcf&);

lcf cosh_taylor(lcf&, unsigned int n = _SIG_TERMS_);

lcf cosh(lcf&);

lcf sinh_taylor(lcf&, unsigned int n = _SIG_TERMS_);

lcf sinh(lcf&);

lcf tanh(lcf&);

lcf ln_taylor(lcf&, unsigned int n = _SIG_TERMS_);

lcf ln(lcf&);

lcf arcarg(lcf&);

lcf arcsin(lcf&);
```

```
lcf arccos(lcf&);

lcf arctanarg(lcf&);

lcf arctan(lcf&);

//Test Functions
lcf examp1(lcf&);

lcf examp2(lcf&);

lcf examp3(lcf&);

lcf conj1(lcf&);//integral from 0 to 1 conjectured to be pi^2 *(2-
    sqrt(2))/32

lcf conj2(lcf&);//integral from 0 to pi/4 conjectured to be -pi^2/16
    + pi*ln(2)/4 +G(catalan's constant)

lcf conj3(lcf&);//integral from 0 to pi conjectured to be pi^2/4

lcf testfunc0(lcf&);

lcf testfunc1(lcf&);//integral from 0 to 1 is 1/4

lcf testfunc3(lcf&);//integral from 0 to pi/2 is (e^(pi/2)-1)/2

lcf testfunc8(lcf&);//integral from 0 to 1 is 2
```

```
  lcf testfunc9(lcf&);//integral from 0 to pi/2 is -pi*ln(2)/2

  lcf test1(lcf&);

  lcf test2(lcf&);

  //polynomials (used for Darboux Integration)

  lcf bernoulli_poly(lcf&, unsigned int);

  lcf bernoulli_phi(lcf&);

  lcf lb_taylor_poly(lcf&, unsigned int);

  lcf ub_taylor_poly(lcf&, unsigned int);

  lcf experimental_poly(lcf&, unsigned int);

  lcf experimental_phi(lcf&);

  lcf test_poly(lcf&, unsigned int);

  lcf simp_poly(lcf&);

  lcf trap_poly(lcf&);

}
#endif
```

A.3 Source Files

A.3.1 rational.cpp

```
//rational.cpp

#include "rational.h"
#include <math.h>
#include <algorithm>//floor function

using namespace tulliotools;

unsigned int tulliotools::gcd(unsigned int x, unsigned int y) //Euclids
    Algorithm
{
  unsigned int temp;
  while (y != 0)
  {
    temp = y;
    y = x % temp;
    x = temp;
  }
  return x;
}

unsigned int tulliotools::lcm(unsigned int x, unsigned int y)//Recall
    that lcm(x,y)*gcd(x,y)=x*y
{
  return (x*y) / gcd(x, y);
```

```
}

void rat::operator= (const int &x)
{
  if (x < 0)
  {
    this->numerator = x*(-1);
    this->sign = true;
  }
  else
  {
    this->numerator = x;
    this->sign = false;
  }

  this->denominator = 1;
}

void rat::operator= (const float &x)
{
  if (x < 0)
  {
    this->sign = true;
    this->numerator = floor(((-1)*x)*(std::pow(10, _SIG_DIGS_)));//Find
        the numerator to the appropriate percission
  }
  else
  {
    this->sign = false;
```

```
    this->numerator = floor(x*(std::pow(10, _SIG_DIGS_)));//Find the
        numerator to the appropriate percission
  }
  this->denominator = std::pow(10, _SIG_DIGS_);//Find the denominator
      to the appropriate percission
  this->simplify();//simplify the result so it is in lowest form.
}

rat::operator int()
{
  int result = (this->numerator / this->denominator);
  if (2 * (this->denominator % this->numerator) > this->denominator)//
      If this condition is satisfied then we want to round up.
  {
    result = result + 1;//rounding up
  }

  if (sign == true) return (-1)*result;//This instruction ensures that
      the sign is correct
  else return result;
}

rat::operator float()
{
  float result = float(this->numerator) / float(this->denominator);

  if (sign == true) return (-1)*result;//Ensure that the sign is
      correct
  else return result;
```

```
}

unsigned int rat::get_num() const
{
  return this->numerator;
}

unsigned int rat::get_den() const
{
  return this->denominator;
}

bool rat::get_sign() const
{
  return this->sign;
}

rat rat::rat_add(const rat &x) const
{
  rat result;
  bool sign;
  unsigned int numerator, denominator;
  int temp1 = lcm(this->denominator, x.denominator);//Find the lcm of
      the denominators
  int temp2 = this->numerator*(temp1 / this->denominator);//set temp2
      so that temp2/lcm = this->numerator/this->denominator. We do the
      samething for x in the next line.
  int temp3 = x.numerator*(temp1 / x.denominator);//After this step
      both rat's will have the same denominator.
```

```
  if (this->sign == true) temp2 = temp2*(-1); //Account for the signs
  if (x.sign == true) temp3 = temp3*(-1); //Dito

  int temp4 = temp2 + temp3; //Since both rat;'s have a common
     denominator we may simply add their numerators.
  if (temp4 < 0) sign = true, numerator = (unsigned)(temp4*(-1)); //Set
     the sign and numerator if the numerator is negitive
  else sign = false, numerator = (unsigned)temp4; //Set the sign and
     numerator if the numerator is possitive

  denominator = (unsigned)temp1; //Denominator is the lcm

  result = rat(sign, numerator, denominator); //creat the new number
  result.simplify(); //ensure the new number is in lowest form
  return result;
}

rat rat::rat_mult(const rat &x) const
{
  rat result;
  bool sign = ((this->sign) != (x.sign));
  int temp1 = this->numerator*x.numerator;
  int temp2 = this->denominator*x.denominator;
  result = rat(sign, temp1, temp2);
  result.simplify();
  return result;
}
```

```
rat rat::rat_invert() const
{
  rat result = rat(this->sign, this->denominator, this->numerator);
  result.simplify();
  return result;
}

rat rat::rat_dev(const rat &x) const
{
  rat result;
  bool sign = ((this->sign) || (x.sign));

  result = rat(sign, (this->numerator)*(x.denominator), (this->
     denominator)*(x.numerator));//Invert and multiply
  result.simplify();
  return result;
}

rat rat::exp(unsigned int n) const
{
  rat result = rat(false, 1, 1);

  for (unsigned int i = 0; i < n; i++)
  {
    result = this->rat_mult(result);//Multiply by itself the requisite
       number of times
  }
  result.simplify();
  return result;
```

```
}

bool rat::is_greater(const rat &x) const
{
  if ((this->sign == true) && (x.sign == false)) return false;//If this
       is negitive and x is positive then clearly this is not greater
      than x.
  else if ((this->sign == false) && (x.sign == true)) return true;//On
      the other hand, if this is positive and x is negitive then this
      must be greater.
  else if ((this->sign == false) && (x.sign == false))//If both are
      positive we compare directly
  {
    if (this->numerator*x.denominator > x.numerator*this->denominator)
        return true;
    else return false;
  }
  else//The only remaining case is that both are negitive
  {
    if (this->numerator*x.denominator < x.numerator*this->denominator)
        return true;
    else return false;
  }
}

bool rat::is_less(const rat &x) const//Is_less just reverses the order
    of the two numbers. obviously if x>y then y<x.
{
  if (x.is_greater(*this)) return true;
```

```
  else return false;
}

bool rat::is_equal(const rat &x) const//If neither is greater than the
    other they must be equal.
{
  if ((this->is_greater(x) == false) && (this->is_less(x) == false))
     return true;
  else return false;
}

rat::rat() = default;//Constructs a rat with empty values

rat::rat(bool x, unsigned int y, unsigned int z)//Constructs a rat with
     given sign, numerator, and denominator.
{
  this->sign = x;
  this->numerator = y;
  this->denominator = z;
}

rat::rat(const int &x)//Constructs a rat from a given integer
{
  if (x < 0)
  {
    this->numerator = x*(-1);//Since x is negative this will make the
        numerator positive
    this->sign = true;//Set the sign accordingly
  }
```

```
  else
  {
    this->numerator = x;//x is positive so no change needs to be made
    sign = false;
  }

  this->denominator = 1;
}

rat::rat(const unsigned int& x)
{
  this->sign = false;
  this->numerator = x;
  this->denominator = 1;
}

rat::rat(const double &x)
{
  if (x < 0)
  {
    this->sign = true;
    this->numerator = floor(((-1)*x)*(10 ^ (_SIG_DIGS_)));
  }
  else
  {
    this->sign = false;
    this->numerator = floor(x*(10 ^ (_SIG_DIGS_)));
  }
  this->denominator = 10 ^ (_SIG_DIGS_);
```

```
  this->simplify();
}

void rat::simplify()//Puts this into its simplist terms.
{
  unsigned int temp = gcd(this->numerator, this->denominator);//find
      the largest number that devides both the numerator and the
      denominator
  this->numerator = this->numerator / temp;//factor that number out of
      the numerator
  this->denominator = this->denominator / temp;//and again out of the
      denominator
}
```

A.3.2 levicivita.cpp

```
#include "levicivita.h"
#include "rational.h"



using namespace tulliotools;
using std::vector;

rat lcf::get_order() const
{
  return this->support.front();//Return the first element in the
      support vector
}
```

```
rat lcf::get_depth() const
{
  return this->support.back();//Return the last element in the support
      vector
}

double lcf::get_real() const
{
  return this->get_value(rat(false, 0, 1));//Return the value of the
      element at the rational point zero
}

std::vector<double> lcf::get_value() const
{
  return this->value;
}

double lcf::get_value(const rat &x) const
{
  unsigned int size = this->support.size();
  bool found_flag = false;
  double result;

  for (unsigned int i = 0; i < size; i++)//In this loop we search the
      support vecter for the provided support point...
  {
    if ((this->support[i] == x) && (found_flag == false))
    {
      found_flag = true;//...if we find it then we end the loop...
```

```
      result = this->value[i]; //...and return the vale at that support
          point.
    }
  }

  if (found_flag == false) result = 0; //If we don't find the argument
      in the support vecter then we return 0.

  return result;
}

double lcf::get_value(const unsigned int &x) const
{
  if (x < this->value.size()) return this->value[x]; //Check that the
      argument refrences a valid vector entry.
  else return 0;
}

std::vector<rat> lcf::get_support() const
{
  return this->support;
}

rat lcf::get_support(const unsigned int &x) const
{
  if (x < this->support.size()) return this->support[x];
  else return rat(false, 0, 1); //returns zero if x is out of range. Not
       sure this is the best way to do it but it will work for now.
}
```

```
bool lcf::is_greater(const lcf &x) const
{
  bool null_flag_this = false, null_flag_x = false;
  if (this->support.size() == 0) null_flag_this = true;
  if (x.support.size() == 0) null_flag_x = true;

  if (null_flag_this == true && null_flag_x == true) return false;//
     Checkes for two zero elements
  else if (null_flag_this == true || null_flag_x == true)
  {
    if (null_flag_this == true)
    {
      if (x.value[0] < 0) return true;
      else return false;
    }
    else
    {
      if (this->value[0] > 0) return true;
      else return false;
    }
  }
  else
  {
    unsigned int i = 0;
    bool result, finished_flag = false;
    while (finished_flag == false)//finished_flag is changed to TRUE
       when we find a support point where the elements take on
       different values
```

```
{

  if (this->support[i] < x.support[i])//This and the following
      condition selevt for the element with the lesser support point
  {
    result = true;
    finished_flag = true;
  }
  else if (this->support[i] > x.support[i])
  {
    result = false;
    finished_flag = true;
  }
  else//If this condition is reached it means that both elements
      have the same ith support point, we proceed by comparing their
       values at that support point
  {
    if (this->value[i] > x.value[i])
    {
      result = true;
      finished_flag = true;
    }
    else if (this->value[i] < x.value[i])
    {
      result = false;
      finished_flag = true;
    }
    else//If this condition is reached it means that the elements
         the elements share the same value at their ith support point
```

```
{
  if ((i >= this->support.size() - 1) || (i >= x.support.size()
      - 1))//If we have checked all the values of one of the
      elements then we make the dicision based on the next non-
      zero value of the element with more support points, if
      both elements have the same number of support points and
      we have still arived here then they must be equal
  {
    if (this->support.size() > x.support.size())
    {
      if (this->get_value(i + 1) > 0) result = true;
      else result = false;
    }
    else if(x.support.size()>this->support.size())
    {
      if (x.get_value(i + 1) > 0) result = false;
      else result = true;
    }
    else result = false;//here the two elements must be exactly
        equal
    finished_flag = true;
  }
  else
  {
    i++;
  }
}
}
```

```
    }
    return result;
  }
}

bool lcf::is_equal(const lcf &x) const
{
  if (this->is_greater(x) == false && this->is_less(x) == false) return
      true;
  else return false;
}

bool lcf::is_less(const lcf &x) const
{
  return x.is_greater(*this);
}


lcf lcf::lcf_add(const lcf &x) const//Addition
{
  vector<double> temp_val;
  vector<rat>temp_supp;
  unsigned int i = 0, j = 0, size1 = this->support.size(), size2 = x.
      support.size();
  while ((i < size1) && (j < size2))
  {
    if (this->support[i] < x.support[j])
    {
      temp_val.push_back(this->value[i]);
```

```
    temp_supp.push_back(this->support[i]);
    i++;
  }
  else if (this->support[i] == x.support[j])//If the two support
      points are equal then we add the corisponding values and include
       that in the new element
  {
    temp_val.push_back(this->value[i] + x.value[j]);
    temp_supp.push_back(this->support[i]);
    i++;
    j++;
  }
  else
  {
    temp_val.push_back(x.value[j]);
    temp_supp.push_back(x.support[j]);
    j++;
  };
};

if (j < size2)//Add all the remaining terms of the second element if
    there are any
{
  while (j < size2)
  {
    temp_val.push_back(x.value[j]);
    temp_supp.push_back(x.support[j]);
    j++;
  }
```

```
  }
  if (i < size1)//Add all the remaining terms of the first element if
     there are any
  {
    while (i < size1)
    {
      temp_val.push_back(this->value[i]);
      temp_supp.push_back(this->support[i]);
      i++;
    }
  }

  return lcf(temp_val, temp_supp);
}



lcf lcf::lcf_mult(const lcf &x) const//Multiplication
{
  lcf result;
  vector<double> temp_val;
  vector<rat> temp_supp;
  for (unsigned int i = 0; i < this->support.size(); i++)
  {
    for (unsigned int j = 0; j < x.support.size(); j++)
    {
      temp_val.push_back(this->value[i] * x.value[j]);
      temp_supp.push_back((this->support[i]) + (x.support[j]));
    }
  }
```

```
  result = lcf(temp_val, temp_supp);
  result.simplify();//This is an important step as the above process
     may include the same support point multiple times, as part of the
     simplification procedure these will all be collected together
  return result;
}


lcf lcf::lcf_invert(unsigned int n) const//Find the inverse of a given
   number up to depth n
{

  if (this->value.size() == 0)//This insures the program dosent crash
     if we try to devied by zero, in proper operation this condition
     should never evaluate as true
  {
    return lcf();
  }
  else if (this->value.size() == 1)//If there is only one term
     inversion is trivial
  {
    vector<double> new_value;
    new_value.push_back(1 / this->value[0]);

    vector<rat> new_support;
    new_support.push_back(support[0] * rat(true, 1, 1));

    return lcf(new_value, new_support);
  }
```

```
  else//all lcf numbers can be expressed as ad^q(1+s) so inverse is a
      ^{-1}d^{-q}(1-s+s^2-...)
  {
    lcf factor = lcf(double(1) / this->value[0], this->support[0] * rat
        (true, 1, 1));
    lcf s;
    lcf t = lcf1, taylor = lcf1;

    for (unsigned int i = 1; i < this->value.size(); i++)
    {
      s.value.push_back(this->value[i] / this->value[0]);
      s.support.push_back(this->support[i] - this->support[0]);
    }

    s = s*(-1);

    for (unsigned int i = 0; i < n; i++)
    {
      t = t.lcf_mult(s);
      taylor = taylor.lcf_add(t);
    }

    return factor.lcf_mult(taylor);
  }
}

lcf lcf::exp(unsigned int n) const//Exponents
{
  lcf result = lcf1;
```

```
  for (unsigned int i = 0; i < n; i++)
  {
    result = this->lcf_mult(result);
  }
  return result;
}

lcf lcf::abs() const
{
  lcf result;
  if (this->is_less(lcf0)) result = lcf1*(-1);
  else result = lcf1;
  return this->lcf_mult(result);
}



lcf::lcf() = default;//Default constructor



lcf::lcf(std::vector<double> &x, std::vector<rat> &y)//multi-support
    point constructor
{
  this->value = x;
  this->support = y;
  this->simplify();
}

lcf::lcf(const double &x, const rat &y)//single support point
```

```
    constructor.
{
  this->value.push_back(x);
  this->support.push_back(y);
  this->simplify();
}

void lcf::simplify()//This method simplifies its lcf instance in the
    sense that it puts the support points in increasing order and
    ensures that each point ocurres only once in the support all while
    keeping track of the value vector to ensure the element remains
    unchanged. It also removes any entries where the value is zero.
{
  vector<unsigned int> positions;
  vector<double> temp_val;
  vector<rat> temp_supp;
  double val;
  rat min;
  while (support.size() != 0)
  {
    val = 0;
    min = this->support[0];
    for (unsigned int i = 1; i < this->support.size(); i++)//This loop
        finds the minimum support point
    {
      if ((min<support[i]) == false)
      {
        min = this->support[i];
      }
```

```
  }

  for (unsigned int i = 0; i < this->support.size(); i++)//creates a
      vector with the positions of all entries with support point
      equal to min
  {
    if (min == this->support[i]) positions.push_back(i);
  }

  for (unsigned int i = positions.size(); i > 0; i--)//Adds the
      values at the positions found above.
  {
    val += this->value[positions[i - 1]];
    value.erase(value.begin() + positions[i - 1]);
    support.erase(support.begin() + positions[i - 1]);
  }

  if (val != 0)//If val=0 we dont include it in the vector
  {
    temp_val.push_back(val);
    temp_supp.push_back(min);
  }
  positions.clear();
}

if (temp_val.size() > _SIG_SUPP_)//If the result is longer than the
    desired depth then we truncate
{
  temp_val.erase(temp_val.begin() + (_SIG_SUPP_), temp_val.end());
```

```
    temp_supp.erase(temp_supp.begin() + (_SIG_SUPP_), temp_supp.end());
  }

  if (temp_supp.size() == 0)//This condition is included incase our
      simplification proccess reduces the variable to 0.
  {
    temp_val.clear();
    temp_val.push_back(double(0));

    temp_supp.clear();
    temp_supp.push_back(rat0);
  }

  this->value = temp_val;
  this->support = temp_supp;
}

lcf tulliotools::NK_sum(std::vector<lcf> &input)//Employs a modified
    version of Kahan's algorith in an attempt to limit rounding error,
    it doesnt seem to help much
{
  double sum, rem, temp;
  rat order = input[0].get_order();
  std::vector<rat> supp;
  std::vector<double> sig_terms, sum_terms;
  std::vector<vector<double> > terms;
  for (unsigned int i = 1; i < input.size(); i++)
  {
    if (order > input[i].get_order()) order = input[i].get_order();
```

```
  }
  for (unsigned int i = 0; i < input.size(); i++)
  {
    for (int j = 0; j < _SIG_SUPP_; j++)
    {
      sig_terms.push_back(input[i].get_value(order + j));
    }
    terms.push_back(sig_terms);
    sig_terms.clear();
  }
  for (unsigned int i = 0; i < _SIG_SUPP_; i++)
  {
    sum = terms[0][i];
    rem = 0;
    for (unsigned int j = 1; j < terms.size(); j++)
    {
      temp = sum + terms[j][i];
      if (abs(sum) > abs(terms[j][i])) rem += sum - temp + terms[j][i];
      else rem += terms[j][i] - temp + sum;
      sum = temp;
    }
    sum = sum + rem;
    sum_terms.push_back(sum);
    supp.push_back(order + i);
  }
  return lcf(sum_terms, supp);
}
```

A.3.3 difops.cpp

```cpp
#include "difops.h"
#include "functions.h"
#include <functional>

using namespace tulliotools;


std::vector<double> tulliotools::fastdif(std::function<lcf(lcf)> func,
    double x)
{
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg);//Evaluate the given function at the given
      point + d
  int depth = int(temp1.get_depth());
  double temp2;
  std::vector<double> result;
  double fact = double(1);

  for (int i = 0; i < depth; i++)//The coeffecients of f(x+d) are of
      the form f^n(x)/n! in this loop we multiply by n! to obtain the
      derivative
  {
    temp2 = temp1.get_value(rat(i));
    temp2 = temp2 * fact;
    result.push_back(temp2);
    fact = fact*(i + 1);
  }
  return result;
}
```

```
double tulliotools::ndif(std::function<lcf(lcf)> func, double x,
    unsigned int n)
{
  lcf arg = lcf(x), den = lcfd.exp(n), num = lcf0;
  for (unsigned int i = 0; i <= n; i++)
  {
    num = num + func(arg)*bico(n, i)*pow(-1, i);
    arg = arg - lcfd;
  }
  return (num / den).get_real();
}
```

A.3.4 intops.cpp

```
#include "intops.h"
#include "definitions.h"
#include "functions.h"
#include <algorithm>//std::min
#include <vector>

using namespace tulliotools;

double tulliotools::real_trap_int(std::function<lcf(lcf)> func, double
    lb, double ub)//The Trapazoidal rule
{
  lcf l = lcf(lb), u = lcf(ub), h = lcf((ub - lb) / _STEP_NO_);
  double result = func(l).get_real() + func(u).get_real();
  for (unsigned int i = 1; i < _STEP_NO_ - 1; i++)
  {
```

```
    result += 2 * func(l + h*i).get_real();
  }
  return result*(ub - lb) / (2 * _STEP_NO_);
}

double tulliotools::real_simps_int(std::function<lcf(lcf)> func, double
    lb, double ub)//Simpson's rule
{
  lcf l = lcf(lb), u = lcf(ub), h = lcf((ub - lb) / _STEP_NO_);
  double result = func(l).get_real() + func(u).get_real();
  for (unsigned int i = 1; i < _STEP_NO_ - 1; i++)
  {
    if (i % 2 == 0) result += 2 * func(l + h*i).get_real();
    if (i % 2 == 1) result += 4 * func(l + h*i).get_real();
  }
  return result*(ub - lb) / (3 * _STEP_NO_);
}

double tulliotools::real_r_int(std::function<lcf(lcf)> func, double x,
   double l)
{
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg);
  int depth = int(temp1.get_depth());
  double temp2;
  double lpow = l;
  double result = 0;
  for (int i = 0; i <= depth; i++)
  {
```

```
    temp2 = temp1.get_value(rat(i));
    temp2 = temp2 / (i + 1);
    temp2 = temp2*lpow;
    result = result + temp2;
    lpow = lpow*l;
  }
  return result;
}

lcf tulliotools::mixed_r_int(std::function<lcf(lcf)> func, double x,
   lcf& l)
{
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg), result = lcf0, lpow = l, temp2;
  int depth = int(temp1.get_depth());
  for (int i = 0; i <= depth; i++)
  {
    temp2 = lcf(temp1.get_value(rat(i)));
    temp2 = temp2 / (i + 1);
    temp2 = temp2*lpow;
    result = result + temp2;
    lpow = lpow*l;
  }
  return result;
}

double tulliotools::real_l_int(std::function<lcf(lcf)> func, double x,
   double l)
{
```

```
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg);
  int depth = int(temp1.get_depth());
  double temp2;
  double lpow = 1;
  double result = 0;
  for (int i = 0; i <= depth; i++)
  {
    temp2 = temp1.get_value(rat(i));
    temp2 = temp2 / (i + 1);
    temp2 = temp2*lpow;
    result = result + temp2;
    lpow = lpow*(-1);
  }
  return result;
}

lcf tulliotools::mixed_l_int(std::function<lcf(lcf)> func, double x,
   lcf& l)
{
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg), lpow = l, result = lcf0, temp2;
  int depth = int(temp1.get_depth());
  for (int i = 0; i <= depth; i++)
  {
    temp2 = lcf(temp1.get_value(rat(i)));
    temp2 = temp2 / (i + 1);
    temp2 = temp2*lpow;
    result = result + temp2;
```

```
    lpow = lpow*(l*(-1));
  }
  return result;
}

double tulliotools::real_c_int(std::function<lcf(lcf)> func, double x,
   double ll, double rl)//l/rl=left/right length
{
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg);
  int depth = int(temp1.get_depth());
  double temp2;
  double llpow = ll, rlpow = rl;
  double result = 0;
  for (int i = 0; i <= depth; i++)
  {
    temp2 = temp1.get_value(rat(i));
    temp2 = temp2 / (i + 1);
    temp2 = temp2 * (llpow + rlpow);
    result = result + temp2;
    llpow = llpow*(ll*(-1));
    rlpow = rlpow*rl;
  }
  return result;
}

lcf tulliotools::mixed_c_int(std::function<lcf(lcf)> func, double x,
   lcf& ll, lcf& rl)
{
```

```
  lcf arg = lcf(x) + lcfd;
  lcf temp1 = func(arg), llpow = ll, rlpow = rl, result = lcf0, temp2;
  int depth = int(temp1.get_depth());
  for (int i = 0; i <= depth; i++)
  {
    temp2 = lcf(temp1.get_value(rat(i)));
    temp2 = temp2 / (i + 1);
    temp2 = temp2 *(llpow + rlpow);
    result = result + temp2;
    llpow = llpow*(ll*(-1));
    rlpow = rlpow*rl;
  }
  return result;
}

double tulliotools::real_darbint(std::function<lcf(lcf)> poly, std::
   function<lcf(lcf)> func, double lb, double ub)
{
  lcf arglb = lcfd + lb, argub = lcfd + ub, funclb = func(arglb),
     funcub = func(argub);
  std::vector<double> polylb = fastdif(poly, double(0)), polyub =
     fastdif(poly, double(1));
  double sum = double(0), length = ub - lb;

  for (unsigned int i = 0; i < _SIG_SUPP_ - 1; i++)
  {
    sum += pow(-1, i)*pow(length, i + 1)*factorial(i)*(polyub[
       _SIG_SUPP_ - 2 - i] * funcub.get_value(rat(i)) - polylb[
       _SIG_SUPP_ - 2 - i] * funclb.get_value(rat(i)));
```

```
  }
  return sum / polyub[_SIG_SUPP_ - 1];
}

double tulliotools::real_tay_step(std::function<lcf(lcf)> func, double
   lb, double ub)
{
  double h = (ub - lb) / _STEP_NO_, result = 0;

  for (unsigned int i = 0; i < _STEP_NO_ + 1; i++)
  {
    result += real_c_int(func, lb + i*h + h / 2, h / 2, h / 2);
  }
  return result;
}

double tulliotools::real_tay_step_sing(std::function<lcf(lcf)> func,
   double lb, double ub)
{
  double h = (ub - lb) / _STEP_NO_, result = real_l_int(func, lb + h, h
     ) + real_r_int(func, ub - h, h);

  for (unsigned int i = 1; i < _STEP_NO_; i++)
  {
    result += real_c_int(func, lb + i*h + h / 2, h / 2, h / 2);
  }
  return result;
}
```

```
double tulliotools::real_darb_step(std::function<lcf(lcf)> poly, std::function<lcf(lcf)> func, double lb, double ub)
{
  double h = (ub - lb) / _STEP_NO_, result = 0;
  for (unsigned int i = 0; i < _STEP_NO_ + 1; i++)
  {
    result += real_darbint(poly, func, lb + i*h, lb + (i + 1)*h);
  }
  return result;
}

double tulliotools::real_darb_step_sing(std::function<lcf(lcf)> poly, std::function<lcf(lcf)> func, double lb, double ub)
{
  double h = (ub - lb) / _STEP_NO_, result = real_l_int(func, lb + h, h) + real_r_int(func, ub - h, h);
  for (unsigned int i = 1; i < _STEP_NO_ - 1; i++)
  {
    result += real_darbint(poly, func, lb + i*h, lb + (i + 1)*h);
  }
  return result;
}
```

A.3.5 trigfuncs.cpp

```
#include <math.h>
#include "functions.h"
#include "intops.h"

using namespace tulliotools;
```

```
lcf tulliotools::sin_taylor(lcf& x, unsigned int n)
{
  lcf result = lcf0;
  lcf temp;
  lcf xpowers = x;
  lcf xincrease = x.exp(2);
  double factorial = double(1);
  for (unsigned int i = 0; i < n + 1; i++)
  {
    temp = xpowers / factorial;
    temp = temp*pow(-1, i);
    result = result + temp;

    xpowers = xpowers*xincrease;
    factorial = factorial*(2 * (i + 1))*(2 * (i + 1) + 1);
  }
  return result;
}

lcf tulliotools::sin(lcf& x)
{
  double real_part = x.get_real();
  lcf inf_part = x - real_part;
  lcf term1, term2;

  term1 = cos_taylor(inf_part)*std::sin(real_part);
  term2 = sin_taylor(inf_part)*std::cos(real_part);
```

```
  return term1 + term2;
}

lcf tulliotools::cos_taylor(lcf& x, unsigned int n)
{
  lcf result = lcf0;
  lcf temp;
  lcf xpowers = lcf1;
  lcf xincrease = x.exp(2);
  double factorial = double(1);
  for (unsigned int i = 0; i < n + 1; i++)
  {
    temp = xpowers / factorial;
    temp = temp*pow(-1, i);
    result = result + temp;

    xpowers = xpowers*xincrease;
    factorial = factorial*(2 * i + 1)*(2 * i + 2);
  }
  return result;
}

lcf tulliotools::cos(lcf& x)
{
  double real_part = x.get_real();
  lcf inf_part = x - real_part;
  lcf term1, term2;

  term1 = cos_taylor(inf_part)*std::cos(real_part);
```

```
  term2 = sin_taylor(inf_part)*std::sin(real_part);

  return term1 - term2;
}

lcf tulliotools::tan(lcf& x)
{
  return sin(x) / cos(x);
}

lcf tulliotools::exp_taylor(lcf& x, unsigned int n)
{
  lcf result = lcf1;
  lcf temp;
  lcf xpowers = x;
  lcf xincrease = x;
  double factorial = double(1);
  for (unsigned int i = 0; i < n + 1; i++)
  {
    temp = xpowers / factorial;
    result = result + temp;

    xpowers = xpowers*xincrease;
    factorial = factorial*(i + 2);
  }
  return result;
}

lcf tulliotools::exp(lcf& x)
```

```
{
  double real_part = x.get_real();
  lcf inf_part = x - real_part;
  lcf term;

  term = exp_taylor(inf_part);
  term = term*std::exp(real_part);

  return term;
}



lcf tulliotools::cosh_taylor(lcf& x, unsigned int n)
{
  lcf result = lcf0;
  lcf temp;
  lcf xpowers = lcf1;
  lcf xincrease = x.exp(2);
  double factorial = double(1);
  for (unsigned int i = 0; i < n + 1; i++)
  {
    temp = xpowers / factorial;
    result = result + temp;

    xpowers = xpowers*xincrease;
    factorial = factorial*(2 * i + 1)*(2 * i + 2);
  }
  return result;
}
```

```
lcf tulliotools::cosh(lcf& x)
{
  double real_part = x.get_real();
  lcf inf_part = x - real_part;
  lcf term1, term2;

  term1 = cosh_taylor(inf_part)*std::cosh(real_part);
  term2 = sinh_taylor(inf_part)*std::sinh(real_part);

  return term1 + term2;
}


lcf tulliotools::sinh_taylor(lcf& x, unsigned int n)
{
  lcf result = lcf0;
  lcf temp;
  lcf xpowers = x;
  lcf xincrease = x.exp(2);
  double factorial = double(1);
  for (unsigned int i = 0; i < n + 1; i++)
  {
    temp = xpowers / factorial;
    result = result + temp;

    xpowers = xpowers*xincrease;
    factorial = factorial*(2 * (i + 1))*(2 * (i + 1) + 1);
  }
```

```
  return result;
}

lcf tulliotools::sinh(lcf& x)
{
  double real_part = x.get_real();
  lcf inf_part = x - real_part;
  lcf term1, term2;

  term1 = cosh_taylor(inf_part)*std::sinh(real_part);
  term2 = sinh_taylor(inf_part)*std::cosh(real_part);

  return term1 + term2;
}

lcf tulliotools::tanh(lcf& x)
{
  return sinh(x) / cosh(x);
}

lcf tulliotools::ln_taylor(lcf& x, unsigned int n)
{
  lcf result = lcf0;
  lcf temp;
  lcf xpowers = x;
  int sign = 1;
  for (unsigned int i = 0; i < n + 1; i++)
  {
    temp = xpowers / (i + 1);
```

```
    temp = temp*sign;
    result = result + temp;

    xpowers = xpowers*x;
    sign = sign*-1;
  }
  return result;
}

lcf tulliotools::ln(lcf& x)
{
  double real_part = x.get_real();
  lcf inf_part = x - real_part;
  lcf term1, term2;
  term1 = log(real_part);
  term2 = ln_taylor(inf_part / real_part);
  return term1 + term2;
}

lcf tulliotools::arcarg(lcf& x)
{
  lcf result = lcf1 - x.exp(2);
  result = nroot(result, 2);
  result = lcf1 / result;
  return result;
}


lcf tulliotools::arcsin(lcf& x)
```

```
{
  double real_x = x.get_real();
  lcf l = x - real_x;

  return mixed_r_int(arcarg, real_x, l) + asin(real_x);
}

lcf tulliotools::arccos(lcf& x)
{
  double real_x = x.get_real();
  lcf l = x - real_x;

  return mixed_r_int(arcarg, real_x, l)*(-1) + acos(real_x);
}

lcf tulliotools::arctanarg(lcf& x)
{
  lcf term = x.exp(2) + lcf1;
  return lcf1 / term;
}


lcf tulliotools::arctan(lcf& x)
{
  double real_x = x.get_real();
  lcf l = x - real_x;
  return mixed_r_int(arctanarg, real_x, l) + atan(real_x);
}
```

A.3.6 specfuncs.cpp

```
#include "functions.h"
#include <algorithm>//for the min function
#include <functional>//for bind
#include <math.h>//for floor

using tulliotools::lcfd;
using tulliotools::lcf1;
using namespace tulliotools;
using namespace std::placeholders;

lcf tulliotools::bessel_arg(unsigned int n, double x, lcf& t)
{
  return cos(t*n - sin(t)*x);
}

double tulliotools::bessel(unsigned int n, double x)
{
  auto temp_func = std::bind(bessel_arg, n, x, _1);
  return real_r_int(temp_func, 0, M_PI);
}

lcf tulliotools::erf_arg(lcf& t)
{
  return exp(t.exp(2)*(-1));
}

lcf tulliotools::erf(lcf& x)
```

```
{
  return mixed_c_int(erf_arg, 0, x, x) / sqrt(M_PI);
}

double tulliotools::factorial(unsigned int n)
{
  return (n == double(0) || n == double(1)) ? 1 : factorial(n - 1)*n;
}

double tulliotools::bico(unsigned int n, unsigned int k)
{
  k = std::min(k, n - k);
  double num = double(1), den = double(1);
  if (k == 0) return double(1);
  else
  {
    for (unsigned int i = 1; i <= k; i++)
    {
      num *= n + 1 - i;
      den *= i;
    }
    return num / den;
  }
}

std::vector<double> tulliotools::bernoulli(unsigned int n)
{
  std::vector<double> result;
  double sum = 0;
```

```
  for (unsigned int i = 0; i <= n; i++)
  {
    sum = 0;
    for (unsigned int j = 0; j <= i; j++)
    {
      for (unsigned int k = 0; k <= j; k++)
      {
        sum += pow(k, i)*bico(j, k)*pow(-1, k) / (j + 1);
      }


    }
    result.push_back(sum);
  }
  return result;
}

std::vector<double> tulliotools::bernoulli_rec(unsigned int n)
{
  std::vector<double> result;
  double sum = 0;
  for (unsigned int i = 0; i <= n; i++)
  {
    if (i == 0) result.push_back(1);
    else
    {
      sum = 0;
      for (unsigned int j = 0; j < i; j++)
      {
        sum -= bico(i, j)*result[j] / (i - j + 1);
```

```
      }
      result.push_back(sum);
    }
  }
  return result;
}

std::vector<double> tulliotools::bernoulli_gf(unsigned int n)
{
  n = int(n);
  std::vector<double> result;
  lcf arg = lcfd, temp = arg / (exp(arg) - 1);

  for (int i = 0; i < n; i++)
  {
    result.push_back(temp.get_value(rat(i))*factorial(i));
  }
  return result;
}

std::vector<double> tulliotools::bernoulli_tan(unsigned int n)
{
  n = int(n);
  std::vector<double> result;
  lcf arg = lcfd, temp = tan(arg);
  result.push_back(1);
  result.push_back(-0.5);
  for (int i = 1; i < n - 1; i++)
  {
```

```
    result.push_back(temp.get_value(rat(i))*factorial(i + 1) / (pow(-1,
        floor((i + 1) / 2) - 1)*pow(2, i + 1)*(pow(2, i + 1) - 1)));
  }
  return result;
}

std::vector<double> tulliotools::bernoulli_tanh(unsigned int n)
{
  n = int(n);
  std::vector<double> result;
  lcf arg = lcfd, temp = tanh(arg);
  result.push_back(1);
  result.push_back(-0.5);
  for (int i = 1; i < n - 1; i++)
  {
    result.push_back(temp.get_value(rat(i))*factorial(i + 1) / (pow(2,
       i + 1)*(pow(2, i + 1) - 1)));
  }
  return result;
}

std::vector<double> tulliotools::bernoulli_init()
{
  std::vector<double> result;
  result.push_back(_BERN_0_);
  result.push_back(_BERN_1_);
  result.push_back(_BERN_2_);
  result.push_back(double(0));
  result.push_back(_BERN_4_);
```

```
result.push_back(double(0));
result.push_back(_BERN_6_);
result.push_back(double(0));
result.push_back(_BERN_8_);
result.push_back(double(0));
result.push_back(_BERN_10_);
result.push_back(double(0));
result.push_back(_BERN_12_);
result.push_back(double(0));
result.push_back(_BERN_14_);
result.push_back(double(0));
result.push_back(_BERN_16_);
result.push_back(double(0));
result.push_back(_BERN_18_);
result.push_back(double(0));
result.push_back(_BERN_20_);
result.push_back(double(0));
result.push_back(_BERN_22_);
result.push_back(double(0));
result.push_back(_BERN_24_);
result.push_back(double(0));
result.push_back(_BERN_26_);
result.push_back(double(0));
result.push_back(_BERN_30_);
result.push_back(double(0));
result.push_back(_BERN_32_);
result.push_back(double(0));
result.push_back(_BERN_34_);
```

```
  return result;
}

lcf tulliotools::nroot(lcf& x, unsigned int n)//finds the nth root of x
   .
{
  lcf result;
  if (x == lcf0)
  {
    result = 0;
  }
  else if (x.get_order()>=0)//If this condition is satisfied than x is
     at most finite and so we can take its natural log
  {
    if (x > lcf0)
    {
      result = ln(x) / n;
      result = exp(result);
    }
    else if (x < lcf0)
    {
      result = x*(-1);
      result = ln(result) / n;
      result = exp(result);
      result = result*(-1);
    }
    return result;
  }
  else//In this case x must be infinitely large and thus ln(x) is
```

```
    undefined, we factor out the infinitely large part and find the
    nth root of the two factors independently
  {
    rat arg_order = x.get_order(), res_order = arg_order / n;
    lcf divisor = lcf(double(1), arg_order), factor = lcf(double(1),
       res_order), temp = x / divisor;
    result = nroot(temp, n)*factor;//temp should now be at most finite
       so with this argument the function should execute the other
       branch of the conditional statment.
    return result;
  }
}
```

A.3.7 polyfuncs.cpp

```
#include "functions.h"
#include <functional>

using namespace tulliotools;

lcf tulliotools::bernoulli_poly(lcf& x, unsigned int n)
{
  lcf result = lcf0;
  std::vector<double> bern = bernoulli_init();
  for (unsigned int i = 0; i <= n; i++)
  {
    result = result + x.exp(i)*bern[n - i] * bico(n, i);
  }
  return result;
}
```

```
lcf tulliotools::bernoulli_phi(lcf& x)
{
  return bernoulli_poly(x, _SIG_SUPP_-1);
}

lcf tulliotools::lb_taylor_poly(lcf& x, unsigned int n)
{
  return x.exp(n - 1);
}

lcf tulliotools::ub_taylor_poly(lcf& x, unsigned int n)
{
  return (x - 1).exp(n - 1);
}

lcf tulliotools::experimental_poly(lcf& x, unsigned int n)
{
  lcf result = lcf1;
  for (unsigned int i = 1; i < n; i++)
  {
    result = result*(x - double(i) / n);
  }
  return result;
}

lcf tulliotools::experimental_phi(lcf& x)
{
  return experimental_poly(x, _SIG_SUPP_);
```

```
}

lcf tulliotools::test_poly(lcf& x, unsigned int n)
{
  lcf result = lcf1;
  for (unsigned int i = 1; i < n; i++)
  {
    result = result*(x - double(i) / M_PI);
  }
  return result;
}

lcf tulliotools::simp_poly(lcf& x)
{
  return x.exp(3) + 1;
}

lcf tulliotools::trap_poly(lcf& x)
{
  return x*M_PI;
}
```

A.3.8 testfuncs.cpp

```
#include "functions.h"
#include <math.h>

using namespace tulliotools;

lcf tulliotools::examp1(lcf& x)
```

```
{
  lcf result, num1, num2, den1a, den1b, den1, den2;

  num1 = sin(x.exp(3) + x * 2 + 1);

  num2 = x + 1;
  num2 = num2.abs();
  num2 = cos(sin(ln(num2))) + 3;

  den1a = sin(cos(tan(exp(x))));

  den1b = cos(sin(exp(tan(x + 2))));



  den1 = exp(tanh(sinh(cosh(den1a / den1b))));

  den2 = sin(sinh(cos(arctan(ln(exp(x) + x.exp(2) + 3))))) + 2;

  return (num1 + num2 / den1) / den2;
}

lcf tulliotools::examp2(lcf& x)
{
  lcf term1, term2;
  term1 = cos(x);
  term1 = exp(term1);

  term2 = sin(x);
  term2 = exp(term2);
```

```
  return term1 + term2;
}

lcf tulliotools::examp3(lcf& x)
{
  lcf term1, term2, term3;
  term1 = x.exp(4);
  term1 = ln(term1);

  term2 = x.exp(2) + x * 3;
  term2 = cos(term2);

  term3 = x.exp(3) + x.exp(2) * 4;
  term3 = sin(term3);

  return term1*term2*term3;
}

lcf tulliotools::conj1(lcf& x)
{
  lcf num, den;
  num = x.exp(2)*ln(x);
  den = (x.exp(2) - 1)*(x.exp(4) + 1);
  return num / den;
}

lcf tulliotools::conj2(lcf& x)
{
```

```
  lcf num, den;
  num = x.exp(2);
  den = sin(x);
  den = den.exp(2);
  return num / den;
}

lcf tulliotools::conj3(lcf& x)
{
  lcf num, den;
  num = x*sin(x);
  den = cos(x);
  den = den.exp(2);
  den = den + 1;
  return num / den;
}

lcf tulliotools::testfunc0(lcf& x)
{
  lcf arg = x.exp(3) + x + 1;
  arg = arg.abs();
  return exp(sin(cos(ln(arg))));
}

lcf tulliotools::testfunc1(lcf& x)
{
  return x*ln(x + 1);
}
```

```
lcf tulliotools::testfunc3(lcf& x)
{
  return exp(x)*cos(x);
}

lcf tulliotools::testfunc8(lcf& x)
{
  lcf temp = ln(x);
  return temp.exp(2);
}

lcf tulliotools::testfunc9(lcf& x)
{
  return ln(cos(x));
}

lcf tulliotools::test1(lcf& x)
{
  return exp(x.exp(2) - x + 2);
}

lcf tulliotools::test2(lcf& x)
{
  return (sin(sin(sin(sin(sin(x)))))) / (cos(cos(cos(cos(cos(x))))));
}
```

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