Generalized Inverses of Matrices over Skew Polynomial Rings

by

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Abstract

The applications of generalized inverses of matrices appear in many fields like applied mathematics, statistics and engineering [2]. In this thesis, we discuss generalized inverses of matrices over Ore polynomial rings (also called Ore matrices).

We first introduce some necessary and sufficient conditions for the existence of $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses of Ore matrices, and give some explicit formulas for these inverses. Using $\{1\}$-inverses of Ore matrices, we present the solutions of linear systems over Ore polynomial rings. Next, we extend Roth's Theorem 1 and generalized Roth's Theorem 1 to the Ore matrices case. Furthermore, we consider the extensions of all the involutions ψ on $\mathbb{R}(x)$, and construct some necessary and sufficient conditions for ψ to be an involution on $\mathbb{R}(x)[D;\sigma,\delta]$. Finally, we obtain two different explicit formulas for $\{1,3\}$- and $\{1,4\}$-inverses of Ore matrices.

The Maple implementations of our main algorithms are presented in the Appendix.

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Contents

Notation and Terminology

Notations and terminologies which are mentioned in Chapter 2 to 5 are as follows:

$\mathbb{C}$	the field of complex numbers
δ	a σ-derivation of R, p. 3
σ	an automorphism of R, p. 4
R	a division ring, p. 4
A^*	the involution of an Ore matrix A, p. 4
$A^\dagger$	the MP-inverse of an Ore matrix A, p. 7
S^m	the set of all m-component column vectors over S, p. 10
${}^nS$	the set of all n-component row vectors over S, p. 10
$\rho_r(A)$	the row rank of an Ore matrix A, p. 10
$\rho_c(A)$	the column rank of an Ore matrix A, p. 10
$\rho(A)$	the inner rank of an Ore matrix A, p. 10
$S_r^{m\times n}$	the set of all m by n Ore matrices with the inner rank r
$\mathcal{R}(A)$	the range space of an Ore matrix A, p. 47
$\mathcal{N}(A)$	the null space of an Ore matrix A, p. 47
$\mathbb{R}(x)$	the rational function field over the real number field $\mathbb{R}$, p. 72

Chapter 1

Introduction

1.1 Skew Polynomial Rings

Skew polynomial rings were introduced by Noether and Schmeidler [46] in 1920. Jacobson described the paper [46] as one of two papers that began the study of abstract algebra. The systematical researches were done by Ore [47] in 1933. Ore [47] discussed the operations of differentiation and conjugation and extended Euclidean algorithm (Euclid's algorithm) to skew polynomial rings. Ore [47] also gave some basic theorems and specific properties of structures of skew polynomial rings. Due to the contributions of Ore, skew polynomial rings are also called Ore extensions.

As one of the most significant families of noncommutative rings, skew polynomial rings are necessary and crucial for the research of ring theory. For example, it is well-known that the Weyl algebra is a special skew polynomial ring. The properties of skew polynomial rings motivate many researchers, like Lam [38] studied the evaluation theory of skew polynomials in 1986, Osborn and Passman [48] considered the properties of derivations of skew polynomial rings in 1995 and Goodearl [28] discussed prime ideals in skew polynomial rings in 1992.

Skew polynomial rings have been applied in many fields like error-correcting codes [27], functional decompositions of polynomials [26], skew cyclic codes [14] and cryptographic applications [13]. More details can be found in [18], [19] and

[34]. Here we recall the definition and the properties of skew polynomial rings.

Definition 1.1. *([29], Definition, p.33) Let R be a ring and σ be a ring endomorphism of R. A σ-derivation on R is a map $\delta : R \longrightarrow R$ satisfying the following conditions:*

(i) $\delta(u+v) = \delta(u) + \delta(v)$;

(ii) $\delta(uv) = \sigma(u)\delta(v) + \delta(u)v$;

for all u and $v \in R$.

Definition 1.2. *([29], Definition, p.34. [47], p.481) Let σ be a ring endomorphism of a ring R and δ be a σ-derivation on R. If S satisfies the following conditions:*

(i) S is a ring, containing R as a subring;

(ii) x is an element of S;

(iii) S is a free left R-module with the basis $\{1, x, x^2, ...\}$;

(iv) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$;

then S is called a skew polynomial ring over R, denoted by $R[x; \sigma, \delta]$.

For any $a \in S \setminus \{0\}$, a can be written as $a = \sum_{i=0}^{n} r_i x^i$, where $r_i \in R$, $i \in \{0, 1, \ldots, n\}$. The $\max\{i | r_i \neq 0,\ i = 0, 1, ..., n\}$ is called the degree of a, denoted by $\deg(a)$ ([29], Definition, p.36).

For any a and $b \in S$, let σ be injective over a domain R. Then

$$\deg(ab) = \deg(a) + \deg(b). \tag{1.1}$$

The following Leibniz rule is used in lots of proofs in this thesis, and it can be found in many papers and books like [29].

Lemma 1.1. *[29] Let $S = R[x;\sigma,\delta]$. If $\sigma\delta = \delta\sigma$, then*

$$x^k r = \sum_{i=0}^{k} \binom{k}{i} \sigma^i \delta^{k-i}(r) x^i, \tag{1.2}$$

for all $r \in R$.

Throughout this thesis we fix some notation as follows:
$S = R[x;\sigma,\delta]$ is a skew polynomial ring, where R is a division ring, σ is an automorphism of R, and $\sigma\delta = \delta\sigma$ except Chapter 4. An Ore matrix is a matrix whose entries belong to S and all Ore matrices are finite and nonzero matrices.

1.2 Involutions

In order to construct all kinds of generalized inverses of Ore matrices, we introduce definitions and properties of involutions on S in this section. In Chapter 4, we consider the extensions of the involutions on $\mathbb{R}(x)$ to a special skew polynomial ring $\mathbb{R}(x)[D;\sigma,\delta]$.

The involution on S has been defined and discussed for a long time, see for example, [20], p.326.

Definition 1.3. *[20] An involution on S is an anti-automorphism $*$ with $*^2 = 1$.*

This involution can be extended to $S^{m\times n}$ in the following natural way.
Let $A = (a_{ij}) \in S^{m\times n}$ and $*$ be an involution on S. Then we extend $*$ to A by defining $A^* = (a_{ij}^*)^T \in S^{n\times m}$. Clearly, $*$ is a one-to-one correspondence from $S^{m\times n}$ to $S^{n\times m}$. Also it is easy to check that $*$ satisfies the following two properties which are extended from [7]:

Lemma 1.2. *For any $A \in S^{m\times n}$ and $B \in S^{n\times p}$,*

$$(AB)^* = B^*A^*, \ (A^*)^* = A.$$

Proof. Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \cdots & \sum_{i=1}^{n} a_{1i}b_{ip} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{mi}b_{i1} & \cdots & \sum_{i=1}^{n} a_{mi}b_{ip} \end{bmatrix}.$$

Next,

$$\begin{aligned}(AB)^* &= \begin{bmatrix} \sum_{i=1}^{n} (a_{1i}b_{i1})^* & \cdots & \sum_{i=1}^{n} (a_{mi}b_{i1})^* \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} (a_{1i}b_{ip})^* & \cdots & \sum_{i=1}^{n} (a_{mi}b_{ip})^* \end{bmatrix}, \\ &= \begin{bmatrix} \sum_{i=1}^{n} b_{i1}^* a_{1i}^* & \cdots & \sum_{i=1}^{n} b_{i1}^* a_{mi}^* \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} b_{ip}^* a_{1i}^* & \cdots & \sum_{i=1}^{n} b_{ip}^* a_{mi}^* \end{bmatrix}.\end{aligned}$$

On the other hand,

$$\begin{aligned}B^*A^* &= \begin{bmatrix} b_{11}^* & \cdots & b_{n1}^* \\ \vdots & \ddots & \vdots \\ b_{1p}^* & \cdots & b_{np}^* \end{bmatrix} \begin{bmatrix} a_{11}^* & \cdots & a_{m1}^* \\ \vdots & \ddots & \vdots \\ a_{1n}^* & \cdots & a_{mn}^* \end{bmatrix}, \\ &= \begin{bmatrix} \sum_{i=1}^{n} b_{i1}^* a_{1i}^* & \cdots & \sum_{i=1}^{n} b_{i1}^* a_{mi}^* \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} b_{ip}^* a_{1i}^* & \cdots & \sum_{i=1}^{n} b_{ip}^* a_{mi}^* \end{bmatrix}.\end{aligned}$$

So

$$(AB)^* = B^*A^*.$$

Clearly, $(A^*)^* = A$. $\square$

In the following thesis, without confusion, we use A^* as the extension of the *involution* $*$ *on* S *to* A*, where* $A \in S^{m\times n}$

1.3 Generalized Inverses

In order to analyze some integral equations, Fredholm [25] introduced the concept of a generalized inverse which was called the pseudoinverse by him in 1903. Moore ([44], [45]) defined a generalized inverse which was named the general reciprocal by him when studying the effectiveness of the reciprocal of a matrix. Moore ([44], [45]) proved the uniqueness of this generalized inverse and gave some of its properties. But the complicated notation of Moore's work resulted in little new research on generalized inverses until 1951. Bjerhammar ([9], [10], [11]) found the relationship between linear systems and generalized inverses of matrices and obtained least squares solutions of linear systems by generalized inverses. Penrose [50] proved that the general reciprocal which was defined by Moore is the unique matrix that satisfies following four equations (1)-(4) below. Since Penrose's studies made a great improvement for generalized inverses, the general reciprocal is usually called the Moore-Penrose inverse. The computations of generalized inverses of polynomial matrices can be found in [36], [35] and [37].

The applications of generalized inverses of matrices appear in many areas like network theory, mathematical programming problems and maximum likelihood estimations [52], stochastic processes and Markov chains [16] and parallel sums of matrices [23]. The $\{1,4\}$-inverses of matrices play an important part in analytical dynamics and tomography [56]. The $\{2\}$-inverses of matrices can be applied in constrained least squares estimators [31] and numerical analysis [6]. The $\{1,3\}$-inverses of matrices can help to find least squares solutions of linear systems [54]. More details of generalized inverses of matrices can be found in [7], [51] and [52].

As extensions of inverses of matrices, generalized inverses of matrices do not require that matrices are square or nonsingular [7]. For $A \in S^{m\times n}$ and $X \in S^{n\times m}$, consider the following equations ([7], p.40. [52], vii):

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA. \tag{4}$$

Definition 1.4. *([7], Definition 1.1, p.40. [52], Definition 2, p.21) For any $A \in S^{m\times n}$. Let k be a list of distinct numbers from the set $\{1,2,3,4\}$. $A\{k\}$ denotes the set of matrices $X \in S^{n\times m}$ which satisfy the equations in k. A matrix $X \in A\{k\}$ is called a $\{k\}$-inverse of A, and also denoted by $A^{(k)}$. In particular, $A\{1,2,3,4\}$ is called the MP-inverse of A, denoted by $A^\dagger$.*

More generalized inverses of matrices like group inverses and Drazin inverses of matrices can be found in [7]. We extend properties and theorems of generalized inverses of matrices over fields to S in this thesis.

1.4 The Outline of the Thesis

This thesis is organized as follows:

In Chapter 2, we introduce some definitions of column ranks, row ranks, involutions and Jacobson forms of Ore matrices.

In Chapter 3, we discuss some necessary and sufficient conditions for the existence of $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses of Ore matrices. For all Ore matrices, we can determine whether they have $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses over S. We also construct some explicit formulas for $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses of Ore matrices. Then we extend Roth's Theorem 1 and generalized Roth's Theorem 1 to Ore matrices. Next, we achieve the full rank factorizations of Ore matrices, and find that the full rank factorization of a given Ore matrix is not unique.

In Chapter 4, $\mathbb{R}(x)$ is the rational function field, σ is an automorphism of $\mathbb{R}(x)$ and ψ is an involution on $\mathbb{R}(x)$. We give some necessary and sufficient conditions for ψ to be an involution on $\mathbb{R}(x)[D]$, $\mathbb{R}(x)[D;\delta]$ and $\mathbb{R}(x)[D;\sigma]$ respectively. We also prove that every involution ψ on $\mathbb{R}(x)$ can not be extended to be an involution on $\mathbb{R}(x)[D;\sigma,\delta]$.

In Chapter 5, we discuss solutions of linear systems over S by using $\{1\}$-inverses of Ore matrices. Finally, we give two different explicit formulas for $\{1,3\}$- and $\{1,4\}$-inverses of Ore matrices over S.

The Maple implementations of our main algorithms are given in the Appendix.

Chapter 2

Ore Matrices: Properties

In this chapter, we outline and discuss some properties of Ore matrices. In Section 2.1, we introduce the definitions of column ranks, row ranks and inner ranks of Ore matrices. In Section 2.2, we discuss GCRDs, GCLDs, LCRMs and LCLMs of pairs of elements in S. In Section 2.3, we consider the Jacobson forms of Ore matrices and give a algorithm to get the Jacobson form of a given Ore matrix. In section 2.4, we discuss some necessary properties of row ranks, column ranks and inner ranks of Ore matrices. More details of properties of Ore matrices can be found in [20].

2.1 Column Ranks, Row Ranks and Inner Ranks

In this section, we introduce column ranks, row ranks and inner ranks of Ore matrices. For matrices over fields, ranks play a crucial role. As S is a noncommutative ring, we need to define left and right independence, and thus define row ranks, column ranks and inner ranks of Ore matrices.

Proposition 2.1. *([29], Theorem 2.8, p.39) $S = R[x; \sigma, \delta]$ is a left and right principal ideal domain.*

Let M be a left S-module and $m_1, \dots, m_n \in M$. If for any $s_1, \dots, s_n \in S$

$$\sum_{i=1}^{n} s_i m_i = 0,$$

implies

$$s_1 = \cdots = s_n = 0,$$

then $m_1, \dots, m_n \in M$ are called left linearly independent over S. For a right S-model, the right linearly independence can be defined similarly. A left (right) S-module M which is left (right) generated by a left (right) linearly independent set $(m_i)_{i \in I}$ is called free. If $|I| = k$, then the rank of the module M is equal to k ([20], p.2). Since S is a PID, the rank of the module M is unique ([20], p.2, p.3).

Definition 2.1. *([20], p.80) For any $A \in S^{m \times n}$, the column rank of A is the maximum number of right linearly independent columns of A over S, denoted by $\rho_c(A)$; the row rank of A is the maximum number of left linearly independent rows of A over S, denoted by $\rho_r(A)$.*

Proposition 2.2. *([20], Proposition 5.4.2, p.283) For any $A \in S^{m \times n}$, the rank of the submodule of ${}^nS$ which is left spanned by the m rows of A is called the row rank of A, denoted by $\rho_r(A)$; the rank of the submodule of S^m which is right spanned by the n columns of A is called the column rank of A, denoted by $\rho_c(A)$.*

Definition 2.2. *([20], p.3) For any $A \in S^{m \times n}$, let $A = UV$, where $U \in S^{m \times r}$ and $V \in S^{r \times n}$. The inner rank of A is the least possible value of r, denoted by $\rho(A)$.*

In Section 2.4, we prove that for any $A \in S^{m \times n}$,

$$\rho_r(A) = \rho_c(A) = \rho(A).$$

2.2 GCRDs, GCLDs, LCRMs and LCLMs

Since S is a PID, there exist GCRDs, GCLDs, LCRMs and LCLMs for all pairs of elements in S, which help us to transform all Ore matrices into diagonal matrices.

Definition 2.3. *([20], p.54, p.154. [22], definition 2.1, p.102) For any u and $v \in S$, the monic $d \in S$ that satisfies the following conditions:*

(i) there exist s and $t \in S$ such that $u = ds$ and $v = dt$,

(ii) if $r \in S$ also satisfies (i), then there is a $r_1 \in S$ such that $d = rr_1$,

is called the greatest common left divisor of u and v, denoted by $\mathrm{gcld}(u, v)$, in particular, $\mathrm{gcld}(u, 0) = u$ and $\mathrm{gcld}(0, 0)$ doesn't exist over S.

The gcrd can be defined similarly.

Definition 2.4. *([20], p.54, p.117. [22], definition 2.1, p.102) Let u and $v \in S \setminus \{0\}$. The monic $l \in S$ that satisfies the following conditions:*

(i) there exist s and $t \in S$ such that $l = su = tv$,

(ii) if $r \in S$ also satisfies (i), then there is a $r_1 \in S$ such that $r = r_1 l$,

is called the least common left multiple of u and v, denoted by $\mathrm{lclm}(u, v)$.

The lcrm can be defined similarly. We refer to [20] and [22] for more details about gcrds, gclds, lcrms and lclms.

2.3 Jacobson Forms

Jacobson defined and discussed a kind of canonical form of matrices over skew polynomial rings [34] in 1943. These canonical forms play a similar role as Smith forms of matrices over polynomial rings. Now people call these canonical forms Jacobson forms. The history and more details of the Jacobson forms of Ore matrices can be found in [34] and [18]. The Jacobson forms of Ore matrices play the most important part in the whole thesis. For all Ore matrices, we can determine whether they have $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses over S by their Jacobson forms.

Theorem 2.1. *([20], Theorem 1.4.7, p.80) For any $A \in S^{m\times n}$, there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that*

$$PAQ = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.1}$$

where $d_i \in S \setminus \{0\}$ and d_{i-1} is a left divisor of d_i, $i \in \{2,\ 3, \ldots, r\}$. The right side of (2.1) is called the Jacobson form of A.

Proof. The proof of this theorem is well-known. Here we give one proof which starts from a 2 by 2 Ore matrix and then use mathematical induction to finish the proof. We also use this method to implement our algorithm to get the Jacobson form of a given Ore matrix. Choose $A \in S^{2\times 2}$ and write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If $a_{11} = a_{12} = a_{21} = 0$, then the Jacobson form of A can be got by interchanging columns and rows. So, we consider other cases. If $a_{11} = 0$, then we can use elementary transformations to get the new $a_{11} \neq 0$. Now, we can assume that $a_{11} \neq 0$. When $a_{21} \neq 0$, since S is a PID, there exists a $g_1 \in S$ such that $\mathrm{gcrd}(a_{11},\ a_{21}) = g_1$. There also exist $s_1,\ t_1,\ k_1$ and $l_1 \in S$ such that

$$s_1 a_{11} + t_1 a_{21} = g_1,$$

and

$$k_1 a_{11} = l_1 a_{21} = \operatorname{lclm}(a_{11}, a_{21}).$$

Let

$$E_1 = \begin{bmatrix} s_1 & t_1 \\ k_1 & -l_1 \end{bmatrix}.$$

Then

$$E_1 A = \begin{bmatrix} g_1 & b_{12} \\ 0 & b_{22} \end{bmatrix}, \tag{2.2}$$

where b_{12} and b_{22} are corresponding entries of $E_1 A$. If $a_{21} = 0$, then $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the required matrix $\begin{bmatrix} g_1 & b_{12} \\ 0 & b_{22} \end{bmatrix}$. When $b_{12} \neq 0$, write

$$g_2 = \operatorname{gcld}(g_1, b_{12}), \tag{2.3}$$

$$F_1 = \begin{bmatrix} s_2 & k_2 \\ t_2 & -l_2 \end{bmatrix},$$

where

$$g_1 s_2 + b_{12} t_2 = g_2,$$

and

$$g_1 k_2 = b_{12} l_2 = \operatorname{lcrm}(g_1, b_{12}).$$

Next,

$$E_1 A F_1 = \begin{bmatrix} g_2 & 0 \\ c_{21} & c_{22} \end{bmatrix},$$

where c_{21} and c_{22} are corresponding entries of $E_1 A F_1$. By ([20], p.80), E_1 and F_1

are invertible matrices over S. If $b_{12} = 0$, then $\begin{bmatrix} g_1 & b_{12} \\ 0 & b_{22} \end{bmatrix}$ is the required matrix $\begin{bmatrix} g_2 & 0 \\ c_{21} & c_{22} \end{bmatrix}$. Similarly, let

$$E_2 = \begin{bmatrix} s_3 & t_3 \\ k_3 & -l_3 \end{bmatrix},$$

where

$$s_3 g_2 + t_3 c_{21} = g_3,$$

and

$$k_3 g_2 = l_3 c_{21} = \operatorname{lclm}(g_2, c_{21}).$$

Then

$$E_2 E_1 A F_1 = \begin{bmatrix} g_3 & d_{12} \\ 0 & d_{22} \end{bmatrix},$$

where d_{12} and d_{22} are the corresponding entries of $E_2 E_1 A F_1$. Keep doing this,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} g_1 & b_{12} \\ 0 & b_{22} \end{bmatrix} \to \begin{bmatrix} g_2 & 0 \\ c_{21} & c_{22} \end{bmatrix} \to \cdots \to \begin{bmatrix} g_n & 0 \\ * & * \end{bmatrix}.$$

Clearly,

$$\deg(g_1) \geq \deg(g_2) \geq \cdots \geq \deg(g_n). \tag{2.4}$$

If there exists an equality in (2.4), then we assume that $\deg(g_1) = \deg(g_2)$. By (2.3), there exists a $p_1 \in S$ such that

$$g_2 p_1 = g_1.$$

From (1.1),

$$\deg(p_1) = 0, \;\; p_1 \in R.$$

As g_1 and g_2 are monic, $p_1 = 1$ by (1.2). Next, there is a $p \in S$ such that $b_{12} = g_1 p$. It follows from (2.2) that

$$E_1 A = \begin{bmatrix} g_1 & g_1 p \\ 0 & b_{22} \end{bmatrix}.$$

Furthermore,

$$E_1 A \begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ 0 & b_{22} \end{bmatrix}.$$

If $\deg(g_i) = \deg(g_{i+1})$ never appears in (2.4), then $\deg(g_n)$ must be 0 (i.e., $g_n = 1$). In this case, A is transformed into

$$\begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}.$$

Similarly, for $m \times n$ matrix, there exist invertible matrices $P' \in S^{m\times m}$ and $Q' \in S^{n\times n}$ such that

$$P'AQ' = \begin{bmatrix} e_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & e_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & e_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $e_i \in S \setminus \{0\}$, but e_{i-1} is not a left divisor of e_i, $i \in \{2,\ 3, \ldots, r\}$.

Next, we finish the proof by the following steps.

We also discuss the 2 by 2 Ore matrix. Choose $A \in S^{2\times 2}$ and write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then there are invertible matrices $T_1 \in S^{2\times 2}$ and $U_1 \in S^{2\times 2}$ such that

$$T_1 A U_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

If a_1 or $a_2 = 0$, then the proof is done. We assume that a_1 and $a_2 \in S \setminus \{0\}$. Clearly,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 0 & a_2 \end{bmatrix}. \tag{2.5}$$

There also exist invertible matrices $T_2 \in S^{2\times 2}$ and $U_2 \in S^{2\times 2}$ such that

$$T_2 \begin{bmatrix} a_1 & a_2 \\ 0 & a_2 \end{bmatrix} U_2 = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}.$$

Clearly, there are $x_1,\ x_2, \ldots, x_p$ and $y_1,\ y_2, \ldots, y_q \in S$ such that

$$x_1 x_2 \cdots x_p b_1 y_1 y_2 \cdots y_q = a_1, \tag{2.6}$$

and

$$x_1 x_2 \cdots x_p b_1 y_1 y_2 \cdots y_{q-1} = \text{gcld}(a_1,\ a_2). \tag{2.7}$$

Let

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 0 & b_2 \end{bmatrix}. \tag{2.8}$$

Then there exist invertible matrices $T_3 \in S^{2\times 2}$ and $U_3 \in S^{2\times 2}$ such that

$$T_3 \begin{bmatrix} b_1 & b_2 \\ 0 & b_2 \end{bmatrix} U_3 = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

Clearly, there are $u_1,\ u_2,\ldots,u_g$ and $t_1,\ t_2,\ldots,t_h \in S$ such that

$$u_1u_2\cdots u_g c_1 t_1 t_2\cdots t_h = b_1.$$

Keep doing this,

$$A \to \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \to \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \to \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \to \cdots \to \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix},$$

$$\deg(a_1) \geq \deg(b_1) \geq \cdots \geq \deg(w_1). \tag{2.9}$$

If there exists an equality in (2.9), then we assume that $\deg(a_1) = \deg(b_1)$. By (2.6) and (1.1),

$$\deg(x_1) = \cdots = \deg(x_p) = \deg(y_1) = \cdots = \deg(y_q) = 0.$$

As $x_1x_2\cdots x_p b_1 y_1 y_2\cdots y_{q-1}$ and a_1 are monic,

$$y_q = 1,$$

by (2.6) and (1.2). Similarly,

$$x_1 = x_2 = \cdots = x_p = y_1 = y_2 = \cdots = y_{q-1} = 1.$$

From (2.6) and (2.7),

$$b_1 = a_1 = \text{gcld}(a_1,\ a_2).$$

So a_1 is a left divisor of a_2.

If there doesn't exist an equality in (2.9), then

$$\deg(w_1) = 0.$$

Thus, $w_1 = 1$ which must be a left divisor of w_2.

Similarly, for any $A \in S^{m\times n}$, there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that

$$PAQ = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $d_i \in S \setminus \{0\}$ and d_{i-1} is a left divisor of d_i, $i \in \{2,\ 3, \ldots, r\}$.

If we choose

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ a_2 & a_2 \end{bmatrix}, \tag{2.10}$$

in (2.5), and

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ b_2 & b_2 \end{bmatrix}, \tag{2.11}$$

in (2.8), then d_{i-1} can be a right divisor of d_i. □

The Maple implementations of the above algorithm to get the Jacobson forms of Ore matrices are given in the Appendix.

2.4 Properties of Inner Ranks

In this section, the main result is that the row rank, the column rank and the inner rank are all equal for a given Ore matrix [20], which is applied in a lot of proofs in this thesis.

Proposition 2.3. *([20], p.284) For any* $A \in S^{m\times n}$, $B \in S^{n\times q}$,

$$\rho_c(AB) \leq \rho_c(B), \quad \rho_r(AB) \leq \rho_r(A). \tag{2.12}$$

Proof. Let

$$A = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \alpha_1 B \\ \vdots \\ \alpha_m B \end{bmatrix}.$$

If

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m = \mathbf{0},$$

where $k_i \in S$, $i \in \{1, 2, \ldots, m\}$, then

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m)B = \mathbf{0},$$

$$k_1(\alpha_1 B) + k_2(\alpha_2 B) + \cdots + k_m(\alpha_m B) = \mathbf{0}.$$

Now, if $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_q}$ are left linearly dependent over S, then $\alpha_{i_1}B, \alpha_{i_2}B, \ldots, \alpha_{i_q}B$ are also left linearly dependent over S, where $\{i_1, i_2, \ldots, i_q\}$ is an index of $\{1, 2, \ldots, m\}$. So

$$\rho_r(AB) \leq \rho_r(A).$$

Similarly, $\rho_c(AB) \leq \rho_c(B)$. □

Proposition 2.4. *([20], Proposition 5.4.3, p.284) For any* $A \in S^{m\times n}$,

$$\rho_c(A) \leq \min\{m, n\}, \quad \rho_r(A) \leq \min\{m, n\}. \tag{2.13}$$

Proof. Write

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

When $m > n$, let W be the submodule of ${}^nS$ which is left spanned by m rows of A. Then

$$W \subseteq M = \{k_1e_1 + k_2e_2 + \cdots + k_ne_n |\ k_i \in S\},$$

where

$$e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1).$$

Next,

$$\operatorname{rank}(W) \leq \operatorname{rank}(M) = n,$$

and $\rho_r(A) \leq n$. When $m \leq n$, since there are only m rows, clearly,

$$\rho_r(A) \leq m.$$

So

$$\rho_r(A) \leq \min\{m, n\}.$$

□

By Proposition 2.3 and Proposition 2.4, we can prove Proposition 2.5.

Proposition 2.5. *([20], Theorem 1.4.7, p.80. [20], Proposition 5.4.4, p.285) For any $A \in S^{m\times n}$,*

$$\rho_r(A) = \rho_c(A) = \rho(A). \tag{2.14}$$

Proof. Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

and assume that E_i is the product of elementary matrices which are used to interchange two columns. By Theorem 2.1, there exist invertible matrices $P_1 \in S^{m\times m}$ and $E_1 \in S^{n\times n}$ such that

$$P_1AE_1 = \begin{bmatrix} a_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_1 & B_1 \\ \mathbf{0} & B \end{bmatrix},$$

where $B_1 \in {}^{n-1}S$ and $B \in S^{(m-1)\times(n-1)}$. Similarly, there exist invertible matrices $P_2 \in S^{(m-1)\times(m-1)}$ and $E_2 \in S^{(n-1)\times(n-1)}$ such that

$$P_2BE_2 = \begin{bmatrix} b_2 & c_{23} & \cdots & c_{2n} \\ 0 & c_{33} & \cdots & c_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{m3} & \cdots & c_{mn} \end{bmatrix} = \begin{bmatrix} b_2 & C_1 \\ \mathbf{0} & C \end{bmatrix},$$

where $C_1 \in {}^{n-2}S$ and $C \in S^{(m-2)\times(n-2)}$. Then

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix} P_1AE_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & E_2 \end{bmatrix} = \begin{bmatrix} a_1 & * & * & \cdots & * \\ 0 & b_2 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix}.$$

Keep doing this, there exist invertible matrices $P \in S^{m\times m}$ and $E \in S^{n\times n}$ such that

$$PAE = \begin{bmatrix} a_1 & * & \cdots & * & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & * & \cdots & * \\ 0 & \cdots & 0 & w_r & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where E is the product of elementary matrices which are used to interchange two columns. Let

$$PAE = (\beta_1,\ \beta_2, \ldots, \beta_r,\ \beta_{r+1}, \ldots, \beta_n).$$

Clearly,

$$\rho_c(\beta_1,\ \beta_2, \ldots, \beta_r) = r,$$

then $\rho_c(PAE) \geq r$. Since E just interchanges columns, E doesn't change the column rank of A. So $\rho_c(PA) \geq r$. By (2.12),

$$\rho_c(A) \geq r.$$

If $r < m$, let

$$PAE = \begin{bmatrix} A_r \\ \mathbf{0} \end{bmatrix},$$

where $A_r \in S^{r\times n}$. Then

$$A = P^{-1} \begin{bmatrix} A_r \\ \mathbf{0} \end{bmatrix} E^{-1}.$$

Let

$$P^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

where $\begin{bmatrix} P_1 \\ P_3 \end{bmatrix} \in S^{m\times r}$. Then

$$\begin{aligned} A &= \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} A_r \\ \mathbf{0} \end{bmatrix} E^{-1} = \begin{bmatrix} P_1 A_r \\ P_3 A_r \end{bmatrix} E^{-1} \\ &= \begin{bmatrix} P_1 \\ P_3 \end{bmatrix} A_r E^{-1}. \end{aligned}$$

Write

$$W = A_r E^{-1}.$$

Next,

$$W \in S^{r\times n},$$

$$\rho(A) \leq r.$$

So $\rho_c(A) \geq \rho(A)$. When $r = m$. Clearly, $\rho(A) \leq r$. Thus

$$\rho_c(A) \geq \rho(A).$$

On the other hand, let $\rho(A) = r_1$. Then there exist $M \in S^{m\times r_1}$ and $N \in S^{r_1\times n}$ such that $A = MN$. From (2.12) and (2.13),

$$\rho_c(A) = \rho_c(MN) \leq \rho_c(N) \leq r_1,$$

thus $\rho_c(A) \leq \rho(A)$. Finally,

$$\rho_c(A) = \rho(A).$$

Similarly,

$$\rho_r(A) = \rho(A).$$

□

Next, we give the following Proposition 2.6 which helps us to finish a lot of proofs in this thesis.

Proposition 2.6. *([20], Proposition 5.4.4, Equation (3), p.285) For any* $A \in S^{m\times n}$ *and* $B \in S^{n\times p}$*,*

$$\rho(AB) \leq \min\{\rho(A), \rho(B)\}. \tag{2.15}$$

Proof. By Proposition 2.3

$$\rho_r(AB) \leq \rho_r(A),$$

$$\rho_c(AB) \leq \rho_c(B).$$

From (2.14),

$$\rho_r(AB) \leq \rho_r(B).$$

So

$$\rho_r(AB) \leq \min\{\rho_r(A), \rho_r(B)\}.$$

□

Proposition 2.7. *([20], p.285) If* $P \in S^{n\times n}$ *is invertible over* S*, then*

$$\rho(P) = n.$$

Proof. Since P is invertible, there exists a $P^{-1} \in S^{n\times n}$ such that $PP^{-1} = I$. Then

$$\rho(P) \geq \rho(I) = n.$$

By (2.13), $\rho(P) \leq n$. So $\rho(P) = n$. □

Proposition 2.8. *([20], p.285) If* $P \in S^{m\times m}$ *and* $Q \in S^{n\times n}$ *are both invertible over* S*, then for any* $A \in S^{m\times n}$*,*

$$\rho(PA) = \rho(A) = \rho(AQ).$$

Proof. Evidently, $\rho(PA) \leq \rho(A)$ and $A = P^{-1}PA$. Then

$$\rho(A) = \rho(P^{-1}PA) \leq \rho(PA).$$

So

$$\rho(A) = \rho(PA).$$

Similarly,

$$\rho(A) = \rho(AQ).$$

□

For any $A \in \mathbb{C}^{m\times n}$, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$ which is not true for Ore matrices. But we can extend $\mathrm{rank}(A) = \mathrm{rank}(A^*)$ in [7] to Ore matrices by the following lemma.

Lemma 2.1. *For any $A \in S^{m\times n}$,*

$$\rho(A) = \rho(A^*).$$

Proof. Let

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Then

$$A^* = \begin{bmatrix} \alpha_1^* & \alpha_2^* & \cdots & \alpha_n^* \end{bmatrix}.$$

Assume that $\rho(A) = r$ and $\alpha_1, \ldots, \alpha_r$ are left linearly independent over S. If

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r = \mathbf{0},$$

where $k_i \in S$, $i \in \{1, 2, \ldots, r\}$, then

$$k_1 = k_2 = \cdots = k_r = 0.$$

Let

$$\alpha_1^* l_1 + \alpha_2^* l_2 + \cdots + \alpha_r^* l_r = \mathbf{0},$$

where $l_i \in S$, $i \in \{1, 2, \ldots, r\}$. Then

$$l_1^* \alpha_1 + l_2^* \alpha_2 + \cdots + l_r^* \alpha_r = \mathbf{0},$$

and

$$l_1^* = l_2^* = \cdots = l_r^* = 0.$$

So $l_1 = l_2 = \cdots = l_r = 0$ and $\rho(A^*) = \rho_c(A^*) \geq r = \rho(A)$. Similarly, $\rho(A) \geq \rho(A^*)$. Therefore

$$\rho(A) = \rho(A^*).$$

$\square$

Chapter 3

Ore Matrices: Generalized Inverses

In this chapter, we discuss some necessary and sufficient conditions for the existence of $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses of Ore matrices. For all Ore matrices, we can determine whether they have $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses. We also give some explicit formulas for $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$- and MP-inverses of Ore matrices. Then we extend Roth's Theorem 1 and generalized Roth's Theorem 1 to Ore matrices. Next, we achieve the full rank factorizations of Ore matrices, and find that the full rank factorization of a given Ore matrix is not unique. Over the field $\mathbb{C}$, $A \in \mathbb{C}^{n\times n}$ is invertible if and only if $\text{rank}(A) = n$, which is wrong for Ore matrices. So we could not determine whether an Ore matrix is invertible by its inner rank, which brings us many difficulties in the proofs of generalized inverses over S. The Maple implementations of most above results can be found in the Appendix.

3.1 $\{1\}$-Inverses

Since S is not a field, it is impossible to transform all Ore matrices into diagonal matrices whose all main diagonal entries are 0 or 1. We give some necessary and sufficient conditions for the existence of $\{1\}$-inverses of Ore matrices and some explicit formulas for $\{1\}$-inverses over S, which help us to do further researches of $\{1,2\}$-inverses in Section 3.5, $\{1,3\}$- and $\{1,4\}$-inverses in Section 3.7, MP-inverses in Section 3.8, Roth's Theorem 1 in Section 3.2 and generalized Roth's Theorem 1 in Section 3.3.

Definition 3.1. *[7] Let $A \in S^{n\times n}$. A is called idempotent over S if $A^2 = A$.*

Lemma 3.1. *For any $A \in S^{m\times n}$, if A has a MP-inverse $A^\dagger$ over S, then $A^\dagger$ is unique.*

Proof. It is similar to the proof of Exercise 1.1 in [7]. □

Lemma 3.2. *For any $A \in S_r^{m\times n}$ (i.e., $A \in S^{m\times n}$ and $\rho(A) = r$),*

(a) $(A^{(1)})^* \in A^*\{1\}$.

(b) If A has the inverse over S, then $A^{(1)} = A^{-1}$.

(c) $\rho(A) \leq \rho(A^{(1)})$.

(d) If P and Q both have inverses over S, then $Q^{-1}A^{(1)}P^{-1}$ is a $\{1\}$-inverse of PAQ.

(e) $\rho\left(AA^{(1)}\right) = \rho\left(A^{(1)}A\right) = \rho\left(A\right)$.

Proof. It is similar to the proof of Lemma 1.1 in [7]. □

Lemma 3.3. *For any $A \in S^{m\times n}$, let P and $Q \in A\{1\}$. If $X = PAQ$, then $X \in A\{1,2\}$.*

Proof. It is similar to the proof of Lemma 1.3 in [7]. □

Theorem 3.1. *For any $A \in S^{m\times n}$, if $AXA = A$, then $XAX = X \Longleftrightarrow \rho(X) = \rho(A)$.*

Proof. It is similar to the proof of Theorem 1.2 in [7]. □

Theorem 3.2. *For any $A \in S^{m\times n}$, if $XAX = X$, then $AXA = A \Longleftrightarrow \rho(X) = \rho(A)$.*

Proof. It is similar to the proof of Theorem 1.2 in [7]. □

The following Lemma 3.4 is used in a lot of proofs of generalized inverses of Ore matrices. The original proof applies the result that if $A \in \mathbb{C}^{n\times n}$, then A is invertible if and only if rank$A$=n, which is wrong for Ore matrices.

Lemma 3.4. *For any $A \in S^{m\times n}$, if $A^{(1)}$ exists over S, then*

$$AA^{(1)} = I_m \Longleftrightarrow \rho(A) = m, \tag{3.1}$$

$$A^{(1)}A = I_n \Longleftrightarrow \rho(A) = n. \tag{3.2}$$

Proof. This lemma is obtained by extending Lemma 1.2 in [7] to Ore matrices. We give the proof of (3.2).
($\Longrightarrow$) Clearly,

$$\rho(A) \geq \rho(A^{(1)}A).$$

Since $AA^{(1)}A = A$,

$$\rho(A) \leq \rho(A^{(1)}A).$$

Then

$$\rho(A) = \rho(A^{(1)}A) = \rho(I_n) = \rho_c(I_n) = n.$$

($\Longleftarrow$) Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

and

$$A^{(1)}A = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}.$$

Since

$$AA^{(1)}A = A,$$

the multiplication of the first row of A and the first column of $A^{(1)}A$ implies that

$$a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1} = a_{11},$$

the multiplication of the second row of A and the first column of $A^{(1)}A$ implies that

$$a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + \cdots + a_{2n}b_{n1} = a_{21},$$

the multiplication of the third row of A and the first column of $A^{(1)}A$ implies that

$$a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + \cdots + a_{3n}b_{n1} = a_{31}, \cdots$$

the multiplication of the m-th row of A and the first column of $A^{(1)}A$ implies that

$$a_{m1}b_{11} + a_{m2}b_{21} + a_{m3}b_{31} + \cdots + a_{mn}b_{n1} = a_{m1}.$$

So

$$\begin{cases} a_{11}(b_{11}-1) + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1} = 0, \\ a_{21}(b_{11}-1) + a_{22}b_{21} + a_{23}b_{31} + \cdots + a_{2n}b_{n1} = 0, \\ a_{31}(b_{11}-1) + a_{32}b_{21} + a_{33}b_{31} + \cdots + a_{3n}b_{n1} = 0, \\ \quad\vdots \qquad\qquad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}(b_{11}-1) + a_{m2}b_{21} + a_{m3}b_{31} + \cdots + a_{mn}b_{n1} = 0. \end{cases}$$

Let

$$A = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Then

$$\alpha_1(b_{11}-1) + \alpha_2 b_{21} + \cdots + \alpha_n b_{n1} = \mathbf{0}.$$

Due to $\rho_c(A) = n$,

$$b_{11} = 1,\ b_{21} = b_{31} = \cdots = b_{n1} = 0.$$

Similarly,

$$A^{(1)}A = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = I_n.$$

$AA^{(1)} = I_m \Longleftrightarrow \rho(A) = m$ can be proved by the same method. □

Proposition 3.1. *For any $A \in S^{m\times n}$, let $A = PBQ$, where $P \in S_r^{m\times r}$ and $Q \in S_v^{v\times n}$. If $P^{(1)}$ and $Q^{(1)}$ both exist over S, then $\rho(A) = \rho(B)$.*

Proof. Clearly, $\rho(A) \leq \rho(B)$. By Lemma 3.4, $P^{(1)}AQ^{(1)} = P^{(1)}PBQQ^{(1)} = IBI = B$ and $\rho(A) \geq \rho(B)$. So $\rho(A) = \rho(B)$. This proposition is obtained by extending Exercise 1.7 in [7] to Ore matrices. □

Proposition 3.2. *For any $A \in S^{m\times n}$, if the Jacobson form of A is*

$$\begin{bmatrix} I_m & \mathbf{0} \end{bmatrix},$$

i.e., there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that

$$PAQ = \begin{bmatrix} I_m & \mathbf{0} \end{bmatrix},$$

then $A\{1\} = A\{1, 2, 3\}$.

Proof. Clearly, $A\{1, 2, 3\} \subseteq A\{1\}$. By Proposition 2.8,

$$\rho(A) = m.$$

Then

$$AA^{(1)} = I_m,$$

by Lemma 3.4. Next,

$$A^{(1)}AA^{(1)} = A^{(1)},$$

$$(AA^{(1)})^* = I_m^* = I_m = AA^{(1)}.$$

So $A\{1\} \subseteq A\{1,2,3\}$. $\square$

Similarly, we can get the following Proposition 3.3.

Proposition 3.3. *For any $A \in S^{m\times n}$, if the Jacobson form of A is*

$$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix},$$

then $A\{1\} = A\{1,2,4\}$.

In the following theorems, we find some necessary and sufficient conditions for the existence of $\{1\}$-inverses of Ore matrices. For all Ore matrices, we can determine whether they have $\{1\}$-inverses over S. Furthermore, if they have $\{1\}$-inverses, then we can give all their $\{1\}$-inverses over S.

Theorem 3.3. *For any $A \in S^{m\times n}$, A has a $\{1\}$-inverse over $S \iff$ there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that*

$$A = P \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Q,$$

where $d_i \in R \setminus \{0\}$, $i \in \{1,\ 2, \ldots, r\}$.

Proof. This theorem is obtained by extending Theorem 1 in [55] (p.514) or Theorem 2 in [12] (p.492) to Ore matrices.

($\Longrightarrow$) By Theorem 2.1, there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that

$$A = P \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Q = P \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q,$$

where

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_r \end{bmatrix},$$

and $d_i \in S \setminus \{0\}$. Let X be a $\{1\}$-inverse of A and $W = QXP$. Then

$$X = Q^{-1}WP^{-1}.$$

Write

$$W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}.$$

Since $AXA = A$,

$$P \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} QQ^{-1} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} P^{-1}P \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q = P \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q.$$

Next,

$$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Further,

$$DW_1D = D.$$

Let

$$W_1 = \begin{bmatrix} w_{11} & \cdots & w_{1r} \\ \vdots & \ddots & \vdots \\ w_{r1} & \cdots & w_{rr} \end{bmatrix}.$$

Then

$$d_i w_{ii} d_i = d_i, \ i \in \{1, 2, \dots r\},$$

$$d_i w_{ij} d_j = 0, \ i, \ j \in \{1, 2, \dots r\}, \ i \neq j.$$

By (1.1),

$$\deg(d_i w_{ii} d_i) = \deg(d_i) + \deg(w_{ii}) + \deg(d_i) = \deg(d_i),$$

$$\deg(d_i) = \deg(w_{ii}) = 0.$$

In other words, d_i and $w_{ii} \in R$, where $i \in \{1, 2, \dots r\}$. Clearly,

$$w_{ii} = d_i^{-1}, \ i \in \{1, 2, \dots r\}.$$

As S is a domain and $d_i w_{ij} d_j = 0$,

$$w_{ij} = 0, \ i, \ j \in \{1, 2, \dots r\}, \ i \neq j.$$

So

$$X = Q^{-1} \begin{bmatrix} D^{-1} & W_2 \\ W_3 & W_4 \end{bmatrix} P^{-1}.$$

($\Longleftarrow$) Clearly,

$$Q^{-1} \begin{bmatrix} D^{-1} & W_2 \\ W_3 & W_4 \end{bmatrix} P^{-1},$$

is a {1}-inverse of A. $\square$

The Maple implementations of the above method to find $\{1\}$-inverses of Ore matrices are given in the Appendix. From the proof of Theorem 3.3, it is easy to get the following theorem.

Theorem 3.4. *For any $A \in S^{m\times n}$, A has a $\{1\}$-inverse over $S \iff$ there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that*

$$A = P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q,$$

in which case if X is a $\{1\}$-inverse of A over S, then X can be written as

$$Q^{-1} \begin{bmatrix} I_r & W_2 \\ W_3 & W_4 \end{bmatrix} P^{-1},$$

where $W_2 \in S^{r\times(m-r)}$, $W_3 \in S^{(n-r)\times r}$ and $W_4 \in S^{(n-r)\times(m-r)}$ are arbitrary Ore matrices.

3.2 Roth's Theorems

Two theorems which are called Roth's Theorems now were given by Roth [53] to solve Sylvester's equation in 1952. Roth's Theorems can be applied in the computations of characteristic polynomials of matrices, common eigenvalue problems and inertia methods [21] and Embry's Theorem, solving Lyapunov equation and hyperinvariant subspace problems [8].

Flanders and Wimmer [24] gave alternate proofs of Roth's Theorems by using linear transformations in 1977. Baksalary and Kala [3] provided an intelligent necessary and sufficient condition for Roth's Theorem 1 by using $\{1\}$-inverses of matrices in 1979. Ward [58] gave different proofs for Roth's Theorems in 1993.

Ward and Gerrish [57] introduced constructive proofs of Roth's theorems in 2000, and the most difficult technique is the elementary transformation. Such proofs show an important achievement and give us a feasible method which can be used for Ore matrices [57]. We wish to extend two Roth's theorems and proofs in [57] to Ore matrices. Since not every element in S has an inverse, it is impossible to transform all Ore matrices into diagonal matrices whose all main diagonal entries are 0 or 1, which is the first step in the proof of Ward and Gerrish. By the assumption of the existence of $\{1\}$-inverses of two Ore matrices, we give the proof of Roth's theorem 1 for Ore matrices.

The original Roth's Theorem 1 and Roth's Theorem 2 are as follows:

Theorem 3.5. *([53], Theorem I) For any $G \in \mathbb{C}^{m\times n}$, $E \in \mathbb{C}^{m\times p}$ and $F \in \mathbb{C}^{q\times n}$,*

$$EX - YF = G,$$

is consistent over $\mathbb{C}$ $\Longleftrightarrow$

$$\begin{bmatrix} E & G \\ \mathbf{0} & F \end{bmatrix} \textit{ and } \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix},$$

are equivalent.

Theorem 3.6. *([53], Theorem II) For any $G \in \mathbb{C}^{n\times n}$, $E \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{n\times n}$,*

$$EX - XF = G,$$

is consistent over $\mathbb{C}$ $\Longleftrightarrow$

$$\begin{bmatrix} E & G \\ \mathbf{0} & F \end{bmatrix} \textit{ and } \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix},$$

are similar.

Now, we extend the proof of Roth's Theorem 1 in [57] to Ore matrices by the following Theorem 3.7.

Theorem 3.7. *For any $G \in S^{m\times n}$, if $E \in S^{m\times p}$ and $F \in S^{q\times n}$ both have $\{1\}$-*

inverses over S, then

$$EX - YF = G,$$

is consistent over S $\Longleftrightarrow$

$$\rho\left(\begin{bmatrix} E & G \\ \mathbf{0} & F \end{bmatrix}\right) = \rho\left(\begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix}\right).$$

Proof. It follows from Theorem 3.4 and the proof of Roth's Theorem 1 in [57]. $\square$

In the original Roth's theorem 1, the necessary and sufficient condition is that two matrices are equivalent over $\mathbb{C}$. So we need to introduce the definition for two Ore matrices to be equivalent over S.

Definition 3.2. *([1], p.290. [43], p.77) Two Ore matrices U and $V \in S^{m\times n}$ are called equivalent if $U = MVN$, where $M \in S^{m\times m}$ and $N \in S^{n\times n}$ are two invertible matrices.*

When we extend the proof of Roth's Theorem 1 in [57] to Ore matrices, we get the following theorem.

Theorem 3.8. *For any $G \in S^{m\times n}$, if $E \in S^{m\times p}$ and $F \in S^{q\times n}$ both have $\{1\}$-inverses over S, then*

$$U = \begin{bmatrix} E & G \\ \mathbf{0} & F \end{bmatrix}, \quad V = \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix},$$

are equivalent over S $\Longleftrightarrow$ $\rho(U) = \rho(V)$.

Proof. It follows from the proof of Theorem 3.7. $\square$

Section 3.4 provides some ways to check that if two Ore matrices are equivalent. But these ways are not simple. Theorem 3.7 gives a way to determine whether a matrix equation has solutions over S. But the assumption of Theorem 3.7 is that E and F both have $\{1\}$-inverses over S, which is strong. Next, we discuss Roth's Theorem 2 for Ore matrices.

In the proof of Roth's Theorem 2 in [57], the Kronecker product is necessary, while the Kronecker product for Ore matrices is quit complicated and it is difficult to be used in this case. Flanders and Wimmer [24] gave us another proof of Roth's Theorem 2 by linear transformations. Since S is a noncommutative ring, the maps ϕ that we need to prove Roth's Theorem 2 for Ore matrices always don't satisfy the second condition of the linear transformation (i.e., $\phi(sA) \neq s\phi(A)$, where $s \in S$ and A is an Ore matrix). In 1937, Jacobson introduced a necessary and sufficient condition for two matrices to be similar over a non-commutative field [33], which is helpful for our future research of Roth's Theorem 2.

3.3 Generalized Roth's Theorem 1

The solutions of the matrix equation $AXB+CYD = E$ over fields were discussed by many authors like Baksalary and Kala [4], Chu [17], Hernández and Gassó [30] and Xu, Wei and Zheng [59].

Özgüler considered the solutions of $AXB + CYD = E$ over a commutative PID $\mathcal{T}$ [49], where $A \in \mathcal{T}^{m\times n}$, $B \in \mathcal{T}^{p\times q}$, $C \in \mathcal{T}^{m\times t}$, $D \in \mathcal{T}^{w\times q}$ and $E \in \mathcal{T}^{m\times q}$. In order to discuss solutions of $AXB + CYD = E$, he proved that there exist matrices $M_1 \in \mathcal{T}^{n\times m}$, $M_2 \in \mathcal{T}^{t\times m}$, $M_3 \in \mathcal{T}^{t\times(n+t-m)}$, $M_4 \in \mathcal{T}^{n\times(n+t-m)}$, $M_5 \in \mathcal{T}^{(n+t-m)\times t}$, $M_6 \in \mathcal{T}^{(n+t-m)\times n}$, $M_7 \in \mathcal{T}^{q\times w}$, $M_8 \in \mathcal{T}^{q\times p}$, $M_9 \in \mathcal{T}^{(p+w-q)\times w}$, $M_{10} \in \mathcal{T}^{(p+w-q)\times p}$, $M_{11} \in \mathcal{T}^{p\times(p+w-q)}$ and $M_{12} \in \mathcal{T}^{w\times(p+w-q)}$ such that [49]

$$\begin{bmatrix} A & C \\ M_6 & -M_5 \end{bmatrix} \begin{bmatrix} M_1 & M_4 \\ M_2 & -M_3 \end{bmatrix} = I_{n+t}, \quad \begin{bmatrix} M_9 & M_{10} \\ M_7 & -M_8 \end{bmatrix} \begin{bmatrix} M_{12} & D \\ M_{11} & -B \end{bmatrix} = I_{p+w}.$$

But he didn't show us the method to construct these matrices. In this section, we find the way to achieve the above goal, and extend our method to Ore matrices. We also extend five equivalent statements for $AXB + CYD = E$ to be solvable over $\mathcal{T}$ to S.

Definition 3.3. *([49], p.582) An Ore matrix $A \in S^{m\times n}$ is called right (left)* *unimodular over S if there exists an Ore matrix $B \in S^{n\times m}$ such that $BA = I$ ($AB = I$).*

The following Theorem 3.9 and Theorem 3.10 give us two necessary and sufficient conditions to determine whether a given Ore matrix is left or right unimodular over S.

Theorem 3.9. *For any $A \in S^{m\times n}$, A is left unimodular over $S \iff A \in S_m^{m\times n}$ and A has a $\{1\}$-inverse over S.*

Proof. ($\Longrightarrow$) It is easy to give the proof.
($\Longleftarrow$) By Lemma 3.4, $AA^{(1)} = I$. □

Theorem 3.10. *For any $A \in S^{m\times n}$, A is right unimodular over $S \iff A \in S_n^{m\times n}$ and A has a $\{1\}$-inverse over S.*

Next, we discuss the solutions of $AXB + CYD = E$ over S.

Theorem 3.11. *Let $A \in S^{m\times n}$, $B \in S^{p\times q}$, $C \in S^{m\times t}$, $D \in S^{w\times q}$ and $E \in S^{m\times q}$. If $\begin{bmatrix} A & C \end{bmatrix} \in S_m^{m\times(n+t)}$ and $\begin{bmatrix} A & C \end{bmatrix}$ has a $\{1\}$-inverse over S, then there exist $M_1 \in S^{n\times m}$, $M_2 \in S^{t\times m}$, $M_3 \in S^{t\times(n+t-m)}$, $M_4 \in S^{n\times(n+t-m)}$, $M_5 \in S^{(n+t-m)\times t}$ and $M_6 \in S^{(n+t-m)\times n}$ such that*

$$\begin{bmatrix} A & C \\ M_6 & -M_5 \end{bmatrix} \begin{bmatrix} M_1 & M_4 \\ M_2 & -M_3 \end{bmatrix} = I_{n+t}.$$

Proof. This theorem is obtained by extending Equation (10) in [49] to Ore matrices.
Let $A_1 = \begin{bmatrix} A & C \end{bmatrix}$. By Theorem 3.4, there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{(n+t)\times(n+t)}$ such that

$$PA_1Q = \begin{bmatrix} I_m & \mathbf{0} \end{bmatrix} \in S^{m\times(n+t)}.$$

Next,

$$A_1Q = \begin{bmatrix} P^{-1} & \mathbf{0} \end{bmatrix} \in S^{m\times(n+t)}.$$

Let $W = \begin{bmatrix} \mathbf{0} & V \end{bmatrix} \in S^{(n+t-m)\times(n+t)}$, where $V \in S^{(n+t-m)\times(n+t-m)}$ is an arbitrary invertible Ore matrix. Then

$$\begin{bmatrix} A_1Q \\ W \end{bmatrix} = \begin{bmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} \in S^{(n+t)\times(n+t)},$$

$$\begin{bmatrix} A_1Q \\ W \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & V^{-1} \end{bmatrix} = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & I_{n+t-m} \end{bmatrix},$$

$$\begin{bmatrix} A_1 \\ WQ^{-1} \end{bmatrix} Q \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & V^{-1} \end{bmatrix} = I_{n+t}.$$

□

Similarly, we give the following Theorem 3.12.

Theorem 3.12. *Let* $A \in S^{m\times n}$, $B \in S^{p\times q}$, $C \in S^{m\times t}$, $D \in S^{w\times q}$ *and* $E \in S^{m\times q}$. *If* $\begin{bmatrix} B \\ D \end{bmatrix} \in S_q^{(p+w)\times q}$ *and* $\begin{bmatrix} B \\ D \end{bmatrix}$ *has a* $\{1\}$*-inverse over* S, *then there exist* $M_7 \in S^{q\times w}$, $M_8 \in S^{q\times p}$, $M_9 \in S^{(p+w-q)\times w}$, $M_{10} \in S^{(p+w-q)\times p}$, $M_{11} \in S^{p\times(p+w-q)}$ *and* $M_{12} \in S^{w\times(p+w-q)}$ *such that*

$$\begin{bmatrix} M_9 & M_{10} \\ M_7 & -M_8 \end{bmatrix} \begin{bmatrix} M_{12} & D \\ M_{11} & -B \end{bmatrix} = I_{p+w}.$$

Next, we give five equivalent statements for $AXB + CYD = E$ to be solvable over S.

Theorem 3.13. *Suppose that Ore matrices* A, B, C *and* D *satisfy the conditions of Theorem 3.11 and Theorem 3.12, if* A, B, C *and* D *all have* $\{1\}$*-inverses over* S, *then the following assertions are equivalent:*

(i) The matrix equation

$$AXB + CYD = E,$$

has solutions over S.

(ii) The matrix equation

$$AM_4X_0 + Y_0M_{10}B = EM_8B - AM_1E,$$

has solutions over S.

(iii) The matrix equations

$$AX_1 + Y_1D = E, \qquad X_2B + CY_2 = E,$$

have solutions over S.

(iv)

$$\rho\left(\begin{bmatrix} C & E & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & E \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D \end{bmatrix}\right) = \rho\left(\begin{bmatrix} C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D \end{bmatrix}\right),$$

over S.

(v) The matrix equation

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix} X_3 + Y_3 \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix} = \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix},$$

has solutions over S.

Proof. It follows from the proof of Theorem 5 in [49] and Theorem 3.8. But it is easy to prove Theorem 3.13 by the chain of

$$(iii) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (iii).$$

□

In Theorem 3.13, we also assume that A, B, C and D all have $\{1\}$-inverses over S, since we could not remove the assumption of the existence of $\{1\}$-inverses of Ore matrices in Theorem 3.7.

Theorem 3.14. *Let* $A \in S^{m\times n},\ B \in S^{p\times q},\ C \in S^{m\times t},\ D \in S^{w\times q},\ E \in S^{m\times q}$, $P = \begin{bmatrix} A & C \end{bmatrix}$ *and* $Q = \begin{bmatrix} B \\ D \end{bmatrix}$. *If* $P^{(1)}$ *and* $Q^{(1)}$ *both exist over* S, *then a necessary and sufficient condition for the matrix equation*

$$AXB + CYD = E,$$

to have solutions over S *is that there are* $P^{(1)}$ *and* $Q^{(1)}$ *over* S *such that*

$$PP^{(1)}EQ^{(1)}Q = E,$$

in which case

$$\begin{bmatrix} X & \mathbf{0} \\ \mathbf{0} & Y \end{bmatrix} = P^{(1)}EQ^{(1)} + Z - P^{(1)}PZQQ^{(1)},$$

is the general solution, where $Z \in S^{(n+t)\times(p+w)}$ *is an arbitrary Ore matrix.*

Proof. By Theorem 5.1 and

$$AXB + CYD = E \Longleftrightarrow \begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} X & \mathbf{0} \\ \mathbf{0} & Y \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = E,$$

it is easy to give the proof. □

3.4 Jacobson Normal Forms

In above theorems, a way to determine whether two Ore matrices are equivalent is necessary. By Theorem 3.15, we find that an Ore matrix may have different Jacobson forms over S, so it is difficult to determine whether two Ore matrices are equivalent by their Jacobson forms. So we introduce the Jacobson normal forms of Ore matrices in this section.

Definition 3.4. *([15], Definition 4.9, p.30. [20], p.186) a and $b \in S \setminus \{0\}$ are called left similar if S/Sa and S/Sb are isomorphic as left S-modules (i.e., $S/Sa \cong S/Sb$).*

Definition 3.5. *([15], Definition 4.9, p.30. [20], p.186) a and $b \in S \setminus \{0\}$ are called right similar if S/aS and S/bS are isomorphic as right S-modules (i.e., $S/aS \cong S/bS$).*

Lemma 3.5. *([15], Lemma 4.11, p.31) For any a and $b \in S \setminus \{0\}$,*

$$S/aS \cong S/bS,$$

as right S-modules $\Longleftrightarrow$

$$S/Sa \cong S/Sb,$$

as left S-modules.

Proof. It follows from the proof of Lemma 4.11 in [15] (p.31). $\square$

By Lemma 3.5, a and $b \in S \setminus \{0\}$ are left similar if and only if they are right similar. Next, we introduce the Jacobson normal forms of Ore matrices.

Definition 3.6. *([32], Teichmüller-Nakayama normal form, p.1732. [18], Ch.8. [43], p.77) For any $A \in S^{m\times n}$, there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that*

$$PAQ = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.3}$$

where $s \in S \setminus \{0\}$. The right side of (3.3) is called the Jacobson normal form of A.

Theorem 3.15. *([18], Theorem 8.2.4, p.494. [43], p.78) Let $A = \begin{bmatrix} I_{r_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $B = \begin{bmatrix} I_{r_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, where a and $b \in S \setminus \{0\}$. Then A and B are equivalent over $S \iff r_1 = r_2$ and $S/Sa \cong S/Sb$ as left S-modules (i.e., a and b are left similar).*

Theorem 3.15 shows us a way to determine whether two Ore matrices are equivalent over S. But the method to check that if two skew polynomials are left similar is very complicated. So we introduce the following definitions and theorems.

Definition 3.7. *([39], p.8) For any monic $h \in S$ with $\deg(h) \geq 1$, h can be written as*

$$h = x^n + r_{n-1}x^{n-1} + r_{n-2}x^{n-2} + \cdots + r_0,$$

where $r_i \in R$, $i \in \{0, 1, \ldots, n-1\}$. The n by n matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -r_0 & -r_1 & -r_2 & \cdots & -r_{n-1} \end{bmatrix},$$

is called the companion matrix of h, written C_h.

Definition 3.8. *([39], p.3. [41], Definition 3, p.324) Let r_1 and $r_2 \in R$. r_1 is said to be (σ, δ)-conjugate to r_2 if $r_1 = \sigma(r_3)r_2r_3^{-1} + \delta(r_3)r_3^{-1}$ for a $r_3 \in R \setminus \{0\}$.*

Theorem 3.16. *([39], Theorem 4.10, Lemma 4.9, p.10) Let M and $N \in R^{n\times n}$. M is said to be (σ,δ)-conjugate to N if $M = \sigma(Q)NQ^{-1} + \delta(Q)Q^{-1}$ for an invertible matrix $Q \in R^{n\times n}$.*

Theorem 3.17. *([39], Theorem 4.10, p.10) Two monic g and $h \in S$ with $\deg(g) \geq 1$ and $\deg(h) \geq 1$ are left (or right) similar over $S \iff C_g$ is (σ,δ)-conjugate to C_h.*

Proposition 3.4. *[20] If g and $h \in S$ are left (or right) similar over S, then $\deg(g) = \deg(h)$.*

3.5 $\{1,2\}$-Inverses

In this section, we find a necessary and sufficient condition for the existence of $\{1,2\}$-inverses of Ore matrices. So for all Ore matrices, we can determine whether they have $\{1,2\}$-inverses over S. If they have $\{1,2\}$-inverses, then we can find all their $\{1,2\}$-inverses over S.

Theorem 3.18. *For any $A \in S^{m\times n}$, A has a $\{1,2\}$-inverse over $S \iff$ there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that*

$$A = P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q,$$

in which case if X is a $\{1,2\}$-inverse of A over S, then X can be written as

$$Q^{-1} \begin{bmatrix} I_r & W_2 \\ W_3 & W_3W_2 \end{bmatrix} P^{-1},$$

where $W_2 \in S^{r\times(m-r)}$ and $W_3 \in S^{(n-r)\times r}$ are arbitrary Ore matrices.

Proof. $(\Longrightarrow)$ It is easy to give the proof by Theorem 3.4.

($\Longleftarrow$) Clearly,

$$Q^{-1}\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1},$$

is a $\{1,2\}$-inverse of A.

Let X be a $\{1,2\}$-inverse of A. Then X can be written as

$$Q^{-1}\begin{bmatrix} I_r & W_2 \\ W_3 & W_4 \end{bmatrix} P^{-1},$$

by Theorem 3.4. Since $XAX = X$,

$$W_4 = W_3W_2.$$

So X can be written as

$$Q^{-1}\begin{bmatrix} I_r & W_2 \\ W_3 & W_3W_2 \end{bmatrix} P^{-1}.$$

□

The Maple implementations of the above algorithm to get $\{1,2\}$-inverses of Ore matrices are given in the Appendix.

3.6 The Null Space and the Range Space

For matrices A and B over $\mathbb{C}$, $\mathcal{R}(AB) = \mathcal{R}(A)$ if and only if rankAB=rank$A$, and $\mathcal{N}(AB) = \mathcal{N}(B)$ if and only if rankAB=rank$B$ [7]. In [7], these two results are always used in proofs of generalized inverses of matrices over $\mathbb{C}$. In this section, we prove that $\mathcal{R}(AB) = \mathcal{R}(A)$ if and only if $\rho(AB) = \rho(A)$, and $\mathcal{N}(AB) = \mathcal{N}(B)$ if and only if $\rho(AB) = \rho(B)$, where A and B are two Ore matrices over S. The original proof applies the relationship between ranges of matrices and ranks of matrices over $\mathbb{C}$ and the relationship between nullities of matrices and ranks of matrices over $\mathbb{C}$, which are difficult to prove over S, since S is a noncommutative ring. In this section, we use the Jacobson forms of Ore matrices to prove it.

Lemma 3.6. *For any $A \in S^{m\times n}$, if $Q \in S^{m\times m}$ is invertible over S, then*

$$\mathcal{N}(QA) = \mathcal{N}(A).$$

Proof. This lemma is obtained by extending Exercise 1.10 in [7] to Ore matrices. Clearly,

$$\mathcal{N}(A) \subseteq \mathcal{N}(QA).$$

Let

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in S^n \middle| A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{0} \right\},$$

$$\mathcal{N}(QA) = \left\{ \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} \in S^n \middle| QA \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} = \mathbf{0} \right\}.$$

Choose

$$\begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} \in \mathcal{N}(QA).$$

Then

$$QA \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} = \mathbf{0}.$$

Since Q is invertible,

$$A \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} = \mathbf{0}.$$

Next,

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} \in \mathcal{N}(A).$$

Thus

$$\mathcal{N}(QA) \subseteq \mathcal{N}(A).$$

Therefore

$$\mathcal{N}(QA) = \mathcal{N}(A).$$

□

By Lemma 3.6, we can give the proof of the following Theorem 3.19.

Theorem 3.19. *For any* $A \in S^{m\times n}$ *and* $B \in S^{n\times q}$, $\mathcal{N}(AB) = \mathcal{N}(B) \iff \rho(AB) = \rho(B)$.

Proof. This theorem is obtained by extending Exercise 1.10 in [7] to Ore matrices.

($\Longrightarrow$) Evidently,

$$\rho(AB) \leq \rho(B).$$

Assume that

$$\rho(AB) < \rho(B).$$

Let

$$\rho(AB) = r_1,\ \rho(B) = r_2,\ r_1 < r_2.$$

Then there exist invertible matrices $P_1 \in S^{m\times m},\ Q_1 \in S^{q\times q},\ P_2 \in S^{n\times n}$ and

$Q_2 \in S^{q\times q}$ such that

$$AB = P_1 \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_{r_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Q_1 = P_1 \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_1,$$

$$B = P_2 \begin{bmatrix} f_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{r_2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Q_2 = P_2 \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_2,$$

where

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{r_1} \end{bmatrix},\ F = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f_{r_2} \end{bmatrix},$$

d_i and $f_j \in S \setminus \{0\}$, $i \in \{1, 2, \ldots, r_1\}$, $j \in \{1, 2, \ldots, r_2\}$. Next,

$$\mathcal{N}(AB) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \in S^q \,\middle|\, P_1 \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_1 \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} = \mathbf{0} \right\},$$

$$\mathcal{N}(B) = \left\{ \begin{bmatrix} x_1' \\ \vdots \\ x_q' \end{bmatrix} \in S^q \middle| P_2 \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_2 \begin{bmatrix} x_1' \\ \vdots \\ x_q' \end{bmatrix} = \mathbf{0} \right\}.$$

By Lemma 3.6,

$$\mathcal{N}(AB) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \in S^q \middle| \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_1 \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} = \mathbf{0} \right\},$$

$$\mathcal{N}(B) = \left\{ \begin{bmatrix} x_1' \\ \vdots \\ x_q' \end{bmatrix} \in S^q \middle| \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_2 \begin{bmatrix} x_1' \\ \vdots \\ x_q' \end{bmatrix} = \mathbf{0} \right\}.$$

Let

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = Q_1 \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix}, \quad \begin{bmatrix} y_1' \\ \vdots \\ y_q' \end{bmatrix} = Q_2 \begin{bmatrix} x_1' \\ \vdots \\ x_q' \end{bmatrix}.$$

Then

$$\mathcal{N}(AB) = \left\{ Q_1^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} \in S^q \middle| \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \mathbf{0} \right\},$$

$$\mathcal{N}(B) = \left\{ Q_2^{-1} \begin{bmatrix} y_1' \\ \vdots \\ y_q' \end{bmatrix} \in S^q \middle| \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1' \\ \vdots \\ y_q' \end{bmatrix} = \mathbf{0} \right\}.$$

If

$$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \mathbf{0}, \quad \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1' \\ \vdots \\ y_q' \end{bmatrix} = \mathbf{0},$$

then

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_{r_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{r_1} \\ \vdots \\ \vdots \\ y_q \end{bmatrix} = \mathbf{0},$$

$$\begin{bmatrix} f_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{r_2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1^{'} \\ y_2^{'} \\ \vdots \\ y_{r_2}^{'} \\ \vdots \\ \vdots \\ y_q^{'} \end{bmatrix} = \mathbf{0}.$$

Moreover,

$$y_1 = y_2 = \cdots = y_{r_1} = 0,$$

$$y_1^{'} = y_2^{'} = \cdots = y_{r_2}^{'} = 0.$$

Since $y_{r_1+1}, \cdots, y_q$ and $y_{r_2+1}^{'}, \cdots, y_q^{'}$ are arbitrary,

$$\dim(\mathcal{N}(AB)) = q - r_1, \ \dim(\mathcal{N}(B)) = q - r_2.$$

Due to $r_1 < r_2$,

$$\dim(\mathcal{N}(AB)) > \dim(\mathcal{N}(B)),$$

which contradicts the assumption that $\mathcal{N}(AB) = \mathcal{N}(B)$. So

$$\rho(AB) = \rho(B).$$

From the above proof, we can also get that for any $A \in S^{m\times n}$,

$$\dim(\mathcal{N}(A)) = n - \rho(A).$$

($\Longleftarrow$) Clearly, $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$,

$$\dim(\mathcal{N}(B)) = q - \rho(B),$$

and

$$\dim(\mathcal{N}(AB)) = q - \rho(AB).$$

Since $\rho(B) = \rho(AB)$,

$$\dim(\mathcal{N}(B)) = \dim(\mathcal{N}(AB)).$$

So

$$\mathcal{N}(B) = \mathcal{N}(AB).$$

□

Theorem 3.20. *For any $A \in S^{m\times n}$,*

$$\dim(\mathcal{N}(A)) = n - \rho(A).$$

Proof. It follows from the proof of Theorem 3.19. This theorem is obtained by extending Exercise 1.10 in [7] to Ore matrices. □

Now, we find the relationship between nullities of Ore matrices and inner ranks of Ore matrices. Next, we discuss the ranges of Ore matrices.

Lemma 3.7. *For any $A \in S^{m\times n}$ and $Q \in S^{n\times n}$, if Q is invertible over S, then*

$$\mathcal{R}(A) = \mathcal{R}(AQ).$$

Proof. This lemma is obtained by extending Exercise 1.10 in [7] to Ore matrices.

Clearly, $\mathcal{R}(AQ) \subseteq \mathcal{R}(A)$. Let

$$\mathcal{R}(AQ) = \left\{ AQ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in S^n \right\},$$

$$\mathcal{R}(A) = \left\{ A \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} \middle| \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} \in S^n \right\}.$$

Choose any $A \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} \in \mathcal{R}(A)$, and let $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Q^{-1} \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix}$. Then

$$A \begin{bmatrix} x_1^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} = AQ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Next,

$$\mathcal{R}(A) \subseteq \mathcal{R}(AQ).$$

Finally, $\mathcal{R}(A) = \mathcal{R}(AQ)$. $\square$

By Lemma 3.7, we can prove the following Theorem 3.21.

Theorem 3.21. *For any $A \in S^{m \times n}$ and $B \in S^{n \times q}$,*

$$\mathcal{R}(AB) = \mathcal{R}(A) \Longleftrightarrow \rho(AB) = \rho(A).$$

Proof. This theorem is obtained by extending Exercise 1.10 in [7] to Ore matrices. ($\Longrightarrow$) Clearly, $\rho(AB) \leq \rho(A)$. Suppose that

$$r_1 = \rho(AB) < \rho(A) = r_2.$$

By Theorem 2.1, there exist invertible matrices $P_1 \in S^{m\times m}$, $Q_1 \in S^{q\times q}$, $P_2 \in S^{m\times m}$ and $Q_2 \in S^{n\times n}$ such that

$$AB = P_1 \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_{r_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Q_1 = P_1 \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_1,$$

$$A = P_2 \begin{bmatrix} f_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{r_2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Q_2 = P_2 \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q_2,$$

where

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{r_1} \end{bmatrix}, \ F = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f_{r_2} \end{bmatrix},$$

d_i and $f_j \in S \setminus \{0\}$, $i \in \{1, 2, \ldots, r_1\}$, $j \in \{1, 2, \ldots, r_2\}$. Then

$$\mathcal{R}(AB) = \left\{ AB \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \middle| \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \in S^q \right\}, \ \mathcal{R}(A) = \left\{ A \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} \middle| \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} \in S^n \right\}.$$

By Lemma 3.7,

$$\mathcal{R}(AB) = \left\{ P_1 \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_{r_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r_1} \\ \vdots \\ \vdots \\ x_q \end{bmatrix} \middle| \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r_1} \\ \vdots \\ \vdots \\ x_q \end{bmatrix} \in S^q \right\},$$

$$\mathcal{R}(A) = \left\{ P_2 \begin{bmatrix} f_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{r_2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{r_2}' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix} \middle| \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{r_2}' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix} \in S^n \right\}.$$

Due to $r_1 < r_2$,

$$\dim(\mathcal{R}(AB)) < \dim(\mathcal{R}(A)).$$

which contradicts that $\mathcal{R}(AB) = \mathcal{R}(A)$. So

$$\rho(AB) = \rho(A).$$

($\Longleftarrow$) Clearly,

$$\mathcal{R}(AB) \subseteq \mathcal{R}(A).$$

Let

$$\rho(AB) = \rho(A) = r.$$

Similarly,

$$\dim(\mathcal{R}(AB)) = \dim(\mathcal{R}(A)).$$

So

$$\mathcal{R}(AB) = \mathcal{R}(A).$$

$\square$

For any $A \in \mathbb{C}^{m\times n}$, $\operatorname{rank}(A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A^*A)$, which is wrong for Ore matrices. We extend this result to Ore matrices by adding some conditions.

Lemma 3.8. *If $A \in S_r^{m\times r}$, then*

$$\rho(A) = \rho(AA^*).$$

Proof. This lemma is obtained by extending Lemma 1.4 in [7] to Ore matrices. Let

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

Then

$$A^* = \begin{bmatrix} \alpha_1^* & \alpha_2^* & \cdots & \alpha_m^* \end{bmatrix},$$

and

$$AA^* = \begin{bmatrix} \alpha_1\alpha_1^* & \cdots & \alpha_1\alpha_m^* \\ \vdots & \ddots & \vdots \\ \alpha_m\alpha_1^* & \cdots & \alpha_m\alpha_m^* \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}.$$

Due to $\rho(A) = r$, we assume that $\alpha_1, \ldots, \alpha_r$ are left linearly independent over S. Let

$$k_1\beta_1 + k_2\beta_2 + \cdots + k_r\beta_r = \mathbf{0},$$

where $k_i \in S$, $i \in \{1, 2, \ldots, r\}$. Then

$$k_1(\alpha_1\alpha_1^*, \cdots, \alpha_1\alpha_m^*) + \cdots + k_r(\alpha_r\alpha_1^*, \cdots, \alpha_r\alpha_m^*) = \mathbf{0}.$$

Further,

$$\begin{cases} k_1\alpha_1\alpha_1^* + k_2\alpha_2\alpha_1^* + \cdots + k_r\alpha_r\alpha_1^* = \mathbf{0}, \\ k_1\alpha_1\alpha_2^* + k_2\alpha_2\alpha_2^* + \cdots + k_r\alpha_r\alpha_2^* = \mathbf{0}, \\ \quad\vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots \\ k_1\alpha_1\alpha_m^* + k_2\alpha_2\alpha_m^* + \cdots + k_r\alpha_r\alpha_m^* = \mathbf{0}. \end{cases}$$

Moreover,

$$\begin{cases} (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)\alpha_1^* = \mathbf{0}, \\ (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)\alpha_2^* = \mathbf{0}, \\ \quad\vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots \\ (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)\alpha_m^* = \mathbf{0}. \end{cases}$$

Let

$$w = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r.$$

Then

$$\alpha_1 w^* = \alpha_2 w^* = \cdots = \alpha_m w^* = \mathbf{0},$$

and $Aw^* = \mathbf{0}$. Since

$$\dim(\mathcal{N}(A)) = r - \rho(A), \;\; \rho(A) = r,$$

we get that $w^* = \mathbf{0}$ and

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r = \mathbf{0}.$$

As $\alpha_1, \ldots, \alpha_r$ are left linearly independent over S,

$$k_1 = k_2 = \cdots = k_r = 0.$$

Thus

$$\rho(AA^*) = \rho_r(AA^*) \geq r = \rho(A).$$

Clearly, $\rho(AA^*) \leq \rho(A)$. Therefore

$$\rho(AA^*) = \rho(A).$$

$\square$

Lemma 3.9. *If $A \in S_r^{r\times n}$, then*

$$\rho(A) = \rho(A^*A).$$

Proof. It is similar to the proof of Lemma 3.8. $\square$

Lemma 3.10. *If $A \in S_n^{n\times n}$, then*

$$\rho(A) = \rho(A^*A) = \rho(AA^*).$$

Proof. It is similar to the proof of Lemma 3.8. $\square$

Corollary 3.1. *If $A \in S_r^{m\times r}$, then*

$$\mathcal{R}(A) = \mathcal{R}(AA^*).$$

Proof. It follows from Theorem 3.21 and Lemma 3.8. This corollary is obtained by extending Corollary 1.2 in [7] to Ore matrices. $\square$

Corollary 3.2. *If $A \in S_r^{r\times n}$, then*

$$\mathcal{N}(A) = \mathcal{N}(A^*A).$$

Proof. It follows from Theorem 3.19 and Lemma 3.9. This corollary is obtained by extending Corollary 1.2 in [7] to Ore matrices. $\square$

After proving Corollary 3.1 and Corollary 3.2, we can give the following two theorems.

Theorem 3.22. *If $A \in S_r^{r\times m}$ and $(A^*A)^{(1)}$ exists over S, then $(A^*A)^{(1)}A^*$ is a $\{1,2,3\}$-inverse of A over S.*

Proof. It is similar to the proof of Theorem 1.3 in [7]. $\square$

Theorem 3.23. *If $A \in S_r^{m\times r}$ and $(AA^*)^{(1)}$ exists over S, then $A^*(AA^*)^{(1)}$ is a $\{1,2,4\}$-inverse of A over S.*

Proof. It is similar to the proof of Theorem 1.3 in [7]. $\square$

3.7 $\{1,3\}$- and $\{1,4\}$-Inverses

In this section, we find some necessary and sufficient conditions for the existence of $\{1,3\}$- and $\{1,4\}$-inverses of Ore matrices. For all Ore matrices, we can determine whether they have $\{1,3\}$- and $\{1,4\}$-inverses over S. We also give some explicit formulas for $\{1,3\}$- and $\{1,4\}$-inverses of Ore matrices.

Theorem 3.24. *For any $A \in S^{m\times n}$, there exists a $A^{(1,3)}$ over $S \iff A^*A$ has a $\{1\}$-inverse over S, and $\rho(A) = \rho(A^*A)$, in which case $X = (A^*A)^{(1)}A^*$ is a $\{1,3\}$-inverse of A over S.*

Proof. It is similar to the proof of Proposition 3.10 in [51]. $\square$

The Maple implementations of the above algorithm to get $\{1,3\}$-inverses of Ore matrices are given in the Appendix.

Theorem 3.25. *For any $A \in S^{m\times n}$, there exists a $A^{(1,4)}$ over $S \iff AA^*$ has a $\{1\}$-inverse over S, and $\rho(A^*) = \rho(AA^*)$, in which case $X = A^*(AA^*)^{(1)}$ is a $\{1,4\}$-inverse of A over S.*

Proof. It is similar to the proof of Proposition 3.10 in [51]. $\square$

The Maple implementations of the above algorithm to get $\{1,4\}$-inverses of Ore matrices are given in the Appendix.

3.8 MP-Inverses

For $A \in \mathbb{C}^{n\times n}$, rank$A = n$ if and only if A is invertible, which is always used in proofs of MP-inverses of matrices over $\mathbb{C}$. But it is wrong over S. So we avoid it and choose other proofs in this section. We discuss two necessary and sufficient conditions for the existence of MP-inverses of Ore matrices. For all Ore matrices, we can determine whether they have MP-inverses over S. We also give some explicit formulas to construct MP-inverses of Ore matrices.

Theorem 3.26. *For any $A \in S^{m\times n}$, if $A^{(1,4)}$ and $A^{(1,3)}$ both exist over S, then $A^{(1,4)}AA^{(1,3)}$ is the MP-inverse of A over S.*

Proof. It is similar to the proof of Theorem 1.4 in [7]. □

By Theorem 3.24 and Theorem 3.25, we can determine whether a given Ore matrix A has $\{1,3\}$- and $\{1,4\}$-inverses and find some $\{1,3\}$- and $\{1,4\}$-inverses of A. By Theorem 3.26, we can compute the MP-inverse of A.

Proposition 3.5. *For any $A \in S_r^{m\times n}$, let $A = FG$, where $F \in S_r^{m\times r}$ and $G \in S_r^{r\times n}$. If $G^\dagger$ and $F^\dagger$ both exist over S, then $G^\dagger F^\dagger$ is the MP-inverse of A over S.*

Proof. This proposition is obtained by extending Exercise 1.17 in [7] to Ore matrices.

Let

$$X = G^\dagger F^\dagger.$$

By Lemma 3.4,

$$F^\dagger F = I,\ GG^\dagger = I.$$

Next,

$$AXA = FGG^\dagger F^\dagger FG = FG = A,$$

$$XAX = G^\dagger F^\dagger FGG^\dagger F^\dagger = G^\dagger F^\dagger = X,$$

$$XA = G^\dagger F^\dagger FG = G^\dagger G,$$

$$AX = FGG^\dagger F^\dagger = FF^\dagger.$$

□

Proposition 3.6. *For any $A \in S^{m\times n}$,*

$$(A^\dagger)^\dagger = A,\ (A^*)^\dagger = (A^\dagger)^*.$$

Proof. This proposition is obtained by extending Exercise 1.18 in [7] to Ore matrices.

Clearly, $(A^\dagger)^\dagger$ is the unique MP-inverse of $A^\dagger$. Since $A^\dagger$ is the unique MP-inverse of A, A is the unique MP-inverse of $A^\dagger$. Then

$$(A^\dagger)^\dagger = A.$$

Similarly,

$$(A^*)^\dagger = (A^\dagger)^*.$$

□

Proposition 3.7. *For any $A \in S^{n\times n}$, if $A^* = A$ and $A^2 = A$, then*

$$A^\dagger = A.$$

Proof. This proposition is obtained by extending Exercise 1.20 in [7] to Ore matrices.

Let $X = A$. Then

$$AXA = AAA = AA = A,\ XAX = AAA = AA = A,$$

$$(AX)^* = (AA)^* = A^* = A,\ (XA)^* = (AA)^* = A^* = A.$$

□

Proposition 3.8. *For any $A \in S_r^{m\times n}$, let $A = FG$, where $F \in S_r^{m\times r}$ and $G \in S_r^{r\times n}$. If $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, $G^{(1)}$, $G^{(2)}$ and $G^{(4)}$ all exist over S, then*

$$G^{(i)}F^{(1)} \in A\{i\} \quad (i = 1, 2, 4); \qquad G^{(1)}F^{(j)} \in A\{j\} \quad (j = 1, 2, 3).$$

Proof. It is similar to the proof of Exercise 1.29 in [7]. □

Proposition 3.9. *For any $A \in S_r^{m\times n}$, let $A = FG$, where $F \in S_r^{m\times r}$ and $G \in S_r^{r\times n}$. If $F^\dagger$ and $G^\dagger$ both exist over S, then $G^{(1,4)}F^\dagger = G^\dagger F^{(1,3)}$ is the MP-inverse of A over S.*

Proof. This proposition is obtained by extending Exercise 1.30 in [7] to Ore matrices.

By Lemma 3.4,

$$F^{(1)}F = GG^{(1)} = I.$$

Let $X = G^\dagger F^{(1,3)}$. Then

$$AXA = FGG^\dagger F^{(1,3)}FG = FG = A,$$

$$XAX = G^\dagger F^{(1,3)}FGG^\dagger F^{(1,3)} = G^\dagger F^{(1,3)} = X,$$

$$(AX)^* = (FGG^\dagger F^{(1,3)})^* = (FF^{(1,3)})^* = AX,$$

$$(XA)^* = (G^\dagger F^{(1,3)}FG)^* = (G^\dagger G)^* = XA.$$

□

From Proposition 3.5 to Proposition 3.9, we give some formulas for MP-inverses of Ore matrices under some special conditions. Next, we introduce two necessary and sufficient conditions for the existence of MP-inverses of Ore matrices. For all Ore matrices, we can determine whether they have MP-inverses and find their MP-inverses.

Theorem 3.27. *For any $A \in S^{m\times n}$, there exists the $A^\dagger$ over $S \Longleftrightarrow A^*A$ and AA^* both have $\{1\}$-inverses over S, and $\rho(A) = \rho(AA^*) = \rho(A^*A)$.*

Proof. It is similar to the proof of Proposition 3.10 in [51]. □

Theorem 3.28. *For any $A \in S^{m\times n}$, there exists the $A^\dagger$ over $S \Longleftrightarrow A^*AA^*$ has a $\{1\}$-inverse over S, and $\rho(A) = \rho(AA^*) = \rho(A^*A)$, in which case $X = A^*(A^*AA^*)^{(1)}A^*$ is the unique MP-inverse of A over S.*

Proof. It is similar to the proof of Proposition 3.10 in [51]. □

The Maple implementations of the above algorithm to get MP-inverses of Ore matrices are given in the Appendix.

3.9 The Full Rank Factorizations

We introduce the method to achieve the full rank factorizations of Ore matrices ([7], Lemma 1, p.26) in this section. But we also find that the full rank factorization of a given Ore matrix is not unique.
Let $A \in S_r^{m\times n}$. By Theorem 2.1, there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that

$$PAQ = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} D_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{3.4}$$

where

$$D_r = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_r \end{bmatrix},$$

and $d_i \in S \setminus \{0\}$, $i \in \{1, 2, \ldots, r\}$. The right side of (3.4) is the Jacobson form of A. Then

$$A = P^{-1} \begin{bmatrix} D_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q^{-1}.$$

Next,

$$A = P^{-1} \begin{bmatrix} D_r \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} I_r & \mathbf{0} \end{bmatrix} Q^{-1}.$$

Since invertible matrices don't change the inner rank, $P^{-1} \begin{bmatrix} D_r \\ \mathbf{0} \end{bmatrix} \in S_r^{m \times r}$ and $\begin{bmatrix} I_r & \mathbf{0} \end{bmatrix} Q^{-1} \in S_r^{r \times n}$. Now we achieve the full rank factorization of A ([7], Lemma 1, p.26). We can also write that

$$A = P^{-1} \begin{bmatrix} I_r \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} D_r & \mathbf{0} \end{bmatrix} Q^{-1},$$

where $P^{-1} \begin{bmatrix} I_r \\ \mathbf{0} \end{bmatrix} \in S_r^{m \times r}$ and $\begin{bmatrix} D_r & \mathbf{0} \end{bmatrix} Q^{-1} \in S_r^{r \times n}$. So the full rank factorization of A is not unique.

3.10 Another Way to Construct MP-Inverses

In Section 3.8, we introduced two necessary and sufficient conditions for the existence of MP-inverses of Ore matrices. In this section, we discuss another necessary and sufficient condition by using the full rank factorizations of Ore matrices.

Theorem 3.29. *For any $A \in S_r^{m\times n}$, let $A = FG$, where $F \in S_r^{m\times r}$ and $G \in S_r^{r\times n}$. If F^*F and GG^* are both invertible over S, then $G^*(F^*AG^*)^{-1}F^*$ is the MP-inverse of A over S.*

Proof. It is similar to the proof of Theorem 1.5 in [7]. □

In order to prove Theorem 3.30, we need to give the proofs of Lemma 3.11 and Lemma 3.12.

Lemma 3.11. *For any A and $B \in S^{n\times n}$, if $\rho(A) = \rho(B) = n$, then*

$$\rho(AB) = \rho(A) = \rho(B) = n.$$

Proof. Let

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \ B = \begin{bmatrix} \beta_1 & \beta_2 & \cdots \beta_n \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_n, \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_n, \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n\beta_1 & \alpha_n\beta_2 & \cdots & \alpha_n\beta_n \end{bmatrix}.$$

Let

$$AB = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix},$$

and assume that

$$k_1C_1 + k_2C_2 + \cdots + k_nC_n = \mathbf{0},$$

where $k_i \in S$, $i \in \{1, 2, \ldots, n\}$. Then

$$\begin{cases} k_1\alpha_1\beta_1 + k_2\alpha_2\beta_1 + \cdots + k_n\alpha_n\beta_1 = \mathbf{0}, \\ k_1\alpha_1\beta_2 + k_2\alpha_2\beta_2 + \cdots + k_n\alpha_n\beta_2 = \mathbf{0}, \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ k_1\alpha_1\beta_n + k_2\alpha_2\beta_n + \cdots + k_n\alpha_n\beta_n = \mathbf{0}. \end{cases}$$

Further,

$$\begin{cases} (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)\beta_1 = \mathbf{0}, \\ (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)\beta_2 = \mathbf{0}, \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)\beta_n = \mathbf{0}. \end{cases}$$

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)(\beta_1,\ \beta_2,\ \ldots, \beta_n) = \mathbf{0}.$$

Next,

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)B = \mathbf{0},$$

$$B^*(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)^* = \mathbf{0}.$$

Because of $\rho(B) = \rho(B^*) = n$ and $\dim(\mathcal{N}(B^*)) = n - \rho(B^*) = 0$,

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n)^* = \mathbf{0},$$

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = \mathbf{0}.$$

Since $\rho_r(A) = n$,

$$k_1 = k_2 = \cdots = k_n = 0.$$

So

$$\rho(AB) = \rho_r(AB) = n = \rho(A) = \rho(B).$$

□

Lemma 3.12. *If $A \in S_r^{m\times r}$ and $B \in S_r^{r\times n}$, then*

$$\rho(AB) = \rho(A) = \rho(B) = r.$$

Proof. Let

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}, \ B = \begin{bmatrix} \beta_1 & \beta_2 & \cdots \beta_n \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_n \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m\beta_1 & \alpha_m\beta_2 & \cdots & \alpha_m\beta_n \end{bmatrix}.$$

Let

$$AB = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{bmatrix}.$$

Assume that $\alpha_1,\ \alpha_2, \ldots, \alpha_r$ are left linearly independent over S, and $\beta_1,\ \beta_2, \ldots, \beta_r$ are right linearly independent over S. Write

$$k_1C_1 + k_2C_2 + \cdots + k_rC_r = \mathbf{0},$$

where $k_i \in S$, $i \in \{1, 2, \ldots, r\}$. Then

$$\begin{cases} k_1\alpha_1\beta_1 + k_2\alpha_2\beta_1 + \cdots + k_r\alpha_r\beta_1 = \mathbf{0}, \\ k_1\alpha_1\beta_2 + k_2\alpha_2\beta_2 + \cdots + k_r\alpha_r\beta_2 = \mathbf{0}, \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ k_1\alpha_1\beta_n + k_2\alpha_2\beta_n + \cdots + k_r\alpha_r\beta_n = \mathbf{0}. \end{cases}$$

Next,

$$\begin{cases} (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)\beta_1 = \mathbf{0}, \\ (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)\beta_2 = \mathbf{0}, \\ \quad\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)\beta_n = \mathbf{0}. \end{cases}$$

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_n)(\beta_1,\ \beta_2,\ \ldots, \beta_n) = \mathbf{0}.$$

Moreover,

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)B = \mathbf{0},$$

$$B^*(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)^* = \mathbf{0}.$$

Since $\rho(B) = \rho(B^*) = r$ and $\dim(\mathcal{N}(B^*)) = r - \rho(B^*) = 0$,

$$(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r)^* = \mathbf{0},$$

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r = \mathbf{0}.$$

As $\alpha_1,\ \alpha_2, \ldots, \alpha_r$ are left linearly independent over S,

$$k_1 = k_2 = \cdots = k_r = 0.$$

So

$$\rho(AB) = \rho_r(AB) \geq r.$$

Clearly, $\rho(AB) \leq r$. Therefore

$$\rho(AB) = r.$$

□

By the following Theorem 3.30, we can give another necessary and sufficient condition for the existence of MP-inverses of Ore matrices.

Theorem 3.30. *For any $A \in S_r^{m\times n}$, if the Jacobson form of A is $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, i.e., there exist invertible matrices $P \in S^{m\times m}$ and $Q \in S^{n\times n}$ such that*

$$A = P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q,$$

where $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$, $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$, $P_1 \in S^{m\times r}$ is the first r columns of P and $Q_1 \in S^{r\times n}$ is the first r rows of Q, then $A^\dagger$ exists over $S \Longleftrightarrow$ the Jacobson form of $P_1^* P_1 Q_1 Q_1^*$ is I_r.

Proof. This theorem is obtained by extending Theorem 1.5 in [7] to Ore matrices. By Theorem 3.28, for any $A \in S_r^{m\times n}$, A has the MP-inverse over S $(\Longleftrightarrow)$ A^*AA^* has a $\{1\}$-inverse over S and $\rho(A) = \rho(AA^*) = \rho(A^*A)$.

$$\rho(A) = \rho(AA^*) = \rho(A^*A),$$

$(\Longleftrightarrow)$

$$\rho\left(P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} QQ^* \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^* \right) = \rho\left(Q^* \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^*P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q \right) = r.$$

As P and Q are invertible over S,

$$(\Longleftrightarrow)\ \rho\left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} QQ^* \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \rho\left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^*P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = r.$$

$$(\Longleftrightarrow)\ \rho\left(\begin{bmatrix} Q_1Q_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \rho\left(\begin{bmatrix} P_1^*P_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = r.$$

$$(\Longleftrightarrow)\ \rho(Q_1Q_1^*) = \rho(P_1^*P_1) = r.$$

It is easy to get

$$\begin{aligned} A^*AA^* &= Q^* \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^*P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} QQ^* \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^*, \\ &= Q^* \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_1^* \\ P_2^* \end{bmatrix} \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1^* & Q_2^* \end{bmatrix} \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^*, \\ &= Q^* \begin{bmatrix} P_1^*P_1Q_1Q_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^*. \end{aligned}$$

Since P and Q are invertible over S, A^*AA^* has a $\{1\}$-inverse over S $(\Longleftrightarrow)$ $\begin{bmatrix} P_1^*P_1Q_1Q_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ has a $\{1\}$-inverse over S $(\Longleftrightarrow)$ there exist invertible matrices $E \in S^{r\times r}$ and $F \in S^{r\times r}$ such that

$$EP_1^*P_1Q_1Q_1^*F = \begin{bmatrix} I_{r_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Now, A has the MP-inverse over S

$$(\Longleftrightarrow)\ \rho(Q_1Q_1^*) = \rho(P_1^*P_1) = r,\ EP_1^*P_1Q_1Q_1^*F = \begin{bmatrix} I_{r_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

By Lemma 3.11,

$$(\Longleftrightarrow)\ \rho(Q_1Q_1^*) = \rho(P_1^*P_1) = r,\ EP_1^*P_1Q_1Q_1^*F = I_r,$$

$(\Longleftrightarrow)$ $P_1^*P_1Q_1Q_1^*$ is invertible over S $(\Longleftrightarrow)$ the Jacobson form of $P_1^*P_1Q_1Q_1^*$ is I_r.

□

3.11 Future Work

We wish to prove Roth's Theorem 1 over S by using weaker conditions and extend Roth's Theorem 2 to Ore matrices in future. We also want to construct more efficient algorithms to find $\{1\}$-, $\{1, 2\}$-, $\{1, 3\}$-, $\{1, 4\}$- and MP-inverses of Ore matrices.

Chapter 4

Involutions on Skew Polynomial Rings

In this chapter, we assume that $\mathbb{R}(x)$ is the rational function field, σ is an automorphism of $\mathbb{R}(x)$, $\delta(x) = 1$ (when $\delta \neq 0$) and $\mathbb{R}(x)[D; \sigma, \delta]$ is a skew polynomial ring over $\mathbb{R}(x)$. Firstly, we construct the possible forms of involutions on $\mathbb{R}(x)[D]$, $\mathbb{R}(x)[D; \delta]$ and $\mathbb{R}(x)[D; \sigma]$ respectively. Then we extend all the involutions ψ on $\mathbb{R}(x)$ to $\mathbb{R}(x)[D]$, $\mathbb{R}(x)[D; \delta]$ and $\mathbb{R}(x)[D; \sigma]$ respectively. Next, we give some necessary and sufficient conditions for ψ to be an involution on $\mathbb{R}(x)[D]$, $\mathbb{R}(x)[D; \delta]$ and $\mathbb{R}(x)[D; \sigma]$ respectively. Finally, we prove that every involution ψ on $\mathbb{R}(x)$ can not be extended to be an involution on $\mathbb{R}(x)[D; \sigma, \delta]$.

4.1 Involutions on $\mathbb{R}(x)$

In this section, we give all the involutions on $\mathbb{R}(x)$ by the automorphisms of $\mathbb{R}(x)$. The format of the automorphisms of $\mathbb{R}(x)$ is given in the following lemma.

Lemma 4.1. *[40] Every automorphism of $\mathbb{R}(x)$ can be induced by a map ψ:*

$$\psi(x) = \frac{ax + b}{cx + d},$$

where $a,\ b,\ c,\ d \in \mathbb{R}$ and $ad - bc \neq 0$.

Next, we give the condition for ψ to be an involution on $\mathbb{R}(x)$.

Lemma 4.2. *([5], Involutions, p.244) Every involution on $\mathbb{R}(x)$ can be induced by a map ψ:*

$$\psi(x) = \frac{ax+b}{cx-a},$$

or

$$\psi(x) = x,$$

where $a,\ b,\ c \in \mathbb{R}$ and $a^2 + bc \neq 0$.

Proof. By Lemma 4.1, every automorphism of $\mathbb{R}(x)$ can be induced by a map ψ:

$$\psi(x) = \frac{ax+b}{cx+d},$$

where $a,\ b,\ c,\ d \in \mathbb{R}$ and $ad - bc \neq 0$. Next,

$$\psi\left[\psi(x)\right] = \frac{a^2x + bcx + ab + bd}{acx + cdx + bc + d^2}.$$

If ψ is an involution on $\mathbb{R}(x)$, then

$$\frac{a^2x + bcx + ab + bd}{acx + cdx + bc + d^2} = x.$$

Furthermore,

$$(a^2 + bc)x + (ab + bd) = (ac + cd)x^2 + (bc + d^2)x.$$

$$(a+d)b = (a+d)c = 0,\ (a^2 + bc) = (bc + d^2).$$

If $a + d \neq 0$, then $b = c = 0$ and $a = d$. So

$$\psi(x) = x.$$

If $a + d = 0$, then

$$\psi(x) = \frac{ax + b}{cx - a},$$

where $a^2 + bc \neq 0$. On the other hand, clearly, $\psi(x) = \frac{ax+b}{cx-a}$ and $\psi(x) = x$ are two involutions on $\mathbb{R}(x)$. $\square$

Lemma 4.3. *In $\mathbb{R}(x)$, $\sigma\delta = \delta\sigma \Longleftrightarrow \sigma$ can be written as*

$$\sigma(x) = x + d,$$

where $d \in \mathbb{R}$.

Proof. ($\Longleftarrow$) It is easy to give the proof.

($\Longrightarrow$) Since σ is an automorphism of $\mathbb{R}(x)$, σ can be written as

$$\sigma(x) = \frac{px + q}{ux + v},$$

where $p,\ q,\ u,\ v \in \mathbb{R}$ and $pv - qu \neq 0$. By $\delta\sigma(x) = \sigma\delta(x)$,

$$\frac{pv - qu}{(ux + v)^2} = 1.$$

Then

$$u = 0,\ p = v.$$

So σ can be written as

$$\sigma(x) = x + d,$$

where $d \in \mathbb{R}$. $\square$

From now on, we assume that $\psi(x) = x$ and $\psi(x) = \frac{ax+b}{cx-a}$ are two involutions on $\mathbb{R}(x)$, where $a,\ b,\ c \in \mathbb{R}$ and $a^2 + bc \neq 0$.

4.2 Involutions on $S = \mathbb{R}(x)[D;\sigma,\delta]$

In this section, we extend all the involutions ψ on $\mathbb{R}(x)$ (i.e., $\psi(x) = x$ and $\psi(x) = \frac{ax+b}{cx-a}$) to $\mathbb{R}(x)[D]$, $\mathbb{R}(x)[D;\delta]$ and $\mathbb{R}(x)[D;\sigma]$ respectively. Then we give some necessary and sufficient conditions for ψ to be an involution on $\mathbb{R}(x)[D]$, $\mathbb{R}(x)[D;\delta]$ and $\mathbb{R}(x)[D;\sigma]$ respectively. Finally, we prove that every involution ψ on $\mathbb{R}(x)$ can not be extended to be an involution on $\mathbb{R}(x)[D;\sigma,\delta]$.

4.2.1 $\sigma = 1$ and $\delta = 0$

We extend all the involutions ψ on $\mathbb{R}(x)$ (i.e., $\psi(x) = x$ and $\psi(x) = \frac{ax+b}{cx-a}$) to $\mathbb{R}(x)[D]$ (i.e., $\sigma = 1$ and $\delta = 0$ in $\mathbb{R}(x)[D;\sigma,\delta]$) by Theorem 4.1 and Theorem 4.2.

Lemma 4.4. *Let $S = \mathbb{R}(x)[D]$ and ψ be an involution on S. Then*

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$.

Proof. Let ψ be an involution on S and write

$$\psi(D) = \sum_{i=0}^{n} a_i(x)D^i,$$

where $a_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \dots, n\}$. Then

$$\begin{aligned}
\psi\left[\psi(D)\right] = \psi\left(\sum_{i=0}^{n} a_i(x)D^i\right) &= \sum_{i=0}^{n} \psi(D^i)\psi\left[a_i(x)\right], \\
&= \sum_{i=0}^{n} \psi(D)^i \psi\left[a_i(x)\right], \\
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{n} a_j(x)D^j\right)^i \psi\left[a_i(x)\right].
\end{aligned}$$

Clearly, the degree of $\psi\left[\psi(D)\right]$ is n^2 and the coefficient of D^{n^2} is

$$a_n^n(x)\psi\left[a_n(x)\right].$$

Since $\psi\left[\psi(D)\right] = D$,

$$a_n^n(x)\psi\left[a_n(x)\right] = 0.$$

As S is a domain and ψ is an automorphism of $\mathbb{R}(x)$, $a_n(x) = 0$. Similarly,

$$a_{n-1}(x) = \cdots = a_2(x) = 0,$$

then we can write

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x)$ and $b_1(x) \in \mathbb{R}(x)$. Clearly, $a_1(x) \neq 0$. □

Theorem 4.1. *Let $S = \mathbb{R}(x)[D]$. Then every involution ψ on S with $\psi(x) = x$ can be written as*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[-D + d(x)\right]^i r_i(x),$$

or

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} D^i r_i(x) = \sum_{i=0}^{n} r_i(x)D^i,$$

where $r_i(x)$ and $d(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

Proof. Let ψ be an involution on S. By Lemma 4.4,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$. Next,

$$\begin{aligned}\psi\left[\psi(D)\right] = \psi\left[a_1(x)D + b_1(x)\right] &= \psi(D)a_1(x) + b_1(x),\\ &= \left[a_1(x)D + b_1(x)\right]a_1(x) + b_1(x),\\ &= a_1(x)Da_1(x) + b_1(x)a_1(x) + b_1(x),\\ &= a_1^2(x)D + b_1(x)a_1(x) + b_1(x).\end{aligned}$$

So

$$a_1(x)^2 = 1,$$

and

$$b_1(x)a_1(x) + b_1(x) = 0.$$

Then

$$\psi(D) = -D + d(x),$$

or

$$\psi(D) = D,$$

where $d(x) \in \mathbb{R}(x)$. Thus

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}[\psi(D)]^i\psi(r_i(x)) = \sum_{i=0}^{n}\left[-D + d(x)\right]^i r_i(x), \tag{4.1}$$

or

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} D^i r_i(x) = \sum_{i=0}^{n} r_i(x)D^i, \tag{4.2}$$

where $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \dots, n\}$. Clearly, (4.2) is an involution on S. We prove that (4.1) is an involution on S by the following steps.

(1) It is easy to check that

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i + \sum_{i=0}^{n} t_i(x)D^i\right) = \psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) + \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right), \tag{4.3}$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0,1,\ldots,n\}$.

(2) For any $r(x)$ and $t(x) \in \mathbb{R}(x)$,

$$\begin{aligned}\psi\left[(r(x)D^m)(t(x)D^n)\right] &= \psi\left[r(x)t(x)D^{m+n}\right],\\ &= \left[-D+d(x)\right]^{m+n} r(x)t(x),\end{aligned}$$

and

$$\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] = \left[-D+d(x)\right]^n t(x)\left[-D+d(x)\right]^m r(x).$$

Then

$$\psi[(r(x)D^m)(t(x)D^n)] = \psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right].$$

So

$$\psi\left(\left(\sum_{i=0}^{n} r_i(x)D^i\right)\left(\sum_{j=0}^{n} t_j(x)D^j\right)\right) = \psi\left(\sum_{j=0}^{n} t_j(x)D^j\right)\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right),$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0,1,\ldots,n\}$.

(3) For any $r(x) \in \mathbb{R}(x)$,

$$\begin{aligned}\psi\left(\psi[r(x)D^m]\right) &= \psi\left(\left[-D+d(x)\right]^m r(x)\right),\\ &= r(x)D^m.\end{aligned}$$

From (4.3),

$$\psi\left(\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right)\right) = \sum_{i=0}^{n} r_i(x)D^i, \tag{4.4}$$

for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0,1,\ldots,n\}$.

(4) Suppose

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[-D+d(x)\right]^i r_i(x) = 0.$$

Then the coefficient of D^n in $\sum\limits_{i=0}^{n}\left[-D+d(x)\right]^i r_i(x)$ is $(-1)^n r_n(x)$. Next, $r_n(x) =$

0. Similarly,

$$r_{n-1}(x) = r_{n-2}(x) = \cdots = r_0(x) = 0.$$

So ψ is one to one.

(5) Using (4.4), we can get that ψ is onto. $\square$

Next, we extend the involution $\psi(x) = \frac{ax+b}{cx-a}$ on $\mathbb{R}(x)$ to $\mathbb{R}(x)[D]$ by the following Theorem 4.2.

Theorem 4.2. *Let $S = \mathbb{R}(x)[D]$, $a,\ b,\ c \in \mathbb{R}$ and $a^2 + bc \neq 0$. Then the unique form of the involution ψ on S with $\psi(x) = \frac{ax+b}{cx-a}$ is*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} \left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$, $b_1(x)$ and $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$. ψ is an involution on $S \Longleftrightarrow a_1(x)$ and $b_1(x)$ satisfy

$$a_1(x)\psi[a_1(x)] = 1,$$

and

$$b_1(x)\psi(a_1(x)) + \psi(b_1(x)) = 0.$$

Proof. Let ψ be an involution on S. By Lemma 4.4,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$.

Now, we prove that the unique form of the involution on S is

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}[\psi(D)]^i\psi(r_i(x)) = \sum_{i=0}^{n} \left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)).$$

We give the proof of the necessary and sufficient condition for ψ to be an involution on S by the following steps.

$(\Longrightarrow)$

$$\begin{aligned}\psi\left[\psi(D)\right] &= \psi\left[a_1(x)D + b_1(x)\right] = \psi(D)\psi(a_1(x)) + \psi(b_1(x)),\\ &= \left[a_1(x)D + b_1(x)\right]\psi\left[a_1(x)\right] + \psi(b_1(x)),\\ &= a_1(x)D\psi(a_1(x)) + b_1(x)\psi(a_1(x)) + \psi(b_1(x)),\\ &= a_1(x)\psi[a_1(x)]D + b_1(x)\psi(a_1(x)) + \psi(b_1(x)).\end{aligned}$$

Then

$$a_1(x)\psi[a_1(x)] = 1,$$

and

$$b_1(x)\psi(a_1(x)) + \psi(b_1(x)) = 0. \tag{4.5}$$

$(\Longleftarrow)$ Next, we prove that

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[a_1(x)D + b_1(x)\right]^i\psi(r_i(x)),$$

is an involution on S, where $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(1) Evidently,

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i + \sum_{i=0}^{n} t_i(x)D^i\right) = \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right) + \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right), \tag{4.6}$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(2) For any $r(x)$ and $t(x) \in \mathbb{R}(x)$,

$$\begin{aligned}\psi[(r(x)D^m)(t(x)D^n)] &= \psi[r(x)t(x)D^{m+n}],\\ &= \left[a_1(x)D + b_1(x)\right]^{m+n}\psi[r(x)t(x)],\end{aligned}$$

and

$$\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] = \left[a_1(x)D + b_1(x)\right]^n \psi(t(x))\left[a_1(x)D + b_1(x)\right]^m \psi(r(x)).$$

Then

$$\psi\left[(r(x)D^m)(t(x)D^n)\right] = \psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right].$$

So

$$\psi\left(\left(\sum_{i=0}^{n} r_i(x)D^i\right)\left(\sum_{j=0}^{n} t_j(x)D^j\right)\right) = \psi\left(\sum_{j=0}^{n} t_j(x)D^j\right)\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right), \tag{4.7}$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(3) In the above proof, if

$$a_1(x)\psi[a_1(x)] = 1,$$

and

$$b_1(x)\psi(a_1(x)) + \psi(b_1(x)) = 0,$$

then

$$\psi(\psi(D)) = D.$$

By (4.7),

$$\psi(r_i(x)D^i) = \psi(D^i)\psi(r_i(x)).$$

Next,

$$\psi\left[\psi(r_i(x)D^i)\right] = \psi\left[\psi(D^i)\psi(r_i(x))\right] = \psi\left[\psi(r_i(x))\right]\psi\left[\psi(D^i)\right] = r_i(x)D^i.$$

It follows from (4.6) that

$$\psi\left(\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right)\right) = \sum_{i=0}^{n} r_i(x)D^i, \tag{4.8}$$

for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(4) Suppose

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} [a_1(x)D + b_1(x)]^i \, \psi(r_i(x)) = 0.$$

It is easy to check that the coefficient of D^n in

$$\sum_{i=0}^{n} [a_1(x)D + b_1(x)]^i \, \psi(r_i(x)),$$

is $a_1^n(x)\psi(r_n(x))$. Then $a_1^n(x)\psi(r_n(x)) = 0$, and

$$r_n(x) = 0.$$

Similarly,

$$r_{n-1}(x) = r_{n-2}(x) = \cdots = r_0(x) = 0.$$

So ψ is one to one.

(5) Using (4.8), we get that ψ is onto. □

4.2.2 $\sigma = 1$ and $\delta \neq 0$

We extend all the involutions ψ on $\mathbb{R}(x)$ (i.e., $\psi(x) = x$ and $\psi(x) = \frac{ax+b}{cx-a}$) to $\mathbb{R}(x)[D;\delta]$ (i.e., $\sigma = 1$ and $\delta \neq 0$ in $\mathbb{R}(x)[D;\sigma,\delta]$) by Theorem 4.3 and Theorem 4.4.

Lemma 4.5. *Let $S = \mathbb{R}(x)[D,\delta]$ and ψ be an involution on S. Then*

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$.

Proof. It is similar to the proof of Lemma 4.4. □

In order to prove Theorem 4.3 and Theorem 4.4, we need to give the proofs of the following lemmas.

Lemma 4.6. *([29]) Let $S = \mathbb{R}(x)[D;\delta]$. For any $a(x) \in \mathbb{R}(x)$,*

$$D^h\,[a(x)] = \sum_{i=0}^{h} \binom{h}{i} \delta^{h-i}\,[a(x)]\,D^i, \tag{4.9}$$

and

$$a(x)D^h = \sum_{i=0}^{h} (-1)^{h-i} \binom{h}{i} D^i \delta^{h-i}\,[a(x)]. \tag{4.10}$$

Lemma 4.7. *Let $S = \mathbb{R}(x)[D;\delta]$. For any $b(x)$ and $d(x) \in \mathbb{R}(x)$,*

$$b(x)\,[-D + d(x)]^m = \sum_{i=0}^{m} \binom{m}{i} [-D + d(x)]^i\,\delta^{m-i}\,[b(x)]. \tag{4.11}$$

Proof. We prove (4.11) by mathematical induction.

(1) For $m = 1$, clearly,

$$b(x)\,[-D + d(x)] = \delta\,[b(x)] + [-D + d(x)]\,b(x). \tag{4.12}$$

(2) Assume that (4.11) is true for $m = k$.

(3) For $m = k + 1$,

$$b(x)\,[-D + d(x)]^{k+1} = \left(\sum_{i=0}^{k} \binom{k}{i} [-D + d(x)]^i\,\delta^{k-i}\,[b(x)] \right) [-D + d(x)]\,.$$

By (4.12),

$$\delta^{k-i}\,[b(x)]\,[-D + d(x)] = \delta^{k+1-i}\,[b(x)] + [-D + d(x)]\,\delta^{k-i}\,[b(x)]\,.$$

Then

$$\begin{aligned}
&b(x)\left[-D+d(x)\right]^{k+1},\\
&=\sum_{i=0}^{k}\binom{k}{i}\left[-D+d(x)\right]^{i}\delta^{k+1-i}\left[b(x)\right]+\left[-D+d(x)\right]^{i+1}\delta^{k-i}\left[b(x)\right],\\
&=\sum_{i=0}^{k+1}\binom{k+1}{i}\left[-D+d(x)\right]^{i}\delta^{k+1-i}\left[b(x)\right].
\end{aligned}$$

□

Lemma 4.8. *Let $S = \mathbb{R}(x)[D; \delta]$. For any $a(x)$ and $d(x) \in \mathbb{R}(x)$,*

$$\left[-D+d(x)\right]^{m}a(x)=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\delta^{m-i}\left[a(x)\right]\left[-D+d(x)\right]^{i}. \tag{4.13}$$

Proof. We prove (4.13) by mathematical induction.

(1) For $m = 1$, clearly,

$$\left[-D+d(x)\right]a(x)=-\delta\left[a(x)\right]+a(x)\left[-D+d(x)\right]. \tag{4.14}$$

(2) Assume that (4.13) is true for $m = k$.

(3) For $m = k + 1$,

$$\left[-D+d(x)\right]^{k+1}a(x)=\left[-D+d(x)\right]\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\delta^{k-i}\left[a(x)\right]\left[-D+d(x)\right]^{i}\right).$$

From (4.14),

$$\left[-D+d(x)\right]\delta^{k-i}\left[a(x)\right]=-\delta^{k+1-i}\left[a(x)\right]+\delta^{k-i}\left[a(x)\right]\left[-D+d(x)\right].$$

Then

$$
\begin{aligned}
&[-D+d(x)]^{k+1}\,a(x)\\
&=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(-\delta^{k+1-i}\,[a(x)]+\delta^{k-i}\,[a(x)]\,[-D+d(x)]\right)[-D+d(x)]^{i},\\
&=\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^{k+1-i}\delta^{k+1-i}\,[a(x)]\,[-D+d(x)]^{i}.
\end{aligned}
$$

□

Lemma 4.9. *Let $S = \mathbb{R}(x)[D;\delta]$. If $a_1(x) = \frac{(cx-a)^2}{a^2+bc} \in \mathbb{R}(x)$ and $\psi(x) = \frac{ax+b}{cx-a}$, where $a,\ b,\ c \in \mathbb{R}$ and $a^2+bc \neq 0$, then*

$$
\psi\,[t(x)]\,[a_1(x)D+b_1(x)]^m = \sum_{i=0}^{m}\binom{m}{i}\,[a_1(x)D+b_1(x)]^i\,\psi[\delta^{m-i}(t(x))], \tag{4.15}
$$

for any $b_1(x)$ and $t(x) \in \mathbb{R}(x)$.

Proof. We prove (4.15) by mathematical induction.

(1) For $m=1$,

$$
\begin{aligned}
&\sum_{i=0}^{m}\binom{m}{i}\,[a_1(x)D+b_1(x)]^i\,\psi[\delta^{m-i}(t(x))],\\
&=[a_1(x)D+b_1(x)]\,\psi(t(x))+\psi[\delta(t(x))],\\
&=a_1(x)\psi(t(x))D+a_1(x)\delta[\psi(t(x))]+b_1(x)\psi(t(x))+\psi[\delta(t(x))].
\end{aligned}
$$

Since

$$
a_1(x)=\frac{(cx-a)^2}{a^2+bc},\ \ \psi(x)=\frac{ax+b}{cx-a}, \tag{4.16}
$$

$$
a_1(x)\delta[\psi(t(x))]+\psi[\delta(t(x))]=0. \tag{4.17}
$$

Then

$$[a_1(x)D + b_1(x)]\,\psi(t(x)) + \psi[\delta(t(x))] = \psi(t(x))\,[a_1(x)D + b_1(x)]\,. \qquad (4.18)$$

(2) Assume that (4.15) is true for $m = k$.

(3) For $m = k + 1$,

$$\begin{aligned}
&\psi(t(x))\,[a_1(x)D + b_1(x)]^{k+1}\,, \\
&= \psi(t(x))\,[a_1(x)D + b_1(x)]^{k}\,[a_1(x)D + b_1(x)]\,, \\
&= \left\{\sum_{i=0}^{k}\binom{k}{i}\,[a_1(x)D + b_1(x)]^{i}\,\psi[\delta^{k-i}(t(x))]\right\}[a_1(x)D + b_1(x)]\,.
\end{aligned}$$

From (4.18),

$$\psi[\delta^{k-i}(t(x))]\,[a_1(x)D + b_1(x)] = \psi[\delta^{k+1-i}(t(x))] + [a_1(x)D + b_1(x)]\,\psi[\delta^{k-i}(t(x))].$$

So

$$\psi(t(x))\,[a_1(x)D + b_1(x)]^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}\,[a_1(x)D + b_1(x)]^{i}\,\psi[\delta^{k+1-i}(t(x))].$$

□

Next, we extend the involution $\psi(x) = x$ on $\mathbb{R}(x)$ to $\mathbb{R}(x)[D;\delta]$ by the following Theorem 4.3.

Theorem 4.3. *Let $S = \mathbb{R}(x)[D;\delta]$. Then every involution ψ on S with $\psi(x) = x$ can be written as*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\,[-D + d(x)]^{i}\,r_i(x),$$

where $r_i(x)$ and $d(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

Proof. Let ψ be an involution on S. By Lemma 4.5,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$. Next,

$$\begin{aligned}
\psi\left[\psi(D)\right] = \psi\left[a_1(x)D + b_1(x)\right] &= \psi(D)a_1(x) + b_1(x), \\
&= \left[a_1(x)D + b_1(x)\right]a_1(x) + b_1(x), \\
&= a_1(x)Da_1(x) + b_1(x)a_1(x) + b_1(x), \\
&= a_1^2(x)D + a_1(x)\delta[a_1(x)] + b_1(x)a_1(x) + b_1(x).
\end{aligned}$$

So

$$a_1^2(x) = 1,$$

and

$$a_1(x)\delta[a_1(x)] + b_1(x)a_1(x) + b_1(x) = 0.$$

Clearly,

$$\begin{aligned}
\psi\left[(xD)x\right] = \psi\left[x(xD + 1)\right] = \psi(x^2D + x) &= \psi(x^2D) + \psi(x), \\
&= \left[a_1(x)D + b_1(x)\right]x^2 + x, \\
&= a_1(x)x^2D + 2a_1(x)x + b_1(x)x^2 + x,
\end{aligned}$$

and

$$\begin{aligned}
\psi(x)\psi(xD) &= x[a_1(x)D + b_1(x)]x, \\
&= x[a_1(x)xD + a_1(x) + b_1(x)x], \\
&= a_1(x)x^2D + a_1(x)x + b_1(x)x^2.
\end{aligned}$$

Since $\psi(x)\psi(xD) = \psi\left[(xD)x\right]$,

$$a_1(x) = -1.$$

Then

$$\psi(D) = -D + d(x),$$

where $d(x) \in \mathbb{R}(x)$. Thus

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}[\psi(D)]^i\psi(r_i(x)) = \sum_{i=0}^{n}\left[-D + d(x)\right]^i r_i(x),$$

where $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(1) It is easy to check that

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i + \sum_{i=0}^{n} t_i(x)D^i\right) = \psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) + \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right), \quad (4.19)$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(2) For any $r(x)$ and $t(x) \in \mathbb{R}(x)$,

$$\begin{aligned}
\psi\left[(r(x)D^m)(t(x)D^n)\right] &= \psi\left(r(x)\sum_{i=0}^{m}\binom{m}{i}\delta^{m-i}(t(x))D^iD^n\right), \\
&= \psi\left(\sum_{i=0}^{m}\binom{m}{i}r(x)\delta^{m-i}(t(x))D^{i+n}\right), \\
&= \sum_{i=0}^{m}\binom{m}{i}\left[-D + d(x)\right]^{i+n} r(x)\delta^{m-i}(t(x)),
\end{aligned}$$

and

$$\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] = \left[-D + d(x)\right]^n t(x)\left[-D + d(x)\right]^m r(x).$$

It follows from (4.11) that

$$\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] = \sum_{i=0}^{m}\binom{m}{i}\left[-D + d(x)\right]^{n+i}\delta^{m-i}(t(x))r(x).$$

Then

$$\psi[(r(x)D^m)(t(x)D^n)] = \psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right].$$

So

$$\psi\left(\left(\sum_{i=0}^{n} r_i(x)D^i\right)\left(\sum_{j=0}^{n} t_j(x)D^j\right)\right) = \psi\left(\sum_{j=0}^{n} t_j(x)D^j\right)\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right),$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(3) For any $r(x) \in \mathbb{R}(x)$,

$$\psi\left(\psi[r(x)D^m]\right) = \psi\left(\left[-D + d(x)\right]^m r(x)\right).$$

By (4.13),

$$\begin{aligned}\psi\left(\psi[r(x)D^m]\right) &= \psi\left(\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\delta^{m-i}(r(x))\left[-D + d(x)\right]^i\right),\\ &= \sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}(D)^i\delta^{m-i}(r(x)).\end{aligned}$$

By (4.10),

$$\psi\left(\psi[r(x)D^m]\right) = r(x)D^m.$$

From (4.19),

$$\psi\left(\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right)\right) = \sum_{i=0}^{n} r_i(x)D^i, \tag{4.20}$$

for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(4) Suppose

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[-D + d(x)\right]^i r_i(x) = 0.$$

Then the coefficient of D^n in $\sum_{i=0}^{n}\left[-D + d(x)\right]^i r_i(x)$ is $(-1)^n r_n(x)$ by (4.13). Next,

$r_n(x) = 0$. Similarly,

$$r_{n-1}(x) = r_{n-2}(x) = \cdots = r_0(x) = 0.$$

So ψ is one to one.

(5) Using (4.20), we can get that ψ is onto. $\square$

We extend the involution $\psi(x) = \frac{ax+b}{cx-a}$ on $\mathbb{R}(x)$ to $\mathbb{R}(x)[D; \delta]$ by the following Theorem 4.4.

Theorem 4.4. *Let $S = \mathbb{R}(x)[D; \delta]$, $a, b, c \in \mathbb{R}$ and $a^2 + bc \neq 0$. Then the unique form of the involution ψ on S with $\psi(x) = \frac{ax+b}{cx-a}$ is*

$$\psi\left(\sum_{i=0}^{n} r_i(x) D^i\right) = \sum_{i=0}^{n} \left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$, $b_1(x)$ and $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$. ψ is an involution on $S \Longleftrightarrow$

$$a_1(x) = \frac{(cx-a)^2}{a^2+bc},$$

and $b_1(x)$ satisfies

$$\frac{-2c}{cx-a} + b_1(x)\frac{a^2+bc}{(cx-a)^2} + \psi(b_1(x)) = 0.$$

Proof. Let ψ be an involution on S. By Lemma 4.5,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$.

Now, we prove that the unique form of the involution on S is

$$\psi\left(\sum_{i=0}^{n} r_i(x) D^i\right) = \sum_{i=0}^{n} [\psi(D)]^i \psi(r_i(x)) = \sum_{i=0}^{n} \left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)).$$

Next, we give the proof of the necessary and sufficient condition for ψ to be an involution on S.

($\Longrightarrow$)

$$\begin{aligned}
\psi\left[\psi(D)\right] &= \psi\left[a_1(x)D + b_1(x)\right] = \psi(D)\psi(a_1(x)) + \psi(b_1(x)),\\
&= \left[a_1(x)D + b_1(x)\right]\psi\left[a_1(x)\right] + \psi(b_1(x)),\\
&= a_1(x)D\psi(a_1(x)) + b_1(x)\psi(a_1(x)) + \psi(b_1(x)),\\
&= a_1(x)(\psi(a_1(x))D + \delta[\psi(a_1(x))]) + b_1(x)\psi(a_1(x)) + \psi(b_1(x)),\\
&= a_1(x)\psi[a_1(x)]D + a_1(x)\delta[\psi(a_1(x))] + b_1(x)\psi(a_1(x)) + \psi(b_1(x)).
\end{aligned}$$

Then

$$a_1(x)\psi[a_1(x)] = 1,$$

and

$$a_1(x)\delta[\psi(a_1(x))] + b_1(x)\psi(a_1(x)) + \psi(b_1(x)) = 0. \tag{4.21}$$

Since

$$\psi(x) = \frac{ax+b}{cx-a},$$

it is easy to get

$$\begin{aligned}
\psi\left[(xD)x\right] &= \psi\left[x(xD+1)\right] = \psi(x^2D + x) = \psi(x^2D) + \psi(x),\\
&= \left[a_1(x)D + b_1(x)\right]\left(\frac{ax+b}{cx-a}\right)^2 + \frac{ax+b}{cx-a},
\end{aligned}$$

and

$$\begin{aligned}
&\psi(x)\psi(xD),\\
&= \left(\frac{ax+b}{cx-a}\right)\left([a_1(x)D + b_1(x)]\,\frac{ax+b}{cx-a}\right),\\
&= \left(\frac{ax+b}{cx-a}\right)\left(a_1(x)D\frac{ax+b}{cx-a} + b_1(x)\frac{ax+b}{cx-a}\right),\\
&= \left(\frac{ax+b}{cx-a}\right)\left(a_1(x)\frac{ax+b}{cx-a}D + a_1(x)\left(\frac{-a^2-bc}{(cx-a)^2}\right) + b_1(x)\frac{ax+b}{cx-a}\right),\\
&= a_1(x)\left(\frac{ax+b}{cx-a}\right)^2 D + a_1(x)\left(\frac{ax+b}{cx-a}\right)\left(\frac{-a^2-bc}{(cx-a)^2}\right) + b_1(x)\left(\frac{ax+b}{cx-a}\right)^2.
\end{aligned}$$

By $\psi(x)\psi(xD) = \psi\left[(xD)x\right]$,

$$a_1(x)\left(\frac{ax+b}{cx-a}\right)\left(\frac{-a^2-bc}{(cx-a)^2}\right) + \frac{ax+b}{cx-a} = 0.$$

Then

$$a_1(x)\left(\frac{-a^2-bc}{(cx-a)^2}\right) + 1 = 0.$$

Next,

$$a_1(x) = \frac{(cx-a)^2}{a^2+bc} \neq 0,$$

and

$$\psi(a_1(x)) = \frac{a^2+bc}{(cx-a)^2}.$$

Moreover,

$$\delta(\psi(x)) = \frac{-a^2-bc}{(cx-a)^2}.$$

From (4.21),

$$\frac{(cx-a)^2}{a^2+bc}\delta\left(\frac{a^2+bc}{(cx-a)^2}\right) + b_1(x)\frac{a^2+bc}{(cx-a)^2} + \psi(b_1(x)) = 0,$$

$$\frac{(cx-a)^2}{a^2+bc}\left(-2c\frac{a^2+bc}{(cx-a)^3}\right) + b_1(x)\frac{a^2+bc}{(cx-a)^2} + \psi(b_1(x)) = 0,$$

$$\left(\frac{-2c}{cx-a}\right) + b_1(x)\frac{a^2+bc}{(cx-a)^2} + \psi(b_1(x)) = 0.$$

($\Longleftarrow$) Assume that

$$\frac{-2c}{cx-a} + b_1(x)\frac{a^2+bc}{(cx-a)^2} + \psi(b_1(x)) = 0,$$

$$a_1(x) = \frac{(cx-a)^2}{a^2+bc} \neq 0.$$

We prove that

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)),$$

is an involution on S by the following steps, where $r_i(x) \in \mathbb{R}(x)$, $i \in \{0,1,\ldots,n\}$.

(1) Evidently,

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i + \sum_{i=0}^{n} t_i(x)D^i\right) = \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right) + \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right), \quad (4.22)$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0,1,\ldots,n\}$.

(2) For any $r(x)$ and $t(x) \in \mathbb{R}(x)$,

$$\begin{aligned}\psi[(r(x)D^m)(t(x)D^n)] &= \psi\left(r(x)\sum_{i=0}^{m}\binom{m}{i}\delta^{m-i}(t(x))D^iD^n\right),\\ &= \psi\left(\sum_{i=0}^{m}\binom{m}{i}r(x)\delta^{m-i}(t(x))D^{i+n}\right),\\ &= \sum_{i=0}^{m}\binom{m}{i}\left[a_1(x)D + b_1(x)\right]^{i+n}\psi[r(x)\delta^{m-i}(t(x))],\end{aligned}$$

and

$$\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] = \left[a_1(x)D + b_1(x)\right]^n\psi(t(x))\left[a_1(x)D + b_1(x)\right]^m\psi(r(x)).$$

From (4.15),

$$\psi\left[(r(x)D^m)(t(x)D^n)\right] = \psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right].$$

So

$$\psi\left(\left(\sum_{i=0}^{n} r_i(x)D^i\right)\left(\sum_{j=0}^{n} t_j(x)D^j\right)\right) = \psi\left(\sum_{j=0}^{n} t_j(x)D^j\right)\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right), \tag{4.23}$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(3) In the above proof, if

$$a_1(x) = \frac{(cx-a)^2}{a^2+bc},$$

and $b_1(x)$ satisfies

$$\frac{-2c}{cx-a} + b_1(x)\frac{a^2+bc}{(cx-a)^2} + \psi(b_1(x)) = 0,$$

then

$$\psi(\psi(D)) = D.$$

By (4.23),

$$\psi(r_i(x)D^i) = \psi(D^i)\psi(r_i(x)).$$

Next,

$$\psi\left[\psi(r_i(x)D^i)\right] = \psi\left[\psi(D^i)\psi(r_i(x))\right] = \psi\left[\psi(r_i(x))\right]\psi\left[\psi(D^i)\right] = r_i(x)D^i.$$

It follows from (4.22) that

$$\psi\left(\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right)\right) = \sum_{i=0}^{n} r_i(x)D^i, \tag{4.24}$$

for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(4) Suppose

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} \left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)) = 0.$$

It is easy to check that the coefficient of D^n in

$$\sum_{i=0}^{n} \left[a_1(x)D + b_1(x)\right]^i \psi(r_i(x)),$$

is $a_1^n(x)\psi(r_n(x))$. Then $a_1^n(x)\psi(r_n(x)) = 0$, and

$$r_n(x) = 0.$$

Similarly,

$$r_{n-1}(x) = r_{n-2}(x) = \cdots = r_0(x) = 0.$$

So ψ is one to one.

(5) Using (4.24), we get that ψ is onto. □

By Theorem 4.3 and Theorem 4.4, we can give two examples of involutions on $\mathbb{R}(x)[D; \delta]$.

Example 4.1. *Let $S = \mathbb{R}(x)[D; \delta]$. Then*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} D^i r_i(-x),$$

is an involution on S, where $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

Example 4.2. *Let $S = \mathbb{R}(x)[D; \delta]$. For any $d \in \mathbb{R}$,*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} D^i r_i(-x + d),$$

is an involution on S, where $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

By Theorem 4.3 and Theorem 4.4, the Maple implementations of some examples of involutions on $\mathbb{R}(x)[D; \delta]$ are given in the Appendix.

4.2.3 $\sigma \neq 1$ and $\delta = 0$

We extend all the involutions ψ on $\mathbb{R}(x)$ (i.e., $\psi(x) = x$ and $\psi(x) = \frac{ax+b}{cx-a}$) to $\mathbb{R}(x)[D;\sigma]$ (i.e., $\sigma \neq 1$ and $\delta = 0$ in $\mathbb{R}(x)[D;\sigma,\delta]$) by Theorem 4.5 and Theorem 4.6.

Lemma 4.10. *Let $S = \mathbb{R}(x)[D,\sigma]$ and ψ be an involution on S. Then*

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$.

Proof. Let ψ be an involution on S and define

$$\psi(D) = \sum_{i=0}^{n} a_i(x)D^i,$$

where $a_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

$$\begin{aligned}
\psi\left[\psi(D)\right] =& \psi\left(\sum_{i=0}^{n} a_i(x)D^i\right) = \sum_{i=0}^{n} \psi(D^i)\psi(a_i(x)), \\
=& \sum_{i=0}^{n} \psi(D)^i \psi(a_i(x)), \\
=& \sum_{i=0}^{n} \left(\sum_{j=0}^{n} a_j(x)D^j\right)^i \psi(a_i(x)).
\end{aligned}$$

Clearly, the degree of $\psi\left[\psi(D)\right]$ is n^2 and the coefficient of D^{n^2} is

$$a_n(x)\sigma^n\left[a_n(x)\right]\sigma^{2n}\left[a_n(x)\right]\cdots\sigma^{n^2-n}\left[a_n(x)\right]\sigma^{n^2}\left[\psi(a_n(x))\right].$$

Since $\psi\left[\psi(D)\right] = D$,

$$a_n(x)\sigma^n\left[a_n(x)\right]\sigma^{2n}\left[a_n(x)\right]\cdots\sigma^{n^2-n}\left[a_n(x)\right]\sigma^{n^2}\left[\psi(a_n(x))\right] = 0.$$

As S is a domain and σ is an automorphisms of $\mathbb{R}(x)$, $a_n(x) = 0$. Similarly,

$$a_{n-1}(x) = \cdots = a_2(x) = 0.$$

Next, we can write

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x)$ and $b_1(x) \in \mathbb{R}(x)$. Clearly, $a_1(x) \neq 0$.

$\square$

Theorem 4.5. *Let $S = \mathbb{R}(x)[D;\sigma]$. Then the unique form of the involution ψ on S with $\psi(x) = x$ is*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} [a_1(x)D]^i\, r_i(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$. A necessary and sufficient condition for ψ to be an involution on $S = \mathbb{R}(x)[D;\sigma]$ is that

$$a_1(x)\sigma(a_1(x)) = 1,\ \sigma^2(x) = x.$$

Proof. Let ψ be an involution on S. By Lemma 4.10,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$. Clearly,

$$\begin{aligned}\psi\left[\psi(D)\right] =& \psi\left[a_1(x)D + b_1(x)\right] = \psi(D)\psi(a_1(x)) + \psi(b_1(x)),\\ &= \left[a_1(x)D + b_1(x)\right] a_1(x) + b_1(x),\\ &= a_1(x)Da_1(x) + b_1(x)a_1(x) + b_1(x),\\ &= a_1(x)\sigma\left[a_1(x)\right] D + b_1(x)a_1(x) + b_1(x).\end{aligned}$$

By $\psi\left[\psi(D)\right] = D$,

$$a_1(x)\sigma\left[a_1(x)\right] = 1,\ \ b_1(x)a_1(x) + b_1(x) = 0.$$

Similarly,

$$\begin{aligned}
\psi\left[(xD)x\right] = \psi(x\sigma(x)D) = \psi\left[\sigma(x)D\right]\psi(x) &= \psi(D)\sigma(x)x,\\
&= \left[a_1(x)D + b_1(x)\right]\sigma(x)x,\\
&= a_1(x)D\sigma(x)x + b_1(x)\sigma(x)x,\\
&= a_1(x)\sigma(x)\sigma^2(x)D + b_1(x)\sigma(x)x,
\end{aligned}$$

and

$$\begin{aligned}
\psi(x)\psi(xD) &= x\left[a_1(x)D + b_1(x)\right]x,\\
&= xa_1(x)Dx + xb_1(x)x,\\
&= xa_1(x)\sigma(x)D + xb_1(x)x.
\end{aligned}$$

Since $\psi(x)\psi(xD) = \psi\left[(xD)x\right]$ and $a_1(x) \neq 0$,

$$\sigma^2(x) = x,$$

and

$$b_1(x)\sigma(x) = b_1(x)x.$$

If $b_1(x) \neq 0$, then $\sigma(x) = x$, which has already been discussed in Theorem 4.1. So $b_1(x) = 0$. Now, we prove that the unique form of the involution on S is

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[a_1(x)D\right]^i r_i(x),$$

and a necessary condition for ψ to be an involution on S is

$$a_1(x)\sigma(a_1(x)) = 1,\ \sigma^2(x) = x.$$

We give the proof of the sufficient condition by the following steps.

(1) Evidently,

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i + \sum_{i=0}^{n} t_i(x)D^i\right) = \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right) + \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right), \quad (4.25)$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(2) For any $r(x)$ and $t(x) \in \mathbb{R}(x)$,

$$\begin{aligned}\psi\left[(r(x)D^m)(t(x)D^n)\right] &= \psi\left[r(x)\sigma^m(t(x))D^{m+n}\right],\\ &= \left[a_1(x)D\right]^{m+n} r(x)\sigma^m\left[t(x)\right],\end{aligned}$$

$$\begin{aligned}\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] &= \left[a_1(x)D\right]^n t(x)\left[a_1(x)D\right]^m r(x),\\ &= \left[a_1(x)D\right]^{m+n}\sigma^{-m}\left[t(x)\right]r(x).\end{aligned}$$

Since $\sigma^2(x) = x$,

$$\psi[(r(x)D^m)(t(x)D^n)] = \psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right].$$

So

$$\psi\left(\left(\sum_{i=0}^{n} r_i(x)D^i\right)\left(\sum_{j=0}^{n} t_j(x)D^j\right)\right) = \psi\left(\sum_{j=0}^{n} t_j(x)D^j\right)\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right), \quad (4.26)$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(3) In the above proof, if

$$a_1(x)\sigma(a_1(x)) = 1,$$

then

$$\psi\left[\psi(D)\right] = D.$$

By (4.26), for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$,

$$\psi\left[r_i(x)D^i\right] = \psi(D^i)r_i(x).$$

Next,

$$\psi[\psi(r_i(x)D^i)] = \psi[\psi(D^i)r_i(x)] = \psi\left[r_i(x)\right]\psi\left[\psi(D^i)\right] = r_i(x)D^i.$$

From (4.25), for any $r_i(x) \in \mathbb{R}(x)$

$$\psi\left(\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right)\right) = \sum_{i=0}^{n} r_i(x)D^i. \tag{4.27}$$

(4) Suppose

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}(a_1(x)D)^i r_i(x) = 0.$$

It is easy to check that the coefficient of D^n in $\sum\limits_{i=0}^{n}(a_1(x)D)^i r_i(x)$ is

$$a_1(x)\sigma^1\left[a_1(x)\right]\sigma^2\left[a_1(x)\right]\cdots\sigma^{n-1}\left[a_1(x)\right]\sigma^n(r_n(x)).$$

Then

$$r_n(x) = 0.$$

Similarly,

$$r_{n-1}(x) = r_{n-2}(x) = \cdots = r_0(x) = 0.$$

So ψ is one to one.

(5) By (4.27), it is easy to get that ψ is onto. $\square$

By the above Theorem 4.5, we can extend the involution $\psi(x) = x$ on $\mathbb{R}(x)$ to $\mathbb{R}(x)[D;\sigma]$. Next, we extend the involution $\psi(x) = \frac{ax+b}{cx-a}$ on $\mathbb{R}(x)$ to $\mathbb{R}(x)[D;\sigma]$

by the following Theorem 4.6.

Theorem 4.6. *Let $S = \mathbb{R}(x)[D;\sigma]$, $a,\ b,\ c \in \mathbb{R}$ and $a^2 + bc \neq 0$. Then the unique form of the involution ψ on S with $\psi(x) = \frac{ax+b}{cx-a}$ is*

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} [a_1(x)D]^i\,\psi(r_i(x)),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$. A necessary and sufficient condition for ψ to be an involution on $S = \mathbb{R}(x)[D;\sigma]$ is that

$$\sigma\left(\psi\left[\sigma(x)\right]\right) = \psi(x),\ a_1(x)\sigma[\psi(a_1(x))] = 1.$$

Proof. Let ψ be an involution on S. By Lemma 4.10,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$. Next,

$$\begin{aligned}
\psi\left[\psi(D)\right] =& \psi\left[a_1(x)D + b_1(x)\right] = \psi(D)\psi(a_1(x)) + \psi(b_1(x)),\\
&= \left[a_1(x)D + b_1(x)\right]\psi(a_1(x)) + \psi(b_1(x)),\\
&= a_1(x)D\psi(a_1(x)) + b_1(x)\psi(a_1(x)) + \psi(b_1(x)),\\
&= a_1(x)\sigma\left(\psi\left[a_1(x)\right]\right)D + b_1(x)\psi(a_1(x)) + \psi(b_1(x)).
\end{aligned}$$

Due to $\psi\left[\psi(D)\right] = D$, $a_1(x)\sigma\left(\psi\left[a_1(x)\right]\right) = 1$.

$$\begin{aligned}
\psi\left[(xD)x\right] &= \psi(x\sigma(x)D) = \psi(D)\psi\left[x\sigma(x)\right],\\
&= \left[a_1(x)D + b_1(x)\right]\psi\left[\sigma(x)\right]\psi(x),\\
&= a_1(x)D\psi\left[\sigma(x)\right]\psi(x) + b_1(x)\psi\left[\sigma(x)\right]\psi(x),\\
&= a_1(x)\sigma\left[\psi(x)\right]\sigma\left(\psi\left[\sigma(x)\right]\right)D + b_1(x)\psi\left[\sigma(x)\right]\psi(x),
\end{aligned}$$

and

$$\begin{aligned}\psi(x)\psi(xD) &= \psi(x)\psi(D)\psi(x),\\ &= \psi(x)\left[a_1(x)D + b_1(x)\right]\psi(x),\\ &= \psi(x)a_1(x)D\psi(x) + b_1(x)\left[\psi(x)\right]^2,\\ &= \psi(x)a_1(x)\sigma\left[\psi(x)\right]D + b_1(x)\left[\psi(x)\right]^2.\end{aligned}$$

Since $\psi\left[(xD)x\right] = \psi(x)\psi(xD)$,

$$\sigma\left(\psi\left[\sigma(x)\right]\right) = \psi(x),$$

and

$$b_1(x)\psi\left[\sigma(x)\right]\psi(x) = b_1(x)\left[\psi(x)\right]^2.$$

If $b_1(x) \neq 0$, then $\sigma(x) = x$, which has already been discussed in Theorem 4.2. So $b_1(x) = 0$. Now, we prove that the unique form of the involution on S is

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n}\left[a_1(x)D\right]^i\psi(r_i(x)),$$

and a necessary condition for ψ to be an involution on $S = \mathbb{R}(x)[D;\sigma]$ is

$$\sigma\left(\psi\left[\sigma(x)\right]\right) = \psi(x),\ a_1(x)\sigma(\psi[a_1(x)]) = 1.$$

We give the proof of the sufficient condition by the following steps.

(1) Evidently,

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i + \sum_{i=0}^{n} t_i(x)D^i\right) = \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right) + \psi\left(\sum_{i=0}^{n} t_i(x)D^i\right), \quad (4.28)$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(2) For any $r(x)$ and $t(x) \in \mathbb{R}(x)$,

$$\begin{aligned}\psi\left[(r(x)D^m)(t(x)D^n)\right] &= \psi\left[r(x)\sigma^m(t(x))D^{m+n}\right], \\ &= [a_1(x)D]^{m+n}\,\psi\left[r(x)\right]\psi\left[\sigma^m(t(x))\right],\end{aligned}$$

and

$$\begin{aligned}\psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right] &= [a_1(x)D]^n\,\psi[t(x)]\,[a_1(x)D]^m\,\psi[r(x)], \\ &= [a_1(x)D]^{m+n}\,\sigma^{-m}\left(\psi[t(x)]\right)\psi[r(x)].\end{aligned}$$

Since $\sigma\left(\psi\left[\sigma(x)\right]\right) = \psi(x)$,

$$\sigma^m\left(\psi\left[\sigma^m(x)\right]\right) = \psi(x),$$

$$\psi\left[\sigma^m(x)\right] = \sigma^{-m}\left[\psi(x)\right].$$

Then

$$\psi\left[(r(x)D^m)(t(x)D^n)\right] = \psi\left[t(x)D^n\right]\psi\left[r(x)D^m\right].$$

So

$$\psi\left(\left(\sum_{i=0}^{n} r_i(x)D^i\right)\left(\sum_{j=0}^{n} t_j(x)D^j\right)\right) = \psi\left(\sum_{j=0}^{n} t_j(x)D^j\right)\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right), \tag{4.29}$$

for any $r_i(x)$ and $t_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$.

(3) In the above proof, if

$$a_1(x)\sigma(\psi[a_1(x)]) = 1,$$

then

$$\psi(\psi(D)) = D.$$

By (4.29), for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$,

$$\psi(r_i(x)D^i) = \psi(D^i)\psi\left(r_i(x)\right).$$

Next,

$$\psi\left[\psi(r_i(x)D^i)\right] = \psi\left[\psi(D^i)\psi\left(r_i(x)\right)\right] = \psi\left[\psi\left(r_i(x)\right)\right]\psi\left[\psi(D^i)\right] = r_i(x)D^i.$$

From (4.28), for any $r_i(x) \in \mathbb{R}(x)$, $i \in \{0, 1, \ldots, n\}$,

$$\psi\left(\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right)\right) = \sum_{i=0}^{n} r_i(x)D^i. \tag{4.30}$$

(4) Suppose

$$\psi\left(\sum_{i=0}^{n} r_i(x)D^i\right) = \sum_{i=0}^{n} \left[a_1(x)D\right]^i \psi(r_i(x)) = 0.$$

It is easy to check that the coefficient of D^n in $\sum\limits_{i=0}^{n} \left[a_1(x)D\right]^i \psi(r_i(x))$ is

$$a_1(x)\sigma^1\left[a_1(x)\right]\sigma^2\left[a_1(x)\right]\cdots\sigma^{n-1}\left[a_1(x)\right]\sigma^n\left[\psi(r_n(x))\right].$$

Then

$$r_n(x) = 0.$$

Similarly,

$$r_{n-1}(x) = r_{n-2}(x) = \cdots = r_0(x) = 0.$$

So ψ is one to one.

(5) By (4.30), it is clearly that ψ is onto. $\square$

4.2.4 $\sigma \neq 1$ and $\delta \neq 0$

We prove that every involution ψ on $\mathbb{R}(x)$ (i.e., $\psi(x) = x$ or $\psi(x) = \frac{ax+b}{cx-a}$) can not be extended to be an involution on $\mathbb{R}(x)[D;\sigma,\delta]$ by Theorem 4.7 and Theorem 4.8, where $\sigma \neq 1$, $\delta \neq 0$ and $\sigma\delta = \delta\sigma$.

Lemma 4.11. *Let $S = \mathbb{R}(x)[D,\sigma,\delta]$ and ψ be an involution on S, where $\sigma \neq 1$, $\delta \neq 0$ and $\sigma\delta = \delta\sigma$. Then*

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$.

Proof. It is similar to the proof of Lemma 4.10. □

Theorem 4.7. *Let $S = \mathbb{R}(x)[D;\sigma,\delta]$. If $\psi(x) = x$, $\sigma \neq 1$, $\delta \neq 0$ and $\sigma\delta = \delta\sigma$, then ψ can not be extended to be an involution on S.*

Proof. Let ψ be an involution on S. By Lemma 4.11,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$. Clearly,

$$\psi\left[(xD)x\right] = a_1(x)\sigma(x)\sigma^2(x)D + [\text{lower terms}],$$

and

$$\psi(x)\psi(xD) = xa_1(x)\sigma(x)D + [\text{lower terms}].$$

Since $\psi(x)\psi(xD) = \psi\left[(xD)x\right]$,

$$\sigma^2(x) = x.$$

Then $\sigma(x)$ can be written as

$$\sigma(x) = \frac{ax+b}{cx-a},$$

or

$$\sigma(x) = x,$$

where $a,\ b,\ c \in \mathbb{R}$ and $a^2 + bc \neq 0$. When

$$\sigma(x) = \frac{ax+b}{cx-a},$$

by $\delta\sigma(x) = \sigma\delta(x)$,

$$\frac{-a^2 - bc}{(cx-a)^2} = 1,$$

which is impossible. Thus

$$\sigma(x) = x,$$

which has been discussed in Theorem 4.3. $\square$

Theorem 4.8. *Let $S = \mathbb{R}(x)[D;\sigma,\delta]$. If $\psi(x) = \frac{ax+b}{cx-a}$, $\sigma \neq 1$, $\delta \neq 0$ and $\sigma\delta = \delta\sigma$, where $a,\ b,\ c \in \mathbb{R}$ and $a^2 + bc \neq 0$, then ψ can not be extended to be an involution on S.*

Proof. Let ψ be an involution on S. By Lemma 4.11,

$$\psi(D) = a_1(x)D + b_1(x),$$

where $a_1(x) \in \mathbb{R}(x) \setminus \{0\}$ and $b_1(x) \in \mathbb{R}(x)$. Since

$$\delta\sigma = \sigma\delta,$$

σ can be written as

$$\sigma(x) = x + d,$$

where $d \in \mathbb{R}$ by Lemma 4.3.

Clearly,

$$\psi\left[(xD)x\right] = a_1(x)\sigma\left[\psi(\sigma(x))\right]\sigma\left[\psi(x)\right]D + [\text{lower terms}],$$

and

$$\psi(x)\psi(xD) = a_1(x)\psi(x)\sigma\left[\psi(x)\right]D + [\text{lower terms}].$$

Next,

$$\sigma\left[\psi(\sigma(x))\right] = \psi(x),$$

$$\sigma\left[\psi(x+d)\right] = \frac{ax+b}{cx-a},$$

$$\sigma\left(\frac{ax+ad+b}{cx+cd-a}\right) = \frac{ax+b}{cx-a},$$

$$\frac{ax+2ad+b}{cx+2cd-a} = \frac{ax+b}{cx-a},$$

$$(a^2+bc)d = 0.$$

So

$$d = 0,\ \ \sigma(x) = x,$$

which has been discussed in Theorem 4.4. □

4.3 Restrictions in $\mathbb{R}(x)$

Lemma 4.12. *([5], Involutions, p.244) Let* $S = \mathbb{R}(x)[D; \sigma, \delta]$, $\sigma \neq 1$, $\delta \neq 0$ *and* $\sigma\delta = \delta\sigma$. *If* ψ *is an involution on* S*, then the image of* ψ *in* $\mathbb{R}(x)$ *is*

$$\psi(x) = x,$$

or

$$\psi(x) = \frac{ax+b}{cx-a},$$

where $a,\ b,\ c \in \mathbb{R}$ *and* $a^2+bc \neq 0$.

Proof. By Lemma 4.2, it is easy to give the proof. □

Chapter 5

Applications of Generalized Inverses of Ore Matrices

In this chapter, we consider how to find the solutions of linear systems by using $\{1\}$-inverses of Ore matrices. Furthermore, we give two different explicit formulas for $\{1,3\}$- and $\{1,4\}$-inverses of Ore matrices. We also introduce some formulas for $\{2\}$-, $\{2,3\}$- and $\{2,4\}$-inverses with prescribed inner ranks of Ore matrices, and projectors over S.

5.1 Solutions of Linear Systems

In Section 3.1, we have already found some necessary and sufficient conditions for the existence of $\{1\}$-inverses of Ore matrices and their explicit formulas. In this section, we solve linear systems over S by using $\{1\}$-inverses of Ore matrices.

Theorem 5.1. *Let $A \in S^{m\times n}$, $B \in S^{p\times q}$ and $D \in S^{m\times q}$. If $A^{(1)} \in S^{n\times m}$ and $B^{(1)} \in S^{q\times p}$ both exist over S, then a necessary and sufficient condition for*

$$AXB = D,$$

to have solutions over S is that there are $A^{(1)}$ and $B^{(1)}$ over S such that

$$AA^{(1)}DB^{(1)}B = D,$$

in which case

$$X = A^{(1)}DB^{(1)} + Z - A^{(1)}AZBB^{(1)}, \tag{5.1}$$

is the general solution, where $Z \in S^{n\times p}$ is an arbitrary Ore matrix.

Proof. It is similar to the proof of Theorem 2.1 in [7]. □

Corollary 5.1. *Let $A \in S^{m\times n}$. If $A^{(1)}$ exists over S, then*

$$A\{1\} = \{A^{(1)} + Y - A^{(1)}AYAA^{(1)} : \ Y \in S^{n\times m}\}.$$

Proof. It is similar to the proof of Corollary 2.1 in [7]. □

By Theorem 3.4 and Corollary 5.1, we can give another formula to find all $\{1\}$-inverses of Ore matrices.

Corollary 5.2. *Let $A \in S^{m\times n}$ and $b \in S^m$. If $A^{(1)}$ exists over S, then a necessary and sufficient condition for*

$$AX = b,$$

to have solutions over S is that there is a $A^{(1)}$ such that

$$AA^{(1)}b = b,$$

in which case

$$X = A^{(1)}b + (I - A^{(1)}A)s,$$

is the general solution, where $s \in S^n$ is an arbitrary Ore matrix.

Proof. It is similar to the proof of Corollary 2.2 in [7]. □

5.2 Other Formulas for $\{1,3\}$- and $\{1,4\}$-Inverses

In this section, we give two different explicit formulas for $\{1,3\}$- and $\{1,4\}$-inverses of Ore matrices.

Theorem 5.2. *For any $A \in S^{m\times n}$, X is a $\{1,3\}$-inverse of A over $S \Longleftrightarrow X$ is a solution of*

$$AX = AA^{(1,3)},$$

where $A^{(1,3)}$ is an arbitrary $\{1,3\}$-inverse of A.

Proof. It is similar to the proof of Theorem 2.3 in [7]. □

Corollary 5.3. *For any $A \in S^{m\times n}$, if $A^{(1,3)}$ exists over S, then*

$$A\{1,3\} = \{A^{(1)}AA^{(1,3)} + (I - A^{(1)}A)Y : \ Y \in S^{n\times m}\}.$$

Proof. This corollary is obtained by extending Corollary 2.3 in [7] to Ore matrices. By Theorem 5.1, the general solution of

$$AX = AA^{(1,3)},$$

is

$$X = A^{(1)}AA^{(1,3)} + (I - A^{(1)}A)Y.$$

By Theorem 5.2,

$$A\{1,3\} = \{A^{(1)}AA^{(1,3)} + (I - A^{(1)}A)Y : \ Y \in S^{n\times m}\}.$$

□

In Theorem 3.24, we give a explicit formula to compute a $\{1,3\}$-inverse of a given Ore matrix. Now, by Corollary 5.3, we can find all $\{1,3\}$-inverses of a given Ore matrix.

Theorem 5.3. *For any $A \in S^{m\times n}$, X is a $\{1,4\}$-inverse of A over $S \Longleftrightarrow X$ is a solution of*

$$XA = A^{(1,4)}A,$$

where $A^{(1,4)}$ is an arbitrary $\{1,4\}$-inverse of A.

Proof. It is similar to the proof of Theorem 2.4 in [7]. □

Corollary 5.4. *For any $A \in S^{m\times n}$, if $A^{(1,4)}$ exists over S, then*

$$A\{1,4\} = \{A^{(1,4)}AA^{(1)} + Z(I - AA^{(1)}) : \; Z \in S^{n\times m}\}.$$

Proof. It is similar to the proof of Corollary 5.3. This corollary is obtained by extending Corollary 2.4 in [7] to Ore matrices. □

In Theorem 3.25, we give a explicit formula to compute a $\{1,4\}$-inverse of a given Ore matrix. Now, by Corollary 5.4, we can find all $\{1,4\}$-inverses of a given Ore matrix.

Proposition 5.1. *For any $A \in S^{m\times n}$, if $A^{(1,4)}$ and $A^{(1,3)}$ both exist over S, then $A^{(1,4)}AA^{(1,3)}$ is the MP-inverse of A over S.*

Proof. This proposition is obtained by extending Exercise 2.10 in [7] to Ore matrices.
We give the proof by Theorem 5.2 and Theorem 5.3. Write

$$X = A^{(1,4)}AA^{(1,3)}.$$

Then

$$AX = AA^{(1,4)}AA^{(1,3)} = AA^{(1,3)},$$

$$XA = A^{(1,4)}AA^{(1,3)}A = A^{(1,4)}A.$$

So $X \in A\{1,3,4\}$ by Theorem 5.2 and Theorem 5.3. Clearly,

$$\begin{aligned} XAX &= A^{(1,4)}AA^{(1,3)}AA^{(1,4)}AA^{(1,3)} = A^{(1,4)}AA^{(1,4)}AA^{(1,3)}, \\ &= A^{(1,4)}AA^{(1,3)} = X. \end{aligned}$$

□

5.3 Generalized Inverses with Prescribed Inner Ranks

In this section, we give some formulas for $\{2\}$-, $\{2,3\}$- and $\{2,4\}$-inverses with prescribed inner ranks of Ore matrices. $A\{i,j\}_r$ is the set of all $\{i,j\}$-inverses of an Ore matrix A with the prescribed inner rank r.

The following Theorem 5.4 provides us a formula to compute $\{2\}$-inverses with prescribed inner ranks of Ore matrices.

Theorem 5.4. *For any $A \in S_r^{m\times n}$ and any integer $t \in (0,\ r]$. If $U^{(1)} \in S^{t\times n}$ and $V^{(1)} \in S^{m\times t}$ both exist over S, then*

$$A\{2\}_t = \{UV :\ U \in S^{n\times t},\ V \in S^{t\times m},\ VAU = I_t\}.$$

Proof. It is similar to the proof of Theorem 2.5 in [7]. □

Corollary 5.5. *For any $A \in S_r^{m\times n}$. If $U^{(1)} \in S^{r\times n}$ and $V^{(1)} \in S^{m\times r}$ both exist over S, then*

$$A\{1,2\} = \{UV :\ U \in S^{n\times r},\ V \in S^{r\times m},\ VAU = I_r\}.$$

Proof. It is similar to the proof of Corollary 2.5 in [7]. □

The following Theorem 5.5 gives us a formula to compute $\{2,3\}$-inverses with prescribed inner ranks of Ore matrices.

Theorem 5.5. *For any $A \in S_r^{m\times n}$ and any integer $t \in (0,\ r]$. If $(AU)^\dagger \in S_t^{t\times m}$ exists over S, then*

$$A\{2,3\}_t = \{U(AU)^\dagger:\ U \in S^{n\times t},\ AU \in S_t^{m\times t}\}.$$

Proof. It is similar to the proof of Theorem 2.6 in [7]. □

The following Theorem 5.6 gives us a formula to compute $\{2,4\}$-inverses with prescribed inner ranks of Ore matrices.

Theorem 5.6. *For any $A \in S_r^{m\times n}$ and any integer $t \in (0,\ r]$. If $(UA)^\dagger \in S_t^{n\times t}$ exists over S, then*

$$A\{2,4\}_t = \{(UA)^\dagger U:\ U \in S^{t\times m},\ UA \in S_t^{t\times n}\}.$$

Proof. It is similar to the proof of Theorem 2.7 in [7]. □

5.4 Projectors

In this section, we discuss idempotent matrices, direct sums, complementary spaces, projectors and some related important theorems over S, which are crucial for the future study of generalized inverses of Ore matrices.
The following Lemma 5.1 and Lemma 5.2 give us some properties of idempotent Ore matrices.

Lemma 5.1. *Let $A \in S^{m\times m}$ be idempotent (i.e., $A^2 = A$). Then*

(a) $(A^*)^2 = A^*$ and $(I - A)^2 = I - A$.

(b) $(I - A)A = A(I - A) = \mathbf{0}$.

(c) $As = s \Longleftrightarrow s \in \mathcal{R}(A)$, where $s \in S^m$.

(d) A is a $\{1,2\}$-inverse of A over S.

(e) $\mathcal{R}(I-A)=\mathcal{N}(A)$.

Proof. This lemma is obtained by extending Lemma 2.1 in [7] to Ore matrices.

(a)

$$A^*A^*=(A^2)^*=A^*,$$

$$(I-A)^2=I-2A+A^2=I-A.$$

(b) It is easy to give the proof.

(c) ($\Longrightarrow$) It is easy to give the proof. ($\Longleftarrow$) Since $s\in\mathcal{R}(A)$, there exists a $s_1\in S^m$ such that $s=As_1$. So

$$As=AAs_1=As_1=s.$$

(d) It is easy to give the proof.

(e) Since $A\in A\{1\}$, the general solution of $AX=\mathbf{0}$ is

$$X=(I-AA)z,$$

by Corollary 5.2, where $z\in S^m$. Then

$$\mathcal{N}(A)\subseteq\mathcal{R}(I-A).$$

On the other hand, for any $X\in\mathcal{R}(I-A)$,

$$X=(I-A)y,$$

where $y\in S^m$, then

$$AX=A(I-A)y=Ay-A^2y=\mathbf{0}.$$

So

$$\mathcal{R}(I-A)\subseteq\mathcal{N}(A).$$

□

Lemma 5.2. *For any $A \in S_r^{m\times m}$, let $A = FG$ be a full rank factorization over S. If $F^{(1)} \in S^{r\times m}$ and $G^{(1)} \in S^{m\times r}$ both exist over S, then $A^2 = A \Longleftrightarrow GF = I$.*

Proof. It is similar to the proof of Lemma 2.2 in [7]. □

Clearly, S^n is a left (or right) linear space over S. So we can define the left (or right) subspace of S^n over S. Now, we introduce the definition and properties of the direct sum of two Ore matrices.

Definition 5.1. *([7], 1.3, p.6) Let M and N be two sets of S^n.*

$$M + N = \{m + n : \ m \in M, \ n \in N\},$$

is called the sum of M and N.

It is clearly that if M and N are two left (or right) subspaces of S^n over S, then $M + N$ is a left (or right) subspace of S^n over S.

Definition 5.2. *([7], 1.3, p.6) Let M and N be two left (or right) subspaces of S^n over S. $M + N$ is called the direct sum if any $s \in S^n$ can be written uniquely as*

$$s = m + n \ (m \in M, \ n \in N),$$

denoted by $M \oplus N$.

Proposition 5.2. *If M and N are two left (or right) subspaces of S^n over S, then the following statements are equivalent:*

(a) $M + N$ is called the direct sum of M and N.

(b) $M \cap N = \{\mathbf{0}\}$.

(c) If $m + n = \mathbf{0}$, where $m \in M$ and $n \in N$, then $m = n = \mathbf{0}$.

Proof. It is easy to give the proof. This proposition is obtained by extending Exercise 0.1 in [7] to Ore matrices. □

Next, we give the definition of the projectors over S.

Definition 5.3. *([7], 1.3, p.6) Let $S^n = M \oplus N$. Then M and N are called complementary over S. For any $s \in S^n$, m is called the projection of s on M along N if $s = m + n$ $(m \in M,\ n \in N)$.*

Definition 5.4. *([7], p.59) The transformation that carries any $s \in S^n$ into its projection on M along N is called the projector on M along N, denoted by $P_{M,N}$.*

Clearly, $P_{M,N}$ implies that $S^n = M \oplus N$ and $M \cap N = \{\mathbf{0}\}$. So

$$P_{M,N} S^n = M.$$

The following Theorem 5.7 helps us to finish the proofs of Corollary 5.6 and Proposition 5.6.

Theorem 5.7. *If $A \in S^{n\times n}$ and $A^2 = A$, then*

$$S^n = \mathcal{R}(A) \oplus \mathcal{N}(A),$$

and

$$A = P_{\mathcal{R}(A),\mathcal{N}(A)}.$$

If $S^n = M \oplus L$, then there exists a unique idempotent $P_{M,L}$ such that

$$\mathcal{R}(P_{M,L}) = M,\ \mathcal{N}(P_{M,L}) = L.$$

Proof. It is similar to the proof of Theorem 2.8 in [7]. $\square$

For any $A \in \mathbb{C}^{m\times n}$, $\mathcal{N}(A^{(1)}A) = \mathcal{N}(A)$ and $\mathcal{R}(AA^{(1)}) = \mathcal{R}(A)$. We extend this result to S by the following Proposition 5.3.

Proposition 5.3. *For any $A \in S^{m\times n}$,*

$$\mathcal{N}(A^{(1)}A) = \mathcal{N}(A),\ \mathcal{R}(AA^{(1)}) = \mathcal{R}(A),$$

$$\mathcal{N}((AA^{(1)})^*) = \mathcal{N}(A^*), \ \mathcal{R}((A^{(1)}A)^*) = \mathcal{R}(A^*).$$

Proof. This proposition is obtained by extending Exercise 1.9 in [7] to Ore matrices.

$$\mathcal{R}(AB) = \mathcal{R}(A) \Longleftrightarrow \rho(AB) = \rho(A),$$

$$\mathcal{N}(AB) = \mathcal{N}(B) \Longleftrightarrow \rho(AB) = \rho(B),$$

and

$$\rho(AA^{(1)}) = \rho(A^{(1)}A) = \rho(A), \ \rho(A^*) = \rho(A).$$

□

Corollary 5.6. *For any $A \in S^{m\times n}$, if $AXA = A$ and $XAX = X$, then*

$$AX = P_{\mathcal{R}(A),\mathcal{N}(X)}, \ XA = P_{\mathcal{R}(X),\mathcal{N}(A)}.$$

Proof. This corollary is obtained by extending Corollary 2.7 in [7] to Ore matrices. Since $(AX)^2 = AX$, we can get $S^n = \mathcal{R}(AX) \oplus \mathcal{N}(AX)$ and $AX = P_{\mathcal{R}(AX),\mathcal{N}(AX)}$ by Theorem 5.7. Due to

$$\mathcal{R}(AX) = \mathcal{R}(A), \ \mathcal{N}(AX) = \mathcal{N}(X),$$

$$S^n = \mathcal{R}(A) \oplus \mathcal{N}(X), \ AX = P_{\mathcal{R}(A),\mathcal{N}(X)}.$$

Similarly,

$$S^n = \mathcal{R}(X) \oplus \mathcal{N}(A), \ XA = P_{\mathcal{R}(X),\mathcal{N}(A)}.$$

□

Finally, we extend some other results of the generalized inverses over $\mathbb{C}$ to S.

Proposition 5.4. *For any $A \in S^{n\times n}$ and $X \in S^{n\times n}$, if $A^2 = A$, $XAX = X$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, then $X^2 = X$.*

Proof. This proposition is obtained by extending Exercise 2.18 in [7] to Ore matrices.

Since $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, $X \in \mathcal{R}(A)$. From Lemma 5.1 (c),

$$X = AX.$$

Since $XAX = X$,

$$X^2 = X.$$

□

Proposition 5.5. *For any $A \in S^{m\times n}$,*

$$AP_{M,N} = A \Longleftrightarrow \mathcal{N}(A) \supseteq N, \tag{5.2}$$

$$P_{M,N}A = A \Longleftrightarrow \mathcal{R}(A) \subseteq M. \tag{5.3}$$

Proof. This proposition is obtained by extending Exercise 2.20 in [7] to Ore matrices. We give the proof of (5.3).

($\Longleftarrow$) Since $P_{M,N}M = M$ and $\mathcal{R}(A) \subseteq M$,

$$P_{M,N}A = A.$$

($\Longrightarrow$) Clearly,

$$\mathcal{R}(P_{M,N}A) \subseteq \mathcal{R}(P_{M,N}) = M.$$

So

$$\mathcal{R}(A) \subseteq \mathcal{R}(P_{M,N}) = M.$$

□

Proposition 5.6. *For any $A \in S^{m\times n}$ and $B \in S^{n\times q}$, if $(AB)^{(1)}$ exists over S,* *then*

$$B(AB)^{(1)}AB = B \Longleftrightarrow \rho(AB) = \rho(B), \tag{5.4}$$

$$AB(AB)^{(1)}A = A \Longleftrightarrow \rho(AB) = \rho(A). \tag{5.5}$$

Proof. This proposition is obtained by extending Exercise 2.21 in [7] to Ore matrices. We give the proof of (5.5).

($\Longrightarrow$) Since $AB(AB)^{(1)}A = A$,

$$\rho(AB) \geq \rho(A).$$

It is clearly that $\rho(AB) \leq \rho(A)$, then

$$\rho(AB) = \rho(A).$$

($\Longleftarrow$) From Theorem 3.21,

$$\mathcal{R}(AB) = \mathcal{R}(A),$$

and

$$\mathcal{R}(AB) = \mathcal{R}(AB(AB)^{(1)}).$$

So

$$\mathcal{R}(AB(AB)^{(1)}) = \mathcal{R}(A).$$

As $AB(AB)^{(1)}$ is idempotent,

$$AB(AB)^{(1)} = P_{\mathcal{R}(AB(AB)^{(1)}),\mathcal{N}(AB(AB)^{(1)})},$$

by Theorem 5.7. Since $\mathcal{R}(A) \subseteq \mathcal{R}(AB(AB)^{(1)})$,

$$P_{\mathcal{R}(AB(AB)^{(1)}),\mathcal{N}(AB(AB)^{(1)})}A = A,$$

by Proposition 5.5. So

$$AB(AB)^{(1)}A = A.$$

$\square$

Appendix A

Maple Implementations

A.1 Maple Codes

1 Maple Codes in the Rational Function Field

In this chapter, we construct procedures to compute Jacobson forms and generalized inverses of polynomial matrices over the rational function field.

1.1 Compute the Jacobson Form of a Given Polynomial Matrix Over the Rational Function Field

```
> restart
> with(LinearAlgebra) :
```

We can get three results by the following procedure, the first result is the Jacobson form of a m by n polynomial matrix B, the second result $totalE$ and the third result $totalF$ are two polynomial matrices such that $totalE \cdot B \cdot totalF$= the Jacobson form of B.

```
> JF := proc(B, m, n)#the Jacobson Form
   local w, A, totalE, totalinE, totalF, totalinF, e, E, inE, F, inF, i, k, l, b11, b21;
  local inE12, inE22, EE, tt, j, b12, inF21, inF22, FF, tt1;
   w := max(m, n) :# we achieve Jacobson Form of nosquare matrix by this code
  A := B :
  totalE := Matrix(IdentityMatrix(m)) : totalinE := Matrix(IdentityMatrix(m)) : totalF
       := Matrix(IdentityMatrix(n)) :
  totalinF := Matrix(IdentityMatrix(n)) :
  for e from 0 by 1 to w - 2 do
  E := Matrix(IdentityMatrix(m)) : inE := Matrix(IdentityMatrix(m)) :
  F := Matrix(IdentityMatrix(n)) : inF := Matrix(IdentityMatrix(n)) :
  for i from 1 by 1 to m - e - 1 do
  if e ≥ min(m, n) then
          break
      end if:# we achieve Jacobson Form of nosquare matrix by this code
  if A[1 + e, 1 + e] ≠ 0 and A[2 + e, 1 + e] ≠ 0 then
  gcdex(simplify(A[1 + e, 1 + e]), simplify(A[2 + e, 1 + e]), x,'s','t') :
  k := simplify(lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[1 + e, 1 + e]) : l
       := simplify(lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[2 + e, 1 + e]) :
  E(1 + e, 1 + e) := s : E(1 + e, 2 + e) := t : E(2 + e, 1 + e) := k : E(2 + e, 2 + e) :=
      -l :
   elif A[1 + e, 1 + e] = 0
```

```
then
  E(1 + e, 1 + e) := 0 : E(1 + e, 2 + e) := 1 : E(2 + e, 1 + e) := 1 : E(2 + e, 2
    + e) := 0 :
 elif A[2 + e, 1 + e] = 0# we do not need to do anything
 then E(1 + e, 1 + e) := 1 : E(1 + e, 2 + e) := 0 : E(2 + e, 1 + e) := 0 : E(2
    + e, 2 + e) := 1 :
 end if:
totalE := simplify(E.totalE);
A := simplify(E.A) : if i = m - e - 1 then break end if:
EE := Matrix(IdentityMatrix(m)) :
tt := A[2 + e, 1 ..-1] :
A[2 + e, 1 ..-1] := A[i + 2 + e, 1 ..-1] :
A[i + 2 + e, 1 ..-1] := tt :
tt := EE[2 + e, 1 ..-1] :
EE[2 + e, 1 ..-1] := EE[i + 2 + e, 1 ..-1] :
EE[i + 2 + e, 1 ..-1] := tt :
totalE := simplify(EE.totalE);
end do:
for j from 1 by 1 to n - e - 1 do
if e ≥ min(m, n) then
      break
   end if:# we achieve Jacobson Form of nosquare matrix by this code
if A[1 + e, 1 + e] ≠ 0 and A[1 + e, 2 + e] ≠ 0  then
 gcdex(simplify(A[1 + e, 1 + e]), simplify(A[1 + e, 2 + e]), x,'s','t') :
k := simplify( lcm(A[1 + e, 1 + e], A[1 + e, 2 + e]) / A[1 + e, 1 + e] ) : l
    := simplify( lcm(A[1 + e, 1 + e], A[1 + e, 2 + e]) / A[1 + e, 2 + e] ) :
F(1 + e, 1 + e) := s : F(1 + e, 2 + e) := k : F(2 + e, 1 + e) := t : F(2 + e, 2 + e) := -l :

elif A[1 + e, 1 + e] = 0
then
  F(1 + e, 1 + e) := 0 : F(1 + e, 2 + e) := 1 : F(2 + e, 1 + e) := 1 : F(2 + e, 2
    + e) := 0 :
 elif A[1 + e, 2 + e] = 0# we do not need to do anything
  then F(1 + e, 1 + e) := 1 : F(1 + e, 2 + e) := 0 : F(2 + e, 1 + e) := 0 : F(2
    + e, 2 + e) := 1 :
 end if:
totalF := simplify(totalF.F);
```

```
A := simplify(A.F) : if j = n − e − 1 then break end if:
FF := Matrix(IdentityMatrix(n)) :
tt1 := A[1 ..-1, 2 + e] :
A[1 ..-1, 2 + e] := A[1 ..-1, j + 2 + e] :
A[1 ..-1, j + 2 + e] := tt1 :
tt1 := FF[1 ..-1, 2 + e] :
FF[1 ..-1, 2 + e] := FF[1 ..-1, j + 2 + e] :
FF[1 ..-1, j + 2 + e] := tt1 :# this is the inverse of an elementary matrix
totalF := simplify(totalF.FF);
end do:
end do:
print(A);
print(totalE) :
print(totalF) :
end proc:
```

Example 1

> $B := Matrix([[1, x^2+x, 3x^3, x, x+1], [x^3, x^2, 6x^2, x, x+4], [x^3+x, x^2+1, 6x^2, x, x+5]])$

$$B := \begin{bmatrix} 1 & x^2+x & 3x^3 & x & x+1 \\ x^3 & x^2 & 6x^2 & x & x+4 \\ x^3+x & x^2+1 & 6x^2 & x & x+5 \end{bmatrix} \quad \textbf{(1)}$$

The Jacobson form of B:

> $JF(B, 3, 5)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2x^2+5x & -2x^3+x^2-4x+6 & 2x^3-x^2+2x-5 \\ -x^3+x^2+4x & -x^4-x^3-x^2+5 & x^4+x^3-x-4 \end{bmatrix}$$

$$\left[\left[1, x+1, \frac{39}{76}x-\frac{5}{4}, -\frac{117}{76}x^7-\frac{27}{38}x^6+\frac{633}{76}x^5-\frac{144}{19}x^4+\frac{144}{19}x^3-\frac{3}{2}x^2, x^2+3x\right],\right. \quad \textbf{(2)}$$

$$\left[0, 0, \frac{39}{76}x^3 - \frac{15}{19}x^2 - \frac{13}{76}x - \frac{1}{4}, -\frac{3}{76}\left(39x^3 - 60x^2 - 13x - 19\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), x^4 - 3x^2 - x\right],$$

$$\left[0, 0, 0, -1, 0\right],$$

$$\left[0, 0, -\frac{39}{76}x^4 + \frac{15}{19}x^3 + \frac{91}{76}x^2 + \frac{55}{76}x - \frac{71}{76}, \frac{3}{76}\left(39x^4 - 60x^3 - 91x^2 - 55x + 71\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), -x^5 + 5x^3 + 5x^2 - x - 4\right],$$

$$\left[0, -1, -\frac{39}{76}x^3 + \frac{21}{76}x^2 - \frac{11}{19}x + \frac{5}{4}, \frac{117}{76}x^9 - \frac{45}{19}x^8 - \frac{39}{76}x^7 - \frac{3}{4}x^6 + \frac{117}{76}x^5 - \frac{45}{19}x^4 + \frac{39}{38}x^3 + \frac{3}{2}x^2, -x^4 - x^3 + x\right]\Big]$$

> $totalE := \begin{bmatrix} 1 & 0 & 0 \\ -2x^2 + 5x & -2x^3 + x^2 - 4x + 6 & 2x^3 - x^2 + 2x - 5 \\ -x^3 + x^2 + 4x & -x^4 - x^3 - x^2 + 5 & x^4 + x^3 - x - 4 \end{bmatrix}$

$$totalE := \begin{bmatrix} 1 & 0 & 0 \\ -2x^2 + 5x & -2x^3 + x^2 - 4x + 6 & 2x^3 - x^2 + 2x - 5 \\ -x^3 + x^2 + 4x & -x^4 - x^3 - x^2 + 5 & x^4 + x^3 - x - 4 \end{bmatrix} \tag{3}$$

> $totalF := \Big[\left[1, x + 1, \frac{39}{76}x - \frac{5}{4}, -\frac{117}{76}x^7 - \frac{27}{38}x^6 + \frac{633}{76}x^5 - \frac{144}{19}x^4 + \frac{144}{19}x^3 - \frac{3}{2}x^2, x^2 + 3x\right],$

$$\left[0, 0, \frac{39}{76}x^3 - \frac{15}{19}x^2 - \frac{13}{76}x - \frac{1}{4}, -\frac{3}{76}\left(39x^3 - 60x^2 - 13x - 19\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), x^4 - 3x^2 - x\right],$$

$$\left[0, 0, 0, -1, 0\right],$$

$$\left[0, 0, -\frac{39}{76}x^4 + \frac{15}{19}x^3 + \frac{91}{76}x^2 + \frac{55}{76}x - \frac{71}{76}, \frac{3}{76}\left(39x^4 - 60x^3 - 91x^2 - 55x + 71\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), -x^5 + 5x^3 + 5x^2 - x - 4\right],$$

$$\left[0, -1, -\frac{39}{76}x^3 + \frac{21}{76}x^2 - \frac{11}{19}x + \frac{5}{4}, \frac{117}{76}x^9 - \frac{45}{19}x^8 - \frac{39}{76}x^7 - \frac{3}{4}x^6 + \frac{117}{76}x^5 - \frac{45}{19}x^4 + \frac{39}{38}x^3 + \frac{3}{2}x^2, -x^4 - x^3 + x\right]\Big]:$$

Check that $totalE \cdot B \cdot totalF$=the Jacobson form of B :

```
> simplify(totalE.B.totalF)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (4)$$

Example 2

```
> B := Matrix([[0, 0, 0, 0], [x^3, x^2, 6 x^2, x], [x^3 + x, x^2 + 1, 6 x^2, x]])
```

$$B := \begin{bmatrix} 0 & 0 & 0 & 0 \\ x^3 & x^2 & 6x^2 & x \\ x^3+x & x^2+1 & 6x^2 & x \end{bmatrix} \quad (5)$$

The Jacobson form of B:

```
> JF(B, 3, 4)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & x^2+1 & -x^2 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -x & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6x \end{bmatrix} \quad (6)$$

```
>
```

1.2 Construct a {1}-Inverse of a Given Polynomial Matrix Over the Rational Function Field

We can get a {1}-inverse of the above m by n polynomial matrix B by the following procedure. The parameter AA is the Jacobson form of B, parameters $totalE$ and $totalF$ are two polynomial matrices such that $totalE \cdot B \cdot totalF$=the Jacobson form of B.

```
> INV1 :=proc(AA, totalE, totalF, m, n)# {1}-inverse
  local AAA, q, i;
```

```
AAA := Matrix(n, m) :
q := min(m, n) :
for i from 1 by 1 to q do
if AA[i, i] ≠ 0 then
AAA[i, i] := 1/AA[i, i] :
end if:
end do:
#print(AAA);
print(totalF.AAA.totalE);
end proc:
```

```
> A := Matrix([[1, x^2 + x, 3 x^3, x, x + 1], [x^3, x^2, 6 x^2, x, x + 4], [x^3 + x, x^2 + 1, 6 x^2, x, x
    + 5]])
```

$$A := \begin{bmatrix} 1 & x^2+x & 3x^3 & x & x+1 \\ x^3 & x^2 & 6x^2 & x & x+4 \\ x^3+x & x^2+1 & 6x^2 & x & x+5 \end{bmatrix} \tag{7}$$

We construct a {1}-inverse of A by the following steps, and the method is from Theorem 3.3 in our thesis.

Step 1: Compute the Jacobson form of A.

The Jacobson form of A :

```
> JF(A, 3, 5)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2x^2+5x & -2x^3+x^2-4x+6 & 2x^3-x^2+2x-5 \\ -x^3+x^2+4x & -x^4-x^3-x^2+5 & x^4+x^3-x-4 \end{bmatrix}$$

$$\Big[\Big[1, x+1, \frac{39}{76}x - \frac{5}{4}, -\frac{117}{76}x^7 - \frac{27}{38}x^6 + \frac{633}{76}x^5 - \frac{144}{19}x^4 + \frac{144}{19}x^3 - \frac{3}{2}x^2, x^2 + 3x\Big], \tag{8}$$

$$\Big[0, 0, \frac{39}{76}x^3 - \frac{15}{19}x^2 - \frac{13}{76}x - \frac{1}{4}, -\frac{3}{76}(39x^3 - 60x^2 - 13x - 19)x^2(x^4 - x^3 - 2x^2 + 2x - 2), x^4 - 3x^2 - x\Big],$$

$\left[0, 0, 0, -1, 0\right]$,

$\left[0, 0, -\frac{39}{76}x^4 + \frac{15}{19}x^3 + \frac{91}{76}x^2 + \frac{55}{76}x - \frac{71}{76}, \frac{3}{76}\left(39x^4 - 60x^3 - 91x^2 - 55x + 71\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), -x^5 + 5x^3 + 5x^2 - x - 4\right]$,

$\left[0, -1, -\frac{39}{76}x^3 + \frac{21}{76}x^2 - \frac{11}{19}x + \frac{5}{4}, \frac{117}{76}x^9 - \frac{45}{19}x^8 - \frac{39}{76}x^7 - \frac{3}{4}x^6 + \frac{117}{76}x^5 - \frac{45}{19}x^4 + \frac{39}{38}x^3 + \frac{3}{2}x^2, -x^4 - x^3 + x\right]\Big]$

$$> AA := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} : totalE$$

$$:= \begin{bmatrix} 1 & 0 & 0 \\ -2x^2 + 5x & -2x^3 + x^2 - 4x + 6 & 2x^3 - x^2 + 2x - 5 \\ -x^3 + x^2 + 4x & -x^4 - x^3 - x^2 + 5 & x^4 + x^3 - x - 4 \end{bmatrix}:$$

$> totalF := \Big[\left[1, x + 1, \frac{39}{76}x - \frac{5}{4}, -\frac{117}{76}x^7 - \frac{27}{38}x^6 + \frac{633}{76}x^5 - \frac{144}{19}x^4 + \frac{144}{19}x^3 - \frac{3}{2}x^2, x^2 + 3x\right]$,

$\left[0, 0, \frac{39}{76}x^3 - \frac{15}{19}x^2 - \frac{13}{76}x - \frac{1}{4}, -\frac{3}{76}\left(39x^3 - 60x^2 - 13x - 19\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), x^4 - 3x^2 - x\right]$,

$\left[0, 0, 0, -1, 0\right]$,

$\left[0, 0, -\frac{39}{76}x^4 + \frac{15}{19}x^3 + \frac{91}{76}x^2 + \frac{55}{76}x - \frac{71}{76}, \frac{3}{76}\left(39x^4 - 60x^3 - 91x^2 - 55x + 71\right)x^2\left(x^4 - x^3 - 2x^2 + 2x - 2\right), -x^5 + 5x^3 + 5x^2 - x - 4\right]$,

$\left[0, -1, -\frac{39}{76}x^3 + \frac{21}{76}x^2 - \frac{11}{19}x + \frac{5}{4}, \frac{117}{76}x^9 - \frac{45}{19}x^8 - \frac{39}{76}x^7 - \frac{3}{4}x^6 + \frac{117}{76}x^5 - \frac{45}{19}x^4 + \frac{39}{38}x^3 + \frac{3}{2}x^2, -x^4 - x^3 + x\right]\Big]:$

Step 2: Construct a {1}-inverse of A by Theorem 3.3 in our thesis.

$> INV1(AA, totalE, totalF, 3, 5)$

$\Big[\Big[1 + (x + 1)\left(-2x^2 + 5x\right) + \left(\frac{39}{76}x - \frac{5}{4}\right)\left(-x^3 + x^2 + 4x\right), (x + 1)\left(-2x^3 + x^2\right.$ **(9)**

$$-4x+6)+\left(\frac{39}{76}x-\frac{5}{4}\right)(-x^4-x^3-x^2+5),\ (x+1)\,(2x^3-x^2+2x-5)$$
$$+\left(\frac{39}{76}x-\frac{5}{4}\right)(x^4+x^3-x-4)\Big],$$
$$\Big[\left(\frac{39}{76}x^3-\frac{15}{19}x^2-\frac{13}{76}x-\frac{1}{4}\right)(-x^3+x^2+4x),\ \left(\frac{39}{76}x^3-\frac{15}{19}x^2-\frac{13}{76}x\right.$$
$$\left.-\frac{1}{4}\right)(-x^4-x^3-x^2+5),\ \left(\frac{39}{76}x^3-\frac{15}{19}x^2-\frac{13}{76}x-\frac{1}{4}\right)(x^4+x^3-x-4)\Big],$$
$$[0,0,0],$$
$$\Big[\left(-\frac{39}{76}x^4+\frac{15}{19}x^3+\frac{91}{76}x^2+\frac{55}{76}x-\frac{71}{76}\right)(-x^3+x^2+4x),\ \left(-\frac{39}{76}x^4\right.$$
$$\left.+\frac{15}{19}x^3+\frac{91}{76}x^2+\frac{55}{76}x-\frac{71}{76}\right)(-x^4-x^3-x^2+5),\ \left(-\frac{39}{76}x^4+\frac{15}{19}x^3\right.$$
$$\left.+\frac{91}{76}x^2+\frac{55}{76}x-\frac{71}{76}\right)(x^4+x^3-x-4)\Big],$$
$$\Big[2x^2-5x+\left(-\frac{39}{76}x^3+\frac{21}{76}x^2-\frac{11}{19}x+\frac{5}{4}\right)(-x^3+x^2+4x),\ -6+2x^3-x^2$$
$$+4x+\left(-\frac{39}{76}x^3+\frac{21}{76}x^2-\frac{11}{19}x+\frac{5}{4}\right)(-x^4-x^3-x^2+5),\ 5-2x^3+x^2$$
$$-2x+\left(-\frac{39}{76}x^3+\frac{21}{76}x^2-\frac{11}{19}x+\frac{5}{4}\right)(x^4+x^3-x-4)\Big]\Big]$$

The {1}-inverse of A :

$$> X := \Big[\Big[1+(x+1)\,(-2x^2+5x)+\left(\frac{39}{76}x-\frac{5}{4}\right)(-x^3+x^2+4x),\ (x+1)\,(-2x^3$$
$$+x^2-4x+6)+\left(\frac{39}{76}x-\frac{5}{4}\right)(-x^4-x^3-x^2+5),\ (x+1)\,(2x^3-x^2+2x-5)$$
$$+\left(\frac{39}{76}x-\frac{5}{4}\right)(x^4+x^3-x-4)\Big],$$
$$\Big[\left(\frac{39}{76}x^3-\frac{15}{19}x^2-\frac{13}{76}x-\frac{1}{4}\right)(-x^3+x^2+4x),\ \left(\frac{39}{76}x^3-\frac{15}{19}x^2-\frac{13}{76}x\right.$$
$$\left.-\frac{1}{4}\right)(-x^4-x^3-x^2+5),\ \left(\frac{39}{76}x^3-\frac{15}{19}x^2-\frac{13}{76}x-\frac{1}{4}\right)(x^4+x^3-x-4)\Big],$$
$$[0,0,0],$$
$$\Big[\left(-\frac{39}{76}x^4+\frac{15}{19}x^3+\frac{91}{76}x^2+\frac{55}{76}x-\frac{71}{76}\right)(-x^3+x^2+4x),\ \left(-\frac{39}{76}x^4+\frac{15}{19}x^3\right.$$
$$\left.+\frac{91}{76}x^2+\frac{55}{76}x-\frac{71}{76}\right)(-x^4-x^3-x^2+5),\ \left(-\frac{39}{76}x^4+\frac{15}{19}x^3+\frac{91}{76}x^2\right.$$
$$\left.+\frac{55}{76}x-\frac{71}{76}\right)(x^4+x^3-x-4)\Big],$$
$$\Big[2x^2-5x+\left(-\frac{39}{76}x^3+\frac{21}{76}x^2-\frac{11}{19}x+\frac{5}{4}\right)(-x^3+x^2+4x),\ -6+2x^3-x^2$$

$$+4x+\left(-\frac{39}{76}x^3+\frac{21}{76}x^2-\frac{11}{19}x+\frac{5}{4}\right)(-x^4-x^3-x^2+5), 5-2x^3+x^2-2x$$
$$+\left(-\frac{39}{76}x^3+\frac{21}{76}x^2-\frac{11}{19}x+\frac{5}{4}\right)(x^4+x^3-x-4)\Big]\Big]:$$

Check that $AXA = A$:

```
> simplify(A.X.A) − A
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10)$$

2 Maple Codes in the Ore_algebra Package

In this chapter, we construct Ore matrices and skew polynomials by the Ore_algebra package. We achieve algorithm implementations to find Jacobson forms and involutions of Ore matrices. Since there exist some problems (please see Section 2.1 Problems in the Ore_algebra Package) about GCRDs, GCLDs, LCLMs, LCRMs in the Ore_algebra package, we could not transform all Ore matrices into diagonal matrices, which makes us could not get generalized inverses of Ore matrices. But we can still give some involutions of Ore matrices.

2.1 Problems in the Ore_algebra Package

```
> restart; with(Ore_algebra) :
> A := diff_algebra([Dx, x]) :
```

Problem 1

```
> G := skew_gcdex(4/5, 3/2, Dx, A, "monic")
```

$$G := \left[1, 0, \frac{1}{15}, 15, -8\right] \quad (11)$$

According to the description of the skew_gcdex in the skew_gcdex package, the following two results should both be 0.

```
> simplify(skew_product(G[2], 4/5, A) + skew_product(G[3], 3/2, A) − G[1])
```

$$-\frac{9}{10} \quad (12)$$

$$-\frac{9}{10} \quad \textbf{(12)}$$

```
> simplify(skew_product(G[4], 4/5, A) + skew_product(G[5], 3/2, A))# right lcm
```

$$0 \quad \textbf{(13)}$$

According to the description of the skew_gcdex in the skew_gcdex package, the first entry of the following result should be x.

```
> skew_gcdex(x, x, Dx, A, "monic")# right gcd
```

$$[1, 0, 1, 1, -1] \quad \textbf{(14)}$$

Problem 2

We could not get the GCRD of two skew polynomials, if one skew polynomial is a fractional polynomial, which is necessary in the algorithm of calculating the Jacobson form of a given Ore matrix.

```
> skew_gcdex(1/x, x, Dx, A, "monic")# right gcd
```

Error, invalid input: Ore_algebra:-skew_gcdex expects its 1st argument, P, to be of type polynom, but received 1/x

Problem 3

According to the description of the skew_gcdex in the skew_gcdex package, the first entries of the following two results should both be 1.

```
> G := skew_gcdex(1, x, Dx, A)
```

$$G := [x, 0, 1, x, -1] \quad \textbf{(15)}$$

```
> G := skew_gcdex(x, 1, Dx, A)
```

$$G := [1, 0, 1, 1, -x] \quad \textbf{(16)}$$

2.2 Compute the Jacobson Form of a Given Ore Matrix

```
> restart;
  with(Ore_algebra) :
  WW := diff_algebra(diff = [Dx, x]) :# or skew_algebra(diff = [Dx, x]) :
     or diff_algebra([Dx, x]) :
  with(LinearAlgebra) :
```

2.2.1 The Multiplication of Two Ore Matrices

The following procedure is the multiplication of two Ore matrices, where parameter A is a m by p Ore matrix and parameter B is a p by n Ore matrix.

```
> Multli := proc(A, B, m, n, p) #the product of two Ore matrices
   local i, j , k, AB := Matrix(1..m, 1..n, 0);
   for i from 1 by 1 to m # m is the number of rows of A
  do
   for j from 1 by 1 to n # n is the number of columns of B
   do
   for k from 1 by 1 to p
     # p is the number of columns of A=the number of rows of B
   do
  AB[i, j] := simplify(AB[i, j] + skew_product(A[i, k], B[k, j], WW));
   # print(AB[i, j])
  end do:
  end do:
   end do:
   AB
   end proc:
```

2.2.2 The Jacobson Form of a Given Ore Matrix

We can get three results by the following procedure, the first result is the Jacobson form of a m by n Ore matrix B, the second result $totalE$ and the third result $totalF$ are two Ore matrices such that $totalE \cdot B \cdot totalF$= the Jacobson form of B.

```
> JF := proc(B, m, n)#   the Jacobson Form
  local inE12, inE22, EE, tt, j, b12, inF21, inF22, FF, tt1, H, H1, p;
   local w, A, totalE, totalinE, totalF, totalinF, e, E, inE, F, inF, i, k, l, b11, b21;
    A := B :
  w := max(m, n) :# we achieve Jacobson Form of nosquare matrix by this code
  totalE := Matrix(IdentityMatrix(m)) :
     totalF := Matrix(IdentityMatrix(n)) :
  w := min(m, n) :
  for e  from 0 by 1 to w - 2 do
   E := Matrix(IdentityMatrix(m)) :
```

```
F := Matrix(IdentityMatrix(n)) :
 for i from 1 by 1 to m - e - 1 do
 p := m :#we achieve the Jacobson Form of a nonsquare matrix by this code
if e ≥ min(m, n) then
        break
    end if:# we achieve Jacobson Form of nosquare matrix by this code
 if A[1 + e, 1 + e] ≠ 0 and A[2 + e, 1 + e] ≠ 0   then
#gcdex(simplify(A[1 + e, 1 + e]), simplify(A[2 + e, 1 + e]), x,'s','t') :
 #H:=ExtendedGCD_'right'(A[1 + e, 1 + e], A[2 + e, 1 + e], WW,'c1','c2') :
   # H is gcrd
 H := skew_gcdex(A[1 + e, 1 + e], A[2 + e, 1 + e], Dx, WW) :# right gcd
# k:=simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[1 + e, 1 + e] ) : l
     := simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[2 + e, 1 + e] ) :
 E(1 + e, 1 + e) := H[2] : E(1 + e, 2 + e) := H[3] : E(2 + e, 1 + e) := H[4] :
   E(2 + e, 2 + e) := H[5] :
 elif A[1 + e, 1 + e] = 0
then
  E(1 + e, 1 + e) := 0 : E(1 + e, 2 + e) := 1 : E(2 + e, 1 + e) := 1 : E(2 + e,
   2 + e) := 0 :
 elif A[2 + e, 1 + e] = 0# we do not need to do anything
 then E(1 + e, 1 + e) := 1 : E(1 + e, 2 + e) := 0 : E(2 + e, 1 + e) := 0 : E(2
    + e, 2 + e) := 1 :
 end if:
#b11:=simplify( A[1 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[2 + e, 1 + e]) ) :
 #b21:=simplify( A[2 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[2 + e, 1 + e]) ) :
 # gcdex(k, -1, x,'p','q') :
 #inE12:=simplify(p - b11·s·p - b11·t·q) :
#inE22:=simplify(q - b21·s·p - b21·t·q) :
 #inE(1 + e, 1 + e) := b11 : inE(1 + e, 2 + e) := inE12 : inE(2 + e, 1 + e)
     := b21 : inE(2 + e, 2 + e) := inE22 :
```

```
#totalinE:= simplify(totalinE.inE);
 totalE := Multli(E, totalE, m, m, p) :
 A := Multli(E, A, m, n, p) :
#print(A);#print(E);
if i = m - e - 1 then break end if:
EE := Matrix(IdentityMatrix(m)) :
tt := A[2 + e, 1 .. -1] :
 A[2 + e, 1 .. -1] := A[i + 2 + e, 1 .. -1] :
 A[i + 2 + e, 1 .. -1] := tt :
tt := EE[2 + e, 1 .. -1] :
 EE[2 + e, 1 .. -1] := EE[i + 2 + e, 1 .. -1] :
 EE[i + 2 + e, 1 .. -1] := tt :
 totalE := Multli(EE, totalE, m, m, p) :
 #totalinE:= simplify(totalinE.EE) :
end do:
 for j from 1 by 1 to n - e - 1 do
 p := n :#we achieve the Jacobson Form of a nonsquare matrix by this code
if e ≥ min(m, n) then
        break
    end if:# we achieve Jacobson Form of nosquare matrix by this code
  if A[1 + e, 1 + e] ≠ 0 and A[1 + e, 2 + e] ≠ 0  then
H1 := skew_gcdex(A[1 + e, 1 + e], A[1 + e, 2 + e], Dx, WW, "left") :
    #` `left gcd
F(1 + e, 1 + e) := H1[2] : F(1 + e, 2 + e) := H1[4] : F(2 + e, 1 + e)
    := H1[3] : F(2 + e, 2 + e) := H1[5] :
 elif A[1 + e, 1 + e] = 0
then
  F(1 + e, 1 + e) := 0 : F(1 + e, 2 + e) := 1 : F(2 + e, 1 + e) := 1 : F(2
    + e, 2 + e) := 0 :
 elif A[1 + e, 2 + e] = 0# we do not need to do anything
  then F(1 + e, 1 + e) := 1 : F(1 + e, 2 + e) := 0 : F(2 + e, 1 + e) := 0 : F(2
    + e, 2 + e) := 1 :
end if:
 totalF := Multli(totalF, F, n, n, p) :
```

```
A := Multli(A, F, m, n, p) :
#print(A);
if j = n − e − 1 then break end if:
FF := Matrix(IdentityMatrix(n)) :
tt1 := A[1 .. -1, 2 + e] :
A[1 .. -1, 2 + e] := A[1 .. -1, j + 2 + e] :
A[1 .. -1, j + 2 + e] := tt1 :
tt1 := FF[1 .. -1, 2 + e] :
FF[1 .. -1, 2 + e] := FF[1 .. -1, j + 2 + e] :
FF[1 .. -1, j + 2 + e] := tt1 :
  # this is the inverse of an elementary matrix
totalF := Multli(totalF, FF, n, n, p) :
end do:
end do:
print(A);
print(totalE);
print(totalF);
end proc:
```

2.2.3 An Example

```
> A := Matrix([[1, Dx^2 + x, 0, 0], [0, 1, 0, 0], [x, 0, 1, 1]]);
    # a 3 by 4 Ore matrix
```

$$A := \begin{bmatrix} 1 & Dx^2 + x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x & 0 & 1 & 1 \end{bmatrix} \tag{17}$$

```
> JF(A, 3, 4)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & Dx^2 x + x^2 & x & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ x & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -x & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \tag{18}$$

Due to the problems in the Ore_algebra package, we could not transform A into a diagonal matrix.

2.3 Involutions

2.3.1 Some Involutions of Skew Polynomials

The methods that we use to construct involutions of skew polynomials are from Theorem 4.3 and Theorem 4.4 in our thesis.

```
> with(PolynomialTools) :
```

Example 1

```
> Conjugate1 := proc(f) # Involution of skew polynomials
   local i, n, a, bb, b, j;
  a := CoefficientVector(f, Dx);
  n := numelems(a);
   b := 0;
      for i from 0 by 1 to n -1# n is the number of  coefficients
  do
   bb := 1;
   for j from 1 by 1 to i# n is the number of  coefficients
   do
    bb := simplify(skew_product(bb, (x^2·Dx + x), WW));
  end do:
    b := simplify(b + skew_product(bb, subs(x = 1/x, a[1 + i])
   , WW));
```

```
end do:
b;
end proc:
```

(1) Check that $(f1 \cdot f2)^* = (f2)^* \cdot (f1)^*$

```
> f1 := x Dx; f2 := Dx^5 x^3 - Dx^2 x^2 + Dx x; f3 := skew_product(f1, f2, WW)
```

$$f1 := Dx\,x$$
$$f2 := Dx^5 x^3 - Dx^2 x^2 + Dx\,x$$
$$f3 := Dx^6 x^4 + 3\,Dx^5 x^3 - Dx^3 x^3 - Dx^2 x^2 + Dx\,x \tag{19}$$

$(f2)^* \cdot (f1)^*$ **:**

```
> f4 := simplify(skew_product(Conjugate1(f2), Conjugate1(f1), WW))
```

$$f4 := Dx^6 x^8 + 15\,Dx^5 x^7 + 60\,Dx^4 x^6 + 60\,Dx^3 x^5 - Dx^3 x^3 - Dx^2 x^2 + Dx\,x \tag{20}$$

$(f1 \cdot f2)^*$ **:**

```
> f5 := Conjugate1(f3)
```

$$f5 := Dx^6 x^8 + 15\,Dx^5 x^7 + 60\,Dx^4 x^6 + 60\,Dx^3 x^5 - Dx^3 x^3 - Dx^2 x^2 + Dx\,x \tag{21}$$

$(f1 \cdot f2)^* - (f2)^* \cdot (f1)^*$ **:**

```
> f4 - f5
```

$$0 \tag{22}$$

(2) Check that $\left(f^*\right)^* = f$

```
> f := x^4·Dx^5 + Dx^5 x^3 - Dx^2
```

$$f := Dx^5 x^4 + Dx^5 x^3 - Dx^2 \tag{23}$$

f^* **:**

```
> ff1 := Conjugate1(f);
```

$$ff1 := Dx^5 x^7 + Dx^5 x^6 + 10\,Dx^4 x^6 + 5\,Dx^4 x^5 + 20\,Dx^3 x^5 - Dx^2 x^4 - 4\,Dx\,x^3 - 2\,x^2 \tag{24}$$

$\left(f^*\right)^* - f$**:**

```
> simplify(Conjugate1(ff1) - f)
```

$$0 \tag{25}$$

Example 2

```
> Conjugate2 := proc(f) # Involution of skew polynomials
   local i, n, a, bb, b, j;
  a := CoefficientVector(f, Dx);
  n := numelems(a);
   b := 0;
       for i from 0 by 1 to n -1# n is the number of  coefficients
  do
   bb := 1;
   for j from 1 by 1 to i# n is the number of  coefficients
   do
   bb := simplify(skew_product(bb, (x^2·Dx + (1 + x - x^2) ), WW));
  end do:
   b := simplify(b + skew_product(bb, subs(x = 1/x, a[1 + i])
   , WW));
   end do:
  b;
   end proc:
```

(1) Check that $(f1 \cdot f2)^* = (f2)^* \cdot (f1)^*$

```
> f1 := x Dx; f2 := Dx^5 x^3 - Dx^2 x^2 + Dx x; f3 := skew_product(f1, f2, WW)
```

$$f1 := Dx\, x$$

$$f2 := Dx^5\, x^3 - Dx^2\, x^2 + Dx\, x$$

$$f3 := Dx^6\, x^4 + 3\, Dx^5\, x^3 - Dx^3\, x^3 - Dx^2\, x^2 + Dx\, x \tag{26}$$

$(f2)^* \cdot (f1)^*$:

```
> f4 := simplify(skew_product(Conjugate2(f2), Conjugate2(f1), WW))
```

$$f4 := \frac{1}{x^4}\,(Dx^6\, x^{12} - 6\, Dx^5\, x^{12} + 15\, Dx^5\, x^{11} + 15\, Dx^4\, x^{12} + 6\, Dx^5\, x^{10} - 75\, Dx^4\, x^{11} \tag{27}$$
$$- 20\, Dx^3\, x^{12} + 30\, Dx^4\, x^{10} + 150\, Dx^3\, x^{11} + 15\, Dx^2\, x^{12} + 45\, Dx^4\, x^9 - 180\, Dx^3\, x^{10}$$
$$- 150\, Dx^2\, x^{11} - 6\, Dx\, x^{12} + 15\, Dx^4\, x^8 - 120\, Dx^3\, x^9 + 300\, Dx^2\, x^{10} + 75\, Dx\, x^{11}$$
$$+ x^{12} + 90\, Dx^2\, x^9 - 210\, Dx\, x^{10} - 15\, x^{11} + 29\, Dx^3\, x^7 - 90\, Dx^2\, x^8 + 54\, x^{10}$$

$+20\,Dx^3 x^6 - 87\,Dx^2 x^7 + 120\,Dx\,x^8 - 15\,x^9 - 61\,Dx^2 x^6 + 87\,Dx\,x^7 - 45\,x^8$
$- 33\,Dx^2 x^5 + 62\,Dx\,x^6 - 29\,x^7 + 15\,Dx^2 x^4 + 67\,Dx\,x^5 - 21\,x^6 + 34\,Dx\,x^4 - 34\,x^5$
$- 48\,Dx\,x^3 - 49\,x^4 + 6\,Dx\,x^2 - 15\,x^3 + 59\,x^2 - 16\,x + 1)$

$(\boldsymbol{f1}\cdot\boldsymbol{f2})^*$ **:**

```
> f5 := Conjugate2(f3)
```

$$f5 := \frac{1}{x^4}\big(Dx^6 x^{12} - 6\,Dx^5 x^{12} + 15\,Dx^5 x^{11} + 15\,Dx^4 x^{12} + 6\,Dx^5 x^{10} - 75\,Dx^4 x^{11} - 20\,Dx^3 x^{12} + 30\,Dx^4 x^{10} + 150\,Dx^3 x^{11} + 15\,Dx^2 x^{12} + 45\,Dx^4 x^9 - 180\,Dx^3 x^{10} - 150\,Dx^2 x^{11} - 6\,Dx\,x^{12} + 15\,Dx^4 x^8 - 120\,Dx^3 x^9 + 300\,Dx^2 x^{10} + 75\,Dx\,x^{11} + x^{12} + 90\,Dx^2 x^9 - 210\,Dx\,x^{10} - 15\,x^{11} + 29\,Dx^3 x^7 - 90\,Dx^2 x^8 + 54\,x^{10} + 20\,Dx^3 x^6 - 87\,Dx^2 x^7 + 120\,Dx\,x^8 - 15\,x^9 - 61\,Dx^2 x^6 + 87\,Dx\,x^7 - 45\,x^8 - 33\,Dx^2 x^5 + 62\,Dx\,x^6 - 29\,x^7 + 15\,Dx^2 x^4 + 67\,Dx\,x^5 - 21\,x^6 + 34\,Dx\,x^4 - 34\,x^5 - 48\,Dx\,x^3 - 49\,x^4 + 6\,Dx\,x^2 - 15\,x^3 + 59\,x^2 - 16\,x + 1\big) \tag{28}$$

$(\boldsymbol{f1}\cdot\boldsymbol{f2})^* - (\boldsymbol{f2})^*\cdot(\boldsymbol{f1})^*$ **:**

```
> f4 - f5
```

$$0 \tag{29}$$

(2) Check that $\left(f^*\right)^* = f$

```
> f := x^4·Dx^5 + Dx^5 x^3 − Dx^2
```

$$f := Dx^5 x^4 + Dx^5 x^3 - Dx^2 \tag{30}$$

$\boldsymbol{f}^*$ **:**

```
> ff1 := Conjugate2(f);
```

$$ff1 := \frac{1}{x^4}\big(Dx^5 x^{11} + Dx^5 x^{10} - 5\,Dx^4 x^{11} + 5\,Dx^4 x^{10} + 10\,Dx^3 x^{11} + 10\,Dx^4 x^9 - 30\,Dx^3 x^{10} - 10\,Dx^2 x^{11} + 5\,Dx^4 x^8 - 20\,Dx^3 x^9 + 50\,Dx^2 x^{10} + 5\,Dx\,x^{11} - 35\,Dx\,x^{10} - x^{11} + 10\,Dx^3 x^7 - 31\,Dx^2 x^8 + 20\,Dx\,x^9 + 9\,x^{10} + 10\,Dx^3 x^6 - 30\,Dx^2 x^7 + 42\,Dx\,x^8 - 10\,x^9 - 30\,Dx^2 x^6 + 26\,Dx\,x^7 - 16\,x^8 - 20\,Dx^2 x^5 + 28\,Dx\,x^6 - 6\,x^7 + 10\,Dx^2 x^4 + 40\,Dx\,x^5 - 10\,x^6 + 20\,Dx\,x^4 - 22\,x^5 - 35\,Dx\,x^3 - 31\,x^4 + 5\,Dx\,x^2 - 5\,x^3 + 45\,x^2 - 14\,x + 1\big) \tag{31}$$

$\left(\boldsymbol{f}^*\right)^* - \boldsymbol{f}$ **:**

```
> simplify(Conjugate2(ff1) − f)
```

$$0 \tag{32}$$

Example 3

```
> Conjugate3 := proc(f) # Involution of skew polynomials
   local i, n, a, bb, b, j;
  a := CoefficientVector(f, Dx);
   n := numelems(a);
   b := 0;
       for i from 0 by 1 to n -1# n is the number of  coefficients
  do
   bb := 1;
   for j from 1 by 1 to i# n is the number of  coefficients
  do
   bb := simplify(skew_product(bb, (1/100·x^2·Dx + (-300 + 1/100·x + 3
       · x^2) ), WW));
   end do:
  b := simplify(b + skew_product(bb, subs(x = 100/x, a[1 + i])
   , WW));
  end do:
  b;
   end proc:
```

(1) Check that $(f1 \cdot f2)^* = (f2)^* \cdot (f1)^*$

```
> f1 := x Dx; f2 := Dx^5 x^3 - Dx^2 x^2 + Dx x; f3 := skew_product(f1, f2, WW)
```

$$f1 := Dx\,x$$

$$f2 := Dx^5\,x^3 - Dx^2\,x^2 + Dx\,x$$

$$f3 := Dx^6\,x^4 + 3\,Dx^5\,x^3 - Dx^3\,x^3 - Dx^2\,x^2 + Dx\,x \tag{33}$$

$(f2)^* \cdot (f1)^*$:

```
> f4 := simplify(skew_product(Conjugate3(f2), Conjugate3(f1), WW))
```

$$f4 := \frac{1}{10000}\,\frac{1}{x^4}\left(Dx^6 x^{12} + 1800\,Dx^5 x^{12} + 15\,Dx^5 x^{11} + 1350000\,Dx^4 x^{12} - 180000\,Dx^5 x^{10} + 22500\,Dx^4 x^{11} + 540000000\,Dx^3 x^{12} - 269999940\,Dx^4 x^{10} + 13500000\,Dx^3 x^{11} + 121500000000\,Dx^2 x^{12} - 1350000\,Dx^4 x^9 - 161999928000\,Dx^3 x^{10} + 4050000000\,Dx^2 x^{11} + 14580000000000\,Dx\,x^{12} + 13500000000\,Dx^4 x^8 - 1619999940\,Dx^3 x^9 - 48599967600000\,Dx^2 x^{10} + 607500000000\,Dx\,x^{11} + 729000000000000\,x^{12} + 16199998200000\,Dx^3 x^8 - 728999946000\,Dx^2 x^9 - 7289993520000000\,Dx\,x^{10} + 36450000000000\,x^{11} + 26999990000\,Dx^3 x^7 + 7289998380000000\,Dx^2 x^8 - 145799983800000\,Dx\,x^9 - 437399514000000000\,x^{10} - 540000000000000\,Dx^3 x^6 + 24299991000000\,Dx^2 x^7 + 1457999514000000000\,Dx\,x^8 - 10934998380000000\,x^9 - 486000000000010000\,Dx^2 x^6 + 7289997300000000\,Dx\,x^7 + 10934995140000000000\,x^8 + 810000900000000\,Dx^2 x^5 - 1458000000000006000000\,Dx\,x^6 + 728999730000000000\,x^7 + 121500000000000000000\,Dx^2 x^4 + 486000540000010000\,Dx\,x^5 - 14580000000000900000000\,x^6 + 7289998379998800000000\,Dx\,x^4 + 72900081000003000000\,x^5 - 364500270000000000000\,Dx\,x^3 + 10934995139999640000000000\,x^4 - 14580000000000000000000\,Dx\,x^2 - 109350064799991000000000\,x^3 - 4373995139999550000000000\,x^2 + 3645002700000000000000000\,x + 729000000000000000000000000\right) \tag{34}$$

$(\boldsymbol{f1}\cdot\boldsymbol{f2})^*$ **:**

```
> f5 := Conjugate3(f3)
```

$$f5 := \frac{1}{10000}\,\frac{1}{x^4}\left(Dx^6 x^{12} + 1800\,Dx^5 x^{12} + 15\,Dx^5 x^{11} + 1350000\,Dx^4 x^{12} - 180000\,Dx^5 x^{10} + 22500\,Dx^4 x^{11} + 540000000\,Dx^3 x^{12} - 269999940\,Dx^4 x^{10} + 13500000\,Dx^3 x^{11} + 121500000000\,Dx^2 x^{12} - 1350000\,Dx^4 x^9 - 161999928000\,Dx^3 x^{10} + 4050000000\,Dx^2 x^{11} + 14580000000000\,Dx\,x^{12} + 13500000000\,Dx^4 x^8 - 1619999940\,Dx^3 x^9 - 48599967600000\,Dx^2 x^{10} + 607500000000\,Dx\,x^{11} + 729000000000000\,x^{12} + 16199998200000\,Dx^3 x^8 - 728999946000\,Dx^2 x^9 - 7289993520000000\,Dx\,x^{10} + 36450000000000\,x^{11} + 26999990000\,Dx^3 x^7 + 7289998380000000\,Dx^2 x^8 - 145799983800000\,Dx\,x^9 - 437399514000000000\,x^{10} - 540000000000000\,Dx^3 x^6 + 24299991000000\,Dx^2 x^7 + 1457999514000000000\,Dx\,x^8 - 10934998380000000\,x^9 - 486000000000010000\,Dx^2 x^6 \right. \tag{35}$$

$$+ 7289997300000000\, Dx\, x^7 + 1093499514000000000\, x^8$$
$$+ 810000900000000\, Dx^2\, x^5 - 1458000000000006000000\, Dx\, x^6$$
$$+ 728999730000000000\, x^7 + 1215000000000000000\, Dx^2\, x^4$$
$$+ 486000540000010000\, Dx\, x^5 - 1458000000000900000000\, x^6$$
$$+ 7289998379998800000000\, Dx\, x^4 + 729000810000030000000\, x^5$$
$$- 36450027000000000000\, Dx\, x^3 + 1093499513999640000000000\, x^4$$
$$- 145800000000000000000000\, Dx\, x^2 - 109350064799991000000000\, x^3$$
$$- 4373995139995500000000000\, x^2 + 3645002700000000000000000\, x$$
$$+ 729000000000000000000000000)$$

$(\boldsymbol{f1} \cdot \boldsymbol{f2})^* - (\boldsymbol{f2})^* \cdot (\boldsymbol{f1})^*$ **:**

```
> f4 − f5
```

$$0 \tag{36}$$

(2) Check that $\left(f^*\right)^* = f$

```
> f := x^4·Dx^5 + Dx^5 x^3 − Dx^2
```

$$f := Dx^5\, x^4 + Dx^5\, x^3 - Dx^2 \tag{37}$$

$\boldsymbol{f}^*$ **:**

```
> ff1 := Conjugate3(f);
```

$$ff1 := \frac{1}{10000}\, \frac{1}{x^4} \big(Dx^5\, x^{11} + 100\, Dx^5\, x^{10} + 1500\, Dx^4\, x^{11} + 150010\, Dx^4\, x^{10} \tag{38}$$
$$+ 900000\, Dx^3\, x^{11} - 149500\, Dx^4\, x^9 + 90012000\, Dx^3\, x^{10} + 270000000\, Dx^2\, x^{11}$$
$$- 15000000\, Dx^4\, x^8 - 179399980\, Dx^3\, x^9 + 27005400000\, Dx^2\, x^{10}$$
$$+ 40500000000\, Dx\, x^{11} - 18000600000\, Dx^3\, x^8 - 80729982000\, Dx^2\, x^9$$
$$+ 4051080000000\, Dx\, x^{10} + 2430000000000\, x^{11} + 9000000000\, Dx^3\, x^7$$
$$- 8100540000001\, Dx^2\, x^8 - 16145994600000\, Dx\, x^9 + 243081000000000\, x^{10}$$
$$+ 900000000000\, Dx^3\, x^6 + 810000000000\, Dx^2\, x^7 - 1620162000000600\, Dx\, x^8$$
$$- 1210949460000000\, x^9 + 81000000000000\, Dx^2\, x^6 + 2429999999999996\, Dx\, x^7$$
$$- 121516200000090000\, x^8 - 272700000000000\, Dx^2\, x^5$$
$$+ 243000000000060000\, Dx\, x^6 + 242999999999998800\, x^7$$
$$- 27000000000000000\, Dx^2\, x^4 - 163620000000000000\, Dx\, x^5$$
$$+ 24300000000017999998\, x^6 - 16199454600000000000\, Dx\, x^4$$
$$- 24542999999999940000\, x^5 + 4158000000000000000\, Dx\, x^3$$
$$- 242983638000090000000\, x^4 + 405000000000000000000\, Dx\, x^2$$

$+ 1247399454600000000000\,x^3 + 1214917380000000000000\,x^2$
$- 255150000000000000000\,x - 24300000000000000000000)$

$\left(f^*\right)^* - f$:

```
> simplify(Conjugate3(ff1) - f)
```

$$0 \quad (39)$$

2.3.2 An Involution of Ore Matrices

The following procedure is an involution of skew polynomials:

```
> Conjugate := proc(f) # involution of skew polynomials
   local i, n, a, bb, b, j;
  a := CoefficientVector(f, Dx);
   n := numelems(a);
   b := 0;
      for i from 0 by 1 to n -1# n is the number of  coefficients
  do
  b := simplify(b + skew_product((-1)^i·Dx^i, a[1 + i], WW));
  end do:
   b;
   end proc:
```

The following procedure is an involution of Ore matrices, and the parameter *A* is a *m* by *n* Ore matrix.

```
> Involution := proc(A, m, n) # the involution of Ore matrices
   local i, j , k, B := Matrix(1..n, 1..m, );
      for i from 1 by 1 to n # m is the number of rows of A
  do
   for j from 1 by 1 to m # n is the number of columns of A
  do
   B[i, j] := Conjugate(A[j, i]);
   end do:
   end do:
   B
   end proc:
```

2.3.3 Check the Involution in Section 2.3.2

```
> A := Matrix([[x^2, Dx^2, 0], [x^2·Dx^2 + 1, 0, 0], [x·Dx^4 + 2 Dx^2 + 1, Dx^3 x
      + 3 Dx^2 + Dx x + 1, 1]]);
```

$$A := \begin{bmatrix} x^2 & Dx^2 & 0 \\ Dx^2\, x^2 + 1 & 0 & 0 \\ Dx^4\, x + 2\, Dx^2 + 1 & Dx^3\, x + 3\, Dx^2 + Dx\, x + 1 & 1 \end{bmatrix} \tag{40}$$

(1) A^* **:**

```
> AA := Involution(A, 3, 3);
```

$$AA := \begin{bmatrix} x^2 & Dx^2\, x^2 + 4\, Dx\, x + 3 & Dx^4\, x + 4\, Dx^3 + 2\, Dx^2 + 1 \\ Dx^2 & 0 & -Dx^3\, x - Dx\, x \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

(2) $(A^*)^*$ **:**

```
> Involution(AA, 3, 3);# check that A** = A
```

$$\begin{bmatrix} x^2 & Dx^2 & 0 \\ Dx^2\, x^2 + 1 & 0 & 0 \\ Dx^4\, x + 2\, Dx^2 + 1 & Dx^3\, x + 3\, Dx^2 + Dx\, x + 1 & 1 \end{bmatrix} \tag{42}$$

$$> B := \begin{bmatrix} x^2 & x^2 \cdot Dx^2 - 4\, Dx\, x + 3 & Dx^4\, x - 4\, Dx^3 + 2\, Dx^2 + 1 \\ Dx^2 & 0 & Dx^3\, x + Dx\, x \\ 0 & 0 & 1 \end{bmatrix}$$

$$B := \begin{bmatrix} x^2 & Dx^2\, x^2 - 4\, Dx\, x + 3 & Dx^4\, x - 4\, Dx^3 + 2\, Dx^2 + 1 \\ Dx^2 & 0 & Dx^3\, x + Dx\, x \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

(3) B^* **:**

```
> BB := Involution(B, 3, 3);
```

$$BB := \begin{bmatrix} x^2 & Dx^2 & 0 \\ Dx^2\,x^2 + 8\,Dx\,x + 9 & 0 & 0 \\ Dx^4\,x + 8\,Dx^3 + 2\,Dx^2 + 1 & -Dx^3\,x - 3\,Dx^2 - Dx\,x - 1 & 1 \end{bmatrix} \qquad (44)$$

(4) AB :

```
> CC := Multli(A, B, 3, 3, 3)
```

$$CC := [[Dx^4 + x^4, Dx^2\,x^4 - 4\,Dx\,x^3 + 3\,x^2, Dx^4\,x^3 + Dx^5\,x - 4\,Dx^3\,x^2 + 2\,Dx^4 + Dx^3\,x + 2\,Dx^2\,x^2 + 2\,Dx^2 + x^2], [Dx^2\,x^4 + 4\,Dx\,x^3 + 3\,x^2, Dx^4\,x^4 - 2\,Dx^2\,x^2 - 4\,Dx\,x + 3, Dx^6\,x^3 - 2\,Dx^5\,x^2 + 2\,Dx^4\,x^2 + Dx^4\,x + Dx^2\,x^2 - 4\,Dx^3 + 2\,Dx^2 + 1], [Dx^4\,x^3 + Dx^5\,x + 8\,Dx^3\,x^2 + 3\,Dx^4 + Dx^3\,x + 2\,Dx^2\,x^2 + 12\,Dx^2\,x + Dx^2 + 8\,Dx\,x + x^2 + 4, Dx^6\,x^3 + 4\,Dx^5\,x^2 + 2\,Dx^4\,x^2 - Dx^4\,x + Dx^2\,x^2 - 6\,Dx^2 - 4\,Dx\,x + 3, Dx^8\,x^2 + Dx^6\,x^2 + 4\,Dx^6\,x + 6\,Dx^5\,x + 2\,Dx^4\,x^2 - 4\,Dx^5 + 2\,Dx^4\,x + 10\,Dx^4 + 8\,Dx^3\,x + Dx^2\,x^2 - 4\,Dx^3 + 10\,Dx^2 + 2\,Dx\,x + 2]] \qquad (45)$$

(5) $(AB)^*$:

```
> Z1 := Involution(CC, 3, 3);
```

$$Z1 := [[Dx^4 + x^4, Dx^2\,x^4 + 4\,Dx\,x^3 + 3\,x^2, Dx^4\,x^3 - Dx^5\,x + 4\,Dx^3\,x^2 - 2\,Dx^4 - Dx^3\,x + 2\,Dx^2\,x^2 - 2\,Dx^2 + x^2], [Dx^2\,x^4 + 12\,Dx\,x^3 + 27\,x^2, Dx^4\,x^4 + 16\,Dx^3\,x^3 + 70\,Dx^2\,x^2 + 92\,Dx\,x + 27, Dx^6\,x^3 + 14\,Dx^5\,x^2 + 2\,Dx^4\,x^2 + 49\,Dx^4\,x + 16\,Dx^3\,x + Dx^2\,x^2 + 36\,Dx^3 + 18\,Dx^2 + 8\,Dx\,x + 9], [Dx^4\,x^3 - Dx^5\,x + 16\,Dx^3\,x^2 - 3\,Dx^4 - Dx^3\,x + 2\,Dx^2\,x^2 + 60\,Dx^2\,x - Dx^2 + 8\,Dx\,x + x^2 + 48\,Dx + 4, Dx^6\,x^3 + 20\,Dx^5\,x^2 + 2\,Dx^4\,x^2 + 111\,Dx^4\,x + 16\,Dx^3\,x + Dx^2\,x^2 + 168\,Dx^3 + 26\,Dx^2 + 4\,Dx\,x + 3, Dx^8\,x^2 + 16\,Dx^7\,x + Dx^6\,x^2 + 4\,Dx^6\,x + 56\,Dx^6 + 6\,Dx^5\,x + 2\,Dx^4\,x^2 + 28\,Dx^5 + 2\,Dx^4\,x + 10\,Dx^4 + 8\,Dx^3\,x + Dx^2\,x^2 + 12\,Dx^3 + 10\,Dx^2 + 2\,Dx\,x + 2]] \qquad (46)$$

(6) $B^* A^*$:

```
> Z2 := Multli(BB, AA, 3, 3, 3)
```

$$Z2 := [[Dx^4 + x^4, Dx^2\,x^4 + 4\,Dx\,x^3 + 3\,x^2, Dx^4\,x^3 - Dx^5\,x + 4\,Dx^3\,x^2 - 2\,Dx^4 - Dx^3\,x \qquad (47)$$

$+2\,Dx^2\,x^2-2\,Dx^2+x^2],$
$[Dx^2\,x^4+12\,Dx\,x^3+27\,x^2, Dx^4\,x^4+16\,Dx^3\,x^3+70\,Dx^2\,x^2+92\,Dx\,x+27, Dx^6\,x^3$
$+14\,Dx^5\,x^2+2\,Dx^4\,x^2+49\,Dx^4\,x+16\,Dx^3\,x+Dx^2\,x^2+36\,Dx^3+18\,Dx^2$
$+8\,Dx\,x+9],$
$[Dx^4\,x^3-Dx^5\,x+16\,Dx^3\,x^2-3\,Dx^4-Dx^3\,x+2\,Dx^2\,x^2+60\,Dx^2\,x-Dx^2+8\,Dx\,x$
$+x^2+48\,Dx+4, Dx^6\,x^3+20\,Dx^5\,x^2+2\,Dx^4\,x^2+111\,Dx^4\,x+16\,Dx^3\,x+Dx^2\,x^2$
$+168\,Dx^3+26\,Dx^2+4\,Dx\,x+3, Dx^8\,x^2+16\,Dx^7\,x+Dx^6\,x^2+4\,Dx^6\,x+56\,Dx^6$
$+6\,Dx^5\,x+2\,Dx^4\,x^2+28\,Dx^5+2\,Dx^4\,x+10\,Dx^4+8\,Dx^3\,x+Dx^2\,x^2+12\,Dx^3$
$+10\,Dx^2+2\,Dx\,x+2]]$

(7) Check that $(AB)^*=B^*A^*$:

```
> Z1 - Z2
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(48)}$$

We check that $(A^*B)^*=B^*A$ in the following steps.

(7) $(A^*B)^*$:

```
> q := Involution(Multli(AA, B, 3, 3, 3), 3, 3)
```

$q := [[Dx^4\,x^2+4\,Dx^3\,x+x^4+3\,Dx^2, x^2\,Dx^2, 0],$ **(49)**
$[Dx^2\,x^4+12\,Dx\,x^3+27\,x^2, Dx^4\,x^2+8\,Dx^3\,x+9\,Dx^2, 0],$
$[-Dx^5\,x^3+Dx^4\,x^3-9\,Dx^4\,x^2-Dx^3\,x^3+Dx^4\,x+16\,Dx^3\,x^2-19\,Dx^3\,x-Dx^2\,x^2$
$+60\,Dx^2\,x-7\,Dx^2+7\,Dx\,x+x^2+48\,Dx+4, Dx^6\,x+8\,Dx^5+2\,Dx^4+Dx^3\,x$
$+4\,Dx^2+Dx\,x+1, 1]]$

(8) B^*A :

```
> qq := Multli(BB, A, 3, 3, 3)
```

$qq := [[Dx^4\,x^2+4\,Dx^3\,x+x^4+3\,Dx^2, x^2\,Dx^2, 0],$ **(50)**
$[Dx^2\,x^4+12\,Dx\,x^3+27\,x^2, Dx^4\,x^2+8\,Dx^3\,x+9\,Dx^2, 0],$
$[-Dx^5\,x^3+Dx^4\,x^3-9\,Dx^4\,x^2-Dx^3\,x^3+Dx^4\,x+16\,Dx^3\,x^2-19\,Dx^3\,x-Dx^2\,x^2$
$+60\,Dx^2\,x-7\,Dx^2+7\,Dx\,x+x^2+48\,Dx+4, Dx^6\,x+8\,Dx^5+2\,Dx^4+Dx^3\,x$

$+4\,Dx^2 + Dx\,x + 1, 1]]$

(9) $(A^* B)^* - B^* A$ **:**

```
> q − qq
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (51)$$

So, we have checked that $(A^* B)^* = B^* A$ **.**

3 Maple Codes in the OreAlgebra Package

In this chapter, we construct Ore matrices and skew polynomials by the OreAlgebra package. We achieve algorithm implementations to get Jacobson forms, Jacobson normal forms, generalized inverses and involutions of Ore matrices.

3.1 Compute the Jacobson Form of a Given Ore Matrix

```
> restart;
  with(OreTools) : WW := SetOreRing(x, 'differential')
```

$$WW := UnivariateOreRing(x, differential) \quad (52)$$

3.1.1 The Multiplication of Two Ore Matrices

The following procedure is the multiplication of two Ore matrices, where parameter *A* is a *m* by *p* Ore matrix and parameter *B* is a *p* by *n* Ore matrix.

```
> Multli := proc(A, B, m, n, p) # the product of two matrices
   local i, j , k, AB := OrePoly(0)·Matrix(1 ..m, 1 ..n, 1)#Matrix([[OrePoly(0),
      OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(0)], [OrePoly(0),
      #OrePoly(0), OrePoly(0)]]);
   for i from 1 by 1 to m # m is the number of rows of A
  do
   for j from 1 by 1 to n # n is the number of columns of B
   do
   for k  from 1 by 1 to p # p is the number of columns of A=the number of rows of B
   do
  AB[i, j] := Add(AB[i, j], Multiply(A[i, k], B[k, j], WW));
   # print(AB[i, j])
  end do:
```

```
end do:
 end do:
 AB
 end proc:
```

3.1.2 The Jacobson Form of a Given Ore Matrix

We can get three results by the following procedure, the first result is the Jacobson form of a *m* by *n* Ore matrix *B*, the second result *totalE* and the third result *totalF* are two Ore matrices such that *totalE* · *B* · *totalF*= the Jacobson form of *B*.

```
> Jac := proc(B, m, n)#` `Jacobson Form
  local inE12, inE22, EE, tt, j, b12, inF21, inF22, FF, tt1, H, H1, totalE111, totalF111,
      E111, F111, EE111, FF111;
   local w, A, totalE, totalinE, totalF, totalinF, e, E, inE, F, inF, i, k, l, b11, b21, p;
   A := B :
  p := m;
   totalE111 := Matrix(m, shape = identity) :
    totalF111 := Matrix(n, shape = identity) :
   totalE := subs(1 = OrePoly(1), 0 = OrePoly(0), totalE111) :
      # m by m Ore identity matrix
    totalF := subs(1 = OrePoly(1), 0 = OrePoly(0), totalF111) :
      # n by n Ore identity matrix
   w := max(m, n) :# we achieve the  Jacobson  Form  of  a  nonsquare matrix
      by this code
  for e  from 0 by 1 to w − 2 do
   E111 := Matrix(m, shape = identity) :
    F111 := Matrix(n, shape = identity) :
   E := subs(1 = OrePoly(1), 0 = OrePoly(0), E111) :# m by m Ore identity matrix
   F := subs(1 = OrePoly(1), 0 = OrePoly(0), F111) :# n by n Ore identity matrix
   for i from 1 by 1 to m − e − 1 do
   p := m :#we achieve the  Jacobson  Form  of  a  nonsquare matrix by this code
  if e ≥ min(m, n) then
          break
      end if:#we achieve the  Jacobson  Form  of  a  nonsquare matrix by this code
  if A[1 + e, 1 + e] ≠ OrePoly(0) and A[2 + e, 1 + e] ≠ OrePoly(0)  then
  #gcdex(simplify(A[1 + e, 1 + e]), simplify(A[2 + e, 1 + e]), x,'s','t') :
   H := ExtendedGCD_'right'(A[1 + e, 1 + e], A[2 + e, 1 + e], WW,'c1','c2') :# H is gcrd
  # k:=simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[1 + e, 1 + e] ) : l
       := simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[2 + e, 1 + e] ) :
   E(1 + e, 1 + e) := c1[1] : E(1 + e, 2 + e) := c2[1] : E(2 + e, 1 + e) := c1[2] : E(2
      + e, 2 + e) := c2[2] :
```

```
elif A[1 + e, 1 + e] = OrePoly(0)
then
 E(1 + e, 1 + e) := OrePoly(0) : E(1 + e, 2 + e) := OrePoly(1) : E(2 + e, 1 + e)
    := OrePoly(1) : E(2 + e, 2 + e) := OrePoly(0) :
elif A[2 + e, 1 + e] = OrePoly(0)# we do not need to do anything
then E(1 + e, 1 + e) := OrePoly(1) : E(1 + e, 2 + e) := OrePoly(0) : E(2 + e, 1
    + e) := OrePoly(0) : E(2 + e, 2 + e) := OrePoly(1) :
end if:
#b11:=simplify( A[1 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[2 + e, 1 + e]) ) :
#b21:=simplify( A[2 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[2 + e, 1 + e]) ) :
# gcdex(k, -l, x,'p','q') :
#inE12:=simplify(p - b11·s·p - b11·t·q) :
#inE22:=simplify(q - b21·s·p - b21·t·q) :
#inE(1 + e, 1 + e) := b11 : inE(1 + e, 2 + e) := inE12 : inE(2 + e, 1 + e) := b21 :
    inE(2 + e, 2 + e) := inE22 :
#totalinE:= simplify(totalinE.inE);
totalE := Multli(E, totalE, m, m, p) :
A := Multli(E, A, m, n, p) :
#print(A);
if i = m - e - 1 then break end if:
 EE111 := Matrix(m, shape = identity) :
 EE := subs(1 = OrePoly(1), 0 = OrePoly(0), EE111) :# m by m Ore identity matrix
 tt := A[2 + e, 1 ..-1] :
 A[2 + e, 1 ..-1] := A[i + 2 + e, 1 ..-1] :
 A[i + 2 + e, 1 ..-1] := tt :
tt := EE[2 + e, 1 ..-1] :
 EE[2 + e, 1 ..-1] := EE[i + 2 + e, 1 ..-1] :
 EE[i + 2 + e, 1 ..-1] := tt :
 totalE := Multli(EE, totalE, m, m, p) :
 #totalinE:= simplify(totalinE.EE) :
 end do:
 for j from 1 by 1 to n - e - 1 do
 p := n :#``we achieve the Jacobson Form of a nonsquare matrix by this code
 if e ≥ min(m, n) then
        break
    end if:#we achieve the Jacobson Form of a nonsquare matrix by this code
#gcdex(simplify(A[1 + e, 1 + e]), simplify(A[1 + e, 2 + e]), x,'s','t') :
#k:=simplify( lcm(A[1 + e, 1 + e], A[1 + e, 2 + e]) / A[1 + e, 1 + e] ) : l
    := simplify( lcm(A[1 + e, 1 + e], A[1 + e, 2 + e]) / A[1 + e, 2 + e] ) :
 if A[1 + e, 1 + e] ≠ OrePoly(0) and A[1 + e, 2 + e] ≠ OrePoly(0) then
```

```
#gcdex(simplify(A[1 + e, 1 + e]), simplify(A[2 + e, 1 + e]), x,'s','t') :
H1 := ExtendedGCD_'left'(A[1 + e, 1 + e], A[1 + e, 2 + e], WW,'d1','d2');# H is gcrd
# k:=simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[1 + e, 1 + e] ) : l
    := simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[2 + e, 1 + e] ) :
# E(1 + e, 1 + e) := c1[1] : E(1 + e, 2 + e) := c2[1] : E(2 + e, 1 + e) := c1[2] :
   E(2 + e, 2 + e) := c2[2] :
 F(1 + e, 1 + e) := d1[1] : F(1 + e, 2 + e) := d1[2] : F(2 + e, 1 + e) := d2[1] : F(2
    + e, 2 + e) := d2[2] :
#F(1 + e, 1 + e) := d2[1] : F(1 + e, 2 + e) := d1[2] : F(2 + e, 1 + e) := d1[1] :
   F(2 + e, 2 + e) := d2[2] :
 elif A[1 + e, 1 + e] = OrePoly(0)
then
 F(1 + e, 1 + e) := OrePoly(0) : F(1 + e, 2 + e) := OrePoly(1) : F(2 + e, 1 + e)
    := OrePoly(1) : F(2 + e, 2 + e) := OrePoly(0) :
 elif A[1 + e, 2 + e] = OrePoly(0)# we do not need to do anything
 then F(1 + e, 1 + e) := OrePoly(1) : F(1 + e, 2 + e) := OrePoly(0) : F(2 + e, 1
    + e) := OrePoly(0) : F(2 + e, 2 + e) := OrePoly(1) :
end if:

   # in theory, the first F is right, but in the real code, we can find that we need the
   second F,
#b11:=simplify( A[1 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[1 + e, 2 + e]) ) :
#b12:=simplify( A[1 + e, 2 + e] / gcd(A[1 + e, 1 + e], A[1 + e, 2 + e]) ) :
# gcdex(simplify(k), simplify(-1), x,'p','q') :
#inF21:=simplify(p - b11·s·p - b11·t·q) :
# inF22:=simplify(q - b12·s·p - b12·t·q) :
#inF(1 + e, 1 + e) := b11 : inF(1 + e, 2 + e) := b12 : inF(2 + e, 1 + e) := inF21 :
   inF(2 + e, 2 + e) := inF22 :
#totalinF:=simplify( inF. totalinF) :
totalF := Multli(totalF, F, n, n, p) :
 A := Multli(A, F, m, n, p) :
#print(A);
 if j = n - e - 1 then break end if:
 FF111 := Matrix(n, shape = identity) :
FF := subs(1 = OrePoly(1), 0 = OrePoly(0), FF111) :# n by n Ore identity matrix
tt1 := A[1 ..-1, 2 + e] :
A[1 ..-1, 2 + e] := A[1 ..-1, j + 2 + e] :
A[1 ..-1, j + 2 + e] := tt1 :
 tt1 := FF[1 ..-1, 2 + e] :
FF[1 ..-1, 2 + e] := FF[1 ..-1, j + 2 + e] :
```

```
FF[1..-1, j+2+e] := tt1 :# this is inverse of an elementary matrix
 totalF := Multli(totalF, FF, n, n, p) :
# totalinF:=simplify( FF. totalinF) :
end do:
end do:
print(A);
print(totalE);
print(totalF);
end proc:
```

3.1.3 The Inner Rank of the Jacobson Form of a Given Ore Matrix

```
> rank := proc(A, m, n)
local i, w;
w := min(m, n);
 for i from 1 by 1 to w
do

if A[i, i] = OrePoly(0)  then break end if:

end do:
print(i - 1);
end proc:
```

3.1.4 An Example

```
> with(LinearAlgebra) :
```

> $C := Matrix\left(\left[\left[OrePoly(x), OrePoly\left(\frac{1}{x}\right), OrePoly\left(\frac{1}{x}\right)\right], [OrePoly(x), OrePoly(x, x), OrePoly(x, x, x)]\right]\right);$

$$C := \begin{bmatrix} OrePoly(x) & OrePoly\left(\frac{1}{x}\right) & OrePoly\left(\frac{1}{x}\right) \\ OrePoly(x) & OrePoly(x, x) & OrePoly(x, x, x) \end{bmatrix} \tag{53}$$

> $Jac(\mathrm{C}, 2, 3)$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(0) & OrePoly(1) \\ OrePoly(1) & OrePoly(-1) \end{bmatrix}$$

(54)

$$\left[\left[OrePoly\left(\frac{1}{x}\right), OrePoly\left(\frac{x^7-3x^5+5x^4+3x^3-4x^2-x-1}{(x^4-2x^2+2x+1)^2}, -\frac{x}{x^4-2x^2+2x+1}\right), OrePoly\left(\frac{6x^2(x^8+3x^6-6x^5-7x^4+2x^3+3x^2+4x+2)}{(x^4-2x^2+2x+1)^4}, -\frac{2x^3(2x^4+2x^2-5x-4)}{(x^4-2x^2+2x+1)^3}, \frac{x^4}{(x^4-2x^2+2x+1)^2}\right)\right], \left[OrePoly(0), OrePoly\left(-\frac{x(x^5-x^4+2x^3-2x^2-5x-1)}{(x^4-2x^2+2x+1)^2}, \frac{x^3}{x^4-2x^2+2x+1}\right), OrePoly\left(-\frac{1}{(x^4-2x^2+2x+1)^4}(x^4(x^{10}-2x^9+x^8+8x^7+26x^6-48x^5-52x^4+24x^3+25x^2+22x+11)), -\frac{x^5(x^5-4x^4-2x^3-2x^2+11x+8)}{(x^4-2x^2+2x+1)^3}, -\frac{x^6}{(x^4-2x^2+2x+1)^2}\right)\right], \left[OrePoly(0), OrePoly\left(-\frac{x^3}{x^4-2x^2+2x+1}\right), OrePoly\left(\frac{x^4(x^6-2x^5-3x^4+2x^3+5x^2-1)}{(x^4-2x^2+2x+1)^3}, \frac{x^6}{(x^4-2x^2+2x+1)^2}\right)\right]\right] \quad \textbf{(54)}$$

The Jacobson form of C:

$$> BBB := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix}$$

$$BBB := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(55)}$$

The inner rank of C:

$> rank(BBB, 2, 3)$

$$2 \quad \textbf{(56)}$$

(57)

3.2 Construct a {1}-Inverse of a Given Ore Matrix

```
> with(OreTools[Utility]) : with(LinearAlgebra) :
```

We can get a {1}-inverse of the above m by n Ore matrix B by the following procedure. The parameter AA is the Jacobson form of B, parameters $totalE$ and $totalF$ are two Ore matrices such that $totalE \cdot B \cdot totalF$=the Jacobson form of B.

```
> INV11 :=proc(AA, totalE, totalF, m, n)
  local AAA, q, i, AAB, QQ, AAC;
  AAC := Matrix(n, m) :
  AAA := subs(0 = OrePoly(0), AAC) :
  q := min(m, n) :
  for i from 1 by 1 to q do
  if AA[i, i] ≠ OrePoly(0) then
  AAB := 1/Coefficient(AA[i, i], 0) :
  AAA[i, i] := OrePoly(AAB) :
  end if:
  end do:
  #print(AAA);
  Multli(Multli(totalF, AAA, n, m, n), totalE, n, m, m);
  end proc:
```

```
> with(LinearAlgebra) :
```

```
> A := Matrix([[OrePoly(x), OrePoly(0), OrePoly(0) , OrePoly(0)], [OrePoly(x, x),
     OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(1, 1), OrePoly(0), OrePoly(1),
     OrePoly(0)]]);
```

$$A := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(1, 1) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(58)}$$

We construct a {1}-inverse of A by the following steps, and the method is from Theorem 3.3 in our thesis.

Step 1: Compute the Jacobson form of A.

```
> Jac(A, 3, 4)
```

$$\begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x}, -1\right) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x^2}, -\frac{1}{x}\right) & OrePoly(0) & OrePoly(1) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad \textbf{(59)}$$

The Jacobson form of A :

> $AA := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix}:$

Step 2: Construct a {1}-inverse of A by Theorem 3.3 in our thesis.
Note that *totalE* and *totalF* are two Ore matrices such that
***totalE*$\cdot A \cdot$*totalF*= the Jacobson form of A.**

> $totalE := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x}, -1\right) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x^2}, -\frac{1}{x}\right) & OrePoly(0) & OrePoly(1) \end{bmatrix} : totalF$
> $:= \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix}:$

> $INV11(AA, totalE, totalF, 3, 4)$

(60)

$$\begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(0) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x}, -1\right) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x^2}, -\frac{1}{x}\right) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \tag{60}$$

The {1}-inverse of A:

> $X := \begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(0) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x}, -1\right) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x^2}, -\frac{1}{x}\right) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix}$ ###a {1}-inverse of A

$$X := \begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(0) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x}, -1\right) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x^2}, -\frac{1}{x}\right) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \tag{61}$$

(1) Check that $AXA = A$:

> $Multli(Multli(A, X, 3, 3, 4), A, 3, 4, 3)$### check that AXA=A

$$\begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(1, 1) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \tag{62}$$

(2) We also find that $XAX = X$:

> $Multli(Multli(X, A, 4, 4, 3), X, 4, 3, 4)$# check XAX=X

(63)

$$\begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(0) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x}, -1\right) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x-1}{x^2}, -\frac{1}{x}\right) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (63)$$

So X is a {1,2}-inverse of A.

3.3 An Example of an Ore Matrix That Doesn't Have {1}-Inverses

The following A doesn't have {1}-inverses.

```
> A := Matrix([[OrePoly(1), OrePoly(0), OrePoly(0) , OrePoly(0)], [OrePoly(x, x, x),
    OrePoly(x, x), OrePoly(0), OrePoly(0)], [OrePoly(1, 1), OrePoly(0), OrePoly(1),
    OrePoly(x, x)]]);
```

$$A := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x, x) & OrePoly(x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(1, 1) & OrePoly(0) & OrePoly(1) & OrePoly(x, x) \end{bmatrix} \quad (64)$$

The first matrix of Jac(A,3, 4) is the Jacobson form of A :

```
> Jac(A, 3, 4)
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(x, x) & OrePoly(0) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(-1, -1) & OrePoly(0) & OrePoly(1) \\ OrePoly(-x, -x, -x) & OrePoly(1) & OrePoly(0) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(-x, -x) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (65)$$

The Jacobson form of A :

$$> QWW := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(x,x) & OrePoly(0) \end{bmatrix}$$

$$QWW := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(x,x) & OrePoly(0) \end{bmatrix} \tag{66}$$

Since $OrePoly(x,x)$ represents the operator $x+xD$, A doesn't have {1}-inverses by Theorem 3.3 in our thesis.

3.4 Involutions

3.4.1 An Involution of Skew Polynomials

```
> with(OreTools) :
> with(OreTools[Converters]) :
```

The following procedure is an involution of skew polynomials, and we construct this procedure by Theorem 4.3 and Theorem 4.4 in our thesis.

```
> Conjugate1 := proc(ff)# Involution of skew polynomials
   local i, n, a, b, c, d, e;
  a := Coefficients(ff) :# coefficient of ff
   n := Degree(ff) :
     for i from 0 by 1 to n # n is the number of  degree
   do
   b[i+1] := (-1)^i d^i/dx^i f(x); # change OrePoly to usual poly, and add (-1)^i, b[i
       +1] is the degree
  end do:
   for i from 0 by 1 to n # n is the number of  degree
  do
    c[i+1] := FromLinearEquationToOrePoly(b[i+1], f, WW);
      # change OrePoly to usual poly,
   d[i+1] := FromLinearEquationToOrePoly(a[i+1]· d^0/dx^0 f(x), f, WW);
      # change OrePoly to usual poly, this is coefficient, we must time d^0/dx^0 f(x)
   end do:
  e := OrePoly(0) :
   for i from 0 by 1 to n # n is the number of  degree
```

```
do
 e := Add(e, Multiply(c[i + 1], d[i + 1], WW) );#``
 end do:
  if n = 0 then e := ff; end if:
 e;
end proc:
```

3.4.2 An Involution of Ore Matrices

The following procedure is an involution of Ore matrices, and the parameter A is a m by n Ore matrix.

```
> Invol := proc(A, m, n) # Involution of Ore matrices
  local i, j , k, B := OrePoly(0)·Matrix(1 ..n, 1 ..m, 1);
   for i from 1 by 1 to n # m is the number of rows of A
  do
   for j from 1 by 1 to m # n is the number of columns of A
   do
  B[i, j] := Conjugate1( A[ j, i]);
   end do:
   end do:
    B
   end proc:
```

Check the involution of Ore matrices:

> $A := Matrix\left(\left[\left[OrePoly(x, x, x), OrePoly(x^2), OrePoly\left(\frac{1}{x}\right)\right], \left[OrePoly(4, 3, 4), OrePoly(5, x, x, 3), OrePoly(x^2, x^3)\right]\right]\right);$

$$A := \begin{bmatrix} OrePoly(x, x, x) & OrePoly(x^2) & OrePoly\left(\frac{1}{x}\right) \\ OrePoly(4, 3, 4) & OrePoly(5, x, x, 3) & OrePoly(x^2, x^3) \end{bmatrix} \tag{67}$$

(1) A^* :

> $AA := Invol(A, 2, 3)\#A^*$

$$AA := \begin{bmatrix} OrePoly(x-1, -x+2, x) & OrePoly(4, -3, 4) \\ OrePoly(x^2) & OrePoly(4, -x+2, x, -3) \\ OrePoly\left(\frac{1}{x}\right) & OrePoly(-2x^2, -x^3) \end{bmatrix} \tag{68}$$

(2) $\left(A^*\right)^*$:

> $Invol(AA, 3, 2)$# check that $A^{**} = A$

$$\begin{bmatrix} OrePoly(x, x, x) & OrePoly(x^2) & OrePoly\left(\frac{1}{x}\right) \\ OrePoly(4, 3, 4) & OrePoly(5, x, x, 3) & OrePoly(x^2, x^3) \end{bmatrix} \quad (69)$$

> $B := Matrix\left(\left[\left[OrePoly(1, 1), OrePoly(x), OrePoly\left(\frac{1}{x}\right)\right], [OrePoly(4), OrePoly(5, x, 3), OrePoly(x^2)], [OrePoly(4\,x), OrePoly(3), OrePoly(x^2, x)]\right]\right);$

$$B := \begin{bmatrix} OrePoly(1, 1) & OrePoly(x) & OrePoly\left(\frac{1}{x}\right) \\ OrePoly(4) & OrePoly(5, x, 3) & OrePoly(x^2) \\ OrePoly(4\,x) & OrePoly(3) & OrePoly(x^2, x) \end{bmatrix} \quad (70)$$

(3) B^* **:**

> $BB := Invol(B, 3, 3)$#B^*

$$BB := \begin{bmatrix} OrePoly(1, -1) & OrePoly(4) & OrePoly(4\,x) \\ OrePoly(x) & OrePoly(4, -x, 3) & OrePoly(3) \\ OrePoly\left(\frac{1}{x}\right) & OrePoly(x^2) & OrePoly(x^2 - 1, -x) \end{bmatrix} \quad (71)$$

(4) AB**:**

> $CC := Multli(A, B, 2, 3, 3)$#this is AB

$$CC := \left[\left[OrePoly(4\,x^2 + x + 4, 2\,x, 2\,x, x), OrePoly\left(\frac{6\,x^3 + x^2 + 3}{x}, x^3 + x^2 + 2\,x, 4\,x^2\right), OrePoly\left(\frac{x^6 + x^3 + x^2 - x + 2}{x^2}, \frac{2\,(x-1)}{x}, 1\right)\right], \left[OrePoly(8\,x^3 + 24, 4\,x^4 + 4\,x + 7, 7 + 4\,x, 16), OrePoly(3\,x^2 + 4\,x + 28, 3\,x^3 + 14\,x + 8, x^2 + 11\,x + 15, x^2 + 3\,x + 24, 6\,x, 9), OrePoly\left(\frac{3\,x^7 + 7\,x^5 + 2\,x^4 + 4\,x^2 - 3\,x + 8}{x^3}, \frac{x^7 + 3\,x^5 + 4\,x^4 + 18\,x^2 + 3\,x - 8}{x^2}, \frac{x^5 + x^4 + 18\,x^2 + 4}{x}, 3\,x^2\right)\right]\right] \quad (72)$$

(5) $(AB)^*$ **:**

> $Invol(CC, 2, 3)$#this is $(AB)^*$

$$[[OrePoly(4x^2+x+2, -2x+4, 2x-3, -x), OrePoly(-8x^3+20, -4x^4-4x+1, 7+4x, -16)], \left[OrePoly\left(\frac{3x^3-x^2+6x+3}{x}, -x^3-x^2+14x, 4x^2\right), OrePoly(-6x^2+4x+16, -3x^3-10x+8, x^2+5x+6, -x^2-3x, 6x, -9)\right], \left[OrePoly\left(\frac{x^5+x^2+x-1}{x}, -\frac{2(x-1)}{x}, 1\right), OrePoly\left(-\frac{2(x^5-5x^3-2)}{x}, -\frac{x^6-5x^4-2x^3+3}{x}, \frac{x^5+x^4+4}{x}, -3x^2\right)\right]] \tag{73}$$

(6) B^*A^* **:**

> $Multli(BB, AA, 3, 2, 3)$# this is B^*A^*

$$[[OrePoly(4x^2+x+2, -2x+4, 2x-3, -x), OrePoly(-8x^3+20, -4x^4-4x+1, 7+4x, -16)], \left[OrePoly\left(\frac{3x^3-x^2+6x+3}{x}, -x^3-x^2+14x, 4x^2\right), OrePoly(-6x^2+4x+16, -3x^3-10x+8, x^2+5x+6, -x^2-3x, 6x, -9)\right], \left[OrePoly\left(\frac{x^5+x^2+x-1}{x}, -\frac{2(x-1)}{x}, 1\right), OrePoly\left(-\frac{2(x^5-5x^3-2)}{x}, -\frac{x^6-5x^4-2x^3+3}{x}, \frac{x^5+x^4+4}{x}, -3x^2\right)\right]] \tag{74}$$

So, we have checked that $(AB)^* = B^*A^*$**.**

We check that $\left(A^*B\right)^* = B^*A$ **by the following steps.**

> $A := Matrix\left(\left[\left[OrePoly(x, x, x), OrePoly(x^2), OrePoly\left(\frac{1}{x}\right)\right], [OrePoly(4, 3, 4), OrePoly(5, x, x, 3), OrePoly(x^2, x^3)], \left[OrePoly(x, 1), OrePoly(x^2+1), OrePoly\left(\frac{1}{x}\right)\right]\right]\right);$

(75)

$$A := \begin{bmatrix} OrePoly(x, x, x) & OrePoly(x^2) & OrePoly\left(\frac{1}{x}\right) \\ OrePoly(4, 3, 4) & OrePoly(5, x, x, 3) & OrePoly(x^2, x^3) \\ OrePoly(x, 1) & OrePoly(x^2+1) & OrePoly\left(\frac{1}{x}\right) \end{bmatrix} \quad (75)$$

(7) A^* **:**

```
> AA := Invol(A, 3, 3)#A^*
```

$$AA := \begin{bmatrix} OrePoly(x-1, -x+2, x) & OrePoly(4, -3, 4) & OrePoly(x, -1) \\ OrePoly(x^2) & OrePoly(4, -x+2, x, -3) & OrePoly(x^2+1) \\ OrePoly\left(\frac{1}{x}\right) & OrePoly(-2x^2, -x^3) & OrePoly\left(\frac{1}{x}\right) \end{bmatrix} \quad (76)$$

(8) $\left(A^* B\right)^*$ **:**

```
> Invol(Multli(AA, B, 3, 3, 3), 3, 3)
```

$$\Big[\Big[OrePoly(4x^2+x+15, 4x+11, 15, -x), OrePoly(4x^3+x^2+2x+20, -x^2 +4x, 4x, 12), OrePoly\left(\frac{4x^4+4x^2+x+1}{x^2}, \frac{4x^4-1}{x}\right)\Big], \quad (77)$$

$$\Big[OrePoly(x^2+3x+16, x^2-4x+15, x^2-3x+28, 9-4x, 12), OrePoly(x^3 +3x^2+23, -2x, -x^2+3x+21, -x^2+3x+18, 0, 9),$$

$$OrePoly\left(\frac{2x^3+7x+3}{x}, 30x, -x^4+21x^2, 3x^3\right)\Big],$$

$$\Big[OrePoly(x^3+4x^2-2x+1, 3x^2, 4x^2-x+1), OrePoly(x^4+3x^2+x-1, -x, x^3, 3x^2), OrePoly\left(\frac{x^6+x^3+1}{x^2}, x^5-1\right)\Big]\Big]$$

(9) $B^* A$ **:**

```
> Multli(BB, A, 3, 3, 3)
```

$$\Big[\Big[OrePoly(4x^2+x+15, 4x+11, 15, -x), OrePoly(4x^3+x^2+2x+20, -x^2 +4x, 4x, 12), OrePoly\left(\frac{4x^4+4x^2+x+1}{x^2}, \frac{4x^4-1}{x}\right)\Big], \quad (78)$$

$$\left[OrePoly(x^2+3x+16, x^2-4x+15, x^2-3x+28, 9-4x, 12), OrePoly(x^3 +3x^2+23, -2x, -x^2+3x+21, -x^2+3x+18, 0, 9), OrePoly\left(\frac{2x^3+7x+3}{x}, 30x, -x^4+21x^2, 3x^3\right)\right],$$
$$\left[OrePoly(x^3+4x^2-2x+1, 3x^2, 4x^2-x+1), OrePoly(x^4+3x^2+x-1, -x, x^3, 3x^2), OrePoly\left(\frac{x^6+x^3+1}{x^2}, x^5-1\right)\right]\Bigg]$$

From (8) and (9), we find that $(A^* B)^* = B^* A$.

3.5 Construct a {1,3}-Inverse of a Given Ore Matrix

```
> B := Matrix([[OrePoly(x), OrePoly(0), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(x, x, x), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x, x) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad (79)$$

We construct a {1,3}-inverse of B by the following steps, and the method is from Theorem 3.23 in our thesis.

Step 1: Compute the Jacobson form of $B^* B$.

B^* :

```
> Bstar := Invol(B, 3, 4)
```

$$Bstar := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(x-1, -x+2, x) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (80)$$

$B^* B$:

```
> Q1 := Multli(Bstar, B, 4, 4, 3) # B^*B
```

$$Q1 := [[OrePoly(2x^2-2x+2, 2+2x, x^2+2x+2, 4x, x^2), OrePoly(0), OrePoly(x \quad (81)$$

```
   - 1, -x + 2, x), OrePoly(0) ],
  [OrePoly(0), OrePoly(1), OrePoly(0), OrePoly(0) ],
  [OrePoly(x, x, x), OrePoly(0), OrePoly(1), OrePoly(0) ],
  [OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(0) ]]
```

The Jacobson form of $B^{*}B$:

```
> Jac(Q1, 4, 4) :
```

$$\begin{bmatrix} OrePoly(x^2) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix}$$

$$\Bigg[\Bigg[OrePoly(1), OrePoly(0), OrePoly(-x+1, x-2, -x), OrePoly(0)\Bigg],$$

$$\Bigg[OrePoly\left(-\frac{x^2-2x+6}{x^3}, -\frac{x-4}{x^2}, -\frac{1}{x}\right), OrePoly(0),$$

$$OrePoly\left(\frac{2\,(x^3-x^2+2x-3)}{x^3}, -\frac{2\,(x-6)}{x^3}, \frac{x^2+2x-6}{x^2}, 0, 1\right), OrePoly(0)\Bigg],$$

$$\Bigg[OrePoly(0), OrePoly(1), OrePoly(0), OrePoly(0)\Bigg],$$

$$\Bigg[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(1)\Bigg]\Bigg]$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \qquad \textbf{(82)}$$

$$> QQQ := \begin{bmatrix} OrePoly(x^2) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} :$$

$$> totalE := \Bigg[\Bigg[OrePoly(1), OrePoly(0), OrePoly(-x+1, x-2, -x), OrePoly(0)\Bigg],$$

$$\Bigg[OrePoly\left(-\frac{x^2-2x+6}{x^3}, -\frac{x-4}{x^2}, -\frac{1}{x}\right), OrePoly(0),$$

$$OrePoly\left(\frac{2\,(x^3-x^2+2\,x-3)}{x^3}, -\frac{2\,(x-6)}{x^3}, \frac{x^2+2\,x-6}{x^2}, 0, 1\right), OrePoly(0)\Big],$$
$$\Big[OrePoly(0), OrePoly(1), OrePoly(0), OrePoly(0)\Big],$$
$$\Big[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(1)\Big]\Big]:$$

> $totalF := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix}:$

Step 2: Construct a {1}-inverse of B^*B.

> $INV11(QQQ, totalE, totalF, 4, 4)$# a {1}-inverse of B^*B

$$\Big[\Big[OrePoly\left(\frac{1}{x^2}\right), OrePoly(0), OrePoly\left(-\frac{x-1}{x^2}, \frac{x-2}{x^2}, -\frac{1}{x}\right), OrePoly(0)\Big], \tag{83}$$
$$\Big[OrePoly(0), OrePoly(1), OrePoly(0), OrePoly(0)\Big],$$
$$\Big[OrePoly\left(-\frac{x^2-2\,x+6}{x^3}, -\frac{x-4}{x^2}, -\frac{1}{x}\right), OrePoly(0),$$
$$OrePoly\left(\frac{2\,(x^3-x^2+2\,x-3)}{x^3}, -\frac{2\,(x-6)}{x^3}, \frac{x^2+2\,x-6}{x^2}, 0, 1\right), OrePoly(0)\Big],$$
$$\Big[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(0)\Big]\Big]$$

> $A3 := \Big[\Big[OrePoly\left(\frac{1}{x^2}\right), OrePoly(0), OrePoly\left(-\frac{x-1}{x^2}, \frac{x-2}{x^2}, -\frac{1}{x}\right), OrePoly(0)\Big],$

$$\Big[OrePoly(0), OrePoly(1), OrePoly(0), OrePoly(0)\Big],$$
$$\Big[OrePoly\left(-\frac{x^2-2\,x+6}{x^3}, -\frac{x-4}{x^2}, -\frac{1}{x}\right), OrePoly(0),$$
$$OrePoly\left(\frac{2\,(x^3-x^2+2\,x-3)}{x^3}, -\frac{2\,(x-6)}{x^3}, \frac{x^2+2\,x-6}{x^2}, 0, 1\right), OrePoly(0)\Big],$$
$$\Big[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(0)\Big]\Big]:$$

B :

```
> B := Matrix([[OrePoly(x), OrePoly(0), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(x, x, x), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#``
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x, x) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \tag{84}$$

B^* :

```
> Bstar := Invol(B, 3, 4)
```

$$Bstar := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(x-1, -x+2, x) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \tag{85}$$

Step 3: Construct a {1,3}-inverse of B.

```
> X := Multli(A3, Bstar, 4, 3, 4)# a {1,3}−inverse of B
```

$$X := \begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x^2-x+2}{x^2}, -\frac{x-2}{x}, -1\right) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \tag{86}$$

Check that X is a {1,3}-inverse of B.

B :

```
> B := Matrix([[OrePoly(x), OrePoly(0), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(x, x, x), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#``
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x, x) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \tag{87}$$

(1) Check that $BXB = B$:

```
> Multli(Multli(B, X, 3, 3, 4), B, 3, 4, 3)# check BXB=B
```

$$\begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(x, x, x) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad (88)$$

(2) Check that $XBX = X$:

```
> Multli(Multli(X, B, 4, 4, 3), X, 4, 3, 4)# check XBX=X
```

$$\begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly\left(-\frac{x^2-x+2}{x^2}, -\frac{x-2}{x}, -1\right) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (89)$$

(3) Check that $(BX)^* = BX$:

***BX*:**

```
> Multli(B, X, 3, 3, 4)# check (BX)^* = BX
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (90)$$

$(BX)^*$:

```
> Invol(Multli(B, X, 3, 3, 4), 3, 3)
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (91)$$

(4) Check that $(XB)^* = XB$:

***XB*:**

```
> Multli(X, B, 4, 4, 3)# check (XB)^* = XB
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (92)$$

$(XB)^*$:

```
> Invol(Multli(X, B, 4, 4, 3), 4, 4)
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (93)$$

We find that X is a {1,3}-inverse and a MP-inverse of B.

3.6 Construct a {1,4}-Inverse of a Given Ore Matrix

```
> B := Matrix([[OrePoly(x), OrePoly(x, x, x), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#``
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad (94)$$

We construct a {1,4}-inverse of B by the following steps, and the method is from Theorem 3.24 in our thesis.

Step 1: Compute the Jacobson form of BB^*.

B^* :

```
> Bstar := Invol(B, 3, 4)
```

$$Bstar := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) \\ OrePoly(x-1, -x+2, x) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (95)$$

BB^* :

```
> Q1 := Multli(B, Bstar, 3, 3, 4)# BB*
```

$$Q1 := \begin{bmatrix} OrePoly(2x^2, 2x, x^2, 4x, x^2) & OrePoly(x, x, x) & OrePoly(0) \\ OrePoly(x-1, -x+2, x) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (96)$$

The Jacobson form of BB^* :

```
> Jac(Q1, 3, 3)
```

$$\begin{bmatrix} OrePoly(x^2) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(-x, -x, -x) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly\left(-\frac{x^2+x+2}{x^3}, \frac{x+2}{x^2}, -\frac{1}{x}\right) & OrePoly(2, 0, 1, 0, 1) & OrePoly(0) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(97)}$$

$$> A1 := \begin{bmatrix} OrePoly(x^2) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} :$$

$$> totalE := \begin{bmatrix} OrePoly(1) & OrePoly(-x, -x, -x) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly\left(-\frac{x^2+x+2}{x^3}, \frac{x+2}{x^2}, -\frac{1}{x}\right) & OrePoly(2, 0, 1, 0, 1) & OrePoly(0) \end{bmatrix} :$$

$$> totalF := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} :$$

Step 2: Construct a {1}-inverse of BB^* .

```
> INV11(A1, totalE, totalF, 3, 3)# a {1}-inverse of BB*
```

(98)

$$\begin{bmatrix} OrePoly\left(\frac{1}{x^2}\right) & OrePoly\left(-\frac{1}{x}, -\frac{1}{x}, -\frac{1}{x}\right) & OrePoly(0) \\ OrePoly\left(-\frac{x^2+x+2}{x^3}, \frac{x+2}{x^2}, -\frac{1}{x}\right) & OrePoly(2, 0, 1, 0, 1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad \textbf{(98)}$$

> $A3 :=$

$$\begin{bmatrix} OrePoly\left(\frac{1}{x^2}\right) & OrePoly\left(-\frac{1}{x}, -\frac{1}{x}, -\frac{1}{x}\right) & OrePoly(0) \\ OrePoly\left(-\frac{x^2+x+2}{x^3}, \frac{x+2}{x^2}, -\frac{1}{x}\right) & OrePoly(2, 0, 1, 0, 1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} :$$

***B* :**

```
> B := Matrix([[OrePoly(x), OrePoly(x, x, x), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#``
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(99)}$$

B^* :

```
> Bstar := Invol(B, 3, 4)
```

$$Bstar := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) \\ OrePoly(x-1, -x+2, x) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \textbf{(100)}$$

Step 3: Construct a {1,4}-inverse of *B*.

```
> X := Multli(Bstar, A3, 4, 3, 3)# a {1,4}-inverse of B
```

$$X := \begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(-1, -1, -1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \textbf{(101)}$$

Check that X is a {1,4}-inverse of B.

B:

```
> B := Matrix([[OrePoly(x), OrePoly(x, x, x), OrePoly(0), OrePoly(0)], [OrePoly(0),
  OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(1),
  OrePoly(0)]]);#``
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(102)}$$

(1) Check that $BXB = B$:

```
> Multli(Multli(B, X, 3, 3, 4), B, 3, 4, 3)# check BXB=B
```

$$\begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(103)}$$

(2) Check that $XBX = X$:

```
> Multli(Multli(X, B, 4, 4, 3), X, 4, 3, 4)# check XBX=X
```

$$\begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(-1, -1, -1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \textbf{(104)}$$

(3) Check that $(BX)^* = BX$:

BX:

```
> Multli(B, X, 3, 3, 4)# check (BX)* = BX
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad \textbf{(105)}$$

$(BX)^*$:

```
> Invol(Multli(B, X, 3, 3, 4), 3, 3)
```

(106)

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (106)$$

(4) Check that $(XB)^* = XB$:

XB :

```
> Multli(X, B, 4, 4, 3)# check (XB)^* = XB
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (107)$$

$(XB)^*$:

```
> Invol(Multli(X, B, 4, 4, 3), 4, 4)
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (108)$$

We find that X is a {1,4}-inverse and a MP-inverse of B .

3.7 Construct a MP-Inverse of a Given Ore Matrix

```
> B := Matrix([[OrePoly(x), OrePoly(x, x, x), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#` `
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad (109)$$

We construct the unique MP-inverse of B by the following steps, and the method is from Theorem 3.27 in our thesis.

Step 1: Compute the Jacobson form of $B^* B B^*$.

B^* :

```
> Bstar := Invol(B, 3, 4)
```

$$Bstar := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) \\ OrePoly(x-1, -x+2, x) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (110)$$

$B^* B$:

```
> Multli(Bstar, B, 4, 4, 3)
```

$$[[OrePoly(x^2), OrePoly(x^2, x^2, x^2), OrePoly(0), OrePoly(0)], \quad (111)$$
$$[OrePoly(x^2 - 2x + 2, -x^2 + 4x, x^2), OrePoly(x^2 - 2x + 3, 2 + 2x, x^2 + 2x + 2, 4x, x^2), OrePoly(0), OrePoly(0)],$$
$$[OrePoly(0), OrePoly(0), OrePoly(1), OrePoly(0)],$$
$$[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(0)]]$$

$B^* BB^*$:

```
> Q1 := Multli(Multli(Bstar, B, 4, 4, 3), Bstar, 4, 3, 4) # B^*BB^*
```

$$Q1 := [[OrePoly(2x^3, 2x^2, x^3, 4x^2, x^3), OrePoly(x^2, x^2, x^2), OrePoly(0)], \quad (112)$$
$$[OrePoly(2x^3 - 6x^2 + 13x - 1, -2x^3 + 14x^2 - 5x + 6, 3x^3 - 5x^2 + 15x, -x^3 + 12x^2 - 8x + 8, 2x^3 - 7x^2 + 22x, -x^3 + 10x^2, x^3), OrePoly(x^2 - 2x + 3, 2 + 2x, x^2 + 2x + 2, 4x, x^2), OrePoly(0)],$$
$$[OrePoly(0), OrePoly(0), OrePoly(1)],$$
$$[OrePoly(0), OrePoly(0), OrePoly(0)]]$$

The Jacobson form of $B^* BB^*$:

```
> Jac(Q1, 4, 3)
```

$$\begin{bmatrix} OrePoly(x^3) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix}$$

$$\left[\left[OrePoly(x^2 + 1, 0, x^2, 0, x^2), OrePoly(-x^2, -x^2, -x^2), OrePoly(0), OrePoly(0)\right],\right.$$
$$\left[OrePoly(0), OrePoly(0), OrePoly(1), OrePoly(0)\right],$$
$$\left[OrePoly\left(-\frac{2x^4 + x^2 + 2x + 6}{x^4}, \frac{2x^3 + x + 4}{x^3}, -\frac{3x^2 + 1}{x^2}, 1, -2, 1, -1\right),\right.$$

$$OrePoly(2, 0, 1, 0, 1), OrePoly(0), OrePoly(0)\Big],$$

$$\Big[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(1)\Big]\Big]$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \mathbf{(113)}$$

$$> AA := \begin{bmatrix} OrePoly(x^3) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix}:$$

$$> totalE := \Big[\Big[OrePoly(x^2+1, 0, x^2, 0, x^2), OrePoly(-x^2, -x^2, -x^2), OrePoly(0),$$

$$OrePoly(0)\Big],$$

$$\Big[OrePoly(0), OrePoly(0), OrePoly(1), OrePoly(0)\Big],$$

$$\Big[OrePoly\Big(-\frac{2x^4+x^2+2x+6}{x^4}, \frac{2x^3+x+4}{x^3}, -\frac{3x^2+1}{x^2}, 1, -2, 1, -1\Big),$$

$$OrePoly(2, 0, 1, 0, 1), OrePoly(0), OrePoly(0)\Big],$$

$$\Big[OrePoly(0), OrePoly(0), OrePoly(0), OrePoly(1)\Big]\Big]:$$

$$> totalF := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix}:$$

Step 2: Construct a {1}-inverse of $B^* BB^*$.

$$> A3 := INV11(AA, totalE, totalF, 4, 3)\# \text{ a \{1\}-inverse of } B^*BB^*$$

$$A3 := \Big[\Big[OrePoly\Big(\frac{x^2+1}{x^3}, 0, \frac{1}{x}, 0, \frac{1}{x}\Big), OrePoly\Big(-\frac{1}{x}, -\frac{1}{x}, -\frac{1}{x}\Big), OrePoly(0), \quad \mathbf{(114)}$$

$$OrePoly(0)\Big],$$

$$\left[OrePoly\left(-\frac{2x^4+x^2+2x+6}{x^4}, \frac{2x^3+x+4}{x^3}, -\frac{3x^2+1}{x^2}, 1, -2, 1, -1\right), OrePoly(2,0,1,0,1), OrePoly(0), OrePoly(0)\right], \left[OrePoly(0), OrePoly(0), OrePoly(1), OrePoly(0)\right]\right]$$

***B* :**

```
> B := Matrix([[OrePoly(x), OrePoly(x, x, x), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(x,x,x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad \textbf{(115)}$$

B^* :

```
> Bstar := Invol(B, 3, 4)
```

$$Bstar := \begin{bmatrix} OrePoly(x) & OrePoly(0) & OrePoly(0) \\ OrePoly(x-1, -x+2, x) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \textbf{(116)}$$

Step 3: Construct the unique MP-inverse of *B*.

```
> X := Multli(Multli(Bstar, A3, 4, 4, 3), Bstar, 4, 3, 4)# the {MP}-inverse of B
```

$$X := \begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(-1,-1,-1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \textbf{(117)}$$

Check the MP-inverse.

***B* :**

```
> B := Matrix([[OrePoly(x), OrePoly(x, x, x), OrePoly(0), OrePoly(0)], [OrePoly(0),
    OrePoly(1), OrePoly(0), OrePoly(0)], [OrePoly(0), OrePoly(0), OrePoly(1),
    OrePoly(0)]]);#` `
```

$$B := \begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad (118)$$

(1) $BXB = B$ **:**

```
> Multli(Multli(B, X, 3, 3, 4), B, 3, 4, 3)# check BXB=B
```

$$\begin{bmatrix} OrePoly(x) & OrePoly(x, x, x) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \end{bmatrix} \quad (119)$$

(2) $XBX = X$ **:**

```
> Multli(Multli(X, B, 4, 4, 3), X, 4, 3, 4)# check XBX=X
```

$$\begin{bmatrix} OrePoly\left(\frac{1}{x}\right) & OrePoly(-1, -1, -1) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad (120)$$

(3) BX **:**

```
> Multli(B, X, 3, 3, 4)# check (BX)* = BX
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (121)$$

$(BX)^*$ **:**

```
> Invol(Multli(B, X, 3, 3, 4), 3, 3)
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) \end{bmatrix} \quad (122)$$

(4) XB **:**

```
> Multli(X, B, 4, 4, 3)# check (XB)* = XB
```

(123)

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \mathbf{(123)}$$

$(XB)^*$:

```
> Invol(Multli(X, B, 4, 4, 3), 4, 4)
```

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0) & OrePoly(0) \end{bmatrix} \quad \mathbf{(124)}$$

3.8 Compute the Jacobson Normal Form of a Given Ore Matrix

We can get three results by the following procedure, the first result is the Jacobson normal form of a *m* by *n* Ore matrix *B*, the second result *totalE* and the third result *totalF* are two Ore matrices such that *totalE*·*B*·*totalF*= the Jacobson normal form of *B*.

```
> NORMJac := proc(B, m, n)#` Jacobson Normal Form
  local inE12, inE22, EE, tt, j, b12, inF21, inF22, FF, tt1, H, H1, totalE111, totalF111, E111,
      F111, EE111, FF111;
  local w, A, totalE, totalinE, totalF, totalinF, e, E, inE, F, inF, i, k, l, b11, b21, p, ee;
  A := B :
  p := m;
  totalE111 := Matrix(m, shape = identity) :
  totalF111 := Matrix(n, shape = identity) :
  totalE := subs(1 = OrePoly(1), 0 = OrePoly(0), totalE111) :# m by m Ore identity matrix
  totalF := subs(1 = OrePoly(1), 0 = OrePoly(0), totalF111) :# n by n Ore identity matrix
  w := max(m, n) :# we achieve the Jacobson Form of a nonsquare matrix by this code
  for ee from 1 by 1 to w - 1 do
  A[ee, 1 + ee] := A[1 + ee, 1 + ee]# this is the difference between` Jacobson normal form
     and Jacobson form
  end do:
  for e from 0 by 1 to w - 2 do
  E111 := Matrix(m, shape = identity) :
  F111 := Matrix(n, shape = identity) :
  E := subs(1 = OrePoly(1), 0 = OrePoly(0), E111) :# m by m Ore identity matrix
  F := subs(1 = OrePoly(1), 0 = OrePoly(0), F111) :# n by n Ore identity matrix
  for i from 1 by 1 to m - e - 1 do
```

```
p := m :#we achieve the Jacobson Form of a nonsquare matrix by this code
if e ≥ min(m, n) then
        break
    end if:#we achieve the Jacobson Form of a nonsquare matrix by this code
if A[1 + e, 1 + e] ≠ OrePoly(0) and A[2 + e, 1 + e] ≠ OrePoly(0) then
#gcdex(simplify(A[1 + e, 1 + e]), simplify(A[2 + e, 1 + e]), x,'s','t') :
H := ExtendedGCD_'right'(A[1 + e, 1 + e], A[2 + e, 1 + e], WW,'c1','c2') :# H is gcrd
# k:=simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[1 + e, 1 + e] ) : l
    := simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[2 + e, 1 + e] ) :
E(1 + e, 1 + e) := c1[1] : E(1 + e, 2 + e) := c2[1] : E(2 + e, 1 + e) := c1[2] : E(2
    + e, 2 + e) := c2[2] :
 elif A[1 + e, 1 + e] = OrePoly(0)
then
E(1 + e, 1 + e) := OrePoly(0) : E(1 + e, 2 + e) := OrePoly(1) : E(2 + e, 1 + e)
    := OrePoly(1) : E(2 + e, 2 + e) := OrePoly(0) :
 elif A[2 + e, 1 + e] = OrePoly(0)# we do not need to do anything
 then E(1 + e, 1 + e) := OrePoly(1) : E(1 + e, 2 + e) := OrePoly(0) : E(2 + e, 1 + e)
    := OrePoly(0) : E(2 + e, 2 + e) := OrePoly(1) :
end if:
#b11:=simplify( A[1 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[2 + e, 1 + e]) ) :
#b21:=simplify( A[2 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[2 + e, 1 + e]) ) :
# gcdex(k, -l, x,'p','q') :
#inE12:=simplify(p - b11·s·p - b11·t·q) :
#inE22:=simplify(q - b21·s·p - b21·t·q) :
#inE(1 + e, 1 + e) := b11 : inE(1 + e, 2 + e) := inE12 : inE(2 + e, 1 + e) := b21 : inE(2
    + e, 2 + e) := inE22 :
#totalinE:= simplify(totalinE.inE);
totalE := Multli(E, totalE, m, m, p) :
A := Multli(E, A, m, n, p) :
#print(A);
if i = m - e - 1 then break end if:
 EE111 := Matrix(m, shape = identity) :
EE := subs(1 = OrePoly(1), 0 = OrePoly(0), EE111) :# m by m Ore identity matrix
tt := A[2 + e, 1 ..-1] :
A[2 + e, 1 ..-1] := A[i + 2 + e, 1 ..-1] :
A[i + 2 + e, 1 ..-1] := tt :
tt := EE[2 + e, 1 ..-1] :
EE[2 + e, 1 ..-1] := EE[i + 2 + e, 1 ..-1] :
EE[i + 2 + e, 1 ..-1] := tt :
 totalE := Multli(EE, totalE, m, m, p) :
```

```
#totalinE := simplify(totalinE.EE) :
 end do:
 for j from 1 by 1 to n − e − 1 do
p := n :#` `we achieve the Jacobson Form of a nonsquare matrix by this code
if e ≥ min(m, n) then
        break
    end if:#we achieve the Jacobson Form of a nonsquare matrix by this code
#gcdex(simplify(A[1 + e, 1 + e]), simplify(A[1 + e, 2 + e]), x,'s','t') :
#k:=simplify( lcm(A[1 + e, 1 + e], A[1 + e, 2 + e]) / A[1 + e, 1 + e] ) : l
      := simplify( lcm(A[1 + e, 1 + e], A[1 + e, 2 + e]) / A[1 + e, 2 + e] ) :
 if A[1 + e, 1 + e] ≠ OrePoly(0) and A[1 + e, 2 + e] ≠ OrePoly(0)  then
 #gcdex(simplify(A[1 + e, 1 + e]), simplify(A[2 + e, 1 + e]), x,'s','t') :
H1 := ExtendedGCD_'left'(A[1 + e, 1 + e], A[1 + e, 2 + e], WW,'d1','d2');# H is gcrd
# k:=simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[1 + e, 1 + e] ) : l
      := simplify( lcm(A[1 + e, 1 + e], A[2 + e, 1 + e]) / A[2 + e, 1 + e] ) :
# E(1 + e, 1 + e) := c1[1] : E(1 + e, 2 + e) := c2[1] : E(2 + e, 1 + e) := c1[2] : E(2
      + e, 2 + e) := c2[2] :
 F(1 + e, 1 + e) := d1[1] : F(1 + e, 2 + e) := d1[2] : F(2 + e, 1 + e) := d2[1] : F(2
      + e, 2 + e) := d2[2] :
#F(1 + e, 1 + e) := d2[1] : F(1 + e, 2 + e) := d1[2] : F(2 + e, 1 + e) := d1[1] : F(2
      + e, 2 + e) := d2[2] :
  elif A[1 + e, 1 + e] = OrePoly(0)
then
 F(1 + e, 1 + e) := OrePoly(0) : F(1 + e, 2 + e) := OrePoly(1) : F(2 + e, 1 + e)
      := OrePoly(1) : F(2 + e, 2 + e) := OrePoly(0) :
 elif A[1 + e, 2 + e] = OrePoly(0)# we do not need to do anything
 then F(1 + e, 1 + e) := OrePoly(1) : F(1 + e, 2 + e) := OrePoly(0) : F(2 + e, 1 + e)
      := OrePoly(0) : F(2 + e, 2 + e) := OrePoly(1) :
end if:
# in theory, the first F is right, but in the real code, we can find that we need the second F,
#b11:=simplify( A[1 + e, 1 + e] / gcd(A[1 + e, 1 + e], A[1 + e, 2 + e]) ) :
#b12:=simplify( A[1 + e, 2 + e] / gcd(A[1 + e, 1 + e], A[1 + e, 2 + e]) ) :
# gcdex(simplify(k), simplify(-1), x,'p','q') :
#inF21:=simplify(p − b11·s·p − b11·t·q) :
# inF22:=simplify(q − b12·s·p − b12·t·q) :
#inF(1 + e, 1 + e) := b11 : inF(1 + e, 2 + e) := b12 : inF(2 + e, 1 + e) := inF21 : inF(2
      + e, 2 + e) := inF22 :
#totalinF:=simplify( inF. totalinF) :
```

```
totalF := Multli(totalF, F, n, n, p) :
 A := Multli(A, F, m, n, p) :
#print(A);
 if j = n − e − 1 then break end if:
 FF111 := Matrix(n, shape = identity) :
FF := subs(1 = OrePoly(1), 0 = OrePoly(0), FF111) :# n by n Ore identity matrix
tt1 := A[1 ..-1, 2 + e] :
A[1 ..-1, 2 + e] := A[1 ..-1, j + 2 + e] :
A[1 ..-1, j + 2 + e] := tt1 :
 tt1 := FF[1 ..-1, 2 + e] :
FF[1 ..-1, 2 + e] := FF[1 ..-1, j + 2 + e] :
FF[1 ..-1, j + 2 + e] := tt1 :# this is inverse of an elementary matrix
 totalF := Multli(totalF, FF, n, n, p) :
# totalinF:=simplify( FF. totalinF) :
end do:
end do:
print(A);
print(totalE);
print(totalF);
end proc:
with(LinearAlgebra) :
```

> $A := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(x, x^2) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(x) \end{bmatrix}$

$$A := \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(x, x^2) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(x) \end{bmatrix} \quad \textbf{(125)}$$

> $NORMJac(A, 3, 3)$:

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0, -x) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(-1) & OrePoly(1) \end{bmatrix}$$

(126)

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0, -x) \\ OrePoly(0) & OrePoly(0) & OrePoly\left(\frac{1}{x}\right) \\ OrePoly(0) & OrePoly\left(\frac{1}{x}\right) & OrePoly(0, -1) \end{bmatrix} \quad (126)$$

The Jacobson normal form of A :

$$> \begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0, -x) \end{bmatrix}$$

$$\begin{bmatrix} OrePoly(1) & OrePoly(0) & OrePoly(0) \\ OrePoly(0) & OrePoly(1) & OrePoly(0) \\ OrePoly(0) & OrePoly(0) & OrePoly(0, -x) \end{bmatrix} \quad (127)$$

3.9 Problems in the OreAlgebra Package

Problem 1

According to the description of the ExtendedGCD in the ExtendedGCD **package, the following result should be $OrePoly(1)$.**

$$> H := ExtendedGCD_{'right'}\left(OrePoly(1), OrePoly\left(-\frac{1}{2}\,x\right), WW, 'c1', 'c2'\right) \# \ H \text{ is gcrd}$$

$$H := OrePoly\left(-\frac{1}{2}\,x\right) \quad (128)$$

Problem 2

According to the description of the ExtendedGCD in the ExtendedGCD **package, the following result should be $OrePoly(x)$.**

$$> H := ExtendedGCD_{'right'}\left(OrePoly(x), OrePoly\left(-\frac{1}{2}\,x\right), WW, 'c1', 'c2'\right) \# \ H \text{ is gcrd}$$

$$H := OrePoly\left(-\frac{1}{2}\,x\right) \quad (129)$$

Problem 3

According to the description of the ExtendedGCD in the ExtendedGCD **package, the following result should be $OrePoly(x)$.**

$$> H := ExtendedGCD_{'right'}(OrePoly(x), OrePoly(2\cdot x), WW, 'c1', 'c2') \# \ H \text{ is gcrd}$$

$$H := OrePoly(2\,x) \quad (130)$$

Problem 4

According to the description of the ExtendedGCD in the ExtendedGCD package, the following result should be *OrePoly*(x).

> $H := ExtendedGCD_{'left'}(OrePoly(\mathrm{x}), OrePoly(2 \cdot x), WW, 'c1', 'c2')$# H is gcld

$$H := OrePoly(1) \tag{131}$$

>

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