

Fixed Point Iterative Methods for Linear Complementarity  
Problems in American Option Pricing

by

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## **Abstract**

The linear complementarity formulation for American option pricing is studied. In particular, fixed point solution methods from literature are considered. Here simplified proofs for convergence criterion are provided. Further generalizations and modifications are also suggested alongside more tractable convergence analysis. Alternative formulations for the option pricing problem are also surveyed. These will be in the form of the Chapman-Kolmogorov lattice methods and the Modified Mellin transform framework. Some numerical experiments are then conducted for comparison purposes.

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# 1

## Introduction

The theory of pricing financial instruments dates back to the work of L.Bachelier [2] in his paper, "The theory of speculation". Since then, significant strides have been taken in designing and pricing numerous financial derivative instruments. In finance a *derivative* is a financial security whose value depends on some underlying asset. For example, an option gives the holder the right (not obligation) to buy (call option) or sell (put option) an underlying asset at a predetermined price, known as the strike price, at an agreed upon future date also known as the maturity date.

While the valuation of many of these derivatives has historically been based on the Risk Neutral Valuation technique, cases where complications arose have warranted different approaches. Among these is the pricing of American style derivatives. An American option, for example, gives the holder the freedom to exercise at any time up to the maturity date, unlike a European option which stipulates exercise only at maturity. This extra flexibility offered by the American option represents a free boundary problem because the maturity date is unknown a priori. Assuming rationality, the holder will exercise at date that will maximize their profit out of the contract, hence valuation of derivatives of this nature are also known as optimal



stopping problems. As illustrated in [6], we can price perpetual American Options analytically and further works such as [33], have demonstrated analytic solutions when variations are taken into account. Modern approaches to price American options with a finite maturity date mainly involve numerical methods (see for example [21]). By implementing a finite difference scheme, a common and classical iterative method known as the Projected Successive Over-relaxation Method (PSOR) developed by Cryer [5] allows for solving the free boundary problem when set up as a linear complementarity problem (LCP) [27]. This is a useful technique as we do not have to state the unknown boundaries explicitly.

While PSOR methods proved to be useful and popular, better efficiency has always been sought. Notably, extensive research has been carried out to develop higher performing iterative methods. Among these is the Modulus Based Matrix Splitting Method (MBMSM) by Bai [3]. In literature, the main criticism of PSOR methods is their two-step procedure where, in order to satisfy the inequalities involved in the LCP setup, a second step is implemented to bound each iterate. This can be costly computationally. As such, single step procedures have received more attention in recent times. Bai's method achieves this by proving equivalence of an implicit fixed point method to the original LCP and then deducing a single step iterative procedure based on the fixed point problem. Following on this, Bai's work has been seen as a template to further works in recent times, focussed on further enhancing the efficiency of the MBMSM. For example Zheng and Yin [35] proposed accelerating parameters which reduced CPU computational time and costs. Even more instrumental was the work of Li [20] who generalized the matrix formulations in the original MBMSM, providing convergence analysis along the way. This widens flexibility in fine-tuning parameters for faster and less costly convergence towards the unique solution. While Bai's work has proved to be a solid breakthrough here, the nature of the implicit fixed point equations can be long and make convergence

analysis complicated. Shi, Yang and Huang [28] propose an explicit fixed point algorithm that certainly involves less parameters, and for matrices with good properties such as being an  $M$ -Matrix, convergence analysis can be more tractable compared to the MBMSM. In [28], the proposed algorithm is for a specific problem in American option pricing. They also introduce a relaxation parameter designed to accelerate convergence and determine the range for which the iteration will converge. However, their determination of this range is still strenuous as it involves computing long matrix inverses and manually checking the components of the resulting iterative matrix in convergence testing. In this thesis, we will develop the American option pricing problem Shi, Yang and Huang study and provide stability analysis of the numerical scheme. We propose a new proof for convergence criterion of the SYH method which is significantly more tractable and avoids computation of complicated inverses resulting from iterative matrices. As a consequence of this proof, we propose a convergence domain for a system in which, unlike the symmetric tri-diagonal matrix system in [28]), caters for general matrix formulations. This is of strong importance in option pricing, particularly when non-uniform grids are used to minimize oscillations due to lack of smoothness of the pay-off function and time-stepping numerical schemes such as the Rannacher scheme [9] are employed. The second new generalization developed in this thesis involves widening the scope of Shi, Yang and Huang's fixed point algorithm. Specifically, using ideas from the MBMSM, we generalize this fixed point algorithm via an introduction of some arbitrary positive diagonal matrices. As a special case, we show that Shi, Yang and Huang's method can be retained. Such a general method, we believe, can be viewed in two ways: a generalization of Shi, Yang and Huang's fixed algorithm and a modification of the Modulus Based Matrix Splitting Method by Bai, due to its development and tractability in the convergence results. Outside of solving for LCPs in American option pricing, other formulations of these free boundary problems exist. As such, we discuss two of these formulations,

particularly popular in industry. These are lattice methods and a more analytical approach via an integral representation.

## Interest Rate Theory

The interest rate on an investment is the rate at which that initial investment grows over a given time period. For example, on an initial investment of  $A_0$  for  $n$  years, the formula for simple interest is

$$A_s = A_0(1 + ni)$$

where  $i$  is the annualised simple interest rate. More commonly used is the compounded interest rate given by the following formula

$$A_c = A_0(1 + i)^n$$

for an accumulation  $A_c$  after  $n$  years. With compound interest, interest is applied on the accumulated amount at the end of each year unlike in simple interest where interest is applied on the initial investment.

We will be more concerned with continuously compounded interest. Here the accumulation is given by

$$A_c = A_0 e^{\int_{t_0}^t r_s ds}$$

where  $r_s$  is the continuously compounded interest rate from time  $t_0$  to  $t$ . Similarly, we can discount back future cash flows to today's terms, known as the present value. For example, discounting a unit payment  $t$  years from now using a continuously compounded rate of  $r_s$  we obtain  $e^{-\int_{t_0}^t r_s ds}$ . In this thesis,  $r_s$  will be some constant value and, unless specified,  $r$  will represent the risk free interest rate. From the

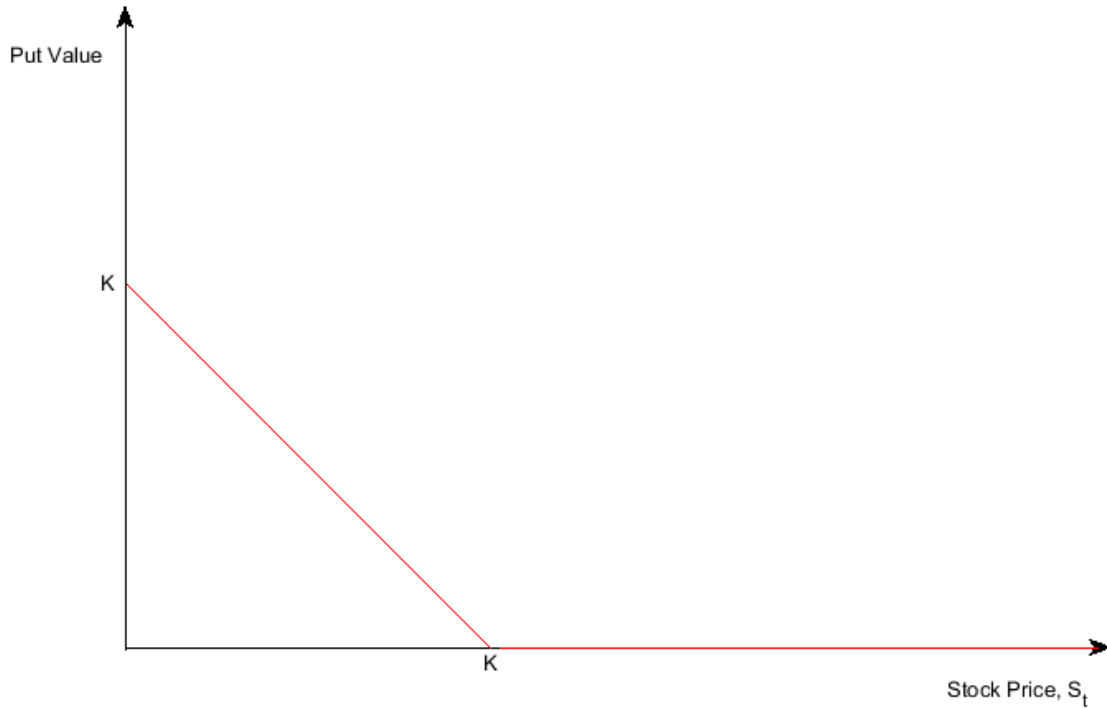


Figure 1.1: Put Option Value at Maturity  $T$

name, this is the interest rate that is guaranteed by a bank account deposit with a zero probability of default. A common proxy for the risk free rate are yields on US short-term treasuries.

## Put and Call Options

A financial derivative has a value that depends on some underlying security. One of the simplest derivative is a *future* which gives the holder an obligation to buy a security,  $s$ , at some future time  $T$ , for an agreed upon price  $K$ . A future's payoff is given by

$$s_T - K.$$

As already noted, a European option differs from a futures contract in the sense

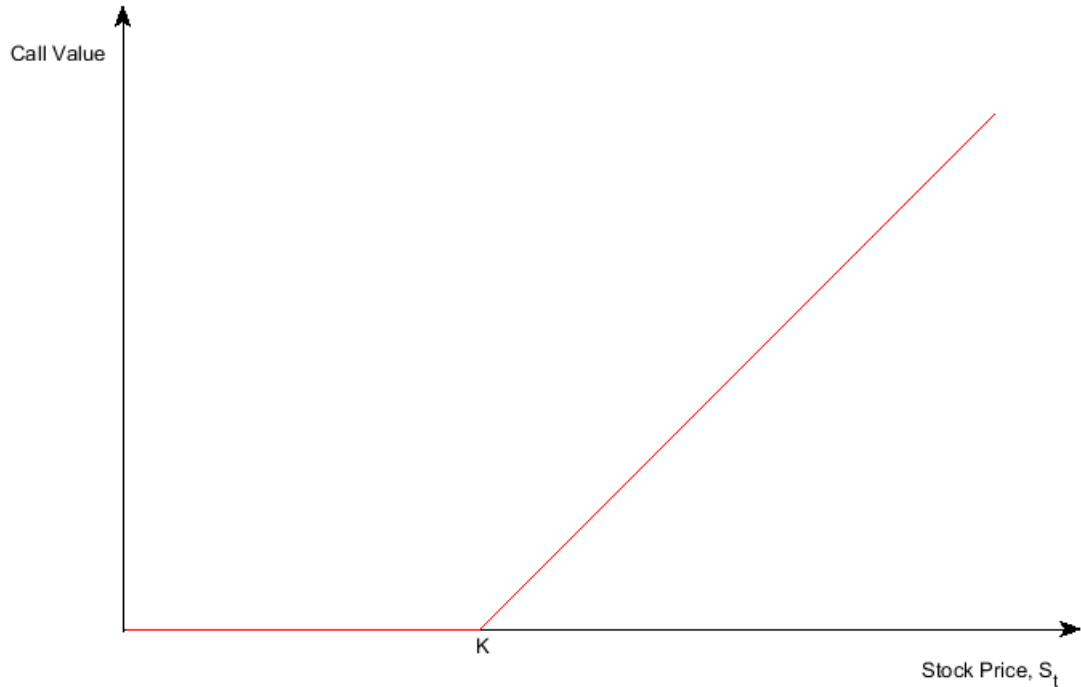


Figure 1.2: Call Option Value at Maturity  $T$

that the holder has the freedom to buy (call) or sell (put) at maturity date  $T$ . Since a rational investor would exercise only if the contract is profitable at maturity, the call option pay off is given by

$$\max(s_T - K, 0) = (s_T - K)^+.$$

and that of a put option given by

$$\max(K - s_T, 0) = (K - s_T)^+.$$

## Arbitrage and Put-Call Parity

We begin by the concept arbitrage. In finance, particularly in derivative pricing, the concept of no arbitrage is crucial for a compact framework.

**Definition 1.1.** An arbitrage opportunity exists if, by starting with an initial net account  $v_0 = 0$  then

$$\mathbb{P}(v_t > 0) > 0, \mathbb{P}(v_t < 0) = 0$$

for some future time  $t$ , where  $\mathbb{P}$  represents a real world probability measure.

Thus an arbitrage opportunity guarantees, by starting with a net account of zero, a non-zero probability of a risk free profit. We will assume no arbitrage opportunities exist in our work. However, in practice, arbitrage opportunities do exist for short periods of time before being wiped out. Of great use is a result giving the relationship between European Call and Put Options.

**Lemma 1.2.** (Put-Call Parity, [30]) Let  $c_t$  and  $p_t$  be the values of European call and put options respectively, with strike price  $K$  on an underlying asset  $s_t$  and maturity  $T$ . Then the following holds for all  $t \in [0, T]$ ,

$$c_t - p_t = s_t e^{-qt} - K e^{-rt}, \tag{1.1}$$

where  $q$  is the continuously compounded dividend yield on the underlying security.

With it, given the price of Euro Call we can obtain that of the corresponding Euro Put (similar underlying, strike price and maturity) and vice versa.

## Probability Theory

We begin by defining the notion of a random variable.

**Definition 1.3.** A random variable  $X$  is a measurable function defined as

$$X : \mathbb{R} \rightarrow \mathbb{R}.$$

Of interest will be how we can compute the expected values and measurements of deviation from such expectations for these random variable. Before we get into that, we will define a few more terms.

**Definition 1.4.** A  $\sigma$ -field on a set  $\Omega$  is a set of subsets (denoted  $F$ ) of  $\Omega$  such that the following are true,

- (i)  $\Omega \in F$
- (ii) If  $A \in F$ , then its complement  $A^c \in F$
- (iii) If  $(A_k)$  for  $k = 1, 2, \dots$  is in  $F$  for all  $k$ , then  $\bigcup_{k=1}^{\infty} A_k \in F$ .

In probability theory, we can view the  $\sigma$ -field as a collection of all possible events.

**Definition 1.5.** Let  $X$  be a random variable. A function  $f$  is a probability density function for  $X$  if,

- (i)  $\int_{\mathbb{R}} f(x)dx = 1$
- (ii) For all closed intervals  $[a, b]$ ,  $\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$ , where  $\mathbb{P}$  is a probability measure.

With this we can now define the expectation of a random variable  $X$ .

**Definition 1.6.** Let  $X$  be a random variable defined on the Probability Space  $(\Omega, F, \mathbb{P})$ . Then the expectation of  $X$  is given by,

$$\begin{aligned} \mathbb{E}(X) &= \int_{\Omega} X d\mathbb{P} \\ &= \int_{-\infty}^{\infty} xf(x)dx. \end{aligned}$$

In addition to the expectation, the variance of  $X$ ,  $Var(X)$ , is a measure of dispersion (how far spread) from the mean (expectation) value of  $X$ . Its formula is given by  $\mathbb{E}[X - \mathbb{E}(X)]^2$ .

One of the most commonly used density functions in probability theory and numerous applications is that of a Normally Distributed Random Variable. If  $X$  is normally distributed, with mean  $\mu$  and variance  $\sigma^2$ , then its density function,  $f_X$ , is given by,

$$f_X(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We denote  $X \sim \mathcal{N}(\mu, \sigma^2)$  to imply a normally distributed random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$ . The normal distribution will be an important subject of this thesis.

## Thesis Outline

This thesis proceeds as follows: Following the introduction in chapter 1, in chapter 2 we review some well-known background material in mathematical finance, including the derivation of the famous Black-Scholes PDE, which will be central our work. We will also emphasize on the martingale formulation of the discounted price of a derivative, which will help obtain an intuitive understanding of the difficulty presented in analytically valuing American options (except for rare cases).

The main results achieved in this thesis are presented in chapter 3. Here, we initially develop the linear complementarity formulation for American option pricing in the form of finite difference numerical schemes, for which we provide stability analysis. We then introduce solution methods from literature: an explicit fixed point algorithm by Shi, Yang and Huang (SYH) [28] and the Modulus Based Matrix Splitting Method by Bai [3]. The first new enhancement to the work done in [28] will be in the form of a simplified proof for the convergence of SYH's algorithm. We suggest



an approach that avoid strenuous and long matrix multiplications, especially in this case with the involvement of inverses. This proof is extended to the case where the resulting matrix formulation in the original LCP is not a symmetric tri-diagonal matrix as studied in [28]. As a modification to the Modulus Based Matrix Splitting Method proposed by Bai, we generalize the SYH method and highlight the special case in which the original algorithm presented in [28] is retained. For this, we also carry out some convergence analysis.

Chapter 4 surveys alternative formulations to the American option pricing problem. These are in the form of the Chapman-Kolmogorov lattice methods [1] proposed by Aluigi, Corradini and Gheno as well as a Modification of the Mellin transform to price call options in the form of an integral representation. The latter methodology, proposed by Frontczak and Schobel [8], solves the integral expressions using the Gauss-Laguerre quadrature.

In chapter 5 we will carry out some numerical experiments to price some options using the generalized SYH method and compare results with other procedures and formulations.

## 2

# Preliminaries

In this chapter we will review some well known results about option pricing. In particular, risk neutral valuation and the optimal stopping problem in American option pricing. First we will define and look at a few key terms.

**Definition 2.1.** A stochastic process is a sequence (continuous or discrete) of random variables on the interval  $[0, \infty)$  and is denoted by  $(X_t)_{t \geq 0}$ .

Unless specified, we will consider continuous stochastic processes in this thesis. Stochastic processes play an important role in the valuation of derivatives. The basis here will be an assumption that the evolution of an underlying asset will resemble that of a certain stochastic process. Of great concern to us will be a process known as Brownian Motion.

**Definition 2.2.** A stochastic process  $(\omega_t)_{t \geq 0}$  is said to be Brownian if the following hold:

- $\omega_0 = 0$ ;
- For all  $t \geq s$ ,  $\omega_t - \omega_s$  is normally distributed with mean 0 and variance  $t - s$ ;
- $\omega_t$  has independent increments, that is, the process  $\omega_t - \omega_s$  depends only on  $t - s$ ;

- $\omega_t$  is continuous.

Also crucial in risk neutral valuation is the concept of a martingale.

**Definition 2.3.** A martingale is a stochastic process  $(x_t)_{t \geq 0}$  which satisfies the following property,

$$\mathbb{E}[x_t | x_s] = x_s$$

for all  $t \geq s$ , where  $\mathbb{E}[x_t | x_s]$  represents the conditional expectation of  $x_t$  given  $x_s$ .

Intuitively, a martingale says that the stochastic process  $x_t$  does not have a systematic tendency to move up or down. The expected future value is equal to the value known presently. It is well known that Brownian Motion a martingale. To see this, we note that  $\mathbb{E}[\omega_t | \omega_s] = \mathbb{E}[(\omega_t - \omega_s) + \omega_s | \omega_s] = \mathbb{E}[(\omega_t - \omega_s) | \omega_s] + \mathbb{E}[\omega_s | \omega_s]$ . It is clear that  $\mathbb{E}[\omega_s | \omega_s] = \omega_s$ . Also notice that since  $\omega_t - \omega_s$  is a Brownian increment, by property of Brownian Motion, it must have zero mean. Hence Brownian Motion is a Martingale. Of great importance to continuous time finance and the derivation of the Black-Scholes equation is Ito's lemma stated below,

$$\partial f(x, t) = \left( \frac{\partial f}{\partial t} + \mu(x, t) \frac{\partial f}{\partial x} + \frac{\sigma^2(x, t)}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma(x, t) \frac{\partial f}{\partial x} d\omega.$$

where  $f$  is a twice continuously differentiable function and  $dx = \mu(x, t)dt + \sigma(x, t)d\omega$ .

## 2.1 Risk Neutral Valuation

**Definition 2.4.** Two probability measures  $\mathbb{P}$  and  $\mathbb{Q}$  are said to be equivalent if

$$\mathbb{P}(A) = 1 \iff \mathbb{Q}(A) = 1$$

for any event  $A$ .

In financial mathematics, we can derive a measure equivalent to the real world probability measure and such that under this new measure, the expected rate of return of all assets in our market model is just the risk free interest rate. This measure,  $\mathbb{Q}$  is known as the Risk Neutral Measure or Equivalent Martingale Measure. Under  $\mathbb{Q}$ , it is well known (see for example [14]) that all discounted asset prices are martingales,

$$\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}A_T|\mathcal{F}_t] = A_t$$

for some asset price  $A$ .

Here  $\mathcal{F}_t$ , known as a filtration, represents the price history of asset  $A_t$  up to time  $t$ . As we will see, this is very instrumental in the valuation of derivatives, appropriately named as Martingale Pricing. For the pricing of a contingent claim, we then have the following,

**Lemma 2.5.** Let  $f(t, s_t)$  be a contingent claim dependent on time  $t$  and security  $s_t$ . Then its value at  $t$  is given by

$$f(t, s_t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f(T, s_T)|\mathcal{F}_t]. \quad (2.1)$$

Here  $f$  describes the pay-off function of our contingent claim. Also note that the Martingale Property of discounted asset prices (in this scenario  $e^{-r(T-t)}f(T, s_T)$  is the martingale) is preserved. This crucially allows us to find the numerical value of  $f$  by computing the expectation on the right hand side of (2.1).

As an example, we will compute the value of a forward contract whose payoff  $f$  is given by

$$f(T, s_T) = s_T - K,$$

for some strike price  $K$ . Thus using our martingale pricing approach, we obtain

$$\begin{aligned}
f(t, s_t) &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f(T, s_T) | \mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} (s_T - K) | \mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} s_T | \mathcal{F}_t] - \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} K | \mathcal{F}_t] \\
&= s_t e^{-r(t-t)} - K e^{-r(T-t)} \\
&= s_t - K e^{-r(T-t)},
\end{aligned}$$

since  $s_T e^{-r(T-t)}$  is a discounted asset price and  $K e^{-r(T-t)}$  is a deterministic function.

### 2.1.1 Valuation of European Options

In this subsection we will determine the value of plain vanilla Euro calls using martingale pricing. The price of corresponding puts is easily obtained from the put-call parity relation.

Recall that the pay-off function of a call option on a security  $s$ , strike price  $K$  and maturity  $T$  is given by  $(s_T - K)^+$ . Hence its value at time  $t$  can be obtained by computing the expectation

$$c(t, s_t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} (s_T - K)^+ | \mathcal{F}_t].$$

By definition of expectations, we know that this can be evaluated from the integral,

$$c(t, s_t) = \int_{-\infty}^{\infty} e^{-r(T-t)} (s_T - K)^+ g(s_T) ds_T, \quad (2.2)$$

where  $g$  is the probability density function of our random variable, in this case  $s_T$ . This is general. Our problem, can however, be simplified a little further. We first recall that we assumed  $s$  to evolve stochastically under Geometric Brownian Mo-

tion (GBM). For our analysis, under  $\mathbb{Q}$ ,  $s$  obeys the following stochastic differential equation,

$$ds_t = rs_t dt + \sigma s_t d\omega_t^{\mathbb{Q}}, \quad (2.3)$$

where  $\omega_t^{\mathbb{Q}}$  is a Brownian motion under the equivalent martingale measure  $\mathbb{Q}$ .

While the task of solving SDEs is generally a complex one, the above equation can be solved by taking an intuitive guess of the solution and applying Ito's Lemma. By noticing resemblance to deterministic equations of the form  $\frac{dx}{x} = dt$ , this educated guess comes in the form of

$$s_t = e^h, \quad (2.4)$$

for some function  $h$  which we wish to determine using Ito's Lemma. Since  $h(t, s_t) = \log(s_t)$ , we have from Ito's lemma,

$$\begin{aligned} dh &= \sigma s_t h'(s_t) d\omega_t^{\mathbb{Q}} + (rs_t h'(s_t) + \frac{\sigma^2 s_t^2}{2} h''(s_t)) dt \\ &= \sigma s_t \left(\frac{1}{s_t}\right) d\omega_t^{\mathbb{Q}} + \left(rs_t \left(\frac{1}{s_t}\right) + \frac{\sigma^2 s_t^2}{2} \left(\frac{-1}{s_t^2}\right)\right) dt \\ &= \sigma d\omega_t^{\mathbb{Q}} + \left(r - \frac{\sigma^2}{2}\right) dt. \end{aligned}$$

Integrating both sides from current time  $t$  to maturity date  $T$ , we obtain,

$$\begin{aligned} \int_t^T dh &= \int_t^T \sigma d\omega_s^{\mathbb{Q}} - \int_t^T \left(r - \frac{\sigma^2}{2}\right) dt \\ h(s_T) - h(s_t) &= \sigma(\omega_T^{\mathbb{Q}} - \omega_t^{\mathbb{Q}}) - \left(r - \frac{\sigma^2}{2}\right)(T - t) \\ \log(s_T) - \log(s_t) &= \sigma(\omega_T^{\mathbb{Q}} - \omega_t^{\mathbb{Q}}) - \left(r - \frac{\sigma^2}{2}\right)(T - t). \end{aligned}$$

We recall two items. Firstly,  $\omega_T^{\mathbb{Q}} - \omega_t^{\mathbb{Q}}$  is just a Brownian increment, thus by definition

$$\omega_T^{\mathbb{Q}} - \omega_t^{\mathbb{Q}} \sim \mathcal{N}(0, T - t).$$

Also for some random variable  $Y$  with density  $f$ ,

$$\mathbb{E}[g(y)] = \int_{-\infty}^{\infty} g(y)f(y)dy.$$

With this information, we notice that the integral in (2.2) vanishes whenever  $s < K$ . Hence by monotonicity of the logarithmic function, we are only concerned with the case  $\log(s_T) > \log(K)$ . To this point we will consider a standard normal variable  $Y$ , that is

$$Y \sim \mathcal{N}(0, 1).$$

It is easy to observe that

$$\omega_T^{\mathbb{Q}} - \omega_t^{\mathbb{Q}} = \sqrt{T - t}Y.$$

Hence we can re-write the solution of our SDE as

$$\begin{aligned} \log(s_T) &= \log(s_t) + \left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma\sqrt{T - t}Y \\ s_T &= s_t e^{\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma\sqrt{T - t}Y}. \end{aligned}$$

Also, since our valuation integral does not vanish whenever  $\log(s_T) > \log(K)$ , the following is an equivalent condition,

$$\begin{aligned} \log(s_t) + (r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}Y &> \log(K) \\ \iff Y &> \frac{\log(\frac{K}{s_t}) - (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ &= y_0. \end{aligned}$$

Thus with  $g(Y) = (s_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}Y} - K)^+$ , we immediately recognize that our valuation integral becomes

$$\begin{aligned} c(t, s_t) &= \int_{-\infty}^{\infty} e^{-r(T - t)} g(y) f(y) dy \\ &= \int_{y_0}^{\infty} e^{-r(T - t)} (s_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}Y} - K) f(y) dy \\ &= \int_{y_0}^{\infty} e^{-r(T - t)} (s_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}Y}) f(y) dy - \int_{y_0}^{\infty} e^{-r(T - t)} K f(y) dy. \end{aligned}$$

Recall that the density function of a standard normal variable  $Y$  is given by  $\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ .

Then the well known cumulative distribution function  $\Phi$  for  $Y$  is defined as

$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ . Thus our integral becomes,

$$\begin{aligned} c(t, s_t) &= \frac{s_t e^{-r(T - t)}}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}Y - \frac{y^2}{2}} dy - \frac{K e^{-r(T - t)}}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \frac{s_t e^{-r(T - t)}}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}y - \frac{y^2}{2}} dy - K e^{-r(T - t)} \Phi(-y_0) \\ &= \frac{s_t e^{-r(T - t)}}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{-\frac{1}{2}(Y - \sigma\sqrt{T - t})^2 + r(T - t)} dy - K e^{-r(T - t)} \Phi(-y_0). \end{aligned}$$



Let  $u = y - \sigma\sqrt{T-t}$ . Then

$$\begin{aligned} c(t, s_t) &= \frac{s_t e^{-r(T-t)+r(T-t)}}{\sqrt{2\pi}} \int_{y_0 - \sigma\sqrt{T-t}}^{\infty} e^{-\frac{u^2}{2}} du - K e^{-r(T-t)} \Phi(-y_0) \\ &= s_t \Phi(\sigma\sqrt{T-t} - y_0) - K e^{-r(T-t)} \Phi(-y_0). \end{aligned}$$

This gives the value of a European Call Option. The value of a Put can be easily obtained from the Put-Call Parity.

### 2.1.2 Derivation of the Black Scholes PDE

We are now ready to derive the famous option pricing partial differential equation. We will assume that the underlying, in this case the stock price  $s_t$ , follows the Geometric Brownian Motion :

$$\frac{ds_t}{s_t} = \mu dt + \sigma d\omega_t, \quad (2.5)$$

where  $\mu$ , the mean return, and  $\sigma$ , the volatility of the stock, are both constant parameters. As we have already indicated, we will also assume the risk free interest rate  $r$  to be constant.

Now, assuming that the derivative, on an underlying asset  $s_t$ , to be valued is given by  $v(s_t, t)$  and has maturity  $T$ , we can see from Ito's formula that, imposing  $v$  to be continuously twice differentiable,

$$dv(s_t, t) = \sigma s_t v'(s_t) d\omega_t + (\mu s_t v'(s_t) + \frac{1}{2} \sigma^2 s_t^2 v''(s_t) + v'(t)) dt. \quad (2.6)$$

Our aim now is to construct a riskless portfolio (a basket of securities which grows at a known constant rate  $r$ ). So we consider the following portfolio,  $\pi$ ,

- Buy one unit of derivative  $v$  (Long  $v$ )
- Borrow  $\delta$  units of the underlying  $s_t$  (Short  $\delta$  units of  $s$ )

Hence the value of our portfolio  $\pi$  is given by

$$\pi = v - \delta s. \quad (2.7)$$

Hence we have,

$$\begin{aligned} d\pi &= dv - \delta ds_t \\ &= dv - \delta(\mu s_t dt + \sigma s_t d\omega_t) \\ &= \sigma s_t v'(s_t) d\omega_t + (\mu s_t v'(s_t) + \frac{1}{2} \sigma^2 s_t^2 v''(s_t) + v'(t)) dt - \delta(\mu s_t dt + \sigma s_t d\omega_t). \end{aligned}$$

Collecting terms the  $dt$  and  $d\omega_t$  terms together we obtain,

$$d\pi = \sigma s_t (v'(s_t) - \delta) d\omega_t + (\mu s_t v'(s_t) + \frac{1}{2} \sigma^2 s_t^2 v''(s_t) + v'(t) - \mu \delta s_t) dt. \quad (2.8)$$

Notice that our portfolio has been divided into two parts: the  $dt$  component which evolves deterministically and the stochastic component  $d\omega_t$  which ultimately makes our portfolio risky. Hence to eliminate this randomness, we can choose  $\delta$  to be  $\frac{\partial v}{\partial s_t}$  and we immediately remain with

$$d\pi = \left( \frac{1}{2} \sigma^2 s_t^2 \frac{\partial^2 v}{\partial s_t^2} + \frac{\partial v}{\partial t} \right) dt. \quad (2.9)$$

But since  $\pi$  will now evolve risk free, this means

$$\begin{aligned} r\pi dt &= d\pi \\ &= \left( \frac{1}{2} \sigma^2 s_t^2 \frac{\partial^2 v}{\partial s_t^2} + \frac{\partial v}{\partial t} \right) dt. \end{aligned}$$

But recall that  $\pi = v - \delta s_t$  and  $\delta = \frac{\partial v}{\partial s_t}$ . Thus we can re-write the above as

$$r(v - s_t \frac{\partial v}{\partial s_t})dt = (\frac{1}{2}\sigma^2 s_t^2 \frac{\partial^2 v}{\partial s_t^2} + \frac{\partial v}{\partial t})dt$$

and thus we arrive at the Black Scholes PDE:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s_t^2 \frac{\partial^2 v}{\partial s_t^2} + r s_t \frac{\partial v}{\partial s_t} - rv = 0. \quad (2.10)$$

## 2.2 American Options

**Definition 2.6.** An American derivative is a financial contract with the option to exercise at any time up to maturity

Suppose we have a maturity time  $T$  with  $t \in [0, T]$ , underlying security  $s_t$  and derivative payoff given by  $\Phi(s_t, t)$ . Then, building on our work from risk neutral valuation, the fair value of the American derivative at time  $t$  is given by

$$v_{t_0}^a = \sup_{t \in [t_0, T]} \mathbb{E}^{\mathbb{Q}}[e^{-r(t-t_0)} \Phi(s_t, t) | \mathcal{F}_{t_0}].$$

Because the option holder exercises at the time that maximizes their profit, we consider the supremum of all possible exercise times. Recall that a European derivative on the same underlying, maturity and payoff is given by

$$v_{t_0}^e = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t_0)} \Phi(s_T, T) | \mathcal{F}_{t_0}].$$

From this, we can clearly see that the condition  $v_t^a \geq v_t^e$  holds at all times since  $t \in [0, T]$ . Of course, from financial intuition, we expected this to be the case due to the extra flexibility in holding an American derivative.

## 2.2.1 Stopping and Continuation Regions

For the purposes of clear illustration, we are going to focus on American put options in this subsection. Recall its pay off function is given by  $(K - s_T)^+$  where  $K$  is the exercise price.

To motivate this subsection, suppose the optimal stopping time for pricing an American put with maturity  $T$  is denoted by  $t^*$ . Then the value of the put is given by

$$p_t^a = \mathbb{E}_{\mathbb{Q}}[e^{-(t^*-t)}(K - s_{t^*})^+ | \mathcal{F}_t].$$

However, since the value of the Put is unknown apriori, this equation is of no use to us as we cannot infer the value of  $t^*$ . As such we need to consider other approaches.

**Definition 2.7.** Let  $v^a$  be the value of an American Derivative with pay-off function  $\Phi(s_T, T)$  and contract period  $[t_0, T]$ . Then the Stopping and Continuation Regions,  $R_s$  and  $R_c$ , are defined as

$$R_s := \left\{ (s_t, t) \mid \forall t \in [t_0, T], (s_t, t) \in \mathbb{R}^2 \text{ and } v^a(s_t, t) = \Phi(s_t, t) \right\}$$

and

$$R_c := \left\{ (s_t, t) \mid \forall t \in [t_0, T], (s_t, t) \in \mathbb{R}^2 \text{ and } v^a(s_t, t) > \Phi(s_t, t) \right\}.$$

The first noticeable scenario missing here is the case when  $v^a < \Phi(s_t, t)$ . We recall that an American derivative can be exercised at any time up to maturity. Hence, whenever the value of an American Option is less than its payoff, one would purchase it and immediately exercise for a profit of  $\Phi(s_t, t) - v^a$ , thus creating an arbitrage opportunity. To put this into perspective, consider our American Put. When the

value of the option equals its pay off function, it is immediately optimal to exercise. The following theorem ensures that the value of the put will remain the same as the pay off until maturity from the time this first happens.

**Theorem 2.8.** (Smooth Pasting Condition, [22]). Let  $v_t^a$  be the value of an American Derivative at time  $t$  with pay-off function  $\Phi(s_T, T)$ . Then the following holds

$$\frac{\partial v^a(s_t, t)}{\partial s} \Big|_{s=s_f(t)} = \Phi'(s_f(t)),$$

where  $s_f(t)$  is the optimal stopping value of the derivative valued at time  $t$

With this, it is then desirable for the holder of the put to exercise immediately when this stopping time is reached. That way they can maximize their profits by exercising at  $K$  and investing the proceeds at the risk free interest rate for the remaining time for a profit of  $Ke^{r(T-t_0)} - K$ .

As such it is desirable to exercise whenever this stopping time is hit and any other time after that up to maturity. Hence the term stopping region. The time-dependent value of the stock when this stopping time is reached is the stopping value which we will denote as  $s_f(t)$ . From intuition and by considering the curvature of puts, we can infer that this stopping value decreases as time to maturity increases. So now that we have determined that it is ideal to exercise in the stopping region, considering the situation when the put value is greater than the pay off makes it an easier task. This is because exercising in this situation will result into an immediate loss since  $p^a + s \geq K$ . Thus whenever

$$p_t^{am} > (K - s_t)^+,$$

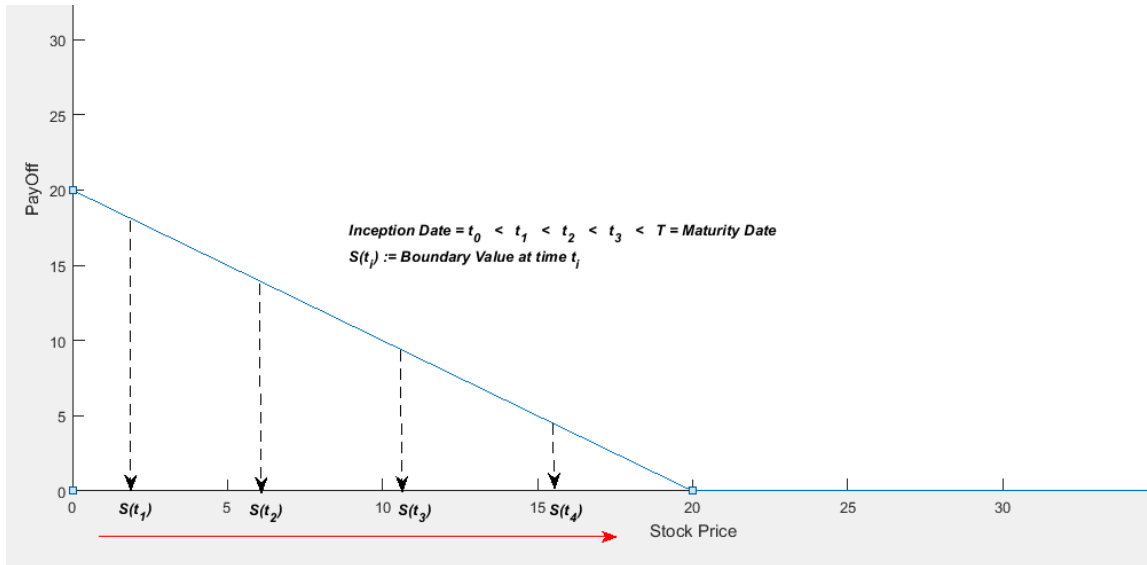


Figure 2.1: Stopping Values at Varying times to Maturity

the rational investor will not exercise and thus we remain in the continuation region. The boundary that separates the continuation and stopping regions is called the exercise boundary. Note that, apriori, this is unknown, hence the desire to invoke the theory of free boundary problems.

### 2.2.2 The Black-Scholes Inequality for American Options

Now that we have motivated the free boundary pricing methodology, we will introduce the PDE central to our model. Recall the Black - Scholes PDE for a dividend paying stock is

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 v}{\partial s^2} + (r - q)s \frac{\partial v}{\partial s} - rv = 0.$$

We note that in the continuation region, for the case  $p^a > K - s$ , the Black Scholes equation holds. It is interesting to consider what happens when we hit the exercise

boundary when equality holds. Note from the Smooth Pasting Condition,

$$\frac{\partial v}{\partial s_t} = \frac{\partial}{\partial s_t}(K - s_t) = -1.$$

Similarly, it is easy to derive  $\frac{\partial v}{\partial t} = 0$  and  $\frac{\partial^2 v}{\partial s^2} = 0$ . Hence, in the stopping region, we can obtain, by substituting into the left hand side the Black - Scholes PDE,

$$(r - q)s_t(-1) - r(K - s_t) = qs_t - rK,$$

for all  $s_t \leq s_f(t)$ . We will recall this in chapter 4. The boundary condition for the American put Option is given by, as shown in [27],

$$\lim_{t \rightarrow T} s_b(t) = K \min(1, \frac{r}{q}),$$

and that the stopping time increases as  $t \rightarrow T$ . Then for all  $t \in [0, T]$ , it is easy to see that

$$s_t \leq s_f(t) < \lim_{t \rightarrow T} s_f(t) \leq \frac{r}{q}K,$$

leading to  $qs_t - rK < 0$ . Hence over the stopping region, the Black Scholes equation becomes

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}s^2 \frac{\partial^2 v}{\partial s^2} + (r - q)s \frac{\partial v}{\partial S} - rv < 0,$$

and combining the continuation and stopping regions gives the *Black Scholes Inequality*,

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}s^2 \frac{\partial^2 v}{\partial s^2} + (r - q)s \frac{\partial v}{\partial S} - rv \leq 0.$$

This is more general. With  $q = 0$ , then clearly  $qs_t - rK < 0$  and the Black-Scholes Inequality would follow.

A similar analysis on an American Call on a dividend paying stock yields the same inequality. In what follows, we will discuss computational methods surveyed from literature for American option pricing.



# 3

## Linear Complementarity Problems in Option Pricing

We begin by recalling the Black - Scholes partial differential equation in option pricing (without dividends)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s_t^2 \frac{\partial^2 v}{\partial s_t^2} + r s_t \frac{\partial v}{\partial s_t} - r v = 0. \quad (3.1)$$

Defining an operator  $\mathcal{L}v := \frac{1}{2}\sigma^2 s_t^2 \frac{\partial^2 v}{\partial s_t^2} + r s_t \frac{\partial v}{\partial s_t} - r v$ , we can write this as  $\partial_t v + \mathcal{L}v = 0$ .

### 3.1 Black Scholes PDE Transformation

For computational purposes, it is desirable to consider a transformed version of the Black-Scholes PDE. We can show that this is equivalent to the equation:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}, \quad (3.2)$$

where  $s = Ke^x$ ,  $t = T - \frac{2\tau}{\sigma^2}$ ,  $v(s_t, t) = v(Ke^x, T - \frac{2\tau}{\sigma^2}) = v(x, \tau)$ .

*Proof.* We first note that using the  $t = T - \frac{2\tau}{\sigma^2}$  and  $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial t}$ , we have  $\frac{\partial v}{\partial t} = \frac{-\sigma^2}{2} \frac{\partial v}{\partial \tau}$ .

Similarly, from  $s = Ke^x$  we can deduce that  $\partial x = \frac{1}{s}\partial s$ . Hence the following relations,

$$\frac{\partial v}{\partial s} = \frac{1}{s} \frac{\partial v}{\partial x} \quad (3.3)$$

and

$$\frac{\partial^2 v}{\partial s^2} = \frac{-1}{s^2} \left( \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} \right). \quad (3.4)$$

Inserting these into (3.1) - the Black Scholes Equation, we obtain

$$\begin{aligned} -\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2} s^2 \left( \frac{1}{s^2} \left( \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} \right) + rs \left( \frac{1}{s} \frac{\partial v}{\partial x} \right) - rv \right) &= 0 \\ \iff -\frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial x^2} + \left( \frac{2r}{\sigma^2} - 1 \right) \frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v &= 0. \end{aligned}$$

Further, we consider the transformation  $v = Ke^{ax+b\tau}y(x, \tau)$  for some constants  $a, b$ .

Differentiating, we obtain,

- $\frac{\partial v}{\partial \tau} = Ke^{ax+b\tau} \left( by + \frac{\partial y}{\partial \tau} \right);$
- $\frac{\partial v}{\partial x} = Ke^{ax+b\tau} \left( ay + \frac{\partial y}{\partial x} \right);$
- $\frac{\partial^2 v}{\partial x^2} = Ke^{ax+b\tau} \left( \frac{\partial^2 y}{\partial x^2} + 2a \frac{\partial y}{\partial x} + a^2 y \right).$

From which, inserting in the above equation, we obtain,

$$\frac{\partial^2 y}{\partial x^2} + \left[ 2a - \left( 1 - \frac{2r}{\sigma^2} \right) \right] \frac{\partial y}{\partial x} + \left[ \frac{2r}{\sigma^2} + a^2 - b - a \left( 1 - \frac{2r}{\sigma^2} \right) \right] y = \frac{\partial y}{\partial \tau}. \quad (3.5)$$

Our goal is to vanish the  $\frac{\partial y}{\partial x}$  and  $y$  terms, hence we need to solve the linear system

$$\begin{aligned} 2a - \left( 1 - \frac{2r}{\sigma^2} \right) &= 0 \\ \frac{2r}{\sigma^2} + a^2 - b - a \left( 1 - \frac{2r}{\sigma^2} \right) &= 0 \end{aligned}$$

for  $a$  and  $b$  yielding  $a = \frac{1}{2} - \frac{r}{\sigma^2}$  and  $b = -\frac{1}{4} \left( \frac{2r}{\sigma^2} - 1 \right)^2 - \frac{2r}{\sigma^2}$ . Thus we have transformed

the Black Scholes equation into a more tractable PDE,

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}. \tag{3.6}$$

□

An identical transformation of the Black-Scholes inequality,  $\partial_t v + \mathcal{L}v \leq 0$ , that will be key for our linear complementarity setup for American Option pricing gives,

$$\frac{\partial y}{\partial \tau} \geq \frac{\partial^2 y}{\partial x^2}.$$

### 3.1.1 Boundary Conditions

Now that we have transformed the original Black-Scholes PDE into a simplified heat equation for computational purposes, it is also necessary to look at how the boundary conditions are consequently transformed.

Recall that  $t = T - \frac{2\tau}{\sigma^2}$ . This implies,

$$\tau = \frac{\sigma^2}{2}(T - t).$$

Thus, up to a scaling of  $\frac{\sigma^2}{2}$ , we can see that  $\tau$  represents the time left to maturity of the derivative contract. Thus for example, when  $t = 0$ , up to a scaling of  $\frac{\sigma^2}{2}$ ,  $\tau$  represents the full duration of the contract. It is thus the case that the terminal conditions in the original Black-Scholes PDE set up are transformed into initial conditions under the PDE transformation with  $y$  and  $\tau$ . That is we obtain,

$$\begin{aligned} v(s, T) &= v(x, 0) \\ &= Ke^{-x(\frac{\tau}{\sigma^2} - \frac{1}{2})}y(x, 0). \end{aligned}$$

But recall that for a call option at maturity,  $v^c(s, T) = (s - K)^+ = (Ke^x - K)^+ = K(e^x - 1)^+$  and for a put we obtain, a similar derivation as above,  $v^p(s, T) = (K - s)^+ = K(1 - e^x)^+$ . Thus we can deduce initial conditions of call and put to be

$$y^c(x, 0) = e^{-x(\frac{1}{2} - \frac{r}{\sigma^2})}(e^x - 1)^+ \quad (3.7)$$

and

$$y^p(x, 0) = e^{-x(\frac{1}{2} - \frac{r}{\sigma^2})}(1 - e^x)^+ \quad (3.8)$$

respectively. The boundary conditions require special consideration. First we note that for a call option,

$$v^c(s, t) = 0 \quad (3.9)$$

when  $s = 0$ . However looking at our transformation  $s = Ke^x$  we need  $s$  to be non-zero by definition of the exponential function. Thus when  $\lim_{s \rightarrow 0^+} s = 0$ ,  $x \rightarrow -\infty$  and

$$y^c(x, \tau) \rightarrow 0. \quad (3.10)$$

And for a put option,

$$v^p(s, t) = 0 \quad (3.11)$$

when  $s \rightarrow \infty$ . Thus, when  $x \rightarrow \infty$ ,

$$y^p(x, \tau) \rightarrow 0. \quad (3.12)$$

These cases hold for all time  $\tau$ . It remains to consider the cases when  $s \rightarrow \infty$  and  $s = 0$  for the call and put options respectively. For this we will recall the Put-Call Parity,

$$v^c - v^p = s_t - Ke^{-r(T-t)}.$$

Hence as  $s \rightarrow 0$ , the value of the call is negligible and the put value is  $v^p \rightarrow Ke^{-r(T-t)} - s_t$ . Thus as  $x \rightarrow -\infty$ ,

$$y^p(x, \tau) \rightarrow Ke^{-\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)x + \left(\frac{1}{4}\left(\frac{2r}{\sigma^2} - 1\right) + \frac{2r}{\sigma^2}\right)\tau} (Ke^{-r(T-t)} - s_t).$$

Similarly, for the call option, when  $s \rightarrow \infty$ , the value of the put becomes negligible (see [24]) and the call becomes  $v^c = s_t - Ke^{-r(T-t)}$ . Re-adjusting to our transformation, this is the case when  $x \rightarrow \infty$ , hence

$$y^c(x, \tau) \rightarrow Ke^{-\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)x + \left(\frac{1}{4}\left(\frac{2r}{\sigma^2} - 1\right) + \frac{2r}{\sigma^2}\right)\tau} (s_t - Ke^{-r(T-t)}).$$

### 3.1.2 Discretization of the Transformed Black-Scholes PDE

Discretization of (3.2) means dividing the  $x$  and  $\tau$  intervals into discrete points (for our purposes the distance between the discretized points, or mesh sizes, will be equal). Expanding on the analysis given in [27], from our transformation of the Black-Scholes PDE, recall that  $t = T - \frac{2\tau}{\sigma^2}$ . This shows that the maximum value of  $\tau$  occurs at  $\frac{\sigma^2 T}{2} = \tau_{\max}$ . Hence for the time discretization we can consider  $\Delta\tau = \frac{\tau_{\max}}{N}$  where the most suitable  $N$  value is not necessarily known a priori. For the  $x$  discretization, it will be necessary to truncate the interval  $-\infty < x < \infty$  to some  $x_{\min} < x < x_{\max}$  such that  $\Delta x = \frac{x_{\max} - x_{\min}}{M}$  for a suitable value of  $M$ . We will get back to the actual values of  $x_{\min}$  and  $x_{\max}$  in the actual computations. Hence, notation-wise, we have:

- $\tau_n = n \cdot \Delta\tau$  for  $n = 0, 1, 2, \dots, N$

- $x_i = x_{\min} + i\Delta x$  for  $i = 0, 1, 2, \dots, M$
- $y_i^n = y(i\Delta x, n\Delta\tau)$

From simple Taylor expansions and taking central differences(see [27]), we can thus re-write (3.2) under what is known as the Crank-Nicolson Method,

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} + O(\Delta\tau^2) = \frac{1}{2\Delta x^2}(y_{i+1}^n - 2y_i^n + y_{i-1}^n + y_{i+1}^{n+1} - 2y_i^{n+1} + y_{i-1}^{n+1}) + O(\Delta x^2). \quad (3.13)$$

Dropping the  $O(\Delta\tau^2)$  and  $O(\Delta x^2)$  and changing the notation from  $y$  to  $f$ , we obtain a more computationally useful expression,

$$\frac{f_i^{n+1} - f_i^n}{\Delta\tau} = \frac{1}{2\Delta x^2}(f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}) \quad (3.14)$$

which we can, as illustrated in [27], further re-write as

$$-\frac{\lambda}{2}f_{i-1}^{n+1} + (1 + \lambda)f_i^{n+1} - \frac{\lambda}{2}f_{i+1}^{n+1} = \frac{\lambda}{2}f_{i-1}^n + (1 - \lambda)f_i^n + \frac{\lambda}{2}f_{i+1}^n \quad (3.15)$$

where  $\lambda = \frac{\Delta\tau}{\Delta x^2}$ . In general, (3.12) can be written as,

$$\frac{f_i^{n+1} - f_i^n}{\Delta\tau} = \theta \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{\Delta x^2} + (1 - \theta) \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta x^2}, \quad (3.16)$$

where  $\theta = 0$  represents the explicit method,  $\theta = 0.5$  the Crank-Nicolson Method and  $\theta = 1$  the backward difference (or fully-implicit method). Separating the two time levels, this can be further reduced into,

$$(1 + 2\lambda\theta)f_i^{n+1} - \lambda\theta(f_{i-1}^{n+1} + f_{i+1}^{n+1}) = (1 - 2\lambda(1 - \theta))f_i^n + \lambda(1 - \theta)(f_{i-1}^n + f_{i+1}^n). \quad (3.17)$$

In what follows, we will use the above formulation to conduct stability analysis of the three different schemes.

### 3.1.3 Stability Analysis

The notion of stability is defined in numerous ways in literature. We will work with the following definition,

**Definition 3.1.** The difference scheme  $f^{n+1} = A(\Delta t, \Delta x)f^n$  is said to be stable with respect to the norm  $\|\cdot\|$  if there exist non-negative constants  $K$  and  $\beta$  so that

$$\|f^n\| \leq K e^{n\Delta t} \|f^0\|,$$

for  $n\Delta t \leq \beta$ .

The main punchline of stability is that the inequality in the above definition remains bounded as  $\Delta t \rightarrow 0$ . Before continuing we recall the definition of the discrete Fourier Transform of a function  $f$ ,

**Definition 3.2.** The discrete Fourier transform of a  $f \in l_2$  is the function  $\hat{f} \in L_2[-\pi, \pi]$  defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} f_m$$

for  $\xi \in [-\pi, \pi]$ .

Using the fact that  $\{f^n\}$  is stable in  $l_2 \iff \{\hat{f}^n\}$  is stable in  $L_2[-\pi, \pi]$  [31], we consider the stability of the  $\theta$ -scheme. Taking the discrete Fourier transforms of both sides of equation (3.17), we obtain

$$(1 + 2\lambda\theta)\hat{f}^{n+1}(\xi) - \lambda\theta \left[ \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} (f_{m-1}^{n+1} + f_{m+1}^{n+1}) \right] =$$

$$(1 - 2\lambda(1 - \theta))\hat{f}^n + \lambda(1 - \theta) \left[ \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} (f_{m-1}^n + f_{m+1}^n) \right].$$

By setting  $j = m + 1$ , it is easy to show that  $\frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} f_{m\pm 1}^n = e^{\pm i\xi} \hat{f}^n$ . Hence,

$$(1 + 2\lambda\theta)\hat{f}^{n+1}(\xi) - \lambda\theta(e^{-i\xi} + e^{i\xi})\hat{u}^{n+1}(\xi) = (1 - 2\lambda(1 - \theta))\hat{f}^n(\xi) + \lambda(1 - \theta)(e^{-i\xi} + e^{i\xi})\hat{f}^n(\xi).$$

We then observe that  $e^{-i\xi} + e^{i\xi} = \cos(-\xi) + i\sin(\xi) + \cos(\xi) + i\sin(\xi) = 2\cos\xi$ . This then gives us,

$$[1 + 2\lambda\theta - 2\lambda\theta\cos\xi]\hat{f}^{n+1} = [1 - 2\lambda(1 - \theta) + 2\lambda(1 - \theta)\cos\xi]\hat{f}^n(\xi). \quad (3.18)$$

For the explicit method ( $\theta = 0$ ),

$$\begin{aligned} \hat{f}^{n+1}(\xi) &= [1 - 2\lambda(1 - \cos\xi)]\hat{f}^n \\ &= [1 - 2\lambda(1 - (1 - 2\sin^2\frac{\xi}{2}))]\hat{f}^n \\ &= (1 - 4\lambda\sin^2\frac{\xi}{2})\hat{f}^n \\ &= (1 - 4\lambda\sin^2\frac{\xi}{2})^{n+1}\hat{f}^{(0)}. \end{aligned}$$

Notice that when  $|1 - 4\lambda\sin^2\frac{\xi}{2}|^{n+1} \leq 1$ , stability is ensured with a choice of  $K = 1$  and  $\beta = 0$ . Otherwise, since  $(n + 1)\Delta t$  is bounded, we can find a large enough  $n$  for any  $K$  and  $\beta$  such that the scheme will not be bounded by the  $Ke^{\beta(n+1)\Delta t}$  term. Hence to satisfy stability, the explicit method requires the condition  $|1 - 4\lambda\sin^2\frac{\xi}{2}|^{n+1} \leq 1 \iff -1 \leq 1 - 4\lambda\sin^2\frac{\xi}{2} \leq 1$ . Observing that the sin term is always positive and bounded above by 1, we immediately recognise that if  $\lambda = \frac{\Delta\tau}{\Delta x^2} \in [0, \frac{1}{2}]$ , stability is ensured. Similarly, for the backward difference method ( $\theta = 1$ ), we obtain  $(1 + 4\lambda\sin^2\frac{\xi}{2})\hat{f}^{n+1}(\xi) = \hat{f}^n$ . Repeating the difference scheme  $n + 1$  times, we obtain,

$$\hat{f}^{n+1} = \frac{1}{(1 + 4\lambda\sin^2\frac{\xi}{2})^{n+1}}\hat{f}^{(0)}.$$

As such, in this case, we require  $|1 + 4\lambda\sin^2\frac{\xi}{2}| \geq 1$  for stability, which is clearly always



true, regardless of the value of  $\lambda$  (which we already know to be positive). Hence, unlike the explicit method, the backward difference method offers unconditional stability (as there are no constraints on  $\Delta\tau$  and  $\Delta x$ ). Finally we discuss the stability of the Crank-Nicolson Method ( $\theta = 0.5$ ). This yields

$$(1 + 2\lambda \sin^2 \frac{\xi}{2}) \hat{f}^{n+1} = (1 - 2\lambda \sin^2 \frac{\xi}{2}) \hat{f}^n,$$

from which we require  $\left| \frac{1 - 2\lambda \sin^2 \frac{\xi}{2}}{1 + 2\lambda \sin^2 \frac{\xi}{2}} \right| \leq 1$ . This is, however, clearly always true and we can conclude that the Crank-Nicolson Method is also unconditionally stable.

## 3.2 American Option Pricing and Linear Complementarity

We will now derive the American option pricing problem as a linear complementarity. As already discussed, an American option satisfies  $\partial_t v + \mathcal{L}v \leq 0$  ( $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0$  under our transformations) with  $v \geq (K - s)^+$  for an American put and  $v \geq (s - K)^+$  for an American call with dividends. From our transformation, the American put (the same derivation holds for the call) satisfies

$$v = K e^{(\frac{1}{2} - \frac{r}{\sigma^2})x - (\frac{1}{4}(\frac{2r}{\sigma^2} - 1) + \frac{2r}{\sigma^2})\tau} y(x, \tau).$$

Recall that the associated terminal condition is given by  $v^a \geq (K - s)^+ = K(1 - e^x)^+$ .

Thus we have

$$K e^{(\frac{1}{2} - \frac{r}{\sigma^2})x - (\frac{1}{4}(\frac{2r}{\sigma^2} - 1) + \frac{2r}{\sigma^2})\tau} y(x, \tau) \geq K(1 - e^x)^+ \quad (3.19)$$

yielding,

$$y(x, \tau) \geq e^{-\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)x + \left(\frac{1}{4}\left(\frac{2r}{\sigma^2} - 1\right) + \frac{2r}{\sigma^2}\right)\tau} (1 - e^x)^+ := g(x, \tau). \quad (3.20)$$

Notice that by (3.8) we have  $y(x, 0) = g(x, 0)$ . Similarly, observe that when  $x \rightarrow \infty$ ,  $g = 0$  since it vanishes whenever  $x \geq 0$  and by (3.11),  $y(\infty, \tau) = 0$ . Also, when  $x \rightarrow -\infty$ ,  $g$  blows up to infinity. Hence we can write the limits of  $y$  as,

$$\lim_{x \rightarrow \pm\infty} y(x, \tau) = \lim_{x \rightarrow \pm\infty} g(x, \tau).$$

The infinity terms in the limits will be truncated to finite terms for the purposes of numerical computation such that we end up with

$$y(x_{\min}, \tau) = g(x_{\min}, \tau)$$

and

$$y(x_{\max}, \tau) = g(x_{\max}, \tau).$$

We then consider the two scenarios for an American option. In the stopping region,  $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0$  and  $y = g$  (since the option equals the pay-off function). For the continuation region,  $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} = 0$  and  $y > g$ . Putting everything together, we can state the system to price the American put as a linear complementarity problem (see

for example [28]),

$$\left\{ \begin{array}{l} (\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2})(y - g) = 0 \\ y(x_{\max}, \tau) = g(x_{\max}, \tau) \text{ and } y(x_{\min}, \tau) = g(x_{\min}, \tau) \\ y(x, 0) = g(x, 0) \\ y \geq g \\ \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0 \end{array} \right. \quad (3.21)$$

The discretization under the  $\theta$ -scheme for  $y_\tau \geq y_{xx}$  is thus given by

$$(1 + 2\lambda\theta)f_i^{n+1} - \lambda\theta(f_{i-1}^{n+1} + f_{i+1}^{n+1}) \geq (1 - 2\lambda(1 - \theta))f_i^n + \lambda(1 - \theta)(f_{i-1}^n + f_{i+1}^n). \quad (3.22)$$

Thus, to this extent, we introduce useful notation as well as pay attention to the boundary conditions of our problem. Similar notation can be found in, for example, [27, 28]. The boundary conditions exist at nodes  $g_0^n$  and  $g_M^n$ .

- $c_i^n = (1 - 2\lambda(1 - \theta))f_i^n + \lambda(1 - \theta)(f_{i-1}^n + f_{i+1}^n)$  for  $i = 2, \dots, M - 2$
- $c_1^n = f_1^n + \lambda(1 - \theta)(f_0^n - 2f_1^n + f_2^n) + \lambda\theta f_0^{n+1}$
- $c_{M-1}^n = \lambda\theta g_M^{n+1} + f_{M-1}^n + \lambda(1 - \theta)(g_M^n - 2f_{M-1}^n + f_{M-2}^n)$
- $c^n = (c_1^n, \dots, c_{M-1}^n)$
- $f^n = (f_1^n, \dots, f_{M-1}^n)$
- $g^n = (g_1^n, \dots, g_{M-1}^n),$

and (3.22) can be represented in matrix form as

$$A f^{n+1} \geq c^n, \quad (3.23)$$

where

$$A = \begin{bmatrix} 1 + 2\lambda\theta & -\lambda\theta & & & 0 \\ -\lambda\theta & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda\theta \\ 0 & & & -\lambda\theta & 1 + 2\lambda\theta \end{bmatrix} \in \mathbb{R}^{(M-1) \times (M-1)}.$$

Returning to our linear complementarity formulation (3.21), we note that the matrix system above is solving for the  $(n + 1)$ th node. Thus we have,

$$\begin{cases} (Af^{n+1} - c^n)^T (f^{n+1} - g^{n+1}) = 0 \\ f_0^n = g_0^n \text{ and } f_i^n = g_i^n \\ f_M^{(0)} = g_M^{(0)} \\ f^n \geq g^n \end{cases} \quad (3.24)$$

where  $x^T$  denotes the transpose of the column vector  $x$ .

### 3.3 Shi, Yang and Huang Explicit Fixed Point Iteration

In this section we will now focus on the solution methods of the option pricing LCP developed. In particular, we will work with the fixed point methods via Shi, Yang and Huang [28] and Bai [3], from which new generalizations and modifications will be presented. The theory will, however, be developed from the following general

LCP,

$$\begin{cases} (Au - b)^T u = 0 \\ u \geq 0 \\ w = Au - b \geq 0, \end{cases} \quad (3.25)$$

where  $u, b \in \mathbb{R}^n$  are column vectors and  $A \in \mathbb{R}^{n \times n}$ . We will denote this LCP for the rest of this and subsequent chapters with  $\mathcal{A}(A, u, b)$ .

We give a few definitions. Let  $M = (M_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ . Then  $M$  is said to be diagonally dominant if for all  $i = 1, \dots, n$ ,  $M_{ii} \geq \sum_{j=1, j \neq i}^n |M_{ij}|$  (strictly diagonally dominant if the inequality is strict). The elements of a matrix  $|M|$  are defined to be  $|M|_{ij} = |M_{ij}|$  for all  $i, j = 1, \dots, n$ . Moreover,  $M$  is said to be non-negative if and only if it is non-negative element-wise, that is  $M_{ij} \geq 0$  for all  $i, j = 1, \dots, n$ . If  $M_{ii} > 0$  for all  $i = 1, \dots, n$ ,  $M_{ij} \leq 0$  for  $i, j = 1, \dots, n$ ,  $i \neq j$ ,  $M$  is non-singular and  $M^{-1}$  is non-negative, then  $M$  is said to be an  $M$ -Matrix.  $M$  is called a  $P$ -Matrix if all of its principal minors are positive. Lastly, we define the  $l_\infty$ -norm on  $\mathbb{R}^n$  as  $\|u\|_\infty = \max_i |u_i|$  for  $u = (u_1, \dots, u_n)$ .

Shi, Yang and Huang (see [28]) presented an equivalence between the generalized LCP,  $\mathcal{A}$ , and an explicit fixed point algorithm. We will assume that  $\mathcal{A}(A, u, b)$  has a unique solution  $u^*$  (it is well known that if  $A$  is a  $P$ -Matrix, then  $\mathcal{A}(A, u, b)$  admits a unique solution, see for example [19]).

Before presenting the equivalence, we define a few terms. For  $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$ , define  $u_+, u_- \in \mathbb{R}^n$  to be such that  $(u_\pm)_i = \max(\pm u_i, 0)$  for all  $i = 1, \dots, n$ . Then, it is easy to observe that  $u = u_+ - u_-$ . Thus, if  $u^*$  is the convergent solution of the iteration given by,

$$u^{k+1} = u_+^k - \alpha(Au_+^k - b), \quad (3.26)$$

$u_+^*$  is a solution of  $\mathcal{A}$ . Here, some positive relaxation parameter,  $\alpha$ , designed to accelerate the iteration convergence is introduced. This iterative process comes from the fixed point equation given by  $u = u_+ - \alpha(Au_+ - b)$ . Notice that  $u_-^* = u_+^* - u^* = u_+^* - (u_+^* - \alpha(Au_+^* - b)) = \alpha(Au_+^* - b)$ . Also observe that  $u_- u_+^T = \sum_i^n \max(u_i, 0)(\max(-u_i, 0)) = 0$ . Thus  $\alpha(Au_+^* - b)(u_+^*)^T = 0 \iff (Au_+^* - b)(u_+^*)^T = 0$  and, since by definition  $u_-, u_+ \geq 0$  and  $\alpha > 0$ , the equivalence is immediate.

### 3.3.1 Convergence

Shi, Yang and Huang consider  $A = D - U - L$ , where  $D$  is the diagonal part,  $U$  and  $L$  are the upper and lower triangular parts of  $A$ . The component-wise iteration is thus presented as follows,

$$u_i^{k+1} = (u_+^k)_i - \alpha[(D - U)u_+^k - Lu_+^{k+1} - b]_i \quad (3.27)$$

for all  $i = 1, \dots, n$  and  $k = 0, 1, \dots$

The computation of  $u_{i-1}^{k+1}$  precedes that of  $u_i^{k+1}$ . Thus, because of the structure of matrix - vector multiplication with a lower triangular matrix, the  $Lu_+^{k+1}$  part of the above algorithm provides the most up to date information about  $u$ , better than the plain component-wise version of (3.26). Further,  $A$  is a symmetric tri-diagonal matrix defined as,

$$\begin{cases} a_{ii} = 1 + 2\lambda\theta \\ a_{ij} = -\lambda\theta \text{ for } i = j + 1 \text{ and } j = 1, \dots, n - 1 \\ a_{ij} = -\lambda\theta \text{ for } i = j - 1 \text{ and } j = 2, \dots, n \\ a_{ij} = 0 \text{ otherwise.} \end{cases} \quad (3.28)$$

Notice that this is the matrix we deduced in the linear complementarity formulation for the the American option pricing problem under the  $\theta$ -scheme. Before stating the convergence result for such a matrix, we give the following lemma useful to the proof presented by Shi, Yang and Huang,

**Lemma 3.3.** [12] Let  $A \in \mathbb{R}^{n \times n}$ . Then  $\rho(A) \leq \|A\|_\infty = \max_i \sum_{j=1}^n |(A)_{ij}|$  for all  $i = 1, \dots, n$ , where  $\rho(A)$  is the spectral radius of  $A$ .

By definition, the spectral radius of  $A$  is given by  $\rho(A) = \max_i |\lambda_i|$  for  $i = 1, \dots, n$  where  $\lambda_i$  is an eigenvalue of  $A \in \mathbb{R}^{n \times n}$ . We now present the convergence criterion for (3.27),

**Theorem 3.4.** [28] Let  $A$  be as described in (3.28) and  $b \in \mathbb{R}^n$ . If  $\alpha \in (0, \frac{2}{1+4\lambda\theta})$ , then the sequence  $\{u^k\}$  generated by (3.27) converges to the unique solution of the fixed point equation given by,

$$u = u_+ - \alpha(Au_+ - b).$$

See [28] for proof of the theorem. The first stage of the proof deduces the inequality of the form  $\|u^{k+1} - u^k\|_2 \leq \rho((\mathbb{I} - \alpha|L|)^{-1}|\mathbb{I} - \alpha(D - U)|) \|u^k - u^{k-1}\|_2$  through some algebraic manipulations. The authors then manually compute the matrix product  $(\mathbb{I} - \alpha|L|)^{-1}|\mathbb{I} - \alpha(D - U)|$  and, according to lemma 3.3, they deduce the range of values for  $\alpha$  which ensure that  $\|(\mathbb{I} - \alpha(D - U))\|_\infty$  is strictly bounded by 1.

In what follows, we will develop an alternative approach to obtain the range of  $\alpha$  for the convergence of (3.27).

**Lemma 3.5.** [25] Let  $M, N$  be a regular splitting of matrix  $J \in \mathbb{R}^{n \times n}$  ( $J = M - N$ ). Then  $\rho(M^{-1}N) < 1 \iff J$  is non-singular and  $J^{-1} \geq 0$ .

We now consider the proof for convergence of (3.27) from an overview perspective.

**Theorem 3.6.** Let  $A$  be as described in (3.28) and  $b \in \mathbb{R}^n$ . If  $\alpha \in (0, \frac{1}{1+2\lambda\theta})$ , then the sequence  $\{u^k\}$  generated by (3.27) converges to the unique solution of the fixed point equation given by,

$$u = u_+ - \alpha(Au_+ - b).$$

*Proof.* Let  $T(u) = u = u_+ - \alpha(Au_+ - b)$ . Then,

$$\begin{aligned} T(u) - T(v) &= u_+ - v_+ - \alpha(Au_+ - Av_+) \\ &= (\mathbb{I} - \alpha A)(u_+ - v_+) \end{aligned}$$

From this, we observe that  $\|T(u) - T(v)\|_\infty \leq \|I - \alpha A\|_\infty \|u - v\|_\infty$ . But notice that  $\|\mathbb{I} - \alpha A\|_\infty = |1 - \alpha - \alpha 2\lambda\theta| + \alpha 2\lambda\theta = 1 - \alpha < 1$  whenever  $\alpha \in (0, \frac{1}{1+2\lambda\theta})$ . Thus, by the Banach fixed point theorem [15], since  $T$  is a contraction mapping, it admits a unique fixed point. Further, from the proof of the Banach fixed point theorem, any general iteration  $T(u^k) = u^{k+1}$  associated with this fixed point problem converges to the unique fixed point whenever  $\alpha \in (0, \frac{1}{1+2\lambda\theta})$ .  $\square$

As a consequence, we have the following corollary,

**Corollary 3.7.** Let  $A \in \mathbb{R}^{n \times n}$  be a tri-diagonal  $M$ -Matrix. Then whenever  $\alpha \in (0, \min_i \frac{1}{a_{ii}})$ , the iterative algorithm (3.27) converges to its unique solution.

*Proof.* The proof follows directly from theorem 3.6. Here,  $\|\mathbb{I} - \alpha A\|_\infty = |1 - \alpha a_{ii}| + \alpha(|a_{i,i+1}| + |a_{i,i-1}|)$ , with  $a_{i,n+1} = 0 = a_{i,0}$ . Hence, since  $A$  is an  $M$ -matrix (thus  $|a_{ii}| \geq |a_{i,i+1}| + |a_{i,i-1}|$ ), whenever  $\alpha \in (0, \frac{1}{a_{ii}})$ ,  $\|\mathbb{I} - \alpha A\|_\infty < 1$  for all  $i = 1, \dots, n$ . Hence

$$\alpha \in (0, \min_i \frac{1}{a_{ii}}).$$



is a sufficient condition of the iteration (3,27) which concludes our proof.  $\square$

We now present an alternative proof of theorem (3.4) which will match Shi, Yang and Huang's result ( $\alpha \in (0, \frac{2}{1+4\lambda\theta})$ ) and improve the convergence range deduced in theorem 3.6. Our approach, different from that presented in [28], will open doors for more complicated matrix formulations.

*Proof.* As already noted, through algebraic manipulations, we can show that  $\|u^{k+1} - u^k\|_2 \leq \rho((\mathbb{I} - \alpha|L|)^{-1}|\mathbb{I} - \alpha(D - U)|)\|u^k - u^{k-1}\|_2$ . Instead of computing the iteration matrix  $(\mathbb{I} - \alpha|L|)^{-1}(|\mathbb{I} - \alpha(D - U)|)$  and estimating the absolute row sums as in [28], we set  $M = \mathbb{I} - \alpha|L|$  and  $N = |\mathbb{I} - \alpha(D - U)|$  and define the matrix  $(J_{ij})_{n \times n} = J = M - N$ . Thus, we have the elements of  $J$  given by,

$$J_{ij} = \begin{cases} 1 - |1 - \alpha(1 + 2\lambda\theta)| & \text{for } i = j \\ -\alpha\lambda\theta & \text{for } j = i + 1 \text{ and } i = 1, \dots, n - 1 \\ -\alpha\lambda\theta & \text{for } j = i - 1 \text{ and } i = 2, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

All we need for convergence is for  $J$  to be an  $M$ -Matrix. By setting  $1 - \alpha 2\lambda\theta > 0$ , notice,

$$\begin{aligned} 1 - |1 - \alpha(1 + 2\lambda\theta)| &> \alpha 2\lambda\theta \\ \iff \alpha 2\lambda\theta - 1 < 1 - \alpha(1 + 2\lambda\theta) < 1 - \alpha 2\lambda\theta \\ \iff 0 < \alpha < \frac{2}{1 + 4\lambda\theta}. \end{aligned}$$

In addition, define the matrix  $\langle J \rangle$  by,

$$\langle J \rangle_{ij} = \begin{cases} |J_{ii}| & \text{for } i = j \\ -|J_{ij}| & \text{otherwise.} \end{cases} \quad (3.29)$$

Then clearly,  $\langle J \rangle = J$ . Also,  $J$  has generalized diagonal dominance (see [18]) if there exists a positive vector  $x \in \mathbb{R}^n$  such that  $|J_{ii}|x_i > \sum_{j \neq i} |J_{ij}|x_j$  for all  $i = 1, \dots, n$ . Using the equivalence shown in [18],  $J$  has generalized diagonal dominance  $\iff \langle J \rangle$  is an  $M$ -Matrix. Hence whenever  $\alpha \in (0, \frac{2}{1+4\lambda\theta})$ , since we already know that  $\lambda\theta \geq 0$ ,  $J$  is diagonally dominant (hence, with  $x = (1, \dots, 1)$ , possess general diagonal dominance). Thus  $\langle J \rangle = J$  must be an  $M$ -Matrix and, as such,  $\rho(M^{-1}N) < 1$ . This completes the proof.  $\square$

Compared to the proof provided in [28], which involved a long manual computation of  $(\mathbb{I} - \alpha|L|)^{-1}|\mathbb{I} - \alpha(D - U)|$ , our proof is easily tractable and useful for cases in which the manual computations are complicated. This is illustrated in the following corollary,

**Corollary 3.8.** Let  $A \in \mathbb{R}^{n \times n}$  be a general strictly diagonally dominant matrix with a non-negative diagonal and non-positive off-diagonal elements. Then whenever  $\alpha \in (0, \min_{i=1, \dots, n} \frac{2}{a_{ii} + \sum_{j=1, j \neq i} |a_{ij}|})$ , the iteration given in (3.27) converges to its unique solution.

*Proof.* Following on from theorem 3.8, with the same choice of  $M$  and  $N$ , in the

general setting  $J$  becomes,

$$J_{ij} = \begin{cases} 1 - |1 - \alpha a_{ii}| & \text{for } i = j \\ \alpha a_{i,i+1} & \text{for } j = i + 1 \text{ and } i = 1, \dots, n - 1 \\ \alpha a_{i,i-1} & \text{for } j = i - 1 \text{ and } i = 2, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

Setting  $1 - \alpha \max_i \sum_{j=1, j \neq i} |a_{ij}| > 0$ , for  $J$  to be diagonally dominant, we require  $1 - |1 - \alpha a_{ii}| > \alpha \sum_{j=1, j \neq i} |a_{ij}| = f(i) \iff \alpha f(i) - 1 < 1 - \alpha a_{ii} < 1 - \alpha f(i)$ . The lower bound gives  $\alpha < \frac{2}{a_{ii} + f(i)}$ . From the upper bound, we have  $0 < \alpha(a_{ii} - f(i))$ . But since  $A$  is diagonally dominant,  $a_{ii} > f(i)$ , which means we need  $\alpha > 0$ . Also note that  $J = \langle J \rangle$ . Hence (3.27) converges whenever,

$$\alpha \in \left(0, \min_{i=1, \dots, n} \frac{2}{a_{ii} + \sum_{j=1, j \neq i} |a_{ij}|}\right),$$

and this completes the proof.  $\square$

This result is more powerful since the (off-)diagonal elements of  $A$  are not restricted to be equal and it does not necessarily have to be a symmetric tri-diagonal. In the following section we introduce a modulus-based matrix splitting implicit fixed point method. Based on our understanding of Shi, Yang and Huang's method, we provide a new modification of this modulus-based approach. This modification can also be viewed as a generalization of the fixed point iteration discussed in this section.

### 3.4 The Modulus-Based Matrix Splitting Method

The LCP  $\mathcal{A}$  normally requires some inequalities to be solved. For example, Cryer [5] introduced the now classical projected successive over-relaxation method requiring a two-step iterative process to satisfy the bounds for  $\mathcal{A}$ . We have seen in the preceding

section that  $\mathcal{A}$  can be set up as a explicit fixed point iteration by a method of equivalence. This has computational cost advantages.

Similarly, Bai [3] introduces an implicit fixed point method requiring a single iterative step and has equivalence to  $\mathcal{A}$ . We present this method (without proof, which is a straightforward checking that the propositions match) in the following theorem.

**Theorem 3.9.** [3] Let  $A = M - N$  be a matrix splitting with  $A \in \mathbb{R}^{n \times n}$ ,  $\Omega_1, \Omega_2 \in \mathbb{R}^{n \times n}$  are non-negative diagonal matrices and  $\Omega$  and  $\Gamma$  are  $n \times n$  positive diagonal matrices such that  $\Omega = \Omega_1 + \Omega_2$ . Then for the LCP  $\mathcal{A}$ , the following are true.

- (a) If  $(w, u)$  is a solution of  $\mathcal{A}$ , then  $z = \frac{1}{2}(\Gamma^{-1}u - \Omega^{-1}w)$  satisfies the following implicit fixed point equation,

$$(M\Gamma + \Omega_1)z = (N\Gamma - \Omega_2)z + (\Omega - A\Gamma)|z| + b. \quad (3.30)$$

- (b) If  $z$  satisfies the implicit condition in (a), then

$$u = \Gamma(|z| + z) \quad w = \Omega(|z| - z)$$

is a solution of  $\mathcal{A}$ .

This implicit fixed point equation contains numerous matrices, hence can result in complications when conducting convergence analysis. Since  $\Omega_1$ ,  $\Omega_2$  and  $\Gamma$  are arbitrary, Bai makes the following choices,

$$\Omega_2 = 0 \quad \Gamma = \frac{1}{\gamma}\mathbb{I}, \quad (3.31)$$

where  $\gamma > 0$ . Hence  $\Omega = \Omega_1$  (thus must be positive diagonal). These selections

simplify the implicit fixed point equation into,

$$(M + \Omega)z = Nz + (\Omega - A)|z| + \gamma b. \quad (3.32)$$

From this, we present Bai's Modulus Based Matrix Splitting Method (MBMSM) for  $A$  in Algorithm 1. This algorithm can take various forms depending on the choices

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**1** *Algorithm 1* Let  $A = M - N$  be a splitting of the matrix  $A \in \mathbb{R}^{n \times n}$ . Suppose that  $z^{(0)}$  is an initial estimate, for subsequent iterative steps until  $\{u^k\}$  converges, compute  $z^{k+1}$  by solving the following linear iterative system,

$$(M + \Omega)z^{k+1} = Nz^k + (\Omega - A)|z^k| + \gamma b, \quad (3.33)$$

and set  $u^{k+1} = \frac{1}{\gamma}(|z^{k+1}| + z^{k+1})$ .

---

of splitting matrix  $A$ . For example, by introducing two relaxation parameters,  $\alpha$  and  $\beta$ , [3], and setting  $M = \frac{1}{\alpha}(D - \beta L)$ ,  $N = \frac{1}{\alpha}[(1 - \alpha)D + (\alpha - \beta)L + \alpha U]$  and  $\gamma = 2$ , the algorithm becomes,

$$(D + \Omega - \beta L)z^{k+1} = [(1 - \alpha)D + (\alpha - \beta)L + \alpha U]z^k + (\Omega - \alpha A)|z^k| + 2\alpha b, \quad (3.34)$$

with  $u^{k+1} = \frac{1}{2}(|z^{k+1}| + z^{k+1})$ . This is known as the Modulus-Based Accelerated Over-relaxation Method (MAOR).

### 3.4.1 Generalization of Shi, Yang and Huang's Fixed Point Method

While Bai and other subsequent works here (see for example Zheng, Li and Vong [34] who introduce a two-step relaxation algorithm in the linear system. This is however not projecting the inequalities to be satisfied, as in the PSOR) carry out some convergence analysis for different forms of the matrix  $A$ , such as it being positive definite, we present a generalization of the Shi, Yang and Huang's fixed point method

already seen in the subsequent section. We believe this will form a new and more parsimonious template for further works that followed Bai's iterative method. To this end we generalize (3.27) with this new result, based on the modulus-based matrix splitting method,

**Theorem 3.10.** Let  $A = M - N$  be a matrix splitting with  $A \in \mathbb{R}^{n \times n}$  and  $\Omega$  and  $\Gamma$  are  $n \times n$  positive diagonal matrices. Then for the LCP  $\mathcal{A}$ , the following are true.

- (a) If  $(w, u)$  is a solution of  $\mathcal{A}$ , then  $z = \Gamma^{-1}u - \Omega^{-1}w$  satisfies the following implicit fixed point equation,

$$\Omega z = \Omega z_+ - [(M - N)\Gamma z_+ - b]. \quad (3.35)$$

- (b) If  $z$  satisfies the implicit condition in (a), then

$$u = \Gamma z_+ \quad w = \Omega z_-$$

is a solution of  $\mathcal{A}$ .

*Proof.* We begin by proving part (a). We write  $u = \Gamma z_+$  and  $w = \Omega z_-$  for some  $z \in \mathbb{R}^n$ . This maintains the non-negativity of  $u$  and  $w$ . Also, since  $(u, w)$  is a solution of the  $\mathcal{A}$ , it must be true that  $w^T u = 0$ . As we have already seen, this is always true

from the definition of  $z_+$  and  $z_-$ . Further,  $w = Au - b$ , which is true if and only if,

$$\begin{aligned} \Omega z_- &= A\Gamma z_+ - b \\ \iff \Omega z_- &= (M - N)\Gamma z_+ - b \\ \iff \Omega(z_+ - z) &= (M - N)\Gamma z_+ - b \\ \iff \Omega z &= \Omega z_+ - [(M - N)\Gamma z_+ - b]. \end{aligned}$$

The solution of this implicit problem for  $z$  must then be, since we already have determined  $(u, w)$ ,  $z = z_+ - z_- = \Gamma^{-1}u - \Omega^{-1}w$ . This concludes proof of part (a). The proof of part (b) is immediate since, from the proof of part (a), we know that the implicit equation can be written as  $\Omega z_- = A\Gamma z_+ - b$ , which gives us  $w = Au - b$ . Also as already noted,  $w^T u = 0$ , hence  $(u, w)$  is a solution for  $\mathcal{A}$  and this concludes our proof.  $\square$

Based on the above, we can present the generalized solution algorithm (GSM),

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1 *Algorithm 2* Let  $A = M - N$  be a splitting of the matrix  $A \in \mathbb{R}^{n \times n}$ . Suppose that  $z^{(0)}$  is an initial estimate, for subsequent iterative steps until  $\{u^k\}$  converges, compute  $z^{k+1}$  by solving the following linear iterative system,

$$\Omega z^{k+1} = \Omega z_+^k - ((M - N)\Gamma z_+^k - b), \quad (3.36)$$

and set  $u^{k+1} = \Gamma z_+^{k+1}$ .

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### 3.4.2 Convergence of the Generalized Method

We will assume that  $A$  is strictly diagonally dominant  $n \times n$  tri-diagonal  $M$ -Matrix. For the matrix splitting  $A = D - U - L$ , we now formally present a generalization of Shi, Yang and Huang's fixed point method,

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1 *Algorithm 3* For an initial estimate  $z^{(0)} \in \mathbb{R}^n$ , until convergence of  $\{z^k\}$ , solve for the fixed point problem  $\Omega z = \Omega z_+ - (A\Gamma z_+ - b)$ , through the following component-wise iteration,

$$z_i^{k+1} = (z_+^k)_i - [(\Omega^{-1}D\Gamma - \Omega^{-1}U\Gamma)z_+^k - \Omega^{-1}L\Gamma z_+^{k+1} - \Omega^{-1}b]_i \quad (3.37)$$

for  $i = 1, \dots, n$ .

---

The convergence analysis of Algorithm 3 follows from what we have seen already. Specifically, by denoting  $J_* = \Omega^{-1}J\Gamma$  for some  $J \in \mathbb{R}^{n \times n}$ , we have

$$z_i^{k+1} = (z_+^k)_i - [(D_* - U_*)z_+^k - L_*z_+^{k+1} - \Omega^{-1}b]_i.$$

Since  $\Omega$  and  $\Gamma$  are positive diagonal matrices it is easy to check that

$$(D_*)_{ij} = \begin{cases} \frac{a_{ii}\Gamma_{ii}}{\Omega_{ii}} & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \quad (U_*)_{ij} = \begin{cases} \frac{a_{ij}\Gamma_{ii}}{\Omega_{ii}} & \text{for } i = j - 1 \text{ and } j = 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

$$(L_*)_{ij} = \begin{cases} \frac{a_{ij}\Gamma_{ii}}{\Omega_{ii}} & \text{for } i = j + 1 \text{ and } j = 1, \dots, n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, as before, to obtain  $\rho((\mathbb{I} - |L_*|)^{-1}|\mathbb{I} - (D_* - U_*)|) < 1$ , we set  $M = \mathbb{I} - |L_*|$  and  $N = |\mathbb{I} - (D_* - U_*)|$ . Then  $M - N = J$  is such that,

$$J_{ij} = \begin{cases} 1 - |1 - \frac{a_{ii}\Gamma_{ii}}{\Omega_{ii}}| & \text{for } i = j \\ \frac{a_{i,i+1}\Gamma_{ii}}{\Omega_{ii}} & \text{for } i = 1, \dots, n - 1 \text{ and } j = i + 1 \\ \frac{a_{i,i-1}\Gamma_{ii}}{\Omega_{ii}} & \text{for } i = 2, \dots, n \text{ and } j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$



Setting  $1 - \frac{\Gamma_{ii}}{\Omega_{ii}}(|a_{i,i+1}| + |a_{i,i-1}|) > 0$ , for  $J$  to be diagonally dominant, we need,

$$\begin{aligned}
& 1 - \left|1 - \frac{a_{ii}\Gamma_{ii}}{\Omega_{ii}}\right| > \frac{\Gamma_{ii}}{\Omega_{ii}}(|a_{i,i+1}| + |a_{i,i-1}|) \\
\iff & \frac{\Gamma_{ii}}{\Omega_{ii}}(|a_{i,i+1}| + |a_{i,i-1}|) - 1 < 1 - \frac{a_{ii}\Gamma_{ii}}{\Omega_{ii}} < 1 - \frac{\Gamma_{ii}}{\Omega_{ii}}(|a_{i,i+1}| + |a_{i,i-1}|) \\
& \iff \frac{\Gamma_{ii}}{\Omega_{ii}} \in \left(0, \frac{2}{a_{ii} + |a_{i,i+1}| + |a_{i,i-1}|}\right).
\end{aligned}$$

for all  $i = 1, \dots, n$ . Notice, since  $A$  is diagonally dominant, we need  $\frac{\Gamma_{ii}}{\Omega_{ii}} > 0$ . We also set  $a_{i,n+1} = 0$  and  $a_{i,0} = 0$ . Further, we observe that  $J = \langle J \rangle$ . Thus  $\frac{\Gamma_{ii}}{\Omega_{ii}} \in \left(0, \frac{2}{a_{ii} + |a_{i,i+1}| + |a_{i,i-1}|}\right)$  is a sufficient convergence criterion.

To retain Shi, Yang and Huang's original algorithm for some  $\alpha > 0$ , we can select  $\Gamma = \mathbb{I}$  and  $\Omega = \frac{1}{\alpha}\mathbb{I}$ . Hence this generalized method offers more avenues to explore better convergence rates and errors by widening the class of trial matrices,  $\Omega$  and  $\Gamma$ . This is crucial in managing computational costs and inefficiencies. Furthermore, the convergence criterion is easily tractable.

# 4

## Lattice and Integral Frameworks

In this chapter we will discuss alternative methods for pricing American options surveyed in literature. These will be the lattice method and the integral representation framework. Of main focus will be recent works by Aluigi, Corradini and Gheno [1] who present a highly efficient lattice framework based on the Chapman-Kolmogorov equation and by Frontczak and Schobel [8] who discuss the integral formulation derived from modifying the Mellin transforms.

### 4.1 The Chapman-Kolmogorov Lattice Method

We begin by reviewing the fundamental methodology of lattice (or tree) methods for option pricing. In continuous time, under the risk neutral measure, it is well known that the discounted prices of all assets in our market model are martingales. Crucially, as already discussed, this enables us to give the value of a financial derivative at time  $t$ , with a maturity pay-off  $v(s_T, T)$ , as

$$v_t(s_t, t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}v(s_T, T)|\mathcal{F}_t], \quad (4.1)$$

where the filtration  $\mathcal{F}_t$  represent the underlying asset's price history up to the current time  $t$ . In general, lattice methods attempt to discretize this problem by confining  $s_t$  to a finite range of possible values. However, as more accurate convergence is sought by narrower time-step discretizations, computational costs tend to increase. Before discussing recent and efficient works here, in the form of Chapman - Kolmogorov lattice trees (CKL Method) by Aluigi, Corradini and Gheno [1], we recall the lattice option pricing framework through binomial and trinomial trees. As we will see, these can also be used for our free boundary problem in American option evaluation.

#### 4.1.1 The Lattice Method Framework

We consider the derivative contract period  $[0, T]$ , which is partitioned into  $M$  sub-intervals of equal length  $\Delta t = \frac{T}{M}$ . Setting the price of the underlying  $s_t$  at time  $0 = t_0$  to be  $S_0$ , we confine its value at the next time-step to be

$$s_{t_1} = \begin{cases} us_0 & \text{with probability } q \\ ds_0, & \text{with probability } 1 - q \end{cases} \quad (4.2)$$

where  $u > 1$  represents an up jump and  $0 < d < 1$  the corresponding downward jump amplitude of the security. It is common in literature to set  $d = \frac{1}{u}$ . Confining the security's dynamics to these two options in subsequent time steps up to maturity describes what is known to be a Binomial Tree. We illustrate a two step binomial model in diagram 4.1. The trinomial tree provides an additional possibility between successive time steps, where the security remains constant. Thus the corresponding

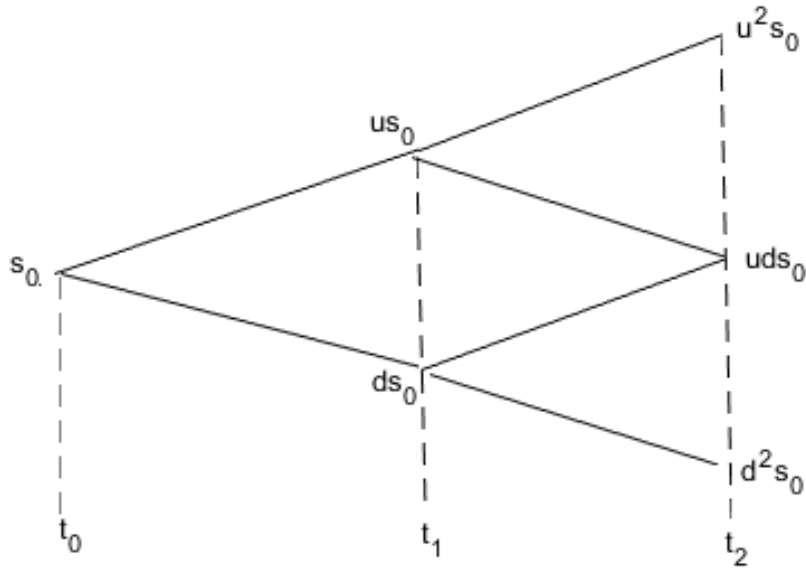


Figure 4.1: Two Step Binomial Lattice

possibilities at  $t_1$  are given by,

$$s_{t_1} = \begin{cases} us_0 & \text{with probability } q_u \\ s_0 & \text{with probability } q_m \\ ds_0, & \text{with probability } q_d \end{cases} \quad (4.3)$$

where  $q_u + q_m + q_d = 1$ .

This framework, for a suitably large number of time-steps, has proved to be popular methodology for derivative pricing. To illustrate the pricing methodology for European call option  $v_{t_0}$  with strike price  $K$ , for example, we reconsider the two-step binomial model. Working backwards, evaluations of the pay-offs are done at each node at the maturity date time step ( $t_M = T$ ). The next step is to compute

the value of the option at time-step  $t_1$  as present values (which is the fundamental basis of fair value valuation) of the prices obtained in time step 2. For example, the top node  $v_{t_1}^{top}$  is valued as,

$$v_{t_1}^{top} = \begin{cases} e^{-r\Delta t}(u^2s_0 - K)^+ & \text{with probability } q \\ e^{-r\Delta t}(uds_0 - K)^+ & \text{with probability } 1 - q. \end{cases} \quad (4.4)$$

These are the only possibilities, since from this node, the next possible destinations for the security are  $u^2s_0$  and  $uds_0$ . Thus combining everything, we have  $v_{t_1}^{top} = e^{-r\Delta t}[q(u^2s_0 - K)^+ + (1 - q)(uds_0 - K)^+]$ . This procedure is repeated for the lower node, to obtain  $v_{t_1}^{down}$ . Finally we evaluate the price of the derivative at  $t_0$  as,

$$v_{t_0} = qv_{t_1}^{top} + (1 - q)v_{t_1}^{down}.$$

In general this can be done for an  $M$ -step tree, and for the trinomial method, three possibilities are considered during the valuation at each node.

This method allows for the pricing of American options. We recall that, for an American derivative, early exercise is allowable. This gives rise to what is known as early exercise premium, making it more valuable than its European counterpart with identical contract parameters. By arbitrage arguments, since the derivative holder can exercise at some time step before maturity, the value at a node must then be the larger of the pay-off at that node and discounted value determined from nodes in time-steps ahead. For example, at the top node (time step  $t_1$ ) in our two-step binomial model, the value of the American call on a dividend paying underlying (which is a requirement for existence of early exercise premium, only for the call case) is given by,

$$c_{t_1}^{top} = \max\{(us_0 - K)^+, e^{-r\Delta t}[q(u^2s_0 - K)^+ + (1 - q)(uds_0 - K)^+]\}.$$

We do the same for the lower node  $c_{t_1}^{down}$  to then obtain the value of the American Call at  $t_0$  as,

$$c_{t_0} = \max\{(s_0 - K)^+, e^{-r\Delta t}[qc_{t_1}^{top} + (1 - q)c_{t_1}^{down}]\}.$$

In what follows, we discuss the CKL Method which is more comprehensive than the binomial and tri-nomial trees.

### 4.1.2 The CKL Method by Aluigi, Corradini and Gheno

Aluigi, Corradini and Gheno develop the CKL method by considering a general case when the security can make  $N$  different jumps in a single time step (which provides a more practical model) over  $M$  time steps. Depending on the security's characteristics (perhaps its price history or industry comparables), they bound it with some  $s_{\min}$  and  $s_{\max}$ . More subtle detail is the that, from an node in the tree before the maturity time-step, the jump can be to any of the  $N$  possible values of  $s$ . As we will see, such a bounded price process (with a countable finite state space characteristic) will allow us to implement the Chapman Kolmogorov equation into a more efficient lattice framework. To this point, we define a few terms before introducing the model.

**Definition 4.1.** A stochastic process  $(x_t)_{t \geq 0}$  on some finite state space  $\Omega = \{1, \dots, n\}$  is called a Markov process if, given the process history up to time  $s$ ,  $\mathcal{F}_s$ , the probability of transitioning from state  $i$  at time  $s$  to  $j$  at time  $t > s$  is given by,

$$p_{ij}(s, t) = p(x_t = j | \mathcal{F}_s) = p(x_t = j | x_s = i).$$

This relation is known as the Markov property. Indeed Brownian Motion is an example of a Markov process. Central to the CKL Method is the Chapman-

Kolmogorov equation from probability theory,

$$p(x_{t_3}|x_{t_1}) = \int_{x_{t_2}} p(x_{t_3}|x_{t_2})p(x_{t_2}|x_{t_1})dx_{t_2} \quad (4.5)$$

where  $p(x_{t_j}|x_{t_i})$  is the conditional probability (distribution) of transitioning from state  $x_{t_i}$  to state  $x_{t_j}$ . This is true if and only  $x_t$  is a Markov process. For an intuitive understanding of the right hand side of the above equation, we are integrating over all the possible state spaces  $x_{t_2}$  the process occupies at time  $t_1$  from the specified state space  $x_{t_0}$  before transitioning to the specified state  $x_{t_2}$  at time  $t_2$ .

We now recall that, in the risk neutral world, the underlying security  $s_t$  evolves according to  $ds_t = rdt + \sigma d\omega_t$  under the Black-Scholes. The CKL Method considers the discrete version of this process,

$$s_{i+1} = s_i + r\Delta t + \sigma\sqrt{\Delta t}\epsilon_i \quad (4.6)$$

where  $i = 0, \dots, M-1$  discretizes variables during contract period  $[0, T]$  with  $\Delta t = \frac{T}{M}$  and  $\epsilon_i \sim \mathcal{N}(0, 1)$ . Defining a bound  $s_{t_i} \in [s_{\min}, s_{\max}]$ , the price process can take the any of the following  $N$  values,

$$\{s_{\min} + k\Delta s\}_k \quad \text{for } k = 0, \dots, N-1$$

with  $\Delta s = \frac{s_{\max} - s_{\min}}{N-1}$  and  $s_{\min} + (N-1)\Delta s = s_{\max}$ . For the valuation of the a derivative  $v_0$  at  $t = 0$ , by definition, (4.1) can be written in the integral form

$$v_0 = e^{-rT} \int_{\mathbb{R}^+} v(s_T, T) f(s_T|s_0) ds,$$

where  $f(s_T|s_0)$  is the probability density function of the random variable  $s_T$  given the price history at time  $t_0$ . In discrete form, Aluigi, Corradini and Gheno state

that, from the Chapman-Kolmogorov equation, we have

$$f(s_T|s_0) = \frac{1}{\Delta s} \sum_{s_{M-1} \dots s_1} \prod_{i=0}^{M-1} f(s_{i+1}|s_i). \quad (4.7)$$

We provide a proof of how they achieve this general relation. In a general sense, Equation (4.5) takes the discrete form, for  $f$ ,  $f(s_T|s_0) = \sum_{s_{t_i}} f(s_T|s_{t_i})f(s_{t_i}|s_0)\Delta s$  for some  $t_i < T$  and  $s_{t_i}$  takes the values  $s_{\min} + k\Delta s$  for all  $k = 0, \dots, N - 1$  (that is, we are summing a product of probabilities for all possible state spaces the process can take in the 'intermediate' state space  $s_{t_i}$  during its transition from  $s_{t_0}$  to  $s_T$ ). This is true for all  $i = 0, \dots, M - 1$ ,

$$\begin{aligned} f(s_M|s_0) &= \sum_{s_{M-1}} f(s_M|s_{M-1})f(s_{M-1}|s_0)\Delta s \\ &= \sum_{s_{M-1}} f(s_M|s_{M-1})\left(\sum_{s_{M-2}} f(s_{M-1}|s_{M-2})f(s_{M-2}|s_0)\right)(\Delta s)^2 \\ &\cdot \\ &\cdot \\ &\cdot \\ &= \sum_{s_{M-1}} f(s_M|s_{M-1})\left[\sum_{s_{M-2}} f(s_{M-1}|s_{M-2})\left[\sum_{s_{M-3}} f(s_{M-2}|s_{M-3})\left[\dots\left[\sum_{s_1} \right. \right. \right. \right. \right. \\ &f(s_2|s_1)f(s_1|s_0)\dots\left.\left.\left.\left.\right]\right]\right]\right](\Delta s)^{M-1} \\ &:= \sum_{s_{M-1} \dots s_1} f(s_M|s_{M-1})f(s_{M-1}|s_{M-2})\dots f(s_2|s_1)f(s_1|s_0)(\Delta s)^{M-1} \\ &= \frac{1}{\Delta s} \sum_{s_{M-1} \dots s_1} \prod_{i=0}^{M-1} [f(s_{i+1}|s_i)\Delta s], \end{aligned}$$

which is a computation of  $N \times (M - 1)$  sums. For an  $M$  large enough, we can approximate  $f$  with normal distribution (since  $\Delta t \rightarrow 0$  as  $M$  gets larger). Thus, noticing the expectation of  $s_{i+1}$  is  $r$  and variance is  $\sigma^2\Delta t$ , we yield  $s_{i+1} \sim \mathcal{N}(s_i +$



$r, \sigma^2 \Delta t$ ) with,

$$f(s_{i+1}|s_i) = \frac{1}{\sigma\sqrt{2\pi\Delta t}} e^{-\frac{(s_{i+1}-s_i-r\Delta t)^2}{2\sigma^2\Delta t}}. \quad (4.8)$$

Finally, the derivative price can be approximated with

$$v_0 = e^{-rT} \sum_{s_M} v(s_T, T) \sum_{s_{M-1} \dots s_1} \prod_{i=0}^{M-1} f(s_{i+1}|s_i) \Delta s \quad (4.9)$$

with the  $f$  approximated to be normally distributed. This evaluates a European type derivative for the initial node,  $t_0$ . Identical computations are done at the other nodes in the tree, subject to adjustment of the time-frame to be considered. Additionally, as in the binomial model example, pricing an American derivative would require considering the maximum of the present value of the price in nodes ahead and the payoff at the node for which the price is to be determined.

The CKL method proves to be an efficient lattice method of recent times. The main cost in carrying out computations of this nature comes increasing the number of time steps. In their convergence analysis, Horasanh [11] notes that for an analytical price (which we showed exists for European options) of 25.70902424 on a European call with spot price of 120, strike price 100, 0.25 volatility, 0.10 risk free rate and contract length of 0.5, the binomial method fairs worse than the trinomial with an error of 0.000163150. This is almost double the reported error of the trinomial of 0.000094508 for an identical number of timesteps, 5000 in this case and with the error being the differential from the true (analytical) price. Notably, however, for the 5000 timesteps, the binomial method yields a total of 12,507,501 nodes in the tree while the trinomial has 25,010,001, almost double that of the binomial. Hence a clear trade-off between accuracy and computational cost. Importantly though, we observe that the price differentials are relatively negligible from the analytical prices.

This is the reason why in practice, for cases where analytical prices do not exist, such as in the American option case studied in this thesis, prices from lattice methods are seen as proxies to the true prices despite high computational costs.

This brings us to the convergence analysis and accuracy of the CKL method. For the same option used to compare the binomial and trinomial methods, the CKL method yields a comparable price of 25.7090 in only 500 time-steps. Hence for a significantly smaller time step procedure, the CKL method appears to be highly efficient compared to the other two models. To put results into further perspective, Aluigi, Corradini and Gheno report a price of 25.7093 for only 100 time steps, which is still impressive convergence towards the Black-Scholes analytical price. The CKL method thus appears to be a strong alternative in American option pricing.

## 4.2 The Modified Mellin Transform Method

We now develop an integral representation to price derivatives as presented by Frontczak and Schobel [8].

**Definition 4.2.** Let  $f$  be some locally integrable function and  $z$  be a complex number. Then we define the Mellin transform of  $f$  by the following complex valued function,

$$M(f(x), z) := \int_0^{\infty} f(x)x^{z-1}dx,$$

where we can use the notation  $\tilde{f} = M(f, z)$ .

The case of convergence (or existence) of the integral in the above definition depends on how the function  $f$  behaves at 0 and  $\infty$ . On the complex plane, the largest vertical strip denoted  $\langle a, b \rangle$ , where  $a, b \in \mathbb{R}$  on which convergence of the integral occurs is known as the fundamental strip. In general, however, how  $f$

behaves asymptotically at the integral bounds can provide us with a guaranteed strip in which convergence occurs. This is summarised in the following lemma,

**Lemma 4.3.** [7] Let  $f$  be a locally integrable function such that  $\lim_{x \rightarrow 0^+} f(x)$  is  $\mathcal{O}(x^{-u})$  and  $\lim_{x \rightarrow \infty} f(x)$  is  $\mathcal{O}(x^{-v})$ , then if  $u < v$ , the Mellin transform of  $f$  exists, that is the integral formulation of  $\tilde{f}$  converges on  $\langle u, v \rangle$ .

We can re-obtain  $f$  from the inverse of the Mellin transform which, for a real value  $c \in \langle a, b \rangle$  where  $f(c + it)$  is integrable, is defined as

$$f(x) = \tilde{f}^{-1}(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z)x^{-z} dz.$$

### 4.2.1 Modification for Call Option Valuation

Returning to our option pricing problem, we notice some problems for the call option case ( $u^e = u^e(s, t)$  from here on) case which Frontczak and Schobel study. Because for a European call we have  $\lim_{s \rightarrow 0} u^e(s, t) = 0$  and  $\lim_{s \rightarrow \infty} u^e(s, t) = s$ , notice that  $u = 0$  and  $v = -1$  according to lemma 4.3. As such, Frontczak and Schobel suggested the following modification of the Mellin transform to circumvent this problem,

$$M(u^e(s, t), -z) := \int_0^\infty f(x)x^{-(z+1)} dx,$$

with  $M(u^e(s, t), -z) = \tilde{u}^e(z)$ . Here the modification was in the form of replacing  $z$  for  $-z$ . With this, the inverse then becomes,

$$u^e = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{u}^e(z)s^z dz.$$

A useful property of the Mellin transform is that  $M[(x \frac{d}{dx})^k f(x), z] = (-1)^k z^j \tilde{f}(z)$  [7]. Thus, taking the Modified Mellin transforms for the Black-Scholes PDE on a

dividend paying underlying  $s_t$  for the valuation of our European call,  $u^e$ , we obtain,

$$\begin{aligned} \partial_t \tilde{u}^e(s, t) + \frac{1}{2} \sigma^2 z^2 \tilde{u}^e(z) - (r - q) z \tilde{u}^e(z) - r \tilde{u}^e(z) &= \partial_t \tilde{u}^e(z) + \frac{1}{2} \sigma^2 \left( z^2 - \frac{2(r - q)}{\sigma^2} z - \frac{2r}{\sigma^2} \right) \tilde{u}^e(z) \\ &= \partial_t \tilde{u}^e(z) + \frac{1}{2} \sigma^2 P(z) \tilde{u}^e(z) \\ &= 0, \end{aligned}$$

where  $P(z) = \frac{1}{2} \sigma^2 \left( z^2 - \frac{2(r - q)}{\sigma^2} z - \frac{2r}{\sigma^2} \right) \tilde{u}^e(z)$ . This is simple linear ODE with a solution of the form,

$$\tilde{u}^e(z) = \psi(z) e^{-\frac{1}{2} \sigma^2 P(z) t}.$$

Recall that at maturity  $T$ , the value of the call is equivalent to its payoff. Hence the Mellin transform at  $t = T$  becomes,

$$\begin{aligned} \tilde{u}^e(z, T) &= \int_0^\infty (s - K)^+ s^{-(z+1)} dz \\ &= \int_{s \geq K} (s - K)^+ s^{-(z+1)} dz \\ &= \int_{s \geq K} s^{-z} - K s^{-(z+1)} dz \\ &= K^{-z+1} \left( \frac{1}{z-1} - \frac{1}{z} \right). \end{aligned}$$

Thus the constant term  $\psi(z) = K^{-z+1} \left( \frac{1}{z-1} - \frac{1}{z} \right) e^{\frac{1}{2} \sigma^2 P(z) T}$ . We thus give a proposition providing the equivalence between the Black-Scholes analytical solution for European option pricing and the Modified Mellin transform method.

**Proposition 4.4.** [8] Let  $z \in \mathbb{C}$  be such that  $Re(z) \in (1, \infty)$ . Then for a constant

$c \in (1, \infty)$ ,

$$u^e(s, t) = \frac{1}{2\pi i} \int_{c+i\infty}^{c-i\infty} K^{-z+1} \left( \frac{1}{z-1} - \frac{1}{z} \right) e^{\frac{1}{2}\sigma^2 P(z)(T-t)} s^z dz \quad (4.10)$$

$$= s e^{-q(T-t)} \Phi(d_1(s, K, T-t)) - X e^{-r(T-t)} \Phi(d_2(s, K, T-t)), \quad (4.11)$$

$$(4.12)$$

where  $d_1(x, y, t) = \frac{\log \frac{x}{y} + (r-q+\frac{\sigma^2}{2})(t)}{\sigma\sqrt{t}}$ ,  $d_2 = d_1 - \sigma\sqrt{t}$  and  $\Phi$  is the normal density function whose value can be obtained from a normal distribution table.

The comprehensive proof of this proposition in [8] uses a key step early on by choosing  $z = c + iy$  (the same  $c$  in the integral limits, which satisfies the requirement for the real part of  $z$  to be strictly larger than 1 and less than infinity). With this, transforming the variables in the resulting integral to take the form of the error function  $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2} du$  and noticing its relationship with the normal distribution function given by  $\phi(x) = 2\Phi(\sqrt{2}x) - 1$  leads to the result.

## 4.2.2 The Gauss-Laguerre Quadrature for the American Call

As expected, with the existence of early exercisability, the American call option case is more complicated. Frontczak and Schobel, however, come up with clever technique of stating the problem as a PDE by defining a function  $g$  that caters for what happens to the original PDE on both sides of the free boundary (which, as we recall, we denoted with  $s_f(t)$ ). Recall that the value of an American option equals its pay-off whenever it hits the free boundary and it becomes optimal to exercise. For the American call option, this stopping region is such that  $s_f(t) \leq s_t < \infty$ . We also saw that for this situation, the left hand side of the Black Scholes PDE becomes  $rK - qs$ . The complementary domain is the continuation region ( $0 \leq s_t \leq s_f(t)$ ) in which the value of the American call is equal to its European counterpart (thus

satisfying the Black Scholes PDE). Summarising this, with  $u^a(s, t)$  denoting the value of the American call, we have

$$\partial_t u^a + \frac{1}{2} \sigma^2 s^2 u_{ss}^a + (r - q) s u_s^a - r u^a = g(s, t),$$

where  $g$  is defined as

$$g(s, t) = \begin{cases} rK - qs & \text{for } s_f(t) \leq s_t < \infty \\ 0 & \text{for } 0 \leq s_t < s_f(t). \end{cases}$$

We can take the Modified Mellin transform for  $g$  which turns out to be,

$$\tilde{g}(z, t) = \frac{rK}{z} (s_f(t))^{-z} - \frac{q}{z-1} (s_f(t))^{-z+1}.$$

As in the European call case, taking the Modified transforms of the Modified PDE system for the American call leads to  $\partial_t \tilde{u}^a(z, t) + \frac{1}{2} P(z) \tilde{u}^a(z, t) = \tilde{g}(z, t)$ , which is a non-linear homogeneous ODE with the general solution,

$$\tilde{u}^a(z, t) = k(z) e^{-\frac{1}{2} \sigma^2 P(z) t} - \int_t^T \tilde{g}(z, t) e^{\frac{1}{2} P(z)(w-t)} dw,$$

When  $t = T$ , the integral term vanishes and, as before, the constant term becomes  $K^{-z+1}(\frac{1}{z-1} - \frac{1}{z})$ , thus the ODE system becomes,

$$\begin{aligned} \tilde{u}^a(z, t) = & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (K^{-z+1}(\frac{1}{z-1} - \frac{1}{z}) e^{\frac{1}{2} \sigma^2 P(z)(T-t)} s^z dz + \\ & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{q s_f(w)}{z-1} \left(\frac{s}{s_f(t)}\right)^z e^{\frac{1}{2} \sigma^2 P(z)(w-t)} dw dz - \\ & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{rK}{z} \left(\frac{s}{s_f(t)}\right)^z e^{\frac{1}{2} \sigma^2 P(z)(w-t)} dw dz. \end{aligned}$$

Frontczak and Schobel notice that the first integral in the above expression is actually the value of the European call option on the same underlying, strike and maturity. The residual integral terms thus represent the value of flexibility (early exercisability) in the American option and is non-negative. With this, they show that their analytical representation of the American call is equivalent to a previously determined version by Kim [16],

**Proposition 4.5.** [8] The Modified Mellin representation for the American call option is equivalent to

$$u^a(s, \tau) = u^e(z, \tau) + \int_0^\tau q s e^{-q(\tau-\xi)} \Phi(d_1(s, s_f(\xi), \tau - \xi)) d\xi - \int_0^\tau r K e^{-r(\tau-\xi)} \Phi(d_2(s, s_f(\xi), \tau - \xi)) d\xi$$

where  $\tau = T - t$  and  $s = s(\tau) \leq s_f(\tau)$ .  $d_1$  and  $d_2$  are defined as before.

By changing variables to  $\tau = T - t$  and  $\xi = \tau - w$ , the proof follows as in the previous case for the European call option. These appropriate variable changes are in the form of setting  $w = c + iy$ ,  $\zeta = \frac{1}{2}\sigma^2\xi$  and  $\alpha = \log\left(\frac{s(\tau)}{s_f(\tau-\xi)} + \zeta\left(2c + \frac{2(r-q)}{\sigma^2}\right)\right)$  for the residual terms in Modified formulation, to obtain,

$$\begin{aligned} I_1(\xi) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{q s_f(\tau - \xi)}{z - 1} \left(\frac{s}{s_f(\tau - \xi)}\right)^z e^{\frac{1}{2}\sigma^2 P(z)\xi} dz \\ &= q s_f(\tau - \xi) e^{-r\xi + \zeta c^2 + c(\alpha - 2c\zeta)} \frac{1}{2\pi} \int_0^\infty \frac{c - 1 - iy}{(c - 1)^2 + y^2} e^{-\zeta y^2 + iy\alpha} dy, \end{aligned}$$

and

$$\begin{aligned} I_2(\xi) &= \frac{rK}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{z} \left(\frac{s}{s_f(\tau - \xi)}\right)^z e^{\frac{1}{2}\sigma^2 P(z)\xi} dw dz \\ &= rK e^{-r\xi + \zeta c^2 + c(\alpha - 2c\zeta)} \frac{1}{2\pi} \int_{-\infty}^\infty \frac{c - iy}{c^2 + y^2} e^{-\zeta y^2 + iy\alpha} dy. \end{aligned}$$

The American call can be written as  $u^a(s, t) = u^e(s, t) + \int_0^\tau I_1(\xi) d\xi - \int_0^\tau I_2(\xi) d\xi$  and

the equivalence follows.

Numerically solving the residual integral terms thus suffices to obtain the value of the American call. For this, Frontczak and Schobel propose the Gauss-Laguerre Quadrature method, which is popular for  $[0, \infty)$  integral domains. For such domains, this quadrature is superior as it avoids truncation which introduces further errors. In particular, exact accuracy, with appropriate parameters, can be obtained for integrals of the form  $\int_0^\infty x^n e^{-x} dx$  [29], (exp-poly integrals from here on). The Gauss-Laguerre method takes for form

$$\int_0^\infty f(x) dx = \sum_{i=1}^n \omega_i f(\chi_i),$$

where  $\omega$  and  $\chi$  are the weights and abscissae (abscissa singular) of the quadrature respectively. For a choice of  $\omega$  and  $\chi$  that give exact accuracy for exp-poly integrals of the form  $\int_0^\infty x^{2n-1} e^{-x} dx$ , we solve for the system  $\sum_{i=1}^n \omega_i \chi_j^i = f(j)$  for all  $j = 1, \dots, 2n - 1$  and  $f(j) = \int_0^\infty x^j e^{-x} dx$ . For such exact accuracy in exp-poly integrals,  $\{\omega_i, \chi_i\}_{i=1, \dots, n}$  is known as an  $n$ -point Gauss-Laguerre Quadrature. In fact, there exist closed-form formulations of polynomials known as Laguerre polynomials [23],  $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n)$ , from which the weights ( $\omega_i = \frac{1}{(n+1)^2 (L_{n+1}(\chi_i))^2}$ ) and abscissae (roots of the  $n$ -th degree equation  $L_n(x) = 0$ ) of the quadrature can be obtained.

As such, Frontczak and Schobel, from trigonometric relations and integral reduction formulae, convert  $I_1(\xi)$  and  $I_2(\xi)$  into the form  $\int_0^\infty e^{-x} f(x) dx$  and propose the Gauss-Laguerre quadrature. These formulations are summarized as,

$$I_1(\xi) = \frac{qs f(\tau - \xi) e^{-r\xi - \zeta c^2 + c\alpha}}{2\sqrt{\pi\zeta}} \int_0^\infty e^{-(c-1)w} e^{-\frac{(\alpha-w)^2}{4\zeta}} dw,$$

and

$$I_2(\xi) = \frac{rK e^{-r\xi - \zeta c^2 + c\alpha}}{2\sqrt{\pi\zeta}} \int_0^\infty e^{-cw} e^{-\frac{(\alpha-w)^2}{4\zeta}} dw.$$



For an  $n$ -point Gauss - Laguerre quadrature, the integral terms of  $I_1(\xi)$  and  $I_2(\xi)$  can thus be estimated with  $\frac{1}{c-1} \sum_{i=1}^n \omega_i f(\frac{x_i}{c-1})$  and  $\frac{1}{c} \sum_{i=1}^n \omega_i f(\frac{x_i}{c})$  respectively, where  $f(x) = e^{-\frac{(\alpha-x)^2}{4\zeta}}$ .

The results of the Modified Mellin method, where the integral representation is solved for using the Gauss-Laguerre quadrature and the trapezoidal rule, are comparable to other methods such as the lattice method. One major drawback is in the estimation of the free boundaries (the  $s_f(t)$  terms) which Frontczak and Schobel assume to be known for all  $t$ . Hence, further research into understanding the values of these free boundaries, possibly aligning them to detailed characteristics of the underlying, such as price history, implied market data and comparables.

# 5

## Numerical Experiments

In this chapter we will conduct some numerical experiments to evaluate the performance of our generalized fixed point method. The first task is to convert our option pricing linear complementarity problem (3.24) into the form  $\mathcal{A}(u, w, b)$ . This can be achieved by assigning,

$$u = f - g \quad b = c - Ag,$$

to obtain,

$$\begin{cases} (Au - b)^T u = 0 \\ u \geq 0 \\ w = Au - b \geq 0. \end{cases} \quad (5.1)$$

Unless specified, we will perform computations for an American call option for a dividend paying underlying  $s$ , with  $\sigma = 0.30$  (volatility),  $r = 0.07$  (risk free rate),  $T = 0.5$  (contract length from  $t = 0$ ) and dividend yield of 0.03. It is common to set some convergence tolerance as a stopping rule for iterative processes. In our experiments, iterations will be halted whenever  $\|u^{k+1} - u^k\| \leq 10^{-4}$ . Further we

also need to truncate the spatial domain. It is common in literature to consider multiples of the strike price. In [4], for example, Casaban, Company and Romero consider the case when the maximum value allowable is 3 times the strike. In our experiments we consider a multiplier of 20, that is  $s_{\max} = 20K$  and  $s_{\min} = \frac{K}{20}$ . From the transformations already carried out in chapter 3, this translates to  $x_{\max} = \log 20$  and  $x_{\min} = \log 0.05$ .

All our experiments are conducted in the MATLAB environment.

## 5.1 Performance of the Generalized SYH Method

In their numerical experiments, Shi, Yang and Huang considered the case when  $\alpha = \frac{1.2+0.1\lambda\theta}{1+2\lambda\theta}$  for their original explicit fixed point method. As such, we will consider a form of the GSYH (Generalized Shi, Yang and Huang) Method by introducing an accuracy moderator in the form of  $\gamma$  taking various values. More precisely,  $\Omega = \alpha\mathbb{I}$  and  $\Gamma = \gamma\mathbb{I}$  in (3.37). Notice that when  $\gamma = 1$ , we retain Shi, Yang and Huang's original fixed point method. Prices obtained from a binomial tree with 10000 time-steps will be viewed as proxies to the true option prices. For example, an American call on an underlying with spot price,  $s_0$ , of 80 has a true price of 1.6644. Under the GSYH Method with  $\gamma = 1$ , the valuation was 1.6709. In addition, several fine-tuning experiments appear to reveal a relationship between the so-called moneyness (difference between the underlying spot price and strike. We say the option is in the money when  $s_0 - K > 0$ , out of the money when  $s_0 - K < 0$  and at the money for  $s_0 = K$ ) of the option and a  $\gamma$  that matches the true price. A further exploration of this relationship, while keeping all other parameters constant ( $r$ ,  $\sigma$  and  $q$ ), is demonstrated in Figure 5.1. From this, we propose an exponential approximation

for the relation ship between  $s$  and  $\gamma$ ,

$$\gamma = 1 + 0.000032e^{k(s-80)}.$$

We can use one of the combinations of  $\gamma$  and  $s$  from Table 5.1 to find the value of the constant  $k$ . For example, with  $s = 82.5$ , we have  $k = \frac{1}{2.5} \log\left(\frac{0.235}{0.32}\right) = -0.1234941$ . Thus, we reconsider the performance of the Generalized SYH with  $\gamma$  exhibiting exponential decay. This is illustrated in Table 5.2. The proximity to the true prices under this exponential model for  $\gamma$  appears to be smaller (hence better price estimation) in comparison to the performance of the case when  $\gamma = 1$ . Here price computations were conducted over a wider range of spot prices, expanding outside the range used to determine the formulation of  $\gamma$ . We also introduce a further formulation, for comparison purposes, given by  $\gamma = 1.00001000 + \frac{0.000001000}{\sqrt{|s_0 - K|}}$ .

## 5.2 Effect of Varying Volatility on Choice of Gamma

Perhaps the most important parameter in financial modelling and option pricing is the volatility of the underlying security,  $\sigma$  and a well researched area in stochastic volatility modelling exists (see for example [10, 13]). Here, while keeping the spot price  $s = 120$ ,  $r = 0.07$ ,  $T = 0.5$  and  $K = 100$  all constant, we investigate the discrepancy of option prices derived from the GSYH model. As always, prices from a 10 000 step Binomial Model will be a proxy for the true prices. Further, we keep  $\Omega = \alpha\mathbb{I}$  with  $\alpha = \frac{1.2+0.1\lambda\theta}{1+2\lambda\theta}$  and  $\Gamma = \gamma\mathbb{I}$  where we let  $\gamma = 0.999991561$ . The results of the experiment are presented in Table 5.3. The results highlight an enhancement in accuracy when the GSYH is implemented with an accuracy moderator, as the volatility varies. While we have kept the other parameters constant, the experiment, crucially, also kept the accuracy moderator constant at 0.999991561 and, as such, can be applied for many other options exhibiting similar characteristics in spot price, risk

Spot Price	True Price	GSYH Price for $\Gamma = \gamma \mathbb{I}$	$\gamma$
80	1.6644	1.6640	1.00003200
82.5	3.6116	3.6116	1.0000235
85	2.8488	2.8488	1.0000174
87.5	3.6116	3.6116	1.0000142
90	4.4947	4.4947	1.0000122
92.5	5.5004	5.5004	1.00000981
95	6.6302	6.6302	1.00000240

Table 5.1: Gamma constants matching true prices.

free rate, contract maturity and dividend yield but varying volatility. A variation in volatility is the most important difference across underlying securities.

s

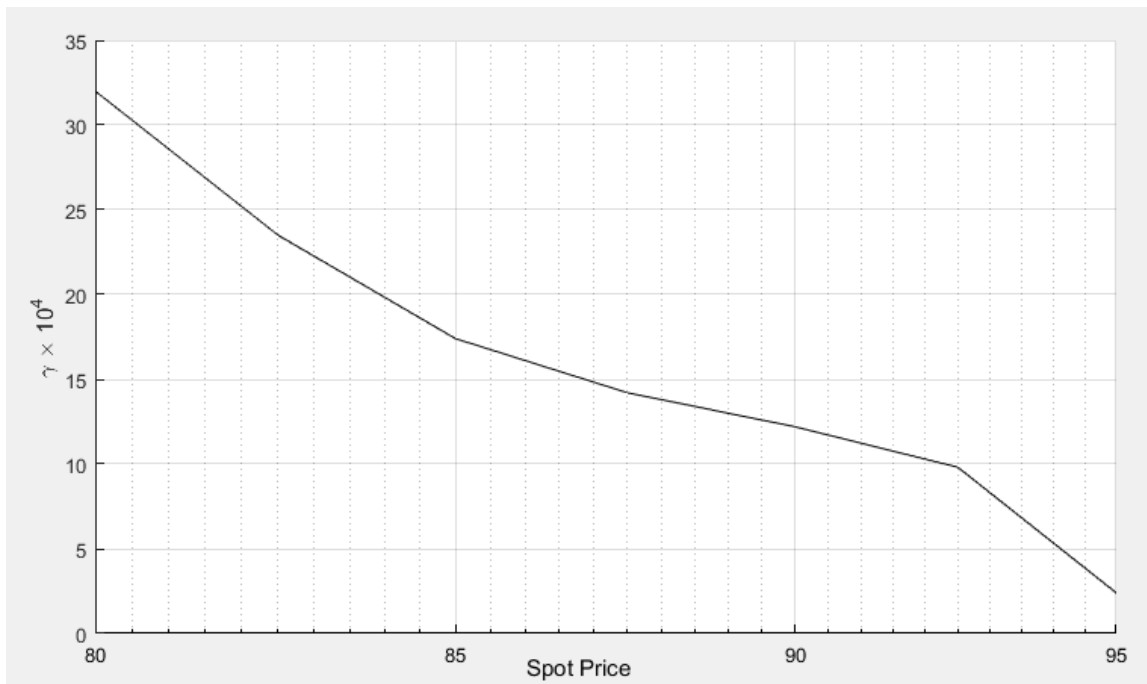


Figure 5.1: Relationship between the Gamma constant and Spot price.

Spot Price	True Price	GSYH with $\gamma = 1.00001000 + \frac{0.000001000}{\sqrt{ s_0 - K }}$	GSYG with $\gamma = 1 + 0.0000032e^{-0.1235(s-80)}$	GSYH with $\gamma = 1$
80	1.6644	1.6687	1.6640	1.6709
85	2.8488	2.8511	2.8488	2.8546
90	4.4947	4.4956	4.4961	4.5004
105	12.3236	12.3234	12.3280	12.3289
110	15.7977	15.7900	15.7953	15.7958
115	19.6123	19.6159	19.6207	19.6209
120	23.7061	23.6984	23.7026	23.7026

Table 5.2: Accuracy Moderation of the GSYH Method with varying spot prices

$\sigma$	True Price	GSYH with $\gamma = 0.999991561$	GSYH with $\gamma = 1$
0.30	23.7061	23.7061	23.7026
0.35	24.7203	24.7199	24.7156
0.40	25.8363	25.8359	25.8307
0.45	27.0241	27.0243	27.0182
0.50	28.2652	28.2652	28.2581

Table 5.3: Effect of Variation in Volatility on Specific Choice of  $\gamma$

$q$	True Price	GSYH with $\gamma = 0.999991561$	GSYH with $\gamma = 1$
0.01	24.7286	24.7314	24.7271
0.02	24.2145	24.2158	24.2120
0.03	23.7061	23.7061	23.7026
0.04	23.2054	27.2042	23.2011
0.05	22.7389	22.7366	22.7339
0.06	22.3554	22.3507	22.3483

Table 5.4: Effect of Variation in Dividend Yield on Specific Choice of  $\gamma$

### 5.3 Effect of Varying The Dividend Yield on the Choice of Gamma

Another major variation across different underlying securities is the dividend pay-out to security holders. As, such it is of interest to consider the variation in the dividend yield  $q$  while keeping other parameters constant. As in the previous sections, we will keep the spot price  $s = 120$ ,  $r = 0.07$ ,  $T = 0.5$  and  $K = 100$  all constant. Similarly, we will have  $\alpha = \frac{1.2+0.1\lambda\theta}{1+2\lambda\theta}$  and  $\gamma = 0.999991561$  as the accuracy moderator. The results of the experiment are provided in Table 5.4. Notably, an improvement is shown through the accuracy moderation as a better match to the true price is exhibited when  $\gamma = 0.999991561$ . This appears to show that the accuracy moderator is unaffected by the alteration in dividend yield over the tested range, which has strong practical importance when, for example, the true dividend yield declared is unknown.

## 6

# Conclusion

In this thesis, we have interpreted the American option (for finite maturity) pricing problem as a linear complementarity problem (LCP). Considering a general LCP, we showed its equivalence to two fixed point problems central to the work done in this thesis: Shi, Yang and Huang's fixed point method [28] (the SYH Method) and the Modulus Based Matrix Splitting Method [3] (MBMS Method). The SYH Method was characterized by some positive relaxation parameter  $\alpha$  and in their work, Shi, Yang and Huang proved that whenever  $\alpha < \frac{2}{1+4\lambda\theta}$ , then convergence is guaranteed. This proof required a long matrix multiplication, involving complicated inverses. However since the matrix considered in this case was a sparse matrix, the computation was not necessarily difficult in this scenario, since the terms the diagonal and off-diagonals concerned were constant. A general matrix can be more difficult to deal with in-order to obtain a closed form of the matrix product. As such, we suggested a simplified method which matches Shi, Yang and Huang's result and provides a more immediate insight into the convergence characteristics of the iterative system. We extended this to a more general matrix, where (off)-diagonal elements are not necessarily constant.

In option pricing, this is a useful generalization. While, we mainly considered uniform



grids in discretizing the original linear complementarity problem (with the PDE system), it is common to carry out pricing with a non-uniform grid, which leads to general tri-diagonal iterative matrices. This is mainly due to the oscillations that occur in derivative pricing near expiry or the strike price. As such special refinement is often preferred near these regions. For example, in jump diffusion models where the underlying evolves, for example, according to,  $\frac{ds_t}{s_t} = \mu dt + \sigma d\omega_t + (\eta - 1)dp$  where  $\eta$  is a random variable representing the jump size amplitude and  $dp$  represents the jump process, we obtain Partial Integro Differential Equations (PIDEs) of the form,

$$\frac{\partial u}{\partial t} + (r - \lambda\kappa)s \frac{\partial u}{\partial s} + \frac{s^2\sigma^2}{2} \frac{\partial^2 u}{\partial s^2} - (r + \lambda)u + \int_{\mathbb{R}^+} u(s\eta, t)g_\eta d\eta = 0.$$

and as such, semi-discrete problems of the form  $u_t + (G + J)u = 0$  where  $G$ , a tri-diagonal matrix, represents the derivative part of the PIDE and  $J$  the integral part, computed using some integral approximations, depending on the nature of the density function for the jumps in the model. One of these densities, and widely applied in recent years, comes under Kou's model [17]. Instead of computing the full matrix  $J$  in [32], Toivanen derives a formula which give the result of multiplying matrix  $J$  with a vector. For illustrative purposes, let us denote this as  $(Jv)_i = \zeta(v_i, t)$ . Toivanen actually further discusses that a recursive formula (while controlling rounding errors) can be used to further accelerate the results of multiplication by  $J$ . As such, since  $G$  is a tri-diagonal, it is an open area of future work to integrate the recursive formula into the  $b$  term of our general LCP  $\mathcal{A}$ . Salmi and Toivanen [26] actually compute the  $G$  for a special case, by making modifications (due to a non uniform grid) to the volatility to ensure that the off-diagonal elements  $G$  are non-positive, hence making it an  $M$ -matrix. Because of the generality of  $G$ , tractable convergence results and insights could be useful, and methods such as the Generalized SYH can be applied.

We also conducted numerical experiments in the MATLAB environment to test out

the performance of the Generalized SYH method. We considered the valuation of American calls on a dividend paying underlying. With reference to 10000 step binomial prices as proxies to the true option prices, we considered the idea of an accuracy moderator to try to improve the proximity of prices under the SYH method. In particular, we considered the parameter  $\alpha = \frac{1.2+0.1\lambda\theta}{1+2\lambda\theta}$  which Shi, Yang and Huang used in their numerical experiments. By introducing a further parameter  $\gamma$  in the generalized method, (with  $\gamma = 1$  retaining the original SYH method), accuracy was shown to be improved. While it is not immediate which choice of  $\gamma$  provides the best accuracy, we conducted numerical experiments for various spot prices and the respective  $\gamma$  values that yield prices matching (or nearly matching) the true prices, from which a closed form estimation of  $\gamma$  was derived. We emphasize that this was illustrative and it is an open area of research to investigate more accurate formulations for  $\gamma$ . However, without changing  $\gamma \neq 1$ , we also conducted further experiments to consider what happens to the value of the American call when the volatility and dividend yield vary. These are perhaps the most distinct parameters across different underlying securities, so it is useful to consider performance of the Generalized SYH for the same spot price but varying volatility and dividend yield (representing a variation in the underlying security). Our results for the specified ranges appeared to show stronger performance when  $\gamma \neq 1$  over the case when  $\gamma = 1$ . We also remark that the computational time in seconds for true prices under the binomial model ranged between 178.0000s to 190.0000s, while under the GSYH Method, prices were obtained within 0.5000s to 1.1000s. Hence the significantly larger computational time under the binomial model justifies a search for faster pricing schemes such as the GSYH Method. This thesis also discussed other capable formulations for the American option pricing problem: The Chapman - Kolmogorov Lattice Method and the Modified Mellin Transform Method for American call options.

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