ON X-NORMED SPACES AND OPERATOR THEORY ON c_0 OVER A FIELD WITH A KRULL VALUATION OF ARBITRARY RANK

by

Angel Barría Comicheo

A thesis submitted to the Faculty of Graduate Studies of

The University of Manitoba

in partial fulfillment of the requirements of the degree of

Doctor of Philosophy

Department of Mathematics

University of Manitoba

Winnipeg

Abstract

Between 2013 and 2015 Aguayo et al. developed an operator theory on the space c_0 of null sequences in the complex Levi-Civita field $\mathcal{C}$ by defining an inner product on c_0 that induces the supremum norm on c_0 and then studying compact and self-adjoint operators on c_0, thus presenting a striking analogy between c_0 over $\mathcal{C}$ and the Hilbert space ℓ^2 over $\mathbb{C}$. In this thesis, we try to obtain these results in the most general case possible by considering a base field with a Krull valuation taking values in an arbitrary commutative group. This leads to the concept of X-normed spaces, which are spaces with norms taking values in a totally ordered set X not necessarily embedded in $\mathbb{R}$. Two goals are considered in the thesis: (1) to present and contribute to a theory of X-normed spaces, and (2) to develop an operator theory on c_0 over a field with a Krull valuation of arbitrary rank. In order to meet the goal (1), a systematic study of valued fields, G-modules and X-normed spaces is conducted in order to satisfy the generality of the settings required. For the goal (2), we identify the major differences between normed spaces over fields of rank 1 and X-normed spaces over fields of higher rank; and we try to find the right conditions for which the techniques employed in the rank-1 case can be used in the higher rank case. For (1) the author develops a new tool to work with transfinite induction simplifying the techniques employed in X-normed spaces, thus accomplishing a Generalized Baire Category Theorem that allows the proof of an Open Mapping theorem for X-normed spaces. Regarding (2), we show that an operator can be identified as

compact with adjoint by studying the behavior of the image of any base of c_0. Although characterizations are obtained for some linear operators on c_0, it is still unknown whether the spectral theorem holds for compact self-adjoint operators in the non-Archimedean case.

Acknowledgements

I want to thank my wife Marianela Oliva Contreras for her support and encouragement, and my parents and siblings for their confidence on me. Also I want to thank my thesis advisor Dr. Khodr Shamseddine who has been an invaluable source of wisdom and trust during my years as a doctoral student and that kindly helped me to overcome the difficulties that arose in this journey.

I express my gratitude to the members of the Examiner Committee for their insights that helped me to improve this thesis.

This thesis was supported by a doctoral scholarship for the author from the Formation of Advanced Human Capital Program of the Chilean National Commission for Scientific and Technological Research (CONICYT-PFCHA/Doctorado en el Extranjero/72130551).

To my wife.

Contents

List of Tables

Introduction

In non-Archimedean Analysis, i.e. when we consider a non-Archimedean valued field as our base field to define normed spaces, we obtain new results that do not hold in classical Functional Analysis (i.e. when our base field is $\mathbb{R}$ or $\mathbb{C}$), however we lose some other results, thus producing a whole new theory that sometimes demands a totally new intuition. In this new setting, the results that are possible to obtain depend greatly on the non-Archimedean valued field we choose to work with and also depend on the structure of the normed space that we consider. For example, let's consider the concept of orthomodularity. Let K be a field with an involution $a \mapsto a^*$. A vector space E over K with a Hermitian form $(\cdot, \cdot) : E \times E \to K$ (linear in the first variable and $(x, y) = (y, x)^*$ for all $x, y \in E$) is called **orthomodular** if the projection theorem holds: every subspace H of E satisfies: $H = H^{\perp\perp} \Rightarrow E = H \oplus H^{\perp}$. Notice that Hilbert spaces can be characterized in terms of orthomodularity. In fact, if E is an orthomodular space over K admitting an orthonormal sequence, then $K = \mathbb{R}$ or $K = \mathbb{C}$ and E is linearly homeomorphic to a Hilbert space [42]. In general, if a non-Archimedean valued field K has finite rank, then there is no infinite-dimensional Banach space over K that is orthomodular [30, 4.4.6]. However, if the non-Archimedean valued field K has infinite rank, then we can find an infinite-dimensional Banach space E over K that is orthomodular (where the norm satisfies $||x||^2 = (x, x)$ for all $x \in E$) [30, 4.4.9].

In the first chapter of the thesis, a deep analysis of the metric and algebraic structures of non-Archimedean valued fields is presented, providing among other things, descriptions of the most commonly used non-Archimedean valued fields. Additionally, the relationship between the order structures of ordered fields and their structure as valued fields is explained. The culmination of this introductory chapter is the classification of all the valued fields in a meaningful manner that is beneficial to non-experts and experts in the field alike.

In the Chapters 2–4 of this thesis, we systematically study several aspects of non-Archimedean Functional Analysis in order to determine the conditions for which we can conceive compact and adjoint operators in a Banach space equipped with a Schauder basis in the most general settings possible. Since every Banach space (over a non-Archimedean valued field K) with a Schauder basis is linearly homeomorphic to the space c_0 of null sequences in K ([32, 2.3.9]), we will focus on the study of operators defined on c_0. As we will see, the properties of c_0 vary depending on whether the valuation takes values in $\mathbb{R}$ or in an arbitrary commutative ordered group; those two cases will be referred to as the rank-1 case and the higher-rank case, respectively.

In order to understand the impact of the choice of the valued field we will begin in Chapter 1 with an introduction to ultrametric spaces, where we will study the metric and topological properties of a non-Archimedean valued field. Then, a variety of non-Archimedean valued fields will be presented and their properties collected in an extensive catalog (1.9). We will review the basic structures of valued fields and will develop the tools that will allow us to fully classify them in 6 distinctive categories (1.10). Among other results (see List of contributions), we will prove that the order topology of any non-Archimedean ordered field coincides with the topology induced by their 'natural' valuation (1.8.9). Note that most of Chapter 1 (with the exception of a couple of new results) is a selection of results which have recently been published in a refereed paper [8] in Contemporary Mathematics of the American Mathematical

Society.

In Chapter 2 we will study Banach spaces over non-Archimedean valued fields of rank 1 and consider their similarities and differences with Banach spaces in classical Functional Analysis. We will also review linear operators defined on normed spaces and compare the different norms that can be defined for continuous linear operators in the non-Archimedean context. Then, we review the concepts of form-orthogonality and its more general alternative: norm-orthogonality. These concepts will be of great importance in the subsequent chapters. We will finish the chapter with the concept that generalizes the concept of separability on normed spaces: being of countable type. This concept is the key to characterize those Banach spaces that have a Schauder basis.

Further, in Chapter 3 we will define norms on a vector space E over K that take values in an ordered set X rather than in $\mathbb{R}$. In this new context, the valuations can take values in an arbitrary commutative ordered group G which acts on X turning the latter into a G-module. With this new object, we will be able to conceive equalities of the form $||\lambda x|| = |\lambda|||x||$ for $\lambda \in K$, $|\lambda| \in G$, $x \in E$ and $||x|| \in X$. In this general setting we will study continuous linear operators, new Banach spaces and consider the necessary conditions to have a Schauder basis in every closed subspace of c_0. Here the author develops new results that complement the study of this abstract area.

In Chapter 4 we will develop an operator theory on the space c_0 over a field K with valuation of arbitrary rank. For this we will use an inner product on c_0 which will induce the supremum norm of c_0 and therefore we will be able to work with both form-orthogonality and norm-orthogonality to characterize those closed subspaces that can be complemented. Additionally, we will study compact operators on c_0

as well as those operators that admit an adjoint. In particular, we will discuss the difficulties to conceive compact operators on c_0 when the valued field has higher rank and the author will offer an alternative definition trying to capture what is expected from a compact operator. We will present the work done to generalize several results regarding operators in the rank-1 case to the higher rank case exhibiting partial results and some surprising ones.

The contributions made in Chapters 2, 3 and 4 are going to be prepared as a paper for submission after the distribution of this thesis.

Finally, in Chapter 5 we will briefly discuss future research projects that could deepen the knowledge reached and address some of the open problems that were identified in this thesis.

List of contributions

Throughout this thesis several topics will be discussed and a large number of results will be presented. In order to differentiate between known results and contributions made by the author of this thesis, the labels **Lemma**, **Theorem** and **Corollary** will be used to refer to the former kind of results, while the label **Proposition** will be reserved for the latter.

In the list below, a contribution in the form of a result will be classified as *novelty* (when the result has not been found in the bibliography nor in the literature), *original proof* (when the result is known but the presented proof uses an original approach) or *extension* (when only a particular case of the same result has been proved before without reaching the scope of the presented result). Moreover, throughout the thesis the author's most significant contributions will the labeled by ★**Proposition** and are listed below using the symbol ★.

Not all the contributions of this thesis are in the form of a proposition. While most of the results in Chapters 1 and 2 are known, the author wishes to emphasize that a significant amount of time and effort was invested in researching existing knowledge about non-Archimedean valued fields and organizing it in the way presented in Chapter 1, which culminated in a classification of all non-Archimedean valued fields that is useful to both the experts and non-experts in non-Archimedean Analysis. No such presentation and classification have been done before in the literature. The work presented in Chapter 1 is a selection of results that have recently been published in a refereed paper [8] which received a very positive report from an expert in the field:

> "A few good examples are proposed. For example, conditions to have a complete ultrametric field that is or is not spherically complete, or to obtain the completion, an immediate extension."
>
> "Generalizations are defined in ultrametric spaces which are not so much known and can be very useful when we study algebras of continuous functions defined in such an ultrametric space, with values in a complete ultrametric field. This way, the paper is a very good presentation of the non-Archimedean topologies that can be very useful to students or researchers beginning working in ultrametric analysis. Moreover, the presentation of generalizations of ultrametric values of rank superior to one are far from well known and can be useful to many specialists."

The contributions of this thesis in the form of a result are listed as follows:

Contributions in Chapter 1: Non-Archimedean valued fields.

- 1.6.5 on p. 31: *novelty.*
- 1.7.3 on p. 34: *extension.*
- 1.7.5 on p. 35: *extension.* The implication (a)⇒(c) is only found for $K = \mathbb{R}$ using a different and much longer proof in [40, p. 218].
- 1.7.7 on p. 35: *extension.*
- 1.7.8 on p. 36: *extension.*
- 1.8.7 on p. 39: *extension.*
- 1.8.8 on p. 40: *extension.*

★ 1.8.9 on p. 40: *extension.*

Contributions in Chapter 2: Banach Spaces over fields with rank 1 valuations.

- 2.1.6 on p. 50: *extension.* The part (b) is new.
- 2.2.7 on p. 54: *extension.* The parts (b) and (c) are new.

★ 2.4.15 on p. 65: *novelty.* It answers a posed problem [32, p. 33].

★ 2.4.16 on p. 66: *novelty.*

Contributions in Chapter 3: Normed spaces over fields with valuations of higher rank.

★ 3.1.6 on p. 71: *novelty.*

- 3.4.5 on p. 81: *extension.* Parts (f)-(k) are added to [30, 1.1.4].
- 3.4.17 on p. 85: *extension.* Part (c) is added to [30, 2.2.2].

★ 3.4.19 on p. 87: *original proof.*

★ 3.4.20 on p. 87: *original proof.* The statement corrects [30, 2.2.5].

★ 3.4.22 on p. 89: *novelty.*

- 3.4.23 on p. 90: *original proof.* Part (c) is added to [30, 1.4.1].
- 3.4.28 on p. 91: *novelty.*
- 3.4.29 on p. 92: *novelty.*
- 3.4.31 on p. 92: *extension.* It generalizes the results 2.2.4, 2.2.5 and 2.2.7 to the higher-rank case.
- 3.4.38 on p. 94: *original proof.*

★ 3.4.39 on p. 95: *novelty.* It generalizes the Baire Category theorem.

★ 3.4.40 on p. 96: *extension.* It generalizes part of the Open Mapping theorem to the higher-rank case.

- 3.4.41 on p. 96: *novelty.*

Contributions in Chapter 4: Operator theory on c_0 over fields with valuation of arbitrary rank.

- 4.1.8 on p. 103: *extension.* It generalizes [29, 7.1] to the higher-rank case.

- 4.2.7 on p. 105: *extension.* It generalizes the result to the higher-rank case. [1, Proposition 3] to the higher-rank case.

- 4.4.6 on p. 114: New definition for compact operators on c_0 over a Krull valued field of higher rank.

- 4.4.7 on p. 114: *extension.* It generalizes [45, 4.40 $\varepsilon \Rightarrow \alpha$] to the higher-rank case.

★ 4.4.9 on p. 115: *extension.*

- 4.4.10 on p. 118: *original proof* and *extension.* It generalizes [14, 1.3] to the higher-rank case.

★ 4.4.13 on p. 120: *original proof* and *extension.* It generalizes [14, 2.7] to the higher-rank case.

- 4.5.5 on p. 128: *extension.*

- 4.5.6 on p. 129: *extension.*

- 4.5.7 on p. 130: *extension.*

★ 4.5.9 on p. 131: *extension.*

List of symbols

Normed spaces

Operators spaces

Miscellaneous

Chapter 1

Non-Archimedean valued fields

In this thesis we will be discussing different aspects of Functional Analysis when considering scalars in a non-Archimedean valued field. Before we start the development of the main theory we will briefly discuss this kind of fields to help the reader realize that there are big differences with the classical Functional Analysis rooted in the choice of the basis field. The most remarkable features of this chapter are the catalog of fields 1.9 and the classification of fields 1.10. For another type of classification see 1.9.1.

Most of the proofs of the results presented in this chapter will be omitted in order to avoid the use of a significant amount of space in topics that are out of the scope of this thesis. For a broader scope and context of these results as well as a more detailed discussion about them see [8]. Nonetheless some of the results obtained by the author will be presented here with a proof, including 1.7.7 and 1.7.8 which were obtained after the publication of [8].

1.1 Ultrametric Spaces

A valued field is a mathematical entity with a topological and an algebraic structures that will be defined and studied in Section 1.2. In this section we will discuss the notion of an ultrametric space which is a metric space used to study the metric and topological properties of a non-Archimedean valued field without worrying about its algebraic structure.

Let's begin reviewing the concept of ultrametric and the consequences of the strong triangle inequality.

1.1.1 Definition. *A* ***metric*** *on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in X$:*

(a) $d(x, y) \geq 0$*, and* $d(x, y) = 0$ *if and only if* $x = y$*,*

(b) $d(x, y) = d(y, x)$*,*

(c) $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality).*

The pair (X, d) is called a ***metric space****.*

1.1.2 Notations. *Let (X, d) be a metric space, $a \in X$ and $r > 0$. The sets $\mathbf{B(a, r)} := \{x \in X : d(x, a) < r\}$ and $\mathbf{B[a, r]} := \{x \in X : d(x, a) \leq r\}$ are called the* ***open and closed balls of center*** $\mathbf{a}$ ***and radius*** $\mathbf{r}$*, respectively. The family of open balls forms a basis of neighbourhoods for a uniquely determined Hausdorff topology on X. This topology is called the* ***topology induced*** *by d on X. With respect to this topology the open balls are open sets and the closed balls are closed sets in X.*

The diameter of a non-empty set $Y \subset X$ is $\mathbf{diam(Y)} := \sup\{d(x, y) : x, y \in Y\}$ and the distance between two non-empty sets $Y, Z \subset X$ is $\mathbf{dist(Y, Z)} := \inf\{d(y, z) : y \in Y, z \in Z\}$. The set of values of a metric $d : X \times X \to \mathbb{R}$ is denoted by $\mathbf{d(X \times X)}$ and defined as $\{d(x, y) : x, y \in X\}$.

1.1.3 Definition. *A metric $d : X \times X \to \mathbb{R}$ is called an* ***ultrametric*** *when it satisfies the so-called* ***strong triangle inequality*** $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ *for all $x, y, z \in X$. In this case, the pair (X, d) is called an* ***ultrametric space****.*

1.1.4 Examples. *(a) Let X be a set and $d : X \times X \to \mathbb{R}$ be the discrete metric, i.e. $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for all $x, y \in X$. Then (X, d) is an ultrametric space. In this case, for $x \in X$, we have that $B(x, 1) = B(x, r)$ for all $0 < r < 1$. Also, for each y in $B(x, r)$, we have that $B(x, r) = B(y, r)$. As we will see, it is not uncommon to find a ball of an ultrametric space with infinitely many radii, and where each point of a ball is a center of the ball.*

(b) Let p be a fixed prime. The p-adic metric on $\mathbb{Z}$ is defined by $d(n, m) = 0$ if $n = m$, and for $n \neq m$, $d(n, m) = p^{-r}$ where r is the largest non-negative integer such that p^r divides $m - n$. The pair $(\mathbb{Z}, d)$ is an ultrametric space.

(c) Let $\mathbb{R}[x]$ be the ring of all polynomials with real coefficients. For each nonzero polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ in $\mathbb{R}[x]$, put $\lambda(p) = \min\{i : a_i \neq 0\}$. Thus the map $d : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$ defined by

$$d(p, q) := \begin{cases} e^{-\lambda(p-q)} & , \text{ if } p \neq q \\ 0 & , \text{ if } p = q \end{cases}$$

is an ultrametric on $\mathbb{R}[x]$.

(d) Let $\mathbb{N}$ be the set of positive integers and $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be the map defined by

$$d(m, n) := \begin{cases} \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}\} & , \text{ if } m \neq n \\ 0 & , \text{ if } m = n. \end{cases}$$

Then $(\mathbb{N}, d)$ is an ultrametric space.

(e) Any subset of a non-Archimedean valued field $(K, |\ |)$ with the map $(x, y) \mapsto |x - y|$ constitutes an ultrametric space (non-Archimedean valued fields will be presented later in this chapter). Notice that with this example we have listed

all ultrametric spaces, since W. Schikhof proved in [36] that any ultrametric space can isometrically be embedded into a non-Archimedean valued field.

Let (X, d) be an ultrametric space. If $x, y, z \in X$ are such that $d(x, z) \neq d(z, y)$, then $d(x, y) = \max\{d(x, z), d(z, y)\}$, i.e. every triangle with vertices in X is isosceles. This condition is called the ***Isosceles triangle principle***.

1.1.5 Theorem ([32, p. 3], [45, 2.A])**.** *Let (X, d) be a metric space. The metric d is an ultrametric if and only if it satisfies the Isosceles triangle principle.*

The following theorem collects the most remarkable results of an ultrametric space, all direct consequences of the strong triangle inequality.

1.1.6 Theorem ([8, 1.6])**.** *Let (X, d) be an ultrametric space. Then the following properties hold:*

(a) Each point of a ball is a center of the ball.

(b) Each ball in X is closed and open in the topology induced by the ultrametric.

(c) Each ball has an empty boundary.

(d) Two balls are either disjoint, or one is contained in the other.

(e) Let $a \in Y \subset X$. Then $diam(Y) = \sup\{d(x, a) : x \in Y\}$.

(f) The radii of a ball B form the set $\{r \in \mathbb{R} : r_1 \leq r \leq r_2\}$, where $r_1 = diam(B)$, $r_2 = dist(B, X \backslash B)$ ($r_2 = \infty$ if $B = X$). It may happen that $r_1 < r_2$, so that a ball may have infinitely many radii.

(g) If B_1, B_2 are disjoint balls, then $dist(B_1, B_2) = d(x, y)$ for all $x \in B_1, y \in B_2$.

(h) Let $U \neq \emptyset$ be an open subset of X. Given a sequence $(r_n)_n$ in $(0, \infty)$, strictly decreasing and convergent to 0, there exists a partition of U formed by balls of the form $B[a, r_n]$, with $a \in U$ and $n \in \mathbb{N}$.

(i) *Let* $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. *For* $x, y \in X$ *the relation* $d(x,y) < \varepsilon$ *is an equivalence relation and induces a partition of* X *into open balls of radius* ε. *Analogously for* $d(x,y) \leq \varepsilon$ *and closed balls.*

(j) *Let* $Y \subset X$, B *a ball in* X, $B \cap Y \neq \emptyset$. *Then,* $B \cap Y$ *is a ball in* Y.

(k) *Let* $(x_n)_n$ *be a sequence in* X *converging to* $x \in X$, *then for each* $a \in X \setminus \{x\}$, *there exists* $N \in \mathbb{N}$ *such that* $d(x_n, a) = d(x, a)$ *for all* $n \geq N$.

(l) *There are not new values of an ultrametric after completion, i.e. if* $(X^\wedge, d^\wedge)$ *is the completion of an ultrametric space* (X, d), *then* $d(X \times X) = d^\wedge(X^\wedge \times X^\wedge)$.

(m) *A sequence* $(x_n)_n$ *in* X *is Cauchy if and only if* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Recall that a metric space is said to be Cauchy complete if every Cauchy sequence is convergent, or equivalently, if each nested sequence of closed balls whose radius approaches 0, has a non-empty intersection. This motivates the following:

1.1.7 Definition. *An ultrametric space is called* ***spherically complete*** *if each nested sequence of balls has a non-empty intersection.*

1.1.8 Remark. *The concept of spherical completeness plays a key role as a necessary and sufficient condition for the validity of the Hahn-Banach theorem in the non-Archimedean context ([45, 4.10, 4.15]). Spherical completeness also has implications related to fixed points ([34, 2.3]) and best approximations ([38, 21.1, 21.2]).*

1.1.9 Theorem ([45, 2.3])**.** *Let* (X, d) *be an ultrametric space. The following statements are equivalent:*

(a) (X, d) *is spherically complete.*

(b) *For any collection* $(B_i)_{i\in I}$ *of balls in* X *such that* $B_i \cap B_j \neq \emptyset$ *for any* $i, j \in I$, *we then have* $\bigcap_{i\in I} B_i \neq \emptyset$.

(c) Every sequence of balls $B[a_1, \varepsilon_1] \supset B[a_2, \varepsilon_2] \supset \cdots$ *for which* $\varepsilon_1 > \varepsilon_2 > \cdots$ *has a nonempty intersection.*

It is clear that a spherically complete ultrametric space is Cauchy complete, but the converse is not always true. For instance, the space of the example 1.1.4(d) is a complete ultrametric space that is not spherically complete.

1.2 Non-Archimedean valued fields

In Section 1.1 we have presented the most important metric and topological properties of an ultrametric space, which are satisfied by any non-Archimedean valued field. In the rest of this chapter we will analyze these properties together with the algebraic structure of non-Archimedean valued fields and will present the most commonly used families of non-Archimedean valued fields. We will see that unlike the classical case, there are several Cauchy complete non-Archimedean fields, some of which are algebraically closed and others real closed. Similarly to Section 1.1, most of the proofs of the results in the reminding sections are omitted with the exception of some of the results obtained by the author.

1.2.1 Definition. *Let* K *be a field. A* ***valuation*** *on* K *is a map* $|\cdot| : K \to \mathbb{R}$ *satisfying the following axioms for all* $x, y \in K$*:*

(a) $|x| \geq 0$*, and* $|x| = 0$ *if and only if* $x = 0$*,*

(b) $|xy| = |x||y|$*,*

(c) $|x + y| \leq |x| + |y|$*.*

The pair $(K, |\cdot|)$ *is called a* ***valued field****.*

It is not hard to see that $|1_K| = 1$, $|-x| = |x|$ and $|x^{-1}| = |x|^{-1}$ for $x \neq 0$. In the rest of the document we will denote the set $K \setminus \{0\}$ by K^*.

1.2.2 Definition. *A valuation* $|\cdot|$ *on* K *is called* ***non-Archimedean*** *if it satisfies the* ***strong triangle inequality*** $|x+y| \leq \max\{|x|,|y|\}$ *for all* $x, y \in K$*. Otherwise it is called* ***Archimedean****.*

1.2.3 Theorem. *[[45, 1.1], [38, lemma 8.2]]*

Let $(K, |\cdot|)$ *be a valued field. The following conditions are equivalent:*

(a) $|\cdot|$ *is non-Archimedean,*

(b) If $a, b \in K$ *and* $|a| < |b|$*, then* $|b-a| = |b|$ *(Isosceles triangle principle),*

(c) the set $\{|n1_K| : n \in \mathbb{N}\}$ *is bounded,*

(d) $|n1_K| \leq 1$ *for every* $n \in \mathbb{N}$*,*

(e) $|2 \cdot 1_K| \leq 1$ *.*

1.2.4 Examples. *(a) The usual absolute values in* $\mathbb{R}$ *and* $\mathbb{C}$ *are valuations, and since the set* $\{|n| : n \in \mathbb{N}\}$ *is unbounded, they are Archimedean.*

(b) Let K *be a field. The map defined by* $|x| = 1$ *for* $x \neq 0$ *and* $|0| = 0$ *is called* ***trivial valuation*** *and is a non-Archimedean valuation.*

(c) If K *is a finite field, then the trivial valuation is the only valuation on* K*. In fact, if there is* $x \in K^*$ *with* $|x| \neq 1$*, then the set* $\{|x^n| : n \in \mathbb{Z}\}$ *is infinite.*

(d) Suppose the characteristic of K *is finite, i.e. there is* $n \in \mathbb{N}$ *such that* $n1_K = 0$*. This is denoted by* $char(K) \neq 0$*. Then any valuation on* K *is non-Archimedean. Indeed, in this case the prime subfield of* K *(the subfield of* K *generated by* 1_K*) is finite. Thus the set* $\{|n1_k| : n \in \mathbb{N}\}$ *is bounded.*

(e) Let p *be a prime number. The* ***p-adic valuation*** $|\quad|_p$ *on* $\mathbb{Q}$ *is defined by* $|0|_p = 0$ *and*

$$\left| p^v \frac{m}{n} \right| = \frac{1}{e^v}$$

where e is the basis of the natural logarithms, $v \in \mathbb{Z}$, and $n, m \in \mathbb{Z} \setminus \{0\}$ are not divisible by p. Since the set $\{|n1|_p : n \in \mathbb{Z}\} = \{e^{-v} : v \in \mathbb{N}\} \cup \{0\}$ is bounded in $\mathbb{R}$, the p-adic valuation on $\mathbb{Q}$ is non-Archimedean.

(f) *Let p be an irreducible polynomial in $K[x]$, where K is any field. The p-adic valuation $|\ \ |_p$ on the rational function field $K(x)$ (the quotient field of the ring of polynomials $K[x]$) is defined by $|0|_p = 0$ and*

$$\left| p^v \frac{f}{g} \right| = \frac{1}{e^v}$$

where $v \in \mathbb{Z}$, and $f, g \in K[x] \setminus \{0\}$ are not divisible by p. Since the set $\{|n1|_p : n \in \mathbb{Z}\} = \{0, 1\}$, the p-adic valuation on $K(x)$ is non-Archimedean. For a description of all valuations on $K(x)$ see [35, 3.1.K] and [4].

A valuation $|\cdot|$ on a field K defines the metric $d(x, y) := |x - y|$ for $x, y \in K$. In particular, any valuation on K induces a metrizable topology on K. If the valuation $|\cdot|$ is non-Archimedean, then the induced metric d is an ultrametric.

1.3 Completion of valued fields

In this section we will give a brief construction of our first nontrivial examples of Cauchy complete, non-Archimedean valued fields. Note that in a field K with the trivial valuation, every Cauchy sequence is eventually constant, and thus K is complete. We are not interested in this case. Notice that $\mathbb{Q}$ is not Cauchy complete with respect to the usual absolute value $|\cdot|_0$ nor with respect to any p-adic valuation since $\mathbb{Q}$ is not a Baire space. Similarly the field of rational functions $K(x)$ is not Cauchy complete with respect to the x-adic valuation (this is not trivial and it is proved in section 2.5 of [8]).

1.3.1 Theorem ([16, 1.1.4])**.** *Let $(K, |\cdot|)$ be a valued field. There exist a Cauchy complete valued field $(\widehat{K}, \widehat{|\cdot|})$ and an embedding $i : K \to \widehat{K}$, such that $|x| = \widehat{|i(x)|}$ for all $x \in K$, and the image $i(K)$ is dense in $\widehat{K}$. If $(\widehat{K'}, \widehat{|\cdot|}', i')$ is another such trio, then there exists a unique isomorphism $\varphi : \widehat{K} \to \widehat{K'}$ satisfying $\widehat{|\varphi(x)|}' = \widehat{|x|}$ for all $x \in \widehat{K}$ and making the following diagram commutative:*

$$
\begin{tikzcd}
\widehat{K} \arrow[rr, "\varphi"] & & \widehat{K'} \\
& K \arrow[ul, "i"] \arrow[ur, "i'"'] &
\end{tikzcd}
$$

1.3.2 Definition. *A pair $(\widehat{K}, \widehat{|\cdot|})$ as in Theorem 1.3.1 is called a* ***completion*** *of the valued field $(K, |\cdot|)$.*

1.3.3 Remark. *Let $(K, |\cdot|)$ be a valued field and $(\widehat{K}, \widehat{|\cdot|})$ its completion with embedding $i : K \to \widehat{K}$ such that $|x| = \widehat{|i(x)|}$ for all $x \in K$. Then*

$$\{|n1_K| : n \in \mathbb{N}\} = \{\widehat{|i(n1_K)|} : n \in \mathbb{N}\} = \{\widehat{|n1_{\widehat{K}}|} : n \in \mathbb{N}\}.$$

Therefore the completion of an Archimedean valued field is an Archimedean valued field and the completion of a non-Archimedean valued field is a non-Archimedean valued field. In the latter case, we have that $|K| := \{|x| : x \in K\} = \{\widehat{|x|} : x \in \widehat{K}\} =: \widehat{|\widehat{K}|}$, by Theorem 1.1.6(l).

It is well-known that every Archimedean valued field $(K, |\cdot|)$ is Cauchy complete if and only if there exist $\lambda \in (0, 1]$ and a field monomorphism $\sigma : K \to \mathbb{C}$, such that $|x| = (|\sigma(x)|_0)^\lambda$ for all $x \in K$, satisfying either $\sigma(K) = \mathbb{R}$ or $\sigma(K) = \mathbb{C}$ where $|\cdot|_0$ denotes the usual absolute value on $\mathbb{C}$ (see [8, 2.15], [41, 15.2.2]). As we will see, in the non-Archimedean case there are different families of Cauchy complete non-Archimedean valued fields.

Let $(K, |\cdot|)$ be a non-Archimedean valued field. Then $K^* = \{\lambda \in K : \lambda \neq 0\}$ is a multiplicative group and so is the **value group** of $(K, |\cdot|)$ defined by $|K^*| = \{|\lambda| : \lambda \in K^*\}$. The value group can either be dense in $(0, \infty)$ or discrete, where, for the second possibility, if the valuation is not trivial, then the group $|K^*|$ is cyclic with generator $\rho := \max\{r \in |K^*| : r < 1\}$ ([41, B.5.2]). Any element $\pi \in K$ for which $|\pi| = \rho$ is called a **uniformizer** for $|\cdot|$. The 'closed' unit disk in K, $B[0,1]$, is a ring and $B(0,1)$, the 'open' unit disk in K, is a maximal ideal of $B[0,1]$ ([45, p. 4]).

1.3.4 Definition. *Let $(K, |\cdot|)$ be a non-Archimedean valued field. The quotient field $B[0,1]/B(0,1)$ is called the **residue class field** of K. Moreover, if $|K^*|$ is discrete in $(0, \infty)$, then the valuation $|\cdot|$ is said to be **discrete** and if $|K^*|$ is dense in $(0, \infty)$, then the valuation is said to be **dense**.*

1.3.5 Examples. *Fields with discrete valuation.*

(a) Consider $(\mathbb{Q}, |\cdot|_p)$ for a prime number p. In this case, $|\mathbb{Q}^*| = \{e^{-n} : n \in \mathbb{Z}\} = \langle e^{-1} \rangle = \langle |p|_p \rangle$ and therefore p is a uniformizer for $|\cdot|_p$.

$$\begin{aligned} B[0,1] &= \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \text{ is not divisible by } p \right\} \\ \text{and} \qquad B(0,1) &= \left\{ \frac{pm}{n} : m, n \in \mathbb{Z},\ n \text{ is not divisible by } p \right\}. \end{aligned}$$

Notice that $B[0,1] = (\mathbb{Z} \setminus (p))^{-1}\mathbb{Z}$, i.e. it is the localization of the ring $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$, and $B(0,1) = pB[0,1]$ is the maximal ideal of $B[0,1]$. Using the first isomorphism theorem for rings, we conclude that $B[0,1]/B(0,1) \simeq \mathbb{Z}/p\mathbb{Z}$. In other words, the residue class field of $(\mathbb{Q}, |\cdot|_p)$ is isomorphic to $\mathbb{F}_p$ (the only field with p elements). For a simpler but longer proof see [35, p. 62].

(b) Consider $(K(x), |\cdot|_p)$ for an irreducible polynomial $p \in K[x]$, where K is a

field. In this case, p is a uniformizer for $|\cdot|_p$,

$$
\begin{aligned}
& B[0,1]=\left\{\frac{f}{g}: f, g \in K[x],\ g \text{ is not divisible by } p\right\} \\
\text{and}\qquad & B(0,1)=\left\{\frac{pf}{g}: f, g \in K[x],\ g \text{ is not divisible by } p\right\}.
\end{aligned}
$$

Analogously to the previous example, $B[0,1]$ *is the localization of the ring* $K[x]$ *at the prime ideal* (p)*, and* $B(0,1)=pB[0,1]$ *is the maximal ideal of* $B[0,1]$*. Thus the residue class field of* $(K(x),|\cdot|_p)$ *is isomorphic to* $K[x]/(p)$*. In particular, if* $p(x)=x$*, then the residue class field of* $(K(x),|\cdot|_p)$ *is isomorphic to* K*. For another proof see [35, p. 88].*

1.3.6 Theorem ([16, 1.3.4])**.** *Let* $(K,|\cdot|)$ *be a non-Archimedean valued field and* $(\widehat{K},\widehat{|\cdot|})$ *its completion. If* k *and* $\widehat{k}$ *are their respective residue class fields, then* $k \simeq \widehat{k}$ *and* $|K^*|=|\widehat{K^*}|$.

The following result is going to bring us an explicit description of the completions of $(\mathbb{Q},|\cdot|_p)$ and $(K(x),|\cdot|_p)$.

1.3.7 Theorem ([16, 1.3.5])**.** *Let* $|\cdot|$ *be a discrete valuation on the field* K*, with uniformizer* π *and residue class field* k*. Then every element* $x \in K^*$ *can be written uniquely as a convergent series*

$$
x = r_v\pi^v + r_{v+1}\pi^{v+1} + r_{v+2}\pi^{v+2} + \cdots = \lim_{n\to\infty}\sum_{i=v}^{n} r_i\pi^i
$$

where $v=\log_{|\pi|}|x|$*,* $r_v \neq 0$*, and the coefficients* r_i *are taken from a set* $R \subset B[0,1]$ *of representatives of the residue classes in* k *(i.e., the canonical map* $B[0,1]\to k$ *induces a bijection of* R *onto* k*).*

1.3.8 Remark. *If* π *is a uniformizer then* $|K^*|=\{|\lambda| : \lambda\in K^*\}=\{|\pi|^n : n\in\mathbb{Z}\}$*. Thus, for every* $x\in K^*$*, there is* $v\in\mathbb{Z}$ *such that* $|x|=|\pi|^v$*. Hence* $\log_{|\pi|}|x|\in\mathbb{Z}$*.*

1.3.9 Definition. *Let p be a prime number. The completion of the field $(\mathbb{Q}, |\cdot|_p)$ is called the **field of p-adic numbers** and is denoted by $\mathbb{Q}_p$.*

We checked in 1.3.5 that p is a uniformizer for $(\mathbb{Q}, |\cdot|_p)$ and that $\mathbb{F}_p$ is its residue class field. By 1.3.6, $\mathbb{Q}_p$ has the same residue class field and p as a uniformizer. According to 1.3.7, by taking $R = \{0, 1, \dots, p-1\}$, this field has the following description

$$\mathbb{Q}_p = \left\{ \sum_{i=v}^{\infty} r_i p^i : v \in \mathbb{Z}, r_i \in R,\ r_v \neq 0 \right\} \cup \{0\},$$

where the valuation on $\mathbb{Q}_p$ is defined as $|0| = 0$ and $\left| \sum_{i=v}^{\infty} r_i p^i \right| = e^{-v}$ when $r_v \neq 0$.

The closed disk in $\mathbb{Q}_p$ of center 0 and radius 1, is the **ring of p-adic integers**:

$$B[0,1] = \mathbb{Z}_p := \left\{ \sum_{i=0}^{\infty} r_i p^i : r_i = 0, 1, \dots, p-1 \right\}.$$

Be aware of the fact that addition of two 'series' of the form $\sum_{i=v}^{\infty} r_i p^i$ is not coefficientwise, as the set $\{0, 1, \dots, p-1\}$ is not closed under addition. As a simple example observe that (choosing $p = 7$)

$$5p^i + 4p^i = 5p^i + 2p^i + 2p^i = 7p^i + 2p^i = p^{i+1} + 2p^i.$$

1.3.10 Definition. *Let K be a field and set $p = x \in K[x]$. The completion of $(K(x), |\cdot|_p)$ is called the **field of formal Laurent series** and is denoted by $K((x))$.*

In 1.3.5, we saw that x is a uniformizer for $(K(x), |\cdot|_x)$ and that K is the residue class field of this valued field. By 1.3.6, $K((x))$ has x as a uniformizer and K as the residue class field as well. By 1.3.7, this field has the following description

$$K((x)) = \left\{ \sum_{i=v}^{\infty} r_i x^i : v \in \mathbb{Z}, r_i \in K,\ r_v \neq 0 \right\} \cup \{0\},$$

where the valuation on $K((x))$ is defined as $|0| = 0$ and $\left|\sum_{i=v}^{\infty} r_i x^i\right| = e^{-v}$ when $r_v \neq 0$. The closed disk in $K((x))$ of center 0 and radius 1, is the **ring of formal power series**:

$$B[0,1] = K[[x]] := \left\{\sum_{i=0}^{\infty} r_i x^i : r_i \in K\right\}.$$

Notice that $K((x))$ is the quotient field of $K[[x]]$ ([35, 3.1.L]). Here the addition of two such series is defined coefficientwise and the multiplication of two nonzero series is defined as follows:

$$\left(\sum_{j=v_1}^{\infty} s_j x^j\right) \cdot \left(\sum_{k=v_2}^{\infty} t_k x^k\right) = \sum_{i=v}^{\infty} r_i x^i,$$

with $v = v_1 + v_2$, $r_i = \sum_{j+k=i} s_j t_k = \sum_{j=v_1}^{\infty} s_j t_{i-j} = \sum_{k=v_2}^{\infty} s_{i-k} t_k$, where $s_j = 0$ for $j < v_1$ and $t_k = 0$ for $k < v_2$.

The following theorem shows how different a Cauchy complete non-Archimedean valued field is from their Archimedean analogous $\mathbb{R}$ and $\mathbb{C}$.

1.3.11 Theorem ([7, II.1.1])**.** *Let $(K, |\cdot|)$ be a Cauchy complete non-Archimedean valued field. If $(x_n)_n$ is a sequence of elements of K, then*

$$\sum_{n=1}^{\infty} x_n \textit{ is convergent in } K \Leftrightarrow \lim_{n\to\infty} x_n = 0.$$

1.4 Ordered fields

In this section we will present some examples of ordered fields and will discuss the concept of Archimedean extension of a field created by Hans Hahn in 1907 [19].

Formally real fields

By a ring, we will mean a commutative ring with unit $1 \neq 0$. Let A be a ring that is an ordered set such that its additive group $(A, +)$ is an ordered group (it has a total ordering which is compatible with the addition). The ring A is ordered if for all $x, y \in A$, $x > 0$ and $y > 0$ implies $xy > 0$. Note that an ordered ring is necessarily an integral domain. A field that is an ordered ring will be called an **ordered field**.

1.4.1 Definition. *A field K is* ***formally real*** *if it satisfies the following condition: given $a_1, \ldots, a_n \in K$ such that $\sum_{i=1}^{n} a_i^2 = 0$, then $a_1 = \cdots = a_n = 0$.*

The next result identifies the formally real fields as the fields that can be ordered.

1.4.2 Theorem ([5, 1.70(5) and 1.71(6)])**.**

Let K be a field. The following conditions are equivalent:

(a) K is formally real,

(b) -1 is not a sum of squares in K,

(c) There exists an order $\leq$ on K such that $(K, \leq)$ is an ordered field.

1.4.3 Examples. *(a) If $char(K) \neq 0$ then, there is $n \in \mathbb{N}$ such that $0 = \sum_{i=1}^{n} 1^2$. Hence K is not formally real. Thus if K is formally real, then $char(K) = 0$.*

(b) The field of complex numbers $\mathbb{C}$ cannot be an ordered field, since $-1 = i^2$ and therefore it is not formally real.

(c) If K is an ordered field then we can define an order in $K((x))$, which is compatible with the addition and multiplication. Thus $K((x))$ can be ordered, and therefore it is formally real. Such order is defined as follows: for every $z \in k((x))$ there are $r_i \in K$ such that $z = \sum_{i=v}^{\infty} r_i x^i$. We say that $z > 0$ if $z \neq 0$ and $r_v > 0$. Then $z_1 > z_2$ if $z_1 - z_2 > 0$.

(d) $\mathbb{Q}_p$ is not formally real because if $p = 2$, then -7 is a square and if $p > 2$ then $1 - p$ is a square ([35, p. 144]). Recall that in a formally real field the squares are non-negative elements. Since $\mathbb{Q} \subset \mathbb{Q}_p$, $char(\mathbb{Q}_p) = 0$.

General Hahn fields and the Embedding theorem

Let's review the concept of Archimedean extension of a field and the so called general Hahn fields which play a fundamental role in the classification of valued fields, since every ordered field can be embedded in a general Hahn field as we will see later.

1.4.4 Definition. *Let S be an ordered group. Two elements $x, y \in S^*$ are **comparable** if there exist $n, m \in \mathbb{N}$ such that $|x|_0 < n|y|_0$ and $|y|_0 < m|x|_0$, where*

$$|a|_0 := \begin{cases} a & , a \geq 0 \\ -a & , a < 0 \end{cases}$$

*Let K be an ordered field. The relation of being comparable is an equivalence relation on K^* and to denote 'x and y are comparable' we write $x \sim y$. This relation defines a partition of K^* into equivalence classes, which are called the **Archimedean classes of** K. The equivalence class of $x \in K$ is denoted by $[x]$. Let's denote the class of all the Archimedean classes by G_K.*

1.4.5 Theorem. *Let K be an ordered field. The class G_K is an ordered abelian group under the order $\prec$ and addition $+$ defined as follows: for every $x, y \in K^*$*

(a) $[x] \prec [y]$ *if and only if* $\forall n \in \mathbb{N}, n|y|_0 < |x|_0$, *or equivalently, if* $y \not\sim x$ *and* $|y|_0 < |x|_0$.

(b) $[x] + [y] := [xy]$

In this group, the neutral element is $[1]$, and $-[x] = [x^{-1}]$.

1.4.6 Definition. *An ordered field K is **Archimedean** if $G_K = \{[1]\}$, i.e. when any two elements in K^* are comparable.*

It is easily proved in [8, 3.7] that an ordered field K is Archimedean if and only if for every $x \in K$, there exists $n \in \mathbb{N}$ such that $|x|_0 < n1_K$.

Thus for every ordered field K, the group G_K measures the 'Archimedicity' or the 'non-Archimedicity' of K. The field $\mathbb{R}$ of real numbers (the only ordered, Dedekind complete field up to isomorphism) is characterized by the fact that each Archimedean ordered field can be embedded in $\mathbb{R}$ ([20, 3.5]). Hans Hahn in [19] (1907) generalized this property (see 1.4.11) and by doing so he ended up with ordered fields that contain all the ordered fields with a given 'level of Archimedicity'.

1.4.7 Definition. *Let E/K be an extension of ordered fields, where the order on E restricted to K coincides with that of K. We say that E is an **Archimedean extension** of K if for every $x \in E$, there exists $y \in K$ such that x and y are comparable in E. In that case, G_E and G_K are isomorphic ordered groups. An ordered field K is **Archimedean complete** if it has no proper Archimedean extension fields.*

1.4.8 Definition. *Let K be an ordered field. If G is an ordered abelian group isomorphic to G_K, then we say that K **is of type** G and G is called an **Archimedean group of** K.*

Notice that $\mathbb{R}$ is (up to isomorphism) the only Archimedean complete, ordered field of type $\{0\}$ (see [8, 3.10]). Now let's present the general Hahn fields.

1.4.9 Theorem ([5, 6.20, 6.21, 7.32], [13, 2.15], [19])**.**

Let K be a field (not necessarily ordered) and G an ordered abelian group. The set $K((G)) := \{f : G \to K : supp(f) \text{ is well-ordered}\}$, where $supp(f) := \{x \in G : f(x) \neq 0\}$, is a field under the addition and multiplication defined as follows: for every $f, g \in K((G))$ and $x \in G$,

(a) $(f+g)(x) := f(x) + g(x)$,

(b) $fg(x) := \sum_{a+b=x} f(a)g(b)$

*Fields of the form $K((G))$ are called **general Hahn fields**.*

When K is an ordered field we can define an order on $K((G))$ generalizing the definition of the order in $K((x))$ (see 1.4.3).

1.4.10 Definition (Ordered general Hahn fields). *Let K be an ordered field and consider $\lambda : K((G))^* \to G$, $\lambda(f) = \min\{supp(f)\}$. For $f, g \in K((G))$ we define:*

$$f < g \Leftrightarrow f \neq g \text{ and } (g - f)(\lambda(g - f)) > 0.$$

Then $(K((G)), \leq)$ is an ordered field.

Theorem 1.4.11 is crucial for the classification of ordered structures.

1.4.11 Theorem ([12], [21, 3.1], [5, 1.64], [13, 1.35], [19] (Hahn's Embedding Theorem)). *If K is an ordered field, then for every Archimedean group G of K, there exists an order-preserving field monomorphism σ from K into $\mathbb{R}((G))$ and $\mathbb{R}((G))$ is an Archimedean extension of $\sigma(K)$.*

Moreover, we have that $\mathbb{R}((G))$ is a generalization of $\mathbb{R}$ in the following sense.

1.4.12 Theorem ([12, pp. 862–863], [21, 3.2], [19] (Hahn's Completeness Theorem)). *If G is an ordered abelian group, then the field $\mathbb{R}((G))$ is (up to isomorphism) the only Archimedean complete, ordered field of type G.*

1.5 Hahn Fields and Levi-Civita fields

In this section a non-Archimedean valuation will be defined in some general Hahn fields and the family of the Levi-Civita fields will be presented.

1.5.1 Definition. *A* ***Hahn field*** *is a general Hahn field $K((G))$ (1.4.10) for which G is a subgroup of $(\mathbb{R}, +)$ and K is any field.*

The distinctive characteristic of a Hahn field is that we can define in a natural way a non-Archimedean valuation on them.

1.5.2 Theorem ([38, A.9 pp. 288–292], [39, II.6 corollary, p. 51])**.** *Let G be a subgroup of $(\mathbb{R},+)$ and let K be any field. If the map $|\cdot|: K((G)) \to \mathbb{R}$ is defined by*

$$|f| := \begin{cases} e^{-\min\{supp(f)\}} & , f \neq 0 \\ 0 & , f = 0, \end{cases}$$

then $(K((G)), |\cdot|)$ is a Cauchy complete non-Archimedean valued field with residue class field isomorphic to K and value group $|K((G))^*| = \{e^g \in \mathbb{R} : g \in G\}$. Moreover, it is spherically complete.

In 1.3.7 we showed that when a field has a discrete valuation, then every nonzero element can be written in a unique way as a limit of a convergent power series. However in some Hahn fields this also is possible when the valuation is dense.

1.5.3 Theorem. *Let K be any field. Consider in the field $K((\mathbb{Q}))$ the element $d : \mathbb{Q} \to K$ defined by*

$$d(x) := \begin{cases} 1 & , x = 1 \\ 0 & , x \neq 1. \end{cases}$$

Then, for any $r \in \mathbb{Q}$, we have that

$$d^r(x) = \begin{cases} 1 & , x = r \\ 0 & , x \neq r. \end{cases}$$

The value group of $(K((\mathbb{Q})), |\cdot|)$ is $\{e^{-r} = |d^r| = |d|^r : r \in \mathbb{Q}\}$. Furthermore, every nonzero element f in $K((\mathbb{Q}))$ is the sum of a convergent generalized power series with respect to the valuation on $K((\mathbb{Q}))$, specifically:

$$f = \sum_{r \in \mathbb{Q}} f(r) d^r = \sum_{r \in supp(f)} f(r) d^r.$$

Additionally, every generalized power series of the form $\sum_{r \in \mathbb{Q}} a_r d^r$ for which $\{r \in$

$\mathbb{Q} : a_r \neq 0\}$ *forms a well ordered subset of* $\mathbb{Q}$, *is convergent in* $K((\mathbb{Q}))$, *and if two series of such form differ in at least one coefficient then their sums are different.*

Let's present another interesting family of valued fields: the Levi-Civita fields.

1.5.4 Definition. *Let* K *be any field and let* G *be a subgroup of* $(\mathbb{R}, +)$. *The set* $L[G, K] := \{f : G \to K \mid supp(f) \cap (-\infty, n]$ *is finite for every* $n \in \mathbb{Z}\}$, *is a subfield of* $K((G))$ *([45, 1.3]). When we restrict the valuation of* $K((G))$ *to* $L[G, K]$, *the latter becomes a Cauchy complete, non-Archimedean valued field with residue class field isomorphic to* K *and value group* $|L[G, K]^*| = \{e^g : g \in G\}$. *Fields of the form* $L[G, K]$ *are called* ***Levi-Civita fields***.

1.5.5 Theorem ([8, 3.19])**.** *Let* K *be any field and let* G *be a subgroup of* $(\mathbb{R}, +)$. *The following statements hold:*

(a) The fields $K((G))$ *and* $L[G, K]$ *coincide if and only if* G *is discrete.*

(b) The field $L[G, K]$ *is spherically complete if and only if* G *is discrete.*

(c) If K *is an ordered field, then* $K((G))$ *is an Archimedean extension of* $L[G, K]$ *with respect to the order defined in 1.4.10. If in addition* K *is Archimedean, then both* $K((G))$ *and* $L[G, K]$ *are of type* G *(see 1.4.8).*

1.5.6 Example. *If* $G = (\mathbb{Z}, +)$ *and* K *is any field, then we have that* $L[\mathbb{Z}, K] = K((\mathbb{Z}))$ *is isomorphic to the field of Laurent series by the isomorphism* $\varphi : K((x)) \to L[\mathbb{Z}, K]$ *defined by*

$$\varphi\left(\sum_{i=m}^{\infty} a_i x^i\right) = \sum_{i=m}^{\infty} a_i d^i.$$

with $|q|_x = |\varphi(q)|$ *for every* $q \in K((x))$, *where* $|\cdot|_x$ *is the* x*-adic valuation in* $K((x))$.

1.5.7 Example. *The fields* $\mathbb{F}_p((x)) = \mathbb{F}_p((\mathbb{Z})) = L[\mathbb{Z}, \mathbb{F}_p]$ *and* $\mathbb{Q}_p$ *are Cauchy complete with respect to their valuations, both have the same value group* $\{e^n : n \in \mathbb{Z}\}$, *and their residue class fields are isomorphic to* $\mathbb{F}_p$. *However, these fields are not isomorphic since* $L[\mathbb{Z}, \mathbb{F}_p]$ *has characteristic* p *while* $\mathbb{Q}_p$ *has characteristic* 0.

1.6 Real-closed field extensions of $\mathbb{R}$

In this section we will study real closed field extensions of $\mathbb{R}$ with non-Archimedean valuations, in particular, we will see that under some conditions, the smallest among such (proper) extensions is a Levi-Civita field. Recall that a field K is **algebraically closed** if every polynomial in $K[x]$ has a root in K, hence every $p \in K[x], p \neq 0$, has $deg(p)$ roots in K, counting algebraic multiplicity. If L/K is a field extension and $a \in L$, then a is **algebraic** over K if it is the root of a polynomial $p \in K[x]$. If every element of L is algebraic over K, then L is said to a be an **algebraic extension** of K. Also, K is **real-closed** if K is formally real and does not admit a proper algebraic extension that is formally real.

1.6.1 Theorem ([24, Chapter XI], [5, 1.71(21),1.71(22)], [11, 5.4.4], [10, Chapter 5, Section 4, Lemma 4.1])**.**

Let K be a field. The following conditions are equivalent:

(a) K is real-closed,

(b) $x^2 + 1$ is irreducible in K and $K(i)$ is algebraically closed ($i^2 = -1$),

(c) K is an ordered field, each positive element of K has a square root and every $p \in K[x]$ of odd degree has a root in K,

(d) any sentence in the first-order language of fields is true in K if and only if it is true in $\mathbb{R}$,

(e) K is an ordered field and the intermediate value theorem holds for all polynomials over K.

The previous theorem shows that if we want to develop a theory of Calculus over ordered fields for which the intermediate value theorem holds, then our basis field has to be real-closed at least.

1.6.2 Definition. *Let (x_n) be a sequence in an ordered field K. We say (x_n) is **Cauchy** if for every 0-neighborhood U with respect to the order topology in K, there exists $N \in \mathbb{N}$ such that $x_m - x_n \in U$ for all $m, n \geq N$. Also, (x_n) is **convergent** to $x \in K$ if for every 0-neighborhood U with respect to the order topology in K, there exists $N \in \mathbb{N}$ such that $x_n - x \in U$ for all $n \geq N$. The field K is **Cauchy complete** if every Cauchy sequence of K is convergent in the order topology.*

The Levi-Civita field is the smallest non-Archimedean real-closed field extension of $\mathbb{R}$ that is Cauchy complete in the following sense:

1.6.3 Theorem ([8, 3.27])**.** *Let $K/\mathbb{R}$ be a field extension where K is a Cauchy complete real-closed field such that:*

(a) the order in K extends the one in $\mathbb{R}$,

(b) there exists $\delta \in K$ such that $0 < \delta < r$ for every $r \in \mathbb{R}^+$ and (δ^n) converges to 0 in the order topology.

If d is the element of $L[\mathbb{Q}, \mathbb{R}]$ defined in 1.5.3, then there exists an order-preserving field monomorphism $\sigma : L[\mathbb{Q}, \mathbb{R}] \to K$ defined by

$$\sigma(f) = \sigma\left(\sum_{q \in supp(f)} f(q) d^q \right) = \sum_{q \in supp(f)} f(q) \delta^q.$$

1.6.4 Remark. *An embedding from $L[\mathbb{Q}, \mathbb{R}]$ into K may exist even when the field K does not satisfy the condition (b). For example, the field $\mathbb{R}((\mathbb{Q}[x]))$ extends $L[\mathbb{Q}, \mathbb{R}]$, but it does not satisfy the condition (b). In fact, we assert the following:*

1.6.5 Proposition. *If K is an ordered field such that G_K has an infinite subset of pairwise non-comparable elements that is cofinal in G_K (1.4.4), then for each $x \in K^*$, the sequence $(x^n)_n$ does not converge to 0 in the order topology.*

For example, the field $\mathbb{R}((\mathbb{Z}[x]))$ satisfies the hypothesis of the proposition but $\mathbb{R}((\mathbb{Z}^3))$ does not.

Proof. By the Hahn's embedding theorem 1.4.11, the field K can be embedded in $\mathbb{R}((G_K))$ so we can consider the Hahn valuation on K (see 1.8.6, (c)). Let $x \in K^*$ and $\lambda(x) = \min\{supp(x)\} \in G_K$. Then $\lambda(x^n) = n\lambda(x) \in G_K$. By hypothesis we can choose $\varepsilon \in K$, $\varepsilon > 0$ such that $\lambda(\varepsilon) > n\lambda(x)$ for all $n \in \mathbb{N}$. If $|x|_0 = \max\{x, -x\}$, then $\lambda(x^n) = \lambda((|x|_0)^n) = \lambda((|x|_0)^n - \varepsilon)$ for all $n \in \mathbb{N}$. Thus $((|x|_0)^n - \varepsilon)(\lambda((|x|_0)^n - \varepsilon)) = ((|x|_0)^n - \varepsilon)(\lambda((|x|_0)^n)) = (|x|_0)^n(\lambda((|x|_0)^n)) > 0$. Therefore $(|x|_0)^n > \varepsilon$ for all $n \in \mathbb{N}$. In other words, $x^n \notin (-\varepsilon, \varepsilon)$ for all $n \in \mathbb{N}$. Hence, $(x^n)_n$ does not converge to 0 in the order topology. ■

1.7 Algebraic closure of valued fields and their completions

In this section we will present non-Archimedean valued fields that are the algebraic closures of other valued fields, like certain Puiseux series fields or the algebraic closure of $\mathbb{Q}_p$. Also we will review the completion of algebraic closures like the field of p-adic complex numbers $\mathbb{C}_p$, or the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}]$ as the completion of a Puiseux series field.

Recall that an **algebraic closure** of a field K, from now on denoted by $\mathbf{K^a}$, is an algebraically closed algebraic extension of K. Each field has an algebraic closure and any two algebraic closures of a field K are isomorphic by means of an isomorphism leaving K pointwise fixed ([26, I.8.25], [23, 66], [24, V 2.5, 2.9]).

Consider the field $\mathbb{Q}_p^a$ with the only valuation that extends $|\cdot|_p$ on $\mathbb{Q}_p$, also denoted by $|\cdot|_p$. It its known (e.g. [8, 4.6]) that $\mathbb{Q}_p^a$ satisfies the following properties:

(a) $\mathbb{Q}_p^a$ is a proper extension of $\mathbb{Q}_p$,

(b) the residue class field of $\mathbb{Q}_p^a$ is $\mathbb{F}_p^a$,

(c) $|(\mathbb{Q}_p^a)^*| = \{e^r : r \in \mathbb{Q}\}$,

(d) $(\mathbb{Q}_p^a, |\cdot|_p)$ is not locally compact,

(e) $(\mathbb{Q}_p^a, |\cdot|_p)$ is not Cauchy complete,

(f) $\mathbb{Q}_p^a$ is an infinite dimensional vector space over $\mathbb{Q}_p$.

1.7.1 Definition. *The completion of the field $(\mathbb{Q}_p^a, |\cdot|_p)$ is called the **field of p-adic complex numbers** and it is denoted by $\mathbb{C}_p$. The valuation on $\mathbb{C}_p$ which extends $|\cdot|_p$ will also be denoted by $|\cdot|_p$.*

It is known (e.g. [8, 4.8]) that the field $\mathbb{C}_p$ satisfies the following properties:

(a) the residue class field of $\mathbb{C}_p$ is $\mathbb{F}_p^a$,

(b) $|\mathbb{C}_p^*| = \{e^r : r \in \mathbb{Q}\}$,

(c) $(\mathbb{C}_p, |\cdot|_p)$ is not locally compact,

(d) $\mathbb{C}_p$ is algebraically closed,

(e) $\mathbb{C}_p$ is an infinite dimensional vector space over $\mathbb{Q}_p$,

(f) $\mathbb{C}_p$ is separable,

(g) $\mathbb{C}_p$ and $\mathbb{C}$ are isomorphic as fields,

(h) $\mathbb{C}_p$ is not spherically complete.

1.7.2 Definition. *Let K be a field. The set*

$$\begin{aligned} K\langle\langle x\rangle\rangle &:= \bigcup_{n=1}^{\infty} K((x^{\frac{1}{n}})) \\ &= \left\{\sum_{i=v}^{\infty} r_i x^{\frac{i}{n}} : v \in \mathbb{Z}, r_i \in K,\ r_v \neq 0,\ \textit{for some } n \in \mathbb{N}\right\} \cup \{0\}, \end{aligned}$$

is a field when we adopt the convention $x^{\frac{\ell p}{\ell q}} = x^{\frac{p}{q}}$ for all $\ell \in \mathbb{Z}$. In fact, if $a \in K((x^{\frac{1}{n}}))$ and $b \in K((x^{\frac{1}{m}}))$, then both a and b are elements of $K((x^{\frac{1}{nm}}))$, and therefore $a+b$ and ab are well-defined in $K((x^{\frac{1}{nm}}))$ and hence in $K\langle\langle x\rangle\rangle$.

The mapping $\theta : K\langle\langle x\rangle\rangle \to \bigcup_{n=1}^{\infty} K\left(\left(\frac{1}{n}\mathbb{Z}\right)\right)$ *defined by*

$$\theta\left(\sum_{i=v}^{\infty} r_i x^{\frac{i}{n}}\right) = \sum_{i=v}^{\infty} r_i d^{\frac{i}{n}},$$

is an isomorphism such that $|\theta(f)| = |f|$ *for all* $f \in K\langle\langle x\rangle\rangle$*. Therefore* $K\langle\langle x\rangle\rangle$ *can be considered as a subfield of* $K((\mathbb{Q}))$*. When we restrict the valuation of* $K((\mathbb{Q}))$ *(see below of 1.3.10) to* $K\langle\langle x\rangle\rangle$*, the latter becomes a non-Archimedean valued field with residue class field isomorphic to* K *and value group equal to* $\{e^r : r \in \mathbb{Q}\}$*. Fields of the form* $K\langle\langle x\rangle\rangle$ *are called* ***Puiseux series fields****.*

With this field we obtain the following chain of field extensions:

$$K \subsetneqq K(x) \subsetneqq K((x)) \subsetneqq K\langle\langle x\rangle\rangle \subsetneqq L[\mathbb{Q}, K] \subsetneqq K((\mathbb{Q})).$$

1.7.3 Proposition. *If* K *is any field, then* $L[\mathbb{Q}, K]$ *is the completion of the Puiseux series field* $(K\langle\langle x\rangle\rangle, |\cdot|)$*.*

Proof. Consider the identification $K\langle\langle x\rangle\rangle = \bigcup_{n=1}^{\infty} K\left(\left(\frac{1}{n}\mathbb{Z}\right)\right)$. By 1.5.3, every element of the Levi-Civita field $L[\mathbb{Q}, K]$ has the form $f = \sum_{i=v}^{\infty} r_i d^{\alpha_i}$ where $v \in \mathbb{Z}$, $r_i \in K$ and $(\alpha_i)_i$ is a strictly increasing sequence in $\mathbb{Q}$ such that $\{\alpha_i : i = v, v{+}1, \dots\} \cap (-\infty, n]$ is finite for every $n \in \mathbb{Z}$. Note that the partial sum sequence $(\sum_{i=v}^{n} r_i d^{\alpha_i})_n$ is Cauchy in $\bigcup_{n=1}^{\infty} K\left(\left(\frac{1}{n}\mathbb{Z}\right)\right)$ with limit f. Hence the theorem holds by 1.3.1, since $\bigcup_{n=1}^{\infty} K\left(\left(\frac{1}{n}\mathbb{Z}\right)\right)$ is dense in $L[\mathbb{Q}, K]$ and the latter is Cauchy complete (1.5.4). ■

The next result together with 1.7.3 shows the interesting analogy of the trios ($\mathbb{Q}_p$, $\mathbb{Q}_p^a$, $\mathbb{C}_p$) and ($K((x))$, $K\langle\langle x\rangle\rangle$, $L[\mathbb{Q}, K]$) when K satisfies the following conditions:

1.7.4 Theorem ([8, 4.11])**.** *If* K *is an algebraically closed field of characteristic* 0*, then* $K((x))^a = K\langle\langle x\rangle\rangle$ *and* $L[\mathbb{Q}, K]$ *is algebraically closed.*

Now we are able to determine when the Intermediate Value Theorem is valid for polynomials over $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$.

1.7.5 Proposition. *The following statements are equivalent:*

(a) K is real-closed.

(b) $K\langle\langle x\rangle\rangle$ is real-closed.

(c) $L[\mathbb{Q}, K]$ is real-closed.

Proof. (a) $\Longrightarrow$ (b) follows from [9, 2.6 Theorem 2.91] while (b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a) are proved in [5, 6.23 (1)]. Let's prove (a) $\Longrightarrow$ (c): if K is real-closed, then $K(i)$ is an algebraically closed filed of characteristic 0, where i is a root for $x^2+1=0$ (1.6.1). By 1.7.4 the field $K(i)\langle\langle x\rangle\rangle$ is algebraically closed and hence so is $L[\mathbb{Q}, K](i) = L[\mathbb{Q}, K(i)]$ ([8, 4.5], 1.7.3). Finally, $L[\mathbb{Q}, K]$ is real-closed by 1.6.1. ■

We know that the analogy of the trios breaks down when the characteristic of K is positive, because in that case $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ are not algebraically closed ([8, 4.13]).

Due to the following theorem we will be able to prove interesting field isomorphisms that are far from well-known.

1.7.6 Theorem ([43, p. 125], [22, 1.12, p. 317])**.** *If K_1 and K_2 are two algebraically closed fields of the same characteristic and their cardinality $\#K_1 = \#K_2$ is uncountable, then K_1 and K_2 are isomorphic as fields.*

1.7.7 Proposition. *If K is algebraically closed of characteristic 0 and cardinality $\#K = 2^{\aleph_0}$, then the fields K, $K\langle\langle x\rangle\rangle$, $L[\mathbb{Q}, K]$ and $K((\mathbb{Q}))$ are isomorphic to $\mathbb{C}$.*

Proof. By [8, 4.15], it follows that $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ are algebraically closed fields. Additionally, by [8, 6.18, 6.20] we have that $K((\mathbb{Q}))$ is algebraically closed. Since each of these fields contains K and is contained in the space of functions $K^{\mathbb{Q}}$, we conclude that all these fields have cardinality equal to $\#K = 2^{\aleph_0}$. Thus by 1.7.6, it follows that these fields are isomorphic to $\mathbb{C}$. ■

1.7.8 Remark. *Notice that*

$$\mathbb{C}\langle\langle x\rangle\rangle = \mathbb{R}\langle\langle x\rangle\rangle + i\mathbb{R}\langle\langle x\rangle\rangle$$

$$L[\mathbb{Q},\mathbb{C}] = L[\mathbb{Q},\mathbb{R}] + iL[\mathbb{Q},\mathbb{R}]$$

$$\mathbb{C}((\mathbb{Q})) = \mathbb{R}((\mathbb{Q})) + i\mathbb{R}((\mathbb{Q}))$$

Furthermore, the fields $\mathbb{R}\langle\langle x\rangle\rangle$ and $L[\mathbb{Q},\mathbb{R}]$ are real-closed by 1.7.5, and $\mathbb{R}((\mathbb{Q}))$ is real closed by [8, 3.23]. Although we just proved that $\mathbb{C}\langle\langle x\rangle\rangle$, $L[\mathbb{Q},\mathbb{C}]$ and $\mathbb{C}((\mathbb{Q}))$ are isomorphic to $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, none of the fields $\mathbb{R}\langle\langle x\rangle\rangle$, $L[\mathbb{Q},\mathbb{R}]$ and $\mathbb{R}((\mathbb{Q}))$ is isomorphic to $\mathbb{R}$. In fact, by [5, 1.71.1], for any real-closed field K we have that for all $x \in K$, $x > 0$ if and only if $x = y^2$ for some $y \in K$. Thus, if $\phi : K_1 \to K_2$ is a field isomorphism between real-closed fields K_1 and K_2, then $x > 0$ implies $\phi(x) > 0$ for all $x \in K_1$. Hence ϕ is order preserving. Therefore, if $a > 0$ is an infinitesimal in K_1, then $\phi(a) > 0$ is an infinitesimal in K_2.

1.8 General valuations and the higher rank case

Note that in 1.5.2 we have restricted our attention to those Hahn fields of Archimedean group embedded in $(\mathbb{R}, +)$ in order to define a valuation with values in $\mathbb{R}$. However it is possible to define a valuation with values in any ordered abelian group. If $|\cdot| : K \to \mathbb{R}$ is a valuation on a field K and we consider the function $v : K \to \mathbb{R}\cup\{\infty\}$ defined by $v(x) = -\ln|x|$, then we are shifting our attention to the additive structure of $\mathbb{R}$ rather than the multiplicative one of $(0,\infty)$. In particular, if $|\cdot|$ is a non-Archimedean valuation, then the strong triangle inequality now has the form: $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$. In general, if we redefine valuation as a function of the form v it is possible to rewrite all the previous results concerning valuations in terms of v and the additive group structure of $\mathbb{R}$ without losing any result in the process. In this section we will use this approach to define a generalization of a non-Archimedean valuation.

1.8.1 Definition. *Let K be a field and let $(G,+)$ be an ordered abelian group. A map $v: K \to G \cup \{\infty\}$ is a* ***general valuation*** *(or* ***Krull valuation****) on K if:*

(a) v is onto,

(b) $v(x) = \infty$ if and only if $x = 0$,

(c) $v(xy) = v(x) + v(y)$,

(d) $v(x+y) \geq \min\{v(x), v(y)\}$,

where ∞ is a symbol that satisfies, for all $g \in G$, the following axioms:

$$g < \infty \text{ and } \infty = \infty + \infty = g + \infty = \infty + g.$$

The group $G = v(K^*)$ is called the ***value group*** *of (K, v) and the quotient $\{x \in K : v(x) \geq 0\}/\{x \in K : v(x) > 0\}$ is the* ***residue class field*** *of (K, v). If G is order-isomorphic to a subgroup of $(\mathbb{R},+)$, then we say that v has* ***rank* 1***. Otherwise we say that v is of* ***higher rank****. The general valuation v is called* ***discrete*** *if G is cyclic.*

When the general valuation v is discrete and G is not trivial, then it is isomorphic to $(\mathbb{Z},+)$ and hence v has rank 1.

1.8.2 Theorem ([7, Chapter III, section 3])**.** *An ordered group $(G,+)$ is order-isomorphic to a subgroup of $(\mathbb{R},+)$ if and only if it is Archimedean, i.e. for any $a, b \in G$, $b > 0$, there exists $n \in \mathbb{N}$ such that $a < nb$. In particular, every Archimedean ordered group is abelian.*

1.8.3 Definition. *Let K be a field with general valuation $v: K \to G \cup \{\infty\}$. The sets of the form*

$$U_v[a,g] := U[a,g] := \{x \in K : v(x-a) \geq g\}$$

*for $a \in K, g \in G$, form a basis for a topology on K called the **valuation topology** induced by v. When G is not trivial, the sets of the form*

$$U_v(a,g) := U(a,g) := \{x \in K : v(x-a) > g\}$$

also form a basis for the valuation topology induced by v. The field K equipped with this topology becomes a Hausdorff topological field ([5, 7.64]).

1.8.4 Definition. *Let K be an ordered field and let G_K be the ordered group of the Archimedean classes of K (1.4.4, 1.4.5). The map $\mu : K \to G_K \cup \{\infty\}$ defined by*

$$\mu(x) := \begin{cases} [x] & x \neq 0 \\ \infty & x = 0 \end{cases}$$

*is a general valuation on K and it is called **the order valuation** of K ([5, 1.61]).*

Notice that if K is an Archimedean ordered field then the valuation topology induced by the order valuation coincides with the discrete topology.

1.8.5 Theorem ([5, 7.63, 7.64])**.** *Let K be a non-Archimedean ordered field and let μ be the order valuation on K. Then the order topology on K coincides with the valuation topology induced by μ.*

1.8.6 Examples. *(a) If $|\cdot| : K \to \mathbb{R}$ is a non-Archimedean valuation on a field K, then the function $v : K \to \mathbb{R} \cup \{\infty\}$ defined by $v(x) = -\ln|x|$, when $x \neq 0$ and $v(0) = \infty$, is a general valuation (after a suitable restriction of the codomain). The topology induced by $|\cdot|$ coincides with the valuation topology induced by v, since $B(x, e^{-r}) = U(x,r)$ and $B[x, e^{-r}] = U[x,r]$ for every $x \in K$ and $r \in \mathbb{R}$.*

(b) If $v : K \to G \cup \{\infty\}$ is a general valuation of rank 1 on K and $i : G \to (\mathbb{R}, +)$

is an order-preserving homomorphism, then the map $|\cdot|: K \to \mathbb{R}$ *defined by*

$$|x| := \begin{cases} e^{-i(v(x))} & , x \neq 0 \\ 0 & , x = 0, \end{cases}$$

is a non-Archimedean valuation on K. *The topology induced by* $|\cdot|$ *coincides with the valuation topology induced by* v, *since* $B(x, |y|) = U(x, v(y))$ *and* $B[x, |y|] = U[x, v(y)]$ *for every* $x, y \in K$.

(c) Let K *be a field and let* G *be any ordered abelian group. The map* $\lambda : K((G)) \to G \cup \{\infty\}$ *defined by*

$$\lambda(f) := \begin{cases} \min\{supp(f)\} & , f \neq 0 \\ \infty & , f = 0, \end{cases}$$

is a general valuation on the general Hahn field $K((G))$ *called* ***the Hahn valuation*** *on* $K((G))$ *(compare with the valuation of a Hahn field in 1.5.2).*

1.8.7 Proposition. *Let* K *be an Archimedean ordered field,* G *a nontrivial ordered abelian group,* μ *the order valuation on* $K((G))$ *and* λ *the Hahn valuation on* $K((G))$. *The following statements hold.*

(a) $\mu(f) = [f] = \{g \in K((G)) : \lambda(g) = \lambda(f)\}$ *for every* $f \neq 0$,

(b) the valuation topologies induced by μ *and* λ *coincide with the order topology on* $K((G))$ *(defined in 1.4.10),*

(c) If G *is a subgroup of* $(\mathbb{R}, +)$, *then the valuation* $|\cdot|$ *of the Hahn field* $K((G))$ *defined in 1.5.2, induces the order topology on* $K((G))$.

Proof. To prove the first statement it is enough to note that f and g are comparable if and only if $\lambda(f) = \lambda(g)$. The second statement follows from 1.8.5 and the equalities

$U_\mu(f,\mu(g)) = U_\lambda(f,\lambda(g))$ and $U_\mu[f,\mu(g)] = U_\lambda[f,\lambda(g)]$ for all $f,g \in K((G))$. The last statement follows from part (b) and from example 1.8.6(a). ■

1.8.8 Proposition. *Let F/K be a field extension and let $v : F \to G \cup \{\infty\}$ be a general valuation on F. The restriction $v|_K$ of v to K is a general valuation on K. Moreover, if τ_v ($\tau_{v|_K}$) denotes the valuation topology on F (on K) induced by v (by $v|_K$ respectively), and $\tau_v \cap K$ denotes the subspace topology on K induced by (F, τ_v), then*

$$\tau_{v|_K} = \tau_v \cap K.$$

Proof. After a suitable restriction of the codomain the map $v|_K$ is a general valuation on K. The second statement follows from the following equalities valid for all $x, y \in K$: $U_{v|_K}(x, v(y)) = \{z \in K : v(x-z) > v(y)\} = U_v(x, v(y)) \cap K$. ■

1.8.9 ★ Proposition. *Let K be a non-Archimeden ordered field. By 1.4.11, K can be embedded in $\mathbb{R}((G_K))$ where G_K is the ordered group of Archimedean classes of K. Let*

- *τ_K be the order topology on K,*
- *τ_μ the valuation topology on K induced by the order valuation μ of K,*
- *τ_0 the order topology on $\mathbb{R}((G_K))$,*
- *τ_η the valuation topology on $\mathbb{R}((G_K))$ induced by the order valuation η of $\mathbb{R}((G_K))$,*
- *τ_λ the valuation topology on $\mathbb{R}((G_K))$ induced by the Hahn valuation λ of $\mathbb{R}((G_K))$.*

Then $\quad \tau_K = \tau_\mu = \tau_{\lambda|_K} = \tau_{\eta|_K} = \tau_\lambda \cap K = \tau_0 \cap K = \tau_\eta \cap K.$

Proof. The equality $\tau_K = \tau_\mu$ follows from 1.8.5 while the equalities $\tau_\lambda \cap K = \tau_0 \cap K = \tau_\eta \cap K$ follow by 1.8.7. The equalities $\tau_{\lambda|_K} = \tau_\lambda \cap K$ and $\tau_{\eta|_K} = \tau_\eta \cap K$ follow from

1.8.8. Finally we will show that $\tau_\mu = \tau_\eta \cap K$. For every $f \in K$, $\mu(f) = \{g \in K : g \text{ is comparable to } f\}$. Since K is embedded in $\mathbb{R}((G_K))$ we have that $K \subset \mathbb{R}((G_K))$. Hence $\mu(f) \subset \eta(f) = \{g \in \mathbb{R}((G_K)) : \lambda(g) = \lambda(f)\}$. It follows that $\mu(f) = \mu(g)$ if and only if $\eta(f) = \eta(g)$ for all $f, g \in K$. Additionally, $\mu(f) < \mu(g)$ if and only if $\eta(f) < \eta(g)$. Hence $U_\eta(f, \eta(g)) \cap K = U_\mu(f, \mu(g))$ for all $f, g \in K$. ■

1.8.10 Remark. *Let $(A, <)$ be an ordered set with order topology τ_A and let B be a subset of A. The order in A induces an order in B which induces a topology $\tau_<$ on B. In general the subspace topology $\tau_A \cap B$ on B may be different from the topology $\tau_<$ ([27, p. 90]). Also, it is well-known that if B is a convex subset of A, then $\tau_A \cap B = \tau_<$ ([27, 16.4]). The surprising thing about the previous result is that for every non-Archimedean ordered field K its order topology τ_K (which is the topology induced by the order induced by the order on $\mathbb{R}((G_K))$) always coincides with the subspace topology $\tau_0 \cap K$, even when K is not convex in $\mathbb{R}((G_K))$. For example, $L[\mathbb{Q}, \mathbb{R}]$ is not a convex subfield of $\mathbb{R}((\mathbb{Q}))$ and by 1.8.9, the order topology on $L[\mathbb{Q}, \mathbb{R}]$ coincides with the subspace topology inherited from $\mathbb{R}((\mathbb{Q}))$.*

More details about general valuations can be found in [35, 13.1] and [33].

1.9 Catalog of fields

In the following table we summarize the fields and their properties presented so far. Let G be an ordered abelian group, let K be any field and let p be a positive prime integer. Assume that each of the listed fields has the usual non-Archimedean valuation (or general valuation) defined in this chapter and it is equipped with the respective induced topology. Denote the cardinalities of K and G by c and g respectively.

Notice that the first 4 fields of the table form a chain of field extensions in the mixed characteristic case, i.e. each of the fields has a characteristic different than the characteristic of its residue class field:

$$\mathbb{Q} \subset \mathbb{Q}_p \subset \mathbb{Q}_p^a \subset \mathbb{C}_p. \tag{1.1}$$

The last 5 fields of the table form a chain of field extensions in the equal characteristic case, i.e. each of the fields has a characteristic equal to the characteristic of its residue class field. Now consider the chain:

$$K(x) \subset K((x)) \subset K\langle\langle x\rangle\rangle \subset L[\mathbb{Q}, K]. \tag{1.2}$$

Note that the chains (1.1) and (1.2) are similar in several ways, for example the construction of the fields in the i-th position from the fields in the $(i-1)$-th position is identical (under certain conditions when it is necessary). Other metric, topological or algebraic similarities are easily seen from the table.

If the m-th property is satisfied by the n-th field of the table, then the symbol ✓ will appear in the entry (m, n). Otherwise the symbol ✗ will take place. When a number (n) appears instead, the property is satisfied under certain conditions specified below the table.

1.9.1 Remark. *The study of maximal immediate extension fields has allowed the author to add the p-adic Mal'cev-Neumann fields (the p-adic analogues of the general Hahn fields) to the right side of the chain (1.1) and their properties can be found in a larger catalog presented in [8]. These fields also play an important role in the classification of non-Archimedean valued fields. In fact, it can be proved that any non-Archimedean valued field can be embedded into either a general Hahn field or into a p-adic Mal'cev-Neumann field (see [8, 6.13, 7.2, 7.17]) and therefore the structure of any non-Archimedean valued field is not as diverse as one initially may think. These fields have been omitted here because they will not be used in the subsequent chapters and their mere presentation needs a significant amount of space.*

	$\mathbb{Q}$	$\mathbb{Q}_p$	$\mathbb{Q}_p^a$	$\mathbb{C}_p$	$K(x)$	$K((x))$	$K\langle\langle x\rangle\rangle$	$L[\mathbb{Q}, K]$	$K((G))$
Totally disconnected	✓	✓	✓	✓	✓	✓	✓	✓	✓
Separable	✓	✓	✓	✓	(1)	(1)	(1)	(1)	(1)
Cauchy complete	✗	✓	✗	✓	✗	✓	✗	✓	✓
Spherically complete	✗	✓	✗	✗	✗	✓	✗	✗	✓
Locally compact	✗	✓	✗	✗	✗	(2)	✗	✗	(2)
Algebraically Closed	✗	✗	✓	✓	✗	✗	(3)	(3)	(4)
Formally real	✓	✗	✗	✗	(5)	(5)	(5)	(5)	(5)
Real-closed	✗	✗	✗	✗	✗	✗	(6)	(6)	(7)
Archimedean complete	✗	✗	✗	✗	✗	(8)	✗	✗	(8)
Cardinality	$\aleph_0$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$c^{\aleph_0}$	$c^{\aleph_0}$	$c^{\aleph_0}$	$c^{\aleph_0}$	c^g
Residue class field	$\mathbb{F}_p$	$\mathbb{F}_p$	$\mathbb{F}_p^a$	$\mathbb{F}_p^a$	K	K	K	K	K

Table 1.1: Catalog of fields

(1) Under the assumption that G is a subgroup of $(\mathbb{R}, +)$, the field $K((G))$ is separable if and only if K and G are countable ([8, 1.9]). Since $K \subset K(x) \subset K((x)) \subset K\langle\langle x\rangle\rangle \subset L[\mathbb{Q}, K] \subset K((\mathbb{Q}))$, each of these fields is separable if and only if K is separable.

(2) Under the assumption that G is a subgroup of $(\mathbb{R}, +)$, the field $K((G))$ is locally compact if and only if K is finite and G is cyclic ([38, 12.2], 1.5.2). In particular, $K((x))$ is locally compact if and only if K is finite.

(3) Each of the fields $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ is algebraically closed if and only if K is algebraically closed of characteristic 0 ([8, 4.15]).

(4) The field $K((G))$ is algebraically closed if and only if K is algebraically closed and G is divisible, i.e. for every $a \in G$ and $n \in \mathbb{Z}$, there exists and element $b \in G$ such that $a = nb$ ([8, 6.18, 6.20]).

(5) The field $K((G))$ is formally real if and only if K is formally real (1.4.10). In particular, since $K \subset K(x) \subset K((x)) \subset K\langle\langle x\rangle\rangle \subset L[\mathbb{Q}, K] \subset K((\mathbb{Q}))$, each of these fields is formally real if and only if K is formally real.

(6) Each of the fields $K\langle\langle x\rangle\rangle$ and $L[\mathbb{Q}, K]$ is real-closed if and only if K is real-closed (1.7.5).

(7) The field $K((G))$ is real-closed if and only if K is real-closed and G is divisible ([5, 6.23 (1)-(2)]). In particular, $K((x))$ is real-closed for no field K.

(8) $K((G))$ is Archimedean complete of type G if and only if K is isomorphic to $\mathbb{R}$ as ordered field (1.4.12).

1.10 Classification of fields

The following diagram classifies the fields that admit a valuation or general valuation (nontrivial valuation when the field is infinite) in 6 non-overlapping classes. With a suitable choice of some of these classes it is possible to obtain partitions for: the class of ordered fields, the class of Archimedean valued fields, and the class of non-Archimedean valued fields. Below each of the 6 classes, there are some examples of their members.

Archimedean valued fields

non-Archimedean valued fields

Non-formally real Archimedean valued fields	Formally real Archimedean valued fields	Formally real fields where $G_K \neq \{0\}$ is non-Archimedean	Formally real non-Archimedean valued fields	Non-formally real non-Archimedean valued fields	Non-formally real fields with general valuation of higher rank
$\mathbb{C}$ $\mathbb{Q}(i)$ $\mathbb{Q}(\sqrt{2}, i)$	$\mathbb{R}$ $\mathbb{Q}$ $\mathbb{Q}(\sqrt{3})$ $\mathbb{Q}(\pi)$	$\mathbb{R}((\mathbb{Z}^3))$ $L[\mathbb{R}^2, \mathbb{Q}]$ $\mathbb{Q}((x, y))$ $\mathbb{R}((L[\mathbb{Z}, \mathbb{Q}]))$	$\mathbb{R}(x)$ $\mathbb{R}((x))$ $\mathbb{Q}(e)\langle\langle x\rangle\rangle$ $L[\mathbb{Q}, \mathbb{R}]$ $\mathbb{Q}((\mathbb{Q}))$	$\mathbb{Q}_p$, $\mathbb{C}_p$ $\mathbb{F}_q((x))$ $\mathbb{Q}^a\langle\langle x\rangle\rangle$ $\mathbb{Q}_p((\mathbb{Q}))$ $L[\mathbb{Q}, \mathbb{C}]$	$\mathbb{Q}_p((\mathbb{Z}^5))$ $\mathbb{F}_q((x, y))$ $\mathbb{C}((\mathbb{R} \oplus \mathbb{Q}))$ $L[\mathbb{Z}^2, \mathbb{Q}_p]$

Ordered fields

Fields with general valuations

Table 1.2: Classification of valued fields

In the following, the 6 classes are ordered from left to right.

1. By [41, 15.2.2] the second class can be described as the collection of all the fields that are isomorphic to a subfield of $\mathbb{R}$ or equivalently as the collection of all the Archimedean ordered fields.

2. By 1.8.2 the third class can be described as the collection of all the non-Archimedean ordered fields K for which G_K cannot be embedded in $(\mathbb{R}, +)$ ($G_K \neq \{0\}$ is not an Archimedean ordered group).

3. By 1.8.2 the fourth class can be described as the collection of all the non-Archimedean ordered fields K for which G_K can be embedded in $(\mathbb{R}, +)$ ($G_K \neq \{0\}$ is an Archimedean ordered group).

Chapter 2

Banach spaces over fields with rank-1 valuations

In this chapter we will study Banach spaces over non-Archimedean valued fields of rank 1 and consider their similarities and differences with Banach spaces in classical Functional Analysis. Also, linear operators will be considered on normed spaces over non-Archimedean valued fields and the different norms that can be defined for continuous linear operators will be compared. Then, we will describe the concepts of form-orthogonality, norm-orthogonality and spaces of countable type that will be used in subsequent chapters.

2.1 Preliminaries

2.1.1 Definition. *Let $(K, |\cdot|)$ be a non-Archimedean valued field and let E be a vector space over K. A **norm** on E is a map $||\cdot|| : E \to [0, \infty)$ such that:*

(a) $||x|| = 0$ *if and only if* $x = 0$,

(b) $||\lambda x|| = |\lambda| \, ||x||$,

(c) $||x + y|| \leq \max\{||x||, ||y||\}$,

for all $x, y \in E$. We call $(E, ||\cdot||)$ a ***normed space*** *over K and if it is Cauchy complete with respect to the induced ultrametric $(x, y) \mapsto ||x - y||$, then $(E, ||\cdot||)$ is called a* ***Banach*** *space. The topology induced by this ultrametric is a vector topology, i.e. addition of vectors of E and scalar multiplication are continuous functions. The* ***closed unit ball*** *of a normed space E is the set*

$$B_E := \{x \in E : ||x|| \leq 1\}.$$

2.1.2 Remark. *An* ***A-norm*** *is a map $||\cdot|| : E \to [0, \infty)$ satisfying $(a), (b)$ and the triangle inequality $||x + y|| \leq ||x|| + ||y||$, $(x, y \in E)$, but not (c). For example, if $|\cdot|$ is a nontrivial valuation on K, then $(\lambda_1, \lambda_2) \mapsto \sqrt{|\lambda_1|^2 + |\lambda_2|^2}$ is an A-norm on K^2, whereas $(\lambda_1, \lambda_2) \mapsto \max\{|\lambda_1|, |\lambda_2|\}$ is a norm in the sense of 2.1.1. According to C.Perez-Garcia and W.H.Schikhof ([32, p. 16]), although A-norms can be a source for interesting topological problems, all spaces that appear naturally in duality theory carry norms or seminorms in the sense of 2.1.1. Also we have the following result noted by A.C.M van Rooij in [45, p. 88]: the dual space of a vector space with respect to an A-norm can be obtained as the dual space of a vector space with respect to a norm. In fact, let $||\cdot||$ be an A-norm on a vector space E over K. This A-norm induces a topology on E and we can consider the space $E^{\triangledown}$ of all the linear maps $f : E \to K$ continuous with respect to this topology. On this $E^{\triangledown}$ we can define a norm (not merely an A-norm) by*

$$||f||_0 := \sup_{x \neq 0} \frac{|f(x)|}{||x||}, \ (f \in E^{\triangledown}).$$

Among all seminorms on E that are $\leq ||\cdot||$, there is a largest one, say v. Set $N := \{x \in E : v(x) = 0\}$. Then N is a subspace of E and v induces a norm on E/N in a natural way. We have that $(E/N)'$ and $E^{\triangledown}$ are isomorphic Banach spaces.

2.1.3 Examples ([45, pp. 47–49]). *Consider a Cauchy complete non-Archimedean valued field $(K, |\ \ |)$ with nontrivial valuation.*

(a) *Let $e_1, \ldots, e_n$ be a basis of an n-dimensional space over K. Then $\sum_{i=1}^{n} \lambda_i e_i \mapsto \max_i |\lambda_i|$ is a norm for which the space is Banach.*

(b) *Let ℓ^∞ be the space of all bounded sequences in K equipped with termwise operations. For $x = (\lambda_1, \lambda_2, \ldots) \in \ell^\infty$, define $||x||_\infty := \sup\{|\lambda_n| : n \in \mathbb{N}\}$. With this norm ℓ^∞ is a Banach space.*

(c) *$c_0 := \{(\lambda_1, \lambda_2, \ldots) \in \ell^\infty : \lim_{n\to\infty} \lambda_n = 0\}$ is a closed subspace of ℓ^∞, hence a Banach space. Notice that for each $x \in c_0$, $||x||_\infty = \max\{|\lambda_n| : n \in \mathbb{N}\}$.*

(d) *Let X be any set. The bounded maps $f : X \to K$ form a vector space over K denoted by $\ell^\infty(X)$ which is a Banach space under the norm $||f||_\infty := \sup\{|f(x)| : x \in X\}$. Notice that $\ell^\infty = \ell^\infty(\mathbb{N})$.*

(e) *$c_0(X) := \{f \in \ell^\infty(X) : \forall \varepsilon > 0, \{x \in X : |f(x)| > \varepsilon\}$ is finite$\}$ is a closed subspace of $\ell^\infty(X)$, hence a Banach space. Notice that for each $f \in c_0(X)$, $||f||_\infty = \max\{|f(x)| : x \in X\}$ and that $c_0 = c_0(\mathbb{N})$.*

(f) *Let X be a topological space and $BC(X)$ the space of all continuous and bounded functions $f : X \to K$. Then $BC(X)$ is a closed subspace of $\ell^\infty(X)$ with respect to the supremum norm $||\cdot||_\infty$ hence a Banach space.*

(g) *Let F be a complete valued field containing K as a subfield. Then F is a Banach space over K. In particular, the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}]$ (see 1.5.4) is a Banach space over $\mathbb{R}((x)) = \mathbb{R}((\mathbb{Z}))$. Observe that $|L[\mathbb{Q}, \mathbb{R}]^*| = \{e^r : r \in \mathbb{Q}\}$ is a proper superset of $|\mathbb{R}((x))^*| = \{e^r : r \in \mathbb{Z}\}$. This implies that nonzero vectors cannot always be normalized. For example, $d^{1/2} \in L[\mathbb{Q}, \mathbb{R}]$ does not have a multiple of norm 1. In fact, if $\lambda \in \mathbb{R}((x))^*$ then $|\lambda| = e^n$ for some*

$n \in \mathbb{Z}$ and hence $|\lambda d^{1/2}| = e^{n-1/2} \neq 1$ (for the definition of d see 1.5.3). This leads us to the following open problem.

PROBLEM (1960): Let $(E, ||\cdot||)$ be a Banach space over K. Does there exist an equivalent norm $||\cdot||_0$ such that $||E||_0 = |K|$, i.e. for each nonzero $x \in E$ there exists $\lambda \in K$ such that $||\lambda x|| = 1$? There are some partial answers and reductions, but the complete answer is so far unknown!

In order to overcome the possible non-existence of vectors of norm δ for some particular $\delta \in \mathbb{R}^+$, we have the following lemma which is appropriate when we only need the existence of vectors of norm close to δ.

2.1.4 Lemma. *Let $(E, ||\cdot||)$ be a normed space over a non-Archimedean valued field $(K, |\cdot|)$ where $|\cdot|$ is not trivial. The following statements hold:*

(a) There exists $\pi \in K^*$ such that $0 < |\pi| < 1$.

(b) For any $\delta \in \mathbb{R}^+$, $\pi \in K^*$, $0 < |\pi| < 1$ and for any nonzero $x \in E$, there exists a unique $n \in \mathbb{Z}$ such that $|\pi|\delta < ||\pi^n x|| \leq \delta$.

Proof. (a) Since the valuation on K is nontrivial, there exists $y \in K^*$ such that $|y| \neq 1$. Because of the equality $|y||y^{-1}| = 1$, we can choose $|\pi| = \min\{|y|, |y^{-1}|\}$.

(b) Put $s := |\pi|$ and $w := ||x||$. Consider $n = \lceil \log_s(\delta/w) \rceil$. Then n is such that $s^n w$ is as close to δ as possible while still being smaller than or equal to δ, or equivalently $s^n w \leq \delta < s^{n-1} w$. Hence $s\delta < s^n w \leq \delta$. ∎

Lemma 2.1.4 is one of the most used basic tools in non-Archimedean Analysis. For example, it can be found in the proof of the fact that the value group of a valuation is either cyclic or dense in $(0, \infty)$. Also in the proofs of 2.1.6, 2.2.1, 2.2.4, 2.2.5 and 2.2.7.

2.1.5 Definition. *Two norms $||\cdot||_1$ and $||\cdot||_2$ on a vector space E over K are called* ***equivalent*** *if the metrics $(x,y)\mapsto ||x-y||_1$ and $(x,y)\mapsto ||x-y||_2$ induce the same topology on E.*

2.1.6 Proposition. *Let $(K,|\cdot|)$ be a non-Archimedean valued field and E a vector space over K.*

(a) When $|\cdot|$ is nontrivial:

Two norms on E, say $||\cdot||_1$ and $||\cdot||_2$, are equivalent if and only if there exist constants $0<c<C$ such that

$$c||\cdot||_1 \leq ||\cdot||_2 \leq C||\cdot||_1. \quad (\clubsuit)$$

If K is Cauchy complete and E is finite-dimensional, then all norms on E are equivalent, and the space E is a Banach space with respect to each norm.

(b) When $|\cdot|$ is trivial:

If E is finite-dimensional, then all norms are equivalent to the trivial one, and the space E is a Banach space with respect to each norm.

If E is an infinite-dimensional space, then there exist equivalent norms on E that don't satisfy the inequality $(\clubsuit)$ for any positive constants c and C.

Proof. When $|\cdot|$ is nontrivial: for the first statement see [32, 2.1.7] and for the second statement see [7, IV.1.2 and IV.1.3].

When $|\cdot|$ is trivial: for the first statement suppose that E has dimension $n\in\mathbb{N}$ and let $\{e_1,\ldots,e_n\}$ be a basis for E. We will show that there exist positive constants c and C such that for all nonzero $x\in E$, $c\leq ||x||\leq C$.

If $C=:\sum_{i=1}^n ||e_i||$ then for any $x=\sum_{i=1}^n \lambda_i e_i\in E$, $||x||\leq \sum_{i=1}^n |\lambda_i|||e_i||\leq C$. For each $i\in\mathbb{N}$, let $X_i\subseteq E$ be the span of all the vectors $x\in E$ such that $||x||<1/i$.

We will show that $X_k = \{0\}$ for some $k \in \mathbb{N}$ and hence we can put $c := 1/k$. Let $m = \min\{\dim(X_i) : i \in \mathbb{N}\}$ and let $k \in \mathbb{N}$ such that $\dim(X_k) = m$. Since $X_{i+1} \subset X_i$ for all $i \in \mathbb{N}$, $\dim(X_i) = m$ for all $i \geq k$. Thus $X_k = X_i$ for $i \geq k$, so $X_k = \bigcap\{X_i : i \geq k\}$. Now suppose $x \in X_k$. Then for each $i \geq k$, x can be written as a linear combination of vectors of norm less than $1/i$. But this implies that $||x|| < m/i$. Since i is arbitrarily large, this means that $||x|| = 0$ and hence $x = 0$. Thus $X_k = \{0\}$, as desired.

For the second statement suppose that E is infinite-dimensional with basis B. Let $||\cdot||'$ be the trivial norm on E. Consider the map $||\cdot|| : E \to [0, \infty)$ where $||x||$ is defined as the number of nonzero coefficients when we write x as a linear combination of elements of B. Then $||\cdot||$ is a norm on E that is equivalent to $||\cdot||'$, but the quotient $||x||/||x||'$ is unbounded on $E \setminus \{0\}$. ■

By mimicking the proof of Theorem 1.3.11 we obtain the following generalization:

2.1.7 Theorem. *Let E be a Banach space over a non-Archimedean valued field K. If $(x_n)_n$ is a sequence of elements in E, then*

$$\sum_{n=1}^{\infty} x_n \text{ is convergent in } E \Leftrightarrow \lim_{n\to\infty} x_n = 0.$$

In fact, more is true:

2.1.8 Theorem. *Let E be a normed space over a non-Archimedean valued field K. Then E is a Banach space if and only if for each sequence $(x_n)_n$ on E that is convergent to 0, the series $\sum_{n=1}^{\infty} x_n$ is convergent in E.*

Proof. One side of the equivalence is a consequence of 2.1.7. Now suppose that for each null sequence $(x_n)_n$ on E, the series $\sum_{n=1}^{\infty} x_n$ is convergent in E. Let $(s_n)_n$ be a Cauchy sequence in E. Let $m_1 \in \mathbb{N}$ such that $||s_m - s_{m_1}|| \leq 1$ for all $m \geq m_1$. Define inductively $m_2, m_3, \ldots$ such that $m_n < m_{n+1}$ and $||s_m - s_{m_n}|| \leq \frac{1}{n}$ for all

$m \geq m_n$. Thus the sequence $x_n := s_{m_{n+1}} - s_{m_n}$ is convergent to 0 and hence $\sum_{n=1}^{\infty} x_n$ is convergent by assumption. Since

$$s_{m_n} = s_{m_1} + (s_{m_2} - s_{m_1}) + \cdots + (s_{m_n} - s_{m_{n-1}}) = s_{m_1} + \sum_{i=1}^{n-1} x_i,$$

it follows that $(s_{m_n})_n$ is a convergent subsequence of $(s_m)_m$. Hence the latter is convergent on E. ■

2.2 Linear Operators

2.2.1 Theorem. *Let E and F be normed spaces over a non-Archimedean valued field K with a nontrivial valuation and let $T : E \to F$ be a linear operator. The following statements are equivalent:*

(a) T is continuous.

(b) T is continuous at 0.

(c) There exists $M \in \mathbb{R}^+$ such that $||T(x)|| \leq M||x||$ for all $x \in E$.

(d) The set $\{||T(x)|| : ||x|| \leq 1\}$ is bounded.

Proof. The implications (a) $\Longrightarrow$ (b), (c) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (d) are trivial. The implication (b) $\Longrightarrow$ (c) is proved in [32, 2.1.4] and the implication (d) $\Longrightarrow$ (c) is proved in [29, 3.3 Lemma 1] with the following correction: where it says "$\frac{1}{|\alpha|^{n+1}} < ||z|| \leq \frac{1}{|\alpha|^n}$" it should instead say $\frac{1}{|\alpha|^{n-1}} < ||z|| \leq \frac{1}{|\alpha|^n}$. ■

2.2.2 Definition. *Let E and F be normed vector spaces over K. The vector space of all continuous linear maps $T : E \to F$ will be denoted by $L(E, F)$. We write $E' := L(E, K)$ and $L(E) := L(E, E)$. The space E' is called the **dual space** of E.*

2.2.3 Definition. *Let E and F be normed vector spaces over a non-Archimedean valued field with a nontrivial valuation. The map $||\cdot|| : L(E,F) \to \mathbb{R}$ defined by*

$$||T|| := \inf\{c \in [0,\infty) : ||T(x)|| \leq c||x|| \text{ for all } x \in E\};$$

is a norm on $L(E,F)$.

2.2.4 Theorem. *Let E and F be normed vector spaces over a non-Archimedean valued field with a nontrivial valuation. If $E \neq \{0\}$ and $T \in L(E,F)$ then:*

(a) $||T||_1 := \sup\left\{\dfrac{||T(x)||}{||x||} : x \neq 0\right\} = ||T||.$

(b) $||T||_2 := \sup\left\{\dfrac{||T(x)||}{||x||} : 0 < ||x|| \leq 1\right\} = ||T||.$

Proof. (a): As in the classical case, $||T(x)|| \leq ||T||_1||x||$ for all $x \in E$. Thus by definition of $||\cdot||$ it follows that $||T|| \leq ||T||_1$. On the other hand, for all $\varepsilon > 0$ we have that $||T(x)|| \leq (||T|| + \varepsilon)||x||$ for all $x \in E$. Hence $||T||_1 \leq ||T||$.

(b): By (a) it is enough to show that $||T||_2 = ||T||_1$. By their definition it is immediate that $||T||_2 \leq ||T||_1$. For the converse, let $x \in E \setminus \{0\}$. By 2.1.4 there exists $\alpha \in K^*$ such that $||\alpha x|| \leq 1$. Then

$$\frac{||T(x)||}{||x||} = \frac{||T(\alpha x)||}{||\alpha x||} \leq ||T||_2.$$

Therefore $||T||_1 \leq ||T||_2$. ∎

2.2.5 Theorem. *Let E and F be normed vector spaces over a non-Archimedean valued field K with a nontrivial valuation. If $||E|| := \{||e|| : e \in E\} = \{|\lambda| : \lambda \in K\} := |K|$ and $T \in L(E,F)$ then:*

$$||T||_3 := \sup\{||T(x)|| : ||x|| = 1\} = ||T||.$$

Proof. It is clear that $||T||_3 \leq ||T||_1 = ||T||$. To prove the converse, let $x \in E \setminus \{0\}$. By hypothesis there exists $\alpha \in K^*$ such that $||\alpha x|| = 1$. Hence

$$\frac{||T(x)||}{||x||} = \frac{||T(\alpha x)||}{||\alpha x||} = ||T(\alpha x)|| \leq ||T||_3.$$

Therefore $||T|| = ||T||_1 \leq ||T||_3$. Thus, we have proved that $||T|| = ||T||_3$. ∎

2.2.6 Definition. *Let E and F be normed vector spaces over a non-Archimedean valued field with nontrivial valuation. The map $||\cdot||_4 : L(E,F) \to [0,\infty)$ defined by $||T||_4 := \sup\{||T(x)|| : ||x|| \leq 1\}$, is a norm on $L(E,F)$.*

In Theorem 2.2.1 we can see that for a given linear operator $T : E \to F$, $||T||$ exists if and only if $||T||_4$ does. Unlike the classical case, these norms do not always coincide but still they are equivalent on $L(E,F)$ when K is not trivially valued. Their explicit relation is characterized in the following:

2.2.7 Proposition. *Let E and F be normed vector spaces over a non-Archimedean valued field K with a nontrivial valuation $|\cdot|$. Let $T : E \to F$ be a linear map. The following statements hold.*

(a) *T is continuous $\Leftrightarrow ||T|| < \infty \Leftrightarrow ||T||_4 < \infty$.*

(b) *$||T(x)|| \leq ||T||\,||x||$ and $||T(x)|| \leq ||T||_4\,|\alpha|\,||x||$ for all $T \in L(E,F)$, $x \in E$ and all $\alpha \in K$ such that $|\alpha| > 1$.*

(c) *$|\pi|\,||T|| \leq ||T||_4 \leq ||T||$ for all $T \in L(E,F)$ and all $\pi \in K^*$ such that $|\pi| < 1$, i.e. $||\cdot||$ and $||\cdot||_4$ are equivalent.*

(d) *If $|\cdot|$ is a dense valuation on K, then $||T|| = ||T||_4$ for all $T \in L(E,F)$.*

(e) *If $||E|| = |K|$, then $||T|| = ||T||_4$ for all $T \in L(E,F)$.*

Proof. (a): It follows from 2.2.1.

(b): For the first inequality it is enough to notice that for $x \in E \setminus \{0\}$, $\frac{||T(x)||}{||x||} \leq ||T||_1 = ||T||$. For the second inequality let $x \in E \setminus \{0\}$. Since 2.1.4, for any given $\pi \in K$, $0 < |\pi| < 1$, there exists $n \in \mathbb{Z}$ such that $|\pi| < ||\pi^n x|| \leq 1$. Thus $||T(\pi^n x)|| \leq ||T||_4$. Hence

$$\frac{||T(x)||}{||x||} = \frac{||T(\pi^n x)||}{||\pi^n x||} \leq \frac{||T||_4}{||\pi^n x||} \leq ||T||_4 |\pi|^{-1}.$$

(c): Let $T \in L(E, F)$ and $\pi \in K$ such that $0 < |\pi| < 1$. The relation $|\pi|\, ||T|| \leq ||T||_4$ follows from the part (b). Now let's show that $||T||_4 \leq ||T||$. If $T = 0$, then $||T||_4 = ||T|| = 0$. Suppose that $E \neq \{0\}$ and $T \neq 0$. For $x \in E$ such that $0 < ||x|| \leq 1$ we have that $||T(x)|| \leq ||T(x)||/||x||$. Hence, by 2.2.4, $||T||_4 \leq ||T||_2 = ||T||$.

(d): It is enough to consider a sequence $(\alpha_n)_n$ in K such that $|\alpha_1| < |\alpha_2| < \cdots < 1$ and $\lim_{n\to\infty} |\alpha_n| = 1$. Then the result follows from (c).

(e): If $||E|| = |K|$, then by 2.2.5, $||T|| = ||T||_3 \leq ||T||_4$. The converse inequality follows from (c). ■

2.2.8 Remark. *When the valuation $|\cdot|$ on K is nontrivial and discrete, the norms $||\ ||$ and $||\ ||_4$ may differ. For instance, consider $E = \mathbb{Q}_p \times \mathbb{Q}_p$ as a $\mathbb{Q}_p$-vector space with the norm $||(x, y)|| := \max\{2|x|_p, 2|y|_p\}$ and let $T : E \to E$ be such that $T(x, y) = (x, y)$. Then T is continuous and $||T|| = 1$ but $||T||_4 = \sup\{||T(x, y)|| : ||(x, y)|| \leq 1\} = \sup\{||(x, y)|| : ||(x, y)|| \leq 1\} = 2e^{-1}$. Here we can verify that 2.2.7(c) is satisfied: $e^{-n}\, ||T|| < ||T||_4 < ||T||$ for all $n \in \mathbb{N}$. Notice that in this case, $||E|| \cap |K| = \{0\}$.*

2.2.9 Remark. *All the theorems presented in this section so far (2.2.1, 2.2.4, 2.2.5 and 2.2.7) fail to be true when the valuation $|\cdot|$ on K is the trivial one. For instance, consider a field K with the trivial valuation and put $E = (K[x], ||\cdot||_a)$ and $F =$*

$(K[x], ||\cdot||_b)$ where $||\cdot||_a$ is the trivial norm on $K[x]$ and $||\cdot||_b$ is defined by

$$\left|\left|\sum_{n=0}^{N}\alpha_n x^n\right|\right|_b := \begin{cases} N+1 & \text{if } \alpha_N \neq 0 \\ 0 & \text{if } \alpha_n = 0 \text{ for all } n. \end{cases}$$

The identity map $I : E \to F$ is continuous but none of the expressions $||I||$ and $||I||_i$, for $i \in \{1, \dots, 4\}$, is well-defined. Indeed, $\frac{||x^n||_b}{||x^n||_a} = n+1$ for all $n \in \mathbb{N}$. For another example, consider a field K of characteristic 0 with the trivial valuation and let $E = (K[x], ||\cdot||_c)$ where $||\cdot||_c$ is defined by

$$\left|\left|\sum_{n=0}^{N}\alpha_n x^n\right|\right|_c := \begin{cases} e^N & \text{if } \alpha_N \neq 0 \\ 0 & \text{if } \alpha_n = 0 \text{ for all } n. \end{cases}$$

Let $T : E \to E$ be the differential operator

$$T\left(\sum_{n=0}^{N}\alpha_n x^n\right) := \begin{cases} \sum_{n=0}^{N} n\alpha_n x^{n-1} & \text{if } N \geq 1 \\ 0 & \text{if } N = 0. \end{cases}$$

Then $||T|| = ||T||_1 = e$ but $||T||_2 = ||T||_3 = ||T||_4 = 0$ and hence $||\cdot||_2, ||\cdot||_3$ and $||\cdot||_4$ are no longer norms.

2.2.10 Theorem ([45, 3.M])**.** *Let E, F be normed vector spaces.*

(a) If F is Banach then $L(E, F)$ is Banach.

(b) If E is a Banach space and D is a closed subspace of E then E/D is a Banach space.

2.2.11 Example ([45, 3.Q])**.** *In complex functional analysis the dual space of c_0 is isomorphic to ℓ^1. In the non-Archimedean theory the situation is radically different:*

For $x \in c_0$, $y \in \ell^\infty$, say $x = (\lambda_1, \lambda_2, \dots), y = (\mu_1, \mu_2, \dots)$, put

$$\langle x, y \rangle := \sum_{n=1}^{\infty} \lambda_n \mu_n.$$

For each $y \in \ell^\infty$ define $Ty \in c_0'$ by $(Ty)(x) := \langle x, y \rangle$ for $x \in c_0$. It can be proved that T maps ℓ^∞ linearly and isometrically onto c_0'.

The following three theorems are analogous to theorems in the Archimedean case and the same techniques are used in their proofs.

2.2.12 Theorem. *([45, 3.5] Closed Graph Theorem) Let T be a linear map from a Banach space E into a Banach space F such that its graph $\{(x, Tx) : x \in E\}$ is a closed subset of $E \times F$. Then T is continuous.*

2.2.13 Theorem. *([45, 3.11] Open Mapping Theorem) If E and F are Banach spaces and if $T \in L(E, F)$ is surjective, then the image under T of any open subset of E is open in F.*

2.2.14 Theorem. *([45, 3.12] Uniform Boundedness Theorem). Let E be a Banach space and F a normed vector space. If S is a subset of $L(E, F)$ such that for every $x \in E$ the set $\{Tx : T \in S\}$ is bounded in F, then S is a bounded set in $L(E, F)$.*

2.3 Orthogonality

An **inner product** in a vector space E over $\mathbb{C}$ is a map $(\cdot, \cdot) : E \times E \to \mathbb{C}$ that satisfies:

(i) $x \mapsto (x, y)$ is linear in x, for each y,

(ii) $(x, y) = \overline{(y, x)}$,

(iii) $(x, x) > 0$ whenever $x \neq 0$.

In the non-Archimedean case, we replace complex conjugation by a field automorphism:

2.3.1 Definition. *Let E be a vector space over K and consider a field automorphism $\lambda \mapsto \lambda^*$ such that $\lambda^{**} = \lambda$ and $|\lambda^*| = |\lambda|$, for all $\lambda \in K$. (We allow * to be the identity).*

*A map $(\cdot,\cdot) : E \times E \to K$ is an **inner product** (with respect to *) if*

(a) $x \mapsto (x, y)$ *is linear, for each* $y \in E$*,*

(b) $(x, y) = (y, x)^*$ *for all* $x, y \in E$*,*

(c) $(x, x) \neq 0$ *whenever* $x \neq 0$*.*

If the field K is ordered then we replace the condition (c) with the following:

(c') $(x, x) > 0$ *whenever* $x \neq 0$*.*

2.3.2 Remark. *This definition of inner product for arbitrary valued fields is a natural generalization of the classical case. In a vector space over $\mathbb{C}$, any form $(\cdot,\cdot)$ satisfying (i), (ii) and (c) is either an inner product or $-(\cdot,\cdot)$ is an inner product.*

If E is a vector space over K with inner product $(\cdot,\cdot)$, then the map $||\cdot|| : E \to \mathbb{R}$ defined by $||x|| := \sqrt{|(x, x)|}$, for $x \in E$, is a norm on E.

Apparently we can have Hilbert spaces as we do in the classical way, but the next result shows that we cannot proceed with the theory like in the classical case.

2.3.3 Theorem ([32, 2.4.5])**.** *Let $(\cdot,\cdot)$ be an inner product on a Banach space E over K such that $|(x, x)| = ||x||^2$ for all $x \in E$. Suppose that for each closed subspace D of E, there exists a subspace F of E such that $E = D + F$ and $(x, y) = 0$ for all $x \in D$ and $y \in F$. Then* $\dim E < \infty$*.*

An alternative to inner products, is a powerful non-Archimedean concept, valid in any normed space:

2.3.4 Definition. *Let x, y be elements of a normed space E over K. We say that the vector x is* ***(norm-)orthogonal*** *to y and write $x \perp y$ if*

$$||x|| = \min\{||x - \lambda y|| : \lambda \in K\} = dist(x, Ky).$$

2.3.5 Theorem ([32, 2.2.1, 2.2.3])**.** *Let E be a normed space and $x, y \in E$.*

(a) If $||x - y|| \geq ||x||$ then $||x - y|| \geq ||y||$. (van Rooij Principle)

(b) If $x \perp y$ then $y \perp x$. (Symmetry)

(c) $x \perp y$ if and only if for every $\lambda, \mu \in K$, $||\lambda x + \mu y|| = \max\{||\lambda x||, ||\mu y||\}$.

2.3.6 Remark. *In classical analysis the relation $\perp$ behaves differently than its counterpart in non-Archimedean analysis. In classical analysis the relation $\perp$ is symmetric only on spaces where the norm is induced by an inner product ([6, Theorem 4.6]). In that case, $x \perp y$ is equivalent to $(x, y) = 0$. This equivalence makes norm-orthogonality a natural alternative to (form-)orthogonality in the non-Archimedean context. Another difference is illustrated in the following example.*

2.3.7 Example. *Let $(K, |\cdot|)$ be a non-Archimedean valued field and let K^2 be normed by $(\lambda_1, \lambda_2) \mapsto \max\{|\lambda_1|, |\lambda_2|\}$. The set of vectors $\{(1,0), (0,1), (1,1)\}$ is pairwise orthogonal.*

2.3.8 Lemma ([37, 2.C])**.** *Let E be a normed space.*

(a) The relation $\not\perp$ (not orthogonal) is an equivalence relation on $E \setminus \{0\}$.

(b) If $x \perp y$ and $y \not\perp z$ then $x \perp z$.

(c) If $T \subset E$ is a set of pairwise orthogonal vectors and there are $x \in T$ and $z \in E$ such that $x \not\perp z$, then $z \perp y$ for all $y \in T \setminus \{x\}$.

Another difference with classical orthogonality is demonstrated in the following.

2.3.9 Theorem ([32, 2.2.9]). *(Perturbation theorem) Let E be a normed space. If $\{e_n : n \in \mathbb{N}\} \subset E$ is pairwise orthogonal and $f_1, f_2, \cdots \in E$ are such that $||f_n - e_n|| < ||e_n||$ for all n, then $\{f_n : n \in \mathbb{N}\}$ is pairwise orthogonal.*

2.3.10 Definition. *Let E be a normed space.*

*(a) For two sets $X_1, X_2 \subset E$ we say that X_1 and X_2 are **(norm-)orthogonal**, and write $X_1 \perp X_2$, if $x_1 \perp x_2$ for all $x_1 \in X_1, x_2 \in X_2$. For $x \in E$ we write $x \perp X_2$ instead of $\{x\} \perp X_2$.*

*(b) For two subspaces $D_1, D_2 \subset E$ we say that D_2 is an **(norm-)orthogonal complement** of D_1 if $D_1 \perp D_2$ and $E = D_1 + D_2$. A subspace D is called **(norm-) orthocomplemented** if it has an orthogonal complement.*

2.3.11 Remark. *Notice that a subspace may have more than one orthogonal complement. In fact, in example 2.3.7 the subspaces $K(0,1)$ and $K(1,1)$ are both orthogonal complements of $K(1,0)$*

2.3.12 Theorem ([32, p. 24]). *Let D_1, D_2 be subspaces of E with $E = D_1 + D_2$.*

(a) D_1 and D_2 are orthogonal complements of each other if and only if

$$||d_1 + d_2|| = \max\{||d_1||, ||d_2||\}$$

for each $d_1 \in D_1$, $d_2 \in D_2$.

(b) for $x \in E$ we have $x \perp D_2$ if and only if $||x|| = dist(x, D_2)$.

(c) Orthocomplemented subspaces are closed. (The converse is not always true!)

2.3.13 Notation. *If $S \neq \emptyset$ is a subset of a normed space, then $[S]$ will denote the subspace generated by S.*

2.3.14 Definition. *A subset X of a normed space E, $0 \notin X$, is called an **(norm-) orthogonal system** if for each $x \in X$ we have $x \perp [X \setminus \{x\}]$.*

2.3.15 Theorem ([32, p. 25 and 2.2.10]).

Let E be a normed space and $X \subset E$, $0 \notin X$.

(a) X is an orthogonal system if and only if each finite subset of X is an orthogonal system.

(b) Any orthogonal system is linearly independent.

(c) Every orthogonal system is contained in a maximal one.

(d) Two maximal orthogonal systems in a normed space have the same cardinality.

2.4 Spaces of Countable type

In classical Analysis, every Hilbert space $(H, (\cdot,\cdot))$ has an orthonormal basis $\{e_\alpha : \alpha \in I\}$ i.e. each $x \in E$ can be written uniquely as $x = \sum_{\alpha\in I}(x, e_\alpha)e_\alpha$ where the series converges to x in the norm induced by the inner product $(\cdot,\cdot)$ and the set $\{\alpha \in I : (x, e_\alpha) \neq 0\}$ is countable. In particular, we have that $\overline{[\{e_\alpha : \alpha \in I\}]} = E$.

In this section we are interested in the characterization of Banach spaces E that contain a countable set X such that $\overline{[X]} = E$.

2.4.1 Definition. *A normed space E is of **countable type** if it contains a countable set X such that $\overline{[X]} = E$.*

2.4.2 Remark. *In the non-Archimedean case it is preferable to work with the concept of 'being of countable type' rather than of 'separability'. This is because if E is a separable vector space over K, then K has to be separable. Thus the concept of separability for vector spaces over K is not of use when K is not separable. However, for vector spaces over K 'being of countable type' is a property that doesn't add restrictions over the field K. Furthermore, as the next result implies, 'being of countable type' is a natural generalization of separability in the classical case.*

2.4.3 Theorem ([32, p. 28])**.** *Let E be a normed vector space over K and suppose that K is separable. Then E is of countable type if and only if E is separable.*

2.4.4 Examples. *(a) Let K be a non-Archimedean valued field. Then K^n with the supremum norm is of countable type.*

(b) The space $(c_0, ||\cdot||_\infty)$ defined in 2.1.3 is of countable type since

$$c_0 = \overline{[\{e_n : n \in \mathbb{N}\}]}$$

where $e_n(i) = \delta_{i,n}$ for $n, i \in \mathbb{N}$.

Now we present a generalization of the concept of orthogonality required to characterize the spaces of countable type.

2.4.5 Definition. *Let E be a normed space and let $t \in (0, 1]$.*

*(a) A vector $x \in E$ is **t-orthogonal** to $y \in E$ (with respect to $||\cdot||$) if*

$$dist(x, Ky) \geq t||x||.$$

*(b) Let $X \subset E$ such that $0 \notin X$. X is called a **t-orthogonal system** (with respect to $||\cdot||$) if for each $x \in X$, $y \in [X \setminus \{x\}]$, x is t-orthogonal to y.*

Notice that t-orthogonality is symmetric and that 1-orthogonality is the concept of orthogonality defined in 2.3.4.

2.4.6 Example. *The system $\{e_n : n \in \mathbb{N}\}$ of canonical vectors in $(c_0, ||\cdot||_\infty)$ is orthogonal.*

2.4.7 Lemma ([32, p. 27])**.** *Let E be a normed space, $t \in (0, 1]$ and $X \subset E$, $0 \notin X$. The following conditions are equivalent:*

(a) X is t-orthogonal;

(b) *each finite subset of X is t-orthogonal;*

(c) *for each $n \in \mathbb{Z}$, each distinct $e_1, \dots, e_n \in X$, and each $\lambda_1, \dots, \lambda_n \in K$,*

$$||\lambda_1 e_1 + \cdots + \lambda_n e_n|| \geq t \max_{1 \leq i \leq n} ||\lambda_i e_i||.$$

2.4.8 Definition. *Let E be a normed space, let $t \in (0, 1]$. A sequence $e_1, e_2, \dots \in E$ is called a **t-orthogonal basis** of E if $\{e_1, e_2, \dots\}$ is t-orthogonal and every $x \in E$ has an expansion $x = \sum_{n=1}^{\infty} \lambda_n e_n$, where $\lambda_n \in K$. The basis $e_1, e_2, \dots$ is **orthogonal** if $t = 1$.*

2.4.9 Theorem ([32, p. 30])**.** *If $e_1, e_2, \dots \in E$ is a t-orthogonal basis of E then every $x \in E$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} \lambda_n e_n$, for $\lambda_n \in K$.*

Proof. Let $x \in E$. If $x = \sum_{n=1}^{\infty} \lambda_n e_n$ then by continuity of the norm we have that:

$$||x|| = \lim_{m \to \infty} \left|\left| \sum_{n=1}^{m} \lambda_n e_n \right|\right| \geq t \lim_{m \to \infty} \max_{1 \leq n \leq m} ||\lambda_n e_n|| = t \max_{n \in \mathbb{N}} ||\lambda_n e_n||,$$

so if $x = \sum_{n=1}^{\infty} \mu_n e_n$ for certain $\mu_n \in K$, then $0 = \sum_{n=1}^{m} (\lambda_n - \mu_n) e_n$, so that $t \max_{n \in \mathbb{N}} ||(\lambda_n - \mu_n) e_n|| = 0$, i.e., $\lambda_n = \mu_n$ for all $n \in \mathbb{N}$. ∎

2.4.10 Example. *The system $\{e_n : n \in \mathbb{N}\}$ of canonical vectors in $(c_0, ||\cdot||_\infty)$ is an orthogonal basis for c_0.*

2.4.11 Theorem ([32, 2.3.7])**.** *Let E be a normed space.*

(a) *If E has a t-orthogonal basis for some $t \in (0, 1]$, then E is of countable type.*

(b) *If E is of countable type then E has a t-orthogonal basis for each $t \in (0, 1)$.*

Finally we obtain the characterization that we wanted:

2.4.12 Theorem ([32, 2.3.9])**.** (a) *Each infinite-dimensional Banach space of countable type is linearly homeomorphic to c_0.*

(b) Each normed space of countable type is linearly homeomorphic to a subspace of c_0.

Proof. (a) Suppose that E is an infinite-dimensional Banach space of countable type. Let $t \in (0,1)$. By 2.4.11 there exists a t-orthogonal basis $e_1, e_2, \ldots$ of E. Let $\pi \in K$ such that $0 < |\pi| < 1$. By using the same technique used in the claim of 2.2.1, without loss of generality we can assume that $|\pi| \leq ||e_n|| \leq 1$ for all $n \in \mathbb{N}$. Consider the map $T : c_0 \to E$ defined by

$$T(\lambda_1, \lambda_2, \ldots) = \sum_{n=1}^{\infty} \lambda_n e_n.$$

By definition T is linear and since $e_1, e_2, \ldots$ is a t-orthogonal basis of E, T is bijective. For $x = (\lambda_1, \lambda_2, \ldots) \in c_0$, we have that

$$||T(x)|| = \left|\left| \sum_{n=1}^{\infty} \lambda_n e_n \right|\right| \leq \max_{n\in\mathbb{N}} ||\lambda_n e_n|| \leq \max_{n\in\mathbb{N}} |\lambda_n| = ||x||.$$

By t-orthogonality,

$$||T(x)|| = \left|\left| \sum_{n=1}^{\infty} \lambda_n e_n \right|\right| \geq t \max_{n\in\mathbb{N}} ||\lambda_n e_n|| \geq t|\pi| \max_{n\in\mathbb{N}} |\lambda_n| = t|\pi|||x||,$$

so that T is a homeomorphism. ■

2.4.13 Remark. *This result shows that, for a given K, c_0 is (up to linear homeomorphisms) the only infinite-dimensional Banach space over K of countable type. This is in contrast to classical Analysis, where there exist separable Banach spaces without a Schauder basis as Enflo proves in [15], and where two Banach spaces that have a Schauder basis need not be linearly homeomorphic (e.g. ℓ^1 and ℓ^2).*

The next result highlights a remarkable difference between non-Archimedean and classical Analysis.

2.4.14 Theorem ([32, 2.3.13])**.** *Let E be a Banach space of countable type. Then each closed subspace D of E is* ***complemented****, i.e. there is a subspace F of E such that $E = D + F$. In fact, for every $M > 1$ there exists a continuous $P \in L(E)$ onto D such that $P^2 = P$ and $||P|| \leq M$.*

Notice that in classical Analysis this is not true. For each $p \neq 2$, the separable Banach space ℓ^p has a closed subspace that is not complemented (see [28]).

The following is an open problem asked in [32, p. 33]: Let E be a non-complete normed space of countable type. Is every closed subspace D of E complemented?

As is shown in the following result, the answer is positive.

2.4.15 ★ Proposition. *Let E be a normed space of countable type. Every closed subspace D of E complemented.*

Proof. Let E be a normed space of countable type and let D be a closed subspace of E. If the space E is complete, then the result follows from 2.4.14. Now let's suppose that E is not complete. Consider $\hat{E}$ the completion of E and $\overline{D}$ the closure of D in $\hat{E}$. First, we will show that $\hat{E}$ is of countable type. By hypothesis, there is a set $\{y_n \in E : n \in \mathbb{N}\}$ such that for each $e \in E$, and for every neighborhood $U \subset E$ of e, we have that $U \cap [\{y_n : n \in \mathbb{N}\}] \neq \emptyset$. Since E is dense in $\hat{E}$, it follows that for every $x \in \hat{E}$, and every neighborhood $V \subset \hat{E}$ of x, $V \cap E \neq \emptyset$. Let $e_0 \in V \cap E$. Then $V \cap E \subset E$ is a neighborhood of e_0 and thus $V \cap E \cap [\{y_n : n \in \mathbb{N}\}] \neq \emptyset$. Hence $V \cap [\{y_n : n \in \mathbb{N}\}] \neq \emptyset$, so $\hat{E} = \overline{[\{y_n : n \in \mathbb{N}\}]}$. Now by 2.4.14, there exists $P \in L(\hat{E})$ such that $P^2 = P$ and $P(E) = \overline{D}$. Since D is a closed subspace of E, $D = E \cap \overline{D}$ and since P is the identity on $\overline{D}$, it follows that $P|_E \in L(E)$ is a projection onto D and therefore D is complemented on E. ■

A map $P \in L(E)$ is called **projection** when $P^2 = P$. Before we finish this section let's present the existing relation between the norm of a projection and the t-orthogonality of its kernel and range denoted by $\ker(P)$ and $R(P)$ respectively.

2.4.16 ★ Proposition. *Let E be a normed space and let $P \in L(E)$ be a projection. If $||P|| \leq M$ for some $M \in \mathbb{R}$ (and hence $M \geq 1$), then x and y are M^{-1}-orthogonal for all $x \in Ker(P)$ and all $y \in R(P)$.*

Proof. Since $P = P^2$, $1 \leq ||P|| \leq M$. Let $x \in Ker(P)$ and let $y \in R(P)$. We have that $dist(y, Kx) \geq M^{-1}||y||$. Otherwise, the inequality $dist(y, Kx) < M^{-1}||y||$ implies the existence of a scalar $\beta \in K$ such that $||y - \beta x|| < M^{-1}||y||$. Notice that $y = P(y) = P(y - \beta x)$. Thus $M||y - \beta x|| < ||P(y - \beta x)||$ and hence $M < ||P||$ contradicting our hypothesis. ■

Chapter 3

Normed spaces over fields with valuations of higher rank

In this chapter the theory of X-normed spaces over non-Archimedean valued fields with valuations of higher rank is presented. We will identify the difficulties that arise when we move from the rank-1 case to the higher rank case and in the process, some new standalone results will be presented. The main contributions made in this chapter can be found in subsections 3.4.3, 3.4.4 and 3.4.7.

Although this chapter will help to present the context for the next chapter where we will study operators on c_0 over a non-Archimedean valued field with a valuation of higher rank, several results presented here will be independent of what we will do in Chapter 4; but they are included for their own merit and mathematical value as well as for the completion of the theory developed in this chapter.

We will begin this chapter with the presentation of a generalization of ultrametric spaces where the metrics take values in an arbitrary totally ordered set X. Such ultrametric spaces are called scaled spaces and will describe the metric properties of a generalized concept of normed space presented later on: X-normed spaces, where the norms take values in X. An X-normed space will have a base field with a general valuation taking values in an arbitrary ordered commutative group G, and in order to have a proper scalar multiplication for these X-normed spaces we will need an action map $G \times X \to X$ which will equip X with a G-module structure.

Next, in the pursuit of a formalization for the norm of a bounded linear operator defined in an X-normed space, we will need to study the Dedekind completion of G-modules in order to use the concept of supremum needed to define the norm of a bounded linear operator as well as the norm on a quotient space in this general context.

Then in 3.4.3, the author presents sufficient conditions to prove that the quotient space of a spherically complete X-normed space is spherically complete. After that in 3.4.4, the relations of the cofinality and coinitiality between a group G and a G-module X are studied and their consequences are presented.

Later the concept of a Banach space is defined for this general context and some related results are presented.

Subsequently in 3.4.7, I present generalized versions of the Baire category theorem and the Open Mapping Theorem suitable for this new context.

Finally, major properties of spaces of countable type are discussed and an idea to solve an open problem regarding the existence of bases is presented.

3.1 Scaled Spaces

Before we studied valued fields in Section 1.2 we studied ultrametric spaces in Section 1.1 in order to have a better understanding of the topologies involved in valued fields without getting distracted by their algebraic structure. We will do the same for X-normed spaces defined in this chapter. Thus we will briefly discuss abstract structures called scaled spaces that will describe the topology of an X-normed space.

3.1.1 Definition. *Let M be a set, let X be a totally ordered set and let 0 be a symbol* *such that* $0 < x$ *for all* $x \in X$*. An* X***-valued scale*** *on* M *is a map* $d : M \times M \to X \cup \{0\}$ *such that for all* $x, y, z \in X$*:*

(a) $d(x, y) = 0 \Leftrightarrow x = y$*,*

(b) $d(x, y) = d(y, x)$*, and*

(c) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$*.*

The space (M, X, d) *is called a* ***scaled space*** *(ultrametric space if* $X \subset (0, \infty)$*).*

3.1.2 Notation. *Let* (M, X, d) *be a scaled space and consider* $a \in M$ *and* $r \in X$*. The sets* $B_M(a, r) := B(a, r) := \{b \in M : d(b, a) < r\}$ *and* $B_M[a, r] := B[a, r] := \{b \in X : d(b, a) \leq r\}$ *are called the* ***open and closed balls of center*** a ***and radius*** r*, respectively. The family of open balls forms a basis of neighborhoods for a uniquely determined Hausdorff topology on* M*. This topology is called the* ***topology induced*** *by* d *on* M*.*

By noting that the major properties of ultrametric spaces are consequences of the strict triangle inequality independently of the values of the ultrametric, we can obtain the result below:

3.1.3 Theorem. *Let* (M, X, d) *be a scaled space. Then the following properties are satisfied:*

(a) Each point of a ball is a center of the ball.

(b) Each ball in M *is both closed and open in the topology induced by* d*.*

(c) Each ball has an empty boundary.

(d) Two balls are either disjoint, or one is contained in the other.

(e) If two balls B_1, B_2 *are disjoint, then* $dist(B_1, B_2) = d(a, b)$ *for all* $a \in B_1, b \in B_2$*.*

(f) Let $\varepsilon \in X$. For $a, b \in M$ the relation $d(a,b) < \varepsilon$ is an equivalence relation and induces a partition of M into open balls of radius ε. Analogously for $d(a,b) \leq \varepsilon$ and closed balls.

(g) Let $Y \subset X$, B a ball in X, $B \cap Y \neq \emptyset$. Then, $B \cap Y$ is a ball in Y.

(h) Let $(a_n)_n$ be a sequence in M converging to $a \in M$, then for each $b \in M \setminus \{a\}$, there exists $N \in \mathbb{N}$ such that $d(a_n, b) = d(a, b)$ for all $n \geq N$.

(i) A sequence $(a_n)_n$ in M is Cauchy if and only if $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$.

Proof. Analogous to the proof of 1.1.6. ■

3.1.4 Definition. *A **nest of balls** in a scaled space is a nonempty collection of balls that is linearly ordered by inclusion.*

In the rank 1 case the condition of spherical completeness is needed in results regarding: extension of maps like the Hahn-Banach theorem ([45, 4.10, 4.15]), fixed points ([34, 2.3]) and best approximations ([38, 21.1, 21.2]). We will see in this chapter that we do not always have a first countability condition in the set where a scale takes values and therefore the condition of Cauchy completeness is inappropriate to guarantee convergence of nets. It is natural to require stronger conditions when the context has reached this level of generality and here the concept of spherical completeness proved to be a natural condition for convergence in scaled spaces. The next definition is a generalization of the definition 1.1.7.

3.1.5 Definition. *A scaled space M is **spherically complete** if each nest of balls in M has a nonempty intersection.*

As we will see later, the next result is a useful tool to work with spherically complete scaled and normed spaces in the higher-rank case, and allows a natural transition from induction on $\mathbb{N}$ to transfinite induction.

3.1.6 ★ Proposition. *A scaled space M is spherically complete if and only if every nest of balls in M indexed by a limit ordinal has a nonempty intersection.*

Proof. If M is spherically complete then every nest of balls in M has a nonempty intersection.

To prove the other implication suppose that every nest of balls indexed by a limit ordinal has a nonempty intersection. Let $\{B_i : i \in I\}$ be a nest of balls in M, i.e. $I \neq \emptyset$ is a totally ordered set such that for each $i \in I$, B_i is a ball in M and for $i, j \in I$, $i < j$ implies $B_j \subset B_i$. If I has a maximum, say k, then $\cap_{i \in I} B_i = B_k \neq \emptyset$. Now suppose that I does not have a maximum. It is enough to prove the following claim.

Claim: There exist a limit ordinal β and a set $\{i_\alpha \in I : \alpha < \beta\}$ that is cofinal on I.

In fact, if the claim is true, then by hypothesis $\emptyset \neq \cap_{\alpha < \beta} B_{i_\alpha} = \cap_{i \in I} B_i$. Let's prove the claim. Since $I \neq \emptyset$, there exists $i_0 \in I$. Since $i_0 \neq \max I$, we have that $\{i \in I : i > i_0\} \neq \emptyset$. So we can choose some $i_1 \in I$ such that $i_1 > i_0$. Suppose that for an ordinal δ, for every $\alpha < \delta$ we choose i_α such that $\{i_\alpha \in I : \alpha < \delta\}$ is strictly increasing. If $\{i_\alpha \in I : \alpha < \delta\}$ is cofinal on I, then we are done. Otherwise, we can choose i_δ from $\{i \in I : i > i_\alpha \text{ for all } \alpha < \delta\}$, since it is nonempty. Because of the axiom of choice, the process has to terminate, otherwise we found an injection from the proper class of all the ordinals into a set. ■

The concept of spherical completeness for scaled spaces offers a result regarding best approximations similar to [38, 21.1, 21.2] in ultrametric spaces. Specifically, we have the following:

3.1.7 Theorem ([30, 1.2.3])**.** *Let (M, X, d) be a scaled space and let $V \subset M$ be a spherically complete subspace. Then each $x \in M$ has a best approximation in V i.e.* $\min\{d(x, v) : v \in V\}$ *exists.*

The next result characterizes those scaled spaces that can be considered as ultrametric spaces:

3.1.8 Theorem ([30, 1.2.4]). *Let (M, X, d) be a scaled space. The following statements are equivalent:*

*(a) M is **ultrametrizable**, i.e. the topology on M induced by d is also induced by some ultrametric $d' : M \times M \to \mathbb{R}$.*

(b) M is discrete or there exist $s_1 > s_2 > \dots$ in X such that $\lim_n s_n = 0$.

3.2 Valued fields with valuations of higher rank

In 1.8.1 we defined Krull valuations with values in an additive ordered abelian group $(G, +)$. In this chapter we will define normed spaces over fields with Krull valuations for which it is convenient to consider G as a multiplicative group $(G, \cdot)$ instead. While an additive group $(G, +)$ satisfying $a + b = b + a$ for all $a, b \in G$, is said to be **abelian**, a multiplicative group $(G, \cdot)$ satisfying $ab = ba$ for all $a, b \in G$, is said to be **commutative**. Once the terms abelian or commutative are used, then the adjectives additive and multiplicative will be omitted. Hence, the definition of Krull valuation is rephrased as follows: given a field K and an arbitrary ordered commutative group G, a **Krull valuation** on K is a map $| \cdot | : K \mapsto G \cup \{0\}$ satisfying:

(a) $| \cdot |$ is onto,

(b) $|x| = 0$ iff $x = 0$,

(c) $|x + y| \leq max\{|x|, |y|\}$,

(d) $|xy| = |x||y|$.

where 0 is a symbol such that $0 < g$ and $0 = 0g = g0 = 00$ for all $g \in G$. With this new definition, the map $|\cdot|$ is a Krull valuation if and only if the map $v(x) = |x|^{-1}$

is a valuation defined as in 1.8.1 but using a multiplicative group instead and putting $0^{-1} = \infty$. Notice that the map $d : K \times K \to G \cup \{0\}$, $d(x, y) = |x - y|$ is a scale on K. The topology induced by this scale coincides with the valuation topology induced by v (see 1.8.3). We will avoid the case when $| \cdot |$ induces the discrete topology on K, this is when $| \cdot |$ is the trivial valuation, i.e. $G = \{1\}$.

In this new setting, the general Hahn field defined in 1.4.9 is adjusted as follows:

3.2.1 Definition. *Given a field K and an arbitrary ordered commutative group G, the general Hahn field $K((G))$ is the field of all $\lambda \in K^G$, for which $supp(\lambda)$ is dually well-ordered in the order of G (i.e. each non-empty subset of $supp(\lambda)$ has a largest element), equipped with pointwise addition, i.e. $(\lambda_1 + \lambda_2)(g) = \lambda_1(g) + \lambda_2(g)$ for all $g \in G$, and with the multiplication defined by $(\lambda_1\lambda_2)(g) = \sum_{fh=g} \lambda_1(f)\lambda_2(h)$ for all $g \in G$.*

The Hahn valuation defined on $K((G))$ (1.8.6.c) now is presented as the map $| \cdot | : K((G)) \to G \cup \{\infty\}$ defined by

$$|\lambda| := \begin{cases} \max\{supp(\lambda)\} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$$

In 1.8.1 we said that G has rank 1 when it can be embedded in $(\mathbb{R}, +)$. Since now we are using multiplicative groups, we will say that G has rank 1 when it can be embedded in $((0, \infty), \cdot)$ and that it has higher rank otherwise. It is necessary to give a precise definition for the rank of G when its rank is not 1.

3.2.2 Definition. *A subset H of a totally ordered group G is* ***convex*** *if $x, y \in H$, $z \in G$, $x \leq z \leq y$ implies $z \in H$. Each proper convex subgroup is bounded from below and from above. The set of convex subgroups is totally ordered by inclusion. A convex subgroup H is called* ***principal*** *if there is an $a \in G$ such that H is the smallest convex subgroup of G containing a. The order type of the set of all principal* *nontrivial subgroups is called the **rank** of G.*

3.2.3 Examples. *(a) Let $F := \{e^n \in \mathbb{R} : n \in \mathbb{Z}\}$ the cyclic group generated by the real number e (the group operation is the multiplication in $\mathbb{R}$). The commutative group $G := \{(g_n)_n \in \prod_{n=1}^{\infty} F : \{n : g_n \neq 1\}$ is finite$\}$ with the anti-lexicographic order (see 3.3.3.b) has rank ω (the first infinite ordinal) and the nontrivial principal subgroups of G are of the form $H_n := \prod_{k=1}^{n} F \times \prod_{k=n+1}^{\infty} \{1\}$, for $n \in \mathbb{N}$.*

(b) Given $n \in \mathbb{N}$, $\mathbb{Z}^n$ has rank n.

3.2.4 Notation. *From now on, G will be a totally ordered commutative group. If $|\cdot| : K \to G \cup \{0\}$ is a Krull valuation, then we will say that G is the value group of K.*

3.2.5 Definition. *Let $(K, |\cdot|)$ be a valued field with value group G. The valued field $(K, |\cdot|)$ is **Cauchy complete** when each Cauchy net of K is convergent in K.*

3.2.6 Theorem ([30, 1.4.1])**.** *Let $(K, |\cdot|)$ be a valued field with value group G. The following statements are equivalent:*

(a) $(K, |\cdot|)$ is metrizable.

(b) $(K, |\cdot|)$ is ultrametrizable.

(c) G has a coinitial sequence $(g_n)_{n\in\mathbb{N}}$, i.e. for each $g \in G$, there exists n such that $g_n \leq g$.

(d) G has a cofinal sequence $(g_n)_{n\in\mathbb{N}}$, i.e. for each $g \in G$, there exists n such that $g \leq g_n$.

(e) $K \setminus \{0\}$ contains a countable set C for which $0 \in \overline{C}$.

(f) K contains a countable subset that is not closed.

3.3 G-modules

In the new setting, a norm can take values in an ordered set X in which a group G acts. Let's define the range of this generalized concept of norm.

3.3.1 Definition. *Let G be a totally ordered commutative group. A totally ordered set X is called a G-**module** if there exists a map $G \times X \to X$, written $(g, x) \mapsto gx$, such that for all $g, g_1, g_2 \in G$ and all $x, x_1, x_2 \in X$ we have:*

(a) $g_1(g_2x) = (g_1g_2)x$;

(b) $1x = x$;

(c) $g_1 \leq g_2 \Rightarrow g_1x \leq g_2x$;

(d) $x_1 \leq x_2 \Rightarrow gx_1 \leq gx_2$;

(e) Gx *is coinitial in* X;

(f) X *has no smallest element.*

*The map $(g, x) \mapsto gx$ is called a G-**action** on X.*

The axioms that define a G-action on X produce the following consequences:

3.3.2 Lemma ([30, pp. 246–247])**.** *If X is a G-module, then for all $x, x_1, x_2 \in X$ and $g, g_1, g_2 \in G$:*

(c′) *the implication* $g_1 < g_2 \Rightarrow g_1x < g_2x$ *does not hold for some* G *and* X,

(d′) $x_1 < x_2 \Rightarrow gx_1 < gx_2$,

(e′) Gx *is cofinal in* X,

(f′) X *has no largest element.*

3.3.3 Examples. *(a) G is a G-module when we consider the multiplication on G as a G-action on G.*

(b) *Let β be an ordinal. For each $\alpha < \beta$, put $G_\alpha = G$. Define*

$$X := \left\{ (g_\alpha)_{\alpha<\beta} \in \prod_{\alpha<\beta} G_\alpha : \{\alpha : g_\alpha \neq 1\} \text{ is finite} \right\}$$

and consider the antilexicographic order on X, i.e. for $x = (x_\alpha)$ and $y = (y_\alpha)$ in X, $x < y$ if and only if $x \neq y$ and $1 < yx^{-1}_{\phi(yx^{-1})}$ where $yx^{-1} = (y_\alpha x_\alpha^{-1})_{\alpha<\beta} \in X$ and $\phi(x) = \max\{\alpha : x_\alpha \neq 1\}$ (compare with the order defined in 1.4.10).

With the action of G on X defined by $gx := (gx_\alpha)_{\alpha<\beta}$ for all $g \in G$ and $x = (x_\alpha)_{\alpha<\beta}$ in X, we have that X is a G-module.

(c) *Let β be an ordinal. For each $\alpha < \beta$, let $G_\alpha := \{g_\alpha : g \in G\}$ be a copy of G such that $G_0 = G$ and $G_{\alpha_1} \cap G_{\alpha_2} = \emptyset$ for $\alpha_1 < \alpha_2 < \beta$. Let*

$$X := \bigcup_{\alpha<\beta} G_\alpha$$

be ordered by stating that for all $s, t \in G$, such that $t < s$, we put $t < s_{\alpha_2} < s_{\alpha_1} < s$ for $0 < \alpha_1 < \alpha_2 < \beta$. Then X becomes a G-module by extending the multiplication of G by $gs_\alpha := (gs)_\alpha$.

3.3.4 Notation. *From now on K will be a field with Krull valuation $|\cdot| : K \to G \cup \{0_G\}$, where G is a totally ordered commutative group and 0_G is an element adjoined to G satisfying: $0_G = 0_G g = g0_G = 0_G 0_G$ for all $g \in G$. Additionally, to each G-module X we adjoin an element 0_X for which, $0_X = 0_G 0_X = 0_G x < x$ for each $x \in X$. However, we will denote 0_G and 0_X by 0.*

3.3.5 Definition. *Let E be a vector space over $(K, |\cdot|)$ and let X be a G-module. An X**-norm** on E is a map $||\cdot|| : E \to X \cup \{0\}$ such that for all $x, y \in E$, $\lambda \in K$:*

(a) $||x|| = 0 \Leftrightarrow x = 0$;

(b) $||\lambda x|| = |\lambda|||x||$;

(c) $||x+y|| \leq \max\{||x||, ||y||\}$.

The space $(E, ||\cdot||)$ *is called an* X***-normed space****.*

Some examples of X-normed spaces using the G-modules presented in 3.3.3:

3.3.6 Examples. *Let* K *be a field with Krull valuation* $|\cdot| : K \to G \cup \{0\}$.

(a) *The space* $c_0 := \left\{(x_n)_n \in \prod_{n\in\mathbb{N}} K : \lim_n x_n = 0\right\}$ *with the norm* $||(x_n)_n|| := \max\{|x_n| : n \in \mathbb{N}\}$ *is an* X*-normed space where* $X = G$*. Here, for* $(x_n)_n \in \prod_{n\in\mathbb{N}} K$, $\lim_n x_n = 0$ *if and only if for all* $g \in G$*, there exists* $N \in \mathbb{N}$ *such that* $n \geq N \Rightarrow |x_n| < g$.

(b) *Let* β *be an ordinal and let* X *be the* G*-module of 3.3.3.b. The space* $c_{00} := \left\{(x_\alpha)_\alpha \in \prod_{\alpha<\beta} K : \{\alpha : x_\alpha \neq 0\} \text{ is finite}\right\}$ *with the norm* $||(x_\alpha)_\alpha|| := (|x_\alpha|)_{\alpha<\beta}$ *is an* X*-normed space.*

(c) *Consider* $K = \mathbb{R}((x))$ *with the valuation* $|\cdot|$ *defined after 1.3.10. Hence* $G = \{e^n \in \mathbb{R} : n \in \mathbb{Z}\}$*. Let* β *be an ordinal and let* X *be the* G*-module of 3.3.3.c. Define* $\mathbb{I}$ *as the set of all the finite sets of ordinals smaller than* β*. The space* $K(Y_\alpha : \alpha < \beta) := \{\sum_{\alpha\in J} \lambda_\alpha Y_\alpha^{n_\alpha} : J \in \mathbb{I}, \lambda_\alpha \in K, n_\alpha \in \mathbb{N}\}$ *is an* X*-normed space with the norm defined as* $||\sum_{\alpha\in J} \lambda_\alpha Y_\alpha^{n_\alpha}|| := ||\lambda_\delta Y_\delta^{n_\delta}|| := |\lambda_\delta| e_\alpha^{n_\delta}$ *where* $\delta = \max(J)$.

In the rank-1 case, the topology induced by a norm is a vector topology, i.e. the addition of vectors and the scalar multiplication are continuous maps with respect to the topology induced by the norm. In the result below we verify that these facts are satisfied by every X-normed space among other properties.

3.3.7 Theorem. *Let* $(E, ||\cdot||)$ *be an* X*-normed space where* X *is a* G*-module.*

(a) *the map* $d : E \times E \to X \cup \{0\}$, $d(e, f) := ||e - f||$ *is an* X*-valued scale.*

(b) *For every* $x \in E \setminus \{0\}$*, the norm is constant on the ball* $B(x, ||x||)$.

(c) *The norm* $||\cdot|| : E \to X \cup \{0\}$ *is continuous with respect to the topology induced by* d.

(d) *The addition map* $+ : E \times E \to E$ *is continuous.*

(e) *The scalar multiplication* $\cdot : K \times E \to E$ *is continuous.*

(f) *For all* $\lambda \in K \setminus \{0\}$, $T : E \to E$, $T(e) = \lambda e$ *is a homeomorphism.*

(g) *For any subset* $A \subset E$ *and any* $\lambda \in K$, $\lambda\overline{A} = \overline{\lambda A}$.

Proof. (a): Analogous to the standard proof of a norm inducing a metric.

(b): For any $y \in B(x, ||x||)$, we have that $||y - x|| < ||x||$. Thus by the strong triangular inequality of the norm, it follows that $||x|| = ||y||$.

(c): Let $(e_i)_{i\in I}$ be a net on E convergent to $e \in E$. If $e = 0$, this means that for all $s \in X$, there exists $j \in I$ such that $i \geq j \Rightarrow ||e_i|| < s$. Thus the net $(||e_i||)_{i\in I}$ on X is convergent to $0 = ||e||$. Now, if $e \neq 0$, there exists $j \in I$ such that $i \geq j \Rightarrow ||e_i - e|| < ||e||$. By 3.3.7.b, $||e_i|| = ||e||$ for $i \geq j$. Therefore $(||e_i||)_{i\in I}$ is convergent to $||e||$.

(d): For all $x, y, x', y' \in E$, we have that $||(x + y) - (x' + y')|| \leq \max\{||x - x'||, ||y - y'||\}$. Thus $+(B(x, s) \times B(y, s)) \subset B(x + y, s)$ for all $s \in X$.

(e): Let $(\lambda_i, e_i)_{i\in I}$ be a net on $K \times E$ convergent to $(\lambda, e) \in K \times E$. $||\lambda_i e_i - \lambda e|| = ||\lambda_i e_i - \lambda e_i + \lambda e_i - \lambda e|| \leq \max\{|\lambda_i - \lambda|\, ||e_i||, |\lambda|\, ||e_i - e||\}$. By 3.3.7.b, either $||e_i|| \to 0$ or $||e_i||$ is eventually equal to the constant $||e||$. Either way, $\lambda_i e_i \to \lambda e$.

The statements (f) and (g) have standard proofs. ∎

3.4 Continuous linear operators and Dedekind completions

In this section we will study continuous linear operators between normed spaces in the higher rank case and will develop the necessary means to conceive norms for bounded operators and for quotient spaces. Throughout this section we will let X, Y be G-modules, let E be an X-normed space and let F be a Y-normed space.

3.4.1 Theorem. *Let $T : E \to F$ be a continuous linear map. For every $t \in X$, the set $\{||Te|| : ||e|| \leq t\}$ is bounded above in Y.*

Proof. By continuity of T, for each $y \in Y$, there exists $x \in X$ such that $||Te|| \leq y$ whenever $||e|| \leq x$. Let $t \in X$. Since Gt is coinitial in X, there exists $g \in G$ such that $gt < x$. Choose $\lambda \in K$ such that $|\lambda| = g$ and let $e \in E$ be such that $||e|| \leq t$. Then $||\lambda e|| = |\lambda|\,||e|| \leq gt < x$. Hence $||T(\lambda e)|| \leq y$ and thus $||Te|| \leq g^{-1}y$. ■

A natural candidate for the norm of a continuous linear map $T : E \to F$ would be a map of the form $||\cdot||_t : L(E, F) \to Y$, $||T||_t := \sup\{||Te|| : ||e|| \leq t\}$ for $t \in X$. The problem is that the supremum may not exist in Y. To solve this problem we invoke the concept of Dedekind completion.

3.4.1 Dedekind completion of an ordered set

In order to define a norm that requires the notion of supremum or infimum we introduce the following:

3.4.2 Definition. *A totally ordered set C is called **Dedekind complete** if each non-empty subset of C that is bounded above has a supremum.*

Notice that a totally ordered set C is Dedekind complete if and only if each nonempty subset of C that is bounded below has an infimum. In the subsequent paragraphs we will study the Dedekind completion of an ordered set and verify that any ordered set can be embedded in a natural way in a Dedekind complete ordered set.

3.4.3 Notation. *Let $A \subset B \subset C$ where C is a totally ordered set. We say that $s = sup_B A$ if $s \in B$, $s \geq a$ for all $a \in A$, and if $b \in B$ satisfies $b \geq a$ for all $a \in A$, then $b \geq s$. Similarly, we define $\inf_B A$. If both $\sup_B A$ and $\sup_C A$ exist, then $\sup_B A \geq \sup_C A$, but we do not always have equality. When there is no possible confusion, we will sometimes write $\sup A$ instead of $\sup_B A$.*

The next construction mimics the construction of the ordered set of real numbers from the rational numbers set as its Dedekind completion.

3.4.4 Definition. *Let C be a totally ordered set. A subset S of C is called a* ***cut*** *if*

(a) $S \neq \emptyset$, S is bounded above;

(b) if $x \in S$, $y < x$ then $y \in S$;

(c) if $\sup S$ exists then $\sup S \in S$.

Let $C^{\#}$ be the collection of all cuts of C. With the ordering by inclusion $C^{\#}$ becomes a totally ordered set. Let's verify that $C^{\#}$ is Dedekind complete. Let $A \subset C^{\#}$ be nonempty and bounded above. There is a cut T such that $S \subset T$ for all $S \in A$. Then $V := \bigcup_{S \in A} S$ is nonempty and bounded above by T, and by adding $\sup_C V$ (if it exists) to V we obtain a cut equal to $\sup_{C^{\#}} A$. We have the natural order embedding $\varphi : C \to C^{\#}$ given by

$$\varphi(x) := \{a \in C : a \leq x\},$$

which is strictly increasing (order-preserving). Often we shall identify C and $\varphi(C)$. The ordered set $C^{\#}$ is called the ***Dedekind completion*** *of C, and it is the smallest Dedekind complete ordered set containing C in the sense that for any given Dedekind* *complete ordered set D and any order embedding $f : C \to D$, there exists a unique order embedding $\overline{f} : C^{\#} \to D$ such that $\overline{f} \circ \varphi = f$. Additionally, $C^{\#}$ is unique up to order-isomorphism. For details see ([44, 5.2]).*

In the next result we summarize the main properties that can be obtained from the intrinsic relation between an ordered set C and its Dedekind completion $C^{\#}$.

3.4.5 Proposition. *Let C be a totally ordered set with Dedekind completion $C^{\#}$. Then the following statements hold:*

(a) C is Dedekind complete if and only if $C = C^{\#}$.

(b) C is cofinal and coinitial in $C^{\#}$.

(c) For every $s \in C^{\#}$, $\{c \in C : c \leq s\}$ is a cut in C and every cut in C has this form.

(d) If $s, t \in C^{\#}$, $s < t$, then there exist $x, y \in C$ with $s \leq x < t$ and $s < y \leq t$.

(e) For each $s \in C^{\#}$, $s = \sup_{C^{\#}}\{c \in C : c \leq s\} = \inf_{C^{\#}}\{c \in C : c \geq s\}$.

(f) For each $s \in C^{\#}$, $s_0 := \max\{c \in C : c < s\}$ exists if and only if $s_1 := \max\{c \in C^{\#} : c < s\}$ exists. In that case, $s_0 = s_1$.

(g) For each $s \in C^{\#}$, $s = \sup_{C^{\#}}\{c \in C : c < s\}$ if and only if $s = \sup_{C^{\#}}\{c \in C^{\#} : c < s\}$.

(h) For each $s \in C^{\#}$, $\sup_{C^{\#}}\{c \in C : c < s\} = \sup_{C^{\#}}\{c \in C^{\#} : c < s\}$.

(i) For each $s \in C^{\#}$, $s_0 := \min\{c \in C : s < c\}$ exists if and only if $s_1 := \min\{c \in C^{\#} : s < c\}$ exists. In that case, $s_0 = s_1$.

(j) For each $s \in C^{\#}$, $s = \inf_{C^{\#}}\{c \in C : s < c\}$ if and only if $s = \inf_{C^{\#}}\{c \in C^{\#} : s < c\}$.

(k) For each $s \in C^{\#}$, $\inf_{C^{\#}}\{c \in C : s < c\} = \inf_{C^{\#}}\{c \in C^{\#} : s < c\}$.

(l) If $A \subset C$, $s = \sup_C A$, *then* $s = \sup_{C^{\#}} A$. *If* $t = \inf_C A$, *then* $t = \inf_{C^{\#}} A$.

Proof. The statements (a), (b), (c), (d), (e) and (l) are proved in [30, 1.1.4].

(f): Let $s \in C^{\#}$ and suppose that $s_0 := \max\{c \in C : c < s\}$ exists. Then $s_0 \in G$ and $s_0 \leq \sup\{c \in C^{\#} : c < s\} := s_1$. If the inequality were strict, then by 3.4.5.d, $s_1 \in G$. Thus, by the definition of supremum, there would exist $t_1, t_2 \in G^{\#}$ such that $s_0 < t_1 < t_2 < s_1$. Then by 3.4.5.d there would be a $g \in G$ such that $s_0 < t_1 \leq g < t_2 < s_1 < s$ contradicting the definition of s_0. Hence $s_0 = s_1 = \max\{c \in C^{\#} : c < s\}$. Now suppose that $s_1 := \max\{c \in C^{\#} : c < s\}$ exists. Then $s_1 < s$ and by 3.4.5.d we have that $s_1, s \in G$. Hence $s_1 = \max\{c \in C : c < s\}$.

(g): if $s = \sup_{C^{\#}}\{c \in C : c < s\}$ and $s_1 = \sup_{C^{\#}}\{c \in C^{\#} : c < s\}$ then $s \leq s_1 \leq s$. Now suppose that $s = \sup_{C^{\#}}\{c \in C^{\#} : c < s\}$ and let $s_0 := \sup_{C^{\#}}\{c \in C : c < s\}$. Then $s_0 \leq s$. If $s_0 < s$, then by definition of supremum there exist $t_1, t_2 \in G^{\#}$ such that $s_0 < t_1 < t_2 < s_1$. Then by 3.4.5.d there would be a $g \in G$ such that $s_0 < t_1 \leq g < t_2 < s_1 < s$ contradicting the definition of s_0. Therefore $s_0 = s$.

(h): the supremum $M := \sup_{C^{\#}}\{c \in C^{\#} : c < s\}$ can take only two values: either $M = \max\{c \in C^{\#} : c < s\}$ or $M = s$. In the former case, by (f) it follows that $M = \max\{c \in C : c < s\}$; in the latter case, by (g) we have that $M = \sup_{C^{\#}}\{c \in C : c < s\}$.

The proofs of (i), (j) and (k) are analogous to the ones of (f), (g) and (h) respectively. ■

3.4.2 The Dedekind completion $X^{\#}$ as a G-module

For a norm on $L(E, F)$ we need to make sense of the equality $||\lambda T|| = |\lambda|\,||T||$ for any $T \in L(E, F)$ and any $\lambda \in K$. For this end we need a G-module structure in $X^{\#}$. First, let's present the behavior of infima and suprema in a G-module with respect to its G-action.

3.4.6 Theorem ([30, 1.5.3])**.** *Let X be a G-module.*

(a) Let $V \subset X$, $g \in G$. If $\sup_X V$ exists, then $g \sup_X V = \sup_X gV$. If $\inf_X V$ exists, then $g \inf_X V = \inf_X gV$. If V is not bounded above (below), then neither is gV.

(b) Let $W \subset G$, $s \in X$. If $\sup_G W$ and $\sup_X Ws$ exist, then $\sup Ws \leq (\sup_G W)s$. If $\inf_G W$ and $\inf_X Ws$ exist, then $\inf Ws \geq (\inf_G W)s$. The set W is bounded above (below) if and only if Ws is bounded above (below).

3.4.7 Remark. *(a) To express the fact that some subset V of a G-module X is not bounded below we sometimes write $\inf V = 0$ which can be interpreted as the infimum taken in $X \cup \{0\}$.*

(b) For an example in which the inequalities in (b) above are strict see [30, 1.5.5 (c)].

The next result defines a structure of G-module on $X^\#$ when X is a G-module.

3.4.8 Theorem ([30, 1.5.4])**.** *Let X be a G-module. The G-action $G \times X \to X$ can uniquely be extended to a G-action $G \times X^\# \to X^\#$ making of $X^\#$ a G-module. Specifically, the map $(g, s) \mapsto gs := \sup\{gx : x \in X, x \leq s\}$ is the only G-action on $X^\#$ extending the one on X.*

Now we are in position to define a norm in $L(E, F)$.

3.4.9 Definition. *For each $t \in X$, the map $||\cdot||_t : L(E, F) \to Y^\# \cup \{0\}$, $||T||_t := \sup\{||Te|| : ||e|| \leq t\}$ defines a $Y^\#$-norm on $L(E, F)$ called a* ***uniform norm****.*

3.4.10 Remark. *(a) For any given $s, t \in X$, there are $g, h \in G$ such that*

$$g||\cdot||_s \leq ||\cdot||_t \leq h||\cdot||_s.$$

(b) *The induced topology on the space $L(E, F)$ is the topology of uniform convergence on bounded sets (see [30, p. 257]).*

3.4.11 Definition. *Suppose that $X = Y$ and let $Lip(E, F)$ be the space of all $T \in L(E, F)$ such that there is a $g \in G$ satisfying $||Te|| \leq g||e||$ for all $e \in E$, i.e. T is a linear* ***Lipschitz map*** *or* ***bounded map****. The map $|| \cdot || : Lip(E, F) \to G^{\#}$, $||T|| := \inf\{g \in G : ||Te|| \leq g||e|| \text{ for all } e \in E\}$ defines the* ***Lipschitz norm*** *on $Lip(E, F)$.*

3.4.12 Remark. *Unlike the $rank(G) = 1$ case, it is possible to have $Lip(E, F) \subsetneq L(E, F)$. See for example [17] and [18, XI].*

3.4.3 Quotient spaces of X-normed spaces

In this subsection we will focus on the structure of $X^{\#}$-normed space of a quotient space and study sufficient conditions to have spherically complete quotient spaces.

3.4.13 Definition. *Let E be an X-normed space where X is a G-module. Let D be a closed subspace of E and let $\pi : E \to E/D$ be the canonical map. The map $||\pi(e)|| := \inf\{||e - d|| : d \in D\}$ defines an $X^{\#}$-norm on E/D called the* ***quotient norm****.*

The following result is mentioned in [30, p. 257] with no proof.

3.4.14 Theorem. *Let X, E, D, E/D and π be as in definition 3.4.13. The following statements hold:*

(a) $||\pi(e)|| \leq ||e||$ *for all $e \in E$,*

(b) *The norm topology on E/D is equivalent to the quotient topology induced by π.*

(c) *If F is an $X^{\#}$-normed space, $T \in L(E, F)$ and $D = \ker(T)$, then the map $T_1 : E/D \to F$ defined by $T_1(\pi(e)) := T(e)$ belongs to $L(E/D, F)$ and for each*

$s, t \in X^{\#}$, with $s < t$ we have $||T||_s \leq ||T_1||_s \leq ||T||_t$. In particular, if T is Lipschitz then so is T_1 and $||T|| = ||T_1||$.

Proof. The proofs of (a) and (b) are analogous to their classical counterparts. In (c) to prove that T_1 is a well-defined continuous linear map it is enough to use the fact that $D = \ker(T)$ and the definition of the norm in E/D. Let $s, t \in X^{\#}$, with $s < t$. Let $e \in E$ be such that $||e|| \leq s$. Then by (a), $||\pi(e)|| \leq s$. Hence $||Te|| = ||T_1\pi(e)||$. Thus $\{||Te|| : ||e|| \leq s\} \subset \{||T_1\pi(e)|| : ||\pi(e)|| \leq s\}$ and therefore $||T||_s \leq ||T_1||_s$. Now suppose $||\pi(e)|| \leq s < t$. Then there exists $d \in D$ such that $||\pi(e)|| \leq ||e - d|| < t$. Since $||T_1\pi(e)|| = ||Te|| = ||T(e-d)||$, it follows that $\{||T_1\pi(e)|| : ||\pi(e)|| \leq s\} \subset \{||Te|| : ||e|| \leq t\}$ and therefore $||T_1||_s \leq ||T||_t$. Finally, suppose that T is Lipschitz. Then there exists $g \in G$ such that $||Te|| \leq g||e||$ for all $e \in E$. Hence we have that for any $d \in D$, $||T_1\pi(e)|| = ||T(e-d)|| \leq g||e-d||$ which implies that $||T_1\pi(e)|| \leq g||\pi(e)||$. Thus T_1 is Lipschitz and $\{g \in G : ||Te|| \leq g||e|| \text{ for all } e \in E\} \subset \{g \in G : ||T_1\pi(e)|| \leq g||\pi(e)|| \text{ for all } \pi(e) \in E/D\}$. Now let $g \in G$ be such that $||T_1\pi(e)|| \leq g||\pi(e)||$ for all $e \in E$. Then $||Te|| = ||T_1\pi(e)|| \leq g||\pi(e)|| \leq g||e||$ for all $e \in E$. Therefore $\{g \in G : ||Te|| \leq g||e|| \text{ for all } e \in E\} = \{g \in G : ||T_1\pi(e)|| \leq g||\pi(e)|| \text{ for all } \pi(e) \in E/D\}$ and hence $||T_1|| = ||T||$. ■

3.4.15 Definition. *Let X, E, F, D, E/D, π, T and T_1 be as in 3.4.14.c. The space F is called a **quotient** of E if $T \in L(E, F)$ can be chosen such that the map T_1 is an isometrical isomorphism. Such a T is called a **quotient map**. A quotient map $T \in L(E, F)$ is a **strict quotient map** (and F is called a **strict quotient** of E) if for all $f \in F$ we have $||f|| = \min\{||e|| : Te = f\}$.*

3.4.16 Remark. *A surjective $T \in L(E, F)$ is a quotient map if and only if for each $f \in F$ we have $||f|| = \inf\{||e|| : Te = f\}$.*

The next result characterizes the quotient maps and strict quotient maps.

3.4.17 Proposition. *Let X be a G-module, let E, F be X-normed spaces and let $\pi : E \to F$ be a linear map. Then the following statements hold.*

(a) π is a quotient map if and only if, for each $s \in X$, $\pi(B_E(0, s)) = B_F(0, s)$.

(b) π is a strict quotient map if and only if π is a quotient map and for each $s \in X$, $\pi(B_E[0, s]) = B_F[0, s]$.

(c) π is a strict quotient map if and only if, for each $s \in X$, $\pi(B_E[0, s]) = B_F[0, s]$.

Proof. The statements (a) and (b) are proved in [30, 2.2.2]. Let's prove (c). If π is a strict quotient map, then by (b), $\pi(B_E[0, s]) = B_F[0, s]$ for each $s \in X$. Now suppose that for every $s \in X$, $\pi(B_E[0, s]) = B_F[0, s]$. There are two mutually exclusive cases.

Case 1: $s_0 := \max\{x \in X : x < s\}$ exists. In this case, $B_E(0, s) = B_E[0, s_0]$ and therefore $\pi(B_E(0, s)) = \pi(B_E[0, s_0]) = B_F[0, s_0] = B_F(0, s)$.

Case 2: $s = \sup\{x \in X : x < s\}$. In this case, $B_E(0, s) = \bigcup_{x<s} B_E[0, x]$ and thus $\pi(B_E(0, s)) = \bigcup_{x<s} \pi(B_E[0, x]) = \bigcup_{x<s} B_F[0, x] = B_F(0, s)$.

Therefore by (a), π is a quotient map and by (b), π is a strict quotient map. ■

3.4.18 Corollary ([30, 2.2.3])**.** *Let X be a G-module, let E, F be X-normed spaces, let $\pi : E \to F$ be a linear map and let $f \in F$ and $s \in X$.*

(a) If π is a quotient map and $B = B_F(f, s)$ then $\pi(B_E(e, s)) = B$ for each $e \in \pi^{-1}(B)$.

(b) If π is a strict quotient map and $B = B_F[f, s]$ then $\pi(B_E[e, s]) = B$ for each $e \in \pi^{-1}(B)$.

The next result shows how strict quotients of spherically complete spaces are spherically complete while demonstrating the usefulness of our tool, Proposition 3.1.6 allowing the use of transfinite induction. Although this result was first stated in [30, 2.2.5], the proof offered there is not a clear one and the author thinks that a rigorous proof is needed. The proof offered here uses transfinite induction which is a new approach.

3.4.19 ★ Proposition. *Strict quotients of spherically complete spaces are spherically complete.*

Proof. Let X be a G-module and let E, F be X-normed spaces. Let $\pi : E \to F$ be a strict quotient map and suppose that E is spherically complete. By 3.1.6 it is enough to prove that every nest of balls in F indexed by a limit ordinal has a nonempty intersection. Let β be a limit ordinal and let $\{B_\alpha : \alpha < \beta\}$ be a collection such that for each $\alpha < \beta$, $B_\alpha = B_F[f_\alpha, s_\alpha]$ for some $f_\alpha \in F$ and $s_\alpha \in X$ satisfying $s_{\alpha_2} < s_{\alpha_1}$ and $B_{\alpha_2} \subset B_{\alpha_1}$ whenever $\alpha_1 < \alpha_2 < \beta$. By 3.4.18 $B_\alpha = \pi(B_E[a, s_\alpha])$ for all $a \in \pi^{-1}(B_\alpha)$ and for every $\alpha < \beta$. Let's introduce our hypothesis of transfinite induction. Suppose that for some $\delta < \beta$, we have a nest of balls in E, say $\{B_E[a_\alpha, s_\alpha] : \alpha < \delta\}$ such that $B_\alpha = \pi(B_E[a_\alpha, s_\alpha])$. Let $s := \inf\{s_\alpha : \alpha < \delta\}$. Then $s \geq s_\delta$. Since E is spherically complete, there exists an element a in $\bigcap_{\alpha<\delta} B_E[a_\alpha, s_\alpha]$. By 3.1.3.a $B_E[a, s_\alpha] = B_E[a_\alpha, s_\alpha]$ and thus $B_E[a, s] = \bigcap_{\alpha<\delta} B_E[a, s_\alpha] = \bigcap_{\alpha<\delta} B_E[a_\alpha, s_\alpha]$ and by 3.4.18 $B_F[f_\alpha, s_\alpha] = \pi(B_E[a_\alpha, s_\alpha]) = \pi(B_E[a, s_\alpha]) = B_F[\pi(a), s_\alpha]$. Hence $\pi(B_E[a, s]) = B_F[\pi(a), s] = \bigcap_{\alpha<\delta} B_F[\pi(a), s_\alpha] = \bigcap_{\alpha<\delta} B_F[f_\alpha, s_\alpha] \supset B_F[f_\delta, s_\delta]$. Then there exists a_δ in $B_E[a, s]$ such that $\pi(a_\delta) = f_\delta$. Thus $B_F[f_\delta, s_\delta] = \pi(B_E[a_\delta, s_\delta])$ and $B_E[a_\delta, s_\delta] \subset B_E[a_\delta, s] = B_E[a, s]$. Now by transfinite induction we can build a nest of balls $\{B_E[a_\alpha, s_\alpha] : \alpha < \beta\}$ such that $B_\alpha = \pi(B_E[a_\alpha, s_\alpha])$. Since there exists an element b in $\bigcap_{\alpha<\beta} B_E[a_\alpha, s_\alpha]$, it follows that $\pi(b) \in \bigcap_{\alpha<\beta} B_\alpha$. ■

In Proposition 3.4.19 the hypothesis over the quotient space of being strict can be dropped when we assume a certain first countability condition on $X^{\#}$.

3.4.20 ★ Proposition. *Let X be a G-module and let E be a spherically complete X-normed space. If for every $s \in X^{\#} \cup \{0\}$, such that $s = \inf\{x \in X : s < x\}$, there* *exists a sequence* $(t_n)_{n\in\mathbb{N}}$ *in* X *such that* $s < t_n$ *for all* $n \in \mathbb{N}$ *and* $s = \inf\{t_n : n \in \mathbb{N}\}$, *then every quotient space* F *of* E *is spherically complete.*

Proof. Let F be a quotient space of E. By 3.1.6 it is enough to prove that every nest of balls in F indexed by a limit ordinal has a nonempty intersection. Although F is a $X^{\#}$-normed space, it is enough to consider nests of balls with radii in X by 3.4.5.d. Let β be a limit ordinal and let $\{B_\alpha : \alpha < \beta\}$ be a collection such that for each $\alpha < \beta$, $B_\alpha = B_F(f_\alpha, s_\alpha)$ for some $f_\alpha \in F$ and $s_\alpha \in X$ satisfying $s_{\alpha_2} < s_{\alpha_1}$ and $B_{\alpha_2} \subset B_{\alpha_1}$ whenever $\alpha_1 < \alpha_2 < \beta$. Let $s := \inf\{s_\alpha : \alpha < \beta\}$. By hypothesis, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in X such that $s = \inf\{t_n : n \in \mathbb{N}\}$. We can assume that $(t_n)_{n\in\mathbb{N}}$ is strictly decreasing (by taking a subsequence if it is necessary). Let $x_0 = s_0$ and for each $n \in \mathbb{N}$, let $x_n \in \{s_\alpha : \alpha < \beta\}$ be such that $x_n < \min\{t_n, x_{n-1}\}$. Then there is a strictly increasing sequence of ordinals $\alpha_n < \beta$ such that $x_n = s_{\alpha_n}$. Thus $s = \inf\{x_n : n \in \mathbb{N}\}$. Put $f_n := f_{\alpha_n}$ and $B_n = B_F(f_n, x_n)$ for each $n \in \mathbb{N}$. Let $\pi : E \to F$ be a quotient map and let $a_1 \in \pi^{-1}(B_1)$. Then by 3.4.18, $\pi(B_E(a_1, x_1)) = B_1$. Suppose that for $k \in \mathbb{N}$, we have a nest of balls $\{B_E(a_n, x_n) : 1 \le n \le k\}$ such that $\pi(B_E(a_n, x_n)) = B_n$ for each $n \in \{1, \dots, k\}$. Since $B_{k+1} \subset B_k = \pi(B_E(a_k, x_k))$, there exists $a_{k+1} \in B_E(a_k, x_k)$ such that $\pi(a_{k+1}) = f_{k+1}$. Hence by 3.1.3.a, we have that $B_E(a_{k+1}, x_{k+1}) \subset B_E(a_{k+1}, x_k) = B_E(a_k, x_k)$ and by 3.4.18 it follows that $\pi(B_E(a_{k+1}, x_{k+1})) = B_{k+1}$. Therefore, by induction there is a nest of balls $\{B_E(a_n, x_n) : n \in \mathbb{N}\}$ such that $\pi(B_E(a_n, x_n)) = B_n$ for all $n \in \mathbb{N}$. Since E is spherically complete, we can choose an element a in $\bigcap_{n=1}^{\infty} B_E(a_n, x_n)$. Then $\pi(a) \in \bigcap_{n=1}^{\infty} \pi(B_E(a_n, x_n)) = \bigcap_{n=1}^{\infty} B_n = \bigcap_{\alpha<\beta} B_\alpha$. ∎

Proposition 3.4.20 is an improvement to [30, 2.2.5] where it is supposed that the countability condition in $G^{\#}$ is enough. This is false and it is proved in 3.4.28.

3.4.4 Coinitiality and Cofinality of G-modules

In this subsection we will study the coinitiality and cofinality of G-modules and will see their topological consequences.

3.4.21 Definition. *The **cofinality** (**coinitiality**) of an ordered set C is the least ordinal β such that there is a cofinal (coinitial) subset of C of the form $\{c_\alpha \in C : \alpha < \beta\}$. It is denoted by* $\operatorname{cof}(C)$ *(*$\operatorname{coi}(C)$*, respectively).*

3.4.22 ★ Proposition. *Let X be a G-module. Then* $\operatorname{coi}(X^\#) = \operatorname{coi}(X) = \operatorname{coi}(G) = \operatorname{cof}(G) = \operatorname{cof}(X) = \operatorname{cof}(X^\#)$.

Proof. $\operatorname{coi}(G) = \operatorname{cof}(G)$: It is enough to notice that a set $\{g_\alpha \in G : \alpha < \beta\}$ is coinitial in G if and only if the set $\{g_\alpha^{-1} \in G : \alpha < \beta\}$ is cofinal in G.

$\operatorname{cof}(G) = \operatorname{cof}(X)$: Let $\beta := \operatorname{cof}(G)$. Then there exists a set $\{g_\alpha \in G : \alpha < \beta\}$ cofinal in G. Let's prove that the set $\{g_\alpha x \in X : \alpha < \beta\}$ is cofinal in X for any $x \in X$. Let $x \in X$. For each $s \in X$, there exists $g \in G$ such that $s \leq gx$ (3.3.2.e'). Then there is an ordinal $\alpha < \beta$ such that $s \leq gx \leq g_\alpha x$. Therefore $\operatorname{cof}(X) \leq \beta = \operatorname{cof}(G)$. Now put $\delta := \operatorname{cof}(X)$ and suppose that $\{x_\alpha \in X : \alpha < \delta\}$ is cofinal in X. Let $x \in X$. For every $\alpha < \delta$, there exists $h_\alpha \in G$ such that $x_\alpha \leq h_\alpha x$ (3.3.2.e'). Thus $\{h_\alpha x \in X : \alpha < \delta\}$ is cofinal in X and hence $\{h_\alpha \in G : \alpha < \delta\}$ is cofinal in G. Otherwise, there exists $h \in G$ such that $h_\alpha < h$ for all $\alpha < \delta$. Then $h_\alpha x \leq hx$ for all $\alpha < \delta$ and thus $hx = \max(X)$ which contradicts 3.3.2.f'. Therefore $\operatorname{cof}(G) \leq \delta = \operatorname{cof}(X)$.

$\operatorname{cof}(X) = \operatorname{cof}(X^\#)$: immediate from 3.4.5.c.

$\operatorname{coi}(X^\#) = \operatorname{coi}(X) = \operatorname{coi}(G)$: analogous to the proof of $\operatorname{cof}(G) = \operatorname{cof}(X) = \operatorname{cof}(X^\#)$. ■

The next result characterizes the G-modules associated with ultrametrizable valued fields.

3.4.23 Proposition. *Let $(K, |\cdot|)$ be a valued field and let X be a G-module. The following statements are equivalent:*

(a) $(K, |\cdot|)$ *is ultrametrizable,*

(b) there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ *in* K *such that* $\lim_n \lambda_n = 0$,

(c) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ *in* X *such that* $\lim_n x_n = 0$.

Proof. By 3.2.6 the statements (a) and (b) are equivalent to $\mathrm{coi}(G) = \omega$ (the first infinite ordinal). By Proposition 3.4.22, this is equivalent to $\mathrm{coi}(X) = \omega$, which in turn is equivalent to the existence of a countable coinitial subset $C \subset X$. Now notice that a countable subset $C \subset X$ is coinitial if and only if C contains a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_n x_n = 0$. ■

We will see in the subsection 3.5 that in order to obtain some important results on X-normed spaces, the order topology on $G^{\#}$ is asked to be 1st countable. In the rest of the subsection we will characterize the valued fields that satisfy this condition.

3.4.24 Definition. *Let $(K, |\cdot|)$ be a valued field and let $B_K = B_K[0,1]$. A subset A of K is* ***absolutely convex*** *if it is a B_K-submodule of K, in other words, if $0 \in A$ and $x, y \in A$, $\lambda, \mu \in B_K$ implies $\lambda x + \mu y \in A$. If in addition, there exists a countable subset $R \subset A$ such that every $a \in A$ can be written as $a = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$, for some $n \in \mathbb{N}$, $\lambda_i \in B_K$ and $a_i \in R$, then A is said to be* ***countably generated****.*

The following result describes the absolutely convex subsets of K.

3.4.25 Theorem ([30, 1.4.3])**.** *Let $(K, |\cdot|)$ be a valued field. The sets $\{0\}$, K, $B(0,r) = \{\lambda \in K : |\lambda| < r\}$ and $B[0,r] = \{\lambda \in K : |\lambda| \leq r\}$ for $r \in G^{\#}$, are absolutely convex. Each absolutely convex subset of K is of one of these forms.*

The **order topology** on a totally ordered set C is defined as the topology generated by the sets of the form $\{c \in C : c > s\}$ and $\{c \in C : c < s\}$ where $s \in C$.

3.4.26 Theorem ([30, 1.4.4])**.** *Let $(K, |\cdot|)$ be a valued field. The following statements are equivalent:*

(a) *Each absolutely convex subset of K is countably generated as a B_K-module.*

(b) *G has a cofinal sequence. For each $s \in G^\#$, there are $g_1, g_2, \dots \in G$, $g_n < s$ for all $n \in \mathbb{N}$ such that $\sup_{G^\#}\{t \in G^\# : t < s\} = \sup_{G^\#}\{g_n : n \in \mathbb{N}\}$.*

(c) *G has a coinitial sequence. For each $s \in G^\#$, there are $g_1, g_2, \dots \in G$, $g_n > s$ for all $n \in \mathbb{N}$ such that $\inf_{G^\#}\{t \in G^\# : s < t\} = \inf_{G^\#}\{g_n : n \in \mathbb{N}\}$.*

(d) *The order topology on $G^\#$ satisfies the first axiom of countability. $G^\#$ has a cofinal sequence.*

3.4.27 Remark. *Notice that the condition imposed on $G^\#$ in 3.4.26.c coincides with the condition on the G-module $X^\#$ in 3.4.20 when we replace $X^\#$ by $G^\#$ in 3.4.20. But this does not mean that we have an implication in general as is shown in the following result.*

3.4.28 Proposition. *If a valued field $(K, |\cdot|)$ satisfies the statements of Theorem 3.4.26, then a G-module X does not have to satisfy 3.4.26.c, i.e. for some G-module X there exists an element $s \in X^\#$ such that $\inf_{X^\#}\{t \in X^\# : s < t\} < \inf_{X^\#}\{g_n : n \in \mathbb{N}\}$ for every sequence $(g_n)_n$ in X with $g_n > s$.*

Proof. Consider $G = \{e^n \in \mathbb{R} : n \in \mathbb{Z}\}$ with the usual multiplication. Since G is countable, it satisfies 3.4.26.c. Now consider the G-module X defined in 3.3.3.c with $\beta := \omega_1$ (the first uncountable ordinal). For $n \in \mathbb{Z}$, let $s := e^n$. We have that $s = \inf\{x \in X : s < x\} = \inf\{e_\alpha^{n+1} : \alpha < \beta\}$. If there are $x_1, x_2, \dots \in X$ such that $s < x_n$ for all $n \in \mathbb{N}$, then there exists $\alpha < \beta$ such that $e_\alpha^{n+1} < x_n$ for all $n \in \mathbb{N}$. Otherwise $\mathrm{cof}(\beta) = \omega$, which contradicts the definition of β. Therefore, $s < e_\alpha^{n+1} \le \inf\{x_n : n \in \mathbb{N}\}$. ∎

3.4.29 Remark. *Notice that the G-module X of the previous proposition also shows that the statements 3.4.26.b and 3.4.26.c are not equivalent in every G-module. In fact, X satisfies 3.4.26.b (every $e^m_\alpha \in X$ is the successor of $e^m_{\alpha+1}$ and therefore $e^m_{\alpha+1} =$* $\sup\{x \in X : x < e^m_\alpha\}$*. Hence $X = X^\#$ and thus it is enough to choose $g_n = e^m_{\alpha+1}$ for all $n \in \mathbb{N}$). But X does not satisfy 3.4.26.c as is shown in 3.4.28.*

3.4.5 Equivalence of norms in $Lip(E, F)$ and $Lip_Y(E, F)$

In section 2.2, we studied the equivalence of 5 norms on $L(E, F)$ in the $rank(G) = 1$ case. Now in the general case, as we commented earlier, not every continuous linear operator is bounded, so we will restrict ourselves to the case in which the operators are bounded and where we are able to define the usual norms for these operators.

3.4.30 Definition. *Let G be a totally ordered commutative group and let K be a valued field with value group G. Let E be a G-normed space over K and let F be a Y-normed space over K where Y is a G-module. Consider the set $Lip_Y(E, F)$ of all the continuous linear maps $T : E \to F$ for which there is a $y \in Y$ such that $||Te|| \leq ||e||y$ for all $e \in E$. This set is a vector space over K. Notice that for $Y = G$, we have that $Lip_Y(E, F) = Lip(E, F)$. Given $T \in Lip_Y(E, F)$, the expression $\frac{||Te||}{||e||}$ will denote $||e||^{-1}||Te|| \in Y$ for $||e|| \in G$ and $||Te|| \in Y$.*

3.4.31 Proposition. *Let G, $(K, |\cdot|)$, E, F, Y and $Lip_Y(E, F)$ be as in definition 3.4.30. Let $T \in Lip_Y(E, F)$. The following formulas define $Y^\#$-norms on*

$Lip_Y(E,F)$:

$$||T|| := \inf_{Y^{\#}\cup\{0\}} \{y \in Y : ||Te|| \leq ||e||y,\ \textit{for all } e \in E\}$$

$$||T||_{\#} := \inf_{Y^{\#}\cup\{0\}} \{y \in Y^{\#} : ||Te|| \leq ||e||y,\ \textit{for all } e \in E\}$$

$$||T||_1 := \sup_{Y^{\#}\cup\{0\}} \left\{\frac{||Te||}{||e||} \in Y : e \in E\setminus\{0\}\right\}$$

$$||T||_2 := \sup_{Y^{\#}\cup\{0\}} \left\{\frac{||Te||}{||e||} \in Y : 0 < ||e|| \leq 1\right\}$$

$$||T||_3 := \sup_{Y^{\#}\cup\{0\}} \{||Te|| \in Y : ||e|| = 1\}$$

$$||T||_4 := \sup_{Y^{\#}\cup\{0\}} \{||Te|| \in Y : ||e|| \leq 1\}.$$

Furthermore, $||T|| = ||T||_{\#} = ||T||_1 = ||T||_2 = ||T||_3 = ||T||_4$ *for all* $T \in Lip_Y(E,F)$.

Proof. Let $T \in Lip_Y(E,F)$, $T \neq 0$. The verification of the fact that the formulas above define $Y^{\#}$-norms is straightforward. In order to prove their equality consider the set $A = \left\{\frac{||Te||}{||e||} \in Y : e \in E\setminus\{0\}\right\}$. Then $||T||_{\#} = \inf\{y \in Y^{\#} : a \leq y,$ for all $a \in A\} = \inf\{y \in Y^{\#} : \sup A \leq y\} = \sup A = ||T||_1$ where the penultimate equality is due to 3.4.5.e. By definition, we have $||T||_{\#} \leq ||T||$. Suppose that $||T||_{\#} < ||T||$. Then by 3.4.5.d, there exists $x \in Y$ such that $a \leq ||T||_{\#} \leq x < ||T||$ for all $a \in A$, contradicting the definition of $||T||$. Therefore $||T||_{\#} = ||T||$. Let $i \in \{2,3,4\}$. By definition, we have that $||T||_i \leq ||T||_1$. For $e \in E\setminus\{0\}$, let $\lambda \in K$ such that $|\lambda| = ||e||^{-1}$. Thus $\frac{||Te||}{||e||} = \frac{||T(\lambda e)||}{||\lambda e||} = ||T(\lambda e)|| \leq ||T||_i$. Therefore $||T||_1 = ||T||_i$. ∎

3.4.6 Banach spaces and linear operators with finite rank

In the next chapter we will be working with a Banach space over a field of higher rank so it is appropriate to mention some known results regarding Banach spaces to see the similarity with the rank-1 case.

3.4.32 Definition. *Let K be Cauchy complete valued field and let X be a G-module.*

An X-normed space is ***Banach*** *when each Cauchy net of E is convergent in E.*

3.4.33 Example. *Let X be a G-module and let $s : \mathbb{N} \to X$. Then $c_0(\mathbb{N}, s)$ is the space of all the sequences $(\lambda_n)_n$ in K for which $\lim_n |\lambda| s(n) = 0$ with coordinatewise operations and with X-norm $||(\lambda_1, \lambda_2, \dots)|| = \max_n |\lambda_n| s(n)$. If $X = G$ and $s(n) = 1$ for all $n \in \mathbb{N}$ we have that $c_0(\mathbb{N}, s) = c_0$. If K is Cauchy complete then the space $c_0(\mathbb{N}, s)$ is Banach. The proof is standard.*

3.4.34 Theorem. *Let E be a Banach space over a Cauchy complete non-Archimedean valued field K of higher rank. If $(x_n)_n$ is a sequence of elements in E, then*

$$\sum_{n=1}^{\infty} x_n \text{ is convergent in } E \Leftrightarrow \lim_{n\to\infty} x_n = 0.$$

Proof. Analogous to the proof of 2.1.7. ■

3.4.35 Theorem ([30, 2.3.5])**.** *Let K be Cauchy complete and let X be a Dedekind complete G-module. Every continuous linear map from an X-normed space E into a finite-dimensional X-normed space F is Lipschitz.*

3.4.36 Theorem ([30, 2.3.6])**.** *Let E, F be normed spaces over K. If F is a Banach space, then so is $L(E, F)$. If, in addition, E, F are both X-normed spaces for some G-module X that is Dedekind complete, then $Lip(E, F)$ is a Banach space.*

3.4.37 Corollary ([30, 2.3.7])**.** *Let K be Cauchy complete and let E, F be normed spaces over K. If F is finite-dimensional, then $L(E, F)$ is a Banach space. In particular, E' is a Banach space.*

3.4.38 Proposition. *Suppose that K is ultrametrizable. Let X be a G-module, let E be an X-normed space and let F be a quotient space of E. If E is Banach, then F is Banach.*

Proof. Since K is ultrametrizable, then there exists a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ in X such that $\inf\{t_n : n \in \mathbb{N}\} = 0$. Now the proof follows as in 3.4.20 with $s = 0$. ■

3.4.7 Generalizing the Open Mapping Theorem

If the field K is assumed to be ultrametrizable and Cauchy complete, then the Open Mapping Theorem, the Closed Graph Theorem and the Uniform Boundedness Theorem are proven in [30, 2.5.4, 2.5.6]. Recall that if K is ultrametrizable, then $\mathrm{cof}(G)$ must be countable (see 3.4.21 and 3.4.23).

In this section we will present the progress made in the pursuit of an Open Mapping Theorem without restrictions on the group G. First, let's present our generalization of the Baire Category Theorem for a spherically complete scaled space.

3.4.39 ★ Proposition. *Let X be a totally ordered set without minimal element. Suppose that $\mathrm{coi}(X) = \beta$. Let Z be a spherically complete scaled space with scale $d : Z \times Z \to X$. For any ordinal $\beta' \leq \beta$, and any collection $\{A_\alpha : \alpha < \beta'\}$ of closed subsets of Z with empty interior, the union $\bigcup_{\alpha<\beta'} A_\alpha$ has empty interior.*

Proof. Given an open ball $U \subset Z$, we will prove that $U \not\subset \bigcup_{\alpha<\beta'} A_\alpha$. In view of $A_0^o = \emptyset$, there exists $y \in U$ such that $y \notin A_0$, and since A_0 is closed, we can choose an open ball $U_0 \subset Z$ such that $U_0 \cap A_0 = \emptyset$ and $y \in U_0 \subset U$. Suppose that for $\alpha < \beta'$, we have a collection $\{U_\gamma : \gamma < \alpha\}$ of open balls in Z such that $\gamma_1 < \gamma_2 < \alpha \Rightarrow U_{\gamma_2} \subset U_{\gamma_1}$ and $U_\gamma \cap A_\gamma = \emptyset$ for all $\gamma < \alpha$. For each $\gamma < \alpha$, choose $a_\gamma \in Z$ and $\delta_\gamma \in X$ such that $U_\gamma = B(a_\gamma, \delta_\gamma)$. Since Z is spherically complete, we can choose $b \in \bigcap_{\gamma<\alpha} U_\gamma$. Notice that $\alpha < \beta' \leq \beta$ implies that $\{S_\gamma : \gamma < \alpha\}$ is not coinitial. Thus, there exists $\delta_\alpha \in X$, satisfying $\delta_\alpha < \delta_\gamma$ for all $\gamma < \alpha$. Then $V_\alpha := B(b, \delta_\alpha) \subset B(b, \delta_\gamma) = U_\gamma$ for all $\gamma < \alpha$. In view of $A_\alpha^o = \emptyset$, we can choose $a \in V_\alpha \setminus A_\alpha$, and since A_α is closed, there exists an open ball $U_\alpha \subset V_\alpha$ such that $U_\alpha \cap A_\alpha = \emptyset$. By using transfinite induction we have obtained a nest of balls $\{U_\alpha : \alpha < \beta'\}$ in Z such that $\alpha_1 < \alpha_2 < \beta' \Rightarrow U_{\alpha_2} \subset U_{\alpha_1}$

and $U_\alpha \cap A_\alpha = \emptyset$ for all $\alpha < \beta'$. Since Z is spherically complete, there exists $x \in \bigcap_{\alpha<\beta'} U_\alpha \subset U_0 \subset U$ such that $x \notin \bigcup_{\alpha<\beta'} A_\alpha$. Hence $U \not\subset \bigcup_{\alpha<\beta'} A_\alpha$. ■

3.4.40 ★ Proposition. *Let E and F be X-normed spaces for some G-module X. If F is spherically complete and $T \in L(E,F)$ is surjective, then $\overline{T(B)}$ is an open 0-neighborhood in F, for each ball $B := B(0,\varepsilon)$ in E.*

Proof. Put $\beta := \mathrm{cof}(X)$. By 3.4.22, we have that $\mathrm{coi}(X) = \beta$. Let $\{\varepsilon_\alpha \in X : \alpha < \beta\}$ be a cofinal set in X. Then $E = \bigcup_{\alpha<\beta} B(0,\varepsilon_\alpha)$ and $F = T(E) = \bigcup_{\alpha<\beta} T(B(0,\varepsilon_\alpha)) = \bigcup_{\alpha<\beta} \overline{T(B(0,\varepsilon_\alpha))}$. By 3.4.39, there exists $\alpha < \beta$ such that $\overline{T(B(0,\varepsilon_\alpha))}$ has a non-empty interior. Hence we can find an open ball $B(x,r) \subset \overline{T(B(0,\varepsilon_\alpha))}$. Notice that by the strong triangle inequality, $B(0,\varepsilon_\alpha)$ is an additive group and hence $T(B(0,\varepsilon_\alpha))$ and $\overline{T(B(0,\varepsilon_\alpha))}$ are also additive groups. Thus, for every $z \in \overline{T(B(0,\varepsilon_\alpha))}$, $z + B(0,r) = z - x + B(x,r) \subset \overline{T(B(0,\varepsilon_\alpha))} + \overline{T(B(0,\varepsilon_\alpha))} + \overline{T(B(0,\varepsilon_\alpha))} \subset \overline{T(B(0,\varepsilon_\alpha))}$. Therefore $\overline{T(B(0,\varepsilon_\alpha))}$ is open in F. ■

An immediate consequence of the previous result is the following:

3.4.41 Proposition. *Let E and F be X-normed spaces for some G-module X. If F is spherically complete and $T \in L(E,F)$ is surjective and closed, then T is open.*

Proof. By hypothesis $T(B(0,\varepsilon)) = \overline{T(B(0,\varepsilon))}$ since $B(0,\varepsilon)$ is clopen in E for all $\varepsilon \in X$. Now by 3.4.40 we conclude that $T(B(0,\varepsilon))$ is open and therefore T is open. ■

The standard proof of the Open Mapping Theorem uses convergent series to show that $T(B(0,\varepsilon)) = \overline{T(B(0,\varepsilon))}$ for any surjective $T \in L(E,F)$, where E and F are Banach spaces. In our case, this technique is not enough when $\mathrm{cof}(X)$ is uncountable (when K is not ultrametrizable) because we cannot properly define a sum of an uncountable number of vectors in a Banach space.

3.5 Spaces of countable type and orthogonality

The concepts of norm-orthogonality (2.3.4, 2.3.5), orthogonal system (2.3.14), orthogonal complements (2.3.10) and countable type (2.4.1) can be defined in the general case in the same way. From now on, when referring to norm-orthogonality we will just say orthogonality.

Regarding spaces of countable type we have to following result.

3.5.1 Theorem ([31, 2.5.1]). *Let E be a Banach space of countable type with base field K. Suppose:*

(a) the topology on K induced by the valuation is not metrizable, or

(b) G is principal (see 3.2.2), or

(c) each B_K-submodule of K is countably generated.

Then each closed subspace of E is of countable type.

However it is still unknown whether being of countable type is a hereditary property, i.e. it is unknown whether a subspace of a space of countable type is of countable type when none of the conditions (a), (b) and (c) are satisfied.

3.5.2 Definition. *Let E be an X-normed space. An **orthogonal basis** for E is an orthogonal countable set $(x_n)_n$ in E such that for each $x \in E$, there is a unique sequence $(\lambda_n)_n$ in K satisfying $x = \sum_{n=1}^{\infty} \lambda_n x_n$.*

H. Ochsenius and W. H. Schikhof proved the following characterization of those X-normed spaces that admit a basis under certain countability condition.

3.5.3 Theorem ([30, 3.4.1]). *Let E be a Banach space of countable type and suppose the order topology on $G^{\#}$ is first countable and that $G^{\#}$ has a cofinal sequence. Then the following statement are equivalent:*

(a) E has an orthogonal basis.

(b) Each closed subspace of E has an orthogonal basis.

(c) Each finite-dimensional subspace of E has an orthogonal basis.

(d) Each one-dimensional subspace of E is orthocomplemented.

(e) Each finite-dimensional subspace of E is orthocomplemented.

(f) For each finite-dimensional subspace D of E and $a \in E$ the set $\{||a - d|| : d \in D\}$ has a minimum.

Notice that the condition imposed over $G^{\#}$ in Theorem 3.5.3 has equivalent expressions stated in 3.4.26. The next result will be used in the next chapter.

3.5.4 Corollary. *Suppose that the order topology on $G^{\#}$ is first countable and that $G^{\#}$ has a cofinal sequence. Then each closed subspace of c_0 has an orthogonal basis.*

Proof. By 3.5.3, it is enough to notice that c_0 has an orthogonal basis. The canonical vectors $e_n = (\delta_{n,m})_{m \in \mathbb{N}}$ form a basis for c_0. In fact, for each $x = (\lambda_n)_n \in c_0$ we have that $x = \sum_{n=1}^{\infty} \lambda_n e_n$ where the uniqueness of the scalars is clear. Since $|| \sum_{n=1}^{k} \alpha_n e_n|| = \max\{||\alpha_n e_n|| : 1 \leq n \leq k\}$ for all $k \in \mathbb{N}$, it follows that $\{e_n : n \in \mathbb{N}\}$ is orthogonal. ∎

Whether every closed subspace of an X-normed space with an orthogonal basis also has a basis is an open problem. In particular, it is unknown whether every closed subspace of c_0 has an orthogonal basis when there are no countability conditions over G. Both answers are positive in the rank-1 case.

However, it is my view that the proof of the rank-1 case with the approach of Gruson - van der Put [45, 5.7] using projective spaces can be applied to the higher rank case provided that every quotient space of a spherically complete space is spherically complete. Thus the contributions 3.4.19 and 3.4.20 aim in that direction.

Chapter 4

Operator theory on c_0 over a field of arbitrary rank

In recent works ([2], [3] 2013-2015), Aguayo, J., Nova, M., and Shamseddine, K. provided characterizations of several types of operators on c_0 over the rank-1 field $L[\mathbb{Q}, \mathbb{C}]$ (1.5.4). In this chapter we will present the efforts of the author to generalize these results to a non-Archimedean valued field of arbitrary rank, the general Hahn field (3.2.1) equipped with a Krull valuation taking values in an arbitrary ordered commutative group G. As we will see, only partial results have been accomplished. First, we will start with the most distinctive difference between the non-Archimedean and Archimedean cases: there exists an inner product that induces the supremum norm on c_0. Thus the space c_0 will be our analogue for ℓ^2 in the Archimedean case. Then we will review the characterization of those closed subspaces of c_0 that admit a normal complementation while studying the projections associated with these spaces. After that, a brief characterization of the operators that admit an adjoint will be presented to be complemented later with a study of compact operators on c_0 emphasizing the differences with such operators in the rank-1 case. Finally, an analysis on spectral theory results is developed for some operators in c_0.

4.1 Notation. *From now on, K will denote a real-closed field (1.6). If i is a root of the polynomial x^2+1 (which is irreducible on K by 1.6.1), then the extension $K(i)$ of K is an algebraically closed field and therefore it is the algebraic closure of K.*

The general Hahn field $K((G))$ will be denoted by $\mathcal{H}$. Notice that the general Hahn field $K(i)((G)) = \mathcal{H} + i\mathcal{H} = \mathcal{H}(i)$. For each nonzero $z = x + iy \in \mathcal{H}(i)$ ($x, y \in \mathcal{H}$) the Hahn valuation defined in Section 3.2.1 satisfies:

$$|z| = \max\{\mathrm{supp}(z)\} = \max\{\max\{\mathrm{supp}(x)\}, \max\{\mathrm{supp}(y)\}\} = \max\{|x|, |y|\}.$$

The involution $x + iy \mapsto \overline{x+iy} := x - iy$ is an automorphism on $\mathcal{H}(i)$ such that $|z| = |\overline{z}|$ and $z\overline{z} \in \mathcal{H}$ for all $z \in \mathcal{H}(i)$.

4.1 Inner product in c_0

In this section we will study an inner product on $c_0 := c_0(\mathcal{H}(i)) := \{(\lambda_j)_{j\in\mathbb{N}} : \lambda_j \in \mathcal{H}(i)$, for all $j \in \mathbb{N}$ such that $\lim_j \lambda_j = 0\}$ that induces the usual norm on c_0. Notice that $c_0 = c_0(\mathcal{H}) \oplus ic_0(\mathcal{H})$, i.e. for each $\mathrm{z} = (z_n) \in c_0$, there are unique $\mathrm{x} = (x_n)$ and $\mathrm{y} = (y_n)$ in $c_0(\mathcal{H})$ such that $\mathrm{z} = \mathrm{x} + i\mathrm{y}$ and the G-norm on c_0 satisfies:

$$||\mathrm{z}|| = \max_{n\in\mathbb{N}} |z_n| = \max_{n\in\mathbb{N}} \max\{|x_n|, |y_n|\} = \max\{||\mathrm{x}||, ||\mathrm{y}||\}.$$

4.1.1 Theorem. *Consider the form $\langle\cdot,\cdot\rangle : c_0 \times c_0 \to \mathcal{H}(i)$, $\langle \mathrm{z}, \mathrm{w}\rangle = \sum_{n=1}^{\infty} z_n\overline{w_n}$. The statements below hold for all $\mathrm{z}, \mathrm{z}', \mathrm{w} \in c_0$ and $\alpha, \beta \in \mathcal{H}(i)$.*

(a) $\langle\cdot,\cdot\rangle$ *is well-defined.*

(b) $\langle \mathrm{z}, \mathrm{z}\rangle = 0 \Leftrightarrow \mathrm{z} = 0$

(c) $\langle \alpha\mathrm{z} + \beta\mathrm{z}', \mathrm{w}\rangle = \alpha\langle \mathrm{z}, \mathrm{w}\rangle + \beta\langle \mathrm{z}', \mathrm{w}\rangle$

(d) $\langle \mathrm{z}, \mathrm{w}\rangle = \overline{\langle \mathrm{w}, \mathrm{z}\rangle}$

(e) $|\langle \mathrm{z}, \mathrm{w}\rangle| \leq ||\mathrm{z}||\,||\mathrm{w}||$

(f) $\langle \mathrm{z}, \mathrm{w}\rangle = 0$ *for all* $\mathrm{w} \in c_0$ *if and only if* $\mathrm{z} = 0$.

Proof. *(a)*: since $\mathrm{z}, \mathrm{w} \in c_0$, we have that $\lim_n z_n\overline{w_n} = 0$. By Cauchy completeness of $\mathcal{H}(i)$ it follows that $\sum_{n=1}^{\infty} z_n\overline{w_n}$ is convergent in $\mathcal{H}(i)$.

(b): Since $\mathcal{H}$ is an ordered field (see 1.4.10), $0 < x^2 + y^2 = z\overline{z}$ for all $z = x + iy \in \mathcal{H}(i) \setminus \{0\}$ $(x, y \in \mathcal{H})$. Thus for $\mathrm{z} = (z_n) \in c_0 \setminus \{0\}$, there exists $k \in \mathbb{N}$ such that $z_k \neq 0$. Therefore, $0 < z_k\overline{z_k} \leq \langle \mathrm{z}, \mathrm{z}\rangle$. The other direction is obvious.

The statements *(c)* and *(d)* have standard proofs.

(e): $|\langle \mathrm{z}, \mathrm{w}\rangle| = |\sum_{n=1}^{\infty} z_n\overline{w_n}| \leq \max_n\{|z_n\overline{w_n}|\} \leq ||\mathrm{z}||\,||\mathrm{w}||$.

(f): It is enough to consider for each $m \in \mathbb{N}$, $\mathrm{w} = \mathrm{e_m} = (w_n)_n \in c_0$ with $w_n = \delta_{m,n}$. ■

From now on $\langle\cdot,\cdot\rangle$ will denote the inner product on c_0 defined in the previous theorem.

4.1.2 Lemma. *Let* $(F, |\cdot|)$ *be a valued field with residue class field* k *and value group* G. *The field* k *is formally real if and only if*

$$|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2| = \max\{|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2\}$$

for all $n \in \mathbb{N}$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$.

Proof. Suppose that k is formally real and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$. Put $M = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} \in G$ and $I = \{i : |\lambda_i| = M\}$. For every $\lambda \in B[0,1] \subset F$, let λ^* be its class in k.

Case 1: $M = 1$. In this case we have that $(\sum_{i\in I}\lambda_i^2)^* = \sum_{i\in I}(\lambda_i^*)^2 \neq 0^*$ because k is formally real. Hence $|\sum_{i\in I}\lambda_i^2| = 1$. Since $|\lambda_i| < 1$ for each $i \notin I$, $|\sum_{i\notin I}\lambda_i^2| < |\sum_{i\in I}\lambda_i^2|$. Thus by the Isosceles triangle principle we have that $|\sum_{i=1}^{n}\lambda_i^2| = |\sum_{i\notin I}\lambda_i^2 + \sum_{i\in I}\lambda_i^2| = |\sum_{i\in I}\lambda_i^2| = 1 = M^2$.

Case 2: $0 \neq M \neq 1$. In this case, we choose $\alpha \in F$ such that $|\alpha| = M \in G$ and apply the proof of the case 1 to the collection $\alpha^{-1}\lambda_1, \alpha^{-1}\lambda_2, \ldots, \alpha^{-1}\lambda_n \in F$.

Now for the converse, suppose that $|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2| = \max\{|\lambda_1|^2, \ldots, |\lambda_n|^2\}$ for all $n \in \mathbb{N}$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n \in k$ be such that $\sum_{i=1}^n (\sigma_i)^2 = 0^*$. For each $i \in \{1, \ldots, n\}$, let $\lambda_i \in B[0,1]$ be a representative of σ_i, i.e. $\lambda_i^* = \sigma_i$. Then

$$\left(\sum_{i=1}^n \lambda_i^2\right)^* = \sum_{i=1}^n (\lambda_i^*)^2 = 0^*.$$

Then $1 > |\sum_{i=1}^n \lambda_i^2| = \max\{|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2\} \geq |\lambda_i^2|$ for all $i \in \{1, \ldots, n\}$. Therefore $\sigma_i = \lambda_i^* = 0^*$ for all $i \in \{1, \ldots, n\}$. ∎

4.1.3 Lemma. *For all $n \in \mathbb{N}$ and $z_1, z_2, \ldots, z_n \in \mathcal{H}(i)$,*

$$|z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}| = \max\{|z_1\overline{z_1}|, |z_2\overline{z_2}|, \ldots, |z_n\overline{z_n}|\}.$$

Proof. For $k \in \{1, \ldots, n\}$, let $x_k, y_k \in \mathcal{H}$ be such that $z_k = x_k + iy_k$. Then $z_k\overline{z_k} = x_k^2 + y_k^2$. Since the residue class field of $\mathcal{H}$ is K, which is formally real by choice (see 4.1), the result follows from the previous lemma. ∎

4.1.4 Theorem. *The inner product $\langle \cdot, \cdot \rangle$ induces the norm on c_0, i.e. for all $z \in c_0$:*

$$|\langle \mathrm{z}, \mathrm{z} \rangle| = ||\mathrm{z}||^2.$$

Proof. By Lemma 4.1.3, for any $\mathrm{z} = (z_n) \in c_0$ we have $|\langle \mathrm{z}, \mathrm{z} \rangle| = |\sum_{k=1}^\infty z_k\overline{z_k}| = \lim_n |\sum_{k=1}^n z_k\overline{z_k}| = \lim_n \max\{|z_1\overline{z_1}|, |z_2\overline{z_2}|, \ldots, |z_n\overline{z_n}|\} = \max_{n\in\mathbb{N}} |z_n|^2 = ||\mathrm{z}||^2$. ∎

Notice that the previous result justifies the analogy between c_0 and the separable Hilbert space ℓ^2. However, this analogy is not as profound as one ideally would like to have because of the following result.

4.1.5 Theorem ([31, 2.6.5])**.** *Let K be metrizable and let G be not principal. Then* *there is a closed subspace D in c_0 without closed complement, i.e. there does not exist a projection $P \in L(c_0)$ such that $P(c_0) = D$.*

4.1.6 Notation. *From now on we will assume that G is divisible, i.e. for every $g \in G$ and $n \in \mathbb{N}$ there exists $h \in G$ such that $h^n = g$. With this assumption now we can write $\sqrt{|\langle \mathrm{z}, \mathrm{z} \rangle|} = ||\mathrm{z}||$ for all $\mathrm{z} \in c_0$*

4.1.7 Definition. *A subset D of c_0 such that for all $\mathrm{x}, \mathrm{y} \in D$, $\mathrm{x} \neq \mathrm{y} \Rightarrow \langle \mathrm{x}, \mathrm{y} \rangle = 0$, is called a **normal family**. When a normal family is countable, we say that is a **normal sequence**. A normal sequence of unit vectors is called an **orthonormal sequence**. Notice that if $(\mathrm{x}_n)_n$ is an orthonormal sequence in c_0, then $1 = ||\mathrm{x}_n||^2 = |\langle \mathrm{x}_n, \mathrm{x}_n \rangle|$ but it is not necessary to have $\langle \mathrm{x}_n, \mathrm{x}_n \rangle = 1$. A **basis** of c_0 is an orthonormal sequence $(\mathrm{y}_n)_n$ in c_0 such that for each $\mathrm{x} \in c_0$, there exists a unique $\lambda = (\lambda_n)_n \in c_0$ satisfying $\mathrm{x} = \sum_{n=1}^{\infty} \lambda_n \mathrm{y}_n$.*

Notice that if $(\mathrm{y}_n)_n$ is a basis of c_0, then for every $\mathrm{x} \in c_0$ we have that

$$\mathrm{x} = \sum_{n=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_n \rangle}{\langle \mathrm{y}_n, \mathrm{y}_n \rangle} \mathrm{y}_n.$$

We can orthonormalize any linearly independent sequence of c_0 by the Gram-Schmidt procedure, which holds in our general case.

4.1.8 Proposition. *If $(\mathrm{z}_n)_n$ is a sequence of linearly independent vectors in c_0, then there exists an orthonormal sequence $(\mathrm{y}_n)_n$ in c_0 such that*

$$[\{\mathrm{z}_1, \mathrm{z}_2, \ldots, \mathrm{z}_n\}] = [\{\mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_n\}]$$

for every $n \in \mathbb{N}$.

Proof. The proof is analogous to the standard proof. Having obtained the set

$\{y_1, y_2, \ldots, y_{n-1}\}$ for $n > 1$, compute

$$w_n = z_n - \sum_{i=1}^{n-1} \frac{\langle z_n, y_i\rangle}{\langle y_i, y_i\rangle} y_i$$

and let $y_n = \lambda w_n$ for some $\lambda \in \mathcal{H}(i)$ such that $|\lambda| = ||w_n||^{-1} \in G$. ■

The next result shows that every normal family of vectors in c_0 is an orthogonal family (see 2.3.4 and 2.3.15).

4.1.9 Theorem. *Let* $(x_n)_n$ *be a normal sequence in* c_0 *and let* $(\lambda_n)_n$ *be an arbitrary sequence in* $\mathcal{H}(i)$. *Then, for all* $k \in \mathbb{N}$, *we have that*

$$\left\| \sum_{n=1}^{k} \lambda_n x_n \right\| = \max\{||\lambda_n x_n|| : 1 \leq n \leq k\}.$$

Proof. The proof of [2, Lemma 2] can be adapted to our case. ■

However, the converse of this result is false (see [29, Example 3.1]). Therefore if $(y_n)_n$ is a basis of c_0 then it is an orthogonal basis of c_0 (see 2.4.8), but the converse is not always true.

4.2 Normal complement subspaces of c_0

In this section we will characterize the linear functionals in c_0' that admit a Riesz representation and then we will identify those closed subspaces of c_0 that are normal complemented. For both results, the Riemann-Lebesgue property will be key.

4.2.1 Definition. *Given a subspace* M *of* c_0, *the space of all* $y \in c_0$ *such that* $\langle x, y\rangle = 0$ *for all* $x \in M$ *will be denoted by* M^p. *When* $c_0 = M \oplus M^p$, *we say that* M *is* ***normal complemented*** *and* M^p *is called the* ***normal complement*** *of* M.

4.2.2 Definition. *A sequence* $(z_n)_n$ *of nonzero vectors of* c_0 *has the* ***Riemann-Lebesgue property (RLP)*** *if for all* $w \in c_0$, $\lim_n\langle z_n, w\rangle = 0$.

4.2.3 Example. *Any basis of c_0 has the RLP. In fact, let $(\mathrm{z}_n)_n$ be a basis of c_0, i.e. for each $\mathrm{w} \in c_0$, $\mathrm{w} = \sum_{n=1}^{\infty} \langle \mathrm{w}, \mathrm{z}_n \rangle \langle \mathrm{z}_n, \mathrm{z}_n \rangle^{-1} z_n$. Hence*

$$\lim_n |\langle \mathrm{z_n}, \mathrm{w} \rangle| = \lim_n |\langle \mathrm{w}, \mathrm{z}_n \rangle \langle \mathrm{z}_n, \mathrm{z}_n \rangle^{-1}| = 0.$$

4.2.4 Theorem. *If $S \subset c_0$ is a finite orthonormal subset, say $\{\mathrm{z}_1, \mathrm{z}_2, \ldots, \mathrm{z}_n\}$, or is an orthonormal sequence $(\mathrm{z}_n)_n$, which satisfies the RLP, then S can be extended to a basis for c_0; that is, there exists a countable orthonormal sequence $(\mathrm{w}_n)_n$ (possibly finite) such that $S \cup \{\mathrm{w}_n : n \in \mathbb{N}\}$ is a basis for c_0.*

Proof. The proof of [29, 8.1] can be adapted to our case. ■

4.2.5 Theorem. *Let E be an infinite-dimensional Banach space over K. For an orthogonal sequence $e_1, e_2, \ldots$ in E the following are equivalent:*

(a) $\{e_n : n \in \mathbb{N}\}$ *is an orthogonal basis for E.*

(b) $e_n \neq 0$ *for each $n \in \mathbb{N}$ and $\overline{[\{e_n : n \in \mathbb{N}\}]} = E$.*

Proof. (a)⇒(b) is obvious. To prove (b)⇒(a) we define a linear map $T : c_0(\mathbb{N}, s) \to E$ as follows:

$$T(\lambda_1, \lambda_2, \ldots) = \sum_{n=1}^{\infty} \lambda_n e_n,$$

where $s(n) := ||e_n||$ for each $n \in \mathbb{N}$. Since K is Cauchy complete, so is $c_0(\mathbb{N}, s)$ (see 3.4.33). Notice that T is an isometry because of the orthogonality of $\{e_n : n \in \mathbb{N}\}$. Hence $\mathrm{Im}(T)$ is Banach so it is closed in E. Therefore, the inclusions $[\{e_n : n \in \mathbb{N}\}] \subset \mathrm{Im}(T) \subset E$ imply that T is onto after considering their closures, proving (a). ■

4.2.6 Notation. *From now on we will assume that G is divisible, G has a countable cofinal subset and that the order topology on $G^{\#}$ is first countable (see 3.4.26).*

4.2.7 Proposition. *Every closed subspace D of c_0 admits a basis.*

Proof. Let D be a closed subspace of c_0. By 3.5.4, D has an orthogonal basis S. Since S is countable, we can apply 4.1.8 to obtain a countable orthonormal sequence $\{e_n : n \in \mathbb{N}\}$ such that $\overline{[\{e_n : n \in \mathbb{N}\}]} = \overline{[S]} = E$. Therefore, $\{e_n : n \in \mathbb{N}\}$ is a basis, by 4.2.5. ■

To study continuous linear functionals on c_0, we need to present the space ℓ^∞.

4.2.8 Notation.

$$\ell^\infty := \ell^\infty(\mathcal{H}(i)) := \left\{ (\lambda_j)_{j\in\mathbb{N}} : \lambda_j \in \mathcal{H}(i),\ \textit{for all } j \in \mathbb{N}, \sup_j |\lambda_j| < \infty \right\}.$$

The formula $||(\lambda_1, \lambda_2, \dots)|| := \sup_j |\lambda_j|$ defines a $G^\#$-norm on ℓ^∞ for which ℓ^∞ becomes a Banach space. The proof of this statement is standard. Notice that $\ell^\infty = \ell^\infty(\mathcal{H}) \oplus i\ell^\infty(\mathcal{H})$, i.e. for each $\mathrm{z} = (z_n) \in \ell^\infty$, there are unique $\mathrm{x} = (x_n)$ and $\mathrm{y} = (y_n)$ in $\ell^\infty(\mathcal{H})$ such that $\mathrm{z} = \mathrm{x} + i\mathrm{y}$ and the $G^\#$-norm on ℓ^∞ satisfies:

$$||\mathrm{z}|| = \sup_{n\in\mathbb{N}} |z_n| = \sup_{n\in\mathbb{N}} \max\{|x_n|, |y_n|\} = \max\{||\mathrm{x}||, ||\mathrm{y}||\}.$$

In the Archimedean world the dual space of c_0 is isometrically isomorphic to ℓ^1. In our case, however, c_0' is isometrically isomorphic to ℓ^∞, as we will see below.

4.2.9 Lemma. *For each* $\mathrm{x} = (x_k)_k \in c_0$ *and* $\mathrm{z} = (z_k)_k \in \ell^\infty$, *the formula*

$$B(\mathrm{x}, \mathrm{z}) := \sum_{k=1}^{\infty} x_k z_k$$

defines a bilinear form $\langle\cdot,\cdot\rangle : c_0 \times \ell^\infty \to \mathcal{H}(i)$ *such that* $B(\cdot, \overline{\cdot})$ *extends the inner product on* c_0 *defined in 4.1.1.*

Proof. Let's prove that B is well-defined. Let $z \in \ell^\infty$ and $\mathrm{x} \in c_0$. Since, $|x_k z_k| \leq |x_k| \|z\|$ for each $k \in \mathbb{N}$, we have that $\lim_k |x_k z_k| = \lim_k |x_k| \|z\| = 0$, by 3.4.6.b.

Hence $(\sum_{k=1}^{n} x_k z_k)_n$ is a Cauchy sequence, and thus convergent since $\mathcal{H}(i)$ is Cauchy complete. Clearly, $\langle \cdot, \cdot \rangle$ is bilinear. If $z \in c_0$, then $B(\mathrm{x}, \overline{\mathrm{z}}) = \sum_{k=1}^{\infty} x_k \overline{z_k} = \langle \mathrm{x}, \mathrm{z} \rangle$. ■

4.2.10 Theorem ([32, 2.5.11])**.** *The map $\Upsilon : \ell^\infty \to c_0'$ defined by $\Upsilon(\mathrm{z}) := B(\cdot, \mathrm{z})$ $(\mathrm{z} \in \ell^\infty)$, is an isometrical isomorphism.*

Proof. For each $\mathrm{x} \in c_0$, $|\Upsilon(\mathrm{z})(\mathrm{x})| = |B(\mathrm{x}, \mathrm{z})| \leq \|\mathrm{x}\|\|\mathrm{z}\|$. Therefore Υ is well-defined and $||\Upsilon(\mathrm{z})|| \leq \|\mathrm{z}\|$. Since B is bilinear, it follows that Υ is linear. Applying $\Upsilon(\mathrm{z})$ to the canonical basic vectors $e_n = (\delta_{nk})_k \in \ell^\infty$ we obtain $||\Upsilon(\mathrm{z})|| \geq |\Upsilon(\mathrm{z})(e_n)| = |z_n|$ for all $n \in \mathbb{N}$. Therefore $||\Upsilon(\mathrm{z})|| = ||\mathrm{z}||$. To prove surjectivity, let $g \in c_0'$. Since $|g(e_n)| \leq ||g||||e_n|| = ||g||$ for all $n \in \mathbb{N}$, we deduce that $\mathrm{z} := (g(e_k))_k \in \ell^\infty$. Let $\mathrm{x} := (x_k)_k = \sum_{k=1}^{\infty} x_k e_k$. Then, by linearity and continuity, $\Upsilon(z)(x) = \sum_{k=1}^{\infty} x_k g(e_k) = g\left(\sum_{k=1}^{\infty} x_k e_k\right) = g(\mathrm{x})$. ■

4.2.11 Definition. *A functional $f \in c_0'$ is called a* ***Riesz functional*** *if there exists $\mathrm{z} \in c_0$, such that $f(\mathrm{x}) = \langle \mathrm{x}, \mathrm{z} \rangle$ for all $\mathrm{x} \in c_0$.*

The next result characterizes the continuous linear functionals c_0 that are Riesz functionals.

4.2.12 Theorem. *Let f be a nontrivial linear functional on c_0. The following statements are equivalent:*

(a) *f is a Riesz functional,*

(b) *every basis of* $\ker(f)$ *has the RLP,*

(c) $\ker(f)$ *has a basis with the RLP,*

(d) $\ker(f)^p \neq \{0\}$,

(e) $\ker(f) \oplus \ker(f)^p = c_0$.

(f) *$f \in \overline{[e_n' : n \in \mathbb{N}]} \cong c_0$, where $e_n' \in c_0'$ is defined by $e_n'(\mathrm{x}) = \lambda_n$, for $\mathrm{x} = \sum_{n=1}^{\infty} \lambda_n \mathrm{e}_n$, where $\{\mathrm{e}_n : n \in \mathbb{N}\}$ is the canonical basis of c_0.*

Proof. The proofs are analogous to the ones of the rank-1 case referenced below. For (a)$\Rightarrow$(b) see [1, Theorem 5]. The implication (b)$\Rightarrow$(c) is trivial. For (c)$\Rightarrow$(a) see [29, 9.2]. For (a)$\Leftrightarrow$(d) and (a)$\Rightarrow$(e) see [29, 9.1]. Since $f \neq 0$, the implication (e)$\Rightarrow$(d) is trivial. For (a)$\Leftrightarrow$(f) see [1, Proposition 3]. ∎

4.2.13 Definition. *A continuous linear operator $P \in L(c_0)$ is said to be a* ***normal projection*** *if it satisfies the following statements:*

(a) $P^2 = P$,

(b) $\langle \mathrm{x}, \mathrm{y} \rangle = 0$, *for all* $\mathrm{x} \in \ker(P)$ *and* $\mathrm{y} \in \mathrm{Im}(P)$.

4.2.14 Lemma. *If $P \neq 0$ is a normal projection on c_0, then $||P|| = 1$.*

Proof. If $P = 0$ or $P = Id$ (the identity operator), then we are done. Now suppose that $0 \neq P \neq Id$. For any $\mathrm{z} \in c_0$, $\mathrm{z} - P\mathrm{z} \in \ker(P)$, and hence $\langle \mathrm{z} - P\mathrm{z}, P\mathrm{z} \rangle = 0$. Then $||P\mathrm{z}||^2 = |\langle P\mathrm{z}, P\mathrm{z} \rangle| = |\langle \mathrm{z}, P\mathrm{z} \rangle| \leq ||\mathrm{z}||\, ||P\mathrm{z}||$. Thus $||P|| \leq 1$. On the other hand, for $\mathrm{z} \in c_0 \setminus \ker(P)$, we have that $||P(P\mathrm{z})|| = ||P\mathrm{z}||$. Hence $||P|| \geq 1$, by 3.4.31. ∎

As it is shown in [1, p. 790] not every subspace of c_0 has a normal complement. By 4.2.4, it follows that every finite-dimensional subspace of c_0 has a normal complement. The next result characterizes the infinite-dimensional closed subspaces of c_0 that are normal complemented.

4.2.15 Theorem ([29, 8.2],[2, Proposition 4])**.** *Let M be an infinite-dimensional closed subspace of c_0. The following statements are equivalent:*

(a) M is normal complemented,

(b) there exists a normal projection $P \in L(c_0)$ such that $\ker(P) = M$.

(c) every basis of M has the RLP,

(d) M has a basis with the RLP.

Proof. (a)⇒(b): Suppose that $c_0 = M \oplus M^p$. Then for each $\mathrm{z} \in c_0$, there exists a unique pair $(\mathrm{x_z}, \mathrm{y_z}) \in M \times M^p$ such that $\mathrm{z} = \mathrm{x}_z + \mathrm{y}_z$. Then $P : c_0 \to c_0$, $P\mathrm{z} = \mathrm{y_z}$ is a map such that $P^2 = P$, $\ker(P) = M$ and $\mathrm{Im}(P) = M^p$. Additionally, $||P\mathrm{z}|| = ||\mathrm{y}_z|| \leq \max\{||\mathrm{x_z}||, ||\mathrm{y_z}||\} = ||\mathrm{z}||$, by 4.1.9. Therefore, P is a normal projection.

(b)⇒(c): Let $\mathrm{z} \in c_0$. By 4.2.7, M has a basis $\{\mathrm{z}_n : n \in \mathbb{N}\}$. If $\mathrm{z} \in \ker(P)$, then $z = \sum_{n=1}^{\infty} \langle \mathrm{z}, \mathrm{z}_n \rangle \langle \mathrm{z}_n, \mathrm{z}_n \rangle^{-1} \mathrm{z}_n$. Hence,

$$\lim_n |\langle \mathrm{z}, \mathrm{z}_n \rangle| = \lim_n |\langle \mathrm{z}, \mathrm{z}_n \rangle \langle \mathrm{z}_n, \mathrm{z}_n \rangle^{-1}| = 0.$$

If $\mathrm{z} \notin \ker(P)$, then $\mathrm{z} - P\mathrm{z} \in \ker(P)$ and $\langle P\mathrm{z}, \mathrm{z}_n \rangle = 0$. Thus $\langle \mathrm{z}, \mathrm{z}_n \rangle = \langle \mathrm{z} - P\mathrm{z}, \mathrm{z}_n \rangle + \langle P\mathrm{z}, \mathrm{z}_n \rangle = \langle \mathrm{z} - P\mathrm{z}, \mathrm{z}_n \rangle$, and therefore, $\lim_n |\langle \mathrm{z}, \mathrm{z}_n \rangle| = \lim_n |\langle \mathrm{z} - P\mathrm{z}, \mathrm{z}_n \rangle| = 0$. The implication (c)⇒(d) is trivial. (d)⇒(a): Suppose that M has a basis $B = \{\mathrm{z}_n : n \in \mathbb{N}\}$ with the RLP. By 4.2.4, there exists a countable orthonormal set $D = \{\mathrm{w}_1, \mathrm{w}_2, \dots\}$ (possibly finite) such that $B \cup D$ is a basis for c_0. Hence $c_0 = \overline{[B]} \oplus \overline{[D]}$ where $M = \overline{[B]}$ and $M^p = \overline{[D]}$. ■

4.3 Operators on c_0 admitting an adjoint operator

In this section we will present the concepts of adjoint and self-adjoint operators and provide a useful description for those operators on c_0 that have an adjoint operator.

4.3.1 Definition. *A linear operator $S : c_0 \to c_0$ is said to be* ***adjoint*** *of a given operator $T \in L(c_0)$ if $\langle T\mathrm{x}, \mathrm{y} \rangle = \langle \mathrm{x}, S\mathrm{y} \rangle$ for all $\mathrm{x}, \mathrm{y} \in c_0$. If $S = T$, we say that T is* ***self-adjoint****. By standard techniques it is shown that if $T \in L(c_0)$ has an adjoint S, then S is unique and belongs to $L(c_0)$. The adjoint operator of T will be denoted by T^*.*

The next result characterizes the continuous linear operators on c_0 that admit an adjoint operator as those that preserve the RLP generalizing the scope of the result

[3, Lemma 4].

4.3.2 Theorem ([3, Lemma 4]). *For each $T \in L(c_0)$ the following statements are equivalent.*

(a) T admits an adjoint operator.

(b) $\{T\mathrm{y}_n : n \in \mathbb{N}\}$ has the RLP for any set $\{\mathrm{y}_n \in c_0 : n \in \mathbb{N}\}$ with the RLP.

(c) $\{T\mathrm{y}_n : n \in \mathbb{N}\}$ has the RLP for any basis $\{\mathrm{y}_n : n \in \mathbb{N}\}$ of c_0.

(d) There is a basis $\{\mathrm{y}_n : n \in \mathbb{N}\}$ of c_0 such that $\{T\mathrm{y}_n : n \in \mathbb{N}\}$ has the RLP.

Proof. (a)⇒(b): If T has an adjoint operator T^* and $\{\mathrm{y}_n \in c_0 : n \in \mathbb{N}\}$ has the RLP, then $\lim_n \langle T\mathrm{y}_n, \mathrm{x}\rangle = \lim_n \langle \mathrm{y}_n, T^*\mathrm{x}\rangle = 0$ for all $\mathrm{x} \in c_0$, since $\{\mathrm{y}_n : n \in \mathbb{N}\}$ has the RLP. The implication (b)⇒(c) is clear since every basis satisfy the RLP (see 4.2.3). The implication (c)⇒(d) is immediate. (d)⇒(a): Suppose that there exists a basis $\{\mathrm{y}_n : n \in \mathbb{N}\}$ of c_0 such that $\{T\mathrm{y}_n : n \in \mathbb{N}\}$ has the RLP. Hence $\lim_n \left|\frac{\langle \mathrm{z}, T\mathrm{y}_n\rangle}{\langle \mathrm{y}_n, \mathrm{y}_n\rangle}\right| = \lim_n |\langle T\mathrm{y}_n, \mathrm{z}\rangle| = 0$ for all $\mathrm{z} \in c_0$. Therefore the linear operator $S : c_0 \to c_0$,

$$S\mathrm{z} := \sum_{k=1}^{\infty} \frac{\langle \mathrm{z}, T\mathrm{y}_k\rangle}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle} \mathrm{y}_k$$

is well-defined for all $\mathrm{z} \in c_0$. For any $\mathrm{x}, \mathrm{z} \in c_0$ we have that:

$$\langle T\mathrm{x}, \mathrm{z}\rangle = \left\langle T\left(\sum_{k=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_k\rangle}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle}\mathrm{y}_k\right), \mathrm{z}\right\rangle = \left\langle \sum_{k=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_k\rangle}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle} T\mathrm{y}_k, \mathrm{z}\right\rangle = \sum_{k=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_k\rangle}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle}\langle T\mathrm{y}_k, \mathrm{z}\rangle.$$

Similarly we have that:

$$\langle \mathrm{x}, S\mathrm{z}\rangle = \left\langle \mathrm{x}, \sum_{k=1}^{\infty} \frac{\langle \mathrm{z}, T\mathrm{y}_k\rangle}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle}\mathrm{y}_k\right\rangle = \sum_{k=1}^{\infty} \frac{\overline{\langle \mathrm{z}, T\mathrm{y}_k\rangle}}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle}\langle \mathrm{x}, \mathrm{y}_k\rangle = \sum_{k=1}^{\infty} \frac{\langle T\mathrm{y}_k, \mathrm{z}\rangle}{\langle \mathrm{y}_k, \mathrm{y}_k\rangle}\langle \mathrm{x}, \mathrm{y}_k\rangle.$$

Thus $\langle T\mathrm{x}, \mathrm{z}\rangle = \langle \mathrm{x}, S\mathrm{z}\rangle$ for all $\mathrm{x}, \mathrm{z} \in c_0$. Therefore $S = T^*$. ■

4.3.3 Theorem ([2, Theorem 6]). *Let $P \in L(c_0)$ be such that $P^2 = P$. Then P is a normal projection, if and only if P is self-adjoint.*

In the next result the author clarifies and summarize the presentation of [2, Theorem 7] providing a clearer proof.

4.3.4 Theorem ([2, Theorem 7])**.** *Let S be a nonempty orthonormal subset of c_0. The operator $P \in L(c_0) \setminus \{0\}$ is a normal projection and $S = \{\mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_n\}$ is a basis for* $\mathrm{Im}(P)$ *if and only if, for all* $\mathrm{x} \in c_0$*,*

$$P\mathrm{x} = \sum_{i=1}^{n} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i.$$

The operator $P \in L(c_0) \setminus \{0\}$ is a normal projection and $S = \{\mathrm{y}_n : n \in \mathbb{N}\}$ is a basis for $\mathrm{Im}(P)$ *if and only if, S has the RLP and for all* $\mathrm{x} \in c_0$*,*

$$P\mathrm{x} = \sum_{i=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i.$$

Proof. Suppose that S is an infinite basis and that $P \in L(c_0) \setminus \{0\}$ is a normal projection. Notice that, $\mathrm{Im}(P) = \ker(I - P)$ is a closed subspace of c_0. Hence, S has the RLP by 4.2.15. Since S is a basis for $\mathrm{Im}(P)$ (by 4.2.7 there is such a basis), for each $\mathrm{x} \in c_0$, there exists a unique $\alpha = (\alpha_n)_n$ in c_0 such that $P\mathrm{x} = \sum_{n=1}^{\infty} \alpha_n \mathrm{y}_n$. Now since P is self-adjoint (4.3.3) we have $\langle \mathrm{x}, \mathrm{y}_n \rangle = \langle \mathrm{x}, P\mathrm{y}_n \rangle = \langle P\mathrm{x}, \mathrm{y}_n \rangle = \sum_{m=1}^{\infty} \langle \alpha_m \mathrm{y}_m, \mathrm{y}_n \rangle = \alpha_n \langle \mathrm{y}_n, \mathrm{y}_n \rangle$. Thus, for each $n \in \mathbb{N}$, $\alpha_n = \langle \mathrm{x}, \mathrm{y}_n \rangle \langle \mathrm{y}_n, \mathrm{y}_n \rangle^{-1}$. Conversely, suppose that $P : c_0 \to c_0$ is such that $P\mathrm{x} = \sum_{i=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$, for all $\mathrm{x} \in c_0$. Since S has the RLP, the map P is well-defined. First, let us show that $P \in L(c_0) \setminus \{0\}$. For any $\mathrm{x} \in c_0$, we have that:

$$||P\mathrm{x}|| \leq \sup_{i \in \mathbb{N}} \left\| \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i \right\| = \sup_{i \in \mathbb{N}} |\langle \mathrm{x}, \mathrm{y}_i \rangle| \leq ||\mathrm{x}||.$$

Clearly, P is nonzero and linear. Therefore, $P \in L(c_0) \setminus \{0\}$.

Second, let us verify that P is a projection. For $\mathrm{x} \in c_0$:

$$P^2\mathrm{x} = \sum_{i=1}^{\infty} \frac{\langle P\mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_j \rangle}{\langle \mathrm{y}_j, \mathrm{y}_j \rangle} \frac{\langle \mathrm{y}_j, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \right) \mathrm{y}_i = \sum_{i=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i = P\mathrm{x}.$$

Third, let us show that P is self-adjoint. For $\mathrm{x}, \mathrm{z} \in c_0$:

$$\langle P\mathrm{x}, \mathrm{z}\rangle = \sum_{i=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y_i}\rangle}{\langle \mathrm{y_i}, \mathrm{y_i}\rangle} \langle \mathrm{y_i}, \mathrm{z}\rangle = \sum_{i=1}^{\infty} \frac{\langle \mathrm{y_i}, \mathrm{z}\rangle}{\langle \mathrm{y_i}, \mathrm{y_i}\rangle} \langle \mathrm{x}, \mathrm{y_i}\rangle = \langle \mathrm{x}, P\mathrm{z}\rangle.$$

Therefore, by 4.3.3, we conclude that P is a normal projection. If S is finite, the proof follows similarly. ■

4.4 Compact operators on c_0

As we will see shortly, the concept of compact operator in non-Archimedean functional Analysis is slightly different from the classical one due to the fact that the compactness of subsets in a vector space is not appropriate to define what is intended by compact operator in cases where the base field is not locally compact. However, the non-Archimedean version of this concept allows the approximation of this kind of operators by operators of finite rank. First we will review the non-Archimedean concept of compact operator when the base field has rank 1, and then we will introduce a proper adaptation for this concept in the higher rank case.

In order to define a compact operator in the non-Archimedean context we need the following:

4.4.1 Definition. *Let E be a Banach space over a valued field K and let B_K be the unit disc of K. A nonempty subset A of E is **absolutely convex** if it is a B_K-submodule of E, i.e. if $\lambda x + \mu y \in A$ for all $x, y \in A$ and $\lambda, \mu \in B_K$. A subset A of E is called **compactoid** if for each $\varepsilon > 0$ there is a finite set $F \subset E$ such that $A \subset B(0, \varepsilon) + aco\, F$, where*

$$aco\, F := \left\{ \sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}, \lambda_i \in B_K, a_i \in F, \text{ for each } i \right\}$$

*and 'aco' stands for **absolutely convex hull**.*

The next result shows that compactoidness is the natural generalization of precompactness.

4.4.2 Theorem ([45],4.S(vii)). *Let E be a normed space over a locally compact valued field. A subset of E is compactoid if and only if it is precompact.*

Now we have the necessary terminology to present the concept of non-Archimedean compact operator when the base field has rank 1.

4.4.3 Definition. *Let E and F be two normed spaces over a non-Archimedean valued field of rank 1. A linear map $T : E \to F$ is said to be* ***compact*** *if $T(B_E)$ is compactoid, where $B_E = \{x \in E : ||x|| \leq 1\}$.*

Notice that if $T : E \to F$ is compact, then it is also continuous.

4.4.4 Remark. *In the Archimedean case, if E and F are Banach spaces then a linear map $T : E \to F$ is called compact if $\overline{T(B_E)}$ is compact, or equivalently, if $T(B_E)$ is precompact. In the non-Archimedean context when defining a compact operator it is more appropriate to use the concept of compactoidness instead of precompactness. There are two reasons for that:*

(a) compactoidness is the natural generalization of precompactness (see 4.4.2). For the same reason, if K is a locally compact non-Archimedean valued field, then T is compact if and only if $\overline{T(B_E)}$ is compact.

(b) The lack of nontrivial absolutely convex compact subspaces in a Banach space when the base field is not locally compact. In fact, let F be a Banach space over a valued field K that is not locally compact. If $B \subset F$ is absolutely convex and compact, then $B = \{0\}$. To prove this suppose on the contrary that there exists $x \in B \setminus \{0\}$. If B_K is the unit disk of K, then xB_K is a closed subspace of B. Thus $B_K \cong xB_K$ is compact and hence K is locally compact, a contradiction. Therefore, if $T \in L(E, F)$ and $T(B_E)$ is precompact, then $T(B_E) \subset \overline{T(B_E)} = \{0\}$. Hence $T = 0$.

When the non-Archimedean field K has rank 1, we have a satisfactory characterization of compact operators that reads as follows:

4.4.5 Theorem ([45, 4.39, 4.40]). *Let E and F be two normed spaces over a non-Archimedean valued field of rank 1. The following conditions are equivalent for all $T \in L(E, F)$:*

(a) T is compact,

(b) for each $\varepsilon > 0$, there exists $S \in L(E, F)$ of finite-dimensional range such that $||T - S|| < \varepsilon$,

(c) there are vectors $\mathrm{a}_1, \mathrm{a}_2, \cdots \in F$, and functionals $g_1, g_2, \cdots \in E'$ such that $\lim_k ||g_k||||\mathrm{a}_k|| = 0$ and $T = \sum_{k=1}^{\infty} g_k \mathrm{a}_k$, i.e. the sequence $\left(\sum_{k=1}^{n} g_k(\cdot)\mathrm{a}_k\right)_{n\in\mathbb{N}}$ converges uniformly to T.

The author has recently come to the conclusion that the proof of the previous characterization cannot be replicated in the higher rank case because, unlike the rank 1 case, not every compactoid is contained in the closure of the absolutely convex hull of a sequence tending to 0 (see [30, 3.5.7]). The author has been unable to overcome this additional difficulty and chose to redefine a compact operator for our higher rank case as follows:

4.4.6 Definition. *An operator $T \in L(c_0)$ is called **compact** if there are vectors $\mathrm{a}_1, \mathrm{a}_2, \cdots \in c_0$, and functionals $g_1, g_2, \cdots \in c_0'$ such that $\lim_i ||g_i||||\mathrm{a}_i|| = 0$ and $T = \sum_{i=1}^{\infty} g_i \mathrm{a}_i$, i.e. the sequence $\left(\sum_{i=1}^{n} g_i(\cdot)\mathrm{a}_i\right)_{n\in\mathbb{N}}$ converges uniformly to T.*

4.4.7 Proposition. *If $T \in c_0$ is a compact operator, then $T(B)$ is a compactoid, where $B := \{\mathrm{x} \in c_0 : ||\mathrm{x}|| \leq 1\}$.*

Proof. There are vectors $\mathrm{a}_1, \mathrm{a}_2, \cdots \in c_0$, and functionals $g_1, g_2, \cdots \in c_0'$ such that $\lim_i ||g_i||||\mathrm{a}_i|| = 0$ and $T = \sum_{i=1}^{\infty} g_i \mathrm{a}_i$

For each $i \in \mathbb{N}$, choose $\varepsilon_i \in \mathcal{H}(i)$ such that $||\varepsilon_i g_i|| = 1$ whenever $g_i \neq 0$ and $\varepsilon_i = 1$ otherwise. Putting $h_i := \varepsilon_i g_i$ and $\mathrm{z}_i := \varepsilon_i^{-1}\mathrm{a}_i$ we have that $||h_i|| \leq 1$ for all $i \in \mathbb{N}$, and $T = \sum_{i=1}^{\infty} h_i \mathrm{z}_i$. For any given $g \in G$, there is an $N \in \mathbb{N}$ such that $||T\mathrm{x} - \sum_{i=1}^{n} h_i(\mathrm{x})\mathrm{z}_i|| < g$, for all $\mathrm{x} \in c_0$, whenever $n \geq N$. Hence, $T(B) \subset B(0,g) + aco\{\mathrm{z}_1, \mathrm{z}_2, \ldots, \mathrm{z}_n\}$. ∎

It is not known whether the converse of Proposition 4.4.7 holds in general.

We will show that if $T \in L(c_0)$ is compact then for any given basis $\{\mathrm{y}_n : n \in \mathbb{N}\}$ of c_0 we are able to find suitable functionals f_i such that $(\sum_{i=1}^{n} f_i(\cdot)\mathrm{y}_i)_{n\in\mathbb{N}}$ converges uniformly to T. First let's consider the following lemma.

4.4.8 Lemma. *For any given basis $\{\mathrm{y}_1, \mathrm{y}_2, \ldots\}$ of c_0, the projection map $\mathrm{y}'_j : c_0 \to \mathcal{H}(i)$ defined by*

$$\mathrm{y}'_j(\mathrm{x}) := \frac{\langle \mathrm{x}, \mathrm{y}_j \rangle}{\langle \mathrm{y}_j, \mathrm{y}_j \rangle}$$

is a member of c'_0 for all $i \in \mathbb{N}$. Furthermore, $\mathrm{y}'_j(\mathrm{y}_i) = \delta_{ij}$, $||\mathrm{y}'_i|| = 1$ for all $i \in \mathbb{N}$, $\mathrm{x} = \sum_{i=1}^{\infty} \mathrm{y}'_i(\mathrm{x})\mathrm{y}_i$ and $||\mathrm{x}|| = \max_{i\in\mathbb{N}} |\mathrm{y}'_i(\mathrm{x})|$ for all $\mathrm{x} \in c_0$.

Proof. All the statements have a standard proof except by the last equality. If $\mathrm{x} = 0$ then we are done. For $\mathrm{x} \in c_0 \setminus \{0\}$ we have that $\mathrm{x} = \sum_{i=1}^{\infty} \mathrm{y}'_i(\mathrm{x})\mathrm{y}_i$. Then $\lim |\mathrm{y}'_i(\mathrm{x})| = \lim ||\mathrm{y}'_i(\mathrm{x})\mathrm{y}_i|| = 0$ (3.4.34). Hence, for $g \in G$, $g < ||x||$, there exists $n_0 \in \mathbb{N}, n_0 > 1$ such that $|\mathrm{y}'_i(\mathrm{x})| < g$ for all $i \geq n_0$. Thus, by the strong triangle inequality

$$\left\| \sum_{i \geq n_0}^{\infty} \mathrm{y}'_i(\mathrm{x})\mathrm{y}_i \right\| \leq \sup_{i \geq n_0} ||\mathrm{y}'_i(\mathrm{x})\mathrm{y}_i|| = \sup_{i \geq n_0} |\mathrm{y}'_i(\mathrm{x})| \leq g < ||x||$$

and therefore by the Isosceles triangle principle

$$||x|| = \left\| \mathrm{x} - \sum_{i \geq n_0}^{\infty} \mathrm{y}'_i(\mathrm{x})\mathrm{y}_i \right\| = \left\| \sum_{i=1}^{n_0 - 1} \mathrm{y}'_i(\mathrm{x})\mathrm{y}_i \right\| = \max_{1 \leq i \leq n_0 - 1} ||\mathrm{y}'_i(\mathrm{x})\mathrm{y}_i|| = \max_{i \in \mathbb{N}} |\mathrm{y}'_i(\mathrm{x})|.$$

∎

4.4.9 ★ Proposition. *For any compact operator $T \in L(c_0)$ and any given basis $\{y_1, y_2, \dots\}$ of c_0, there are functionals $f_1, f_2, \dots \in c_0'$ such that $\lim_i ||f_i|| = 0$ and $T = \sum_{i=1}^{\infty} f_i y_i$.*

Proof. By using 4.4.8 together with the linearity and continuity of T, we have that for each $x \in c_0$,

$$\begin{aligned} Tx = \sum_{i=1}^{\infty} \frac{\langle Tx, y_i\rangle}{\langle y_i, y_i\rangle} y_i = \sum_{i=1}^{\infty} \frac{\left\langle T\left(\sum_{j=1}^{\infty} \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle} y_j\right), y_i\right\rangle}{\langle y_i, y_i\rangle} y_i &= \sum_{i=1}^{\infty} \frac{\left\langle \sum_{j=1}^{\infty} \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle} Ty_j, y_i\right\rangle}{\langle y_i, y_i\rangle} y_i \\ &= \sum_{i=1}^{\infty} \frac{\sum_{j=1}^{\infty} \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle} \langle Ty_j, y_i\rangle}{\langle y_i, y_i\rangle} y_i \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{\langle x, y_j\rangle}{\langle y_j, y_j\rangle} \frac{\langle Ty_j, y_i\rangle}{\langle y_i, y_i\rangle} \right) y_i \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} y_i'(Ty_j) y_j'(x) \right) y_i. \end{aligned}$$

It remains to show that, for each $i \in \mathbb{N}$, the map $f_i : c_0 \to \mathcal{H}(i)$, defined by $f_i(x) := \sum_{j=1}^{\infty} y_i'(Ty_j) y_j'(x)$ is a linear functional on c_0 such that $\lim_i ||f_i|| = 0$. It is clear that f_i is a linear map. Let's show it is also continuous. Let $x \in c_0$. Since $||y_i'|| = 1$ for all $i \in \mathbb{N}$, it follows that

$$|f_i(x)| \leq \sup_{j \in \mathbb{N}} \left| y_i'(Ty_j) y_j'(x) \right| \leq \sup_{j \in \mathbb{N}} |y_i'(Ty_j)| ||x|| \leq ||x|| ||T||$$

Thus, $f_i \in c_0'$ and in particular $||f_i|| \leq \sup_{j \in \mathbb{N}} |y_i'(Ty_j)|$ for all $i \in \mathbb{N}$. Since $|y_i'(Ty_j)| = |f_i(y_j)| \leq ||f_i||$ for all $j \in \mathbb{N}$, we conclude that $||f_i|| = \sup_{j \in \mathbb{N}} |y_i'(Ty_j)|$. Now if we prove that $\lim_i ||f_i|| = 0$, then we are done.

By hypothesis, there are vectors $a_1, a_2, \dots \in c_0$, and functionals $g_1, g_2, \dots \in c_0'$ such that $\lim_i ||g_i|| ||a_i|| = 0$ and $T = \sum_{i=1}^{\infty} g_i a_i$. Without loss of generality, we can assume that $a_i \neq 0$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose $\varepsilon_i \in \mathcal{H}(i)$ such that $||\varepsilon_i a_i|| = 1$. Putting $h_i := \varepsilon_i^{-1} g_i$ and $z_i := \varepsilon_i a_i$ we have that $||z_i|| = 1$, $\lim_i ||h_i|| =$

$\lim_i ||h_i|| ||\mathrm{z}_i|| = \lim_i ||g_i|| ||\mathrm{a}_i|| = 0$ and $T = \sum_{i=1}^{\infty} h_i \mathrm{z}_i$. For each $n \in \mathbb{N}$, let $\beta_n := (\beta_{ni})_{i\in\mathbb{N}} \in c_0$ be such that $\mathrm{z}_n = \sum_{i=1}^{\infty} \beta_{ni}\mathrm{y}_i$. Hence $||\beta_n|| = ||\mathrm{z}_n|| = 1$ for all $n \in \mathbb{N}$.

On the one hand, for all $j \in \mathbb{N}$,

$$T\mathrm{y}_j = \sum_{n=1}^{\infty} h_n(\mathrm{y}_j)\mathrm{z}_n = \sum_{n=1}^{\infty} h_n(\mathrm{y}_j)\left(\sum_{i=1}^{\infty} \beta_{ni}\mathrm{y}_i\right) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} h_n(\mathrm{y}_j)\beta_{ni}\mathrm{y}_i\right).$$

On the other hand, by continuity of the projection maps (4.4.8), for all $k \in \mathbb{N}$,

$$\mathrm{y}'_k(T\mathrm{y}_j) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} h_n(\mathrm{y}_j)\beta_{ni}\mathrm{y}'_k(\mathrm{y}_i)\right) = \sum_{n=1}^{\infty} h_n(\mathrm{y}_j)\beta_{nk}.$$

Therefore, for all $i \in \mathbb{N}$, we can write

$$||f_i|| = \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T\mathrm{y}_j)| = \sup_{j\in\mathbb{N}} \left|\sum_{n=1}^{\infty} h_n(\mathrm{y}_j)\beta_{ni}\right|.$$

Since $\lim_i ||h_i|| = 0$, there exists $M \in G$ such that $||h_i|| \le M$ for all $i \in \mathbb{N}$. Given $\varepsilon \in G$, there exists $N \in \mathbb{N}$ such that $n \ge N \Rightarrow ||h_n|| < \varepsilon$. Let $N_1 \in \mathbb{N}$ such that $N_1 \ge N$ and $i \ge N_1 \Rightarrow |\beta_{ni}| < M^{-1}\varepsilon$, for all $n \in \{1, 2, \ldots, N\}$. Then, for any $j \in \mathbb{N}$ and $i \ge N_1$, we have that

$$\begin{aligned}
|\mathrm{y}'_i(T\mathrm{y}_j)| = \left|\sum_{n=1}^{\infty} h_n(\mathrm{y}_j)\beta_{ni}\right| &= \left|\sum_{n=1}^{N} h_n(\mathrm{y}_j)\beta_{ni} + \sum_{n=N+1}^{\infty} h_n(\mathrm{y}_j)\beta_{ni}\right| \\
&\le \max\left\{\left|\sum_{n=1}^{N} h_n(\mathrm{y}_j)\beta_{ni}\right|, \left|\sum_{n=N+1}^{\infty} h_n(\mathrm{y}_j)\beta_{ni}\right|\right\} \\
&\le \max\left\{\max_{1\le n\le N} |h_n(\mathrm{y}_j)\beta_{ni}|, \max_{n\ge N+1} |h_n(\mathrm{y}_j)\beta_{ni}|\right\} \\
&< \max\{MM^{-1}\varepsilon, \varepsilon\} \\
&\le \varepsilon
\end{aligned}$$

Therefore, $||f_i|| < \varepsilon$ whenever $i \ge N_1$. ■

An immediate consequence is that the identity operator on c_0 is not a compact operator. Another consequence of the previous result is the following characterization of compact operators.

4.4.10 Proposition. *Let $T \in L(c_0)$ and let $(\mathrm{y}_i)_{i\in\mathbb{N}}$ be a basis of c_0. Then T is compact if and only if* $\lim_i \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T\mathrm{y}_j)| = 0$.

Proof. In the proof of 4.4.9, we proved that $\lim_i \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T\mathrm{y}_j)| = 0$, whenever T is compact. Now suppose that $\lim_i \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T\mathrm{y}_j)| = 0$. Again, by the proof of 4.4.9, for each $i \in \mathbb{N}$, the map $f_i : c_0 \to \mathcal{H}(i)$, defined by $f_i(\mathrm{x}) := \sum_{j=1}^{\infty} \mathrm{y}'_i(T\mathrm{y}_j)\mathrm{y}'_j(x)$, is a continuous linear functional on c_0 such that $||f_i|| = \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T\mathrm{y}_j)|$ and $T\mathrm{x} = \sum_{i=1}^{\infty} f_i(\mathrm{x})\mathrm{y}_i$ for all $\mathrm{x} \in c_0$. By hypothesis, $\lim_i ||f_i||||\mathrm{y}_i|| = \lim_i ||f_i|| = 0$, which implies that the sequence $(\sum_{i=1}^{n} f_i(\cdot)\mathrm{y}_i)_n$ uniformly converges to T, i.e. T is compact. ∎

At the beginning of the proof 4.4.9 we saw that if $\{\mathrm{y}_i : i \in \mathbb{N}\}$ is a basis of c_0 and $T \in L(E, F)$ then

$$T\mathrm{x} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \mathrm{y}'_i(T\mathrm{y}_j)\mathrm{y}'_j(x) \right) \mathrm{y}_i.$$

As in the classical case, this implies that any $T \in L(c_0)$ (not necessarily compact) is completely determined by the infinite matrix $[T] := (\mathrm{y}'_i(T\mathrm{y}_j))_{i,j\in\mathbb{N}}$. Actually, more can be said about this:

4.4.11 Theorem. *Let $\{\mathrm{y}_i : i \in \mathbb{N}\}$ be a basis of c_0. The correspondence which associates to any operator $T \in L(c_0)$ the infinite matrix $[T] = (\mathrm{y}'_i(T\mathrm{y}_j))_{i,j\in\mathbb{N}}$ defines an isometric linear isomorphism of $L(c_0)$ onto the Banach space of infinite matrices $A = (\alpha_{ij})_{i,j\in\mathbb{N}}$ with entries in $\mathcal{H}(i)$ satisfying $||A|| := \sup\{|\alpha_{ij}| : i, j \in \mathbb{N}\} \in G^{\#}$ and* $\lim_{i\in\mathbb{N}} |\alpha_{ij}| = 0$ *for all $j \in \mathbb{N}$.*

Proof. The proof of [14, 1.1] can be easily adapted to our case. ∎

4.4.12 Remark. *It follows that any continuous linear operator $T \in L(c_0)$, can be identified with a bounded matrix whose column vectors are in c_0:*

$$[T] = \begin{array}{c} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow & \\ 0 & 0 & 0 & \end{matrix} \end{array}$$

where $\alpha_{ij} = \mathrm{y}_i'(T\mathrm{y}_j)$ *for all* $i, j \in \mathbb{N}$.

Regarding operators that admit adjoint we can do the following analysis. If $T \in L(c_0)$ *has an adjoint* T^**, then for a given basis* $\{\mathrm{y}_i : i \in \mathbb{N}\}$ *of* c_0*, we have that*

$$\mathrm{y}_i'(T^*\mathrm{y}_j) = \frac{\langle T^*\mathrm{y}_j, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle} = \frac{\langle \mathrm{y}_j, T\mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle} = \frac{\overline{\langle T\mathrm{y}_i, \mathrm{y}_j\rangle}}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle} = \overline{\mathrm{y}_j'(T\mathrm{y}_i)}\frac{\langle \mathrm{y}_j, \mathrm{y}_j\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}$$

Since $\langle \mathrm{y}_i, \mathrm{y}_i\rangle$ *is not necessarily equal to* 1*, the matrix* $[T^*]$ *is not necessarily* $\overline{[T]}^t$*. However,* $|\mathrm{y}_i'(T^*\mathrm{y}_j)| = |\mathrm{y}_j'(T\mathrm{y}_i)|$ *and by 4.3.2 and 4.4.11 we conclude that* $T \in L(c_0)$ *admit an adjoint if and only if the row vectors of* $[T]$ *are in* c_0*. In this case we have:*

$$[T] = \begin{array}{c} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{matrix} \to 0 \\ \to 0 \\ \to 0 \\ \end{matrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow & & \\ 0 & 0 & 0 & & \end{matrix} \end{array}$$

where $\alpha_{ij} = \mathrm{y}_i'(T\mathrm{y}_j)$ *for all* $i, j \in \mathbb{N}$.

Regarding compact operators, according to 4.4.10, an operator $T \in c_0$ *is compact if and only if the row vectors of* $[T]$ *form a null sequence in* ℓ^∞*, i.e.*

$$[T] = \begin{array}{c} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{matrix} \|r_1\| \\ \|r_2\| \\ \|r_3\| \\ \downarrow \\ 0 \end{matrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow & & \\ 0 & 0 & 0 & & \end{matrix} \end{array}$$

where $\alpha_{ij} = \mathrm{y}_i'(T\mathrm{y}_j)$ *for all* $i, j \in \mathbb{N}$*, and* $r_i = (\alpha_{i1}, \alpha_{i2}, \dots) \in \ell^\infty$ *is the* i*-th row* *vector of* $[T]$.

With this characterization in mind, the author has developed the following proof for the next result:

4.4.13 ★ Proposition. *Let* $T \in L(c_0)$ *be an operator with adjoint operator* T^**. If* T *is compact, so is* T^**.*

Proof. Let $(\mathrm{y}_i)_{i\in\mathbb{N}}$ be an orthonormal sequence in c_0 and suppose that $T \in L(c_0)$ is a compact operator with adjoint operator T^*. According to 4.4.10, it is enough to prove that $\lim_i \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T^*\mathrm{y}_j)| = 0$. By 4.4.10, we know that T satisfies $\lim_j \sup_{i\in\mathbb{N}} |\mathrm{y}'_j(T\mathrm{y}_i)| = 0$. Also notice that

$$|\mathrm{y}'_i(T^*\mathrm{y}_j)| = \left|\frac{\langle T^*\mathrm{y}_j, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\right| = \left|\frac{\langle \mathrm{y}_j, T\mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\right| = \left|\frac{\langle T\mathrm{y}_i, \mathrm{y}_j\rangle}{\langle \mathrm{y}_j, \mathrm{y}_j\rangle}\right| = |\mathrm{y}'_j(T\mathrm{y}_i)|$$

Therefore, $\lim_j \sup_{i\in\mathbb{N}} |\mathrm{y}'_i(T^*\mathrm{y}_j)| = \lim_j \sup_{i\in\mathbb{N}} |\mathrm{y}'_j(T\mathrm{y}_i)| = 0.$ (1)

Since $T\mathrm{y}_i = \sum_{j=1}^{\infty} \mathrm{y}'_j(T\mathrm{y}_i)\mathrm{y}_j$, we have that $\lim_j |\mathrm{y}'_i(T^*\mathrm{y}_j)| = \lim_j |\mathrm{y}'_j(T\mathrm{y}_i)| = 0$ for all $i \in \mathbb{N}$. Therefore $R_i := \{n \in \mathbb{N} : |\mathrm{y}'_i(T^*\mathrm{y}_n)| = \sup_j |\mathrm{y}'_i(T^*\mathrm{y}_j)|\}$ is a nonempty finite set. Suppose for a moment that $\lim_i \sup_{j\in\mathbb{N}} |\mathrm{y}'_i(T^*\mathrm{y}_j)| \neq 0$. We will show that this leads to a contradiction. By our assumption, there exists $g \in G$ such that for all $N \in \mathbb{N}$, there exists $k_N \in \mathbb{N}$ satisfying $k_N \geq N$ and $\sup_j |y'_{k_N}(T^*\mathrm{y}_j)| \geq g.$ (2)

By (1), there is $M \in \mathbb{N}$ such that $j \geq M \Rightarrow \sup_i |\mathrm{y}'_i(T^*\mathrm{y}_j)| < g$. Since $|y'_{k_N}(T^*\mathrm{y}_n)| \geq g$ for every $n \in R_{k_N}$, we conclude that $R_{k_N} \subset \{1, 2, \ldots M\}$ for all $N \in \mathbb{N}$. (3)

Since $\lim_j |\mathrm{y}'_i(T^*\mathrm{y}_j)| = 0$ for all $i \in \mathbb{N}$, there exists $Q \in \mathbb{N}$ such that $n \geq Q \Rightarrow |\mathrm{y}'_i(T^*\mathrm{y}_n)| < g$ for $i \in \{1, 2, \ldots M\}$. (4)

But (2) implies $|\mathrm{y}'_{k_Q}(T^*\mathrm{y}_n)| \geq g$ for all $n \in R_{k_Q}$ where $k_Q \geq Q$ and $R_{k_Q} \subset \{1, 2, \ldots M\}$ by (3). This contradicts (4). ■

By using the previous analysis on the matrix representation of an operator and 4.4.13, we can conclude that an operator $T \in L(c_0)$ with adjoint T^* is compact whenever the row vectors and column vectors of $[T]$ form null sequences in c_0, i.e. T is identified by a matrix of the form:

$$\begin{array}{cccc}
 & \begin{array}{cccc} \|c_1\| & \|c_2\| & \|c_3\| & \to 0 \end{array} & & \\
[T] = & \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} & \begin{array}{c} \to 0 \\ \to 0 \\ \to 0 \\ \end{array} & \begin{array}{c} \|r_1\| \\ \|r_2\| \\ \|r_3\| \\ \downarrow \\ 0 \end{array} \\
 & \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \\ 0 & 0 & 0 & \end{array} & &
\end{array}$$

where $\alpha_{ij} = \mathrm{y}_i'(T\mathrm{y}_j)$ for all $i, j \in \mathbb{N}$, $r_i = (\alpha_{i1}, \alpha_{i2}, \dots) \in c_0$ is the i-th row vector of $[T]$, and $c_j = (\alpha_{1j}, \alpha_{2j}, \dots) \in c_0$ is the j-th column vector of $[T]$. Notice that $||c_j|| = ||T\mathrm{y}_j||$ for all $j \in \mathbb{N}$ by 4.4.8. We can summarize this characterization in the following result.

4.4.14 Theorem. *Let $T \in L(c_0)$ be an operator with adjoint operator T^* and let $(\mathrm{y}_j)_{j\in\mathbb{N}}$ be a basis of c_0. The following statements are equivalent:*

(a) T is compact,

(b) $\lim_{j\to\infty} ||T\mathrm{y}_j|| = 0$,

For an alternative proof of this result, we can adapt the proof of [3, Theorem 8] to our case.

4.5 Aspects of non-Archimedean Spectral Theory

One of the objectives of this thesis is to generalize the results of [2] and [3] to the case where the base field has a Krull valuation of higher rank, in particular, to obtain a spectral theorem for compact self-adjoint operators on c_0 over $\mathcal{H}(i)$. However, the author of this thesis has recently found a problem with the proof of the spectral theorem presented in [3, Theorem 10] and together with the authors of that paper has tried to save the result without success until the date of submission of this thesis. Therefore, it is not known whether the spectral theorem holds even when the base field has a valuation of rank 1. When trying to prove the spectral theorem for compact self-adjoint operators in the rank 1 case, the author has come to the conclusion that the proof of that result in the Archimedean case cannot be adapted to the non-Archimedean case because in the classical case the proof is based on the following facts:

(a) the spectrum of a compact self-adjoint operator is non-empty, which is proved by using Liouville's Theorem. In the non-Archimedean case Liouville's Theorem holds for functions $f : K \to K$ that admit a power series expansion. But it is unknown whether a function $f : K \to K$ that is differentiable has a power series expansion. In the classical case this is proved by using the Cauchy's Theorem which heavily depends on the connectedness of $\mathbb{C}$. In our case, any non-Archimedean valued field is totally disconnected.

(b) $||T||$ or $-||T||$ is an eigenvalue for T, when T is a compact self-adjoint operator on a Hilbert space. It is unknown in the non-Archimedean context whether for any given compact self-adjoint operator there exists an eigenvalue λ such that $|\lambda| = ||T||$. In the Archimedean case, the key point in the proof of this fact is that any sequence in a compact set has a convergent subsequence. In the non-Archimedean context we have an analogous result for compactoid sets but only stated for orthogonal sequences.

The pursuit of a spectral theorem for compact self-adjoint operators in the non-Archimedean case proved to be a very difficult task and the author has decided to continue with it in the near future as part of a postdoctoral research.

Now let's present some examples of operators on c_0, and try to find their eigenvalues to see how the non-Archimedean context may or may not affect the nature of the procedures employed, and how the results may differ depending on the rank of the base field.

4.5.1 Example. *Consider the Banach space c_0 over the Levi-Civita field $L[\mathbb{Q},\mathbb{C}]$. Let $T \in L(c_0)$ be the weighted shift operator defined by $T(x_1, x_2, x_3, \dots) = (0, dx_1, d^2x_2, \dots)$ (for the definition of d see 1.5.3). The matrix that defines this operator relative to the canonical basis of c_0 is:*

$$[T] = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ d & 0 & 0 & 0 & \cdots \\ 0 & d^2 & 0 & 0 & \cdots \\ 0 & 0 & d^3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

By our analysis in matrix representation done in section 4.4 we conclude that T is a compact operator because the norm of the i-th row vector of $[T]$ is $|d^{i-1}|$ and $\lim_{i\to\infty} |d^{i-1}| = 0$. Also, since each row vector of $[T]$ is in c_0 we conclude that T admits an adjoint T^* which is also compact and satisfies $[T^*] = [T]^t$. This operator does not have eigenvalues. Otherwise, there is $\lambda \in L[\mathbb{Q},\mathbb{C}]$ and a non-zero vector $\mathrm{x} = (x_n)_n \in c_0$ such that $T\mathrm{x} = \lambda\mathrm{x}$. On the one hand, $\lambda \neq 0$ since T is clearly injective. On the other hand, if the first non-zero coordinate of x is in the k-th position, then by definition of T, the first non-zero coordinate of $T\mathrm{x}$ is in the $(k+1)$-th position contradicting the equality $T\mathrm{x} = \lambda\mathrm{x}$.

Notice that if we replace d by any other scalar $\alpha \in L[\mathbb{Q},\mathbb{C}]$ such that $0 < |\alpha| < 1$ then T will still be a compact operator with adjoint and without eigenvalues. This is no longer true in the higher rank case. In fact, if the base field is $\mathcal{H}(i)$ with $G = \mathbb{Q}[x]$, then $\lim_{n\to\infty} |\alpha|^n \neq 0$ for all $\alpha \in \mathcal{H}(i) \setminus \{0\}$ because of 1.6.5. Hence T is no longer continuous.

Regarding the Archimedean context, if we replace c_0 by ℓ^2 and choose any $d \in \mathbb{C}$ such that $0 < |d| < 1$, then T will be a compact operator in ℓ^2 with adjoint and without eigenvalues. The proof of this fact is similar to the one presented above.

4.5.2 Example. *Consider the operator $T \in c_0$ defined by $T(x_1, x_2, x_3, \dots) = (x_2, x_1 + x_3, x_2 + x_4, \dots, x_n + x_{n+2}, \dots)$ over the field $\mathcal{H}(i)$. Notice that $T = S + S^*$ where S is the right-shift operator and S^* is the left-shift operator. The matrix that defines this operator relative to the canonical basis of c_0 is:*

$$[T] = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

By our analysis in matrix representation done in section 4.4 we conclude that T is not a compact operator because the norm of each row vector of $[T]$ is 1. Also, since each row vector of $[T]$ is in c_0 we conclude that T admits an adjoint T^*. Since $[T]$ is a symmetric matrix with entries in $\mathcal{H}$, it follows that T is self-adjoint. This operator does not have eigenvalues. Otherwise, there is $\lambda \in \mathcal{H}(i)$ and a non-zero vector $\mathrm{x} = (x_n)_n \in c_0$ such that $T\mathrm{x} = \lambda\mathrm{x}$. This implies that $x_2 = \lambda x_1$ and $x_{n+2} = \lambda x_{n+1} - x_n$ for all $n \in \mathbb{N}$. If $x_1 = 0$ then $\mathrm{x} = (x_n)_n = 0$ contradicting the hypothesis that x is an eigenvector of T. Thus $x_1 \neq 0$. Without lost of generality, let's assume that $x_1 = 1$ (if x is an eigenvector, so is $x_1^{-1}\mathrm{x}$) and hence $x_2 = \lambda$. We will show that we have a contradiction for all the possible values of λ.

Case 1: $|\lambda| < 1$. In this case, let's prove by induction that $|x_{n+2}| \leq 1$ and $|x_{2n+1}| = 1$ for all $n \in \mathbb{N}$. Once we do that we have $\mathrm{x} \notin c_0$ which is a contradiction.

For $n = 1$, we have that $x_{1+2} = x_3 = \lambda x_2 - x_1 = \lambda^2 - 1$. Since $|\lambda^2| < 1$, we have that $|x_3| = 1$ by the Isosceles triangle principle (see 1.2.3.b). Suppose that $|x_{k+2}| \leq 1$ for all $k \in \{1, \dots, m\}$. Then $|x_{m+3}| = |\lambda x_{m+2} - x_{m+1}| \leq \max\{|\lambda x_{m+2}|, |x_{m+1}|\} \leq 1$. Thus $|x_{n+2}| \leq 1$ for all $n \in \mathbb{N}$. Now suppose that $|x_{2k+1}| = 1$ for some $k > 1$. By the previous induction, we have that $|x_{2k+1}| \leq 1$ which implies that $|\lambda x_{2(k+1)}| < 1 = |x_{2k+1}|$. Thus $|x_{2(k+1)+1}| = |\lambda x_{2(k+1)} - x_{2k+1}| = |x_{2k+1}| = 1$ by 1.2.3.b. Therefore $|x_{2n+1}| = 1$ for all $n \in \mathbb{N}$.

Case 2: $|\lambda| > 1$. *In this case, let's prove by induction that* $|x_n| < |x_{n+1}|$ *for all* $n \in \mathbb{N}$. *Once we do that we have* $\mathrm{x} \notin c_0$ *which is a contradiction.*

For $n = 1$, *we have that* $x_{1+2} = x_3 = \lambda x_2 - x_1 = \lambda^2 - 1$. *Since* $1 < |\lambda| < |\lambda^2|$, *we have that* $|x_3| = |\lambda^2|$ *by 1.2.3.b. Suppose that* $|x_k| < |x_{k+1}|$ *for some* $k > 1$. *Then* $|x_k| < |x_{k+1}| < |\lambda x_{k+1}|$ *and thus we have that* $|x_{k+2}| = |\lambda x_{k+1} - x_k| = |\lambda x_{k+1}| > |x_{k+1}|$. *Thus* $|x_{n+2}| \leq 1$ *for all* $n \in \mathbb{N}$. *Therefore* $|x_n| < |x_{n+1}|$ *and for all* $n \in \mathbb{N}$.

Case 3: $|\lambda| = 1$. *In this case, we can prove in a similar way to the case 1 that* $|x_n| \leq 1$ *for all* $n \in \mathbb{N}$. *Let's prove by induction that* $1 \in \{|x_n|, |x_{n+1}|\}$ *for all* $n \in \mathbb{N}$. *Once we do that we have* $\mathrm{x} \notin c_0$ *which is a contradiction.*

Since $x_1 = 1$, *the statement is true for* $n = 1$. *Now suppose that* $1 \in \{|x_k|, |x_{k+1}|\}$ *for some* $k \in \mathbb{N}$. *Let's show that* $1 \in \{|x_{k+1}|, |x_{k+2}|\}$. *If* $|x_{k+1}| = 1$ *then we are done. Otherwise we have that* $|x_{k+1}| < 1$ *and by our inductive assumption we have* $|x_k| = 1$. *Thus* $|\lambda x_{k+1}| = |x_{k+1}| < |x_k|$. *Then by 1.2.3.b we have that* $|x_{k+2}| = |\lambda x_{k+1} - x_k| = |x_k| = 1$. *Therefore* $1 \in \{|x_n|, |x_{n+1}|\}$ *for all* $n \in \mathbb{N}$.

Notice that when we study the same operator T *in* ℓ^2 *over* $\mathbb{C}$, *it is also true that* T *is a self-adjoint non-compact operator without eigenvalues but in that case the proof of the last statement is not as simple as the previous one because there we do not have the Isosceles triangle principle 1.2.3.b. In that case the proof that the author came up with involves the resolution of the linear recurrence relation* $x_{n+2} = \lambda x_{n+1} - x_n$ *and showing that* $\lim_{n\to\infty} x_n \neq 0$ *for all* $\lambda \in \mathbb{C}$.

4.5.3 Example. *Although there is not yet a proof for a spectral theorem for compact self-adjoint operators in the non-Archimedean case, there are such operators that admit a spectral decomposition. Consider the Banach space* c_0 *over the Levi-Civita field* $L[\mathbb{Q}, \mathbb{C}]$. *Let* $T \in L(c_0)$ *be defined by*

$$T(x_1, x_2, x_3, \dots) = \left(\sum_{i=2}^{\infty} x_i d^{i-2}, x_1, dx_1, d^2x_1 \dots\right)$$

(for the definition of d see 1.5.3). The matrix that defines this operator relative to the canonical basis of c_0 *is:*

$$[T] = \begin{pmatrix} 0 & 1 & d & d^2 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ d & 0 & 0 & 0 & \cdots \\ d^2 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

By our analysis in matrix representation done in section 4.4 we conclude that T *is a compact operator because the norm of the i-th row vector of* $[T]$ *is* $|d^{i-2}|$ *for* $i \geq 2$ *and* $\lim_{i\to\infty} |d^{i-2}| = 0$*. Also, since each row vector of* $[T]$ *is in* c_0 *we conclude that* T *admits an adjoint. Furthermore,* T *is self-adjoint because* $[T]$ *is symmetric with entries in* $\mathcal{H}$*. Let's find the eigenvalues of* T*. Suppose* $\lambda \in \mathcal{H}(i)$ *is such that* $T\mathrm{x} = \lambda\mathrm{x}$ *for some non-zero* $\mathrm{x} = (x_n)_n \in c_0$*. Then we have the following relations:*

$$\lambda x_1 = \sum_{i=2}^{\infty} x_i d^{i-2} \qquad \textit{and} \qquad \lambda x_n = x_1 d^{n-2} \quad \textit{for all } n \geq 2$$

If $\lambda \neq 0$*, then* $x_1 \neq 0$ *because* $\mathrm{x} \neq 0$*. Hence* $x_n = \lambda^{-1}d^{n-2}x_1$ *for all* $n \geq 2$ *and by substituting this into the equality* $\lambda x_1 = \sum_{i=2}^{\infty} x_i d^{i-2}$ *we obtain* $\lambda^2 = \sum_{i=0}^{\infty} d^{2i}$*. Since the Levi-Civita field* $L[\mathbb{Q},\mathbb{R}]$ *is real-closed, we can take the square root and define* $\sigma := \sqrt{\sum_{i=0}^{\infty} d^{2i}}$*. Hence* σ *and* $-\sigma$ *are eigenvalues of* T *with eigenvectors* $\mathrm{y}_1 = (\sigma, 1, d, d^2, \ldots)$ *and* $\mathrm{y}_1 = (-\sigma, 1, d, d^2, \ldots)$ *respectively. Notice that the eigenspaces corresponding to* σ *and* $-\sigma$ *are* 1*-dimensional.*

If $\lambda = 0$*, then we conclude that the conditions* $x_1 = 0$ *and* $\sum_{i=2}^{\infty} x_i d^{i-2} = 0$ *determine the kernel of* T*. There are several vectors in* $\ker(T)$*, for instance, for each* $n \in \mathbb{N}$*, the vector* $\mathrm{z}_n := \left(0, 1, d, d^2, \ldots, d^{n-3}, \frac{d^{2(n-2)}-\sigma^2}{d^{n-2}}, d^{n-1}, \ldots\right) \in \ker(T)$*. In other words, the vectors* z_n *are* 0*-eigenvectors of* T*. Also it can be proved that* $\{\mathrm{z}_n : n \in \mathbb{N}\}$ *is linearly independent. Notice that* $\overline{span\{\mathrm{y}_1, \mathrm{y}_2, \mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3 \ldots\}} = c_0$*. In* *fact, if e_n is n-th canonical vector of c_0 then*

$$e_1 = \frac{1}{2\sigma}\mathrm{y}_1 - \frac{1}{2\sigma}\mathrm{y}_2 \qquad \textit{and}$$

$$e_n = \frac{d^{n-2}}{2\sigma^2}\mathrm{y}_1 + \frac{d^{n-2}}{2\sigma^2}\mathrm{y}_2 - \frac{d^{n-2}}{\sigma^2}\mathrm{z}_n \qquad \textit{for all } n \geq 2.$$

By using the Gram-Schmidt orthogonalization process in $\{\mathrm{z}_n : n \in \mathbb{N}\}$, and by appending y_1 and y_2 to the resulting collection, we obtain a basis of c_0 (after normalization) for which the operator T has the following matrix representation:

$$[T] = \begin{pmatrix} \sigma & 0 & 0 & 0 & \cdots \\ 0 & -\sigma & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Diagonal Operators

Since a spectral theorem has not been obtained yet, the author has decided to restrict the study of operators to the family of diagonal operators on c_0 over $\mathcal{H}(i)$ to check whether the theory of diagonal operators is affected in the non-Archimedean case over a valued field of higher rank.

4.5.4 Definition. *An operator $T \in L(c_0)$ is called* ***diagonal*** *if there exists basis $\{\mathrm{y}_n : n \in \mathbb{N}\}$ for c_0 formed by eigenvectors of T.*

It can be easily verified that if $T \in L(c_0)$ is diagonal and if the set of eigenvectors $\{\mathrm{y}_n : n \in \mathbb{N}\}$ is a basis for c_0 then, the eigenvalues of T form a sequence $\lambda = (\lambda_i)_i$ in ℓ^∞ such that, for each $\mathrm{x} \in c_0$:

$$T\mathrm{x} = \sum_{i=1}^{\infty} \lambda_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$$

The matrix representation of T with respect to the basis $\{\mathrm{y}_n : n \in \mathbb{N}\}$ will be the diagonal matrix:

$$[T] = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_3 & 0 & \cdots \\ 0 & 0 & 0 & \lambda_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

By 4.4.11, it follows that $||T|| = \sup\{|\lambda_i| : i \in \mathbb{N}\} = ||\lambda||$. By 4.4.12, we can conclude that T admits adjoint and $[T^*] = \overline{[T]^t}$. In fact, even when $\langle \mathrm{y_i}, \mathrm{y_i} \rangle \neq 1$ we have that $\lambda_i = \mathrm{y}'_i(T^*\mathrm{y}_i) = \overline{\mathrm{y}'_i(T\mathrm{y}_i)}\frac{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle} = \overline{\mathrm{y}'_i(T\mathrm{y}_i)} = \overline{\lambda_i}$ (compare with 4.4.12). Hence for each $\mathrm{x} \in c_0$ we have that

$$T^*\mathrm{x} = \sum_{i=1}^{\infty} \overline{\lambda_i} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$$

and therefore T is self-adjoint if and only if $\lambda \in c_0(\mathcal{H})$, i.e. $\lambda_i = \overline{\lambda_i}$ for all $i \in \mathbb{N}$.

Similarly, by 4.4.12 we have that T is compact if and only if $\lambda \in c_0$.

Now we are in position to present our decomposition theorem for diagonal operators on c_0.

4.5.5 Proposition. *For every diagonal operator $T \in L(c_0)$, there are unique self-adjoint operators $T_1, T_2 \in L(c_0)$ such that $T = T_1 + iT_2$, $T^* = T_1 - iT_2$ and $||T|| = \max\{||T_1||, ||T_2||\}$.*

Proof. Let $T \in L(c_0)$ be a diagonal operator and let $\{\mathrm{y}_1, \mathrm{y}_2, \ldots\}$ be a basis for c_0 formed by eigenvectors of T. Then there exists a unique $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in \ell^\infty$ such that $T\mathrm{x} = \sum_{i=1}^{\infty} \lambda_i \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i$, for all $\mathrm{x} \in c_0$. There are unique $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ and $\beta = (\beta_i)_{i \in \mathbb{N}}$ in $\ell^\infty(\mathcal{H})$, such that $\lambda = \alpha + i\beta$ (see 4.2.8). Define T_1 and T_2 as follows: $T_1\mathrm{x} = \sum_{i=1}^{\infty} \alpha_i \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i$, and $T_2\mathrm{x} = \sum_{i=1}^{\infty} \beta_i \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i$. Thus T_1 and T_2 are self-adjoint diagonal operators on c_0. It is clear that $T = T_1 + iT_2$ and $T^* = T_1 - iT_2$. Finally, $||T|| = ||\lambda|| = \max\{||\alpha||, ||\beta||\} = \max\{||T_1||, ||T_2||\}$ by 4.2.8. ■

The next result shows the basic properties regarding the composition of diagonal operators. Notice the special considerations made in the proof of (e) since the inequality takes place in $G^{\#}$.

4.5.6 Proposition. *Let* $\{\mathrm{y}_i : i \in \mathbb{N}\}$ *be a basis of* c_0 *and let* $S, T \in L(c_0)$ *be diagonal operators defined by*

$$S\mathrm{x} := \sum_{i=1}^{\infty} \alpha_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i \qquad \textit{and} \qquad T\mathrm{x} := \sum_{i=1}^{\infty} \beta_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$$

for all $\mathrm{x} \in c_0$ *and for some* $\alpha = (\alpha)_{i\in\mathbb{N}}$ *and* $\beta = (\beta_i)_{i\in\mathbb{N}}$ *in* ℓ^∞. *Then the following statements hold.*

(a) $(ST)\mathrm{x} = \sum_{i=1}^{\infty} \alpha_i \beta_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$, *for all* $\mathrm{x} \in c_0$.

(b) $ST = TS$

(c) If $T \neq 0$ *satisfies* $T^n = T$ *for some* $n \geq 2$, *then* $|\beta_i| = 1$ *for every* $i \in \mathbb{N}$. *In particular,* T^{n-1} *is a normal projection.*

(d) T *has an inverse in* $L(c_0)$ *if and only if* $\beta_i \neq 0$ *for all* $i \in \mathbb{N}$, $(\beta_i^{-1})_i \in \ell^\infty$. *In this case* $T^{-1}\mathrm{x} = \sum_{i=1}^{\infty} \beta_i^{-1} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$, *for all* $\mathrm{x} \in c_0$.

(e) T *is invertible if and only if* $\inf_i |\beta_i| > 0$.

Proof. (a): for any $\mathrm{x} \in c_0$, we have that:

$$(ST)\mathrm{x} = \sum_{i=1}^{\infty} \alpha_i \frac{\langle T\mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i = \sum_{i=1}^{\infty} \alpha_i \frac{\langle \sum_{j=1}^{\infty} \beta_j \frac{\langle \mathrm{x}, \mathrm{y}_j \rangle}{\langle \mathrm{y}_j, \mathrm{y}_j \rangle} \mathrm{y}_j, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i = \sum_{i=1}^{\infty} \alpha_i \beta_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i.$$

The statement *(b)* follows immediately from *(a)*.

(c): By *(a)* we have that for every $n \in \mathbb{N}$ and $\mathrm{x} \in c_0$,

$$T^n\mathrm{x} = \sum_{i=1}^{\infty} \beta_i^n \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i.$$

Let $A := \{i \in \mathbb{N} : \beta_i \neq 0\}$. If $T^n = T \neq 0$ for $n \geq 2$, then $\beta_i^n = \beta_i$ and hence $|\beta_i| = 1$ for every $i \in A$. In particular, we have that that $\beta_i^{n-1} = 1$ for each $i \in A$. Then

$$T^{n-1}\mathrm{x} = \sum_{i \in A} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$$

and therefore, the result follows from 4.3.4.

(d): Suppose that T has an inverse in $L(c_0)$. Then, for each $i \in \mathbb{N}$, $T\mathrm{y}_i = \beta_i \mathrm{y}_i$ implies that $\beta_i \neq 0$, otherwise T is not injective. Hence $T^{-1}\mathrm{y}_i = \beta_i^{-1}\mathrm{y}_i$ and thus $|\beta_i^{-1}| = ||\beta_i^{-1}\mathrm{y}_i|| = ||T^{-1}\mathrm{y}_i|| \leq ||T^{-1}||$. Therefore $(\beta_i^{-1})_i \in \ell^\infty$. By continuity of T^{-1}, for every $\mathrm{x} \in c_0$ we have that:

$$T^{-1}\mathrm{x} = T^{-1}\left(\sum_{i=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i\right) = \sum_{i=1}^{\infty} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} T^{-1}\mathrm{y}_i = \sum_{i=1}^{\infty} \beta_i^{-1} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i.$$

(e): In the proof of *(d)* we showed that, if T is invertible, then $0 < |\beta_i^{-1}| \leq ||T^{-1}||$ for all $i \in \mathbb{N}$. Since G is cofinal in $G^\#$, we can choose $h \in G$ such that $||T^{-1}|| \leq h$. Hence, $0 < h^{-1} \leq |\beta_i|$ for all $i \in \mathbb{N}$ and therefore, $\inf_i |\beta_i| \geq h^{-1} > 0$. Conversely, suppose that $\inf_i |\beta_i| > 0$. Hence, $\beta_i \neq 0$ for all $i \in \mathbb{N}$. Put $\alpha := (\beta_i^{-1})_i$. Since G is coinitial in $G^\#$, we can choose $g \in G$ such that $g < \inf_i |\beta_i|$. Hence, $0 < g \leq |\beta_i|$ for all $i \in \mathbb{N}$. Then, it follows that $|\beta_i^{-1}| \leq g^{-1}$ for all $i \in \mathbb{N}$ and therefore, $\sup_i |\beta_i^{-1}| < \infty$. Hence, $\alpha \in \ell^\infty$ and by using *(a)*, we can verify that $S\mathrm{x} := \sum_{i=1}^{\infty} \beta_i^{-1} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$ is the inverse operator of T. Thus we have proved *(d)* and *(e)*. ∎

The next result characterizes those compact diagonal operators on c_0 that are cyclic of finite order, i.e. those whose compositions form a finite magma.

4.5.7 Proposition. *Let $\{\mathrm{y}_i : i \in \mathbb{N}\}$ be a basis of c_0. Consider a nonzero compact diagonal operator $T \in L(c_0)$ defined by*

$$T\mathrm{x} := \sum_{i=1}^{\infty} \beta_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$$

for all $\mathrm{x} \in c_0$ *and for some* $\beta = (\beta_i)_{i\in\mathbb{N}}$ *in* c_0. *If* $T^n = T$ *for some* $n \geq 2$, *then* $\mathrm{Im}(T)$ *is finite dimensional, the set* $A = \{n \in \mathbb{N} : \beta_n \neq 0\}$ *is finite and* $|\beta_n| = 1$ *for every* $n \in A$. *In particular,* T^{n-1} *is a normal projection.*

Proof. By 4.5.6.a, we have that for every $n \in \mathbb{N}$ and $\mathrm{x} \in c_0$,

$$T^n\mathrm{x} = \sum_{i=1}^{\infty} \beta_i^n \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i.$$

If $T^n = T \neq 0$, then $\beta_i^n = \beta_i$ and hence $|\beta_i| = 1$ for every $i \in A$. Since $\lim_i \beta_i = 0$, A must be finite. Then $\mathrm{Im}(T) = [y_n : n \in A]$. The rest follows from 4.5.6.c. ■

Let's finish the study of diagonal operators with an inspection of their spectra.

4.5.8 Definition. *Let* $T \in L(c_0)$. *The* ***spectrum*** *of* T *is the set*

$$\sigma(T) := \{\mu \in \mathcal{H}(i) : T - \mu I \textit{ is not invertible}\}.$$

The ***spectral radius*** *of* T *is defined as*

$$r_\sigma(T) := sup\{|\mu| : \mu \in \sigma(T)\}.$$

4.5.9 ★ Proposition. *Let* $T \in L(c_0)$ *be a diagonal operator defined by*

$$T\mathrm{x} := \sum_{i=1}^{\infty} \lambda_i \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i,$$

for each $\mathrm{x} \in c_0$, *where* $(\mathrm{y}_i)_{i\in\mathbb{N}}$ *is a basis of* c_0 *formed by eigenvectors of* T *and* $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ *in* ℓ^∞. *Then the spectrum of* T *is* $\sigma(T) = \overline{\{\lambda_n : n \in \mathbb{N}\}}$ *and the spectral radius of* T *is* $||T||$.

Proof. Let $\mu \in \ell^\infty$. For each $\mathrm{x} \in c_0$ we have that:

$$(T - \mu I)\mathrm{x} = \sum_{i=1}^{\infty} \lambda_i \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i - \sum_{i=1}^{\infty} \mu \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i = \sum_{i=1}^{\infty}(\lambda_i - \mu) \frac{\langle \mathrm{x}, \mathrm{y}_i\rangle}{\langle \mathrm{y}_i, \mathrm{y}_i\rangle}\mathrm{y}_i.$$

On the one hand, by 4.5.6.e, $\mu \in \sigma(T)$ if and only if $\inf_{n\in\mathbb{N}} |\lambda_n - \mu| = 0$. On the other hand, $\inf_{n\in\mathbb{N}} |\lambda_n - \mu| = 0$ if and only if there exists a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $\lim_k \lambda_{n_k} = \mu$. This is equivalent to $\mu \in \overline{\{\lambda_n : n \in \mathbb{N}\}}$. Furthermore, we have that $||T|| = \sup\{|\lambda_n| : n \in \mathbb{N}\} \leq r_\sigma(T)$. Suppose that the inequality is strict. Then there exists $\mu \in \overline{\{\lambda_n : n \in \mathbb{N}\}}$ such that $|\lambda_n| < |\mu|$ for all $n \in \mathbb{N}$. Hence $0 < |\mu| = |\mu - \lambda_n| = \inf_n |\mu - \lambda_n|$ and therefore $\mu \notin \sigma(T)$ (4.5.6.e), a contradiction. ■

Notice that although Proposition 4.5.9 also holds in the Archimedean case, the proof presented above makes use of the Isosceles triangle principle.

Conclusions

The author has verified that the presented results for diagonal operators hold also in the Archimedean case, where sometimes the difference is only detected in the proofs.

In the same vein, the author has verified (see idea below) that when the non-Archimedean valued field has rank 1, the spectrum of an arbitrary operator $T \in L(c_0)$ is closed and bounded by $||T||$. Notice that every closed bounded subset of a field K is compact if and only if K is locally compact. In our case, $\mathcal{H}(i)$ is not locally compact (for a complete characterization of locally compact valued fields see [8, 2.32, 2.33]). Additionally, the space of invertible operators is open in $L(c_0)$: if $T, S \in L(c_0)$, T is invertible and $||T - S|| < ||T||$ then S is also invertible. The idea of the proofs in the non-Archimedean context are very similar to the ones in the Archimedean case, where the differences are mainly related to:

(a) the use of the inequality $||\sum_n T^n|| \leq \sup ||T^n||$ instead of $||\sum_n T^n|| \leq \sum_n ||T^n||$,

(b) the convergence of $\sum_n T^n$ is obtained from the convergence $||T^n|| \to 0$, while in the Archimedean case a comparison with a convergent geometric series takes place.

A more substantial difference is visible when we try to obtain the mentioned results using a non-Archimedean valued field of higher rank. The main difficulty in this case is the lack of the inequality $||TS|| \leq ||T||\,||S||$ for $T, S \in L(c_0)$. This inequality requires the existence of a product operation defined on $G^{\#}$ that extends the action of G on $G^{\#}$. The problem is that the only two operations that meet the requirements are not well-behaved, in the sense that $G^{\#}$ will not be a group and there are elements $\tau \in G^{\#}$ such that $s\tau = \tau$ no matter how small $s \in G$ is chosen. Nonetheless, not all is lost. If we consider $S, T \in L(c_0)$ to be compact and self-adjoint, we have that $||S||$ and $||T||$ are elements of G, i.e. the suprema are attained by the valuation of some entry in their respective associated matrix. In that case the inequality $||TS|| \leq ||S||||T||$ is back in the game and therefore the results mentioned above also hold for the higher rank case for compact and self-adjoint operators.

Chapter 5

Future work

While preparing this thesis, the author came across several interesting problems that would require a significant amount of time to try to solve and hence did not fit in the time line of producing this thesis. Since the resolution of those problems will improve the current understanding of non-Archimedean Functional Analysis, the author plans to work on them in his postdoctoral research.

5.1 Is there a spectral theorem for compact and self-adjoint operators on c_0?

As it is indicated in Section 4.5, it is unknown whether there is an spectral theorem for compact and self-adjoint operators on c_0 even in the rank-1 case. So far, the compact and self-adjoint operators in $L(c_0)$ that the author has studied can be separated in two groups: (1) operators that are hard to study and it is unclear whether they have any eigenvalue or not, and (2) operators that satisfy the spectral theorem. Due to the difficulty to find a counterexample the author is now more inclined to the possibility that the spectral theorem holds in the rank-1 case.

5.2 Characterization of Banach spaces with the complementation property

Let us say that a Banach space E has the **complementation property** if every closed subspace of E is complemented, i.e. for each closed subspace D of E, there exits another closed subspace F of E such that $E = D + F$ and $D \cap F = \{0\}$. Notice that orthogonality is not involved in the complementation property.

In classical Functional Analysis, J. Lindenstrauss and L. Tzafriri have proved [25] that a Banach space over $\mathbb{R}$ or $\mathbb{C}$ has the complementation property if and only if it is linearly homeomorphic to a Hilbert space.

An interesting question is the following: what Banach spaces over a non-Archimedean valued field have the complementation property? In a way, this question tries to identify the "Hilbert spaces" in the non-Archimedean context.

There are some partial answers to this question:

(a) If the valuation of the base field is discrete then every Banach space has the complementation property (see [45, 4.7]).

(b) If the valuation of the base field is dense, and if the Banach space is of countable type, then the latter has the complementation property (see 2.4.15).

The author of this thesis conjectures that if the valuation of the base field is dense, then a Banach space has the complementation property if and only if it is of countable type.

5.3 Is being of countable type an inheritable property for normed spaces?

In the case $rank(G) = 1$, the subspaces of a normed space of countable type are again of countable type (see [32, 2.3.14]). However, when we are dealing with valuations of higher rank, the question does not have a complete answer. According to 3.5.1 if E is a Banach space of countable type with basis field K, then each closed subspace of E is of countable type whenever one of the following conditions is satisfied:

(a) K is non-metrizable, or

(b) G is principal, or

(c) each B_K-submodule of K is countably generated.

The answer otherwise is unknown, this is the case when G is the union of a strictly increasing sequence of convex subgroups, and G has a countable coinitial set and not all the B_K-submodules of K are countably generated.

As H. Ochsenius and W.H. Schikhof pointed out in [31, 2.5] a major step in solving the problem would be the answer to the question as to whether closed subspaces of c_0 are of countable type.

5.4 Is the quotient space of a spherically complete normed space spherically complete?

In the case $rank(G) = 1$, the answer is positive as it is proved in [45, 4.2]. Such a proof relies heavily on the countability of the coinitiality of the value group G. In the higher-rank case, the answer is unknown. But if the answer is positive then there is a strategy to prove that every subspace of a normed space of countable type is of countable type. This would be done by following the ideas of the proof of the following result:

5.4.1 Theorem ([45, 5.9])**.** *Let E be a Banach space with an orthogonal basis. Then every linear subspace of E has an orthogonal basis.*

5.5 Topological approach to Hilbert-like spaces

In classical Analysis, a Hilbert space $(H, (\cdot, \cdot))$ satisfies the following distinctive properties:

(a) The inner product $(\cdot, \cdot)$ is continuous with respect to the topology induced by the norm.

(b) H is a Banach space.

(c) There exists an orthonormal basis $\{e_\alpha : \alpha \in I\}$ such that every $x \in H$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} \lambda_{\alpha_n} e_{\alpha_n}$, where $\lambda_{\alpha_n} \in \mathbb{R}$ (or $\mathbb{C}$). If the basis is countable, then H is of countable type (see definition in 2.4.1 and characterization in 2.4.11).

(d) Every closed subspace D of H is normal complemented ($H = D \oplus D^{\perp}$).

(e) The topological dual of H is isometrically isomorphic to H (Riesz representation theorem).

If H is infinite-dimensional and has a countable orthogonal basis then H is linearly homeomorphic to ℓ^2.

Our objective is to find a characterization of spaces satisfying these properties in the context of non-Archimedean analysis using locally convex topologies.

Let $K = (K, |\cdot|)$ be a Cauchy complete non-Archimedean valued field. The valuation $|\cdot|$ is assumed to be nontrivial. Since we want to define a space satisfying properties that are analogous to (b) and (c), let's consider an infinite-dimensional Banach space E over K of countable type (see definitions in 2.1.1 and 2.4.1). By Theorem 2.4.12, E is linearly homeomorphic to $(c_0, ||\cdot||_\infty)$ (see definition in 2.1.3). Now we need an inner product $(\cdot,\cdot)$ on c_0 such that $||x||_\infty = \sqrt{|(x,x)|}$ (see 2.3.1). Narici proved in [29] that the inner product $(\cdot,\cdot) : c_0 \times c_0 \to K$ defined by

$$(x,y) := \sum_{n=1}^{\infty} x_n y_n$$

satisfies the equality $||x||_\infty = \sqrt{(x,x)}$ for every $x \in c_0$, if and only if the residue class field of K is formally real (see definitions in 1.3.4 and 1.4.1).

So let's assume that the residue class field of K is formally real.

As we mentioned in 2.3.3, it is impossible for the space $(c_0, ||\cdot||_\infty)$ to satisfy the property (d). But we still can ask if there exists a locally convex topology σ on c_0 such that:

- $(\cdot,\cdot)$ is σ-continuous,
- c_0 is of countable type with respect to σ,
- D is a σ-closed subspace of c_0 if and only if D is a complemented $||\cdot||_\infty$-closed subspace,
- $(c_0,\sigma)'$ is linearly isomorphic to c_0

In other words, we could look for a topology on c_0 that characterizes all the $||\cdot||_\infty$-closed subspaces of c_0 that admit orthocomplements.

Since we wish to have a result that is analogous to the Riesz representation theorem on (c_0,σ) (i.e. to satisfy (e)) let's consider the functionals of the form $x \mapsto (x,y)$ where $x, y \in c_0$. Then the function $p_y(x) := |(x,y)|$ is a continuous seminorm on c_0. If $R := \{p_y : y \in c_0\}$ then a good candidate to study would be the locally convex topology on c_0 induced by the family R, i.e $\sigma := \sigma(c_0, R)$.

Bibliography

[1] Aguayo, J., and Nova, M. Non-Archimedean Hilbert like spaces. *Bulletin of the Belgian Mathematical Society-Simon Stevin 14*, 5 (2007), 787–797.

[2] Aguayo, J., Nova, M., and Shamseddine, K. Characterization of compact and self-adjoint operators on free Banach spaces of countable type over the complex Levi-Civita field. *Journal of Mathematical Physics 54*, 2 (2013), 023503.

[3] Aguayo, J., Nova, M., and Shamseddine, K. Inner product on B^*-algebras of operators on a free Banach space over the Levi-Civita field. *Indagationes Mathematicae 26*, 1 (2015), 191–205.

[4] Alexandru, V., Popescu, N., Zaharescu, A., et al. All valuations on $K(X)$. *Journal of Mathematics of Kyoto University 30*, 2 (1990), 281–296.

[5] Alling, N. L. *Foundations of analysis over surreal number fields*, vol. 141 of *North-Holland Mathematics Studies.* Elsevier, 1987.

[6] Alonso, J., Martini, H., and Wu, S. On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces. *Aequationes mathematicae 83*, 1-2 (2012), 153–189.

[7] Bachman, G. *Introduction to p-adic numbers and valuation theory.* Academic paperbacks. Academic Press. Inc, 1964.

[8] Barria Comicheo, A., and Shamseddine, K. Summary on non-Archimedean valued fields. In *Advances in Ultrametric Analysis* (2018), A. Escassut, C. Perez-Garcia, and K. Shamseddine, Eds., vol. 704 of *Contemporary Mathematics*, Providence, RI: American Mathematical Society, pp. 1–36.

[9] Basu, S., Pollack, R., and Roy, M.-F. *Algorithms in real algebraic geometry, volume 10 of Algorithms and Computation in Mathematics*, vol. 10 of *Algorithms and Computation in Mathematics*. Springer Berlin Heidelberg, 2006.

[10] Bell, J. L., and Machover, M. *A course in mathematical logic*. North-Holland, 1977.

[11] Chang, C. C., and Keisler, H. J. *Model theory*, vol. 73 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1990.

[12] Clifford, A. H. Note on Hahn's theorem on ordered abelian groups. *Proceedings of the American Mathematical Society 5*, 6 (1954), 860–863.

[13] Dales, H. G., and Woodin, W. H. *Super-real fields: totally ordered fields with additional structure*. No. 14 in London Mathematical Society Monographs. Oxford University Press, 1996.

[14] Diarra, B. Bounded linear operators on ultrametric Hilbert spaces. *Afr. Diaspora J. Math 8*, 2 (2009), 173–181.

[15] Enflo, P. A counterexample to the approximation problem in Banach spaces. *Acta Mathematica 130*, 1 (1973), 309–317.

[16] Engler, A. J., and Prestel, A. *Valued fields*. Springer Monographs in Mathematics. Springer Science & Business Media, 2005.

[17] Fässler-Ullmann, A. On non-classical Hilbert spaces. *Expositiones Matematicae 3* (1983), 275–277.

[18] Gross, H., and Künzi, U.-M. On a class of orthomodular quadratic spaces. *Enseign. Math 31*, 187-212 (1985), 184.

[19] Hahn, H. Über die nichtarchimedischen größensysteme. In *Hans Hahn Gesammelte Abhandlungen Band 1/Hans Hahn Collected Works Volume 1.* Springer, 1995, pp. 445–499.

[20] Hall, J. F. Completeness of ordered fields. *arXiv preprint arXiv: 1101.5652* (2011).

[21] Hausner, M., and Wendel, J. G. Ordered vector spaces. *Proceedings of the American Mathematical Society 3*, 6 (1952), 977–982.

[22] Hungerford, T. W. *Algebra, volume 73 of graduate texts in mathematics.* Springer-Verlag, New York, 1980.

[23] Kaplansky, I. *Fields and rings.* Chicago Lectures in Mathematics. University of Chicago Press, 1972.

[24] Lang, S. *Algebra revised third edition*, vol. 211. Springer Science and Media, 2002.

[25] Lindenstrauss, J., and Tzafriri, L. On the complemented subspaces problem. *Israel Journal of Mathematics 9*, 2 (1971), 263–269.

[26] McCarthy, P. J. *Algebraic extensions of fields.* Dover Publications, Inc., 1991.

[27] Munkres, J. R. *Topology.* Prentice Hall, 2000.

[28] Murray, F. On complementary manifolds and projections in spaces L_p and l^p. *Transactions of the American Mathematical Society 41*, 1 (1937), 138–152.

[29] Narici, L., and Beckenstein, E. A non-Archimedean inner product. *Contemporary Mathematics 384* (2005), 187–202.

[30] Ochsenius, H., and Schikhof, W. H. Banach spaces over fields with an infinite rank valuation. In *p-adic Functional Analysis* (1999), J. Kakol, N. D. G.-D. Kimpe, and C. Perez-Garcia, Eds., vol. 207 of *Lecture Notes in Pure and Appl. Math.*, Marcel Dekker, pp. 233–293.

[31] Ochsenius, H., and Schikhof, W. H. Norm Hilbert spaces over Krull valued fields. *Indagationes Mathematicae 17*, 1 (2006), 65–84.

[32] Perez-Garcia, C., and Schikhof, W. H. *Locally Convex Spaces over Non-Archimedean Valued Fields*, vol. 119 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2010.

[33] Perrin, Y. Fonctions analytiques dans les corps valués de rang supérieur à un. *Compositio Mathematica 49*, 1 (1983), 51–74.

[34] Ribenboim, P. The new theory of ultrametric spaces. *Periodica Mathematica Hungarica 32*, 1-2 (1996), 103–111.

[35] Ribenboim, P. *The theory of classical valuations.* Springer Monographs in Mathematics. Springer Science & Business Media, 1999.

[36] Schikhof, W. H. Isometrical embeddings of ultrametric spaces into non-Archimedean valued fields. In *Indagationes Mathematicae (Proceedings)* (1984), vol. 87, Elsevier, pp. 51–53.

[37] Schikhof, W. H. *A crash course in p-adic Analysis.* P. Universidad Católica de Chile, Fac. Matemáticas, 2003.

[38] Schikhof, W. H. *Ultrametric Calculus: an introduction to p-adic analysis*, vol. 4 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2007.

[39] Schilling, O. F. G. *The Theory of Valuations*. Mathematical surveys. American Mathematical Society, 1950.

[40] Shamseddine, K. M., and Berz, M. Analysis on the Levi-Civita field, a brief overview. *Contemp. Math 508* (2010), 215–237.

[41] Shell, N. *Topological fields and near valuations*, vol. 135 of *Monographs and textbooks in pure and applied mathematics*. Dekker, 1990.

[42] Soler, M. P. Characterization of Hilbert spaces by orthomodular spaces. *Communications in Algebra 23*, 1 (1995), 219–243.

[43] Steinitz, E. Algebraische Theorie der Körper. *Journal für die reine und angewandte Mathematik 137* (1910), 167–309.

[44] Truss, J. K. *Foundations of mathematical analysis*. Oxford University Press, 1997.

[45] Van Rooij, A. *Non-Archimedean Functional Analysis*, vol. 51 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, 1978.