PROBLEMS IN EXTREMAL GRAPH THEORY AND EUCLIDEAN RAMSEY THEORY

by

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Abstract

This thesis addresses problems of three types.

The first type is finding extremal numbers for unions of graphs, each with a colour-critical edge (joint work with V. Nikiforov). In 1968, Simonovits found extremal numbers $ex(n, H)$ for graphs with a colour-critical edge for large n (without specifying how large). A similar result for unions of graphs, each with a colour-critical edge, can be deduced from Simonovits' 1974 work. Nikiforov and I improved this result, giving a precise bound for n.

The second type of problem considered is maximizing the number of cycles in a graph (joint work with A. Arman and D. Gunderson). It is proved that for sufficiently many vertices, the complete balanced bipartite graph is the unique triangle-free graph with the maximum number of cycles, thus answering a conjecture posed by Durocher et al. Other results include upper and lower bounds on the maximum number of cycles in graphs and multigraphs with a given number of edges, or with a given number of vertices and edges. The lower bounds in some cases come from random graphs; the asymptotics for the expected number of cycles in the random graph $G(n, m)$ is found for all possible relations between n and m.

The final chapter is dedicated to Euclidean Ramsey theory. Two results about two-colouring of Euclidean spaces are given. One of the results answers in the affirmative a question asked in 1973 by Erdős and others: if the Euclidean plane is coloured in red and blue, are there either two red points at distance one or five blue points on a line with distance one between consecutive points? The second result (joint work with A. Arman) answers the similar question for six points in 3-dimensional space.

Contents

Chapter 1

Introduction

1.1 Notation

Let $\mathbb{Z}$ be the set of integers, and let $\mathbb{Z}^+$ be the set of positive integers. For $k \in \mathbb{Z}^+$, define $[k] = \{i \in \mathbb{Z} : 1 \leq i \leq k\}$ and for a set S, let $[S]^k = \{T \subseteq S : |T| = k\}$. Denote $[[n]]^k$ by simply $[n]^k$.

A *graph* G is an ordered pair of sets (V, E) such that $V \neq \emptyset$ and $E \subseteq [V]^2$. Elements of $V = V(G)$ are called *vertices* of G and elements of $E = E(G)$ are called *edges* of G. An edge $\{x, y\} \in E(G)$ may be denoted by simply xy. A vertex $v \in V(G)$ and an edge $e \in E(G)$ are said to be *incident* if and only if $v \in e$. Two graphs G and H are called *isomorphic* if and only if there is a bijection $\phi : V(G) \to V(H)$ such that for any $x, y \in V(G)$, $\{x, y\} \in E(G)$ if and only if $\{\phi(x), \phi(y)\} \in E(H)$.

The *neighbourhood* of any vertex $v \in V(G)$ is $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, and the *degree* of x is $\deg_G(x) = |N_G(x)|$. When it is clear what G is, subscripts are deleted, using only $N(x)$ and $\deg(x)$. Denote the maximum degree of a graph by $\Delta(G)$, and the minimum degree by $\delta(G)$.

A graph $G = (V, E)$ is called complete if and only if $E = [V]^2$. A complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is called *r-partite* if and only if there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_r$, each $V_i \neq \emptyset$, so that for any $i \in [r]$, $[V_i]^2 \cap E = \emptyset$. A 2-partite graph is called *bipartite*. For a bipartite graph G with partition $V = A \cup B$, if $E = \{\{x, y\} : x \in A, y \in B\}$, then G is called the *complete bipartite graph* on partite sets A and B, denoted $G = K_{|A|,|B|}$. The *balanced complete bipartite graph* on n vertices is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

For a graph G, vertices $v_0, \ldots, v_m \in V(G)$ and edges $e_1, \ldots, v_m$, the sequence of alternating vertices and edges $v_0, e_1, v_1, e_2, \ldots, e_m, v_m$ is a *walk* of length m if and only if for any $i \in [m]$, $e_m = v_{m-1}v_m$; such walk is called *closed* if and only if $v_0 = v_m$ and *open* otherwise. A *trail* is a walk without repeated edges. A *path* is an open trail without repeated vertices. A *cycle* is a closed trail without repeated vertices (except the first vertex, which also appears as the last one). The *girth* of a graph is the length of a shortest cycle contained in the graph.

For $m \in \mathbb{Z}$, let P_m denote a graph isomorphic to a path of length m (P_m has $m + 1$ vertices and m edges), and let C_m denote a graph isomorphic to a cycle of length m.

If G and H are graphs, say that H is a (weak) *subgraph* of G if and only if

$V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is called *induced* if and only if $E(H) = E(G) \cap [H]^2$. For a graph G and $U \subseteq V(G)$, the *subgraph induced by* U is the subgraph with the vertex set U and the edge set $[U]^2 \cap V(G)$; the subgraph of G induced by the set U is denoted by $G[U]$ and the subgraph $G[V(G) \backslash U]$ is denoted by $G - U$.

This thesis contains several results about the number of cycles in a graph. The "number of cycles in a graph G" here means the number of subgraphs of G that are isomorphic to a cycle, so cycles that only differ in orientation or labelling are counted as one cycle.

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, [V]^2 \backslash E)$. For a graph H and $s \in \mathbb{Z}^+$, write sH for the disjoint union of s copies of H. For graphs G and H with disjoint vertex sets, the *join* of G and H is the graph $G \vee H$ with $V(G \vee H) = V(G) \cup V(H)$, $E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$ (see Figure 1.1 for an example).

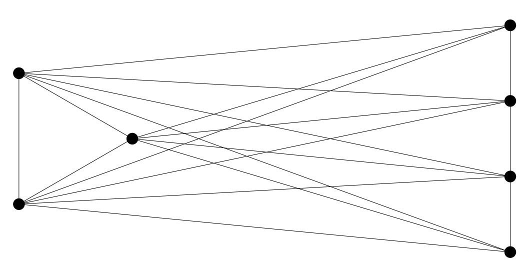


Figure 1.1: $K_3 \vee P_3$

A *multigraph* is an ordered pair (V, E), where V is a set and E is a multiset of two-element subsets of V. The elements of V are called *vertices*, and the elements of E are called *edges*. The *degree* $deg_G(V)$ of a vertex $v \in V(G)$ is the number of edges incident to v. For two vertices $u, v \in V(G)$, denote by $E(u, v)$ the set of all edges containing u and v. Two edges containing the same pair of vertices are called *parallel edges*. For a vertex $v \in V(G)$, the *neigbourhood* of v, denoted by $N(v)$, is the set of all vertices adjacent to v (by some edge). For $k \geq 2$, a *cycle of length* k in a multigraph G is an alternating sequence of k distinct vertices and k distinct edges $\{v_0, e_1, v_1, e_2, \ldots, e_k, v_k = v_0\}$, where for each $i \in [k]$, $v_i \in V(G)$, and $e_i \in E(v_{i-1}, v_i)$.

For a set A and for $r \in \mathbb{Z}^+$, an r*-colouring* of A is any function $\Delta : A \to [r]$. Such a function induces partition of A into classes of form $\Delta^{-1}(k), k \in [r]$, which are called *colour classes*. Elements of the same colour class are said to have the same colour, or to be *monochromatic*. By "colouring" I mean "r-colouring" for some r (i.e. only finite colourings are considered in this thesis).

For a graph G, a colouring of $V(G)$ is called *proper (good)* if and only if there is no edge in $E(G)$ connecting two vertices of the same colour. The *chromatic number of a graph* G, denoted by $\chi(G)$, is the least number $k \in \mathbb{Z}^+$ such that there exist a proper k-colouring of $V(G)$.

Standard Landau notation is used in this thesis: for two functions $f, g : \mathbb{Z}^+ \to \mathbb{R}$, write $f(n) \sim g(n)$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, write $f(n) = o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, and write $f(n) = O(g(n))$ if and only if there exist $c > 0$ and

$n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$, $f(n) \leq cg(n)$. Note that $o(g(n))$ and $O(g(n))$ describe classes of functions, so $f = o(g(n))$ ($f = O(g(n))$) is not an equality in a usual sense; rather, it means that f is an element of the corresponding class.

1.2 Background

The results in this thesis are in two fields: extremal graph theory and Euclidean Ramsey theory.

In general, extremal results in graph theory are about maximizing (or minimizing) a quantity among all graphs from some class. For a graph H and $n \in \mathbb{Z}^+$, define the *extremal number* $ex(n, H)$ to be the maximum number of edges in a graph on n vertices that does not contain H as a subgraph. Finding extremal numbers is a typical question in extremal graph theory; often the extremal numbers can only be approximated by a function depending on n. A comprehensive survey of the extremal graph theory is the book by Bollobás [12].

An example of an extremal result is

Proposition 1.2.1. *Let G be a graph on n vertices. If G has more than $\lfloor \frac{n}{2} \rfloor$ edges, then G contains a P_2 (a path on 3 vertices).*

Proof. Suppose that G does not contain P_2 and has at least $\lfloor \frac{n}{2} \rfloor + 1$ edges. Then all the edges of G are disjoint, therefore the number of vertices in G is at least $2(\lfloor \frac{n}{2} \rfloor + 1) > n$, which contradicts the assumption. □

A notion very closely related to tht of an extremal result is that of a "Ramsey-type result". A typical Ramsey-type result states that for any finite collection of colours, if elements or (small) subsets of some large mathematical structure are coloured in one of the colours, then there is a (medium) substructure with all elements (subsets) having the same colour. Here is one version of the classical Ramsey theorem.

Theorem 1.2.2 (Ramsey, 1930 [68]). *For any $k, m, r \in \mathbb{Z}^+$, there is a least integer $n \in \mathbb{Z}^+$ such that for any set S with n elements and for any r-colouring $\Delta : [S]^k \to [1, r]$, there is a set $T \in [S]^m$ such that $[T]^k$ is monochromatic.*

Some extremal results imply Ramsey results. For instance, Proposition 1.2.1 implies that for any $(n-2)$-colouring of $E(K_n)$, there is a monochromatic P_2. Indeed, since K_n has $\frac{n(n-1)}{2}$ edges, by the pigeonhole principle, at least one colour class contains more than $\frac{n}{2}$ edges, and therefore contains a P_2. A notable example of the connection between extremal and Ramsey results is the one between the famous van der Waerden's and Szemerédi's density theorems.

Theorem 1.2.3 (van der Waerden, 1927 [87]). *Let $r, k \in \mathbb{Z}^+$. There exists a least $N = W(r, k) \in \mathbb{Z}^+$ such that if the elements of $[N]$ are r-coloured, then there is a monochromatic arithmetic progression of length k.*

Theorem 1.2.4 (Szemerédi, 1969 [79, 80], finite version). *Let $k \in \mathbb{Z}^+$, $\delta \in (0, 1]$. There exists a least $N = N(k, \delta) \in \mathbb{Z}^+$ such that every subset of $[N]$ with at least δN elements contains an arithmetic progression of length k.*

Szemerédi's theorem, which is an extremal result, implies van der Waerden's theorem – a Ramsey result. Indeed, if $[N(k, \frac{1}{r})]$ is coloured in r colours, then one of the colour classes has at least $\frac{1}{r}N(k, \frac{1}{r})$ elements and hence contains an arithmetic progression of length k. Therefore, $W(r, k) \leq N(k, \frac{1}{r})$. For more information on van der Waerden's theorem and related results see Karen Gunderson's thesis [45].

The Ramsey-type results of this thesis are in Euclidean Ramsey theory, which is, in a sense, a combination of Ramsey theory with discrete geometry. Euclidean Ramsey theory is mostly concerned with colouring points of a Euclidean space $\mathbb{E}^d$ and existence of a particular monochromatic point configuration. Many questions in this area look quite simple, but have not been answered yet. For example, it is possible to colour the points of E^2 in red and blue so that there is no monochromatic set of three points forming a unit equilateral triangle [27]. It has been conjectured in [27] that for any non-equilateral triangle T and for any red-blue colouring of E^2, there is a red or blue congruent copy of T (see Section 4.1 for the definition of congruency). The conjecture has remained open for over 40 years, although it has been proved for many special triangles. Another previously open question from the same paper [27] is answered in Section 4.2.

1.3 Structure of this thesis

Apart from this introductory chapter, this thesis contains three chapters.

Chapter 2 contains results about extremal numbers for different graphs; some classic results are listed in Section 2.1. A generalization of a theorem of Simonovits is given in Section 2.2 (this is joint work with Vladimir Nikiforov).

Chapter 3 is dedicated to different extremal problems on counting cycles. Section 3.1 contains a result about the maximum number of cycles in a triangle-free graph with given number of vertices, and based on a paper with Andrii Arman and David Gunderson [10].

Section 3.2 is based on joint work with Andrii Arman [9] and includes bounds on the maximum number of cycles in graphs with a fixed number of edges and in graphs with fixed numbers of vertices and edges.

In Section 3.3, the number of cycles in random graphs with given number of edges is considered. The results of Section 3.3 were discovered independently by Andrii Arman and myself.

Chapter 4 contains results in Euclidean Ramsey theory. Section 4.1 is an overview of known results in Euclidean Ramsey theory, with the focus on asymmetric problems. Two of such problems are solved in Sections 4.2 and 4.3. The first one is based on my recent paper [83], and the second on a joint paper with Andrii Arman [8].

Chapter 2

Extremal graph theory

2.1 Basic results in extremal graph theory

A central type of problem in extremal graph theory asks for a maximum number of edges in a graph on a fixed number of vertices that does not contain a certain subgraph H (or a family of subgraphs). Some major results are reviewed in this section. One of the first and most famous theorems in extremal graph theory is Mantel's Theorem.

Theorem 2.1.1 (Mantel, 1907 [56])**.** *Let G be a graph on $n \geq 3$ vertices. If G has more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, then G contains a triangle.*

Mantel's Theorem can be rephrased in the following way: the maximum number of edges in a graph on n vertices that does not contain a triangle is at most $\lfloor \frac{n^2}{4} \rfloor$. The maximum is achieved by the complete balanced bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

For $n \in \mathbb{Z}^+$ and a graph H, define the extremal number $ex(n, H)$ to be the maximum number of edges in a graph on n vertices that does not contain H as a subgraph. A graph on n vertices with $ex(n, H)$ edges that does not contain H is called an *extremal graph* for H, and the set of all extremal graphs for H on n vertices is denoted by $EX(n, H)$. Mantel's Theorem states that $ex(n, K_3) \leq \lfloor \frac{n^2}{4} \rfloor$.

The exact extremal numbers are known for very few graphs. One example of an exact result is Turán's theorem, that gives the exact extremal numbers for $H = K_r$.

For positive integers n, r, define the Turán graph $T(n, r)$ to be the complete r-partite graph on n vertices with partition sets having sizes "as equal as possible", namely $V(T(n, r)) = V_1 \cup V_2 \cup \cdots \cup V_r,$, and for $i \neq j$, $|V_i - V_j| \leq 1$. If n is divisible by r, then every vertex in $T(n, r)$ has degree $\frac{r-1}{r}n$, hence $|E(T(n, r)| = \frac{r-1}{2r}n^2$. In general, if q is the remainder of n modulo r, then the number of edges in the Turán graph is $t(n, r) = |E(T(n, r))| = (1 - \frac{1}{r})\frac{(n-q)^2}{2} + \frac{q(r-1)(n-q)}{r} + \binom{q}{2} = (1 - \frac{1}{r})\frac{n^2 - q^2}{2} + \binom{q}{2}$. Note that $t(n, r) = (1 + o(1))(1 - \frac{1}{r})\frac{n^2}{2}$.

Theorem 2.1.2 (Turán, 1941 [84], [85])**.** *Let G be a K_r-free graph on n vertices. Then the number of edges in G is at most $t(n, r-1)$. Furthermore, the only K_r-free graph G that has the maximum number of edges is $T(n, r-1)$ (that is, $EX(n, K_r) = \{T(n, r-1)\}$).*

The following result gives asymptotic values for extremal numbers.

Theorem 2.1.3 (Erdős-Simonovits, 1966 [32])**.** *Let H be a graph with chromatic* *number* $\chi(H) \geq 2$. *Then*

$$\lim_{n\to\infty} \frac{ex(n,H)}{n^2} = \frac{1}{2}\left(1 - \frac{1}{\chi(H)-1}\right).$$

The Erdős-Simonovits Theorem does not give exact numbers, but provides the asymptotics for $ex(n, H)$ when H is not bipartite ($\chi(H) \geq 3$). The asymptotic values of $ex(n, H)$ for most bipartite graphs are not known, which causes the division of Turán problems into two types: "degenerate problems" (for bipartite graphs) and "non-degenerate problems". An extensive survey of the degenerate case is the chapter by Füredi and Simonovits [37].

Another type of graph with known exact extremal numbers is an odd cycle. The exact extremal numbers for odd cycles can be found in a paper by Füredi and Gunderson [36] (that paper also gives the complete description of the extremal graphs). Extremal numbers for paths were found by Erdős and Gallai [25].

An edge e of a graph is called *colour-critical* if and only if deleting e reduces the chromatic number of the graph. Simonovits [73] found extremal numbers $ex(n, G)$ for large n for G with a colour-critical edge. In the same paper, extremal numbers and extremal graphs for several different classes of graphs were found, including the union of k disjoint complete graphs of the same size. The results of Simonovits and generalizations thereof are discussed in Section 2.2.

Finding extremal numbers for different graphs is one way to extend the Mantel's theorem. Another extension is maximizing the number of different subgraphs in a triangle-free graph. One such problem is maximizing the number of cycles in a triangle-free graph, which is considered in Section 3.1. Note that the maximum number of cycles in a graph on n vertices without restrictions is given by the complete graph K_n.

2.2 Extremal results for graphs with a colour-critical edge

2.2.1 Background and the main result

As mentioned in Section 2.1, Simonovits proved the following theorem about the extremal graphs of graphs with a colour-critical edge:

Theorem 2.2.1 (Simonovits, 1968 [73]). *Let H be a graph with chromatic number r that has a colour-critical edge. Then there exists n_0 so that for all $n \geq n_0$, $ex(n; H) = t(n, r-1)$. Furthermore, $EX(n, H) = \{T(n, r-1)\}$.*

Recall from Section 1.1 that the join of two graphs G and H is the graph $G \vee H$ obtained from the union of G and H by adding all edges between vertices of G and H. A generalization of Theorem 2.2.1 for the union of s copies of graphs with colour-critical edges can be found in Simonovits' Ph.D. thesis [76]. A further generalization is the following:

Theorem 2.2.2 (Simonovits, 1974 [74]). *Let H be a graph, $\chi(H) = r + 1$, and $s \in \mathbb{Z}^+$. If omitting any $s - 1$ vertices of H does not change the chromatic number* *of H, but omitting s suitable edges of H decreases the chromatic number by 1, then for n sufficiently large, $EX(n,H) = \{K_{s-1} \vee T(n-s+1,r)\}$.*

The aim of this chapter is to prove a more precise result about extremal graphs for the union of s isomorphic graphs, each with a colour-critical edge.

Let $K(s_1,...,s_r)$ be the complete r-partite graph with partition sets of sizes $s_1,...,s_r$, and denote $K(\underbrace{p,...,p}_{r})$ by $K_r(p)$. For $p \geq 2$, let $K_r^+(p)$ denote $K_r(p)$ with an edge within one of the partition sets (Figure 2.1).

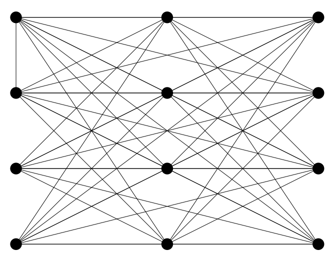


Figure 2.1: $K_3^+(4)$

The following theorem is the main result of this section.

Theorem 2.2.3 (Nikiforov-Tsaturian, not published)**.** *Let s, r and n be positive integers such that $r \geq 2$, $\frac{2}{\ln n} \leq c = r^{-(r+7)(r+1)}$, and $n \geq \frac{4s}{c}$. If G is a graph with n vertices and*

$$|E(G)| \geq |E(K_{s-1} \vee T(n-s+1,r))|,$$

then G contains $sK_r^+(\lfloor \frac{c\ln n}{2s^2} \rfloor)$, unless $G = K_{s-1} \vee T(n-s+1,r)$.

Theorem 2.2.3 is proved below, after some preliminaries.

Note that any graph on k vertices with chromatic number $r+1$ that has a colour-critical edge is contained in $K_r^+(k)$ for some k, so Theorem 2.2.3 implies the following.

Corollary 2.2.4. *Let s, r and n be positive integers such that $r \geq 2$, $\frac{2}{\ln n} \leq c = r^{-(r+7)(r+1)}$, and $n \geq \frac{4s}{c}$. Let H be a graph with chromatic number $r+1$ and at most $\left\lfloor \frac{c \ln n}{2s^2} \right\rfloor$ vertices that has a colour-critical edge. If G is a graph with n vertices and*

$$|E(G)| \geq |E(K_{s-1} \vee T(n-s+1,r))|,$$

then G contains sH unless $G = K_{s-1} \vee T(n-s+1,r)$.

2.2.2 Preliminaries

For a graph G, an r-clique of G is a subgraph of G isomorphic to K_r. Note that I used the letter r earlier in this section to denote the chromatic number of a graph. Write $Cl_r(G)$ for the set of r-cliques in G and write $k_r(G) = |Cl_r(G)|$.

Denote the set of s-cliques contained in members of a set $M \subset Cl_r(G)$ by $Cl_s(M)$. An *r-joint* of size t is the union of t distinct r-cliques that have (at least one) common edge. Write $js_r(G)$ for the maximum size of an r-joint in G.

An inequality stated by Moon and Moser in [60], and first proved by Khadžiivanov and Nikiforov [49], is needed.

Theorem 2.2.5 (Moon–Moser, 1962 [60], Khadžiivanov–Nikiforov, 1978 [49]). *Let $1 \leq s < t < n$, and let G be a graph of order n, with $k_t(G) > 0$. Then*

$$\frac{(t+1)\,k_{t+1}(G)}{tk_t(G)} - \frac{n}{t} \geq \frac{(s+1)\,k_{s+1}(G)}{sk_s(G)} - \frac{n}{s}. \tag{2.1}$$

As shown in [49], inequality (2.1) implies the following result:

Corollary 2.2.6 (Khadžiivanov–Nikiforov, 1978 [49]). *Let $r \geq 2$ and G be a graph of order n. If $k_r(G)$ is the number of r-cliques in G, and*

$$|E(G)| > \left(1 - \frac{1}{r}\right) n^2/2,$$

then

$$k_r(G) > \frac{n^r}{r^r}.$$

The following theorem gives a bound on the number of cliques in a $K_r(\lfloor c^r \ln n \rfloor)$-free graph.

Theorem 2.2.7 (Nikiforov, 2008 [64]). *Let $r \geq 2$, let c and n be such that $0 < c < 1/r!$ and $n \geq \exp(c^{-r})$, and let G be a graph on n vertices. If $k_r(G) > cn^r$, then G contains a $K_r(\lfloor c^r \ln n \rfloor)$.*

Recall from the introduction that $t(n,r) = |E(T(n,r))|$. The following theorem provides a bound on the extremal number $ex(n, K_r^+(\lfloor c \ln n \rfloor))$.

Theorem 2.2.8 (Nikiforov, 2010 [65]). *Let $r \geq 2$, $c = r^{-(r+7)(r+1)}$, $\ln n \geq 2/c$, and let G be a graph of order n. If $|E(G)| \geq t(n,r)$, then G contains a $K_r^+(\lfloor c \ln n \rfloor)$, unless $G = T(n,r)$.*

Note that if q is the remainder of n modulo r, then

$$t(n,r) = \frac{r-1}{2r}\left(n^2 - q^2\right) + \binom{q}{2},$$

hence

$$\frac{r-1}{2r}n^2 - \frac{r}{8} \leq t(n,r) \leq \frac{r-1}{2r}n^2. \tag{2.2}$$

2.2.3 Proof of Theorem 2.2.3

Direct counting shows that

$$\left|E\left(K_{s-1} \vee T\left(n-s+1,r\right)\right)\right| = (s-1)\,n - \binom{s}{2} + t(n-s+1,r).$$

Using the bounds (2.2), it can be found that

$$\begin{aligned}(s-1)\,n - \binom{s}{2} + t(n-s+1,r) &\geq (s-1)\,n - \frac{s(s-1)}{2} + \frac{r-1}{2r}(n-s+1)^2 - \frac{r}{8} \\ &\geq \frac{r-1}{2r}n^2 + \frac{n\,(s-1)}{r} - \frac{(s-1)^2}{2r} - \frac{s-1}{2} - \frac{r}{8}.\end{aligned}$$

Hence, the conditions $\frac{2}{\ln n} \leq c = r^{-(r+7)(r+1)}$, and $n \geq \frac{4s}{c}$ imply that

$$\left|E\left(G\right)\right| \geq \left|E\left(K_{s-1} \vee T\left(n-s+1,r\right)\right)\right| > \frac{r-1}{2r}n^2. \tag{2.3}$$

Another point to make is that if $s \geq 2$, then for any vertex u of G,

$$\begin{aligned}\left|E\left(G-u\right)\right| &\geq \left|E\left(K_{s-1} \vee T\left(n-s+1,r\right)\right)\right| - n + 1 \\ &= \left|E\left(K_{s-2} \vee T\left(n-s+1,r\right)\right)\right|.\end{aligned}$$

Finally, set $H = K_r^+(\lfloor (c/s) \ln n \rfloor)$, and call an edge e of G a *base edge* if and only if there is a $K_r^+(\lfloor c \ln n \rfloor) \subset G$ that contains e in the first partition set. Applying Theorem 2.2.8 repeatedly, the following can be obtained:

Corollary 2.2.9. *Let $r \geq 2$, $c = r^{-(r+7)(r+1)}$, $\ln n \geq \frac{2}{c}$, and let G be a graph on n vertices. If $|E(G)| \geq t(n,r) + k$, then G contains at least k base edges.*

Proof of Theorem 2.2.3. Suppose that r, s, n and G satisfy the premises of Theorem 2.2.3. The proof is by induction on s. The base case $s = 1$ follows directly from Theorem 2.2.8, so assume that $s \geq 2$, and that the statement is true for all $s' < s$.

Case 1. Suppose that G has a vertex u with degree at least $n - \frac{1}{2r^{r^2}} \ln n$. In view of (2.3), Corollary 2.2.6 implies that $k_r(G) > r^{-r} n^r$, and so, Theorem 2.2.7 implies that there exists a $K_r(\lceil r^{-r^2} \ln n \rceil) \subset G$. The vertex u is joined to all but at most $(r^{-r^2}/2) \ln n$ vertices of this $K_r(\lceil r^{-r^2} \ln n \rceil)$, and therefore, there exists an $F = K_r^+(\lceil (r^{-r^2}/2) \ln n \rceil) \subset G$, such that u belongs to F and is connected to all vertices in the partition set of u.

By (2.2.3), the graph $G' = G - u$ satisfies

$$|E(G')| \geq |E(K_{s-2} \vee T(n-s+1, r))|, \tag{2.4}$$

and by the induction hypothesis, either $(s-1) H \subset G'$ or equality holds in (2.4) and

$$G' = K_{s-2} \vee T(n-s+1, r).$$

If $(s-1) H \subset G'$, in view of

$$|V((s-1) H)| \leq (s-1) r (c/s) \ln n < rc \ln n < (r^{-r^2}/4) \ln n,$$

it follows that G contains an sH. If equality holds in (2.4) and $G' = K_{s-2} \vee T(n-s+1,r)$, then u has degree $n-1$ and so $G = K_{s-1} \vee T(n-s+1,r)$. This completes the proof if there is a vertex u with degree at least $n - \frac{1}{2r^{r^2}} \ln n$.

Case 2. Assume that degree of every vertex u of G is less than $n - \frac{1}{2r^{r^2}} \ln n$.

Write B for the set of all base edges of G and suppose that

$B_0 = \{u_1v_1, \ldots, u_kv_k\}$ is a maximal matching in B.

Claim. $k \geq s$.

Proof of claim. Assume that $k \leq s-1$. Set $U = V(G) \setminus \left(\cup_{i=1}^{k} \{u_i, v_i\}\right)$ and observe that U does not contain edges from B because B_0 is a maximal matching in B. Moreover, for any $\ell \in [k]$,

$$|N_B(u_\ell) \cap U| \leq 2 \quad \text{or} \quad |N_B(v_\ell) \cap U| \leq 2. \tag{2.5}$$

Indeed, otherwise there would be two base edges $u_\ell w'$ and $v_\ell w''$ such that $w', w'' \in U$ and $w' \neq w''$; hence B_0 is not a maximal matching in B. By symmetry, in view of (2.5), for any $\ell \in [k]$ assume that $|N_B(v_\ell) \cap U| \leq 1$.

Let $G' = G - \{u_1, \cdots, u_k\}$ and note that (a crucial fact) G' contains at most $k(k+1)/2$ edges of B. Indeed, U contains no edges of B at all, and $|N_B(v_\ell) \cap U| \leq 1$ for any $\ell \in [k]$; hence

$$|E(G') \cap B| \leq \binom{k}{2} + k = k(k+1)/2. \tag{2.6}$$

On the other hand,

$$
\begin{aligned}
|E(G')| &> |E(G)| - \sum_{\ell \in [k]} d(u_\ell) > |E(G)| - k\left(n - \frac{1}{2r^{r^2}} \ln n\right) \\
&> t(n-k,r) - \binom{k}{2} + \frac{k}{2r^{r^2}} \ln n \\
&> t(n-k,r) + k(k+1)/2.
\end{aligned}
$$

Hence, Corollary 2.2.9 implies that G' contains more than $k(k+1)/2$ base edges, contradicting (2.6) and thus proving the claim.

The claim implies that G contains s independent (not sharing a vertex) base edges $u_1v_1, \dots, u_sv_s$. To finish the proof, it remains to show that these s base edges can be extended to a sH.

For $i \in [s]$, denote a copy of $K_r^+(\lfloor c \ln n \rfloor)$ that contains u_iv_i by F_i. Let F_i' be the graph obtained from F_i by deleting for all $j \neq i$ the vertices u_j, v_j. Note that there are less than $2s$ vertices deleted from every F_i, and so F_i' contains a $K_r^+(\lfloor c \ln n \rfloor - 2s)$, which for $n \geq \frac{4s}{c}$ contains $K_r^+(\lfloor \frac{c \ln n}{2} \rfloor)$. It is proved below that there are s disjoint copies of H that are subgraphs of $F_1', \dots, F_s'$. Suppose that the maximum number of disjoint copies of H that are subgraphs of $F_1', \dots, F_s'$ is $m \leq (s-1)$, and no copy is a subgraph of, say, F_1. Then the total number of vertices in the m copies is $ms\lfloor \frac{c \ln n}{2s^2} \rfloor \leq (s-1)\lfloor \frac{c \ln n}{2s} \rfloor$. Therefore there are at least $\lfloor \frac{c \ln n}{2} \rfloor - (s-1)\lfloor \frac{c \ln n}{2s} \rfloor > \lfloor \frac{c \ln n}{2s^2} \rfloor$ vertices left in each partition class of F_1, forming a copy of H, disjoint from the chosen m copies, which contradicts the maximality of m and therefore $m \geq s$. $\square$

Chapter 3

Counting cycles

3.1 Triangle-free graphs with maximum number of cycles

3.1.1 Background

This section is based on joint work with Andrii Arman and David Gunderson [10].

Durocher, Gunderson, Li and Skala [22] considered the maximum number of cycles in a triangle-free graph. It was asked which triangle-free graphs contain the maximum number of cycles; this question arose from the study of path-finding algorithms [14].

Conjecture 3.1.1 (Durocher–Gunderson–Li–Skala, 2014 [22])**.** *For each $n \geq 4$, the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ contains more cycles than any other*

n-vertex triangle-free graph.

The authors [22] confirmed Conjecture 3.1.1 when $4 \le n \le 13$, and made progress toward this conjecture in general. For example, they showed the conjecture to be true when restricted to "nearly regular graphs", that is, for each positive integer k and sufficiently large n, $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ has more cycles than any other triangle-free graph on n vertices whose minimum degree and maximum degree differ by at most k.

In Theorems 3.1.10 and 3.1.11 below, it is shown that Conjecture 3.1.1 holds true for $n \ge 141$. Theorem 3.1.5 gives a useful estimate for the number of cycles in $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. In Lemma 3.1.9, an upper bound is given for the number of Hamiltonian cycles in a triangle-free graph.

3.1.2 Preliminaries

Stirling's approximation formula says that as $n \to \infty$,

$$n! = (1 + o(1))\sqrt{2\pi n}(n/e)^n. \tag{3.1}$$

In 1955, Robbins [71] proved that

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Slightly more convenient bounds (valid for all $n \ge 1$) are freely used in this section (*e.g.*, in the proof of Theorem 3.1.5).

$$\sqrt{2\pi} \cdot \sqrt{n}\left(\frac{n}{e}\right)^n < n! \le e \cdot \sqrt{n}\left(\frac{n}{e}\right)^n. \tag{3.2}$$

Two modified Bessel functions (see, *e.g.*, [1]) are used:

$$I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2}; \tag{3.3}$$

$$I_1(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k+1}k!(k+1)!}. \tag{3.4}$$

In particular, when $x = 2$ is used in either modified Bessel function, useful approximations are obtained:

$$2.27958 \leq \sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) \leq 2.279586; \tag{3.5}$$

$$1.5906 \leq \sum_{k=0}^{\infty} \frac{k}{(k!)^2} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) \leq 1.59064. \tag{3.6}$$

The following shows that among all bipartite graphs, the balanced one has the most cycles.

Lemma 3.1.2 ([22])**.** *For $n \geq 4$, $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is the unique graph that has the greatest number of cycles among all bipartite graphs on n vertices.*

So, to settle Conjecture 3.1.1, it is then sufficient to prove that a cycle-maximal triangle-free graph is bipartite. To this end, the following result is essential:

Theorem 3.1.3 (Andrásfai, 1964 [6])**.** *Any triangle-free graph G on n vertices with minimum degree greater than $2n/5$ is bipartite.*

See also [7] for an English proof of Theorem 3.1.3 and related results. Theorem 3.1.3 is sharp because of C_5 (or a “blow-up” of C_5).

For a graph G, let $c(G)$ denote the number of cycles in G.

Lemma 3.1.4 ([22]). *For $n \geq 4$, the number of cycles in the balanced complete bipartite graph is*

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = \sum_{k=2}^{\lfloor n/2\rfloor} \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2k(\lfloor n/2\rfloor - k)!(\lceil n/2\rceil - k)!}. \tag{3.7}$$

The following form for the number of cycles in $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ gives a way to estimate the right hand side of (3.7) in Lemma 3.1.4:

Theorem 3.1.5 (Arman–Gunderson–Tsaturian, 2016 [10]). *For $n \geq 12$,*

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \geq \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot \begin{cases} I_0(2) & \text{if } n \text{ is even,} \\ I_1(2) & \text{if } n \text{ is odd,} \end{cases} \tag{3.8}$$

$$\geq \pi \left(\frac{n}{2e}\right)^n \cdot \begin{cases} I_0(2) & \text{if } n \text{ is even,} \\ I_1(2) & \text{if } n \text{ is odd.} \end{cases} \tag{3.9}$$

As $n \to \infty$,

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = (1+o(1)) \begin{cases} I_0(2)\pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is even,} \\ I_1(2)\pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is odd.} \end{cases} \tag{3.10}$$

Proof: Using (3.2), the proof that (3.9) follows from (3.8) is elementary, and so is omitted.

By Lemma 3.1.4, write

$$\begin{aligned} c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) &= \sum_{k=2}^{\lfloor n/2\rfloor} \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2k(\lfloor n/2\rfloor-k)!(\lceil n/2\rceil-k)!} \\ &= \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor}\cdot\sum_{k=2}^{\lfloor n/2\rfloor}\frac{\lfloor n/2\rfloor}{k(\lfloor n/2\rfloor-k)!(\lceil n/2\rceil-k)!}. \end{aligned} \tag{3.11}$$

Case 1 (n even): Suppose that for $\ell \geq 2$, $n = 2\ell$, and set

$$a_\ell = \sum_{k=2}^{\ell}\frac{\ell}{k((\ell-k)!)^2} = \sum_{i=0}^{\ell-2}\frac{\ell}{(\ell-i)(i!)^2}.$$

Then equation (3.11) becomes

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor}\cdot a_\ell. \tag{3.12}$$

Claim: For $\ell \geq 4$, $a_{\ell+1} < a_\ell$. (This claim is needed later only for $\ell \geq 6$.)

Proof of Claim:

$$\begin{aligned} a_\ell - a_{\ell+1} &= \sum_{i=0}^{\ell-2}\left(\frac{\ell}{\ell-i}-\frac{\ell+1}{\ell+1-i}\right)\frac{1}{(i!)^2} - \frac{\ell+1}{2((\ell-1)!)^2} \\ &= \sum_{i=0}^{\ell-2}\frac{i}{(\ell+1-i)(\ell-i)(i!)^2} - \frac{\ell+1}{2((\ell-1)!)^2} \\ &= \sum_{i=2}^{\ell-2}\frac{i}{(\ell+1-i)(\ell-i)(i!)^2} + \frac{1}{\ell(\ell-1)} - \frac{\ell+1}{2((\ell-1)!)^2} \\ &> \frac{1}{\ell(\ell-1)} - \frac{\ell+1}{2((\ell-1)!)^2} \\ &= \frac{2((\ell-1)!)^2-(\ell+1)\ell(\ell-1)}{2((\ell-1)!)^2\ell(\ell-1)} \\ &\geq 0 \qquad\qquad (\text{for } \ell \geq 4), \end{aligned}$$

finishing the proof of the claim.

Since the sequence $\{a_\ell\}$ is non-increasing and bounded below (by 0, for example), $\lim_{\ell\to\infty} a_\ell$ exists. To find this limit, write

$$a_\ell = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell-i)(i!)^2} = \sum_{i=0}^{\ell-2} \frac{1}{(i!)^2} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)(i!)^2}.$$

Put $b_\ell = \displaystyle\sum_{i=0}^{\ell-2} \frac{1}{(i!)^2}$ and $c_\ell = \displaystyle\sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)(i!)^2}$. Then

$$\begin{aligned}
c_\ell &= \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)(i!)^2} \\
&= \sum_{i=0}^{3} \frac{i}{(\ell-i)(i!)^2} + \sum_{i=4}^{\ell-2} \frac{i}{(\ell-i)(i!)^2} \\
&\le \frac{3}{\ell-3} + \frac{1}{\ell} \sum_{i=4}^{\ell-2} \frac{1}{i!} && \left(\text{since } \tfrac{i}{(\ell-i)i!} \le \tfrac{1}{\ell} \text{ for } \ell \ge 4\right) \\
&\le \frac{3}{\ell-3} + \frac{e}{\ell},
\end{aligned}$$

where the last line is a very loose bound based on $e = \sum_{i=0}^{\infty} \frac{1}{i!}$.

Therefore, $\lim_{\ell\to\infty} c_\ell = 0$, and so

$$\begin{aligned}
\lim_{\ell\to\infty} a_\ell &= \lim_{\ell\to\infty} (b_\ell + c_\ell) \\
&= \lim_{\ell\to\infty} b_\ell \\
&= \sum_{i=0}^{\infty} \frac{1}{(i!)^2} \\
&= I_0(2) && \text{(by (3.5))}.
\end{aligned}$$

Since a_ℓ is non-increasing for $\ell \ge 6$ (and $n = 2\ell$),

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \ge \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot I_0(2),$$

which proves the even case of (3.8). By (3.5), as $n \to \infty$,

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = (1 + o(1)) \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor} \cdot I_0(2),$$

and by Stirling's approximation (3.1), the proof of the even case of (3.10) is complete.

Case 2 (n odd): Suppose that for $\ell \geq 6$, $n = 2\ell + 1$. The proof follows the even case, and so is only outlined. Put

$$a_\ell = \sum_{k=2}^{\ell} \frac{\ell}{k(\ell - k)!(\ell + 1 - k)!} = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell - i)i!(i + 1)!}.$$

Claim: For $\ell \geq 4$, $a_{\ell+1} < a_\ell$.

Proof of claim: Letting $\ell \geq 4$,

$$\begin{aligned} a_\ell - a_{\ell+1} &= \sum_{i=0}^{\ell-2} \frac{i}{(\ell + 1 - i)(\ell - i)} \cdot \frac{1}{i!(i + 1)!} - \frac{\ell + 1}{2(\ell - 1)!\ell!} \\ &= \sum_{i=2}^{\ell-2} \frac{i}{(\ell + 1 - i)(\ell - i)} \cdot \frac{1}{i!(i + 1)!} + \frac{1}{2(\ell - 1)\ell} - \frac{\ell + 1}{2(\ell - 1)!\ell!} \\ &> \frac{(\ell - 2)!(\ell - 1)! - (\ell + 1)}{2(\ell - 1)!\ell!} \\ &\geq 0, \end{aligned}$$

finishing the proof of the claim.

Therefore, $\lim_{\ell \to \infty} a_\ell$ exists. To find this limit, write

$$a_\ell = \sum_{i=0}^{\ell-2} \frac{1}{i!(i + 1)!} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)i!(i + 1)!}.$$

Letting $b_\ell = \sum_{i=0}^{\ell-2} \frac{1}{i!(i + 1)!}$ and $c_\ell = \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)i!(i + 1)!}$, observe that

$$c_\ell = \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)i!(i + 1)!} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)i!(i + 1)!} \leq \frac{3}{\ell - 3} + \frac{e}{\ell},$$

and so $\lim_{\ell\to\infty} c_\ell = 0$. Thus,

$$\lim_{\ell\to\infty} a_\ell = \lim_{\ell\to\infty} b_\ell = \sum_{i=0}^{\infty} \frac{1}{i!(i+1)!} = \sum_{i=0}^{\infty} \frac{i+1}{((i+1)!)^2} = \sum_{i=0}^{\infty} \frac{i}{(i!)^2},$$

which, by (3.6), is equal to $I_1(2)$. Then again

$$\begin{aligned}
c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) &\geq \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot I_1(2) \\
&= \frac{\ell!(\ell+1)!}{2\ell} \cdot I_1(2) \\
&= \frac{(\ell!)^2}{2\ell}(\ell+1) \cdot I_1(2) \\
&= (1+o(1))\pi \left(\frac{\ell}{e}\right)^{2\ell} (\ell+1) \cdot I_1(2) && \text{(by (3.1))} \\
&> (1+o(1))\pi \left(\frac{\ell}{e}\right)^{2\ell} (\ell-1) \cdot I_1(2) \\
&= (1+o(1))\pi \left(\frac{n-1}{2e}\right)^{n-1} \left(\frac{n-1}{2}\right) \cdot I_1(2) \\
&= (1+o(1))\pi e \left(\frac{n-1}{2e}\right)^{n} \cdot I_1(2) \\
&= (1+o(1))\pi e \left(\frac{n-1}{n}\right)^{n} \left(\frac{n}{2e}\right)^{n} \cdot I_1(2) \\
&= (1+o(1))\pi \left(\frac{n}{2e}\right)^{n} \cdot I_1(2),
\end{aligned}$$

and as $n \to \infty$,

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = (1+o(1))\pi \left(\frac{n}{2e}\right)^{n} \cdot I_1(2).$$

This completes the proof for odd n, and so the proof of the theorem. □

Lemma 3.1.6 (Arman–Gunderson–Tsaturian, 2016 [10])**.** *Let H be a triangle-free graph on 6 vertices with $x, y \in V(H)$. Then there are at most 9 different x–y paths.*

Proof: If H is empty, there are no x–y paths in H. If H is not empty, then consider two cases.

Case 1: H contains no copy of C_5. Then H contains no odd cycle, and so is bipartite. Without loss of generality, add edges to H to make H a complete bipartite graph. There are only 3 different complete bipartite graphs on six vertices, namely $K_{1,5}$, $K_{2,4}$, and $K_{3,3}$. By inspection, in any of these, the maximum number of paths between any two vertices is at most 9 (which is realized for $K_{3,3}$).

Case 2: H contains a copy of C_5. Suppose that $x_1, x_2, x_3, x_4, x_5, x_1$ forms a cycle C, and that x_6 is the remaining vertex. Then x_6 is adjacent to at most two vertices of C. If x_6 is adjacent to fewer than two vertices of C, then add an extra edge or two so that x_6 is adjacent to precisely two vertices of C; without loss of generality, suppose that x_6 is adjacent to x_1 and x_3. Then the maximum number of paths between any two vertices is 4 (for example, between x_2 and x_6). □

3.1.3 Counting types of cycles

In this section, upper bounds are given for the number of cycles through a fixed edge and for the number of Hamiltonian cycles in a graph.

Lemma 3.1.7 (Arman–Gunderson–Tsaturian, 2016 [10]). *There exists $n_0 \in \mathbb{Z}^+$ so that for every even integer $n \geq n_0$, if G is a triangle-free graph on n vertices, and $x_1x_2 \in E(G)$, then the number of cycles containing the edge x_1x_2 is at most $10\pi \frac{n^{n-1}}{(2e)^n}$.*

Proof: Let G be a triangle-free graph on n vertices, and let $x_1x_2 \in E(G)$. For each $k = 4, \ldots, n$, let c_k denote the number of cycles of length k that contain the edge x_1x_2. The goal is to give an upper bound for $\sum_{k=4}^{n} c_k$.

Let $2 \leq i \leq \frac{n-4}{2}$; an upper bound on $c_{2i} + c_{2i+1}$ is first calculated; to do so, count all possible cycles of the form $x_1, x_2, \ldots, x_{2i}$ or $x_1, x_2, \ldots, x_{2i+1}$; the counting is accomplished by building a cycle vertex by vertex. For each $j > 1$, put $d_j = |N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}|$. Then for any j, there are at most d_j ways to choose an x_{j+1}. Note that $N(x_j) \cap N(x_{j+1}) = \emptyset$, since otherwise a triangle is formed containing x_j and x_{j+1}. Also,

$$|(N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}) \cup (N(x_{j+1})\backslash\{x_1, \ldots, x_j\})| \leq |V(G)\backslash\{x_1, \ldots, x_j\}| = n - j.$$

Therefore,

$$\begin{aligned} d_j + d_{j+1} &\leq |N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}| + |N(x_{j+1})\backslash\{x_1, \ldots, x_j\}| \\ &= |(N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}) \cup (N(x_{j+1})\backslash\{x_1, \ldots, x_j\})| \\ &\leq n - j, \end{aligned}$$

and thus

$$d_j d_{j+1} \leq \left\lfloor \frac{n-j}{2} \right\rfloor \cdot \left\lceil \frac{n-j}{2} \right\rceil. \tag{3.13}$$

Using (3.13), the number of ways to choose vertices $x_3, x_4, \ldots, x_{2i}$ so that $x_1, x_2, x_3, x_4, \ldots, x_{2i}$ form a path is at most

$$\prod_{j=2}^{2i-1} d_j = \prod_{j=1}^{i-1} (d_{2j} d_{2j+1}) \leq \prod_{j=1}^{i-1} \left(\left\lfloor \frac{n-2j}{2} \right\rfloor \cdot \left\lceil \frac{n-2j}{2} \right\rceil \right) = \prod_{j=1}^{i-1} \left(\frac{n-2j}{2} \right)^2. \tag{3.14}$$

If $x_1, x_2, \dots x_{2i}$ is a path and there is an edge $x_{2i}x_1 \in E(G)$, there is one cycle $x_1, x_2, \dots, x_{2i}$ of length $2i$, and no cycles of the form $x_1, x_2, \dots, x_{2i+1}$ because otherwise, x_1, x_{2i}, x_{2i+1} form a triangle. So, in total, there is exactly one cycle that contains the path $x_1, x_2, \dots, x_{2i}$ and has length $2i$ or $2i+1$. If there is no edge $x_{2i}x_1$, then there is no cycle $x_1, \dots, x_{2i}$ and at most $n-2i$ cycles of the form $x_1, \dots, x_{2i}, x_{2i+1}$. In any case, there are at most $n-2i$ cycles of length $2i$ or $2i+1$ containing the path $x_1, \dots, x_{2i}$.

By these observations and inequality (3.14),

$$c_{2i} + c_{2i+1} \leq (n-2i) \prod_{j=1}^{i-1} \left(\frac{n-2j}{2}\right)^2. \tag{3.15}$$

To evaluate $\sum_{k=4}^{n} c_k$, separate the sum into two parts:

$$\begin{aligned}
\sum_{k=4}^{n-5} c_k &= \sum_{i=2}^{(n-6)/2} (c_{2i} + c_{2i+1}) \\
&\leq \sum_{i=2}^{(n-6)/2} \left((n-2i) \prod_{j=1}^{i-1} \left(\frac{n-2j}{2}\right)^2\right) && \text{(by (3.15))} \\
&= \sum_{i=2}^{(n-6)/2} (n-2i) \left(\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-2i}{2}\right)!}\right)^2 \\
&= \left(\left(\frac{n-2}{2}\right)!\right)^2 \sum_{j=3}^{\frac{n-4}{2}} \frac{2j}{(j!)^2} \\
&= \left(\left(\frac{n-2}{2}\right)!\right)^2 \left(\sum_{j=1}^{\frac{n-4}{2}} \frac{2j}{(j!)^2} - \frac{2}{(1!)^2} - \frac{2 \cdot 2}{(2!)^2}\right) \\
&\leq \left(\left(\frac{n-2}{2}\right)!\right)^2 (2 \cdot (1.591) - 3) && \text{(by (3.6))} \\
&< 0.19 \left(\left(\frac{n-2}{2}\right)!\right)^2.
\end{aligned} \tag{3.16}$$

To count $\sum_{k=n-4}^{n} c_k$, note that by (3.13), there are at most

$$\prod_{i=2}^{n-5} d_i \leq \prod_{j=1}^{\frac{n-6}{2}} \left(\frac{n-2j}{2}\right)^2$$

ways to choose a path $x_1, x_2, \ldots, x_{n-4}$, and by Lemma 3.1.6, there are at most 9 paths that connect x_{n-4} and x_1 in the graph $G[\{x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_1\}]$; that is, there are at most 9 ways to complete the path $x_1, x_2, \ldots, x_{n-4}$ to a cycle. Therefore,

$$\sum_{k=n-4}^{n} c_k \leq 9 \prod_{j=1}^{\frac{n-6}{2}} \left(\frac{n-2j}{2}\right)^2 = 9 \cdot \frac{\left(\left(\frac{n-2}{2}\right)!\right)^2}{(2!)^2} = \frac{9}{4}\left(\left(\frac{n-2}{2}\right)!\right)^2. \tag{3.17}$$

Adding equations (3.16) and (3.17),

$$\sum_{k=4}^{n} c_k \leq 0.19\left(\left(\frac{n-2}{2}\right)!\right)^2 + \frac{9}{4}\left(\left(\frac{n-2}{2}\right)!\right)^2 = 2.44\left(\left(\frac{n-2}{2}\right)!\right)^2. \tag{3.18}$$

By Stirling's approximation, as $n \to \infty$,

$$\begin{aligned}
2.44\left(\left(\frac{n-2}{2}\right)!\right)^2 &= (1+o(1))2.44\left(\frac{n-2}{2e}\right)^{n-2} \cdot \pi(n-2) \\
&= (1+o(1))2.44\pi\frac{n^{n-1}}{(2e)^n}(2e)^2\left(\frac{n-2}{n}\right)^{n-1} \\
&= (1+o(1))2.44\pi\frac{n^{n-1}}{(2e)^n}4e^2 \cdot \frac{1}{e^2} \\
&= (1+o(1))9.76\pi\frac{n^{n-1}}{(2e)^n} \\
&< 10\pi\frac{n^{n-1}}{(2e)^n} \qquad \text{(for } n \text{ suff. large)}
\end{aligned}$$

completing the proof of the lemma. □

Lemma 3.1.8 (Arman–Gunderson–Tsaturian, 2016 [10]). *There exists $n_0 \in \mathbb{Z}^+$ so that for every odd integer $n \geq n_0$, if G is a triangle-free graph on n vertices, and*

$x_1x_2 \in E(G)$ with $\deg_G(x_2) \le \frac{2}{5}n$, then the number of cycles containing the edge x_1x_2 is at most $7.81\pi\frac{n^{n-1}}{(2e)^n}$.

Proof: The proof is similar to that of Lemma 3.1.7. Let G be a triangle-free graph on n vertices, and let $x_1x_2 \in E(G)$, where $\deg(x_2) \le \frac{2}{5}n$. For each $k = 4, \ldots, n$, let c_k denote the number of cycles of length k that contain the edge x_1x_2.

For $3 \le i \le \frac{n-5}{2}$, an upper bound on $c_{2i-1} + c_{2i}$ is first calculated; to do so, count all possible cycles of the form $x_1, x_2, \ldots, x_{2i-1}$ or $x_1, x_2, \ldots, x_{2i}$. As in Lemma 3.1.7, for each $j > 1$, put $d_j = |N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}|$. There are at most d_j ways to choose an x_{j+1}, and

$$d_j d_{j+1} \le \left\lfloor \frac{n-j}{2} \right\rfloor \cdot \left\lceil \frac{n-j}{2} \right\rceil. \tag{3.19}$$

Using (3.19) and the fact that $d_2 \le \frac{2}{5}n$, the number of ways to choose vertices $x_3, x_4, \ldots, x_{2i-1}$ so that $x_1, x_2, x_3, x_4, \ldots, x_{2i-1}$ form a path is at most

$$\begin{aligned}\prod_{j=2}^{2i-2} d_j = d_2 \prod_{j=3}^{2i-2} d_j &\le \frac{2}{5}n \prod_{j=1}^{i-2} (d_{2j+1} d_{2j+2}) \\ &\le \frac{2}{5}n \prod_{j=1}^{i-2} \left(\left\lfloor \frac{n-2j-1}{2} \right\rfloor \cdot \left\lceil \frac{n-2j-1}{2} \right\rceil \right) \\ &= \frac{2}{5}n \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2} \right)^2. \end{aligned} \tag{3.20}$$

If $x_{2i-1}x_1 \in E(G)$, there is one cycle of length $2i-1$ and no cycles of length $2i$; if there is no such edge, there are no cycles of length $2i-1$ and at most $n - 2i + 1$ cycles of length $2i$. By these observations and (3.20),

$$c_{2i-1} + c_{2i} \le (n - 2i + 1)\frac{2}{5}n \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2} \right)^2. \tag{3.21}$$

To evaluate $\sum_{k=4}^{n} c_k$, separate the sum into three parts:

$$\sum_{k=4}^{n} c_k = c_4 + \sum_{k=5}^{n-5} c_k + \sum_{k=n-4}^{n} c_k.$$

First,

$$c_4 \leq d_2 d_3 < n \cdot n = n^2. \tag{3.22}$$

Next,

$$\begin{aligned}
\sum_{k=5}^{n-5} c_k &= \sum_{i=3}^{\frac{n-5}{2}} (c_{2i-1} + c_{2i}) \\
&\leq \sum_{i=3}^{\frac{n-5}{2}} \left[(n-2i+1)\frac{2}{5}n \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2}\right)^2 \right] && \text{(by (3.21))} \\
&= \frac{2}{5}n \sum_{i=3}^{\frac{n-5}{2}} \left[(n-2i+1) \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2}\right)^2 \right] \\
&= \frac{2}{5}n \sum_{i=3}^{\frac{n-5}{2}} (n-2i+1) \left(\frac{\left(\frac{n-3}{2}\right)!}{\left(\frac{n-2i+1}{2}\right)!} \right)^2 \\
&= \frac{2}{5}n \left(\left(\frac{n-3}{2}\right)! \right)^2 \sum_{j=3}^{\frac{n-5}{2}} \frac{2j}{(j!)^2} \\
&= \frac{2}{5}n \left(\left(\frac{n-3}{2}\right)! \right)^2 \left(\sum_{j=1}^{\frac{n-5}{2}} \frac{2j}{(j!)^2} - \frac{2}{(1!)^2} - \frac{2 \cdot 2}{(2!)^2} \right) \\
&< \frac{2}{5}n \left(\left(\frac{n-3}{2}\right)! \right)^2 (3.19 - 3) && \text{(by (3.6))} \\
&= 0.076n \left(\left(\frac{n-3}{2}\right)! \right)^2 . && (3.23)
\end{aligned}$$

To count $\sum_{k=n-4}^{n} c_k$, note that by (3.22), there are at most

$$\prod_{i=2}^{n-5} d_i = d_2 \cdot \prod_{j=1}^{(n-7)/2} d_{2j+1} d_{2j+2} \leq \frac{2}{5}n \prod_{j=1}^{\frac{n-7}{2}} \left(\frac{n-2j-1}{2}\right)^2$$

ways to choose a path $x_1, x_2, \ldots, x_{n-4}$, and by Lemma 3.1.6, there are at most 9 ways to complete the path to a cycle (by paths that connect x_{n-4} and x_1) in the graph $G[\{x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_1\}]$. Therefore,

$$\sum_{k=n-4}^{n} c_k \leq 9 \cdot \frac{2}{5} n \prod_{j=1}^{\frac{n-7}{2}} \left(\frac{n-2j-1}{2}\right)^2 = 9 \cdot \frac{2}{5} n \cdot \frac{\left(\left(\frac{n-3}{2}\right)!\right)^2}{(2!)^2} = \frac{9}{10} n \left(\left(\frac{n-3}{2}\right)!\right)^2. \tag{3.24}$$

Adding (3.22), (3.23), and (3.24), as $n \to \infty$,

$$\begin{aligned}
\sum_{k=4}^{n} c_k &\leq n^2 + 0.076n \left(\left(\frac{n-3}{2}\right)!\right)^2 + \frac{9}{10} n \left(\left(\frac{n-3}{2}\right)!\right)^2 \\
&= n^2 + 0.976n \left(\left(\frac{n-3}{2}\right)!\right)^2 \qquad (3.25) \\
&= n^2 + (1+o(1))0.976n(n-3)\pi \left(\frac{n-3}{2e}\right)^{n-3} \\
&= (1+o(1))0.976\pi n \cdot \frac{n^{n-2}}{(2e)^n} \left(\frac{n-3}{n}\right)^{n-2} (2e)^3 \\
&= (1+o(1))0.976\pi \cdot \frac{n^{n-1}}{(2e)^n} \frac{1}{e^3} (2e)^3 \\
&= (1+o(1))7.808\pi \cdot \frac{n^{n-1}}{(2e)^n} \\
&< 7.81\pi \frac{n^{n-1}}{(2e)^n} \qquad \text{(for } n \text{ sufficiently large)},
\end{aligned}$$

completing the proof. □

The next lemma is of independent interest and might be previously known, but a search failed to find a source. The bound obtained is asymptotically tight for the case of even k, as it gives the number of Hamiltonian cycles in the balanced bipartite graph on k vertices.

Lemma 3.1.9 (Arman–Gunderson–Tsaturian, 2016 [10])**.** *Let H be a triangle-free graph on k vertices. Then H has at most $e^2\left(\frac{k}{2e}\right)^k$ Hamiltonian cycles.*

Proof: Let x_1 be the first vertex of a Hamiltonian cycle. As in previous two lemmas, for each $i \geq 1$, put $d_i = |N(x_i)\backslash\{x_1, \ldots, x_i\}|$. For any i, there are at most d_i ways to choose a vertex x_{i+1}. Note that $N(x_i) \cap N(x_{i+1}) = \emptyset$ because if the intersection contains some vertex v, then v, x_i, and x_{i+1} form a triangle. Also,

$$|N(x_i)\backslash\{x_1, \ldots, x_i\} \cup N(x_{i+1})\backslash\{x_1, \ldots, x_{i+1}\}| \leq |V(H)\backslash\{x_1, \ldots, x_i\}| = k - i.$$

Therefore,

$$\begin{aligned} d_i + d_{i+1} &= |N(x_i)\backslash\{x_1, \ldots, x_i\}| + |N(x_{i+1})\backslash\{x_1, \ldots, x_{i+1}\}| \\ &= |N(x_i)\backslash\{x_1, \ldots, x_i\} \cup N(x_{i+1})\backslash\{x_1, \ldots, x_{i+1}\}| \\ &\leq k - i, \end{aligned}$$

and thus $d_i d_{i+1} \leq \left\lfloor \frac{k-i}{2} \right\rfloor \cdot \left\lceil \frac{k-i}{2} \right\rceil$.

When k is odd, the number of Hamiltonian cycles is at most

$$\begin{aligned} \prod_{i=1}^{k-1} d_i = \prod_{j=1}^{\frac{k-1}{2}} d_{2j-1} d_{2j} &\leq \prod_{j=1}^{\frac{k-1}{2}} \left\lfloor \frac{k-2j+1}{2} \right\rfloor \cdot \left\lceil \frac{k-2j+1}{2} \right\rceil \\ &= \prod_{j=1}^{\frac{k-1}{2}} \left(\frac{k-2j+1}{2}\right)^2 \\ &= \left(\left(\frac{k-1}{2}\right)!\right)^2, \end{aligned}$$

and by (3.2), this number is at most

$$\left(\frac{\left(\frac{k-1}{2}\right)^{\frac{k-1}{2}+\frac{1}{2}}}{e^{\frac{k-1}{2}-1}}\right)^2 = \frac{\left(\frac{k-1}{2}\right)^k}{e^{k-3}} = e^3\left(\frac{k-1}{k}\right)^k\left(\frac{k}{2e}\right)^k = e^3\frac{1}{\left(1+\frac{1}{k-1}\right)^k}\left(\frac{k}{2e}\right)^k$$
$$\leq e^2\left(\frac{k}{2e}\right)^k,$$

completing the proof for odd k.

When k is even, similarly obtain

$$\prod_{i=1}^{k-1} d_i = \left(\prod_{j=1}^{\frac{k-2}{2}} d_{2j-1}d_{2j}\right)\cdot d_{k-1} \leq \left(\prod_{j=1}^{\frac{k-2}{2}}\left\lfloor\frac{k-2j+1}{2}\right\rfloor\cdot\left\lceil\frac{k-2j+1}{2}\right\rceil\right)\cdot 1$$
$$= \prod_{j=1}^{\frac{k-1}{2}}\left(\frac{k-2j}{2}\right)\left(\frac{k-2j+2}{2}\right) = \frac{k}{2}\left(\left(\frac{k-2}{2}\right)!\right)^2 \leq \frac{k}{2}\left(\frac{\left(\frac{k-2}{2}\right)^{\frac{k-2}{2}+\frac{1}{2}}}{e^{\frac{k-2}{2}}}\right)^2$$
$$= k\frac{(k-2)^{k-1}}{e^{k-4}2^k} = e^4\left(\frac{k-2}{k}\right)^{k-1}\left(\frac{k}{2e}\right)^k = e^4\frac{1}{\left(1+\frac{2}{k-2}\right)^{k-1}}\left(\frac{k}{2e}\right)^k$$
$$\leq e^2\left(\frac{k}{2e}\right)^k,$$

completing the proof for even k, and hence for the lemma. □

3.1.4 Main theorems

In Theorem 3.1.10 below, Conjecture 3.1.1 is proved for sufficiently large n. Then in Theorem 3.1.11, a lower bound on such n is given.

Theorem 3.1.10 (Arman–Gunderson–Tsaturian, 2016 [10]). *There exists $n_0 \in \mathbb{Z}^+$ so that for any $n \geq n_0$, the triangle-free graph on n vertices with the largest number of cycles is $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.*

Proof: Let G be a triangle-free graph on n vertices. It is first shown that if G contains a vertex of small degree, then G has far fewer cycles than does $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Let $x \in V(G)$, and assume that $\deg(x) \leq \frac{2}{5}n$. Cycles in G are counted according to whether or not they contain x.

First, the number of cycles not containing x are counted. Any cycle in $G - x$ is a Hamiltonian cycle for some subgraph, and so the number of cycles in G not containing x is loosely bounded above by

$$\sum_{Y \subseteq V(G) \setminus x} (\text{number of Ham. cycles in } G[Y]) \tag{3.26}$$

$$\begin{aligned}
&\leq \sum_{k=4}^{n-1} \binom{n-1}{k} e^2 \left(\frac{k}{2e}\right)^k && \text{(by Lemma 3.1.9)} \\
&< e^2 \sum_{k=4}^{n-1} \binom{n-1}{k} \left(\frac{n-1}{2e}\right)^k \\
&< e^2 \left(1 + \frac{n-1}{2e}\right)^{n-1} && \text{(by binomial expansion of } \left(1 + \tfrac{n-1}{2e}\right)^{n-1}) \\
&= e^2 \left(\frac{n+2e-1}{2e}\right)^{n-1} \\
&= e^2 \left(\frac{n}{2e}\right)^{n-1} \left(\frac{n+2e-1}{n}\right)^{n-1} \\
&< e^2 \left(\frac{n}{2e}\right)^{n-1} \left(1 + \frac{2e-1}{n}\right)^{n} \\
&\leq e^2 \left(\frac{n}{2e}\right)^{n-1} e^{2e-1} \\
&= \frac{2e^{2e+2}}{n} \left(\frac{n}{2e}\right)^{n}. && (3.27)
\end{aligned}$$

Next, the number of cycles containing x is counted. Each cycle C containing x

has exactly two edges (in C) incident with x, and so the number of cycles containing x is

$$\frac{1}{2}\sum_{y\in N(x)}(\text{number of cycles containing } xy). \tag{3.28}$$

By Lemma 3.1.7, for even n, expression (3.28) is at most

$$\frac{1}{2}\cdot\frac{2}{5}n\cdot 10\pi\frac{n^{n-1}}{(2e)^n}=2\pi\left(\frac{n}{2e}\right)^n.$$

In this case, for n sufficiently large, the total number of cycles in G is at most

$$2\pi\left(\frac{n}{2e}\right)^n+\frac{2e^{2e+2}}{n}\left(\frac{n}{2e}\right)^n=\left(2\pi+\frac{2e^{2e+2}}{n}\right)\left(\frac{n}{2e}\right)^n\le 2.01\pi\left(\frac{n}{2e}\right)^n.$$

However, by (3.8), the number of cycles in $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is (for n even) at least $2.27958\pi\left(\frac{n}{2e}\right)^n$.

Let n be odd; then by Lemma 3.1.8, the expression (3.28) is at most

$$\frac{1}{2}\cdot\frac{2}{5}n\cdot 7.81\pi\frac{n^{n-1}}{(2e)^n}=1.562\pi\left(\frac{n}{2e}\right)^n. \tag{3.29}$$

Thus, for odd n sufficiently large, by (3.29) and (3.27) the total number of cycles in G is at most

$$1.562\pi\left(\frac{n}{2e}\right)^n+\frac{2e^{2e+2}}{n}\left(\frac{n}{2e}\right)^n\le 1.57\pi\left(\frac{n}{2e}\right)^n.$$

By (3.8) in Theorem 3.1.5, the number of cycles in $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ for n odd is at least $1.5906\pi\left(\frac{n}{2e}\right)^n$.

In both the even and odd case, if G contains a vertex of degree at most $\frac{2}{5}n$, then G has fewer cycles than does $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

So assume that $\delta(G) > \frac{2}{5}n$. Then by Theorem 3.1.3, G is bipartite. By Lemma 3.1.2, the number of cycles in G is maximized by $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. □

Theorem 3.1.11 (Arman–Gunderson–Tsaturian, 2016 [10])**.** *The statement of Theorem 3.1.10 with* $n_0 = 141$ *is true.*

Proof: To show that the statement holds for $n_0 = 141$, further estimations on $c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})$ are needed. Both when n is even and when n is odd, (3.12) holds (but the expression for a_ℓ changes). Since each (one for odd, one for even) sequence of a_ℓs are non-increasing for $n \geq 140$, by (3.12),

$$c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \leq \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot \begin{cases} a_{71} & \text{for } n \text{ even,} \\ a_{70} & \text{for } n \text{ odd,} \end{cases}$$

$$\leq \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot \begin{cases} 2.302786 & \text{for } n \text{ even,} \\ 1.60067 & \text{for } n \text{ odd.} \end{cases} \tag{3.30}$$

(The values of a_{70} and a_{71} were calculated by computer.) With these estimates in hand, now Theorem 3.1.10 is proved with $n_0 = 141$. Let G be a triangle-free graph on $n \geq 141$ vertices. First, it is proved that $c(G) \leq 6 \cdot c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})$ for G having an odd number of vertices. This result is then used to prove the statement of Theorem 3.1.11 for an even number of vertices in G. And finally, Theorem 3.1.11 is verified for an odd number of vertices in G.

Without loss of generality, assume that there is a vertex of degree at most $\frac{2}{5}n$ (since otherwise, the theorem is proved by Theorem 3.1.3 and Lemma 3.1.2). In the following calculations, bounds given in (3.2) and Theorem 3.1.5 are used freely.

Case 1: Let $n \geq 141$ be odd. By (3.25) from the proof of Lemma 3.1.8, the number of cycles passing through an edge xy in G is at most $n^2 + 0.976n\left(\left(\frac{n-3}{2}\right)!\right)^2$. Then the number of cycles in G is bounded by

$$\begin{aligned}
c(G) &\leq \frac{1}{2}\cdot\frac{2}{5}n\cdot\left[n^2 + 0.976n\left(\left(\frac{n-3}{2}\right)!\right)^2\right] + \frac{2e^{2e+2}}{n}\left(\frac{n}{2e}\right)^n \\
&= \frac{\frac{n-1}{2}!\frac{n+1}{2}!}{n-1}\cdot I_1(2)\cdot\left(\frac{\frac{n}{5}\left[n^2 + 0.976n\left(\left(\frac{n-3}{2}\right)!\right)^2\right](n-1)}{\frac{n-1}{2}!\frac{n+1}{2}!\cdot I_1(2)}\right. \\
&\qquad\qquad \left. + I_1(2)\cdot\pi\left(\frac{n}{2e}\right)^n\left(\frac{2e^{2e+2}}{n\cdot I_1(2)\pi}\right)\right. \\
&\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\cdot\left(10^{-10} + \frac{8}{5}\cdot(0.976)\left(\frac{n^2}{n^2-1}\right) + \frac{2e^{2e+2}}{n\pi}\right)\cdot\frac{1}{I_1(2)} \\
&\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\cdot 6.
\end{aligned}$$

Note that the constant 6 above is a rough estimate, but using a better bound does not give a better bound on n in the statement of the theorem.

Case 2: Let n be even and $n \geq 142$. Then by (3.18), the proof of Theorem 3.1.10, and by the result in Case 1,

$$\begin{aligned}
c(G) &\leq \frac{1}{2}\cdot\frac{2}{5}n\cdot 2.44\left(\left(\frac{n-2}{2}\right)!\right)^2 + 6\cdot c(K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil}) \\
&= \frac{\frac{n}{5}2.44\left(\left(\frac{n-2}{2}\right)!\right)^2}{\frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor}\cdot I_0(2)}\cdot\frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor}\cdot I_0(2) \\
&\qquad\qquad + c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\frac{6\cdot c(K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil})}{c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})} \\
&\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\cdot\left(\frac{\frac{4}{5}\cdot 2.44}{I_0(2)} + \frac{6\cdot 1.60067\cdot\frac{\lfloor\frac{n-1}{2}\rfloor!\lceil\frac{n-1}{2}\rceil!}{2\lfloor\frac{n-1}{2}\rfloor}}{I_0(2)\frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor}}\right) \qquad \text{(by (3.30))} \\
&= c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\left(\frac{\frac{4}{5}\cdot 2.44}{I_0(2)} + \frac{6\cdot 1.60067}{I_0(2)}\cdot\frac{2}{n}\right) \\
&\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \quad (\text{for } n \geq 142).
\end{aligned}$$

Returning to the case when n is odd, using equation (3.30) again,

$$\begin{aligned}
c(G) &\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\cdot\left(\frac{10^{-10}+\frac{8}{5}(0.976)\left(\frac{n^2}{n^2-1}\right)}{I_1(2)} + \frac{2.302786\cdot\frac{\lfloor\frac{n-1}{2}\rfloor!\lceil\frac{n-1}{2}\rceil!}{2\lfloor\frac{n-1}{2}\rfloor}}{I_1(2)\cdot\frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2\lfloor n/2\rfloor}}\right) \\
&\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\cdot\left(\frac{10^{-10}+\frac{8}{5}(0.976)\left(\frac{n^2}{n^2-1}\right)}{I_1(2)} + \frac{2.302786}{I_1(2)\cdot(n+1)}\right) \\
&\leq c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})\cdot 0.9947 \\
&< c(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}).
\end{aligned}$$

This completes the proof of the theorem for $n \geq 141$. □

3.1.5 Concluding remarks

A few natural extensions of Theorem 3.1.10 and Conjecture 3.1.1 can be considered. For example, for $k > 1$, what C_{2k+1}-free graphs have the most number of cycles?

It is well known (see, *e.g*, [12], p. 150, [36], or [75]) that for n large enough, the unique C_{2k+1}-free n-vertex graph with the maximum number of edges is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. Also, Balister, Bollobás, Riordan, and Schelp [11] showed that for a fixed n and k and $\frac{n}{2} < \Delta < n - k$, any maximal C_{2k+1}-free graph with maximum degree Δ on n vertices is the complete bipartite graph $K_{\Delta, n-\Delta}$. Since any C_{2k+1}-free graph with a maximum number of cycles is also edge-maximal, it might then seem reasonable to pose the following:

Conjecture 3.1.12 (Arman–Gunderson–Tsaturian, 2016 [10])**.** *For any $k > 1$, if an n-vertex C_{2k+1}-free graph has the maximum number of cycles, then $G = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

In support of Conjecture 3.1.12, by duplicating the proofs in this section, one can show that the order of magnitude for edges in a C_{2k+1}-free graph is correct:

Theorem 3.1.13 (Arman–Gunderson–Tsaturian, unpublished)**.** *For any $k \geq 2$, there exists a constant $\alpha_{2k+1} \geq 1$ so that if G is a C_{2k+1}-free graph on n vertices with the maximum number of cycles, then $c(G) \leq \alpha_{2k+1} \cdot c(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor})$.*

One might entertain other questions related to Conjecture 3.1.1. For example:

Question 3.1.14. *What is the maximum number of cycles in a graph on n vertices with girth at least g?*

The case $g = 3$ is trivial and this section addresses this question for $g = 4$; however, there seems to be little known for $g \geq 5$. Another question that might be interesting is:

Question 3.1.15. *For $k \geq 4$ what is the maximum number of cycles in a K_k-free graph on n vertices? Could it be that the cycle-maximal K_k-free graphs are indeed Turán graphs?*

A type of stability result also follows from the techniques given in this section. Let $n \to \infty$. Theorem 3.1.10 shows that among all triangle-free graphs with n vertices and $m = \left\lfloor \frac{n^2}{4} \right\rfloor$ edges, $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ has the most number of cycles. Let $\ell = o(n)$, and set $m = \left\lfloor \frac{n^2}{4} \right\rfloor - \ell$. If G has n vertices and m edges, and has the most cycles among all triangle-free n-vertex graphs with m edges, then same argument as in the proof of Theorem 3.1.10 implies that G is bipartite. By the maximality of the number of cycles, one can show that G is a subgraph of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

For $14 \leq n \leq 140$, Conjecture 3.1.1 remains open. With a bit more care, it appears that with the techniques in this section, one might be able to prove Conjecture 3.1.1 for the even n to $n \geq 100$ or so, but the techniques used here do not seem to leave much room for the odd n. Skala [77] has suggested that Lemma 3.1.6 might be proved for graphs with slightly more vertices, and such an improvement might yield modest improvements for the bound on n for which Theorem 3.1.10 holds.

3.2 Maximum number of cycles in graphs with fixed number of edges

3.2.1 Background and main results

This section is based on joint work with Andrii Arman [9]. As mentioned in section 3.1, counting the number of cycles in a graph was studied for different classes of the graphs: graphs with given cyclomatic number, planar graphs, 3-regular and 4-regular graphs, and many others. However, only a few general bounds for number of cycles that use basic graph parameters are known. In this chapter bounds on the number of cycles in a graph as a function of number of vertices and edges are presented.

Let $C(G)$ denote the number of cycles in a graph G. In 1897, Ahrens [2] proved that for a graph G with n vertices, m edges and k components,

$$m - n + k \leq C(G) \leq 2^{m-n+k} - 1. \tag{3.31}$$

The lower bound in (3.31) is tight; for example, it is achieved by any disjoint union of cycles and trees. The tightness of the upper bound in (3.31) was shown by Mateti and Deo [57] and the only graphs for which the upper bound is tight are K_3, K_4, $K_{3,3}$ and $K_4 - e$ (Figure 3.1).

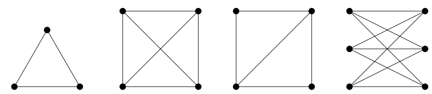


Figure 3.1

Aldred and Thomassen [4] improved the upper bound in (3.31) by showing that for a connected graph G,

$$C(G) \leq \frac{15}{16} 2^{m-n+1}. \tag{3.32}$$

Entringer and Slater [23] considered $C(G)$ for the class of connected graphs with fixed cyclomatic number $r = m - n + 1$. It follows from the results of [23] that there is a 3-regular connected graph G for which $C(G) > 2^{r-1}$. Shi [72] presented an example of an outer-planar 3-regular Hamiltonian graph G with $C(G) = 2^{r-1} + r - 1$.

Király [50] investigated $C(G)$ for several classes of graphs: the union and the sum of two trees, 3-regular and 4-regular graphs, and graphs with average degree 4. Király also conjectured that there is a constant c, such that for any graph G that has m edges,

$$C(G) \leq c(1.4)^m.$$

Aldred and Thomassen [4] studied $C(G)$ for the class of planar graphs. In this chapter, $C(G)$ is investigated for two other classes of graphs: those with n vertices and m edges, and those with m edges.

Theorem 3.2.7 implies that if graph G has n vertices and m edges, then

$$C(G) \leq \begin{cases} \frac{3}{4}\Delta(G)(\frac{m}{n-1})^{n-1}, \text{ if } \frac{m}{n-1} \geq 3, \\ \\ \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m, \text{ if } \frac{m}{n-1} < 3. \end{cases} \tag{3.33}$$

The bound in (3.33) is better than in (3.32) for graphs with sufficiently large number of edges and average degree at least 4.25.

For $m \in \mathbb{Z}^+$ let $C(m)$ be the maximum number of cycles in a graph with m edges. In Corollary 3.2.8 it is shown that

$$C(m) < 8.25(\sqrt[3]{3})^m,$$

which for $m > 4056$ implies

$$C(m) < 1.443^m.$$

Theorem 3.2.7 and Corollary 3.2.8 are proved in Subsection 3.2.3.

In Subsection 3.2.2 it is shown that extremal graphs for $C(m)$ have bounded degrees. Namely, it is shown (Theorem 3.2.3) that if G is a graph with m edges with $C(G) = C(m)$, then $\Delta(G) \leq 11$.

In Subsection 3.2.4, for m sufficiently large, a graph G with m edges is constructed, with

$$C(G) \geq (2 + \sqrt{8})^{\frac{m}{5}-1} \geq 1.37^m.$$

Corollary 3.2.8 and the result of Subsection 3.2.4 imply that for $m > 4056$,

$$1.37^m \leq C(m) \leq 1.443^m. \tag{3.34}$$

In Subsection 3.2.5, the problems of maximizing number of cycles in multigraphs with given number of edges or given number of vertices and edges are considered. It is shown (Theorem 3.2.11) that if G is a multigraph that has the most cycles among all multigraphs with m edges, then

$$\frac{9}{10}(\sqrt[3]{3})^m \leq C(G) \leq 8.25(\sqrt[3]{3})^m.$$

3.2.2 Maximum degree of graphs with $C(m)$ cycles

Recall that, for $m \in \mathbb{Z}^+$, $C(m)$ is the maximum number of cycles in a graph with m edges.

The main result of this section is Theorem 3.2.3, which states that the maximum degree of a graph with $C(m)$ cycles is at most 11. The proof relies on the following two lemmas.

Lemma 3.2.1 (Arman–Tsaturian [9]). *Let $k \geq 6$ be a positive integer. For $1 \leq i < j \leq k$, let $w_{i,j}$ be a non-negative real number, and let $S = \sum_{1 \leq i < j \leq k} w_{i,j}$. Then there exists a 6-element set $D \subseteq [k]$ such that*

$$\sum_{\substack{1 \leq i < j \leq k \\ i \notin D, j \notin D}} w_{i,j} \geq \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.$$

Proof. The proof relies on an averaging argument. For $1 \leq i < j \leq k$, define $w_{j,i} = w_{i,j}$, and for each $i \in [k]$ set $w_i = \sum_{j \in [k], j \neq i} w_{i,j}$. Note that

$$\sum_{i \in [k]} w_i = 2S.$$

Let X be the collection of all 6-element subsets of $[k]$, i.e. $X = [k]^6$. For $D \in X$ let

$$
\begin{aligned}
S(D) &= \sum_{\substack{1 \le i < j \le k \\ i \notin D, j \notin D}} w_{i,j} \\
&= S - \sum_{i \in D} \left(\sum_{j \in [k], j \neq i} w_{i,j} \right) + \sum_{i,j \in D, i<j} w_{i,j} \\
&= S - \sum_{i \in D} w_i + \sum_{i,j \in D, i<j} w_{i,j}.
\end{aligned}
$$

Let $\overline{S}$ be the average of $S(D)$ over all $D \in X$; then

$$
\begin{aligned}
\overline{S} &= \frac{\sum_{D \in X} \left(S - \sum_{i \in D} w_i + \sum_{i,j \in D, i<j} w_{i,j} \right)}{\binom{k}{6}} \\
&= S - \frac{\sum_{i \in [k]} \sum_{D \in X, i \in D} w_i}{\binom{k}{6}} + \frac{\sum_{1 \le i < j \le k} \sum_{D \in X: i,j \in D} w_{i,j}}{\binom{k}{6}} \\
&= S - \frac{\sum_{i \in [k]} \binom{k-1}{5} w_i}{\binom{k}{6}} + \frac{\sum_{1 \le i < j \le k} \binom{k-2}{4} w_{i,j}}{\binom{k}{6}} \\
&= S - \frac{\binom{k-1}{5} \cdot 2S}{\binom{k}{6}} + \frac{\binom{k-2}{4} \cdot S}{\binom{k}{6}} \\
&= \left(1 - \frac{6(2k-7)}{k(k-1)} \right) S.
\end{aligned}
$$

There exists $D \in X$, such that $S(D) \ge \overline{S}$, i.e.,

$$
\sum_{\substack{1 \le i < j \le k \\ i \notin D, j \notin D}} w_{i,j} \ge \left(1 - \frac{6(2k-7)}{k(k-1)} \right) S.
$$

$\square$

Lemma 3.2.2 (Arman–Tsaturian [9]). *Let $k \geq 2$ be a positive integer. For $1 \leq i < j \leq k$, let $w_{i,j}$ be a non-negative real number, and let $S = \sum_{1\leq i<j\leq k} w_{i,j}$. Then there exists a partition $\pi = A_1 \cup A_2 \cup A_3 \cup A_4$ of $[k]$, such that*

$$\sum_{1\leq \ell<m\leq 4} \sum_{i \in A_\ell, j \in A_m} w_{i,j} \geq \left(\frac{3k^2-4}{4k(k-1)}\right) S.$$

Proof. For all $\ell \in [4]$ let $a_\ell = \lfloor \frac{k+\ell-1}{4} \rfloor$ (note that $a_1 + a_2 + a_3 + a_4 = k$). Let X be the collection of all ordered partitions $\pi = A_1 \cup A_2 \cup A_3 \cup A_4$ of $[k]$ such that for all $\ell \in [4]$, $|A_\ell| = a_\ell$. Note that

$$|X| = \frac{k!}{a_1!a_2!a_3!a_4!}.$$

For $\pi = (A_1 \cup A_2 \cup A_3 \cup A_4) \in X$ define

$$\begin{aligned} S(\pi) &= \sum_{1\leq \ell<m\leq 4} \sum_{i\in A_\ell, j\in A_m} w_{i,j} \\ &= S - \sum_{\ell\in[4]} \sum_{i<j, i,j\in A_\ell} w_{i,j}. \end{aligned}$$

Let $\overline{S}$ be the average of $S(\pi)$ over all possible choices of π.

$$\begin{aligned} \overline{S} &= \frac{\sum_{\pi\in X}(S - \sum_{\ell\in[4]} \sum_{i,j\in A_\ell, i<j} w_{i,j})}{|X|} \\ &= S - \frac{\sum_{\ell\in[4]} \sum_{\pi\in X} \sum_{i,j\in A_\ell, i<j} w_{i,j}}{|X|} \\ &= S - \frac{\sum_{\ell\in[4]} \sum_{1\leq i<j\leq k} \sum_{\pi\in X:\, i,j\in A_\ell} w_{i,j}}{|X|} \end{aligned}$$

Note that for any choice of $\ell \in [4]$ and any choice of i, j, such that $1 \leq i < j \leq k$

there are exactly

$$\frac{(k-2)!a_\ell(a_\ell-1)}{a_1!a_2!a_3!a_4!}$$

quadruples $\pi \in X$, such that $i, j \in A_\ell$. Then,

$$\begin{aligned}
\overline{S(\pi)} &= S - \left(\sum_{\ell \in [4]} \sum_{1 \le i < j \le k} \frac{(k-2)!a_\ell(a_\ell-1)}{a_1!a_2!a_3!a_4!} w_{i,j} \right) / |X| \\
&= S - \left(\sum_{\ell \in [4]} \frac{(k-2)!a_\ell(a_\ell-1)}{a_1!a_2!a_3!a_4!} \cdot S \right) \cdot \frac{1}{|X|} \\
&= S - \left(\sum_{\ell \in [4]} \frac{(k-2)!a_\ell(a_\ell l-1)}{a_1!a_2!a_3!a_4!} \right) \cdot S \cdot \frac{a_1!a_2!a_3!a_4!}{k!} \\
&= S - \left(\sum_{\ell \in [4]} \frac{\lfloor \frac{k+\ell-1}{4} \rfloor (\lfloor \frac{k+\ell-1}{4} \rfloor - 1)}{k(k-1)} \right) \cdot S \\
&= S \left(1 - \frac{1}{k(k-1)} \right) \cdot \begin{cases} \frac{k(k-4)}{4}, & \text{if } k \equiv 0 \bmod 4 \\ \frac{(k-1)(k-3)}{4}, & \text{if } k \equiv \pm 1 \bmod 4 \\ \frac{(k-2)^2}{4}, & \text{if } k \equiv 2 \bmod 4 \end{cases} \\
&\ge S \left(1 - \frac{(k-2)^2}{4k(k-1)} \right).
\end{aligned}$$

There exists a $\pi = (A_1 \cup A_2 \cup A_3 \cup A_4) \in X$, such that $S(\pi) \ge \overline{S}$; therefore the partition $A_1 \cup A_2 \cup A_3 \cup A_4$ satisfies the statement of Lemma 3.2.2. □

Theorem 3.2.3 (Arman–Tsaturian, unpublished [9]). *If G is a graph with m edges such that $C(G) = C(m)$, then $\Delta(G) \le 11$.*

Proof. Let m be a positive integer and G be a graph with m edges. To prove Theorem 3.2.3, it is sufficient to prove that if $\Delta(G) \geq 12$, then there is a graph H with m edges and with $C(H) > C(G)$.

Let $\Delta(G) = k \geq 12$ and u be a vertex of maximal degree in G. Let $N(u) = \{u_1, u_2, \ldots, u_k\}$ be the neighbourhood of u. For $1 \leq i < j \leq k$, define $w_{i,j}$ to be the number of paths from vertex u_i to vertex u_j in the graph $G - u$. Then the number of cycles in graph G that pass through vertex u is $S = \sum_{1 \leq i < j \leq k} w_{i,j}$. By Lemma 3.2.1, there is a 6-element set D, such that

$$\sum_{\substack{1 \leq i < j \leq k \\ i \notin D, j \notin D}} w_{i,j} \geq \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{3.35}$$

Suppose, upon re-indexing, that $D = \{k-5, k-4, \ldots, k-1, k\}$. Applying Lemma 3.2.2 to the collection of real numbers $w_{i,j}$ with $1 \leq i < j \leq k-6$ gives a partition $A_1 \cup A_2 \cup A_3 \cup A_4 = [k-6]$ with

$$\sum_{1 \leq l < m \leq 4} \sum_{i \in A_l, j \in A_m} w_{i,j} \geq \left(\frac{3(k-6)^2 - 4}{4(k-6)(k-7)}\right)\left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{3.36}$$

For each $i \in [4]$, let $U_i = \{u_j \ : \ j \in A_i\}$. Define the graph H formed from G by (see Figure 3.2):

- deleting u (a vertex of a maximum degree);
- adding four new vertices v_1, v_2 , v_3, v_4;
- for each $i = 1, 2, 3, 4$ adding edges from v_i to each vertex of U_i;

- for all $1 \leq i < j \leq 4$ adding edges $v_i v_j$.

Then $|E(H)| = |E(G)|$.

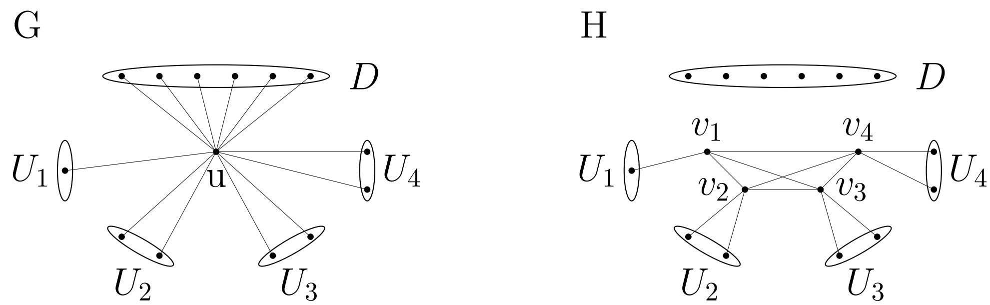


Figure 3.2: Constructing the graph H from G.

To count the number of cycles in H, note the following four points:

- Every cycle in G that does not pass through the vertex u is still a cycle in H. There are $C(G) - S$ such cycles.

- Let C be a cycle in G that for some $1 \leq i < j \leq k-6$ contains

  the path $u_i u u_j$. If for some $l \in [4]$, u_i and u_j are in the same class U_l, then C corresponds to the cycle in H that uses the path $u_i v_l u_j$ instead of $u_i u u_j$. If $u_i \in U_l$ and $u_j \in U_m$ for some $1 \leq l < m \leq 4$, the cycle C corresponds to the cycle in H that uses the path $u_i v_l v_m u_j$ instead of $u_i u u_j$. By (3.35), there are at least

$$\left(1 - \frac{6(2k-7)}{k(k-1)}\right) S$$

  cycles in G that use path $u_i u u_j$ with $u_i, u_j \in N(u) \backslash D$.

- For all $l, m \in [4], l \neq m$, and $i \in A_l, j \in A_m$, every cycle in G that contains the path $u_i u u_j$ gives rise to additional four cycles in H (except the one containing $u_i v_l v_m u_j$). For example, if $l = 1$ and $m = 2$, then the four new cycles contain paths $u_i v_1 v_3 v_2 u_j$, $u_i v_1 v_4 v_2 u_j$, $u_i v_1 v_3 v_4 v_2 u_j$ and $u_i v_1 v_4 v_3 v_2 u_j$ instead of $u_i u u_j$. By (3.36), there are at least

$$\left(\frac{3(k-6)^2-4}{4(k-6)(k-7)}\right)\left(1-\frac{6(2k-7)}{k(k-1)}\right)S = \left(\frac{3k^2-36k+104}{4k(k-1)}\right)S$$

such cycles in G.

- There are 7 new cycles in H spanned by the vertices v_1, v_2, v_3, v_4.

By the observations above, the number of cycles in H is

$$\begin{aligned} C(H) &\geq C(G) - S + \left(1-\frac{6(2k-7)}{k(k-1)}\right)S + 4\left(\frac{3k^2-48k+104}{4k(k-1)}\right)S + 7 \\ &= C(G) + 7 + S\left(\frac{3(k-4)(k-12)}{k(k-1)}\right) \\ &> C(G). \end{aligned}$$

Therefore, H has more cycles than G. □

3.2.3 Cycles in graphs or multigraphs with fixed number of vertices and edges

Multigraphs are defined as in Section 1.1. Denote the number of cycles of length at least 3 in a multigraph G by $C(G)$. No loop can be a part of a cycle, hence only multigraphs without loops are considered.

The main result of this subsection Theorem 3.2.7, which provides an upper bound for number of cycles in a graph (or multigraph) with fixed number of vertices and edges. To prove Theorem 3.2.7, some definitions and lemmas are needed.

Definition 3.2.4. *Let G be a multigraph without loops on n vertices. For vertices $v_1, \dots, v_i \in V(G)$, define $F(v_1, v_2, \dots, v_i) = N(v_i) \backslash \{v_1, \dots, v_{i-1}\}$ and define $f(v_1, \dots, v_i) = \max\{deg_{G-\{v_2,\dots,v_{i-1}\}}(v_i), 1\}$. Denote the number of cycles in G that contain the path $v_1e_1v_2 \dots e_{i-1}v_i$ by $C(v_1e_1v_2 \dots e_{i-1}v_i)$, and define $C(v_1)$ to be the number of cycles containing the vertex v_1. For brevity, write $F_i = F(v_1, \dots, v_i)$, $f_i = f(v_1, \dots, v_i)$, $C_i = C(v_1e_1 \dots e_{i-1}v_i)$.*

Lemma 3.2.5 (Arman–Tsaturian [9]). *Let G be a multigraph with no loops on $n \geq 2$ vertices, $k \in [n-1]$, and $v_1e_1v_2e_2 \dots v_k$ be a path in G. For $i \in [n-1]$, let F_i, f_i, C_i be defined as in Definition 3.2.4. If $F_k \neq \emptyset$, then*

$$C_k \leq f_k \cdot \max_{\substack{k+1 \leq t \leq n \\ v_{k+1} \in F_k \\ \vdots \\ v_t \in F_{t-1}}} \{f_{k+1} \cdot f_{k+2} \cdot \dots \cdot f_t\}.$$

Proof. Fix $n \geq 2$. Let G be a multigraph on n vertices. The proof is by downwards induction on k.

Base case. Let $k = n-1$. Let $v_1e_1 \dots v_{n-1}$ be a path in G; C_{n-1} is to be bounded. The condition $F_{n-1} \neq \emptyset$ means that $F_{n-1} = \{v_n\}$ and it remains to be proved that $C_{n-1} \leq f_{n-1}f_n$. Let s be the number of edges between v_{n-1} and v_1. Then

$C_{n-1} \leq s + (f_{n-1} - s)f_n$. By definition, $f_n \geq 1$; therefore $s + (f_{n-1} - s)f_n \leq sf_n + (f_{n-1} - s)f_n = f_{n-1}f_n$, which proves the base case.

Inductive step. Let $i \in [n-1]$. Assume that the statement of the lemma holds for $k = n-i$, and prove it for $k = n-i-1$, in other words, let $v_1e_1 \dots v_{n-i-1}$ be a path in G, and $C_{n-i-1} = C(v_1e_1 \dots e_{n-i-2}v_{n-i-1})$ is to be bounded.

Let s be the number of edges between v_{n-i-1} and v_1. Then

$$C_{n-i-1} = s + \sum_{\substack{v_{n-i} \in F_{n-i} \\ e_{n-i-1} \in E(v_{n-i}, v_{n-i-1})}} C(v_1e_1 \dots v_{n-i-1}e_{n-i-1}v_{n-i}).$$

For all possible choices of v_{n-i} and e_{n-i-1}, according to inductive hypothesis,

$$C(v_1e_1 \dots e_{n-i-1}v_{n-i}) \leq \begin{cases} f_{n-i} \max_{\substack{n-i+1 \leq t \leq n \\ v_{n-i+1} \in F_{n-i} \\ \vdots \\ v_t \in F_{t-1}}} \{f_{n-i+1} \dots f_t\}, & \text{if } F_{n-i} \neq \emptyset \\ f_{n-i}, & \text{if } F_{n-i} = \emptyset \end{cases}$$

$$\leq \max_{\substack{n-i \leq t \leq n \\ v_{n-i} \in F_{n-i-1} \\ \vdots \\ v_t \in F_{t-1}}} \{f_{n-i} \dots f_t\}.$$

Therefore,

$$C_{n-i-1} \leq s + (f_{n-i-1} - s) \cdot \max_{\substack{n-i \leq t \leq n \\ v_{n-i} \in F_{n-i-1} \\ \vdots \\ v_t \in F_{t-1}}} \{f_{n-i} \dots f_t\}$$

$$\leq f_{n-i-1} \cdot \max_{\substack{n-i \leq t \leq n \\ v_{n-i} \in F_{n-i-1} \\ \vdots \\ v_t \in F_{t-1}}} \{f_{n-i} \dots f_t\}.$$

This proves that the statement of the lemma holds for $l = i + 1$, completing the inductive step. By induction, the statement holds for all $l \in [n-1]$. □

Lemma 3.2.6 (Arman–Tsaturian [9]). *Let G be a multigraph with $n \geq 3$ vertices and m edges and no loops, and let v_1 be a vertex in G of degree $\Delta(G)$.*

- *If $\frac{m}{n-1} \geq 3$, and $\left\lfloor \frac{m}{n-1} \right\rfloor = s$, $\frac{m}{n-1} - s = \alpha$, then there are at most $\frac{\Delta(G)}{2}(s^{1-\alpha}(s+1)^{\alpha})^{n-1}$ cycles in G that contain v_1.*

- *If $\frac{m}{n-1} < 3$, then there are at most $\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m$ cycles in G that contain v_1.*

Proof. Let G be a multigraph with $n \geq 3$ vertices and m edges without loops, and v_1 be a vertex with degree $\Delta(G)$.

For any edge $e = v_1v_2$ incident to v_1, by Lemma 3.2.5, the number of cycles that contain e is at most

$$f_2 \cdot \max_{\substack{3 \leq t \leq n \\ v_3 \in F_2 \\ \vdots \\ v_t \in F_{t-1}}} \{f_3 \dots f_t\} \leq \max_{\substack{2 \leq t \leq n \\ v_2 \in F_1 \\ \vdots \\ v_t \in F_{t-1}}} \{f_2 \dots f_t\}.$$

Every cycle through v_1 contains two such edges, therefore the number of cycles that contain v_1 is at most

$$\frac{\Delta(G)}{2} \cdot \max_{\substack{2 \leq t \leq n \\ v_2 \in F_1 \\ \vdots \\ v_t \in F_{t-1}}} \{f_2 \dots f_t\} \tag{3.37}$$

Let $v_2, \dots v_t$ be a collection of vertices that give the maximum in (3.37) with the smallest possible t. Then $f_t \geq 2$ (otherwise remove all $f_i = 1$ after the last $f_k \geq 2$ to obtain the smaller collection of vertices that gives maximum in (3.37)). Then for all $2 \le i \le t$,

$$f_i = deg_{G-\{v_2,\dots,v_{i-1}\}}(v_i).$$

For $2 \le i \le t$, all the edge sets $\{v_i u \in E(G) : u \in V(G) \backslash \{v_2, \dots, v_i\}\}$ are mutually disjoint, so $f_2 + \dots + f_t \le m$. Therefore,

$$\frac{\Delta(G)}{2} f_2 \cdot \ldots \cdot f_t \le \frac{\Delta(G)}{2} \cdot \max_{\substack{2 \le t \le n \\ x_2 + \ldots + x_t \le m, \\ \forall i \in [2,t], x_i \in \mathbb{Z}^+}} \{x_2 \cdot x_3 \cdot \ldots \cdot x_t\}.$$

So the number of cycles in G that contain v_1 is at most

$$\frac{\Delta(G)}{2} \cdot \max_{\substack{2 \le t \le n \\ x_2 + \ldots + x_t \le m, \\ \forall i \in [2,t], x_i \in \mathbb{Z}^+}} \{x_2 \cdot x_3 \cdot \ldots \cdot x_t\}. \tag{3.38}$$

For a fixed t, the product $x_2 \dots x_t$ in (3.38) obtains its maximum when the x_is ($i \ge 2$) are as equal as possible (for all $i, j \in [t] \backslash \{1\}$ $|x_i - x_j| \le 1$), and their sum is equal to m. Let $\lfloor \frac{m}{n-1} \rfloor = s$, $\frac{m}{n-1} = s + \alpha$.

If $s \ge 3$ (which is equivalent to $\frac{m}{n-1} \ge 3$), let the maximum in (3.38) be achieved for some $t \le n$ and let $x_2, \dots, x_t$ be a collection of x_is that gives the maximum in (3.38). If $t < n$, then $s \ge 3$ implies that either for some $i \in [t] \backslash \{1\}$, $x_i \ge 5$, or for two different $i, j \in [t] \backslash \{1\}$, $x_i = x_j = 4$. In the first case replacing x_i by $x_i - 2$ and setting $x_{t+1} = 2$ gives a collection of x_is with a bigger product. In the second case setting $x_i = x_j = 3$ and $x_{t+1} = 2$ increases the product of x_is. Hence, the maximum in (3.38) is achieved when $t = n$. For all $2 \le i \le n$, $x_i = s$ or $x_i = s + 1$. Then the number of cycles in G that pass through v_1 is at most

$$\frac{\Delta(G)}{2}x_2 \dots x_n = \frac{\Delta(G)}{2}s^{(1-\alpha)(n-1)}(s+1)^{\alpha(n-1)} = \frac{\Delta(G)}{2}(s^{1-\alpha}(s+1)^{\alpha})^{n-1}.$$

If $s < 3$, let the maximum of (3.38) be achieved for some $2 \leq t \leq n$ and let $x_2, \dots, x_t$ be the collection of x_is that gives the maximum in (3.38). Recall that for all $i, j \in [t]\backslash\{1\}$, $|x_i - x_j| \leq 1$. If for two different $i, j \in [t]\backslash\{1\}$, $x_i = x_j > 3$ holds, then $m > 6 + 3(t-2) = 3t$, and $s < 3$ implies that $t < n$. Replacing x_i by $x_i - 1$, x_j by $x_j - 1$ and setting $x_{t+1} = 2$ increases the product $x_2 \dots x_n$. Therefore, there is at most one i such that $x_i = 4$. If there is an i such that $x_i = 1$, then replacing any x_j $(j \neq i)$ by $x_j + 1$ and deleting x_i increases the product $x_2 \dots x_n$. If for some $i, j, k \in [t]$, $x_i = x_j = x_k = 2$ holds, then replacing x_i by 3, x_j by 3 and deleting x_k increases the product $x_2 \dots x_n$. Therefore, $\{x_2, \dots, x_t\} \in \{\{3, 3, \dots, 3, 2, 2\}, \{3, 3, \dots, 3, 4\}, \{3, 3, \dots, 3, 2\}, \{3, 3, \dots, 3\}\}$. Then $x_2 \dots x_t$ is at most $3^{\frac{m}{3}}$, so the number of cycles that pass through v_1 is at most

$$\frac{\Delta(G)}{2}x_2 \dots x_t \leq \frac{\Delta(G)}{2}3^{\frac{m}{3}}.$$ □

Theorem 3.2.7 (Arman–Tsaturian [9])**.** *Let G be a multigraph with $n \geq 2$ vertices and m edges.*

If $\frac{m}{n-1} < 3$, then

$$C(G) < \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\frac{m}{n-1} \geq 3$, and $\lfloor \frac{m}{n-1} \rfloor = s$, $\alpha = \frac{m}{n-1} - s$, then

$$C(G) < \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{n-1} = \frac{3}{4}\Delta(G)((s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}})^m.$$

Proof. The proof is by mathematical induction on n.

Base case. If $n = 2$, there is only one multigraph on n vertices with m edges – two vertices connected by m edges. In this case $s = \frac{m}{n-1} = m$, and G has $\max\{\binom{m}{2}, 0\}$ cycles, which is less than $\frac{3}{4}m(\sqrt[3]{3})^m$ (for the case $m < 3$), and less than $\frac{3}{4}m \cdot m$ (for the case $m \geq 3$).

Inductive step. Let $k \geq 3$ be an integer, and suppose that the statement of the theorem is proved for $n = k - 1$. Let G be a multigraph with k vertices, m edges and let v_1 be a vertex of maximal degree in G.

Case 1. Suppose that $\frac{m}{k-1} < 3$.

If $\Delta(G) \leq 2$, then every edge is contained in at most one cycle, and every cycle contains at least two edges, so the number of cycles in G is at most

$$\frac{m}{2} \leq \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\Delta(G) \geq 3$, then the multigraph $G - v_1$ has at most $m - 3$ edges, $\Delta(G - v_1) \leq \Delta(G)$ and $\frac{|E(G-v_1)|}{|V(G-v_1)|-1} \leq \frac{m}{k-1} < 3$; therefore, by inductive assumption, the number of cycles in $G - v_1$ is at most $\frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m-3}$. By Lemma 3.2.6, the number of cycles that contain v_1 is at most $\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m$. Therefore the total number of cycles in G is at most

$$\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m + \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m-3} = \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

Case 2. Suppose that $\frac{m}{k-1} \geq 3$.

Let $s = \lfloor \frac{m}{k-1} \rfloor$, $\alpha = \frac{m}{k-1} - \lfloor \frac{m}{k-1} \rfloor$. Note that $\Delta(G - v_1) \leq \Delta(G)$ and let

$$y = \frac{|E(G - v_1)|}{|V(G - v_1)| - 1} \leq \frac{m}{k-1}.$$

The function $f(x) = (\lfloor x \rfloor)^{1-x+\lfloor x \rfloor}(\lfloor x \rfloor + 1)^{x - \lfloor x \rfloor}$ is non-decreasing on every interval $[a, a+1], a \in \mathbb{Z}_{\geq 0}$ (and hence on $\mathbb{R}^+$), therefore

$$s^{1-\alpha}(s+1)^{\alpha} \geq f(3) = 3. \tag{3.39}$$

If $y \geq 3$, then, by the induction hypothesis,

$$\begin{aligned}|E(G - v_1)| &\leq \frac{3}{4}\Delta(G)((\lfloor y \rfloor)^{1-y+\lfloor y \rfloor}(\lfloor y \rfloor + 1)^{y-\lfloor y \rfloor})^{k-2} \\ &\leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{k-2}.\end{aligned}$$

If $y < 3$, then $|E(G - v_1)| < 3(k-2)$, and by the induction hypothesis

$$\begin{aligned}|E(G - v_1)| &\leq \frac{3}{4}\Delta(G)(\sqrt[3]{3})^{|E(G-v_1)|} < \frac{3}{4}\Delta(G)(\sqrt[3]{3})^{3(k-2)} \\ &= \frac{3}{4}\Delta(G) \cdot 3^{k-2} \leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{k-2}.\end{aligned}$$

Hence, for any y, $|E(G - v_1)| \leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{k-2}$, which together with Lemma 3.2.6 and (3.39) implies that

$$\begin{aligned}C(G) &= \frac{3\Delta(G)}{4}(s^{1-\alpha}(s+1)^{\alpha})^{k-2} + \frac{\Delta(G)}{2}(s^{1-\alpha}(s+1)^{\alpha})^{k-1} \\ &\leq \frac{3\Delta(G)}{4}(s^{1-\alpha}(s+1)^{\alpha})^{k-1},\end{aligned}$$

which proves the inductive step and hence the theorem. □

Corollary 3.2.8 (Arman–Tsaturian [9])**.** *For any integer m*

$$C(m) < 8.25(\sqrt[3]{3})^m.$$

Proof. Let G be a graph with n vertices and m edges, such that $C(G) = C(m)$. Suppose that $\frac{m}{n-1} \geq 3$. Let $f(s,\alpha) = (s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}}$, then for any $s > 0$, $f(s,\alpha)$ is monotone in α and $\max_{s\in\mathbb{Z}_+,\alpha\in[0,1)} f(s,\alpha) = \max_{s\in\mathbb{Z}_+} s^{\frac{1}{s}} = \sqrt[3]{3}$. This, together with Theorem 3.2.3 and Theorem 3.2.7, implies that for $s = \lfloor \frac{m}{n-1} \rfloor$ and $\alpha = \frac{m}{n-1} - \lfloor \frac{m}{n-1} \rfloor$

$$C(m) = C(G) < \frac{3}{4}\Delta(G)((s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}})^m \leq 8.25(\sqrt[3]{3})^m.$$

If $\frac{m}{n-1} < 3$, then, by Theorem 3.2.3 and Theorem 3.2.7 ,

$$C(m) = C(G) < \frac{3}{4}\Delta(G)(\sqrt[3]{3})^m \leq 8.25(\sqrt[3]{3})^m.$$

□

3.2.4 Example of a graph with $(1.37)^m$ cycles

For $n \geq 1$ let H_n be the graph on $2n+2$ vertices with

$$V(H_n) = \{u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots v_{n+1}\} \qquad \text{and}$$

$$E(H_n) = \{u_i v_j : i, j \in [n+1], |i-j| \leq 1\} \cup \{u_i u_{i+1} : i \in [n]\} \cup \{v_i v_{i+1} : i \in [n]\}.$$

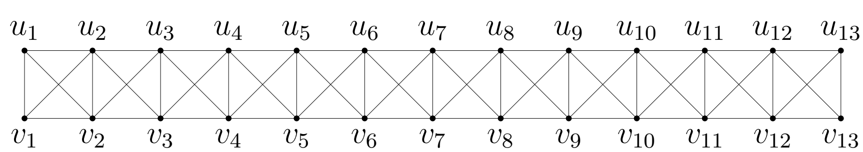


Figure 3.3: The graph H_{12}.

For $n \geq 1$, denote by $P(n)$ the number of paths from the vertex u_1 to the vertex

u_{n+1} in H_n. Note that $P(n)$ is also equal to the number of paths from u_1 to v_{n+1} in H_n.

Claim 3.2.9. *For all $n \geq 2$,*

$$P(n) = 4P(n-1) + 4P(n-2). \tag{3.40}$$

Proof sketch. The proof of the claim relies on an inductive argument and an observation that each path from u_1 to u_{n+1} in H_n corresponds to exactly one of the following paths:

- the path from u_1 to u_n in H_{n-1} followed by the path $u_n u_{n+1}$ or by the path $u_n v_{n+1} u_{n+1}$;
- the path from u_1 to v_n in H_{n-1} followed by the path $v_n u_{n+1}$ or by the path $v_n v_{n+1} u_{n+1}$;
- the path from u_1 to u_{n-1} in H_{n-2} followed by the path $u_{n-1} u_n v_{n+1} v_n u_{n+1}$ or by the path $u_{n-1} v_n v_{n+1} u_n u_{n+1}$.;
- the path from u_1 to v_{n-1} in H_{n-2} followed by the path $v_{n-1} u_n v_{n+1} v_n u_{n+1}$ or by the path $v_{n-1} v_n v_{n+1} u_n u_{n+1}$. □

Solving the recurrence relation 3.40 leads to the inequality $P(n) \geq (2 + 2\sqrt{2})^n$. Define the graph G_n by identifying vertices u_1 and u_n in H_n. Then G_n has $2n + 1$ vertices, $m = 5n + 1$ edges and $C(G_n) \geq (2 + 2\sqrt{2})^n$.

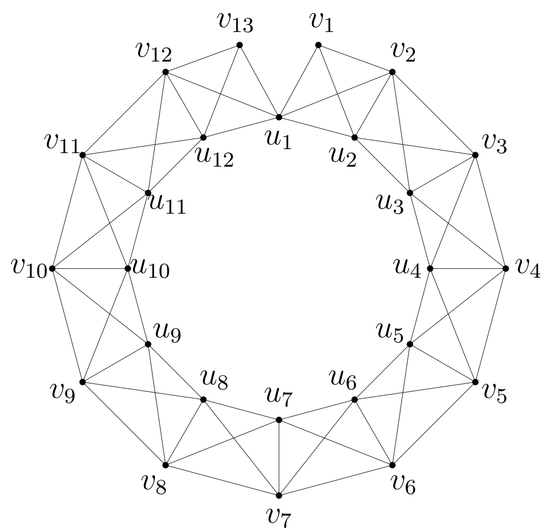


Figure 3.4: G_{12} with 25 vertices and 61 edges.

For an integer m let graph G be obtained from $G_{\lfloor\frac{m-1}{5}\rfloor}$ by adding $(m-5\lfloor\frac{m-1}{5}\rfloor-1)$ edges. Then G has m edges and for m large enough

$$C(G) \geq C(G_{\lfloor\frac{m-1}{5}\rfloor}) \geq (2+2\sqrt{2})^{\lfloor\frac{m-1}{5}\rfloor} \geq (2+2\sqrt{2})^{\frac{m}{5}-1} > 1.37^m.$$

3.2.5 Maximum number of cycles in multigraphs

The problems of maximizing the number of cycles with fixed number of edges or fixed average degree can be also considered for multigraphs. Using the techniques presented in this chapter, the following two results can be proved (but are not proved here).

Theorem 3.2.10 (Arman–Tsaturian [9])**.** *Let $n \geq 2$ and $m \geq 3$ be integers. Let G be a multigraph that has the maximum number of cycles among all the multigraphs* *with n vertices and m edges. Let* $\lfloor \frac{m}{n-1} \rfloor = s$, $\alpha = \frac{m}{n-1} - s$.

If $\frac{m}{n-1} \geq 3$, *then*

$$\frac{8}{27} s (s^{1-\alpha}(s+1)^{\alpha})^{n-1} \leq C(G) \leq \frac{3}{4} \Delta(G) (s^{1-\alpha}(s+1)^{\alpha})^{n-1}.$$

If $\frac{m}{n-1} \leq 3$, *then*

$$4(\sqrt[3]{3})^{m-4} \leq C(G) < \frac{3}{4} \Delta(G) \cdot (\sqrt[3]{3})^{m}.$$

The upper bounds in Theorem 3.2.10 follow from Theorem 3.2.7. For the lower bounds, define $C_{n,m}$ to be the multigraph obtained from the cycle C_n by replacing each of some $m - \lfloor \frac{m}{n} \rfloor n$ consecutive edges with $\lfloor \frac{m}{n} \rfloor + 1$ parallel edges and the rest $\lfloor \frac{m}{n} \rfloor n - m + n$ edges with $\lfloor \frac{m}{n} \rfloor$ parallel edges. The lower bound for the case $\frac{m}{n-1} \geq 3$ is achieved by the graph $C_{n,m}$. The lower bound for the case $\frac{m}{n-1} \leq 3$ is achieved by the graph $C_{\lfloor \frac{m+1}{3} \rfloor, m}$ with extra $n - \lfloor \frac{m+1}{3} \rfloor$ isolated vertices.

Theorem 3.2.11 (Arman–Tsaturian [9])**.** *Let G be a multigraph with $m \geq 3$ edges that has the maximum number of cycles among all the multigraphs with m edges. Then*

$$\frac{9}{10}(\sqrt[3]{3})^{m} < 4(\sqrt[3]{3})^{m-4} \leq C(G) \leq 8.25(\sqrt[3]{3})^{m}.$$

The upper bound in Theorem 3.2.11 can be obtained by repeating the argument of Corollary 3.2.8 and a version of Theorem 3.2.3, modified for multigraphs (the proof is omitted). The example for the lower bound is the same as for the second case of Theorem 3.2.10.

Theorems 3.2.10 and 3.2.11 answer both questions for multigraphs up to a constant factor.

3.2.6 Concluding remarks

Theorem 3.2.7 gives an upper bound for the number of cycles in a graph G with n vertices and average degree d. It is shown in section 3.3 that if $\lim_{n\to\infty} d = \infty$, then there exists a graph with n vertices and m edges that has at least $(1+o(1))^n \left(\frac{2m}{en}\right)^n = (1+o(1))^n \left(\frac{d}{e}\right)^n$ cycles. Therefore, if G is a graph with the maximal number of cycles among all graphs with n vertices and average degree $d \to \infty$, then for n large enough

$$\left(\frac{d}{e}\right)^n (1+o(1))^n \leq C(G) \leq (1+o(1))^n \left(\frac{d}{2}\right)^n.$$

This inequality and the fact that $C(K_n) \approx \frac{c}{\sqrt{n}} \left(\frac{n}{e}\right)^n$ for some constant c motivates the following conjecture.

Conjecture 3.2.12 (Arman–Tsaturian [9]). *For any $\alpha \in (0,1]$ and for sufficiently large integer n, any graph G on n vertices with average degree $d = \alpha n$ satisfies*

$$C(G) \leq (1+o(1))^n \left(\frac{d}{e}\right)^n.$$

Theorem 3.2.8 and the result of Section 3.2.4 imply that $1.37^m \leq C(m) \leq 1.443^m$. Theorem 3.2.7 and the lower bound of 1.37^m imply that if G has m edges and the maximum number of cycles, then $3.66 < d(G) < 10.54$.

Király [50] conjectured that $C(m) < 1.4^m$, which is close to the bound for graphs with average degree $d = 4$ provided by Theorem 3.2.7; in this case, $C(G) < 44(\sqrt{2})^m$.

However, the upper bound in Theorem 3.2.8 is $(\sqrt[3]{3})^m$ and comes from the case when $d = 6$. Considering the fact that the maximum number of cycles in $G(n, m)$ is achieved for graphs with average degree $6.53\ldots$ (see the end of Section 3.3), the following conjectures were posed [9]:

Conjecture 3.2.13 (Arman–Tsaturian [9])**.** *For sufficiently large k, a graph G that has the most cycles among all the graphs with $3k$ edges, is 6-regular.*

Conjecture 3.2.14 (Arman–Tsaturian [9])**.** *For sufficiently large m, there exists a graph G with m edges and at least $(\sqrt{2})^m$ cycles.*

3.3 Number of cycles in random graphs

Let $G(n, m)$ denote the random graph on n vertices with m edges selected uniformly from $\binom{n}{2}$ possible edges. Note that there is another commonly used model of a random graph – $G(n, p)$, the random graph on n vertices where every edge is included with probability p. My goal in this chapter is to find the expected number of cycles in $G(n, m)$.

The expected number of cycles in $G(n, m)$ is the average of the number of cycles over all graphs with n vertices and m edges, and it can be used to get an intuition about the relations between n and m which maximize the number of cycles. Since there is always a graph that has at least the average number of cycles, random graphs also provide lower bounds on the maximum number of cycles. Some of the bounds are worse than the bounds from examples found, but some bounds are conjectured to be the best possible (Conjecture 3.2.12).

As it was mentioned by Takács [82], it follows from the work of Erdős and Rényi [31] that the expected number of cycles in $G(n,m)$ when $m = cn, c \in (0, \frac{1}{2})$ is asymptotically $\frac{1}{2}\ln(\frac{1}{1-2c} - c) - c^2$. Takács [82] also found the limit distribution of the number of cycles of different length in $G(n,m)$ when $m = cn, c \in (0, \frac{1}{2}]$, as well as in $G(n,p)$ (the random graph on n vertices with each edge included with probability p) when $p = \frac{c}{n}, c \in (0, 1]$. Here I mostly focus on the case of $G(n,m)$, $c \geq \frac{1}{2}$ and $m = cn$ for the purposes of maximizing the number of cycles.

Theorem 3.3.1 (Tsaturian, unpublished)**.** *Let $n \to \infty$. If $c \in [\frac{1}{2}, \infty)$ and $m = cn$, then for $G \in G(n,m)$ and $\alpha = \frac{c+1-\sqrt{c^2-2c+3}}{2}$, the expected number of cycles in G is*

$$\mathbb{E}(G) = \left(\frac{2^\alpha c^c}{e^{2\alpha}(c-\alpha)^{c-\alpha}(1-\alpha)^{1-\alpha}} + o(1) \right)^n. \tag{3.41}$$

If $\frac{m}{n} \to \infty$, then for $G \in G(n,m)$, the expected number of cycles is

$$\mathbb{E}(G) = (1 + o(1))^n \left(\frac{2m}{en} \right)^n. \tag{3.42}$$

Proof. Throughout this proof, several facts are used without stating them explicitly:

- $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$; $\lim_{x\to\infty}(1-\frac{1}{x})^x = e^{-1}$;
- for any $a > 0$, as $x \to \infty$, $(1+o(1))^x(a+o(1))^x = (a+o(1))^x$;
- for any function $f(n)$ such that $\lim_{n\to\infty}(f(n))^{\frac{1}{n}} = 1$, it can be written that $f(n) = (1+o(1))^n$ (in particular, this holds if $f(n)$ is a polynomial with degree

greater than 0). Together with the previous fact, for any $a > 0$, $f(n) \cdot (a + o(1))^n = (a + o(1))^n$.

Let $k \in [n]$, $k \geq 3$. For any collection of k possible edges (in particular, for a cycle of length k), there are $\binom{\binom{n}{2}-k}{m-k}$ ways to select m edges that contain the given k edges, therefore the probability that all k edges belong to $G(n,m)$ is equal to $\frac{\binom{\binom{n}{2}-k}{m-k}}{\binom{\binom{n}{2}}{m}}$.

For any k vertices in K_n, there are $\frac{k!}{2k}$ k-cycles on those vertices, therefore the total number of k-cycles in K_n is $\binom{n}{k}\frac{k!}{2k}$.

Let $C_k(G(n,m))$ be the expected number of k-cycles in $G(n,m)$. Then, by linearity of expectation, it follows that

$$C_k(G(n,m)) = \frac{\binom{\binom{n}{2}-k}{m-k}}{\binom{\binom{n}{2}}{m}} \binom{n}{k} \frac{k!}{2k} = \frac{\left(\binom{n}{2}-k\right)!m!n!}{(m-k)!\binom{n}{2}!(n-k)!2k}, \tag{3.43}$$

and the expected total number of cycles in $G(n,m)$ is equal to

$$\sum_{k=3}^{n} C_k(G(n,m)) = \sum_{k=3}^{n} \frac{(\binom{n}{2}-k)!m!n!}{(m-k)!\binom{n}{2}!(n-k)!2k}. \tag{3.44}$$

To estimate which summand in the sum (3.44) is the largest, consider

$$\frac{C_{k+1}(G(n,m))}{C_k(G(n,m))} = \frac{\frac{\left(\binom{n}{2}-k-1\right)!m!n!}{(m-k-1)!\binom{n}{2}!(n-k-1)!2(k+1)}}{\frac{\left(\binom{n}{2}-k\right)!m!n!}{(m-k)!\binom{n}{2}!(n-k)!2k}} = \frac{(m-k)(n-k)k}{(\binom{n}{2}-k)(k+1)}.$$

The condition $C_{k+1}(G(n,m)) \geq C_k(G(n,m))$ is equivalent to

$$(m-k)(n-k)k - \left(\binom{n}{2} - k\right)(k+1) \geq 0. \tag{3.45}$$

Two cases will be considered: $m = cn, c \in [\frac{1}{2}, \infty)$ (graphs with average degree $2c \geq 1$) and $n = o(m)$.

Let $m = cn, c \in [\frac{1}{2}, \infty)$. Let $k = \alpha n$, $\alpha \in [0,1]$ (note that α is a function of n and m). The inequality (3.45) can be written in the following way:

$$(cn - \alpha n)(n - \alpha n)\alpha n - \left(\frac{n(n-1)}{2} - \alpha n\right)(\alpha n + 1) \geq 0,$$

which is equivalent to

$$(c-\alpha)(1-\alpha)\alpha n^3 - \frac{1}{2}\alpha n^3 + o(n^3) \geq 0, \text{or}$$

$$(1-\alpha)(c-\alpha) - \frac{1}{2} + o(1) \geq 0. \tag{3.46}$$

The corresponding quadratic equation for α has solutions $\alpha = \frac{c+1\pm\sqrt{c^2-2c+3}}{2} + o(1)$. For $c \geq \frac{1}{2}$, only the smaller solution is in the interval $[0,1]$; therefore a_k is increasing when $k \in [3, \frac{c+1-\sqrt{c^2-2c+3}}{2}n + o(n))$ and decreasing when $k \in (\frac{c+1-\sqrt{c^2-2c+3}}{2}n + o(n), n]$. Hence, the largest term in the sum (3.44) is the one with $k = \frac{c+1-\sqrt{c^2-2c+3}}{2}n + o(n)$.

When $k = \alpha n$, and $m = cn$, the general term (3.43) can be rewritten as

$$\frac{\left(\binom{n}{2} - \alpha n\right)!(cn)!n!}{((c-\alpha)n)!\binom{n}{2}!\,((1-\alpha)n)!2\alpha n}. \tag{3.47}$$

Using Stirling's approximation [71], it can be found that (3.47) is

$$\frac{(1+0(1))\sqrt{2\pi(\binom{n}{2} - \alpha n)}\left(\frac{\binom{n}{2}-\alpha n}{e}\right)^{(\binom{n}{2}-\alpha n)}\sqrt{2\pi cn}\left(\frac{cn}{e}\right)^{cn}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}}{\sqrt{2\pi(c-\alpha)n}\left(\frac{(c-\alpha)n}{e}\right)^{(c-\alpha)n}\sqrt{2\pi\binom{n}{2}}\left(\frac{\binom{n}{2}}{e}\right)^{\binom{n}{2}}\sqrt{2\pi(1-\alpha)n}\left(\frac{(1-\alpha)n}{e}\right)^{(1-\alpha)n}2\alpha n}$$

$$= \frac{1+o(1)}{2\alpha n e^{\alpha n}}\sqrt{\frac{(\binom{n}{2} - \alpha n)c}{\binom{n}{2}(c-\alpha)(1-\alpha)}}\frac{(\binom{n}{2} - \alpha n)^{(\binom{n}{2}-\alpha n)}(cn)^{cn}}{\binom{n}{2}^{\binom{n}{2}}((c-\alpha)n)^{(c-\alpha)n}}\frac{n^n}{((1-\alpha)n)^{(1-\alpha)n}}. \tag{3.48}$$

Since

$$\left(\frac{\binom{n}{2}}{\binom{n}{2}-\alpha n}\right)^{\binom{n}{2}-\alpha n} = \left(\left(1+\frac{\alpha n}{\binom{n}{2}-\alpha n}\right)^{\frac{\binom{n}{2}-\alpha n}{\alpha n}}\right)^{\alpha n} \sim e^{\alpha n}$$

and

$$(n-1)^{\alpha n} = \left(\left(1-\frac{1}{n}\right)^{n}\right)^{\alpha} n^{\alpha n} \sim \frac{n^{\alpha n}}{e^{\alpha}},$$

it follows that

$$\frac{\left(\binom{n}{2}-\alpha n\right)^{\binom{n}{2}-\alpha n}}{\binom{n}{2}^{\binom{n}{2}}} = \frac{1}{\left(\binom{n}{2}\right)^{\alpha n}\left(\frac{\binom{n}{2}}{\binom{n}{2}-\alpha n}\right)^{\binom{n}{2}-\alpha n}}$$
$$\sim \frac{1}{\left(\frac{n}{2}\right)^{\alpha n}(n-1)^{\alpha n}e^{\alpha n}} \sim \frac{2^{\alpha n}e^{\alpha}}{n^{2\alpha n}e^{\alpha n}} = e^{\alpha}\left(\frac{2^{\alpha}}{n^{2\alpha}e^{\alpha}}\right)^{n}. \tag{3.49}$$

Furthermore,

$$\frac{(cn)^{cn}}{((c-\alpha)n)^{(c-\alpha)n}} = \left(\frac{c^c n^c}{(c-\alpha)^{c-\alpha}n^{c-\alpha}}\right)^n = \left(\frac{c^c n^{\alpha}}{(c-\alpha)^{c-\alpha}}\right)^n, \tag{3.50}$$

and

$$\frac{n^n}{((1-\alpha)n)^{(1-\alpha)n}} = \left(\frac{n}{(1-\alpha)^{1-\alpha}n^{1-\alpha}}\right)^n = \left(\frac{n^{\alpha}}{(1-\alpha)^{1-\alpha}}\right)^n. \tag{3.51}$$

Using (3.49), (3.50), and (3.51) in (3.48), $C_k(G(n,m))$ is

$$\frac{1+o(1)}{2\alpha n e^{\alpha n}}\sqrt{\frac{(\binom{n}{2}-\alpha n)c}{\binom{n}{2}(c-\alpha)(1-\alpha)}}\,e^{\alpha}\left(\frac{2^{\alpha}}{n^{2\alpha}e^{\alpha}}\right)^n\left(\frac{c^c n^{\alpha}}{(c-\alpha)^{c-\alpha}}\right)^n\left(\frac{n^{\alpha}}{(1-\alpha)^{1-\alpha}}\right)^n$$
$$= \frac{1+o(1)}{2\alpha n}e^{\alpha}\sqrt{\frac{(\binom{n}{2}-\alpha n)c}{\binom{n}{2}(c-\alpha)(1-\alpha)}}\left(\frac{2^{\alpha}c^c}{e^{2\alpha}(c-\alpha)^{c-\alpha}(1-\alpha)^{1-\alpha}}\right)^n$$
$$= \left(\frac{2^{\alpha}c^c}{e^{2\alpha}(c-\alpha)^{c-\alpha}(1-\alpha)^{1-\alpha}}+o(1)\right)^n. \tag{3.52}$$

This implies that the largest term in the sum representing the expected number of cycles in $G(n,m)$ when $m = cn$ is the expression (3.52) with $\alpha = \frac{c+1-\sqrt{c^2-2c+3}}{2}$.

Since the ratio of the sum and the largest term is between 1 and $n-2$ (which is $(1+o(1))^n$), the expected number of cycles in $G(n,m)$ can be represented by the same expression, namely by

$$\left(\frac{2^{\frac{c+1-\sqrt{c^2-2c+3}}{2}}c^c+o(1)}{e^{c+1-\sqrt{c^2-2c+3}}\left(c-\frac{c+1-\sqrt{c^2-2c+3}}{2}\right)^{c-\frac{c+1-\sqrt{c^2-2c+3}}{2}}\left(1-\frac{c+1-\sqrt{c^2-2c+3}}{2}\right)^{1-\frac{c+1-\sqrt{c^2-2c+3}}{2}}}\right)^n.$$

Now, let $n=o(m)$. Let $k=\alpha n$, $\alpha\in[0,1]$ (note that α is a function of n and m). Inequality (3.45) can be rewritten in the following way:

$$\left(\frac{m}{n}-\alpha n\right)(n-\alpha n)\,\alpha n-\left(\frac{n(n-1)}{2}-\alpha n\right)(\alpha n+1)\geq 0,$$

which is equivalent to

$$\left(\frac{m}{n}-\alpha\right)(1-\alpha)\,\alpha n^3-\frac{1}{2}\alpha n^3+\left(\alpha^2+\frac{1}{2}\alpha-\frac{1}{2}\right)n^2-\left(\alpha+\frac{1}{2}\right)n\geq 0,\text{or}$$
$$(1-\alpha)\left(\frac{m}{n}-\alpha\right)-\frac{1}{2}+o(1)\geq 0.$$

The corresponding quadratic equation for α has roots $\alpha=\frac{\frac{m}{n}+1\pm\sqrt{(\frac{m}{n}-1)^2+2-o(1)}}{2}$, only the smaller of which is in the interval $(0,1]$. Therefore the largest term in the sum (3.44) corresponds to $k=\frac{\frac{m}{n}+1-\sqrt{(\frac{m}{n}-1)^2+2-o(1)}}{2}n\geq\frac{\frac{m}{n}+1-\sqrt{(\frac{m}{n}-1)^2+2}}{2}n=(1-\frac{1}{\frac{m}{n}-1+\sqrt{(\frac{m}{n}-1)^2+2}-1})n=(\left(1+O\left(\frac{n}{m}\right)\right)n.$

Using Stirling's approximation [71], the k-th term in (3.44) is

$$(1+o(1))\sqrt{\frac{(\binom{n}{2}-k)mn}{(m-k)\binom{n}{2}(n-k)}}\frac{e^{m+m+\binom{n}{2}-2k}}{2ke^{m+n+\binom{n}{2}-k}}\frac{(\binom{n}{2}-k)^{\binom{n}{2}-k}m^m n^n}{(m-k)^{m-k}\binom{n}{2}^{\binom{n}{2}}(n-k)^{n-k}}$$

$$\sim \frac{1+o(1)}{2ke^k}\sqrt{\frac{n}{n-k}}\left(\frac{\binom{n}{2}-k}{\binom{n}{2}}\right)^{\binom{n}{2}-k}\left(\frac{m}{m-k}\right)^{m-k}\left(\frac{n}{n-k}\right)^{n-k}\left(\frac{mn}{\binom{n}{2}}\right)^{k}. \quad (3.53)$$

Since $\left(\frac{m}{m-k}\right)^{m-k} = \left(1+\frac{1}{m-k}\right)^{m-k} \sim e$, and $\left(\frac{\binom{n}{2}}{\binom{n}{2}-k}\right)^{\binom{n}{2}-k} \sim e$, RHS of (3.53) is

$$(1+o(1))\sqrt{\frac{n}{n-k}}\frac{e^2}{2ke^k}\left(\frac{n}{n-k}\right)^{n-k}\left(\frac{mn}{\binom{n}{2}}\right)^{k}. \quad (3.54)$$

Using the fact that the function $f(x) = x^{\frac{1}{x}}$ is decreasing on (e,∞), $k = n + O(\frac{n^2}{m})$ and $n = o(m)$, for some $b \in \mathbb{R}$,

$$\left(\frac{n}{n-k}\right)^{n-k} = \left(\left(\frac{n}{n-k}\right)^{\frac{n-k}{n}}\right)^{n} \geq \left(\left(\frac{bm}{n}\right)^{\frac{n}{bm}}\right)^{n} = (1+o(1))^n.$$

Since $\sqrt{\frac{n}{n-k}}\frac{e^2}{2ke^k} = \sqrt{\frac{n}{n-k}}\frac{e^2}{2k}e^{-n}e^{O(\frac{n^2}{m})} = (1+o(1))^n e^{-n}e^{o(n)} = (1+o(1))^n e^{-n}$, the expression in (3.54) becomes

$$\begin{aligned}(1+o(1))^n\left(\frac{mn}{\binom{n}{2}}\right)^{n+O(\frac{n^2}{m})}e^{-n} &= (1+o(1))^n\left(\frac{2m}{en}\right)^{n+O(\frac{n^2}{m})}\\ &= (1+o(1))^n\left(\frac{2m}{en}\right)^{n}\left(\frac{2m}{en}\right)^{O(\frac{n^2}{m})}\\ &= (1+o(1))^n\left(\frac{2m}{en}\right)^{n}\left(\left(\frac{2m}{en}\right)^{\frac{en}{2m}}\right)^{O(n)}\\ &= (1+o(1))^n\left(\frac{2m}{en}\right)^{n}(1+o(1))^{O(n)}\\ &= (1+o(1))^n\left(\frac{2m}{en}\right)^{n}.\end{aligned}$$

Since the ratio of the sum (3.52) and its largest term is between 1 and $n-2$ (which is $(1+o(1))^n$), the expected number of cycles in $G(n,m)$ is $(1+o(1))^n\left(\frac{2m}{en}\right)^n$. □

Theorem 3.3.1 implies that in the case of dense graphs (when $\frac{m}{n} \to \infty$), for n sufficiently large, there exists a graph on n vertices and m edges with at least $(1+o(1))^n \left(\frac{2m}{en}\right)^n$ cycles. This bound is conjectured to be tight (Conjecture 3.2.12).

A lower bound on the maximum number of cycles in a graph with given number of edges can be deduced from Theorem 3.3.1. The function f defined by

$$f(c) = \frac{2^{\frac{c+1-\sqrt{c^2-2c+3}}{2}} c^c}{e^{c+1-\sqrt{c^2-2c+3}} \left(c - \frac{c+1-\sqrt{c^2-2c+3}}{2}\right)^{c-\frac{c+1-\sqrt{c^2-2c+3}}{2}} \left(\frac{1-c+\sqrt{c^2-2c+3}}{2}\right)^{\frac{1-c+\sqrt{c^2-2c+3}}{2}}}$$

has maximum over $[\frac{1}{2}, \infty)$ at $c = 3.2576\ldots$, with $f(c) = 1.3238\ldots$ (assisted by a computer). Therefore, if $m = (3.2576 + o(1))n$ (hence $n = (0.306... + o(1))m$), then $C(G(n,m)) = (1.3238\ldots + o(1))^n = (1.0899\ldots + o(1))^m$, which implies that for m sufficiently large, there exists a graph on m edges with at least $(1.0899\ldots + o(1))^m$ cycles. Note that if $\frac{m}{n} \to \infty$, the number of cycles in $G(n,m)$ is $(1+o(1))^n \left(\frac{2m}{en}\right)^n = (1+o(1))^n \left(\left(\left(\frac{2m}{en}\right)^{\frac{ne}{2m}}\right)^{\frac{2}{e}}\right)^m = (1+o(1))^n(1+o(1))^m = (1+o(1))^{m+o(m)} = ((1+o(1))^{1+o(1)})^m = (1+o(1))^m$, so for fixed number of edges dense graphs have less cycles.

Although the lower bound obtained for the graphs with fixed number of edges is worse than the bound that follows from the example in subsection 3.2.4, Theorem 3.3.1 suggests that graphs have the most number of cycles when the average degree is close to $2 \cdot 3.2576 \sim 6.53$.

Chapter 4

Euclidean Ramsey theory

4.1 Results in Euclidean Ramsey theory

This section contains a brief overview of known results and problems in Euclidean Ramsey theory; two of the problems mentioned are solved in Sections 4.2 and 4.3. For more results, see the classic survey of the area by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus [27–29] and a more recent survey by Graham [39].

Many problems in Euclidean Ramsey theory ask, for some $d \in \mathbb{Z}^+$, if the d-dimensional Euclidean space $\mathbb{E}^d$ is coloured with $r \geq 2$ colours, does there exist a colour class containing some desired geometric structure?

One of the simplest configurations that can be considered is two points distance one apart. Let $\chi(\mathbb{E}^d)$ be the minimal number of colours needed to colour all of the points of Euclidean space $\mathbb{E}^d$, so that there are no monochromatic points distance one apart. The problem of determining $\chi(\mathbb{E}^d)$ is over seventy years old and is usually referred to as the Hadwiger-Nelson problem (see Soifer [78]). Even the case $n = 2$ remains open. For over fifty years, the best known bounds for the chromatic number of the plane have been $4 \leq \chi(\mathbb{E}^2) \leq 7$. The lower bound is due to Leo Moser and Willy Moser [61] and the upper bound is claimed by Soifer [78] to be due to John Isbell. Only recently, in 2018, Aubrey de Grey [20] proved that $\chi(\mathbb{E}^2) \geq 5$.

In case of three dimensions, $6 \leq \chi(\mathbb{E}^3) \leq 15$, with the lower bound due to Nechushtan [63] and the upper bound due to Coulson [19] and to Radoičić and Tóth [66]. The best known general bounds are

$$(1.239... + o(1))^d \leq \chi(\mathbb{E}^d) \leq (3 + o(1))^d,$$

where the upper bound is due to Larman and Rogers [54] and the lower bound is due to Raigorodskii [67].

Say that two geometric configurations are congruent if and only if there exists an isometry (distance preserving bijection) between them. For $n, r \in \mathbb{Z}^+$ and a geometric configuration X, write $\mathbb{E}^n \to (X)_r$ if and only if for any r-colouring of $\mathbb{E}^n$, there exists a monochromatic congruent copy of X. This notation is called the "Ramsey arrow".

For a positive integer i, denote by ℓ_i the configuration of i collinear points with distance 1 between consecutive points. The problem of finding $\chi(\mathbb{E}^d)$ can be rephrased in terms of the "Ramsey arrow notation"; for example, the Nechushtan's result

$\chi(\mathbb{E}^3 \geq 6)$ [63] reads as $\mathbb{E}^3 \to (\ell_2)_5$. What if larger geometric configurations (instead of ℓ_2) are considered?

One of the first results in this direction was proved by Erdős et al. [27] in 1973 and states that $\mathbb{E}^2 \to (\ell_3)_2$. The authors [27] also conjectured that for any non-equilateral triangle T (here by a triangle, I mean the set of vertices of a triangle), $\mathbb{E}^2 \to (T)_2$. This conjecture still remains open in general, although it has been verified for many special cases (see [39]). If the same question is asked for $\mathbb{E}^3$, namely if it is true that $\mathbb{E}^3 \to (T)_2$, the question becomes much easier, and was answered in the affirmative [27]. A natural question to ask is for what geometric configurations X and for what numbers of colours r there is a large enough dimension d such that $\mathbb{E}^d \to (X)_r$? This question leads to the notion of Ramsey sets.

A geometric configuration X is called *Ramsey* if and only if for any $r \in \mathbb{Z}^+$, there exists $d \in \mathbb{Z}^+$ such that $\mathbb{E}^d \to (X)_r$. Erdős et al. [27] proved that ℓ_2 is Ramsey (for a proof, it is enough to consider a unit simplex on $2r + 1$ points in $\mathbb{E}^{2r}$), and the orthogonal Cartesian product of two Ramsey sets is Ramsey, which implies that any rectangular parallelepiped is Ramsey. Frankl and Rödl [33] proved that any triangle is Ramsey. The same authors [34] showed that the set of vertices of any simplex is Ramsey. Křiž [46] showed the Ramsey property for configurations with transitive group of isometries, which has a solvable subgroup with at most two orbits. This result, in particular, implies that vertex sets of regular polygons and Platonic solids are Ramsey.

A set of points X is called spherical if and only if there exists a $d \in \mathbb{Z}^+$ such that $X \subseteq \mathbb{S}^d$. It has been shown [27] that all Ramsey sets are spherical. There are two main conjectures regarding which sets are Ramsey: Graham [39] conjectured that all spherical sets are Ramsey, and Leader, Russell and Walters [55] conjectured that any Ramsey sets is a subset of a set with a transitive group of isometries. The simplest question that separates the two conjectures (and still remains open) is whether or not the set of vertices of any cyclic quadrilateral is Ramsey.

So far, the only problems that have been considered concentrate on finding a monochromatic copy of one configuration. As in the study of off-diagonal Ramsey numbers, one can also investigate when one structure F_1 is found in the first colour or a second structure F_2 is found in the second colour. For $d \in \mathbb{Z}^+$, and geometric configurations F_1, F_2, let the notation $\mathbb{E}^d \to (F_1, F_2)$ mean that for any red-blue coloring of $\mathbb{E}^d$, either the red points contain a congruent copy of F_1, or the blue points contain a congruent copy of F_2.

One of the results of Erdős et al. [28] states that $\mathbb{E}^2 \to (\ell_2, \ell_4)$; a short proof is given below.

Theorem 4.1.1 (Erdős–Graham–Montgomery–Rothschild–Spencer–Straus, 1975 [28]). *Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there is no two red points distance 1 apart. Then there exist four blue points that form an ℓ_4.*

Proof. Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. First, it is proved that there is a blue ℓ_3. Let O be any red point (if there are no red points, there is a blue ℓ_4). Consider the circle C_1 with center O and radius 1 (Figure 4.1).

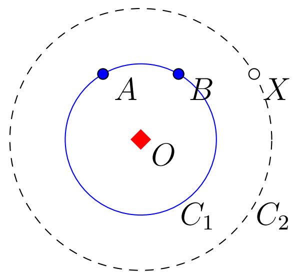


Figure 4.1: Red points are denoted by diamonds, blue points are denoted by discs.

Since any point on C_1 is at distance 1 from O, C_1 is blue. Let C_2 be the circle with center O and radius $\sqrt{3}$. If all points of C_2 are red, then C_2 contains a pair of red points at distance 1, a contradiction. If any point X on C_2 is blue, then there are two points A and B on C_1 that together with X form a blue ℓ_3.

The rest of the proof follows Erdős et al. [28]. Suppose that there is no blue ℓ_4. Let A, B and C be any three blue points forming an ℓ_3. Consider the part of the unit triangular lattice shown in Figure 4.2.

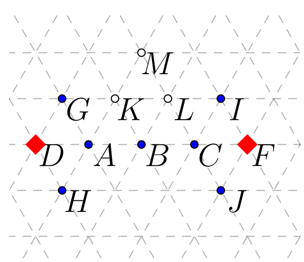


Figure 4.2

The point D is red; otherwise $DABC$ is a red ℓ_4. Similarly, F is red. Then the points G, H, I and J are blue, since each of them is at distance 1 from a red point (D or F). One of the points K and L, say, K, is red (otherwise a blue ℓ_4 $GKLI$ is formed). Then L is at distance 1 from K, and therefore is blue. If M is blue, then $MLCJ$ is a blue ℓ_4; therefore M is red. Then M and K are two red points at distance 1, which contradicts the assumption. □

In the same paper [28], it was asked if $\mathbb{E}^2 \to (\ell_2, \ell_5)$, or perhaps a weaker result holds: $\mathbb{E}^3 \to (\ell_2, \ell_5)$.

Let ℓ_i denote the configuration of i collinear points with distance one between any two consecutive points. The results of Erdős et al. [27–29] include:

- If T_i is a configuration of i points, then $\mathbb{E}^2 \to (T_2, T_3)$.
- If T denotes the isosceles right triangle with unit catheti (legs) and Q^2 denotes the unit square, then $\mathbb{E}^3 \to (T, Q^2)$;
- $\mathbb{E}^4 \to (\ell_2, \ell_5)$.

Theorem 4.1.1 was generalised by Juhász [48], who proved that if T_4 is any configuration of 4 points, then $\mathbb{E}^2 \to (\ell_2, T_4)$. Juhász [47] informed me that Iván's thesis [44] contains a proof that for any configuration T_5 of 5 points, $\mathbb{E}^3 \to (\ell_2, T_5)$ (which implies that $\mathbb{E}^3 \to (\ell_2, \ell_5)$). This result of Iván [44] also follows from results of Szlam [81] and Nechushtan [63], as well as the result of Juhász now follows from results of Szlam [81] and de Grey [20]. Section 4.2 contains a proof of the fact that $\mathbb{E}^2 \to (\ell_2, \ell_5)$. Section 4.3 contains a proof of the fact that $\mathbb{E}^3 \to (\ell_2, \ell_6)$.

The existence of a k, such that $\mathbb{E}^2 \not\to (\ell_2, \ell_k)$, was first noted by Erdős and Graham [26], who mention the upper bound of "10000000, more or less". A more precise bound of $k = 10^{10}$ follows from a recent result of Conlon and Fox [18], who showed that for all $n \geq 2$, $\mathbb{E}^n \not\to (\ell_2, \ell_{10^{5n}})$.

It was shown by Szlam [81] that if $T_{\chi(\mathbb{E}^d)-1}$ is any configuration of $\chi(\mathbb{E}^d) - 1$ points, then $\mathbb{E}^d \to (\ell_2, T_{\chi(\mathbb{E}^d)-1})$, which together with the result of Raigorodskii [67] implies that for high dimensions, $\mathbb{E}^d \to (\ell_2, \ell_{(1.239+o(1))^d})$.

4.2 Proof of $\mathbb{E}^2 \to (\ell_2, \ell_5)$

This section contains the proof of my result that shows that $\mathbb{E}^2 \to (\ell_2, \ell_5)$.

Theorem 4.2.1 (Tsaturian, 2017 [83]). *Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Then there exist five blue points that form an ℓ_5.*

The proof is by contradiction; it is assumed that there are no five blue points forming an ℓ_5. The following lemmas are needed.

Lemma 4.2.2 (Tsaturian, 2017 [83]). *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. If there is no blue ℓ_5, then there are no three blue points forming an equilateral triangle with side length 3 and with a red centre.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose that blue points A, B and C form an equilateral triangle with side length 3 and with red centre O. Consider the part of the unit triangular lattice shown in Figure 1(a). The points D, E, F, G are blue, since they are distance 1 apart from O. The point X is red; otherwise $XADEB$ is a red ℓ_5. Similarly, Y is red (to prevent red $YAFGC$). Then X and Y are two red points distance 1 apart, which contradicts the assumption. □

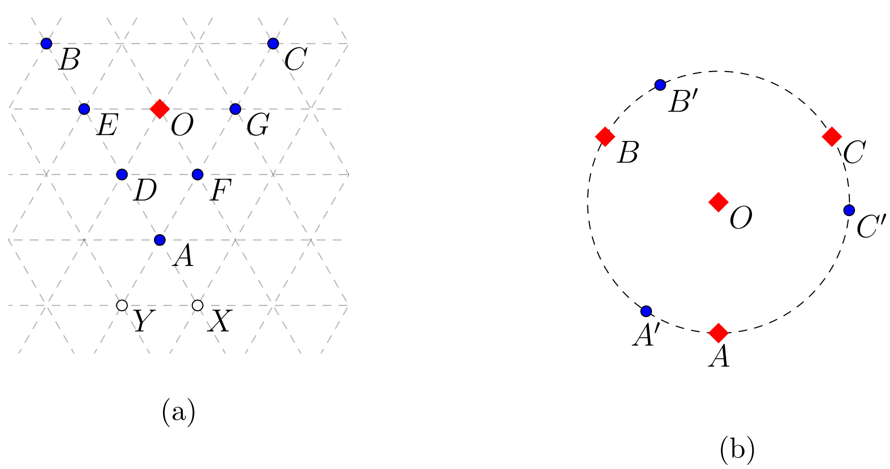


Figure 4.3: Red points are denoted by diamonds, blue points are denoted by discs.

Lemma 4.2.3 (Tsaturian, 2017 [83])**.** *Let* $\mathbb{E}^2$ *be coloured in red and blue so that there is no red* ℓ_2*. If there is no blue* ℓ_5*, then there are no three red points forming an equilateral triangle with side length* 3 *and with a red centre.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose that blue points A, B and C form an equilateral triangle with side length 3 and with red centre O. Let A', B', C' be the images of A, B and C, respectively, under a rotation about O so that $AA' = BB' = CC' = 1$ (see Figure 1(b)). Then A', B', C' are blue and form an equilateral triangle with side length $\sqrt{3}$ and red center O, which contradicts the result of Lemma 4.2.2. $\square$

Define $\mathfrak{T}_3$, $\mathfrak{T}_4$, $\mathfrak{T}_5$, $\mathfrak{T}_6$, $\mathfrak{T}_7$ to be the configurations of three, four, five, six and seven points (respectively) depicted in Figure 4.4 (all the smallest distances between the points are equal to $\sqrt{3}$).

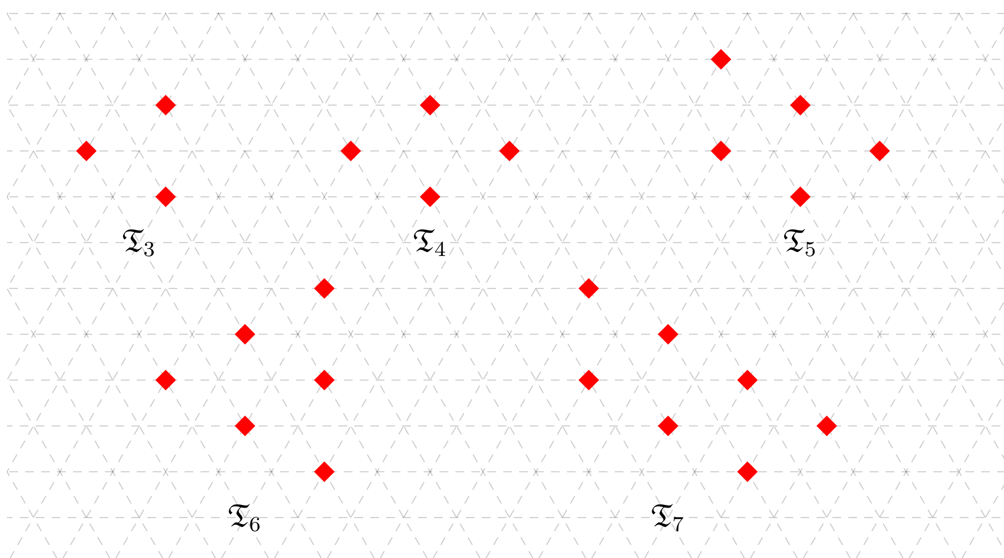


Figure 4.4

Lemma 4.2.4 (Tsaturian, 2017 [83])**.** *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. If there is no blue ℓ_5, then there are no seven red points forming a $\mathfrak{T}_7$.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose that A, B, C, D, E, F and G are red points forming a $\mathfrak{T}_7$ (as in Figure 4.5).

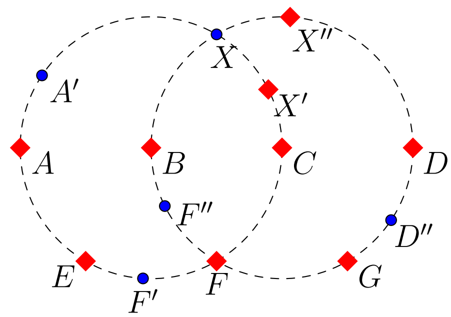


Figure 4.5

Let X be the reflection of F in BC. Let X', A', F' be the images of X, A, F, respectively, under the clockwise rotation about B such that $XX' = AA' = FF' = 1$. Since A and F are red, A' and F' are blue. If X' is blue, then $X'A'F'$ is a blue equilateral triangle with side length 3 and red center B, which contradicts the result of Lemma 4.2.2. Therefore, X' is red.

Let X'', D'', F'' be the images of X, D, F, respectively, under the clockwise rotation about C such that $XX'' = DD'' = FF'' = 1$. Since D and F are red, D'' and F'' are blue. If X'' is blue, then $X''D''F''$ is a blue equilateral triangle with side length 3 and red center C, which contradicts the result of Lemma 4.2.2. Therefore, X'' is red. Since X' can be obtained from X'' by the clockwise rotation through $60°$ about X, $XX'X''$ is a unit equilateral triangle, hence $X'X''$ is a red ℓ_2, which contradicts the assumption of the lemma. □

Lemma 4.2.5 (Tsaturian, 2017 [83]). *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. Let A, B, C be three red points forming a $\mathfrak{T}_3$. If there is no blue ℓ_5, then there exists a red $\mathfrak{T}_6$ that contains $\{A, B, C\}$ as a subset.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Let A, B, C be three red points forming a $\mathfrak{T}_3$. Consider the unit triangular lattice depicted in Figure 4.6.

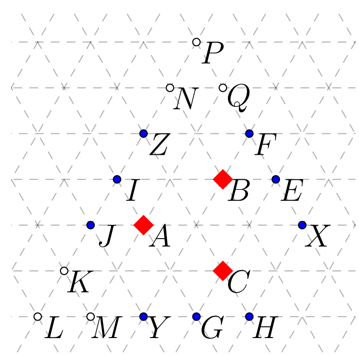


Figure 4.6

Suppose that there is no red point D such that A, B, C, D form a $\mathfrak{T}_4$. Then points X, Y, Z are blue. Points E, F, G, H, I, J are blue, since each of them is distance 1 from a red point. If the point K is red, then the points L and M are blue and $LMYGH$ is a blue ℓ_5. Therefore, K is blue. Then N is red (otherwise $KJIZN$ is a blue ℓ_5), hence P and Q are blue, which leads to a blue ℓ_5 $PQFEX$. A contradiction is obtained, therefore there exists a red point D such that A, B, C, D form a $\mathfrak{T}_4$.

Let A, B, C, D form a red $\mathfrak{T}_4$. Consider the part of the unit triangular lattice depicted in Figure 4.7.

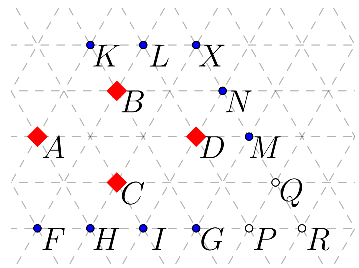


Figure 4.7

Suppose that there is no red point E such that A, B, C, D, E form a $\mathfrak{T}_5$. Then the points X, F and G are blue. Points H, I, K, L, M, N are blue, since each of them is distance 1 from a red point. Point P is red (otherwise $FHIGP$ is a blue ℓ_5); therefore Q and R are blue. Then X, N, M, Q, R form a blue ℓ_5, which gives a contradiction. Hence, there exists a red point E such that A, B, C, D, E form a $\mathfrak{T}_5$.

Let A, B, C, D, E form a $\mathfrak{T}_5$ (see Figure 4.8).

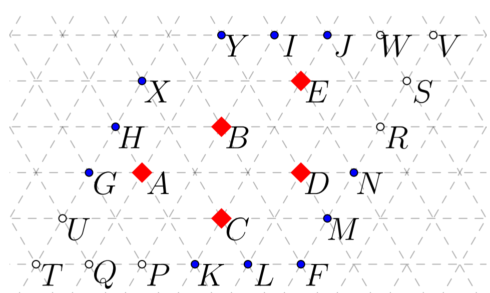


Figure 4.8

Suppose that F is blue. By Lemma 4.2.3, points X and Y are blue (otherwise

X, E, C (Y, A, D) form a red triangle with side length 3 and red center B). Points G, H, I, J, K, L, M, N are blue, since each one of them is at distance 1 from a red point. If P is blue, then Q is red (otherwise $QPKLF$ is a blue ℓ_5); U and T are blue and form a blue ℓ_5 with points G, H and X. Therefore, P is red. Similarly, R is red (otherwise S is red and $VWJIY$ is a blue ℓ_5). Then A, B, C, D, E, P and R form a red $\mathfrak{T}_7$, which is not possible by Lemma 4.2.4. Therefore, F is red and A, B, C, D, E, F form a red $\mathfrak{T}_6$.

□

Lemma 4.2.6 (Tsaturian, 2017 [83]). *Let* $\mathbb{E}^2$ *be coloured in red and blue so that there is no red* ℓ_2*. Let* $\mathfrak{L}$ *be a unit triangular lattice that contains three red points forming a* $\mathfrak{T}_3$*. If there is no blue* ℓ_5*, then the colouring of* $\mathfrak{L}$ *is unique (up to translation or rotation by a multiple of* $60°$*), and is depicted in Figure 4.9.*

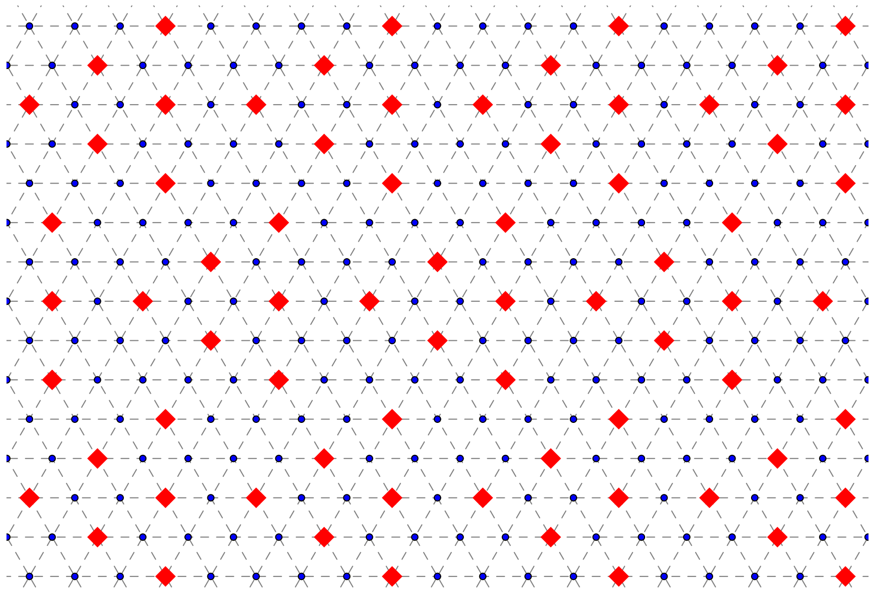


Figure 4.9

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose there exist three red points of $\mathfrak{L}$ that form a $\mathfrak{T}_3$. By Lemma 4.2.5, it may be assumed that there is a red $\mathfrak{T}_6$. Denote the points of this red $\mathfrak{T}_6$ by A, B, C, D, E, F (see Figure 4.10). Below, it is proved that the translate $A'B'C'D'E'F'$ of $ABCDEF$ by the vector of length 5 collinear to $\overrightarrow{AD}$ is red.

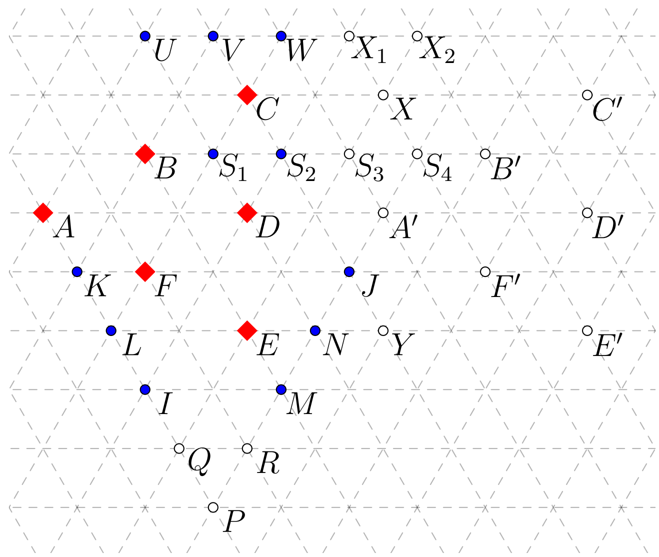


Figure 4.10

Consider the points shown in Figure 4.10. Since A, D and F are red, by Lemma 4.2.3, I is blue. Since C, F and D are red, by Lemma 4.2.3, J is blue. Points K, L, M, N are blue, since each one is distance 1 apart from a red point. If R is red, then both P and Q are blue and form a blue ℓ_5 with K, L and I. Therefore R is blue. Then the point A' is red (otherwise $A'JNMR$ is a red ℓ_5).

Since S_1, S_2, S_3, S_4 are blue (they have distance 1 from red points D and A'), B' is red. Similarly, F' is red. Points V and W are blue as they are distance 1 apart from C. Points U is blue by Lemma 4.2.3 (since A, D and B are red). If X is red, then X_1 and X_2 are blue and a blue ℓ_5 $UVWX_1X_2$ is formed. Therefore, X is blue. Similarly, Y is blue. By Lemma 4.2.5, $A'B'F'$ must be contained in a red $\mathfrak{T}_6$, and since X and Y are blue, the only possible such $\mathfrak{T}_6$ is $A'B'C'D'E'F'$. Hence, A', B',

C', D', E', F' are blue.

Similarly, the translates of $ABCDEF$ by vectors of length 5 collinear to $\overrightarrow{EB}$ and $\overrightarrow{CF}$ are red. By repeatedly applying the same argument to the new red translates, it can be seen that all the translates of $ABCDEF$ by a multiple of 5 in $\mathfrak{L}$ are red. All the other points are blue, as each one is distance 1 apart from a red point. Hence, the colouring as in Figure 4.9 is obtained. □

Lemma 4.2.7 (Tsaturian, 2017 [83])**.** *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. Let $\mathfrak{L}$ be a unit triangular lattice that does not contain three red points forming a $\mathfrak{T}_3$. If there is no blue ℓ_5, then the colouring of $\mathfrak{L}$ is unique (up to translation or rotation by a multiple of* $60°$*), and is depicted in Figure 4.11.*

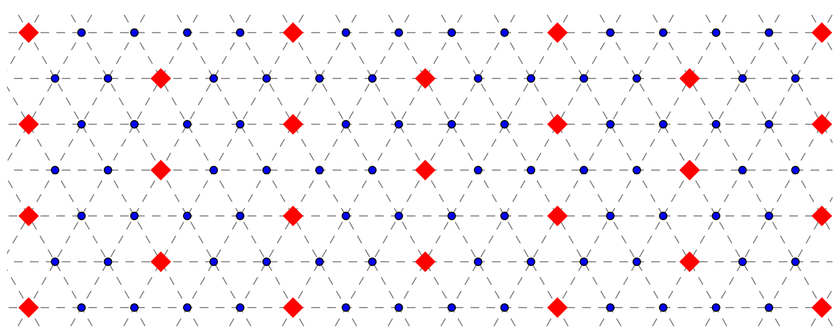


Figure 4.11

Proof. Suppose that there is no blue ℓ_5. If $\mathfrak{L}$ does not contain a red point, then any ℓ_5 is blue, therefore $\mathfrak{L}$ contains a red point A. By Lemma 4.2.2, one of the points of $\mathfrak{L}$ at distance $\sqrt{3}$ to A is red (otherwise the three such points form a blue triangle with side length 3 and red centre A). Denote this point by B (Figure 4.12).

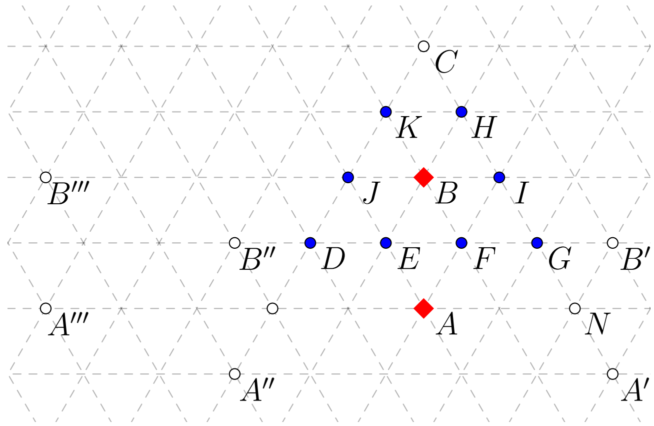


Figure 4.12

Since $\mathfrak{L}$ does not contain a red $\mathfrak{T}_3$, the points D and G are blue. Points E, F, I, H, K, J are blue, since they are distance 1 apart from B. Then the point B' is red (otherwise blue ℓ_5 $DEFGB'$ is formed). Point N is 1 apart from B', hence blue. Then C and A' are red (otherwise a blue ℓ_5 is formed).

By repeating the same argument for points B and C, B and A (instead of A and B), and so on, it can be shown that any node of $\mathfrak{L}$ on the line AB is red. Similarly, since A' and B' are both red, any node of $\mathfrak{L}$ on the line $A'B'$ is red. By the same argument, A'', B'' and any node on the line containing them is red; A''', B''' and any node on the line containing them is red, and so on. By colouring all points distance 1 apart form red points blue, the colouring in Figure 4.11 is obtained. □

Proof of Theorem 4.2.1. Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Suppose that there are no five blue points that form an ℓ_5. Then there is a red point A. Consider two points B and

C, both distance 5 apart from A, such that $|BC| = 1$. At least one of the points B and C (say, B) is blue. Consider the unit triangular lattice $\mathfrak{L}$ that contains A and B. By Lemma 4.2.6 and Lemma 4.2.7, $\mathfrak{L}$ is coloured either as in Figure 4.9 or as in Figure 4.11. But neither one of the colourings contains two points of different colour distance 5 apart, which gives a contradiction. Therefore, there exist five blue points that form an ℓ_5. □

4.3 Proof of $\mathbb{E}^3 \to (\ell_2, \ell_6)$

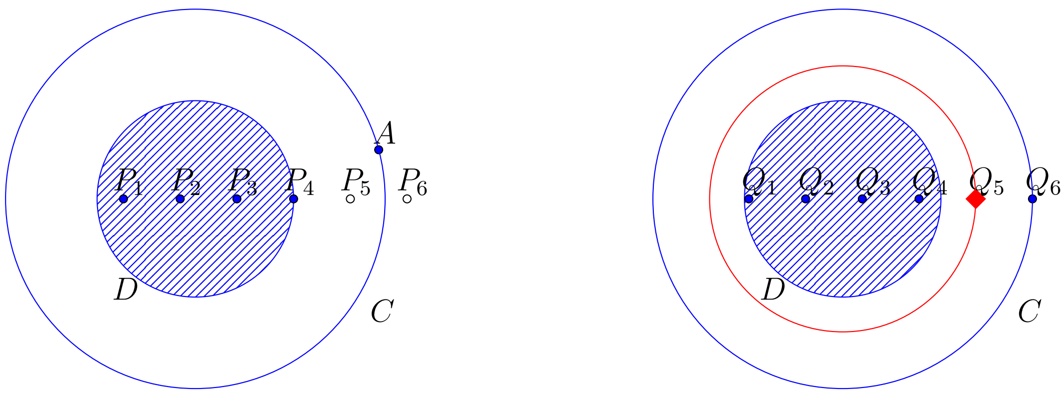
This section contains the proof of the following theorem:

Theorem 4.3.1 (Arman–Tsaturian, 2018 [8]). *Let the Euclidean space $\mathbb{E}^3$ be coloured in red and blue so that there are no two red points distance 1 apart. Then there exist six blue points that form an ℓ_6.*

The following four lemmas are needed.

Lemma 4.3.2 (Arman–Tsaturian, 2018 [8]). *Let $\mathbb{E}^3$ be coloured in red and blue so that there are no two red points distance 1 apart. If there are no six blue points forming an ℓ_6, then there is no disk with radius $\sqrt{3}$, such that all of its points (including interior) are blue.*

Proof. Suppose that there exists a blue disk D with center O and radius $\sqrt{3}$. Consider a rectangular coordinate system on the plane containing D, centered at O (then $D = \{x^2 + y^2 \leq 3, x, y \in \mathbb{R}\}$). Let P_4 be any point on the boundary of D (for simplicity, let the coordinates of P_4 be $(\sqrt{3}, 0)$. Then points P_4, $P_3(\sqrt{3}-1, 0)$, $P_2(\sqrt{3}-2, 0)$, $P_1(\sqrt{3}-3, 0)$ belong D, and therefore are blue (see Figure 4.13a). Consider points $P_5(\sqrt{3}+1, 0)$, $P_6(\sqrt{3}+2, 0)$, and a point A (say, $(\sqrt{3}+\frac{3}{2}, \frac{\sqrt{3}}{2})$) at distance 1 to both P_5 and P_6. If A is red, then both P_5 and P_6 are blue and $P_1, P_2, P_3, P_4, P_5, P_6$ form a blue ℓ_6. Therefore A is blue. When the point P_4 is rotated around the center, the point A (when rotated) spans a blue circle C with radius $\sqrt{(\sqrt{3}+\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{6+3\sqrt{3}}$.



(a) (b)

Figure 4.13

Consider any point Q_6 on C (for simplicity, let the coordinates of Q_6 be $(\sqrt{6+3\sqrt{3}}, 0)$; see Figure 4.13b). Since $-\sqrt{3} < \sqrt{6+3\sqrt{3}} - 5$ and $\sqrt{6+3\sqrt{3}} - 2 < \sqrt{3}$, points $Q_4(\sqrt{6+3\sqrt{3}}-2, 0)$, $Q_3(\sqrt{6+3\sqrt{3}}-3, 0)$, $Q_2(\sqrt{6+3\sqrt{3}}-4, 0)$, $Q_1(\sqrt{6+3\sqrt{3}}-5, 0)$ are all inside D, and therefore are blue. Then the point $Q_5(\sqrt{6+3\sqrt{3}}-1, 0)$ is red (otherwise $Q_1Q_2Q_3Q_4Q_5Q_6$ is a blue ℓ_6). If the point Q_6

is chosen arbitrarily on C, Q_5 spans a red circle with radius $\sqrt{6+3\sqrt{3}}-1>1$, that contains two red points distance 1 apart, a contradiction. $\square$

Lemma 4.3.3 (Arman–Tsaturian, 2018 [8]). *Let $\mathbb{E}^3$ be coloured in red and blue so that there are no two red points distance 1 apart. If there are no six blue points forming an ℓ_6, then there are no two red points distance 2 apart.*

Proof. Assume that points A and B are red and $|AB|=2$. Choose a rectangular coordinate system centered at A so that B has coordinates $(2,0,0)$. Then the circles $\{(-\frac{1}{2},y,z) : y^2+z^2=\frac{3}{4}\}$, $\{(\frac{1}{2},y,z) : y^2+z^2=\frac{3}{4}\}$, $\{(\frac{3}{2},y,z) : y^2+z^2=\frac{3}{4}\}$ and $\{(\frac{5}{2},y,z) : y^2+z^2=\frac{3}{4}\}$ are blue. Consider any line ℓ parallel to AB that intersects all the blue circles (for simplicity, let $\ell=\{(x,-\frac{\sqrt{3}}{2},0), x\in\mathbb{R}\}$), and let P_1,P_2,P_3,P_4 be the points of intersection (see Figure 4.14, in this case $P_1(-\frac{1}{2},-\frac{\sqrt{3}}{2},0)$, $P_2(\frac{1}{2},-\frac{\sqrt{3}}{2},0)$, $P_3(\frac{3}{2},-\frac{\sqrt{3}}{2},0)$, $P_4(\frac{5}{2},-\frac{\sqrt{3}}{2},0)$).

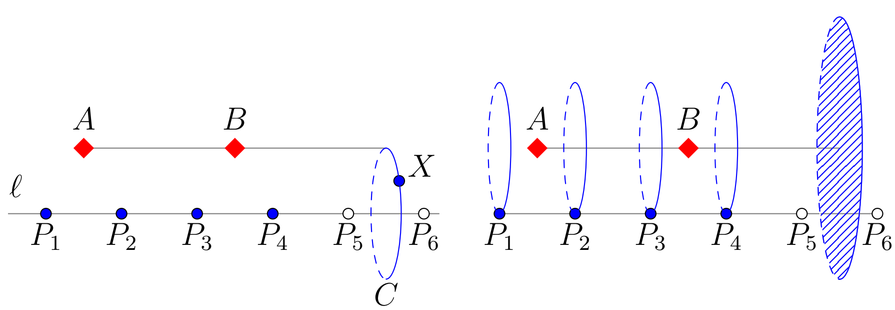


Figure 4.14

Let X be any point on the circle C with radius $\frac{\sqrt{3}}{2}$ centered at the point on ℓ

with x-coordinate 4 (in Figure 4.14, the center has coordinates $(4, -\frac{\sqrt{3}}{2}, 0)$). If X is red, then the points P_5 and P_6 on ℓ with x-coordinates $\frac{7}{2}$ and $\frac{9}{2}$ (on the Figure 4.14, $P_5(\frac{7}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_6(\frac{9}{2}, -\frac{\sqrt{3}}{2}, 0)$) are both at distance 1 to X. Then P_5 and P_6 are blue, and $P_1P_2P_3P_4P_5P_6$ is a blue ℓ_6. Therefore, X is blue, hence the circle C is blue. When ℓ is rotated around AB, the circle C spans a blue disk with radius $\sqrt{3}$, which contradicts the statement of Lemma 4.3.2. □

Lemma 4.3.4 (Arman–Tsaturian, 2018 [8])**.** *Let $\mathbb{E}^3$ be coloured in red and blue so that there are no two red points distance* 1 *apart. If there are no six blue points forming an ℓ_6, then there are no two red points distance* 4 *apart.*

Proof. The proof is similar to that of Lemma 4.3.3. Assume that points A and B are red and $|AB| = 4$. Choose a rectangular coordinate system centered at A so that B has coordinates $(4, 0, 0)$. Then the circles $\{(-\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, $\{(\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, $\{(\frac{7}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ and $\{(\frac{9}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ are blue. Consider any line ℓ parallel to AB that intersects all the blue circles (for simplicity, let $\ell = \{(x, -\frac{\sqrt{3}}{2}, 0) : x \in \mathbb{R}\}$), and let P_1, P_2, P_5, P_6 be the points of intersection (see Figure 4.15, in this case $P_1(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_2(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_5(\frac{7}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_6(\frac{9}{2}, -\frac{\sqrt{3}}{2}, 0)$). Let X be any point on the circle C with radius $\frac{\sqrt{3}}{2}$ centered at the point on ℓ with x-coordinate 2 (in Figure 4.15, the center has coordinates $(2, -\frac{\sqrt{3}}{2}, 0)$). If X is red, then the points P_3 and P_4 on ℓ with x-coordinates $\frac{3}{2}$ and $\frac{5}{2}$ (in Figure 4.15 $P_3(\frac{3}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_4(\frac{5}{2}, -\frac{\sqrt{3}}{2}, 0)$) are both at distance 1 to X. Then P_3 and P_4 are blue, and $P_1P_2P_3P_4P_5P_6$ is a blue ℓ_6. Therefore, X is blue, hence the circle C is blue.

When ℓ is rotated around AB, the circle C spans a blue disk with radius $\sqrt{3}$, which contradicts the statement of Lemma 4.3.2. □

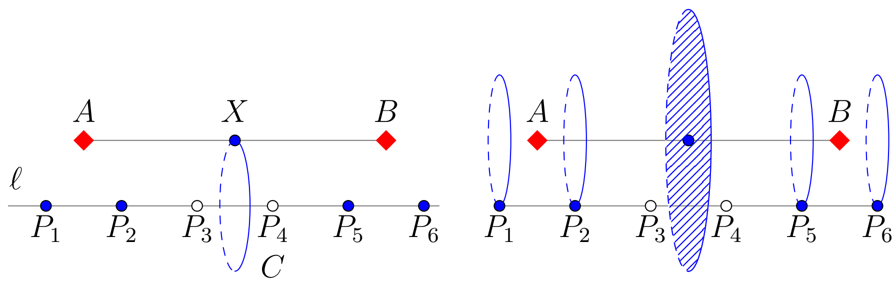


Figure 4.15

Lemma 4.3.5 (Arman–Tsaturian, 2018 [8])**.** *Let $\mathbb{E}^3$ be coloured in red and blue so that there are no two red points distance* 1 *apart. If there are no six blue points forming an ℓ_6, then there are no two red points distance* 3 *apart.*

Proof. Assume that points A and B are red and $|AB| = 3$. Choose a rectangular coordinate system so that A has coordinates $(1, 0, 0)$ and B has coordinates $(4, 0, 0)$. Then, by Lemma 4.3.3, any point at distance 2 to A is blue; in particular, the circles $\{(0, y, z) : y^2 + z^2 = 3\}$ and $\{(2, y, z) : y^2 + z^2 = 3\}$ are blue. By the same argument, circles $\{(3, y, z) : y^2 + z^2 = 3\}$ and $\{(5, y, z) : y^2 + z^2 = 3\}$ are blue, since their points are at distance 2 from B. Consider any line ℓ parallel to AB that intersects all the blue circles (for simplicity, let $\ell = \{(x, -\sqrt{3}, 0) : x \in \mathbb{R}\}$), and let P_1, P_3, P_4, P_6 be the points of intersection (see Figure 4.16, in this case $P_1(0, -\sqrt{3}, 0)$, $P_3(2, -\sqrt{3}, 0)$,

$P_4(3,-\sqrt{3},0)$, $P_6(5,-\sqrt{3},0)$).

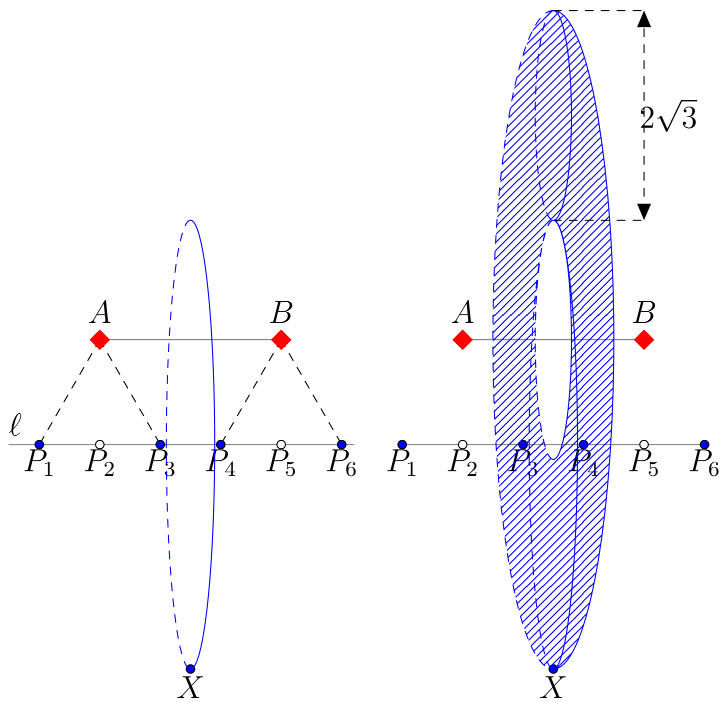


Figure 4.16

Let X be any point on the circle C with radius $\frac{\sqrt{55}}{2}$ centered at the point on ℓ with x-coordinate $\frac{7}{2}$ (on the Figure 4.16, the center has coordinates $(\frac{7}{2},-\sqrt{3},0)$). If X is red, then the points P_2 and P_5 on ℓ with x-coordinates 1 and 4 (on the Figure 4.16, $P_2(1,-\sqrt{3},0)$, $P_5(4,-\sqrt{3},0)$) are both at distance 4 to X. Then, by Lemma 4.3.4, P_2 and P_5 are blue, and $P_1P_2P_3P_4P_5P_6$ is a blue ℓ_6. Therefore, X is blue, hence the circle C is blue. When ℓ is rotated around AB, the circle C spans the blue annulus bounded by circles of radii $\frac{\sqrt{55}}{2}+\sqrt{3}$ and $\frac{\sqrt{55}}{2}-\sqrt{3}$, which contains a blue disk of radius $\sqrt{3}$. By Lemma 4.3.2, this contradicts the fact that there are no two red points distance 1 apart. □

Lemma 4.3.6 (Arman–Tsaturian, 2018 [8]). *Let $\mathcal{L}$ be a unit triangular lattice on a plane. Let points of $\mathcal{L}$ be coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_6. If $\mathcal{L}$ contains two red points distance $\sqrt{3}$ apart, then $\mathcal{L}$ does not contain a blue ℓ_5.*

Proof. Suppose that A_1 and A_2 are two red nodes of $\mathcal{L}$ distance $\sqrt{3}$ apart. First, it is proved that the node A_3 symmetric to A_1 in A_2 is also red. Consider the part of $\mathcal{L}$ depicted in Figure 4.17a. Points Q_3 and Q_6 are at distance 3 from A_1, and therefore are blue by Lemma 4.3.5. Points Q_4 and Q_5 are at distance 1 from A_2, therefore blue. Then point P_1 is blue; otherwise points Q_1 and Q_2 are both blue and form a blue ℓ_6 with points Q_3, Q_4, Q_5, Q_6. Points P_2 and P_6 are at distance 4 from A_1, therefore are blue by Lemma 4.3.4. Points P_3 and P_5 are at distance 2 from A_2, therefore are blue by Lemma 4.3.3. Then the point A_3 has to be red in order to prevent the blue $P_1P_2P_3A_3P_5P_6$.

Using the same argument, it can be proved that the node A_4 symmetric to A_2 in A_3 is red, and similarly for any $k \in \mathbb{Z}$ a point on the line A_1A_2 at distance $k\sqrt{3}$ from A_1 is red.

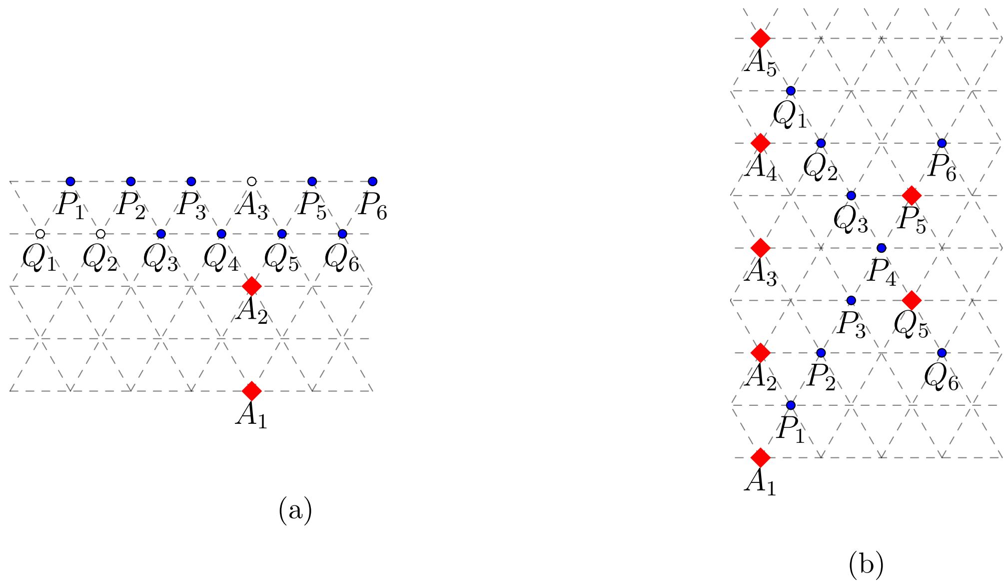


Figure 4.17

Consider five consecutive red nodes A_1, A_2, A_3, A_4, A_5 on the line A_1A_2 and the part of the lattice depicted in Figure 4.17b. By Lemmas 4.3.3, 4.3.5, 4.3.4, the points P_1, P_2, P_3, P_4 are blue, since they are at distance 1, 2, 3 and 4 from A_1, respectively. By Lemma 4.3.5, the point P_6 is blue (since it is 3 apart from A_4). Then point P_5 is red (otherwise P_1, P_2, P_3, P_4, P_6 form a blue ℓ_6). Similarly, the points Q_1, Q_2, Q_3, Q_6 are blue; hence Q_5 is red. Then P_5 and Q_5 are two red nodes distance $\sqrt{3}$ apart, which forces all the nodes of $\mathcal{L}$ on the line P_5Q_5 to be red.

If the colouring is expanded further in a similar way, the plane is then coloured in the unique way, shown in Figure 4.18. Every ℓ_5 that belongs to $\mathcal{L}$ contains a red point, and therefore the colouring does not contain a blue ℓ_5. $\square$

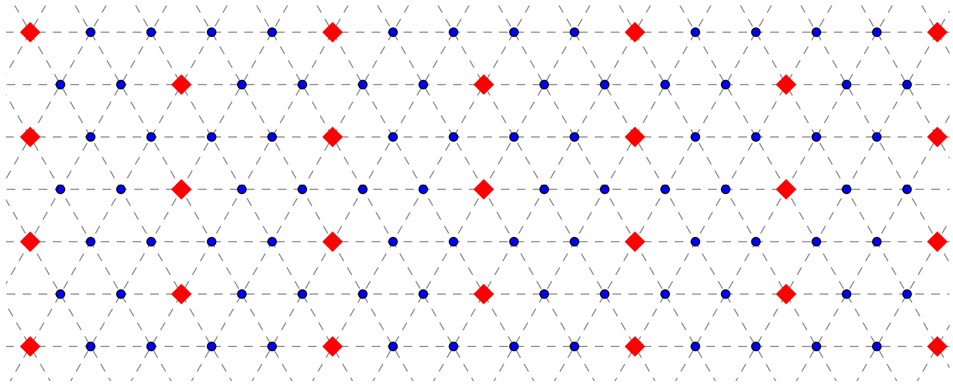


Figure 4.18

Proof of the Theorem 4.3.1. The proof is by contradiction. Suppose that $\mathbb{E}^3$ is coloured in two colours so that there is no red ℓ_2 and no blue ℓ_6. By Theorem 4.2.1, there is a blue ℓ_5, say, $X_1X_2X_3X_4X_5$. Consider any unit triangular lattice $\mathcal{L}$ such that X_1, X_2, X_3, X_4, X_5 are nodes of $\mathcal{L}$. Since $\mathcal{L}$ does not contain a blue ℓ_6, there is a red node A in $\mathcal{L}$. Consider the part of $\mathcal{L}$ depicted in Figure 4.19.

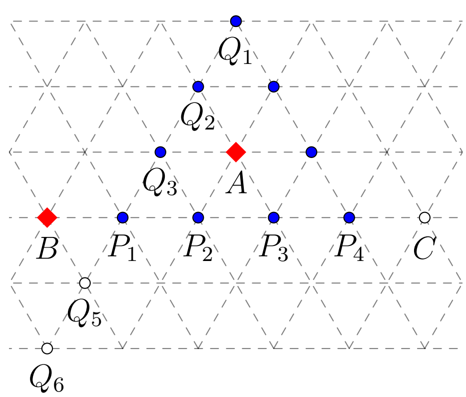


Figure 4.19

The points P_2 and P_3 are blue, since they are distance 1 apart from A. Since $\mathcal{L}$

contains a blue ℓ_5, by Lemma 4.3.6, there are no two red nodes of $\mathcal{L}$ distance $\sqrt{3}$ apart, therefore points P_1 and P_4 are blue. Then points B and C can not be both blue (otherwise a blue ℓ_6 is formed), therefore one of them, say, B, is red. Then points Q_5 and Q_6 are at distance 1 and $\sqrt{3}$ from B, hence blue. The points Q_3, Q_2, Q_1 are distance 1, 1, $\sqrt{3}$ apart from A, respectively, therefore blue. Hence, the points Q_1, Q_2, Q_3, P_1, Q_5, Q_6 form a blue ℓ_6, which contradicts the initial assumption. $\square$

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