Hamiltonian vector fields on a space of curves on the 3-sphere

by

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A thesis submitted to the Faculty of Graduate Studies of

The University of Manitoba

in partial fulfilment of the requirements of the degree of

MASTER OF SCIENCE

Department of Mathematics

University of Manitoba

Winnipeg

Abstract

This thesis reviews aspects related to the integrability of a Hamiltonian system on a space of arc length parametrized curves on the unit sphere S^3 in $\mathbb{R}^4$ of a fixed length L. In particular, we find that the flow of the Hamiltonian vector field corresponding to the total torsion function $X(s) \mapsto \int_o^L \tau(s)ds$ generates the curve shortening equation. Additionally, we show that the total torsion function belongs to a hierarchy of Poisson commuting functions.

Acknowledgements

I would first like to thank my thesis supervisor Dr. Derek Krepski. I am and will forever be grateful for his valuable guidance, support and encouragement throughout my studies. He steered me in the right direction when I needed.

I would also like to thank Dr. Adam Clay and Dr. Saman Muthukumarana for being a part of my thesis committee and providing valuable comments on this thesis.

I would also like to thank the Department of Mathematics and my supervisors' NSERC grants for their financial support during my studies here in the University of Manitoba.

Finally, I would like to express my profound gratitude to my parents for their love, affection and support throughout my life. I would also like to thank my wife Lubana Yasmin for motivating me throughout my years of study.

This thesis is dedicated to my beloved parents

Contents

Introduction

Let us begin with the notion of Hamiltonian mechanics. Classical mechanics has two main points of view, Lagrangian formalism and Hamiltonian formalism. Here, we intend to focus only on Hamiltonian formalism. Consider n particles $A_1, \ldots, A_n$ moving in $\mathbb{R}^3$ under the influence of respective forces $F_1, \ldots, F_n$. Recall, Newton's second law of motion states that

$$F_i = m_i a_i, \qquad i = 1, \ldots, n,$$

where m_i denotes the mass of the i-th particle and a_i denotes its acceleration due to the force F_i acting on it. For simplicity, we assume the force is conservative, that is, there exist functions $V_i : \mathbb{R}^3 \to \mathbb{R},\ i = 1, \ldots, n$ typically called potentials such that $F_i = -\nabla V_i$. Letting q_i denote the position of the i-th particle, Newton's second law takes the form

$$-\nabla V_i = m_i \frac{d^2 q_i}{dt^2}, \qquad i = 1, \ldots, n, \tag{1}$$

where $\dfrac{d^2 q_i}{dt^2} = a_i$, acceleration of the i-th particle. Equation (1) represents a system of second order differential equations and a unique solution to these equations can be obtained given the initial position and velocity of the particles. A convenient way to deal with the system (1) is to introduce momentum coordinates $p_i = m_i \dfrac{dq_i}{dt}$, where $\dfrac{dq_i}{dt}$ denotes the velocity of the i-th particle. Let $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n)$

and introduce a function

$$H(q,p) = \sum_{i=1}^{n} \left(\frac{\|p_i\|^2}{2m_i} + V_i(q_i) \right)$$

called the Hamiltonian, which represents the sum of kinetic energy and potential energy of the system. Then Newton's second law (1) is equivalent to Hamilton's equations

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{p_i}{m_i} = \dfrac{\partial}{\partial p_i}\left(\dfrac{\|p_i\|^2}{2m_i} + V_i(q_i)\right) = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = m_i \dfrac{d^2 q_i}{dt^2} = -\nabla V_i(q_i) = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{2}$$

which is a system of first order differential equations. In the discussion above, the parameter space of possible positions $(q_1, \ldots, q_n) \in \mathbb{R}^{3n}$ of the n particles is known as the *configuration space*, while the position-momentum space $(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ is known as the *phase space* of the system.

The dynamical system given by (2) is known as a Hamiltonian system with the Hamiltonian H. There are a few advantages of the Hamiltonian formulation of the system (1). One of these advantages can readily be seen as Hamilton's equations are first order differential equations and hence relatively easier to solve than second order differential equations given by the system (1). Next, we will discuss how the Hamiltonian approach generalizes to other phase spaces.

Let us choose the configuration space N of a dynamical system to be an m-dimensional differentiable manifold. Then the phase space is given by the cotangent bundle T^*N of the manifold N. In the discussion above, $N = \mathbb{R}^{3n}$ and $T^*N = \mathbb{R}^{3n} \times \mathbb{R}^{3n}$. The cotangent bundle T^*N naturally has a symplectic structure, that is, T^*N is a $2m$-dimensional differentiable manifold equipped with a closed non-degenerate differential 2-form. See Example 1.1.2. This is the starting point of symplectic geometry, which generalizes the above discussion to 'phase spaces' that are not necessarily the cotangent bundle of some configuration space manifold.

Solutions to the system (2) can be viewed as curves $\gamma(t) = (q(t), p(t))$ on the phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$, whose tangent vectors must agree with the vector field

$$X_H = \Big(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}\Big).$$

Notice that the function H determines the vector field X_H on the right side of the equation (2). In the more general setting of symplectic geometry, how does a Hamiltonian $H : M \to \mathbb{R}$ determine a vector field on a "phase space" given by a symplectic manifold M? The answer to this question is precisely what a symplectic structure achieves. The Hamiltonian vector field X_H corresponding to the function H is determined via the relation

$$dH(V) = \omega(X_H, V), \quad \forall \text{ vector fields } V, \tag{3}$$

where ω is the symplectic form on M and $dH(V)$ denotes the directional derivative of H in the direction of V. In this way, one may write down Hamilton's equations as

$$\frac{d}{dt}(\gamma(t)) = X_H(\gamma(t)). \tag{4}$$

Then the solutions to (4) are integral curves of the Hamiltonian vector field X_H.

A Hamiltonian system (4) may incorporate conserved quantities. For example, the function H itself is conserved, i.e. H remains constant along the integral curves of X_H. This is easy to see in case of (2), since

$$\begin{aligned}
\frac{d}{dt}\Big(H(q,p)\Big) &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} \\
&= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\Big(\frac{\partial H}{\partial p_i}\Big) + \frac{\partial H}{\partial p_i}\Big(-\frac{\partial H}{\partial q_i}\Big) \\
&= 0.
\end{aligned}$$

Also H is conserved in the more general setting as it is immediate from equation (3)

$$dH(X_H) = \omega(X_H, X_H) = 0.$$

More generally, for any other function f, $df(X_H)$ represents the directional derivative of f in the direction X_H of the evolution of the system. Hence, f is conserved along the flow of the system if and only if $df(X_H) = 0$. In this case, we say the functions f and H Poisson commute (see section 1.4).

Other conserved quantities, also known as integrals of motion, may exist and if there are sufficiently many (see Definition 1.17) of these, then we say that the system (4) is completely integrable. Existence of enough integrals of motion allows one to introduce new coordinates in which the system is solvable by quadrature, where quadrature stands for integrals, definite or indefinite. Sometimes it is not possible to obtain an analytic solution to the system. In this case, numerical methods are often used to solve the system and the system is still considered integrable.

In case of infinite dimensional phase spaces, we can use a similar approach as in the finite dimensional case to write down Hamilton's equations and to look for integrals of motion. The focus of this thesis is to study the complete integrability of a Hamiltonian system defined on a particular infinite dimensional symplectic manifold.

Literature review

For the purpose of this thesis, we closely follow the paper [6] authored by V. Jurdjevic. In [6], V. Jurdjevic showed that the integral curves of Hamiltonian vector fields corresponding to respective integrals of motion generate some well known equations in physics and also discussed their integrability. More precisely, in [6], V. Jurdjevic used a symplectic structure on an infinite dimensional manifold $\mathfrak{M}$ of arc length parametrized curves $X(s)$ on the 3-sphere $S^3 = \{x \in \mathbb{R}^4 : \|x\| = 1\}$ to compute flows of Hamiltonian vector fields. We will return to the precise description of $\mathfrak{M}$ in Chapter 2. In what follows, we describe the formalism that lead to the equations under consideration.

Consider the function $f_2 : \mathfrak{M} \to \mathbb{R}$ given by $X(s) \mapsto \frac{1}{2}\int_0^L \kappa^2(s)ds$, where $\kappa(s)$ denotes the curvature of $X(s)$, and let X_{f_2} denote its Hamiltonian vector field. The flow of X_{f_2} defines a "curve" in $\mathfrak{M}$, which is a variation of curves $Y(s,t)$, with $Y(s,0) = X(s)$. As we will see in more detail in Chapter 2, the tangent vectors $\Lambda(s,t)$ along each curve $s \mapsto Y(s,t)$ are, roughly speaking, in $\mathbb{R}^3$. In [6], it is shown that $\Lambda(s,t)$ evolves according to the equation

$$\frac{\partial \Lambda}{\partial t} = \frac{\partial^2 \Lambda}{\partial s^2} \times \Lambda, \tag{5}$$

which is known as Heisenberg's magnetic equation (HME) in the literature (see for instance [17], [8], [6]). Jurdjevic also computes the Hamiltonian vector field corresponding to the function $f_3 = \int_0^L \left(\frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{4}\kappa^2\right)ds$ (see Theorem 7 in [6]), where $\tau(s)$ denotes torsion, and claims that the integral curves of the Hamiltonian vector field associated with the total torsion function $f_1 = \int_0^L \tau \; ds$ generate the equation

$$\frac{\partial \Lambda}{\partial t} = \kappa N \tag{6}$$

where N denote the unit normal vectors to the curves in $Y(s,t)$.

As shown in [6], the functions f_1, f_2, f_3 and

$$f_4 = \int_0^L \frac{1}{2}\Big((\kappa')^2 + \frac{1}{2}\kappa^2\tau^2(s) + \frac{1}{2}\kappa^2\tau(s) + \frac{1}{8}\kappa^2 - \frac{1}{8}\kappa^4\Big)ds$$

pairwise Poisson commute, which suggests they may be a part of an infinite list of Poisson commuting functions, that is, a completely integrable system.

Functions similar to the above appear in the paper [10] by J. Langer and R.

Perline, which considers a similar space of curves in Euclidean space $\mathbb{R}^3$. The authors of [10] showed that the families of curves $\gamma(s,t)$ in $\mathbb{R}^3$ evolve according to the filament equation $\dfrac{\partial \gamma}{\partial t} = \kappa B$ may be viewed as a completely integrable Hamiltonian system. Complete integrability of the filament equation implies that the space of curves possesses infinitely many functions which are pairwise Poisson commuting. The successive functions in this infinite list are obtained through the repeated application of a recursion operator. The recursion operator is constructed using two compatible Poisson structures on the space of curves.

The functions f_1, f_4 are also mentioned in [16], where the authors show the non-linear Schrödinger equation is a completely integrable Hamiltonian system.

In this thesis, we consider an infinite dimensional manifold $\mathfrak{M}$, a space of arc length parametrized curves on S^3. The manifold $\mathfrak{M}$ admits a symplectic structure, and we compute the Hamiltonian vector fields corresponding to the functions f_1, f_2, f_4 defined on $\mathfrak{M}$ and show that f_1, f_2, f_3, f_4 pairwise Poisson commute. It seems plausible to have an infinite list of Poisson commuting functions on $\mathfrak{M}$. However, at this point, we do not have any mechanism like the recursion operator mentioned in [10] to produce successive functions in this infinite list.

Outline of the thesis

This thesis consists of three chapters. In Chapter 1, we summarize the overall concepts that are necessary to understand this thesis. Introduction to symplectic manifolds, Hamiltonian vector fields and integrable systems are provided in this chapter.

Chapter 2 describes the space of curves of our interest along with some basic properties of the Lie group SU_2, the set of all 2×2 unitary matrices having determinant 1.

Chapter 3 contains all the results of this thesis. In particular, we establish the following result:

Proposition 3.1. Let $f_1 : \mathfrak{M} \to \mathbb{R}$ be the function given by $f_1 = \int_0^L \tau \, ds$ and let $\mathcal{X}_{f_1}$ denote its Hamiltonian vector field. If $g(t,s)$ denote the integral curves of $\mathcal{X}_{f_1}$ with tangent vectors $\frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t)$, then $\Lambda(s,t)$ evolves according to the *Curve Shortening equation* (CSE),

$$\frac{\partial \Lambda}{\partial t}(s,t) = \kappa(s,t)N(s,t). \tag{7}$$

Similarly, in Proposition 3.3, we give the evolution equation for the Hamiltonian vector field corresponding to the function f_4.

Finally, we show that the functions f_1, f_2, f_3, f_4 on $\mathfrak{M}$ pairwise Poisson commute. Suggestions for further research are described.

1

Mathematical Background

In this chapter, we present some required background for this study. The reader is assumed to be familiar with introductory differential geometry. In Section 1.1, we focus on discussing finite dimensional and infinite dimensional symplectic manifolds along with some basic definitions. In Section 1.2, we discuss Hamiltonian vector fields and their integral curves with examples. Section 1.3 reviews the notion of Lie bracket, Lie algebra and Poisson bracket. Section 1.4 contains the notion of complete integrability of a Hamiltonian system. We followed [1], [2], [9], [12] [14], [15] for the material presented in this chapter.

1.1 Symplectic manifolds

Let V be an N-dimensional vector space over $\mathbb{R}$, and let $\omega : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$ be a k-linear map. That is,

$$\omega(v_1, \ldots, \alpha v_i, \ldots, v_k) = \alpha\omega(v_1, \ldots, v_i, \ldots, v_k)$$

$$\omega(v_1, \ldots, v_i + v_i', \ldots, v_k) = \omega(v_1, \ldots, v_i, \ldots, v_k) + \omega(v_1, \ldots, v_i', \ldots, v_k)$$

for $\alpha \in \mathbb{R}$ and $v_1, \ldots, v_i, v_i', \ldots, v_k \in V$. The k-linear map ω is said to be skew-symmetric if for any pair of arguments

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

Definition 1.1. Let $\mathcal{M}$ be a smooth manifold. Let $k > 0$ be an integer. A *differential k-form* ω is a smooth assignment $p \mapsto \omega_p$, where $p \in \mathcal{M}$ and ω_p is a skew-symmetric k-linear map

$$\omega_p : \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{k} \to \mathbb{R}$$

where $T_p\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at p.

Definition 1.2. Let $\mathcal{M}$ be a smooth manifold and ω a differential k-form. The *exterior derivative* $d\omega$ is a differential $(k+1)$-form defined by

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \; X_i\Big(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})\Big) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \; \omega\Big([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}\Big) \end{aligned}$$

where $X_1, \ldots, X_{k+1}$ are smooth vector fields on $\mathcal{M}$, and the entries with a hat sign are absent in the expression. [14],[15].

Remark 1.1.1. For finite dimensional manifolds, the definition of exterior derivative as in Definition 1.2 is known to be equivalent to the local form in coordinates. That is, given a chart $(U, x_1, \ldots, x_m)$ on $\mathcal{M}$, the exterior derivative of the k-form $\omega = \sum_I f_I dx_I$ where I denotes multi-index $1 \le i_1 < \cdots < i_k \le m$ and $f_I \in \mathcal{C}^\infty(U)$ are functions is defined as

$$d\omega = \sum_I \sum_{l=1}^{m} \frac{\partial f_I}{\partial x_l} dx_l \wedge dx_I.$$

See Theorem 13 on page 213 in [14] for the proof of equivalence. On infinite dimensional manifolds, the local formula is inadequate and we use the global formula as a suitable substitute. Section 33 in Chapter 7 in [9] has a careful analysis of differential forms on infinite dimensional manifolds. As noted in Remark 33.22 therein (see also the Lemma following Remark 33.22) Definition 1.1 is compatible with the global definition given in Definition 1.2.

Let ω be a k-form and ξ be a l-form on $\mathcal{M}$. Then the exterior derivative operator d satisfies the following properties,

(a) d is linear.

(b) $d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^k\, \omega \wedge d\xi$.

(c) $d(d\omega) = 0$; in short $d^2 = 0$.

Definition 1.3. A differential k-form ω is said to be *exact* if $\omega = d\alpha$, where α is a $(k-1)$ form and it is said to be closed if $d\omega = 0$.

All exact forms are closed but not all closed forms are exact.

Definition 1.4. A differential 2-form ω is said to be *symplectic* if it is closed and non-degenerate. Here, non-degeneracy means that for all $p \in \mathcal{M}$, if there exists a vector $u \in T_p\mathcal{M}$ such that $\omega_p(u, v) = 0$ for all $v \in T_p\mathcal{M}$, then $u = 0$.

Definition 1.5. A *symplectic manifold* is a pair $(\mathcal{M}, \omega)$ where $\mathcal{M}$ is a manifold and ω is a symplectic form on $\mathcal{M}$. Sometimes, we refer to the symplectic form as a symplectic structure in this thesis.

On a finite dimensional manifold $\mathcal{M}$, the form ω being symplectic implies that dim $T_p\mathcal{M}$= dim $\mathcal{M}$ must be even. This is due to the fact that non-degenerate skew-symmetric bilinear forms can exist only on even dimensional vector spaces.

Example 1.1.1. The manifold $\mathcal{M} = \mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic manifold.

To check ω is closed, notice that

$$\begin{aligned}\omega &= \sum_{i=1}^{n} dx_i \wedge dy_i \\ &= d\Big(\sum_{i=1}^{n} x_i \; dy_i\Big)\end{aligned}$$

where $\sum_{i=1}^{n} x_i \; dy_i$ is a 1-form and hence ω is exact. Therefore ω is closed.

To prove the non-degeneracy of ω, let $u = \sum_{i=1}^{n} \Big(a_i \left.\frac{\partial}{\partial x_i}\right|_p + b_i \left.\frac{\partial}{\partial y_i}\right|_p \Big)$ be a vector in $T_p\mathcal{M}$, the tangent space to $\mathcal{M}$ at p. The form ω is non-degenerate at p if $\omega_p(u, v) = 0$ for all $v = \sum_{i=1}^{n} \Big(f_i \left.\frac{\partial}{\partial x_i}\right|_p + g_i \left.\frac{\partial}{\partial y_i}\right|_p \Big)$ in $T_p\mathcal{M}$, then $u = 0$. Now,

$$\begin{aligned}\omega_p(u, v) &= \sum_{i=1}^{n} \Bigg(dx_i\Big(a_i \left.\frac{\partial}{\partial x_i}\right|_p + b_i \left.\frac{\partial}{\partial y_i}\right|_p \Big) dy_i\Big(f_i \left.\frac{\partial}{\partial x_i}\right|_p + g_i \left.\frac{\partial}{\partial y_i}\right|_p \Big)\Bigg) \\ &\quad - \sum_{i=1}^{n} \Bigg(dy_i\Big(a_i \left.\frac{\partial}{\partial x_i}\right|_p + b_i \left.\frac{\partial}{\partial y_i}\right|_p \Big) dx_i\Big(f_i \left.\frac{\partial}{\partial x_i}\right|_p + g_i \left.\frac{\partial}{\partial y_i}\right|_p \Big)\Bigg) \\ &= \sum_{i=1}^{n} \Big(a_i g_i - b_i f_i \Big)\end{aligned}$$

So, $\sum_{i=1}^{n} \Big(a_i g_i - b_i f_i \Big) = 0, \forall f_i, g_i$. Let $g_i = a_i$ and $f_i = -b_i$. Then $\sum_{i=1}^{n} \Big(a_i^2 + b_i^2 \Big) = 0$ implies $a_i, b_i = 0$ and consequently $u = 0$. Thus if $\omega_p(u, v) = 0$ for any choice of v, then $u = 0$. Therefore, ω is non-degenerate.

Example 1.1.2. Let N be an n-dimensional manifold. Then its cotangent bundle $\mathcal{M} = T^*N$ naturally has a symplectic structure.

Let $\pi : T^*N \to N$ be the projection map and θ be a co-vector in T^*N. Then the 1-form α at θ is the map $\alpha_\theta : T_\theta(T^*N) \to \mathbb{R}$ given by $\alpha_\theta(v) = \theta\Big((T_\theta\pi)(v)\Big)$, where $v \in T_\theta(T^*N)$ and $T_\theta\pi$ denotes the tangent map of π at θ.

Again, let us choose a chart $\left(T^*U, q_1, \ldots, q_n, p_1, \ldots, p_n\right)$ on T^*N. Then the 1-form in terms of coordinates is defined as $\beta = \sum_{i=1}^{n} p_i \wedge dq_i$. The definition of 1-form is actually independent of the choice of charts on T^*N. This can be seen from the following calculation as both α and β agree on a basis $\left\{\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}\right\}$, $i = 1, \ldots, n$ for the tangent space of T^*N at any point θ. So,

(i) $\alpha_\theta\left(\frac{\partial}{\partial q_i}\right) = \theta\left(T_\theta\pi(\frac{\partial}{\partial q_i})\right) = \sum_{i=1}^{n} p_i \wedge dq_i\left(\frac{\partial}{\partial q_i}\right) = p_i$, since θ itself is a co-vector and $\beta\left(\frac{\partial}{\partial q_i}\right) = \sum_{i=1}^{n} p_i \wedge dq_i\left(\frac{\partial}{\partial q_i}\right) = p_i$.

(ii) $\alpha_\theta\left(\frac{\partial}{\partial p_i}\right) = \theta\left(T_\theta\pi(\frac{\partial}{\partial p_i})\right) = 0$, since $T_\theta\pi$ sends $\frac{\partial}{\partial p_i}$ to zero and $\beta\left(\frac{\partial}{\partial p_i}\right) = \sum_{i=1}^{n} p_i \wedge dq_i\left(\frac{\partial}{\partial p_i}\right) = 0$.

Set $\omega = -d\beta = \sum_{i=1}^{n} dq_i \wedge dp_i$. Then ω is closed and non-degenerate as shown in the previous example. Such a form ω is called the canonical symplectic form on T^*N.

In this thesis, we will work with an infinite dimensional manifold. In general, infinite dimensional manifolds are manifolds modeled on an infinite dimensional locally convex vector space as the finite dimensional manifolds are modeled on $\mathbb{R}^n$. There are various models of infinite dimensional manifolds in the literature such as Banach manifold, Hilbert manifold, Fréchet manifold etc. We will be using the Fréchet manifold here.

The focus of this thesis is to work on a particular Fréchet manifold which will be defined in Chapter 2. The differential forms on an infinite dimensional Fréchet manifold are defined in the same manner as defined on the finite dimensional manifolds. As a result, the formulas associated with the differential forms (such as, exterior derivative, Cartan's magic formula) on finite dimensional manifolds still hold on Fréchet manifold.

For every $p \in \mathcal{M}$ the symplectic form ω induces a map $\tilde{\omega}_p : T_p\mathcal{M} \to (T_p\mathcal{M})^*$ which is injective, by non-degeracy of $\omega_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. When this map is also surjective, ω is said to be strongly non-degenerate. In finite dimensions, these are equivalent; however in infinite dimensions, this is not the case. Thus sometimes ω is called 'weakly' non-degenerate.

1.2 Hamiltonian vector fields

Let $(\mathcal{M}, \omega)$ be a finite dimensional symplectic manifold and let $h : \mathcal{M} \to \mathbb{R}$ be a smooth function. The differential dh of h is a 1-form. By non-degeneracy of ω there exists a unique vector field X_h on $\mathcal{M}$ such that $\iota_{X_h}\omega = \omega(X_h, -) = dh$.

For infinite dimensional symplectic manifolds, it is possible that the 1-form dh of a function h does not equal $\omega(X, -)$ for any vector field X. This is due to the weak non-degeneracy of the symplectic form. In spite of this fact, Hamiltonian vector fields are defined in the same manner as in the finite dimensional case.

Definition 1.6. A vector field X_h on $(\mathcal{M}, \omega)$ for which $\omega(X_h, -) = dh$ is called a Hamiltonian vector field for the function $h : \mathcal{M} \to \mathbb{R}$.

Example 1.2.1. Let $(\mathbb{R}^2, \omega)$ be a symplectic manifold with the symplectic form $\omega = dx \wedge dy$. Also let the Hamiltonian function $h : \mathbb{R}^2 \to \mathbb{R}$ be given by $h(x, y) = x^2 + y^2$. We will compute the Hamiltonian vector field X_h at a point $p = (a, b)$ on $\mathbb{R}^2$. We need to show that $\omega(X_h, V) = dh(V)$, where X_h and V are vector fields of the form $X_h = f(x, y)\dfrac{\partial}{\partial x} + g(x, y)\dfrac{\partial}{\partial y}$ and $V = v_1(x, y)\dfrac{\partial}{\partial x} + v_2(x, y)\dfrac{\partial}{\partial y}$ respectively. Let $\gamma(t) = (a + tv_1, b + tv_2)$ be the flow generated by V. Then

$$
\begin{aligned}
dh_p(V) &= \left.\frac{d}{dt}\right|_0 h(a + tv_1, b + tv_2) \\
&= \left.\frac{d}{dt}\right|_0 [(a + tv_1)^2 + (b + tv_2)^2]
\end{aligned}
$$

$$= 2av_1 + 2tv_1^2 + 2bv_2 + 2tv_2^2\Big|_{t=0}$$

$$= 2av_1 + 2bv_2 \tag{1.1}$$

On the other hand,

$$\omega(X_h, V) = (dx \wedge dy)\Big(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y},\ v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\Big)$$

$$= dx(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y})\ dy(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y})$$

$$- dx(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y})\ dy(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y})$$

$$= fv_2 - gv_1 \tag{1.2}$$

From equation (1.1) and (1.2), we have $f = 2b$ and $g = -2a$. Therefore, the Hamiltonian vector field is $X_h = 2b\frac{\partial}{\partial x} - 2a\frac{\partial}{\partial y}$ corresponding to the Hamiltonian h.

Definition 1.7. Let X be a vector field on $\mathcal{M}$. For every point $p \in \mathcal{M}$, suppose there exists a unique curve $\Phi_t(p)$ such that $\Phi_0(p) = p$ and $\frac{d}{dt}[\Phi_t(p)] = X[\Phi_t(p)]$ for all t. Then the curve $\Phi_t(p)$ is said to be an *integral curve* or *local flow* of X. Then $\Phi_t(\Phi_s(p)) = \Phi_{t+s}(p)$ whenever both sides are defined.

Example 1.2.2. Let $X = 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$ be a vector field on $\mathcal{M} = \mathbb{R}^2$. Then the system of differential equations generated by X is,

$$\begin{cases} \dfrac{dx}{dt} = 2x \\[2ex] \dfrac{dy}{dt} = -2y \end{cases}$$

Solving the above equations by separation of variables with initial condition $p = (x_0, y_0)$, we obtain $x(t) = x_0e^{2t}$ and $y(t) = y_0e^{-2t}$. Thus an integral curve of X is

$\Phi_t(p) = (x_0 e^{2t}, y_0 e^{-2t})$.

1.3 Brackets

We can view a vector field on $\mathcal{M}$ as a linear map $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that satisfies the product rule,

$$X(fg) = X(f)g + fX(g)$$

for $f, g \in C^\infty(\mathcal{M})$. That is, X is a derivation of the algebra $C^\infty(\mathcal{M})$ of smooth functions. The vector field X acting on a smooth function f is given by,

$$X(f) = df(X)$$

where $df(X)$ is the directional derivative of f in the direction of X.

Example 1.3.1. For a pair of vector fields X_1, X_2 on $\mathcal{M}$, the commutator

$$[X_1, X_2] = X_1 \circ X_2 - X_2 \circ X_1 : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$$

is again a vector field. To show $[X_1, X_2]$ is a vector field, we verify that it satisfies the product rule stated above by direct calculation. We obtain,

$$\begin{aligned}(X_1 \circ X_2)(fg) &= X_1\Big(X_2(f)g + fX_2(g)\Big) \\ &= X_1(X_2(f))g + fX_1(X_2(g)) + X_2(f)X_1(g) + X_1(f)X_2(g) \qquad (1.3)\end{aligned}$$

Similarly,

$$(X_2 \circ X_1)(fg) = X_2(X_1(f))g + fX_2(X_1(g)) + X_1(f)X_2(g) + X_2(f)X_1(g) \qquad (1.4)$$

Subtracting equation (1.4) from (1.3) we get,

$$[X_1, X_2](fg) = X_1(X_2(f))g + fX_1(X_2(g)) - X_2(X_1(f))g - fX_2(X_1(g))$$

$$= (X_1 \circ X_2 - X_2 \circ X_1)(f)g + f(X_1 \circ X_2 - X_2 \circ X_1)(g)$$

$$= [X_1, X_2](f)g + f[X_1, X_2](g)$$

Therefore, $[X_1, X_2]$ is a vector field.

Definition 1.8. The vector field $[X_1, X_2] = X_1 \circ X_2 - X_2 \circ X_1$ is called the *Lie bracket* of the vector fields X_1, X_2.

Definition 1.9. The Lie derivative of a differential k-form ω is defined as

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega \tag{1.5}$$

This identity is known as *Cartan's magic formula.* See Theorem (4.3.3) in [12].

Proposition 1.10. If X_1 and X_2 are Hamiltonian vector fields on a symplectic manifold $(\mathcal{M}, \omega)$, then their Lie bracket $[X_1, X_2]$ is a Hamiltonian vector field with Hamiltonian $\omega(X_2, X_1)$. [2].

Proof. For the symplectic form ω we have,

$$\iota_{[X_1,X_2]}\omega = \mathcal{L}_{X_1}\iota_{X_2}\omega - \iota_{X_2}\mathcal{L}_{X_1}\omega$$

$$= d\iota_{X_1}\iota_{X_2}\omega + \iota_{X_1}d\iota_{X_2}\omega - \iota_{X_2}d\iota_{X_1}\omega - \iota_{X_2}\iota_{X_1}d\omega \quad \text{[Using (1.5)]}$$

$$= d\Big(\omega(X_2, X_1)\Big)$$

The second and third term vanish since $\iota_{X_2}\omega$ and $\iota_{X_1}\omega$ are exact forms and thus their exterior derivatives are zero. The fourth term also vanishes since ω is a symplectic form. $\square$

Definition 1.11. A *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* such that it satisfies the following properties:

(i) $[X_1, X_2] = -[X_2, X_1], \ \forall\ X_1, X_2 \in \mathfrak{g}$,

(ii) $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0, \ \forall\ X_1, X_2, X_3 \in \mathfrak{g}$

That is, the Lie bracket is skew-symmetric and satisfies the Jacobi identity.

Definition 1.12. Let $(\mathcal{M}, \omega)$ be a symplectic manifold. Let $f, g : \mathcal{M} \to \mathbb{R}$ be two smooth functions on $\mathcal{M}$. Then the *Poisson bracket* of f and g is defined as,

$$\{f, g\} = \omega(X_f, X_g).$$

1.4 Integrable systems

Definition 1.13. A *Hamiltonian system* is a triple $(\mathcal{M}, \omega, h)$, where $(\mathcal{M}, \omega)$ is a symplectic manifold and $h : \mathcal{M} \to \mathbb{R}$ is a smooth function, called the *Hamiltonian.* [2].

Theorem 1.14. Let $(\mathcal{M}, \omega, h)$ be a Hamiltonian system. Let $f \in C^\infty(\mathcal{M})$. Then $\{f, h\} = 0$ if and only if f is constant along integral curves of X_h. [2].

Proof. From the definition of Poisson bracket we have,

$$\begin{aligned}\{f, h\} &= -\ \omega(X_f, X_h) \\ &= -\ df(X_h)\end{aligned}$$

That is, $\{f, h\}$ is the directional derivative of f along the integral curve of X_h. Hence $\{f, h\} = 0$, if and only if f is constant along the integral curves of X_h. $\square$

Definition 1.15. Let $(\mathcal{M}, \omega)$ be a symplectic manifold. Two functions $f, g \in C^\infty(\mathcal{M})$ are said to *Poisson commute* if $\{f, g\} = 0$.

Definition 1.16. A function f as in Theorem 1.14 is said to be an *integral of motion* for the Hamiltonian system $(\mathcal{M}, \omega, h)$. [2].

A symplectic vector space is a pair (V, φ), where V is a vector space over $\mathbb{R}$ and $\varphi : V \times V \to \mathbb{R}$ is a non-degenerate bilinear skew-symmetric form. Let U be subspace of V and $U^\perp = \{v \in V : \varphi(v, u) = 0, \forall\ u \in U\}$ be the orthogonal complement subspace of U. Then $V = U \oplus U^\perp$. The subspace U is called *isotropic* if $\varphi|_U = 0$, i.e. $U \subseteq U^\perp$. By non-degeneracy, dim $V =$ dim $U +$ dim $U^\perp$. Since V is $2n$ dimensional, the dimension of U can be at most n.

The functions $f_1, f_2, \cdots, f_n$ on $\mathcal{M}$, where $n = \frac{1}{2}\text{dim}(\mathcal{M})$ are said to be independent if their differentials $(df_1)_p, (df_2)_p, \cdots, (df_n)_p$ are linearly independent at every point p in some open dense subset of $\mathcal{M}$. Let $f_1, f_2, \cdots, f_n$ be pairwise Poisson commuting integrals of motion on the symplectic manifold $(\mathcal{M}, \omega)$. Let $U_p \subseteq T_p\mathcal{M}$ be a subspace at p spanned by the vectors $X_1|_p, \cdots, X_n|_p$. For any two vectors $u_1 = \sum_i a_i\ X_i|_p$ and $u_2 = \sum_j b_j\ X_j|_p$ in U_p,

$$\omega_p(u_1, u_2) = \sum_{i,j} a_i b_j\ \omega(X_i|_p, X_j|_p) = \sum_{i,j} a_i b_j \{f_i, f_j\}(p) = 0$$

Therefore, U_p is an isotropic subspace of the tangent space $T_p\mathcal{M}$ at p and U_p can be at most n dimensional. See [2] for details.

Definition 1.17. A Hamiltonian system $(\mathcal{M}, \omega, h)$ is *completely integrable* if there exists $n = \frac{1}{2}\text{dim}(\mathcal{M})$ independent integrals of motion $f_1 = h, f_2, \cdots, f_n$ on $\mathcal{M}$, which are pairwise Poisson commuting with respect to the Poisson bracket. That is, $\{f_i, f_j\} = 0$, for all i, j. [2]. If $\mathcal{M}$ is infinite dimensional, we will say that $(\mathcal{M}, \omega, h)$ is completely integrable whenever there exist infinitely many independent functions $f_1, f_2, f_3, \cdots$ which are pairwise Poisson commuting.

The existence of n independent Poisson commuting functions allows one to write down Hamilton's equations in a new coordinate system. In this case, Hamilton's equations are linear and solutions to these equations can be obtained analytically. For further details, please see Liouville's Theorem on page 272 in [1].

2

A Space of Curves on the 3-Sphere

In this chapter, we develop some theory that will enable us to carry out future computations. We begin with recalling some elementary facts about the Lie group SU_2, the set of all 2×2 unitary matrices of determinant 1. The covariant derivative on SU_2 obtained through the identification of SU_2 with S^3 will be used later in this chapter to describe the tangent space of the manifold of our interest $\mathfrak{M}$. The manifold $\mathfrak{M}$ is defined as a certain space of arc length parametrized curves. Finally, we define the symplectic form on $\mathfrak{M}$ as used in [6]. The contents of this chapter are relying upon [4], [5], [6], [7], [13].

2.1 Preliminaries

A *Lie group* G is a smooth manifold with a group structure so that the group operations are smooth maps, i.e. the multiplication $(g, h) \mapsto gh$ and inverse $g \mapsto g^{-1}$ are smooth maps.

Here are few examples of matrix Lie groups:

(i) The general linear group, $GL_n(\mathbb{C}) = \{A \in \mathcal{M}_n(\mathbb{C}) : \det(A) \neq 0\}$.

(ii) The special linear group, $SL_n(\mathbb{C}) = \{A \in \mathcal{M}_n(\mathbb{C}) : \det(A) = 1\}$.

(iii) The special unitary group, $SU_n = \{A \in \mathcal{M}_n(\mathbb{C}) : \det(A) = 1, AA^* = I\}$.

In this thesis, we are interested in the group SU_2, the set of all 2×2 unitary matrices having determinant 1. Any matrix $X \in SU_2$ is of the form $X = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$, where $u, v \in \mathbb{C}$ and $u\bar{u} + v\bar{v} = 1$.

To any Lie group G, one can associate a Lie algebra $\mathfrak{g}$. See Definition 1.11. For $G = SU_2$ the corresponding Lie algebra will be denoted by $\mathfrak{g} = \mathfrak{su}_2$. It consists of all 2×2 skew-Hermitian matrices of trace zero with bracket $[A, B] = BA - AB$.

Definition 2.1. For any two matrices A, B in $\mathfrak{su}_2$, let $\langle A, B\rangle = -2\ \mathrm{Trace}(AB)$.

Proposition 2.2. $\langle A, B\rangle$ defines an inner product on $\mathfrak{su}_2$.

Proof. Let $A = \begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & -ia_1 \end{pmatrix}$, $B = \begin{pmatrix} ib_1 & b_2 + ib_3 \\ -b_2 + ib_3 & -ib_1 \end{pmatrix}$ and $C = \begin{pmatrix} ic_1 & c_2 + ic_3 \\ -c_2 + ic_3 & -ic_1 \end{pmatrix}$ be matrices in $\mathfrak{su}_2$. Also let $\alpha \in \mathbb{R}$ be a scalar. We will show that $\langle A, B\rangle$ satisfies the properties of an inner product.

(i) *Symmetry:*

$$\langle B, A\rangle = -2\ \mathrm{Trace}(BA) = -2\ \mathrm{Trace}(AB) = \langle A, B\rangle$$

(ii) *Linearity:*

$$\langle \alpha A, B\rangle = 4\alpha(a_1b_1 + a_2b_2 + a_3b_3) = -\alpha\ 2\ \mathrm{Trace}(AB) = \alpha\langle A, B\rangle$$

and

$$\langle A + B, C\rangle = -2\ \mathrm{Trace}\Big((A + B)C\Big)$$

$$
\begin{aligned}
&= 4(a_1c_1 + a_2c_2 + a_3c_3) + 4(b_1c_1 + b_2c_2 + b_3c_3) \\
&= -2 \text{ Trace}(AC) - 2 \text{ Trace}(BC) \\
&= \langle A, C\rangle + \langle B, C\rangle
\end{aligned}
$$

(iii) *Positive definiteness:*

$$\langle A, A\rangle = -2 \text{ Trace}(AA) = 4(a_1^2 + a_2^2 + a_3^2) \geq 0$$

and $\langle A, A\rangle = 0$ if and only if A is a zero matrix.

$\square$

Lemma 2.3. The Lie algebra $\mathfrak{su}_2$ is isomorphic to $\mathbb{R}^3$ with Lie bracket given by cross product. Moreover, under this isomorphism, the inner product in Definition 2.1 corresponds to the dot product.

Proof. Let $\Phi : \mathfrak{su}_2 \to \mathbb{R}^3$ be the linear map given by $\Phi(A) = a$, where $A = \begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & -ia_1 \end{pmatrix} \in \mathfrak{su}_2$ and $a = 2(a_1, a_2, a_3) \in \mathbb{R}^3$.

Let $B = \begin{pmatrix} ib_1 & b_2 + ib_3 \\ -b_2 + ib_3 & -ib_1 \end{pmatrix}$ be a matrix in $\mathfrak{su}_2$ and $b = 2(b_1, b_2, b_3)$ be a vector in $\mathbb{R}^3$. Then,

$$[A, B] = \begin{pmatrix} i2(a_3b_2 - a_2b_3) & 2(a_1b_3 - a_3b_1) + i2(a_2b_1 - a_1b_2) \\ -2(a_1b_3 - a_3b_1) + i2(a_2b_1 - a_1b_2) & -i2(a_3b_2 - a_2b_3) \end{pmatrix}$$

On the other hand, $b \times a = 2\Big(2(a_3b_2 - a_2b_3), 2(a_1b_3 - a_3b_1), 2(a_2b_1 - a_1b_2)\Big)$, which shows that $\Phi([A, B]) = b \times a$.

Additionally, $\langle A, B\rangle = 4(a_1b_1 + a_2b_2 + a_3b_3) = a \cdot b$. $\square$

Lemma 2.4. For any matrices $A, B, C \in \mathfrak{su}_2$, we have $\langle A, [B, C]\rangle = \langle [A, B], C\rangle$.

Proof. Under the isomorphism between $\mathfrak{su}_2$ and $\mathbb{R}^3$ given by Lemma 2.3, the above Lemma coincides with the formula $a \cdot (b \times c) = (a \times b) \cdot c$ in $\mathbb{R}^3$. $\square$

As with any Lie group G, the *tangent bundle* of SU_2, denoted by TSU_2 is trivializable. That is, $TSU_2 \cong SU_2 \times \mathfrak{su}_2$. So for any $g \in SU_2$ and $A \in \mathfrak{su}_2$, the map $(g, A) \mapsto (T_I L_g)(A)$ is a diffeomorphism, where $L_g : G \to G$ is the map given by $L_g(h) = gh$, $\forall g, h \in G$ and $T_I L_g$ denotes its tangent map at the identity I. Any tangent vector P in $T_g SU_2$ will be denoted by $P = gA$, for some $A \in \mathfrak{su}_2$.

2.2 A Space of Curves in SU_2

Let $X(s)$ in SU_2 be an arc length parametrized curve defined on a fixed interval $[0, L]$. A curve $X(s)$ satisfying $X(0) = I$ will be called *anchored*. Suppose the tangent vectors of $X(s)$ are defined by $\frac{dX}{ds}(s) = X(s)\Lambda(s)$, where $\Lambda(s) \in \mathfrak{su}_2$ satisfy the conditions $\langle \Lambda(s), \Lambda(s)\rangle = 1$ and $\Lambda(0) = \Lambda(L)$, where $\Lambda(0) = A$ for some fixed $A \in \mathfrak{su}_2$.

Definition 2.5. The space of arc length parametrized anchored curves satisfying the above conditions will be denoted by $\mathfrak{M}$.

The space of all smooth maps

$$\mathrm{Map}\Big([0, L], SU_2\Big) = \{\text{smooth maps } X : [0, L] \to SU_2\}$$

is a Fréchet manifold by Corollary 2.3.2 in [5]. As argued in section 3 in [6], $\mathfrak{M}$ is a smooth Fréchet submanifold of $\mathrm{Map}\Big([0, L], SU_2\Big)$.

Remark 2.2.1. Consider a matrix, $X = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in SU_2$ for $u, v \in \mathbb{C}$. Then by definition $\det(X) = u\bar{u} + v\bar{v} = 1$.

Set $u = x_0 + ix_1$ and $v = x_2 + ix_3$ for $x_0, x_1, x_2, x_3 \in \mathbb{R}$. Then $(x_0 + ix_1)(x_0 - ix_1) + (x_2 + ix_3)(-x_2 + ix_3) = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. Hence we may view SU_2 as the 3-sphere $S^3 \subset \mathbb{R}^4$.

Lemma 2.6. For any matrices A, B, C in $\mathfrak{su}_2$, we have

$$[A, [B, C]] = \langle A, C \rangle B - \langle A, B \rangle C.$$

Proof. Under the isomorphism between $\mathfrak{su}_2$ and $\mathbb{R}^3$ given by Lemma 2.3, the above Lemma coincides with the vector triple product formula in $\mathbb{R}^3$. □

Now we will define the covariant derivative on SU_2 as follows.

Definition 2.7. The *covariant derivative* of a curve of tangent vectors $v(s) = X(s)U(s)$ along a curve $X(s)$ in SU_2 is given by

$$\frac{D}{ds}\big(v(s)\big) = X(s)\Big(\frac{dU}{ds}(s) + \frac{1}{2}[U(s), \Lambda(s)]\Big) \tag{2.1}$$

where $\Lambda(s) = X^{-1}(s)\dfrac{dX}{ds}(s)$ is in $\mathfrak{su}_2$. [6], [7].

Proposition 2.8. The covariant derivative on SU_2 corresponds to the covariant derivative on $S^3 \subset \mathbb{R}^4$.

Proof. Let $x(s)$ be a curve in S^3 and $v(s) \in T_{x(s)}S^3$ a field of tangent vectors along the curve $x(s)$. Then the covariant derivative of $v(s)$ is given by

$$\frac{D}{ds}(v) = \frac{dv}{ds} - \Big(\frac{dv}{ds} \cdot x\Big)x.$$

For any two matrices P, Q in SU_2 corresponding to $p = (p_0, p_1, p_2, p_3), q = (q_0, q_1, q_2, q_3)$ in $S^3 \subset \mathbb{R}^4$, we can write $p \cdot q = \dfrac{1}{2}$ Trace(PQ^{-1}). See Lemma A.1 in Appendix A.

Under this correspondence between S^3 and SU_2 we can write

$$\begin{aligned}\frac{D}{ds}(V) &= \frac{dV}{ds} - \frac{1}{2}\ \mathrm{Trace}\Big(\frac{dV}{ds}X^{-1}\Big)X \\ &= X\Big[X^{-1}\frac{dV}{ds} - \frac{1}{2}\ \mathrm{Trace}\Big(\frac{dV}{ds}X^{-1}\Big)I\Big] \end{aligned} \tag{2.2}$$

where $X \in SU_2$ given in Remark 2.2.1 corresponds to $x \in S^3$ and $V = XU \in T_X SU_2$, where $U \in \mathfrak{su}_2$ corresponds to v.

Now the right hand side of equation (2.1) yields

$$\frac{dU}{ds}(s) + \frac{1}{2}[U(s), \Lambda(s)] = X^{-1}\frac{dV}{ds} - \frac{1}{2}X^{-1}\frac{dX}{ds}X^{-1}V - \frac{1}{2}X^{-1}VX^{-1}\frac{dX}{ds} \tag{2.3}$$

where $U = X^{-1}V$ and $\Lambda = X^{-1}\dfrac{dX}{ds}$. Comparing equations (2.2) and (2.3), it suffices to show

$$\frac{dX}{ds}X^{-1}V + VX^{-1}\frac{dX}{ds} = \mathrm{Trace}\Big(\frac{dV}{ds}X^{-1}\Big)X \tag{2.4}$$

Now the right hand side of equation (2.4) can be written as

$$\begin{aligned}\mathrm{Trace}\ \Big(\frac{dV}{ds}X^{-1}\Big)X &= \mathrm{Trace}\ \Big[\Big(\frac{dX}{ds}U + X\frac{dU}{ds}\Big)X^{-1}\Big]X \\ &= \mathrm{Trace}\ \Big[\Big(X\Lambda UX^{-1} + X\frac{dU}{ds}X^{-1}\Big)\Big]X \\ &= \Big[\mathrm{Trace}\ (X\Lambda UX^{-1}) + \mathrm{Trace}\ (X\frac{dU}{ds}X^{-1})\Big]X \\ &= \mathrm{Trace}\ (\Lambda U)X \end{aligned}$$

because $\mathrm{Trace}\Big(\dfrac{dU}{ds}\Big) = 0$. Expressing the left hand side of equation (2.4) in terms of Λ and U we get

$$\frac{dX}{ds}X^{-1}V + VX^{-1}\frac{dX}{ds} = X\Lambda U + XU\Lambda$$

$$= X(\Lambda U + U\Lambda)$$

$$= \text{Trace }(\Lambda U)X$$

because $\Lambda, U \in \mathfrak{su}_2$. See Lemma A.2 in Appendix A.

Therefore, equation (2.4) holds. This proves the proposition. $\square$

Lemma 2.9. Suppose that $X(s,t)$ is a field of curves in SU_2 with its infinitesimal directions

$$Z(s,t) = X^{-1}(s,t)\frac{\partial X}{\partial s}(s,t) \quad \text{and } W(s,t) = X^{-1}(s,t)\frac{\partial X}{\partial t}(s,t).$$

Then

$$\frac{\partial Z}{\partial t} - \frac{\partial W}{\partial s} + [Z,W] = 0. \tag{2.5}$$

[6], [7].

Proof. Let $\frac{D}{dt}\Big(\frac{\partial X}{\partial s}\Big)$ be the covariant derivative along the curve $t \mapsto X(s,t)$ and $\frac{D}{ds}\Big(\frac{\partial X}{\partial t}\Big)$ be the covariant derivative along the curve $s \mapsto X(s,t)$. Since the covariant derivatives on SU_2 are symmetric [13], we have

$$\frac{D}{dt}\Big(\frac{\partial X}{\partial s}\Big) = \frac{D}{ds}\Big(\frac{\partial X}{\partial t}\Big)$$

$$X\Big(\frac{\partial Z}{\partial t} + \frac{1}{2}[Z,W]\Big) = X\Big(\frac{\partial W}{\partial s} + \frac{1}{2}[W,Z]\Big)$$

Simplifying we get

$$\frac{\partial Z}{\partial t} - \frac{\partial W}{\partial s} + [Z,W] = 0$$

$\square$

The tangent space of $\mathfrak{M}$ is defined by the following proposition.

Proposition 2.10. The tangent space $T_X\mathfrak{M}$ at an arc length parametrized curve $X(s)$ with $\Lambda(s) = X(s)^{-1}\dfrac{dX}{ds}(s)$ consists of curves $v(s) = X(s)V(s)$ with $V(s)$ the solution of

$$\frac{dV}{ds}(s) = [\Lambda(s),\ V(s)] + U(s) \tag{2.6}$$

such that $V(0) = 0$, where $U(s)$ is a curve in $\mathfrak{su}_2$ subject to the conditions that $U(0) = 0$ and $\langle \Lambda(s),\ U(s)\rangle = 0$. [6], [7].

Proof. Let $Y(s,t)$ be a family of anchored arc length parametrized curves such that $Y(s,0) = X(s)$. Then $v(s) = \left.\dfrac{\partial Y}{\partial t}(s,t)\right|_{t=0}$ is a tangent vector at $X(s)$ for which $v(0) = 0$ since the curves in $Y(s,t)$ are anchored.

Let $Z(s,t)$ and $W(s,t)$ be two tangent vectors defined by

$$Z(s,t) = Y(s,t)^{-1}\frac{\partial Y}{\partial s}(s,t) \quad \text{and} \quad W(s,t) = Y(s,t)^{-1}\frac{\partial Y}{\partial t}(s,t)$$

Let $\Lambda(s) = Z(s,0)$ and $V(s) = W(s,0)$. Then from equation (2.5) we have

$$\frac{\partial Z}{\partial t}(s,0) - \frac{\partial W}{\partial s}(s,0) + [Z(s,0), W(s,0)] = 0$$

which implies $\dfrac{dV}{ds}(s) = \Big[\Lambda(s), V(s)\Big] + U(s)$, where $U(s) = \dfrac{\partial Z}{\partial t}(s,0)$.

Since the curves $s \mapsto Y(s,t)$ are arc length parametrized for each t, we have $\langle Z(s,t), Z(s,t)\rangle = 1$ and $Z(0,t) = A$. Therefore

$$\left\langle Z(s,0), \frac{\partial Z}{\partial t}(s,0)\right\rangle = 0 \quad \text{and} \quad \frac{\partial Z}{\partial t}(0,t) = 0$$

which implies $\langle \Lambda(s), U(s)\rangle = 0$ and $U(0) = 0$.

Now, suppose that $V(s)$ is a solution of (2.6) generated by a curve $U(s) \in \mathfrak{su}_2$

with $U(0) = 0$ satisfying $\langle \Lambda(s), U(s) \rangle = 0$ and $U(0) = 0$. Let us define,

$$Z(s,t) = \frac{1}{\sqrt{1 + t^2 \langle U(s), U(s) \rangle}} \Big(\Lambda(s) + tU(s) \Big) \tag{2.7}$$

Then we have

$$\begin{aligned}
\Big\langle Z(s,t), Z(s,t) \Big\rangle &= \frac{1}{1 + t^2 \langle U(s), U(s) \rangle} \Big[\langle \Lambda(s), \Lambda(s) \rangle + 2t \langle \Lambda(s), U(s) \rangle + \\
&\qquad\qquad\qquad\qquad\qquad\qquad t^2 \langle U(s), U(s) \rangle \Big] \\
&= \frac{1}{1 + t^2 \langle U(s), U(s) \rangle} \Big[1 + t^2 \langle U(s), U(s) \rangle \Big] \\
&= 1
\end{aligned}$$

Therefore $Y(s,t)$, the solution of the initial value problem

$$\frac{\partial Y}{\partial s}(s,t) = Y(s,t) Z(s,t), \quad Y(0,t) = I$$

corresponds to the family of anchored arc length parametrized curves. Let $\dfrac{\partial Y}{\partial t}(s,t) = Y(s,t)W(s,t)$. Then we get

$$\frac{\partial Z}{\partial t}(s,0) - \frac{\partial W}{\partial s}(s,0) + [Z(s,0), W(s,0)] = 0 \tag{2.8}$$

Now differentiating $Z(s,t)$ with respect to t we obtain

$$\begin{aligned}
\frac{\partial Z}{\partial t}(s,t) &= \frac{-1}{2} \Big(1 + t^2 \langle U(s), U(s) \rangle \Big)^{\frac{-3}{2}} \Big(2t \langle U(s), U(s) \rangle \Big) \Lambda(s) \\
&\quad + \frac{\sqrt{1 + t^2 \langle U(s), U(s) \rangle} + t^2 \Big(1 + t^2 \langle U(s), U(s) \rangle \Big)^{\frac{-1}{2}} \langle U(s), U(s) \rangle}{1 + t^2 \langle U(s), U(s) \rangle} \; U(s)
\end{aligned}$$

which implies $\left.\dfrac{\partial Z}{\partial t}(s,t)\right|_{t=0} = U(s)$. Thus equation (2.8) reduces to

$$\frac{dW}{ds}(s,0) = \Big[\Lambda(s), W(s,0)\Big] + U(s)$$

Therefore, $W(s,0)$ solves the above equation and thus is equivalent to $V(s)$. $\square$

In this thesis we will use the following symplectic form that is used by V. Jurdjevic in [6] on the space of curves $\mathfrak{M}$.

Definition 2.11. Let $v_1(s) = X(s)V_1(s)$ and $v_2(s) = X(s)V_2(s)$ be two tangent vectors at a curve $X(s)$. Then according as in equation (2.6) write

$$U_i(s) = \frac{dV_i(s)}{ds} - [\Lambda(s), V_i(s)], \quad i = 1, 2$$

such that $U_i(0) = 0$ and $\langle \Lambda(s), U_i(s)\rangle = 0$. Let ω be the 2-form on $\mathfrak{M}$ defined by

$$\omega_X(V_1, V_2) = -\int_0^L \Big\langle \Lambda(s), [U_1(s), U_2(s)]\Big\rangle ds. \tag{2.9}$$

3

Hamiltonian vector fields and Complete integrability

Consider a curve $X(s)$ on the space of curves $\mathfrak{M}$ with unit tangent vector $\frac{dX}{ds} = X\Lambda$. Then the curvature $\kappa(s)$ of $X(s)$ is defined by $\left\|\frac{D}{ds}(\frac{dX}{ds})\right\|$. According to the definition of covariant derivative given by the equation (2.1), we have

$$\frac{D}{ds}\left(\frac{dX}{ds}\right)(s) = X(s)\left(\frac{d\Lambda}{ds} + \frac{1}{2}[\Lambda(s), \Lambda(s)]\right) = X(s)\frac{d\Lambda}{ds}(s)$$

since $[\Lambda(s), \Lambda(s)] = 0$. Therefore $\kappa(s) = \left\|\frac{d\Lambda}{ds}\right\|$.

Now the expression for the torsion $\tau(s)$ corresponding to $X(s)$ depends on how one chooses the frame above $X(s)$. For our purpose, we choose the Serret-Frenet frame. The Serret-Frenet frame along the curve $X(s)$ is defined via the following relations:

$$\frac{D}{ds}(X\Lambda) = \kappa XN, \quad \frac{D}{ds}(XN) = -\kappa X\Lambda + \tau XB, \quad \frac{D}{ds}(XB) = -\tau XN \tag{3.1}$$

where, Λ, N and B are skew-Hermitian matrices of trace zero that correspond to the tangent, normal and bi-normal unit vectors to the curve $X(s)$. The derivation of the above Serret-Frenet formulas is similar to the derivation in $\mathbb{R}^3$.

Let us consider the following four functions on $\mathfrak{M}$. Here, the ordering of these functions follows from the literature.

1. $f_1 = \int_0^L \tau(s)ds$
2. $f_2 = \frac{1}{2}\int_0^L \kappa^2\ (s)ds$
3. $f_3 = \int_0^L \big(\frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{4}\kappa^2\big)ds$
4. $f_4 = \int_0^L \frac{1}{2}\Big((\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) + \frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{8}\kappa^2(s) - \frac{1}{8}\kappa^4(s)\Big)ds$

The flow of the Hamiltonian vector field corresponding to the function f_2 generates 'Heisenberg's magnetic equation' given by $\frac{\partial\Lambda}{\partial t}(s,t) = \Big[\frac{\partial^2\Lambda}{\partial^2 s}(s,t),\ \Lambda(s,t)\Big]$ and the flow of the Hamiltonian vector field corresponding to f_1 generates the 'curve shortening equation' given by $\frac{\partial\Lambda}{\partial t}(s,t) = \kappa(s,t)N(s,t)$. Also, as we will show, the above functions pairwise Poisson commute. It seems plausible to have a list of infinitely many pairwise Poisson commuting functions on $\mathfrak{M}$, in which f_1, f_2, f_3, f_4 are the top four functions. The subsequent functions in such an infinite list are yet to be determined, and this raises the question of complete integrability of this Hamiltonian system. A similar problem has been handled in [10]. The authors of [10] have shown that the filament equation defined on a space of curves in $\mathbb{R}^3$ is a completely integrable Hamiltonian system. That is, the space of curves possesses infinitely many functions which are pairwise Poisson commuting. The successive functions in this infinite list are obtained through the repeated application of a recursion operator. The recursion operator is constructed using two compatible Poisson structures on the space of curves.

In this chapter, we will compute the Hamiltonian vector fields $\mathcal{X}_{f_i},\ i = 1, 2, 4$ corresponding to the functions f_1, f_2 and f_3 via the symplectic structure on $\mathfrak{M}$. The computations of the Hamiltonian vector fields associated with the functions f_1 and

f_4 are new, while those with f_2 have been reproduced here from [6]. On the other hand, computation of the Hamiltonian vector field corresponding to the function f_3 can be found in [6].

For computational convenience we will rewrite the functions f_1, f_2, f_4 in terms of the tangent vector Λ using the Serret-Frenet equations. Thus the functions take the following forms

1. $f_1 = \int_0^L \langle \Lambda', \Lambda' \rangle^{-1} \ \langle \Lambda'', [\Lambda(s), \Lambda'] \rangle ds$
2. $f_2 = \frac{1}{2} \int_0^L \langle \Lambda', \ \Lambda' \rangle ds$
3. $f_4 = \int_0^L \left(\frac{1}{2} \langle \Lambda'', \Lambda' \rangle^2 \langle \Lambda', \Lambda' \rangle^{-1} + \frac{1}{2} \langle \Lambda'', [\Lambda, \Lambda'] \rangle^2 \langle \Lambda', \Lambda' \rangle^{-1} - \frac{1}{8} \langle \Lambda', \Lambda' \rangle^2 \right) ds$

where Λ', Λ'' denote first and second derivative of Λ respectively with respect to s. See Appendix B for detailed calculation.

Towards the end of this chapter, we will show that the functions f_1, f_2, f_3, f_4 pairwise Poisson commute, suggesting they may belong to a list of infinitely many Poisson commuting functions, which would show the complete integrability of the corresponding Hamiltonian system. Section 3.5 discusses the potential of further research in this topic.

3.1 Hamiltonian vector field of $\int_0^L \tau ds$

Proposition 3.1. Let $f_1 : \mathfrak{M} \to \mathbb{R}$ be the function given by $f_1 = \int_0^L \tau \ ds$ and let $\mathcal{X}_{f_1}$ denotes its Hamiltonian vector field. If $g(s,t)$ denote the integral curves of $\mathcal{X}_{f_1}$ with tangent vectors $\frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t)$, then $\Lambda(s,t)$ evolves according to the *Curve Shortening equation* (CSE)

$$\frac{\partial \Lambda}{\partial t}(s,t) = \kappa(s,t) N(s,t). \tag{3.2}$$

Proof. Let $Y(s,t)$ be a family of anchored arc length parametrized curves that satisfies $Y(s,0) = X(s)$ and $v(s) = \left.\frac{\partial Y}{\partial t}(s,t)\right|_{t=0} = X(s)V(s)$.

Also let $Z(s,t)$ denote the matrices defined by $\frac{\partial Y}{\partial s}(s,t) = Y(s,t)Z(s,t)$. It follows that $\Lambda(s) = Z(s,0)$ and that $V(s)$ solves $\frac{dV}{ds} = [\Lambda(s), V(s)] + U(s)$ with $U(s) = \frac{\partial Z}{\partial t}(s,0)$.

Then the directional derivative of f_1 at $X(s)$ along V is given by

$$
\begin{aligned}
df_{1X}(V) &= \frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t),\ [Z, \frac{\partial Z}{\partial s}(s,t)]\right\rangle ds\Bigg|_{t=0} \\
&= \int_0^L \frac{\partial}{\partial t}\left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t),\ [Z, \frac{\partial Z}{\partial s}(s,t)]\right\rangle ds\Bigg|_{t=0} \\
&+ \int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} \frac{\partial}{\partial t}\left\langle \frac{\partial^2 Z}{\partial s^2}(s,t),\ [Z, \frac{\partial Z}{\partial s}(s,t)]\right\rangle ds\Bigg|_{t=0} \\
&= -2\int_0^L \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \left\langle \frac{d\Lambda}{ds}(s),\ \frac{dU}{ds}(s)\right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s),\ \frac{d\Lambda}{ds}(s)]\right\rangle ds \\
&+ \int_0^L \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-1} \left\langle \frac{d^2 U}{ds^2}(s),\ [\Lambda(s),\ \frac{d\Lambda}{ds}(s)]\right\rangle ds \\
&+ \int_0^L \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s),\ \frac{dU}{ds}(s)]\right\rangle ds \\
&+ \int_0^L \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2}(s),\ [U(s),\ \frac{d\Lambda}{ds}(s)]\right\rangle ds \\
&= I_1 + I_2 + I_3 + I_4 \quad \text{(say)}
\end{aligned} \tag{3.3}
$$

Using *integration by parts* multiple times and considering the fact that $\Lambda(0) = \Lambda(L)$ we find

$$
df_{1X}(V) = -\int_0^L \left\langle G(s), U(s)\right\rangle ds \tag{3.4}
$$

See Appendix B.1 for detailed calculation regarding equation (3.4). The Hamiltonian vector field $\mathcal{X}_{f_1}$ associated with f_2 is of the form

$$\mathcal{X}_{f_1}(X(s)) = X(s)F_1(s)$$

for some curve $F_1(s) \in \mathfrak{su}_2$.

Since $\mathcal{X}_{f_1}(X)$ is in $T_X\mathfrak{M}$, $F_1(s)$ is the solution of

$$\frac{dF_1}{ds}(s) = \Big[\Lambda(s), V(s)\Big] + U_{f_1}(s), \quad F_1(0) = 0 \tag{3.5}$$

for some curve $U_{f_1}(s) \in \mathfrak{su}_2$ satisfying the conditions $U_{f_1}(0) = 0$ and $\langle \Lambda(s),\ U_{f_1}(s)\rangle = 0$.

Then the curve U_{f_1} satisfies the symplectic form

$$\omega_X(F_1, F) = -\int_0^L \Big\langle \Lambda(s),\ [U_{f_1}(s),\ U(s)]\Big\rangle\ ds \tag{3.6}$$

where, $U(s)$ is an arbitrary curve in $\mathfrak{su}_2$ that satisfies $U(0) = 0$ and $\langle \Lambda(s),\ U(s)\rangle = 0$.

Now by definition from equation (3.4) and(3.6) we obtain

$$\int_0^L \Big\langle G(s),\ U(s)\Big\rangle\ ds = \int_0^L \Big\langle \Lambda(s),\ [U_{f_1}(s),\ U(s)]\Big\rangle\ ds$$

Which implies

$$\begin{aligned}
0 &= \int_0^L \Big[\Big\langle G(s),\ U(s)\Big\rangle - \Big\langle \Lambda(s),\ [U_{f_1}(s),\ U(s)]\Big\rangle\Big]\ ds \\
&= \int_0^L \Big[\Big\langle G(s),\ U(s)\Big\rangle - \Big\langle [\Lambda(s),\ U_{f_1}(s)],\ U(s)\Big\rangle\Big]\ ds \quad \text{[Using Lemma 2.4]} \\
&= \int_0^L \Big\langle G(s) - [\Lambda(s),\ U_{f_1}(s)],\ U(s)\Big\rangle\ ds \qquad (3.7)
\end{aligned}$$

Substituting $U(s) = [\Lambda(s), C(s)]$ in equation (3.28) for some curve $C(s)$ that satisfies $C(0) = 0$,

$$0 = \int_0^L \Big\langle G(s) - [\Lambda(s),\ U_{f_1}(s)],\ [\Lambda(s),\ C(s)] \Big\rangle\, ds$$

$$= \int_0^L \Big\langle \Big[G(s) - [\Lambda(s),\ U_{f_1}(s)],\ \Lambda(s)\Big],\ C(s) \Big\rangle\, ds$$

$$= \int_0^L \Big\langle \Big[G(s), \Lambda(s)\Big] - \Big[[\Lambda(s),\ U_{f_1}(s)],\ \Lambda(s)\Big],\ C(s) \Big\rangle\, ds$$

Recall, if $f(x)g(x) = 0$ for all $x \in [a, b]$ and for any arbitrary continuous function $g(x)$ then $f(x) = 0$ on $[a, b]$. Thus since $C(s)$ is arbitrary,

$$0 = \Big[G(s),\ \Lambda(s)\Big] - \Big[[\Lambda(s),\ U_{f_1}(s)],\ \Lambda(s)\Big]$$

$$= \Big[G(s),\ \Lambda(s)\Big] - \Big[\langle \Lambda(s),\ \Lambda(s)\rangle U_{f_1}(s) - \langle U_{f_1}(s),\ \Lambda(s)\rangle \Lambda(s)\Big] \quad \text{[Using Lemma 2.6]}$$

$$= \Big[G(s),\ \Lambda(s)\Big] - U_{f_1}(s) \quad [\text{since } \langle U_{f_1}(s),\ \Lambda\rangle = 0]$$

and therefore

$$U_{f_1} = -\Big[\Lambda(s),\ G(s)\Big] \tag{3.8}$$

Now expressing $G(s)$ in terms of *curvature* and *torsion* using the Serret-Frenet equations we obtain

$$\begin{aligned} G(s) &= -2\kappa^{-2}\kappa'\tau N - 2\tau\Lambda + 2\kappa^{-1}\tau^2 B + 2k^{-1}\tau' N + 2\kappa^{-3}(\kappa')^2 B - \kappa^{-2}\kappa'' B \\ &\quad + 2\kappa^{-2}\kappa'\tau N + \tau\Lambda - \kappa'\tau N - k^{-1}\tau^2 B - 2\kappa^{-3}(\kappa')^2 B + \kappa^{-2}\kappa'' B - \kappa^{-1}\tau' N \\ &\quad - \kappa^{-1}\tau^2 B + \tau\Lambda + \kappa B + \tau\Lambda \\ &= \kappa B + \tau\Lambda \end{aligned}$$

Thus

$$U_{f_1} = -[\Lambda,\ \kappa B + \tau\Lambda] = \kappa N$$

The integral curves $t \mapsto X(s,t)$ of $\mathcal{X}_{f_1}$ are the solutions of

$$\frac{\partial X}{\partial t} = X(s,t)F_1(s,t) \quad \text{and} \quad \frac{\partial X}{\partial s} = X(s,t)\Lambda(s,t)$$

where $F_1(s,t)$ is the solution of

$$\frac{\partial F_1}{\partial s}(s,t) = [\Lambda(s,t),\ F_1(s,t)] + \kappa N \tag{3.9}$$

According to the Lemma 2.9 we have

$$\frac{\partial \Lambda}{\partial t}(s,t) - \frac{\partial F_1}{\partial s}(s,t) + [\Lambda(s,t), F_1(s,t)] = 0 \tag{3.10}$$

Using the equation (3.9) in (3.10) we get,

$$\frac{\partial \Lambda}{\partial t}(s,t) - [\Lambda(s,t),\ F_1(s,t)] - \kappa(s,t)N(s,t) + [\Lambda(s,t),\ F_1(s,t)] = 0$$

Therefore,

$$\frac{\partial \Lambda}{\partial t}(s,t) = \kappa(s,t)N(s,t)$$

which is known as *Curve shortening equation* (CSE). See [3], [6]. □

3.2 Hamiltonian vector field of $\frac{1}{2}\int_0^L \kappa^2\ ds$

Proposition 3.2. Let $f_2 : \mathfrak{M} \to \mathbb{R}$ be the function given by $f_2 = \frac{1}{2}\int_0^L \kappa^2 ds$ and let $\mathcal{X}_{f_2}$ denote its Hamiltonian vector field. If $g(t,s)$ denote the integral curves of $\mathcal{X}_{f_2}$ with tangent vectors $\frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t)$, then $\Lambda(s,t)$ evolves according to

Heisenberg's magnetic equation(HME)

$$\frac{\partial \Lambda}{\partial t}(s,t) = \left[\frac{\partial^2 \Lambda}{\partial s^2}(s,t),\ \Lambda(s,t)\right]. \tag{3.11}$$

Proof. Suppose that a family of anchored arc length parametrized curves denoted by $Y(s,t)$ satisfies $Y(s,0) = X(s)$ and $v(s) = \left.\frac{\partial Y}{\partial t}(s,t)\right|_{t=0} = X(s)V(s)$.

Let $Z(s,t)$ denote the matrices defined by $\frac{\partial Y}{\partial s}(s,t) = Y(s,t)\ Z(s,t)$. It follows from Proposition 2.10 that $\Lambda(s) = Z(s,0)$ and $V(s)$ is the solution of $\frac{dV}{ds} = [\Lambda(s),\ V(s)] + U(s)$ with $U(s) = \left.\frac{\partial Z(s,t)}{\partial t}\right|_{t=0}$.

Then the directional derivative of f_2 at $X(s)$ in the direction of V is given by

$$\begin{aligned}
df_{2X}(V) &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial Z}{\partial s}(s,t)\ \right\rangle ds\Big|_{t=0} \\
&= \frac{1}{2}\cdot 2\int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial}{\partial s}\frac{\partial Z}{\partial t}(s,t)\right\rangle ds\Big|_{t=0} \\
&= \int_0^L \left\langle \frac{\partial Z}{\partial s}(s,0),\ \frac{\partial}{\partial s}\frac{\partial Z}{\partial t}(s,0)\right\rangle ds \\
&= \int_0^L \left\langle \frac{d\Lambda}{ds}(s),\ \frac{dU}{ds}(s)\right\rangle ds \\
&= \left\langle \frac{d\Lambda}{ds}(s),\ U(s)\right\rangle\Big|_{s=0}^{s=L} - \int_0^L \left\langle \frac{d^2\Lambda}{ds^2}(s),\ U(s)\right\rangle ds
\end{aligned}$$

The boundary term $\left\langle \frac{d\Lambda}{ds}(s),\ U(s)\right\rangle\Big|_{s=0}^{s=L}$ equals zero due to the assumption $\Lambda(0) = \Lambda(L)$. Consequently

$$df_{2X}(V) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}(s),\ U(s)\right\rangle ds \tag{3.12}$$

The Hamiltonian vector field $\mathcal{X}_{f_2}$ corresponding to f_2 is of the form

$$\mathcal{X}_{f_2}\big(X(s)\big) = X(s)F_2(s)$$

for some curve $F_2(s) \in \mathfrak{su}_2$.

Since $\mathcal{X}_{f_2}(X) \in T_X\mathfrak{M}, F(s)$ is the solution of

$$\frac{dF_2}{ds}(s) = \Big[\Lambda(s),\ F_2(s)\Big] + U_{f_2}(s), \quad F_2(0) = 0 \tag{3.13}$$

for some curve $U_{f_2}(s) \in \mathfrak{su}_2$ satisfying the conditions $U_{f_2}(0) = 0$ and $\langle \Lambda(s), U_{f_2}(s)\rangle = 0$.

Then the curve U_{f_2} satisfies the following equation

$$\omega_X(F_2(s),\ F(s)) = -\int_0^L \Big\langle \Lambda(s),\ [U_{f_2}(s),\ U(s)]\Big\rangle\, ds \tag{3.14}$$

where, $U(s) \in \mathfrak{su}_2$ is an arbitrary curve that satisfies $U(0) = 0$ and $\langle \Lambda(s),\ U(s)\rangle = 0$.

From equation (3.12) and(3.14) we obtain

$$\int_0^L \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ U(s)\Big\rangle\, ds = \int_0^L \Big\langle \Lambda(s),\ [U_{f_2}(s),\ U(s)]\Big\rangle\, ds$$

Which implies

$$\begin{aligned} 0 &= \int_0^L \Big(\Big\langle \frac{d^2\Lambda}{ds^2}(s),\ U(s)\Big\rangle - \Big\langle [\Lambda(s),\ U_{f_2}(s)],\ U(s)\Big\rangle\Big)\, ds \quad \text{[Using Lemma 2.4]} \\ &= \int_0^L \Big\langle \frac{d^2\Lambda}{ds^2}(s) - [\Lambda(s),\ U_{f_2}(s)],\ U(s)\Big\rangle\, ds \end{aligned} \tag{3.15}$$

Replacing $U(s)$ by $[\Lambda(s),\ C(s)]$ for some curve $C(s)$ that satisfies $C(0) = 0$, equation (3.15) becomes

$$\begin{aligned} 0 &= \int_0^L \Big\langle \frac{d^2\Lambda}{ds^2}(s) - [\Lambda(s),\ U_{f_2}(s)],\ [\Lambda(s), C(s)]\Big\rangle\, ds \\ &= \int_0^L \Big\langle \Big[\frac{d^2\Lambda}{ds^2}(s) - [\Lambda(s), U_{f_2}(s)], \Lambda(s)\Big], C(s)\Big\rangle\, ds \quad \text{[Using Lemma 2.4]} \end{aligned}$$

$$= \int_0^L \left\langle \left[\frac{d^2\Lambda}{ds^2}(s), \Lambda(s)\right] - \Big[[\Lambda(s), U_{f_2}(s)], \Lambda(s)\Big], C(s) \right\rangle \, ds.$$

Since $C(s)$ is arbitrary, we have

$$0 = \left[\frac{d^2\Lambda}{ds^2}(s), \Lambda(s)\right] - \Big[[\Lambda(s),\ U_{f_2}(s)], \Lambda(s)\Big]$$

$$= \left[\frac{d^2\Lambda}{ds^2}(s),\ \Lambda(s)\right] - \Big[\langle \Lambda(s),\ \Lambda(s)\rangle U_{f_2}(s) - \langle U_{f_2}(s),\ \Lambda(s)\rangle \Lambda(s)\Big] \quad \text{[Using Lemma 2.6]}$$

$$= \left[\frac{d^2\Lambda}{ds^2}(s),\ \Lambda(s)\right] - U_{f_2}(s) \quad [\text{since } \langle U_{f_2}(s),\ \Lambda(s)\rangle = 0]$$

and therefore

$$U_{f_2}(s) = -\left[\Lambda(s),\ \frac{d^2\Lambda}{ds^2}(s)\right] \tag{3.16}$$

The integral curves $t \mapsto X(s,t)$ of $\mathcal{X}_{f_2}$ are the solutions of

$$\frac{\partial X}{\partial t} = X(s,t)\ F_2(s,t) \quad \text{and} \quad \frac{\partial X}{\partial s} = X(s,t)\ \Lambda(s,t)$$

where $F_2(s,t)$ is the solution of

$$\frac{\partial F_2}{\partial s}(s,t) = [\Lambda(s,t),\ F_2(s,t)] - \left[\Lambda(s,t),\ \frac{\partial^2\Lambda}{\partial^2 s}(s,t)\right] \tag{3.17}$$

Now according to Lemma 2.9 we have

$$\frac{\partial \Lambda}{\partial t}(s,t) - \frac{\partial F_2}{\partial s}(s,t) + [\Lambda(s,t), F_2(s,t)] = 0 \tag{3.18}$$

Using the equation (3.17) in (3.18) we get

$$\frac{\partial \Lambda}{\partial t}(s,t) - [\Lambda(s,t),\ F_2(s,t)] + \left[\Lambda(s,t),\ \frac{\partial^2\Lambda}{\partial^2 s}(s,t)\right] + [\Lambda(s,t),\ F_2(s,t)] = 0$$

After the cancellation we are left with

$$\frac{\partial \Lambda}{\partial t}(s,t) = \left[\frac{\partial^2 \Lambda}{\partial^2 s}(s,t),\ \Lambda(s,t)\right]$$

which is known as *Heigenberg's magnetic equation.* See [6], [8],[17]. Therefore, $\Lambda(s,t)$ evolves according to (HME). □

3.3 Hamiltonian vector field of $\int_0^L \Big((\kappa')^2 + \frac{1}{2}\kappa^2\tau^2 + \frac{1}{2}\kappa^2\tau + \frac{1}{8}\kappa^2 - \frac{1}{8}\kappa^4\Big)ds$

Proposition 3.3. Let $f_4 : \mathfrak{M} \to \mathbb{R}$ be the function given by

$$f_4 = \int_0^L \Big((\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) + \frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{8}\kappa^2(s) - \frac{1}{8}\kappa^4(s)\Big)ds$$

and let $\mathcal{X}_{f_4}$ denote its Hamiltonian vector field. If $g(t,s)$ denote the integral curves of $\mathcal{X}_{f_4}$ with tangent vectors $\frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t)$, then $\Lambda(s,t)$ evolves according to,

$$\frac{\partial \Lambda}{\partial t}(s,t) = (-\kappa\tau^3 + 3\kappa''\tau + 3\kappa'\tau' + \frac{3}{2}\kappa^3\tau + \kappa\tau'')N + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')B \tag{3.19}$$

where the primes denote derivatives with respect to the arc length s.

Proof. Let $Y(s,t)$ be a family of anchored arc length parametrized curves that satisfies $Y(s,0) = X(s)$ and $v(s) = \frac{\partial Y}{\partial t}(s,t)\Big|_{t=0} = X(s)V(s)$.

Also let $Z(s,t)$ be the matrices defined by $\frac{\partial Y}{\partial s}(s,t) = Y(s,t)Z(s,t)$. It follows that $\Lambda(s) = Z(s,0)$ and that $V(s)$ is the solution of $\frac{dV}{ds} = [\Lambda(s),\ V(s)] + U(s)$ with $U(s) = \frac{\partial Z}{\partial t}(s,0)$.

For computational convenience, we split the function f_4 into three different functions, namely

(i) $I_1 = \int_0^L \frac{1}{2}\langle \Lambda'', \ \Lambda' \rangle^2 \ \langle \Lambda', \ \Lambda' \rangle^{-1} \ ds$

(ii) $I_2 = \int_0^L \frac{1}{2}\langle \Lambda'', \ [\Lambda, \ \Lambda'] \rangle^2 \ \langle \Lambda', \ \Lambda' \rangle^{-1} \ ds$

(iii) $I_3 = -\int_0^L \frac{1}{8}\langle \Lambda', \ \Lambda' \rangle^2 \ ds$

The directional derivative corresponding to I_1, $dI_{1X}(V)$ at X in the direction of V is given by

$$
\begin{aligned}
dI_{1X}(V) = {} & \frac{\partial}{\partial t}\int_0^L \frac{1}{2}\left\langle \frac{\partial^2 Z}{\partial s^2}(s,t), \ \frac{\partial Z}{\partial s}(s,t)\right\rangle^2 \left\langle \frac{\partial Z}{\partial s}(s,t), \ \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} ds\Bigg|_{t=0} \\
= {} & -\int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-2} \left\langle \frac{\partial Z}{\partial s}(s,t), \frac{\partial}{\partial t}\frac{\partial Z}{\partial s}(s,t)\right\rangle \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle^2 \Bigg|_{t=0} \\
& + \int_0^L \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle \left\langle \frac{\partial}{\partial t}\frac{\partial^2 Z}{\partial s^2}(s,t), \ \frac{\partial Z}{\partial s}(s,t)\right\rangle \left\langle \frac{\partial Z}{\partial s}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} \Bigg|_{t=0} \\
& + \int_0^L \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t), \frac{\partial}{\partial t}\frac{\partial Z}{\partial s}(s,t)\right\rangle \left\langle \frac{\partial Z}{\partial s}(s,t), \ \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} \Bigg|_{t=0} \\
= {} & -\int_0^L \left\langle \frac{d\Lambda}{ds}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \left\langle \frac{d\Lambda}{ds}(s), \ \frac{dU}{ds}(s)\right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle^2 ds \\
& + \int_0^L \left\langle \frac{d^2\Lambda}{ds^2}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \left\langle \frac{d^2 U}{ds^2}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle \left\langle \frac{d\Lambda}{ds}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle^{-1} ds \\
& + \int_0^L \left\langle \frac{d^2\Lambda}{ds^2}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \left\langle \frac{d^2\Lambda}{ds^2}(s), \ \frac{dU}{ds}(s)\right\rangle \left\langle \frac{d\Lambda}{ds}(s), \ \frac{d\Lambda}{ds}(s)\right\rangle^{-1} ds \\
= {} & I_{11} + I_{12} + I_{13} \quad \text{(say)}
\end{aligned} \tag{3.20}
$$

Using integration by parts and $\Lambda(0) = \Lambda(L)$, we obtain

$$dI_{1X}(V) = -\int_0^L \Big\langle G_1(s), \ U(s) \Big\rangle ds \tag{3.21}$$

See Appendix B.2 for calculations associated with the above equation.

Similarly, the directional derivative corresponding to I_2 at X along V is given by

$$dI_{2X}(V) = \frac{\partial}{\partial t}\int_0^L \frac{1}{2}\left\langle \frac{\partial^2 Z}{\partial s^2}(s,t),\ \Big[Z(s,t),\ \frac{\partial Z}{\partial s}(s,t)\Big]\right\rangle^2 \left\langle \frac{\partial Z}{\partial s}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} ds\Big|_{t=0}$$

$$= -\int_0^L \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \left\langle \frac{d\Lambda}{ds}(s),\ \frac{dU}{ds}(s)\right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big]\right\rangle^2 ds$$

$$+ \int_0^L \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2}(s), \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big]\right\rangle \left\langle \frac{d^2U}{ds^2}(s), \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big]\right\rangle ds$$

$$+ \int_0^L \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2}(s), \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big]\right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s), \Big[\Lambda(s), \frac{dU}{ds}(s)\Big]\right\rangle ds$$

$$+ \int_0^L \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2}(s), \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big]\right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s), \Big[U(s), \frac{d\Lambda}{ds}(s)\Big]\right\rangle ds$$

$$= I_{21} + I_{22} + I_{23} + I_{24} \quad \text{(say)} \tag{3.22}$$

Using integration by parts few times, we obtain

$$dI_{2X}(V) = -\int_0^L \Big\langle G_2(s),\ U(s)\Big\rangle ds \tag{3.23}$$

See Appendix B.2 for calculations regarding the above equation.

Finally, we compute that the directional derivative corresponding to I_3 is given by

$$dI_{3X}(V) = \frac{\partial}{\partial t}\int_0^L -\frac{1}{8}\left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial Z}{\partial s}(s,t)\right\rangle^2 ds\Big|_{t=0}$$

$$= -\frac{1}{2}\int_0^L \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle \left\langle \frac{d\Lambda}{ds}(s),\ \frac{dU}{ds}(s)\right\rangle ds$$

Again, using integration by parts we obtain

$$dI_{3X}(V) = \int_0^L \left\langle \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\right\rangle \frac{d\Lambda}{ds}(s),\ U(s)\right\rangle ds$$

$$+\frac{1}{2}\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle\ \frac{d^2\Lambda}{ds^2}(s),\ U(s)\Big\rangle\ ds$$

$$=\int_0^L \Big\langle G_3(s),\ U(s)\Big\rangle ds \tag{3.24}$$

See Appendix B.2 for calculations regarding the above equation. Therefore, the directional derivative corresponding to f_4 at X along V is given by

$$df_{4X}(V) = dI_{1X}(V) + dI_{2X}(V) + dI_{3X}(V)$$

$$= -\int_0^L \Big\langle G_1(s) + G_2(s) - G_3(s),\ U(s)\Big\rangle ds$$

$$= -\int_0^L \Big\langle R(s),\ U(s)\Big\rangle ds \tag{3.25}$$

where, $R(s) = G_1(s) + G_2(s) - G_3(s)$. See Appendix B.2 for details.

The Hamiltonian vector field $\mathcal{X}_{f_4}$ associated with f_4 is of the form

$$\mathcal{X}_{f_4}(s) = X(s)F_4(s)$$

for some curve $F_4(s) \in \mathfrak{su}_2$.

Since $\mathcal{X}_{f_4}(X)$ is in $T_X\mathfrak{M}$, $F_4(s)$ is the solution of

$$\frac{dF_4}{ds}(s) = \Big[\Lambda(s),\ V(s)\Big] + U_{f_4}(s), \quad F_4(0) = 0 \tag{3.26}$$

for some curve $U_{f_4}(s) \in \mathfrak{su}_2$ that satisfies the conditions $U_{f_4}(0) = 0$ and $\langle \Lambda(s),\ U_{f_4}(s)\rangle = 0$.

Then the curve U_{f_4} satisfies the following equation

$$\omega_X(F_4(s),\ F(s)) = -\int_0^L \Big\langle \Lambda(s),\ [U_{f_4}(s),\ U(s)]\Big\rangle\ ds \tag{3.27}$$

where, $U(s)$ is an arbitrary curve in $\mathfrak{su}_2$ that satisfies $U(0) = 0$ and $\langle \Lambda(s),\ U(s)\rangle = 0$.

From equation (3.25) and(3.27) we obtain

$$\int_0^L \Big\langle R(s),\ U(s)\Big\rangle\, ds = \int_0^L \Big\langle \Lambda(s),\ [U_{f_4}(s),\ U(s)]\Big\rangle\, ds$$

Which implies

$$\begin{aligned}
0 &= \int_0^L \Big[\Big\langle R(s),\ U(s)\Big\rangle - \Big\langle \Lambda(s),\ [U_{f_4}(s),\ U(s)]\Big\rangle\Big]\, ds \\
&= \int_0^L \Big[\Big\langle R(s),\ U(s)\Big\rangle - \Big\langle [\Lambda(s),\ U_{f_4}(s)],\ U(s)\Big\rangle\Big]\, ds \quad \text{[Using Lemma 2.4]} \\
&= \int_0^L \Big\langle R(s) - [\Lambda(s),\ U_{f_4}(s)],\ U(s)\Big\rangle\, ds \qquad (3.28)
\end{aligned}$$

Substituting $U(s) = [\Lambda(s), C(s)]$ for some curve $C(s)$ that satisfies $C(0) = 0$, equation (3.28) becomes

$$\begin{aligned}
0 &= \int_0^L \Big\langle R(s) - [\Lambda(s),\ U_{f_4}(s)],\ [\Lambda(s),\ C(s)]\Big\rangle\, ds \\
&= \int_0^L \Big\langle \Big[R(s) - [\Lambda(s),\ U_{f_4}(s)],\ \Lambda(s)\Big],\ C(s)\Big\rangle\, ds \\
&= \int_0^L \Big\langle \Big[R(s),\ \Lambda(s)\Big] - \Big[[\Lambda(s),\ U_{f_4}(s)],\ \Lambda(s)\Big],\ C(s)\Big\rangle\, ds
\end{aligned}$$

Since $C(s)$ is arbitrary

$$\begin{aligned}
0 &= \Big[R(s),\ \Lambda(s)\Big] - \Big[[\Lambda(s),\ U_{f_4}(s)],\ \Lambda(s)\Big] \\
&= \Big[R(s),\ \Lambda(s)\Big] - \Big[\langle \Lambda(s),\ \Lambda(s)\rangle U_{f_4}(s) - \langle U_{f_4}(s),\ \Lambda(s)\rangle \Lambda(s)\Big] \quad \text{[Using Lemma 2.6]} \\
&= \Big[R(s),\ \Lambda(s)\Big] - U_{f_4}(s) \quad [\text{since } \langle U_{f_4}(s),\ \Lambda(s)\rangle = 0]
\end{aligned}$$

and therefore

$$U_{f_4}(s) = -\Big[\Lambda(s),\ R(s)\Big]$$

Expressing $R(s)$ in terms of curvature and torsion, we get

$$\begin{aligned} R(s) = \kappa''' N - \kappa\kappa''\Lambda + 3\kappa''\tau B - 3\kappa'\tau^2 N + 2\kappa^2\tau^2\Lambda - \kappa\tau^3 B - 3\kappa\tau\tau' N + \\ + 3\kappa'\tau' B + \kappa\tau'' B + \frac{3}{2}\kappa^2\kappa' N + \frac{1}{2}\kappa^4\Lambda + \frac{3}{2}\kappa^3\tau B \end{aligned} \tag{3.29}$$

See Appendix B.2 for detail calculations. Therefore

$$\begin{aligned} U_{f_4}(s) &= -[\Lambda(s), R(s)] \\ &= -(-3\kappa''\tau + \kappa\tau^3 - \frac{3}{2}\kappa^3\tau - 3\kappa'\tau' - \kappa\tau'')N \\ &\quad - (\kappa''' - 3\kappa'\tau^2 - 3\kappa\tau\tau' + \frac{3}{2}\kappa^2\kappa')B \end{aligned} \tag{3.30}$$

The integral curves $t \mapsto X(s,t)$ of $\mathcal{X}_{f_4}$ are the solutions of

$$\frac{\partial X}{\partial t} = X(s,t)F_4(s,t) \text{ and } \frac{\partial X}{\partial s} = X(s,t)\Lambda(s,t)$$

where $F_4(s,t)$ is the solution of

$$\begin{aligned} \frac{\partial F_4}{\partial s}(s,t) = [\Lambda(s,t), F_4(s,t)] + (3\kappa''\tau - \kappa\tau^3 + \frac{3}{2}\kappa^3\tau + 3\kappa'\tau' + \kappa\tau'')N \\ + (-\kappa''' + 3\kappa'\tau^2 + 3\kappa\tau\tau' - \frac{3}{2}\kappa^2\kappa')B \end{aligned} \tag{3.31}$$

From Lemma 2.9 we know

$$\frac{\partial \Lambda}{\partial t}(s,t) - \frac{\partial F_4}{\partial s}(s,t) + [\Lambda(s,t), F_4(s,t)] = 0 \tag{3.32}$$

From equations (3.30), (3.31) and (3.32) we obtain,

$$\frac{\partial \Lambda}{\partial t}(s,t) - [\Lambda(s,t),\ F_4(s,t)] - U_{f_4}(s) + [\Lambda(s,t),\ F_4(s,t)] = 0$$

After cancellation we get,

$$\begin{aligned}\frac{\partial \Lambda}{\partial t}(s,t) = &\Big(-\kappa\tau^3 + 3(\kappa'\tau)' + \frac{3}{2}\kappa^3\tau + \kappa\tau''\Big)N \\ &+ \Big(-\kappa''' + 3\kappa'\tau^2 + 3\kappa\tau\tau' - \frac{3}{2}\kappa^2\kappa'\Big)B\end{aligned}$$

This concludes the proof. □

3.4 Poisson commuting functions

Theorem 3.4. The functions f_1, f_2, f_3, f_4 on $\mathfrak{M}$ pairwise Poisson commute.

Proof. Here, we will show that the functions f_2 and f_4 Poisson commute.

The Poisson bracket of the functions f_2 and f_4 is defined as

$$\{f_2, f_4\} = \omega_X(F_2(s), F_4(s)) = -\int_0^L \Big\langle \Lambda(s), [U_{f_2}(s), U_{f_4}(s)]\Big\rangle ds \tag{3.33}$$

where $U_{f_2}(s)$ and $U_{f_4}(s)$ are Hamiltonian vector fields associated with f_2 and f_4 respectively. We need to show that $\{f_2, f_4\} = 0$.

From equation (3.16) we have

$$U_{f_2} = [\Lambda'', \Lambda] = [\kappa' N - \kappa^2\Lambda + \kappa\tau B,\ \Lambda] = -\kappa' B + \kappa\tau N$$

and from equation (3.30) we have

$$U_{f_4} = (-\kappa\tau^3 + 3\kappa''\tau + 3\kappa'\tau' + \frac{3}{2}\kappa^3\tau + \kappa\tau'')N + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')B$$

Taking the Lie bracket of these two vector fields we obtain

$$[U_{f_2}(s), U_{f_4}(s)] = \Big(\kappa\kappa'''\tau - 3\kappa^2\tau^2\tau' - 3\kappa\kappa'\tau^3 + \frac{3}{2}\kappa^3\kappa'\tau\Big)\ \Lambda + \Big(\kappa\kappa'\tau^3 - 3\kappa'\kappa''\tau - 3(\kappa')^2\tau' - \frac{3}{2}\kappa^3\kappa'\tau - \kappa\kappa'\tau''\Big)\ \Lambda.$$

Then

$$\langle \Lambda(s),\ [U_{f_2}(s), U_{f_4}(s)]\rangle = \Big(\kappa\kappa'''\tau - 3\kappa^2\tau^2\tau' - 3\kappa\kappa'\tau^3 + \frac{3}{2}\kappa^3\kappa'\tau\Big) + \Big(\kappa\kappa'\tau^3 - 3\kappa'\kappa''\tau - 3(\kappa')^2\tau' - \frac{3}{2}\kappa^3\kappa'\tau - \kappa\kappa'\tau''\Big). \tag{3.34}$$

From equation (3.33) and (3.34) we conclude

$$\begin{aligned}\{f_2, f_4\} &= -\int_0^L [\kappa\kappa'''\tau - 3\kappa^2\tau^2\tau' - 3\kappa\kappa'\tau^3 + \frac{3}{2}\kappa^3\kappa'\tau] + [\kappa\kappa'\tau^3 - 3\kappa'\kappa''\tau \\ &\qquad - 3(\kappa')^2\tau' - \frac{3}{2}\kappa^3\kappa'\tau - \kappa\kappa'\tau''] \ ds \\ &= -\int_0^L \frac{d}{ds}\Big[- \kappa^2\tau^3 - \frac{3}{2}(\kappa')^2\tau - \kappa\kappa'\tau' - \frac{1}{2}(\kappa')^2\tau + \kappa\kappa''\tau\Big]\ ds \\ &= 0\end{aligned}$$

Similar proofs for the remaining pair of functions are given in the Appendix B.3.

Therefore, f_2 Poisson commutes with f_4. □

Remark 3.4.1. Since the functions f_1, f_2, f_3 and f_4 Poisson commute with each other, it suggests that they may be a part of an infinite list of Poisson commuting functions. See [10]. If this was the case, then the Hamiltonian systems corresponding to the above functions would be completely integrable.

3.5 Future research

As we have mentioned earlier, the authors of [10] have shown that the filament equation is a completely integrable system on the space of curves in $\mathbb{R}^3$. That is, there exist an infinite list of pairwise Poisson commuting functions on the space of curves under consideration. In [10], the authors constructed a recursion operator using two compatible Poisson structures on the space of curves to compute the successive functions in this infinite list. In this thesis, we have used the following symplectic structure given by equation (26) in [6],

$$\omega_X(V_1, V_2) = -\int_0^L \Big\langle \Lambda(s), [U_1(s), U_2(s)] \Big\rangle ds$$

on the space of curves $\mathfrak{M}$. There might exist other symplectic structures on $\mathfrak{M}$ which are compatible with ω. One such candidate is

$$\Omega_X(V_1, V_2) = \int_0^L \Big\langle \Lambda(s), [V_1(s), V_2(s)] \Big\rangle ds$$

as indicated by V. Jurdjevic in [6]. Perhaps the Poisson brackets resulting from these symplectic structures can be used to construct a recursion operator to obtain the complete integrability for the systems which are Hamiltonian with respect to both structures [11], similar to the approach of Joel Langer and Ron Perline in [10].

Appendix A

Lemmas in Chapter 2

Lemma A.1. For any two matrices P, Q in SU_2, $2(p \cdot q) = \text{Trace}(PQ^{-1})$, where p, q are vectors in $\mathbb{R}^4$.

Proof. Let $p = (p_0, p_1, p_2, p_3)$ and $q = (q_0, q_1, q_2, q_3)$ be vectors in $\mathbb{R}^4$. Then

$$p \cdot q = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_2 q_2 + p_3 q_3$$

Let

$$P = \begin{pmatrix} p_0 + ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & p_0 - ip_1 \end{pmatrix}, \quad Q = \begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{pmatrix}$$

be matrices in SU_2. Then $Q^{-1} = \begin{pmatrix} q_0 - iq_1 & -q_2 - iq_3 \\ q_2 - iq_3 & q_0 + iq_1 \end{pmatrix}$ and

$$\begin{aligned} \text{Trace}(PQ^{-1}) &= (p_0 + ip_1)(q_0 - iq_1) + (p_2 + ip_3)(q_2 - iq_3) \\ &+ (-p_2 + ip_3)(-q_2 - iq_3) + (p_0 - ip_1)(q_0 + iq_1) \\ &= 2(p_0 q_0 + p_1 q_1 + p_2 q_2 + p_2 q_2 + p_3 q_3) \end{aligned}$$

$\square$

Lemma A.2. For any two matrices Λ and U in $\mathfrak{su}_2$, $\Lambda U + U\Lambda = \text{Trace}(\Lambda U)I$.

Proof. Let

$$\Lambda = \begin{pmatrix} i\lambda_1 & \lambda_2 + i\lambda_3 \\ -\lambda_2 + i\lambda_3 & -i\lambda_1 \end{pmatrix}, \qquad U = \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix}$$

be matrices in $\mathfrak{su}_2$. Then

$$\Lambda U = \begin{pmatrix} -\lambda_1 u_1 - \lambda_2 u_2 - \lambda_3 u_3 + i(\lambda_2 u_3 - \lambda_3 u_2) & -\lambda_1 u_3 + \lambda_3 u_1 + i(\lambda_1 u_2 - \lambda_2 u_1) \\ \\ \lambda_1 u_3 - \lambda_3 u_1 + i(\lambda_1 u_2 - \lambda_2 u_1) & -\lambda_1 u_1 - \lambda_2 u_2 - \lambda_3 u_3 + i(\lambda_3 u_2 - \lambda_2 u_3) \end{pmatrix}$$

and

$$U\Lambda = \begin{pmatrix} -\lambda_1 u_1 - \lambda_2 u_2 - \lambda_3 u_3 + i(\lambda_3 u_2 - \lambda_2 u_3) & \lambda_1 u_3 - \lambda_3 u_1 + i(\lambda_2 u_1 - \lambda_1 u_2) \\ \\ \lambda_3 u_1 - \lambda_1 u_3 + i(\lambda_2 u_1 - \lambda_1 u_2) & -\lambda_1 u_1 - \lambda_2 u_2 - \lambda_3 u_3 + i(\lambda_2 u_3 - \lambda_3 u_2) \end{pmatrix}$$

Adding ΛU and $U\Lambda$, we obtain

$$\begin{aligned} \Lambda U + U\Lambda &= \begin{pmatrix} -2(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3) & 0 \\ \\ 0 & -2(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3) \end{pmatrix} \\ &= -2(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)I \end{aligned}$$

On the other hand, $\text{Trace}(\Lambda U) = -2(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)$. Therefore, $\Lambda U + U\Lambda = \text{Trace}(\Lambda U)I$. $\square$

Appendix B

Computations in Chapter 3

The functions f_1, f_2, f_3, f_4 in terms of Λ: To rewrite the functions f_1, f_2, f_3, f_4 in terms of the tangent vector Λ, we need the expressions for κ, τ and κ' in terms of Λ. To do so we will make use of the Serret-Frenet formulas given in terms of ordinary derivative of Λ, N, B. The formulas are given by

$$\frac{d\Lambda}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa\Lambda + (\tau + \frac{1}{2})B, \quad \frac{dB}{ds} = -(\tau + \frac{1}{2})N$$

The expression for torsion in the above formulas is in fact $\tau + \frac{1}{2}$, whereas the torsion in the Serret-Frenet formulas using covariant derivative is τ.

The expression for κ is obtained by

$$\langle \Lambda', \ \Lambda' \rangle = \langle \kappa N, \kappa N \rangle = \kappa^2$$

Now, $\Lambda'' = \kappa' N - \kappa^2 \Lambda + \kappa\tau B + \frac{1}{2}\kappa B$. So $\kappa' = \dfrac{1}{\kappa}\langle \Lambda'', \Lambda' \rangle = \dfrac{\langle \Lambda'', \Lambda' \rangle}{\sqrt{\langle \Lambda', \ \Lambda' \rangle}}$. By definition of torsion we have

$$\langle N', B \rangle = \left\langle \frac{1}{\kappa}\Lambda'' - \frac{1}{\kappa^2}\kappa'\Lambda', \ [\Lambda, \frac{1}{\kappa}\Lambda'] \right\rangle$$

$$\tau + \frac{1}{2} = \Big\langle \frac{1}{\kappa}\Lambda'', \ [\Lambda, \frac{1}{\kappa}\Lambda'] \Big\rangle$$

$$\tau = \frac{1}{\kappa^2}\Big\langle \Lambda'', \ [\Lambda, \Lambda'] \Big\rangle - \frac{1}{2}$$

Using the above information we obtain

1. $f_1 = \int_0^L \tau(s) ds = \int_0^L \langle \Lambda', \Lambda' \rangle^{-1} \ \langle \Lambda'', [\Lambda(s), \Lambda'] \rangle ds.$

2. $f_2 = \frac{1}{2} \int_0^L \kappa^2 \ (s) ds = \frac{1}{2} \int_0^L \langle \Lambda', \ \Lambda' \rangle ds.$

   Here we ignore the term $-\int_0^L \frac{1}{2} ds$, because it does not affect the computation of Hamiltonian vector field.

3. $f_3 = \int_0^L \Big[\frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{4}\kappa^2\Big] ds = \int_0^L \frac{1}{2} \langle \Lambda'', [\Lambda, \Lambda'] \rangle \ ds.$

4. $f_4 = \int_0^L \frac{1}{2}\Big[(\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) + \frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{8}\kappa^2(s) - \frac{1}{8}\kappa^4(s)\Big] ds$

$$= \int_0^L \Bigg(\frac{1}{2}\langle \Lambda'', \Lambda' \rangle^2 \langle \Lambda', \Lambda' \rangle^{-1} + \frac{1}{2}\langle \Lambda'', [\Lambda, \Lambda'] \rangle^2 \langle \Lambda', \Lambda' \rangle^{-1} - \frac{1}{2}\Big\langle \Lambda'', \ [\Lambda, \Lambda'] \Big\rangle +$$

$$\frac{1}{8}\langle \Lambda', \ \Lambda' \rangle + \frac{1}{2}\Big\langle \Lambda'', \ [\Lambda, \Lambda'] \Big\rangle - \frac{1}{4}\langle \Lambda', \ \Lambda' \rangle + \frac{1}{8}\langle \Lambda', \ \Lambda' \rangle \langle \Lambda', \ \Lambda' \rangle - \frac{1}{8}\langle \Lambda', \ \Lambda' \rangle^2 \Bigg) ds$$

$$= \int_0^L \Big(\frac{1}{2}\langle \Lambda'', \Lambda' \rangle^2 \langle \Lambda', \Lambda' \rangle^{-1} + \frac{1}{2}\langle \Lambda'', [\Lambda, \Lambda'] \rangle^2 \langle \Lambda', \Lambda' \rangle^{-1} - \frac{1}{8}\langle \Lambda', \Lambda' \rangle^2 \Big) ds.$$

B.1 Calculations in Section 3.1

In this section we provide the calculations related to the Proposition 3.1.

Steps for obtaining equation (3.4)**:** We will use integration by parts to evaluate the integrals I_1, I_2, I_3.

I_1**:** Let

$$dV = \frac{dU}{ds} ds \quad \text{and} \quad U = -2\Big\langle \frac{d\Lambda}{ds}, \ \frac{d\Lambda}{ds} \Big\rangle^{-2} \ \frac{d\Lambda}{ds} \ \Big\langle \frac{d^2\Lambda}{ds^2}, \ [\Lambda, \frac{d\Lambda}{ds}] \Big\rangle$$

Then

$$
\begin{aligned}
I_1 &= \Big[UV\Big]_0^L - \int_0^L V\,dU \\
&= 0 - 8\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}\Big\rangle^{-3} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\ \Big\rangle\ \frac{d\Lambda}{ds}(s)\Big\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s), \frac{d\Lambda}{ds}(s)]\Big\rangle, \\
&U(s)\Big\rangle\ ds + 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \frac{d^2\Lambda}{ds^2}(s)\ \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s), \frac{d\Lambda}{ds}(s)]\Big\rangle,\ U(s)\Big\rangle\ ds \\
&+ 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \frac{d\Lambda}{ds}(s)\ \Big\langle \frac{d^3\Lambda}{ds^3}(s),\ [\Lambda(s), \frac{d\Lambda}{ds}(s)]\Big\rangle,\ U(s)\Big\rangle\ ds \\
&+ 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \frac{d\Lambda}{ds}(s)\ \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s), \frac{d^2\Lambda}{ds^2}(s)]\Big\rangle,\ U(s)\Big\rangle\ ds \\
&= I_{11} + I_{12} + I_{13} + I_{14} \quad \text{(say)} \qquad \text{(B.1)}
\end{aligned}
$$

The term $\Big[UV\Big]_0^L$ vanishes since $\Lambda(0) = \Lambda(L)$.

I_2**:** Let

$$
dV = \frac{d^2U}{ds^2}ds \quad \text{and} \quad U = \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1}\Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]
$$

Then

$$
\begin{aligned}
I_2 &= 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big],\ \frac{dU}{ds}(s)\Big\rangle\ ds \\
&- \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\Big\rangle^{-1}\ \Big[\Lambda(s), \frac{d^2\Lambda}{ds^2}(s)\Big],\ \frac{dU}{ds}(s)\Big\rangle\ ds \\
&= I_{21} + I_{22} \qquad \text{(say)} \qquad \text{(B.2)}
\end{aligned}
$$

Using integration by parts again we obtain

$$
I_{21} = 8\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-3} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle^2\ \Big[\Lambda(s),\ \frac{d\Lambda}{ds}(s)\Big],\ U(s)\Big\rangle\ ds
$$

$$- 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big],\ U(s)\Big\rangle\ ds$$

$$- 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^3\Lambda}{ds^3}(s)\Big\rangle \Big[\Lambda(s),\ \frac{d\Lambda}{ds}(s)\Big],\ U(s)\Big\rangle\ ds$$

$$- 2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle \Big[\Lambda(s), \frac{d^2\Lambda}{ds^2}(s)\Big],\ U(s)\Big\rangle\ ds \tag{B.3}$$

and

$$\begin{aligned}
I_{22} = &-2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle \Big[\Lambda(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big],\ U(s)\Big\rangle\ ds \\
&+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big[\Lambda(s),\ \frac{d^3\Lambda}{ds^3}(s)\Big],\ U(s)\Big\rangle\ ds \\
&+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-1} \Big[\frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big],\ U(s)\Big\rangle\ ds
\end{aligned} \tag{B.4}$$

I_3 returns the same outcome as I_{22} when evaluated using integration by parts. So

$$I_3 = I_{22} \tag{B.5}$$

and the integral I_4 takes the form

$$I_4 = \int_0^L \Big\langle - \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-1} \Big[\frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s)\Big],\ U(s)\Big\rangle\ ds \tag{B.6}$$

Therefore

$$\begin{aligned}
df_{1X}(V) &= I_{11} + I_{12} + I_{13} + I_{14} + I_{21} + 2I_{22} + I_4 \\
&= -\int_0^L \Big\langle G(s), U(s)\Big\rangle\ ds
\end{aligned} \tag{B.7}$$

where,

$$
\begin{aligned}
G(s) = {}& 8\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}\Big\rangle^{-3} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\ \Big\rangle\ \frac{d\Lambda}{ds}(s)\Big\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s), \frac{d\Lambda}{ds}(s)]\Big\rangle \\
&+ 2\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \frac{d^2\Lambda}{ds^2}(s)\ \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s), \frac{d\Lambda}{ds}(s)]\Big\rangle \\
&+ 2\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \frac{d\Lambda}{ds}(s)\Big\langle \frac{d^3\Lambda}{ds^3}(s),\ [\Lambda(s), \frac{d\Lambda}{ds}(s)]\Big\rangle \\
&+ 2\langle\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2}\ \frac{d\Lambda}{ds}(s)\ \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ [\Lambda(s), \frac{d^2\Lambda}{ds^2}(s)]\Big\rangle \\
&+ 8\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-3} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle^{2} \Big[\Lambda(s),\ \frac{d\Lambda}{ds}(s)\Big] \\
&- 2\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle \Big[\Lambda(s), \frac{d\Lambda}{ds}(s)\Big] \\
&- 2\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^3\Lambda}{ds^3}(s)\Big\rangle \Big[\Lambda(s),\ \frac{d\Lambda}{ds}(s)\Big] \\
&- 4\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big\rangle \Big[\Lambda(s), \frac{d^2\Lambda}{ds^2}(s)\Big] \\
&+ 2\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big[\Lambda(s),\ \frac{d^3\Lambda}{ds^3}(s)\Big] \\
&+ 2\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-1} \Big[\frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\Big] \\
&- \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\Big\rangle^{-1} \Big[\frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s)\Big]
\end{aligned}
$$

Now we will express $G(s)$ in terms of curvature and torsion using Serret-Frenet equations. Before doing so here we provide a list of computations that will help us rewrite expressions like $G(s)$ with respect to curvature and torsion.

1. $\Lambda' = \kappa N$

2. $\Lambda'' = \kappa' N - \kappa^2 \Lambda + \kappa\tau B$

3. $\Lambda''' = \kappa'' N - 3\kappa\kappa'\Lambda + 2\kappa'\tau B - \kappa^3 N + \kappa\tau' B - \kappa\tau^2 N$

4. $\Lambda^{\text{iv}} = \kappa''' N - 4\kappa\kappa''\Lambda + 3\kappa''\tau B - 3(\kappa')^2\Lambda - 6\kappa^2\kappa' N + 3\kappa'\tau' B - 3\kappa'\tau^2 N$

   $+ \kappa^4\Lambda - \kappa^3\tau B + \kappa\tau'' B - 3\kappa\tau\tau' N + \kappa^2\tau^2\Lambda - \kappa\tau^3 B$

5. $\langle \Lambda', \Lambda' \rangle = \kappa^2$

6. $\langle \Lambda', \Lambda'' \rangle = \kappa\kappa''$

7. $\langle \Lambda', \Lambda''' \rangle = \kappa\kappa'' - \kappa^4 - \kappa^2\tau^2$

8. $\langle \Lambda', \Lambda^{\text{iv}} \rangle = \kappa\kappa''' - 6\kappa^3\kappa' - 3\kappa\kappa'\tau^2 - 3\kappa^2\tau\tau'$

9. $\langle \Lambda'', \Lambda'' \rangle = (\kappa')^2 + \kappa^4 + \kappa^2\tau^2$

10. $\langle \Lambda'', \Lambda''' \rangle = \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa\kappa'\tau^2 + \kappa^2\tau\tau'$

11. $[\Lambda, \Lambda'] = \kappa B$

12. $[\Lambda, \Lambda''] = \kappa' B - \kappa\tau N$

13. $[\Lambda, \Lambda'''] = \kappa'' B - 2\kappa'\tau N - \kappa^3 B - \kappa\tau' N - \kappa\tau^2 B$

14. $[\Lambda', \Lambda''] = \kappa^3 B + \kappa^2\tau\Lambda$

15. $\langle \Lambda'', [\Lambda, \Lambda'] \rangle = \kappa^2\tau$

16. $\langle \Lambda'', [\Lambda, \Lambda''] \rangle = 0$

17. $\langle \Lambda'', [\Lambda', \Lambda''] \rangle = 0$

18. $\langle \Lambda'', [\Lambda, \Lambda'''] \rangle = -2(\kappa')^2\tau - \kappa\kappa'\tau' + \kappa\kappa''\tau - \kappa^4\tau - \kappa^2\tau^3$

19. $\langle \Lambda''', [\Lambda, \Lambda'] \rangle = 2\kappa\kappa'\tau + \kappa^2\tau'$

20. $\langle \Lambda''', [\Lambda, \Lambda''] \rangle = 2(\kappa')^2\tau + \kappa\kappa'\tau' - \kappa\kappa''\tau + \kappa^4\tau + \kappa^2\tau^3$

21. $\langle \Lambda^{\text{iv}}, [\Lambda, \Lambda'] \rangle = 3\kappa\kappa''\tau + 3\kappa\kappa'\tau' - \kappa^4\tau + \kappa^2\tau'' - \kappa^2\tau^3$

Therefore

$$G(s) = -8\kappa^{-2}\kappa'\tau N + 2\kappa^{-2}\tau\kappa' N - 2\tau\Lambda + 2\kappa^{-1}\tau^2 B + 4\kappa^{-2}\kappa'\tau N + 2\kappa^{-1}\tau' N$$

$$+ 4\kappa^{-3}(\kappa')^2 B - 2\kappa^{-2}\kappa'' B + 2\kappa^{-2}\kappa'\tau N - 2\kappa^{-3}(\kappa')^2 B + \kappa^{-2}\kappa'' B - \kappa^{-1}\tau' N$$

$$- \kappa^{-1}\tau^2 B + \tau\Lambda + \kappa B + \tau\Lambda$$

B.2 Calculations in Section 3.3

The calculations related to the Proposition 3.3 are provided in this section.

Steps to obtain equation (3.21)**:** Using integration by parts as before we obtain,

I_{11}**:** Let

$$dV = \frac{dU}{ds}ds \quad \text{and} \quad U = -\left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \frac{d\Lambda}{ds}(s) \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^2$$

Then

$$\begin{aligned}
I_{11} = &-4\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-3} \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s)\right\rangle^3 \frac{d\Lambda}{ds}(s),\ U(s)\right\rangle ds \\
&+ \int_0^L \left\langle \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s)\right\rangle^2,\ U(s)\right\rangle ds \\
&+ 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \frac{d\Lambda}{ds}(s) \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d\Lambda}{ds}(s)\right\rangle \left\langle \frac{d^3\Lambda}{ds^3}(s), \frac{d\Lambda}{ds}(s)\right\rangle, U(s)\right\rangle ds \\
&+ 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s)\right\rangle^{-2} \frac{d\Lambda}{ds}(s) \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d\Lambda}{ds}(s)\right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d^2\Lambda}{ds^2}(s)\right\rangle, U(s)\right\rangle ds
\end{aligned} \tag{B.8}$$

I_{12}: Let

$$dV = \frac{d^2U}{ds^2}ds \quad \text{and} \quad U = \left\langle \frac{d\Lambda^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1}$$

Then we get

$$\begin{aligned}
I_{12} = &- \int_0^L \left\langle \left\langle \frac{d^3\Lambda}{ds^3}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ \frac{dU}{ds}(s) \right\rangle ds \\
&- \int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^2\Lambda}{ds^2}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ \frac{dU}{ds}(s) \right\rangle ds \\
&- \int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ \frac{dU}{ds}(s) \right\rangle ds \\
&+ 2\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^2 \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-2},\ \frac{dU}{ds}(s) \right\rangle ds \\
&= I_{121} + I_{122} + I_{123} + I_{124}
\end{aligned} \tag{B.9}$$

Using integration by parts again for each of the above integrals in (B.9) we get

$$\begin{aligned}
I_{121} = &\int_0^L \left\langle \left\langle \frac{d^4\Lambda}{ds^4}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds \\
&+ \int_0^L \left\langle \left\langle \frac{d^3\Lambda}{ds^3}(s),\ \frac{d^2\Lambda}{ds^2}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds \\
&+ \int_0^L \left\langle \left\langle \frac{d^3\Lambda}{ds^3}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds \\
&- 2\int_0^L \left\langle \left\langle \frac{d^3\Lambda}{ds^3}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s) \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds}(s), \frac{d^2\Lambda}{ds^2}(s) \right\rangle, U(s) \right\rangle ds
\end{aligned}$$

$$I_{122} = 2\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^3\Lambda}{ds^3}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$+\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^2\Lambda}{ds^2}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$-\,2\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d^2\Lambda}{ds^2}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s) \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds}(s), \frac{d^2\Lambda}{ds^2}(s) \right\rangle, U(s) \right\rangle ds$$

$$I_{123} = \int_0^L \left\langle \left\langle \frac{d^3\Lambda}{ds^3}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$+\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^2\Lambda}{ds^2}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$+\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d^3\Lambda}{ds^3}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$-\,2\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^2 \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-2},\ U(s) \right\rangle ds$$

and

$$I_{124} = -4\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d\Lambda}{ds}(s) \right\rangle \left\langle \frac{d^3\Lambda}{ds^3}(s), \frac{d\Lambda}{ds}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s) \right\rangle^{-2}, U(s) \right\rangle ds$$

$$-\,4\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d\Lambda}{ds}(s) \right\rangle \left\langle \frac{d^2\Lambda}{ds^2}(s), \frac{d\Lambda}{ds}(s) \right\rangle \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s), \frac{d\Lambda}{ds}(s) \right\rangle^{-2}, U(s) \right\rangle ds$$

$$+\,8\int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^3 \frac{d\Lambda}{ds}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-3},\ U(s) \right\rangle ds \tag{B.10}$$

I_{13}: Let

$$dV = \frac{dU}{ds} ds \quad \text{and} \quad U = \left\langle \frac{d^2\Lambda\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1}$$

Consequently I_{13} becomes

$$I_{13} = -\int_0^L \left\langle \left\langle \frac{d^3\Lambda}{ds^3}(s),\ \frac{d\Lambda}{ds}(s) \ \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$- \int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d^2\Lambda}{ds^2}(s) \right\rangle \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$- \int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle \frac{d^3\Lambda}{ds^3}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-1},\ U(s) \right\rangle ds$$

$$+ 2 \int_0^L \left\langle \left\langle \frac{d^2\Lambda}{ds^2}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^2 \frac{d^2\Lambda}{ds^2}(s) \left\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \right\rangle^{-2},\ U(s) \right\rangle ds \qquad \text{(B.11)}$$

Thus we obtain

$$\begin{aligned} dI_1\Lambda(V) &= I_{11} + I_{12} + I_{13} \\ &= - \int_0^L \left\langle G_1(s), U(s) \right\rangle ds \end{aligned}$$

where $G_1(s)$ is the sum of all the terms in each of the above integrals excluding $U(s)$.

Expression of $G_1(s)$ in terms of curvature and torsion becomes

$$G_1(s) = \kappa''' N - \kappa\kappa'' \Lambda + \kappa'' \tau B$$

Steps to obtain equation (3.23)**:**

$$\begin{aligned} I_{21} = -4 \int_0^L & \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-3} \left\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2} \right\rangle \frac{d\Lambda}{ds} \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle^2,\ U \right\rangle ds \\ + \int_0^L & \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \frac{d^2\Lambda}{ds^2} \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle^2,\ U \right\rangle ds \\ + 2 \int_0^L & \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \frac{d\Lambda}{ds} \left\langle \frac{d^3\Lambda}{ds^3},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle,\ U \right\rangle ds \\ + 2 \int_0^L & \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \frac{d\Lambda}{ds} \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d^2\Lambda}{ds^2}\right] \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle,\ U \right\rangle ds \end{aligned}$$

$$I_{22} = 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2} \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda,\ \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda,\ \frac{d\Lambda}{ds}\right],\ \frac{dU}{ds} \right\rangle ds$$

$$- \int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-1} \left\langle \frac{d^3\Lambda}{ds^3},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda, \frac{d\Lambda}{ds}\right],\ \frac{dU}{ds} \right\rangle ds$$

$$- \int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda,\ \frac{d^2\Lambda}{ds^2}\right] \right\rangle \left[\Lambda,\ \frac{d\Lambda}{ds}\right],\ \frac{dU}{ds} \right\rangle ds$$

$$- \int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-1} \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda, \frac{d^2\Lambda}{ds^2}\right],\ \frac{dU}{ds} \right\rangle ds$$

$$= I_{221} + I_{222} + I_{223} + I_{224} \quad \text{(say)} \tag{B.12}$$

Again, we evaluate each of the integrals in equation (B.12) by letting $dV = \dfrac{dU}{ds}\, ds$ and the rest as U. Thus we get

$$I_{221} = 8\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-3} \left\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2} \right\rangle^2 \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda,\ \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda,\ \frac{d\Lambda}{ds}\right],\ U \right\rangle ds$$

$$- 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \left\langle \frac{d^2\Lambda}{ds^2},\ \frac{d^2\Lambda}{ds^2} \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda, \frac{d\Lambda}{ds}\right],\ U \right\rangle ds$$

$$- 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds},\ \frac{d^3\Lambda}{ds^3} \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda, \frac{d\Lambda}{ds}\right],\ U \right\rangle ds$$

$$- 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2} \right\rangle \left\langle \frac{d^3\Lambda}{ds^3},\ \left[\Lambda,\ \frac{d\Lambda}{ds}\right] \right\rangle \left[\Lambda, \frac{d\Lambda}{ds}\right],\ U \right\rangle ds$$

$$- 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2} \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d^2\Lambda}{ds^2}\right] \right\rangle \left[\Lambda,\ \frac{d\Lambda}{ds}\right],\ U \right\rangle ds$$

$$- 2\int_0^L \left\langle \left\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds} \right\rangle^{-2} \left\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2} \right\rangle \left\langle \frac{d^2\Lambda}{ds^2},\ \left[\Lambda, \frac{d^2\Lambda}{ds^2}\right] \right\rangle \left[\Lambda, \frac{d^2\Lambda}{ds^2}\right],\ U \right\rangle ds \tag{B.13}$$

$$
\begin{aligned}
I_{222} = -2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2}\Big\rangle \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
- \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^4\Lambda}{ds^4},\ \Big[\Lambda, \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\Lambda, \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
- \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
- \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big],\ U\Big\rangle\, ds \qquad \text{(B.14)}
\end{aligned}
$$

$$
\begin{aligned}
I_{223} = -2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2}\Big\rangle \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda, \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda, \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda, \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda, \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda, \frac{d^3\Lambda}{ds^3}\Big]\Big\rangle \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda, \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big],\ U\Big\rangle\, ds \qquad \text{(B.15)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{224} = -2\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2}\Big\rangle \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda, \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big],\ U\Big\rangle\, ds \\
+ \int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big],\ U\Big\rangle\, ds
\end{aligned}
$$

$$+\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big],\ U \Big\rangle\, ds$$

$$+\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda, \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\Lambda,\ \frac{d^3\Lambda}{ds^3}\Big],\ U \Big\rangle\, ds$$

$$+\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2}, \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2}\Big],\ U \Big\rangle\, ds \tag{B.16}$$

Similarly I_{23} gives us

$$\begin{aligned}
I_{23} = 2\int_0^L &\Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-2} \Big\langle \frac{d\Lambda}{ds},\ \frac{d^2\Lambda}{ds^2}\Big\rangle \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\frac{d^2\Lambda}{ds^2},\ \Lambda\Big],\ U \Big\rangle\, ds \\
-\int_0^L &\Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^3\Lambda}{ds^3},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\frac{d^2\Lambda}{ds^2},\ \Lambda],\ U \Big\rangle\, ds \\
-\int_0^L &\Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d^2\Lambda}{ds^2}\Big]\Big\rangle \Big[\frac{d^2\Lambda}{ds^2},\ \Lambda],\ U \Big\rangle\, ds \\
-\int_0^L &\Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\frac{d^2\Lambda}{ds^2},\ \frac{d\Lambda}{ds}\Big],\ U \Big\rangle\, ds \\
-\int_0^L &\Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\frac{d^3\Lambda}{ds^3},\ \Lambda],\ U \Big\rangle\, ds
\end{aligned} \tag{B.17}$$

and finally

$$\begin{aligned}
I_{24} &= -\int_0^L \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\frac{d\Lambda}{ds},\ U\Big]\Big\rangle\, ds \\
&= -\int_0^L \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big\langle \Big[\frac{d^2\Lambda}{ds^2},\ \frac{d\Lambda}{ds}\Big],\ U\Big\rangle\, ds \\
&= -\int_0^L \Big\langle \Big\langle \frac{d\Lambda}{ds},\ \frac{d\Lambda}{ds}\Big\rangle^{-1} \Big\langle \frac{d^2\Lambda}{ds^2},\ \Big[\Lambda,\ \frac{d\Lambda}{ds}\Big]\Big\rangle \Big[\frac{d^2\Lambda}{ds^2},\ \frac{d\Lambda}{ds}\Big],\ U \Big\rangle\, ds
\end{aligned} \tag{B.18}$$

Therefore, we obtain

$$dI_2\Lambda(V) = I_{21} + I_{22} + I_{23} + I_{24}$$

$$= -\int_0^L \Big\langle G_2(s), U(s) \Big\rangle \, ds$$

where $G_2(s)$ is the sum of all the terms in each of the above integrals excluding $U(s)$.

Now, $G_2(s)$ in terms of curvature and torsion takes the following form

$$\begin{aligned} G_2(s) &= -3\kappa'\tau^2 N + 2\kappa^2\tau^2\Lambda - \kappa\tau^3 B - 3\kappa\tau\tau' N + \kappa^3\tau B + 3\kappa'\tau' B \\ &\quad + 2\kappa''\tau B + \kappa\tau'' B \end{aligned}$$

Steps to obtain equation (3.24)**:**

$$\begin{aligned} G_3(s) &= \Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d^2\Lambda}{ds^2}(s) \Big\rangle \ \frac{d\Lambda}{ds}(s) \\ &\quad + \frac{1}{2}\Big\langle \frac{d\Lambda}{ds}(s),\ \frac{d\Lambda}{ds}(s) \Big\rangle \ \frac{d^2\Lambda}{ds^2}(s) \end{aligned}$$

Expressing $G_3(s)$ in terms of curvature and torsion we get

$$G_3(s) = \frac{3}{2}\kappa^2\kappa' N + \frac{1}{2}\kappa^4\Lambda + \frac{1}{2}\kappa^3\tau B \tag{B.19}$$

B.3 Calculations in Section 3.4

In this section we discuss the Poisson commuting functions mentioned in the Theorem 3.4.

Theorem B.1. The function $f_1 = \displaystyle\int_0^L \tau(s) ds$ Poisson commutes with the functions

(a) $f_2 = \dfrac{1}{2}\displaystyle\int_0^L \kappa^2(s)\ ds.$

(b) $f_3 = \dfrac{1}{2}\displaystyle\int_0^L \kappa^2(s)\tau(s)\ ds.$

(c) $f_4 = \displaystyle\int_0^L \Big(\frac{1}{2}(\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) - \frac{1}{8}\kappa^4(s)\Big)\ ds.$

Proof. The Poisson bracket of two functions f_a and f_b is given by the formula

$$\{f_a, f_b\} = \omega_X(F_a(s), F_b(s)) = -\int_0^L \langle \Lambda(s), [U_{f_a}(s), U_{f_b}(s)] \rangle ds$$

where $U_{f_a}(s)$ and $U_{f_b}(s)$ are Hamiltonian vector fields corresponding to f_a and f_b respectively.

The Hamiltonian vector fields of f_1, f_2 and f_3 are $U_{f_1} = \kappa(s,t)N(s,t)$, $U_{f_2} = [\Lambda'', \Lambda]$ and $U_{f_3} = [\Lambda''' - \langle \Lambda''', \Lambda \rangle \Lambda] - \frac{3}{2}\langle \Lambda, \Lambda'' \rangle \Lambda'$ respectively. Here, $\Lambda', \Lambda'', \Lambda'''$ denote the first, second and third derivatives of Λ with respect to s.

Proof of (a): We are required to show that

$$\{f_2, f_1\} = \omega_X(F_2(s), F_1(s)) = -\int_0^L \langle \Lambda(s), [U_{f_2}(s), U_{f_1}(s)] \rangle ds = 0$$

Now

$$U_{f_2} = [\Lambda'', \Lambda] = [\kappa' N - \kappa^2 \Lambda + \kappa\tau B, \Lambda] = -\kappa' B + \kappa\tau N,$$

so

$$[U_{f_2}(s),\ U_{f_1}(s)] = [-\kappa' B + \kappa\tau N, \kappa N] = \kappa\kappa' \Lambda.$$

Then

$$\begin{aligned}
\{f_2, f_1\} &= -\int_0^L \langle \Lambda,\ \kappa\kappa' \Lambda \rangle ds \\
&= -\int_0^L \kappa\kappa' ds \\
&= -\int_0^L \frac{d}{ds}(\frac{1}{2}\kappa^2) ds \\
&= 0.
\end{aligned}$$

Therefore, the function f_2 Poisson commutes with f_1.

Proof of (b): The proof is similar to the proof in part(a). Here we need to show that

$$\{f_1, f_3\} = \omega_X(F_1(s), F_3(s)) = -\int_0^L \langle \Lambda(s), [U_{f_1}(s), U_{f_3}(s)]\rangle ds = 0$$

Now

$$\Lambda''' = \kappa'' N - 3\kappa\kappa'\Lambda + 2\kappa'\tau B - \kappa^3 N + \kappa\tau' B - \kappa\tau^2 N,$$

$$\langle \Lambda''', \Lambda\rangle\Lambda = -3\kappa\kappa'\Lambda,$$

$$\langle \Lambda, \Lambda''\rangle = \langle \Lambda, \kappa' N - \kappa^2\Lambda + \kappa\tau B\rangle = -\kappa^2,$$

$$\frac{3}{2}\langle \Lambda, \Lambda''\rangle\Lambda' = -\kappa^2 \cdot \kappa N\rangle = -\frac{3}{2}\kappa^3 N.$$

So

$$U_{f_3} = \Big(\Lambda''' - \langle \Lambda''', \Lambda\rangle\Lambda\Big) - \frac{3}{2}\langle \Lambda, \Lambda''\rangle\Lambda'$$

$$= \kappa'' N + 2\kappa'\tau B + \frac{1}{2}\kappa^3 N + \kappa\tau' B - \kappa\tau^2 N.$$

Then the Lie bracket of U_{f_1} and U_{f_3} gives

$$[U_{f_1}, U_{f_3}] = -2\kappa\kappa'\tau\Lambda - \kappa^2\tau'\Lambda.$$

Consequently

$$\{f_3, f_1\} = -\int_0^L \langle \Lambda,\ (-2\kappa\kappa'\tau - \kappa^2\tau')\Lambda\rangle ds$$

$$= -\int_0^L (-2\kappa\kappa'\tau - \kappa^2\tau') ds$$

$$= \int_0^L \frac{d}{ds}(\kappa^2\tau)ds$$

$$= 0.$$

Therefore, the function f_1 Poisson commutes with f_3.

Proof of (c): We need to show that

$$\{f_1, f_4\} = \omega_X(F_1(s), F_4(s)) = -\int_0^L \langle \Lambda(s), [U_{f_1}(s), U_{f_4}(s)]\rangle ds = 0$$

Now

$$U_{f_4} = (-\kappa\tau^3 + 3\kappa''\tau + 3\kappa'\tau' + \frac{3}{2}\kappa^3\tau + \kappa\tau'')N + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')B.$$

So

$$[U_{f_1}(s), U_{f_4}(s)] = [(-\kappa\tau^3 + 3\kappa''\tau + 3\kappa'\tau' + \frac{3}{2}\kappa^3\tau + \kappa\tau'')N + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2$$

$$- \frac{3}{2}\kappa^2\kappa')B,\ \kappa N]$$

$$= (-\kappa\kappa''' + 3\kappa^2\tau\tau' + 3\kappa\kappa'\tau^2 - \frac{3}{2}\kappa^3\kappa')\Lambda$$

Then taking the inner product with Λ we get

$$\langle \Lambda, [U_{f_1}, U_{f_4}]\rangle = -\kappa\kappa''' + 3\kappa^2\tau\tau' + 3\kappa\kappa'\tau^2 - \frac{3}{2}\kappa^3\kappa'.$$

As a consequence

$$\{f_1, f_4\} = -\int_0^L (-\kappa\kappa''' + 3\kappa^2\tau\tau' + 3\kappa\kappa'\tau^2 - \frac{3}{2}\kappa^3\kappa')ds$$

$$= -\int_0^L \Big[\frac{d}{ds}\Big(- \kappa\kappa'' + \frac{1}{2}(\kappa')^2\Big) + \frac{d}{ds}\Big(\frac{3}{2}\kappa^2\tau^2\Big) + \frac{d}{ds}\Big(\frac{3}{8}\kappa^4\Big)\Big]ds$$

$= 0.$

Therefore, the function f_1 Poisson commutes with f_4. $\square$

Theorem B.2. The function $f_2 = \frac{1}{2}\int_0^L \kappa^2(s)\ ds$ Poisson commutes with the function $f_3 = \frac{1}{2}\int_0^L \kappa^2(s)\tau(s)\ ds$.

Proof. See Theorem 7 in [6]. $\square$

Theorem B.3. The function $f_2 = \frac{1}{2}\int_0^L \kappa^2(s)\ ds$ Poisson commutes with the function $f_4 = \int_0^L \Big[\frac{1}{2}(\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) - \frac{1}{8}\kappa^4(s)\Big]ds$.

Proof. See Theorem 3.4. $\square$

Theorem B.4. The function $f_3 = \frac{1}{2}\int_0^L \kappa^2(s)\tau(s)\ ds$ Poisson commutes with the function $f_4 = \int_0^L \Big[(\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) - \frac{1}{8}\kappa^4(s)\Big]ds$.

Proof. We need to show that the Poisson bracket of the functions f_3 and f_4 is equal to zero. That is,

$$\{f_4,\ f_3\} = 0$$

which is defined by the symplectic form.

The Hamiltonian vector fields associated with f_3 and f_4 in terms of curvature and torsion are

$$U_{f_3}(s) = \kappa''N + 2\kappa'\tau B + \frac{1}{2}\kappa^3 N + \kappa\tau' B - \kappa\tau^2 N$$

and

$$U_{f_4}(s) = (-\kappa\tau^3 + 3\kappa''\tau + 3\kappa'\tau' + \frac{3}{2}\kappa^3\tau + \kappa\tau'')N + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')B$$

The inner product of the Lie bracket of these two vector fields with $\Lambda(s)$ generates

$$\Big\langle \Lambda, [U_{f_3}(s), U_{f_4}(s)] \Big\rangle = -\frac{3}{2}\kappa^2\kappa'\kappa'' - \frac{1}{2}\kappa^3\kappa''' - 3\kappa'\kappa''\tau^2 + \kappa\kappa'''\tau^2 - \frac{3}{4}\kappa^5\kappa' - \kappa''\kappa'''$$

$$- 3\kappa\kappa'(\tau')^2 - \kappa^2\tau'\tau'' - 6(\kappa')^2\tau\tau' - \kappa\kappa'\tau^4 - 2\kappa^2\tau^3\tau' - 2\kappa\kappa'\tau\tau''$$

Thus the Poisson bracket

$$\begin{aligned}\{f_4,\ f_3\} = &-\int_0^L \Big(-\frac{3}{2}\kappa^2\kappa'\kappa'' - \frac{1}{2}\kappa^3\kappa''' - 3\kappa'\kappa''\tau^2 + \kappa\kappa'''\tau^2 - \frac{3}{4}\kappa^5\kappa' - \kappa''\kappa''' \\ &- 3\kappa\kappa'(\tau')^2 - \kappa^2\tau'\tau'' - 6(\kappa')^2\tau\tau' - \kappa\kappa'\tau^4 - 2\kappa^2\tau^3\tau' - 2\kappa\kappa'\tau\tau''\Big)\ ds \\ = &-\int_0^L \frac{d}{ds}\Big(-\frac{1}{2}\kappa^3\kappa'' + \kappa\kappa''\tau^2 - \frac{1}{8}\kappa^6 - \frac{1}{2}(\kappa'')^2 - \frac{1}{2}\kappa^2(\tau')^2 \\ &- 2\tau^2(\kappa')^2 - \frac{1}{2}\kappa^2\tau^4 - 2\kappa\kappa'\tau\tau'\Big)\ ds \\ = &\ 0.\end{aligned}$$

Therefore, the function f_3 Poisson commutes with the function f_4. $\square$

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