Construction of Optimal Foldover Designs with the General Minimum Lower-Order Confounding

by

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Abstract

Fractional factorial designs are widely used in industry and agriculture. Over the years much research work has been done to study these designs. Foldover fractional factorial designs can de-alias effects of interest so that the effects can be estimated without ambiguities. We consider optimal foldover designs using general minimum lower-order confounding criterion. Some Properties of such designs are investigated. A catalogue of 16- and 32-run optimal foldover designs is constructed and tabulated for practical use. A comparison is made between the general minimum lower-order confounding optimal foldover designs and other optimal foldover designs obtained using minimum aberration and clear effect criteria.

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Dedication

I dedicate this work to my Dad, Mr Salifu Yusif and my Mum, Miss Rafa Nuhu and my late grandparents.

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Chapter 1

Introduction

Design of experiment is a methodical and organized way of conducting tests that allow us to determine as well as assess our factors and their interactions so that we can easily achieve a desired output. Experimental designs provides practitioners with statistical methods that assist them to find which factors are important and those that have notable influence on the response. This helps to set the factors in such a way that we achieve our output in satisfaction. It also enable an experimenter to optimize the output. Design of experiments are applied in different areas including natural science, social science, industries, pharmaceutical and manufacturing companies (Rekab and Shaikh, 2005).

For an experiment to be a successful one, we need to be careful and attentive in choosing the right experiment. A desirable experiment is the one that leads to minimal cost as well as saves time, since the main aim of industries and companies is to make profit. Some experimenters may even prefer to trade-off some efficiency to save cost and time. Experimenters often employ factorial designs to study the effects of two or more factors at the same time. However, as the number of factors increases, the number of runs increases exponentially (Li and Lin, 2003). Fractional factorials designs are often used where a portion of the complete replicate of the runs is used instead of all the runs.

Different designs may have the same cost and time, but some are relatively more optimal than others in different ways. Optimal experimental designs have been studied extensively recently in the literature. This helps us to choose the designs that have the higher ability to minimize the confounding between effects so that they can be studied and estimated without ambiguities. The popular optimality criteria for fractional factorial designs include minimum aberration and clear effects criteria. Zhang et al. (2008) proposed the general minimum lower-order confounding criterion. Basically Experimenters will be interested in design that are optimal.

One problem an experimenter is likely to face by employing a fractional factorial design is that, some effects may be confounded with others. This creates ambiguities about the analysis and estimation of the factors. Hence, sometimes there is the need for additional runs to clarify these ambiguities (Ye and Li, 2003b ; Edwards and Brooks, 2014). A follow up method, called a foldover, is often used to solve the problem.

A foldover design is constructed by reversing the signs of one or more factors (Li and Lin, 2003). The classical foldover design involves the signs of all factors being reversed. However, subsequent studies revealed that reversing all the signs does not always give the optimal designs. Hence, there is the need to find optimal foldover designs by considering the reverse of each of the factors and their possible combinations.

Some optimality criteria have been used to search optimal foldover designs. Li and Lin (2003) constructed optimal foldover designs using the minimum aberration criterion for 16- and 32-run designs. Wang et al. (2010) used the clear effects criterion to find optimal foldover designs. For more work about optimal foldover designs, see Ou et al. (2015), Li and Lin (2015), Elsawah and Qin (2016), etc.

Although, some research has been done with regards to finding optimal foldover designs, none of them has focused on using the general minimum lower-order confounding criterion to find optimal foldover designs. The general minimum lower-order confounding criterion was proposed by Zhang et al. (2008). It is based on the aliased effect number pattern, which reveals all the basic information and the nature of confounding between all the effects in a design. Unlike the general minimum lower-order confounding criterion, the minimum aberration and clear effect criteria makes use of only a part of aliased effect number pattern. The general minimum lower-order confounding criterion is more refined. It expatiates more, provides further details and helps to achieve the objective of lower confounding in a clear way (Zhang and Mukerjee, 2009).

The main objective of this research is to find the optimal foldover plans using the general minimum lower-order confounding criterion. We then compare our results to the optimal foldover designs under the minimum aberration and clear effects criteria.

In chapter 2, the concept of factorial designs and fractional factorial designs are introduced. Existing work in literature as well as examples are given to explain these designs. The settings and situations in which factorial and fractional factorial designs are employed are explained in this chapter. Some advantages and drawbacks of these designs are also introduced. Furthermore, the concept, reasons and importance of foldover designs are introduced. Demonstrations with examples why there is the need for an experimenter to consider foldover designs are given as well. Some basic terms in design of experiments in relation to factorial, fractional factorial and foldover designs are also introduced and defined. In the end, the optimality criteria including minimum aberration, clear effects, general minimum lower order confounding and their properties are stated and explained. Some relationships between these criteria are mentioned as well as their limitations and drawbacks.

The results and findings of the research are presented in Chapter 3. Some of the observations made are also stated in this chapter. A detailed demonstrations of how to choose optimal foldover designs using the general minimum lower-order confounding is stated. We show how to find aliased effect number pattern of combined foldover designs and how to choose the optimal foldover designs based on the aliased effect number patterns.

In Chapter 4, a catalogue of 16- and 32-run combined optimal foldover designs based on the general minimum lower-order confounding, minimum aberration and clear effect criteria are tabulated. A comparison is made between the general minimum lower-order confounding combined optimal foldover designs and other optimal foldover designs and their respective initial designs.

The summary and conclusion are stated in chapter 5. Some future directions and work of this thesis are also stated.

Chapter 2

Fractional Factorial Designs and Optimality Criteria

2.1 Factorial Designs

In design of experiments we often come across situations where there are two or more factors. These factors can usually take more than one value (level). An experimenter may choose to study these factors one at a time by keeping the others constant or study all of them in one setup. The designs that allows for the later is called *factorial designs*.

One of the draw backs of *one factor at a time designs* (where each factor is studied separately) is that, they only give estimates of a single factor effect at a specified and constant condition. These estimates are essentially good when the effect is the same at all settings of the other factors, that is they behave additively. On the other hand, factorial designs are better in precision when this is the case. Also even when the factors do not have this additive behavior, factorial designs take a step further to measure interaction effects. Unfortunately one factor at time designs can not do this (Box et al., 2013). In factorial designs, one is able to study two or more factors each at different levels. With these designs, we are able to study not only the main effects but also the interaction effects of the factors.

According to Box et al. (2013), factorial designs are desirable in problem solving and scientific findings due to the fact that they help in the determination of the impact of a specific factor on a specific response. They stated that "such designs can be of great empirical importance, dimensional information of this kind can lead to suggestions of physical explanations for the effect, often providing valuable and unexpected directions for further inquiry".

A *full factorial design* refers to a design where all factor level combinations are studied (Montgomery, 2013, and Box et al., 2013). For example, when there are two factors each at two levels, then there are $2 \times 2 = 2^2$ level combinations; and when there are two factors each at three levels, then there are $3 \times 3 = 3^2$ level combinations. For a full factorial design with factors $1, 2, ..., n$ each with $m_1, m_2, ..., m_n$ levels, respectively, we need $m_1 \times m_2 \times ... \times m_n$ runs for a complete replicate. In this thesis, we assume that each factor takes only two levels. A full factorial design is denoted as 2^n design, where n is the number of factors each at two levels, denoted by + and -. There are 2^n observations in the design.

Example 2.1 Consider a 2^2 full factorial design. This is a design with 2 factors 1 and 2, each at two levels denoted by + and -. The factor level combinations are as follows:

observations/formulations	1	2	12
1	-	-	+
2	-	+	-
3	+	-	-
4	+	+	+

The first columns represent the formulations whiles the second and third represent the levels of factors 1 and 2, respectively. The last column represents the levels of the interaction between factors 1 and 2.

We can obtain the same for any 2^n designs. For instance, similarly, for a 2^3 design with 3 factors (1, 2 and 3), we obtain the factor level combinations as follows:

observations/formulations	1	2	3	12	13	23	123
1	-	-	-	+	+	+	-
2	-	-	+	+	-	-	+
3	-	+	-	-	+	-	+
4	-	+	+	-	-	+	-
5	+	-	-	-	-	+	+
6	+	-	+	-	+	-	-
7	+	+	-	+	-	-	-
8	+	+	+	+	+	+	+

2.2 Fractional Factorial Designs

For a 2^n factorial design, we need 2^n runs for a complete replicate. Clearly, we see that the number of required runs increases as the number of factors n increases. Experimenters often have a situation when they have to make a choice as to which factors are important among other factors. In such instances, when the number of factors is large it becomes not only time consuming but extremely expensive. One of the way out of this situation is to employ fraction of the full factorial design. For example, designs 2^2, 2^3, 2^4, ..., and 2^{10} requires 4, 8, 16,... and 1024 runs, respectively, for a complete replicate. Imagine the time and cost involved in 1024 runs. Sometimes running all the full factorial could be wasteful mostly if there is no change in the response, hence it may be more beneficial to run a portion of it (Anderson, 2014).

Any portion of the full factorial design can be employed but will depend on the experimenter as well as the resources available (Holland and Cravens, 1973). Experimenters would like to choose designs that allow them to estimate as many factors as possible. Popular fractional factorial designs are constructed through generators. For example, suppose we want to create a design with five factors 1, 2, 3, 4 and 5. We can choose to let factor 4=12 and factor 5=13. Then, we get the levels of factors 4 and 5 as shown in Table 2.1. Factors 4 and 5 are said to be *generators* of the design. One can see that, instead of running the full factorial which has $2^5 = 32$ runs we are taking a fraction of the 32 runs which has $2^{5-2} = 8$ runs. In general, if a fractional factorial design has n factors with p generators, the design is denoted as a 2^{n-p} *design*.

Table 2.1: The 2^{5-2} Design with the Generators 4=12 and 5=13 in Example 2.2

Runs/Formulations	1	2	3	4=12	5=13
1	-	-	-	+	+
2	-	-	+	+	-
3	-	+	-	-	+
4	-	+	+	-	-
5	+	-	-	-	-
6	+	-	+	-	+
7	+	+	-	+	-
8	+	+	+	+	+

Example 2.2 Consider the design with generators 4=12 and 5=13. We can write 4=12 as I=124 and 5=13 as I=135. Since I·I=I, I·124=124, I·135=135 and 124·135=2345, we have I=124=135=2345 which is called *the defining relation* of the design. The defining relation consists of the generators as well as their products. The terms 124, 135 and 2345 in the defining relation are called the *words* of the design and each number in a word is called a *letter*. The number of letters in each word is referred to as the *word length* of that word. The length of the shortest word in the defining relation of a design is called the *resolution* of that design. In this case the shortest word has a length of 3. Hence this design is a resolution III design.

Fractional factorial designs are widely used in different kind of experimentations, especially in various stages of product manufacturing as well as quality improvement

(Dean and Voss, 1999). These designs allows experimenters to study large number of factors with relatively small number of experimental runs. Even though some information is sacrificed for cost and time, nonetheless fractional factorial designs yields as much information as full factorial design in most cases (Holland and Cravens, 1973). This is a matter of trade off of information and experimental runs but in rare cases. They are most efficient and effective methods when it comes to studying several factors at the same time. Fractional factorial designs are created to study complex experiments under time constraints and when we are interested in several factors (Young, 1995). So many researches have been done with regards to fractional factorial designs in order to get the best out them.

One of the drawbacks of the fractional factorial designs is that, effects may have influence on each other. When this happen, we are often unable to tell which of the factors is really causing the effect on the response we are measuring and hence there are ambiguities in the results. In these case, we say these effects are *aliased* or *confounded* with others. Many effects being aliased is one of the prices we pay for taking small number of runs in consideration.

Considering the *effect hierarchy principle* which states that, *lower order effects* (main effects and two-factor interactions) are most likely to be more important than higher order interactions (Wu and Michael, 2000), we can choose to run a fraction/portion of the full factorial experiment to obtain information about the main effects and lower order interactions. This is the notion behind *fractional factorial* designs.

Example 2.3 Consider the design in Example 2.2. We can obtain the *alias structures* of the design as shown in Table 2.2. To get the equations of the alias structure, we multiply each factor and interaction words in the defining relation. For example to obtain 1=24=35=12345, we multiply 1 by I, 124, 135 and 2345 in the defining relation. We therefore have 1·I=1, 1·124=24, 1·135=35 and 1·2345=12345. Similarly, we obtain the other equations.

Table 2.2: Alias Structure of the 2^{5-2} Design with Generators 4=12 and 5=13

I=124=135=2345
1=24=35=12345
2=14=1235=345
3=1234=15=245
4=12=1345=235
5=1245=13=234
25=145=123=34
45=125=134=23

The equations in Table 2.2 shows the confounding between effects. For instance, 1=24=35=12345 means that effects 1, 24, 35 and 12345 are aliased with each other. This implies that, factor 1 is confounded with interactions 24 and 35 and 12345. Similarly, 25=145=123=34 implies that, interactions 25, 145, 123, and 34 are confounded (aliased) with each other. We see from Table 2.2 that, some two-factor interactions are aliased with two-factor interactions. Main effects are aliased with two-factors interactions.

Example 2.4 Consider a resolution IV design with two generators $5 = 123$ and $6 = 124$. The defining relation is I=1235=1246=3456. The alias structure is given in Table 2.3.

Table 2.3: Alias Structure of the 2^{6-2} Design With Generators 5=123 and 6=124

I=1235=1246=3456

1=235=246=13456

2=135=146=23456

3=125=12346=456

4=12345=126=356

5=123=12456=346

6=12356=124=345

12=35=46=123456

13=25=2346=1456

14=2345=26=1356

15=23=2456=1346

16=2356=24=1345

34=1245=1236=56

36=1256=1234=45

134=245=236=156

136=256=234=145

From Table 2.3, we see that all two-factor interactions and main effects are aliased with two-factor interactions and three-factors interactions, respectively. Like we see in Table 2.2, lower order effects are aliased with each other. This is not something we desired to see. We want to have as many lower order effects being clear as possible. In a worst case, we want to see lower order factors been aliased with higher order interactions according to the effect hierarchy principle assuming effects of higher order interactions are negligible.

2.3 Foldover Designs

One of the downsides of fractional factorial designs is that, the effects of the influential factors or the effects one maybe interested in can be influenced by two or more other factor effects. That is, effects maybe confounded with each other (Box et al., 2013). When we are confronted with this problem one of the way forward is to conduct a follow up experiment.

The most popular follow-up method is the *foldover* which is one of the most useful strategyies that can be used to de-alias effects. *Foldover designs* are obtained by reversing the signs of one or more factors (Ye and Li, 2003a). According to Li and Lin (2003), the set of factors whose signs are to be negated in a foldover design is referred to as *foldover plan*. Example 2.5 explains how to get a foldover design.

Example 2.5 Consider the design in Example 2.2, when we choose a foldover plan where we reverse the signs of factors 4 and 5, all the words in the defining relation that contain 4 but not 5 or contain 5 but not 4 will be negated. If both 4 and 5 are contained in a word, it will not be negated since two negatives multiplied by each other give positive. The defining relation of the new fraction is I=-124=-135=2345. Table 2.4 shows the new fraction when we reverse the signs of factors 4 and 5.

Table 2.4: The New Fraction after Reversing the Signs of Factors 4 and 5 in Example 2.5

Runs/Formulations	1	2	3	-4=12	-5=13
1	-	-	-	-	-
2	-	-	+	-	+
3	-	+	-	+	-
4	-	+	+	+	+
5	+	-	-	+	+
6	+	-	+	+	-
7	+	+	-	-	+
8	+	+	+	-	-

The combination of the initial design and the new fraction is called a *combined foldover design* or simply a *foldover design*. The defining relation of the combined foldover design is I=2345, which is a resolution IV design. The alias structure is shown in Table 2.5.

Table 2.5: Alias Structure of the Combined Foldover Design in Example 2.5

1=12345	12=1345	123=145
2=345	13=1245	124=135
3=245	14=1235	125=134
4=235	15=1234	
5=234	23=45	
	24=35	
	25=34	

We see by comparing Table 2.5 and Table 2.2 we see that all the main effects are de-aliased from two-factor interactions.

Example 2.6 Consider the resolution IV design in Example 2.4, when we reverse the signs of factors 5 and 6, the defining relation of the design is I=-1235=-1246=3456. The resulting combined foldover design has I=3456 as its defining relation. We obtain the following aliasing for the main effects and two factor interactions: 1=13456, 2=23456, 3=456, 4=356, 5=346, 6=345, 12=123456, 13=1456, 14=1356, 16=1345, 23=2456, 24=2356, 25=2346, 26=2345, 34=56, 35=46, 36=45.

Compared to the initial design, we see that some two-factor interactions are de-aliased from two-factor interactions after folding. Furthermore we see that main effects 1 and 2 are not confounded with any three-factor interactions which is not the case in the initial design. Generally, if I and I' are the defining relations of the initial design and the new fraction, respectively, then the defining relation of the combined foldover design is $\frac{1}{2}$(I+I') (Montgomery, 2013).

This is the idea behind foldover designs. However, different foldover plans mays give different defining relations, hence result in different combined foldover designs as well as alias structures. We demonstrate this with the examples below.

Example 2.7 Consider Example 2.2, when we change the sign of factor 2 , we obtain a design with defining relation I=-124=135=-2345. The defining relation of the combined foldover design is I=135. The alias structure for the lower order effects are 1=35, 2=1235, 3=15, 4=1345, 5=13, 12=235, 14=345, 23=125, 24=12345, 25=123, 34=145 and 45=134. The main effects 2 and 4 are de-aliased from their aliased two-factor interactions.

In addition, we reverse the sign of the factor 5, the defining relation of the new fraction is I=124=-135=-2345 and the defining relation of the combined foldover design is I=124. The alias structure of the combined foldover design is 1=24, 2=14, 3=1234, 4=12, 5=1245, 13=234, 15=245, 23=13, 25=145, 34=123, 35=12345, 45=125. Here, the main effects 3 and 5 are de-aliased from two-factor interactions.

When we change the signs of all the five factors, the defining relations of the foldover design is I=2345 and all the main effects are de-aliased from two-factor interactions.

Example 2.8 When we change the sign of only 5 in Example 2.6, we get the defining relation of the resulting combined foldover design as I=1246. One can see that, all the two-factor interactions that contain 5 are de-aliased from the other two-factor interactions.

In the same way when we reverse the sign of only 6 we obtain a combined foldover design with defining relation as I=1235 and all the two-factor interactions that contain 6 are de-aliased. However, when we reverse the signs of all the factors in this case the defining relation remains unchanged, it will still be I=1235=1246=3456.

It is well known that, for a resolution III design, when we change the signs of all the factors the resulting combined foldover design is a resolution IV design. This can be seen in Example 2.7. For a resolution IV design, Montgomery and Runger (1996) pointed out that, when we are interested in estimating a factor, then after changing the signs of that factor, all the two-factor interactions that include the factor of interest are de-aliased. This can be seen in Example 2.8. Foldover designs allow for effects to be estimated without any bias and dependency on any active two-factor interactions (Errore et al., 2015).

In some situations we may encounter a case where relatively many effects are significant hence its becomes difficult to choose which effects to de-alias. In this case, the way forward is to choose a foldover plan that reduces the general aliasing in the combined foldover design by de-aliasing large number of lower order effects (Wang et al., 2010). When we are clueless about which lower order effects are significant, then one of the best option is to design the experiment so that it can cater for any case. For more research on foldover designs, see Li and Mee (2002), Jacroux (2006) and Li and Lin (2015).

There is the need for the search of optimal foldover plans. Since different foldover plans results in different foldover designs, the best practice is to consider all foldover plans as possible without leaving any one out. For an initial design, there are 2^n foldover plans. However, from Theorem 2.1 we do not have to consider all foldover plans since some are equivalent to others. Hence considering those is the same as considering their equivalent ones. Li and Lin (2003) defined *core foldover plans* as foldover plans containing only the generators and their possible combinations.

Theorem 2.1 (Li and Lin, 2003) For a 2^{n-p} design, any foldover plan is equivalent to a core foldover plan and moreover, for every core foldover plan, there are 2^{n-p} foldover plans that are equivalent to it.

This implies that, once we consider all the foldover plans composed by the generators, we have covered all the foldover plans.

For example, consider the design in Example 2.4, there are $2^6 = 64$ foldover plans in total. Instead of reversing the signs of each of the 6 factors and their possible combinations, we only reverse the signs of 5, 6 and their combination (that is 5, 6 and 56). For any core foldover plan $0, 5, 6, 56$, where 0 means no factors is reversed, there are $2^4 = 16$ foldover plans that are equivalent to it. For instance, in Example 2.8, reversing the sign of factor 5 is the same as reversing the sign of factor 3, hence they are equivalent. Reversing the signs of either of these factors lead to I=-1235=1246=-3456 with the combined foldover design having defining relation I=1246.

Optimal foldover designs have been studied over the last decade. Different criteria such as minimum aberration (Li and Lin, 2003), clear effects criteria (Wang et al., 2010) etc., were used to search optimal designs to meet different requirements. Some of these criteria are discussed in the next section.

2.4 Optimality Criteria

In this section, we introduce several optimality criteria. Note that all this criteria depends on effect hierarchy principle. Wu and Michael (2000) stated that effect hierarchy principle is one of the paramount if not the most principles of design of experiments. For this reason, every good design should be able to minimize confounding between lower order effects. Over past years, few criteria were introduced and used in choosing and finding optimal designs. This include maximum resolution criterion, minimum aberration criterion, clear effect criterion and the general minimum lower-order confounding criterion.

According to Chen and Cheng (2012), even though the main objective of all these criteria is to select designs with high ability to estimate lower order effects, they sometimes have different interpretations. This sometimes lead to inconsistencies and even contradictions. Nonetheless they do have a lot of relationships

We will consider the minimum aberration criterion, clear effect criterion as well as the general minimum lower-order confounding criterion. In this section, we will introduce each criterion. The general minimum lower-order confounding criterion is the focal point of this research, hence we will throw more light on it.

2.4.1 Maximum Resolution(MR)

The resolution criterion maximizes the length of the shortest word in the defining relation. In other words, it chooses designs with maximum resolution. Under the maximum resolution criterion, designs with the same resolution are considered equivalent, hence can not be distinguished.

2.4.2 Minimum Aberration (MA)

The minimum aberration criterion was introduced by Hunter and Fries (1980) and it has remained one of the popular criteria in choosing optimal designs. This criterion is used to choose designs in cases where there is very small or lack of knowledge about the likely important effects. Before the introduction of minimum aberration the *resolution criterion* was used. Designs that are considered equivalent according to the resolution criterion can be distinguished by the minimum aberration criterion (Fries and Hunter, 1980). This criteria depends on the *word length pattern* which is defined as $(A_1, A_2, A_3 ...)$, where $A_i, i = 1, 2, ...$ represent the number of words with length i in the defining relation of the design. For instance, for a design with defining relation I=124=135=2345, the word length pattern is (0, 0, 2, 1, 0...). This implies that, there is no word with one or two letters, there are two words with three letters and one word with four letters, etc.

Definition 2.1 (Mukerjee and Wu, 2006). Let d_1 and d_2 represent two different 2^{k-p} designs with $A_i(d_1)$ and $A_i(d_2)$ respectively. Let q be the smallest integer such that $A_q(d_1) \neq A_q(d_2)$. Then d_1 is said to have less aberration than d_2 if $A_q(d_1) < A_q(d_2)$. A design is called a *minimum aberration design* if no other design has less aberration than it.

Example 2.9 Consider 2^{7-2} designs in Table 3 of Li and Lin (2003). The first design 7.2.1 has generators 6=1234 and 7=1245, the second design 7.2.2 has generators 6=123 and 7=145 whiles the third design 7.2.3 has 6=123 and 7=124 as its generators. Therefore their defining relations are I=12346=12457=3567, I=1236=1457=234567 and I=1236=1247=3467, respectively. Clearly we can see that, the designs are of resolution IV, hence all the main effects are not confounded with any main effects and two-factor interactions. However, some two-factor interactions are confounded with two-factor interactions as shown in Table 2.6.

Table 2.6: Confounding Between Two-factor Interactions of 2^{7-2} Designs in Table 3 of Li and Lin (2003)

Design 7.2.1	Design 7.2.2	Design 7.2.3
35=67	12=36	12=36=47
36=57	13=26	13=26
37=56	14=57	14=27
	15=47	16=23
	16=23	17=24
	17=45	34=67
		37=46

From Table 2.6 we see that, design 7.2.1 has 3 two-factor interactions confounded with other two-factor interactions, design 7.2.2 has 6 confounded two-factor interactions whiles design 7.2.3 has such 7 two-factor interactions. The amount of confounding is relatively low in design 7.2.1, for this reason, it is better than design

7.2.2 and 7.2.3. However, if one had an idea or any prior knowledge about which factors are potentially important, then we would rather make our choice based on those factors. With minimum aberration, when you do not have any prior knowledge at least you are sure of making a choice of designs with minimum confounding.

Furthermore, the word length pattern of designs 7.2.1, 7.2.2 and 7.2.3 are (0, 0, 0, 1, 2, 0, 0), (0, 0, 0, 2, 0, 1, 0) and (0, 0, 0, 3, 0, 0, 0), respectively. We see that design 7.2.1 has one word with four letters, while designs 7.2.2 and 7.2.3 has two and three, respectively. According to Definition 2.1, design 7.2.1 has the minimum aberration relative to the other two designs.

A lot of work has been done in the construction of factorial designs using minimum aberration since its invention. For example, Chen and Hedayat (1996) constructed fractional factorial design with weak minimum aberration which combines the maximum resolution and minimum aberration criteria. A design has *weak minimum aberration* if it has the maximum resolution and at the same time the smallest number of words with length equal to the maximum resolution. Chen (1992) gave some results and searched for minimum aberration designs for fractional factorial designs, Franklin (1984), Chen and Wu (1991) and Tichon et al. (2012) did some work of the construction of minimum aberration designs fractional factorial designs and split-plot designs, respectively. Cheng et al. (1999) also looked at minimum aberration and model robustness for factorial designs.

Cheng and Tang (2005) and Tang and Deng (1999) also studied the properties of minimum aberration criterion in terms of λ_1, λ_j and N_j, where λ_1 and λ_j represent the set of main and a set of jth-order effects, respectively and N_j is the number of given effects in λ_j, $j = 2, 3, ...$, that are aliased with effects in λ_1. For instance, N_2

stands the number of main effects aliased with two-factor interactions.

Generally, $N_j = (j+1)A_{j+1} + (n-j+1)A_{j-1}$, where $2 \leq j \leq n-1$ and $N_n = A_{m-1}$. In this case, this is equivalent to the usual minimum aberration criteria. Hence minimizing $N_2, N_3, ..., N_j$ is equivalent to minimizing the usual word length pattern (Cheng and Tang, 2005).

2.4.3 Clear Effects (CE)

One of the major drawbacks of the minimum aberration criterion is that, sometimes it is unable to maximize the number of some clear lower order effects especially two-factor interactions (Murat et al., 2006). *Clear effects* are effects that are not confounded with lower-order effects (main effects and two-factor interactions). If a main effect is unconfounded with any other main effects and two-factor interactions or a two-factor interaction is unconfounded with any main effects and other two-factor interactions, we say it is *clear*. One of the criteria that takes care of this situation is the *clear effects criterion* which is based on the the number of clear main effects and number of clear two-factor interactions. This criterion sequentially maximizes the number of clear lower-order effects.

Definition 2.2 Consider a design with $CE = (C_{main}, C_{2fi})$, where C_{main} and C_{2fi} are the number of clear main effects and two-factor interactions, respectively.Suppose $CE(d_1)$ and $CE(d_2)$ are the CE's of two designs d_1 and d_2, respectively and CE_m is mth component of CE . Let CE_m be the first component for which $CE(d_1)$ and $CE(d_2)$ differ. If $CE_m(d_1) > CE_m(d_2)$, then d_1 has greater number of clear lower order effects than d_2. In other words, the clear effect criterion sequentially maximizes CE.

Example 2.10 Consider the 2^{9-4} designs, 9.4.1, 9.4.2 and 9.4.3, in Table 3 in Zhang et al. (2008). Their generators are 6=123, 7=124, 8=125, 9=1345; 6=123, 7=124, 8=134, 9=2345; and 6=123, 7=234, 8=134, 9=124, respectively. The clear effects of design 9.4.1 is (9, 8) and that of clear effects of design 9.4.2 is (9, 15) whiles design 9.4.3 is (9, 8). The main effects are clear in all the three designs. However, design 9.4.1 has 8 clear two-factor interactions and so is design 9.4.3 but design 9.4.2 has 15 clear two-factor interactions. Hence according to the clear effect criterion, design 9.4.2 is the best.

According to Chen et al. (1993), designs with relatively great number of clear two-factor interactions may be better than designs with minimum aberration. However there are situations where clear effect designs are the same as minimum aberration designs. Wu and Wu (2002) proposed different relationships between minimum aberration designs and the number clear two-factor interactions. They referred to a design with maximum number of clear two-factor interactions as *MaxC2 design*. They proved that, if the A_4 of a minimum aberration design is 1 or 2, then minimum aberration is MaxC2 design and its number of clear two-factor interactions is $n(n-1)/2 - 6A_4$, where n is the number of factors.

The problem with clear effects criterion is that, it is unable to differentiate between designs when there are no clear effects. Another limitation is that, it is also not able to further consider designs with the same number of clear effects.

2.4.4 General Minimum Lower-Order Confounding (GM-LOC)

The general minimum lower-order confounding criterion (GMLOC or GMC) was introduced by Zhang et al. (2008). The purpose of this criterion is not totally different from other criteria except that it is more refined and has some advantages over others. It expatiates more, provides further details as well as achieve the objective in an explicit way (Zhang and Mukerjee, 2009). Like the minimum aberration which depends on the word length pattern, the general minimum lower-order confounding criterion depends on the *aliased effect number pattern* which we will introduce later.

Interestingly, Zhang et al. (2008) showed that, the word length pattern and the numbers of clear lower order effects are functions of the aliased effect number pattern, which makes it unique and more important relative to other criteria.

We will introduce the aliased effect number pattern using the same notations as in Zhang et al. (2008). For any given ith-order effects and jth-order effects, we can find how ith-order effects and jth-order effects are aliased with each other. When ith-order effects is aliased with k jth-order effects, we say that the *degree* of ith-order effects being aliased with jth-order effects is k. The number of ith-order effects that are aliased with jth-order effects at degree k is denoted by ${}_{i}^{\#}C_{j}^{(k)}$, which tells us how severe ith-order effects are aliased with jth-order effects. At the same time, it also tells how many ith-order effects are aliased with jth-order effects. It reveals the general confounding that exists between effects. All the ${}_{i}^{\#}C_{j}^{(k)}$'s in a design forms a set

$$\{{}_{i}^{\#}C_{j}^{(k)}, i = 0, 1, 2, ..., n, j = 0, 1, 2, ...n, k = 0, 1, 2, ..., \binom{n}{j}\}. \tag{2.1}$$

The smaller the degree at which an ith-order effect is aliased with another effect, the less difficult it becomes in estimating the effect. Moreover, the smaller ${}_{i}^{\#}C_{j}^{(0)}$, the greater the severity of the confounding between ith-order effects and jth-order effects. On the other way, when the value of ${}_{i}^{\#}C_{j}^{(0)}$ is greater, the less severity of ith-order effects are aliased with jth-order effects. Eventually, when we are subjected to a situation of maximizing ${}_{i}^{\#}C_{j}^{(0)}$, relatively great size of ${}_{i}^{\#}C_{j}^{(1)}$ implies less severity of the confounding, in that order (Zhang et al., 2008).

As the degree k increases, the severity of aliasing increases, hence the ${}_{i}^{\#}C_{j}^{(k)}$s' are not the same from the confounding point of view. We can arrange them as

$${}_{i}^{\#}C_{j} = ({}_{i}^{\#}C_{j}^{(0)}, {}_{i}^{\#}C_{j}^{(1)}, ..., {}_{i}^{\#}C_{j}^{(v)}), \tag{2.2}$$

where $v = \binom{n}{j}$. For each design with n factors, we present only the first n elements of Equation 2.2 since all $(n+1)$th to v are zero. Equation 2.2 gives the total number of ith-order effects aliased with jth-order effects at various degrees starting from the least to the greatest in terms of severity. The set of numbers

$${}^{\#}C = ({}_{1}^{\#}C_{1}, {}_{0}^{\#}C_{2}, {}_{1}^{\#}C_{2}, {}_{2}^{\#}C_{1}, {}_{2}^{\#}C_{2}, {}_{0}^{\#}C_{3}, {}_{1}^{\#}C_{3}, {}_{2}^{\#}C_{3}, {}_{3}^{\#}C_{1}, {}_{3}^{\#}C_{2}, {}_{3}^{\#}C_{3}, {}_{0}^{\#}C_{4}, {}_{1}^{\#}C_{4}, ...) \tag{2.3}$$

is called the *aliased effect number pattern* (AENP). ${}_{0}^{\#}C_{0}$, ${}_{0}^{\#}C_{1}$ and ${}_{1}^{\#}C_{0}$ are ignored because they the same for any 2^{n-p} designs. The elements of Equation 2.3 are placed in using a rule. The rule is that, ${}^{\#}_{i}C_j$ comes before ${}^{\#}_{q}C_r$ if the maximum of (i, j) is less than the maximum of (q, r), that is, $max(i, j) < max(q, r)$. If the maximum of (i, j) is the same as that of (q, r) and i is less than q (that is $max(i, j) = max(q, r)$ and $i < q$), then ${}^{\#}_{i}C_j$ is placed before ${}^{\#}_{q}C_r$. Again, if the maximum of (i, j) is the same as that of (q, r) and i is equal to q and j is less than r (that is $max(i, j) = max(q, r)$ and $i < q$), then ${}^{\#}_{i}C_j$ is placed before ${}^{\#}_{q}C_r$. For more research on aliased effect number patterns see Wei et al. (2010) and Li et al. (2011), etc.

Now we demonstrate how to find the aliased effect number pattern using Example 2.11.

Example 2.11 Consider the design with generators $4 = 12$ and $5 = 13$. The defining relation is I=124=135=2345. For easy understanding and computation of the alias effect number pattern, we compute the alias structure in terms of each effect starting from main effects to the five-factor interactions. We obtain the alias structures as follows.

Table 2.7: Alias Structure of the Design with Generators 4=12 and 5=13

1=24=35=12345	12=4=235=1345	123=34=25=145	1234=3=245=15	12345=35=24=1
2=14=1235=345	13=234=5=1245	124=I=2345=135	1235=345=2=14	
3=1234=15=245	14=2=345=1235	125=45=23=134	2345=135=124=I	
4=12=1345=235	15=245=3=1234	234=13=1245=5	1345=235=4=12	
5=1245=13=234	23=134=125=45	235=1345=12=4	1245=5=234=13	
	24=1=12345=35	345=1235=14=2		
	25=145=123=34	134=23=45=125		
	34=123=145=25	135=2345=I=124		
	35=12345=1=24	145=25=34=123		
	45=125=134=23	245=15=1234=3		

Recall that the number of ith-order effect aliased with jth-order effect at degree k is ${}^{\#}_{i}C_j^{(k)}$. From Table 2.7, when i=2 (two-factor interactions), we see that there are four two-factor interactions that are not aliased with main effects, hence ${}_{2}^{\#}C_{1}^{0}$=(4). In addition, there are six two-factor interactions that are aliased with only one main effects, hence ${}_{2}^{\#}C_{2}^{1}$=(6). There are no two-factor interactions that aliased with other two or more two-factor interactions, hence ${}_{2}^{\#}C_{1}^{k}$=(0) for k=2, 3, 4, 5. We therefore have the following for i=2 and j=1: ${}_{2}^{\#}C_{1}^{0}$=(4), ${}_{2}^{\#}C_{1}^{1}$=(6), ${}_{2}^{\#}C_{1}^{2}$=(0), ${}_{2}^{\#}C_{1}^{3}$=(0), ${}_{2}^{\#}C_{1}^{4}$=(0) and ${}_{2}^{\#}C_{1}^{5}$=(0), which yields ${}_{2}^{\#}C_{1} = (4, 6, 0, 0, 0, 0)$.

Furthermore, there are two three-factor interactions (j=3) in the defining relations, hence ${}_{0}^{\#}C_{3}^{k}$=(0) for all k except when k=2. Therefore, we have ${}_{0}^{\#}C_{3}^{0}$=(0), ${}_{0}^{\#}C_{3}^{1}$=(0), ${}_{0}^{\#}C_{3}^{2}$=(1), ${}_{0}^{\#}C_{3}^{3}$=(0), ${}_{0}^{\#}C_{3}^{4}$=(0), and ${}_{0}^{\#}C_{3}^{5}$=(0). Putting all the above together we get ${}_{0}^{\#}C_{3}$ is $(0, 0, 1, 0, 0, 0)$.

Similarly from Table 2.7, when i=2 (two-factor interactions), we see that there are four two-factor interactions that are not aliased with other two-factor interactions hence ${}_{2}^{\#}C_{2}^{0}$=(4). In addition, there are six two-factor interactions that are aliased with only one two-factor interactions, hence ${}_{2}^{\#}C_{2}^{1}$=(6). There are no two-factor interactions that aliased with other two or more two-factor interactions, hence ${}_{2}^{\#}C_{2}^{k}$=(0) for k=2, 3, 4, 5. We therefore have the following for i=2 and j=2: ${}_{2}^{\#}C_{2}^{0}$=(4), ${}_{2}^{\#}C_{2}^{1}$=(6), ${}_{2}^{\#}C_{2}^{2}$=(0), ${}_{2}^{\#}C_{2}^{3}$=(0), ${}_{2}^{\#}C_{2}^{4}$=(0) and ${}_{2}^{\#}C_{2}^{5}$=(0), which yields ${}_{2}^{\#}C_{2} = (4, 6, 0, 0, 0, 0)$. In the same manner we obtain all the ${}_{i}^{\#}C_{j}^{k}$'s as shown below.

$$
\begin{array}{lll}
{}^{\#}_{0}C_0 = (1,0,0,0,0,0) & {}^{\#}_{1}C_0 = (5,0,0,0,0,0) & {}^{\#}_{2}C_0 = (10,0,0,0,0,0) \\
{}^{\#}_{0}C_1 = (1,0,0,0,0,0) & {}^{\#}_{1}C_1 = (5,0,0,0,0,0) & {}^{\#}_{2}C_1 = (4,6,0,0,0,0) \\
{}^{\#}_{0}C_2 = (1,0,0,0,0,0) & {}^{\#}_{1}C_2 = (0,4,1,0,0,0) & {}^{\#}_{2}C_2 = (4,6,0,0,0,0) \\
{}^{\#}_{0}C_3 = (0,0,1,0,0,0) & {}^{\#}_{1}C_3 = (1,4,0,0,0,0) & {}^{\#}_{2}C_3 = (2,4,4,0,0,0) \\
{}^{\#}_{0}C_4 = (0,1,0,0,0,0) & {}^{\#}_{1}C_4 = (1,4,0,0,0,0) & {}^{\#}_{2}C_4 = (6,4,0,0,0,0) \\
{}^{\#}_{0}C_5 = (1,0,0,0,0,0) & {}^{\#}_{1}C_5 = (4,1,0,0,0,0) & {}^{\#}_{2}C_5 = (8,2,0,0,0,0)
\end{array}
$$

$$
\begin{array}{lll}
{}^{\#}_{3}C_0 = (8,2,0,0,0,0) & {}^{\#}_{4}C_0 = (4,1,0,0,0,0) & {}^{\#}_{5}C_0 = (1,0,0,0,0,0) \\
{}^{\#}_{3}C_1 = (6,4,0,0,0,0) & {}^{\#}_{4}C_1 = (1,4,0,0,0,0) & {}^{\#}_{5}C_1 = (0,1,0,0,0,0) \\
{}^{\#}_{3}C_2 = (2,4,4,0,0,0) & {}^{\#}_{4}C_2 = (1,4,0,0,0,0) & {}^{\#}_{5}C_2 = (0,0,1,0,0,0) \\
{}^{\#}_{3}C_3 = (4,6,0,0,0,0) & {}^{\#}_{4}C_3 = (0,4,1,0,0,0) & {}^{\#}_{5}C_3 = (1,0,0,0,0,0) \\
{}^{\#}_{3}C_4 = (4,6,0,0,0,0) & {}^{\#}_{4}C_4 = (5,0,0,0,0,0) & {}^{\#}_{5}C_4 = (1,0,0,0,0,0) \\
{}^{\#}_{3}C_5 = (10,0,0,0,0,0) & {}^{\#}_{4}C_5 = (5,0,0,0,0,0) & {}^{\#}_{5}C_5 = (1,0,0,0,0,0)
\end{array}
$$

Now putting all the above together and writing them in order we have the aliased effect number pattern ${}^{\#}C = ({}^{\#}_{1}C_1, {}^{\#}_{0}C_2, {}^{\#}_{1}C_2, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2, {}^{\#}_{0}C_3, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, ...)$ as in Equation 2.3.

Having introduced the concept of aliased effect number pattern, we can now introduce the general minimum lower-order confounding criterion which depends on aliased effect number pattern as mentioned earlier. Again, our general interest and desire is to have designs with less low-order effects been confounded with other low-order effects. Maximizing the entries of Equation 2.3 will achieve this desire. That is the idea behind the general minimum lower-order confounding .

Definition 2.3 Suppose ${}^{\#}C(d_1)$ and ${}^{\#}C(d_2)$ are the aliased effect number patterns of two designs, d_1 and d_2, respectively and ${}^{\#}C_m$ is mth component of ${}^{\#}C$. Let ${}^{\#}C_m$ be the first component for which ${}^{\#}C_m(d_1)$ and ${}^{\#}C_m(d_2)$ differ. If ${}^{\#}C_m(d_1) > {}^{\#}C_m(d_2)$ we say that d_1 has less general lower order confounding relative to d_2.

A design is said to be a *general minimum lower-order confounding* (GMLOC or GMC) design, if it has minimum general lower order confounding relative to other designs. There are many relationships between the general minimum lower-order confounding and other criteria. Some of these criteria uses part of the aliased effect number pattern in selecting optimal designs.

Some of the relations between the minimum aberration and aliased effect number pattern are briefly discussed below. Zhang and Park (2000) showed that if $i \leq j$, then

$$_iC_j = \sum_{q=0}^{i} \binom{n-(j-i+2q)}{i-q}\binom{j-i+2q}{q} A_{j-i+2q}, \quad i,j = 1,2,...,n, \qquad (2.4)$$

where $\binom{r}{0} = 1$, $\binom{r}{s} = 0$ for $r < s$ or $r < 0$ and $A_i = 0$ for $i \leq 2$ or $i > n$ and $_iC_j$ is the total number ith-order effects aliased with jth-order effects. In addition, from (2.4), Zhang et al. (2008) obtained the Equation 2.5 below.

$$_iC_j = \begin{cases} \sum_{k=1}^{m} \frac{k_i^{\#}C_j^{(k)}}{2}, & i = j \\ \sum_{k=1}^{m} k_i^{\#}C_j^{(k)}, & \text{otherwise.} \end{cases} \qquad (2.5)$$

From (2.4) and (2.5) they deduced that, if a design has resolution III or more, the A_is' of a 2^{n-p} design is a function of $\{{}^{\#}_{i}C_{j}^{(k)}, i = 0, 1, 2, ..., n, j = 0, 1, 2, ...n, k = 0, 1, 2, ..., \binom{n}{j}\}$ in two ways, that is, ${}^{\#}_{i}C_{0}^{(0)} = \binom{n}{i} - A_i$ or ${}^{\#}_{i}C_{0}^{(1)} = A_i$ and any of the A_is' is a function of ${}^{\#}_{q}C_{r}^{(k)}$ where $q, r = 1, 2, ..., n$ with $k = 1, 2, ..., v$. Again, they showed that designs with different word length patterns must have different aliased effect number patterns. However, the converse of this is not true.

The following connections exists between the A_i's and the ${}^{\#}_{i}C_{j}^{(k)}$'s

$$A_3 = \frac{1}{3}\sum_{k=0}^{m_2} k_1^{\#}C_2^{(k)}, \quad A_4 = \frac{1}{6}\sum_{k=0}^{m_2} k_2^{\#}C_2^{(k)}, \quad A_5 = \frac{1}{10}\left(\sum_{k=1}^{m_3} k_2^{\#}C_3^{(k)} - (n-3)\sum_{k=0}^{m_2} k_1^{\#}C_2^{(k)}\right), \tag{2.6}$$

and so on, where $m_i = \binom{n}{i}$ (Zhang and Cheng, 2010). We see that word length is related to a portion of aliased effect number pattern. However, the general minimum lower-order confounding considers every single element of aliased effect number pattern. One of their differences is that, they differ in terms of which one is relevant in which situation.

It has been noticed over the years that, a good design should be able to minimize A_3 from the effect hierarchy principle point of view but this has not been proven yet (Hu and Zhang, 2011). Chen and Hedayat (1998) and Hu and Zhang (2011) proved that a general minimum lower-order confounding design have the least A_3 among the designs with the same parameters. Even though the minimum aberration and general minimum lower-order confounding are able to minimize A_3, the two criteria differ in terms of optimality in practice (Hu and Zhang, 2011). The minimum aberration minimizes A_4, which results in the reduction of confounding between two-factor interactions. On the other hand, general minimum lower-order confounding considers each individual confounding between two-factor interactions, which eventually leads to having as many clear two-factor interactions as possible. This is one of the key difference between the two criteria according to Hu and Zhang (2011).

Example 2.12 Consider the 2^{15-9} designs 15.9.1 and 15.8.27 in Table 4 of Zhang et al. (2008). The aliased effect number pattern $({}_1^{\#}C_2\ ;\ {}_2^{\#}C_2\ )$ of designs 15.9.1 and 15.8.27 are (15 ; 27, 0^3, 60, 18) and (15 ; 0, 60, 30, 0, 15), respectively, and their word length patterns are (0, 55, 22, 96) and (0, 30, 60, 60), respectively. The general minimum lower-order confounding design is design 15.9.1 while the design with minimum aberration design is design 15.9.27 among all 2^{15-9} designs. The general minimum lower-order confounding minimized A_3. We see that both designs have the same number of clear main effects but different number of clear two-factor interactions. The general minimum lower-order confounding design has 27 clear two-factor interactions while the minimum aberration design has no clear two-factor interactions.

According to Wei et al. (2010), the general minimum lower-order confounding focuses precisely at giving all the informations about confounding between effects and this is one of the advantages it has over other criteria. Despite the fact that the minimum aberration does not sometimes make the estimations of two-factor interactions possible (in other words sacrifices some two-factor interactions) one can still choose use them when to find average estimability of effects or when we are interested in model robustness (Cheng et al., 1999).

Like the minimum aberration, the clear effects criterion makes use of part of the aliased effect number pattern. The number of clear main effects $C_{\text{main}} = {}_1^{\#}C_2^{(0)}$ while the number of clear two-factor interactions $C_{2fi} = {}_2^{\#}C_2^{(0)} - {}_1^{\#}C_2^{(1)}$ (Hu and Zhang, 2011).

Chapter 3

Methods and Properties

3.1 Aliased Effect Number Patterns (AENP) of Foldover Designs

In chapter 2 we introduced the concept of aliased effect number pattern of designs. Now we demonstrate how to find the aliased number effect number pattern of combined foldover designs obtained from the various foldover plans as in Example 3.1 below.

Example 3.1 Consider the design discussed in Example 2.2 , that is the resolution III design with two generators $4 = 12$ and $5 = 13$. The defining relation is I=124=135=2345. We showed the aliased effect number pattern of this design in Example 2.11. For this design there are three core foldover plans, 4, 5 and 45. We call them foldover plan I, foldover plan II and foldover plan III, respectively. Again we refer their corresponding combine designs as combine foldover designs 1, 2 and 3, respectively. We shall consider all the three cases, foldover plan I, II and III.

Case I: Aliased effect number pattern of the combined foldover design 1. When we reverse the sign of the generator 4 (foldover plan 1), we obtain a design with defining relation I=-124=135=-2345. The resulting combined foldover design from this foldover plan has the defining relation I=135 since all the words that with negative sign are eliminated. The alias structure is shown in Table 3.1.

Table 3.1: Alias Structure of Combined Foldover Design 1

1=35	12=235	123=25	1234=245	12345=24
2=1235	13=5	124=2345	1235=2	
3=155	14=345	125=23	1245=234	
4=1345	15=3	134=45	1345=4	
5=13	23=125	135=I	2345=124	
	24=12345	145=34		
	25=123	234=1245		
	34=145	235=12		
	35=1	245=1234		
	45=134	345=14		

From Table 3.1, the numbers of ith-order effects aliased with jth-order effects at various degrees k (that is ${}_{i}^{\#}C_{j}^{k}$'s) are obtained as shown below:

$$
\begin{array}{lll}
{}^{\#}_{2}C_0 = (10,0,0,0,0,0) & {}^{\#}_{1}C_1 = (5,0,0,0,0,0) & {}^{\#}_{2}C_1 = (7,3,0,0,0,0) \\
{}^{\#}_{0}C_2 = (1,0,0,0,0,0) & {}^{\#}_{1}C_2 = (2,3,0,0,0,0) & {}^{\#}_{2}C_2 = (10,0,0,0,0,0) \\
{}^{\#}_{0}C_3 = (0,1,0,0,0,0) & {}^{\#}_{1}C_3 = (5,0,0,0,0,0) & {}^{\#}_{2}C_3 = (4,6,0,0,0,0) \\
{}^{\#}_{0}C_4 = (1,0,0,0,0,0) & {}^{\#}_{1}C_4 = (3,2,0,0,0,0) & {}^{\#}_{2}C_4 = (10,0,0,0,0,0) \\
{}^{\#}_{0}C_5 = (1,0,0,0,0,0) & {}^{\#}_{1}C_5 = (5,0,0,0,0,0) & {}^{\#}_{2}C_5 = (9,1,0,0,0,0) \\
{}^{\#}_{3}C_1 = (10,0,0,0,0,0) & {}^{\#}_{4}C_1 = (3,2,0,0,0,0) & {}^{\#}_{5}C_1 = (1,0,0,0,0,0) \\
{}^{\#}_{3}C_2 = (4,6,0,0,0,0) & {}^{\#}_{4}C_2 = (4,1,0,0,0,0) & {}^{\#}_{5}C_2 = (0,1,0,0,0,0) \\
{}^{\#}_{3}C_3 = (10,0,0,0,0,0) & {}^{\#}_{4}C_3 = (2,3,0,0,0,0) & {}^{\#}_{5}C_3 = (1,0,0,0,0,0) \\
{}^{\#}_{3}C_4 = (7,3,0,0,0,0) & {}^{\#}_{4}C_4 = (5,0,0,0,0,0) & {}^{\#}_{5}C_4 = (1,0,0,0,0,0) \\
{}^{\#}_{3}C_5 = (10,0,0,0,0,0) & {}^{\#}_{4}C_5 = (5,0,0,0,0,0) & {}^{\#}_{5}C_5 = (1,0,0,0,0,0)
\end{array}
$$

Now putting all the above together and writing them in order we have the aliased effect number pattern as in Equation 2.3. Thus the aliased effect number pattern is ${}^{\#}C = ({}^{\#}_{1}C_1,\, {}^{\#}_{0}C_2,\, {}^{\#}_{1}C_2,\, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2,\, {}^{\#}_{0}C_3,\, {}^{\#}_{1}C_3,\, {}^{\#}_{2}C_3, ...)$.

Case II: Alias effect number pattern of the combined foldover design 2. When we reverse the sign of the generator 5, The defining relation becomes I=124=-135=-2345. The defining relation of the combined design is I=124. The alias structure of the combined foldover design is shown in the Table 3.2 below.

Table 3.2: Alias Structure of Combined Foldover Design 2

1=24	12=4	123=34	1234=3	12345=35
2=14	13=234	124=I	1235=345	
3=1234	14=2	125=45	1345=235	
4=12	15=245	134=23	1245=5	
5=1245	23=134	135=2345	2345=135	
	24=1	145=25		
	25=145	234=13		
	34=123	235=1345		
	35=12345	245=15		
	45=125	345=1235		

Similarly, from the alias structure in Table 3.2 we obtain ${}_{i}^{\#}C_{j}^{k}$ of the resulting combined foldover design as follows.

$$
\begin{array}{lll}
{}_{2}^{\#}C_0 = (10,0,0,0,0,0) & {}_{1}^{\#}C_1 = (5,0,0,0,0,0) & {}_{2}^{\#}C_1 = (7,3,0,0,0,0) \\
{}_{0}^{\#}C_2 = (1,0,0,0,0,0) & {}_{1}^{\#}C_2 = (2,3,0,0,0,0) & {}_{2}^{\#}C_2 = (10,0,0,0,0,0) \\
{}_{0}^{\#}C_3 = (0,1,0,0,0,0) & {}_{1}^{\#}C_3 = (5,0,0,0,0,0) & {}_{2}^{\#}C_3 = (4,6,0,0,0,0) \\
{}_{0}^{\#}C_4 = (1,0,0,0,0,0) & {}_{1}^{\#}C_4 = (3,2,0,0,0,0) & {}_{2}^{\#}C_4 = (10,0,0,0,0,0) \\
{}_{0}^{\#}C_5 = (1,0,0,0,0,0) & {}_{1}^{\#}C_5 = (5,0,0,0,0,0) & {}_{2}^{\#}C_5 = (9,1,0,0,0,0) \\
{}_{3}^{\#}C_1 = (10,0,0,0,0,0) & {}_{4}^{\#}C_1 = (3,2,0,0,0,0) & {}_{5}^{\#}C_1 = (1,0,0,0,0,0) \\
{}_{3}^{\#}C_2 = (4,6,0,0,0,0) & {}_{4}^{\#}C_2 = (4,1,0,0,0,0) & {}_{5}^{\#}C_2 = (0,1,0,0,0,0) \\
{}_{3}^{\#}C_3 = (10,0,0,0,0,0) & {}_{4}^{\#}C_3 = (2,3,0,0,0,0) & {}_{5}^{\#}C_3 = (1,0,0,0,0,0) \\
{}_{3}^{\#}C_4 = (7,3,0,0,0,0) & {}_{4}^{\#}C_4 = (5,0,0,0,0,0) & {}_{5}^{\#}C_4 = (1,0,0,0,0,0) \\
{}_{3}^{\#}C_5 = (10,0,0,0,0,0) & {}_{4}^{\#}C_5 = (5,0,0,0,0,0) & {}_{5}^{\#}C_5 = (1,0,0,0,0,0)
\end{array}
$$

Again, the aliased effect number pattern is ${}^{\#}C = ({}^{\#}_{1}C_1, {}^{\#}_{0}C_2, {}^{\#}_{1}C_2, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2, {}^{\#}_{0}C_3, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, ...)$ as in Equation 2.3.

Case III: Aliased effect number pattern of the combined foldover design 3. When we reverse the sign of both generators 4 and 5, we obtain a design with defining relation I=-124=-135=2345 and the combined foldover design has the defining relation I=2345. The alias structure is shown in Table 3.3 below:

Table 3.3: Alias Structure of Combined Foldover Design 3

1=12345	12=1345	123=145	1234=15	12345=1
2=345	13=1245	124=135	1235=14	
3=245	14=1235	125=134	1345=12	
4=235	15=1234	134=125	1245=13	
5=234	23=45	135=124	2345=I	
	24=35	145=123		
	25=34	234=5		
	34=25	235=4		
	35=24	245=3		
	45=23	345=2		

The ${}_{i}^{\#}C_{j}^{k}$'s are as follows:

$$
\begin{array}{lll}
{}_{2}^{\#}C_0 = (10,0,0,0,0,0) & {}_{1}^{\#}C_1 = (5,0,0,0,0,0) & {}_{2}^{\#}C_1 = (10,0,0,0,0,0) \\
{}_{0}^{\#}C_2 = (1,0,0,0,0,0) & {}_{1}^{\#}C_2 = (5,0,0,0,0,0) & {}_{2}^{\#}C_2 = (4,6,0,0,0,0) \\
{}_{0}^{\#}C_3 = (1,0,0,0,0,0) & {}_{1}^{\#}C_3 = (1,4,0,0,0,0) & {}_{2}^{\#}C_3 = (10,0,0,0,0,0) \\
{}_{0}^{\#}C_4 = (0,1,0,0,0,0) & {}_{1}^{\#}C_4 = (5,0,0,0,0,0) & {}_{2}^{\#}C_4 = (6,4,0,0,0,0) \\
{}_{0}^{\#}C_5 = (1,0,0,0,0,0) & {}_{1}^{\#}C_5 = (4,1,0,0,0,0) & {}_{2}^{\#}C_5 = (10,0,0,0,0,0) \\
{}_{3}^{\#}C_1 = (6,4,0,0,0,0) & {}_{4}^{\#}C_1 = (5,0,0,0,0,0) & {}_{5}^{\#}C_1 = (0,1,0,0,0,0) \\
{}_{3}^{\#}C_2 = (10,0,0,0,0,0) & {}_{4}^{\#}C_2 = (1,4,0,0,0,0) & {}_{5}^{\#}C_2 = (1,0,0,0,0,0) \\
{}_{3}^{\#}C_3 = (4,6,0,0,0,0) & {}_{4}^{\#}C_3 = (5,0,0,0,0,0) & {}_{5}^{\#}C_3 = (1,0,0,0,0,0) \\
{}_{3}^{\#}C_4 = (10,0,0,0,0,0) & {}_{4}^{\#}C_4 = (5,0,0,0,0,0) & {}_{5}^{\#}C_4 = (1,0,0,0,0,0) \\
{}_{3}^{\#}C_5 = (10,0,0,0,0,0) & {}_{4}^{\#}C_5 = (5,0,0,0,0,0) & {}_{5}^{\#}C_5 = (1,0,0,0,0,0)
\end{array}
$$

In the same way, we can put the above in order of importance to obtain Equation 2.3 of the combined foldover design.

From Example 3.1, we presented the aliased effect number patterns of the combined foldover designs from all the foldover plans of the design with generators 4=12 and 5=13. We see that all the foldover plans improve the initial design in one way or the other. The ${}_{i}^{\#}C_{j}^{k}$'s in most of the cases increased compared to the initial design at degree zero and decreased or remained the same for other degrees.

3.2 Findings

3.2.1 Main Findings

The result I below shows that, when the combined foldover design is obtained from a full foldover plan, then, when $i + j$ is odd, all the entries of ${}^{\#}_{i}C_j$ is zero for a combined foldover design except the first entry which represents degree ${}^{\#}_{i}C_j^{(0)}$.

- **Result I**

  For an initial design, when the signs of all the factors are reversed, then when $i + j$ is odd, the resulting combined foldover design has its ${}^{\#}_{i}C_j^{(k)} = (0)$ for $k = 1, 2, ...$

  Proof

  We prove this by considering the fact that, when we reverse the signs of all the factors, all the words with odd number of factors will no longer be in the defining relation of the combined foldover design. Now since we do not have words with odd number letters in the defining relations, we will not have effects with odd number of factors aliased with effects with even number of factors and the result follows.

Example 3.2 From Example 3.1, we see that when we change the sign of all the factors which is equivalent to changing the signs of all the generators in this case, one will notice that, when $i + j$ is odd, the entries of all ${}^{\#}_{i}C_j$s is *zero* except for the first entry ${}^{\#}_{i}C_j^{(0)}$. This is because foldover designs de-alias factors and ${}^{\#}_{i}C_j^{(0)}$ represents the number of i-th order effects that are not aliased with j-th order effects. After folding, the resulting combined foldover design will have more i-th order effects not aliased with j-th order effects.

Result II below shows that, for k=0, ${}^{\#}_{i}C_j^{(k)}$ of the combine foldover design is greater than or equal to that of the initial design. This is due to the fact that more factors are de-aliased after folding the design.

- **Result II**

  The ${}^{\#}_{i}C_j^{(0)}$ of the initial design is always less than or equal to its respective ${}^{\#}_{i}C_j^{(0)}$ of any combined foldover design. That is

$$\{{}^{\#}_{i}C_j^{(0)}\}_{id} \leq \{{}^{\#}_{i}C_j^{(0)}\}_{cfd} \tag{3.1}$$

  where $\{{}^{\#}_{i}C_j^{(0)}\}_{id}$ and $\{{}^{\#}_{i}C_j^{(0)}\}_{cfd}$ represents ${}^{\#}_{i}C_j^{(k)}$ of the initial design and combine foldover design at degree zero, respectively. That is, the number of the i-order effects that are not aliased with any j-th order effects in the initial design is less than or equal to that of their corresponding combined foldover designs.

Example 3.3 Consider the design with generators $4 = 12$ and $5 = 13$ as in Example 3.1. The defining relation of the initial design will be I=124=135=2345. The Table 3.4 shows some selected ith-order effects that are aliased with jth-order effects at various degrees for the initial design and the combined foldover designs 1, 2 and 3.

Table 3.4: ${}_{i}^{\#}C_{j}^{(k)}$, k=0, 1, 2, of the Initial Design and the Three Combined Foldover Designs

	${}_{i}^{\#}C_{j}^{(k)}$ of ID			${}_{i}^{\#}C_{j}^{(k)}$ of CFD1			${}_{i}^{\#}C_{j}^{(k)}$ of CFD2			${}_{i}^{\#}C_{j}^{(k)}$ of CFD3		
i,j	k=0	k=1	k=2	k=0	k=1	k=2	k=0	k=1	k=2	k=0	k=1	k=2
0,0	1	0	0	1	0	0	1	0	0	1	0	0
0,1	1	0	0	1	0	0	1	0	0	1	0	0
0,2	1	0	0	1	0	0	1	0	0	1	0	0
0,3	0	0	1	0	1	0	0	1	0	1	0	0
0,4	0	1	0	1	0	0	1	0	0	0	1	0
0,5	1	0	0	1	0	0	1	0	0	1	0	0
1,0	5	0	0	5	0	0	5	0	0	5	0	0
1,1	5	0	0	5	0	0	5	0	0	5	0	0
1,2	0	4	1	2	3	0	2	3	0	5	0	0
1,3	1	4	0	5	0	0	5	0	0	1	4	0
1,4	1	4	0	3	2	0	3	2	0	5	0	0
1,5	4	1	0	5	0	0	5	0	0	4	1	0
2,0	10	0	0	10	0	0	10	0	0	10	0	0
2,1	4	6	0	7	3	0	7	3	0	10	0	0
2,2	4	6	0	10	0	0	10	0	0	4	6	0
2,3	2	4	4	4	6	0	4	6	0	10	0	0
2,4	6	4	0	10	0	0	10	0	0	6	4	0
2,5	8	2	0	9	1	0	9	1	0	10	0	0

Table 3.4 Continued

	${}^{\#}_{i}C_j^{(k)}$ of ID			${}^{\#}_{i}C_j^{(k)}$ of CFD1			${}^{\#}_{i}C_j^{(k)}$ of CFD2			${}^{\#}_{i}C_j^{(k)}$ of CFD3		
i,j	k=0	k=1	k=2	k=0	k=1	k=2	k=0	k=1	k=2	k=0	k=1	k=2
3,0	8	2	0	9	1	0	9	1	0	10	0	0
3,1	6	4	0	10	0	0	10	0	0	6	4	0
3,2	2	4	4	4	6	0	4	6	0	10	0	0
3,3	4	6	0	10	0	0	10	0	0	4	6	0
3,4	4	6	0	7	3	0	7	3	0	10	0	0
3,5	10	0	0	10	0	0	10	0	0	10	0	0
4,0	4	1	0	5	0	0	5	0	0	4	1	0
4,1	1	4	0	3	2	0	3	2	0	5	0	0
4,2	1	4	0	4	1	0	4	1	0	1	4	0
4,3	0	4	1	2	3	0	2	3	0	5	0	0
4,4	5	0	0	5	0	0	5	0	0	5	0	0
4,5	5	0	0	5	0	0	5	0	0	5	0	0
5,0	1	0	0	1	0	0	1	0	0	1	0	0
5,1	0	1	0	1	0	0	1	0	0	0	1	0
5,2	0	0	1	0	1	0	0	1	0	1	0	0
5,3	1	0	0	1	0	0	1	0	0	1	0	0
5,4	1	0	0	1	0	0	1	0	0	1	0	0
5,5	1	0	0	1	0	0	1	0	0	1	0	0

The first column of Table 3.4 provides the values of i and j. The second to fourth columns of Table 3.4 represent some selected ${}_{i}^{\#}C_{j}^{(k)}$'s of the initial design (ID) for k=0, 1, and 2 respectively. The fifth to seventh columns represent that of combined foldover design 1 (CFD1) while the eighth to tenth columns represent that of combined foldover design 2 (CFD2). The last three columns are those of combined foldover design 3 (CFD3).

We compared columns two, five, eight and eleven of Table 3.4 which represents ${}_{i}^{\#}C_{j}^{(0)}$ for the initial design, combined foldover design 1, combined foldover design 2 and combined foldover design 3, respectively. We observe that, ${}_{i}^{\#}C_{j}^{(0)}$ remains either the same or increased in the combined foldover designs.

For instance, when i=1 and j=2, for the initial design ${}_{1}^{\#}C_{2}^{(0)}$ is 0: for combined foldover design 1, ${}_{1}^{\#}C_{2}^{(0)}$ is 2; for combined foldover design 2, ${}_{1}^{\#}C_{2}^{(0)}$ is 2; while for combined foldover design 3, ${}_{1}^{\#}C_{2}^{(0)}$ is 5. When $i = 3$ and $j = 0$, ${}_{3}^{\#}C_{0}^{(0)}$ is 8 for the initial design and it is 9, 9 and 10 for combined foldover designs 1, 2 and 3, respectively. This supports the obvious fact that after folding we are increasing the number of clear effects.

3.2.2 Some Other Observations/Findings

- We realized that, in many cases, when there are odd number of generators each with two letters, the defining relation has all of its odd letter words aliased with odd number of these generators in them. We explain these in the Example 3.4 below:

Example 3.4 Consider the following designs

1. A design with generator 5=14. The defining relations is I=145.
2. A design with generators 5=12,6=23 and 7=34, the defining relation is I=125=236=347=1356=2467=14567=1234567.
3. A design with generators 5=12,6=13,7=23,8=14 and 9=24. The defining relation is I = 249 = 148 = 1289 = 237 = 3479 = 123478 = 13789 = 136 = 123469 = 3468 = 23689 = 1267 = 14679 = 24678 = 6789 = 125 = 1459 = 2458 = 589 = 1357 = 1234579 = 34578 = 235789 = 2358 = 34569 = 1234569 = 135689 = 567 = 245679 = 145678 = 1256789.

From Example 3.4 we see that all the designs considered have odd number of generators with each generator having only two letters. We notice that the defining relations of all these designs has odd number of this generators in each odd numbered word.

- It was observed that, for many resolution III designs, when the number of generators with two letters is odd except for one, changing the signs of these generators is equivalent to the full foldover plan. When there are odd number of generators with each generated by two factors, the defining relation has all of its odd letter words with odd number of these generators in them. Changing their signs leads to all the odd number words going away. We suspect this to be probably the reason.

Example 3.5 In Example 3.4, the third design has generators 5=12, 6=13, 7=23, 8=14 and 9=24 with defining relation I = 249 = 148 = 1289 = 237 = 3479 = 123478 = 13789 = 136 = 123469 = 3468 = 23689 = 1267 = 14679 =

$24678 = 6789 = 125 = 1459 = 2458 = 589 = 1357 = 1234579 = 34578 = 235789 = 2358 = 34569 = 1234569 = 135689 = 567 = 245679 = 145678 = 1256789$. We see that when we reverse the signs of all the generators (that is 5,6,7,8) the defining relation becomes $\mathrm{I} = -249 = -148 = 1289 = -237 = 3479 = 123478 = -13789 = -136 = 123469 = 3468 = -23689 = 1267 = -14679 = -24678 = 6789 = -125 = 1459 = 2458 = -589 = 1357 = -1234579 = -34578 = 235789 = 2358 = -34569 = -1234569 = 135689 = -567 = 245679 = 145678 = -1256789$ and the resulting combined foldover design has its defining relation as $\mathrm{I} = 1289 = 3479 = 123478 = 123469 = 3468 = 1267 = -14679 = 6789 = 1459 = 2458 = 1357 = 235789 = 2358 = 135689 = 245679 = 145678$.

We see from Example 3.5 that, the combined foldover design obtained from reversing all the generators does not contain words with odd number of letters. Again from the proof of result I we see that reversing the signs of all the factors yields a combined foldover design with a defining relation containing no odd letter word.

- We observed that for most of the designs when the word length pattern of any two combined foldover designs from the same initial design are the same, their aliased effect number pattern is the same. An example is a resolution IV design with two generators $5 = 123$ and $6 = 124$. The defining relation is I=1235=1246=3456. The combine foldover design obtained from all the possible foldover plans have the same word length pattern with the same aliased effect number pattern.

Chapter 4

Optimal Foldover Designs

4.1 Foldover Plans and AENP

In Chapter 2 we introduced the concept of foldover designs. We also showed why there is the need to choose optimal foldover plans. We as well mentioned different criteria that were used in choosing optimal foldover plans in literature. In this chapter, we search for optimal foldover designs.

In order to be sure we have selected optimal foldover plan we need to consider all the possible foldover plans. This includes reversing the sign of each factor as well as their possible combinations. However, according to Theorem 2.1, once we consider all the core foldover plans, we are assured that all the possible foldover plans are covered. That is, we only need to reverse the signs of each of the generators and their combinations instead of all the factors and their combinations. From these core foldover plans, we choose the best ones based on the general minimum lower order confounding criterion for 16- and 32-run designs in Tables 2 and 3 in Zhang et al. (2008).

As explained earlier the elements of the aliased effect number pattern

$$^{\#}C = ({}_1^{\#}C_1, {}_0^{\#}C_2, {}_1^{\#}C_2, {}_2^{\#}C_1, {}_2^{\#}C_2, {}_0^{\#}C_3, {}_1^{\#}C_3, {}_2^{\#}C_3, ...) \tag{4.1}$$

of a design are in order of importance according to the effect hierarchical principle which states that lower order effects are more important than higher order effects. For this reason we consider the first five elements of $^{\#}C$, that is,

$$({}_1^{\#}C_1, {}_0^{\#}C_2, {}_1^{\#}C_2, {}_2^{\#}C_1, {}_2^{\#}C_2) \tag{4.2}$$

since we are convinced that this will be able to differentiate between important designs for most of the cases. For cases where we had more than one optimal foldover plans based on the Equation 4.2, we further obtain their next 9 elements $({}_0^{\#}C_3, {}_1^{\#}C_3, {}_2^{\#}C_3, ..., {}_2^{\#}C_4)$ for those designs in order to choose more relative optimal plans. In other words we say we did consider the first fourteen elements of Equation 4.1. We select optimal foldover designs using the 16- and 32-run designs in Tables 2 and 3 in Zhang et al. (2008). They selected the deigns using design matrix in Chen et al. (1993). The design matrix can be found in Table 1 in the Appendix.

4.2 Algorithm

We follow the following steps (algorithm) in the construction and selection of optimal designs:

1. Specify our initial designs and then specify as well the additional columns (i.e the generators p) in the design matrix.

2. Find all the foldover plans by reversing the signs of generators and their possible combinations ($2^p - 1$ in all).

3. Obtain the combined foldover designs by combining the initial design and the new fraction.

4. Obtain the alias structure of each of the resulting combined foldover designs.

5. Calculate the aliased effect number pattern for each combined foldover design.

6. Choose the optimal designs based on the general minimum lower-order confounding criterion.

The optimal designs obtained from general minimum lower-order confounding criterion are then compared to the optimal designs obtained using other criteria such as clear effect and minimum aberration. Now lets demonstrate how we obtain optimal foldover plans based on the general minimum lower-order confounding criterion.

Example 4.1 Consider a 2^{6-2} design (design 7.3.1 in Table 4.6) with generators 5=23, 6=24 and 7=34. The defining relation is I = 235 = 246 = 347 = 3456 = 2457 = 2367 = 567. For this design the possible foldover plans are 5, 6, 7, 56, 57, 67 and 567. We find the first five entries of Equation 2.3 for each combined foldover design. That is, we calculate ${}^{\#}C = ({}^{\#}_{1}C_1,\, {}^{\#}_{0}C_2,\, {}^{\#}_{1}C_2,\, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2)$ at degrees k=0,1,...,7. Tables 4.1, 4.2, 4.3, 4.4 and 4.5 represents the ${}^{\#}_{1}C_1$, ${}^{\#}_{0}C_2$, ${}^{\#}_{1}C_2$, ${}^{\#}_{2}C_1$ and ${}^{\#}_{2}C_2$, respectively. In each table column 1 represents the foldover plans whiles colum 2 to column 9 represents ${}^{\#}_{i}C_j^{(k)}$ for k=0,1,.....7, respectively.

Table 4.1: ${}^{\#}_{1}C_1$ of the Combined Foldover Designs

	${}^{\#}_{1}C_1^k$							
Foldover Plan	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
5	7	0	0	0	0	0	0	0
6	7	0	0	0	0	0	0	0
7	7	0	0	0	0	0	0	0
56	7	0	0	0	0	0	0	0
57	7	0	0	0	0	0	0	0
67	7	0	0	0	0	0	0	0
567	7	0	0	0	0	0	0	0

Table 4.2: ${}^{\#}_{0}C_2$ of the Combined Foldover Designs

	${}^{\#}_{0}C_2^k$							
Foldover Plan	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
5	1	0	0	0	0	0	0	0
6	1	0	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0
56	1	0	0	0	0	0	0	0
57	1	0	0	0	0	0	0	0
67	1	0	0	0	0	0	0	0
567	1	0	0	0	0	0	0	0

Table 4.3: ${}_1^{\#}C_2$ of the Combined Foldover Designs

	${}_1^{\#}C_2^k$							
Foldover Plan	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
5	2	4	1	0	0	0	0	0
6	2	4	1	0	0	0	0	0
7	2	4	1	0	0	0	0	0
56	2	4	1	0	0	0	0	0
57	2	4	1	0	0	0	0	0
67	2	4	1	0	0	0	0	0
567	7	0	0	0	0	0	0	0

Table 4.4: ${}_2^{\#}C_1$ of the Combined Foldover Designs

	${}_2^{\#}C_1^k$							
Foldover Plan	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
5	15	6	0	0	0	0	0	0
6	15	6	0	0	0	0	0	0
7	15	6	0	0	0	0	0	0
56	15	6	0	0	0	0	0	0
57	15	6	0	0	0	0	0	0
67	15	6	0	0	0	0	0	0
567	21	0	0	0	0	0	0	0

Table 4.5: ${}^{\#}_{2}C_2$ of the Combined Foldover Designs

	${}^{\#}_{2}C_2^k$							
Foldover Plan	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
5	15	6	0	0	0	0	0	0
6	15	6	0	0	0	0	0	0
7	15	6	0	0	0	0	0	0
56	15	6	0	0	0	0	0	0
57	15	6	0	0	0	0	0	0
67	15	6	0	0	0	0	0	0
567	6	12	3	0	0	0	0	0

Based on Definition 2.3 we can compare the combined designs from all foldover plans. However, in addition to the definition, we consider the fact that the componets of Equation 2.3 are in order of importance. For this reason, we do the comparison by looking for the first component for which all of them are not the same starting with Table 4.1 till we find a component for which they differ.

We see from Table 4.1 and Table 4.2 that, there is no difference in the components. So we proceed to the next table which is Table 4.3. From this table we can see that component for which the fisrt difference occur is on ${}^{\#}_{1}C_2^0$. Foldover plans 5, 6, 7, 56, 57 and 67 all have 2 in that column while foldover plan 567 has 7 which is the greatest. Hence it has the minimum general lower order confounding. Therefore according to the general minimum lower-order confounding criterion foldover plan 567 is the optimum. In a case where two or more foldover plans have equal entries in that column we have to look for the next component for which they differ till we get the design with minimum general lower-order confounding.

4.3 Tables

We found the optimal foldover plans for designs in Tables 2 and 3 of Zhang et al. (2008). The tables in this section summarize our results. The optimal foldover plans (OFP) and the resulting optimal combined foldover designs (OCFD) obtained from the aliased effect number pattern criterion (AENP), clear effect (CE) criterion and minimum abberation criterion (MA) are presented. Table 4.6 presents the optimal foldover designs obtained from 16-run designs based on general minimum lower order confounding criterion. Table 4.7 presents the optimal foldover designs obtained from minimum abberation and clear effect criteria for 16-run designs as well. Furthermore, Tables 4.8, 4.9 and 4.10 presents the optimal foldover designs for the 32-run designs based on the general minimum lower order confounding, minimum aberration and clear effects criteria, respectively.

In each table, first column represent the initial design. For a design $n.k.m$, n represents the total number of factors, k represents the number of generators while m is the number of such designs considered. The idea of this notation is from some previous papers such as Li and Lin (2003) and Zhang et al. (2008). For instance, the first design with six factors and two generators will be written as 6.2.1, the second will be written as 6.2.2 and the third as 6.2.3, in that order. A design with seven factors and three generators will be denoted as 7.3.1. The second column lists additional columns from the design matrix. which represents the generators of each design.

The third column of Tables 4.6 and 4.8 lists the elements of ${}^{\#}C$, i.e, aliased effect number pattern of the initial designs. Zhang et al. (2008) showed the aliased effect number pattern but just these two ${}_{1}^{\#}C_2$ and ${}_{2}^{\#}C_2$ which we reproduced. We computed and listed the first five elements which includes ${}_{1}^{\#}C_2$ and ${}_{2}^{\#}C_2$ for the sake of comparison and confirmation. The first two elements of $({}_{1}^{\#}C_1;{}_{0}^{\#}C_2;{}_{1}^{\#}C_2;\, {}_{2}^{\#}C_1;\, {}_{2}^{\#}C_2)$ have all entries being zero except the first. This is because all the design are of resolution greater than or equal to 3.

The fourth and fifth columns of Tables 4.6 and 4.8 represent the optimal foldover plan based on general minimum lower-order confounding and the aliased effect number pattern of the corresponding optimal combined foldover design, respectively. In order to save space and for simplicity we write, for example, $({}_{1}^{\#}C_1;{}_{0}^{\#}C_2;{}_{1}^{\#}C_2;\, {}_{2}^{\#}C_1;\, {}_{2}^{\#}C_2)$ = (9,0,0,0,0,0,0,0,0,0 ; 1,0,0,0,0,0,0,0,0,0 ; 9,0,0,0,0,0,0,0,0,0 ; 36,0,0,0,0,0,0,0,0,0 ; 21,12,3,0,0,0,0,0,0,0) as $(9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 21, 12, 3, 0^7)$.

The third and fourth columns of Tables 4.7 and 4.9 shows the foldover plans chosen as optimum based on minimum aberration criterion and the word length patterns $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, ..., \mathrm{A}_n)$ of the corresponding optimal combined foldover designs, respectively. Again, for some of the results, to save space we write zeros that follow each other as the zero to the power the number of times zeros appear consecutively. For example, the word length pattern of optimal combined foldover design of design 10.5.1 is (0,0,0,4,8,0,0,3,0,0), but we write it as $(0^3, 4, 8, 0^2, 3, 0^2)$.

The optimal combined foldover designs according to the clear effects criterion and the number of clear effects are shown in the fifth and sixth columns of Table 4.7 and third and fourth columns of Table 4.10. The first number represents the number of clear main effects and the second number represents the number of clear two-factor interactions.

Table 4.6: Optimal Foldover Designs for 16-Run Initial Designs Based on GMLOC Criterion

Design	Additional Columns	AENP of initial design	OFP(GMLOC)	AENP of OCFD
		$({}^{\#}_{1}C_1;\ {}^{\#}_{0}C_2;\ {}^{\#}_{1}C_2;\ {}^{\#}_{2}C_1;\ {}^{\#}_{2}C_2)$		$({}^{\#}_{1}C_1;\ {}^{\#}_{0}C_2;\ {}^{\#}_{1}C_2;\ {}^{\#}_{2}C_1;\ {}^{\#}_{2}C_2)$
6.2.1	7,14	$6, 0^6; 1, 0^6; 6, 0^6; 15, 0^6; 0, 12, 3, 0^4$	5,6,56	$6, 0^6; 1, 0^6 : 6, 0^6; 15, 0^6; 9, 6, 0^5$
6.2.2	6,12	$6, 0^6; 1, 0^6; 1, 4, 1, 0^4; 9, 6, 0^5; 9, 6, 0^5$	56	$6, 0^6; 1, 0^6; 6, 0^6; 15, 0^6; 9, 6, 0^5$
6.2.3	3,6	$6, 0^6; 1, 0^6; 6, 0^6; 9, 6, 0^5; 15, 0^6$	56	$6, 0^6; 1, 0^6 : 6, 0^6; 15, 0^6; 15, 0^6$
7.3.1	7,11,14	$7, 0^7; 1, 0^7; 7, 0^7; 21, 0^7; 0, 0, 21, 0^5$	5,6,7,56,57,67,567	$7, 0^7; 1, 0^7; 7, 0^7; 21, 0^7; 6, 12, 3, 0^5$
7.3.2	6,10,12	$7, 0^7; 1, 0^7; 1, 0, 6, 0^5; 9, 12, 0^6; 6, 12, 3, 0^5$	567	$7, 0^7; 1, 0^7; 7, 0^7; 21, 0^7; 6, 12, 3, 0^5$
7.3.3	3,6,12	$7, 0^7; 1, 0^7; 0, 5, 2, 0^5; 12, 9, 0^6; 9, 12, 0^6$	567	$7, 0^7; 1, 0^7; 7, 0^7; 21, 0^7; 9, 12, 0^6$
8.4.1	7,11,13,14	$8, 0^8; 1, 0^8; 8, 0^8; 28, 0^8; 0^3, 28, 0^5$	5,6,7,8,567,568,578,678	$8, 0^8; 1, 0^8; 8, 0^8; 28, 0^8; 7, 0, 21, 0^6$
8.4.2	3,5,7,14	$8, 0^8; 1, 0^8; 2, 0, 6, 0^6; 16, 12, 0^7; 0, 24, 0, 4, 0^5$	568	$8, 0^8; 1, 0^8; 8, 0^8; 28, 0^8; 13, 12, 3, 0^6$
8.4.3	3,7,11,14	$8, 0^8; 1, 0^8; 1, 6, 0, 1, 0^5; 19, 9, 0^7; 7, 0, 21, 0^6$	58	$8, 0^8; 1, 0^8; 8, 0^8; 28, 0^8; 13, 12, 3, 0^6$
8.4.4	6,10,12,14	$8, 0^8; 1, 0^8; 1, 0, 0, 7, 0^5; 7, 21, 0^7; 7, 0, 21, 0^6$	567	$8, 0^8; 1, 0^8; 8, 0^8; 28, 0^8; 7, 0, 21, 0^6$
8.4.5	3,7,12,14	$8, 0^8; 1, 0^8; 0, 4, 4, 0^6; 16, 12, 0^7; 4, 18, 16, 0^6$	57	$8, 0^8; 1, 0^8; 8, 0^8; 28, 0^8; 4, 18, 6, 0^6$
9.5.1	3,7,11,13,14	$9, 0^9; 1, 0^9; 0, 8, 0, 0, 1, 0^5; 24, 12, 0^8; 8, 0, 0, 28, 0^6$	589	$9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 8, 24, 0, 4, 0^6$
9.5.2	3,6,7,11,14	$9, 0^9; 1, 0^9; 0, 2, 5, 2, 0^6; 18, 18, 0^8; 2, 12, 18, 4, 0^6$	56	$9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 2, 12, 18, 4, 0^6$
9.5.3	3,6,10,12,14	$9, 0^9; 1, 0^9; 0, 2, 0, 6, 1, 0^5; 12, 24, 0^8; 2, 12, 18, 4, 0^6$	5678	$9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 2, 12, 18, 4, 0^6$
9.5.4	3,7,9,12,14	$9, 0^9; 1, 0^9; 0, 0, 9, 0^7; 18, 18, 0^8; 0, 18, 18, 0^7$	578	$9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 0, 18, 18, 0^7$
9.5.5	3,6,7,12,14	$9, 0^9; 1, 0^9; 0, 0, 6, 3, 0^6; 15, 21, 0^8; 0, 18, 18, 0^7$	568	$9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 0, 18, 18, 0^7$
10.6.1	3,6,7,11,13,14	$10, 0^{10}; 1, 0^{10}; 0, 0, 8, 0, 2, 0^6; 21, 24, 0^9; 0, 16, 0, 24, 5, 0^6$	56	$10, 0^{10}; 1, 0^{10}; 10, 0^{10}; 45, 0^{10}; 0, 16, 0, 24, 5, 0^6$
10.6.2	3,5,6,10,12,14	$10, 0^{10}; 1, 0^{10}; 0, 0, 3, 4, 3, 0^6; 15, 30, 0^9; 0, 6, 27, 12, 0^7$	56789	$10, 0^{10}; 1, 0^{10}; 10, 0^{10}; 45, 0^{10}; 0, 6, 27, 12, 0^7$
10.6.3	3,6,7,12,14,15	$10, 0^{10}; 1, 0^{10}; 0, 0, 0, 10, 0^7; 15, 30, 0^9; 0, 0, 45, 0^8$	568$\underline{10}$	$10, 0^{10}; 1, 0^{10}; 10, 0^{10}; 45, 0^{10}; 0, 0, 45, 0^8$
11.7.1	3,6,7,11,12,13,14	$11, 0^{11}; 1, 0^{11}; 0, 0, 0, 8, 3, 0^7; 19, 3, 6, 0^9; 0, 0, 24, 16, 15, 0^7$	569	$11, 0^{11}; 1, 0^{11}; 11, 0^{11}; 55, 0^{11}; 0, 0, 24, 16, 15, 0^7$
11.7.2	3,5,6,7,11,13,14	$11, 0^{11}; 1, 0^{11}; 0, 0, 0, 8, 0, 3, 0^6; 16, 39, 0^10; 0, 0, 24, 16, 15, 0^7$	567	$11, 0^{11}; 1, 0^{11}; 11, 0^{11}; 55, 0^{11}; 0, 0, 24, 16, 15, 0^7$
11.7.3	3,5,6,7,9,12,14	$11, 0^{11}; 1, 0^{11}; 0, 0, 0, 5, 6, 0^7; 16, 39, 0^{11}; 0, 0, 15, 40, 0^8$	5679$\underline{10}$	$11, 0^{11}; 1, 0^{11}; 11, 0^{11}; 55, 0^{11}; 0, 0, 15, 40, 0^8$
12.8.1	3,6,7,9,11,12,13,14	$12, 0^{12}; 1, 0^{12}; 0^4, 12, 0^8; 18, 48, 0^{11}; 0, 0, 0, 48, 0, 18, 0^7$	568$\underline{10}$	$12, 0^{12}; 1, 0^{12}; 12, 0^{12}; 66, 0^{12}; 0, 0, 0, 48, 0, 18, 0^7$

Table 4.7: Optimal Foldover Designs for 16-Run Initial Designs Based on MA and CE Criteria

Design	Additional Columns	OFP(MA)	WLP of OCFD	OFP(CE)	CE of OCFD
6.2.1	7,14	5,6,56	$0^3, 1, 0^2$	5,6,56	6,9
6.2.2	6,12	56	$0^3, 1, 0^2$	56	6,9
6.2.3	3,6	56	$0^5, 1$	56	6,15
7.3.1	7,11,14	5,6,7,56,57,67,567	$0^3, 3, 0^3$	5,6,7,56,57,67,567	7,6
7.3.2	6,10,12	567	$0^3, 3, 0^3$	567	7,6
7.3.3	3,6,12	567	$0^3, 2, 0, 1, 0$	567	7,9
8.4.1	7,11,13,14	56,57,58,67,68,78	$0^3, 6, 0^3, 1$	5,6,7,8,567,568,578,678	8,7
8.4.2	3,5,7,14	568	$0^3, 3, 4, 0^3$	568	8,13
8.4.3	3,7,11,14	58	$0^3, 3, 4, 0^3$	58	8,13
8.4.4	6,10,12,14	567	$0^3, 7, 0^4$	567	8,7
8.4.5	3,7,12,14	57	$0^3, 5, 0, 2, 0^2$	57	8,4
9.5.1	3,7,11,13,14	589	$0^3, 6, 8, 0^2, 1, 0$	5,589	9,8
9.5.2	3,6,7,11,14	56	$0^3, 10, 0, 4, 0, 1, 0$	56	9,2
9.5.3	3,6,10,12,14	5678	$0^3, 10, 0, 4, 0, 1, 0$	5678	9,2
9.5.4	3,7,9,12,14	578	$0^3, 9, 0, 6, 0^3$	578	9,2
9.5.5	3,6,7,12,14	568	$0^3, 9, 0, 6, 0^3$	568	9,0
10.6.1	3,6,7,11,13,14	56	$0^3, 18, 0, 8, 0, 5, 0, 0$	56	10,0
10.6.2	3,5,6,10,12,14	56789	$0^3, 16, 0, 12, 0, 3, 0^2$	56789	10,0
10.6.3	3,6,7,12,14,15	$568\underline{10}$	$0^3, 15, 0, 15, 0^3, 1$	$568\underline{10}$	10,0
11.7.1	3,6,7,11,12,13,14	569	$0^3, 26, 0, 24, 13, 0^3$	569	11,0
11.7.2	3,5,6,7,11,13,14	567	$0^3, 26, 0, 24, 13, 0^3$	567	11,0
11.7.3	3,5,6,7,9,12,14	$5679\underline{10}$	$0^3, 25, 0, 27, 0, 10, 0, 1, 0$	$5679\underline{10}$	11,0
12.8.1	3,6,7,9,11,12,13,14	$568\underline{10}$	$0^3, 39, 0, 48, 0, 39, 0^3, 1$	$568\underline{10}$	12,0

Table 4.8: Optimal Foldover Designs for 32-Run Initial Designs Based on GMLOC Criterion

Design	Additional Columns	AENP of initial design $({}^{\#}_{1}C_2;\ {}^{\#}_{2}C_1;\ {}^{\#}_{2}C_2)$	OFP(GMLOC)	ANEP of OCFD $({}^{\#}_{1}C_2;\ {}^{\#}_{2}C_1;\ {}^{\#}_{2}C_2)$
7.2.1	**7,30**	$7, 0^7; 21, 0^7; 15, 6, 0^6$	**6,67**	$7, 0^7; 21, 0^7; 21, 0^7$
8.3.1	**7,11,30**	$8, 0^8; 28, 0^8; 13, 12, 3, 0^6$	**6,7,67,68,78,678**	$8, 0^8; 28, 0^8; 22, 6, 0^7$
9.4.1	**7,11,13,30**	$9, 0^9; 36, 0^9; 15, 0, 21, 0^7$	**67,68,69,78,79,89,6789**	$9, 0^9; 36, 0^9; 21, 12, 3, 0^7$
9.4.2	**7,11,19,30**	$9, 0^9; 36, 0^9; 8, 24, 0, 4, 0^6$	**67,68,78,679,689,789**	$9, 0^9; 36, 0^9; 24, 12, 0^8$
9.4.3	**14,22,26,28**	$9, 0^9; 36, 0^9; 8, 0, 0, 28, 0^6$	**6,7,8,9,678,679,689,789**	$9, 0^9; 36, 0^9; 15, 0, 21, 0^7$
10.5.1	**7,11,19,29,30**	$10, 0^{10}; 45, 0^{10}; 0, 40, 0, 0, 5, 0^6$	$\mathbf{67,68,69,6\underline{10},78,79,7\underline{10},89,8\underline{10},678,679,67\underline{10},689,68\underline{10},789,78\underline{10},679\underline{10},}$ $\mathbf{689\underline{10},789\underline{10},6789\underline{10}.}$	$10, 0^{10}; 45, 0^{10}; 24, 18, 3, 0^8$
11.6.1	**7,11,14,22,26,28**	$11, 0^{11}; 55, 0^{11}; 0, 0, 24, 16, 15, 0^7$	$\mathbf{68,69,6\underline{10},6\underline{11},78,79,7\underline{10},7\underline{11},678,679,67\underline{10},67\underline{11},89\underline{10},89\underline{11},8\underline{10}\,\underline{11},}$ $\mathbf{689\underline{10},68\underline{10}\,\underline{11},69\underline{10}\,\underline{11},789\underline{10},789\underline{11},7\underline{10}\,\underline{11},6789\underline{11},678\underline{10}\,\underline{11},679\underline{10}\,\underline{11}.}$	$11, 0^{11}; 55, 0^{11}; 12, 18, 21, 4, 0^8$
11.6.2	**7,11,14,19,25,28**	$11, 0^{11}; 55, 0^{11}; 0, 0, 15, 40, 0^8$	$\mathbf{68\underline{10},69\underline{11},6\underline{10}\,\underline{11},89\underline{10},89\underline{11},678\underline{10},679\underline{11},67\underline{10}\,\underline{11},789\underline{10},789\underline{11},689\underline{10}\,\underline{11},}$ $\mathbf{6789\underline{10}\,\underline{11}.}$	$11, 0^{11}; 55, 0^{11}; 10, 30, 15, 0^9$
12.7.1	**7,11,13,14,22,26,28**	$12, 0^{12}; 66, 0^{12}; 0, 0, 0, 48, 0, 18, 0^7$	$\mathbf{6,7,8,9,\underline{10},\underline{11},\underline{12},678,\underline{10}\,\underline{11}\,\underline{12},679\underline{10}\,\underline{11},689\underline{10}\,\underline{12},789\underline{11}\,\underline{12}}$	$12, 0^{12}; 66, 0^{12}; 11, 0, 24, 16, 15, 0^8$
12.7.2	**7,11,13,14,19,25,28**	$12, 0^{12}; 66, 0^{12}; 0, 0, 0, 36, 30, 0^8$	$\mathbf{\underline{10},\underline{11},\underline{12},\underline{10}\,\underline{11}\,\underline{12}}$	$12, 0^{12}; 66, 0^{12}; 11, 0, 24, 16, 15, 0^8$
13.8.1	**7,11,13,14,19,22,26,28**	$13, 0^{13}; 78, 0^{13}; 0, 0, 0, 0, 60, 18, 0^8$	$\mathbf{\underline{10}}$	$13, 0^{13}; 78, 0^{13}; 12, 0, 0, 48, 0, 18, 0^8$
14.9.1	**7,11,13,14,19,21,22,26,28**	$14, 0^{14}; 91, 0^{14}; 0^5, 84, 7, 0^8$	$\mathbf{6,7,8,9,\underline{10},\underline{11},\underline{12},\underline{13},\underline{14},678\underline{10}\,\underline{11},789\underline{13}\,\underline{14},\underline{10}\,\underline{11}\,\underline{12}\,\underline{13}\,\underline{14},679\underline{10}\,\underline{12}\,\underline{13},}$ $\mathbf{689\underline{11}\,\underline{12}\,\underline{14}.}$	$14, 0^{14}; 91, 0^{14}; 13, 0, 0, 0, 60, 18, 0^9$
15.10.1	**7,11,13,14,19,21,22,25,26,28**	$15, 0^{15}; 105, 0^{15}; 0^6, 105, 0^9$	$\mathbf{6,7,8,9,\underline{10},\underline{11},\underline{12},\underline{13},\underline{14},\underline{15},678\underline{10}\,\underline{11}\,\underline{13},679\underline{10}\,\underline{12}\,\underline{14},689\underline{11}\,\underline{12}\,\underline{15},789\underline{13}\,\underline{14}\,\underline{15},}$ $\mathbf{\underline{10}\,\underline{11}\,\underline{12}\,\underline{13}\,\underline{14}\,\underline{15}.}$	$15, 0^{15}; 105, 0^{15}; 14, 0, 0, 0, 0, 84, 7, 0^9$

Table 4.9: OptimalFoldover Designs for 32-Run Initial Designs Based on MA Criterion

Design	Additional Columns	OFP(MA)	WLP of OCFD
7.2.1	7,30	6,67	$0^4, 1, 0^2$
8.3.1	7,11,30	6,7,67,68,78,678	$0^3, 1, 2, 0^3$
9.4.1	7,11,13,30	67,68,69,78,79,89,6789	$0^3, 3, 3, 0^3, 1$
9.4.2	7,11,19,30	67,68,78,679,689,789	$0^3, 2, 4, 0, 0, 1, 0$
9.4.3	14,22,26,28	67,68,69,78,79,89,6789	$0^3, 6, 0^3, 1, 0$
10.5.1	7,11,19,29,30	67,68,69,610,78,79,710,89,810,678,679,6710,689,6810,789,7810,67910, 68910,78910,678910.	$0^3, 4, 8, 0^2, 3, 0^2$
11.6.1	7,11,14,22,26,28	6910,610 11,7910,7911,67911,6710 11	$0^3, 10, 0, 16, 0, 5, 0^3$
11.6.2	7,11,14,19,25,28	6810,6911,610 11,8910,8911,67810,67911,6710 11,78910,78911,68910 11, 678910 11.	$0^3, 10, 0, 16, 0, 5, 0^3$
12.7.1	7,11,13,14,22,26,28	6710 12,6711 12,6810 11,6811 12,7810 11,7810 12	$0^3, 15, 0, 32, 0, 15, 0^3, 1$
12.7.2	7,11,13,14,19,25,28	6712,6810,6811,6911,7911,7912,8910,6789,6710 12,6711 12,6810 12,6811 12,6910 11,6911 12, 7910 11,7910 12,8910 11,8910 12,678911,6710 11 12,6910 11 12,8910 11 12,678910 12,678910 11 12.	$0^3, 16, 0, 30, 0, 15, 0, 2, 0^2$
13.8.1	7,11,13,14,19,22,26,28	6711 13,6712 13,6811 12,6812 13,7811 12,7811 13,6710 11 13,6710 12 13,6810 11 12,6810 12 13, 7810 11 12,7810 11 13.	$0^3, 23, 0, 56, 0, 39, 0, 8, 0, 1, 0$
14.9.1	7,11,13,14,19,21,22,26,28	6712 14,6812 13,6910 14,6911 13,7911 12,8910 12,678912,6710 13 14,6711 13 14,6810 13 14,6811 13 14, 6910 11 12,6912 13 14,7810 12 13,7910 12 14,7811 12 13,7811 12 14,7910 11 13,7910 11 14,8910 11 13, 8910 11 14,678910 14,678911 13,6710 11 12 14,6810 11 13 14,7911 12 1314,8910 12 13 14,678910 11 12 13 14.	$0^3, 33, 0, 96, 0, 91, 0, 32, 0, 3, 0, 0$
15.10.1	7,11,13,14,19,21,22,25,26,28	6711 14 15,6712 13 15,6810 14 15,6812 13 14,6910 13 15,6911 13 14,7810 12 15,7811 12 14,7910 11 15, 7911 12 13,8910 11 14,8910 12 13,678910 15,678911 14,678912 13,6710 11 12 15,6710 13 14 15,6810 11 12 14, 6811 13 14 15,6910 11 12 13,6912 13 14 15,7810 12 13 14,7811 12 13 15,7910 11 13 14,7911 12 14 15, 8910 11 13 15,8910 12 14 15,678910 11 12 13 14 15.	$0^3, 45, 0, 160, 0, 195, 0, 96, 0, 15, 0^3$

Table 4.10: Optimal Foldover Designs for 32-Run Initial Designs Based on CE Criterion

Design	Additional Columns	OFP(AENP)	Number CE of OCFD
7.2.1	7,30	6,67	7,21
8.3.1	7,11,30	6,7,67,68,78,678	8,22
9.4.1	7,11,13,30	6,7,8,67,68,69,78,79,89,678,679,689,789,6789	9,21
9.4.2	7,11,19,30	67,68,78,679,689,789	9,24
9.4.3	14,22,26,28	6,7,8,9,678,679,689,789	9,15
10.5.1	7,11,19,29,30	$67,68,69,6\underline{10},78,79,7\underline{10},89,8\underline{10},678,679,67\underline{10},689,68\underline{10},789,78\underline{10},679\underline{10},$ $689\underline{10},789\underline{10},6789\underline{10}.$	10,24
11.6.1	7,11,14,22,26,28	$68,69,6\underline{10},6\underline{11},78,79,7\underline{10},7\underline{11},678,679,67\underline{10},67\underline{11},89\underline{10},89\underline{11},8\underline{10}\ \underline{11},$ $689\underline{10},68\underline{10}\ \underline{11},69\underline{10}\ \underline{11},789\underline{10},789\underline{11},79\underline{10}\ \underline{11},6789\underline{11},678\underline{10}\ \underline{11},679\underline{10}\ \underline{11}.$	11,12
11.6.2	7,11,14,19,25,28	$6,7,8,9,\underline{10},\underline{11},69,6\underline{10},89,678,67\underline{11},68\underline{10},68\underline{11},69\underline{11},6\underline{10}\ \underline{11},78\underline{10},78\underline{11},79\underline{10},79\underline{11},7\underline{10}\ \underline{11},89\underline{10},89\underline{11},9\underline{10}\ \underline{11},6789,678\underline{10},$ $679\underline{10},679\underline{11},67\underline{10}\ \underline{11},689\underline{10},68\underline{10}\ \underline{11},789\underline{10},789\underline{11},78\underline{10}\ \underline{11},89\underline{10}\ \underline{11},6789\underline{11},679\underline{10}\ \underline{11},689\underline{10}\ \underline{11},6789\underline{10}\ \underline{11}.$	11,10
12.7.1	7,11,13,14,22,26,28	$6,7,8,9,\underline{10},\underline{11},\underline{12},678,\underline{10}\ \underline{11}\ \underline{12},679\underline{10}\ \underline{11},689\underline{10}\ \underline{12},789\underline{11}\ \underline{12}.$	12,11
12.7.2	7,11,13,14,19,25,28	$6,7,8,9,\underline{10},\underline{11},\underline{12},\underline{10}\ \underline{11}\ \underline{12},679\underline{10},689\underline{12},678\underline{10}\ \underline{11},789\underline{11}\ \underline{12}.$	12,11
13.8.1	7,11,13,14,19,22,26,28	$6,7,8,9,\underline{10},\underline{11},\underline{12},\underline{13},678\underline{10},\underline{10}\ \underline{11}\ \underline{12}\ \underline{13},689\underline{11}\ \underline{13},789\underline{12}\ \underline{13},679\underline{10}\ \underline{11}\ \underline{12}$	13,12
14.9.1	7,11,13,14,19,21,22,26,28	$6,7,8,9,\underline{10},\underline{11},\underline{12},\underline{13},\underline{14},678\underline{10}\ \underline{11},789\underline{13}\ \underline{14},\underline{10}\ \underline{11}\ \underline{12}\ \underline{13}\ \underline{14},679\underline{10}\ \underline{12}\ \underline{13},$ $689\underline{11}\ \underline{12}\ \underline{14}.$	14,13
15.10.1	7,11,13,14,19,21,22,25,26,28	$6,7,8,9,\underline{10},\underline{11},\underline{12},\underline{13},\underline{14},\underline{15},678\underline{10}\ \underline{11}\ \underline{13},679\underline{10}\ \underline{12}\ \underline{14},689\underline{11}\ \underline{12}\ \underline{15},789\underline{13}\ \underline{14}\ \underline{15},$ $\underline{10}\ \underline{11}\ \underline{12}\ \underline{13}\ \underline{14}\ \underline{15}.$	15,14

4.4 Explanation and Commentary

Generally, it is well-known that, foldover designs are supposed to have less confounding between effects comparing to initial designs since more runs are added to the initial designs. This has been demonstrated when we search for optimal foldover designs. One can compare AENPs of the foldover designs to that of the corresponding initial designs in Tables 4.6 and 4.8.

For instance, considering design 7.3.2, its aliased effect number pattern, $^{\#}C = ({}^{\#}_{1}C_1, {}^{\#}_{0}C_2, {}^{\#}_{1}C_2, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2)$ is $(7, 0^7; 1, 0^7; 1, 0, 6, 0^5; 9, 12, 0^6; 6, 12, 3, 0^5)$ and that of the optimal combined foldover design is $(7, 0^7; 1, 0^7; 7, 0^7; 21, 0^7; 6, 12, 3, 0^5)$. The optimal combined foldover design has a less general lower-order confounding than the initial design.

4.4.1 16 Run

From Tables 4.6 and 4.7 we see that, although some of the designs had all three criteria choosing the same foldover plans as optimal, it is interesting to note that, the general minimum lower-order confounding criterion can choose a completely different foldover plan or design as the optimal from minimum aberration criterion. For instance, for design 8.4.1, where 8 out of $2^4 - 1$=15 are optimum according to general minimum lower-order confounding and clear effects criteria whiles 6 out of 15 are optimum according to minimum aberration criterion. That is 5, 6, 7, 567, 568, 578, 678 out of 5, 6, 7, 8, 56, 57, 58, 67, 68, 78, 567, 578, 568, 678, 5678 are optimum according the general minimum lower-order confounding and clear effects criteria. Whiles 56, 57, 58, 67, 68, 78 are optimum based on minimum aberration criterion.

Similarly, comparing the optimal foldover plans we obtain from general minimum lower-order confounding and clear effects criteria, we see that, the foldover plans chosen by the former is always a subset of foldover plans chosen by the later. This implies that all the foldover plans chosen by general minimum lower-order confounding criterion are found in that of the clear effects criteria. This is actually true since number of clear effects is related to the aliased effect number pattern as seen in section 2.4.4.

Most of the 16-run designs considered have only one optimal foldover plan except for designs 6.2.1, 7.3.1 and 8.4.1 where more than one foldover plan was chosen as optimum. For instance, for design 9.5.2 the foldover plans considered are 5, 6, 7, 8, 9, 56, 57, 58, 59, 67, 68, 69, 78, 79, 89, 567, 568, 569, 578, 579, 589, 678, 679, 689, 789, 5678, 5679, 5689, 5789, 6789 and 56789 but the optimal plan is 56. All the three criteria choose this design as the optimum. The word length pattern, aliased number effect pattern and number of clear effects of the combined foldover design of this foldover plan are $\{0^3, 10, 0, 4, 0, 1, 0\}$,$\{9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 2, 12, 18, 4, 0^6\}$ and $\{9, 2\}$, respectively.

In some cases, such as designs 6.2.1 and 7.3.1, all the $2^2 - 1$ and $2^3 - 1$ foldover plans are optimal according to the three criteria. None of the foldover plans for these designs is better than the other in terms of optimality when we use the three criteria. The aliased number effect pattern of the combined optimal design of 6.2.1 is $\{6, 0^6; 1, 0^6 : 6, 0^6; 15, 0^6; 9, 6, 0^5\}$ and that of 7.3.1 is $\{7, 0^7; 1, 0^7; 7, 0^7; 21, 0^7;$ $6, 12, 3, 0^5\}$. The respective word length pattern and number of clear effects of 6.2.1

are $\{0^3, 1, 0^2\}$ and $\{6, 9\}$. Design 7.3.1 has word length pattern and number of clear effects as $\{0^3, 3, 0^3\}$ and $\{7, 6\}$, respectively.

General minimum lower-order confounding criteria design and minimum aberration may choose the same optimal foldover plans. For example, for 9.5.1, the minimum aberration and general minimum lower-order confounding criteria choose the same foldover plan as optimal which is 589. The word length pattern of the combined optimal design is $\{0^3, 6, 8, 0^2, 1, 0\}$ and aliased effect number pattern is $\{9, 0^9; 1, 0^9; 9, 0^9; 36, 0^9; 8, 24, 0, 4, 0^6\}$. However, according to the clear effect criterion that is not the only optimal design. According to the clear effect criterion the optimal foldover plans are 5 and 589 with number of clear effects of the combined optimal design being 98.

Table 4.11 below summarizes the number of optimal foldover plans by each criterion as well as the number of combined optimal foldover designs they have in common for the designs with more than one optimal foldover plans.

Table 4.11: Number of Common Optimal Designs by the Criteria for 16-Run Designs

	Number of OFP			Common OFP		
Design	GMLOC	MA	CE	GMLOC and MA	GMLOC and CE	MA and CE
6.2.1	3	3	3	3	3	3
7.3.1	7	7	7	7	7	7
8.4.1	8	7	8	0	8	0
9.5.1	1	1	2	1	1	1

The second, third and fourth columns of Table 4.11 represent the number of foldover plans chosen as optimal by general minimum lower-order confounding, minimum aberration and clear effects criterion, respectively. The fifth, sixth and seventh columns represent the number of common optimal foldover plans between the criteria. For design 9.5.1, the clear effect criterion choose two different plans and one of them is the same as that of minimum aberration and general minimum lower-order confounding criteria. For design 8.4.1, minimum aberration choose a totally different foldover plans from the clear effects and general minimum lower-order confounding criteria.

4.4.2 32 Runs

For 32 runs, most of the designs have more than one foldover plan as the optimum for all the three criteria (see Tables 4.8, 4.9 and 4.10). More than half of the designs (7 out of 13) have the same foldover plans as optimum according to all the criteria. Those are designs 7.2.1, 8.3.1, 9.4.1, 9.4.2, 10.5.1, 11.6.2 and 14.9.1. Table 4.12 below compares the numbers of optimal foldover plans obtained by the general minimum lower-order confounding criterion to the other ones chosen by criteria.

Table 4.12: Number of Common Designs by the Criteria for 32-Run Designs

	Number of OFP			Common OFP		
Design	GMLOC	MA	CE	GMLOC and MA	GMLOC and CE	MA and CE
7.2.1	2	2	2	2	2	2
8.3.1	6	6	6	6	6	6
9.4.1	7	7	14	7	7	7
9.4.2	6	6	6	6	6	6
9.4.3	8	7	8	0	8	0
10.5.1	20	20	20	20	20	20
11.6.1	24	12	24	0	24	0
11.6.2	12	12	38	12	12	12
12.7.1	12	6	12	0	12	0
12.7.2	4	24	12	0	4	0
13.8.1	1	12	13	0	1	0
14.9.1	14	14	14	14	14	14
15.10.1	15	28	15	0	15	0

The first four columns represent the design, number of optimal foldover plans by general minimum lower-order confounding, number of optimal foldover plans chosen by minimum aberration and number of optimal foldover plans chosen by clear effects criterion, respectively. The last three columns represent the number of common optimal foldover plans between the three criteria.

Interestingly, similar to 16-run designs, we again see that number of optimal designs chosen by clear effects criterion is greater than or equal to the number chosen as optimal by the general minimum lower-order confounding criterion. This can be seen in Table 4.12. Hence the second column of Table 4.12 is the same as its fifth column. This is due to the fact that, aliased effect number pattern looks further the higher order interaction (more than two) effects, whereas clear effects criterion mostly considers the clear main effects and clear two-factor interactions.

Furthermore, the optimal foldover designs obtained by minimum aberration in some of the designs are different from that of the general minimum lower-order confounding and clear effects. Those design have their entries in columns six and seven of Table 4.12 being zero. For instance, for design 9.4.3, the optimal foldover plans according to general minimum lower-order confounding and clear effects are 6, 7, 8, 9678, 679, 689 and 789, with its aliased effect number pattern and the number of clear effects of optimal combined foldover design being $\{9, 0^9; 36, 0^9; 15, 0, 21, 0^7\}$ and $\{9, 15\}$, respectively. For minimum aberration, the optimal foldover plans are 67, 68, 69, 78, 79, 89, 6789 and the word length pattern of the combined foldover design is $\{0^3, 6, 0^3, 1, 0\}$. We see that there is no common optimal foldover plans between general minimum lower-order confounding and minimum aberration criteria as well as clear effects and minimum aberration criteria.

In the end, for most of the designs, the optimal foldover plans obtained by general minimum lower-order confounding criterion have less than or equal number of optimal foldover plans compared to minimum aberration criterion, except for designs 12.7.1, 11.6.1 and 9.4.3.

One thing that one may be surprised about is the fact that for the 32-run designs we had many foldover plans as optimal irrespective of the criteria. That makes sense because since we have a large number of foldover plans considered and the number chosen is only a small percentage of those. For example, for design 15.10.1 the total number of optimal foldover plans considered is $2^{10} - 1 = 1023$ and only 15 (1.46%) are optimal according to the general minimum lower-order confounding and clear effects criteria, and 28 (2.74%) being optimal according to minimum aberration.

4.5 Comparing Optimal Combined Foldover Designs of GMLOC and MA

We have seen that the general minimum lower-order confounding criterion may choose totally different optimal designs from those of the minimum aberration. We compare the aliased effect number pattern of the optimal combined foldover designs obtained from the general minimum lower-order confounding criterion and those from minimum aberration criterion. From Tables 4.11 and 4.12, we see that, for designs 8.4.1, 9.4.3, 11.6.1 as well as 13.8.1, the general minimum lower-order confounding criterion and the minimum aberration criteria give different foldover plans as optimal. Tables 4.13, 4.14, 4.15 and 4.16 show the aliased effect number pattern of these designs.

Table 4.13: AENP of Optimal Combined Foldover Designs by GMLOC and MA for Design 8.4.1

i,j	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (GMLOC)	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (MA)
1,1	8, 0, 0, 0, 0, 0, 0, 0, 0	8, 0, 0, 0, 0, 0, 0, 0, 0
0,2	1, 0, 0, 0, 0, 0, 0, 0, 0	1, 0, 0, 0, 0, 0, 0, 0, 0
1,2	8, 0, 0, 0, 0, 0, 0, 0, 0	8, 0, 0, 0, 0, 0, 0, 0, 0
2,1	28, 0, 0, 0, 0, 0, 0, 0, 0	28 0, 0, 0, 0, 0, 0, 0, 0
2,2	7, 0, 21, 0, 0, 0, 0, 0, 0	0, 24, 0, 4, 0, 0, 0, 0, 0

Table 4.14: AENP of Optimal Combined Foldover Designs by GMLOC and MA for Design 9.4.3

i,j	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (GMLOC)	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (MA)
1,1	9, 0, 0, 0, 0, 0, 0, 0, 0, 0	9, 0, 0, 0, 0, 0, 0, 0, 0, 0
0,2	1, 0, 0, 0, 0, 0, 0, 0, 0, 0	1, 0, 0, 0, 0, 0, 0, 0, 0, 0
1,2	9, 0, 0, 0, 0, 0, 0, 0, 0, 0	9, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,1	36, 0, 0, 0, 0, 0, 0, 0, 0, 0	36, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,2	15, 0, 21, 0, 0, 0, 0, 0, 0, 0	8, 24, 0, 4, 0, 0, 0, 0, 0, 0

Table 4.15: AENP of Optimal Combined Foldover Designs by GMLOC and MA for Design 11.6.1

i,j	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (GMLOC)	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (MA)
1,1	11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
0,2	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
1,2	11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,1	55, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	55, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,2	12, 18, 21, 4, 0, 0, 0, 0, 0, 0, 0, 0	9, 36, 6, 4, 0, 0, 0, 0, 0, 0, 0, 0

Table 4.16: AENP of Optimal Combined Foldover Designs by GMLOC and MA for Design 13.8.1

i,j	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (GMLOC)	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design (MA)
1,1	13, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	13, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
0,2	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
1,2	13, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	13, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,1	78, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	78, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,2	12, 0, 0, 48, 0, 18, 0, 0, 0, 0, 0, 0, 0, 0	0, 30, 36, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0

The first columns of Tables 4.13, 4.14, 4.15 and 4.16 represent ${}_{i}^{\#}C_j$ for $(i = 1, j = 1)$, $(i = 0, j = 2)$, $(i = 1, j = 2)$, $(i = 2, j = 1)$ and $(i = 2, j = 2)$. The second columns represents the ${}_{i}^{\#}C_j$ of optimal combined foldover designs obtained from the general minimum lower-order confounding criterion while third columns represents the ${}_{i}^{\#}C_j$ of optimal combined foldover designs obtained from the minimum aberration criterion.

From Table 4.13-4.16, we see that all the ${}_{i}^{\#}C_j$'s are the same for both the general minimum lower-order confounding and minimum aberration except for the last row which represents ${}_{2}^{\#}C_2$. Again, from Table 4.13, it can be seen that, for design 8.4.1, the number of two-factor interactions aliased with other two-factor interactions at degree zero, which is ${}_{2}^{\#}C_2^{(0)}$ is 7 and 0 for the general minimum lower-order confounding and minimum aberration criteria, respectively. This implies that there are 7 clear two-factor interactions in the case of the general minimum lower-order confounding while in the case of minimum aberration there is no clear two-factor interaction.

Similarly, in Table 4.14 we see that, for design 9.4.3, ${}_{2}^{\#}C_2^{(0)}$ is 15 for the general minimum lower-order confounding while it is 8 for the minimum aberration criterion. This means that there are 15 clear two-factor interactions in the case of the general minimum lower-order confounding criterion and 8 clear two-factor interaction in the case of minimum aberration.

For design 11.6.1, ${}_{2}^{\#}C_2^{(0)}$ is 12 and 9 for the general minimum lower-order confounding and minimum aberration, respectively, as seen in Table 4.15. Therefore we can say that, there are 12 clear two-factor interactions for general minimum lower-order confounding criterion and 9 clear two-factor interactions for minimum aberration.

Lastly for design 13.8.1, as in Table 4.16, there are 12 clear two-factor interactions for the general minimum lower-order confounding criterio while there are no clear two-factor interactions for the minimum aberration case.

Since our interest is to make as many lower order effects as clear as possible, the designs with greater number clear main effects and two-factor interactions will be preferred. Hence the optimal designs obtain from general minimum lower-order confounding criterion are obviously relatively better that those obtained from the minimum aberration criterion.

4.6 Comparing Optimal Combined Foldover Designs of GMLOC and CE

As stated earlier, for a specific initial design, the optimal foldover designs obtained by the general minimum lower-order confounding criterion are either the same or a subset of those obtained by the clear effect criterion. For this reason, obviously the common optimal foldover designs will have the same aliased effect number pattern with the remaining (uncommon) designs having different aliased effect number pattern from the common ones. For instance, from design 9.5.1, the general minimum lower-order confounding criterion choose 589 as the optimal foldover plan whereas the clear effect criterion chose 5 and 589 as the optimal foldover plans. For simplicity, we denote these foldover plans as F1 and F2, respectively. Table 4.17 shows the ${}^{\#}_{i}C_{j}$ of the two foldover designs.

Table 4.17: AENP of Optimal Foldover Designs F1 and F2 obtained from Design 9.5.1

i,j	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design F1	${}_{i}^{\#}C_j$ of Optimal Combined Foldover Design F2
1,1	9, 0, 0, 0, 0, 0, 0, 0, 0, 0	9, 0, 0, 0, 0, 0, 0, 0, 0, 0
0,2	1, 0, 0, 0, 0, 0, 0, 0, 0, 0	1, 0, 0, 0, 0, 0, 0, 0, 0, 0
1,2	9, 0, 0, 0, 0, 0, 0, 0, 0, 0	9, 0, 0, 0, 0, 0, 0, 0, 0, 0
2,1	36, 0, 0, 0, 0, 0, 0, 0, 0, 0	36 0, 0, 0, 0, 0, 0, 0, 0, 0
2,2	8, 0, 0, 28, 0, 0, 0, 0, 0, 0	8, 24, 0, 4, 0, 0, 0, 0, 0, 0

We see from Table 4.17 that, the ${}_{i}^{\#}C_j$'s are the same for both foldover designs except ${}_{2}^{\#}C_2$. According to the clear effect criterion both designs are equivalent in terms of optimality since ${}_{1}^{\#}C_2^{(0)} = 9$ and ${}_{2}^{\#}C_2^{(0)} -{}_{1}^{\#} C_2^{(0)} = 8$, which represents the number of clear main effect and clear two-factor interactions, respectively, are the same for both designs.

However, according to the general minimum lower-order confounding criterion, foldover design 589 is the only optimal design. The latter is able to differentiate foldover designs 5 and 589 in terms of optimality. Even though both designs have the same number of clear main effects and two-factor interactions, foldover design 589 have 24 two-factor interactions aliased with only 1 two-factor interactions whiles foldover design 5 have 28 two-factor interactions aliased with 3 two-factor interactions. The confounding in latter is more severe than in the former, thereby making it less preferable.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this research, our objective is to find optimal foldover designs using the general minimum lower-order confounding criterion. We began by introducing the concept and importance of design of experiments. We then discussed factorial and fractional factorial designs and their drawbacks. We went further to introduce foldover designs as strategy to follow up our initial experiment. Furthermore, we introduced different optimality criteria. Some of the existing related work in literature was also introduced. Then, we constructed optimal foldover plans using the general minimum lower-order confounding criterion. In this chapter, we present a summary of our work and some possible future research.

We have found that, when the combine foldover designs are obtained by reversing the signs of all the factors, and the sum of the numbers of factors in two effects is odd, then there is no confounding between the two effects, that is, when $i + j$ is odd, ${}_{i}^{\#}C_{j}$ is zero for all k except for k is zero. We have also shown that, ${}_{i}^{\#}C_{j}^{(0)}$ of a combined foldover design is greater than or equal to that of their respective initial design.

We have studied and written the algorithm for searching optimal foldover designs using the general minimum lower-order confounding criterion. A catalogue of 16- and 32-run designs was created and presented in tabular forms.

For comparison sake, optimal foldover designs were also found using minimum aberration and clear effect criteria. It was noted that, for a specific design, there are instances where the general minimum lower-order confounding criterion and minimum aberration criterion choose completely different foldover plans as optimal. In this case, we found that, the optimal designs based on general minimum lower-order confounding have more clear lower order effects than those from minimum aberration. This makes the general minimum lower-order confounding criterion better and more desirable than the minimum aberration. We also observed that when the word length pattern of two combined foldover designs from the same initial design are the same, then they have the same aliased effect number pattern.

On the other hand, we have found that, optimal foldover designs obtained by the general minimum lower-order confounding are either the same or a subset of those chosen by the clear effects criterion. This confirms the fact that clear effect criterion is just a portion of the general minimum lower-order confounding criterion. The general minimum lower-order confounding criterion therefore further differentiate designs chosen by the clear effect criterion and that makes it a purified criteria relative to the clear effect and minimum aberration criteria.

5.2 Future Work

The immediate future work is to investigate whether some of the observations made can be generalized to all cases if not find the conditions in which they hold. We will also like to proof them mathematically if possible.

In future we will like to extend this research to non-regular designs. We will like to construct optimal foldover designs with the general minimum lower-order confounding criterion for non-regular designs.

We will also like to find optimal foldover plans for two-level fractional factorial split-plot designs using general minimum lower-order confounding criterion. Also, we like to extend this to semi-foldover designs.

Appendices

Table 1: Design Matrix from Chen et al. (1993)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
2	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
3	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

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