Minimizing the Maximum Interference in k-connected Wireless Networks

by

Sahar Mehrpour

A thesis submitted to
The Faculty of Graduate Studies of
The University of Manitoba
in partial fulfillment of the requirements
of the degree of

Master of Science

Department of Computer Science
The University of Manitoba
Winnipeg, Manitoba, Canada
July 2016

Thesis advisor

Stephane Durocher

Author

Sahar Mehrpour

Minimizing the Maximum Interference in k-connected Wireless Networks

Abstract

Given a set P of n points in $\mathbb{R}^d$ ($d \in \{1,2\}$), we consider the *k-connected interference minimization* problem, in which the objective is to assign a transmission radius to each node in P such that the resulting network is k-connected and the maximum interference is minimized.

We show for any n and any $1 \leq k < n$, $\Omega(\sqrt{kn})$ and $\Omega(k \log(n/k))$ are lower bounds on the worst-case minimum maximum interference in the symmetric and asymmetric models, respectively. In the symmetric case, we present polynomial-time algorithms that build a k-connected network on any given set of n nodes with interference $O(\sqrt{kn})$ in one dimension and $O(\min\{k\sqrt{n}, k \log \lambda\})$ in two dimensions, where λ denotes the ratio of the longest to shortest distances between any pair of nodes. In the asymmetric case, we present a polynomial-time algorithm that builds a strongly k-connected network with maximum interference $O(k \log \lambda)$ in two dimensions.

Contents

List of Figures

Acknowledgments

First and Foremost, I would like to express my gratitude Dr. Stephane Durocher, my thesis supervisor, for his endless guidance through my research. His patience and encouragement were a great motivation for me during the years I studied in the University of Manitoba.

I want to express my thanks to Dr. Ben Li (thesis committee member), Dr. Julien Arino (thesis committee member), Dr. Neil Bruce, and Dr. Andrea Bunt for their support. I also thank the members of the computational geometry lab in the University of Manitoba, Dr. Jyoti Mondal, Yeganeh Bahoo, and Dr. Saeed Mehrabi. Our lab meetings were the interesting part of my studies.

I also thank the Department of Computer Science for offering the Guaranteed Funding Package (GFP) during my studies.

Last but not least, I express my heartfelt gratitude to my parents for their endless love and support throughout my life.

Dedicated to my kind and beloved parents.

Chapter 1

Introduction

Wireless networks consist of wireless devices which are equipped with processors and power supplies. Wireless devices can adjust their transmission power to modify their transmission ranges. So, each device has a transmission range, corresponding to a region in space and can transmit wireless signals to those devices that are within this region.

A network must be connected if a multi-hop communication channel is required between every pair of nodes. A wireless network is connected if every pair of network nodes can communicate. Two nodes can communicate if there is a sequence of wireless devices that can receive a message from a previous device and send it to the next device in the sequence, such that the desired destination node eventually receives the message. The trivial approach to build a connected network on a set of wireless nodes is to set the transmission powers large enough such that each node communicates with all other nodes in the network. Nonetheless, apart from being impractical in real situations, the resulting redundancy in connectivity is very high. Furthermore, this network has high *interference*.

Various secondary objectives can be considered in addition to the connectivity requirement, often resulting in an optimization problem to construct a network that meets both criteria. Common additional objectives include minimizing the maximum or average power consumption, sender-receiver route length, node degree, ratio of route length to Euclidean distance, and, of particular relevance to wireless networks, interference [15].

In real world situations, the transmission power in wireless nodes is not uniform and is more powerful in some areas. However, simulating a real transmission signal is not an easy task. So, in this thesis, like in many other studies, we assume transmission power fades uniformly in every direction. A node can receive a signal from a sender if the power of the signal is strong enough when it reaches the node. Figure 1.1 represents the changes of the transmission power for a one-dimensional node. The node at the centre of the plot represents a wireless node transmitting a signal. The power of a signal emanating from this point decreases as it gets far from the sender's location. If the power of the signal exceeds a specific given threshold, the signal is strong enough to be received. In Figure 1.1 the blue line represents the power threshold and any node located in the shaded region can receive signals from the node in the centre.

Wireless networks are often modelled by geometric graphs in which each node corresponds to a wireless node and each wireless node is represented by a point. In these graphs, each (directed or undirected depending on the model) edge represents a communication link between a pair of wireless nodes. There are two common

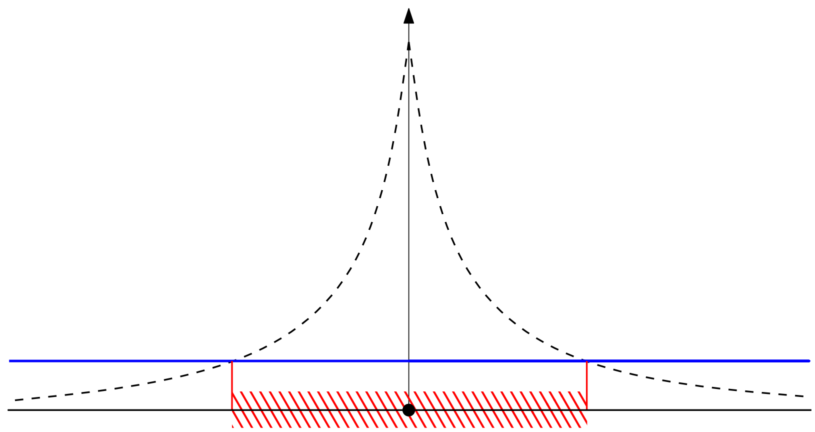


Figure 1.1: This is a location-power transmission plot for a one-dimensional point. The horizontal axis shows the location and the vertical axis shows the transmission power. The threshold for the transmission power is shown by the blue horizontal line. Any point lying in the shaded region receives signals from the point at the centre of the plot.

geometric models for defining interference. In the *sender-based interference model* [5], the interference of a node p is defined to be the number of nodes that receive signals from p. As discussed in [26], this model has some weaknesses in modelling interference since the main interference happens at receiver nodes. von Rickenbach et al. [26] defined *receiver-centric interference model*. In this model, the interference at each point is defined to be the number of nodes whose transmission ranges cover the point. *The maximum interference* is the interference of a node that receives interference from the highest number of nodes among all nodes in the network. *The average interference* is the average interference of all nodes in the network.

In this thesis, we study the *interference minimization problem*. Given a set of points, the problem of minimizing the maximum interference in wireless networks aims to assign a radius to each node such that two conditions are satisfied. The first condition ensures that the resulting network is connected. The second condition verifies that the maximum interference in the network is minimized.

1.1 Motivation

One concern in the maintenance of a wireless network is preserving network connectivity in case of node failure. In this situation, k-connectivity of the network is beneficial, where k parameterizes the network's degree of connectivity. An undirected graph is said to be *k-connected* if it remains connected whenever fewer than k nodes are removed. In directed graphs, the corresponding property is strongly k-connected networks. A directed graph is said to be *strongly k-connected* if there exists k disjoint directed paths between any pair of nodes.

In this thesis, we study the *k-connected interference minimization problem.* This problem is a generalization of the problem of minimizing the maximum interference in wireless networks. In this problem, given a set of points representing node positions the goal is to set the transmission radius of each node to obtain a k-connected network while the maximum interference of the network is minimized. We consider networks in one dimension and two dimensions and study the problem in both symmetric and asymmetric models(defined in Section 1.2.1).

1.2 Definitions

1.2.1 Models and Problem Definition

We represent the position of a wireless node by a point $p_i \in \mathbb{R}^d$. The set $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ represents positions for a set of n nodes. In this thesis, we focus on point sets in one or two dimensions ($d \in \{1, 2\}$).

Each wireless node p has a transmission power and so a transmission range. This range can be defined by a positive real value which is the radius of a d-dimensional ball centred at p. The transmission radius can be represented by a function $r : P \to \mathbb{R}^+$ that assigns a positive real transmission radius to each node.

We use two types of graphs to model a wireless network: directed (asymmetric model) and undirected (symmetric model). In the *symmetric model*, communication in a wireless network is modelled by a symmetric disk graph (SDG). In this graph, there exists an edge between a pair of nodes if the corresponding wireless nodes can communicate with each other (see Figure 1.2) Formally, the *symmetric disk graph* of P with respect to r is an undirected graph with vertex set P and edge set $\{(p, q) \mid \{p, q\} \subseteq P \wedge r(p) \geq \mathrm{dist}(p, q) \wedge r(q) \geq \mathrm{dist}(p, q)\}$, where $\mathrm{dist}(u, v)$ denotes the Euclidean distance between the points u and v in $\mathbb{R}^d$ [1] (see Figure 1.3). When the network is studied in one dimension, the representing model is a one-dimensional graph. In this case, the model is called the *highway model*.

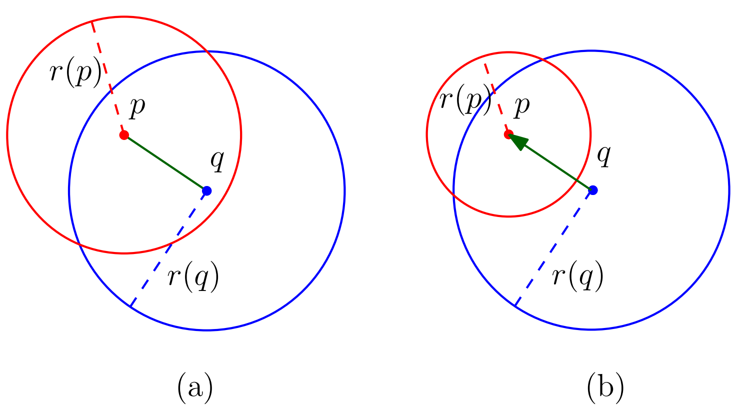


Figure 1.2: (a) An edge in a symmetric network and (b) an edge in the asymmetric network.

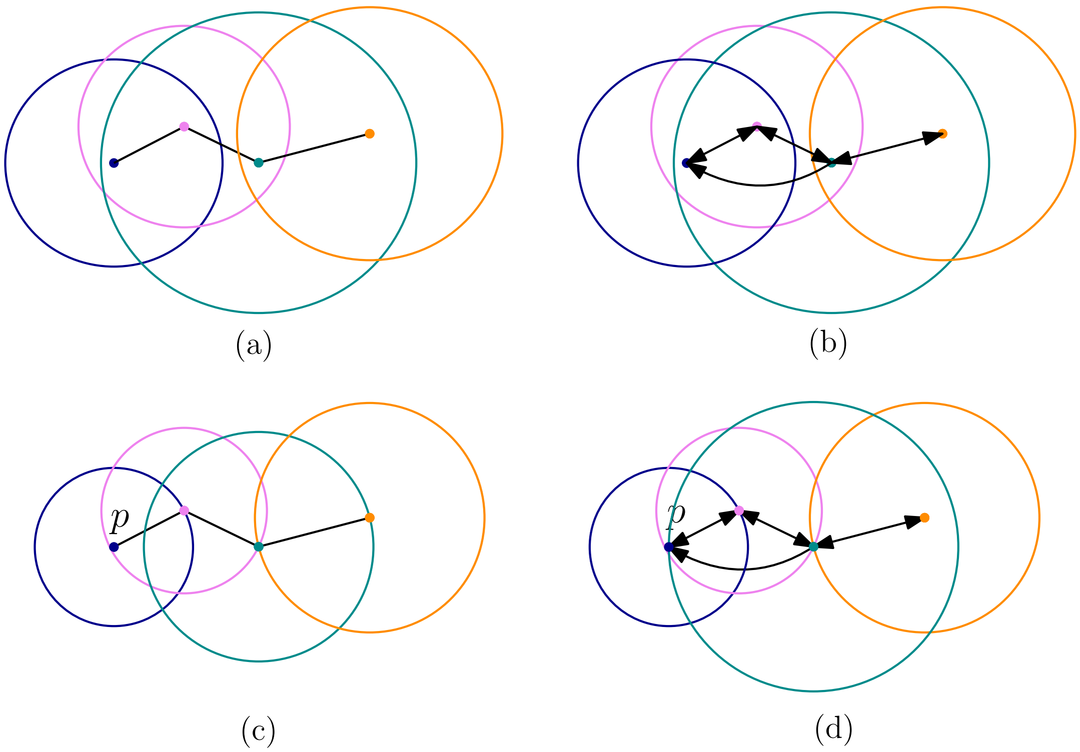


Figure 1.3: Each node and its transmission range has the same color. (a) The black edges shows the symmetric network. (b) The black directed edges shows the asymmetric network. (c) The transmission ranges of a given symmetric network is shown with colored circles. In this network $I(p) = 1$. (c) The transmission ranges of a given asymmetric network is shown with colored circles. In this network $I(p) = 2$.

In the *asymmetric model*, communication in a wireless network is represented by an asymmetric disk graph (DG). In asymmetric disk graphs, there exists a directed edge from vertex p to q if the transmission range of p covers q (see Figure 1.2) Formally, the *asymmetric disk graph* of P with respect to r is a directed graph with vertex set P and edge set $\{(p, q) \mid \{p, q\} \subseteq P \wedge r(p) \geq \mathrm{dist}(p, q)\}$ [1] (see Figure 1.3).

As stated earlier, one objective in topology control is interference optimization. von Rickenbach et al. [26] introduced the receiver-centric interference model. In this model, the *interference* at the node $p \in P$, denoted $I(p)$, is the number of nodes in P

whose transmission range covers p. That is, $I(p) = |\{q \mid q \in P \wedge \mathrm{dist}(p, q) \leq r(q)\}|$. E.g. in Figure 1.3 $I(p) = 1$ in the symmetric network while in the asymmetric one $I(p) = 2$. The *maximum interference* of graph G on point set P with transmission radii given by r is defined as $I(G) = \max_{p \in P} I(p)$.

An undirected graph G is *connected* if there is a path (a sequence of adjacent vertices) joining every pair of vertices in G. A directed graph G is *weakly connected* if there is a path, regardless of the direction of edges, joining every pair of vertices in G. A directed graph G is *strongly connected* if there is a directed path between every pair of vertices in G. An undirected graph G is *k-connected* if there are k disjoint paths joining every pair of vertices in G, or equivalently, G is k-connected if the removal of any j vertices does not disconnect G, for all $j < k$. A directed graph G is *weakly k-connected* if there are k disjoint paths, regardless of the direction of edges, joining every pair of vertices in G. A directed graph G is *strongly k-connected* if there are k directed paths between every pair of vertices in G.

The *maximum interference minimization problem* is defined in both the symmetric and the asymmetric models. This problem aims to assign transmission radii by defining the function r for a given set of points $P \subseteq \mathbb{R}^d$, such that G is connected and $I(G) = \max_{p \in P} I(p)$ is minimized. In the symmetric model, G is a symmetric disk graph and in the asymmetric model, G is an asymmetric disk graph.

The *k-connected maximum interference minimization problem* is a generalization of the maximum interference minimization problem. The goal of the k-connected interference minimization problem is to increase connectivity and minimize the maximum interference. In the symmetric model, the k-connected interference minimization problem tries to define the function r, for a given set of points $P \subseteq \mathbb{R}^d$, such that the corresponding symmetric disk graph G is k-connected and $I(G) = \max_{p \in P} I(p)$ is minimized. Similarly in the asymmetric model, this problem tries to define the assignment function r for a given set of points $P \subseteq \mathbb{R}^d$ such that the resulting asymmetric disk graph G is strongly k-connected and $I(G) = \max_{p \in P} I(p)$ is minimized.

1.2.2 Exponential Chains, Hubs, ϵ-Nets, and Approximation Factors

We now define additional terms and concepts used in the discussion of our results.

Consider an ordered set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{R}$ ordered such that $p_i < p_j$ for all $i < j$. When spacing between consecutive nodes in a set of points P on the line increases exponentially, the interference in any connected graph on P will be large (see Section 2.1.1, e.g., when $P = \{1, 2, 4, 8, 16, \ldots\}$. More generally, the set P contains an *exponential chain* of size m if there exists a subset of indices $\{a_1, \ldots, a_m\} \subseteq \{1, \ldots, n\}$ such that $\max\{\text{dist}(p_{a_i}, p_{a_i - 1}), \text{dist}(p_{a_i}, p_{a_i + 1})\} \geq \max_{j=i+1}^{m} \text{dist}(p_{a_i}, p_{a_j})$ for all $i \in \{1, \ldots, m-1\}$. That is, the transmission range of p_{a_i} in MST(P) covers $\{p_{a_{i+1}}, \ldots, p_{a_m}\}$. For example, the set $P_1 = \{c^i \mid i \in \{0, \ldots, m\}\}$ forms an exponential chain of size m, for and fixed $c \geq 2$. In Figure 1.4 the subset of indices $\{5, 4, 3, 2, 1\}$ defines an exponential chain of size 5 in P_1. Note that the only constraint on $\{a_1, \ldots, a_m\}$ is that $a_i < a_j$ for $i < j$. So the points of an exponential chain do not need to be consecutive or form a perfect geometric sequence. As an example, in Figure 1.4 the subset of indices $\{3, 6, 8, 9, 10\}$ defines an exponential chain of size 5 in P_2.

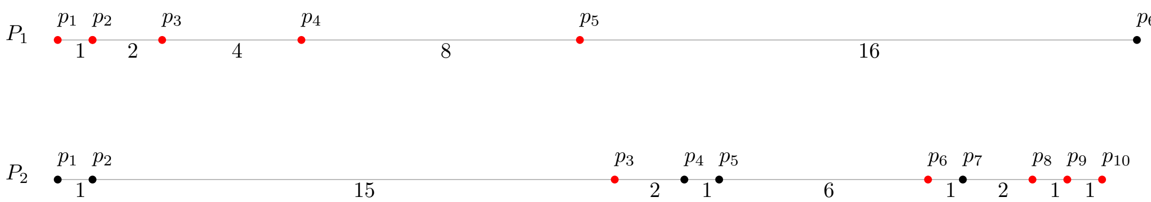


Figure 1.4: $a_1 = 5$, $a_2 = 4$, $a_3 = 3$, $a_4 = 2$, $a_5 = 1$ forms an exponential chain of size 5 in P_1. $a_1 = 3$, $a_2 = 6$, $a_3 = 8$, $a_4 = 9$, $a_5 = 10$ (red points) is an exponential chain of size 5 in P_2.

Given a set $P \subseteq \mathbb{R}$ of n node positions and an assignment of transmission radii corresponding to the symmetric disk graph G on P, von Rickenbach et al. [26] define a *hub* node as any vertex of G that has at least one neighbour to its right; a non-hub node in P has all of its neighbours to its left. For networks in $\mathbb{R}^2$, a subset $H \subseteq P$ may be identified as a set of hubs, where these provide a connected or k-connected backbone to which non-hub nodes connect.

Haussler and Welzl [11] introduced range spaces and ϵ-nets. Halldórsson and Tokuyama [10] defined a family $\mathcal{R}$ of regions as *ranges*. Given a set P of points in $\mathbb{R}^2$ and a family $\mathcal{R}$ of regions (*ranges*) in $\mathbb{R}^2$, the pair $(P, \mathcal{R})$ is a *range space*. For any given $\epsilon \in (0, 1)$, an ϵ-*net* of the range space $(P, \mathcal{R})$ is a subset $S \subseteq P$ such that for any region $R \in \mathcal{R}$, if $|R \cap P| \geq \epsilon n$, then $R \cap S \neq \varnothing$. The family of ranges that Halldórsson and Tokuyama [10] used in their study is the set of all equilateral triangles with one edge parallel to the x-axis. We also apply this family of regions in this thesis.

Consider all k-connected networks on $P \subseteq \mathbb{R}^d$. The minimum maximum interference of all these networks is denoted by $\mathrm{OPT}_k(P)$ and when the set P is implied, the notation OPT_k is used. For $k = 1$, we simply use $\mathrm{OPT}(P)$ and OPT instead of $\mathrm{OPT}_1(P)$ and OPT_1. Also, for a given set $P \subseteq \mathbb{R}^d$, let $\mathrm{MST}(P)$ denote its Euclidean minimum spanning tree and let $\mathrm{DT}(P)$ denote its Delaunay triangulation. Here again, if the point set P is implied, we use MST to denote the minimum spanning tree. It is clear that for one-dimensional point sets, MST is simply the linear network; i.e., every vertex is connected to the immediate vertices on its right and left. For a given set $P \subseteq \mathbb{R}$, Tan [24] defined λ as the ratio of the distances of the closest and the farthest consecutive pairs of nodes. In this thesis, this definition is modified and generalized for higher dimensions. Let $\lambda = d_{\max}/d_{\min}$ be the ratio of the maximum and minimum distances between any pair of points in P, i.e., $d_{\max} = \max_{\{p,q\} \subseteq P} \operatorname{dist}(p, q)$ and $d_{\min} = \min_{\{p,q\} \subseteq P} \operatorname{dist}(p, q)$.

1.3 Overview of Results

We study the problem of maximum interference minimization in two different models: symmetric and asymmetric. The results are summarized as follows.

1.3.1 Symmetric Model

$\Omega(\sqrt{kn})$ Lower Bound: Given a set $P \subseteq \mathbb{R}$ of n points, Rickenbach et al. [26] provided a lower bound of $\Omega(\sqrt{n})$ on the worst case for the minimum maximum interference in any connected network on P. In this thesis we generalize this result to k-connected networks for any $1 \leq k < n$. We establish a lower bound of $\Omega(\sqrt{kn})$ on the worst-case maximum interference among all k-connected networks on a given set of n points in $\mathbb{R}$. This bound applies to point sets in $\mathbb{R}^d$ for any $d \geq 1$.

$O(\sqrt{kn})$ Interference in $\mathbb{R}$: Given a set $P \subseteq \mathbb{R}$ of n points, von Rickenbach et al. [26] provided an algorithm which builds a connected network on P with $O(\sqrt{n})$

interference. We generalize a technique introduced by von Rickenbach et al. and apply it to give an algorithm that builds a k-connected network on P in $O(n \log(n/k))$ time with $O(\sqrt{kn})$ interference, which asymptotically matches the lower bound of $\Omega(\sqrt{kn})$.

$O(k \log \lambda)$ Interference in $\mathbb{R}^2$: Halldórsson and Tokuyama [10] proposed an algorithm based on quadtree decomposition. The network built by this algorithm has interference of $O(\log \lambda)$. Generalization of this technique gives an algorithm that builds a k-connected network on any given two-dimensional point set of size n in $O(n \log \lambda)$ time and with interference in $O(k \log \lambda)$.

$O(k\sqrt{n})$ Interference in $\mathbb{R}^2$: Applying the concept of ϵ-nets, Halldórsson and Tokuyama [10] gave an algorithm that assigns transmission radii to any set P of n nodes in $\mathbb{R}^2$. This algorithm builds a connected network with interference $O(\sqrt{n})$. We applied their technique to introduce an algorithm which builds a k-connected network on any point set $P \subset \mathbb{R}^2$ of n points with interference $O(k\sqrt{n})$. The running time of this algorithm is $O(nk + n \log n + k^3\sqrt{n} \log n)$.

Approximation algorithm in $\mathbb{R}$: We define $\mathrm{Exp}(m, k)$ as a pseudo exponential chain and MST_k as an extension of the standard MST. We show that any network on one dimensional point set P with n points yields interference in $\Omega(I(\mathrm{Exp}(\mathrm{MST}_k(P)/k, k)))$. We conjecture that $I(\mathrm{Exp}(m, k)) \in \Omega(\sqrt{km})$ and so the lower bound on one-dimensional networks on P is $\Omega(\sqrt{I(\mathrm{MST}_k(P))})$. If our conjecture is true, this would give a polynomial-time approximation algorithm with approximation ratio in $O(\sqrt[4]{kn})$.

1.3.2 Asymmetric Model

Lower Bound: Given a set $P \subseteq \mathbb{R}$ of n points, Fussen et al. [9] provided a lower bound of $\Omega(\log n)$ on the worst case for the minimum maximum interference in any weakly connected network on P. In this thesis we generalize this result to strongly k-connected networks for any $1 \leq k < n$. We provide a lower bound of $\Omega(k \log(n/k))$ applying the technique introduced by Tan [24].

Interference in $\mathbb{R}^2$: Similar to the symmetric model, we apply the quadtree decomposition used by Halldórsson and Tokuyama [10] also in the asymmetric model. The resulting algorithm is similar to the symmetric one and runs in $O(k \log n)$. The strongly k-connected network built by this algorithm yields interference $O(k \log \lambda)$.

1.4 Thesis Organization

This thesis is divided into five chapters.

Chapter One: Chapter 1 contains four main sections. The motivation of the problem is described in Section 1.1. The definition used in different parts of the thesis is brought in Section 1.2. The summary of the result is provided in Section 1.3. This section, 1.4, summarizes the organization of the thesis.

Chapter Two: In Chapter 2, we summarize important previous work related to the interference minimization problem. In Section 2.1, some of the results for bounds, algorithms, and complexity of the interference minimization problem in the symmetric model are described briefly. The techniques applied and discussed in this section form the basis and inspiration for the techniques applied in this thesis. Section 2.2 contains brief overview of the work in the asymmetric model.

Chapter Three: Chapter 3 presents the results in the symmetric model. In Section 3.1, we describe our techniques for providing lower bounds on the maximum interference in k-connected networks in one dimension. We also describe our algorithms which build k-connected networks on a given one- and two-dimensional point sets (Sections 3.2 and 3.3). The discussion of an approximation algorithm for building a k-connected one-dimensional network is provided in Section 3.4.

Chapter Four: The asymmetric model is studied in Chapter 4. Section 4.1 contains the discussion and the proof of the lower bound on the maximum interference of asymmetric one-dimensional networks. A polynomial algorithm is proposed in Section 4.2 to build a k-strongly connected network on a given two-dimensional point set.

Chapter Five: Chapter 5 contains conclusions and discussion and future work on the problems we study in this thesis.

Chapter 2

Related Work

In this chapter we provide brief descriptions of the results in the interference minimization problem. Related work for the symmetric model is discussed in Section 2.1 and in Section 2.2 we presented an overview of the related work for the asymmetric model.

2.1 Symmetric Model

The interference minimization problem in the symmetric model has been examined extensively over the past decade. Most of the studies were done under the receiver-based interference model (e.g., [4; 15; 10; 26; 25; 18; 7; 2; 24]). In the following some important results are outlined.

2.1.1 $\Omega(\sqrt{n})$ Lower Bound in $\mathbb{R}$

von Rickenbach et al. [26] studied the problem of minimizing interference in the highway model. In this model, a connected network can be obtained by forming the MST, i.e., simply connecting the nodes linearly such that each nodes communicates with two nodes to its immediate left and right. However, as the authors showed, this solution does not guarantee the minimum interference and results in interference $\Theta(n)$ in exponential chains.

von Rickenbach et al. [26] considered maximum interference in exponential chains. They showed any connected network on an exponential chain with n nodes has interference $\Omega(\sqrt{n})$. The outline of the proof is as follows. Let $P = \{p_1, \ldots, p_n\}$ be an exponential chain in $\mathbb{R}$ indexed from left to right and let G be a network on P with $H \subseteq G$ as a set of hubs. There are two observations: First, $I(G) \geq |H| - 1$ as p_1 is the first hub and is interfered with all other hubs. Second, $I(G) \geq \Delta_G$, where Δ_G is the maximum node degree in G. If $I(G) < \sqrt{n}$, we will have $|P| = |H| + |S| < n$, where $S = P \setminus H$ is the set of non-hub nodes. This contradicts with $|P| = n$ and so, $I(G) \geq \sqrt{n}$. In other words, any connected network on an exponential chain of size n, yields interference $\Omega(\sqrt{n})$.

In the next step, von Rickenbach et al. [26] considered a general set P of n points. The authors showed that any network on P yields interference $\Omega(\sqrt{\gamma})$, where $\gamma = I(\mathrm{MST}(P))$. As in the worst case $\gamma \in \Theta(n)$, any network on a general set of n points has interference $\Omega(\sqrt{n})$.

2.1.2 $O(\sqrt{n})$-Interference in $\mathbb{R}$

von Rickenbach et al. [26] presented an algorithm which builds a one-dimensional connected network on any network by using the concept of hubs (see Section 1.2). Consider $P = \{p_1, \ldots, p_n\}$, a set of n vertices on the line that are indexed from left to right. Let Δ be the maximum node degree of the Uniform Disk Graph on P. In the algorithm proposed by the authors, the leftmost node, p_1, is chosen as the first hub. Then, having the total n vertices, the i-th hub is the vertex with the index of $\lfloor i\sqrt{\Delta} \rfloor$. In the next step, the algorithm connects the hubs linearly and connects every non-hub node to their nearest hubs. The authors showed this algorithm provides a network with interference $O(\sqrt{\Delta})$. In the worst-case $\Delta \in \Theta(n)$ and so, $O(\sqrt{\Delta})$ is asymptotically equal to $O(\sqrt{n})$.

2.1.3 $O(n^{1/4})$-Approximation Algorithm in $\mathbb{R}$

von Rickenbach et al. [26] gave an $O(n^{1/4})$-approximation algorithm for the highway model. Let P be a set of n point in $\mathbb{R}$. Their approximation algorithm first computes $\gamma = I(\text{MST}(P))$ and compares it to $\sqrt{\Delta}$, where Δ is the maximum degree in the Uniform Disk Graph on P. If $\gamma < \sqrt{\Delta}$, it means P is near to being uniformly distributed, and so, MST(P) is selected as the network. Otherwise, P contains an exponential chain of size at least $\sqrt{\Delta}$ and the approximation algorithm applies the hub strategy. The hub strategy of von Rickenbach et al. [26] is described in Section 2.1.2. It selects every $\sqrt{\Delta}$th node as a hub and forms a connected backbone network on the hubs (e.g., their MST), and connects each non-hub node to its nearest hub, giving a network with maximum interference $O(\sqrt{\Delta})$ for any set of n points in $\mathbb{R}$. Applying either strategy provides an approximation ratio $O(\sqrt[4]{\Delta})$.

Note that in the worst case $\Delta \in O(n)$. So, in the worst case the approximation factor is $O(\sqrt[4]{n})$.

2.1.4 Exact Algorithm in $\mathbb{R}$

Another result for one-dimensional networks is obtained by Tan et al. [25]. They provided an algorithm which finds the exact optimal solution by searching among all possible networks. Consider a mapping of a graph on the plane in which the vertices are drawn on a horizontal line and edges are drawn as arcs above or on this line. Given a set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{R}$, assume the arcs $p_i p_k$ and $p_j p_l$ are drawn as in Figure 2.1 where $1 \leq i < j < k < l \leq n$. These two arcs will cross each other. Tan et al. showed it is possible to replace an edge $p_i p_k$ with two shorter edges $p_i p_j$ and $p_j p_k$ such that they do not cross the edge $p_j p_l$ (see Figure 2.1). In addition, this replacement does not increase the interference since no transmission radius is increased. Therefore, with this property, the total number of feasible networks to be considered in an exhaustive search decreases. However, there are still many feasible networks that must be examined to find an optimal network. As a result the running time of this algorithm remains super-polynomial and for a point set P of size n is $O(n^{3+\mathrm{OPT}(P)})$.

2.1.5 Randomized Settings in $\mathbb{R}$

von Rickenbach et al. [26] studied the interference minimization problem in the worst case. Given an exponential chain P of n vertices in $\mathbb{R}$, von Rickenbach et al.

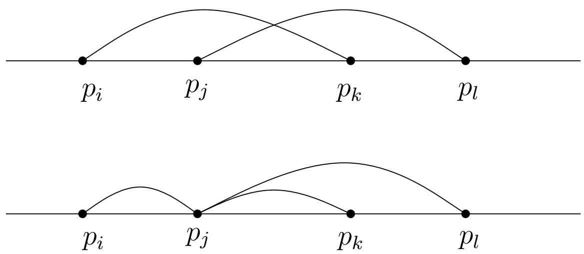


Figure 2.1: The interference in the upper network is equal or greater that than the interference in the lower network.

showed that MST(P) has maximum interference $\Theta(n)$.

Kranakis et al. [18] studied the highway model in the randomized setting. In this setting, the positions of nodes are independently and uniformly selected at random in an open interval $(0, 1)$. Kranakis et al. showed that for a set P of n points in the randomized setting, the maximum interference of MST(P) is $\Theta(\sqrt{\log n})$ with high probability.

2.1.6 NP-Completeness in $\mathbb{R}^2$

The complexity of the problem of minimizing the maximum interference is known for two and higher dimensions. However, the complexity for one-dimensional networks is unknown. It is shown by Buchin [4] that this problem in two dimensions is NP-complete and, therefore, is highly unlikely to be solvable in polynomial time. To prove the hardness of the interference minimization problem, the problem of finding a Hamiltonian path in a grid graph with the maximum degree 3, which is known to be an NP-hard problem [23], is reduced to the problem of minimizing the maximum interference. As a result, one cannot decide whether the optimal maximum interference is at most 3 in polynomial time, unless P = NP. Therefore, one cannot distinguish between the interference 3 and interference 4 in polynomial time, unless P = NP, and so, there is no approximation algorithm for the problem with the approximation ratio less than 4/3, unless P = NP.

2.1.7 $O(\sqrt{n})$-Interference in $\mathbb{R}^2$: ϵ-Nets

Halldórsson and Tokuyama [10] studied the interference minimization problem in two dimensions. They introduced several algorithms that applied different techniques to build connected networks on any given point sets in $\mathbb{R}^2$. One of the algorithms proposed by Halldórsson and Tokuyama [10] applies the ϵ-net theory. See Section 1.2 for the definition of ϵ-nets. The algorithm works as follows.

Given a set P of n points in $\mathbb{R}^2$, the algorithm selects an ϵ-net of size $O(\epsilon^{-1})$. The members of this subset are hubs and form the backbone of this network. Hubs are connected by any connected graph such as the MST or the Local Neighborhood Graph. In the final step, each non-hub node connects to its nearest hub. Each node receives interference from all hubs in the worst case, i.e., at most $O(\epsilon^{-1})$ hubs. Based on the definition of ϵ-nets, each node receives interference from at most $O(n\epsilon)$ non-hub nodes. Consequently, the resulting network has the maximum interference $O(\epsilon n + \epsilon^{-1})$, which corresponds to maximum interference $O(\sqrt{n})$ when $\epsilon = n^{-1/2}$.

Halldórsson and Tokuyama [10] introduced an algorithm that finds an ϵ-net of size $O(\epsilon^{-1})$. However, as it is also stated by the authors, a random sample of size $c\epsilon^{-1} \log \epsilon^{-1}$ is also an ϵ-net with high probability where c is a sufficiently large constant. So selecting random sample of size $\sqrt{n \log n}$, provides an ϵ-net with $\epsilon \in O(\sqrt{n^{-1} \log n})$ with high probability. The network built on P with this random set as hubs has interference $O(\sqrt{n \log n})$ with high probability.

Halldórsson and Tokuyama [10] describe the following algorithm to find an ϵ-net $H \subseteq P$ of size $O(\epsilon^{-1})$. The algorithm begins by greedily constructing a maximal family of disjoint subsets $\{P_1, \ldots, P_l\}$ such that each subset $P_i \subseteq P$ for $i \in \{1, \ldots, l\}$ has the following properties. First, P_i has exactly $\lceil \epsilon n/5 \rceil$ members. Second, there must exist a range $R \in \mathcal{R}$ such that $R \cap P = P_i$. As it is defined earlier in Section 1.2, $\mathcal{R}$ is a set of all equilateral triangles with one edge parallel to x-axis. The algorithm also considers a range $R_0 \in \mathcal{R}$ that is big enough to cover all vertices in P and let P_0 be the three vertices on the corners of R_0. Let $\widetilde{P}$ be the union of the vertices in $\{P_1, \ldots, P_l\}$ and P_0, i.e., $\widetilde{P} = P_0 \cup \bigcup_{i=1}^{l} P_i$.

A pair of points $\{p, q\} \subseteq \widetilde{P}$ is a *Delaunay pair* if there exist a range $R \in \mathbb{R}$ such that only p and q lie on the boundary of R. That is $R \cap \widetilde{P} = \{p, q\}$. The algorithm builds a graph $\mathrm{DT}(\widetilde{P})$ on $\widetilde{P}$ by adding edges between Delaunay pairs.

In the next step, the algorithm colour vertices in $\widetilde{P}$. Consider a set of colours $\{c_1, \ldots, c_{l+3}\}$. For each P_i, the algorithm assigns each vertex in P_i the colour c_i. Each vertex in P_0 is coloured distinctly using the three remaining colours. Note that the points in $P \setminus \widetilde{P}$ remain colourless. Consider two-coloured triangles in $\mathrm{DT}(\widetilde{P})$. If two triangles are adjacent (share an edge) and are coloured with the same colours, they form a chain of two-coloured triangles. We define a *corridor* to be a maximal chain of two-coloured triangles in $\mathrm{DT}(\widetilde{P})$. In the next step, each corridor is greedily partitioned into *subcorridors* such that each subcorridor contains at most $\epsilon n/5$ nodes of P.

In the final step, the set H is defined. The set of endpoints of subcorridors corresponds to the set H. Halldórsson and Tokuyama proved that the set H is an ϵ-net of size $O(\epsilon^{-1})$.

2.1.8 $O(\sqrt{n})$-Interference in $\mathbb{R}^2$: Bucketing

The other technique Halldórsson and Tokuyama [10] applied to build a k-connected network is *bucketing*. The outline of the bucketing method is as follows:

Consider a set P of n points in $\mathbb{R}^2$. Let $G_u(P)$ be the Uniform Disk Graph on P and let $R_{\min}$ be the radius of G_u. The algorithm partitions the plane into squares of size $R_{\min}$ along parallel horizontal and vertical lines. Each of the squares is called a *bucket* positioned uniformly $R_{\min}$ units apart to form a grid. The algorithm then forms a connected network in each bucket by applying the algorithm described in Section 2.1.7. The network built in each bucket has interference $O(\sqrt{\Delta})$, where Δ is the maximum degree in $G_u(P)$. To connect these networks, the algorithm increases a radius of some vertices. It considers $G_u(P)$ and a pair of adjacent buckets B_1 and B_2. If there exist an edge (v_1, v_2) in $G_u(P)$ such that $v_1 \in B_1$ and $v_2 \in B_2$, these buckets are adjacent. For every pair of adjacent buckets, the algorithm selects exactly one edge connecting these two in $G_u(P)$ and increases the transmission radii of its endpoints to $R_{\min}$. The interference of the resulting network is $O(\sqrt{\Delta})$.

2.1.9 $O(\log \lambda)$-Interference in $\mathbb{R}^2$

Halldórsson and Tokuyama [10] presented an algorithm to build a connected network on any point set in two dimensions based on a quadtree decomposition. Given a set P of n points in $\mathbb{R}^2$, a quadtree is built as follows. Initially, the algorithm considers a square B_0 which is big enough to cover all vertices in P. It suffices that the width of B_0 is $w_0 = c \times d_{\max}$, where c is a constant and $d_{\max}$ is the maximum distance between any pair of nodes in P. For a square B_i where $i \geq 0$, let w_i be the width of B_i and let $P_i \subseteq P$ be the set of vertices in B_i.

At some intermediate step, if the square B_i contains more than one node, first, a representative r_i is randomly selected from P_i. Next, the square B_i (parent) is decomposed into four equal size sub-squares (children) and the vertices in $P_i \setminus \{r_i\}$ are partitioned according to the sub-squares.

The algorithm presented by Halldórsson and Tokuyama [10], assigns radii to vertices in P based on the constructed quadtree. Assume the square B_j is the parent of the square B_i for some $i \neq j$ and let p and q be the representatives of B_i and B_j, respectively. The transmission radius of q is set to $\max\{\sqrt{2}w_j, \mathrm{dist}(p,q)\}$ where w_j is the width of B_j. Note that all points are selected as a representative of some square, and so the transmission radii of all vertices are set.

Halldórsson and Tokuyama showed that the interference of the obtained network is $O(\log d^{-1})$, where d is the width of initial square. With a similar argument in the proof of Theorem 3.3, we can show that this algorithm guarantees maximum interference $O(\log \lambda)$ for any set of points P in $\mathbb{R}^2$, where $\lambda = d_{\max}/d_{\min}$.

2.1.10 $O(\mathrm{OPT}^2 \cdot \log n)$-Interference in $\mathbb{R}^2$

Linear programming is another known approach for solving the interference minimization problem. In this method, all required conditions are translated into mathematical constraints. Holec [12] applied this method to find a connected network with minimized maximum interference. He considered all constraints that are necessary for this problem. Some of the constraints concern the properties of the adjacency matrix of the graph corresponding the network. There are also a set of constraints that assures the connectivity of the network. There are also several constraints that relate to interference. However, finding the solution for the exact program is NP-hard. So, a relaxation is applied to the program. At the final step, using randomized rounding gives an approximate solution whose approximation factor is $O(\mathrm{OPT}^2 \cdot \log n)$, where OPT is the minimum interference among the maximum interference of all connected networks on the given set of n points.

2.1.11 $O((\mathrm{OPT} \cdot \log n)^2)$-Interference in $\mathbb{R}^2$

Aslanyan and Rolim [2] proposed an algorithm to find a connected two-dimensional network. They linked this problem to the problem of the Minimum Partial Membership Partial Set Cover (MPMPSC) which is the relaxed format of the Minimum Membership Set Cover problem (MMSC). The problem of the MMSC is proven to be NP-complete [19]. The MPMPSC problem is an extension of the MMSC problem and Aslanyan and Rolim [2] showed there exists a polynomial-time approximation algorithm that solve the MPMPSC problem in polynomial time. Aslanyan and Rolim used the same approximation algorithm to solve the interference minimization problem. Given a set P of n points in $\mathbb{R}^2$, the resulting network has interference $O((\mathrm{OPT} \cdot \log n)^2)$ when the maximum interference of P is bounded by $O(\sqrt{n})$.

2.1.12 Randomized Settings in $\mathbb{R}^2$

The Minimum Spanning Tree is a connected network and is a candidate for the answer to the interference minimization problem. von Rickenbach et al. [26] studied the interference minimization problem and showed given a set P of n points in $\mathbb{R}^d$, in the worst case we have $I(\mathrm{MST}(P)) = \Theta(n)$. So, in the worst case, MST has high interference and does not minimize interference.

Khabbazian et al. [15] studied the interference minimization problem in randomized settings. In this model, locations of nodes are independently and uniformly selected at random in an open interval $(0, 1)^d$. Khabbazian et al. showed for a given set P of n points in randomized settings, $\mathrm{MST}(P)$ has maximum interference $O(\log n)$ with high probability. However, the Minimum Spanning Tree cannot be built using only local information. So, in practice, generating the MST to set transmission radii is not applicable. Therefore, Khabbazian et al. [15] provided an algorithm that assigns a transmission radius to each node using local information to construct a subgraph of MST that has maximum interference $O(\log n)$ with high probability. For node $p \in P$, this information is obtained from those nodes whose connecting path to p has a maximum length of two.

Devroye and Morin [7] improved the results obtained by Khabbazian et al. [14]. They proved that in randomized settings the Minimum Spanning Tree will provide a connected network with expected maximum interference $\Theta((\log n)^{1/2})$ with high probability. The second result they achieved is the improvement on the lower bound on the expected maximum interference to $\Omega((\log n)^{1/4})$. The third result they showed is that there exists a connected network with interference $O((\log n)^{1/3})$ with high probability. This shows that the Minimum Spanning Tree is a suboptimal solution even in the probabilistic setting.

2.2 Asymmetric Model

Although The interference minimization problem is well-studied in the symmetric model, there is much less work in the asymmetric model. In the following we have an overview on some of the work in the asymmetric model.

2.2.1 Exact Quasi-Polynomial Algorithm in $\mathbb{R}$

Brise et al. [3] studied the interference minimization problem in the asymmetric model. Given a set P of n points in $\mathbb{R}$, they characterized the optimal network on P with OPT(P)-Interference. Brise et al. proved that for every directed network on one-dimensional point sets, there exists another network with the same maximum interference and no crossing edges. This result is similar to the result proved by Tan et al. [25] for the symmetric model. Brise et al. also showed that the radius assignment for the simplest structure of the optimal network has the BST-property. An assignment r has the BST-property if for each node p two conditions hold. First, p has at most one child q with $r(p) < r(q)$ and at most one child q' with $r(p) > r(q')$. Second, if q_l and q_r are the descendants of p, all nodes between these two nodes are also descendants of p.

Brise et al. [3] provided a quasi-polynomial algorithm that finds the optimal solution in one dimension by applying dynamic programming. The running time of this algorithm is $n^{O(\log n)}$, or simply, $n^{O(\mathrm{OPT})}$ where OPT is the optimum interference.

They also conjectured that the problem in one dimension is not NP-complete as there exists a quasi-polynomial algorithm for that.

2.2.2 $O(\log n)$-Interference in $\mathbb{R}^d$: Sink Tree

von Rickenbach et al. [27] studied the interference minimization problem in sink trees. This problem tries to find a sink tree with minimized maximum interference. A sink tree is a directed tree such that each of its nodes has a directed path to the sink node. Therefore, sink trees are weakly-connected but they can be strongly-connected by setting the radius of the sink node to infinity which increases the interference of the network by one unit. Fussen et al. [9] showed that there exists a set of n nodes on a line such that any sink tree causes interference at least $\log(n) - 1$.

Given a set P of n points in $\mathbb{R}^d$ ($d \geq 1$), von Rickenbach et al. [27] presented the Nearest Component Connector algorithm which constructs a sink tree with interference $O(\log n)$ in polynomial time. The algorithm works as follows: Let $t \in P$ be a fixed global sink, and let $R_{\max}$ be the maximum permitted transmission radius. The algorithm considers all nodes, including the global sink. Initially, each of the nodes is a single component. In some intermediate step, the algorithm tries to connect the local sink of each component to its nearest neighbor within $R_{\max}$ range outside of its component. If the local sink is unable to connect, the algorithm moves the local sink to its nearest neighbor which is able to connect. If adding edges creates a cycle, the cycle is broken by removing one of its edges. At the end, the sink is moved to node t if the sink of the final tree differs from the predefined global sink.

2.2.3 $O(\log n)$-Interference in $\mathbb{R}^d$: Strongly Connected Network

Korman [17] described two algorithms which build strongly connected directed networks on d-dimensional point sets ($d \geq 1$). In the first algorithm, Layered Nearest Neighbor Network (LLN), there is no limit on the maximum transmission radius. Given a set P of n points in $\mathbb{R}^d$, this algorithm defined P' and initially sets $P' = P$. In each iteration, the algorithm builds the nearest neighbor trees on P'. The root of these directed trees are the input of the algorithm in the next iteration, i.e., P' is the set of roots. In the final step, an infinite radius is assigned to the remaining node. This algorithm runs in $O(n \log n)$ time and has interference $O(\log n)$.

The second algorithm applies the restriction on the maximum transmission radius. The maximum assigned radius is $R_{\min}$, which is the minimum radius for the connected Uniform Disk Graph on the point set. The general structure of the algorithm is similar to the Bucketing algorithm proposed by Halldórsson and Tokuyama [10] (see Section 2.1.8). The algorithm works as follows:

Similar to Section 2.1.8, given a set P of n points in $\mathbb{R}^d$, let $G_u(P)$ be the Uniform Disk Graph with radius $R_{\min}$. Also let $G_u(P')$ be a sub-graph of $G_u(P)$ induced by $P' \subseteq P$. First, the algorithm decomposes the plane into squares of size $R_{\min}$, called buckets. A cluster c is defined to be a set of vertices $P_c \subseteq P$ such that all vertices in P_c belong to the same bucket B and $G_u(P_c)$ is connected. Each buckets is partitioned into four equal squares of size $R_{\min}/2$, called sub-buckets. The algorithm selects hubs from each cluster. For each cluster, if all the vertices of the cluster belong to the same sub-bucket, one vertex is chosen randomly as a hub. Otherwise, the algorithm selects the endpoints of one random edge crossing two sub-buckets. To assure connectivity, the algorithm consider every pair of clusters c_1 and c_2, and if there are two vertices $v_1 \in P(c_1)$ and $v_2 \in P(c_2)$ such that $\mathrm{dist}(v_1, v_2) \leq R_{\min}$, then v_1 and v_2 are also selected as hubs. The radius of hubs are set to be $R_{\min}$. For a non-hub vertex v, let e be the longest edge of v. The radius of v is set to $\min\{R_{\min}, |e|\}$. The resulting network has the interference $O(\log \Delta)$ where Δ is the interference $G_u(P)$.

2.2.4 NP-Completeness in $\mathbb{R}^2$

Buchin [4] showed that the interference minimization problem in two-dimensional symmetric model is NP-complete (see Section 2.1.6). The same result is obtained for the asymmetric model. The complexity of the interference minimization problem in two and higher dimensions is known. However, as there exists a quasi-polynomial algorithm for one-dimensional networks, this problem is unlikely to be NP-complete in one dimension (see Section 2.2.1).

Brise et al. [3] studied the interference minimization problem in asymmetric model. They showed that this problem for two-dimensional planar sensor networks is NP-complete and one cannot decide whether the optimal maximum interference is at most 5 unless P=NP. So there is no algorithm to approximate with a factor less than 6/5 unless P=NP. They reduced the problem of finding a Hamiltonian path in a grid graph of maximum degree 3 to the interference minimization problem. Their technique is very similar to the technique Buchin [4] applied.

Chapter 3

k-Connected Networks in Symmetric Model

In this chapter, we study the problem of interference minimization in the symmetric model. We provide a lower bound on the interference in k-connected networks in Section 3.1. In Sections 3.2 and 3.3 we present algorithms that build k-connected networks in one and two dimensions. We also discuss an approximation algorithm in one dimension in Section 3.4.

3.1 Lower Bounds on Maximum Interference in k-Connected Networks

The following theorem presents a lower bound on the maximum interference in one-dimensional k-connected networks. It is clear that we can extend this result to higher dimensions.

Theorem 3.1. *For every n and every k, $1 \leq k \leq n$, there exists a set of n points $P \subseteq \mathbb{R}$ such that every k-connected network on P has maximum interference $\Omega(\sqrt{kn})$.*

Proof. Consider the set $P = \{p \mid p = 2^i,\ i \in \{0, \ldots, n-1\}\}$ that forms an exponential chain of size n on the line. Consider any k-connected network on P. Let H denote the set of hub vertices and let S denote the set of non-hub vertices, where $|H| + |S| = n$. Since the network is k-connected, all vertices have between k and Δ neighbours, where Δ denotes the maximum vertex degree. Consequently, the first k vertices on the left of the chain are hubs and, furthermore, these k vertices form a clique. Every hub interferes with the leftmost node in the exponential chain. Therefore, the interference at the first node (and, therefore, the maximum interference) is at least $|H| - 1$. Similarly, the maximum interference is at least Δ. That is,

$$I(G) \geq \max\{|H| - 1, \Delta\}. \tag{3.1}$$

Let $E_{S \to H}$ denote the set of edges that join a non-hub vertex to a hub vertex. Similarly, let $E_{H \to H}$ denote the set of edges joining pairs of hubs. This gives,

$$k|S| \leq |E_{S \to H}|. \tag{3.2}$$

Since the first k hubs form a clique, there are $\binom{k}{2}$ edges among these. So we have,

$$\binom{k}{2} \leq |E_{H \to H}|. \tag{3.3}$$

The number of edge endpoints at a hub is bounded by

$$
\begin{aligned}
& & |E_{S\to H}| + 2|E_{H\to H}| &\le |H|\Delta \\
&\Rightarrow & k|S| + 2\binom{k}{2} &\le |H| \cdot I(G) && \text{by (3.1), (3.2) and (3.3)} \\
&\Rightarrow & k(n - |H|) + k(k-1) &\le |H| \cdot I(G) \\
&\Rightarrow & k(n + k - 1) &\le |H|(I(G) + k) \\
& & &\le (I(G) + 1)(I(G) + k) && \text{by (3.1)} \\
&\Rightarrow & I(G) &\ge \frac{\sqrt{(4n-6)k + 5k^2 + 1} - (k+1)}{2}. && (3.4)
\end{aligned}
$$

Next we show that $I(G) \in \Omega(\sqrt{nk})$ for all $n \ge 5$. The result holds trivially for $n \in O(1)$ and, specifically, for $n < 5$. Assume

$$
\begin{aligned}
& & n &\ge 5 && (3.5) \\
&\Rightarrow & 3n &\ge 14 \\
&\Rightarrow & 3n + k &\ge 14 && \text{since } k \ge 1 \\
&\Rightarrow & 3nk + k^2 &\ge 14k \\
&\Rightarrow & 4nk + k^2 &\ge 14k + 3 && \text{by (3.5) and since } k \ge 1 \\
&\Rightarrow & 4nk - 6k + 5k^2 + 1 &\ge 4k^2 + 8k + 4 \\
&\Rightarrow & \sqrt{(4n-6)k + 5k^2 + 1} &\ge 2(k+1) \\
&\Rightarrow & -(k+1) &\ge -\frac{\sqrt{(4n-6)k + 5k^2 + 1}}{2} && (3.6) \\
&\Rightarrow & I(G) &\ge \frac{\sqrt{(4n-6)k + 5k^2 + 1}}{4} && \text{by (3.4) and (3.6)} \\
& & &\ge \frac{\sqrt{2nk + 5k^2}}{4} && \text{by (3.5)} \\
& & &\in \Omega(\sqrt{nk}). \qquad \square
\end{aligned}
$$

In Theorem 3.1 we proved a lower bound for the worst-case maximum interference in a one-dimensional k-connected network (which implies the same lower bound in any higher dimension). In Section 3.2 we present an algorithm that builds a k-connected network with maximum interference $O(\sqrt{kn})$ for any set of n points in $\mathbb{R}$. As we show in Theorem 3.2, the lower bound of Theorem 3.1 is asymptotically tight.

3.2 Building a k-Connected Network in $\mathbb{R}$ with $O(\sqrt{kn})$-Interference

In this section, we present an algorithm that constructs a k-connected network on any set P of n points in $\mathbb{R}$. Our algorithm generalizes the hub technique applied in the algorithm of von Rickenbach et al. [26] to construct a connected network with maximum interference $O(\sqrt{n})$, as discussed in Section 2.1.2.

Instead of every $\sqrt{n}$th node as in [26], we select every $\sqrt{n/(2k+1)}$th node as a hub, resulting in $\lceil \sqrt{n(2k+1)} \rceil$ hubs. Specifically, select the ith node as a hub if $i = \lfloor j\sqrt{n/(2k+1)} \rfloor$ for some $j \in \mathbb{Z}$ (where nodes are numbered $i = 0, \ldots, n-1$). Set each hub node's transmission radius to its furthest point in P (forming a clique on the hubs). Finally, set each non-hub node's transmission radius to the further of the kth hub to its left and the kth hub to its right.

Theorem 3.2. *Given any set P of n points in $\mathbb{R}$ and any $k < n$, transmission radii corresponding to a k-connected network on P with maximum interference $O(\sqrt{kn})$ can be found in $O(n \log(n/k))$ time.*

Proof. First we show that the network produced is k-connected.

$$\begin{aligned}
& & n &> k \\
&\Rightarrow & n &> \frac{k}{2+1/k} \\
&\Rightarrow & \sqrt{n(2k+1)} &> k. \\
&\Rightarrow & \left\lceil \sqrt{n(2k+1)} \right\rceil &> k.
\end{aligned}$$

Therefore, there are at least k hubs. Since the hubs form a clique and each non-hub node is connected to k hubs, the network is k-connected.

Next we bound the maximum interference. Choose any point $p \in P$. The interference at p, denoted $I(p)$, is the sum of the interference it receives from hub and non-hub nodes. Hub nodes define a partition of non-hub nodes into $\lceil \sqrt{n(2k+1)} \rceil$ intervals. Suppose the hub at the left end of each interval belongs to that interval. Let I_i denote the interval that contains p, where intervals are numbered in order from the left. Let h_l and h_r denote the respective hubs at the left and right extremities of I_i. Three types of non-hub nodes interfere with p: nodes in I_i, nodes in I_j for $j < i$ that are connected to h_r, and nodes in I_j for $j > i$ that are connected to h_l. Since each non-hub node connects to its k nearest hubs, p may receive interference from non-hub nodes in k intervals on each side, or $2k$ total intervals, corresponding to at most $\lceil 2k\sqrt{n/(2k+1)} \rceil$ non-hub nodes in other intervals. In addition, p may receive interference from non-hub nodes within its own interval. Finally, p receives interference from at most $\lceil \sqrt{n(2k+1)} \rceil$ hubs. Summing these gives

$$\begin{aligned} I(p) &\leq \left\lceil \sqrt{n(2k+1)} \right\rceil + \left\lceil 2k\sqrt{\frac{n}{2k+1}} \right\rceil + \left\lceil \sqrt{\frac{n}{2k+1}} \right\rceil \\ &< \sqrt{n(2k+1)} + (2k+1)\sqrt{\frac{n}{2k+1}} + 3 \\ &= 2\sqrt{n(2k+1)} + 3 \\ &\in O(\sqrt{kn}). \end{aligned}$$

The hubs can be identified in $O(n \log(n/k))$ time by near-sorting P, e.g., by a partial execution of deterministic quicksort to partition P into blocks of size $\sqrt{n/(2k+1)}$ that returns the partition pivots in sorted order. The list of hubs is traversed in $O(\sqrt{n/k})$ time to assign a transmission radius to each hub, corresponding to the further of the leftmost or rightmost points in P. Non-hub nodes are examined in block sequence, in arbitrary order within a given block. Each non-hub's transmission radius is set to the maximum distance of its kth hub to the left and its kth hub to the right in $O(n)$ total time, achieved by simultaneously traversing the list of hubs and referring to the $(i-k)$th and $(i+k)$th hubs, where i denotes the block index. The total time is dominated by near-sorting, resulting in $O(n \log(n/k))$ time in the worst case. □

This guaranteed $O(\sqrt{kn})$ maximum interference matches the lower bound of $\Omega(\sqrt{kn})$ established in Theorem 3.1, showing that our algorithm is asymptotically optimal in the worst case. Previously, we knew $I(p) \in \Omega(\sqrt{n})$ in the worst case, implied by $k = 1$ [26]. Furthermore, $I(p) \to n-1$ as $k \to n-1$. The interesting implication of Theorem 3.2, however, is for values of k between these two extrema: that the worst-case maximum interference's dependence on k is sublinear for all values of k.

3.3 Building a k-Connected Network in $\mathbb{R}^2$

In this section we present two algorithms that generalize techniques applied in algorithms of Halldórsson and Tokuyama [10] described in Sections 2.1.7 and 2.1.9. Given a set P of n points in $\mathbb{R}^2$, our algorithms construct respective k-connected networks on P with maximum interference $O(k \log \lambda)$ and $O(k\sqrt{n})$, for any k.

3.3.1 $O(k \log \lambda)$-Interference in $\mathbb{R}^2$: Quadtree Decomposition

Theorem 3.3. *Given any set P of n points in $\mathbb{R}^2$ and any $k < n$, transmission radii corresponding to a k-connected network on P with maximum interference $O(k \log \lambda)$ can found in $O(n \log \lambda)$ time, where $\lambda = d_{\max}/d_{\min}$ is the ratio of the maximum and minimum distances between any two points in P.*

Proof. Let B_0 be an axis-parallel square of minimum width $w_0 \leq d_{\max}$ that contains P. Select any set of k points $R_0 \subseteq P$ as representatives for B_0 and set their transmission radii to $\sqrt{2}w_0$. Divide B_0 into four subsquares of width $w_0/2$ and partition $P \setminus R_0$ accordingly. This procedure is applied recursively as follows. Each non-empty square B_i of width w_i contains some set $P_i \subseteq P$. Select a representative set $R_i \subseteq P_i$ arbitrarily, where $|R_i| = \min\{k, |P_i|\}$. Set the transmission radius of each $p \in R_i$ to $\max_{q \in B_j} \operatorname{dist}(p, q)$, where B_j is the parent square to B_i (i.e., q is one of the corners of B_j). The square B_i is divided into four squares of width $w_i/2$ and $P_i \setminus R_i$ is partitioned accordingly. The recursion terminates when $|P_i| \leq k$.

The first k representatives form a k-clique. Each remaining node is connected to the k representatives of its parent square. Consequently, any node forms a k-connected graph with its ancestors in the quadtree. Therefore, the entire network is

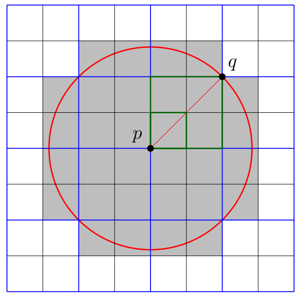


Figure 3.1: A point p is selected as a representative for a quadtree cell, denoted by the smaller bold green square of width w_i. In the worst case, the furthest representative, q, of the parent square of p, denoted by the larger bold green square, is a distance $2\sqrt{2}w_i$ from p. Consequently, p's transmission range interferes with at most 32 cells of the quadtree at its level.

k-connected.

The width of the root square is at most $d_{\max}$. The width of the lowest leaf square in the quadtree is at least $d_{\min}/(2\sqrt{2})$. Therefore, the height of the quadtree is at most $\lceil \log(2\sqrt{2}\lambda) \rceil = \lceil 3/2 + \log \lambda \rceil$. Each representative interferes with at most 32 cells at its level in the quadtree; see Figure 3.1. Therefore, each node $p \in P$ receives interference from at most $32k$ nodes at each level of the tree, for a total interference of at most $32k\lceil 3/2 + \log \lambda \rceil \in O(k \log \lambda)$.

At each node of the quadtree, k representatives are selected and have their transmission radii assigned, and the set P_i is partitioned into four subsets in $O(|P_i|)$ time. Since the quadtree's height is $O(\log \lambda)$, the total time is $O(n \log \lambda)$. $\square$

The algorithm described in the above Theorem is based on a quadtree decomposition. Quadtrees are also defined in higher dimensions. So, it is straightforward to generalize this algorithm to d-dimensions where $d \geq 2$.

3.3.2 $O(k\sqrt{n})$-Interference in $\mathbb{R}^2$: ϵ-Nets

In this section we describe an algorithm that constructs a k-connected network with maximum interference $O(k\sqrt{n})$ for any given set P of n points in $\mathbb{R}^2$. We assume a non-degeneracy condition on points, specifically, that no two points lie on the same line forming an angle of 0, $\pi/3$, or $2\pi/3$ with the x-axis.

This algorithm first selects a set H of $O(k\sqrt{n})$ hubs by finding an $((k\sqrt{n})^{-1})$-net of size $O(k\sqrt{n})$ on P as in the algorithm of Halldórsson and Tokuyama [10] described in Section 2.1.7. Consequently, any range containing at least $O(\sqrt{n}/k)$ points of P must contain a hub. Next, a k-connected backbone is built on the hubs. Finally, each non-hub node is connected to its k nearest hubs.

In general it suffices to k-connect the hubs by taking their complete graph. Although the hubs could be k-connected by applying the algorithm recursively, this does not lead to any asymptotic reduction in the maximum interference. Connecting hubs by a tree, such as the MST or the local neighbourhood graph, does not guarantee k-connectivity after non-hubs connect to their k nearest hubs. For small k (e.g., $k \leq 3$) the Delaunay triangulation provides a good strategy for k-connecting hubs, but a more general strategy is required for larger k.

We analyze the maximum interference of the resulting network. Consider an arbitrary point $p \in P$. Divide the plane around p into six cones $R_1(P), \ldots, R_6(p)$ such that for each i, $R_i(p)$ is the cone consisting of all rays with apex p and angle in $[(i-1)\pi/3, i\pi/3]$. Without loss of generality, we consider the cone $R_1(p)$; analogous results apply to the remaining cones. Let $h_1, \ldots, h_k$ denote the k hubs nearest to p in $R_1(p)$ ordered by increasing distance to p. Let $l_\alpha(p)$ denote the line through p with angle α.

Lemma 3.4. *No point in $R_1(p) \cap (P \setminus H)$ lies on the right of $l_{2\pi/3}(h_k)$ and interferes with p.*

Proof. For the sake of contradiction, assume such a point q exists. Consequently, the transmission radius of q is at least $\text{dist}(p, q)$, and so, q is connected to some hub $h \in H$ where $\text{dist}(p, q) < \text{dist}(q, h)$. However, $\text{dist}(q, h_i) < \text{dist}(p, q) < \text{dist}(q, h)$ for all $i \in \{1, \ldots, k\}$, contradicting the fact that q is connected to its k nearest hubs. $\square$

Lemma 3.5. *There are $O(k\sqrt{n})$ nodes in the area enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$.*

Proof. We decompose the range enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$ into smaller regions and count the vertices in each region. The first region is the range enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_1)$. As this range contains no hub, it contains at most $c\sqrt{n}/k$ nodes of P, for some fixed $c \in \mathbb{R}^+$.

For each $i \in \{1, \ldots, k-1\}$, let Q_i denote the isosceles trapezoidal region enclosed by $l_0(p)$, $l_{\pi/3}(p)$, $l_{2\pi/3}(h_i)$, and $l_{2\pi/3}(h_{i+1})$. We identify ranges in $\mathcal{R}$ that contain no hub whose union covers Q_i. Let H'_1 be a list of the i nearest hubs to p in descending order according to their distance to $l_{\pi/3}(p)$. For each j, let h'_j denote the first hub in the list H'_j. Let A_1 be the range enclosed by $l_0(p)$, $l_{\pi/3}(h'_1)$, and $l_{2\pi/3}(h_k)$. For $j \geq 2$, let $H'_j = H'_{j-1} \setminus \{h'_{j-1}$ and all hubs below $l_0(h'_{j-1})\}$. If $H'_j \neq \varnothing$, let A_j be the range enclosed by $l_0(h'_{j-1})$, $l_{\pi/3}(h'_j)$, and $l_{2\pi/3}(h_i)$. Otherwise, A_{j-1} is the final range necessary to cover Q_i, and we let A_{j-1} be the range enclosed by $l_0(h'_{j-1})$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$. This procedure selects at most $i+1$ ranges whose union covers Q_i, each of which contains no hub in its interior. See Figure 3.2.

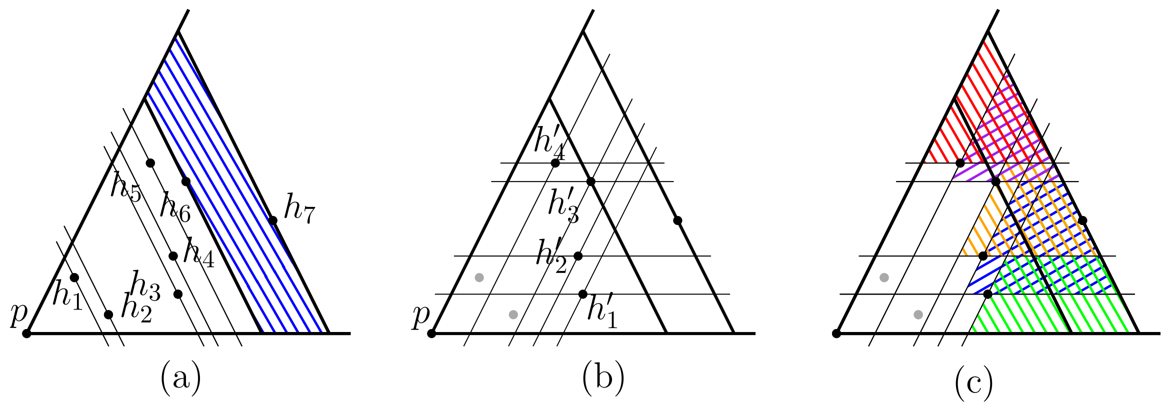


Figure 3.2: (a) The shaded region is the trapezoid Q_6. (b) Four hubs that determine the ranges used to cover Q_6. (c) The five empty ranges whose union covers Q_6.

Along with the first range, the region $\bigcup_{i=1}^{k-1} Q_i$ is exactly the entire region enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$. Since each Q_i can be covered by $i+1$ ranges, each of which contains no hub in its interior, the entire region can be covered by $3k/2 + k^2/2$ ranges. Since each empty range contains at most $c\sqrt{n}/k$ nodes of P, the region enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$ contains at most $ck\sqrt{n} \in O(k\sqrt{n})$ nodes of P. $\square$

Theorem 3.6. *Given any set P of n points in $\mathbb{R}^2$ and any $k < n$, transmission radii corresponding to a k-connected network on P with maximum interference $O(k\sqrt{n})$ can found in $O(nk + n\log n + k^3\sqrt{n}\log n)$ time.*

Proof. We first argue that the resulting network is k-connected. The clique of hubs is k-connected. Each non-hub node is connected to k hubs. Therefore, the entire network is k-connected.

Next we bound the maximum interference. By Lemmas 3.4 and 3.5, for any node $p \in P$, $O(k\sqrt{n})$ non-hub nodes interfere with p in each of the six cones around p. There are $O(k\sqrt{n})$ hubs, each of which may interfere with p. Therefore, $I(p) \in$

$O(k\sqrt{n})$.

Finally we analyze the algorithm's running time. Since this algorithm require running part of the algorithm of Halldórsson and Tokuyama [10] described in Section 2.1.7, we begin by analyzing the time it takes to build the ϵ-net.

Greedily constructing the maximal family of disjoint subsets can be achieved in $O(n \log n)$ time. Similarly, the generalized Delaunay triangulation can be constructed in $O(n \log n)$ time [8] after constructing the Θ-graph (e.g., see [6; 13; 22]). Finding corridors, subcorridors, and their endpoints can done greedily in $O(n)$ time.

In our algorithm we form a clique on the set H of hubs, which can be done in $O(|H| \log |H|)$ time by finding the convex hull of the hubs and setting the transmission radius of each hub to the distance to its furthest hub in $O(\log |H|)$ time per hub using binary search on the boundary of the convex hull, or $O(|H| \log |H|)$ total time. In the final step, we set the transmission radius of each non-hub node to the distance to its kth nearest hub. To do so we can compute a k-nearest neighbour Voronoi diagram of the set H of hubs in $O(k^2|H| \log |H|)$ time [20], upon which a point location data structure (e.g., [16]) is constructed in $O(k|H|(\log k + \log |H|))$ time and applied in $O(k + \log |H|)$ time per non-hub node, or $O(nk + n \log |H|)$ total time. Thus, the running time is dominated by the larger of $O(nk)$, $O(n \log n)$, and $O(k^2|H| \log |H|)$. Since $|H| \in O(k\sqrt{n})$, this gives a total running time of $O(nk + n \log n + k^3\sqrt{n} \log n)$. $\square$

Note that in this Section we define regions by lines passing through nodes. Nonetheless, we consider the area inside the regions for further analysis and we neglect the nodes on the boundary of the region. We have the same assumption for the nodes lying on the lines.

3.4 Approximation Algorithm in $\mathbb{R}$: Discussion

In this section, we present a polynomial-time algorithm that builds a k-connected network on any given set P of n points in $\mathbb{R}$, for any $1 \leq k < n$. We also present a conjecture on the worst-case maximum interference on a point set with certain properties. If this conjecture holds, the resulting network can be shown to have maximum interference $O(\sqrt[4]{kn} \cdot \mathrm{OPT}_k(P))$. That is, our algorithm constructs a k-connected network on P in polynomial time, with approximation factor $O(\sqrt[4]{kn})$ assuming the conjecture.

Consider a set $P = \{p_1, \cdots, p_n\}$ of n vertices with the following properties. Assume $P \setminus \{p_1\}$ is divisible into m distinct consecutive groups P_i ($i \in \{1, \cdots, m\}$) of size at least k such that the following conditions hold:

- Consider a subset $\{a_{i,1}, \ldots, a_{i,k_i}\} \subseteq \{1, \ldots, n\}$, where $k_i \geq k$, giving a subset $P_i = \{p_{a_{i,1}}, \ldots, p_{a_{i,k_i}}\} \subseteq P$ for all $i \in \{1, \ldots, m\}$.
- $P = \cup_{i=1}^{m} P_i$
- $P_i \cap P_j = \emptyset$ for $i \neq j$
- For $i \in \{1, \ldots, m\}$, $\mathrm{dist}(p_{a_{1,1}}, p_{a_{i,1}}) \leq \mathrm{dist}(p_{a_{i,1}}, p_{a_{i,k}})$
- $\cup_{i=1}^{m} p_{a_{i,1}}$ forms an exponential chain (see Figure 3.3)[1].

[1] Note that as the groups are consecutive and the union of them forms P, we have $a_{1,j} = j$, $a_{i,j} = a_{i,1} + j - 1$, and $a_{i+1,1} = a_{i,k_i} + 1$.

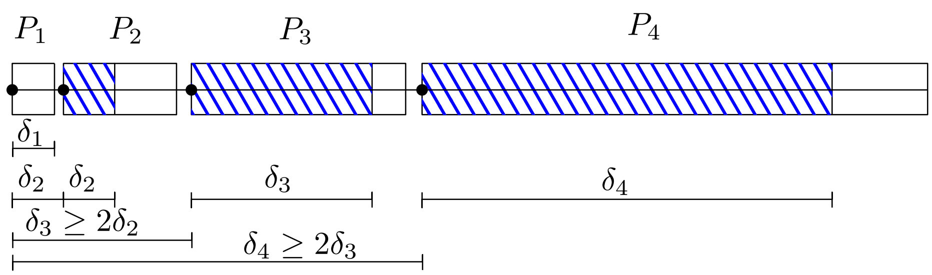


Figure 3.3: Each box denotes one group P_i. The interior of each shaded region contains at most $k-3$ nodes. The point set $P = \cup_{i=1}^{4} P_i$ is $\mathrm{Exp}(4, k)$ for some k.

If all the above properties hold for a point set P, then P is a *pseudo-exponential chain* $\mathrm{Exp}(m, k)$.

Recall the definition of interference from Section 1.2. Given a set P of n points in $\mathbb{R}$ and a network $G = (P, E)$ on P, let $I(p)$ denote the interference at node $p \in P$ yielded by network G and let $I(G) = \max_{p \in P} I(p)$ denote the maximum interference yielded by G. Let $\mathrm{MST}_k(P)$ be a network on a set P of n nodes such that $n > k$ and each node is connected to its $k-1$ nodes on its right plus an edge between the rightmost and the leftmost nodes. It is clear that $\mathrm{MST}_k(P)$ is k-connected.

Lemma 3.7. *Given a set P of n points in $\mathbb{R}$ and any k, $1 \leq k \leq n$, any k-connected network on P has maximum interference $\Omega(\min_{\gamma/k \geq m \geq \gamma - k} I(\mathrm{Exp}(m, k)))$ where $\gamma = \theta(I(\mathrm{MST}_k(P)))$.*

Proof. Let $p \in P$ be a vertex with the highest interference in $\mathrm{MST}_k(P)$, i.e., $I(v) = I(\mathrm{MST}_k(P))$. Let $\{u_1, \ldots, u_\gamma\}$ be the nodes interfering with p lying on its right. Therefore, $I_{\mathrm{MST}_k(P)}(v)/2 \leq \gamma \leq I_{\mathrm{MST}_k(P)}(v)$. Note that at least half of the interfering nodes lies on one side and without loss of generality, we assume they are on the right.

Based on the structure of $\mathrm{MST}_k(P)$, Nodes $\{u_1, \ldots, u_{k-1}\}$ have directed edges to p. As there is no edge between p and u_k, and u_k interferes with p, u_k must be connected to some vertex u' on its right such that $\operatorname{dist}(p, u_k) \leq \operatorname{dist}(u_k, u')$. Without loss of generality, assume u' is the $(k-1)$th node connected to u_k. There are at most $(k-2)$ nodes between u_k and u' that can interfere with p (see Figure 3.4).

Let u_i be the immediate node on the right of u' that interferes with p. With the same argument above, there should be a node u'' on the right of u_i such that $\operatorname{dist}(p, u_i) \leq \operatorname{dist}(u_i, u'')$. We have $\operatorname{dist}(p, u_k) \leq 1/2 \operatorname{dist}(u_k, u_i)$ (see Figure 3.4). The same argument holds for the rest of the interfering nodes.

The sub-chain of P from p to $u_{\gamma+k-1}$ is $\operatorname{Exp}(m, k)$ where m is at least γ/k and at most $\gamma - k$. When $m = \gamma/k$, the first $(k-1)$ interfering nodes form one group, while for the rest of the interfering nodes, every k nodes lies in one group. When $m = \gamma - k$, the first $(k-1)$ interfering nodes form one group, while for the rest of the interfering nodes, each node lies in group (see Figure 3.3). So, the interference of any network on P is at least the interference of this pseudo-exponential chain $\operatorname{Exp}(m, k)$ which is $\min\limits_{\gamma/k \geq m \geq \gamma - k} I(\operatorname{Exp}(m, k))$. $\square$

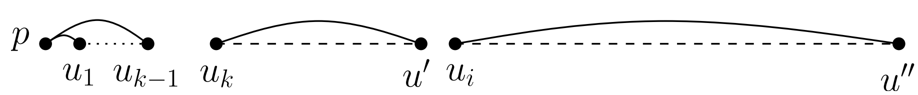


Figure 3.4: Forming $\mathrm{MST}_k(P)$, the first $k-1$ interfering nodes of p have direct edges to p. $k-3$ nodes $\{u_2, \ldots, u_{k-2}\}$ are located on the dotted line and $k-2$ nodes lies on each dashed line.

Note that in $\min_{\gamma/k \geq m \geq \gamma - k} I(\operatorname{Exp}(m, k))$, when $m = \gamma/k$, $\operatorname{Exp}(m, k)$ has the smallest number of groups and the fewest vertices, and so intuitively it has the minimum interference. Therefore, we have $I(\operatorname{Exp}(\gamma/k, k)) = \min_{\gamma/k \geq m \geq \gamma - k} I(\operatorname{Exp}(m, k))$. This statement follows from Conjecture 3.8 as well.

Conjecture 3.8. *Given a pseudo-exponential chain* $\mathrm{Exp}(m, k)$*, any* k*-connected network on this chain has the maximum interference at least* $\Omega(\sqrt{km})$*.*

If Conjecture 3.8 holds, then based on Lemma 3.7 any k-connected network on a given set P of n points has interference at least $\Omega(\sqrt{\gamma})$, where $\gamma = \Theta(I(\mathrm{MST}_k(P)))$. If Conjecture 3.8 holds, we propose the following approximation algorithm.

For a given set P of n points, first the algorithm forms $\mathrm{MST}_k(P)$ on P and computes $I(\mathrm{MST}_k(P))$. If $I(\mathrm{MST}_k(P)) > \sqrt{kn}$, then the approximation algorithm applies the algorithm described in Section 3.2 with interference $O(\sqrt{kn})$. Otherwise, the approximation algorithm returns $\mathrm{MST}_k(P)$ as the final network. If Conjecture 3.8 holds, then this approximation algorithm approximates the optimal network within a factor of $O(k^{1/4}n^{1/4})$:

- If $I(\mathrm{MST}_k(P)) > \sqrt{kn}$, the resulting network built by the algorithm in Section 3.2 has interference $O(\sqrt{kn})$. On the other hand, based on Lemma 3.7, the interference is in $\Omega(\sqrt{I(\mathrm{MST}_k(P))})$. So we have interference $O(\sqrt{kn}/\sqrt{I(\mathrm{MST}_k(P))}) = O(\sqrt{kn}/\sqrt[4]{kn}) \in O(\sqrt[4]{kn})$.

- If $I(\mathrm{MST}_k(P)) \leq \sqrt{kn}$, the network is $\mathrm{MST}_k(P)$ and has interference $I(\mathrm{MST}_k(P))$. Based on Lemma 3.7, the interference is in $\Omega(\sqrt{I(\mathrm{MST}_k(P))})$. So we have interference $O(I(\mathrm{MST}_k(P))/\sqrt{I(\mathrm{MST}_k(P))}) \in O(\sqrt{I(\mathrm{MST}_k(P))}) \in O(\sqrt[4]{kn})$.

Chapter 4

k-Connected Networks in the Asymmetric Model

In this chapter, we study the interference minimization problem in the asymmetric model defined in Section 1.2.1. As we discussed in Chapter 2, for any given set of n points in $\mathbb{R}$, the worst-case maximum interference in connected networks is $\Theta(\sqrt{n})$ in the symmetric model and $\Theta(\log n)$ in the asymmetric model. The interference minimization problem in the symmetric model is studied in Chapter 3, and we showed for any n and any $1 \leq k < n$, $\Omega(k\sqrt{n})$ is a lower bound for the worst-case maximum interference. We also showed for any given set P of n nodes in $\mathbb{R}^2$ and any $1 \leq k < n$ there exists a k-connected network on P with $O(k\sqrt{n})$ maximum interference.

In this chapter, we present a lower bound $\Omega(k \log(n/k))$ on the worst-case maximum interference in strongly k-connected networks. We also describe a polynomial-time algorithm to build a strongly k-connected network with $O(k \log \lambda)$ on any given set of n points in $\mathbb{R}^2$ and any $1 \leq k < n$.

4.1 Lower Bounds on Maximum Interference in Strongly k-Connected Networks

Tan [24] described how to build a recursive graph and analyzed its maximum interference. In this section, we apply a similar technique for providing a lower bound on the worst-case maximum interference in strongly k-connected networks in one dimension. Lemma 4.1 is a generalization of a lemma Tan proved in [24].

Recall from Section 1.2, $\mathrm{OPT}_k(P)$ denotes the minimum maximum interference of all possible strongly k-connected networks on a given point set P. For any two point sets P and P', the shortest distance between P and P' is defined as $\min_{p \in P, p' \in P'} \mathrm{dist}(p, p')$.

Lemma 4.1. *For an arbitrary set P of n points on a line, let l be the distance between the leftmost and the rightmost points of P. Consider another set of points P'. Let G be any strongly k-connected network on $P \cup P'$. If the shortest distance between P and P' is greater than l, then in G, at least $\mathrm{OPT}_k(P)$ nodes of P interfere with any node in P.*

Proof. Suppose P and P' are two sets of points in $\mathbb{R}$ as described in the lemma and the shortest distance between them is at least l. For the sake of contradiction, assume there exist a k-connected network G on $P \cup P'$ such that for each node $v \in P$, fewer than $\mathrm{OPT}_k(P)$ nodes of P interfere with v. We show that we can find a strongly k-connected network G' on P with interference less than $\mathrm{OPT}_k(P)$ deriving a contradiction.

In any strongly k-connected network, there must be k distinct directed paths between any pair of nodes. G is strongly k-connected and for any two nodes in P there are k distinct directed paths in G. We modify these paths such that they pass only through vertices in P. We add the edges of the modified paths to the set of edges in G'.

For any pair of nodes $v_1, v_2 \in P$, we select k distinct directed paths from v_1 to v_2. In each path τ, we remove the vertices in P' and add edges between those pair of nodes in P that are neighbours to the nodes in P' in τ. For example, let $\tau = (v_1, \ldots, u_1, v'_1, \ldots, v'_2, u_2, \ldots, v_2)$ be one of the directed paths from v_1 to v_2 where $u_1, u_2 \in P$ and the path $(v'_1, \ldots, v'_2)$ passes through vertices in P'. We modify the path to $\tau' = (v_1, \ldots, u_1, u_2, \ldots, v_2)$ by removing the nodes in P' and adding the edge (u_1, u_2). For any pair $u, v \in P$ we have $\mathrm{dist}(u, v) \leq l$ and for any $v \in P$ and any $v' \in P'$, we have $\mathrm{dist}(v, v') \geq l$. So $\mathrm{dist}(u_1, u_2) \leq \mathrm{dist}(u_1, v'_1)$. Node u_1 is the only node whose transmission radius is modified: it is decreased. Consequently, the interference of the network G does not increase.

We build a network G' on P by considering the k modified directed paths between any pair of vertices in P. G is a strongly k-connected network on $P \cup P'$ in which fewer than $\mathrm{OPT}_k(P)$ nodes in P interfere with nodes in P and we found a strongly k-connected network G' on P with interference less than $\mathrm{OPT}_k(P)$. This contradicts the definition of $\mathrm{OPT}_k(P)$. □

The following Theorem shows a lower bound on the worst-case maximum interference in strongly k-connected networks in one dimension. This bound also applies in higher dimensions.

Theorem 4.2. *For every n and every k, $1 \leq k \leq n$, there exists a set of n points*

$P \subseteq \mathbb{R}$ such that every strongly k-connected network on P has maximum interference $\Omega(k \log(n/k))$.

Proof. We build a recursive point set P_i using a construction similar to that used by Tan [24]. Let P_1 be a set of $k \geq 1$ nodes located on a line such that consecutive pairs of nodes are unit distance apart. For $i \geq 1$, let X_i be the set of x-coordinates of the vertices in P_i, and let l_i be the distance between the leftmost and the rightmost vertices in P_i.

For simplicity, let P_{i_1} , P_{i_2} , P_{i_3} , and P_{i_4} denote the set of vertices with respectively X_{i-1}, $\{x+l_{i-1}+1 \mid x \in X_{i-1}\}$, $\{x+5l_{i-1}+3 \mid x \in X_{i-1}\}$, and $\{x+6l_{i-1}+4 \mid x \in X_{i-1}\}$ as their x-coordinates (see Figure 4.1). We define X_i as the union of the x-coordinates of the vertices in P_{i_1} , P_{i_2} , P_{i_3} , and P_{i_4}. We have $l_1 = k$ and $l_i = \Theta(k \cdot 9^{i-1})$.

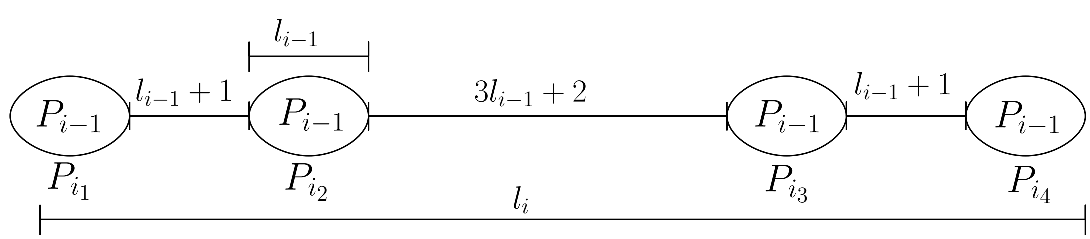


Figure 4.1: The recursive structure of P_i.

We claim $\mathrm{OPT}_k(P_i) = \Theta(i \cdot k)$. First, we show $\mathrm{OPT}_k(P_i) \geq i \cdot k/2$ by induction on i. We have $\mathrm{OPT}_k(P_1) = k > k/2$. Assume the claim holds for $(i-1)$, i.e., $\mathrm{OPT}_k(P_{i-1}) \geq (i-1) \cdot k/2$. We show the claim also holds for i. Since the minimum distance between P_{i_j} $(j \in \{1, \ldots, 4\})$ and $P_i \setminus P_{i_j}$ is greater than the length of P_{i_j}, based on Lemma 4.1, in any strongly k-connected network on P_i at least $\mathrm{OPT}_k(P_{i_j})$ nodes of P_{i_j} interfere with any node in P_{i_j}. To have a strongly k-connected network on

P_i, there must be at least k disjoint directed edges from $P_{i_1} \cup P_{i_2}$ to $P_{i_3} \cup P_{i_4}$. Consider the origins of these edges. We have two cases. In the first case, assume at least $k/2$ of the origins belong to P_{i_2}. These origins also interfere with nodes in P_{i_1}. So, the interference in P_{i_1} is at least $\mathrm{OPT}_k(P_{i_1}) + k/2$. In the second case, assume at most $k/2$ of the origins are in P_{i_2}. So, to have a strongly k-connected network, there must be at least $k/2$ edges from P_{i_2} to P_{i_1}. This set of edges increases the interference in P_{i_1} by at least $k/2$. So, in this case the interference in P_{i_1} is at least $\mathrm{OPT}_k(P_{i_1}) + k/2$. We have $\mathrm{OPT}_k(P_i) \geq \mathrm{OPT}_k(P_{i_1}) + k/2 = \mathrm{OPT}_k(P_{i-1}) + k/2 \geq (i-1) \cdot k/2 + k/2 = i \cdot k/2$.

Next, we observe that $\mathrm{OPT}_k(P_i) \leq \mathrm{OPT}_k(P_{i-1}) + 10k$. Consider having k disjoint directed edges between every pair of P_{i_j} ($j \in \{1, \ldots, 4\}$). The resulting network on P_i is strongly k-connected if P_{i-1} is strongly k-connected. The interference at P_{i_2} will increase by at most $10k$ as there are ten sets of k directed edges whose endpoints can interfere with points in P_{i_2} (see Figure 4.2). Applying this process recursively gives $\mathrm{OPT}_k(P_i) \leq 10(i \cdot k) - 9k \leq 10(i \cdot k)$.

Summarizing the above arguments, we have $i \cdot k/2 \leq \mathrm{OPT}_k(P_i) \leq 10(i \cdot k)$ which is $\mathrm{OPT}_k(P_i) = \Theta(i \cdot k)$. For a given k, we have $|P_i| = 4|P_{i-1}| = k \cdot 4^{i-1}$ which gives $i = 1 + \log_4(|P_i|/k) = \Theta(\log(|P_i|/k))$ and we have $\mathrm{OPT}_k(P_i) = \Theta(k \log(|P_i|/k))$.

$\square$

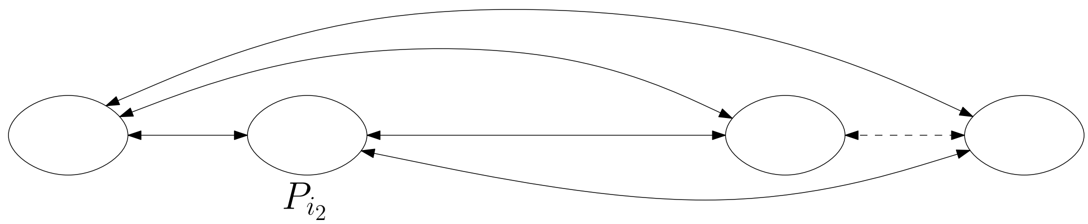


Figure 4.2: The endpoints of the group of edges shown with the dashed line does not interfere with points in P_{i_2}. The endpoints of the directed set of edges shown in solid lines interfere with the nodes in P_{i_2} and may increase the interference.

4.2 Building a Strongly k-Connected Network in $\mathbb{R}^2$ with $O(k \log \lambda)$-Interference

In this section, we present a polynomial-time algorithm that builds a strongly k-connected network on any given set P of n points in $\mathbb{R}^2$. To find such an algorithm, first, we argue that the existing algorithms in the asymmetric model are not straightforward to generalize. However, an algorithm presented for the symmetric model also applies in the asymmetric model.

There are few algorithm proposed to build either strongly or weakly connected networks in the two-dimensional asymmetric model. von Rickenbach et al. [27] studied the problem of finding a sink tree with minimum maximum interference (see Section 2.2.2). Given a set P of n points in $\mathbb{R}^2$ and a fixed sink node $t \in P$, their algorithm first builds a weakly connected network by forming a sink tree and then by setting the radius of the sink node to infinity, makes the network strongly connected. This network has interference $O(\log n)$. One way to generalize this algorithm is to select a set of hubs and to set their radii to infinity. Recall from Section 2.2.2, the algorithm runs in $O(\log n)$ iterations and in each iteration the root of the trees built in the previous iteration are considered to combine the existing trees and form bigger ones. Assume in ith iteration there are at least k trees and in the next iteration there are at most k trees. The root of the trees built in the ith iteration can be selected as hubs and be assigned infinite radii. So there are at least k hubs. Nonetheless, this modification does not make the network strongly k-connected as there are not k distinct paths from non-hub nodes to hubs.

Korman [17] proposed two algorithms for the asymmetric model which are described in Section 2.2.3. The main difference between the algorithms is the bound on the assigned radii. The first algorithm is similar to the algorithm presented by von Rickenbach et al. [27] described in Section 2.2.2. Similar to the discussion above, the algorithm is not straightforward to generalize. The other algorithm proposed by Korman [17] builds a strongly connected network with a limited radius range. This algorithm sets the radius of each node to at most $R_{\min}$, the minimum radius of the connected Uniform Disk Graph. However, this bound prevents networks from being k-connected in some point sets. For example, consider the point set illustrated in Figure 4.3. In this figure there are two clusters of size $k-1$, where $k > 2$, with minimum distance of $R_{\min}$. Even if we set the radius of all nodes to $R_{\min}$, the resulting network is still not k-connected. For $k = 2$, consider three points on a line such that consecutive nodes are at most $R_{\min}$ units apart. There is no strongly or weakly k-connected network with maximum radius $R_{\min}$ on this point set.

Halldórsson and Tokuyama [10] proposed an algorithm to build a symmetric connected network on any two-dimensional point set based on quadtree decomposition (see Section 2.1.9). In Section 3.3.1, we described a generalization of this algorithm

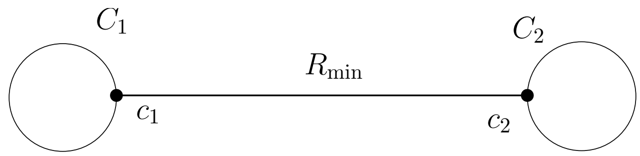


Figure 4.3: C_1 and C_2 are two cluster of size $k-1$, $k > 2$, such that for any $c_1' \in C_1$ and $c_2' \in C_2$, $\mathrm{dist}(c_1', c_2') \geq R_{\min}$. There is a unique pair of points $c_1 \in C_1$ and $c_2 \in C_2$ such that $\mathrm{dist}(c_1, c_2) = R_{\min}$. For this point set, there is no (strongly or weakly) k-connected network with maximum radius $R_{\min}$.

to build a k-connected network in the symmetric model. As we show in Theorem 4.3 a modified version of this algorithm can be applied to build a strongly k-connected network in the asymmetric model.

Recall the definition of λ from Section 1.2: $\lambda = d_{\max}/d_{\min}$ where $d_{\max}$ and $d_{\min}$ are respectively the longest and the shortest distances between any pair of nodes.

Theorem 4.3. *Given any set P of n points in $\mathbb{R}^2$ and any $1 \leq k < n$, transmission radii corresponding to a strongly k-connected network on P with maximum interference $O(k \log \lambda)$ can be found in $O(n \log \lambda)$ time, where $\lambda = d_{\max}/d_{\min}$.*

Proof. Similar to the proof of Theorem 3.3, the algorithm first builds a quadtree. The initial square is B_0 which is chosen to be sufficiently large to cover all vertices in P. For $i \geq 0$, let $P_i \subseteq P$ be the set of vertices in B_i. For square B_i, a representative set $R_i \subseteq P_i$ of size $\min\{k, |P_i|\}$ is randomly selected. Next, if $P_i \setminus R_i$ is non-empty, the square B_i (parent) is decomposed into four equal size sub-squares (children) and the vertices in $P_i \setminus R_i$ are partitioned according to the sub-squares.

For node $p \in R_0$, the transmission radius of p is set to $\sqrt{2}w_0$ where w_0 is the width of B_0. Therefore, each node $p \in R_0$ interferes with all nodes in $P \setminus \{p\}$ and there are directed edges $\{(p, q) \mid q \in P \setminus \{p\}\}$ in the network. The transmission radius of

$p \in B_i$ is set to $\max_{q \in B_j}\{\text{dist}(p,q)\}$, where B_j is the parent of B_i. In other words, there are directed edges $\{(p,q) \mid p \in R_i \wedge q \in R_j\}$.

We claim that the resulting network is strongly k-connected. Consider two points p and q in P where $p \in R_i$ and $q \in R_j$. We show there are k distinct directed path from p to q. If $i = 0$, we have $|R_i| = k$. Note that nodes within the same representative set have directed edges to each other and every node in R_0 has a directed edge to every node in P. So, the k distinct paths from p to q are $\{(p,p',q)|p' \in R_0\}$.

Now assume $i \neq 0$. There is a set of distinct indices $\{a_1,\ldots,a_m\}$, $m < \log\lambda$ and there is a set of paths $\{(p,p_{a_1},\ldots,p_{a_m},q)\}$ where $p_{a_l} \in R_{a_l}$ for $l \in \{1,\ldots m\}$, B_{a_l} is the parent of $B_{a_{l-1}}$ for $l \in \{2,\ldots,m\}$, and B_{a_1} is the parent of B_i. If B_j is an ancestor of B_j, let B_i be the parent of B_{a_m}, otherwise let $a_m = 0$. These paths are valid as there are directed edges from representatives of a child square to representatives of the parent square and also there are directed edges from nodes in R_0 to every node in P. It is clear that there are k distinct directed paths from p to q as $|R_{a_l}| = k$ for $l \in \{1,\ldots m\}$. See Figure 4.4 for illustration.

A similar argument as in the proof of Theorem 3.3 holds for the interference and the running time of the algorithm. The maximum interference of the resulting network is $O(k\log\lambda)$, and the running time of the described algorithm is $O(n\log\lambda)$. □

Similar to the algorithm in Theorem 3.3, it is straightforward to generalize the described algorithm in Theorem 4.3 to higher dimensions.

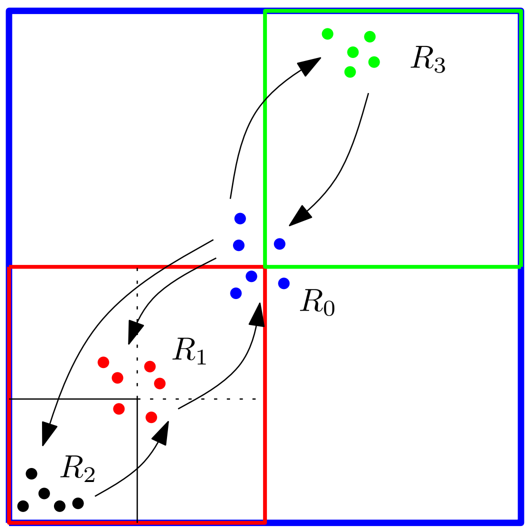


Figure 4.4: Applying the generalized quadtree algorithm for a $k = 6$, representatives of each colored square are illustrated with the same color vertices. k directed paths from $p \in R_0$ to $q \in R_2$ are $\{(p, p', q)|p' \in R_0\}$. k directed paths from $p \in R_2$ to $q \in R_0$ are $\{(p, p_1, q)|p_1 \in R_1\}$ and k directed path from $p \in R_2$ to $q \in R_3$ are $\{(p, p_1, p_0, q|p_1 \in R_1 \wedge p_0 \in R_0\}$.

Chapter 5

Conclusion and Future Work

5.1 Symmetric Model

Lower Bound: Generalizing a technique applied by von Rickenbach et al. [26], we described an $O(n \log(n/k))$-time algorithm that constructs a k-connected network with $O(\sqrt{kn})$ maximum interference for any given set of n points in one dimension and any given k, along with a matching lower bound $\Omega(\sqrt{kn})$ on the worst-case maximum interference for k-connected networks in one dimension for any given n and k. Consequently, these bounds are asymptotically tight in one dimension. Generalizing techniques applied by Halldórsson and Tokuyama [10], we described an $O(nk + n \log n + k^3\sqrt{n} \log n)$-time algorithm that constructs a k-connected network with $O(k\sqrt{n})$ maximum interference for any given set of n points in two dimensions and any given k. The lower bound $\Omega(\sqrt{kn})$ applies in two dimensions, but no higher lower bound is known, leaving open the question of whether a k-connected network with lower maximum interference can be found for any given set of n points in two dimensions and any k. In particular, is maximum interference $O(\sqrt{kn})$ always achievable in two dimensions?

Problem 1. *Given a set P of n points in $\mathbb{R}^2$ and any $1 \leq k < n$, is there a polynomial-time algorithm that builds a k-connected network on P with maximum interference $O(\sqrt{kn})$?*

Approximation Algorithm: von Rickenbach et al. [26] gave a polynomial-time algorithm that builds a connected network with interference at most $O(n^{1/4} \cdot \mathrm{OPT}_1(P))$ for any set P of n points on the line. Their algorithm constructs a network either by applying the hub strategy or returning $\mathrm{MST}(P)$, whichever has lower maximum interference. To bound the approximation factor they rely on a pair of lemmas showing that $\mathrm{OPT}_1(P) \in O(\sqrt{n})$ and $\mathrm{OPT}_1(P) \in \Omega(\sqrt{I(\mathrm{MST}(P))})$. In Section 3.4, we tried to generalized their method to build a k-connected network in one-dimension. In Lemma 3.7 we showed $\Omega(I(\mathrm{Exp}(\gamma/k, k)))$ is a lower bound for the maximum interference in one dimension where $\gamma \in \theta(I(\mathrm{MST}_k(P)))$. The complexity of the interference of $\mathrm{Exp}(m, k)$ remains open. We conjecture $\Omega(\sqrt{km})$ is a lower bound for the worst-case maximum interference in $\mathrm{Exp}(m, k)$ and based on this we proposed a polynomial-time approximation algorithm with approximation factor $O(\sqrt[4]{kn})$.

Problem 2. *Given a set P of n points in $\mathbb{R}^2$ and any $1 < k < n$, is there a polynomial-time $O(\sqrt[4]{kn})$-approximation algorithm that builds a k-connected network on P?*

Problem 3. *Given a set P of n points in $\mathbb{R}^2$ and any $1 \leq k < n$, is there any polynomial-time c-approximation algorithm that builds a strongly k-connected network* *on P where c is a constant?*

Complexity: Finally, the complexity of the interference minimization problem in one dimension remains an important open question. Buchin [4] showed that the problem of finding a connected network that minimizes maximum interference for a given set of n points in two dimensions is NP-complete. This implies hardness for the k-connected problem in two dimensions. However, the complexity of both problems remains unknown in one dimension.

Problem 4. *Is the problem of minimizing the maximum interference for a given set of points in one dimension solvable in polynomial time?*

Recently, Moraes et al. [21] considered different formulations for k-connected networks and provided an experimental study on these formulations to find a formulation which is easy to solve with an integer programming solver.

5.2 Asymmetric Model

One-Dimensional Algorithm: Brise et al. [3] studied the interference minimization in sink trees. They characterized the sink tree with minimized maximum interference and presented a quasi-polynomial algorithm that finds an optimal network. von Rickenbach et al. [27] gave a polynomial-time algorithm that builds a sink tree with $O(\log n)$ interference for any given n points in $\mathbb{R}^d$. As there exists a lower bound $\Omega(\log n)$ on the worst-case maximum interference in one dimension, these bounds are asymptotically tight in one dimension. We generalized the lower bound to strongly k-connected networks and showed for any n and any $k < n$ there exists a set of n

points such that any strongly k-connected network has interference $\Omega(k \log(n/k))$. One open question is whether there exists a polynomial-time algorithm that builds a strongly k-connected network on any given point set with interference matching the lower bound.

Problem 5. *Given a set P of n points in $\mathbb{R}$ and any $1 \leq k < n$, is there any polynomial-time algorithm that builds a strongly k-connected network on P with maximum interference $O(k \log(n/k))$?*

Approximation Algorithm: Korman [17] proposed two algorithms that build strongly connected networks on any set of n points in $\mathbb{R}^d$. The resulting networks built by these polynomial-time algorithms have interference $O(\log n)$ for unbounded transmission radii, and $O(\log \Delta)$ for bounded transmission radii, where Δ is the interference of the Uniform Disk Graph. We described an $O(n \log \lambda)$-time algorithm that builds a strongly k-connected network with $O(k \log \lambda)$ maximum interference by generalizing the technique applied by Halldórsson and Tokuyama [10]. As there is an approximation algorithm in two dimensions for the symmetric model, a natural question is whether there exists a polynomial-time approximation algorithm that builds a strongly k-connected network on any point set in $\mathbb{R}^2$.

Problem 6. *Given a set P of n points in $\mathbb{R}^2$ and any $1 \leq k < n$, is there any polynomial-time c-approximation algorithm that builds a strongly k-connected network on P where c is a constant?*

Complexity: For a given set of n points in two dimensions, Brise et al. [3] showed that finding the strongly-connected network with the minimum maximum interference is NP-Complete. This result also shows the hardness of the interference minimization in strongly k-connected networks in two dimensions. However, the complexity of the problem in one dimension remains open.

Problem 7. *Is the problem of finding a strongly connected network with minimized maximum interference for a given set of points in one dimension solvable in polynomial time?*

Appendix A

Examples

In the following we illustrate examples for the algorithms provided in Chapter 3.

A.1 One Dimension

In this section, we provide an example for the algorithm described in Section 3.2. In this example $k = 2$ and n, the number of nodes, is sufficiently large that $\lceil\sqrt{n/2k+1}\rceil = 3$. In this illustrations $n = 13$.

Figure A.1: P is a set of $n = 13$ points on the line

Ordered the points from left to right, in the first step, the algorithm selects every $\lceil\sqrt{n/2k+1}\rceil$th $= 3$rd node as a hub:

Figure A.2: Every 3 points is selected as a hub shown in blue.

Next, a clique is formed on the set of hubs:

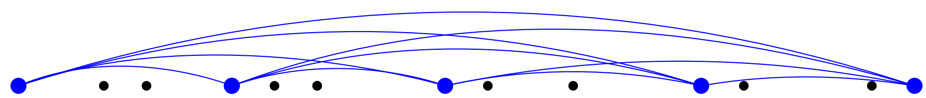


Figure A.3: The hubs are connected with a clique.

Finally, every non-hub node is connected to the k nearest hubs on its right and the k nearest hubs on its left:

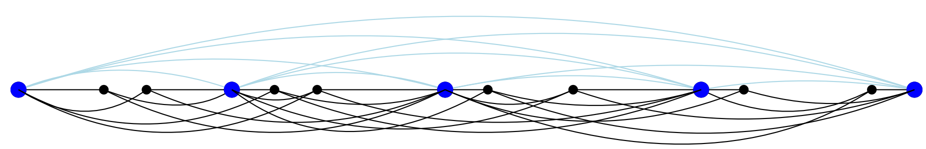


Figure A.4: Non-hub nodes are connected to the two nearest hubs on its left and right.

A.2 Two Dimensions: Quadtree Decomposition

In this section, we provide an example for the algorithm described in Section 3.3.1. In this example $k = 2$ and $n = 45$. The first four steps are illustrated in Figure A.5.

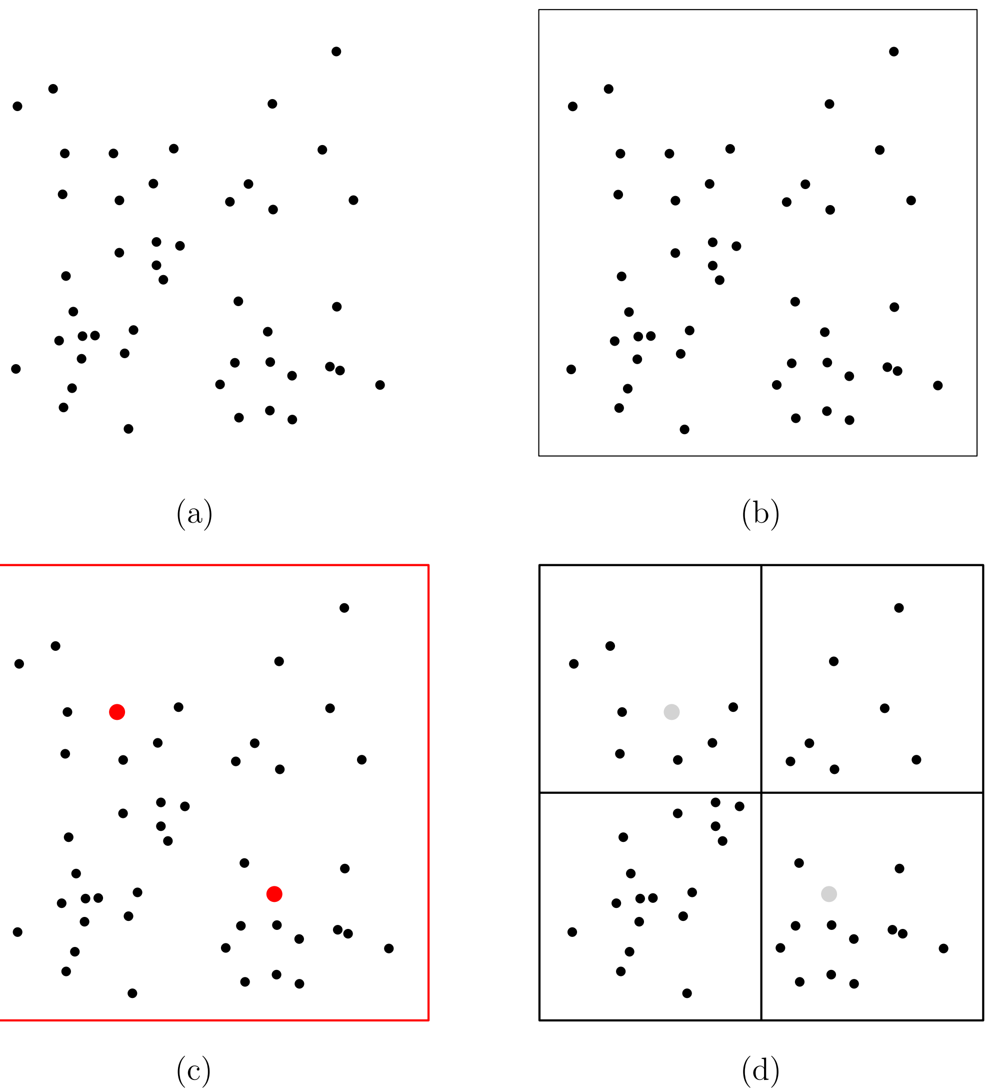
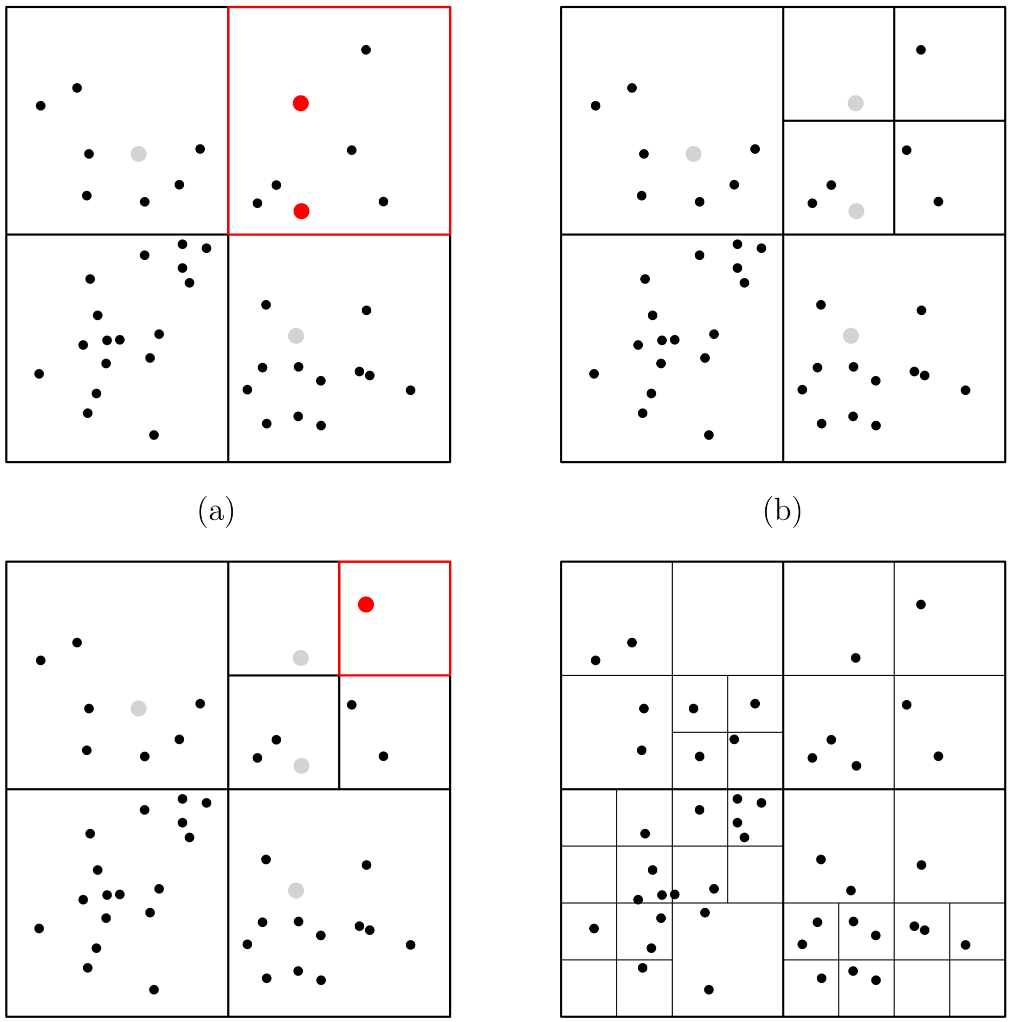


Figure A.5: (a) P is a set of 45 points in $\mathbb{R}^2$. (b) B_0 is a bounding box containing P. (c) The set R_0 is colored in red. (d) B_0 is divided into four equal subsquares and P/R_0 is decomposed accordingly.

Steps 5 to 8 of the algorithm are illustrated in Figure A.6. For each square B_i the algorithm finds R_i. Then it decomposes B_i and P_i/R_i accordingly.

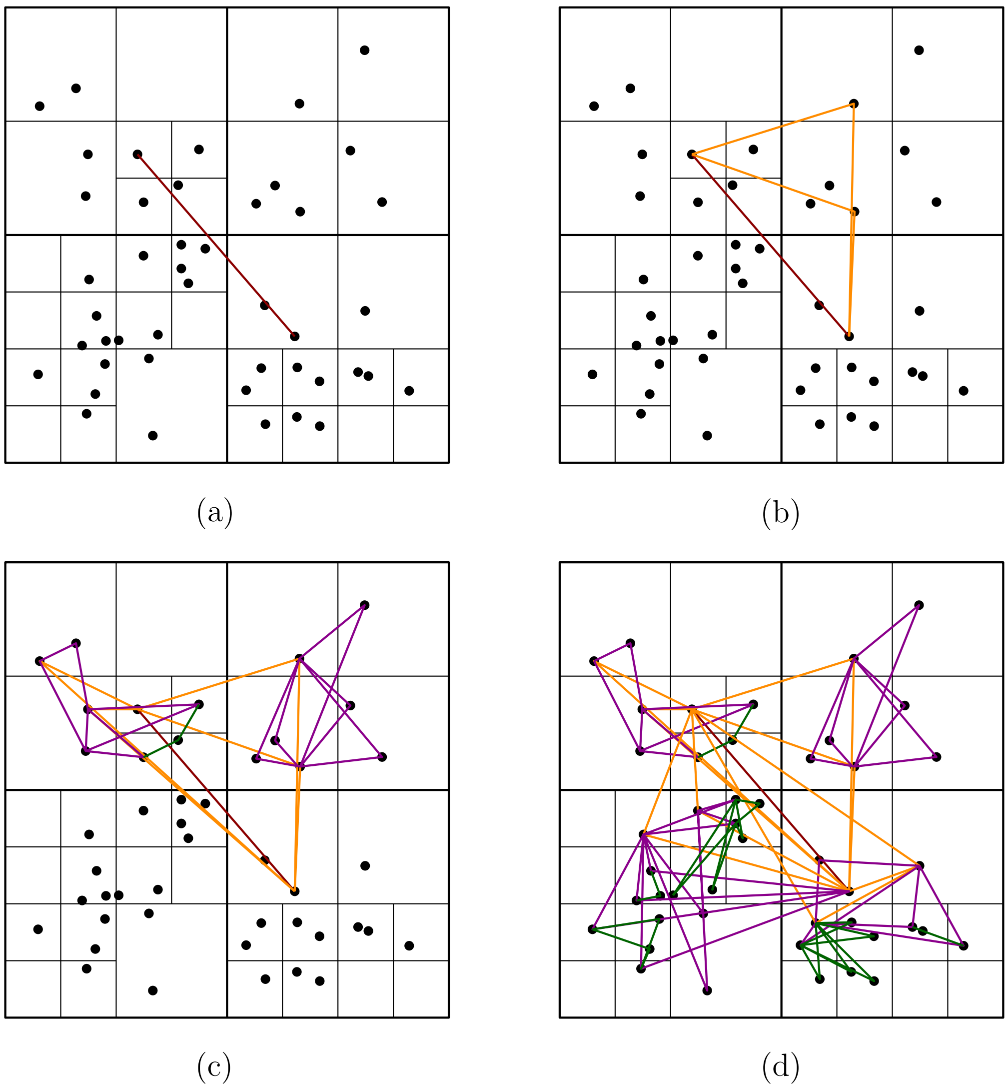


(a) (b)

(c) (d)

Figure A.6: (a) B_i is the red square and R_i is colored in red. (b) B_i is decomposed into 4 subsquares. (c) $B_{i'}$ is the red square and $R_{i'}$ is also colored in red. (d) The final quadtree for P.

In the next step, the algorithm forms a clique on the points in R_0. Then, the representatives of a square B_i are connected to representatives of the parent square of B_i. See Figure A.7 for illustration.



(a) (b) (c) (d)

Figure A.7: (a) A clique is formed on R_0. (b) Orange edges connect the representatives of the square formed in the second level to the points in R_0. (c) Purple (resp. green) edges connect the representatives of the third (resp. forth) level to the ones of the second (resp. third) level. (d) The final 2-connected network on P.

A.3 Two Dimensions: ϵ-Net

In this section, we provide an example for the algorithm described in Section 3.3.2. In this example $k = 2$ and n, the number of nodes, is sufficiently large that $\lceil \epsilon n/5 \rceil = 4$. In the illustrations $n = 24$. The first four steps are illustrated in Figure A.8.

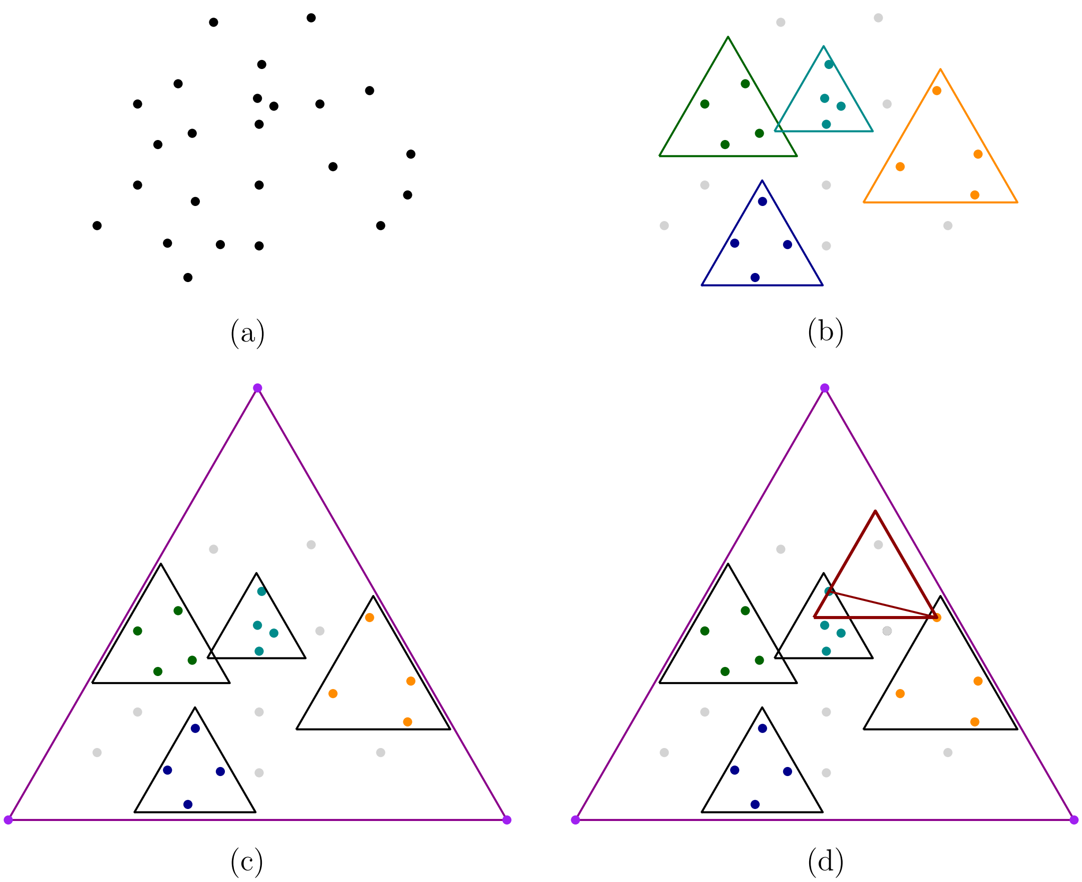


Figure A.8: (a) P a set of 24 points in $\mathbb{R}^2$. (b) Each triangle defines a subset P_i. (c) The big purple triangle is R_0. (d) The brown triangle is an empty range defining a Delaunay pair.

Figure A.9 illustarates steps 5 to 8 of the algorithm.

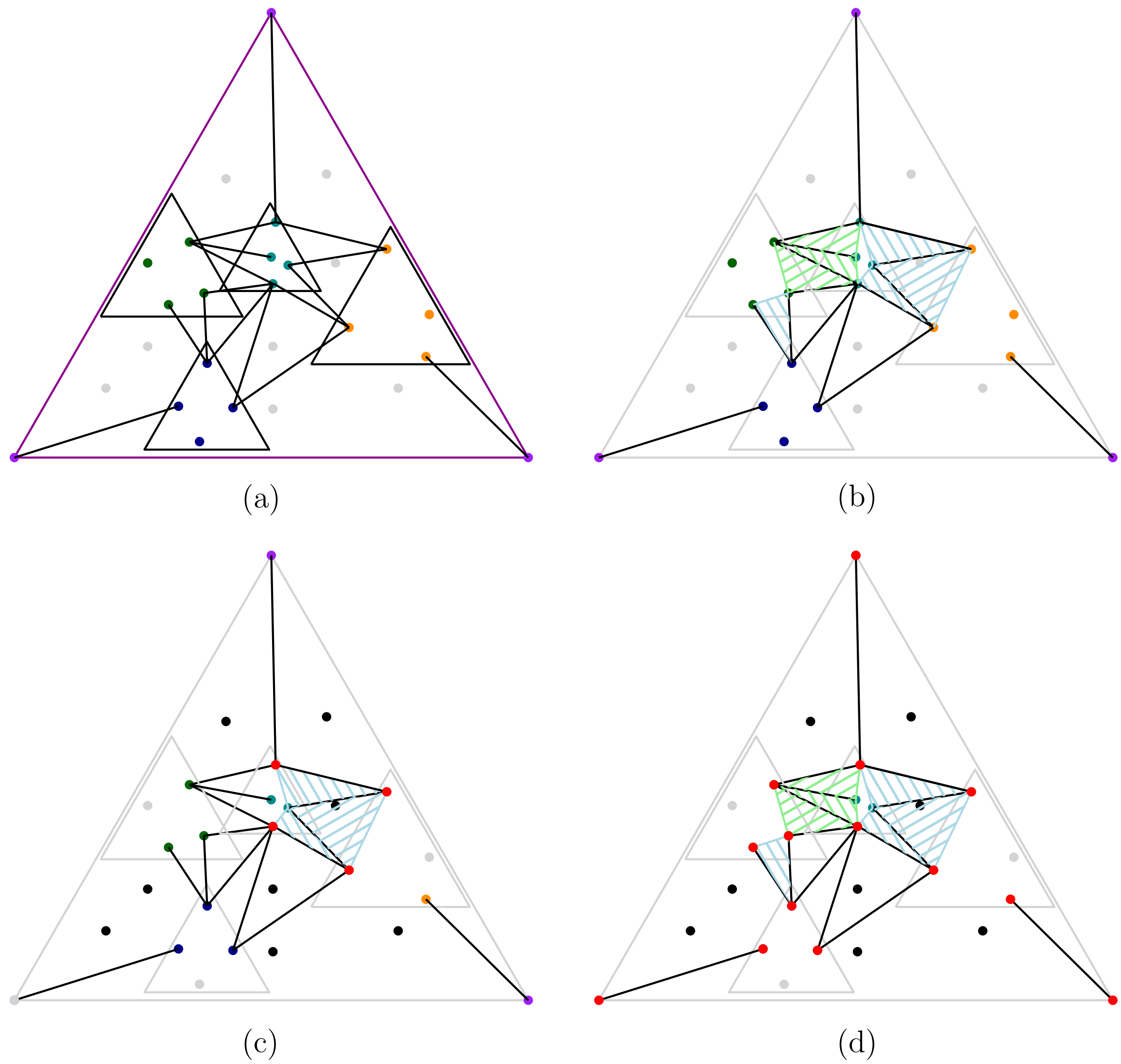


Figure A.9: (a) Delaunay pairs are connected by edges. (b) The shaded regions are corridors. (c) The endpoints of the corridor are selected as hubs. (d) Isolated Delaunay pairs are also selected as hubs.

Figure A.10 illustarates steps 9 to 12 of the algorithm.

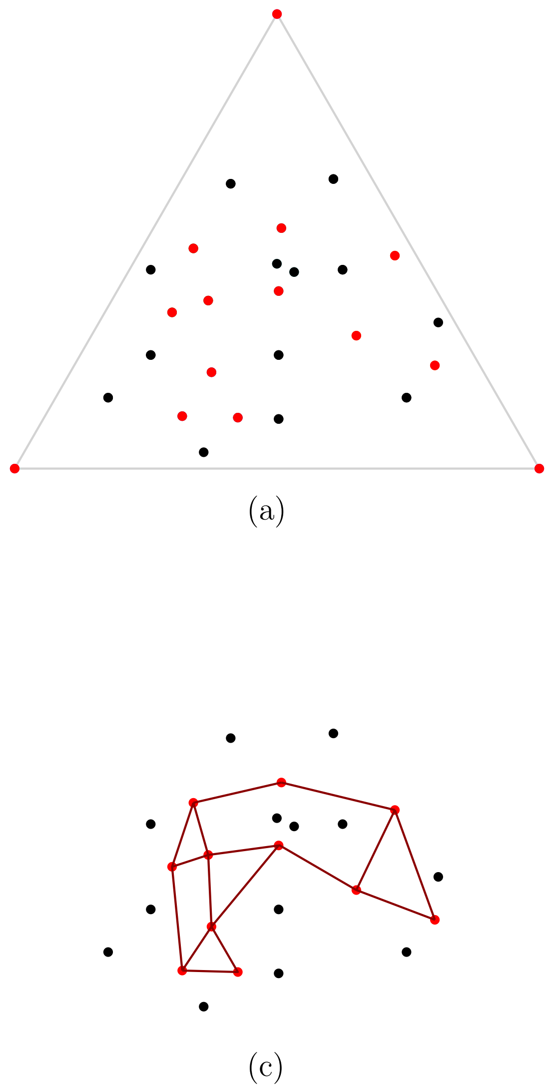


Figure A.10: (a) Hubs are colored in red. (b) The points of R_0 is removed from the set of hubs. (c) A 2-connected network is built on the hubs. (d) Non-hub nodes are connected to their 2 nearest hubs.

Bibliography

[1] A. Karim Abu-Affash, Rom Aschner, Paz Carmi, and Matthew J. Katz. The MST of symmetric disk graphs is light. *Computational Geometry: Theory & Application*, 45(1):54–61, 2012.

[2] Hakob Aslanyan and José D. P. Rolim. Interference minimization in wireless networks. In *Embedded and Ubiquitous Computing (EUC), 2010 IEEE/IFIP 8th International Conference on*, pages 444–449. IEEE, 2010.

[3] Yves Brise, Kevin Buchin, Dustin Eversmann, Michael Hoffmann, and Wolfgang Mulzer. Interference minimization in asymmetric sensor networks. In *International Symposium on Algorithms and Experiments for Sensor Systems, Wireless Networks and Distributed Robotics*, pages 136–151. Springer, 2014.

[4] Kevin Buchin. Minimizing the maximum interference is hard. *arXiv preprint arXiv:0802.2134*, 2008.

[5] Martin Burkhart, Pascal Rickenbach, Rogert Wattenhofer, and Aaron Zollinger. Does topology control reduce interference? In *Proceedings of the 5th ACM international symposium on Mobile ad hoc networking and computing*, pages 9–19. ACM, 2004.

[6] Ken Clarkson. Approximation algorithms for shortest path motion planning. In *Proceedings of the 19th ACM symposium on Theory of computing (STOC)*, pages 56–65. ACM, 1987.

[7] Luc Devroye and Pat Morin. A note on interference in random point sets. In *Canadian Conference on Computational Geometry (CCCG)*, volume 24, pages 201–206, 2012.

[8] Steven Fortune. Voronoi diagrams and Delaunay triangulations. *Computing in Euclidean geometry*, 1:193–233, 1992.

[9] Martin Fussen, Roger Wattenhofer, Aaron Zollinger, Roger Wattenhofer, and Roger Wattenhofer. *On interference reduction in sensor networks.* ETH Zurich, Department of Computer Science, 2004.

[10] Magnús M. Halldórsson and Takeshi Tokuyama. Minimizing interference of a wireless ad-hoc network in a plane. *Theoretical Computer Science*, 402(1):29–42, 2008. Algorithmic Aspects of Wireless Sensor Networks.

[11] David Haussler and Emo Welzl. ϵ-nets and simplex range queries. *Discrete & Computational Geometry*, 2(2):127–151, 1987.

[12] Eric Holec. Using linear programming to minimize interference in wireless networks. Master's thesis, University of Minnesota, 2013.

[13] J. Mark Keil. Approximating the complete Euclidean graph. In *Scandinavian Workshop on Algorithm Theory (SWAT)*, pages 208–213. Springer, 1988.

[14] Majid Khabbazian, Stephane Durocher, and Alireza Haghnegahdar. Bounding interference in wireless ad hoc networks with nodes in random position. In *International Colloquium on Structural Information and Communication Complexity (SIROCCO)*, pages 85–98. Springer, 2012.

[15] Majid Khabbazian, Stephane Durocher, Alireza Haghnegahdar, and Fabian Kuhn. Bounding interference in wireless ad hoc networks with nodes in random position. *IEEE/ACM Transactions on Networking*, 23(4):1078–1091, 2015.

[16] David Kirkpatrick. Optimal search in planar subdivisions. *Society for Industrial and Applied Mathematics (SIAM) Journal on Computing*, 12(1):28–35, 1983.

[17] Matias Korman. Minimizing interference in ad-hoc networks with bounded communication radius. In *International Symposium on Algorithms and Computation*, pages 80–89. Springer, 2011.

[18] Evangelos Kranakis, Danny Krizanc, Pat Morin, Lata Narayanan, and Ladislav Stacho. A tight bound on the maximum interference of random sensors in the highway model. *arXiv preprint arXiv:1007.2120*, 2010.

[19] Fabian Kuhn, Pascal von Rickenbach, Roger Wattenhofer, Emo Welzl, and Aaron Zollinger. Interference in cellular networks: The minimum membership set cover problem. In *International Computing and Combinatorics Conference (COCOON)*, pages 188–198. Springer, 2005.

[20] Der-Tsai Lee. On k-nearest neighbor Voronoi diagrams in the plane. *IEEE Transactions on Computers*, 100(6):478–487, 1982.

[21] Renato E. N. Moraes, Celso C. Ribeiro, and Glaydston M. Ribeiro. Exact formulations for the minimum interference problem in k-connected ad hoc wireless networks. *International Transactions in Operational Research*, 23(6):1113–1139, 2016.

[22] Giri Narasimhan and Michiel Smid. *Geometric spanner networks*. Cambridge University Press, 2007.

[23] Christos H. Papadimitriou and Umesh V. Vazirani. On two geometric problems related to the traveling salesman problem. *Journal of Algorithms*, 5(2):231–246, 1984.

[24] Haisheng Tan. *Minimizing interference in wireless sensor networks*. PhD thesis, University of Hong Kong, 2011.

[25] Haisheng Tan, Tiancheng Lou, Francis C. M. Lau, Yuexuan Wang, and Shiteng Chen. Minimizing interference for the highway model in wireless ad-hoc and sensor networks. In *International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM)*, pages 520–532. Springer, 2011.

[26] Pascal von Rickenbach, Stefan Schmid, Roger Wattenhofer, and Aaron Zollinger. A robust interference model for wireless ad-hoc networks. In *19th IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, pages 8–pp. IEEE, 2005.

[27] Pascal von Rickenbach, Roger Wattenhofer, and Aaron Zollinger. Algorithmic

models of interference in wireless ad hoc and sensor networks. *IEEE/ACM Transactions on Networking (TON)*, 17(1):172–185, 2009.