NORMED SPACES OF CONTINUOUS AND HOLOMORPHIC FUNCTIONS AND WEIGHTED COMPOSITION OPERATORS

by

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Abstract

In the present work we develop a unified way of looking at Normed Spaces of Continuous and Holomorphic Functions (NSCF's and NSHF's) and of Weighted Composition Operators (WCO's) between these spaces.

If $\mathbf{F}$ is a normed space of continuous functions over a topological space X, we relate different properties of $\mathbf{F}$. We show that compactness of the inclusion map $J_{\mathbf{F}}$ from $\mathbf{F}$ into $\mathcal{C}(X)$ is equivalent to continuity of the operation of evaluation on $\overline{B}_{\mathbf{F}} \times X$, where $\overline{B}_{\mathbf{F}}$ is the closed unit ball of $\mathbf{F}$ endowed with the weak topology. Moreover, we show that these conditions follow from mere continuity of the norm of the point evaluations, under the additional assumption that $\mathbf{F}$ is a uniformly smooth Banach space (this was previously known for Hilbert spaces). Also, we show that $\mathbf{F}$ is reflexive if and only if $J_{\mathbf{F}}$ is weakly compact and $\overline{B}_{\mathbf{F}}$ is closed in $\mathcal{C}(X)$ in any equivalent norm on $\mathbf{F}$. On the other hand, we show that some mild conditions on $\mathbf{F}$ imply that X is metrizable.

We also give some sufficient conditions that force WCO's between NSCF's to have continuous symbols (this problem was previously considered only for a specific choice of NSCF's). We provide counterexamples that show that the problem is not trivial. An analogous problem is considered in the holomorphic setting. We also show that on a wide class of NSCF's the only unitary multiplication operators are the multiples of identity.

For a Reproducing Kernel Hilbert Space (RKHS) of holomorphic functions on a complex domain we give a formula that describes the Hermitean metrics on the domain which are pull-backs of some metric on the (dual of) the RKHS via the evaluation map. Then we consider the question when such metrics are invariant with respect to the group of automorphisms of the domain, and obtain some partial results in that direction.

Contents

0.1 Introduction

Normed spaces of functions are ubiquitous in mathematics, especially in various branches of analysis. These spaces can be of a various nature and exhibit different types of behavior, and in this work we discuss some questions related to these spaces from a general, axiomatic viewpoint.

There is a huge body of literature dedicated to certain specific function spaces or families of spaces, such as for example Bergman spaces, Sobolev spaces etc. However, we are more interested in studying such spaces regardless of the specifics of the way they are defined. In fact, there are several separate theories of this general kind, such as the theory of *Function Algebras*, done by using Banach algebra techniques (see e.g. [Dal00]), and the theory of *Spaces of Bounded Continuous and Holomorphic Functions*, done by using so-called strict topologies (see e.g. [Sha71]), or more abstractly, Saks spaces (see [Coo87]). However, our main sources of inspiration are the following two theories.

Reproducing Kernel Hilbert Spaces (RKHS's) theory is a formalism for studying Hilbert spaces whose elements are functions. The roots of the theory go back to the theory of Differential and Integral Equations (Zaremba, Mercer; for a historical overview we refer to the celebrated paper of Aronszajn ([Aro50]), which is also the first systematic exposition of the subject). After these initial studies, RKHS's occupied an important place in Complex Analysis (Bergman, see [Ber70]), Probability (see e.g. [BTA04]), Harmonic Analysis (see e.g. [Sai97]), Machine Learning (see e.g. [ZXZ09]) etc. An important theorem due to Moore and Aronszajn states that there is a one-to-one correspondence between the RKHS's and *positive semi-definite kernels*. One of the common motives in the literature is to establish relations between the properties of RKHS's over topological spaces with the mode of continuity of their kernels (see e.g. [Sch64], [Sza04], [FM13], [CDVT06], the latter of which deals with vector-valued RKHS's).

The second example is the theory of *Banach spaces of holomorphic functions*. This theory was launched as a tool for the Theory of Functions of a Complex Variable to involve methods of Functional Analysis and vice versa (for a good demonstration of this interplay see [Dou98]). Since then a plethora of different spaces were introduced and extensively studied (see e.g. [Hof88], [CM95]). Typically, research in this area is done for some specific pre-chosen space or parametric family of spaces, but there are several instances where the authors consider all possible Banach spaces of holomorphic functions that satisfy certain general axioms (e.g. [Bou14], [MR15], [Zor17a]). On the other hand, there is a significant overlap between the theories of Banach spaces of holomorphic functions and of RKHS, the latter of which is axiomatic by its nature. Namely, one often considers an abstract RKHS of holomorphic functions (see e.g. [BB84], [ARSW11], [GL04], [Zor17b]), or an abstract RKHS of holomorphic functions that satisfies few additional axioms (most notably Nevanlinna-Pick completeness, see [AM02]).

We will use the aforementioned theories as a guide, but mostly leave aside the Hilbert space aspect of the RKHS theory, and the complex analytic aspect of the theory of Banach spaces of holomorphic functions. Namely, we base the present research on the following definition.

A *Normed Space of Continuous Functions* (NSCF) over a Hausdorff topological space X is a linear subspace $\mathbf{F} \subset \mathcal{C}(X)$ equipped with a norm such that the inclusion operator $J_{\mathbf{F}} : \mathbf{F} \to \mathcal{C}(X)$ is continuous. We will commonly refer to X as a *phase space* of $\mathbf{F}$. This definition agrees with existing definitions throughout literature and is suited for immediate applications of the machinery of Functional Analysis. If $\mathbf{F}$ is a NSCF, define the *evaluation map* $\kappa_{\mathbf{F}}$ from X into the dual $\mathbf{F}^*$ of $\mathbf{F}$ by $\langle f, \kappa_{\mathbf{F}}(x)\rangle = f(x)$. Then $\kappa_{\mathbf{F}}(x)$ is the *point evaluation* at x on $\mathbf{F}$. Define also $|\kappa|_{\mathbf{F}}(x) = \|\kappa_{\mathbf{F}}(x)\|$. If X is a complex manifold and $\mathbf{F} \subset \mathcal{H}(X)$, we will say that $\mathbf{F}$ is a *Normed Space of Holomorphic Functions* (NSHF).

In Chapter 1 we find the relations between the following characteristics of NSCF's:

- Topological and geometrical properties of $\mathbf{F}$ as a normed space (e.g. completeness, reflexivity, strict convexity, being a Hilbert space, etc);
- Properties of $\mathbf{F}$ as a collection of functions, in particular, common properties of elements of $\mathbf{F}$ (e.g. boundedness, holomorphicity, Lipschitzness, etc);
- The way $\mathbf{F}$ "sits" in $\mathcal{C}(X)$ (e.g. if the closed unit ball $\overline{B}_{\mathbf{F}}$ of $\mathbf{F}$ is closed in $\mathcal{C}(X)$);
- Properties of the inclusion operator $J_{\mathbf{F}}$ (compactness, weak compactness), and also the operation of evaluation $(f, x) \to f(x)$ of $\mathbf{F} \times X$;
- Properties of the evaluation map $\kappa_{\mathbf{F}}$ and of the function $|\kappa|_{\mathbf{F}}$.

For example, we show (see Theorem 1.1.10) that continuity of the operation of evaluation on $\left(\overline{B}_{\mathbf{F}}, \textit{weak topology}\right) \times X$ is equivalent to compactness of $J_{\mathbf{F}}$, and these conditions follow from mere continuity of $|\kappa|_{\mathbf{F}}$ under the additional assumption that $\mathbf{F}$ is a uniformly smooth Banach space (previously known for Hilbert spaces). Also, we show (see Corollary 1.2.11) that $\mathbf{F}$ is reflexive if and only if $J_{\mathbf{F}}$ is weakly compact and $\overline{B}_{\mathbf{F}}$ is closed in $\mathcal{C}(X)$ in any equivalent norm on $\mathbf{F}$. On the other hand, we show (see e.g. Proposition 1.2.28) that some mild conditions on $\mathbf{F}$ force X to be metrizable and locally compact.

The class of linear operators that capture the very nature of spaces of functions is the class of *weighted composition operators* (WCO). Indeed, these operators are defined (see below) through the operations of multiplication and composition which can be performed on any collection of functions. Furthermore, there is a lot of Banach-Stone-type theorems which show that the WCO's are the only operators that preserve various kinds of structure (see e.g. [FJ03] and [GJ02]). We will informally view these operators as morphisms in the category of NSCF's (NSHF's).

Let $\mathbf{F}$ and $\mathbf{E}$ be NSCF's over locally compact spaces X and Y respectively. A WCO $W_{\varphi,u}$ from $\mathbf{F}$ into $\mathbf{E}$ with multiplicative symbol $u : Y \to \mathbb{C}$ and composition symbol $\varphi : Y \to X$ is a linear transformation defined by $\left[W_{\varphi,u} f\right](y) = u(y) f(\varphi(y))$, $f \in \mathbf{F}$. Note that it can be difficult to determine if $W_{\varphi,u}$ is *well-defined* (i.e. if $W_{\varphi,u}$ indeed maps $\mathbf{F}$ into $\mathbf{E}$) depending on the choice of $\mathbf{F}$, $\mathbf{E}$, u and φ. In the case when $u \equiv 1$ we will call $W_{\varphi,u}$ a *composition* operator, and if $X = Y$ and φ is the identity map, we will call $W_{\varphi,u}$ a *multiplication* operator.

Analogously to studying the very basic properties of NSCF's, in Chapter 2 we consider WCO's between them. We focus on the most fundamental properties of these operators that do not depend on the particular description of the space of functions, or the particular data of the operator, but rather on more general notions such as injectivity, surjectivity, continuity, boundedness, etc of both the operator and the inducing maps. In particular, we ascertain when the "category of the data of the operator" matches the "category of the function space". The latter means that we give some sufficient conditions (Corollary 2.2.4, Theorem 2.2.10 and Theorem 2.2.12 / Proposition 2.3.2, Proposition 2.3.11, Theorem 2.3.3 and Theorem 2.3.9) that force WCO's between NSCF's/NSHF's to have continuous/holomorphic symbols. This problem was previously considered in [SS88] and subsequently in [SM93] for a specific choice of NSCF's.

As for the more quantitative properties of WCO's, we show (Theorem 2.1.30 and Corollary 2.1.31) that on a wide class of NSCF's the only unitary multiplication operators are the multiples of identity. Also we give sufficient conditions for a WCO which is a linear homeomorphism to have a surjective composition symbol (Proposition 2.1.24, which is a generalization of a result from [Zor17b]).

The last topic of this work is the geometric structure which arises from RKHS's of holomorphic functions (i.e. Hilbert NSHF's). Namely, we study the invariance of Hermitean metrics generated by such RKHS's. This subject is somewhat peripheral to the rest of the topics, but it motivated most of the material in Chapter 2. Previously in [Bil17] we obtained an analytic description of the unitary-invariant Riemannian or Hermitean metrics on a Hilbert space, and showed that such metrics have a certain rigidity. In Section 2.4 we study the pull-backs of such metrics from the dual of a RKHS to its phase space via the evaluation map of the RKHS. Given a metric on (the dual of) a RKHS, we express its pull-back in terms of the reproducing kernel (Theorem 2.4.6), and then show that the automorphism group cannot act isometrically via (adjoints) of composition operators on the RKHS (Proposition 2.4.11). On the other hand, we show that the pull-back of the Fubini-Study metric from (the dual of) a RKHS is automorphism-invariant if and only if the RKHS is projectively invariant (Corollary 2.4.15).

Let us now list the main contributions of the thesis including the results already mentioned above.

Section 1.1: Theorem 1.1.10 (part (iv) is a generalization from Hilbert spaces to Uniformly smooth Banach spaces), Proposition 1.1.14.

Section 1.2: Theorem 1.2.10, Corollary 1.2.11, Proposition 1.2.20, Corollary 1.2.22, Proposition 1.2.25 (part (i) is a generalization from Hilbert spaces to reflexive spaces), Proposition 1.2.26, Proposition 1.2.27, Proposition 1.2.28.

Section 1.3: Proposition 1.3.8 (generalization from a Holomorphic Weighted Uniform space to a general Banach space), Theorem 1.3.22 (generalization of several classical results).

Section 2.1: Corollary 2.1.15, Proposition 2.1.24 (generalization from Hilbert NSHF's to general NSCF's), Theorem 2.1.30, Corollary 2.1.31.

Section 2.2: Corollary 2.2.4, Theorem 2.2.10, Theorem 2.2.12, Example 2.2.7.

Section 2.3: Proposition 2.3.2, Proposition 2.3.11, Theorem 2.3.3, Theorem 2.3.9.

Section 2.4: Theorem 2.4.1 (published in [Bil17]), Theorem 2.4.6, Proposition 2.4.11, Corollary 2.4.15 (implicitly stated in [ARSW11] without a proof).

Auxiliary results that also appear to be original: Theorem 2.1.29, Theorem 2.2.8 and Theorem 2.4.14.

The bulk of the results from sections 2.1-2.3 are presented in the article [Bil18b], which was accepted for publication in Complex Analysis and Operator Theory.

0.2 Preliminaries

We begin by introducing some notation. Let $\mathbb{F}$ be the field of scalars, i.e. either $\mathbb{C}$ or $\mathbb{R}$, and $\mathbb{F}^{\times} = \mathbb{F}\setminus\{0\}$. Also, denote $\mathbb{R}^{+} = (0, +\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$. We will also denote the imaginary unit number by $\mathfrak{i}$ in order to reserve the letter i for indexation (typically by integers).

Let X be a set. By $\mathcal{P}(X)$ we denote the collection of all subsets of X, and Id_X is the identity map on X. For a function $f : X \to \mathbb{F}$ and $A \subset X$ we write $\|f\|_\infty^A = \sup_{x \in A} |f(x)|$. If $\alpha \in \mathbb{F}$, we will also denote by α the constant functions with the corresponding value, when its domain is clear from the context.

All topological vector spaces are assumed to be Hausdorff (unless stated otherwise), and manifolds are assumed to be connected and without boundary.

If (X, d) is a pseudo-metric space, $x \in X$ and $r > 0$, then $B_X(x, r)$ $(\overline{B}_X(x, r))$ is the open (closed) ball centered at x with radius r. In the case when X is a linear space and x is the neutral element of X, we use notations $B_X(r)$ and $\overline{B}_X(r)$, or rB_X and $r\overline{B}_X$, where $B_X = B_X(1)$ and $\overline{B}_X = \overline{B}_X(1)$. We also use a special notation $\mathbb{D} = B_{\mathbb{C}}$, i.e. the open unit disk in the complex plane.

Let $\mathcal{F}(X, Y)$ be the set of all maps from a set X into a set Y. If X and Y are topological spaces, then $\mathcal{C}(X, Y)$ is the set of all continuous maps from X into Y. In the case $Y = \mathbb{F}$ we will use the shortened notations $\mathcal{F}(X)$ and $\mathcal{C}(X)$.

0.2.a Some General Topology

We present a short review of the basic topological notions that we will need later on. Our main references are [Eng89], [Bou66] and [Bou98]. Let us start with the following convention.

By saying that a map $\varphi : X \to Y$ is $\tau \to \pi$ continuous we will mean that φ is a continuous map between the topological spaces (X, τ) and (Y, π). In case when the choice of the topology on X (or Y, or both) is clear from the context, we will write that φ is $X \to \pi$ (or $\tau \to Y$, or $X \to Y$) continuous. If the choice of X (or Y) is also clear, we will write that

φ is π (or τ) continuous. If τ is a topology on X and $Z \subset X$, then (Z, τ) is the topological space Z with the induced subspace topology. A map from a topological space Y into X is $Y \to Z$ continuous, if it is $Y \to X$ continuous and its range is contained in Z.

Let X and Y be sets. We say that a collection $\Phi \subset \mathcal{F}(X, Y)$ *separates the points* of X if for every distinct $x, y \in X$ there is $\varphi \in \Phi$ such that $\varphi(x) \neq \varphi(y)$. If Y is a topological space, we say that $\Phi \subset \mathcal{F}(X, Y)$ *generates* a topology τ on X if this topology is the weakest in which all elements of Φ are continuous, i.e. the family

$$\left\{\varphi^{-1}(V) \,|V \text{ is open in } Y,\ \varphi \in \Phi\right\}$$

forms a subbase of τ. If in this case Y is Hausdorff, then τ is Hausdorff if and only if Φ separates the points of X. If X is also a topological space, we say that $\Phi \subset \mathcal{C}(X, Y)$ generates the topology of X if the topology generated by Φ is the original topology of X. Note that if Y is Hausdorff and $\mathcal{C}(X, Y)$ separates the points of X, then X is Hausdorff. One can show that any topology generated on X by $\Phi \subset \mathcal{F}(X, \mathbb{F})$ is completely regular, and so it is Tychonoff, once Φ separates the points of X. In fact, a Hausdorff topological space X is Tychonoff if and only if $\mathcal{C}(X)$ generates its topology.

By definition, the coordinate functions generate the topology of any subset of $\mathbb{F}^d$. Note that not every family of functions that separates the points of X generates its topology. For example, the single function $t \to e^{it}$ does not generate the topology of $[0, 2\pi)$, despite being an injection.

For $A \subset X$ and $\Phi \subset \mathcal{F}(X, Y)$ denote $\Phi|_A$ to be the collection of restrictions of elements of Φ. We will say that Φ generates the topology (separates the points) of A if $\Phi|_A$ generates the topology (separates the points) of A.

We will use the following notation for the limit: if X and Y are topological spaces and $\varphi : X \to Y$, then $\lim\limits_{x} \varphi$ denotes $\lim\limits_{z\to x} \varphi(z)$. Let $\widehat{X}$ denote the one point compactification of X with the ideal element ∞_X. Note that this object makes sense for any non-compact Hausdorff topological space, but is Hausdorff if and only if X is locally compact (see [Eng89, Theorem 3.5.8]). We say that $\lim\limits_{\infty} \varphi = y$ for $y \in Y$ if either X is compact, or $y = \lim\limits_{\infty_X} \widehat{\varphi}$, where $\widehat{\varphi}$ is an extension of φ on $\widehat{X}$. This is equivalent to the fact that for any open neighborhood V of y there is a compact set K in X such that $\varphi(X\backslash K) \subset V$. Additionally, in the case when $Y = \mathbb{R}$, one can define $\lim\limits_{\infty} \varphi = \pm\infty$ in a similar way.

Let X be a topological space embedded into a topological space Y, i.e. $X \subset Y$ and the topology of X is stronger than the topology of Y on X. For $A \subset X \subset Y$ denote the closure of A in X and Y by $\overline{A}^X$ and $\overline{A}^Y$ respectively. The following technical lemma (which is a variation of [Flo80, 1.1. (8)]) is useful when studying sets endowed with multiple topologies.

Lemma 0.2.1.

(i) Let X be a topological space, let Y be a Hausdorff space and let $\varphi \in \mathcal{C}(X, Y)$. If $A \subset X$ is relatively compact, then $\overline{\varphi(A)} = \varphi\left(\overline{A}\right)$ is compact. Moreover, if φ is injective on $\overline{A}$, then $\varphi|_{\overline{A}}$ is a topological embedding (i.e. a homeomorphism onto its image).

(ii) Let X be a topological space embedded into a Hausdorff space Y. Then, if $A \subset X$ is relatively compact in X, then $\overline{A}^Y = \overline{A}^X \subset X$ is compact in Y, and the two topologies coincide on this set.

Corollary 0.2.2. *Let X be a compact (not necessarily Hausdorff) space and let Y be a Hausdorff space. If $\Phi \subset \mathcal{C}(X, Y)$ separates the points of X, then X is Hausdorff and Φ generates its topology.*

A Hausdorff topological space X is called *compactly generated*, or *k-space* whenever each set which has closed intersections with all compact subsets of X is closed itself. Note that this condition is equivalent to the fact that for any Hausdorff space Y and any map $\varphi : X \to Y$, such that $\varphi|_K$ is continuous for every compact $K \subset X$, is continuous on X. It is easy to see that all first countable (including metrizable) and all locally compact Hausdorff spaces are compactly generated. Details concerning the mentioned facts and further information about the introduced class of spaces can be found in [Eng89, 3.3].

Let X be a set and let Y be a a topological space. We will impose a topological structure on $\mathcal{F}(X,Y)$, depending on a chosen family $\mathcal{A} \subset \mathcal{P}(X)$.

The $\mathcal{A}$-*open* topology is the topology on $\mathcal{F}(X,Y)$ with a subbase consisting of all sets of type $\{\varphi : X \to Y \,|\varphi(A) \subset V\}$, where V is an open subset of Y, and $A \in \mathcal{A}$.

Let X be a topological space and let $\mathcal{K}_X$ be the family of all compacts in X. Then the (restriction of) $\mathcal{K}_X$-open topology on $\mathcal{C}(X,Y)$ is called *compact-open* topology. This topology has a number of nice properties. Here we will state one of them. The first part of the next proposition follows from the formula (11) in [Eng89, 2.6.10]; for the second part see [Eng89, Theorem 3.4.2].

Proposition 0.2.3. *Let X, Y, Z be topological spaces. The composition operation from $\mathcal{C}(X,Z) \times \mathcal{C}(Z,Y)$ into $\mathcal{C}(X,Y)$ defined by $(\varphi,\psi) \to \psi \circ \varphi$ is separately continuous with respect to the compact-open topologies on all of these three sets. If furthermore Z is locally compact, then this operation is continuous.*

Now let X be a set again and let Y be metrizable with a metric ρ. The uniform space $\mathcal{F}_{\mathcal{A}}(X,Y)$ is the set $\mathcal{F}(X,Y)$ endowed with $\mathcal{A}$-*uniformity* (of the uniform convergence on members of $\mathcal{A}$), which is determined by the family of pseudometrics $\{\rho_A \,| A \in \mathcal{A}\}$, defined by $\rho_A(\varphi, \psi) = \sup_{x \in A} \rho(\varphi(x), \psi(x))$, for $\varphi, \psi \in \mathcal{F}(X, Y)$. Of course, we can also consider the restriction of this uniformity on the subsets of $\mathcal{F}(X, Y)$. In particular, if X is a topological space, then $\mathcal{C}_{\mathcal{A}}(X, Y)$ will stand for the set $\mathcal{C}(X, Y)$ with $\mathcal{F}_{\mathcal{A}}(X, Y)$ uniformity.

Observe that if $\mathcal{A}_1 \subset \mathcal{A}_2$, then the $\mathcal{A}_2$-open topology ($\mathcal{A}_2$-uniformity) is stronger than the $\mathcal{A}_1$-open topology ($\mathcal{A}_1$-uniformity), since we will have more open sets (pseudometrics) in the generating subbase. However, if $\mathcal{A}_2$ consists only of the finite unions (and subsets) of the elements of $\mathcal{A}_1$, the corresponding topologies (uniformities) will be equivalent. If $\mathcal{A}$ is the collection of all finite subsets of X, we obtain the *pointwise* topology (uniformity) $\mathcal{F}_{\{\{x\}|x \in X\}}(X, Y)$, which we will denote by $\mathcal{F}_X(X, Y)$. It is the weakest natural topological structure on $\mathcal{F}(X, Y)$, and is in fact equal to the product topology (uniformity) on Y^X. The strongest one is $\mathcal{F}_{\{X\}}(X, Y)$ – the uniform convergence on X. If $\bigcup \mathcal{A} = X$ ($\overline{\bigcup \mathcal{A}} = X$), then $\mathcal{F}_{\mathcal{A}}(X, Y)$ ($\mathcal{C}_{\mathcal{A}}(X, Y)$) is a separated uniform space.

Certainly, since the metric defines a topology on Y, we can also consider the $\mathcal{A}$-open topology on $\mathcal{F}(X, Y)$. The uniformity $\mathcal{F}_{\mathcal{A}}(X, Y)$ also defines a topology on $\mathcal{F}(X, Y)$. In general these two topologies are different. It is easy to see that they are equivalent for $\mathcal{F}_X(X, Y)$. We will mostly use the uniformity of $\mathcal{F}_{\mathcal{A}}(X, Y)$, but it is a useful fact that if X is a topological space and $\mathcal{A} \subset \mathcal{K}_X$, then the $\mathcal{A}$-uniformity defines the $\mathcal{A}$-open topology on $\mathcal{C}(X, Y)$ (see [Eng89, Theorem 8.2.6]). In particular, the uniform space $\mathcal{C}_{\mathcal{K}}(X, Y)$ of compact convergence bears the compact-open topology. Note that if X has the discrete topology, then $\mathcal{F}_X(X, Y) = \mathcal{C}_{\mathcal{K}}(X, Y)$.

Let X and Y be sets and let Z be a metric space. Every function $\Sigma : X \times Y \to Z$

generates maps $\kappa_\Sigma : X \to \mathcal{F}(Y,Z)$ and $\iota_\Sigma : Y \to \mathcal{F}(X,Z)$, defined by $\kappa_\Sigma(x) = \Sigma(x,\cdot)$, for $x \in X$, and $\iota_\Sigma(y) = \Sigma(\cdot, y)$, for $y \in Y$. Conversely, any map $\kappa : X \to \mathcal{F}(Y,Z)$ generates $\Sigma^\kappa : X \times Y \to Z$, defined by $\Sigma^\kappa(x,y) = [\kappa(x)](y)$, where $x \in X, y \in Y$. These operations are mutually inverse, in the sense that $\kappa_{\Sigma^\kappa} = \kappa$ and $\Sigma^{\kappa_\Sigma} = \Sigma$ (and the same is true for ι).

The relation of continuity of Σ, κ and ι is connected with the discussion about the so called proper and admissible topologies. The following is a classical result in that direction.

Theorem 0.2.4. [Eng89, Theorem 3.4.1, Theorem 3.4.3 and Theorem 3.4.9] *Let X and Y be topological spaces and let X, Σ, κ and ι be as above.*

(i) If Σ is continuous, then κ and ι are continuous into $\mathcal{C}_\mathcal{K}(Y,Z)$ and $\mathcal{C}_\mathcal{K}(X,Z)$, respectively.

(ii) In the case when X is locally compact or $X \times Y$ is compactly generated (e.g. if both X and Y are metrizable), the $\mathcal{C}_\mathcal{K}(X,Z)$ continuity of ι is equivalent to the continuity of Σ.

0.2.b Some Functional Analysis

Another area whose results and methods will be used extensively in this work is Functional Analysis. Our main references are [BN11] and [Bou87], supplemented by [FHH+11] and [Gro73]. We will start with introducing some notation. Let E be a linear space. We will usually denote its neutral element by 0_E, but in the case when E consists of scalars or scalar functions, we will use 0 instead. If $A \subset \mathbb{F}$ and $B \subset E$, then $AB = \{\alpha e \,|\, \alpha \in A, e \in B\}$. Also, the convex envelope of B is denoted by $\mathrm{conv}B$.

By E' we denote the algebraic dual of E, i.e. the linear space of all linear functionals on E. For $e \in E$ and $\nu \in E'$ denote $\langle e, \nu\rangle = \langle \nu, e\rangle = \nu(e)$. If $D \subset E'$ define the *annihilator* $D^\perp = \{e \in E \,|\, \langle e, \nu\rangle = 0,\ \forall \nu \in D\}$, which is a linear subspace of E. If F is a linear subset of E then $D^{\perp}$ *in* F is $D^{\perp} \cap F$. For example, if $B \subset E$, then $\{\nu \in E' \,|\, \langle e, \nu \rangle = 0,\ \forall e \in B\}$ is $B^{\perp}$ in E'. We will call $D \subset E'$ *separating* if $D^{\perp} = \{0_E\}$. Note that D is separating if and only if D as a collection of functions on E separates the points. If E and F are linear spaces and T is a linear map between them, then the *kernel* Ker T of T is $\{e \in E \,|\, Te = 0_F\}$.

In the case when E is endowed with a norm, the latter usually will be clear from the context and we will denote it simply as $\|\cdot\|$; in the case of ambiguity we will use $\|\cdot\|_E$. If E is an inner product space, we will always specify with a subscript inner product in order to keep it distinct from the natural pairing of E with E'. However, we will usually omit the subscript when denoting the norm generated by the inner product, i.e. $\|e\| = \sqrt{\langle e, e\rangle_E}$, for $e \in E$. If E is a Hilbert space, the orthogonal complement to $B \subset E$ will be denoted by $H \ominus \mathrm{span} B$ in order to prevent confusion with the annihilator.

Let E^* be the topological dual of a normed space E, i.e. the linear space of all continuous linear functionals on E. This space admits multiple natural topological structures, including the pointwise (which in this case is called *weak**) and the compact-open topologies. However, by default we will consider E^* endowed with its natural operator norm. It is well-known that E^* is a Banach space, and that E^{**} contains E as a subspace. Moreover, $\overline{B}_{E^{**}}$ is the closure of B_E in the weak* topology of E^{**} (this fact is called the Goldstine theorem, see [BN11, Theorem 8.4.7]). The restriction of the weak* topology of E^{**} on E is called the *weak* topology. The metrizability of the weak and the weak* topologies depends on separability of E^* and E respectively.

Proposition 0.2.5. [FHH$^+$11, Proposition 2.8, Proposition 3.103 and Proposition 3.106] *If E is a normed space, then:*

(i) E is separable if and only if B_E is weak metrizable.*

(ii) E^* is separable if and only if B_E is weakly metrizable; moreover, these conditions imply that E is also separable.

The following result describes the closure of a subspace in E^* (this is a variation of the Grothendieck's Completion theorem, see [Bou87, III.3.6, Theorem 1]).

Proposition 0.2.6. *Let E be a normed space and let $F \subset E^*$. Then $\overline{F}$ is the set which consists of all $\nu \in E'$ such that $\nu|_{B_E}$ is continuous on B_E with respect to the topology generated by F.*

We will also need some properties of linear operators between locally convex topological vector spaces. Let us start with some characterizations of continuity.

Proposition 0.2.7. *Let E and F be locally convex and let $T : E \to F$ be linear. Then:*

(i) If T is continuous and F is complete, then there is a unique continuous extension $\overline{T}$ of T on the completion $\overline{E}$ of E; this extension is linear.

(ii) Let τ and π be locally convex topologies on E and F respectively, which are weaker than the original topologies of E and F. Assume that F and E satisfy the conditions of the Closed Graph theorem (e.g. E is a Banach space and F is complete) and that T is $\tau \to \pi$ continuous. Then T is $E \to F$ continuous.

Part (i) follows from the fact that every continuous linear map is uniformly continuous, and part (ii) follows from Closed Graph theorem (see [BN11, Theorem 14.3.4]).

An injective continuous map from a metric space into a topological space is called *semi-embedding* if the images of closed balls are closed. Consider a sufficient condition of completeness of normed spaces.

Proposition 0.2.8. [Bou87, III.1.5, Corollary] *If there is a semi-embedding from a normed space E into a complete TVS F, then E is also complete.*

Recall that a linear operator T from a normed space E into a locally convex space F is called (weakly) compact if the image of B_E under T is (weakly) relatively compact in E. Clearly, compactness implies weak compactness, which in turn implies continuity. These properties admit more characterizations.

Proposition 0.2.9. [Gro73, 2.18, Theorem 12 and Theorem 13] *Let E be a normed space, let F be a locally convex space and let $T : E \to F$ be linear. Then:*

(i) If T is compact, then the restriction $T|_{B_E}$ is (uniformly) weak $\to F$ continuous. The converse holds if F is complete.

(ii) T is weakly compact if and only if there exists $T^{**} : E^{**} \to F^{**}$, with range contained in F. In this case T^{**} is also weakly compact, and $T^{**}\overline{B}_{E^{**}} = \overline{T^{**}B_E}$.

Chapter 1

Normed Spaces of Continuous Functions

Recall that a *Normed Space of Continuous Functions* (NSCF) over a topological space X is a linear subspace $\mathbf{F} \subset \mathcal{C}(X)$ equipped with a norm such that the inclusion operator $J_{\mathbf{F}} : \mathbf{F} \to \mathcal{C}(X)$ is continuous. This chapter is dedicated to some basic properties of NSCF's. We will start with a brief summary.

In Section 1.1 we consider a more general construction in order to relate the properties of $J_{\mathbf{F}}$ with the properties of the evaluation map of $\mathbf{F}$ and the operation of evaluation. Namely, we introduce the analogous concepts for linear maps from a generic normed space into $\mathcal{C}(X)$. The modes of continuity of the evaluation map correspond to the properties of $J_{\mathbf{F}}$, and the operation of evaluation (well-known results stated in Theorem 1.1.8, which we expand with Theorem 1.1.10). In particular, we show that continuity of the operation of evaluation on $\left(\overline{B}_{\mathbf{F}}, \textit{weak topology}\right) \times X$ is equivalent to compactness of $J_{\mathbf{F}}$, and that these conditions follow from mere continuity of $|\kappa|_{\mathbf{F}}$ under the additional assumption that $\mathbf{F}$ is a uniformly smooth

Banach space.

Section 1.2 is dedicated to NSCF's as well as Normed Spaces of Functions (NSF's) over sets with no additional structure. We give criterions of reflexivity of NSF's (Theorem 1.2.10) and as a consequence we get a characterization of reflexive NSCF's (Corollary 1.2.11). Another type of results is the interplay between the topological properties of $\mathbf{F}$ and X. The key feature is that $\mathbf{F}$ generates the topology of X if and only if $\kappa_{\mathbf{F}}$ is a topological embedding (Proposition 1.2.26). Using this fact we give criterions for separability and metrizability of X in terms of NSCF's over X (Proposition 1.2.27). On the other hand, we show that a wide range of topological spaces do not support non-separable reflexive NSCF's (Proposition 1.2.25). Finally, we give explicit conditions that guarantee that a NSCF generates the topology of X (Corollary 1.2.22, Proposition 1.2.28).

Section 1.3 is dedicated to Normed spaces of Holomorphic functions. The main goal of this section is to consider the refinements of the results of the preceding sections for the "holomorphic context".

Three families of spaces serve as examples of our constructions: the Weighted Continuous spaces, the Lipschitz spaces and the Bergman spaces. Note that the approach, scope of the consideration, notations and even definitions vary depending on the source, and we aim to present the least restrictive case. For example, we allow the weights to vanish, we do not demand boundedness from the Lipschitz functions, and we consider Bergman spaces over manifolds. The Hardy space is not considered in great details on its own, but it is often used to produce counterexamples (mostly in the next chapter).

1.1 Maps into Spaces of Continuous Functions

The main subject of this section is linear maps $J : F \to \mathcal{C}(X)$, where F is a normed space and X is a topological space. While the nature of F and X is quite different, the presence of the map J brings certain amount of symmetry, and we start with studying this symmetry in the "discontinuous" setting.

1.1.a Maps into Spaces of Functions

Let X be a set. From now on $\mathcal{F}(X)$ is by default equipped with the pointwise topology, i.e. $\mathcal{F}(X) = \mathcal{F}_X(X, \mathbb{F})$ and the natural linear structure given by the pointwise operations. Clearly, $\mathcal{F}(X)$ is a locally convex Hausdorff space with $\left\{ \|\cdot\|_\infty^{\{x\}} | x \in X \right\}$ being a generating family of seminorms. Recall that the pointwise uniformity is the uniformity of the product of the copies of $\mathbb{F}$ parameterised by X. Since $\mathbb{F}$ is a complete metric space, the product uniformity is complete (see [Eng89, Theorem 8.3.9]); in fact it is a Baire space (see [Bou98, p254, 17a]). By Tychonoff theorem, every bounded set in $\mathcal{F}(X)$ is relatively compact. Thus, this space is a Montel space, and so it is reflexive (see [Bou87, IV.2.5, before Proposition 9]).

For $\mathcal{A} \subset \mathcal{P}(X)$ consider the space $\mathcal{BF}_\mathcal{A}(X) = \left\{ f \in \mathcal{F}(X),\ \|f\|_\infty^A < +\infty,\ \forall A \in \mathcal{A} \right\}$, endowed with the collection of seminorms $\left\{ \|\cdot\|_\infty^A | A \in \mathcal{A} \right\}$. This is a complete locally convex space, which is also a uniform subspace of $\mathcal{F}_\mathcal{A}(X, \mathbb{F})$ (note that the latter is not a topological vector space, since the neighborhoods of 0 in that space are not absorbing).

Let F be a linear space. A linear map $J : F \to \mathcal{F}(X)$ generates a function $\Sigma^J : X \times F \to \mathbb{F}$ defined by $\Sigma^J(x, f) = [Jf](x)$. It is clear that Σ^J is linear in the second variable, and conversely, any function $\Sigma : X \times F \to \mathbb{F}$ which is linear in the second variable generates the corresponding J_Σ (see the discussion before Theorem 0.2.4). Also, J generates a map $\kappa_J : X \to F'$ defined by $\langle \kappa_J(x), f\rangle = [Jf](x)$, and any $\kappa : X \to F'$ generates the corresponding J_κ. We will commonly refer to X as the *phase space*, to J as the *embedding* and to κ_J as the *evaluation map*. We will also call $x_J = \kappa_J(x)$, for $x \in X$, the *point evaluation* at x on F relative to J, and denote $A_J = \kappa_J(A)$, for $A \subset X$. In particular, X_J is the set of all point evaluations relative to J and Ker $J = X_J^\perp$.

Example 1.1.1. Let $F = \mathcal{F}(X)$, and let $J = Id_{\mathcal{F}(X)}$. Denote the canonical evaluation map $\kappa_{\mathcal{F}(X)}$, which maps X into $\mathcal{F}(X)'$. If X is clear from the context, we will use the simplified notation $\kappa_{\mathcal{F}}$. For $A \subset X$ denote $A_{\mathcal{F}(X)} = A_{\mathcal{F}} = \kappa_{\mathcal{F}}(A)$. Note that for $x \in X$ the linear functional $\kappa_{\mathcal{F}}(x)$ can be identified with the Dirac's measure δ_x at x. □

Example 1.1.2. Let F be a vector space and let $X \subset F'$. The restriction Σ of the natural pairing $\langle\cdot,\cdot\rangle$ on $X \times F \to \mathbb{F}$ is linear in the second variable. Then $\kappa_{\langle\rangle} : X \to F'$ is the identity on X, and $J_{\langle\rangle}$ is the natural inclusion from F into $\mathcal{F}(X)$. □

Now we will explore the cases when either X or F or both are equipped with a topology. Let us start with some basic properties of continuous maps from normed spaces into $\mathcal{F}(X)$ (the proofs are omitted).

Proposition 1.1.3. *Let F be a normed space and let J be a linear map into $\mathcal{F}(X)$. Then:*

(i) J is continuous if and only if $\kappa_J(X) \subset F^*$.

(ii) If J is continuous, then there is a unique extension $\overline{J} \in \mathcal{L}\left(\overline{F}, \mathcal{F}(X)\right)$ of J on the completion $\overline{F}$ of F. Moreover, $\kappa_{\overline{J}} = \kappa_J$, where F^* is identified with $\overline{F}^*$.

(iii) If J is continuous, then there is $J^{**} \in \mathcal{L}(F^{**}, \mathcal{F}(X))$, and $J^{**}\overline{B}_{F^{**}} = \overline{JB_F}$. Moreover, $\kappa_{J^{**}} = \kappa_J$, where F^* is viewed as a subset of F^{***}.

Consider a variation of Example 1.1.2.

Example 1.1.4. Let F be a normed space and let $X \subset F^*$. We now have that $J_{\langle\rangle}$ is continuous and $J_{\langle\rangle}^{**}$ is the natural inclusion of F^{**} into $\mathcal{F}(X)$. $\square$

We can introduce certain quantities that will characterise the behaviour of the embedding. Let X be a set, let F be a normed space and let $J : F \to \mathcal{F}(X)$ be linear. Consider $|\kappa|_J : X \to [0, +\infty]$, defined by

$$|\kappa|_J(x) = \|\kappa_J(x)\| = \|x_J\| = \sup_{f \in B_F} |[Jf](x)| = \sup_{g \in JB_F} |g(x)|.$$

Equivalently, $|\kappa|_J = \sup\limits_{g \in JB_F} |g|$, and so $|\kappa|_J$ is determined by the closure of JB_F in $\mathcal{F}(X)$.

Clearly, $|\kappa|_J(x) = 0$ means that $[Jf](x) = 0$ for each $f \in F$. Hence, JF is non-trivial if and only if $|\kappa|_J$ has at least one non-zero value. On the other hand $|\kappa|_J(x) < +\infty$ if and only if $\kappa_J(x) \in F^*$, and so J is $\mathcal{F}(X)$ continuous if and only if $|\kappa|_J$ is finite on X.

Since $|\kappa|_J$ is determined by the closure of $\overline{JB_F}^{\mathcal{F}(X)}$, it captures a limited amount of information. Assume that $|\kappa|_J$ is finite. First, observe that $|\kappa|_J$ is "independent" from the kernel of J: if T is the embedding of $E = F/\mathrm{Ker}\ J$ into $\mathcal{F}(X)$ induced by J, then $JB_F = TB_E$, and so $|\kappa|_T = |\kappa|_J$. If E is a dense subset of F, we have that $\overline{B}_F = \overline{B_E}$ (it is easy to see that $\overline{B_E}$ is a closed set that contains B_F), and so $\overline{JB_F} = \overline{JB_E}$. Hence, $|\kappa|_J = |\kappa|_{J|_E}$. Also, from part (ii) of Proposition 0.2.9 we have that $J^{**}\overline{B}_{F^{**}} = \overline{JB_F}$ in $\mathcal{F}(X)$, and so $|\kappa|_{J^{**}} = |\kappa|_J$.

We can also characterize continuity of maps into $\mathcal{BF}_{\mathcal{A}}(X)$ in terms of $|\kappa|_J$.

Proposition 1.1.5. *Let J be a linear map from a normed space F into $\mathcal{F}(X)$, and let $\mathcal{A} \subset \mathcal{P}(X)$. Then:*

(i) J is a continuous map into $\mathcal{BF}_{\mathcal{A}}(X)$ if and only if $|\kappa|_J$ is bounded on every $A \in \mathcal{A}$.

(ii) If F is a Banach space, then J is a continuous map into $\mathcal{BF}_{\mathcal{A}}(X)$ if and only if $JF \subset \mathcal{BF}_{\mathcal{A}}(X)$.

Proof. (i): Since $\| |\kappa|_J \|_\infty^A = \sup\limits_{f \neq 0} \frac{\|Jf\|_\infty^A}{\|f\|}$, for any $A \in \mathcal{A}$, it follows that $\|Jf\|_\infty^A \leq \| |\kappa|_J \|_\infty^A \|f\|$.

(ii): Assume that F is a Banach space and that $JF \subset \mathcal{BF}_{\mathcal{A}}(X)$. For any $A \in \mathcal{A}$ we have that A_J is weak* bounded. Hence, A_J is bounded, due to Banach-Steinhaus theorem (see [BN11, Theorem 11.3.4]), and so $\| |\kappa|_J \|_\infty^A = \sup\limits_{x \in A} \|x_J\| < +\infty$. The proof is completed by applying part (i). □

Aside from $|\kappa|_J$, consider another quantification. Define $d_J : X \times X \to [0, +\infty]$ by $d_J(x, y) = \|x_J - y_J\|$. Analogously to $|\kappa|_J$, this function depends only on $\overline{JB_F}^{\mathcal{F}(X)}$ and is independent of the kernel of J. These two quantities are related by

$$||\kappa|_J(x) - |\kappa|_J(y)| \leq d_J(x, y) \leq |\kappa|_J(x) + |\kappa|_J(y).$$

Clearly, d_J is a pseudometric on X, which is finite if $X_J \subset F^*$, and non-degenerate if and only if JF separates the points of X.

There is a third quantification available exclusively for inner product spaces. Let H be an inner product space and let J be a continuous map from H into $\mathcal{F}(X)$. Define the *kernel* $k^J : X \times X \to \mathbb{F}$ of J by $k^J(x, y) = \langle x_J, y_J \rangle_{H^*}$, $x, y \in X$. We immediately get the following properties:

- $k^J(y, x) = \langle y_J, x_J \rangle_{H^*} = \overline{k^J(x, y)}$;
- $k^J(x, x) = \|x_J\|^2 = |\kappa|_J^2(x) \geq 0$ and $\left|k^J(x, y)\right|^2 \leq k^J(x, x)\, k^J(y, y)$;

- $d_J(x,y)=\sqrt{k^J(x,x)+k^J(y,y)-2\mathrm{Re}\ k^J(x,y)}$.

More generally, k^J is positive semi-definite, i.e. the matrix $\left(k^J(x_i,x_j)\right)_{i,j=1}^n$ is positive semi-definite, for any $x_1,...,x_n\in X$. Indeed, for any $\overrightarrow{\mu}=(\mu_1,...,\mu_n)^T\in\mathbb{F}^n$ and $\overrightarrow{\mu}^*=(\overline{\mu_1},...,\overline{\mu_n})$ we have

$$\overrightarrow{\mu}\left(k^J(x_i,x_j)\right)_{i,j=1}^n\overrightarrow{\mu}^*=\sum_{i,j=1}^n\mu_i\overline{\mu_j}k^J(x_i,x_j)=\left\|\mu_1x_J^1+...+\mu_nx_J^n\right\|\geq 0.$$

Note that k^J is positively definite if and only if X_J is linearly independent set, and $k^J(x,x)=0$ if and only if $k^J(\cdot,x)\equiv 0$ and if and only if $Jf(x)=0$, for every $f\in H$. Also, note that $k^{\overline{J}}|_{\overline{H}\ominus\mathrm{Ker}\ \overline{J}}=k^J$, where $\overline{J}$ is the extension of J on the completion $\overline{H}$ of H.

1.1.b Maps into Spaces of Continuous Functions

In this subsection we investigate analogous questions to the ones considered in the preceding subsection, but in the presence of more structure. Namely, we assume that X is a topological space and study linear maps into $\mathcal{C}(X)$. Let us first mention some properties of the latter space. From now on $\mathcal{C}(X)$ is by default equipped with the compact-open topology, i.e. $\mathcal{C}(X)=\mathcal{C}_{\mathcal{K}}(X,\mathbb{F})$. Since every continuous function is bounded on a compact set, it follows that $\mathcal{C}(X)$ is a linear subspace of $\mathcal{BF}_{\mathcal{K}}(X)=\mathcal{BF}_{\mathcal{K}(X)}(X)$, and so it is a locally convex space. However, $\mathcal{C}(X)$ is not always complete, although it is complete in sufficiently many cases. In particular, this happens if X is compactly generated (combine [Eng89, Theorem 3.3.22] with [BN11, Definition 5.8.6 and Theorem 5.8.8]). Moreover, if X is σ-locally compact (i.e. locally compact and a union of a sequence of compacts), then $\mathcal{C}(X)$ is a Frechet space (see [BN11, Definition 5.8.4 and Theorem 5.8.5]).

Let us start with a variation of a well-known result (see e.g. [Wad61]).

Proposition 1.1.6. *Let X be a topological space, let F be a linear space and let J be a linear map from F into $\mathcal{F}(X)$. Then the evaluation map κ_J is weak* continuous into F' if and only if $JF \subset \mathcal{C}(X)$.*

The following corollary suggests that even if there is no specified topology on X, we can generate one through the pull-back via κ_J.

Corollary 1.1.7. *Let X be a set and let J be a linear map from a linear space F into $\mathcal{F}(X)$. Then:*

(i) The evaluation map κ_J generates the same topology on a set X as the collection $JF \subset \mathcal{F}(X)$.

(ii) Let X be endowed with a Hausdorff topology. The map κ_J is a topological embedding into F' if and only if JF generates the topology of X. In this case X is Tychonoff.

(iii) Let X be endowed with a compact (not necessarily Hausdorff) topology. The map κ_J is a topological embedding into F' if and only if JF separates the points of X. In this case X is Hausdorff.

Proof. From Proposition 1.1.6, the minimal topology on X in which $\kappa_J : X \to F'$ is continuous coincides with the minimal topology on X in which all elements of JF are continuous. Hence, κ_J is a topological embedding if and only if JF separates the points of X and generates the topology of X. Since if a collection of functions generates a Hausdorff topology, then it separates the points, we have (ii). As due to Corollary 0.2.2 if a collection of functions separates the points of a compact space, it also generates the topology, which is Hausdorff, we have (iii). $\square$

The obtained assertion is in fact quite strong: the weak* topology is the weakest natural topology on the dual space. Therefore, if a topology τ on X_J is at least as strong as the weak* topology, κ_J is $X \to \tau$ continuous and JF generates the topology of X, then κ_J is a homeomorphism onto (X_J, τ). Consequently, all such topologies coincide on X_J. In some cases it is easy to verify that JF generates the topology of X. Indeed if X is a compact space, then this takes place whenever JF separates the points of X. A sufficient condition for JF to generate a topology when X is locally compact is given in Corollary 1.2.22.

Also note that if $JF \subset \mathcal{C}(X)$, then $A_J^{\perp} = \overline{A}_J^{\perp}$, for any $A \subset X$, since a continuous function that vanishes on A has to vanish on $\overline{A}$.

Consider the case when both X and F possess a topological structure. We will relate the properties of J, κ_J and the operation of evaluation Σ^J. Aside from continuity, J can have further basic properties of linear operators, and we will consider compactness and weak compactness. Note that in the discontinuous case both of these properties follow automatically from the continuity, since $\mathcal{F}(X)$ is a Montel space. As a starting point in the comparison of the properties of J with the properties of κ_J let us state a variation of a well-known result.

Theorem 1.1.8. *[*[Bar55], [DS58, VI.7, Theorem 1], [Wad61], [Gro73, 3.7, Theorem 5]*] Let X be a Hausdorff space, let F be a normed space and let J be a linear map from F into $\mathcal{F}(X)$. Then:*

(i) J is a continuous operator into $\mathcal{C}(X)$ if and only if κ_J is weak continuous into F^* and $|\kappa|_J$ is bounded on compacts.*

(ii) If F is a Banach space, then J is a continuous operator into $\mathcal{C}(X)$ if and only if κ_J is

weak continuous into* F^**.*

(iii) If κ_J *is* $X \to (F^*, \|\cdot\|)$ *continuous then* J *is a compact operator into* $\mathcal{C}(X)$*. If* J *is a compact operator into* $\mathcal{C}(X)$*, then* $J|_{B_F}$ *is weak* $\to \mathcal{C}(X)$ *continuous. Both converses hold whenever* X *is compactly generated.*

(iv) J *is a weakly compact operator into* $\mathcal{C}(X)$ *if and only if* κ_J *is weakly continuous into* F^**. In this case* $J^{**}\overline{B}_F = \overline{JB_F}$*.*

The following variation of Example 1.1.4 utilizes Theorem 1.1.8 and part (iii) of Corollary 1.1.7.

Example 1.1.9. Let F be a normed space and let $X \subset F^*$. Since F consists of continuous functions on F^*, the natural inclusion $J_{\langle\rangle}$ embeds F into $\mathcal{C}(X)$, when X is endowed with a topology stronger than the weak* topology. Moreover, depending on the choice of the topology $J_{\langle\rangle}$ will have some additional properties, in particular:

- choosing the weak* topology we get that $J_{\langle\rangle}F$ generates the topology of X;
- choosing the weak topology we get that $J_{\langle\rangle}$ is weakly compact;
- choosing the norm topology we get that $J_{\langle\rangle}$ is compact, and X becomes a metric space. $\square$

The upcoming main result of the section adds several equivalences of a similar spirit to the list started in Theorem 1.1.8.

Theorem 1.1.10. *Let* X *be a Hausdorff space, let* F *be a normed space and let* J *be a linear map from* F *into* $\mathcal{F}(X)$*. Then:*

(i) If Σ^J *is continuous on* $X \times F$*, then* J *is a continuous operator into* $\mathcal{C}(X)$*. The converse holds whenever* X *is either locally compact or metrizable, or when* $\dim F < \infty$*.*

(ii) If F *is a Banach space and* X *is either* σ*-locally compact or first countable, then* J *is a continuous operator into* $\mathcal{C}(X)$ *if and only if* κ_J *is weakly* continuous into* F' *and there is a dense set* $A \subset X$ *such that* $A_J \subset F^*$*.*

(iii) If $\Sigma^J\big|_{X \times B_F}$ *is* $X \times$*weak continuous, then* J *is a compact map into* $\mathcal{C}(X)$*, whenever* X *is compactly generated. The converse holds whenever* X *is locally compact, or if* F *is reflexive, or when* X *is metrizable and* F^* *is separable.*

(iv) If F *is a uniformly smooth*[1] *Banach space,* $JF \subset \mathcal{C}(X)$*, and* $|\kappa|_J$ *is continuous, then* J *is a compact map into* $\mathcal{C}(X)$*.*

(v) J *is a weakly compact map into* $\mathcal{C}(X)$ *if and only if* J *is a continuous map into* $\mathcal{C}(X)$ *and* $\overline{JB_F}^{\mathcal{C}(X)} = \overline{JB_F}^{\mathcal{F}(X)}$*.*

Proof. (i): If Σ^J is continuous on $X \times F$, then continuity of J into $\mathcal{C}(X)$ follows immediately from part (i) of Theorem 0.2.4. The converse in all three cases follows from part (ii) of the same theorem, since F is locally compact when $\dim F < \infty$, and if X is metrizable, then $X \times F$ is also metrizable, and so it is also compactly generated.

(ii): First, assume that X is σ-locally compact, and so $\mathcal{C}(X)$ is a Frechet space. Note that $A_J \subset F^*$ means that J is a continuous map from F into $\mathcal{BF}_{\mathcal{A}}(X)$, where $\mathcal{A} = \{\{x\} | x \in A\}$. Since A is dense, this topology is Hausdorff when restricted to $\mathcal{C}(X)$, which contains JF due to weak* continuity of κ_J. Hence, the continuity of J follows from part (ii) of Proposition 0.2.7 applied to F, $\mathcal{C}(X)$ and the introduced topology on the latter.

[1] *see* UF-smooth *in* [FHH+11, before Definition 7.6]

From part (ii) of Theorem 1.1.8 in order to prove the claim in the case when X is first countable it is enough to show that $\{x \in X \,|x_J \in F^*\}$ is closed in X. Since κ_J is $X \to (F', F)$ continuous, it is sequentially continuous with respect to these topologies. As F^* is sequentially closed in (F', F) (see [BN11, 11.9.4] and the discussion after it), $\{x \in X \,|x_J \in F^*\} = \kappa_J^{-1}(F^*)$ is sequentially closed, and since X is first countable, it follows that the latter set is closed.

(iii): Due to part (i) of Theorem 0.2.4, if $\Sigma^J\big|_{X\times B_F}$ is $X \times \mathit{weak}$ continuous, then $J|_{B_F}$ is $\mathit{weak} \to \mathcal{C}(X)$ continuous. Hence, if X is compactly generated, then J is compact according to part (iv) of Theorem 1.1.8. Conversely, if J is compact, then $J|_{B_F}$ is $\mathit{weak} \to \mathcal{C}(X)$ continuous. In fact, $J|_{\overline{B}_F}$ is also $\mathit{weak} \to \mathcal{C}(X)$ continuous. If any of the conditions of the proposition holds, we can use part (ii) of Theorem 0.2.4 to conclude that $\Sigma^J\big|_{X\times B_F}$ is $X \times \mathit{weak}$ continuous. Indeed, either X is locally compact, or F is reflexive and so $\overline{B}_F$ is weakly compact, or $X \times \overline{B}_F$ is metrizable ($\overline{B}_F$ considered in the weak topology), and consequently compactly generated. The last claim follows from the fact that if F^* is separable and so from part (ii) of Proposition 0.2.5 $\overline{B}_F$ is weakly metrizable.

(iv): If F is uniformly smooth, then F^* is uniformly convex (see [FHH$^+$11, Theorem 9.9]). The latter spaces are reflexive (see [BN11, 16.2.6]), and the weak, weak* and the norm topologies coincide on ∂B_{F^*} (see [BN11, 16.2.7]). From part (iii) of Theorem 1.1.8, it is enough to prove that κ_J is $X \to (F^*, \|\cdot\|)$ continuous. Since $JF \subset \mathcal{C}(X)$, it follows from Proposition 1.1.6 that κ_J is weak* continuous. Since $|\kappa|_J$ is continuous, we automatically get the continuity of κ_J at each point of $|\kappa_J|^{-1}(0)$. Hence, we may assume that $0 \notin |\kappa|_J(X)$ and so $\iota = \frac{1}{|\kappa|_J}\kappa_J$ is a weak* continuous map from X into ∂B_{F^*}. But since the weak* and the norm topologies coincide on ∂B_{F^*}, it follows that ι is $X \to (F^*, \|\cdot\|)$ continuous and thus $\kappa_J = |\kappa|_J \, \iota$ is $X \to (F^*, \|\cdot\|)$ continuous.

(v): If J is weakly compact, then $\overline{JB_F}^{\mathcal{C}(X)}$ is weakly compact. Then from Lemma 0.2.1 applied to the weak and pointwise topologies on $\mathcal{C}(X)$ it follows that $\overline{JB}^{\mathcal{C}(X)} = \overline{JB}^{\mathcal{F}(X)}$. Conversely, if $\overline{JB}^{\mathcal{C}(X)} = \overline{JB}^{\mathcal{F}(X)}$, it follows that $\overline{JB}^{\mathcal{C}(X)}$ is a bounded absolutely convex pointwise compact subset of $\mathcal{C}(X)$. Such sets are weakly compact in $\mathcal{C}(X)$ (see [Flo80, 4.3, Corollary 2]), and so J is weakly compact. $\square$

Remark 1.1.11. It follows immediately from part (iii) of Theorem 1.1.8 that if d_J is a continuous pseudometric on X, then J is a compact operator, and the converse holds once X is compactly generated. It is clear, that d_J is continuous $\Rightarrow |\kappa|_J$ is continuous $\Rightarrow |\kappa|_J$ is bounded on compacts $\Rightarrow |\kappa|_J$ is finite, but none of the converses hold in general (the counterexamples can be constructed using Example 1.2.17). However, some of them hold under additional assumptions about F. Indeed, it follows from part (ii) of Theorem 1.1.8, that if $|\kappa|_J$ is finite and F is a Banach space, then $|\kappa|_J$ is bounded on compacts. On the other hand, from part (iv) of Theorem 1.1.10 if F is uniformly smooth, and $|\kappa|_J$ is continuous, then d_J is continuous. $\square$

Remark 1.1.12. The classes of uniformly smooth and uniformly convex spaces contain all Hilbert spaces, all L^p spaces, for $p \in (1, +\infty)$ (see [FHH$^+$11, Theorem 9.10]), and is closed under finite l^p sums (this follows from [FHH$^+$11, Exercise 9.4]). In fact, part (ii) of the theorem was already known for the case when F is a Hilbert space (see e.g. [Sch64]), but the proof was specific for Hilbert spaces. $\square$

Recall that part (ii) of Proposition 1.1.3 is a simple consequence of the completeness of

$\mathcal{F}(X)$. The analogous fact in the continuous case does not always hold, since $\mathcal{C}(X)$ is not always complete. However, it is complete when X is compactly generated. Additionally, the completeness can be replaced with weak compactness of J. We obtain the following result.

Proposition 1.1.13. *Let X be a topological space, let F be a normed space and let $J \in \mathcal{L}(F, \mathcal{C}(X))$. Then there is a unique continuous extension $\overline{J}$ of J on the completion $\overline{F}$ of F, and κ_J is weak* continuous into $\overline{F}^* = F^*$, provided that either X is compactly generated, or J is weakly compact.*

Proof. The uniqueness, as well as the sufficiency of the first condition follows from part (i) of Proposition 0.2.7. The sufficiency of the second condition follows from the fact that F^{**} is a Banach space, containing F as a subspace. Consequently $\overline{F} \subset F^{**}$ and so $J^{**}|_{\overline{F}} = \overline{J}$ is a continuous extension. $\square$

The following fact is useful in the situations when we only know that a part of a normed space is mapped into $\mathcal{C}(X)$.

Proposition 1.1.14. *Let F be a normed space, let X be compactly generated, and let J be a continuous linear map from F into $\mathcal{F}(X)$ such that $|\kappa|_J$ is bounded on every compact set $K \subset X$. If there is a set $D \subset F$, such that $\overline{\mathrm{span} D} = F$ and $JD \subset \mathcal{C}(X)$, then J is $\mathcal{C}(X)$ continuous.*

Proof. From Proposition 1.1.5, $J \in \mathcal{L}(F, \mathcal{BF}_{\mathcal{K}}(X))$, and so we only need to show that $JF \subset \mathcal{C}(X)$. Since $\mathcal{C}(X)$ is a subspace of $\mathcal{BF}_{\mathcal{K}}(X)$, it follows that J maps $E = \mathrm{span} D$ continuously into $\mathcal{C}(X)$. Hence, from part (i) of the Proposition 1.1.13, there is a continuous extension $\overline{J} : \overline{E} \to \mathcal{C}(X)$ of $J|_E$ on the completion $\overline{E}$ of E. Since $F \subset \overline{E}$, both J and $\overline{J}$ are continuous from F into $\mathcal{BF}_{\mathcal{K}}(X)$, coinciding on a dense set, and so they are equal. Hence, $JF \subset \overline{J}\,\overline{E} \subset \mathcal{C}(X)$. $\square$

In case of the inner product spaces the properties of embeddings are largely characterized by the properties of the kernel. The following results are well-known and can be viewed as a variation of Theorem 1.1.8 and Theorem 1.1.10. In particular, part (iv) is a version of part (iv) of Theorem 1.1.10.

Proposition 1.1.15. *[*[BTA04], [CDVT06], [FM13], [Sza04], *and* [Sch64]*] Let H be an inner product space and let $J : H \to \mathcal{F}(X)$ be linear and continuous. Then:*

(i) J is a continuous operator into $\mathcal{C}(X)$ if and only if $JH \subset \mathcal{C}(X)$ and k^J is bounded on compacts. In this case for every $x \in X$ and every orthonormal basis $\{e_i\}_{i\in I}$ of H we have $k^J(\cdot, x) = \sum \overline{Je_i(x)} Je_i$, where the series converges in $\mathcal{C}(X)$.

(ii) If H is complete then J is a continuous operator into $\mathcal{C}(X)$ if and only if $JH \subset \mathcal{C}(X)$. In this case k^J is bounded on compacts and separately continuous.

(iii) If X is compactly generated, then J is a continuous operator into $\mathcal{C}(X)$ if and only if k^J is bounded on compacts and separately continuous.

(iv) If k^J is separately continuous in the points of the diagonal and continuous on the diagonal, then J is a compact operator into $\mathcal{C}(X)$. If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we have $k^J(x, y) = \sum\limits_{i\in I} Je_i(x) \otimes \overline{Je_i(y)}$, for all $x, y \in X$, where the series converges in $\mathcal{C}(X \times X)$.

(v) If X is compactly generated and J is a compact operator into $\mathcal{C}(X)$, then k^J is continuous on $X \times X$.

1.2 Normed Spaces of Continuous Functions

The subject of this section is the normed spaces of continuous functions. In order to distinguish such spaces from "abstract" normed spaces we will use **bold** letters. We start with defining normed spaces of (not necessarily continuous) functions.

1.2.a Normed Spaces of Functions

Throughout this subsection X is a set. Before giving a definition of a *normed space of functions* (NSF) over X let us understand what we expect from it. First of all it should consist of functions, and its linear structure should be consistent with the pointwise operations. Hence, algebraically, a NSF has to be a linear subset of $\mathcal{F}(X)$. The topological requirement that we are looking for also has to "respect" the existing structure on $\mathcal{F}(X)$. This motivates the following definition.

A NSF over X is a linear subspace $\mathbf{F} \subset \mathcal{F}(X)$ endowed with a norm that generates a topology, which is stronger than the subspace topology, i.e. the pointwise topology on $\mathbf{F}$. The last condition is equivalent to the continuity of the inclusion operator $J_{\mathbf{F}} : \mathbf{F} \to \mathcal{F}(X)$, and to the boundedness of $B_{\mathbf{F}}$ in $\mathcal{F}(X)$. Also, it is easy to see that $\mathbf{F}$ is a NSF if and only if its weak topology is stronger than the pointwise topology. If $\mathbf{F}$ is a NSF, we will put $\mathbf{F}$ instead of $J_{\mathbf{F}}$ in the subscript of the notation of all related objects. For example, a point evaluation on $\mathbf{F}$ at $x \in X$ will be denoted by $x_{\mathbf{F}}$. Clearly, the continuity of all point evaluations on $\mathbf{F}$ is another equivalent condition for $\mathbf{F}$ to be a NSF. We will call a complete NSF $\mathbf{F}$ a *Banach space of functions* (BSF).

In order to bridge the situation of a generic map from some normed space F into $\mathcal{F}(X)$

considered in the previous section and the concept of a NSF, we introduce the following notion. We will say that a normed space F is *realized as NSF* over X via $J : F \to \mathcal{F}(X)$ if J is a linear injection, which is continuous with respect to the pointwise topology. Note that the realization (image) $\mathbf{F} = JF$ with its pull-forward norm is a NSF. Clearly, this definition generalizes the concept of a NSF in the sense that each normed space is realized as a NSF via its inclusion.

Example 1.2.1. If J is a continuous linear map from a normed space F into $\mathcal{F}(X)$, then there is a continuous injective linear map $Q : F/\mathrm{Ker}\ J \to \mathcal{F}(X)$, such that $QP = J$, where P is the quotient map from F onto $F/\mathrm{Ker}\ J$. Then JF is the realization of $F/\mathrm{Ker}\ J$ via Q. Consequently, any map $\kappa : X \to F^*$ generates a NSF $J_\kappa F$ with the push-forward structure of $F/\mathrm{Ker}\ J_\kappa = F/\kappa(X)^\perp$. $\square$

Example 1.2.2. Let F and X be as in Example 1.1.4. Then F is realized as a NSF over X via $J_{\langle\rangle}$ if and only if $X^\perp = \{0\}$. In fact, any NSF that separates the points of its phase space can be viewed this way. $\square$

Recall that if F is a normed space, X is a set and $J \in \mathcal{L}(F, \mathcal{F}(X))$, then $J^{**} \in \mathcal{L}(F^{**}, \mathcal{F}(X))$ and $\overline{J} \in \mathcal{L}\left(\overline{F}, \mathcal{F}(X)\right)$. This fact may lead to the impression that we can neglect to distinguish the completeness condition among NSF's, since we can (presumably) always realize the completion via the extension of the inclusion. However, this procedure in general can lead us beyond our definition, since the extension of the embedding is not necessarily injective, as the following example demonstrates.

Example 1.2.3. Let $\{e_n\}_{n=1}^{\infty}$ be a basis in l^2 and let F be the algebraic complement to the vector $f = \sum\limits_{n=1}^{\infty} n^{-1} e_n \in l^2$, which contains $\{e_n\}_{n=1}^{\infty}$. Let J_{l^2} be the inclusion of l^2 into $\mathcal{F}(\mathbb{N})$.

Let $T : l^2 \to l^2$ be the orthogonal projection, parallel to f. Then $J = J_{l^2} T|_F$ realizes F as a NSF over $\mathbb{N}$. However, the completion of F is l^2, and the corresponding extension $\overline{J}$ is equal to $J_{l^2}T$, which is not an injection. Hence, $\overline{J}$ does not realize the completion of F as a NSF over $\mathbb{N}$. $\square$

Let us characterize the possibility of realizing the double dual of a NSCF as a BSCF through the location of the set $X_{\mathbf{F}}$ of the point evaluations on $\mathbf{F}$ in $\mathbf{F}^*$.

Proposition 1.2.4. *Let* $\mathbf{F}$ *be a NSF over a set* X*. Then* $\mathbf{F}^{**}$ *is realized as a NSF via* $J_{\mathbf{F}}^{**}$ *if and only if* $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$ *and if and only if the weak and the pointwise topologies coincide on* $B_{\mathbf{F}}$*. Moreover, in this case* $\overline{\mathbf{F}}$ *is realized as a closed subspace of* $J_{\mathbf{F}}^{**}\mathbf{F}^{**}$*.*

Proof. Since $\mathrm{Ker}\ J_{\mathbf{F}}^{**} = X_{\mathbf{F}}^{\perp}$ in $\mathbf{F}^{**}$, it follows that $J_{\mathbf{F}}^{**}$ is injective if and only if $\overline{\mathrm{span} X_{\mathbf{F}}} = X_{\mathbf{F}}^{\perp\perp} = (\mathrm{Ker}\ J_{\mathbf{F}}^{**})^{\perp} = \{0_{\mathbf{F}}\}^{\perp} = \mathbf{F}^*$. $\square$

1.2.b Normed Spaces of Continuous Functions

We will define *normed spaces of continuous functions* (NSCF) in an analogous way to how we defined NSF's. Let X be a topological space. A NSCF over X is a linear subspace $\mathbf{F} \subset \mathcal{C}(X)$ endowed with a norm that generates a topology, which is stronger than the subspace topology, i.e. the compact-open topology on $\mathbf{F}$. The last condition is equivalent to the continuity of the inclusion operator $J_{\mathbf{F}} : \mathbf{F} \to \mathcal{C}(X)$ and also to the boundedness of $B_{\mathbf{F}}$ in $\mathcal{C}(X)$. Parts (i) and (ii) of Theorem 1.1.8 and Theorem 1.1.10 demonstrate that this is the "correct" definition, in the sense that NSCF's are compatible with the existing structure on $\mathcal{C}(X)$ in many ways. Clearly, any NSCF over X is also a NSF over X. We will call a complete NSCF a *Banach space of continuous functions* (BSCF). We will also call a NSCF

$\mathbf{F}$ (weakly) compactly embedded, if $J_{\mathbf{F}}$ is a (weakly) compact operator, or equivalently, if $B_{\mathbf{F}}$ is (weakly) relatively compact in $\mathcal{C}(X)$.

Similarly to the discontinuous case, a normed space F is called *realized as a NSCF* over X via $J : F \to \mathcal{C}(X)$ if J is a linear injection, which is continuous with respect to the compact-open topology. It follows from part (ii) of Theorem 1.1.8 that if F is a Banach space and J realizes it as a NSF over X with the realization contained in $\mathcal{C}(X)$, then J realizes F as a NSCF over X. In particular, a BSF contained in $\mathcal{C}(X)$ is a BSCF.

Remark 1.2.5. In fact more is true. If $\mathbf{F} \subset \mathcal{C}(X)$ endowed with a semi-norm, such that there is a dense set $A \subset X$ such that $x_{\mathbf{F}}$ is a continuous linear functional for every $x \in A$, then the semi-norm is in fact a norm. Moreover, if this is a complete norm, and X is σ-locally compact, or first countable, then $\mathbf{F}$ is a BSCF, due to part (ii) of Theorem 1.1.10. $\square$

NSCF's have more associated topologies than NSF's. For a NSCF $\mathbf{F}$ over X we have to consider the given topology of $\mathbf{F}$, the weak topology, the compact-open topology and the pointwise topology. In this section we will establish some relations between these topologies. The results from the Subsection 1.1.b yield the following characterization of (weakly) compactly embedded NSCF's.

Proposition 1.2.6. *Let* $\mathbf{F}$ *be a linear subspace of* $\mathcal{C}(X)$ *equipped with a norm.*

(i) If $\mathbf{F}$ *is a compactly embedded NSCF, then the weak topology is stronger than the compact-open topology on* $B_{\mathbf{F}}$. *The converse is true whenever* X *is compactly generated.*

(ii) Assume that $\mathbf{F}$ *is a NSCF. Then* $\mathbf{F}$ *is weakly compactly embedded if and only if* $J_{\mathbf{F}}^{**}\mathbf{F}^{**} \subset \mathcal{C}(X)$ *and if and only if* $\overline{B_{\mathbf{F}}}^{\mathcal{C}(X)} = \overline{B_{\mathbf{F}}}^{\mathcal{F}(X)}$. *In this case the latter closure is also equal to* $J_{\mathbf{F}}^{**}\overline{B_{\mathbf{F}^{**}}}$.

Note that any NSCF over a discrete topological space is compactly embedded due to part (iii) of Theorem 1.1.8. Combining Proposition 1.2.4 with Proposition 1.2.6 gives us the following result.

Corollary 1.2.7. *Let* $\mathbf{F}$ *be a NSCF over* X*. Then:*

(i) If $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$ *and either* X *is compactly generated, or* $\mathbf{F}$ *is a weakly compactly embedded, then* $\overline{\mathbf{F}}$ *is realized as a BSCF over* X *via* $\overline{J_{\mathbf{F}}}$*.*

(ii) $\mathbf{F}^{**}$ *is realized as a BSCF over* X *via* $J_{\mathbf{F}}^{**}$ *if and only if* $\mathbf{F}$ *is weakly compactly embedded and* $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$*.*

(iii) If $\mathbf{F}$ *is compactly embedded then* $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$ *if and only if the weak, pointwise and compact-open topologies coincide on* $B_{\mathbf{F}}$*.*

If $\mathbf{F}$ is a NSCF over X, then the properties of subsets of X impact the properties of their images under $\kappa_{\mathbf{F}}$. For example, if $K \subset X$ is compact, then $K_{\mathbf{F}}$ is weak* compact, since $\kappa_{\mathbf{F}}$ is weak* continuous. On the other hand, if $A \subset X$ is dense, the $A_{\mathbf{F}}$ is separating, since if a continuous function vanishes on a dense set, it vanishes identically. Consequently, $\mathrm{span} A_{\mathbf{F}}$ is weak*-dense.

Let $\mathbf{F}$ be a NSF over a set Y. It is possible to introduce a topology on Y so that $\mathbf{F}$ becomes a NSCF over Y with that topology. Indeed, one can pull back any standard topology from $\mathbf{F}^*$ via $\kappa_{\mathbf{F}}$. Note that if $\mathbf{F}$ separates the points of Y this pull-back will be a Tychonoff topology. A specific choice of the topology on $\mathbf{F}^*$ leads to different properties of $\mathbf{F}$ as a NSCF, which can be deduced from Example 1.1.9.

1.2.c Regular NSF's and NSCF's

In this section we will consider a class of NSF's and a class of NSCF's which satisfy a certain maximality condition.

We will call a NSF $\mathbf{F}$ over a set X *regular* if $\overline{B}_{\mathbf{F}}$ is closed (and then automatically compact) in $\mathcal{F}(X)$. Consider an extension of $\|\cdot\|$ on the entire $\mathcal{F}(X)$ given by $\|f\| = +\infty$, for all $f \in \mathcal{F}(X) \backslash \mathbf{F}$. This extension represents the usual procedure of constructing function spaces, starting with a subadditive homogeneous functional from $\mathcal{F}(X)$ into $[0, +\infty]$ (see examples 1.2.13 and 1.2.14). It is easy to see that the extended $\|\cdot\|$ is a lower semi-continuous functional on $\mathcal{F}(X)$ if and only if $\mathbf{F}$ is regular. It is immediate from Proposition 0.2.8 that every regular NSF is a BSF, but regularity is a stronger condition, as we will see below.

Define the *regularization* $\widehat{\mathbf{F}}$ of a NSF $\mathbf{F}$ over a set X to be the linear subspace of $\mathcal{F}(X)$ endowed with the norm whose unit ball is $\overline{\overline{B}_{\mathbf{F}}}^{\mathcal{F}(X)} = \overline{B_{\mathbf{F}}}^{\mathcal{F}(X)}$. It follows from part (iii) of Proposition 1.1.3 that $\overline{B_{\mathbf{F}}}^{\mathcal{F}(X)} = J_{\mathbf{F}}^{**}\overline{B}_{\mathbf{F}^{**}}$, and so $\widehat{\mathbf{F}} = J_{\mathbf{F}}^{**}\mathbf{F}^{**}$ (as sets). Moreover, since $\mathbf{F}^{**}/\mathrm{Ker}\ J_{\mathbf{F}}^{**} = \mathbf{F}^{**}/X_{\mathbf{F}}^{\perp}$ is isometric to $(\mathrm{span}X_{\mathbf{F}})^{*}$ (see [Bou87, IV.1.4, Proposition 10]), we conclude that $\widehat{\mathbf{F}} = (\mathrm{span}X_{\mathbf{F}})^{*}$ as normed spaces. It is clear that $\mathbf{F}$ and $\widehat{\mathbf{F}}$ coincide (as NSF's) if and only if $\overline{B}_{\mathbf{F}} = \overline{B_{\mathbf{F}}}^{\mathcal{F}(X)}$, i.e. $\overline{B}_{\mathbf{F}}$ is pointwise closed. Thus, we can state the following result.

Proposition 1.2.8. *Let* $\mathbf{F}$ *be a NSF over a set* X*. Then* $\mathbf{F}$ *is regular if and only if* $\mathbf{F} = (\mathrm{span}X_{\mathbf{F}})^{*}$ *(as normed spaces).*

Analogously, we will introduce the concept of regularity for NSCF's. Let X be a topological space and let $\mathbf{F}$ be a NSCF over X. We will call $\mathbf{F}$ *weakly regular* if $\overline{B}_{\mathbf{F}}$ is closed in $\mathcal{C}(X)$, or equivalently if the extension of $\|\cdot\|$ on $\mathcal{C}(X)$ given by $\|f\| = +\infty$, for all $f \in \mathcal{C}(X) \backslash \mathbf{F}$, is a lower semi-continuous functional on $\mathcal{C}(X)$. It is immediate from Proposition 0.2.8 that every weakly regular NSCF over a compactly generated topological space is a BSCF. We will also call a NSCF *regular* if it is a regular NSF, i.e. $\overline{B}_{\mathbf{F}}$ is pointwise compact. Of course, weak regularity is equivalent to regularity in the case when X is a discrete topological space.

Let us clear up the relations between regularity and weak regularity. First, every regular NSCF is weakly regular (since every pointwise closed subset of $\mathcal{C}(X)$ is closed), as well as a regular NSF (by definition). Of course, not any regular NSF is a NSCF, because it does not necessarily consist of continuous functions. However, if it does, it is a regular NSCF, since it is a Banach space, and every complete NSF consisting of continuous functions is a BSCF. More precisely, the following characterizations holds.

Proposition 1.2.9. *Let X be a topological space and let $\mathbf{F}$ be a linear subspace of $\mathcal{C}(X)$ equipped with a norm. Then the following conditions are equivalent:*

(i) $\mathbf{F}$ *is a regular NSCF;*

(ii) $\mathbf{F}$ *is a weakly compactly embedded weakly regular NSCF;*

(iii) $\mathbf{F}$ *is a NSCF and* $\mathbf{F} = (\mathrm{span} X_{\mathbf{F}})^*$ *(as normed spaces).*

Proof. (i)⇔(iii) follows immediately from Proposition 1.2.8. (i)⇔(ii) follows from applying to $\overline{B}_{\mathbf{F}}$ the fact that in the class of bounded absolutely convex subsets of $\mathcal{C}(X)$ pointwise compactness coincides with weak compactness. □

In the next subsection we will see an example of a weakly regular NSCF, which is not regular. It is also worth mentioning that c_0 is a BSCF over $\mathbb{N}$ which is compactly embedded (since its phase space is discrete), but not weakly regular.

Let us go one step further and characterize reflexive NSF's and NSCF's. We start with the former.

Theorem 1.2.10. *Let X be a set and let $\mathbf{F}$ be a linear subspace of $\mathcal{F}(X)$ equipped with a norm. The following are equivalent:*

(i) $\mathbf{F}$ *is a reflexive NSF;*

(ii) $\mathbf{F}$ *is a regular NSF and* $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$*;*

(iii) $\mathbf{F}$ *is a regular NSF in any equivalent norm;*

(iv) Every closed subspace of $\mathbf{F}$ *is a regular NSF.*

Proof. (ii)⇔(i) follows from combining Proposition 1.2.8 with Proposition 1.2.4.

(i)+(ii)⇒(iii)+(iv): If $\mathbf{F}$ is a reflexive NSF, then it is a reflexive NSF in any equivalent norm (and so it is regular), and its every closed subspace is also reflexive (and so regular).

Let us now deal with (iii)⇒(i) and (iv)⇒(i). Both (iii) and (iv) imply that $\mathbf{F}$ is a regular NSF. Let $E = \overline{\mathrm{span} X_{\mathbf{F}}}$. From Proposition 1.2.8 we have $\mathbf{F} = E^*$.

(iii)⇒(i): If (iii) holds, then the closed unit ball of $\mathbf{F}$ is weak* closed in any equivalent norm. Thus, E is reflexive (see [FHH$^+$11, Exercise 3.120] with the solution therein). Hence, $\mathbf{F} = E^*$ is also reflexive.

(iv)⇒(i): If every closed subspace of $\mathbf{F}$ is a regular NSF, then E is a Banach space such that the closed unit ball of every closed subspace of E^* is weak* closed. But then due to Krein-Smulian theorem (see [BN11, Theorem 15.3.10]) every closed subspace of E^* is weak* closed. In particular, the kernel of any continuous functional on E^* is weak* closed, and so $E^{**} = E$, and so $\mathbf{F}$ is reflexive. □

Combining this result with Proposition 1.2.9 we are now able to characterise reflexive NSCF's.

Corollary 1.2.11. *A NSCF* $\mathbf{F}$ *over a topological space* X *is reflexive if and only if it is weakly compactly embedded and any (all) of the following conditions hold:*

(i) $\mathbf{F}$ *is weakly regular and* $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$*;*

(ii) $\mathbf{F}$ *is weakly regular in any equivalent norm;*

(iii) Every closed subspace of $\mathbf{F}$ *is weakly regular.*

Remark 1.2.12. Regular NSF's are complete Saks spaces (see [Coo87]).

1.2.d Examples of NSF's and NSCF's

In this subsection we consider two examples of NSF's, both of which are regular, and three examples of BSCF's, one of which is regular, one is weakly regular, but not regular, while one is not even weakly regular.

Example 1.2.13. Let X be a set and let $\beta : X \to [0, +\infty]$. Define a seminorm on $\mathcal{F}(X)$ in the following way:

$$\|f\|_\infty^\beta = \inf \{a \geq 0, |f(x)| \leq a\beta(x) \ \ \forall x \in X\}.$$

With some abuse of notations, $\|f\|_\infty^\beta = \|\frac{f}{\beta}\|_\infty^X$, where $\frac{0}{0}$ is counted as 0. Note that $\|\cdot\|_\infty^\beta$ is a norm if and only if β is finite. This functional defines a function space

$$\mathcal{F}_\infty^\beta = \left\{f \in \mathcal{F}(X), \|f\|_\infty^\beta < +\infty\right\}.$$

This space is called *Weighted Uniform space*. Note that the similar spaces in the continuous and holomorphic cases are the objects of an extensive study (see [BMS82], [SM93], [BV17]).

In the case when $\beta \equiv 1$ we will use a notation $\mathcal{F}_\infty(X) = \mathcal{F}_\infty^\beta$; for any finite β we have $\mathcal{F}_\infty^\beta = \beta\mathcal{F}_\infty(X)$. Clearly, the closed unit ball $\overline{B}_{\mathcal{F}_\infty^\beta} = \{f \in \mathcal{F}(X), |f(x)| \leq \beta(x)\ \forall x \in X\}$ is pointwise closed in $\mathcal{F}(X)$, and so $\|\cdot\|_\infty^\beta$ is pointwise lower semi-continuous on $\mathcal{F}(X)$.

It is easy to see that

$$|\kappa|_{\mathcal{F}_\infty^\beta} = \beta, \text{ and } d_{\mathcal{F}_\infty^\beta}(x,y) = \beta(x) + \beta(y),\ x \neq y.$$

Hence, $\mathcal{F}_\infty^\beta$ is a (regular) NSF if and only if β is finite, and is a BSF in this case. Note that $\mathcal{F}_\infty(X)$ is not reflexive, unless X is finite. Thus, we have an example of a non-reflexive regular NSF. □

The second example that we will consider is the Lipschitz space, which is itself a subject of a separate theory (see e.g. [Wea99], [Kal04]).

Example 1.2.14. We will need a concept from metric geometry. Let ρ and λ be pseudometrics on sets X and Y respectively. For a map $\varphi : X \to Y$ define the *dilation* of φ by

$$\mathrm{dil}_\lambda^\rho(\varphi) = \inf\{a \geq 0\,|\,\lambda(\varphi(x), \varphi(y)) \leq a\rho(x,y)\ \ \forall x,y \in X\}.$$

It is easy to see that $\mathrm{dil}_\lambda^\rho(\varphi) = 0$ if and only if the image of φ consists of the points which are not distinguishable with respect to λ. If Y is equipped with a topology and λ is continuous, then for any $x, y \in X$, the correspondence $\mathcal{F}_X(X,Y) \ni \varphi \to \lambda(\varphi(x), \varphi(y))$ is (pointwise) continuous. Hence, for each $a \in [0, +\infty)$, the set

$$\{\varphi \in \mathcal{F}(X,Y)\,|\mathrm{dil}_\lambda^\rho(\varphi) \leq a\} = \bigcap_{x,y\in X} \{\varphi \in \mathcal{F}(X,Y)\,|\,\lambda(\varphi(x), \varphi(y)) \leq a\rho(x,y)\}$$

is an intersection of closed sets and therefore, closed itself. Thus, $\mathrm{dil}_\lambda^\rho$ is a lower semi-continuous functional on $\mathcal{F}_X(X,Y)$.

In the case when $Y=\mathbb{F}$ we will denote the corresponding dilation by dil^{ρ}. The *Lipschitz space* over (X,ρ) is the space $\mathrm{Lip}^{\rho}=\{f\in\mathcal{F}(X)\,|\mathrm{dil}^{\rho}(f)<+\infty\}$. The dilation is a semi-norm on Lip^{ρ} and its kernel is the one dimensional subspace consisting of constant functions on X. One can show that $\|\cdot\|_{\rho}^{\{x\}}$ defined by $\|f\|_{\rho}^{\{x\}}=\mathrm{dil}^{\rho}f+|f(x)|$, $f\in\mathrm{Lip}^{\rho}$, is a norm on Lip^{ρ}, for any $x\in X$, and all such norms are equivalent. Moreover, Lip^{ρ} is regular with respect to each of these norms. Also, $d_{\mathrm{Lip}^{\rho}}=\rho$ and $|\kappa_{\mathrm{Lip}^{\rho}}|=\max\{1,\rho(\cdot,x)\}$. □

Let us characterise maps into $\mathcal{F}_{\infty}^{\beta}$ and Lip^{ρ}.

Proposition 1.2.15. *Assume that X is a set, let F be a normed space and let J be a linear map from F into $\mathcal{F}(X)$. Then:*

(i) J is $F\to\mathcal{F}_{\infty}^{\beta}$ continuous, for some $\beta:X\to[0,+\infty]$ if and only if $\||\kappa|_J\|_{\infty}^{\beta}<+\infty$. In this case $\|J\|_{\mathcal{L}\left(F,\mathcal{F}_{\infty}^{\beta}\right)}=\||\kappa|_J\|_{\infty}^{\beta}$. Moreover, J is (weakly) compact into $\mathcal{F}_{\infty}^{\beta}$ if and only if $\left\{\frac{x_J}{\beta(x)}\,|x\in X\right\}$ is (weakly) relatively compact in F^*.

(ii) Let ρ be a finite pseudometrics on X. Then J is $F\to\mathrm{Lip}^{\rho}$ continuous if and only if $\mathrm{dil}_{d_J}^{\rho}\kappa_J<+\infty$. Moreover, J is (weakly) compact into Lip^{ρ} if and only if the set $\left\{\frac{x_J-y_J}{\rho(x,y)}\,|x,y\in X,\ x\neq y\right\}$ is (weakly) relatively compact in F^*.

Proof. The first two claims in part (i) follows from the following chain of equivalences for for $a>0$:

$$\|J\|\le a\Leftrightarrow\sup_{f\in B_F}\|Jf\|_{\infty}^{\beta}\le a\Leftrightarrow|\kappa|_J=\sup_{f\in B_F}|Jf|\le a\beta\Leftrightarrow\||\kappa|_J\|_{\infty}^{\beta}\le a.$$

The last claim follows from versions of Schauder and Gantmacher theorems (see [Gro73, 2.18, Theorem 12 and Theorem 13]). Part (ii) is proven similarly. □

In fact, we have an isometry of spaces $\mathcal{L}\left(F,\mathcal{F}_\infty^\beta\right) \cong \mathcal{F}_\infty^\beta\left(X,F^*\right)$ and $\mathcal{L}\left(F,\mathrm{Lip}^\rho\right) \cong \mathrm{Lip}^\rho\left(X,F^*\right)$, where the right hand sides are the uniform weighted (or Lipschitz) space of F^*-valued functions (we omit the definition). Moreover, the corresponding spaces of the (weakly) compact operators is isometric to the corresponding spaces of vector valued functions with "weighted" (weakly) compact images. As a consequence we get a certain "universality" of $\mathcal{F}_\infty^\beta$ and Lip^ρ.

Corollary 1.2.16. *Let J be a linear map from a normed space F into $\mathcal{F}\left(X\right)$. Then J is a continuous map into $\mathcal{F}\left(X\right)$ if and only if there is a finite β such that $J\in\mathcal{L}\left(F,\mathcal{F}_\infty^\beta\right)$ and if and only if there is a finite pseudometric ρ such that $J\in\mathcal{L}\left(F,\mathrm{Lip}^\rho\right)$.*

Now we will consider examples of BSCF's.

Example 1.2.17. Let X be a non-discrete compactly generated Tychonoff space and let $\beta: X\to(0,+\infty]$ be lower semi-continuous. A *Continuous Weighted Uniform space* $\mathcal{C}_\infty^\beta$ is the linear subspace of $\mathcal{F}_\infty^\beta$ consisting of continuous functions, namely $\mathcal{C}_\infty^\beta=\mathcal{F}_\infty^\beta\cap\mathcal{C}\left(X\right)$, with the seminorm $\|\cdot\|_\infty^\beta$. It is easy to see that $\|\cdot\|_\infty^\beta$ is a norm if and only if β is finite on a dense set. In the case when $\beta\equiv1$ we will use a notation $\mathcal{C}_\infty\left(X\right)$. If $\beta:X\to[0,+\infty)$ is continuous then $\mathcal{C}_\infty^\beta=\beta\mathcal{C}_\infty\left(X\right)$.

It is possible to show that $\overline{B_{\mathcal{C}_\infty^\beta}}^{\mathcal{F}(X)}=\overline{B}_{\mathcal{F}_\infty^\beta}$, from where $|\kappa|_{\mathcal{C}_\infty^\beta}=\beta$ and $d_{\mathcal{C}_\infty^\beta}\left(x,y\right)=\beta\left(x\right)+\beta\left(y\right)$, for all distinct $x,y\in X$. The consequences of the latter three equalities are that $\mathcal{C}_\infty^\beta$ is a (weakly regular) NSCF if and only if β is bounded on compact subsets of X, but is not weakly compactly embedded. Hence, from Proposition 1.2.9 we conclude that $\mathcal{C}_\infty^\beta$ is not regular. $\square$

Consider an example of a BSCF which is not weakly regular.

Example 1.2.18. Assume that X is locally compact, but not compact. For a lower semi-continuous $\beta : X \to (0, +\infty)$ a *Little Weighted Uniform space* is the following closed subspace of $\mathcal{C}_\infty^\beta$:

$$\mathcal{C}_0^\beta = \left\{ f \in \mathcal{C}_\infty^\beta \,|\forall\varepsilon > 0 \;\exists K \in \mathcal{K}(X),\ \ |f(x)| \le \varepsilon\beta(x)\ \ \forall x \notin K \right\}.$$

With the same abuse of notations as in Example 1.2.13, $\mathcal{C}_0^\beta$ consists of continuous functions such that $\lim\limits_{\infty} \frac{f}{\beta} = 0$. It is easy to see that $\mathcal{C}_0^\beta$ contains all continuous scalar functions with compact support, and so $\mathcal{C}_0^\beta$ generates the topology of X. Again in the case when $\beta \equiv 1$ we will use the notation $\mathcal{C}_0(X)$ for $\mathcal{C}_0^\beta$. If β is continuous and does not vanish, then $\mathcal{C}_0^\beta = \beta\mathcal{C}_0(X)$.

One can show that $\mathcal{C}_0^\beta$ is a BSCF if and only if β is locally bounded. In this case $\overline{\overline{B}_{\mathcal{C}_0^\beta}}^{\mathcal{C}(X)} = \overline{B}_{\mathcal{C}_\infty^\beta}$, and consequently $\overline{\overline{B}_{\mathcal{C}_0^\beta}}^{\mathcal{F}(X)} = \overline{B}_{\mathcal{F}_\infty^\beta}$. Hence, $|\kappa|_{\mathcal{C}_0^\beta} = \beta$, $d_{\mathcal{C}_0^\beta}(x, y) = \beta(x) + \beta(y)$, for all distinct $x, y \in X$, and $\overline{B_{\mathcal{C}_0^\beta}}^{\mathcal{F}(X)} = \overline{B}_{\mathcal{F}_\infty^\beta}$.

If X is σ-locally compact, but not compact, one can construct a continuous function in $\overline{B}_{\mathcal{C}_\infty(X)} \backslash \mathcal{C}_0(X)$, and so $\overline{B}_{\mathcal{C}_\infty(X)} = \overline{\overline{B}_{\mathcal{C}_0(X)}}^{\mathcal{C}(X)} \not\subset \mathcal{C}_0(X)$. Hence, in this case $\mathcal{C}_0(X)$ is a BSCF, which is not weakly regular. □

Now consider an example of a compactly embedded NSCF – the Lipschitz space over a topological space.

Example 1.2.19. Let ρ be a continuous pseudo-metric on a topological space X. Then $d_{\mathrm{Lip}^\rho} = \rho$ is continuous, and so Lip^ρ is a regular compactly embedded BSCF, due to part (iii) of Theorem 1.1.8.

If the topology of X is determined by ρ, then Lip^ρ generates the topology of X. Indeed, for any closed $A \subset X$ we have $A = \rho(\cdot, A)^{-1}(0)$, and $\mathrm{dil}^\rho \rho(\cdot, A) \le 1$.

In many cases Lip^ρ is a bi-dual of a NSCF, as described in part (ii) of Corollary 1.2.7.

The role of $\mathbf{F}$ from that proposition is played by the Little Lipschitz space, which is defined analogously to the Little Uniform space (see [Wea99, Theorem 3.3.3], or [JVSVV17] and the references therein). $\square$

It follows that the family of spaces $\mathcal{C}_\infty^\beta$ has "universality" in the class of NSCF's, while the family of spaces Lip^ρ has "universality" in the class of compactly embedded NSCF's. More precisely, we have the following result.

Proposition 1.2.20. *Assume that X is a Tychonoff compactly generated topological space, let F be a normed space and let J be a linear map from F into $\mathcal{F}(X)$. Then:*

(i) J is a continuous map into $\mathcal{C}(X)$ if and only if there is a lower semi-continuous β, which is bounded on compacts in X, and such that $J \in \mathcal{L}\left(F, \mathcal{C}_\infty^\beta\right)$;

(ii) J is compact into $\mathcal{C}(X)$ if and only if there is a continuous pseudometric ρ, such that J is $F \to \mathrm{Lip}^\rho$ continuous.

Proof. (i): If J is a continuous map into $\mathcal{C}(X)$, then $\beta = |\kappa|_J$ is lower semi-continuous, and bounded on compacts due to part (i) of Theorem 1.1.8. Moreover, it is easy to see that J is of norm 1 as an operator from F into $\mathcal{C}_\infty^\beta$. If conversely, there is a lower semi-continuous β, which is bounded on compacts in X, and such that $J \in \mathcal{L}\left(F, \mathcal{C}_\infty^\beta\right)$, then $\mathcal{C}_\infty^\beta$ is a NSCF, and so it is continuously included into $\mathcal{C}(X)$. Hence, J is a continuous map into $\mathcal{C}(X)$.

Part (ii) is proven similarly. $\square$

Recall that if X is compact then $\mathbf{F} \subset \mathcal{C}(X)$ generates the topology of X if and only if it separates the points. It turns out that a similar result for non-compact locally compact spaces involves $\mathcal{C}_0(X)$. We will say that $\mathbf{F} \subset \mathcal{F}(X)$ *strongly separates the points* of X

if it separates the points of X and $0_{\mathbf{F}'} \notin X_{\mathbf{F}}$, i.e. for every $x \in X$ there is $f \in \mathbf{F}$ such that $f(x) \neq 0$. The following proposition can be viewed as a generalization of part (iii) of Corollary 1.1.7 in the case when X is locally compact Hausdorff, but not compact.

Proposition 1.2.21. *Assume that X is a non-compact Hausdorff space X, let F be a vector space and let J be a linear map from F into $\mathcal{F}(X)$. Then $JF \subset \mathcal{C}_0(X)$ if and only if κ_J is continuous and $\lim\limits_{x\to\infty} \kappa_J(x) = 0_{F'}$ with respect to the weak* topology. In this case, the following are equivalent:*

(i) JF strongly separates the points of X;

(ii) JF generates the topology of X;

(iii) X is locally compact, κ_J is a topological embedding into F' and $X_J \sqcup \{0_{F'}\}$ is the one point compactification of X with respect to the weak topology.*

Proof. The first claim follows from Proposition 1.1.6 and the definition of weak* topology. Let us prove the equivalences. Let $\widehat{\kappa_J}$ be the extension of κ_J on the one point compactification $\widehat{X}$ of X given by $\widehat{\kappa_J}(\infty_X) = 0_{F'}$. Since κ_J is continuous and $\lim\limits_{x\to\infty} \kappa_J(x) = 0_{F'}$ with respect to the weak* topology, it follows that $\widehat{\kappa_J}$ is weak* continuous. The condition (iii) is equivalent to the fact that $\widehat{\kappa_J}$ is a topological embedding, while according to part (ii) of Corollary 1.1.7, (ii) is equivalent to the fact that κ_J is a topological embedding. Therefore, (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i): If κ_J is a topological embedding, then it is an injection and so JF separates the points of X. Moreover, if there is $x \in X$ such that $\kappa_J(x) = 0_{F'}$, then $\kappa_J(X) = \widehat{\kappa_J}\left(\widehat{X}\right)$ is compact, and κ_J is a topological embedding onto a Hausdorff compact, which contradicts the assumption that X is non-compact.

(i)⇒(iii): Since JF separates the points of X, it follows that κ_J is an injection, and since $0_{F'} \notin X_J$ we get that $\widehat{\kappa_J}$ is also an injection. In fact, it is an injection from the compact space space $\widehat{X}$ into a Hausdorff space F', and so it is a topological embedding. Since then $\widehat{X}$ is Hausdorff it follows that X is locally compact (see [Eng89, Theorem 3.5.12]). $\square$

Corollary 1.2.22. *Assume that X is a non-compact Hausdorff space. Let $\mathbf{F}$ be a linear subspace of $\mathcal{C}_0(X)$ that strongly separates the points of X. Then $\mathbf{F}$ generates the topology of X and $X_{\mathbf{F}} \sqcup \{0_{\mathbf{F}'}\}$ is the one point compactification of X with respect to the weak* topology.*

Remark 1.2.23. If X is non-compact locally compact, and $\mathbf{F}$ is a NSCF over X such that $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = 0$, then $\mathbf{F} \subset \mathcal{C}_0(X)$, but the converse is wrong, as $|\kappa|_{\mathcal{C}_0(X)} \equiv 1$. $\square$

Remark 1.2.24. There is a lot of literature dedicated to studying spaces with a family of weights (see e.g. [SM93]); some families give rise to so called strict topologies (see e.g. [Sha71], [Coo87]). On the other hand, there is a vast literature dedicated to the spaces $\mathcal{C}(X)$ and $\mathcal{C}_0(X)$ (e.g. [Sem71], [DDLS16]). $\square$

1.2.e Relation Between the Properties of NSCF's and Their Phase Spaces

Topological properties of X put certain limitations on the topological properties of NSCF's over X. Part (i) of the following result appears in [Hil72] for the Hilbert space case.

Proposition 1.2.25. *Let $\mathbf{F}$ be a NSCF over X such that $\overline{\mathrm{span} X_{\mathbf{F}}} = \mathbf{F}^*$ (e.g. $\mathbf{F}$ is reflexive). Then both $\mathbf{F}^*$ and $\mathbf{F}$ are separable whenever one of the following conditions is satisfied:*

(i) X is separable.

(ii) X σ-locally compact and $\mathbf{F}$ is compactly embedded.

Proof. In the light of part (ii) of Proposition 0.2.5 we only need to show that $\mathbf{F}^*$ is separable, or equivalently that $\overline{B}_{\mathbf{F}}$ is weakly metrizable.

If X is separable, then $\mathbf{F}^* = \overline{\mathrm{span}X_{\mathbf{F}}} = \overline{\mathrm{span}_{\mathbb{Q}}X_{\mathbf{F}}} = \overline{\mathrm{span}_{\mathbb{Q}}Y_{\mathbf{F}}}$, where Y is a dense countable subset of X. Hence, $\mathrm{span}_{\mathbb{Q}}Y_{\mathbf{F}}$ is a dense countable subset of $\mathbf{F}^*$.

If $\mathbf{F}$ is compactly embedded and $\overline{\mathrm{span}X_{\mathbf{F}}} = \mathbf{F}^*$ the weak, pointwise and compact-open topology coincide on $\overline{B}_{\mathbf{F}}$ due to Corollary 1.2.7. Recall that if X is σ-locally compact, then $\mathcal{C}(X)$ is metrizable and so is the weak topology on $\overline{B}_{\mathbf{F}}$. $\square$

It is often possible to obtain results in the opposite direction, i.e. to derive topological properties of the phase space from the topological properties of a NSCF over it. The key feature is the following fact, which follows from part (ii) of Corollary 1.1.7.

Proposition 1.2.26. *If X is Hausdorff, then a NSCF $\mathbf{F}$ over X generates the topology of X if and only if $\kappa_{\mathbf{F}}$ is a topological embedding.*

Now we will deduce certain properties of a phase space based on the information about NSCF's it supports.

Proposition 1.2.27. *Let $\mathbf{F}$ be a NSCF over X, which generates its topology. Then:*

(i) If $\mathbf{F}$ is separable, then X is separable. If moreover, $|\kappa|_{\mathbf{F}}$ is bounded (e.g. $\mathbf{F}$ is a Banach space that consists of bounded functions), then X is also metrizable.

(ii) If X is compactly generated and $\mathbf{F}$ is a compactly embedded NSCF, then X is metrizable.

Proof. (i): Let $X^n = |\kappa|_{\mathbf{F}}^{-1}([0,n])$. Since $X^n_{\mathbf{F}} \subset n\overline{B}^*_{\mathbf{F}}$, it follows from part (i) of Proposition 0.2.5 that $X^n_{\mathbf{F}}$ is metrizable and separable in the weak* topology. Since $\mathbf{F}$ generates the topology of X, the map $\kappa_{\mathbf{F}}$ is a homeomorphism with the weak* topology. Hence, X^n is homeomorphic to a metrizable and separable topological space. Since $X = \bigcup_{n\in\mathbb{N}} X^n$, it follows that X is separable; if $|\kappa|_{\mathbf{F}}$ is bounded, then $X = X^n$, for some $n \in \mathbb{N}$, and so in this case X is X is separable and metrizable.

(ii): If X is compactly generated and $\mathbf{F}$ is a compactly embedded NSCF, which generates the topology of X, then $\kappa_{\mathbf{F}}$ is a topological embedding from X into the metric space $\mathbf{F}^*$. Hence, X is homeomorphic to a subset of a metric space, and so it is metrizable. $\square$

Recall that from Corollary 1.2.22 and Remark 1.2.23, if $\mathbf{F}$ is a NSCF over X, which strongly separates the points and such that $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = 0$, then $\mathbf{F}$ generates the topology of X, and X is locally compact (and also separable and metrizable if additionally $\mathbf{F}$ is separable). A similar phenomenon takes place when $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$.

Proposition 1.2.28. *Assume that X is compactly generated, and let $\mathbf{F}$ be a compactly embedded NSCF over X which separates the points of X and such that $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$. Then X is metrizable locally compact and $\mathbf{F}$ generates the topology of X. If additionally $\mathbf{F}$ is separable, then X is separable.*

Proof. Let F be a topological space, which is a one point extension of $\mathbf{F}^*$ defined as follows. The underlying set of F is $\mathbf{F}^* \sqcup \{\infty_{\mathbf{F}^*}\}$, and the open neighborhoods of $\infty_{\mathbf{F}^*}$ are of the form $F\backslash B$, where B is a closed bounded subset of $\mathbf{F}^*$. This construction is an analogue of the one point compactification. It is easy to see that F is indeed a topological space which is Hausdorff.

Since X is compactly generated and $\mathbf{F}$ is compactly embedded, it follows from part (iii) of Theorem 1.1.8 that $\kappa_{\mathbf{F}}$ is continuous, and so $|\kappa|_{\mathbf{F}}$ is continuous. The presence of a continuous function that tends to infinity immediately implies that X is locally compact. Moreover, since $\lim_{\infty} |\kappa|_{\mathbf{F}} = +\infty$ it follows that $\widehat{\kappa_{\mathbf{F}}}$ is continuous from the one point compactification $\widehat{X}$ of X into F, where $\widehat{\kappa_{\mathbf{F}}}$ is an extension of $\kappa_{\mathbf{F}}$ on $\widehat{X}$ given by $\widehat{\kappa_{\mathbf{F}}}(\infty_X) = \infty_{\mathbf{F}^*}$. Hence, $\widehat{\kappa_{\mathbf{F}}}$ is a continuous injection from a compact space into a Hausdorff space, and so it is a topological embedding. Thus, $\kappa_{\mathbf{F}}$ is a topological embedding, and so $\mathbf{F}$ generates the topology of X.

Now metrizability of X and the last claim follow from the Proposition 1.2.27. $\square$

One can ask if we can replace the compact embeddedness of $\mathbf{F}$ with mere continuity of $|\kappa|_{\mathbf{F}}$. A partial answer to this question see in Proposition 2.1.22.

1.2.f Reproducing Kernel Hilbert Spaces

This section is dedicated to the most studied general family of NSF's – RKHS's (see [Aro50], [Sai97], [PS72], [ZXZ09]). We need some preliminary material.

Let us begin with recalling some basic information about inner product spaces. Let H be such space and let $h \in H$. By h^* we will denote the continuous linear functional on H that acts by $\langle g, h^* \rangle = \langle g, h \rangle_H$, for $g \in H$. The correspondence $*$ is an antilinear isometry from H into its dual H^*. We can define an inner product on H^* as the continuous extension of the push-forward $\langle g^*, h^* \rangle_{H^*} = \langle h, g \rangle_H$. This inner product agrees with the natural norm on H^*. If we further assume the completeness of H, i.e. that H is a Hilbert space, then the Riesz theorem ensures that $*$ is a bijection, and so it has an inverse which is also an antilinear isometry. Let ${}^*\nu$ be the element of H, such that $({}^*\nu)^* = \nu$, for $\nu \in H^*$. The established operations have the following relations. Let $f, g \in H$ and $\nu, \mu \in H^*$. Then:

- ${}^*(f^*) = f$, since the two operations are inverses of each other;

- $\langle f, g\rangle_H = \langle f, g^*\rangle = \overline{\langle f^*, g\rangle} = \overline{\langle f^*, g^*\rangle_{H^*}}$;

- $\langle \nu, \mu\rangle_{H^*} = \langle \nu,^* \mu\rangle = \overline{\langle {}^*\nu, \mu\rangle} = \overline{\langle {}^*\nu,^* \mu\rangle_H}$;

- $\langle f, \nu\rangle = \langle f,^* \nu\rangle_H = \overline{\langle f^*, \nu\rangle_{H^*}}$;

- If we identify H^{**} with H, then $f^{**} = f$, since $\langle \nu, f^{**}\rangle = \langle \nu, f^*\rangle_{H^*} = \langle \nu, f\rangle$.

When considering Hilbert function spaces we will often deal with (abstract) functions of two variables. Let us introduce a notation for the simplest such function. If X is a set and $f, g : X \to \mathbb{F}$, define $f \otimes g : X \times X \to \mathbb{F}$ by $[f \otimes g](x, y) = f(x) g(y)$. On the other hand, if $k : X \times X \to \mathbb{F}$, define the *diagonal* function $\widehat{k} : X \to \mathbb{F}$ by $\widehat{k}(x) = k(x, x)$.

The functions of two variables of the most relevance in this subsection are the positive (semi) definite kernels. The set of positive semi-definite functions is a closed cone in $\mathcal{F}(X \times X)$. This cone generates a partial order on $\mathcal{F}(X \times X)$. Namely, if $k, l : X \times X \to \mathbb{F}$, then $k \ll l$ or $l \gg k$ if $l - k$ is positive semi-definite. In particular, the fact that $k : X \times X \to \mathbb{F}$ is positive semi-definite will be denoted by $k \gg 0$. Now consider the simplest example of such a function.

Example 1.2.29. If $w : X \to \mathbb{F}$, then $k = w \otimes \overline{w} \gg 0$. Indeed, for any $x_1, ..., x_n \in X$ the matrix $(k(x_i, x_j))_{i,j=1}^n$ is a product of $\overrightarrow{w} = (w(x_1), ..., w(x_n))$ and $\overrightarrow{w}^*$. Hence

$$\overrightarrow{\mu}\left(k^J(x_i, x_j)\right)_{i,j=1}^n \overrightarrow{\mu}^* = \overrightarrow{\mu}\,\overrightarrow{w}\,\overrightarrow{w}^*\overrightarrow{\mu}^* = |\mu_1 w(x_1) + ... + \mu_n w(x_n)|^2 \geq 0,$$

for any row $\overrightarrow{\mu} = (\mu_1, ..., \mu_n) \in \mathbb{F}^n$. $\square$

We will now finally introduce the titular object of the section. Let $\mathbf{H}$ be a linear subspace of the space of $\mathcal{F}(X)$, equipped with an inner product, turning $\mathbf{H}$ into a Hilbert space.

Then, $\mathbf{H}$ is called a *Reproducing Kernel Hilbert Space (RKHS)* over X if there is a function (*reproducing kernel*) $k : X \times X \to \mathbb{F}$, such that $k_x = k(\cdot, x) \in \mathbf{H}$, for each $x \in X$, and $f(x) = \langle f, k_x \rangle_{\mathbf{H}}$, for each $f \in \mathbf{H}$. Note that the point evaluations on $\mathbf{H}$ are implemented by the scalar product with its elements, and so they are continuous functionals. Hence, $\mathbf{H}$ is a NSF over X. In fact RKHS's are precisely the Hilbert NSF's. Indeed, if $\mathbf{H}$ is a Hilbert NSF's over X, then $k^{\mathbf{H}} = k^{J_{\mathbf{H}}}$ (see) is its reproducing kernel, since $k^{\mathbf{H}}_x = k^{\mathbf{H}}(\cdot, x) =^* x_{\mathbf{H}} \in \mathbf{H}$, and so $\left\langle f, k^{\mathbf{H}}_x \right\rangle_{\mathbf{H}} = \langle f, x_{\mathbf{H}} \rangle = f(x)$.

We will call a Hilbert NSF a *Hilbert Space of Functions* (HSF). We will use this term interchangeably with the term "RKHS", although for the sake of tradition the latter will be used more often. The RKHS's are the special case of the configuration considered in Section 1.2.a. However, Hilbert spaces are remarkable not just as a nice type of Banach spaces, but as a different, richer structure, and so it is reasonable to expect an additional object associated with the RKHS structure, that would resemble the scalar product. This object happens to be the reproducing kernel. We have already seen that a reproducing kernel of a RKHS has to be positive semi-definite (and more generally a kernel of an embedding of an inner product space, see Section 1.1), but it turns out that there is a converse to this fact (see [Aro50]).

Theorem 1.2.30 (Moore-Aronszajn)**.** *Let $k : X \times X \to \mathbb{F}$. Then k is positive semi-definite if and only if there exists a (unique) RKHS over X, having reproducing kernel equal to k.*

Example 1.2.31. Let $w : X \to \mathbb{F}$ and let $k = w \otimes \overline{w} \gg 0$. It is easy to see that $\mathbb{F}w$ endowed with the norm $\|\alpha w\| = |\alpha|$ is a RKHS with the kernel k, and so $\mathbf{H}_k = \mathbb{F}w$ is one-dimensional. On the other hand, if $\mathbf{H}$ is one-dimensional, then $\kappa_{\mathbf{H}}$ maps X into a one-dimensional Hilbert space $\mathbf{H}^*$, and so $\mathbf{H} = \mathbf{H}_{w\otimes\overline{w}}$, where w is $\kappa_{\mathbf{H}}$ considered as a map from X into $\mathbb{F}$. In particular, if k_x is a constant function, for every $x \in X$, we get that $k(y,z) = k(x,z) = \overline{k(z,x)} = \overline{k(x,x)}$, for every $x, y, z \in X$, and so k is a constant, which leads to $\mathbf{H}_k$ being the space of constant functions. $\square$

It is also possible to express some relations between RKHS in terms of their kernels.

Proposition 1.2.32. [Aro50] *Let k, l be positive semi-definite kernels on X. Then $\mathbf{H}_k \subset \mathbf{H}_l$ if and only if there is $\alpha > 0$ such that $k \ll \alpha^2 l$. In this case the inclusion is a bounded operator from $\mathbf{H}_k$ into $\mathbf{H}_l$ with norm at most α.*

Corollary 1.2.33. *Let H be an inner product space, let $J : H \to \mathcal{F}(X)$ be continuous and let k be a positive semi-definite kernel on X. Then J is a bounded operator from H into $\mathbf{H}_k$ with norm at most α if and only if $k^J \ll \alpha^2 k$. It is a linear homeomorphism with the norm of J^{-1} as a map from $\mathbf{H}_k$ onto F at most $\frac{1}{\beta}$ if and only if J is injective and $k^J \gg \beta^2 k$. In particular, J is unitary if and only if it is an injection with $k^J = k$.*

Let $f : X \to \mathbb{F}$ and $k = f \otimes \overline{f}$. Due to Example 1.2.31 we have $\mathbf{H}_k = \mathbb{F}f$ with $\|f\| = 1$. Hence, $\|f\|_l \leq r$, for some $l \gg 0$, if and only if $\mathbf{H}_k$ is included into $\mathbf{H}_l$ with the norm of inclusion being at most r. Thus we obtain a description of the norm and the unit ball of a RKHS solely in terms of its kernel.

Corollary 1.2.34. *Let $\mathbf{H}$ be a RKHS over X. If $f : X \to \mathbb{F}$, then $f \in \mathbf{H}$ if and only if there is $r > 0$ such that $f \otimes \overline{f} \ll r^2 k^{\mathbf{H}}$. In this case the $\|f\| \leq r$. Furthermore, $\overline{B}_{\mathbf{H}} = \left\{ f \in \mathcal{F}(X) \,\middle|\, f \otimes \overline{f} \ll k^{\mathbf{H}} \right\}$.*

Since positive semi-definite kernels are in one-to-one correspondence with RKHS's, it is natural to expect certain properties of kernels to be equivalent to certain properties of their spaces. Let X be a topological space. The following theorem establishes a connection between the mode of continuity of a positive semi-definite kernel on X with the properties of its RKHS as a NSCF.

Theorem 1.2.35. *[*[BTA04], [CDVT06], [FM13], [Sza04], [Sch64]*] Let* $\mathbf{H}$ *be a RKHS over topological space* X*.*

(i) If $\mathbf{H} \subset \mathcal{C}(X)$*, then:*

- $\mathbf{H}$ *is a regular NSCF over* X*, i.e. the inclusion map* $J_{\mathbf{H}} : \mathbf{F} \to \mathcal{C}(X)$ *is continuous;*
- *the kernel* $k^{\mathbf{H}}$ *is separately continuous and bounded on compacts;*
- *for every* $x \in X$ *and every orthonormal basis* $\{e_i\}_{i \in I}$ *of* $\mathbf{H}$ *we have* $k^{\mathbf{H}}(\cdot, x) = \sum\limits_{i \in I} \overline{e_i(x)} e_i$*, where the series converges in* $\mathcal{C}(X)$*, as well as in* $\mathbf{H}$*.*

(ii) If X *is compactly generated, then* $\mathbf{H} \subset \mathcal{C}(X)$ *once one of the following is satisfied:*

- *the kernel* $k^{\mathbf{H}}$ *is separately continuous and bounded on compacts;*
- *there is a collection* $\{e_i\}_{i \in I} \subset \mathcal{C}(X)$ *which is is an orthonormal basis of* $\mathbf{H}$*, such that* $\sum\limits_{i \in I} |e_i|^2$ *is bounded on compacts.*

(iii) If $k^{\mathbf{H}}$ *is separately continuous at the points of the diagonal and continuous on the diagonal, then* $\mathbf{H}$ *is a compactly embedded NSCF, and for any orthonormal basis* $\{e_i\}_{i \in I}$ *we have* $k^{\mathbf{H}} = \sum\limits_{i \in I} e_i \otimes \overline{e_i}$*, where the series converges in* $\mathcal{C}(X \times X)$*.*

(iv) If X *is compactly generated the following are equivalent:*

- $\mathbf{H}$ *is a compactly embedded NSCF;*

- *$k^{\mathbf{H}}$ is continuous on $X \times X$;*

- *$k^{\mathbf{H}}$ is separately continuous at the points of the diagonal and continuous on the diagonal;*

- *there is a collection $\{e_i\}_{i\in I} \subset \mathcal{C}(X)$ such that $k^{\mathbf{H}} = \sum\limits_{i\in I} e_i \otimes \overline{e_i}$, where the sum converges in $\mathcal{C}(X \times X)$;*

- *there is a collection $\{e_i\}_{i\in I} \subset \mathcal{C}(X)$ which is an orthonormal basis of $\mathbf{H}$, such that $\sum\limits_{i\in I} |e_i|^2$ is continuous.*

We will call a Hilbert NSCF a *Hilbert Space of Continuous Functions* (HSCF). Combining the previous theorem with Moore-Aronszajn Theorem, we get a refinement of the latter.

Proposition 1.2.36. *Assume that X is compactly generated and let $k : X \times X \to \mathbb{F}$. Then there is a HSCF over X with the reproducing kernel equal to k if and only if k is positive semi-definite, separately continuous and bounded on the compacts. There is a compactly embedded HSCF over X with the reproducing kernel equal to k if and only if k is positive semi-definite, separately continuous at the points of the diagonal and continuous on the diagonal.*

1.3 Normed Spaces of Holomorphic Functions

This section is dedicated to Normed Spaces of Holomorphic Functions (NSHF's), which form a special class of NSCF's. Of course, in order to discuss holomorphic functions our phase space has to belong to an appropriate category. Let X be a complex manifold and let $\mathcal{H}(X)$ be the linear space of all holomorphic functions on X viewed as a subspace of $\mathcal{C}(X)$. We will call a NSCF $\mathbf{F}$ over X a normed space of holomorphic functions if $\mathbf{F} \subset \mathcal{H}(X)$. Banach and Hilbert Spaces of Holomorphic Functions (BSHF's and HSHF's) are defined similarly. In order to study NSHF's we need to recall some Complex Analysis.

1.3.a Background Complex Analysis

In this subsection X is a complex manifold of dimension $n \in \mathbb{N}$. Certainly, X is a locally compact Hausdorff space. We will state some topological properties of holomorphic functions on X and some properties of the space $\mathcal{H}(X)$. Recall that the latter is a closed subspace of $\mathcal{C}(X)$ due to Weierstrass theorem, while by Montel's theorem a subset of $\mathcal{H}(X)$ is relatively compact if and only if it is bounded (the proofs of these two theorems in [Sch05, 1.4] carry over to complex manifolds).

The root of numerous results that we will use when studying NSHF's is the Open Mapping theorem which states that a non-constant holomorphic function is an open map from X into $\mathbb{C}$ (see [Sch05, Conclusion 1.2.12]). One can easily deduce from this fact the Maximum Principles, which state that if $U \subset X$ is an open relatively compact set and $f \in \mathcal{H}(X)$, then $\max\limits_{\overline{U}} \operatorname{Re} f = \max\limits_{\partial U} \operatorname{Re} f$, $\min\limits_{\overline{U}} \operatorname{Re} f = \min\limits_{\partial U} \operatorname{Re} f$ and $\max\limits_{\overline{U}} |f| = \max\limits_{\partial U} |f|$. Moreover, existence of either one of $\max\limits_{U} \operatorname{Re} f$, $\min\limits_{U} \operatorname{Re} f$ and $\max\limits_{U} |f|$ implies that f is a constant function.

Let us state some consequences of these facts. We will call $A \subset X$ *thin*, if it is contained in a zero-set of a non-zero holomorphic function on X. If $\dim X = 1$, then A is thin if and only if A is discrete (necessity – Uniqueness theorem, see also [Chi89, 1.2.2, Proposition 2]; sufficiency – Weierstrass theorem, see [For91, Theorem 26.7]). There is no analogous description of thin sets if $\dim X > 1$, but it follows immediately from the Open Mapping theorem, that thin sets are always nowhere dense. Another immediate consequence of this theorem is that if X is compact (complex manifold without a boundary), then $\mathcal{H}(X)$ consists of constant functions. Clearly, the latter fact is very specific for the compact complex manifolds, but there is an analogous result in the general case (the proof is based on the Maximum Modulus Principle).

Proposition 1.3.1. *If X is non-compact then $\mathcal{C}_0(X) \cap \mathcal{H}(X) = \{0\}$.*

Since there are very few holomorphic functions on compact complex manifolds we will exclude them from our consideration. On the other hand we will assume that X is second countable (note that this follows automatically if $\dim X = 1$ by Rado's theorem; see [For91, Theorem 23.3]). It is clear that second countable manifolds are separable and σ-locally compact, but it follows from Metrization theorem ([Eng89, 4.4.7]), that they are also metrizable. In fact, the reader may gain sufficient intuition from assuming that X is a *domain*, i.e. an open connected set in $\mathbb{C}^n$. Recall that since X is σ-locally compact $\mathcal{C}(X)$ is a Frechet space, and so in this case $\mathcal{H}(X)$ is a Frechet-Montel space. Such spaces are reflexive (see [Bou87, IV.2.5, before Proposition 9]).

Analogously to $\mathcal{F}(X)$ and $\mathcal{C}(X)$, if X is clear from the context we will denote the evaluation map $\kappa_{\mathcal{H}(X)}$ simply by $\kappa_{\mathcal{H}}$, and if $A \subset X$ we will denote $\kappa_{\mathcal{H}}(A)$ by $A_{\mathcal{H}}$. Combining the Maximum principle with Hahn-Banach theorem (see [Bou87, II.5.3, Corollary 1]) it follows that $U_{\mathcal{H}} \subset \overline{\mathrm{conv}\,(\partial U)_{\mathcal{H}}}$, for any open relatively compact $U \subset X$.

The rest of the section is dedicated to tangent vectors, which can be viewed as both geometric objects and as elements of $\mathcal{H}(X)^*$. The corresponding derivative point evaluations on NSHF's considered later in this section will be extensively used in Section 2.4. For $x \in X$ we will denote by $\mathcal{T}_x$ the holomorphic tangent space to X at x, i.e. $\mathcal{T}_x = T^{10}_x X$ in the standard notations from Complex Geometry; we will also denote $\overline{\mathcal{T}}_x = T^{01}_x X$, which is the antiholomorphic tangent space to X at x. If $\xi = \sum\limits_{i=1}^{n} a_i \left.\frac{\partial}{\partial z_i}\right|_x$ and $\zeta = \sum\limits_{j=1}^{n} b_j \left.\frac{\partial}{\partial z_i}\right|_x$ in some local holomorphic coordinates at x, denote $\overline{\zeta} = \sum\limits_{j=1}^{n} \overline{b_j} \left.\frac{\partial}{\partial \overline{z_j}}\right|_x \in \overline{\mathcal{T}}_x$, which is a Wirtinger derivative (see [Sch05, 1.2.26]), and

$$\xi \otimes \overline{\zeta} = \sum_{i,j=1}^{n} a_i \overline{b_j} \left.\frac{\partial^2}{\partial z_i \partial \overline{z_j}}\right|_x$$

which can be interpreted as a "directional Laplacian".

Remark 1.3.2. Let us justify the notation $\xi \otimes \overline{\zeta}$. We will use some concepts from differential geometry developed in e.g. [War83, pp. 11-14] for the real manifolds, but carry over to the complex manifolds. Let F_x ($\overline{F}_x$) be the algebra of (anti-) holomorphic germs at x, which vanish at x. If F_x^m (or $\overline{F}_x^m$) is the ideal spanned by the m-th power of F_x (or $\overline{F}_x$), for $m \in \mathbb{N}$, then the co-tangent space $\mathcal{T}_x^*$ can be identified with F_x/F_x^2; also $\overline{\mathcal{T}}_x^* = \overline{F_x}/\overline{F_x}^2$. Clearly, $\xi \otimes \overline{\zeta}$ acts bilinearly on $F_x \times \overline{F_x}$, and so we have to show that it is well defined as a functional on $\mathcal{T}_x^* \times \overline{\mathcal{T}_x}^*$, i.e. that it vanishes on $F_x^2 \times \overline{F_x}$ and $F_x \times \overline{F_x}^2$.

In fact, from bilinearity, it is sufficient to check that $\left.\frac{\partial}{\partial z_i}\right|_x \otimes \left.\frac{\partial}{\partial \overline{z_j}}\right|_x$ is well-defined, for $i, j \in \overline{1, n}$, and so we need to confirm that $\left[\left.\frac{\partial}{\partial z_i}\right|_x \otimes \left.\frac{\partial}{\partial \overline{z_j}}\right|_x\right] fg\overline{h} = \left[\left.\frac{\partial}{\partial z_i}\right|_x \otimes \left.\frac{\partial}{\partial \overline{z_j}}\right|_x\right] \overline{fg}h = 0$, where f, g, h are holomorphic functions that vanish at x.

Indeed, if u is holomorphic and v is anti-holomorphic, then $\frac{\partial^2}{\partial z_i \partial \overline{z_j}} uv = \frac{\partial}{\partial z_i} u \frac{\partial}{\partial \overline{z_j}} v$, because both $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \overline{z_j}}$ are derivations and $\frac{\partial}{\partial z_i} v = \frac{\partial}{\partial \overline{z_j}} u = 0$. Thus,

$$\left.\frac{\partial^2}{\partial z_i \partial \overline{z_j}}\right|_x fg\overline{h} = \left.\frac{\partial}{\partial z_i}\right|_x fg \left.\frac{\partial}{\partial \overline{z_j}}\right|_x \overline{h} = \left.\frac{\partial}{\partial \overline{z_j}}\right|_x \overline{h} \left[f(x) \left.\frac{\partial}{\partial z_i}\right|_x g + g(x) \left.\frac{\partial}{\partial z_i}\right|_x f\right] = 0,$$

and similarly $\left[\left.\frac{\partial}{\partial z_i}\right|_x \otimes \left.\frac{\partial}{\partial \overline{z_j}}\right|_x\right] \overline{fg}h = 0$.

The same argument shows that if ξ and ζ are differential expressions of a higher order (evaluated at x), then their "composition" is a tensor product. Note that this is not in fact a "true" composition, since after we evaluate ξ on f we get a number, and ξ is not a priori a value of any canonical vector field, so there is nothing to apply $\overline{\zeta}$ to. $\square$

Using the criterion of holomorphicity (see [GE04]) one can show that the evaluation map $\kappa_{\mathcal{H}} : X \to \mathcal{H}(X)^*$ is holomorphic. Furthermore, we get that $\xi f = \langle f, \xi\kappa_{\mathcal{H}}\rangle$, for any $x \in X$, $\xi \in \mathcal{T}_x$ and $f \in \mathcal{H}(X)$. Here $\xi\kappa_{\mathcal{H}}$ is the limit of elements of $\mathcal{H}(X)^*$, analogous to the limit of scalars, when ξ is applied to usual functions. For example, if $X = \mathbb{C}$, $x = 0$ and $\xi = \frac{d}{dz}\big|_0$, then $\xi\kappa_{\mathcal{H}} = \lim_{z\to 0} \frac{\kappa_{\mathcal{H}}(z)-\kappa_{\mathcal{H}}(0)}{z}$. On the other hand, consider the differential $\mathcal{T}_x\kappa_{\mathcal{H}}$ of $\kappa_{\mathcal{H}}$ at x, which is a linear map from $\mathcal{T}_x$ into $\mathcal{T}_{x_{\mathcal{H}}}\mathcal{H}(X)^*$ identified with $\mathcal{H}(X)^*$. For every $f \in \mathcal{H}(X)$ let f^{**} be the corresponding element of $\mathcal{H}(X)^{**}$. Then f^{**} is a linear, and so holomorphic function on $\mathcal{H}(X)^*$. Also, in order to recover f from f^{**} one can use $f = f^{**} \circ \kappa_{\mathcal{H}}$. Then, from the definition of a differential, for every $\xi \in \mathcal{T}_x$ we have

$$([\mathcal{T}_x\kappa_{\mathcal{H}}]\,\xi)\, f^{**} = \xi\,(f^{**} \circ \kappa_{\mathcal{H}}) = \xi f = \langle f, \xi\kappa_{\mathcal{H}}\rangle\,.$$

Hence, $\xi\kappa_{\mathcal{H}}$ can be identified with the tangent vector $[\mathcal{T}_x\kappa_{\mathcal{H}}]\,\xi$, analogously to the way a differential of a map between Euclidean domains is identified with the Jacobi matrix.

Note that the same fact holds for the higher order differential expressions.

1.3.b Maps into Spaces of Holomorphic Functions

Let X be a second countable complex manifold. Let us start with discussing maps into $\mathcal{H}(X)$ in the spirit of Section 1.1. If F is a normed space and J is a linear continuous (and automatically compact due to Montel's theorem) map from F into $\mathcal{H}(X)$, then $\kappa_J = J^* \circ \kappa_{\mathcal{H}}$ is a holomorphic map from X into F^*. Then $\xi\,(Jf) = \langle f, \xi\kappa_J\rangle$, and $\xi\kappa_J = [\mathcal{T}_x\kappa_J]\,\xi$, for any $x \in X$, $\xi \in \mathcal{T}_x$ and $f \in F$. We will call such functionals *derivative point evaluations* with respect to J. It follows from properties of $\mathcal{H}(X)$ and from the continuity of J^* that if $U \subset X$ is relatively compact, then $U_J \subset \overline{\operatorname{conv}(\partial U)_J}$. Also, since $\mathcal{H}(X)$ is a Frechet-Montel space, the analogue of Proposition 1.1.3 holds for J.

On the other hand, if $\kappa : X \to F^*$ is (weakly) holomorphic, then J_κ is a continuous map from F into $\mathcal{C}(X)$, whose image is contained in $\mathcal{H}(X)$: indeed, Jf is a composition of a holomorphic map κ and a linear map f (viewed as a function on F^*), for every $f \in F$.

Assume again that J is a linear map from F into $\mathcal{H}(X)$. For every $f \in F$ the function $|Jf|$ is pluri-log-subharmonic (i.e. $\log|Jf|$ is pluriharmonic, see [Rud80, Proposition 1.5.4]), and by the Maximum Principle, $\|Jf\|_\infty^{\overline{U}} = \|Jf\|_\infty^{\partial U}$, for any open relatively compact $U \subset X$. Hence, $|\kappa|_J$ is pluri-log-subharmonic as a supremum of pluri-log-harmonic functions with $\| |\kappa|_J \|_\infty^{\overline{U}} = \| |\kappa|_J \|_\infty^{\partial U}$. The continuity of J is equivalent to several conditions.

Proposition 1.3.3. *Let F be a normed space and let $J : F \to \mathcal{F}(X)$ be linear. Then the following conditions are equivalent:*

(i) J is $F \to \mathcal{H}(X)$ continuous (compact);

(ii) κ_J is holomorphic from X into F^*;

(iii) $JF \subset \mathcal{H}(X)$ and $|\kappa|_J$ is continuous;

(iv) $JF \subset \mathcal{H}(X)$ and for any x there is an open relatively compact $U \subset X$, containing x and such that $|\kappa|_J$ is bounded on ∂U;

(v) $JF \subset \mathcal{H}(X)$ and there is a closed set $A \subset X$ such that the closure of any component of $X \backslash A$ is compact, and $|\kappa|_J$ is locally bounded at the points of A.

(vi) There is $D \subset F$ such that $\overline{\operatorname{span} D} = F$, $JD \subset \mathcal{H}(X)$ and $|\kappa|_J$ is bounded on compacts.

Proof. The equivalence of (i) and (ii) was discussed above.

It is clear that (i) is equivalent to the fact that $JF \subset \mathcal{H}(X)$ and J is $F \to \mathcal{C}(X)$ continuous (compact). The latter means that $|\kappa|_J$ is bounded on compacts, due to part (i) of Theorem 1.1.8. From this we get (iii)$\Leftrightarrow$(i)$\Rightarrow$(iv),(v),(vi); the proof of (vi)$\Rightarrow$(i) is analogous to that of Proposition 1.1.14. (iv)$\Rightarrow$(i) follows from the fact that $\| \, |\kappa|_J \, \|_\infty^{\overline{U}} = \| \, |\kappa|_J \, \|_\infty^{\partial U}$, for any open relatively compact $U \subset X$.

(v)$\Rightarrow$(iv): Let $x \in X$. If $x \in A$, then $|\kappa|_J$ is locally bounded at x, and so there is an open relatively compact neighborhood U of x, such that $|\kappa|_J$ is bounded on ∂U. Otherwise, x belongs to U, which is one of the (relatively compact open) components of $X \backslash A$. It is easy to see that $\partial U \subset A$, and so $|\kappa|_J$ is locally bounded at the points of ∂U, and since the latter is compact, it follows that $|\kappa|_J$ is bounded on ∂U. $\square$

Remark 1.3.4. Not every continuous log-subharmonic function is of the form $|\kappa|_J$, for some normed space F and a continuous linear map $J : F \to \mathcal{H}(X)$; see Example 1.3.9. $\square$

Remark 1.3.5. The condition on A is equivalent to the fact that there is no $x \in X$ and continuous path in $\widehat{X}$ that joins x with ∞_X without crossing A. $\square$

Analogously to (iv)$\Rightarrow$(i) one can prove the following result.

Corollary 1.3.6. *Let F be a normed space and let $J : F \to \mathcal{H}(X)$ be linear and such that there is $A \subset X$ such that the closure of any component of $X \backslash \overline{A}$ is compact, and $|\kappa|_J$ is bounded on A. Then J is continuous.*

Remark 1.3.7. It is easy to see that the condition is not necessary. On the other hand it is not clear whether the condition that A is closed and $|\kappa|_J$ is locally bounded on A is sufficient. $\square$

We have the following equivalence in the case when F is a Banach space (the proof is similar to that of part (ii) of Theorem 1.1.10; this is a generalization of a result from [BV17]).

Proposition 1.3.8. *Let F be a Banach space, let $J : F \to \mathcal{H}(X)$ be such that $A_J \subset F^*$, for a non-thin $A \subset X$. Then J is continuous.*

Now we are equipped to deal with NSHF's. It follows from Proposition 1.3.3 that a NSF $\mathbf{F}$ over X is a NSHF if and only if $\kappa_{\mathbf{F}}$ is holomorphic. Moreover, if $\mathbf{F} \subset \mathcal{H}(X)$ is a BSF over X, then it is a BSHF. In fact, Proposition 1.3.8 gives a weaker sufficient condition for a Banach space to be a BSHF than being a NSF (similar to Remark 1.2.5). Of course, every NSHF is a compactly embedded NSCF, and so from part (i) of Proposition 1.2.6 the weak topology is stronger than the compact-open topology on bounded sets.

We will call a NSHF $\mathbf{F}$ over X *regular*, if $\overline{B}_{\mathbf{F}}$ is closed in $\mathcal{H}(X)$ (and then automatically compact), which is equivalent to the fact that $\|\cdot\|_{\mathbf{F}}$, extended by $+\infty$ on $\mathcal{H}(X)\backslash\mathbf{F}$ is lower semi-continuous. It is easy to see that $\mathbf{F}$ is regular if and only if it is weakly regular as a NSCF and regular as a NSF. Moreover, from Proposition 1.2.9, $\mathbf{F}$ is regular, if and only if $\mathbf{F} = (\mathrm{span}X_{\mathbf{F}})^*$ (as normed spaces), and then the pointwise and the compact-open topologies coincide on $\overline{B}_{\mathbf{F}}$. On the other hand from part (ii) of Proposition 1.2.7 it follows that $\mathbf{F}^{**}$ is realized as a (regular) BSHF over X via $J^{**}_{\mathbf{F}}$ if and only if $\overline{\mathrm{span}X_{\mathbf{F}}} = \mathbf{F}^*$ and if and only if the weak, pointwise and compact-open topologies coincide on bounded sets. In this case $\overline{B_{\mathbf{F}}}^{\mathcal{H}(X)} = J^{**}_{\mathbf{F}}\overline{B}_{\mathbf{F}^{**}}$, and also $\mathbf{F}$ is separable, due to Proposition 1.2.25. Finally, it follows that a NSHF is reflexive if and only if it is regular and the weak, pointwise and compact-open topologies coincide on bounded sets.

If $A \subset X$ is non-thin, then $A^{\perp}_{\mathbf{F}} = \{0\}$. In particular, this happens if A is somewhere dense (i.e. such that $\mathrm{int}\overline{A} \neq \varnothing$), or if $\dim X = 1$, and A is non-discrete in X. Also, due to the Maximal Modulus Principle for any open relatively compact $U \subset X$ we have $U_{\mathbf{F}} \subset \overline{\mathrm{conv}\,(\partial U)_{\mathbf{F}}}$.

Now we will consider two closely related families of NSHF's, among which there is an example of a non-separable NSHF, and an example of a (non-reflexive, non-regular) separable BSHF whose bi-dual is a NSHF.

Example 1.3.9. Let $\beta : X \to [0, +\infty]$. A *Holomorphic Weighted Uniform space* $\mathcal{H}^{\beta}_{\infty}$ is the linear subspace of $\mathcal{F}^{\beta}_{\infty}$ consisting of holomorphic functions, namely $\mathcal{H}^{\beta}_{\infty} = \mathcal{F}^{\beta}_{\infty} \cap \mathcal{H}\,(X)$, with the seminorm $\|\cdot\|^{\beta}_{\infty}$. It is easy to see that $\|\cdot\|^{\beta}_{\infty}$ is a norm if and only if β is finite on a non-thin set in X. Also, if β vanishes on a non-thin set, $\mathcal{H}^{\beta}_{\infty} = \{0\}$. In the case when $\beta \equiv 1$ we will use the notation $\mathcal{H}_{\infty}\,(X)$. Furthermore, if $X = \mathbb{D}$ the traditional notation is H^{∞}.

Holomorphic weighted uniform spaces have been extensively studied in the case when β is continuous (see e.g. [BBT98]), however there is not much known for the general case. In particular it is not known in general which choices of β lead to a complete $\mathcal{H}^{\beta}_{\infty}$. If $\mathcal{H}^{\beta}_{\infty}$ is a NSHF, it is regular, and so automatically complete; on the other hand, if $\mathcal{H}^{\beta}_{\infty}$ is a Banach space, then $|\kappa|_{\mathcal{H}^{\beta}_{\infty}} \leq \beta$ is finite on a non-thin set in X, and so, from Proposition 1.3.8, $\mathcal{H}^{\beta}_{\infty}$ is a NSHF (this is a result from [BV17]). Note that it is not true that $|\kappa|_{\mathcal{H}^{\beta}_{\infty}} = \beta$ even if β is continuous pluri-log-subharmonic (see examples in [BBT98]).

It follows that $\mathcal{H}_{\infty}\,(X)$ is a BSHF. Note this space can be non-separable. For example, $\mathcal{H}^{\infty}$ contains an uncountable set $\left\{\exp\left(\frac{\cdot+y}{\cdot-y}\right) | y \in \mathbb{T}\right\}$, such that the distance between two distinct elements of this set is 1.

From Proposition 1.3.3 and Corollary 1.3.6, if there is $A \subset X$ such that the closure of any component of $X\backslash\overline{A}$ is compact, and β is bounded on A, or locally bounded at the points of A, then $\mathcal{H}^{\beta}_{\infty}$ is a BSHF. It turns out that the converse is also true for a specific choice of β. Namely, the next result focuses on $\beta = \mathbb{1}_A^{-1}$, where $\mathbb{1}_A$ is the indicator function of A and we count $\frac{1}{0} = +\infty$.

Theorem 1.3.10 ([AB02])**.** *Let A be a closed set in X and let $\beta = \mathbb{1}_A^{-1}$. Then $\mathcal{H}^{\beta}_{\infty}$ is complete if and only if it is a NSHF and if and only if the closure of any component of $X\backslash A$ is compact.*

Example 1.3.11. The *Little Holomorphic Weighted Uniform space* $\mathcal{H}^{\beta}_{0}$ is defined analogously to $\mathcal{H}^{\beta}_{\infty}$, i.e. as $\mathcal{C}^{\beta}_{0} \cap \mathcal{H}(X)$, with the seminorm of $\|\cdot\|^{\beta}_{\infty}$. If β is locally bounded, $\mathcal{C}^{\beta}_{0}$ is closed in $\mathcal{C}^{\beta}_{\infty}$, and so $\mathcal{H}^{\beta}_{0}$ is a BSHF. Moreover, for a wide class of β the space $\mathcal{H}^{\beta}_{\infty}$ is the bi-dual of $\mathcal{H}^{\beta}_{0}$ (see [BS93]). Note that on the other hand that if β is bounded, then the corresponding $\mathcal{H}^{\beta}_{0}$ is trivial, due to Proposition 1.3.1. $\square$

The following example of a NSHF, is defined in a slightly different manner from the NSHF's considered above.

Example 1.3.12. Assume that X is a domain in $\mathbb{C}^n$. Consider $A(X) = \mathcal{H}(X) \cap \mathcal{C}\left(\overline{X}\right)$, i.e. the subset of $\mathcal{H}(X)$ consisting of functions that admit a continuous extension on $\overline{X}$, or the subset of $\mathcal{C}\left(\overline{X}\right)$ consisting of functions that are holomorphic on X. It is easy to see that $A(X)$ is a closed subset of $\mathcal{C}\left(\overline{X}\right) = \mathcal{C}_{\infty}\left(\overline{X}\right)$ and in particular is a BSHF over X. However, $A(X)$ is not necessarily a regular NSHF over X. Indeed, by Runge's theorem [For91, Theorem 25.5] if $X \subset \mathbb{C}$ is simply connected, then $\overline{B_{A(X)}}^{\mathcal{H}(X)} = \overline{B}_{\mathcal{H}_{\infty}(X)}$, and since $\mathcal{H}_{\infty}(X)\backslash A(X) \neq \varnothing$ we conclude that $\overline{B_{A(X)}}$ is not closed in $\mathcal{H}(X)$. $\square$

1.3.c Hilbert Spaces of Holomorphic Functions

Similarly to the other categories, Hilbert spaces form a class of NSHF's that exhibit certain additional properties. Let X be a complex manifold. We will start with studying a class of functions on $X \times X$ which contains reproducing kernels of HSHF's over X.

A function $l : X \times X \to \mathbb{C}$ is called *sesqui-holomorphic function on* X if it is holomorphic in the first variable and antiholomorphic in the second. We will need the following facts about this class of functions (the proof is briefly discussed in the remark below).

Proposition 1.3.13.

(i) Sesqui-holomorphic functions form a closed linear subspace of $\mathcal{C}(X \times X)$*.*

(ii) If l *and* m *are sesqui-holomorphic on* X*, then the following conditions are equivalent:*

- $l = m$*;*
- $l = m$ *on a somewhere dense subset of* $X \times X$*;*
- $l(\cdot, y) = m(\cdot, y)$ *(or* $l(x, \cdot) = m(x, \cdot)$*) on a somewhere dense subset of* X*, for every* y *(or* x*) in a somewhere dense subset of* X*;*
- $\widehat{l} = \widehat{m}$ *on a somewhere dense subset of* X*, i.e.* m *and* l *coincide on a somewhere dense subset of the diagonal of* $X \times X$*.*

Remark 1.3.14. Let us briefly discuss a general view on the sesqui-holomorphic functions which explains the proof of the claims above. Let X^* be the complex manifold, diffeomorphic to X but with a complex-conjugate holomorphic structure (e.g. if X is a domain, then X^* is its complex conjugate in the usual sense; see [Cal53] for the general case). Then by Hartogs theorem (see [BM48, VII.4, Theorem 4]) l is a sesqui-holomorphic function on X if and only if the function $l' : X \times X^* \to \mathbb{C}$ defined by $l'(x, y) = l(x, \overline{y})$ is holomorphic. Hence, part (i) follows immediately. The equivalence of the first three conditions follows from the definition; the last condition is equivalent to the other due to [BM48, II.4, Theorem 7] (see also [Nik95]). $\square$

If $l : X \times X \to \mathbb{C}$ is conjugate-symmetric it is sesqui-holomorphic if and only if it is holomorphic in the first variable. On the other hand, sesqui-holomorphic function l is conjugate-symmetric if and only if $\widehat{l}$ is real valued. While necessity is obvious, sufficiency follows from part (ii) of Proposition 1.3.13 applied to l and a sesqui-holomorphic function l^* on X defined by $l^*(x, y) = \overline{l(y, x)}$.

Let us furthermore consider positive semi-definite sesqui-holomorphic functions. In fact, a sesqui-holomorphic function is positive semi-definite if it is positive semi-definite on a somewhere dense set in X (see the proof in [BB84]). The simplest examples of positive semi-definite sesqui-holomorphic kernels are functions of the form $f \otimes \overline{f}$, where $f \in \mathcal{H}(X)$. Also note that there are sesqui-holomorphic functions, which are positive on the diagonal but not positive semi-definite. For example, $l(z, w) = 1 - z\overline{w}$ is not positive semi-definite on $\mathbb{D}$, since $1 = \left|l\left(0, \frac{1}{2}\right)\right|^2 \not\leq l(0, 0)\, l\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$. The following theorem describes what kind of RKHS's we obtain from sesqui-holomorphic kernels.

Theorem 1.3.15. *[*[GL04], [Sza04], [BB84]*] For a RKHS* $\mathbf{H}$ *over* X *the following are equivalent:*

(i) $\mathbf{H}$ *consists of holomorphic functions;*

(ii) $\mathbf{H}$ *is a HSHF;*

(iii) $k^{\mathbf{H}}$ *is sesqui-holomorphic;*

(iv) The evaluation map $\kappa_{\mathbf{H}}$ *is holomorphic into* $\mathbf{H}^*$*;*

(v) There is a collection $\{e_i\}_{i\in I} \subset \mathcal{H}(X)$*, which is an orthonormal basis of* $\mathbf{H}$*, such that* $\sum\limits_{i\in I} |e_i|^2$ *is bounded on compacts.*

Combining this result with Moore-Aronszajn theorem we get the following consequence.

Corollary 1.3.16. *For any sesqui-holomorphic kernel* $k : X \times X \to \mathbb{C}$*, which is positive semi-definite on a somewhere dense set in* X *there is a HSHF* $\mathbf{H}_k$ *over* X*, whose kernel is equal to* k*.*

Now let us give some examples of HSHF's. From Corollary 1.3.16, it is enough to specify a sesqui-holomorphic positive semi-definite kernel. Let us consider an example of such a kernel and another class of examples will be studied in the next subsection.

Example 1.3.17. The *Drury-Arveson space of order* m, where m is some cardinal, is the RKHS over the open unit ball B_m of m-dimensional Hilbert space H_m, with the kernel $k_m(z,w) = \left(1 - \langle z, w\rangle_{H_m}\right)^{-1}$. We will denote this space by $DA(m)$. The importance of this family of spaces is determined by the fact that it is maximal for the class of Nevanlinna-Pick complete RKHS's (see [AM02]). If $m \in \mathbb{N}$ then B_m is a complex manifold and $DA(m)$ is a HSHF over it. Let us also indicate that $DA(1)$ is the classical *Hardy space* (usually denoted H^2; see more details in [CM95], [Woj91]) and that $k_1(z,w) = (1 - z\overline{w})^{-1}$ is called the *Szego kernel*.

In order to see that the Szego kernel is positive definite observe that $k_1 = \sum\limits_{n\in\mathbb{N}_0} Id_{\mathbb{D}}^n \otimes \overline{Id_{\mathbb{D}}^n}$, and so the monomials $\{z^n \,|n \in \mathbb{N}_0\}$ form an orthonormal basis of H^2. Similarly, one can see that k_m is positive definite and find the corresponding orthonormal basis, for any m. □

Let us now calculate the derivatives of the diagonals of sesqui-holomorphic functions. For simplicity, assume that $l:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$. Observe that $\widehat{l}=l\circ j_{\mathbb{C}}$, where $j_{\mathbb{C}}=Id_{\mathbb{C}}\oplus Id_{\mathbb{C}}$. Note that $\frac{d}{dz}j_{\mathbb{C}}=1\oplus 1$, $\frac{d}{d\overline{z}}j_{\mathbb{C}}=0\oplus 0$ and $\frac{\partial}{\partial w}l=\frac{\partial}{\partial \overline{z}}l=0$. Then using the chain rule for the Wirtinger derivatives we get that $\frac{d}{dz}\widehat{l}=\left(\frac{\partial}{\partial z}l\right)\circ j_{\mathbb{C}}=\widehat{\frac{\partial}{\partial z}l}$ and $\frac{d}{d\overline{z}}\widehat{l}=\widehat{\frac{\partial}{\partial \overline{w}}l}$. Since $\frac{\partial}{\partial z}l$ and $\frac{\partial}{\partial \overline{w}}l$ are also sesqui-holomorphic, it follows that for $k,m\in\mathbb{N}$ we have

$$\frac{d^{k+m}}{dz^k d\overline{z}^m}\widehat{l}=\widehat{\frac{\partial^{k+m}}{\partial z^k\partial\overline{w}^m}l}.$$

Finally, if $\widehat{l}$ is real-valued (or, equivalently, l is conjugate-symmetric), then $\frac{d}{d\overline{z}}\widehat{l}=\overline{\frac{d}{dz}\widehat{l}}$.

From the formulas above it is easy to deduce that if $x\in X$, $\xi,\zeta\in\mathcal{T}_x$ and l is sesqui-holomorphic on X, then $\xi\widehat{l}=\xi l(\cdot,x)$ and $\xi\otimes\overline{\zeta}\widehat{l}=\xi\overline{\zeta}_2 l=\overline{\zeta}\xi_1 l$, where the subindex indicates the variable with respect to which the differentiation is implemented.

The following result characterizes the mutual position of the point evaluations and derivative point evaluations on a HSHF $\mathbf{H}$.

Proposition 1.3.18. *Let $x,y,z\in X$, let $\xi\in\mathcal{T}_x$ and let $\zeta\in\mathcal{T}_y$. Then:*

(i) $\langle\xi\kappa_{\mathbf{H}},z_{\mathbf{H}}\rangle_{\mathbf{H}^*}=\xi k_z^{\mathbf{H}}$ *and* $[{}^*\xi\kappa_{\mathbf{H}}](z)=\overline{\xi k_z^{\mathbf{H}}}=\overline{\xi}k^{\mathbf{H}}(z,\cdot)$.

(ii) $\langle\xi\kappa_{\mathbf{H}},\zeta\kappa_{\mathbf{H}}\rangle_{\mathbf{H}^*}=\overline{\zeta}\xi_1 k^{\mathbf{H}}$*, where the inner differentiation is with respect to the first variable.*

Proof. We will use properties of the operation $*$ between $\mathbf{H}$ and $\mathbf{H}^*$. Part (i) follows from

$$\overline{[{}^*\xi\kappa_{\mathbf{H}}](z)}=\overline{\langle{}^*\xi\kappa_{\mathbf{H}},z_{\mathbf{H}}\rangle}=\langle\xi\kappa_{\mathbf{H}},z_{\mathbf{H}}\rangle_{\mathbf{H}^*}=\langle\xi\kappa_{\mathbf{H}},{}^*z_{\mathbf{H}}\rangle=\left\langle\xi\kappa_{\mathbf{H}},k_z^{\mathbf{H}}\right\rangle=\xi k_z^{\mathbf{H}}.$$

(ii): We can apply ζ to the function ${}^*\xi\kappa_{\mathbf{H}}\in\mathbf{H}$, calculated in the previous part. We get

$$\zeta\overline{\xi_1 k_{\mathbf{H}}}=\zeta\left({}^*\xi\kappa_{\mathbf{H}}\right)=\langle\zeta\kappa_{\mathbf{H}},{}^*\xi\kappa_{\mathbf{H}}\rangle=\langle\zeta\kappa_{\mathbf{H}},\xi\kappa_{\mathbf{H}}\rangle_{\mathbf{H}^*},$$

and so $\langle\xi\kappa_{\mathbf{H}},\zeta\kappa_{\mathbf{H}}\rangle_{\mathbf{H}^*}=\overline{\zeta\overline{\xi_1 k^{\mathbf{H}}}}=\overline{\zeta}\xi_1 k^{\mathbf{H}}$. ∎

Thus, the functions that represent the derivative point evaluations are the conjugates of the corresponding derivatives of $k^{\mathbf{H}}$ in the first variable. From the preceding proposition and the formulas for derivatives of a diagonal of a sesqui-holomorphic functions we get the following corollary.

Corollary 1.3.19. *Let $x \in X$ and let $\xi, \zeta \in \mathcal{T}_x$. Then $\langle \xi\kappa_{\mathbf{H}}, x_{\mathbf{H}} \rangle_{\mathbf{H}^*} = \xi\widehat{k^{\mathbf{H}}}$ and $\langle \xi\kappa_{\mathbf{H}}, \zeta\kappa_{\mathbf{H}} \rangle_{\mathbf{H}^*} = \xi \otimes \overline{\zeta}\widehat{k^{\mathbf{H}}}$.*

Remark 1.3.20. The same formulas as in the preceding proposition and corollary work for the higher order differential expressions. □

1.3.d Bergman Spaces

An abundant source of Hilbert as well as non-Hilbert examples of NSHF's is the class of Bergman spaces (see [HKZ00], [DS04]). The classical Bergman space was introduced by Stefan Bergman in the first half of the 20th century for the Lebesgue measure on the unit disk in the (complex) plane (see [Ber70]), but we will consider a more general case. While it is more common to work with the Bergman space of differential forms instead of functions when dealing with complex manifolds (see e.g. [Kob59]), we will only consider spaces of functions.

Let X be a second countable complex manifold of dimension n, let μ be a regular measure which is finite on the compact subsets of X and let $p \in [1, +\infty)$. Define a functional $\|\cdot\|_p$ on $\mathcal{F}(X)$ by $\|f\|_p = \left(\int\limits_X |f|^p \, d\mu \right)^{\frac{1}{p}}$, being the usual $L^p(X, \mu)$-norm. This functional defines the *Bergman space* $A^p(X, \mu)$, which is the subset of $\mathcal{H}(X)$ consisting of the L^p-functions, i.e.

$$A^p(X, \mu) = \left\{ f \in \mathcal{H}(X), \|f\|_p < +\infty \right\}.$$

One can show that $\|\cdot\|_p$ is a norm on $A^p(X,\mu)$ if and only if the support of μ is not thin. It is clear that $A^p(X,\mu)$ is a linear subspace of $\mathcal{H}(X)$ which is isometrically embedded into $L^p(X,\mu)$, however, it is not obvious if the image is closed, i.e. if $A^p(X,\mu)$ is complete with respect to $\|\cdot\|_p$.

Remark 1.3.21. In our definition of the Bergman space $A^p(X,\mu)$ the exponent p varied in $[1,+\infty)$. However, we can extend this definition to the case $p=+\infty$ using the usual L^∞ norm. It is easy to see that $A^\infty(X,\mu)=\mathcal{H}^\beta_\infty$, where $\beta=\mathbb{1}^{-1}_{\mathrm{supp}\mu}$. Hence, these spaces (which were also the subject of Theorem 1.3.10) are simultaneously Weighted Holomorphic spaces and a limit case of Bergman spaces. □

It seems that it is too difficult to characterise all measures μ, for which $A^p(X,\mu)$ is a Banach space. Some necessary conditions, sufficient conditions and counterexamples can be found in [AB02] and [PW92]. In particular, it was shown in [AB02], that if μ is a finite measure and $A^p(X,\mu)$ is complete, then $A^q(X,\mu)$ is also complete, for every $q\in[p,+\infty]$, and so from Theorem 1.3.10, every component of $X\backslash\mathrm{supp}\mu$ is relatively compact. On the other hand, we will see that once $A^p(X,\mu)$ is a NSHF, a lot of additional properties follow automatically.

Since X is a manifold, we will study μ locally, via comparison with the push-forward of the Lebesgue measure on the coordinate neighborhoods. Namely, if U and V are open subsets of X and $\mathbb{C}^n$ respectively, and $\varphi:V\to U$ is a biholomorphism, denote the push-forward of the Lebesgue measure via φ by λ_φ.

In the following theorem we generalize well-known properties of the “classical” Bergman spaces.

Theorem 1.3.22.

(i) Assume that there is a closed $A \subset X$ such that the closure of any component of $X \backslash A$ is compact, and for every $x \in A$ there is an open neighborhood U of x and a biholomorphisms $\varphi : B_{\mathbb{C}^n} \to U$, such that λ_φ is absolutely continuous with respect to μ on U with $\int\limits_U \max \left\{ \log \frac{d\lambda_\varphi}{d\mu}, 0 \right\} < +\infty$. Then $A^p(X, \mu)$ is a NSHF, for every $p \in [1, +\infty)$.

(ii) If $p > 1$, and $A^p(X, \mu)$ is a NSHF, then it is reflexive, and the weak, pointwise and the compact-open topologies coincide on its bounded sets.

(iii) If $A^1(X, \mu)$ is a NSHF, then it is regular, and has the Schur Property, i.e. a sequence converges in $A^1(X, \mu)$ if and only if it converges weakly.

In order to prove these results we need some lemmas. First, consider a useful property of $\|\cdot\|_p$.

Lemma 1.3.23. *$\|\cdot\|_p$ is lower semi-continuous on $\mathcal{C}(X)$ (and consequently on $\mathcal{H}(X)$).*

Proof. For a compact $K \subset X$ define a continuous semi-norm $\|\cdot\|_p^K$ on $\mathcal{C}(X)$ by $\|f\|_p = \left(\int\limits_K |f|^p \, d\mu \right)^{\frac{1}{p}}$. Since X is σ-locally compact, for each non-negative measurable function g we have that

$$\int_X g d\mu = \sup \left\{ \int_K g d\mu \,|K \in \mathcal{K}(X) \right\},$$

and so $\|\cdot\|_p = \sup\limits_{K \in \mathcal{K}(X)} \|\cdot\|_p^K$. Hence, $\|\cdot\|_p$ is lower semi-continuous, as a supremum of continuous functionals. □

In the following lemma we estimate the norm of the point evaluations on the Bergman space.

Lemma 1.3.24. *Let $x \in X$, let U be an open neighborhood of x and let φ be a biholomorphism from $B_{\mathbb{C}^n}(y, r)$ onto U, for some $y \in \mathbb{C}^n$ and $r > 0$, such that $\varphi(y) = x$. If λ_φ is absolutely continuous with respect to μ on U with*

$$\alpha = \int\limits_U \max\left\{\log\frac{d\lambda_\varphi}{d\mu}, 0\right\} d\lambda_\varphi < +\infty,$$

then for every $f \in \mathcal{H}(X)$ we have that

$$|f(x)|^p \leq n!\left(\pi r^2\right)^{-n} e^{\alpha n!\left(\pi r^2\right)^{-n}} \|f\|_p^p.$$

Proof. Let $f \in \mathcal{H}(X)$. Let $w = \frac{d\lambda_\varphi}{d\mu}$. Let μ' be the absolutely continuous part of μ under Lebesgue decomposition with respect to λ_φ (see [DS58, III.4, Theorem 14]). Since $\int\limits_U |f|^p\, d\mu' \leq \int\limits_U |f|^p\, d\mu \leq \|f\|_p^p$, it is enough to show that $|f(x)|^p \leq n!\,(\pi r^2)^{-n}\, e^{\alpha n!\left(\pi r^2\right)^{-n}} \int\limits_U |f|^p\, d\mu'$.

Due to the presence of φ we will identify U with $B_{\mathbb{C}^n}(y, r)$ and view λ as a Lebesgue measure on U. We will also identify x with y.

Since $f \in \mathcal{H}(X)$, it follows that $\log|f|^p = p\log|f|$ is subharmonic, and since exp is an increasing function we have

$$\begin{aligned}|f(x)|^p = \exp\log|f(x)|^p &\leq \exp\left(\frac{1}{\lambda(U)}\int_U \log|f|^p\, d\lambda\right)\\ &= \exp\left(\frac{1}{\lambda(U)}\int_U \log\left(|f|^p\frac{1}{w}\right) d\lambda\right)\exp\left(\frac{1}{\lambda(U)}\int_U \log w\ d\lambda\right).\end{aligned}$$

Since $\lambda(U) = \frac{r^{2n}\pi^n}{n!}$, we obtain

$$\exp\left(\frac{1}{\lambda(U)}\int_U \log w d\lambda\right) \leq \exp\left(\frac{d!}{r^{2n}\pi^n}\int\limits_U \max\{\log w, 0\}\, d\lambda\right) = e^{\alpha n!\left(\pi r^2\right)^{-n}},$$

and since exp is a convex function we get that

$$\exp\left(\frac{1}{\lambda(U)}\int_U \log\left(|f|^p\frac{1}{w}\right) d\lambda\right) \leq \frac{1}{\lambda(U)}\int_U |f|^p\frac{1}{w} d\lambda = \frac{n!}{r^{2n}\pi^n}\int_U |f|^p\, d\mu',$$

from where the result follows. ∎

Now we can prove the main theorem.

Proof of Theorem 1.3.22. (i): Let $x \in A$ and let U be an open neighborhood of x for which there is a biholomorphisms $\varphi : B_{\mathbb{C}^n} \to U$, such that λ_φ is absolutely continuous with respect to μ on U with $\alpha = \int\limits_U \max\left\{\log \frac{d\lambda_\varphi}{d\mu}, 0\right\} < +\infty$. Let $z \in \mathbb{C}^n$ be such that $\varphi(z) = x$. Let $r = \frac{1-\|z\|}{2} > 0$. Then φ maps biholomorphically $B(z, r)$ onto an open neighborhood V of x. Let $y \in V$, and let $w = \varphi^{-1}(y)$. Then, φ maps biholomorphically $B(w, r)$ onto an open neighborhood W of y, and $\int\limits_W \max\left\{\log \frac{d\lambda_\varphi}{d\mu}, 0\right\} \leq \alpha$. Hence, it follows from Lemma 1.3.24 that

$$\left\|y_{A^p(X,\mu)}\right\| \leq \left(n!\left(\pi r^2\right)^{-n} e^{\alpha n!\left(\pi r^2\right)^{-n}}\right)^{\frac{1}{p}},$$

and since y was chosen arbitrarily, $|\kappa|_{A^p(X,\mu)}$ is locally bounded at x. Since x was also chosen arbitrarily we conclude that condition (v) of Proposition 1.3.3 applies. Thus, $A^p(X, \mu)$ is a NSHF.

(ii): It follows immediately from Lemma 1.3.23 that if $A^p(X, \mu)$ is a NSHF, then it is regular, and so it is a BSHF. In particular, $A^p(X, \mu)$ is a closed subspace of $L^p(X, \mu)$. The latter is reflexive when $p > 1$ (see [DS58, IV.8, Corollary 2]). Since a closed subspace of a reflexive space is reflexive (see [BN11, Theorem 15.2.7]), $A^p(X, \mu)$ is a reflexive NSHF for $p > 1$. Hence, the weak, pointwise and the compact-open topologies coincide on bounded sets in $A^p(X, \mu)$.

(iii): The strong convergence always implies the weak convergence, while the weak convergence in a NSHF implies the convergence in the compact-open topology. Assume that $\{f_n\}_{n\in\mathbb{N}} \subset A^1(X, \mu)$ converges to 0 weakly. In order to show that $f_n \to 0$ strongly it is sufficient to prove that f_n converges in measure on each subset of finite measure (see [DS58, IV.8, Theorem 12]). Let $A \subset X$ be such set and let $\varepsilon, \delta > 0$. Since X is a union of a sequence of compact sets, there is a compact set K, such that $\mu(A \backslash K) < \varepsilon$. Since $f_n \to 0$ uniformly on K, there exists $m \in \mathbb{N}$, such that for each $n > m$ we have that $\|f_n\|_\infty^K < \delta$. Therefore, $\mu(\{x \in A \,||f_n(x)| > \delta\}) \leq \mu(A \backslash K) < \varepsilon$. From the arbitrariness of ε and δ the convergence in measure follows. $\square$

Here are some concluding remarks about Bergman spaces.

Remark 1.3.25.

- The condition (i) of Theorem 1.3.22 is obtained from modifying the conditions from [PW92] and [AB02] and is less restrictive than both of them.
- It is easy to see that if $A^p(X, \mu)$ is a NSHF, and ν is a measure on X with a compact support, then $A^p(X, \mu + \nu)$ is also a NSHF.
- If $A^2(X, \mu)$ is a NSHF, it is a HSHF.
- If μ is finite, then $\mathcal{H}_\infty(X) \subset A^p(X, \mu)$, for every $p \in [1, +\infty]$.
- It is well-known that $A^1(\mathbb{D}, \lambda)$ has Schur property, where λ is the Lebesgue measure on $\mathbb{D}$, but the proof was given through the isometric isomorphism with l^1 which has this property (see [Woj91, III.A, Theorem 11] and [DS58, IV.8, Corollary 14]). $\square$

Chapter 2

Weighted Composition Operators

Let X and Y be sets, and let $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$. A *weighted composition operator* (WCO) with *composition symbol* φ and *multiplicative symbol* u is a linear map $W_{\varphi,u}$ from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$ defined by

$$[W_{\varphi,u}f](y) = u(y) f(\varphi(y)).$$

Let $\mathbf{F} \subset \mathcal{F}(X)$, $\mathbf{E} \subset \mathcal{F}(Y)$ be linear subspaces. If $W_{\varphi,u}\mathbf{F} \subset \mathbf{E}$, then we say that $W_{\varphi,u}$ is a weighted composition operator from $\mathbf{F}$ into $\mathbf{E}$ (we use the same notation $W_{\varphi,u}$ for what is in fact $W_{\varphi,u}|_{\mathbf{F}}$).

WCO's may be viewed as morphisms in the category of NSF's. For the category of NSCF's we introduce another class of morphisms. Namely, if X and Y are topological spaces and $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ are continuous, then we will say that $W_{\varphi,u}$ is a *continuously induced weighted composition operator* ($\mathrm{WCO}_{\mathcal{C}}$). Analogously, if X and Y are complex manifolds, then a *holomorphically induced weighted composition operator* ($\mathrm{WCO}_{\mathcal{H}}$) between NSHF's is a WCO with holomorphic symbols.

This chapter is mostly dedicated to various basic questions about WCO's, $\mathrm{WCO}_{\mathcal{C}}$'s and $\mathrm{WCO}_{\mathcal{H}}$'s. We will start with a brief summary of the chapter.

In Section 2.1 we consider the most basic properties of WCO's, $\mathrm{WCO}_{\mathcal{C}}$'s and $\mathrm{WCO}_{\mathcal{H}}$'s, mostly relating set-theoretical properties (injectivity, surjectivity, presence of zeros) of the symbols of a WCO with the corresponding properties of that WCO. Also, we show that a WCO between BSF's with a surjective composition symbol is a linear homeomorphism if and only if its adjoint is bounded from below (Proposition 2.1.15), and conversely give sufficient conditions for a WCO which is a linear homeomorphism to have a surjective composition symbol (Corollary 2.1.24).

In Section 2.2 we give some sufficient conditions (Proposition 2.2.4, Theorem 2.2.10 and Theorem 2.2.12), that force a WCO between NSCF's to be a $\mathrm{WCO}_{\mathcal{C}}$. Example 2.2.7 demonstrates that this problem is not trivial. Section 2.3 is dedicated to an analogous problem in the holomorphic context (with Proposition 2.3.2, Proposition 2.3.11, Theorem 2.3.3 and Theorem 2.3.9 providing the corresponding sufficient conditions).

In Section 2.4 we study the pull-backs of unitary-invariant Hermitean metric on the dual of a HSHF via its evaluation map (Theorem 2.4.6), and the possibility of its automorphism-invariance (Proposition 2.4.11 and Corollary 2.4.15).

2.1 Basic Properties of WCO's

In this section we present a variety of results on WCO's that are generalizations to the uniform framework of NSF's and NSCF's of the well-known properties of WCO's on the specific spaces of functions.

In the absence of structure on X there are very few special classes of functions available on X. One of these classes is the class of constant functions. The presence of these functions in $\mathbf{F} \subset \mathcal{F}(X)$ puts a strict restriction on the possible multiplicative symbols of WCO's on $\mathbf{F}$. Namely, if $1 \in \mathbf{F}$, and $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ are such that $W_{\varphi,u}$ maps $\mathbf{F}$ into $\mathbf{E} \subset \mathcal{F}(Y)$, then $u = W_{\varphi,u}1 \in \mathbf{E}$. We will often use this simple observation.

Next we consider two special classes of WCO's.

2.1.a Composition and Multiplication Operators

Let X and Y be sets, let $\varphi : Y \to X$ and let $u : Y \to \mathbb{F}$. In the case when $u \equiv 1$, or $X = Y$ and $\varphi = Id_X$ we use special notations C_φ and M_u, and terms *composition operator* (CO) and *multiplication operator* (MO) respectively for the corresponding WCO's. If Z is another set, $\psi : Z \to Y$ and $v : Z \to \mathbb{F}$, then

$$W_{\psi,v}W_{\varphi,u} = M_v C_\psi M_u C_\varphi = M_{v\cdot u\circ\psi} C_{\varphi\circ\psi} = W_{\varphi\circ\psi, v\cdot u\circ\psi}.$$

In particular, if $X = Y$, then the semi-group of weighted composition operators on $\mathcal{F}(X)$ is the semi-direct product of semi-groups of multiplication and composition operators. Also, note that the identity operator on $\mathcal{F}(X)$ is equal to $W_{Id_X,1} = C_{Id_X} = M_1$, while the zero-operator from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$ is equal to $W_{\varphi,0} = M_0$, where φ is an arbitrary map from Y into X. In particular, we can choose φ to be a constant map.

Any WCO between BSF's is automatically continuous due to part (ii) of Proposition 0.2.7. However, in concrete cases it can be very difficult to determine all CO's, MO's and WCO's between a given pair of NSF's. Moreover, it is possible that $W_{\varphi,u}$ is continuous, while neither M_u nor C_φ are well defined (see Example 2.1.21). Clearly, if both M_u and C_φ are well defined, then $W_{\varphi,u} = M_u C_\varphi$. If in this case $W_{\varphi,u}$ is invertible, then so are M_u and C_φ.

Remark 2.1.1. While CO's are obviously partial cases of WCO's, a lot of WCO's can be viewed as CO's. For $f : X \to \mathbb{F}$ define $\widetilde{f} : \mathbb{F}^\times \times X \to \mathbb{F}$ by $\widetilde{f}(\alpha, x) = \alpha f(x)$, for every $x \in X$ and $\alpha \in \mathbb{F}^\times$. Let $Q : \mathcal{F}(X) \to \mathcal{F}(\mathbb{F}^\times \times X)$ be a linear operator defined by $Qf = \widetilde{f}$. Clearly, Q is an injection, and it is continuous with respect to the pointwise topologies on $\mathcal{F}(X)$ and $\mathcal{F}(\mathbb{F}^\times \times X)$.

For a linear subspace $\mathbf{F}$ of $\mathcal{F}(X)$ define $\widetilde{\mathbf{F}} = Q\mathbf{F}$. If $\mathbf{F}$ is endowed with a norm that turns it into a NSF, then the push-forward of this norm via Q turns $\widetilde{\mathbf{F}}$ into a NSF over $\mathbb{F}^\times \times X$. Now if $\varphi : Y \to X$ and $u : Y \to \mathbb{F}^\times$, then viewing $u \times \varphi$ as a map from Y into $\mathbb{F}^\times \times X$, we get that $C_{u\times\varphi}\widetilde{f} = W_{\varphi,u} f$, for every $f \in \mathbf{F}$. Hence, $W_{\varphi,u} = C_{u\times\varphi} Q$, and so instead of studying $W_{\varphi,u}$ on $\mathbf{F}$ we can deduce most of its properties from the information about $C_{u\times\varphi}$ on $\widetilde{\mathbf{F}}$. $\square$

We will use MO's and CO's to define two operations with NSF's. Let $\mathbf{F}$ be a NSF over a set X. The *rescaling* $v\mathbf{F}$ of $\mathbf{F}$ with respect to the *weight* $v : X \to \mathbb{F}$ is the linear space $M_v\mathbf{F}$ endowed with the push-forward norm from $\mathbf{F}/\mathrm{Ker}\, M_v$ (it follows from Corollary 2.1.4 below that the kernel of any WCO is a pointwise closed set, and so it is closed in $\mathbf{F}$). One can show that $v\mathbf{F}$ is a NSF over X. We will also call $v\mathbf{F}$ a *bounded/non-degenerate* rescaling, if v is bounded/non-vanishing. We will later show that M_v is an injection if v does not vanish, and so a non-degenerate rescaling is an isometric isomorphism.

Let Z be a set and let $\psi : Z \to X$ be an injection. A *restriction* $\mathbf{F} \circ \psi$ of $\mathbf{F}$ along ψ is the linear space $C_\psi\mathbf{F}$ endowed with the push-forward norm from $\mathbf{F}/\mathrm{Ker}\, C_\psi$. Then $\mathbf{F} \circ \psi$ is a NSF over Z. Also, if $\mathbf{F}$ separates the points of X, then $\mathbf{F} \circ \psi$ separates the points of Z. If $\psi(Z)^\perp_{\mathbf{F}} = \{0_{\mathbf{F}}\}$, then the restriction is an isometric isomorphism. In the case when $Z \subset X$, we will use the notation $\mathbf{F}|_Z$.

Naturally, both rescaling and restriction "commute" with CO's and MO's. Let X and Y be sets, let $\mathbf{F} \subset \mathcal{F}(X)$ and $\mathbf{E} \subset \mathcal{F}(Y)$ be linear subspaces, let $\varphi : Y \to X$ be such that $C_\varphi \mathbf{F} \subset \mathbf{E}$, let $v : X \to \mathbb{F}$, let $Z \subset X$ and let $W \subset \varphi^{-1}(Z)$. Then the following diagram commutes

$$\begin{tikzcd}
\mathbf{F}|_Z \arrow[d, "C_{\varphi|_W}"] & \mathbf{F} \arrow[l] \arrow[r] \arrow[d, "C_\varphi"] & v\mathbf{F} \arrow[d, "C_\varphi"] \\
\mathbf{E}|_W & \mathbf{E} \arrow[l] \arrow[r] & v \circ \varphi \mathbf{E}
\end{tikzcd}$$

Similarly, if $\mathbf{F}, \mathbf{E} \subset \mathcal{F}(X)$, $u : X \to \mathbb{F}$ is such that $M_u \mathbf{F} \subset \mathbf{E}$, $v : X \to \mathbb{F}$ and $Z \subset X$, then the following diagram commutes

$$\begin{tikzcd}
\mathbf{F}|_Z \arrow[d, "M_{u|_Z}"] & \mathbf{F} \arrow[l] \arrow[r] \arrow[d, "M_u"] & v\mathbf{F} \arrow[d, "M_u"] \\
\mathbf{E}|_Z & \mathbf{E} \arrow[l] \arrow[r] & v\mathbf{E}
\end{tikzcd}$$

The continuously/holomorphically induced CO and MO ($\mathrm{CO}_\mathcal{C}/\mathrm{CO}_\mathcal{H}$ and $\mathrm{MO}_\mathcal{C}/\mathrm{MO}_\mathcal{H}$) are the CO and MO with continuous/holomorphic symbols.

Let $\mathbf{F}$ be a NSCF over a topological space X. For $v : X \to \mathbb{F}$ we will call $v\mathbf{F}$ a $\mathcal{C}$-rescaling if v is continuous. It is easy to see that $v\mathbf{F}$ is a NSCF in this case. Let Z be a topological space and let $\psi : Z \to X$ be injection. We will call $\mathbf{F} \circ \psi$ a $\mathcal{C}$-restriction if ψ is a topological embedding. In particular, if $Z \subset X$, then $\mathbf{F}|_Z$ is always a $\mathcal{C}$-restriction. It is easy to see that $\mathbf{F} \circ \psi$ is a NSCF. Obviously, if $\mathbf{F}$ generates the topology of X, then $\mathbf{F} \circ \psi$ generates the topology of Z. We will later show that if v vanishes on a nowhere dense set in X (Z is dense in X), then the rescaling (restriction) is an isometric isomorphism.

It is easy to see that if $\mathbf{F}$ is a NSCF over X, then $\widetilde{\mathbf{F}}$ is a NSCF over $\mathbb{F}^\times \times X$.

2.1.b Recognition and Some General Properties of WCO's

Since WCO's may be viewed as morphisms in the category of NSF's, it is important to characterize this class of operators among all continuous linear maps between NSF's. It turns out that this *recognition* problem is simple. Namely, the WCO, CO and MO can be characterized by the way their (algebraic) adjoints act on the point evaluation. Moreover, this can be done with no reference to the target NSF. Indeed, if X, Y are sets, $\mathbf{F} \subset \mathcal{F}(X)$ and $\mathbf{E} \subset \mathcal{F}(Y)$ are linear subspaces, then any linear operator $T : \mathbf{F} \to \mathbf{E}$ can also be viewed as an operator into $\mathcal{F}(Y)$ (by composing with $J_{\mathbf{E}}$). Hence, this operator generates a map $\kappa_T = T' \circ \kappa_{\mathbf{E}}$ from Y to $\mathbf{F}'$ (see Section 1.1). In order to determine if T is a WCO it is enough to look at κ_T.

Proposition 2.1.2. *Let X be a set and let T be a linear map from a linear subspace $\mathbf{F} \subset \mathcal{F}(X)$ into $\mathcal{F}(Y)$. Then $T = W_{\varphi,u}$, for $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ if and only if $\kappa_T = u \cdot \kappa_{\mathbf{F}} \circ \varphi$.*

Proof. The proof follows from the definition of κ_T and $\kappa_{\mathbf{F}}$. Indeed, for every $f \in \mathbf{F}$ and $y \in Y$ we have that $[Tf](y) = \langle f, y_T \rangle = \langle f, \kappa_T(y) \rangle$, and also $\langle f, [u \cdot \kappa_{\mathbf{F}} \circ \varphi](y) \rangle = u(y) \langle f, \varphi(y)_{\mathbf{F}} \rangle = u(y) f(\varphi(y))$. $\square$

Corollary 2.1.3.

(i) T is a WCO if and only if $T'Y_{\mathcal{F}} \subset \mathbb{F}X_{\mathbf{F}}$.

(ii) T is a CO if and only if $T'Y_{\mathcal{F}} \subset X_{\mathbf{F}}$.

(iii) If $X = Y$, then T is a MO if and only if $T'y_{\mathcal{F}} \in \mathbb{F}y_{\mathbf{F}}$, for each $y \in Y$.

In other words, if $W_{\varphi,u}\mathbf{FE}$ then $W'_{\varphi,u}y_{\mathbf{E}} = u(y)\varphi(y)_{\mathbf{F}}$, for every $y \in Y$, and if $T : \mathbf{F} \to \mathbf{E}$ is such that $T'y_{\mathbf{E}} = u(y)\varphi(y)_{\mathbf{F}}$, for every $y \in Y$, then $T = W_{\varphi,u}$. In the same way the adjoint to C_φ is characterised by being a "linear extension" of φ, i.e. $C'_\varphi y_{\mathbf{E}} = \varphi(y)_{\mathbf{F}}$, and if $X = Y$, then $M'_u y_{\mathbf{E}} = u(y)\, y_{\mathbf{F}}$. In particular, if $\mathbf{F} = \mathbf{E}$, then M_u is characterized by the fact that each element of $Y_{\mathbf{F}}$ is an eigenvector of M'_u.

Since the kernel of any operator $T : F \to \mathcal{F}(Y)$ is $Y_T^{\perp}$, we get a representation for the kernel of a WCO.

Corollary 2.1.4. *Let X and Y be sets, let $\mathbf{F} \subset \mathcal{F}(X)$ be a linear subspace and let $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$. Then* Ker $W_{\varphi,u} = \varphi\left(Y \backslash u^{-1}(0)\right)_{\mathbf{F}}^{\perp}$.

Note that in general we cannot reconstruct the symbols of a WCO from its data as a linear operator between certain NSF's, in the sense that the equality of WCO's does not imply equality of their symbols.

Example 2.1.5.

- If $u(y) = 0$, for some $y \in Y$, then $W_{\varphi,u}$ does not depend on $\varphi(x)$, i.e. if $\varphi, \psi : Y \to \mathbb{F}$ coincide outside of y, then $W_{\varphi,u} = W_{\psi,u}$.
- If $\mathbf{F}$ is a NSF over X and $x, y \in X$ are such that $f(x) = f(y)$, for every $f \in \mathbf{F}$, then $Id_{\mathbf{F}} = C_\varphi$, where $\varphi : X \to X$ swaps x with y and fixes the rest of points of X.
- More generally, we can construct a WCO with non-trivial symbols which is equal to the identity on $\mathbf{F}$ if there are two distinct points in X such that the point evaluations on $\mathbf{F}$ at these points are linearly dependent. $\square$

Clearly, if $\mathbf{F}$ separates the points of X then we can reconstruct the symbol of a CO from the CO. In particular, if $Id_{\mathbf{F}} = C_\varphi$, then $\varphi = Id_X$. Also, it is easy to see that it is possible to reconstruct the symbol of a MO whenever $0_{\mathbf{F}^*} \notin X_{\mathbf{F}}$. In order to be able to reconstruct the symbols of a WCO we need to introduce an additional notion.

Let X be a set, let $\mathbf{F} \subset \mathcal{F}(X)$ be a linear subspace, and let $x^1, x^2, ..., x^n \in X$. It is easy to see that the corresponding linear functionals $x^1_{\mathbf{F}}, x^2_{\mathbf{F}}, ..., x^n_{\mathbf{F}}$ are linearly independent if and only if there are $f_1, f_2, ..., f_n \in \mathbf{F}$ such that $f_i(x^j) = \delta_{ij}$, and if and only if for any $\alpha_1, ..., \alpha_n \in \mathbb{F}$ there is $f \in \mathbf{F}$ such that $f(x^i) = \alpha_i$. We will say that $\mathbf{F}$ is *n-independent* if $x^1_{\mathbf{F}}, x^2_{\mathbf{F}}, ..., x^n_{\mathbf{F}}$ are linearly independent for every distinct $x^1, x^2, ..., x^n \in X$. Note that n-independence implies that $\dim \mathbf{F} \geq n$, and if $\dim \mathbf{F} \geq n \in \mathbb{N}$, then there are $x^1, x^2, ..., x^n \in X$ such that $x^1_{\mathbf{F}}, x^2_{\mathbf{F}}, ..., x^n_{\mathbf{F}}$ are linearly independent. In fact, we will only use 1-independence and 2-independence. Note that the former condition is equivalent to $0_{\mathbf{F}'} \notin X_{\mathbf{F}}$, while the latter means that the point evaluations at distinct points are non-collinear. If $\mathbf{F}$ is 2-independent, it strongly separates the points of X, if $\mathbf{F}$ contains non-zero constant functions, it is 1-independent, and if $\mathbf{F}$ contains non-zero constant functions and separates the points, then it is 2-independent. However, none of the converses to these three statements hold. Also, note that $\mathbf{F}$ is 2-independent if and only if $\widetilde{\mathbf{F}}$ separates the points of $\mathbb{F}^\times \times X$.

We will say that $\mathbf{F}$ is *∞-independent*, if $X_{\mathbf{F}}$ is a linearly independent subset of $\mathbf{F}'$, or equivalently, $\mathbf{F}$ is *n-independent*, for every $n \in \mathbb{N}$. It will follow from Proposition 2.1.8 below that both restriction and non-degenerate rescaling preserve the n-independence, for $n \in \mathbb{N} \cup \{\infty\}$. Also, if $\mathbf{F}$ is n-independent and $\mathbf{F} \subset \mathbf{E}$, then $\mathbf{E}$ is n-independent. Finally, a RKHS is ∞-independent if and only if its kernel is positive-definite.

Example 2.1.6. If $X \subset \mathbb{C}^n$ is an infinite set, then the linear space of all algebraic polynomials is ∞-independent due to Interpolation theorem. $\square$

If $\mathbf{F}$ is an n-independent NSF, for $n \in \mathbb{N} \cup \{\infty\}$, and $\mathbf{E}$ is a dense linear subspace of $\mathbf{F}$, then $\mathbf{E}$ is also n-independent. Indeed, if $x^1, x^2, ..., x^n \in X$ and $\alpha_1, ..., \alpha_n \in \mathbb{F}$ are such that $\alpha_1 x^1_{\mathbf{E}} + ... + \alpha_n x^n_{\mathbf{E}} = 0_{\mathbf{E}}$, then $\mathbf{E} \subset \left(\alpha_1 x^1_{\mathbf{F}} + ... + \alpha_n x^n_{\mathbf{F}}\right)^{\perp}$, which is a closed subset of $\mathbf{F}$. Hence, $\mathbf{E}$ can be dense in $\mathbf{F}$ only if $\alpha_1 x^1_{\mathbf{F}} + ... + \alpha_n x^n_{\mathbf{F}} = 0_{\mathbf{F}}$, which is only possible when $\alpha_1 = ... = \alpha_n = 0$.

Example 2.1.7. If $\mathbf{F} \subset \mathcal{C}(X)$ is not 1-independent, we can find its 1-independent restriction on an open subset of X. Indeed, the set $Z = \{x \in X \,|x_{\mathbf{F}} = 0_{\mathbf{F}'}\} = \kappa_{\mathbf{F}}^{-1}(0_{\mathbf{F}'}) = \bigcap\limits_{f \in \mathbf{F}} f^{-1}(0)$ is closed, and $\mathbf{F}|_{X \setminus Z}$ is 1-independent. $\square$

Now let us relate the most basic properties of u, φ and $W_{\varphi,u}$, and also address the issue of recovering the symbols of WCO mentioned before. Most of the facts that we discuss in the rest of the subsection are simple and appear in the literature in various contexts, but we will state them within our general framework. We will also provide the proofs of some of these facts for completeness.

Proposition 2.1.8. *Let X, Y be sets, let $\mathbf{F} \subset \mathcal{F}(X)$ and let $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$. If $W_{\varphi,u}\mathbf{F}$ is 1 (2) independent, then u does not vanish (and φ is an injection). Conversely, if $\mathbf{F}$ is n-independent, for $n \in \mathbb{N} \cup \{\infty\}$, φ is an injection and u does not vanish, then $W_{\varphi,u}\mathbf{F}$ is n-independent.*

Proof. Denote $\mathbf{H} = W_{\varphi,u}\mathbf{F}$.

If $u(y) = 0$, then every element of $\mathbf{H}$ vanishes at y, and so $y_{\mathbf{H}} = 0$, which contradicts the 1-independence. If $\varphi(x) = \varphi(y)$, for some $x, y \in Y$, then $u(y)\, x_{\mathbf{H}} = u(x)\, y_{\mathbf{H}}$. Indeed, for any $g \in \mathbf{H}$ there is $f \in \mathbf{F}$ such that $g = W_{\varphi,u} f$, and so

$$u(y)\, g(x) = u(y)\, u(x)\, f(\varphi(x)) = u(x)\, u(y)\, f(\varphi(y)) = u(x)\, g(y).$$

If $\mathbf{H}$ is 2-independent, then $u(y), u(x) \neq 0$, and so using 2-independence again, we get $x = y$.

Conversely, assume that $\mathbf{F}$ is n-independent, $\varphi : Y \to X$ is an injection and $u : Y \to \mathbb{F}^{\times}$. Let $y^1, y^2, ..., y^n \in Y$ be distinct, and let $\alpha_1, ..., \alpha_n \in \mathbb{F}$ be such that $\alpha_1 y^1_{\mathbf{H}} + ... + \alpha_n y^n_{\mathbf{H}} = 0_{\mathbf{H}}$. Let $x^i = \varphi(y^i)$ and $\beta_i = \alpha_i u(y^i)$, for $i \in \overline{1,n}$. Note that $x^1, x^2, ..., x^n$ are distinct. Then

$$0_{\mathbf{F}'} = W'_{\varphi,u}\left(\alpha_1 y^1_{\mathbf{H}} + ... + \alpha_n y^n_{\mathbf{H}}\right) = \beta_1 x^1_{\mathbf{F}} + ... + \beta_n x^n_{\mathbf{F}},$$

which is only possible when $0 = \beta_i = \alpha_i u(y^i)$, for $i \in \overline{1,n}$. Since u does not vanish we conclude that $\alpha_i = 0$, for $i \in \overline{1,n}$, and so $\mathbf{H}$ is n-independent. $\square$

Proposition 2.1.9. *Let X and Y be sets, let $\mathbf{F}$ be a linear subspace of $\mathcal{F}(X)$ and let $\varphi, \psi : Y \to X$ and $u, v : Y \to \mathbb{F}$. Then:*

(i) If φ is a surjection and u does not vanish, then $W_{\varphi,u}$ is an injection.

(ii) If $\mathbf{F}$ is 2-independent and $W_{\varphi,u} = W_{\psi,v}$ on $\mathbf{F}$, then $u = v$ on Y, and $\varphi = \psi$ on $Y \backslash u^{-1}(0)$. In particular, if either u or v does not vanish, then $u = v$ and $\varphi = \psi$.

(iii) Assume that there is a linear operator $T : \mathbf{F} \to \mathbf{F}$ such that $W_{\psi,v} = W_{\varphi,u} T$. If φ is a surjection and u does not vanish, then there are maps $\theta : X \to X$ and $h : X \to \mathbb{F}$ such that $T = W_{\theta,h}$. If $\mathbf{F}$ is 2-independent, then $\theta \circ \varphi = \psi$ and $u \cdot h \circ \varphi = v$.

Proof. (i): If φ is a surjection and u does not vanish, then from Corollary 2.1.4 we have $\mathrm{Ker}\, W_{\varphi,u} = \varphi(Y \backslash u^{-1}(0))^{\perp}_{\mathbf{F}} = X^{\perp}_{\mathbf{F}} = \{0\}$, since $X_{\mathbf{F}}$ is separating on $\mathbf{F}$.

(ii): Let $Z = Y \backslash u^{-1}(0)$. If $W_{\varphi,u} = W_{\psi,v}$, then $W'_{\varphi,u} = W'_{\psi,v}$, and so $u(y)\varphi(y)_{\mathbf{F}} = v(y)\psi(y)_{\mathbf{F}}$, for any $y \in Y$. Since $\mathbf{F}$ is 2-independent we immediately get that $u^{-1}(0) = v^{-1}(0)$. For every $y \in Z$ we have that $\varphi(y)_{\mathbf{F}} = \frac{v(y)}{u(y)}\psi(y)_{\mathbf{F}}$, and so using 2-independence again we get $\varphi(y)_{\mathbf{F}} = \psi(y)_{\mathbf{F}}$ and $\frac{v(y)}{u(y)} = 1$. Hence, φ and u coincide with ψ and v respectively on Z. Since u and v both vanish outside of Z we conclude that $u = v$.

(iii): If $W_{\psi,v} = W_{\varphi,u}T$ then $T'W'_{\varphi,u} = W'_{\psi,v}$, and so $u(y)\,T'\varphi(y)_{\mathbf{F}} = v(y)\,\psi(y)_{\mathbf{F}}$, for every $y \in Y$. Hence, $T'\varphi(y)_{\mathbf{F}} = \frac{v(y)}{u(y)}\psi(y)_{\mathbf{F}}$, for each $y \in Y$. As φ is a surjection we get that $T'X_{\mathbf{F}} \subset \mathbb{F}X_{\mathbf{F}}$, and so by virtue of part (i) of Corollary 2.1.3, T is a WCO, i.e. $T = W_{\theta,h}$, for some $\theta : X \to X$ and $h : X \to \mathbb{F}$. Since in this case $W_{\psi,v} = W_{\theta\circ\varphi,h\circ\varphi}$, if $\mathbf{F}$ is 2-independent, then it follows from part (ii) that $\theta \circ \varphi = \psi$ and $h \circ \varphi = \frac{v}{u}$. □

Corollary 2.1.10. *Let* $\mathbf{F}$ *be a NSF over* X*, let* $\mathbf{E}$ *be a* 1 *(*2*) independent NSF over* Y*, and let* $\varphi : Y \to X$ *and* $u : Y \to \mathbb{F}$ *be such that* $W_{\varphi,u} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$*. Then* $W^*_{\varphi,u}$ *is injection* $\Leftrightarrow$ $W_{\varphi,u}$ *has dense image* $\Rightarrow$ u *does not vanish (and* φ *is an injection). If moreover* φ *is a surjection, then* $W^*_{\varphi,u}$ *is injection* $\Rightarrow$ $W_{\varphi,u}$ *is an injection.*

Proof. Recall that injectivity of an adjoint operator is equivalent to the density of the image of the original operator (see [Bou87, IV.3, Proposition 5]). The second implication follows from Proposition 2.1.8, since a dense subspace of n-independent NSF is n-independent. If φ is a surjection, then injectivity of $W^*_{\varphi,u}$ implies that u does not vanish, and so $W_{\varphi,u}$ is an injection due to part (i) of Proposition 2.1.9. □

Corollary 2.1.11. *Assume that* $\mathbf{F}$ *and* $\mathbf{E}$ *are* 2*-independent NSF's over sets* X *and* Y *respectively. Let* $\varphi : Y \to X$ *be a surjection and let* $u : Y \to \mathbb{F}$ *be such that* $W_{\varphi,u}$ *is an invertible linear operator between* $\mathbf{F}$ *and* $\mathbf{E}$*. Then* φ *is an injection,* u *does not vanish and* $W^{-1}_{\varphi,u}$ *is a WCO whose symbols are uniquely determined, namely* $W^{-1}_{\varphi,u} = W_{\varphi^{-1},\frac{1}{u\circ\varphi^{-1}}}$.

Slightly modifying the proof of Proposition 2.1.9 yields the following result.

Proposition 2.1.12. *Let X and Y be topological spaces and let $\mathbf{F} \subset \mathcal{C}(X)$ be a linear subspace. Let $\varphi, \psi : Y \to X$ be continuous and let $u, v : Y \to \mathbb{F}$ be such that $W_{\varphi,u}\mathbf{F} \subset \mathcal{C}(Y)$ and $W_{\psi,v}\mathbf{F} \subset \mathcal{C}(Y)$. Then:*

(i) If φ has a dense image and u vanishes on a nowhere dense set, then $W_{\varphi,u}$ is an injection.

(ii) If $\mathbf{F}$ is 2-independent and $W_{\varphi,u} = W_{\psi,v}$, then $u = v$ on Y and $\varphi = \psi$ on $\overline{Y \backslash u^{-1}(0)}$. In particular, if either u or v vanishes on a nowhere dense set, then $u = v$ and $\varphi = \psi$.

(iii) Assume that there is a linear operator $T : \mathbf{F} \to \mathbf{F}$ such that $W_{\psi,v} = W_{\varphi,u}T$. If φ is a surjection, u vanishes on a nowhere dense set, and there is a continuous function $w : Y \to \mathbb{F}$, such that $v = uw$, then there are maps $\theta : X \to X$ and $h : X \to \mathbb{F}$ such that $T = W_{\theta,h}$. If $\mathbf{F}$ is 2-independent, then $\theta \circ \varphi = \psi$ and $h \circ \varphi = w$.

Remark 2.1.13. Parts (i) of Proposition 2.1.9 and Proposition 2.1.12 justify our early claims that certain rescalings and restrictions are isometric isomorphisms. $\square$

Remark 2.1.14. If in the statement of the Proposition 2.1.12 X and Y are complex manifolds and φ, ψ, u, v are holomorphic, then the assumption that u vanishes on a nowhere dense set is fulfilled automatically once $u \not\equiv 0$. Also, in part (i) we can require only that the image of φ is non-thin, e.g. somewhere dense. Moreover, this condition is satisfied automatically if the dimensions of X and Y coincide and φ is a local injection at some point (e.g. if X, Y are 1-dimensional and φ is not a constant).

If $\mathbf{F}$ is a NSHF over a complex manifold X, then a restriction on a non-thin set is an isometric isomorphism. For example, if $\dim X = 1$ and $Z \subset X$ has a limit point, then $\mathbf{F}|_Z$ is isometrically isomorphic to $\mathbf{F}$.

If $\mathbf{F} \neq \{0\}$ and Z is as in Example 2.1.7, then Z is thin, and so $\mathbf{F}|_{X \backslash Z}$ is isometrically isomorphic to $\mathbf{F}$. $\square$

2.1.c Basic Quantitative Properties of WCO's

We will now consider some quantitative properties of WCO's. Combining Corollary 2.1.10 with a variation of the Open Mapping theorem (see [FHH$^+$11, Exercise 2.49] with the solution therein) we obtain the following result.

Corollary 2.1.15. *Assume that* $\mathbf{F}$ *and* $\mathbf{E}$ *are* 1*-independent BSF's over sets* X *and* Y. *Let* $\varphi : Y \to X$ *be a surjection and let* $u : Y \to \mathbb{F}$ *be such that* $W_{\varphi,u} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$. *Then* $W^*_{\varphi,u}$ *is bounded from below (isometry) if and only if* $W_{\varphi,u}$ *is an (isometric) isomorphism.*

Remark 2.1.16. If X is a topological space and $\mathbf{F}$ is a NSCF, then the requirement for φ to be surjective can be replaced with the requirement for φ to have a dense image in X. $\square$

It follows from Proposition 2.1.2 that if $\mathbf{F}$ and $\mathbf{E}$ are NSF's over sets X and Y and $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ are such that $W_{\varphi,u} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, then for every $y \in Y$ we have

$$|u(y)| \, |\kappa|_{\mathbf{F}} (\varphi(y)) = \|u(y) \, \varphi(y)_{\mathbf{F}}\| = \|W^*_{\varphi,u} \kappa_{\mathbf{E}}(y)\| \leq \|W_{\varphi,u}\| \, |\kappa|_{\mathbf{E}}(y),$$

i.e. $|u| \, |\kappa|_{\mathbf{F}} \circ \varphi \leq \|W_{\varphi,u}\| \, |\kappa|_{\mathbf{E}}$, or equivalently $\|W_{\varphi}\| \geq \|u \, |\kappa|_{\mathbf{F}} \circ \varphi\|^{|\kappa|_{\mathbf{E}}}_{\infty}$. Hence, if $|\kappa|_{\mathbf{F}}$ is bounded from below, and $|\kappa|_{\mathbf{E}}$ is bounded from above, then u is bounded. Also, if M_u is bounded (and invertible) on $\mathbf{F}$, then u is bounded (and bounded away from 0).

Example 2.1.17. Let X and Y be Tychonoff spaces and let α and β be nonnegative lower semi-continuous functions on X and Y respectively. Then for continuous $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ the operator $W_{\varphi,u}$ is a bounded from $\mathcal{C}^{\alpha}_{\infty}$ into $\mathcal{C}^{\beta}_{\infty}$ if and only if $\frac{u \cdot \alpha \circ \varphi}{\beta}$ is bounded. Moreover, $\|W_{\varphi,u}\| = \|u \cdot \alpha \circ \varphi\|^{\beta}_{\infty}$. Indeed, for any $f \in \mathcal{C}(X)$ such that $|f| \leq \alpha$, we have that

$|u \cdot f \circ \varphi| \leq u \cdot \alpha \circ \varphi$. Since using Example 1.2.17 one can see that $\|u \cdot \alpha \circ \varphi\|_{\infty}^{\beta} \leq \|W_{\varphi,u}\|$, we conclude that $\|W_{\varphi,u}\| = \|u \cdot \alpha \circ \varphi\|_{\infty}^{\beta}$.

Assume that $a = \inf\limits_{x \in X} \alpha(x) > 0$ and $b = \sup\limits_{y \in Y} \beta(y) < +\infty$. Let $\varphi, \psi \in \mathcal{C}(Y, X)$ be distinct and let $y \in Y$ be such that $\varphi(y) = x \neq z = \psi(y)$. Then,

$$\|C_{\varphi} - C_{\psi}\| = \|C_{\varphi}^{*} - C_{\psi}^{*}\| \geq \frac{\|\left(C_{\varphi}^{*} - C_{\psi}^{*}\right) y_{\mathcal{C}_{\infty}^{\beta}}\|}{\|y_{\mathcal{C}_{\infty}^{\beta}}\|} = \frac{\|x_{\mathcal{C}_{\infty}^{\alpha}} - z_{\mathcal{C}_{\infty}^{\alpha}}\|}{\|y_{\mathcal{C}_{\infty}^{\beta}}\|} = \frac{\alpha(x) + \alpha(z)}{\beta(y)} \geq 2\frac{a}{b},$$

and so the metric on $\mathcal{C}(Y, X)$, generated by the norm of $\mathcal{L}\left(\mathcal{C}_{\infty}^{\alpha}, \mathcal{C}_{\infty}^{\beta}\right)$, is discrete.

Assume that X and Y are compact. Then $\mathcal{C}(X) = \mathcal{C}_{\infty}(X)$ and $\mathcal{C}(Y) = \mathcal{C}_{\infty}(Y)$. It follows that $\|C_{\varphi}\| = 1$, for any $\varphi \in \mathcal{C}(Y, X)$. Hence, every $\mathcal{C}(Y, X)$ can be considered as a subset of $\mathcal{L}(\mathcal{C}(X), \mathcal{C}(Y))$, and the topology induced by the norm of $\mathcal{L}(\mathcal{C}(X), \mathcal{C}(Y))$ is the discrete topology. This is in stark contrast with the compact-open topology that is induced on $\mathcal{C}(Y, X)$ by both compact-open and pointwise topologies on $\mathcal{L}(\mathcal{C}(X), \mathcal{C}(Y))$. $\square$

Another consequence of Proposition 2.1.2 is that if $\mathbf{F}$ and $\mathbf{E}$ are NSF's over sets X and Y and $\varphi : Y \to X$ is such that $C_{\varphi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, then $\|C_{\varphi}\| \geq \mathrm{dil}_{d_{\mathbf{F}}}^{d_{\mathbf{E}}}(\varphi)$. Naturally, the Lipschitz space is the NSF for which this estimate is the most precise.

Example 2.1.18. Let ρ and λ be pseudometrics on sets X and Y respectively and let $\mathbf{F} = \mathrm{Lip}^{\rho}$, $\mathbf{E} = \mathrm{Lip}^{\lambda}$, endowed with the norms $\|\cdot\|_{\mathbf{F}}^{\{x\}}$ and $\|\cdot\|_{\mathbf{E}}^{\{y\}}$ respectively, for some fixed $x \in X$ and $y \in Y$ (see Example 1.2.14). Then, $d_{\mathbf{F}} = \rho$ and $d_{\mathbf{E}} = \lambda$. Since the dilation of a composition of two maps does not exceed the product of their dilations one can deduce that

$$\max\left\{1, \mathrm{dil}_{\rho}^{\lambda}\varphi\right\} \leq \|C_{\varphi}\| \leq \max\left\{1, d_{\mathbf{F}}(\varphi(y), x) + \mathrm{dil}_{\rho}^{\lambda}\varphi\right\}.$$

In particular, continuous CO's are induced precisely by the Lipschitz maps from Y into X, and if $\varphi(y) = x$, then $\|C_{\varphi}\| = \max\left\{1, \mathrm{dil}_{\rho}^{\lambda}\varphi\right\}$. $\square$

Let us consider implications of Proposition 2.1.2 for Reproducing Kernel Hilbert spaces.

Example 2.1.19. If X and Y are sets, k is a positive semi-definite kernel on X and $\varphi : Y \to X$ define $k \circ \varphi : Y \times Y \to \mathbb{F}$ by $[k \circ \varphi](x, y) = k(\varphi(x), \varphi(y))$, for $x, y \in Y$. It follows that if $u : Y \to \mathbb{F}$, then the $k_{W_{\varphi,u}} = u \otimes \overline{u} k \circ \varphi$. In particular, we can find kernels of restrictions and rescalings of a RKHS. Namely, if $v : X \to \mathbb{F}^{\times}$, then $v\mathbf{H}_k = \mathbf{H}_{v \otimes \overline{v} k}$; if Z is a set, and $\psi : Z \to X$ is an injection, then $\mathbf{H}_k \circ \psi = \mathbf{H}_{k \circ \psi}$ (this implies that $k \circ \psi \gg 0$). $\square$

Using Corollary 1.2.33 and part (i) of Proposition 2.1.9 we get a criterion for continuity of WCO's between RKHS's.

Proposition 2.1.20. *Let k and l be positive semi-definite kernels over sets X and Y respectively. Then $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ give rise to a WCO from $\mathbf{H}_k$ into $\mathbf{H}_l$ of norm at most α if and only if $u \otimes \overline{u} k \circ \varphi \ll \alpha^2 l$. If φ is a surjection, u does not vanish and $u \otimes \overline{u} k \circ \varphi \gg \beta^2 l$, then $W_{\varphi,u}$ is a linear homeomorphism with the norm of the inverse at most $\frac{1}{\beta}$. If in this case $\beta = \alpha = 1$, then the $W_{\varphi,u}$ is a unitary.*

Observe that every continuous $\mathrm{MO}_{\mathcal{H}}$'s on NSHF's on $\mathbb{C}^n$ has to be a multiple of identity. Indeed, the symbol of such an operator has to be bounded and holomorphic on $\mathbb{C}^n$, and so it is a constant by Liouville's theorem (see [Sch05, 1.2.12]). This circumstance allows us to construct an example of a unitary $\mathrm{WCO}_{\mathcal{H}}$ on a HSHF such that the corresponding CO and MO are not well-defined.

Example 2.1.21. Let $X = \mathbb{C}$ and let $k : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be defined by $k(z, w) = \exp(z\overline{w})$. Then k is positive definite sesqui-holomorphic and $\mathbf{H}_k$ is the classical *Fock-Segal-Bargmann* space (see [ARSW11]). Define $\varphi : \mathbb{C} \to \mathbb{C}$ and $u : \mathbb{C} \to \mathbb{C}^{\times}$ by $\varphi(z) = z + 1$ and $u(z) = e^{-z-\frac{1}{2}}$, for $z \in \mathbb{C}$. Then $u \otimes \overline{u} k \circ \varphi = k$, and so $W_{\varphi,u}$ is a unitary operator on $\mathbf{H}_k$ due to Proposition

2.1.20. On the other hand, from the comment above, there is no continuous MO$_{\mathcal{H}}$'s on $\mathbf{H}_k$, other than multiples of identity. Moreover, let us show that $C_\varphi \mathbf{H}_k \not\subset \mathbf{H}_k$. Indeed, if $C_\varphi \mathbf{H}_k \subset \mathbf{H}_k$, then due to Proposition 2.1.20, $k \circ \varphi \ll \alpha^2 k$, for some $\alpha > 0$. But then since for $z \in \mathbb{C}$ we have $k(\varphi(z), \varphi(z)) = \exp\left(|z+1|^2\right)$, we get that

$$\left[\alpha^2 k - k \circ \varphi\right](z,z) = \exp\left(|z|^2\right)\left(\alpha^2 - \exp\left(1 + 2\mathrm{Re}\ z\right)\right) > 0.$$

The latter claim is false, and so we have reached a contradiction. For more details on WCO's on $\mathbf{H}_k$ see [Le14]. □

Let us now consider an interesting case of $\mathcal{C}$-rescaling. The following result is analogous to and follows from Corollary 1.2.22.

Proposition 2.1.22. *Let* $\mathbf{F}$ *be a NSCF over a locally compact space* X*, which strongly separates the points, and such that* $|\kappa|_{\mathbf{F}}$ *is continuous (e.g.* $\mathbf{F}$ *is compactly embedded). If* $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$*, then* $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F}$ *generates the topology of* X *and* $\left\{\frac{1}{\|x_{\mathbf{F}}\|} x_{\mathbf{F}} \,|\, x \in X\right\} \sqcup \{0_{\mathbf{F}^*}\}$ *is the one point compactification of* X*.*

Remark 2.1.23. Obviously, if $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$ and $1 \in \mathbf{F}$ then $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$. Conversely, if $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$ and the bounded functions form a dense set in $\mathbf{F}$ then $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$. Indeed, for any $f \in \mathbf{F}$ and $\varepsilon > 0$ there is $g \in \mathbf{F}$ and $\alpha > 0$, such that $\|f - g\| \leq \frac{\varepsilon}{2}$ and $|g(x)| \leq \alpha$, for every $x \in X$. Since $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$, there is a compact set $K \subset X$, such that $|\kappa|_{\mathbf{F}}(x) > \frac{2\alpha}{\varepsilon}$, whenever $x \notin K$. Consequently, for $x \notin K$ we have that

$$\frac{|f(x)|}{|\kappa|_{\mathbf{F}}(x)} \leq \frac{|f(x) - g(x)| + |g(x)|}{|\kappa|_{\mathbf{F}}(x)} \leq \frac{|\langle f - g, x_{\mathbf{F}}\rangle|}{\|x_{\mathbf{F}}\|} + \frac{|g(x)|}{|\kappa|_{\mathbf{F}}(x)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim\limits_{\infty} \frac{|f|}{|\kappa|_{\mathbf{F}}} = 0$, and so $\frac{f}{|\kappa|_{\mathbf{F}}} \in \mathcal{C}_0(X)$. Thus, $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$. □

2.1.d Invertible WCO's on NSCF's

From Corollary 2.1.11 the question of whether the inverse of a bijective WCO is a WCO comes down to the surjectivity of the composition symbol. In this subsection we look for condition that guarantee that if a WCO is a linear homeomorphism, its composition symbol is surjective. Unsurprisingly, such conditions are only present in (at least) the continuous setting. The following is a generalization of a result from [Zor17b].

Proposition 2.1.24. *Assume that* $\mathbf{F}$ *and* $\mathbf{E}$ *are* 2*-independent NSCF's over locally compact spaces* X *and* Y *respectively, with* $\lim\limits_{\infty}|\kappa|_{\mathbf{E}}=+\infty$*. Let* $\varphi:Y\to X$ *be continuous and let* $u:Y\to\mathbb{F}$ *be bounded and such that* $W_{\varphi,u}$ *is a linear homeomorphism from* $\mathbf{F}$ *onto* $\mathbf{E}$*. Then:*

(i) $\varphi(Y)$ *is closed in* X *and* φ *is a topological embedding.*

(ii) In the case when X *is connected an* φ *is and open map, then* φ *is a homeomorphism.*

(iii) In the case when X *and* Y *are topological manifolds with* $\dim X\le\dim Y$*, then* φ *is a homeomorphism.*

Before proving the result consider a technical lemma.

Lemma 2.1.25. *Let* $h:X\to[0,+\infty)$ *be bounded on compacts. If* $\varphi:Y\to X$ *is continuous and such that* $\lim\limits_{\infty}h\circ\varphi=+\infty$*, then* φ *is a closed map.*

Proof. It is easy to show that the conditions of the lemma imply that the preimage of every compact set in X with respect to φ is compact in Y. Since X is compactly generated, this condition is equivalent to the fact that φ is a perfect map (see [Eng89, Theorem 3.7.18]), and consequently, φ is a closed map. $\square$

Now we can proceed with the proof of the proposition.

Proof of Proposition 2.1.24. First, note that if $W_{\varphi,u}$ is bounded and invertible, then so is $W^*_{\varphi,u}$. Also, by Corollary 2.1.10, φ is an injection. Hence, part (iii) follows from part (ii) and Invariance of Domain (see [Hat02, Theorem 2B.3]), and part (ii) follows from part (i). It is left to prove the latter.

Since $\mathbf{F}$ is a NSCF, $|\kappa|_{\mathbf{F}}$ is bounded on the compacts in X. For every $y \in Y$ we have that

$$|\kappa|_{\mathbf{E}}(y) \leq \|W^{*-1}_{\varphi,u}\| \|W^*_{\varphi,u}\kappa_{\mathbf{E}}(y)\| = \|W^{*-1}_{\varphi,u}\| |u(y)| |\kappa|_{\mathbf{F}}(\varphi(y)) \leq \|W^{*-1}_{\varphi,u}\| \alpha |\kappa|_{\mathbf{F}}(\varphi(y)),$$

where $\alpha = \sup\limits_{y \in Y} |u(y)|$. Therefore, $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} \circ \varphi = +\infty$. Hence, from the lemma above, we get that φ is closed. Thus, $\varphi(Y)$ is closed in X and φ is a closed injective map, and so a topological embedding. $\square$

Remark 2.1.26. Note that we do not assume that Y is non-compact. Also note that instead of assuming that $W_{\varphi,u}$ is a linear homeomorphism, it is enough to assume that $W^*_{\varphi,u}$ is bounded from below, although if $\mathbf{F}$ and $\mathbf{E}$ are Banach spaces the latter requirement is equivalent to the former, due to Corollary 2.1.15. $\square$

The proof of part (i) of Proposition 2.1.24 works if the assumptions that $\lim\limits_{\infty} |\kappa|_{\mathbf{E}} = +\infty$ and u is bounded are replaced with the assumption that $\lim\limits_{\infty} \frac{u}{|\kappa|_{\mathbf{E}}} = 0$. However, none of the conditions of the proposition can be dropped completely, as the following series of examples shows.

Example 2.1.27. Consider the Hardy space H^2 (see Example 1.3.17), which is a HSHF over $\mathbb{D}$. We will construct three restrictions of H^2 on subsets of $\mathbb{D}$ and unitary composition operators on them induced by non-bijective symbols. It is known that C_φ is a continuous invertible operator on H^2, for any fractional-linear transformation φ that maps $\mathbb{D}$ onto itself (see [SM93, Theorem 3.2.1]). Let φ be any such transformation, which also maps $X = [0, 1)$ into itself with $\varphi(0) > 0$. For example, let $\varphi(w) = \frac{2w+1}{w+2}$.

Let $\mathbf{F} = H^2|_X$. Since X is not discrete, the restriction operator is an isometric isomorphism. Hence, $C_{\varphi|_X}$ is a continuous invertible operator on $\mathbf{F}$. Finally,

$$\lim_{\infty} |\kappa|_{\mathbf{F}} = \lim_{w\to 1} |\kappa|_{H^2}(w) = \lim_{w\to 1} \frac{1}{1 - |w|^2} = +\infty.$$

However, $\varphi|_X$ is neither a surjection, nor an open map.

Consider another restriction of H^2. Define $\{a_n\}_{n=0}^{\infty}$ recursively by $a_0 = 0$ and $a_{n+1} = \varphi(a_n)$. Since $\varphi|_X$ is a strictly increasing map without fixed points, we get that $\lim\limits_{n\to\infty} a_n = 1$. Then φ^2 maps $Y = \bigcup\limits_{n\geq 0} [a_{2n}, a_{2n+1}]$ into itself, and $\varphi^2|_Y$ is an open map. Let $\mathbf{E} = H^2|_Y$. Then $C_{\varphi^2|_Y}$ is a continuous invertible operator on $\mathbf{E}$ and $\lim\limits_{y\to\infty} |\kappa|_{\mathbf{E}}(y) = \lim\limits_{w\to 1} |\kappa|_{H^2}(w) = +\infty$. However, again $\varphi^2|_Y$ is not a surjection, as $\varphi(Y) \cap [a_0, a_1] = \varnothing$.

For the last restriction, let $Z = \bigcup\limits_{n\geq 0} B_n$, where B_n is an open disk with diameter $[a_n, a_{n+2}]$. Clearly, Z is a connected open set in $\mathbb{C}$. Since φ maps circles into circles, $[0, 1)$ into itself, and preserves angles, it is easy to see that $\varphi(B_n) = B_{n+1}$. Hence, $\varphi(Z) \subset Z$ and $\varphi|_Z$ is an open map. Let $\mathbf{H} = H^2|_Z$. Then $C_{\varphi^2|_Z}$ is a continuous invertible operator on $\mathbf{H}$. However, again $\varphi^2|_Z$ is not a surjection, since $\varphi(Z) \cap B_0 = \varnothing$. The condition of the proposition that fails this time is the one involving the norm of the point-evaluations. Namely, $\lim\limits_{z\to\infty} |\kappa|_{\mathbf{H}}(z) = \lim\limits_{w\to\partial Z} \frac{1}{1-|w|^2}$ does not exist. This result can be modified using rescaling, to construct a continuous invertible WCO$_{\mathcal{C}}$ $W_{\varphi,u}$ on a HSHF $\mathbf{G}$ over Z with $\lim\limits_{z\to\infty} |\kappa|_{\mathbf{G}}(z) = +\infty$, but $\lim\limits_{y\to\infty} \frac{u(y)}{|\kappa|_{\mathbf{G}}(y)} \neq 0$. $\square$

2.1.e Unitary MO's on NSCF's

In this subsection we will assume that **F** is a 1-independent NSCF over a topological space X. We will give several sufficient conditions for **F** to be such that a unitary MO on **F** can only be a multiple of the identity. In other words we are looking for restrictions that guarantee that if $u : X \to \mathbb{F}$ is such that M_u is a unitary, then $u \equiv \alpha$, for some $\alpha \in \mathbb{T}$. Clearly, there is not much that can be said if X is disconnected. Hence, in this subsection we will further assume that X is a connected Hausdorff topological space. A restriction on **F** that would accomplish what we want is not that obvious, and this will be the main subject of our investigation.

Remark 2.1.28. In fact, instead of studying unitary MO's we will mostly consider *co-isometric* MO's (i.e. such that their adjoints are isometries), although if **F** is a Banach space the latter requirement is equivalent to the former, due to Corollary 2.1.15. $\square$

It follows immediately that if M_u is a co-isometry, then $|u| \equiv 1$. We will show in part (i) of Corollary 2.2.4 below that u is continuous. This information is sufficient to conclude that u is a constant function in some cases. Namely, if **F** consists of real-valued functions, it is easy to see that u must be real-valued, and so u is a continuous function on a connected topological space with values ± 1, and so either $u \equiv 1$, or $u \equiv -1$. Similarly, if X is a complex manifold and **F** consists of holomorphic functions, then one can show that u is holomorphic. Hence, u is a constant function, since otherwise u would not have a constant absolute value (due to the Open Mapping theorem).

However, in general there are NSCF's which admit more unitary MO's than just multiples of identity. Indeed, for any topological space X and a continuous function $u : X \to \mathbb{T}$ the operator M_u is an invertible isometry of $\mathcal{C}_\infty(X)$. Hence, if we do not impose restrictions on what kind of functions constitute $\mathbf{F}$, we have to impose other restrictions. The key for finding those restriction is the following general result (see the proof in Appendix A.1).

Theorem 2.1.29. *Let E be a normed space and let $T : E \to E$ be an isometry. Let $D \subset E \backslash \{0_E\}$ consist of eigenvectors of T. Then $T|_D = \alpha Id_D$, for some $\alpha \in \mathbb{T}$ provided that one of the following conditions is satisfied:*

(i) D is connected in the norm topology;

(ii) D is weakly connected and for each $e \in D$ the set $\left\{\nu \in \overline{B}_{E^*}, \langle e, \nu\rangle = \|e\|\right\}$ is finite dimensional.

Now we are equipped to provide some sufficient conditions that guarantee that $\mathbf{F}$ does not support non-trivial unitary MO's.

Theorem 2.1.30. *Let $\mathbf{F}$ be a NSCF over a connected topological space X, and let $u : X \to \mathbb{F}$ be such that M_u is an isometric isomorphism on $\mathbf{F}$. Then $u \equiv \alpha$, for some $\alpha \in \mathbb{T}$ whenever one of the following conditions is satisfied:*

(i) X is compactly generated and $\mathbf{F}$ is a compactly embedded NSCF;

(ii) $\mathbf{F}$ is weakly compactly embedded and for any $x \in X$ the set $\left\{f \in \overline{B_{\mathbf{F}}}^{\mathcal{C}(X)} \middle| f(x) = \|x_{\mathbf{F}}\|\right\}$ is finite-dimensional.

Proof. Let $D = X_{\mathbf{F}}$, and $E = \overline{\mathrm{span} D}$. Since $M_u^* x_{\mathbf{F}} = u(x)\, x_{\mathbf{F}}$, for every $x \in X$, it follows that $T = M_u^*|_E$ maps E into itself. It is enough to verify that T and D satisfy the conditions of Theorem 2.1.29.

From part (iii) of Theorem 1.1.8, if X is compactly generated and $\mathbf{F}$ is compactly embedded, then $\kappa_{\mathbf{F}}$ is norm-continuous, and so $D = \kappa_{\mathbf{F}}(X)$ is connected in the norm topology. Hence, T and D satisfy the condition (i) of the theorem.

If $\mathbf{F}$ is weakly compactly embedded, from part (iv) of Theorem 1.1.8, $\kappa_{\mathbf{F}}$ is weakly continuous, and so D is connected in the weak topology. Recall from the discussion before Proposition 1.2.8 that $E^* = \widehat{\mathbf{F}}$, which is a regular NSCF defined by $\overline{B}_{\widehat{\mathbf{F}}} = \overline{B_{\mathbf{F}}}^{\mathcal{C}(X)}$, and $\langle x_{\mathbf{F}}, f\rangle = f(x)$, for $f \in \widehat{\mathbf{F}}$ and $x \in X$. Since the set $\left\{f \in \overline{B_{\mathbf{F}}}^{\mathcal{C}(X)} \,|\, f(x) = \|x_{\mathbf{F}}\|\right\}$ is finite-dimensional, for every $x \in X$, it follows that T and D satisfy the condition (ii) of the theorem. $\square$

The conditions of part (ii) of the theorem are difficult to apply, and so it is desirable to find stronger conditions which are more readily verifiable. It turns out, such conditions are of geometric nature. Namely, we will introduce a class of normed spaces for which part of the condition (ii) holds automatically. We will call a normed space E *finitely convex* if any (closed) convex subset of ∂B_E is finite dimensional. The class of finitely convex normed spaces contains all finitely-dimensional normed spaces, all uniformly convex normed spaces (or more generally, strictly convex normed spaces, see [FHH+11, Definition 7.6]), and is closed under finite l^p sums, where $l \in (1, +\infty)$ (see Remark A.1.3). We can now state a simplified version of part (ii) of the theorem.

Corollary 2.1.31. *Let $\mathbf{F}$ be a NSCF over a connected topological space X, and let $u : X \to \mathbb{F}$ be such that M_u is an isometric isomorphism on $\mathbf{F}$. Then $u \equiv \alpha$, for some $\alpha \in \mathbb{T}$ whenever one of the following conditions is satisfied:*

(i) $\mathbf{F}$ *is weakly compactly embedded and* $\mathbf{F}^{**}$ *is finitely convex;*

(ii) $\mathbf{F}$ *is reflexive and finitely convex.*

Proof. (i): We only need to verify that the set $L_x = \left\{ f \in \overline{B_{\mathbf{F}}}^{\mathcal{C}(X)} \,|\, f(x) = \|x_{\mathbf{F}}\| \right\}$ is finite-dimensional for any $x \in X$. Indeed, since $\overline{B_{\mathbf{F}}}^{\mathcal{C}(X)} = J_{\mathbf{F}}^{**}\overline{B}_{\mathbf{F}^{**}}$, we have that L_x is the image under $J_{\mathbf{F}}^{**}$ of the set $N_x = \left\{ f \in \overline{B}_{\mathbf{F}^{**}} \,|\langle f, x_{\mathbf{F}}\rangle = \|x_{\mathbf{F}}\| \right\}$. Clearly, $N_x \subset \partial B_{\mathbf{F}^{**}}$ and is a convex set. Since $\mathbf{F}^{**}$ is finitely convex, it follows that N_x is finite-dimensional. Hence, L_x is also finite-dimensional.

(ii): Recall that every reflexive NSCF is weakly compactly embedded. Also, if $\mathbf{F}$ is finitely convex, then $\mathbf{F}^{**} = \mathbf{F}$ is finitely convex as well, and so the conditions of part (i) are satisfied. $\square$

Remark 2.1.32. Note that in part (ii) the conditions on $\mathbf{F}$ were only about its Banach space properties and have nothing to do with its embedding into $\mathcal{C}(X)$. In particular, every HSCF's over a connected Hausdorff space does not support non-trivial unitary MO's. $\square$

Similarly, we can prove an analogous statement for WCO$_{\mathcal{C}}$'s.

Proposition 2.1.33. *Let X be a topological space, let Y be a connected topological space, let $\mathbf{F}$ be a 1-independent NSCF over X that satisfies one of the conditions of Theorem 2.1.30 or Corollary 2.1.31, and let $\mathbf{E}$ be a NSCF over Y. If $\varphi : Y \to X$ is continuous, and $u, v : Y \to \mathbb{F}^{\times}$ are such that there is a co-isometry $S : \mathbf{F} \to \mathbf{F}$ such that $W_{\varphi,u} = W_{\varphi,v}S$ (e.g. if $\mathbf{E}$ is a NSCF and both $W_{\varphi,u}$ and $W_{\varphi,v}$ are co-isometries), then $u = \alpha v$, $\alpha \in \mathbb{T}$.*

Proof. The proof is analogous to that of Theorem 2.1.30, for the same E, $D = \varphi(Y)_{\mathbf{F}}$ and $T = S^*$. Observe that D consists of eigenvectors of T, since from Proposition 2.1.2 we have $S^*\varphi(y)_{\mathbf{F}} = \frac{u(y)}{v(y)}\varphi(y)_{\mathbf{F}}$, for every $y \in Y$, and since $\mathbf{F}$ is 1-independent, $\varphi(y)_{\mathbf{F}} \neq 0_{\mathbf{F}^*}$. $\square$

Remark 2.1.34. Note that the assumption that φ is continuous holds automatically if X and $\mathbf{F}$ satisfy the conditions of the results of Section 2.2. Moreover, this assumption can be replaced with surjectivity of φ, connectedness of X and 2-independence of $\mathbf{F}$, since then from part (iii) of Proposition 2.1.9, it follows that S is a MO, which reduces the problem to Theorem 2.1.30 or Corollary 2.1.31. On the other hand, if φ is continuous, $W_{\varphi,u}$ or $W_{\varphi,v}$ is a co-isometry and $\mathbf{F}$ and $\mathbf{E}$ satisfy the conditions of part (iii) of Proposition 2.1.24, then φ must be a bijection. In fact, if X is a manifold, $\mathbf{F}$ is 2-independent with $|\kappa|_{\mathbf{F}}$ continuous, $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$ and bounded functions form a dense subset of $\mathbf{F}$, then $\mathbf{F}$ is rigid in the following stronger sense: if $\varphi : X \to X$ and $u, v : X \to \mathbb{F}$ are such that $W_{\varphi,u}$ and $W_{\varphi,v}$ are co-isometries, then φ is a self-homeomorphism of X and $u = \alpha v$, for some $\alpha \in \mathbb{T}$, are continuous and non-vanishing. $\square$

2.2 Recognition of WCO$_{\mathcal{C}}$'s on NSCF's

This section is dedicated to the problem of recognition of WCO$_{\mathcal{C}}$'s, i.e. looking for sufficient conditions for a continuous linear map between NSCF's to be a WCO$_{\mathcal{C}}$. The recognition of WCO$_{\mathcal{C}}$'s is a much more subtle problem than the recognition of WCO's, as we will see later.

Throughout this section X and Y are topological spaces and $\mathbf{F} \subset \mathcal{C}(X)$, $\mathbf{E} \subset \mathcal{C}(Y)$ are linear subspaces. Of course, given a continuous linear map T between $\mathbf{F}$ and $\mathbf{E}$, we can apply part (i) of Corollary 2.1.3 and ascertain if there are **any** φ and u such that $T = W_{\varphi,u}$. Hence, the problem of recognizing WCO$_{\mathcal{C}}$'s is reduced to characterizing WCO's, which are WCO$_{\mathcal{C}}$'s. In other words we will be dealing with the following question: if $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ are such that $W_{\varphi,u}\mathbf{F} \subset \mathbf{E}$, when can we guarantee that φ and u are continuous?

Note that we immediately run into the obstruction discussed in Example 2.1.5. Hence, we will mostly stay away from non-2-independent NSCF's, and also rarely allow the multiplicative symbols of WCO's to vanish.

The conditions that we are looking for to guarantee that a WCO is a $\mathrm{WCO}_{\mathcal{C}}$ can be of a very different nature; they can also vary by the object: X, Y, $\mathbf{F}$, $\mathbf{E}$, φ or u. We find it most meaningful to focus on the conditions on $\mathbf{F}$ and $\mathbf{E}$, occasionally demanding u to be bounded or non-vanishing. As we will see, the definite role in the considered question is played by X and $\mathbf{F}$. We will give characterizations in terms of properties of elements of NSCF's, or in terms of norms of point evaluation, etc, and avoid more implicit conditions.

2.2.a Simple Cases of Recognition

Throughout this subsection X and Y are topological spaces. The following proposition is the first step in solving our recognition problem. We start with a result with no reference to the space $\mathbf{E}$.

Proposition 2.2.1. *Let $\mathbf{F}\subset\mathcal{C}(X)$ be a linear subspace. Let $u:Y\to\mathbb{F}$ and $\varphi:Y\to X$ be such that $W_{\varphi,u}\mathbf{F}\subset\mathcal{C}(Y)$. Then:*

(i) If $\mathbf{F}$ is 1-independent and φ is continuous, or if $1\in\mathbf{F}$, then u is continuous.

(ii) If $\mathbf{F}$ generates the topology of X and u is continuous, then φ is continuous outside of $u^{-1}(0)$.

(iii) If $\mathbf{F}$ is 1-independent, the set $u^{-1}(0)$ is closed in Y. If $\mathbf{F}$ is 2-independent, then $\varphi^{-1}(x)\cup u^{-1}(0)$ is closed, for every $x\in X$. Moreover, if u is bounded on K, then $\varphi^{-1}(K)\cup u^{-1}(0)$ is closed, for any compact $K\subset X$.

Proof. (i): If $1 \in \mathbf{F}$, then $u = W_{\varphi,u}1 \in \mathcal{C}(Y)$. Consider the case when $\mathbf{F}$ is 1-independent and φ is continuous. Let $y \in Y$. Since $\varphi(y)_{\mathbf{F}} \neq 0$, there is $f \in \mathbf{F}$ such that $f(\varphi(y)) \neq 0$ and since f and φ are continuous, there is a neighborhood U of y such that $f(\varphi(z)) \neq 0$, for each $z \in U$. Since $W_{\varphi,u}f \in \mathcal{C}(X)$ and $u = \frac{W_{\varphi,u}f}{f\circ\varphi}$ on U, we get that u is continuous on U as a ratio of continuous functions. Hence, u is locally continuous, and thus continuous.

(ii): Since $\mathbf{F}$ generates the topology of X, it follows that $\kappa_{\mathbf{F}}$ is a topological embedding into the weak* topology on $\mathbf{F}'$. Hence, there is $\kappa_{\mathbf{F}}^{-1} : X_{\mathbf{F}} \to X$, which is continuous from the weak* topology on $X_{\mathbf{F}}$ into X. Recall that $W_{\varphi,u}\mathbf{F} \subset \mathcal{C}(Y)$ implies that the map $\lambda = u\cdot\kappa_{\mathbf{F}}\circ\varphi$ is weak* continuous from Y into $\mathbf{F}'$. Hence, on $Y \backslash u^{-1}(0)$ the restriction $\varphi = \kappa_{\mathbf{F}}^{-1} \circ \left(\frac{1}{u} \cdot \lambda\right)$ is given through the operations with continuous maps, and so is continuous itself.

(iii): If $\mathbf{F}$ is 1-independent it follows that $\lambda^{-1}(0_{\mathbf{F}'}) = u^{-1}(0)$. If $\mathbf{F}$ is 1-independent, then $\varphi^{-1}(x) \cup u^{-1}(0) = \lambda^{-1}(\mathbb{F}x_{\mathbf{F}})$. The singleton $\{0_{\mathbf{F}'}\}$ and the $\mathbb{F}$-line $\mathbb{F}x_{\mathbf{F}}$ are weak* closed in $\mathbf{F}'$, and since λ is weak* continuous the preimages $u^{-1}(0)$ and $\varphi^{-1}(x) \cup u^{-1}(0)$ are closed. If K is compact in X, then $K_{\mathbf{F}}$ is weak* compact. If u is bounded on K, the set $B = \overline{u(K)}$ is compact in $\mathbb{F}$, and so $BK_{\mathbf{F}}$ is weak* compact. Therefore, $BK_{\mathbf{F}} \cup \{0_{\mathbf{F}'}\}$ is weak* closed and so $\varphi^{-1}(K) \cup u^{-1}(0) = \lambda^{-1}(BK_{\mathbf{F}} \cup \{0_{\mathbf{F}'}\})$ is closed in Y. $\square$

The following examples show that we can hardly say anything about the behavior of φ near the points where u vanishes.

Example 2.2.2. Let $\mathbf{F} = \mathcal{C}([-1,1])$ which is a BSCF with respect to the supremum norm. Define $u : [-1,1] \to \mathbb{R}$ and $\varphi : [-1,1] \to [-1,1]$ by $u(x) = x$, $\varphi(0) = 0$ and $\varphi(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$. Clearly, φ has no limit at 0. However, it is easy to see that $\|W_{\varphi,u}\| \leq 1$ on $\mathbf{F}$. $\square$

Example 2.2.3. Assume that X is the unit sphere in $\mathbb{R}^3$, punctured in the South Pole, and let

$\mathbf{F}$ be the space of all bounded Lipschitz functions on X. This space is a compactly embedded BSCF with respect to the usual norm. Let u be the distance to the North Pole N, and let $\varphi : X \to X$ be defined by $\varphi(N) = N$ and $\varphi(x) = -x$, $x \neq N$. Again, φ does not have a limit at 0, but one can show that $W_{\varphi,u}$ is continuous on $\mathbf{F}$. $\square$

It follows immediately from Proposition 2.2.1, that every CO (MO, WCO with a non-vanishing multiplicative symbol) is a $\mathrm{CO}_{\mathcal{C}}$ ($\mathrm{MO}_{\mathcal{C}}$, $\mathrm{WCO}_{\mathcal{C}}$) on a wide class of NSCF's.

Corollary 2.2.4. *Let* $\mathbf{F}$ *be a NSCF over* X*. Then:*

(i) If $\mathbf{F}$ *is* 1*-independent, then any MO from* $\mathbf{F}$ *into another NSCF is a* $MO_{\mathcal{C}}$*.*

(ii) If $\mathbf{F}$ *generates the topology of* X*, then any CO from* $\mathbf{F}$ *into into another NSCF is a* $CO_{\mathcal{C}}$*.*

(iii) If $1 \in \mathbf{F}$ *and* $\mathbf{F}$ *generates the topology of* X*, then any WCO from* $\mathbf{F}$ *with a non-vanishing multiplicative symbol into into another NSCF is a* $WCO_{\mathcal{C}}$*.*

The following example shows that we need the 1-independence in part (i).

Example 2.2.5. Let $\mathbf{F} = \{f \in \mathcal{C}([-1,1]) \,|\, f(0) = 0\}$, which is a BSCF over $[-1,1]$ with respect to the supremum norm. Consider $u = \mathbb{1}_{[-1,0]} - \mathbb{1}_{[0,1]}$. This function does not have a limit at 0, but it is easy to see that M_u is bounded on $\mathbf{F}$. $\square$

The following proposition shows how the recognition problem on the rescalings or restrictions of a NSCF is related to that problem on the original NSCF.

Proposition 2.2.6. *Let* $\mathbf{F}$ *and* $\mathbf{E}$ *be NSCF's over* X *and* Y *respectively, such that every continuous WCO from* $\mathbf{F}$ *into* $\mathbf{E}$ *with a (bounded) non-vanishing multiplicative symbol is a* $WCO_{\mathcal{C}}$*. Let* $\mathbf{H}$ *be a (bounded) rescaling or restriction of* $\mathbf{F}$*. Then every continuous WCO from* $\mathbf{H}$ *into* $\mathbf{E}$ *with a (bounded) non-vanishing multiplicative symbol is a* $WCO_{\mathcal{C}}$*.*

Proof. We will only consider the case when $\mathbf{H} = v\mathbf{F}$, for some continuous $v : X \to \mathbb{F}^{\times}$. The other cases are similar. Let $\varphi : Y \to X$ and $u : Y \to \mathbb{F}^{\times}$ be such that $W_{\varphi,u} \in \mathcal{L}(\mathbf{H}, \mathbf{E})$. Since by definition M_v is continuous from $\mathbf{F}$ to $\mathbf{H}$, we get that $W_{\varphi,u\cdot v\circ\varphi} = W_{\varphi,u}M_v \in \mathcal{L}(\mathbf{F}, \mathbf{E})$. From our assumption, it follows that $W_{\varphi,u\cdot v\circ\varphi}$ is a WCO$_{\mathcal{C}}$, and so φ is continuous, as well as $u \cdot v \circ \varphi$. Since v is continuous and non-vanishing, $v \circ \varphi$ is also continuous, and so is u. Thus, $W_{\varphi,u}$ is a WCO$_{\mathcal{C}}$ from $v\mathbf{F}$ into $\mathbf{E}$. $\square$

2.2.b Some Counterexamples

Despite the appeal of part (iii) of Corollary 2.2.4, one can ask what happens if we don't know if the non-zero constant functions belong to our NSCF, or whether the latter generates the topology of the phase space. In fact, it can be very difficult to verify if a given collection of functions generates the topology of their domain, especially if this collection is given implicitly. Also, some natural classes of functions do not contain constants. For example, non-zero constant functions cannot vanish at infinity and are not integrable with respect to an unbounded measure. Thus, we find it important to investigate the situation without the assumption of having constant functions in our function spaces. Consider the following examples of unitary WCO's on a compactly embedded Hilbert space of continuous functions, which fail to be WCO$_{\mathcal{C}}$'s.

Example 2.2.7. Let $X = [0, 2\pi)$. We will first construct a unitary CO on a compactly embedded HSCF's over X, which fails to be a CO$_{\mathcal{C}}$. Then we will modify this construction to obtain similar counterexamples for WCO's on "even nicer" NSCF's over X.

Let $k : X \times X \to \mathbb{C}$ be defined by

$$k(x,y) = \frac{1}{1 - \frac{1}{4}e^{\mathrm{i}(x-y)}} = \frac{1}{1 - \psi(x)\overline{\psi(x)}},$$

where $\psi : [0, 2\pi) \to \mathbb{D}$ is defined by $\psi(x) = \frac{e^{\mathrm{i}x}}{2}$. Since k is a composition of the Szego kernel (see Example 1.3.17) and ψ, it is a continuous positive definite kernel on X (it is even Nevanlinna-Pick complete). Hence, due to Theorem 1.2.35, $\mathbf{H}_k$ is a compactly embedded, ∞-independent HSCF over X. Also, note that $k - 1 = \psi \otimes \overline{\psi} k \gg 0$, and so $1 \in \mathbf{H}_k$. Note however, that $\mathbf{H}_k$ is not a restriction of the Hardy space, since ψ is not a topological embedding.

Define $\varphi : X \to X$ by $\varphi(x) = x + \pi \mod 2\pi\mathbb{Z}$. This is a bijective map which does not have a limit at $x = \pi$. Nevertheless, $k \circ \varphi = k$, and so C_φ is a unitary operator on $\mathbf{H}_k$, due to Proposition 2.1.20. This pathology is only possible because $\mathbf{H}_k$ does not generate the topology of X, which seems to be difficult to verify directly.

For an arbitrary continuous function $v : X \to \mathbb{C}^\times$ consider a rescaling $\mathbf{H}_l = v\mathbf{H}_k$ of $\mathbf{H}_k$, where $l = v \otimes \overline{v} k$. This space also satisfies the nice properties that we have established for the space $\mathbf{H}_k$, and $v \in v\mathbf{H}_k = \mathbf{H}_l$.

Denote $u = \frac{v}{v \circ \varphi}$, which is a scalar-valued function on X with a discontinuity at π. Also, if v is bounded both from above and from below, then so is u. Again, despite discontinuity of u and φ, we see that $W_{\varphi,u}$ is a unitary operator on $\mathbf{H}_l$, since $u \otimes \overline{u} l \circ \varphi = l$. Now we can pick v so that $\mathbf{H}_l$ has certain additional properties. In particular, if v is defined by $v(x) = 1 - \frac{1}{4}e^{\mathrm{i}x}$, then $1 \equiv L(x, 0) \in \mathbf{H}_l$ (but $\mathbf{H}_l$ does not generate the topology of X). Also, for $v(x) = x + \pi$, since $v \in \mathbf{H}_l$, the latter generates the topology of $X \subset \mathbb{R}$ (but does not contain 1).

The nature of the phenomenon above can be informally explained in the following way: $X_{\mathbf{H}_l}$ is a semi-closed curve, whose open and closed ends are aligned, and so it can freely revolve inside the cone which it generates, but this revolution corresponds to a cyclic shift of its points, which is a discontinuous transformation. □

2.2.c Recognition and Cones in TVS's

Throughout this subsection X and Y are topological spaces. Recall that in the light of Remark 2.1.1, WCO's with non-vanishing multiplicative symbol can be viewed as CO's. Hence, from part (ii) of Corollary 2.2.4 we can conclude that every such WCO on a NSCF $\mathbf{F}$ over a locally compact space X is a WCO$_{\mathcal{C}}$, once $\widetilde{\mathbf{F}}$ generates the topology of $\mathbb{F}^{\times} \times X$. Hence, we have to look for sufficient condition of the latter, or some similar topological properties. In order to accomplish this task we need the following general result (see the proof in Appendix A.2).

Theorem 2.2.8. *Let E be a topological vector space, with the operation of scalar multiplication given by $\mu : \mathbb{F} \times E \to E$, and let $K \subset E$ contain no pairs of linearly dependent elements. Then:*

(i) If K is compact, then $\mu|_{\mathbb{F}^{\times} \times K}$ is a topological embedding and $\mu|_{\mathbb{F} \times K}$ is a quotient map onto its image $\mathbb{F}K$.

(ii) Let $B \subset \mathbb{F}^{\times}$ be bounded. If $K \cup \{0_E\}$ is compact, then $\mu|_{B \times K}$ is a topological embedding.

The preceding theorem is a general fact and we will not use it directly. Instead, we will employ the following consequence of that result.

Proposition 2.2.9. *Let E be a topological vector space. Let $\kappa : X \to E\backslash\{0_E\}$ be a topological embedding, let $\lambda : Y \to E$ be continuous, and let $\varphi : Y \to X$ and $u : Y \to \mathbb{F}$ be such that $\lambda(y) = u(y)\kappa(\varphi(y))$, for every $y \in Y$. Then:*

(i) If $\overline{\kappa(X)}$ is compact and contains no pairs of linearly dependent elements, then u is continuous and φ is continuous outside of $u^{-1}(0)$.

(ii) If $\overline{\kappa(X)}$ is compact, $\overline{\kappa(X)}\backslash\{0_E\}$ contains no pairs of linearly dependent elements and u is bounded, then u and φ are continuous outside of $u^{-1}(0)$.

Proof. Let us first address the continuity of u in (i). Since $K = \overline{\kappa(X)}$ contains no pairs of linearly dependent elements, there is a function $\psi : \mathbb{F}K \to \mathbb{F}$, such that $\psi(\mu(\alpha, e)) = \alpha$, for every $\alpha \in \mathbb{F}$ and $e \in K$. Since from Theorem 2.2.8 $\mu|_{\mathbb{F}\times K}$ is a quotient map onto its image $\mathbb{F}K$, while $\psi \circ \mu$ is obviously continuous, it follows that ψ is continuous, and so $u = \psi \circ \lambda$ is also continuous.

Now without loss of generality we may assume that u does not vanish (otherwise replace Y with $Y\backslash u^{-1}(0) = Y\backslash\lambda^{-1}(0_E)$, which is an open subset of Y). Let $K = \overline{\kappa(X)}\backslash\{0_E\}$ and $B = u(Y)$. It follows from Theorem 2.2.8 that if either of the conditions in (i) or (ii) is satisfied, then $\mu|_{B\times K}$ is a topological embedding, and since κ is a topological embedding, we get that the map $\mu' : X \times \mathbb{F} \to E$ defined by $\mu'(\alpha, x) = \alpha\kappa(x) = \mu(\alpha, \kappa(x))$ is also a topological embedding. Therefore, there are continuous maps $\psi : \mu'(B \times X) \to B$ and $\theta : \mu'(B \times X) \to X$, such that $\psi(\mu(\alpha, x)) = \alpha$ and $\theta(\mu(\alpha, x)) = x$, for every $\alpha \in B$ and $x \in X$. Hence, $u = \psi \circ \lambda$ and $\varphi = \theta \circ \lambda$ are also continuous. $\square$

We can now obtain the following two theorems, which are the main results of this section.

Theorem 2.2.10. *Let $\mathbf{F} \subset \mathcal{C}(X)$ be a 2-independent linear subspace. If $u : Y \to \mathbb{F}$ and $\varphi : Y \to X$ are such that $W_{\varphi,u}\mathbf{F} \subset \mathcal{C}(Y)$, then:*

(i) If X is compact, then u is continuous and φ is continuous outside of $u^{-1}(0)$.

(ii) If $\mathbf{F} \subset \mathcal{C}_0(X)$ and u is bounded, then u and φ are continuous outside of $u^{-1}(0)$.

Proof. If $\mathbf{F}$ is 2-independent, then $\kappa_{\mathbf{F}}$ is an injection and $X_{\mathbf{F}}$ contains no pairs of linearly dependent elements. Furthermore, if X is compact, then $X_{\mathbf{F}}$ is weak* compact and $\kappa_{\mathbf{F}}$ is a topological embedding. Else, if X is not compact, but $\mathbf{F} \subset \mathcal{C}_0(X)$, then by virtue of Corollary 1.2.22, $X_{\mathbf{F}} \cup \{0_{\mathbf{F}'}\}$ is compact and $\kappa_{\mathbf{F}}$ is a topological embedding.

Recall that if $W_{\varphi,u}\mathbf{F} \subset \mathcal{C}(Y)$, then the map $\lambda = u \cdot \kappa_{\mathbf{F}} \circ \varphi$ is weak* continuous. Thus, the required continuities follow from applying Proposition 2.2.9 to $\kappa_{\mathbf{F}}$ and λ. □

Remark 2.2.11. Note that 2-independence of $\mathbf{F} \subset \mathcal{C}_0(X)$ forces X to be locally compact. While it is clear that this condition in part (ii) is essential, one is tempted to ask if we need the boundedness of u. Consider $\mathbf{H}_l$ from Example 2.2.7 with $v(x) = 2\pi - x$. Then $\lim\limits_{\infty} |\kappa|_{\mathbf{H}_l} = 0$, and so $\mathbf{H}_l \subset \mathcal{C}_0(X)$, but this NSCF admits a WCO with a non-vanishing multiplicative symbol, which fails to be a $\mathrm{WCO}_{\mathcal{C}}$. □

It is desirable to get rid of the extra conditions of part (ii) of Theorem 2.2.10. This can be done at the expense of generality of the choice of $\mathbf{F}$. Namely, we will require $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$. Recall from Remark 2.1.23 that this condition is fulfilled whenever bounded functions form a dense set in $\mathbf{F}$ and $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$ (the converse does not hold, since $|\kappa|_{\mathcal{C}_0(X)} \equiv 1$).

Theorem 2.2.12. *Let $\mathbf{F}$ and $\mathbf{E}$ be 2-independent NSCF's over X and Y respectively, such that $|\kappa|_{\mathbf{F}}$ and $|\kappa|_{\mathbf{E}}$ are continuous and $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$. If $u : Y \to \mathbb{F}$ and $\varphi : Y \to X$ are such that $W_{\varphi,u} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, then u and φ are continuous outside of $u^{-1}(0)$.*

Proof. First of all, by virtue of Proposition 2.1.22, $\kappa : X \to \mathbf{F}^*$ defined by $\kappa(x) = \frac{1}{|\kappa|_{\mathbf{F}}(x)}\kappa_{\mathbf{F}}(x)$, is a topological embedding into the unit sphere of $\mathbf{F}^*$ with respect to weak* topology.

As usual, the map $u \cdot \kappa_{\mathbf{F}} \circ \varphi = W^*_{\varphi,u} \circ \kappa_{\mathbf{E}}$ is weak* continuous from Y into $\mathbf{F}^*$. Since $|\kappa|_{\mathbf{E}}$ is continuous and non-vanishing, it follows that $\lambda = \frac{u}{|\kappa|_{\mathbf{E}}} \cdot \kappa_{\mathbf{F}} \circ \varphi$ is also weak* continuous. Then $\lambda = v \cdot \kappa \circ \varphi$, where $v = \frac{u|\kappa|_{\mathbf{F}} \circ \varphi}{|\kappa|_{\mathbf{E}}}$ is a bounded function on Y. Indeed, since $\|\kappa \circ \varphi\| \equiv 1$, we have

$$|v(y)| = \|\lambda(y)\| = \frac{\|W^*_{\varphi,u} y_{\mathbf{E}}\|}{\|y_{\mathbf{E}}\|} \leq \|W^*_{\varphi,u}\|,$$

for every $y \in Y$. Hence, from Proposition 2.2.9 both φ and v are continuous outside of $v^{-1}(0) = u^{-1}(0)$. Thus, $u = \frac{v|\kappa|_{\mathbf{E}}}{|\kappa|_{\mathbf{F}} \circ \varphi}$ is also continuous outside of $u^{-1}(0)$. $\square$

Remark 2.2.13. Recall that if $\mathbf{F}$ and $\mathbf{E}$ are compactly embedded, and Y is compactly generated, then $|\kappa|_{\mathbf{F}}$ and $|\kappa|_{\mathbf{E}}$ are continuous. We don't need to require that X is compactly generated since 2-independence of $\mathbf{F}$ and $\frac{1}{|\kappa|_{\mathbf{F}}}\mathbf{F} \subset \mathcal{C}_0(X)$ force X to be locally compact. $\square$

Example 2.2.14. It follows from the preceding theorem that any WCO from $\mathcal{C}_0(X)$ into $\mathcal{C}_0(Y)$ with a non-vanishing multiplicative symbol is a $\mathrm{WCO}_{\mathcal{C}}$. Note that this example is unaccessible neither for Theorem 2.2.10, nor for Corollary 2.2.4.

However, we cannot guarantee continuity of u in the case it vanishes. Let $X = \mathbb{F}$ and let $\mathbf{F} = \mathcal{C}_0(X)$. Define $u : X \to \mathbb{R}$ and $\varphi : X \to X$ by $u(0) = \varphi(0) = 0$, $\varphi(x) = \frac{1}{x}$, $x \neq 0$ and $u(x) = \min\left\{1, \frac{1}{x}\right\}$, $x \neq 0$. Clearly, neither u nor φ is not continuous at 0, but one can show that $\|W_{\varphi,u}\| \leq 1$ on $\mathbf{F}$. $\square$

Combining the obtained results with part (iii) of Proposition 2.1.24, we get the following corollary.

Corollary 2.2.15. *Assume that X and Y are topological manifolds with* $\dim X \leq \dim Y$.

Let $\mathbf{F}$ *and* $\mathbf{E}$ *be* 2*-independent NSCF's over* X *and* Y*, such that the bounded functions form a dense set in* $\mathbf{F}$ *and* $|\kappa|_{\mathbf{F}}$ *and* $|\kappa|_{\mathbf{E}}$ *are both continuous with* $\lim\limits_{\infty}|\kappa|_{\mathbf{F}}=\lim\limits_{\infty}|\kappa|_{\mathbf{E}}=+\infty$*. Let* $\varphi:Y\to X$ *and* $u:Y\to\mathbb{F}^{\times}$ *be such that* u *is bounded and* $W_{\varphi,u}$ *is a linear homeomorphism from* $\mathbf{F}$ *onto* $\mathbf{E}$*. Then* φ *is a homeomorphism.*

Remark 2.2.16. When dealing with compactly embedded NSCF's we can operate with the norm topology on their duals, instead of the weak* topology. However, it is not clear how to guarantee that $X_{\mathbf{F}}$ or $X_{\mathbf{F}}\cup\{0_{\mathbf{F}^*}\}$ are closed, or that $\frac{1}{|\kappa|_{\mathbf{F}}}\cdot\kappa_{\mathbf{F}}$ is a topological embedding with respect to the norm topology. $\square$

2.3 Recognition of $\mathrm{WCO}_{\mathcal{H}}$'s on NSHF's

This section is dedicated to the recognition of $\mathrm{WCO}_{\mathcal{H}}$. Unsurprisingly, this problem turns out to be of a different flavour than its continuous counterpart. Till the end of this section X and Y are (connected) complex manifolds.

The problem of recognition of $\mathrm{WCO}_{\mathcal{H}}$'s comes in two versions. Assume that $u:Y\to\mathbb{F}$ and $\varphi:Y\to X$ are such that $W_{\varphi,u}\in\mathcal{L}(\mathbf{F},\mathbf{E})$, where $\mathbf{F}$ and $\mathbf{E}$ are NSHF's over X and Y respectively. Now we can either ask for the conditions that guarantee that both φ and u are holomorphic, or rely on the results from the previous section and ask for the conditions that guarantee that φ and u are holomorphic, provided that they are both continuous. Yet again, the most meaningful results concern X and $\mathbf{F}$.

2.3.a The Main Results

Let us start with the following variation of parts (i) and (iii) of Proposition 2.2.1.

Proposition 2.3.1. *Let* $\mathbf{F} \subset \mathcal{H}(X)$ *be a linear subspace. Let* $u : Y \to \mathbb{F}$ *and* $\varphi : Y \to X$ *be such that* $W_{\varphi,u}\mathbf{F} \subset \mathcal{H}(Y)$*. Then:*

(i) If $\mathbf{F}$ *is* 1*-independent and* φ *is holomorphic, or if* $1 \in \mathbf{F}$*, then* u *is holomorphic.*

(ii) If $\mathbf{F}$ *is* 1*-independent, then* $u^{-1}(0)$ *is either equal to* Y*, or it is closed and thin in* Y*.*

(iii) If $\mathbf{F}$ *separates the points of* X*,* u *is holomorphic and* $x \in X$*, then either* $\varphi^{-1}(x)$ *is thin in* Y*, or* $Y = \varphi^{-1}(x) \cup u^{-1}(0)$*.*

Proof. The proof of part (i) is exactly the same as of part (i) of Proposition 2.2.1.

(ii): The fact that $u^{-1}(0)$ is closed is given by part (iii) of Proposition 2.2.1. We can assume that there is $y \in Y$ such that $u(y) \neq 0$. Since $\mathbf{F}$ is 1-independent, there is $f \in \mathbf{F}$ such that $f(\varphi(y)) \neq 0$. Then $g = W_{\varphi,u}f$ is a holomorphic function on Y, which does not vanish at y. Hence, $u^{-1}(0) \subset g^{-1}(0)$ is a thin set in Y.

(iii): It is sufficient to consider the case $u \not\equiv 0$. Assume that $\varphi^{-1}(x)$ is not thin in Y. Since $u^{-1}(0)$ is thin, it follows that $\varphi^{-1}(x) \backslash u^{-1}(0)$ is not thin. For every $y \in \varphi^{-1}(x) \backslash u^{-1}(0)$ and $f \in \mathbf{F}$ we have that $[W_{\varphi,u}f](y) = f(x)u(y)$, and so holomorphic functions $W_{\varphi,u}f$ and $f(x)u$ coincide on a non-thin set. Hence, $W_{\varphi,u}f = f(x)u$, for every $f \in \mathbf{F}$, and so $\varphi(y)_{\mathbf{F}} = x_{\mathbf{F}}$ for every $y \in Y\backslash u^{-1}(0)$. Since $\mathbf{F}$ separates the points of X, we conclude that $\varphi(y) = x$ for every $y \in Y\backslash u^{-1}(0)$. $\square$

It follows immediately from part (i) of the proposition, that any MO from a 1-independent NSHF into a NSHF is a $\mathrm{MO}_{\mathcal{H}}$. During our investigation of the recognition problem we routinely avoided WCO's with vanishing multiplicative symbols and non-2-independent NSCF's and NSHF's. The reason for this is that we cannot reconstruct the symbols of a WCO from the data of this WCO, unless its multiplicative symbol does not vanish and the source NSF is 2-independent (see Example 2.1.5). However, in the context of NSHF's this obstruction can be overcome. Let us modify our problem by a slight broadening of the term "WCO$_{\mathcal{H}}$". Let $\mathbf{F}$ and $\mathbf{E}$ be NSHF's over complex manifolds X and Y respectively. Then, for $\varphi : Y \to X$ and $u : Y \to \mathbb{C}$ the operator $W_{\varphi,u}$ is a WCO$_{\mathcal{H}}$ if there are holomorphic $\varphi' : Y \to X$ and $u' : Y \to \mathbb{C}$, such that $W_{\varphi,u} = W_{\varphi',u'}$. We can now solve our main problem for MO's on NSHF's.

Proposition 2.3.2. *Any continuous multiplication operator on a NSHF is a MO$_{\mathcal{H}}$.*

Proof. Let $\mathbf{F} \neq \{0\}$ be a NSHF over X. Due to Remark 2.1.14, the set $Z = \{x \in X \,|x_{\mathbf{F}} = 0_{\mathbf{F}^*}\}$ is closed and thin, and the restriction of $\mathbf{F}$ on $X \backslash Z$ is an isometric isomorphism. Then, $M_{u|_{X\backslash Z}}$ is a continuous MO on a 1-independent NSHF $\mathbf{F}|_{X\backslash Z}$, and so we can conclude from the discussion above that $u|_{X\backslash Z} \in \mathcal{H}(X\backslash Z)$. Also, $u|_{X\backslash Z}$ is bounded by $\|M_u\|$. Hence, by Removable Singularity theorem (see [Chi89, A1.4]) there is a holomorphic function $u' : X \to \mathbb{C}$, which coincides with u on $X\backslash Z$. Since u and u' only differ on Z, we conclude that $M_u = M_{u'}$, and so M_u is a WCO$_{\mathcal{H}}$. $\square$

The rest of the section is dedicated to the analogous problems for CO$_{\mathcal{H}}$ and WCO$_{\mathcal{H}}$. Recall that in the continuous setting the recognition of CO$_{\mathcal{C}}$ was relatively simple, while the recognition of WCO$_{\mathcal{C}}$ involved more sophisticated techniques. The situation in the holomorphic setting is quite opposite: the result for the WCO$_{\mathcal{H}}$ is a consequence of the result for CO$_{\mathcal{H}}$. Let us state the main theorem of the section (the proof constitutes the next subsection).

Theorem 2.3.3.

(i) Any $CO_{\mathcal{C}}$ from a NSHF that separates the points of its phase space into another NSHF is a $CO_{\mathcal{H}}$.

(ii) Any $WCO_{\mathcal{C}}$ from a 2-independent NSHF into another NSHF is a $WCO_{\mathcal{H}}$.

2.3.b The Proof of Theorem 2.3.3

In this subsection X and Y are connected complex manifolds with $\dim X = n$. In order to prove the theorem we need a sequence of lemmas. Let us start with a lemma that shows the infinitesimal behavior of a collection of holomorphic functions that separate the points.

Lemma 2.3.4. *Assume that Z is a domain in $\mathbb{C}^n$ and let $\mathbf{F}$ be a collection of functions in $\mathcal{H}(Z)$ that separate the points of Z. Then there are $f_1, ..., f_n \in \mathbf{F}$, such that* $\det \left[\frac{\partial f_i}{\partial z_j}\right]_{i,j=1}^{n} \not\equiv 0$.

Proof. It is enough to show that there are $x \in Z$ and $f_1, ..., f_n \in \mathbf{F}$, such that the vectors $\nabla f_1(x), ..., \nabla f_n(x) \in \mathbb{C}^n$ are linearly independent. Assume the contrary, i.e. that $m = \max\limits_{x\in Z} \dim \{\nabla f(x) \,|\, f \in \mathbf{F}\} < n$. Let $x \in Z$ be such that $\dim \{\nabla f(x) \,|\, f \in \mathbf{F}\} = m$ and let $f_1, ..., f_m \in \mathbf{F}$, be such that $\nabla f_1(x), ..., \nabla f_m(x) \in \mathbb{C}^n$ are linearly independent.

Without loss of generality we can assume that $h = \det \left[\frac{\partial f_i}{\partial z_j}\right]_{i,j=1}^{m}$ does not vanish at x. Since h is holomorphic on X, it does not vanish on an open connected neighborhood U of x, and so $\nabla f_1(y), ..., \nabla f_m(y)$ are linearly independent, for every $y \in U$. From the choice of m it follows that $\nabla f(y) \in \operatorname{span}\{\nabla f_1(y), ..., \nabla f_m(y)\}$, for any $f \in \mathbf{F}$ and $y \in U$.

From the Implicit Function theorem (see [Chi89, A2.2, Theorem 1]) applied to the vector function $(f_1, ..., f_m)$ there are open sets $V \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^{n-m}$, such that $x \in V \times W \subset U$, and a holomorphic map $\varphi : W \to V$ with the following property: if $y = (y', y'')$, $y' \in V$, $y'' \in W$, then $\varphi(y'') = (y')$ if and only if $f_j(y) = f_j(x)$, for every $j \in \overline{1, m}$.

Define $\psi = \varphi \times Id_W$, which is a holomorphic injection from W into $V \times W$. Let $\mathcal{T}_z\psi$ be the Jacobi matrix of ψ at $z \in W$. Then for every $j \in \overline{1,m}$ we have that $f_j \circ \psi \equiv f_j(x)$, and in particular, it is a constant map on W. Hence, $0_{\mathbb{C}^{n-m}} = \nabla (f_j \circ \psi)(z) = \nabla f_j(\psi(z)) \mathcal{T}_z\psi$, for every $z \in W$.

For every $f \in \mathbf{F}$ we have that $\nabla (f \circ \psi)(z) = \nabla f(\psi(z)) \mathcal{T}_z\psi$ is a linear combination of vectors $\{\nabla f_j(\psi(z)) \mathcal{T}_z\psi\}_{j=1}^m$, and so $\nabla (f \circ \psi) \equiv 0_{\mathbb{C}^{n-m}}$ on W. Thus, $f \circ \psi \equiv f(x)$, for every $f \in \mathbf{F}$, and so $\mathbf{F}$ does not separate the points of the image of ψ. Since $m < n$ and ψ is an injection, its image contains more than one point, and so we have reached a contradiction. $\square$

Lemma 2.3.5. *Let $\psi : X \to \mathbb{C}^n$ be a local biholomorphism at $x \in X$ and let $\varphi : Y \to X$ be continuous and such that $\psi \circ \varphi$ is holomorphic. Then φ is holomorphic at every $y \in Y$ with $\varphi(y) = x$.*

Proof. There is an open neighborhood V of x and open $W \subset \mathbb{C}^n$, such that $\psi|_V$ is a biholomorphism from V onto W. Hence, there is a holomorphic inverse $\psi|_V^{-1}$. Let $y \in Y$ be such that $\varphi(y) = x$. Since φ is continuous, there is an open neighborhood U of x, such that $\varphi(U) \subset V$. Then $\varphi|_U = \psi|_V^{-1} \circ \psi|_V \circ \varphi|_U$ is a composition of holomorphic maps, and so φ is holomorphic at y. $\square$

The following lemma comes from combining Removable Singularity theorem and Rado's theorem (see [Chi89, A1.5]).

Lemma 2.3.6. *Let φ be a continuous map from Y into X. Then:*

(i) If φ is holomorphic on $U = Y \backslash u^{-1}(0)$, for some $u \in \mathcal{C}(Y) \cap \mathcal{H}(Y \backslash u^{-1}(0))$, then either φ is holomorphic, or $u \equiv 0$.

(ii) If φ is holomorphic on $V = Y \backslash \varphi^{-1}(h^{-1}(0))$, for some $h \in \mathcal{H}(X)$, then either φ is holomorphic, or $\varphi(Y) \subset h^{-1}(0)$.

Proof. (i): First, u is holomorphic due to Rado's theorem. If $u \not\equiv 0$, then $Y \backslash U$ is a thin set. Hence, φ is a continuous function, which is holomorphic outside of a thin set. Since φ is continuous, it is locally bounded, and so by Removable Singularity theorem, there is a unique holomorphic extension ψ of $\varphi|_U$ on Y. But since U is dense, and both φ and ψ are continuous, we get that $\varphi = \psi$, and so φ is holomorphic.

(ii) follows from (i) applied to $u = h \circ \varphi$. □

A closed subset Z of X is called *analytic* if for every $z \in Z$ there is an open neighborhood U of z and $f_1, ..., f_m \in \mathcal{H}(U)$ such that $U \cap Z = \bigcap_{i=1}^{m} f^{-1}(0)$. A point $z \in Z$ is called *regular of dimension* m if there is an open neighborhood U of z such that $U \cap Z$ is an embedded complex submanifold of X of dimension m, and the dimension $\dim Z$ of Z is the maximal dimension of its regular points. By Z' we will denote the set of all regular points of dimension $\dim Z$ and let $Z'' = Z \backslash Z'$. We will need a crucial fact (see [Chi89, Chapter 1, 5.2, Theorem 2]) that Z'' is an analytic set in X with $\dim Z'' < \dim Z$. In particular, if $\dim Z = 0$, then $Z = Z'$ is a discrete set. Also note that if $f \in \mathcal{H}(X)$, then $f^{-1}(0)$ is trivially analytic, and its dimension is $n-1$ (see [Chi89, Chapter 1, 2.6, Proposition]). We can now generalize part (ii) of the preceding lemma.

Lemma 2.3.7. *Let Z be an analytic set in X. Let φ be a continuous map from Y into X, which is also holomorphic on $Y \backslash \varphi^{-1}(Z)$. Then either φ is holomorphic, or $\varphi(Y) \subset Z$.*

Proof. Let $G = \varphi^{-1}(Z)$, which is closed in Y, since Z is closed and φ is continuous.

Let $y \in G \backslash \mathrm{int}G$. There is an open neighborhood U of $\varphi(y)$ in X and $f_1, ..., f_m \in \mathcal{H}(U)$ such that $U \cap Z = \bigcap_{i=1}^{m} f^{-1}(0)$. Since φ is continuous, there is an open connected neighborhood V of y such that $V \subset \varphi^{-1}(U)$. Then $V \cap G = \bigcap_{i=1}^{m} [f_i \circ \varphi|_V]^{-1}(0)$. Since $y \notin \mathrm{int}G$, there is $i \in \overline{1,m}$ such that $f_i \circ \varphi|_V \not\equiv 0$, and as $\varphi|_V$ is holomorphic outside $V \cap G \subset [f_i \circ \varphi|_V]^{-1}(0)$, it follows from Lemma 2.3.6 that $\varphi|_V$ is holomorphic, and so $V \cap G$ is thin. In particular, $y \notin \overline{\mathrm{int}G}$, since otherwise an open neighborhood V of y would have a non-empty open intersection with $\mathrm{int}G$, and so $V \cap G$ would not be thin.

Thus, φ is holomorphic outside $\mathrm{int}G$, and since the latter is open, this is only possible if either $\mathrm{int}G = Y$, or $\mathrm{int}G = \varnothing$. In the former case we conclude that $\varphi(Y) \subset Z$, and in the latter we conclude that φ is holomorphic. $\square$

We can now prove the principal ingredient in the proof of our main theorem.

Lemma 2.3.8. *Assume that* $\mathbf{F} \subset \mathcal{H}(X)$ *separates the points of* X*. If* φ *is a continuous map from* Y *into* X *such that* $f \circ \varphi \in \mathcal{H}(Y)$*, for every* $f \in \mathbf{F}$*, then* φ *is holomorphic.*

Proof. We will introduce an additional assumption: we will assume that for $m \in \overline{0,n}$ there is an analytic set Z in X, with $\varphi(Y) \subset Z$ and $\dim Z = m$. Note that a complex manifold is an analytic set in itself, and so this assumption is a tautology for $m = n$. The proof is done via induction over m. In the case when $m = 0$, Z is discrete, and since Y is connected and φ is continuous, it follows that $\varphi(Y)$ is a connected subset of a discrete set. Hence, $\varphi(Y)$ is a singleton, and so φ is a constant map, which is holomorphic.

Assume that the statement is proven for $0, ..., m-1$. We will show that then it is also true for m. Let x be a regular point of Z of dimension m, i.e. there is a neighborhood U of x in X, such that $U \cap Z$ is an embedded complex submanifold of X. Let V be a coordinate neighborhood of $U \cap Z$ at x. Then V is an open set in Z, and since $\varphi(Y) \subset Z$, it follows that $\varphi^{-1}(V)$ is open in Y. Let W be an (open) connected component of $\varphi^{-1}(V)$.

Since $\mathbf{F}$ separates the points of V, using Lemma 2.3.4 one can show that there are $f_1, ..., f_m \in \mathbf{F}$, such that $h = \det \left[\frac{\partial f_i}{\partial z_j}\right]_{i,j=1}^{m} \not\equiv 0$, where $(z_1, ..., z_m)$ are local coordinates on V. At each $z \in V \backslash h^{-1}(0)$ the vector function $(f_1, ..., f_m)$ is a local biholomorphism, and so from the Lemma 2.3.5 φ is holomorphic from $W \backslash \varphi^{-1}(h^{-1}(0))$ into V. Since the inclusion map from V into X is holomorphic, φ is holomorphic from $W \backslash \varphi^{-1}(h^{-1}(0))$ into X. Hence, from part (ii) of Lemma 2.3.6, either $\varphi|_W$ is holomorphic, or $\varphi(W) \subset h^{-1}(0)$. But the last condition implies that $\varphi|_W$ is holomorphic, due to the assumption of induction applied to W, V, $\varphi|_W$ and $h^{-1}(0)$, which is an analytic subset of V of dimension at most $m-1$.

Let Z' be the set of all regular points of Z of dimension m. We have proven that φ is holomorphic on $\varphi^{-1}(Z')$, i.e. outside of $\varphi^{-1}(Z'')$, where $Z'' = Z \backslash Z'$. Since Z'' is analytic of dimension at most $m-1$, using Lemma 2.3.7 and the assumption of induction, we conclude that φ is holomorphic. $\square$

Having established all the lemmas we can now finish the proof of the main theorem.

Proof of Theorem 2.3.3. Part (i) follows directly from Lemma 2.3.8. Let us prove (ii). Let $\mathbf{F}$ be a 2-independent NSHF over X, let $\varphi : Y \to X$ and $u : Y \to \mathbb{C}$ be continuous and such that $W_{\varphi,u}\mathbf{F} \subset \mathcal{H}(Y)$. It is enough to consider the case when $u \not\equiv 0$. Then $u^{-1}(0)$ is closed and thin, due to part (ii) of Proposition 2.3.1, and so $Z = Y \backslash u^{-1}(0)$ is open and non-empty. Since $\mathbf{F}$ is 2-independent, $\widetilde{\mathbf{F}}$ is a collection of holomorphic functions on $\mathbb{C}^{\times} \times X$, which separates the points. Also, $u|_Z \times \varphi|_Z$ is a continuous map from Z into $\mathbb{C}^{\times} \times X$, and $C_{u|_Z \times \varphi|_Z}$ maps $\widetilde{\mathbf{F}}$ into $\mathcal{H}(Z)$ (see Remark 2.1.1). Hence, from part (i), $u|_Z$ and $\varphi|_Z$ are holomorphic. Thus, both u and φ are holomorphic from part of (i) Lemma 2.3.6. □

2.3.c Consequences of Theorem 2.3.3

Combining Theorem 2.3.3 with the results of the previous sections (Corollary 2.2.4 and Theorem 2.2.12 with Remark 2.1.23 respectively) we obtain the following result.

Theorem 2.3.9.

(i) Any continuous CO from a NSHF that generates the topology of its phase space into another NSHF is a $CO_{\mathcal{H}}$.

(ii) Any continuous WCO with a non-vanishing multiplicative symbol from a 2-independent NSHF that generates the topology of its phase space and contains a non-zero constant function, into another NSHF is a $WCO_{\mathcal{H}}$.

(iii) Let $\mathbf{F}$ be a 2-independent NSHF over X, such that the bounded functions form a dense set in $\mathbf{F}$, and $\lim_{\infty} |\kappa|_{\mathbf{F}} = +\infty$. Then any continuous WCO with a non-vanishing multiplicative symbol from $\mathbf{F}$ into another 1-independent NSHF is a $WCO_{\mathcal{H}}$.

Remark 2.3.10. We cannot combine Theorem 2.3.3 with Theorem 2.2.10, because neither X is compact nor $\mathbf{F}$ is contained in $\mathcal{C}_0(X)$, due to Proposition 1.3.1. □

Although the conditions of part (ii) of Theorem 2.3.9 seem to be less restrictive than the conditions of part (iii), the latter has its own advantages. Let us remind the reader that it can be difficult to verify if a NSHF generates the topology of its phase space, and that there are spaces of interest that do not contain non-zero constant functions. For example, a concrete family of deBranges-Rovnyak spaces not containing constants is studied in [JM16].

Finally let us consider a variation of Theorem 2.3.9 for bounded domains in $\mathbb{C}^n$.

Proposition 2.3.11. *Assume that X is a bounded domain $\mathbb{C}^n$, and let $\mathbf{F}$ be a 2-independent NSHF over X. Then any continuous WCO from $\mathbf{F}$ into another 1-independent NSHF is a $WCO_{\mathcal{H}}$, provided that one of the following conditions holds:*

(i) $1 \in \mathbf{F}$ *and there is a finite set Z, such that $\mathbf{F}$ generates the topology of $X \backslash Z$.*

(ii) Bounded functions form a dense set in $\mathbf{F}$, and $\lim\limits_{\infty} |\kappa|_{\mathbf{F}} = +\infty$.

Proof. Let $\mathbf{E}$ be a 1-independent NSHF over Y. Let $u : Y \to \mathbb{F}$ and $\varphi : Y \to X$ be such that $W_{\varphi,u} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$. We may assume $u \not\equiv 0$.

Suppose that (i) holds. Since $1 \in \mathbf{F}$, then u is holomorphic. From Proposition 2.3.1 and part (iii) of Proposition 2.2.1, it follows that either $G = u^{-1}(0) \cup \varphi^{-1}(Z)$ is thin and closed, or $W_{\varphi,u} = W_{\varphi_z,u}$, where $z \in Z$ and $\varphi_z : Y \to X$ is defined by $\varphi_z(y) = z$. In the latter case $W_{\varphi,u}$ is obviously a $\mathrm{WCO}_{\mathcal{H}}$, and so we are left with the former case. From part (ii) of Theorem 2.3.9 applied to restrictions of $\mathbf{F}|_{X\backslash Z}$ and $\mathbf{E}|_{Y\backslash G}$ we get that φ is holomorphic on $Y \backslash G$. Note that φ is bounded, since $\varphi(Y) \subset X$. Hence, by Removable Singularity theorem there is a holomorphic map $\psi : Y \to X$, which coincides with φ on $Y \backslash G$. We will show that $W_{\varphi,u} = W_{\psi,u}$. Indeed, for every $f \in \mathbf{F}$ the functions $W_{\varphi,u}f$ and $W_{\psi,u}f$ are both holomorphic and coincide on a dense set $Y \backslash G$. Hence, $W_{\varphi,u}f = W_{\psi,u}f$, and so $W_{\varphi,u} = W_{\psi,u}$ is a $\mathrm{WCO}_{\mathcal{H}}$.

Analogously, if (ii) holds, it follows from part (iii) of Theorem 2.3.9, that both φ and u are holomorphic outside $u^{-1}(0)$, which is a thin closed set due to part (ii) of Proposition 2.3.1. Again, there is a holomorphic map $\psi : Y \to X$, which coincides with φ on $Y \backslash u^{-1}(0)$, and so $W_{\varphi,u} = W_{\psi,u}$. But then u is also holomorphic due to part (i) of Proposition 2.3.1. Thus, $W_{\varphi,u} = W_{\psi,u}$ is a $\mathrm{WCO}_{\mathcal{H}}$. $\square$

Consider the following examples.

Example 2.3.12. Let $\mathbf{E} = \left\{ f \in \mathcal{C}\left(\overline{\mathbb{D}}\right),\ \ f|_{\mathbb{D}} \in \mathcal{H}\left(\mathbb{D}\right),\ f\left(0\right) = f\left(1\right) \right\}$, endowed with the supremum norm. In fact, $\mathbf{E}$ is a closed subspace of the disk algebra $A\left(\mathbb{D}\right)$ of co-dimension 1, and in particular is a Banach space. Consider the restriction $\mathbf{F} = \mathbf{E}|_{\mathbb{D}}$, which is a BSHF over $\mathbb{D}$. Using the polynomials in $\mathbf{F}$, one can show that $\mathbf{F}$ is ∞-independent, and for every $x \in X$ there is $f \in \mathbf{F}$, which is a local homeomorphism at x. However, the topology generated by $\mathbf{F}$ is the topology of a "folded" disk, tangent to itself at 0 by the "end" that approaches 1. In particular, the sequence $\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 0 in this topology, and so it is not the original topology of $\mathbb{D}$.

Nevertheless, $\mathbf{F}$ contains 1 and generates the topology of $\mathbb{D}\backslash\left\{0\right\}$, and so, by part (i) of the preceding proposition, every continuous WCO from $\mathbf{F}$ into another 2-independent NSHF is a $\mathrm{WCO}_{\mathcal{H}}$. □

Example 2.3.13. Let H^2 be the Hardy space, which is a HSHF over $\mathbb{D}$, and let $\mathbf{F} = \left\{f \in H^2 \left| f'\left(0\right) = 0\right.\right\}$. This is a RKHS over $\mathbb{D}$ with kernel $K\left(z,w\right) = \frac{1 - z\overline{w} + z^2\overline{w}^2}{1 - z\overline{w}}$. Since the identity function does not belong to $\mathbf{F}$ it is not immediately clear if this space generates the topology of $\mathbb{D}$. However, using the polynomials, one can easily show that $\mathbf{F}$ is ∞-independent and that the bounded functions are dense in $\mathbf{F}$. Also, $\lim\limits_{\infty} \left|\kappa\right|_{\mathbf{F}} = \lim\limits_{r\to 1} \sqrt{\frac{1-r^2+r^4}{1-r^2}} = +\infty$, and so $\mathbf{F}$ satisfies the conditions of part (ii) of the preceding proposition. Thus, any continuous WCO from $\mathbf{F}$ into another 1-independent NSHF is a $\mathrm{WCO}_{\mathcal{H}}$. □

Remark 2.3.14. We conclude this section with few more comments.

- It would be desirable to refine part (i) of Proposition 2.3.11. Namely one can ask if it is still true if we allow Z to be locally finite in X or thin. Also, perhaps such Z exists automatically for any 2-independent NSHF.

- In fact, we don't have any example of a continuous WCO between 2-independent NSHF's, which is not a $\text{WCO}_{\mathcal{H}}$ ($\text{WCO}_{\mathcal{C}}$). In particular, we could not adapt the counterexamples from the preceding sections to the holomorphic case.
- It is possible to restate some of the results of this section as a holomorphic version of Proposition 2.2.9.
- Another approach to recognition of WCO's on NSHF's can be found in [MR15]. Somewhat related problems to Proposition 2.1.24 were considered in [Bou14] in the holomorphic setting. $\square$

2.4 Invariant Hermitean Metric Generated By Reproducing Kernel Hilbert Spaces

Geometric objects are largely characterized by their symmetries. Moreover, by geometry of an object we often mean the prescribed class of symmetries. For example, the symmetries of Hilbert spaces are the unitaries, the symmetries of complex manifolds are the biholomorphisms and the symmetries of Hermitean manifolds are biholomorphims that are also isometries. But how to relate the symmetries of a Hilbert space with symmetries of a complex manifold if the former is a RKHS of holomorphic functions over the latter?

A well-known construction of an automorphism-invariant Hermitean metric for bounded domains in $\mathbb{C}^n$ is due to Bergman (see [Ber70]). An intermediate tool for doing that was the Bergman space, which is the space of all square-integrable holomorphic functions on the domain. Note that the latter space is not an invariant object, since it depends on the Lebesgue measure, which is not biholomorphism-invariant. An explanation of why the obtained metric is invariant was suggested by Kobayashi (see [Kob59]), who replaced the Bergman space of functions with the Bergman space of differential forms, which is an invariant object. However, this reasoning was specific to the Bergman space, and cannot be applied to other RKHS's which produce invariant metrics. Kobayashi has also introduced another invariant distance, which is simpler to calculate explicitly. The latter distance received more attention in the works of Skwarczyński ([Skw69]).

Bergman's and Kobayashi's construction can be viewed as follows: take the Fubini-Study metric on the "abstract" separable Hilbert space, identify the latter with (the dual of) the Bergman space and pull back the Fubini-Study metric along the evaluation map, associated with the Bergman space. It is possible to replace the Bergman space with an arbitrary RKHS in this construction. In [CD78] this idea was used to obtain a unitary invariant of multiplication with the free variable on the domains in $\mathbb{C}$ (see also [AM02]). In [ARSW11] the authors did a thorough analysis of the metrics that were obtained this way. One of them then was able (see [Roc16]) to solve a certain embeddability problem using these metrics. However the applicability of his approach is limited, since the metric construction only "matches" some of the RKHS's. To overcome this obstacle one can try to find a "matching" metric for any RKHS, i.e. a replacement of the Fubini-Study metric on the abstract Hilbert space, that would match the properties of the evaluation map somehow.

A natural choice of matching of a metric on an abstract Hilbert space and a RKHS is that the resulting pull-back metric has to be invariant with respect to the automorphisms of the original domain. Since we identify our concrete RKHS with an abstract Hilbert space, the metric considered on the latter should be unitary-invariant, in order for the pull-back not to depend on the identification. In [Bil17] we found an analytic description and basic properties of all such metrics, which we summarize in the following subsection, while in the last two subsections we study the pull-back metric.

2.4.a A Unitary-invariant Hermitean Metric on Hilbert space

Let H be a complex Hilbert space, let $G = H\setminus\{0\}$. We will say that $\sigma : G \times H \times H \to \mathbb{C}$ is *metric* if $\sigma_g = \sigma(g,\cdot,\cdot)$ is conjugate-symmetric and sesquilinear on H, for every $g \in G$. We will call a metric σ *positive (semi) definite* if σ_g is positive (semi) definite for each $g \in G$; we will call it invariant with respect to an injective linear operator T on H if $\sigma_{Tg}(Tf, Th) = \sigma_g(f,h)$ for every $g \in G$ and $f, h \in H$. In this case the length defined by σ is also invariant with respect to T.

The following theorem incorporates several results from [Bil17]: Proposition 2.8, Corollary 3.4, Theorem 3.7 and Remark 3.8. Recall that a linear operator on H is called a *congruency* if it is a scalar multiple of an isometry. We characterise unitary-invariant and congruency-invariant Hermitean metric on G, i.e. such that are invariant with respect to all unitaries and congruencies respectively. The importance of the latter invariance is justified by part (iv) of the theorem.

Theorem 2.4.1. *A non-zero metric σ on H is unitary-invariant if and only if there are (unique) functions $u, v : (0,+\infty) \to \mathbb{R}$, such that*

$$\sigma_g(f,h) = u\left(\|g\|^2\right)\langle f,h\rangle_H + v\left(\|g\|^2\right)\langle f,g\rangle_H \langle g,h\rangle_H,$$

for $g \in G$, and $f, h \in H$. Moreover, in this case the following hold:

(i) σ is positive definite if and only if $u(r) > 0$ and $u(r) + rv(r) > 0$, for every $r > 0$.

(ii) The degree of smoothness of σ coincides with the minimal degree of smoothness of u and v.

(iii) σ is invariant with respect to all congruencies if and only if there are (unique) $a, b \in \mathbb{R}$, such that $u(r) = \frac{a}{r}$ and $v(r) = \frac{b}{r^2}$, for every $r > 0$.

(iv) If σ is invariant with respect to a linear operator $T : H \to H$, then T is a congruency.

We supply the theorem with several of remarks and an example also taken from [Bil17]. In the following subsection they will be adapted to the case when H^* is a RKHS, as well as the theorem above.

Remark 2.4.2. Once u and v are sufficiently smooth, $u(r) > 0$ and $u(r) + rv(r) > 0$, then σ is a Hermitean metric on G. Furthermore, σ is a Kaehler metric if and only if $v = u'$. In this case $w \circ \|\cdot\|^2$ is the potential of this metric, where $w' = u$. □

Remark 2.4.3. The non-strict analogues of the strict inequalities in part (i) correspond to the positive semi-definiteness of σ. In particular, if $u(r) = -rv(r)$, for every $r > 0$, then σ glues elements that are scalar multiples of each other, i.e. factorizes by $\mathbb{C}$-lines. The case when $u(r) = 0$ leads to identifying all elements of norm r. On the other hand, even if $u(r) = -rv(r)$, for some $r > 0$, then for every $g \in r\partial B_H$ the form σ_g is positive definite on any subspace of H that does not contain g. □

Remark 2.4.4. The positive definiteness of congruency-invariant σ is equivalent to $a + b > 0$ in part (iii). The latter contradicts to the necessary condition for σ to be Kaehler, which is reduced to $a = -b$. Thus, there is no Kaehler metrics on G invariant with respect to all congruencies. □

Example 2.4.5. Let $\widetilde{\sigma}$ be a metric on H defined by $\widetilde{\sigma}_g(f,h) = \frac{1}{\|g\|^2}\langle f,h\rangle_H - \frac{1}{\|g\|^4}\langle f,g\rangle_H \langle g,h\rangle_H$. By Remark 2.4.4, this is the unique (up to scalar multiplication) "degenerate Kaehler metric" which is invariant with respect to all congruencies. Using Remark 2.4.3 one can show that it is also the unique (up to scalar multiplication) "degenerate Kaehler metrics" which factorizes by the $\mathbb{C}$-lines. Since $\widetilde{\sigma}$ is the pull-back of the classical Fubini-Study metric on the projective space PH via the natural quotient map, we find it natural to call $\widetilde{\sigma}$ the Fubini-Study metric on G. Note, that $2\log\|\cdot\|$ is the potential of this metric. Inspired by [ARSW11] and following [Kob59], we consider two congruency-invariant pseudodistances on G related to $\widetilde{\sigma}$. For $g,h\in G$ define

$$\delta_1(g,h) = \sin\angle(g,h) = \sqrt{1-\frac{|\langle g,h\rangle|^2}{\|g\|^2\|h\|^2}},\ \delta_2(g,h) = \sin\frac{\angle(g,h)}{2} = \sqrt{2-2\frac{|\langle g,h\rangle|}{\|g\|\|h\|}}.$$

Note that δ_1 and δ_2 also factorise by the $\mathbb{C}$-lines; while the geometric meaning of δ_1 is obvious, δ_2 is the distance between the intersections of the $\mathbb{C}$-lines defined by g,h and the unit sphere. The importance of δ_1 and δ_2 for us is determined by the fact that the length of the curves with respect to δ_1, δ_2 and σ coincide. $\square$

2.4.b Pull-back of a Unitary-invariant Hermitean Metric on a Dual of a RKHS

Using the description from the preceding subsection we can express the pull-back of any unitary-invariant Hermitean metric on the dual of a HSHF via its evaluation map. In this subsection X is a complex manifold and $\mathbf{H}$ is a 1-independent HSHF over X. Then from Theorem 1.3.15 the evaluation map $\kappa=\kappa_{\mathbf{H}}$ is holomorphic from X into $\mathbf{H}^*\setminus\{0_{\mathbf{H}^*}\}$, and the kernel $k=k^{\mathbf{H}}$ is positive on the diagonal and sesqui-holomorphic.

Theorem 2.4.6. *Let σ^k be the pull-back of a smooth unitary-invariant metric σ on $\mathbf{H}^*\setminus\{0_{\mathbf{H}^*}\}$ via κ. Then there are smooth functions $u, v : (0, +\infty) \to \mathbb{R}$, such that for any $x \in X$ and $\xi, \zeta \in \mathcal{T}_x$ we have*

$$\sigma_x^k(\xi,\zeta) = u\left(\widehat{k}(x)\right)\xi \otimes \overline{\zeta\widehat{k}} + v\left(\widehat{k}(x)\right)\xi\widehat{k}\widehat{\overline{\zeta k}}.$$

Proof. Let $G = \mathbf{H}^*\setminus\{0_{\mathbf{H}^*}\}$, and let $\sigma : G \times \mathbf{H}^* \times \mathbf{H}^* \to \mathbb{C}$ be a smooth unitary-invariant function, such that σ_μ is conjugate-symmetric sesquilinear on $\mathbf{H}^*$, for every $\mu \in G$. By virtue of Theorem 2.4.1, there are smooth real functions u, v, such that

$$\sigma_\mu(\lambda,\nu) = u\left(\|\mu\|^2\right)\langle\lambda,\nu\rangle_{\mathbf{H}^*} + v\left(\|\mu\|^2\right)\langle\lambda,\mu\rangle_{\mathbf{H}^*}\langle\mu,\nu\rangle_{\mathbf{H}^*},$$

for $\mu \in G$, and $\lambda, \nu \in \mathbf{H}^*$. Let $x \in X$ and $\xi, \zeta \in \mathcal{T}_x$ be tangent vectors at x. By definition of the pull-back $\sigma_x^k(\xi,\zeta) = \sigma_{\kappa(x)}(\mathcal{T}_x\kappa\xi, \mathcal{T}_x\kappa\zeta)$, where $\mathcal{T}_x\kappa$ is the differential of κ at x. Recall from the end of Subsection 1.3.a and the beginning of Subsection 1.3.b, that $\mathcal{T}_x\kappa\xi = \xi\kappa$ and $\mathcal{T}_x\kappa\zeta = \zeta\kappa$, and so

$$\begin{aligned}\sigma_x^k(\xi,\zeta) = \sigma_{\kappa(x)}(\xi\kappa,\zeta\kappa) &= u\left(\|x_{\mathbf{H}}\|^2\right)\langle\xi\kappa,\zeta\kappa\rangle + v\left(\|x_{\mathbf{H}}\|^2\right)\langle\xi\kappa,x_{\mathbf{H}}\rangle\langle x_{\mathbf{H}},\zeta\kappa\rangle\\ &= u\left(\widehat{k}(x)\right)\xi\otimes\overline{\zeta\widehat{k}} + v\left(\widehat{k}(x)\right)\xi\widehat{k}\overline{v}\widehat{k},\end{aligned}$$

where the last equality follows from from Corollary 1.3.19. □

Remark 2.4.7. In the local coordinates the metric is expressed as

$$\sum_{i,j=1}^{n}\left[u\circ\widehat{k}\frac{\partial^2}{\partial z_i\partial\overline{z}_j}\widehat{k} + v\circ\widehat{k}\frac{\partial}{\partial z_i}\widehat{k}\frac{\partial}{\partial\overline{z}_j}\widehat{k}\right]dz_i\otimes d\overline{z}_j.$$

If u and v are sufficiently smooth and $v = u'$, then σ^k is Kaehler. Using the chain rule one can verify that its potential is $w\circ\widehat{k}$, where $w' = u$, i.e. in local coordinates the metric is $\sum_{i,j=1}^{n}\frac{\partial^2}{\partial z_i\partial\overline{z}_j}\left(w\circ\widehat{k}\right)dz_i\otimes d\overline{z}_j$.

Assume that w can be extended to a holomorphic function in a neighborhood of $(0,+\infty) \subset \mathbb{C}$. If l is another sesqui-holomorphic kernel such that

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \left(w \circ \widehat{k} \right) dz_i \otimes d\overline{z}_j = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \left(w \circ \widehat{l} \right) dz_i \otimes d\overline{z}_j,$$

then locally there is a holomorphic function h such that $w\left(l\left(x,y\right)\right) = w\left(k\left(x,y\right)\right) + h\left(x\right) + \overline{h\left(y\right)}$. This fact is known as Calabi rigidity (see [Cal53]), and we will discuss in more details a global version for a particular choice of w. $\square$

Remark 2.4.8. It follows from part (i) of Theorem 2.4.1 that if $u\left(r\right) > 0$ and $u\left(r\right)+rv\left(r\right) > 0$ for every $r > 0$ then σ^k is positive definite. However, this condition is not necessary. In particular, from Remark 2.4.3, if $u\left(r\right) > 0$ and $u\left(r\right) + rv\left(r\right) \geq 0$ for every $r > 0$, then σ^k is positive definite, unless there are $x \in X$ and $\xi \in \mathcal{T}_x$ such that $\xi\kappa = x_{\mathbf{H}}$, or equivalently, $k_x\left(z\right) = a_1 \left.\frac{\partial}{\partial \overline{w}_1}\right|_x k\left(z,w\right) + ... + a_n \left.\frac{\partial}{\partial \overline{w}_n}\right|_x k\left(z,w\right)$, for every $z \in X$, for some $a_1, ..., a_n \in \mathbb{C}$, with respect to some local coordinates near x.

However, this phenomenon usually does not occur to the spaces of interest. In fact, analogously to Lemma 2.3.4 one can show that if $\mathbf{H}$ is 2-independent, then there is a thin set $Z \subset X$ such that $\xi\kappa = x_{\mathbf{H}}$, for some $x \in X$ and $\xi \in \mathcal{T}_x$, implies $x \in Z$. $\square$

Example 2.4.9. Let us consider the pull-back of the most natural metric on a Hilbert space – the Euclidean metric. This metric corresponds to $u \equiv 1$ and $v \equiv 0$. Then, $\sigma^k_x\left(\xi,\xi\right) = \left\|\xi\kappa\right\|^2 = \xi \otimes \overline{\xi}\widehat{k}$. In fact, σ^k is obtained from turning the distance $d_{\mathbf{H}}$, defined by $d_{\mathbf{H}}\left(x,y\right) = \sqrt{k\left(x,x\right) + k\left(y,y\right) - 2\mathrm{Re}\ k\left(x,y\right)}$, $x, y \in X$, into a length metric. For any $f \in \mathbf{H}$ we have that

$$\left|\xi f\right| = \left|\left\langle f, \xi\kappa\right\rangle\right| \leq \left\|f\right\|\left\|\xi\kappa\right\| = \left\|f\right\|\sqrt{\sigma^k_x\left(\xi,\xi\right)}.$$

Thus, f is a Lipschitz function on X with respect to σ^k, with Lipschitz constant at most

$\|f\|$. This fact may be viewed as a smooth version of Proposition 1.2.15 and Proposition 1.2.20. $\square$

Example 2.4.10 (**Bergman metric**). Assume that X is a bounded domain in $\mathbb{C}^n$. Define a "standard Bergman space" $A^2(X)$ over X as $A^2(X,\lambda)$, where λ is the Lebesgue measure on X. Since polynomials belong to $A^2(X)$ it follows that the latter is ∞-independent. The *Bergman metric* of X is the pull-back $\widetilde{\sigma}^{k^{A^2(X)}}$ of the Fubini-Study metric $\widetilde{\sigma}$ (see Example 2.4.5), whose potential is $\log \widehat{k^{A^2(X)}}$.

If $x=(x_1,x_2,...,x_n)\in X$ and $\xi=\sum\limits_{i=1}^{n} a_i \frac{\partial}{\partial z_i}\Big|_x \neq 0$, then the polynomial f defined by $f(z_1,...,z_n)=\overline{a_1}(z_1-x_1)+...+\overline{a_n}(z_n-x_n)$ belongs to $A^2(X)$ and $f(x)=0$, while $\xi f=|a_1|^2+...+|a_n|^2\neq 0$, and so $\kappa(x)\neq \xi\kappa$. Thus, by virtue of Remark 2.4.8, the Bergman metric is positive definite. We will come back to this metric in the next subsection. $\square$

2.4.c Automorphism-invariance

In this subsection we assume that $\mathbf{H}$ is a 1-independent HSHF over a complex manifold X with a sesqui-holomorphic kernel k which is positive on the diagonal, and with the evaluation map κ, which is holomorphic.

An *automorphism* of X is a biholomorphism of X into itself, i.e. a bijective holomorphic self-map of X (and then its inverse is also holomorphic, see [Chi89, A2.2, Corollary 1]). We will denote the group of all automorphisms of X by $Aut(X)$. We say that X is *homogeneous* if $Aut(X)$ acts transitively on X, i.e. there is $x\in X$ such that the orbit $Aut(X)\cdot x=X$. Consider the following strengthenings of homogeneity: X is called *weakly symmetric* if for any $x,y\in X$ there is $\varphi\in Aut(X)$, such that $\varphi(x)=y$ and $\varphi(y)=x$; X is called *symmetric* if φ in the previous definition can be chosen to be an involution. It is a classic result of Cartan (see [Car35]) that if X is a bounded domain in $\mathbb{C}$, $\mathbb{C}^2$ or $\mathbb{C}^3$, then these concepts coincide, but this is not true in general.

A Hermitean metric on X is called *automorphism-invariant*, if every automorphisms is an isometry with respect to this metric. Automorphism-invariant metrics may be viewed as the most natural, as they only depend on the complex structure of X. In particular, if X is biholomorphic to a domain in $\mathbb{C}^n$, automorphism-invariant metrics do not depend on the realization of X as a particular domain.

This and the following subsections are dedicated to the following question: given a positive semi-definite sesqui-holomorphic kernel k on X, when is it possible to find a unitary-invariant Hermitean metric σ on an (abstract) Hilbert space, such that σ^k is automorphism-invariant? This question is of the most interest when $Aut\,(X)$ is "large", i.e. when X is at least homogeneous.

Assume that we were able to find a unitary-invariant metric σ on $\mathbf{H}^*$, which is invariant with respect to C_φ^*, for every $\varphi \in Aut\,(X)$. Clearly, then σ^k is invariant with respect to all of the automorphisms and our goal is accomplished. However, the following result shows that usually this can only happen in the trivial case.

Proposition 2.4.11. *Assume that X is either weakly symmetric, or homogeneous and biholomorphic to a bounded and balanced*[1] *domain in $\mathbb{C}^n$. If there is a unitary-invariant metric σ on $\mathbf{H}^*$, which is invariant with respect to C_φ^*, for every $\varphi \in Aut\,(X)$, then $\mathbf{H}$ consists of constant functions.*

[1] *a subset B of a complex vector space is called balanced if $\alpha B \subset B$, for any $\alpha \in \overline{\mathbb{D}}$.*

The fact follows immediately from part (iv) of Theorem 2.4.1 and the following lemma.

Lemma 2.4.12. *If X is as in the proposition and C_φ^* is a congruency for every $\varphi \in Aut\,(X)$, then* **H** *is the space of constant functions.*

Proof. If C_φ^* is a congruency, for $\varphi \in Aut\,(X)$, then from Example 2.1.19 we have that $k\left(\varphi\left(x\right),\varphi\left(y\right)\right) = \left\|C_\varphi^*\right\|^2 k\left(x,y\right)$, for every $x, y \in X$. We will have separate proofs of the lemma for the weakly symmetric and bounded balanced cases.

Step 1. Assume that X is weakly symmetric. Fix $x \in X$. Let $y \in X$ and let $\varphi \in Aut\,(X)$ be such that $\varphi\left(x\right) = y$ and $\varphi\left(y\right) = x$. We have that $\overline{k\left(y,x\right)} = \left\|C_\varphi^*\right\|^2 k\left(y,x\right)$, and so $k\left(y,x\right)^2 \in \mathbb{R}$. Since y was chosen arbitrarily we conclude that $k\left(\cdot,x\right)^2$ is a real-valued function. Since this function is holomorphic, due to Open Mapping theorem it follows that k_x is a real constant. Since x was also chosen arbitrarily, we get that k is a real constant, and so **H** is the space of constant functions, according to Example 1.2.31.

Step 2. Let us show now that if X is a balanced domain in $\mathbb{C}^n$, and **H** is such that C_φ^* is a unitary for every $\varphi \in Aut\,(X)$ with $\varphi\left(0_{\mathbb{C}^n}\right) = 0_{\mathbb{C}^n}$, then $k_{0_{\mathbb{C}^n}}$ is a constant.

For $z = \left(z_1, z_2, ..., z_n\right) \in \mathbb{C}^n$ and $I = \left[i_1, i_2, ..., i_n\right] \in \mathbb{N}_0^n$ denote $z^I = z_1^{i_1} z_2^{i_2} ... z_n^{i_n}$ and $|I| = i_1 + i_2 + ... + i_n$. For every $\alpha \in \mathbb{T}$ consider the Taylor expansion

$$k\left(0_{\mathbb{C}^n}, 0_{\mathbb{C}^n}\right) + \sum_{|I|>0} a_I z^I = k_{0_{\mathbb{C}^n}}\left(z\right) = k_{0_{\mathbb{C}^n}}\left(\alpha z\right) = k\left(0_{\mathbb{C}^n}, 0_{\mathbb{C}^n}\right) + \sum_{|I|>0} a_I \alpha^{|I|} z^I,$$

and so $a_I = a_I \alpha^{|I|}$, for every $I \in \mathbb{N}_0^n$. If $\alpha = e^{\mathrm{i}\theta}$, for $\theta \notin \mathbb{Q}$, then $\alpha^{|I|} \neq 1$, which forces $a_I = 0$, whenever $|I| > 0$. Thus, $k_{0_{\mathbb{C}^n}}\left(z\right) \equiv k\left(0_{\mathbb{C}^n}, 0_{\mathbb{C}^n}\right)$.

Step 3. Assume that X is homogeneous and biholomorphic to a bounded and balanced domain in $\mathbb{C}^n$. Define $\alpha : Aut\,(X) \to \mathbb{R}$ by $\alpha\left(\varphi\right) = 2\log\left\|C_\varphi^*\right\|$. It is easy to see that if C_φ^*

is a congruency for every $\varphi \in Aut\,(X)$, then α is a group homomorphism, and the equality $k\left(\varphi\left(x\right), \varphi\left(y\right)\right) = e^{\alpha(\varphi)} k\left(x, y\right)$, for every $x, y \in X$, ensures that α is continuous on $Aut\,(X)$. Since an isotropy group at any point of X is compact (see [GKK11, Corollary 1.3.7]), α maps this group into a compact subgroup of $\mathbb{R}$. The only such subgroup is trivial, and so we get that any automorphism with a fixed point must induce a unitary composition operator.

Since X is homogeneous, for every $x \in X$ we can view X as a balanced domain in $\mathbb{C}^n$ with $x = 0_{\mathbb{C}^n}$. Then, from the lemma above we get that k_x is a constant, but since x was chosen arbitrarily, we conclude that k is a constant. $\square$

Due to Corollary 2.1.15 C_φ^* is a congruency if and only if C_φ is, and so as a byproduct we obtain the following result.

Corollary 2.4.13. *If X is as in the Proposition 2.4.11, and C_φ is a congruency on $\mathbf{H}$, for every $\varphi \in Aut\,(X)$, then $\mathbf{H}$ is the space of constant functions.*

2.4.d Projective Invariance of Sesqui-holomorphic Kernels

In this subsection $\mathbf{H}$ is again a 1-independent HSHF over a complex manifold X with a sesqui-holomorphic kernel k which is positive on the diagonal, and evaluation map κ, which is holomorphic. Proposition 2.4.11 tells that no unitary-invariant Hermitean metric on $\mathbf{H}^*$ can be invariant with respect to the action of $Aut\,(X)$ via the adjoints of the composition operators. However, the pull-back σ^k of such metric still can be automorphism-invariant. In order to see this let us discuss Calabi rigidity for a specific choice of a Hermitean metric on a Hilbert space, as announced in Remark 2.4.7. Namely, we will discuss the possible automorphism invariance of $\widetilde{\sigma}^k$, where $\widetilde{\sigma}$ is the Fubini-Study metric. Before doing so consider the pull-backs δ_1^k and δ_2^k of δ_1 and δ_2 (see Example 2.4.5) through κ. It is easy to see that

$$\delta_1^k(x,y)=\sqrt{1-\frac{|k(x,y)|^2}{k(x,x)\,k(y,y)}},\ \delta_2^k(x,y)=\sqrt{2-2\frac{|k(x,y)|}{\sqrt{k(x,x)\,k(y,y)}}},$$

for any $x,y\in X$. Clearly, if $\mathbf{H}$ is 2-independent, then both δ_1^k and δ_2^k are distances on X. Since the length with respect to $\widetilde{\sigma}$, δ_1 and δ_2 coincide, the same is true for $\widetilde{\sigma}^k$, δ_1^k and δ_2^k. This makes δ_1^k and δ_2^k useful, since they are given explicitly, while the actual distance generated by $\widetilde{\sigma}^k$ can be difficult to calculate. It turns out that each of $\widetilde{\sigma}^k$, δ_1^k and δ_2^k determine k up to a rescaling. This follows immediately from the formulas for $\widetilde{\sigma}^k$, δ_1^k and δ_2^k and the following criterion for rescaling of sesqui-holomorphic functions.

Theorem 2.4.14. [Bil18a, Theorem 6.7] *Let l and m be sesqui-holomorphic functions on X, which are positive on the diagonal. The following are equivalent:*

(i) There is $v:X\to\mathbb{C}^\times$ such that $l=v\otimes\overline{v}m$;

(ii) $\frac{\partial^2}{\partial z_i\partial\overline{z_j}}\log\widehat{l}=\frac{\partial^2}{\partial z_i\partial\overline{z_j}}\log\widehat{m}$, for every $i,j\in\overline{1,\dim X}$, on (an open nonempty set in) X;

(iii) There are $y\in X$ and an open neighborhood U of y such that $\frac{|l(x,y)|^2}{l(x,x)l(y,y)}=\frac{|m(x,y)|^2}{m(x,x)m(y,y)}$, for every $x\in U$.

The map $\varphi\in Aut(X)$ is an isometry with respect to $\widetilde{\sigma}^k$ if and only if the pull-back of $\widetilde{\sigma}$ via κ and via $\kappa\circ\varphi$ is the same, which is equivalent to $\widetilde{\sigma}^k=\widetilde{\sigma}^{k\circ\varphi}$. The last condition means that k and $k\circ\varphi$ are rescalings. Combining this with the result above and with Proposition 2.1.20 we obtain a criterion for $\widetilde{\sigma}^k$ be automorphism invariant (the criterion for δ_1^k and δ_2^k follows from the same reasoning).

Corollary 2.4.15. *The following are equivalent:*

(i) Each (any) of $\widetilde{\sigma}^k$, δ_1^k and δ_2^k is automorphism-invariant;

(ii) For every $\varphi \in Aut(X)$ there is $v_\varphi : X \to \mathbb{C}^\times$ such that W_{φ, v_φ} is a unitary operator on $\mathbf{H}$*;*

(iii) For every $\varphi \in Aut(X)$ there are $y \in X$ and an open neighborhood U of y such that

$$\frac{|k(\varphi(x), \varphi(y))|^2}{k(\varphi(x), \varphi(x))\, k(\varphi(y), \varphi(y))} = \frac{|k(x, y)|^2}{k(x, x)\, k(y, y)},$$

for every $x \in U$.

We will say that $\mathbf{H}$ is *projectively invariant* if it satisfies one of the equivalent conditions of the corollary. Using condition (iii) it is easy to see that a product of projectively invariant kernels is projectively invariant; if k is projectively invariant, then k^s is projectively invariant, for any $s \in \mathbb{R}$, as long as it is well-defined and positive definite. A pointwise limit of projectively invariant kernels is projectively invariant (if it stays sesqui-holomorphic and positive on the diagonal); if the limit is with respect to the compact-open topology, sesqui-holomorphicity is preserved due to part (i) of Proposition 1.3.13. Finally, any kernel of the form $k = h \otimes \overline{h}$, where $h \in \mathcal{H}(X)$ does not vanish, is trivially projectively invariant; hence, any holomorphic rescaling of a sesqui-holomorphic projectively invariant kernel is projectively invariant. Let us consider a typical example.

Example 2.4.16 (Bergman space). Assume that X is a bounded domain in $\mathbb{C}^n$. Let φ be an automorphism X, let $J_\varphi(x)$ be its complex Jacobian at x; then $|J_\varphi(x)|^2$ is the real Jacobian (see [Chi89, A.2.1, Proposition]). Then, for each $f \in A^2(X)$, using the change of variables in the integration, we have

$$\begin{aligned}\|f\|^p_{A^2(X)} &= \int_X |f(y)|^2\, d\mu(y) = \int_X |f \circ \varphi(x)|^2\, |J_\varphi(x)|^2\, d\mu(x) \\ &= \|J_\varphi \cdot f \circ \varphi\|^2_{A^2(X)} = \left\|W_{\varphi, J_\varphi} f\right\|^2_{A^2(X)},\end{aligned}$$

thus W_{φ,J_φ} is an isometry of $A^2(X)$. Then $W_{\varphi^{-1},J_{\varphi^{-1}}}$ is also an isometry of $A^2(X)$, and since

$$W_{\varphi,J_\varphi}W_{\varphi^{-1},J_{\varphi^{-1}}} = Id_{A^2(X)} = W_{\varphi^{-1},J_{\varphi^{-1}}}W_{\varphi,J_\varphi},$$

it follows that both of these operators are unitaries. Thus, $A^2(X)$ is projectively invariant, and so the Bergman metric $\widetilde{\sigma}^{k^{A^2(X)}}$ is automorphism-invariant. More on the subject see in [AF90]. $\square$

Chapter 3

Summary and Future Directions

Before proceeding with the auxiliary facts that constitute the Appendix, let us briefly recap the results of the thesis, and outline some prospectives for further research.

3.1 Summary

The broad topic of this thesis is the application of the axiomatic approach towards Normed Spaces of Continuous Functions and Weighted Composition Operators acting between them.

In Chapter 1 we studied the relations between various properties of NSCF's. Our results include a criterion for reflexivity of a NSCF, sufficient conditions for its completeness, separability and relative compactness of its unit ball in the compact-open topology. The main common theme in our investigation was the importance of the "location" of the point evaluations in the dual of a NSCF. We also considered Reproducing Kernel Hilbert Spaces, Normed Spaces of Holomorphic Functions, as well as three important specific families of NSCF's: Weighted Uniform spaces, Lipschitz spaces and Bergman spaces.

In Chapter 2 we explored some basic questions related to WCO's acting between NSCF's and NSHF's. We found sufficient conditions for NSCF's to be such that the properties of WCO's between them corresponded to the properties of the symbols of that WCO's in a natural way. For example, we considered the questions of when continuity of a WCO implies the continuity and holomorphicity of its symbols, and when invertibility of a WCO implies invertibility of its composition symbol. We also studied the possibility of existence of non-trivial unitary MO's on NSCF's.

The obtained information allowed us to make some progress in determining when a RKHS of holomorphic functions generates an automorphism-invariant Hermitean metric on a domain in $\mathbb{C}^n$.

3.2 Future Directions

Let us mention few problems that remain unsolved as well as some possible directions for the future research. They are related to the following three topics presented in the thesis.

Normed Spaces of Continuous Functions: In the near future I plan to apply the obtained results on NSCF's in the context of the theory Reproducing Kernel Banach Spaces (see e.g. [ZXZ09]). The latter theory can be viewed as an intermediary between the approach developed in Chapter 1 and the more classical RKHS theory.

Weighted Composition Operators: It is desirable to sharpen the conditions in Theorem 2.2.12 and its consequences (part (iii) of Theorem 2.3.9 and part (ii) of Proposition 2.3.11), and in particular construct an example of a continuous WCO between 2-independent NSHF's, which is not a $\mathrm{WCO}_{\mathcal{H}}$ ($\mathrm{WCO}_{\mathcal{C}}$).

It would also be interesting to relax the conditions (i) or (ii) of Theorem 2.1.30. The geometry of $\mathbf{F}$ seems to be of importance here, as is demonstrated by Corollary 2.1.31. Hence, it is possible that relaxing the condition (ii) will lead to some new questions in the geometric Banach space theory.

Invariant Hermitean Metric Generated By Reproducing Kernel Hilbert Spaces: Let us remind the reader the main question regarding this topic. Let X be a homogeneous domain in $\mathbb{C}^n$ and let k be a sesqui-holomorphic positive definite kernel with the corresponding RKHS $\mathbf{H}$. Let σ be a unitary-invariant metric on $\mathbf{H}^*$, and let σ^k be the pull-back of σ via the evaluation map of $\mathbf{H}$. It would be interesting characterize the pairs of σ and k, such that σ^k is automorphism-invariant. Corollary 2.4.15 provides a characterization in the case when σ is the Fubini-Study metric. One can express the necessary condition for the invariance using the formula from Theorem 2.4.6, but the obtained condition is too complicated. My goal is to find a condition along the lines of Corollary 2.4.15, but for a general unitary-invariant metric σ.

Appendix A

Some supplementary results from Functional Analysis

A.1 On Birkhoff Orthogonality

Let E be a normed space. A vector $e \in E$ is called *Birkhoff (or Birkhoff-James) orthogonal* to $f \in E$, if $\|e\| \le \|e + tf\|$ for any $t \in \mathbb{F}$, i.e. $\|e\| = \|Pe\|$, where P is the quotient map onto $E/\text{span}\{f\}$. If E is a Hilbert space, then P is the orthogonal projection onto $E \ominus \text{span} f$, and so the notion of Birkhoff orthogonality coincides with the usual one. Note however, that in general the Birkhoff orthogonality is NOT a symmetric relation, which is one of the crucial differences between these concepts. This inspired our notation $e \vdash f$ for "e is Birkhoff orthogonal to f". There are other generalizations of the notion of orthogonality, some of which are symmetric, but we will only use Birkhoff orthogonality. More details on the subject can be found e.g. in [AMW12] or [FJ03, Section 1.4].

A vector $e \in E$ is called Birkhoff orthogonal to a linear subspace $F \subset E$ if $e \vdash f$, for every $f \in F$, which is equivalent to $\|e\| \le \|e+f\|$ for any $f \in F$. We will denote this relation by $e \vdash E$. A linear subspace $H \subset E$ is Birkhoff orthogonal to a linear subspace $F \subset E$ if $h \vdash F$, for every $h \in H$. This relation will be denoted by $H \vdash F$. Clearly, $H \vdash F$ if and only if $P|_H$ is an isometry, where P is the quotient map onto E/F.

Remark A.1.1. Note that a closed subspace F of E^* is 1-norming if and only if $E \vdash F^\perp$ in F^{**}. $\square$

For $e \in E$ let $e^\parallel = \left\{\nu \in \overline{B}_{E^*}, \langle e, \nu\rangle = \|e\|\right\}$, i.e. $e^\parallel = \overline{B}_{E^*} \bigcap e^{-1}\left(\|e\|\right)$, where e is viewed as a functional on E^*. This set is closed and convex, and it is easy to see that it is in fact included into ∂B_{E^*}. Also note, that $(\alpha e)^\parallel = \frac{\overline{\alpha}}{|\alpha|} e^\parallel$, for any $\alpha \neq 0$. It is well-known that $e \vdash F$ if $e^\parallel \cap F^\perp \neq \varnothing$, where $F^\perp \subset E^*$. Indeed, if $P : E \to E/F$ is a quotient map, then P^* is the isometry from $(E/F)^*$ into $F^\perp$ (see [Bou87, IV.1.4, Proposition 9]), from where $\|Pe\| = \sup\limits_{\nu \in F^\perp \bigcap \overline{B}_{E^*}} |\langle e, \nu\rangle|$. Hence, from the weak* compactness of the balanced set $F^\perp \bigcap \overline{B}_{E^*}$ it follows that $\|Pe\| = \|e\|$ if and only if there exists $\nu \in e^\parallel \cap F^\perp$.

As was already mentioned, the relation $\vdash$ is not symmetric, and so it generates distinct orthogonal complements $e^\vdash = \{f \in E \,|e \vdash f\}$ and ${}^\vdash e = \{f \in E \,|f \vdash e\}$. From the discussion above, ${}^\vdash e$ is the set of all maximal elements of functionals in $e^\perp \subset E^*$, while $e^\vdash = \bigcup\limits_{v \in e^\parallel} v^\perp$.

One of the intuitive aspects of the usual orthogonality is that it separates vectors, i.e. non-zero elements which are "close" cannot be orthogonal. More precisely, for any $e \neq 0$ in a Hilbert space, $e^\perp$ is a hyperplane in E, which is a closed convex (and so weakly closed) set not containing e. It is natural to ask whether the same phenomenon holds in general normed spaces. However, we deal in fact with four different questions, depending on which of the complements and which of the two topologies we choose. It turns out, that for the norm topology the answer is affirmative for both of the complements, since $\{(e,f)\in E\times E\,|e\vdash f\}$ is norm-closed in $E\times E$ (this is easy to see).

The rest of the subsection is dedicated to the question when e is weakly separated from $e^{\vdash}$. Since the latter is a union of hyperplanes, it is not necessarily convex and so it is not necessarily weakly closed. Hence, the question is not trivial, unless $e^{\|}$ contains only one element. Note, that if this condition holds for any $e\in E\backslash\{0_E\}$, then E is called *smooth*. A normed space F is called *strictly convex* if any convex subset of ∂B_F is a singleton. It is trivial, that if E^* is strictly convex, E must be smooth; using Hahn-Banach theorem it is easy to see that if E^* is smooth, E must be strictly convex; neither of the converses do not hold in general. The following proposition gives a definite answer to the question mentioned in the beginning of the paragraph.

Proposition A.1.2. *A non-zero vector $e\in E$ does not belongs to the weak closure of $e^{\vdash}$ if and only if the set $e^{\|}$ is finite-dimensional.*

Proof. Sufficiency. Assume that $e^{\|}$ is finite-dimensional. Since this set is bounded, there is a finite collection $D=\{\nu_1,\nu_2,...,\nu_n\}\subset E^*$, such that $e^{\|}\subset \mathrm{conv}D$. Then the set $U_e=\left\{f\in E\,\middle|\forall j\in\overline{1,n}\;\;|\langle f-e,\nu_j\rangle|<\|e\|\right\}$ is a weakly open neighborhood of e, which is disjoint from $e^{\vdash}$. Indeed, for any $\nu\in e^{\|}$ there are $t_1,...,t_n$, such that $\sum\limits_{j=1}^{n}t_j=1$ and $\nu=\sum\limits_{j=1}^{n}t_j\nu_j$. Then for any $f\in U_e$ we have that

$$|\langle f,\nu\rangle|=\left|\sum_{j=1}^{n}t_j\left\langle f-e,\nu_j\right\rangle+\langle e,\nu\rangle\right|\geq|\langle e,\nu\rangle|-\sum_{j=1}^{n}t_j\left|\langle f-e,\nu_j\rangle\right|>\|e\|-\sum_{j=1}^{n}t_j\|e\|=0,$$

and so $f\notin e^{\vdash}$. Thus, e is weakly separated from $e^{\vdash}$.

Necessity: Assume, there are $\{\nu_1,...,\nu_n\}\subset E^*$, such that $e^{\vdash}$ does not intersect $V_e=\left\{f\in E\,\middle|\forall j\in\overline{1,n}\;\;|\langle f-e,\nu_j\rangle|<1\right\}$. For a non-zero $f\in\{\nu_1,...,\nu_n\}^{\perp}$ we have

$\langle e+tf-e,\nu_j\rangle=0$, for all $j\in\overline{1,n}$, and so $e+tf\in V_e$, for any $t\in\mathbb{F}$. For any $\nu\in e^{\|}$ we have that $\langle e+tf,\nu\rangle=\langle e,\nu\rangle+t\langle f,\nu\rangle=\|e\|+t\langle f,\nu\rangle$. If $\langle f,\nu\rangle\neq 0$, for $t=-\frac{\|e\|}{\langle f,\nu\rangle}$ we have that $\langle e+tf,\nu\rangle=0$, which contradicts to the assumption $V_e\bigcap e^{\vdash}=\varnothing$. Hence $\nu\in f^{\perp}$, and from the arbitrariness of f and ν, we get that $e^{\|}\subset\left(\{\nu_1,...,\nu_n\}^{\perp}\right)^{\perp}=\operatorname{span}\{\nu_1,...,\nu_n\}$. $\square$

The established characterization allows us to prove Theorem 2.1.29.

Proof of Theorem 2.1.29. Since T is an isometry, it follows that for each $e\in K$ there is $u(e)\in\mathbb{T}$ such that $Te=u(e)e$. We need to show that $u(e)$ does not depend on e. Assume that $Te=\alpha e$ and $Tf=\beta\alpha f$, where $\beta\neq 1$. Since T is an isometry, it follows that $\|f+\beta te\|=\|Tf+\beta tTe\|=\|\beta\alpha f+\beta\alpha te\|=\|f+te\|$, for any $t\in\mathbb{C}$. Applying this equality n times we get that $\|f+\beta^n te\|=\|f+te\|$, for any $n\in\mathbb{N}$. From the convexity of the function $t\to\|f+te\|$ and the fact that the convex hull of $\{\beta^n t\}$ contains 0, we get that $\|f\|\leq\|f+te\|$, for any $t\in\mathbb{C}$, i.e. $f\vdash e$.

Hence, $u(e)\neq u(f)$ implies $f\vdash e$, and so if $f\not\vdash e$ we have $u(e)=u(f)$. Since f is topologically disjoint from $f^{\vdash}$ in the norm topology, we have that u is locally a constant with respect to the norm topology. If (i) holds, we get that u is locally a constant on a connected space, and so u is a constant. If (ii) holds, from the preceding proposition f is topologically disjoint from $f^{\vdash}$ in the weak topology, and so u is locally a constant with respect to the weak topology on a connected space. Hence, and so u is a constant. $\square$

Remark A.1.3. Let us show that if E and H are finitely convex normed spaces and $p\in(1,+\infty)$, then the $E\oplus_p H$ is also finitely convex. First, we will show that if $D\subset\partial B_{E\oplus_p H}$ is closed and convex, and D_E (or D_H) is the image of D under the natural projection from $E\oplus_p H$ onto E (or H), then D_E (or D_H) is a closed convex subset of a sphere in E (or H).

It is enough to prove that if $e_1, e_2 \in D_E$, then $\|e_1\| = \|e_2\|$. By definition of D_E there are $h_1, h_2 \in H$ such that $e_1 \oplus h_1, e_2 \oplus h_2 \in D$. Since D is convex it follows that $\frac{e_1+e_2}{2} \oplus \frac{h_1+h_2}{2} \in D \subset \partial B_{E \oplus_p H}$, and so

$$\|e_1\|^p + \|h_1\|^p = \|e_2\|^p + \|h_2\|^p = \left\|\frac{e_1+e_2}{2}\right\|^p + \left\|\frac{h_1+h_2}{2}\right\|^p = 1.$$

Then, using subadditivity of the norm and the inequality between means we get

$$\begin{aligned} 1 = \left\|\frac{e_1+e_2}{2}\right\|^p + \left\|\frac{h_1+h_2}{2}\right\|^p &\leq \left(\frac{\|e_1\| + \|e_2\|}{2}\right)^p + \left(\frac{\|h_1\| + \|h_2\|}{2}\right)^p \\ &\leq \frac{\|e_1\|^p + \|e_2\|^p}{2} + \frac{\|h_1\|^p + \|h_2\|^p}{2} = 1. \end{aligned}$$

Hence, all inequalities are in fact equalities, which is only possible if $\|e_1\| = \|\frac{e_1+e_2}{2}\| = \|e_2\|$, from where $D_E \subset \|e_1\|\partial B_E$. Similarly, D_H is a convex subset of as sphere in H. Thus, $\dim D \leq \dim D_E + \dim D_H < +\infty$, and so $E \oplus_p H$ is finitely convex. $\square$

A.2 Cones in Topological Vector Spaces

Let E be a topological vector space, with the operation of scalar multiplication given by the map $\mu : \mathbb{F} \times E \to E$, i.e. $\mu(a, e) = ae$, for $a \in \mathbb{F}$ and $e \in E$. From the definition of a TVS, μ is continuous and in this section we will establish further topological properties of μ.

Recall that a set $B \subset E$ is called balanced if $aB \subset B$, for each $a \in \mathbb{F}$, such that $|a| \leq 1$. In particular, $0_E \in B$. If $e \in E$ is such that $re \in B$, for some $r \in \mathbb{F}$, then $\overline{B}_{\mathbb{F}}(r)\, e \subset B$.

We will need the following basic property of E (see [BN11, Theorem 4.3.6]): for any open neighborhood U of 0_E there is a balanced open set V, such that $0_E \subset V \subset \overline{V} \subset U$.

The following lemma shows that μ is an "almost" closed map.

Proposition A.2.1. *Let $K \subset E$ and $A \subset \mathbb{F}$ be closed. Then AK is closed, provided that one of the following conditions is satisfied:*

(i) A is bounded and $0 \notin A$;

(ii) $0_E \notin K$ and $0 \notin A$;

(iii) K is bounded and $0_E \notin K$;

(iv) K is bounded and A is bounded.

Proof. **Step 1.** First, observe that (i) implies that A is compact in a topological group $\mathbb{F}^\times$ that acts on E. Since $\mu|_{\mathbb{F}^\times \times E}$ is the action, which is continuous, it follows that in this case the set $AK = \mu|_{\mathbb{F}^\times \times E}(A \times K)$ is closed (see [Bou66, III.4.1, Corollary]).

Step 2. Assume $0_E \notin K$. Since $E \backslash K$ is an open neighborhood of 0_E, there exists a balanced open neighborhood $V \subset E \backslash K$ of 0_E. Since V is balanced, if $V \cap aK \neq \varnothing$, then $|a| < 1$. For any $e \in E$ there is $r > 0$ such that $e \in rV$. Hence rV is an open neighborhood of e disjoint from $(\mathbb{F} \backslash B_{\mathbb{F}}(r))\, K$. Thus, e is topologically disjoint from $(\mathbb{F} \backslash B_{\mathbb{F}}(r))\, K$.

Step 3. Assume that K is bounded. For any $e \neq 0_E$ there is $r > 0$ such that e is topologically disjoint from $\overline{B}_{\mathbb{F}}(r)\, K$. Indeed, consider a balanced open neighborhood V of 0_E, that satisfies $\overline{V} \subset E \backslash \{e\}$ and take $r > 0$ such that $rK \subset V$.

Step 4. Now we show that each of (ii), (iii) and (iv) implies that AK is closed. For $r > 0$ define $A_r^+ = A \backslash B_{\mathbb{F}}(r)$ and $A_r^- = A \cap \overline{B}_{\mathbb{F}}(r)$. Both of these sets are closed and their union is A.

We will show that any $e \notin AK$ is topologically disjoint from AK. In order to do so we will divide A into two or three pieces and show that e is topologically disjoint from BK, for every such piece B. Consider the following cases:

1. If $0_E \notin K$ and $0 \notin A$, then from Step 2, there is $r > 0$, such that e is topologically disjoint from $A_r^+ K$. Since $0 \notin A$, it follows that A_r^- is a closed bounded set, not containing 0, and so $A_r^- K$ is closed by (i). Since $e \notin AK \supset A_r^- K$ we conclude that e is topologically disjoint from $A_r^- K$.

Note that this case covers the situation when $e = 0_E$, and so we can assume that $e \neq 0_E$.

2. If A and K are bounded, by Step 3, there is $r > 0$, such that e is topologically disjoint from $A_r^- K$. Since A is bounded, it follows that A_r^+ is a closed bounded set, not containing 0, and so $A_r^+ K$ is closed by (i). Since $e \notin AK \supset A_r^+ K$ we conclude that e is topologically disjoint from $A_r^+ K$.

3. If K is bounded and $0_E \notin K$, there are $r, R > 0$, such that e is topologically disjoint from $A_r^- K$ and $A_R^+ K$. Then $B = \overline{A \backslash \left(A_r^- \cup A_R^+\right)}$ is a closed bounded set, not containing 0, and so BK is closed by (i). Since $e \notin AK \supset BK$ we conclude that e is topologically disjoint from BK. $\square$

Till the end of the section we will assume that $K \subset E$ is bounded and contains no pairs of linearly dependent elements. Also denote $K_0 = \{0\} \times K$. Then $\mu\left(K_0\right) = \{0_E\}$, while $0_E \notin \mathbb{F}^\times K$ and $\mu|_{\mathbb{F}^\times \times K}$ is an injection. In fact, $\mu|_{\mathbb{F}\times K}^{-1}\left(\mu\left(B\right)\right) = B$, if either $K_0 \subset B$ or $K_0 \cap B = \varnothing$; otherwise $\mu|_{\mathbb{F}\times K}^{-1}\left(\mu\left(B\right)\right) = B \cup K_0$. Instead of proving Theorem 2.2.8 we will prove two slightly more general results.

Theorem A.2.2. *Let $K \subset E$ be closed, bounded and contain no pairs of linearly dependent elements. Then $\mu|_{\mathbb{F}^\times \times K}$ is a topological embedding. Moreover, the following are equivalent:*

(i) $\mu|_{\mathbb{F}\times K}$ *is a closed map;*

(ii) $\mu|_{\mathbb{F}\times K}$ *is a quotient map onto is image;*

(iii) K is countably compact.

Proof. Let us start with the first claim. It is enough to show that $\mu|_{\mathbb{F}^{\times}\times K}$ is an open map onto its image. Let W be an open set in $\mathbb{F}^{\times}\times K$. We will show that $\mu(W)$ is open in $\mathbb{F}K$.

First, assume that $W=V\times U$, where $V\subset\mathbb{F}^{\times}$ and $U\subset K$ are open. Since $0\notin V$, it follows that $K_0\cap W=\varnothing$ and so

$$\mu(W)=\mathbb{F}K\backslash\mu\left[\mathbb{F}\times(K\backslash U)\cup(\mathbb{F}\backslash V)\times K\right]=\mathbb{F}K\backslash\left[\mathbb{F}(K\backslash U)\cup(\mathbb{F}\backslash V)K\right].$$

Since, due to the preceding proposition, both $\mathbb{F}(K\backslash U)$ and $(\mathbb{F}\backslash V)K$ are closed in E we conclude that W is open in $\mathbb{F}K$.

In the general case, by definition of the product topology, there are collections $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ of open sets in K and $\mathbb{F}^{\times}$ respectively, such that $W=\bigcup_{i\in I}V_i\times U_i$. Clearly, $0\notin V_i$, for each $i\in I$, and so V_iU_i is open in $\mathbb{F}K$. Then

$$\mu(W)=\mu\left(\bigcup_{i\in I}V_i\times U_i\right)=\bigcup_{i\in I}\mu(V_i\times U_i)=\bigcup_{i\in I}V_iU_i$$

is a union of open sets, and so open itself.

Let us prove the equivalences. (i)⇒(ii) follows from the fact that any closed surjection is a quotient map (see [Eng89, Corollary 2.4.8]).

(ii)⇒(i): Since from the preceding proposition $\mathbb{F}K$ is closed in E, it is enough to show that if L is closed in $\mathbb{F}\times K$, then $\mu(L)$ is closed in $\mathbb{F}K$. Since $\mu|_{\mathbb{F}\times K}$ is a quotient map onto $\mathbb{F}K$, the latter condition is equivalent to the closeness of $\mu|_{\mathbb{F}\times K}^{-1}(\mu(L))$. This set is equal either to L or to $L\cup K_0$, which are both closed, and so $\mu|_{\mathbb{F}\times K}^{-1}(\mu(L))$ is closed.

(ii)⇒(iii): Let $\{U_n\}_{n\in\mathbb{N}}$ be an open cover of K and let $W=\bigcup_{n\in\mathbb{N}}B_{\mathbb{F}}\left(\frac{1}{n}\right)\times U_n$. Since $K_0\subset W$ it follows that $\mu|_{\mathbb{F}\times K}^{-1}(\mu(W))=W$, which is open, and since μ is a quotient map onto is image, $\mu(W)$ is open in $\mathbb{F}K$. Since $0_E \in \mu(W)$, there is a balanced open neighborhood V of 0_E contained in $\mu(W)$. Since K is bounded there is $r > 0$, such that $rK \subset V$, and since V is balanced, $B_{\mathbb{F}}(r)K \subset V \subset \mu(W)$. Consequently,

$$B_{\mathbb{F}}(r) \times K = \mu|_{\mathbb{F}\times K}^{-1}(B_{\mathbb{F}}(r)K) \subset W = \bigcup_{n\in\mathbb{N}} B_{\mathbb{F}}\left(\frac{1}{n}\right) \times U_n.$$

Hence, $\{U_n\}_{n=1}^{N}$ covers K, where $N = \lfloor \frac{1}{r} \rfloor$. Thus, we found a finite subcover of an arbitrary countable open cover of K, and so (iii) follows.

(iii)$\Rightarrow$(ii): Recall that in the first part of the proof we have showed that $\mu(W)$ is open in $\mathbb{F}K$ for any open $W \subset \mathbb{F}^{\times} \times K$. In a similar way one can show that if V is open in $\mathbb{F}$, then VK is open in $\mathbb{F}K$.

We need to show that $\mu(W)$ is open in $\mathbb{F} \times K$, for any open W in $\mathbb{F} \times K$ such that $W = \mu|_{\mathbb{F}\times K}^{-1}(\mu(W))$. The latter equality means that either $K_0 \cap W = \varnothing$ or $K_0 \subset W$. In the former case, $\mu(W)$ is open since $W \subset \mathbb{F}^{\times} \times K$. In the latter case, for each $e \in K$ there is $r_e > 0$ and an open subset U_e of K, such that $B_{\mathbb{F}}(r_e) \times U_e \subset W$. Define $U_n = \bigcup_{r_e > \frac{1}{n}} U_e$. Clearly $\{U_n\}_{n\in\mathbb{N}}$ is a countable cover of K, and also an increasing sequence of sets. Since K is countably compact, there is $N \in \mathbb{N}$, such that $U_N = K$, and so $B_{\mathbb{F}}\left(\frac{1}{N}\right) \times K \subset W$. Define $W^- = B_{\mathbb{F}}\left(\frac{1}{N}\right) \times K$ and $W^+ = W \backslash \left[\overline{B}_{\mathbb{F}}\left(\frac{1}{2N}\right) \times K\right]$. Then W^+ is an open set in $\mathbb{F}^{\times} \times K$, and so $\mu(W^+)$ is open in $\mathbb{F}K$, while $\mu(W^-)$ is open in $\mathbb{F}K$ due to the comment above. Hence, $\mu(W) = \mu(W^+) \cup \mu(W^-)$ is open in $\mathbb{F}K$. □

Analogously, one can prove the "$K \cup \{0_E\}$ is closed" counterpart of the preceding theorem.

Theorem A.2.3. *Let $K \subset E$ be bounded, contain no pairs of linearly dependent elements and such that $\overline{K} = K \cup \{0_E\}$. Let $B \subset \mathbb{F}$ be a closed disk centered at 0 and let $B' = B \backslash \{0\}$. Then $\mu|_{B'\times K}$ is a topological embedding. Moreover, the following are equivalent:*

(i) $\mu|_{B\times\overline{K}}$ *is a closed map;*

(ii) $\mu|_{B\times\overline{K}}$ *is a quotient map onto is image;*

(iii) $\overline{K}$ *is countably compact.*

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