Developing Conceptual Understanding and Probabilistic Thinking through Tree Diagrams

by Lam Ngoc Nguyen

A Thesis submitted to the Faculty of Graduate Studies of

The University of Manitoba

in partial fulfilment of the requirements of the degree of

MASTER OF EDUCATION

Department of Curriculum Teaching and Learning

University of Manitoba

Winnipeg, Manitoba

Acknowledgements

This thesis would not have been possible without the love, patience, and encouragement from my wife and three incredible daughters. They have been at my side to accompany me on this long journey of learning—for that, I am forever grateful. To my mom and dad whose unwavering support has provided me with all of the motivation I needed to continue towards this goal.

I have benefited immeasurably from the guidance and inspiration of Dr. Ralph Mason. It has been an honour to call myself his student. It is remarkable that he has instilled in me such a passion for understanding how people learn math that we can still be excited about a project that has been in progress for three years.

Thank you to my committee members Dr. Ralph Mason, Dr. Francine Morin, Dr. Paul Betts, and Dr. Reynold Redekopp. Their knowledge, expertise, and dedication have been valuable throughout the process of this project.

Abstract

The nature of probability and uncertainty is complex. Rather than teachers breaking down probability concepts into separate parts, learners can benefit from experiences that engage them in navigating that complexity. This research project explores the experiences of a group of students as they learned the unit on probability in the Grade 12 Applied Mathematics course. I used a practitioner research stance to position myself as both the teacher and researcher. This action-based research enabled me to interpret my observations of the learning as it was happening.

The teaching and learning of the unit centred on the tree diagram—a visual representation of probability experiments. The tree diagram can be a useful learning tool in facilitating rich thinking in the classroom. They not only assist learners in moving towards a conceptual understanding of probability, but are useful as tools in solving problems as well.

Table of Contents

List of Tables

List of Figures

Chapter 1 – Introduction

In this opening chapter, I provide the background of how this project started as well as factors that informed its design. The values that are important to me as a teacher give the context of how the idea for this project started. At its core, this project is about noticing how students learn mathematics. My experiences as a math learner throughout high school and my post-secondary education offer insight into what it means to learn math. A realization of how complex learning math can be comes from my perspectives as both a learner and a teacher. A desire to further understand this complexity led to my research questions and provided the foundation on which the research was designed.

1.1 Opening Comments

This research project was the culmination of the thinking, reading, and conversations I have had about mathematics education throughout my relatively short career as an educator. As an educator, there are many different aspects of education that are important to me on a professional level as well as a personal one. I care about success for all learners. I care about using technology to enable high level learning. I care to disrupt the perception that it is 'OK' to not be a mathematical thinker and that only a particular type of person can do well in mathematics. I care that people see mathematics as more than just numbers, rules, and algorithms.

Although the list of things I value could go on, many of these values return to the same consistent theme; that the inherently interesting nature of mathematics should be apparent in the classroom. For many students, it is not. How this could be accomplished has been the topic of many discussions that I have been involved in over the years. Since my voice is quite present throughout this paper, it is important to acknowledge the basis for which my beliefs were formed.

1.2 My Experience as a Math Learner

I am from a visible minority, originally born in Vietnam and immigrated to Canada as a young child. I grew up in a low social-economic area of Winnipeg and attended elementary and middle school near my home. Although I lived outside of the catchment area for Kelvin High School, I was accepted to the International Baccalaureate program and attended the school for that reason.

During my first few years of high school, the subject that I excelled at and enjoyed the most was mathematics. To many, the idea of being successful at doing math involves efficient recall of mathematical theorems, rules, and algorithms with the end goal of calculating a correct numerical value. In order to excel beyond just calculating, students are expected to apply the learned and practiced rules and algorithms to various situations. I was no different in this regard; in grade 9, I had no problems with the "rise over run" rule. I clearly defined my (x_1, y_1) and (x_2, y_2) coordinates and applied them to any number of formulas successfully to find midpoints, slopes, and distances between points. Since the notation was natural for me to understand, I continued doing well in grades 10 and 11 with more advanced mathematics such as the quadratic formula. I did so well that I insisted to the IB coordinator that the school offer a higher level option for the grade 12 math course.

In this class, I met two very interesting students: Xi and David. While both students were bright and both did well in school, they were drastically different from each other. Xi was organized and meticulous with his schoolwork; he completed all of his calculations with care, was exceptional at memorizing formulas and applying them appropriately, and was very fast at doing all of this. On the other hand, David took his time with problems, mulled them over for a while and usually came up with unique solutions. He was disorganized and almost never completed any schoolwork on time. While he never bothered to memorize any formulas, he was always confident in his understanding of the concepts being learned.

While I looked up to Xi as a much more capable version of myself as a learner, my encounter with David was an enlightening moment with respect to what doing well at mathematics could mean.

At the same time, I was having troubles of my own as I found the course challenging. Trying to remember when to use which rule or formula became overwhelming for topics such as trigonometric identities, differentiation, and integration. Seeing the limitations of the way I was learning mathematics and at the same time seeing David do well in such a different way prompted me to shift my focus in the classroom. I spent less time trying to memorize and more time participating in discussions. The process made learning less frustrating, and actually enjoyable. I went on to pursue a Bachelor of Mathematics at the University of Waterloo where I found even more interesting areas of mathematics; particularly graph theory, combinatorics, and optimization.

There are many students who run into the same problem that I did. For the majority of my secondary school math, I used an approach to learning that relied heavily on remembering algorithms and rules. However, at different points in time, this approach loses its value for several reasons: retention of facts and algorithms is difficult when not practiced and the context of mathematical thinking rarely presents itself as the neat and simplified examples that are used in the classroom. This roadblock happened for me during the higher level class in grade 12, for others it can happen sooner.

I believe that changing a classroom's understanding of what it means to do well in mathematics can enable more *Davids* to do well in school mathematics, leading to more students having positive identities as mathematics learners. By not presenting an experience that is accessible to all learners, the approach to teaching mathematics by calculating with the use of algorithms effectively divides learners into those who can achieve success and those who are not only unsuccessful, but whose identity as a math learner is impacted negatively. Even for those learners who are feeling successful, their success is founded in an ability to follow steps and apply rules, perhaps even solve some problems (but none that stray too far from the ones presented in class). There is a missing component that greatly limits the usefulness of the learning that is happening in this environment: conceptual understanding. Learning needs to occur not only in the details of the calculation, but within context of not only the bigger picture of the concept being learned, but its relation to mathematical thinking and other fields of study.

1.3 My Post-Secondary Education and Early Teaching Experiences

After working for a few years as a software developer, an affinity for working with children and being in a learning environment prompted my decision to return to school to pursue a Bachelor of Education. While it is natural for me to now reflect on my high school learning, I would not be able to do so if not for my experiences in the education program and as a high school teacher. While I was living the experience of learning university mathematics, I was not aware that it was my attempts to understand concepts rather than remember algorithms that allowed me to do well and enjoy learning math. Similarly, while I was living through the education program, I had not yet spent time thinking about what learning math involved nor did I have experience causing that learning to happen.

Upon graduation, I was hired into a full-time permanent position by the Winnipeg School Division and accepted a position at Kelvin High School as the applied mathematics teacher. Returning to the same school in which I was a student only eight years prior meant that many of my teachers were still there and became my colleagues. Like many beginning teachers, I made the most out of the resources available to me. I read through curriculum documents as well as resources, assignments, course notes, and tests left behind by the previous applied teacher. I relied primarily on my organization and planning. Planning however had a very different meaning during my first year than it does for me now. Since I had no experience with students to build on, planning involved thinking about teaching techniques and content without much thought to the learning that would be happening. My lesson plans were detailed to the minute with examples fully worked out and steps for algorithms clearly written to later be transferred to the students.

My actual experiences in the classroom quickly changed the way I prepared for teaching. There were several key problems that I encountered:

- *Students did not enjoy learning mathematics.* They came into my class with a preconceived notion that they would not enjoy it and taking my course did not do anything to change this perception.
- *The small parts did not always add up to the big picture.* By breaking down each unit into individual lessons and assigning each lesson with one new concept, I mistakenly believed that I was organizing the content in a way that helped students eventually piece all of the parts together.
- *Students did not remember anything from class over the long term.* No matter how proficient I observed a student to be at a particular task or algorithm, without enough review before a test or exam, that ability would be lost.

These three problems raised several red flags for me, most notably that there must be something that I could be doing differently to prevent at least some of the issues. These concerns led to questions which eventually led me to apply to the Master of Education program at the University of Manitoba under the Curriculum, Teaching, and Learning department.

The Master of Education program has been a transformative experience in my life. I have particularly enjoyed the dialogue between colleagues and professors, and I recall many key conversations which have shaped my thinking. I was also introduced to journals and articles that opened up a world of research for me that turned out to be a valuable resource. I became better equipped to address the concerns that had originally arisen in my classroom. Rather than spend time planning what my teaching would look like, I spent time thinking about the students' learning experiences. I saw the value of learning for conceptual understanding as opposed to algorithms. I noticed that while it was often a more challenging experience for both me and the student initially, students were more engaged, enjoyed learning math more, and did not need to rely on reviewing before a test to remember important concepts.

1.4 The Study

Seeing the importance of learning mathematics for conceptual understanding, I designed this research project to explore learning tools that move the learner from calculating with algorithms towards conceptual understanding and the bigger picture. While learning tools include constructs that are used by students to calculate a correct answer, the term as it is used in this document builds on that notion. Learning tools must also aid a student's learning by being used in the development of that student's understanding. Learning tools are embedded into the problem solving process and require learners to do considerable thinking and organizing while using them (Johnson, 1983). This project focused on the learning of probability. The learning tool that was used was the tree diagram. As a learning tool, tree diagrams are effective at providing learners with a construct to calculate correct probabilities while also requiring thinking towards an understanding of the concepts and big ideas in probability.

Areas of the research project that were explored focused on the students' experiences as they

- used the tree diagram as a tool for counting and computing probabilities;
- developed a conceptual understanding of probability through the use of the tree diagram;
- used the tree diagram as a platform to express their understanding of concepts in probability and awareness of their own learning; and
- developed their perception of learning mathematics in general for conceptual understanding.

In order to notice the qualities of the students' experiences as they learned and to understand how they learned, I used a qualitative approach to guide the data collection and analysis. However, a blend of several qualitative research methodologies was required to refine the process and to guide my decision making throughout the research. Primarily, a practitioner research stance was used to position myself as both the teacher and researcher in the process of providing an intervention to a group of participants while observing their experiences. A phenomenological approach to the research was used to understand the essence of thinking about probability in order to better understand how to develop that thinking. Elements of a narrative inquiry were used in the data collection process. Interactive writing, field notes, the creation of narrative texts, and closure interviews were all used to develop an understanding of the students' experiences as they learned math differently. The research was guided by the following questions:

1. How can a mathematical tool such as a *tree diagram* assist in moving a student's mathematical learning from computing algorithms to conceptual understanding?
2. Further, what role can the tree diagram have in enabling students to *show* their learning and conceptual understanding?
3. How does an emphasis on conceptual understanding as opposed to a more instrumental approach impact student perception of what it means to understand mathematics?

This document is structured to first examine pedagogical issues that are related to a disconnect between how mathematical content is structured and the way it is learned. I identify the topic of probability as particularly problematic for learners. In chapter two, I explore how learning probability for conceptual understanding through problem-based learning and with the support of a visual learning tool, the tree diagram, can address this disconnect. The use of tree diagrams has potential as a potent strategy for thinking and learning about the topic. In chapter three, I outline and provide support for the blended use of several research methodologies that informed the research design, data analysis, and interpretation of this study. In chapter four, I detail the two week implementation of the action strategy. I include an exploration of the potential value of each of the data types collected while also reflecting on my practice. Chapter five is structured to tell the stories of the purposefully sampled participants through narrative texts constructed from the data. The narrative texts along with the closure interviews provide the basis upon which the data were analyzed. In chapter six, I explore the themes that emerged from the analysis and how they inform the research questions. In chapter seven, I explore an additional theme that emerged later in the research project. The theme relates the professional learning that I experienced while writing this thesis to my position as the teacher-researcher in the study. This role created opportunities for intense listening and noticing of student learning which led to noticing of my own professional learning.

Chapter 2 – The Pedagogical Intervention

Thinking probabilistically accesses the complex mathematical structures of the topic of probability. In this chapter, I argue that when probability is learned as a set of algorithms and procedures, learners are unlikely to gain insight into those structures. I examine the pedagogical issues that arise from a disconnect between the way probability is structured and the way it is learned. Through a problem-based approach, rich learning tasks involving tree diagrams are explored as a way to enable learners to make sense of probability concepts in order to solve problems. This strategy can be potent in fostering the development of probabilistic thinking.

A key characteristic of rich learning tasks is that they allow students to treat situations as problematic, that is to approach them as something to think about rather than a procedure to follow (Hiebert ,1997). Using tree diagrams as a trigger for such tasks includes a process in which students make decisions about the construction, organization, and function of the tree diagram as it relates to the situation they are representing. The term “tree diagram” as it used in this document is constructed with

- a single root node;
- a level of branches for each part of a multiple-part experiment (tossing a coin 5 times will be represented by a 5-level tree);
- multiple branches on each level to represent the number of outcomes for that level;
- the probability of taking a path along a branch written on that branch; and
- the leaf nodes at the end of the path that represent each of the possible outcomes in the sample space.

Figure 2.1 shows a simple tree diagram that represents an experiment where two objects are picked from a bag containing three red balls and three blue squares. A shape is picked from the bag and not put back; a second shape is then picked from the bag.

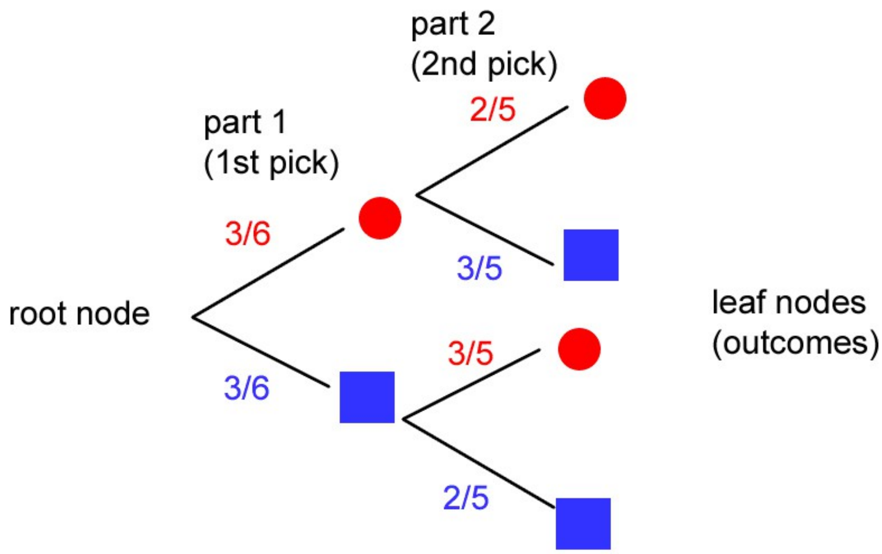


Figure 2.1. A simple tree diagram.

Tree diagrams consist of several components including its construction, labels, probabilities, and leaf nodes. A key goal of this chapter is to explain the way that these components of a tree diagram act as a visual platform for thinking probabilistically.

2.1 Thinking Probabilistically

What does it mean to think probabilistically? Many students do not have as much experience with thinking about uncertainty as they do with data. Before developing these experiences, it is important to first understand why experimentation is ineffective at assisting learners in thinking about and making predictions in situations with uncertainty. Instead, learners should think about these situations by considering probability theory and the mathematical structures of uncertainty. This approach is called probabilistic thinking and can lead to the experiential foundations for developing intuition and a conceptual understanding of probability.

Using experimentation and data to make predictions. We live in a society heavily driven by data. Data is collected, analyzed, and used to make predictions about anything from the economy to national crime rates. Whether in the classroom or out of it, our students have a lot of experience with data. Making measurements in a science class enables them to see the reliability of the scientific method. If a student were to drop a coin from a fixed height and measure the time it takes to fall to the ground, the time measured would be consistent and predictable. However, if the student were to toss that same coin and be asked to predict whether it would come up heads or tails, that student would not be able to say with certainty one answer or the other. This feeling of uncertainty is different from the reliability of experimentation. Even though the student may intuitively think that there is a 50% chance that the coin will come up heads, they still would not feel certain that one hundred tosses would yield exactly fifty heads; nor would that information help them predict the next toss.

The scientific method can be summarized in the following sequence: An observation is made about the world, a question is posed about the observation, a hypothesis is formed about the question, an experiment is conducted, and data is collected to determine if the hypothesis should be accepted or rejected. Students are provided with many opportunities to apply this way of thinking not only in school but outside of it as well. A young child might observe that potato chips taste good and that bubble gum tastes good. He asks himself whether the two foods would taste good when eaten together and makes a hypothesis that they will. However, when he conducts the experiment he realizes that his hypothesis was wrong and rejects it, most likely learning along the way not to haphazardly combine two foods in the future. The idea here is that from a young age, people learn to rely on and trust the information they get from experimentation.

A disconnect between experimentation and uncertainty. Consider applying the same way of thinking to a situation that involves uncertainty. If a student were to randomly pick an ice cream cone out of a box that contained *equal* amounts of three different flavours and she happens to pick chocolate;

would it be reasonable to conclude that there is 100% likelihood of choosing a chocolate cone? If she were to repeat the experiment for ten trials and found that she chose chocolate five times, vanilla three times, and strawberry twice, could she say with confidence that picking chocolate cones is more likely than the other two flavours? Certainly, knowing that there are equal amounts of each flavour in the box, the student's intuition would tell her that her experiment did not accurately portray the likelihood of choosing chocolate. For the first time, experimentation works against the student's intuition and becomes almost completely unreliable, especially with a sample size as small as ten tries. The notion of using the ratio of the number of times an event occurs to the total number of trials is called experimental probability.

For learners who have seen experimentation work so reliably in the past, it is counter-intuitive to see experimental probability work so poorly in helping them make predictions. The Manitoba Curriculum for Mathematics recognizes a disconnect between experimental probability and thinking about uncertainty.

> Events and experiments generate statistical data that can be used to make predictions. It is important that students recognize that these predictions are based upon patterns that have a degree of uncertainty…as students develop their understanding of probability; the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. (MECY, 2009, p. 15)

Once learners see that they cannot rely on experimentation to think about situations that have uncertainty, it is important to develop other ways to think about those situations.

Thinking probabilistically accesses the mathematical structures of probability. In distinguishing between experimental probability and thinking probabilistically, consider a possible framework of steps for thinking probabilistically:

1. Identify a desired outcome of a situation that involves uncertainty.
2. Use theoretical probability to determine the likelihood of that outcome.
3. Then make a decision with a level of confidence gauged directly from that likelihood.

As this framework makes apparent, the key difference between experimental and theoretical probability is that theoretical probability does not depend on an experiment to inform decision making. Instead, it is founded in the notion that the likelihood of an event *E* happening can be determined by the following probability formula:

P(E) = number of ways E can happen / number of all possible outcomes

where *P(E)* represents the likelihood of event *E* occurring. An experiment never actually needs to be carried out because it would not add value to what the experimenter already knows about the likelihood of the event in question.

It is through the ideas of theoretical probability that enable learners who are developing probabilistic thinking to access the mathematical structures of probability. These structures are based in foundational experiences in *counting*. Even as problems become increasingly intricate, the big idea of probability can be linked back to the formula stated above. At its essence, the formula is not a formula in the traditional sense of plugging in values to determine an answer but rather a guiding concept in recognizing that probability stems from an understanding of quantities and thus an ability to count. This is easy at first but many learners become lost when the numbers being counted become too large to visualize. Thus, building the learners experience in counting becomes an essential prerequisite to understanding probability. Fortunately, developing processes for counting systematically can concurrently work to develop the learner's ability to think probabilistically. Lockwood (2012) asserts that "students will become more confident and successful at counting if they develop systematic ways of listing possible results and can clearly tie the generation of those results to counting processes" (p.

134).

Counting. Consider the following question: In a classroom of twelve students, three are chosen to take the positions of President *(P)*, Vice-president *(VP)*, and Treasurer *(T)*. What are the chances that Annabelle, a student in the class, is chosen for any of the three positions?

Before an attempt to think about the probability of the desired event, which is Annabelle being chosen for any position, the learner should recognize the connection between this question and the counting of outcomes. Consider the formula stated earlier:

P(E) = number of ways E can happen / number of all possible outcomes

where *E* now represents: Annabelle being chosen for a position. The *number of ways E can happen* are all of the 3-student teams that include Annabelle and the *number of all possible outcomes* represent all possible 3-student teams that can be made from the class of twelve.

P(Annabelle selected) = 3-student teams with Annabelle / All 3-student teams

Lockwood (2012) establishes three specific ways that learners can develop an awareness of the learning that occurs when they are generating and organizing a list of outcomes: establish what is being counted, focus on a systematic listing, and connect different counting processes to multiple representations of the way outcomes are organized. These three processes all contribute to the understanding of probability.

Establish what is being counted. In this example, while thinking about what three-student teams to include in the possible outcomes, learners may find themselves contemplating whether they should count the following outcomes twice:

Annabelle (P) - Classmate 1 (VP) - Classmate 2 (T)

Classmate 1 (P) – Annabelle (VP) – Classmate 2 (T)

If Annabelle were asked whether being selected as President or Vice-president are two different scenarios, her answer would be an emphatic *yes*. Although both outcomes include the same three students, the order in which they are selected matters, this is known as a *permutation*. In a situation where the teacher is selecting three students to attend Math Camp over the summer, choosing

Annabelle – Classmate 1 – Classmate 2

Classmate 1 – Annabelle – Classmate 2

could actually be considered the same outcome as they both result in the same 3 students attending Math Camp; these situations are called *combinations*.

When learners are faced with these decisions on what possible outcomes to include or not to include, they are naturally guided towards questions of probability. When there are more outcomes to choose from in permutations, probabilities become increasingly unlikely.

Focus on a systematic listing. It would not take long for someone to realize that generating random 3-student teams would be an ineffective method for counting all possible outcomes. It would be particularly difficult to know when all of the outcomes have been covered. This problem can be tackled by creating a systematic listing, which is an organized list of all possible outcomes. A systematic list supports the claim that the list is complete and accurate. There are several different ways to do this, each highlighting different aspects of the outcomes.

If the counter wishes to organize the outcomes by position, he/she may consider the number of possibilities for filling the President position, and then for each of those possibilities, the number of options they would have to fill the Vice President position, and then finally the Treasurer position. This leads to a systematic listing in the form of a tree diagram where each level of the tree represents each position.

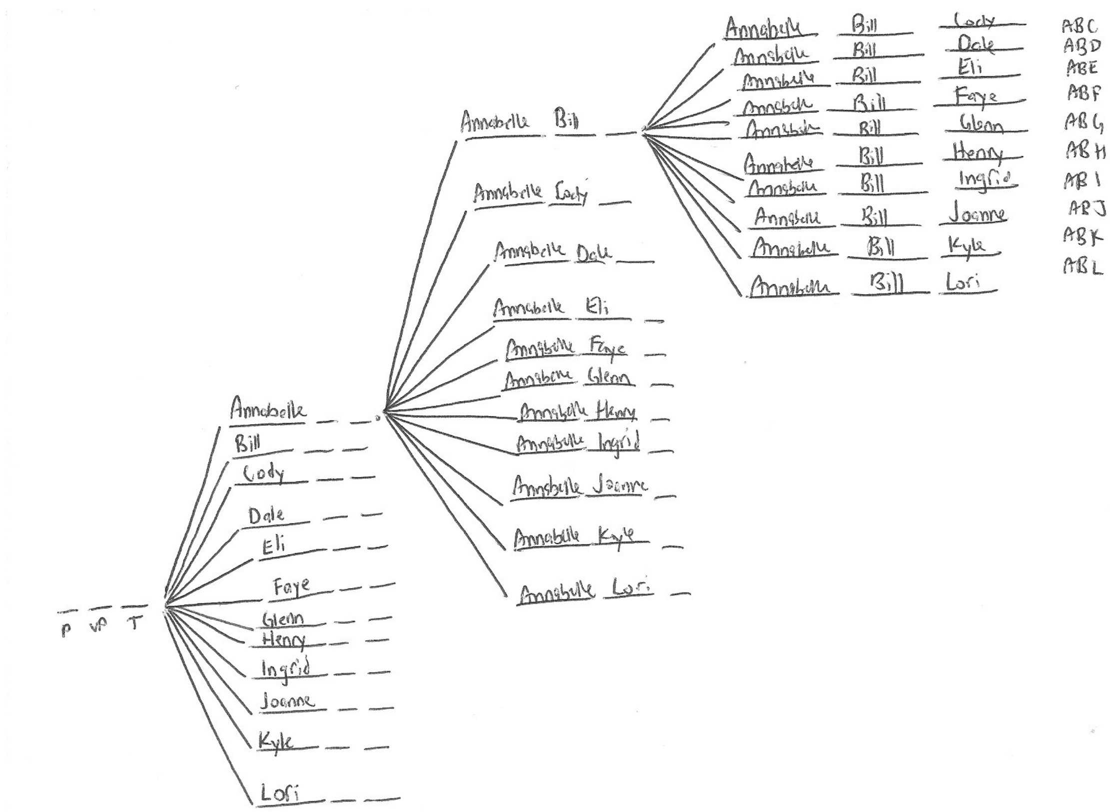


Figure 2.2. A tree diagram showing possible outcomes for a class election.

This figure shows a list of the possible outcomes that have Annabelle as President and Bill as Vice-President. Unlike making an unorganized list of outcomes, the tree diagram gives the student a way of identifying whether all of the outcomes have been covered. Although it would be impractical to draw the rest of this tree, the exercise of drawing a section of it helps students visualize the number of total possible outcomes if the tree were complete. This collection of all outcomes in probability is referred to as the *sample space*. Trees are useful tools in determining sample spaces through organized counting. Since each level of the tree represents a position (president, vice president, and treasurer), then the last branch at the end of each path, called a *leaf node*, represents one outcome of the sample space. All of the leaf nodes as a collective represent the entire sample space. Listing the leaf nodes shows that the counting of the sample space can be organized in a way that accounts for all of the outcomes.

{ABC, ABD, ABE, ABF, ABG, ABH, ABI, ABJ, ABK, ABL, ACB, ACD, ACE, … }

The same method of using tree diagrams for organized counting can be applied to all counting and probability experiments.

Students who gain experience with a few situations like these may soon look for ways to get to the number of outcomes without having to draw the entire tree; this is where a formula such as the Fundamental Counting Principle (FCP) can be introduced. The FCP states that for any number of decisions to be made, if there are m, n, p choices per decision, then the total number of possible outcomes is determined by: $m * n * p$.

In this example, there are 12 possible Presidents, 11 possible Vice-presidents, and 10 possible Treasurers and a total of $12 * 11 * 10 = 1320$ possible three-student teams. However, what the FCP does not do for learners is help them think about different probabilities surrounding the situation, particularly if the situation were to be altered. Without having to learn or memorize a set of rules or formulas, students can use the representation of systematic listing to consider questions such as:

Restrictions on a position (Annabelle must be the President)

Restrictions on students (Annabelle cannot be the President)

Adding students (Michael is a transfer student)

Removing students (Joanne is not eligible for any position)

Students can quickly identify that the first change (restriction on position) has a far greater impact on the total number of outcomes than the second change (restriction on student) by seeing how large the section of the tree would be excluded by the restriction. They can then begin to think probabilistically about how a larger number of outcomes impacts the probability of particular outcomes occurring.

Multiple representations of outcomes. Using the tree diagram helped emphasize the importance of each position and how it affected the total number of outcomes. Using different systematic listings of the outcomes can highlight different aspects of the outcomes; this is where multiple representations play a crucial role in the understanding of probability.

Instead of emphasizing the position, the number of outcomes that include a particular group of students can be emphasized. Consider choosing a group of three students: Annabelle, Bill, and Cody.

Annabelle (P) / Bill (VP) / Cody (T)

Annabelle (P) / Cody (VP) / Bill (T)

Bill (P) / Annabelle (VP) / Cody (T)

Bill (P) / Cody (VP) / Annabelle (T)

Cody (P) / Annabelle (VP) / Bill (T)

Cody (P) / Bill (VP) / Annabelle (T)

Even within this small subset of the class, organized counting shows that there are six possible ways to choose a three-student team that would include Annabelle, Bill, and Cody. This counting process would lead naturally to different questions about how many different groups of three students can be chosen from a class of twelve. This method would also lead naturally into changing the situation from a permutation to a combination as the number of outcomes would be counted as one outcome when they were previously counted as six different outcomes.

2.2 Breaking down Probability

The big ideas of probability are founded in counting and the organization of outcomes. These ideas are then connected to the concept of ratios to determine the likelihood of a desired outcome. Developing probabilistic thinking requires an understanding of how all of these ideas are connected to each other. The ability to think about the connections of multiple concepts can be described as relational understanding (Skemp, 2006).

Relational understanding is based on recognizing connections between various methods and ideas. This understanding enables learners to make decisions on how changes to a problem, or new problems altogether differ from previous problems and what should be done to apply the appropriate methods. For this reason, a relational understanding of multiple probability concepts and how they are connected to each other is more adaptable to new tasks. In contrast, an instrumental understanding refers to the type of learning focused on calculations (Skemp, 2006). Students are presented with rules to use and steps to follow without attention given to the reasoning behind them.

The structures of probability resist an instrumental approach. Although defining relational understanding in contrast to instrumental understanding is not an argument for aiming for only relational understanding, breaking down the learning of probability into singular parts is potentially harmful to developing probabilistic thinking. It is problematic to assume that teaching disconnected rules and formulas to determine the right answer will enable learners to piece the components together at the end of the learning. This problem is magnified for learning probability because word problems in this unit resist fitting into a procedural frame. That is, learning towards an instrumental understanding of probability and how to use each of the separate rules and formulas is not adaptable to new problems.

This characteristic of the topic of probability differentiated the unit from other units in the Grade 12 Applied Mathematics course. Statistics for the provincial standards test for this course have shown over the years that students obtain consistently lower scores on the probability portion of the test when compared to the other seven units. This information was published by MECY as a part of the general comments on the Grade 12 Applied Mathematics Standards Test between 2008 and 2011. Other units in the course may be better fits for a procedural learning frame, at least within the context of a school setting. In units such as matrices, personal finance, and even periodic functions, the main concepts are broken up into subtopics in such a way that instrumental understanding is not disrupted.

Problems do not differ significantly from sample problems that the student has seen previously. The problems in these units are not likely to be unfamiliar to students.

In the Matrices Unit, for example, students learn how to use matrices to model transitions and networks where each problem makes it clear what model is to be used; transition questions will indicate the need for a transition matrix, and network questions will indicate the need for a network matrix. Rather than developing an understanding of the underlying principles of the topic, learning how to do each type of problem algorithmically will suffice. Dick & Childrey (2012) describe why this approach to learning matrices can be detrimental to the ability to expand learning to more advanced topics. They argue that while “students enjoy coming up with these algebraic statements and readily connect the mathematical notation to what is actually happening, when they switch to a matrix description of the transformations, this intuitive connection is often lost” (p. 624). Dick and Childrey propose that teachers approach the learning of matrix multiplication from a simpler perspective that connects to the learners’ intuition. The term 'simpler' here does not refer to an easier concept but rather a focus on a concept that is more visible to students. This approach leads to an understanding of the concept that enables the learner to make connections to more advanced concepts.

> Most important, students truly understand the connection between the algebraic and the matrix descriptions of the transformations. This understanding is fundamental for students not only as they go forward in their studies but also because these matrices are so ubiquitous. (Dick & Childrey, 2012, p. 626)

In a topic such as matrices, learning matrix multiplication as an algorithm enabled some students to perform the calculations but did little to enable them in connecting to the algebraic translations.

The Probability unit differs from other units because there is no ubiquitous classification of the types of problems, particularly with counting possible outcomes. Learning how to do probability problems through the use of algorithms is not only detrimental in making connections to more advanced concepts; it does not even do well in helping students find the correct answers. There is no set example that students can refer to and no template on how to arrive at the final answer. Probability problems are more difficult for students because they could not approach them by only thinking procedurally or by only thinking of examples they have seen. Instead, students required a conceptual understanding in order to apply the different probability formulas or counting techniques to a particular problem. The term conceptual understanding in this context refers to a “learner's understanding of the meaning of a mathematical concept. Knowing that multiplying two negative numbers will yield a positive result is not the same thing as understanding why it is true” (Willingham, 2010, p. 17). In probability, knowing that multiplying the probabilities of two events will yield the probability of both of those events occurring is different than understanding why it is true. This example illustrates how one concept in probability can be understood on multiple levels from an instrumental understanding of calculating the correct probability to a relational understanding of how and why to use an appropriate procedure. Even further, a learner can develop a conceptual understanding of how and why particular probability concepts are true. For the remainder of this document, the term *conceptual understanding* will be used as an umbrella term to describe the understanding of the meaning of a mathematical idea —including the notion of relational understanding.

To illustrate the role of conceptual understanding in learning probability, consider two common formulas for working with probability: the addition rule and the multiplication rule.

The addition rule. The addition rule states: for two events A and B, the probability of A *or* B occurring is the addition of $P(A)$ and $P(B)$.

$$P(A \text{ or } B) = P(A) + P(B)$$

For example, let A be the event of rolling a one on a six-sided die and let B be the event of rolling a two on a six-sided die. The question, "What is the probability of getting a one or a two when rolling a single die?" can be approached with the formula:

P(A) = P(getting a 1) = 1/6

P(B) = P(getting a 2) = 1/6

So, then according to the addition rule, the probability of rolling a 1 or a 2 would be:

P(A or B) = P(1 or 2) = P(getting a 1) + P(getting a 2) = 1/6 + 1/6 = 2/6

However, this version of the formula only works when considering two mutually exclusive events. That is, two events that cannot occur simultaneously. If the question is changed to "What is the probability of getting a one or an odd number when rolling a single die?" the formula no longer yields the correct probability.

P(A) = P(getting a 1) = 1/6

P(B) = P(getting an odd) = 3/6

According to our addition rule,

P(A or B) = P(1 or odd) = P(1) + P(odd) = 1/6 + 3/6 = 4/6

This is incorrect because while either a one or an odd number is acceptable as a successful outcome, the outcome of getting a one is incorrectly counted twice because the two events are not mutually exclusive (the one appears on its own as well as in the set of odd outcomes).

The version of the formula that should have been used is:

P(A or B) = P(A) + P(B) – P(A and B)

The probability of both events occurring at the same time is subtracted in order to eliminate the outcomes that have been counted twice.

P(1 or odd) = P(1) + P(odd) – P(1 and odd) = 1/6 + 3/6 – 1/6 = 3/6

Here, there is one out of the six possible outcomes that will be both one and odd simultaneously, thus 1/6 is subtracted from the addition of the probabilities of the two events.

Although this formula leads to the correct final probability, it leaves the learners with a few challenges. First, learners need to determine whether the events *A* and *B* are mutually exclusive or not. This will lead to a decision about which version of the formula to use.

Mutually exclusive: P(A or B) = P(A) + P(B)

Non mutually exclusive: P(A or B) = P(A) + P(B) – P(A and B)

A deeper conceptual understanding of what it means for two events to be mutually exclusive can lead to the conclusion that only the second formula is necessary. Even if the first formula for non-mutually exclusive formula was used for mutually exclusive events, *P(A and B)* would be equal to 0 by definition as there is no chance that *A* and *B* would both occur; thus, a value of 0 would be subtracted.

For non-mutually exclusive events, learners are then faced with a challenge of determining the value of *P(A and B)*. Determining the value of *P(A and B)* is unlike simply plugging in a value in more commonly used formulas. Rather, determining the value of *P(A and B)* can be very complex, sometimes even involving multiple cases.

However, before any of these challenges, a student must be able to decide when it is appropriate to use the addition rule. The simplest, context-free, advice for using the addition rule is when the key word 'or' is spotted in the problem. Following this advice can be limiting as the context of the use of the word 'or' should be taken into consideration; there are many situations where the word 'or' is used and the addition rule is not appropriate. Clement (2005) identifies the issues of approaching the learning of problem solving through the use of key words and formulas.

> Students view the problem solving process as taking a collection of numbers and finding operations to perform, which are based on the key words in the problem, not on understanding the context…as soon as problems involve unfriendly numbers or more than one operation, these strategies fail them. Students are left with no rule to use and no understanding of the situation,

because sense-making was never a part of the students' problem solving process. (p. 361)

Since probability problems did not fit into a procedural frame, approaching them through decoding would be ineffective.

Alternatively, students can prepare themselves to rely on the fundamental principle underlying the addition rule which is counting. Considering the probability of either event *A* or event *B* happening is really widening the acceptance of what is deemed as a successful outcome. This results in a higher probability as there are more outcomes being counted as successful. This kind of thinking can lead to a more conceptual understanding of when to use the addition rule.

The multiplication rule. The multiplication rule states: for two events *A* and *B*, the probability of both *A and B* occurring is the multiplication of *P(A)* and *P(B)*.

*P(A and B) = P(A)*P(B)*

For example, let *A* be the event of getting a heads on a single coin toss and *B* be the event of getting a heads on a second toss. Consider the question: "What is the probability of getting a heads on the first toss *and* a heads again on the second toss?" The following formula can be used:

P(Heads on first toss) = ½

P(Heads on second toss) = ½

So, according to the multiplication rule:

*P(Heads on first and Heads on second) = P(A)*P(B) = (1/2)*(1/2) = ¼*

Like the addition rule, the multiplication rule is straight forward to use when it is evident to the student when its use is appropriate. Students can feel success in using it to find the final probability of two or more events happening. However, it is also similar to the addition rule in its deficiencies; determining when to use the multiplication rule can be challenging without a conceptual understanding of why *P(A)* and *P(B)* are multiplied together. Using the multiplication rule for more complex situations can be confusing, particularly for events that are *dependent*.

When the probability of event *B* occurring is effected by the outcome of event *A* happening or not, event *B* is *dependent* on event *A*. Consider the following situation: you pick a marble from a bag that contains three red marbles and three blue marbles. You keep the first marble and then reach in the bag to pick a second marble, you then ask:

What is the probability of choosing a red marble on the second pick?

The probability of choosing a red marble for the second pick depends on what colour marble was chosen on the first pick. If the first marble was red, then there are 2/5 red marbles left for the second pick. However, if the first marble was blue, then there are 3/5 red marbles left for the second pick. Jumping directly to using the multiplication rule leads to the following:

*P(1st marble any colour and 2nd marble red) = P(1st marble any colour) * P(2nd marble red)*

This formula highlights that answering the question above requires determining the probability of getting a red marble on the second pick without any interest in what colour marble was picked first. However, it is not possible to fill in the values for the formula as it is; the value for *P(2nd marble red)* can either be 2/5 (first pick was red) or 3/5 (first pick was blue).

Instead, a new version of the multiplication rule ca be used:

*P(A and B) = P(A) * P(B | A)*

The "|" implies "such that" leading to *P(B | A)* representing the probability of event *B* occurring such that *A* has occurred first. Thus, it can be assumed that *A* has occurred and the value for *P(B)* can be filled in accordingly.

To answer the previous question regarding the probability of choosing a red marble on the second pick, the events can be reworded to get:

P(1st pick red or blue and 2nd pick red) =

*P(1st pick red or blue) * P(2nd pick red) =*

*P(1st pick red) * P(2nd pick red | 1st red) + P(1st pick blue) * P(2nd pick red | 1st blue) =*

(3/6)(2/5) + (3/6)(3/5) = 15/30 = ½

An instrumental understanding of the multiplication rule depends on a conceptual understanding of dependent events.

The point is that the type of questions that can be asked in probability is vast and requires conceptual understanding. If there is no template that can be applied to a problem, then the decision on how to proceed cannot revolve around which formula to use but instead must include thinking about the meaning of the problem and the different components in its context. This necessity for understanding is not made evident through the learning of the two formulas presented. In actuality, if the formulas are to be used in any meaningful way, it is most likely in combination with each other and other fundamental counting concepts. Even a slight deviation from the type of sample questions that students are accustomed to will leave them unsure of how to proceed unless supported by a conceptual understanding of probabilistic thinking.

Fundamental counting principle. The term 'basics' refers to the foundation or building blocks for understanding more complex ideas. For example, it is possible to think of the basics for multiplication as addition. Understanding the concept of grouping equal sets repeatedly (multiplication) requires an understanding of grouping two or more sets in the first place (addition). Note here that there is no reference to any of the numerous algorithms for addition or multiplication which serve only to arrive at a final answer and do not add to the understanding of the concepts. It is problematic to identify one algorithm for multiplication to serve as the basis for more complex multiplication applications. Consider the most common algorithm for multiplying two numbers.

1. Arrange the numbers vertically.

2. Start with the right-most digit of the bottom number and multiply to each of the digits in the top number (carrying over any numbers that extend to the next place value).
3. Shift the answer to each of the digits one place value to the left.
4. Repeat for all digits in the bottom number.
5. Add all of the results to get a final answer.

A person can become quite proficient at executing this algorithm without really understanding the concept of multiplication at all. Further, any complex applications of multiplication (percentages, taxes, exponential growth) could not be built conceptually from the proficiency of this algorithm. The example used to show the deficiencies of using a multiplication algorithm as the foundation for understanding multiplication was obvious. Yet, similar logic is often applied to other areas of mathematics with little thought given to the outcome; students are unable to cope with complex problems. In probability, they are unable to think probabilistically because they have never been provided those experiences.

If a conceptual understanding of addition would serve as a better basis for multiplication than the algorithm stated above, perhaps counting can serve as a better basis for probability and probabilistic thinking. This is where the FCP stated above can be useful as it can be used to determine the size of the sample space of an experiment without having to list or draw all of the outcomes themselves. For example, the number of outcomes for an experiment consisting of flipping a coin and tossing a die is:

*Number of outcomes for coin * number of outcomes for die =*

*2 * 6 = 12*

This example is trivial but the fundamental counting principle is applied to many situations (configuring letters, people, lottery numbers, and any other experiment that involves a set of outcomes). This variety of ways in which the FCP can be used requires an understanding of when and why to use it, not just how to fill values into it formulaically.

Authors of the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* (2000) outline the process for learning and thinking about different areas of mathematics from K-12. The NCTM (2000) identifies the importance of creating a foundation for learning probability on experiences with uncertainty and conceptual understanding of sample spaces and counting. “High school students should learn to identify mutually exclusive, joint, and conditional events by drawing on their knowledge of combinations, permutations, and counting to compute the probabilities associated with such events” (NCTM, 2000, p. 331). The foundation of probabilistic thinking is a conceptual understanding of counting possible outcomes and the fundamental counting principle.

In addition to building on knowledge of counting, there is a natural and integral progression in considering sample spaces to build to more advanced probability theory (NCTM, 2000). When a student develops an ability to use organized counting, they are more capable in thinking about not only the size of sample spaces but the way that the outcomes are organized assist them in thinking of which outcomes they are interested in. When a student develops an ability to think of sample spaces, they are able to think about different probability distributions and situations. In the NCTM documents, little attention is given to which formula is applicable to a given situation.

2.3 Teaching Probability Effectively

A disconnect between the mathematical structure of probability and the way it is learned is a pedagogical issue. This issue manifests in the standards test results but also in students' frustrations with probability word problems that resist being approached procedurally. The pedagogical intervention in this study aims to address that disconnect and provide students with a tool to make sense of probability concepts. Thinking about probability through the use of tree diagrams is a rich learning task that gives the learner insight into the structure of probability. Tasks of this nature support students in developing problem-solving strategies and allow for insights into the structure of mathematics (Munter, 2014). A problem-based approach to learning probability builds on the students' experiential foundations and intuition for thinking about uncertainty. Rather than rely on decoding word problems or only using previous examples to solve problems, students learn to think probabilistically by solving authentic and complex probability problems.

A problem-based approach to learning. The term “problem solving” is used in many different ways by the mathematics community at large. These definitions range from seeing problem solving as a skill to be taught in the classroom to being seen as an art form (Stanic & Kilpatrick, 1988). The perspective from which problem solving is viewed has implications for how it is integrated into the classroom. Stanic & Kilpatrick were critical of viewing problem solving as a hierarchy of skills that started from basic mathematical concepts and skills to routine problems and then finally to non-routine problems. Building on this view, Schoenfeld (1992) showed concern for how teaching problem solving as a skill looked in a classroom setting.

> Problem solving techniques (such as drawing diagrams, looking for patterns when n = 1,2,3,4,...) are taught as subject matter, with practice problems assigned so that the techniques can be mastered. After receiving this kind of problem-solving instruction (often a separate part of the curriculum), the students' “mathematical tool kit” is presumed to contain problem-solving skills as well as the facts and procedures they have studied. (p. 338)

These concerns parallel the pedagogical issues that arose from approaching probability problems through procedures or examples of similar problems.

In contrast to the limited view of problem solving as a skill, George Polya (1945) describes problem solving as a process that includes understanding the problem, planning a strategy for solving the problem, executing that strategy, and reviewing the solution. Here, reviewing does not imply a need to check if the solution is correct but rather is a reflection process during which the problem solver can think about what worked, what did not work, and how the thinking used can be extended to future problems. This interpretation of problem solving is in synchronization with both the NCTM standards and the Manitoba Education curriculum documents. The front end of the Manitoba Education curriculum framework of outcomes for grades 9 to 12 (MECY, 2009) include problem solving as one of the seven critical mathematical processes to learning, doing, and understanding mathematics. In this document, problem solving is described as a rich, complex, and authentic activity in which students are not focused on the final solution but on the path to arrive at one possible solution. The document authored by the MECY (2009) defines problem solving in part as “tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers” (p. 10). This view is synchronous with the description of problem-solving found in the NCTM standards, particularly with the notion of problems being non-routine. “Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings” (NCTM, 2000, p. 52).

While Polya's early work provided the foundation for the description of the problem solving process, more recent studies have shifted the focus to attributes of the successful problem solver. These studies have cited planning and monitoring as key discriminators in problem solving success (Schoenfeld, 1992). In this context, planning refers to the contemplation of various solution approaches before they have been implemented and monitoring refers to the mental actions involved in reflecting on the effectiveness of the problem solving. Carlson (2005) builds on this process by observing its cyclical nature. He noticed that once problem solvers “oriented themselves in the problem space, the plan–execute–check cycle was then repeated throughout the remainder of the solution process" (p. 12). Even within the planning phase of that cycle, effective problem solvers engaged in another conjecture–imagine–verify cycle. These attributes require high level thinking processes. An understanding of how to develop these attributes can assist educators in creating rich learning tasks that foster that development.

The cyclical nature of problem solving is not only limited to engaging in the problem solving process, but also applies to the learning of that process. In other words, in solving problems, students learn how to solve problems. This notion supports Clement's assertion about the limitations of problem solving through the decoding of key words. While her alternative approach of quantitative analysis is specific to the context of her study, at the core of the approach is a process that provides students with opportunities to make sense of mathematics. Having opportunities to think within the context of a math problem moves the learner away from survival adaptations like relying on learning from examples to prepare for a procedural test. With regard to developing probabilistic thinking, rich tasks that are designed to foster insights into the structure of probability start by building on the learners' experiences and intuition with uncertainty.

Building intuition about uncertainty. Our intuition comes from our experiences so if our experiences are lacking in variety and depth, then we encounter disconnections between intuition and the mathematical structure of a concept. This is the case with thinking about probability, our experiences both in and out of school do not provide a sufficient foundation for learning to become better at thinking probabilistically.

Our experiences with probability point to chance and uncertainty being haphazard. In an experiment involving uncertainty, it is typically not feasible to see enough repetitions to make sense of the order of results. It is this order in uncertainty that is difficult to observe (Moore, 1990) and which causes misconceptions regarding how uncertainty works. For example, when playing games of chance, making decisions based on exclamations of "I have lost many in a row, I am due for a win" or "beginners luck!" show intuition and in some cases emotion create a road block in thinking probabilistically. A person intuitively knows that there is a 50/50 chance that a flipped coin will come up heads so if a coin is flipped five times and comes up tails each time, then surely it is more likely that the next flip will yield a heads as with every subsequent flip that is not a heads will increase the chances. In the game of roulette, previous winning numbers are posted on a board for all players to view and base their future decisions. If the number thirteen has come up just two turns ago, surely it is less likely than any other number to come up again soon. The misconception here is that while it is unlikely to flip five tails in a row than a more random result, the five tails have no impact on the probability of the next individual toss; a law in probability that is counter-intuitive.

In order to learn to get better at thinking probabilistically, these misconceptions need to be addressed not through formal rules but rather by building from the learners' intuition. Moore (1990) asserts, "Students fail to understand probability because of misconceptions that are not removed by study of formal rules". The formal rules in this case being the addition rule, multiplication rule, and fundamental counting principle introduced previously. Although these are useful in manually determining the correct probability of some particular situations, using them requires a more conceptual understanding of probability and the ability to think probabilistically.

In the *Principles and Standards for School Mathematics* document published by the NCTM, the Pre-Kindergarten to Grade 2 section outlines the recommended initial exposure to learning probability. The NCTM (2000) states that "probability experiences should be informal and often take the form of answering questions about the likelihood of events, using such vocabulary as *more likely* or *less likely"* (p. 51). The answers to such questions usually depend on the context of the situation. The likelihood of a snowfall would be different in Winnipeg than in Los Angeles; as it would be different in Winnipeg in December than it would in July. Building a foundation of experiences with likelihood that includes intuition provides opportunities to discuss and think about the reasons for varying probabilities. When the calculation of exact probabilities is required at later grades, students will be better prepared to think probabilistically in addressing those problems.

The role of tree diagrams. I will now turn to introducing an approach to addressing the intuition and misconceptions surrounding probability: the use of the *tree diagram*. Rich tasks involving tree diagrams are designed by acknowledging a student's intuition as a starting point and then building on conceptual understanding from there. Aspinwall & Shaw (2000) elaborate this point.

> Tree diagrams proved to be enlightening visual tools for students, helping them to modify their thinking and intuitions about probability activities. In other words, they were no longer relying on a quick, intuitive reaction alone but were allowing their intuition to be influenced by more thorough analysis. (p. 219)

They are used to encourage and improve probabilistic thinking all while still acting as a tool to calculating a correct answer.

The following figure shows a tree diagram that represents the following experiment: A bag contains 3 red balls and 3 blue squares. A shape is picked from the bag and not put back; a second shape is then picked from the bag.

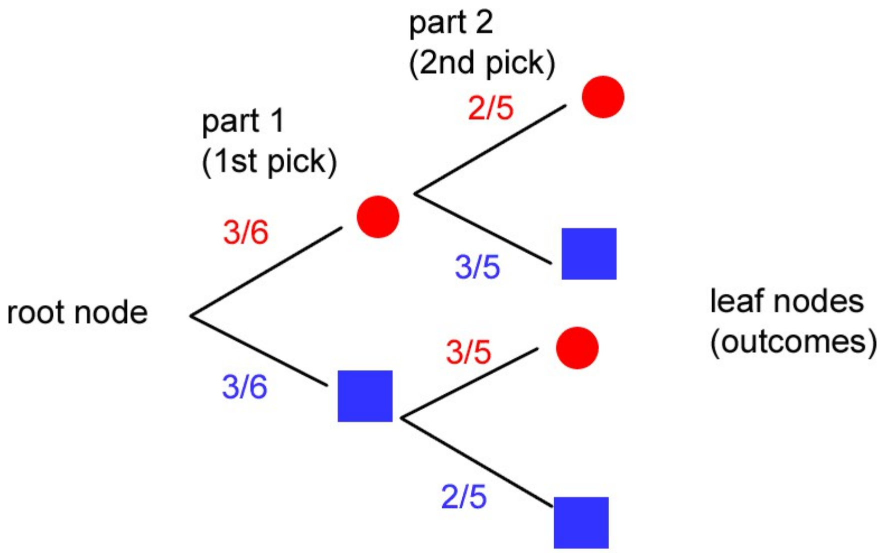


Figure 2.3. A tree diagram representing an experiment of picking two objects.

Even for a small experiment as this, the tree diagram above contains several layers to explore and brings up questions to ask. A glance at the leaf nodes quickly gives us an idea of the size of the sample space; that is, all of the possible outcomes. If counting is considered to be the foundational base for probabilistic thinking, then this provides a visual representation of the likelihood of an outcome occurring that builds on the simple notion of counting. A learner can verify or disprove his/her own intuition about the likelihood of an outcome of interest by counting the leaf nodes that they want as a fraction of all of the leaf nodes. In this example, if the desired outcome is to pick two red marbles, then the tree diagram shows that 1 out of 4 possible outcomes satisfy the condition.

However, building on this counting method has a stipulation: that there is an equal probability of arriving at each of the leaf nodes. This particular tree shows that each path is not weighted equally; a good starting point for conversations around the probability of certain outcomes. This issue can be accentuated further by changing the experiment. Consider that instead of having three red marbles and three blue marbles, there is a bag containing three red marbles and fifty-seven blue marbles. Intuitively, the likelihood of choosing two red marbles decreases significantly. The tree can be used to explore how much the chances have changed. Figure 2.4 shows a tree diagram that represents the change to the experiment.

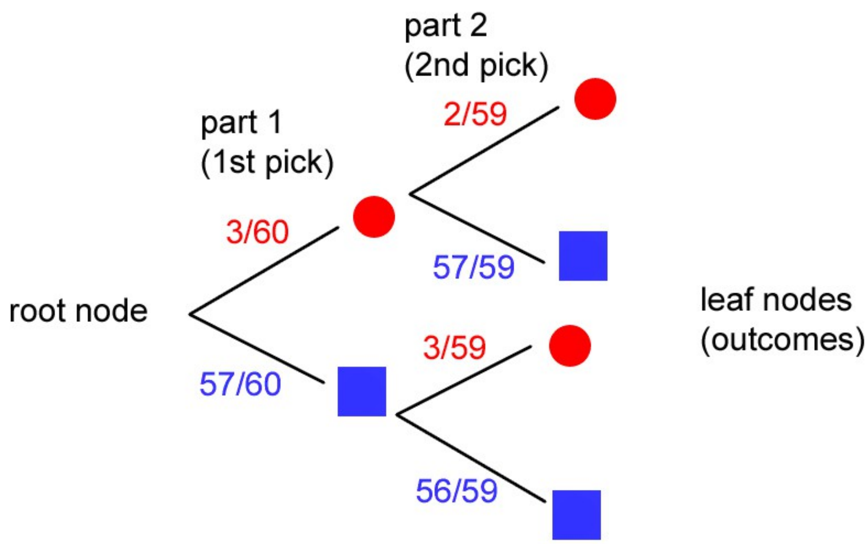


Figure 2.4. A tree diagram for a similar experiment but with more objects to choose from.

The only changes that have been made to the tree are to the probabilities on the branches as this particular change to the experiment did not change the structure (same number of parts and same number of choices per part). The idea that some branches carry more weight than the others can be a good start to conversations about probability. Although impractical, it is possible to draw a tree with all equal branches, but outcomes that would normally have more weight would need to be repeated. For this experiment, the first level of the tree representing the first pick would have sixty branches, three of which would lead to a red marble, and fifty-seven of which would lead to blue squares. Without having to draw such a massive tree, this is a good exercise in probabilistic thinking, especially in visualizing what the second level of the tree would look like and considering then how many leaf nodes there would be.

In order to answer some of these questions without drawing the entire tree, the tree can be condensed by grouping similar outcomes as in Figure 2.4. In taking a closer look at the question:

What is the probability of choosing two red marbles?

This question can be reworded as follows:

What is the probability of choosing a red marble on the first pick and choosing a red marble on the second pick?

This is an example of communication in mathematics being essential to the learning process and not just a means of showing what has been learned. The Manitoba curriculum document (MECY, 2009) also identifies the importance of recognizing the role of language and communication in the learning process. It states that "communication can play a significant role in helping students make connections among concrete, pictorial, graphical, symbolic, verbal, written, and mental representations of mathematical ideas" (p. 8).

In the re-wording of the question, the word 'and' should be emphasized to the learner for discussion. This is an opportunity for the teacher to take a proactive role in turning classroom conversation into cognitive discourse. Cognitive discourse refers to what the teacher says to promote conceptual understanding of the mathematical ideas being learned (Stein, 2007). Students should be challenged to examine the implications that the word 'and' can have on a probability experiment. Wanting both events a *and* b to occur rather than just one of a *or* b to occur leads to a smaller probability as both events need to happen in order to achieve success. It is important to distinguish between this type of discourse and decoding—the key-word approach to problem-solving. A learner who is using a key-word approach does so in place of mathematical reasoning. Engaging in cognitive discourse around the word *and* in this example is encouraging learners to make sense of how the word fits into the problem and not just as a signal to a procedure.

The rewording of the question is also useful in making connections to the visual representation of the experiment in the tree diagram. That is, the question has been separated into parts that match the levels of the tree. By matching the question with the tree, not only the desired outcome (leaf node) becomes evident, but the path with which to get there becomes clear as well—the path with the red marble is taken for the first pick and again for the second pick.

The probability of an outcome could previously be determined by counting the number of desired leaf nodes and dividing that number by the total number of leaf nodes. However, this strategy no longer works due to the weighting of each branch. The use of the word 'and' conjures thoughts of a stricter policy for success. If a person is willing to accept event *A* but then changes his mind to later only accept both event *A* and event *B* occurring, he has effectively made the chances of a successful outcome *less* likely. The tree can be used as a point to show this; the probability of taking the branch to get to event *A* and from there, a second branch and probability to get to event *B*. To satisfy both events occurring, event *B* occurs a fraction of the time that *A* has already occurred; the two probabilities are multiplied to get the probability of taking the entire path.

There is a 3/60 chance of picking a red marble first

There is a 2/59 chance of picking a red marble second

There is a (3/60)(2/59) = 6/3540 chance of picking two red marbles

This thought process captures the essence of the multiplication rule but supports it with an understanding of why it works in arriving at the correct final probability. It is also useful in thinking about how changes to the experiment will change the outcomes. The previous example has shown the changes to the tree when the number of blue marbles in the bag was increased. Students should be encouraged to explore questions about changing the number of marbles picked from the bag, adding a third colour, or replacing the marble in the bag after each pick. These questions can guide a student through thinking probabilistically as there is always an eventual impact on the sample space and change in probability of outcomes.

Basic principles of probability such as complementary events can be explored through experience as learners think about why all branches in a level must sum to 1. Other important concepts such as conditional probability are also raised by making certain changes to the experiment. When the marble is replaced after each pick, the probabilities of the next pick become independent of the previous outcome. This is evident in looking at how the probabilities on the branches of the second level are not changed by the previous node (outcome) they are stemming from.

Another example of a probability question that provides an opportunity for discourse and communication as learning is: *what is the probability of choosing at least one red marble?* While the word *and* implies stricter conditions for a successful outcome, the use of the word *or* implies a willingness to accept *more* outcomes, thus increasing the chances of a successful outcome. For this question, the desired outcome not only includes choosing two red marbles, but it is also acceptable to have only the first pick being red or only the second pick being red. The tree diagram shows that any path that includes one or more red marbles would lead to a successful outcome, ending at three of the four leaf nodes. A connection can be made then to the leaf nodes and what is considered to be a successful outcome; the more conditions that are included with an "*or"* leads to more leaf nodes being included. The probability of each included outcome can be added together. If all of the leaf nodes are included as a successful outcome, then the sum should total 1, which indicates a resulting probability of 100%.

This question can be reworded to reflect the three nodes that are included:

> *What is the probability of choosing a red marble first (and blue second) or choosing a red marble second (and blue first), or choosing a red marble both first and second?*
>
> *P(red 1st or red 2nd or red twice) = P(red 1st) + P(red 2nd) + P(both red) =*
>
> *(3/60)(57/59) + (57/60)(3/59) + (3/60)(2/59) =*

(180 + 171 + 6)/3540 =

357 / 3540

While the above calculation is an example of using the addition rule to effectively determine the correct final probability, a method that uses the tree diagram provides a more complete understanding. The tree diagram addresses the notion of mutual exclusivity here by showing that none of the three outcome nodes would ever happen simultaneously. That is, it is impossible to have a red marble *only* picked first and a red marble *only* picked second in the same two picks.

Consider another question that requires an understanding of mutual exclusivity

When rolling a six-sided die, what is the probability of getting an even number or a number greater than three?

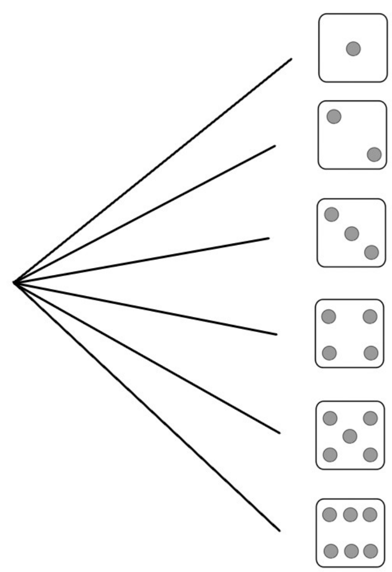


Figure 2.5. A simple tree diagram showing an experiment of rolling a single die.

The outcomes of interest are those that are either *even* or *greater than 3*. The even numbers are circled to indicate the outcomes that satisfy this condition.

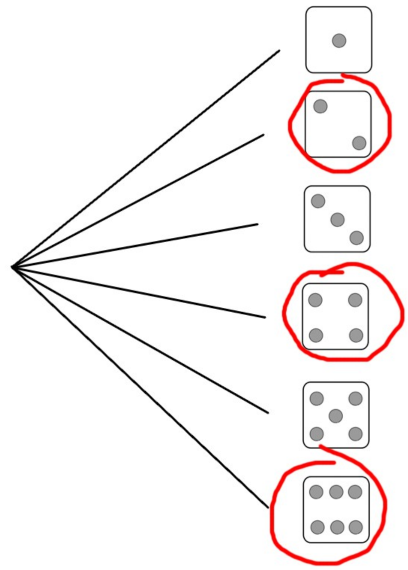


Figure 2.6. The successful outcomes of rolling an even number have been circled.

Boxes are drawn around the numbers that are greater than three to indicate the outcomes that satisfy the second condition.

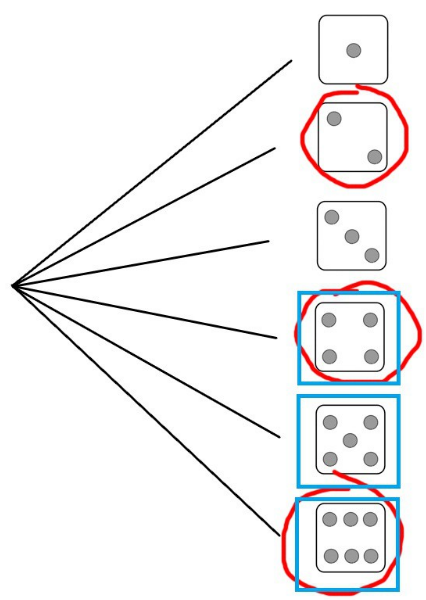


Figure 2.7. A second set of successful outcomes of rolling a number greater than 3 have been boxed.

This provides a visual representation of the concept of non-mutually exclusive events. As the learner counts the outcomes that have been circled and add them to the number of outcomes that have been boxed, it is evident that some of the outcomes have been included more than once. These outcomes need to be subtracted to ensure that any outcomes included in the successful subset have been counted exactly once. It is important to explore the concept of mutual exclusivity not only through the tree diagram but other representations as well such as the Venn diagram. These multiple representations present more meaningful opportunities to ask questions and to think probabilistically than only looking at the addition rule of probability. Rich tasks such as comparing multiple representations, accessed in conjunction with visualization tools can provide opportunities for learners to make sense of math problems. Simmt and colleagues (2012) explain that "this form of inquiry occasions students' image-making as supported by their visualization and deductive reasoning to make meaning of the situation" (p. 11).

Besides being useful for providing visual representation of an abstract concept, fostering probabilistic thinking, and enabling students to think about how changes to a situation affect its probabilities, tree diagrams are useful for instrumental understanding as well. That is, they are useful for calculating final probabilities and determining sample spaces that add to conceptual understanding.

It is possible to use the fundamental counting principle as a calculation tool. The number of ways that each part of the experiment can occur are multiplied to find the total number of ways that the entire experiment can occur. However, this approach may be too abstract for students to understand why it works. Consider the following situation:

If two dice are tossed, how many possible outcomes are there?

The fundamental counting principle would indicate that there are two parts to this experiment (1^{st} die and 2^{nd} die) and that there are 6 ways that each of them can happen.

Therefore, there are 6 x 6 = 36 possible outcomes.

In situations like this and with any other experiment with more parts or branches, drawing a tree diagram is actually not practical as there would be too many branches to fit on a page. However, imagining the tree can still serve in understanding why the FCP works.

How many more outcomes would be created if a 3rd die was included?

While too tedious to draw, imagining how the tree would change adds to an understanding of why the 6 x 6 can be multiplied again by 6 (for every 36 leaf nodes that already exist, the tree would branch out to another six outcomes). Another question that can be prompted with the tree is how fewer outcomes would exist if one of the die had 5 sides instead of 6. Learners can imagine taking off one of the paths and all of the outcomes that stemmed from it.

The tree diagram is both versatile and flexible. It provides a visual representation of most experiments in probability while assisting in answering questions about changes to those experiments. It enables learners to calculate probabilities without having to memorize different key words or templates but through conceptual understanding making the addition rule and multiplication rule unnecessary to learn. Even the creation of a tree diagram is a rich task that taps into a learner's basic understanding of counting and builds from there creating a more meaningful understanding of an experiment and its scope. It then builds on that understanding by showing how more complex concepts such as conditional probability, dependent events, and mutual exclusivity are represented.

2.4 Implications for Learning Mathematics

The benefits of using a tree diagram as an integral part of teaching probability to high school students has implications for the learning of high school mathematics in general. The numerous different stakeholders in the education system bring different priorities and goals. Some parents may care most about the final percentage grade on their child's report card while others may be more concerned with their child's marketability in the job force after graduation. Students themselves may only be concerned with the most efficient way to achieve high marks while balancing other parts of their lives while others may not care about the marks at all and strive to make meaning of the material and experiences that they are being offered in the classroom. Some teachers care most about how many of their students continue on to post-secondary education while others would like to inspire passion in various subject areas.

In a system with so many viewpoints and varying values and priorities, is it possible to approach the content and the curriculum of a mathematics course that enables success for all? That is, can a course with a provincial standards test be taught in a way that those students who care most about marks will do well while at the same time meeting the needs of those students who wish to learn and understand concepts in math in a meaningful way? Intentional or not, any course with a standards test will be taught with preparation for that test in mind. If for no other reason, this claim can be made on the basis of the sample tests provided throughout the course as practice for the real thing.

In many situations, teaching to the test works well if the goal is to achieve a high score on the final standards test; particularly in Manitoba where students are allowed to bring a one-page hand-written study guide into the exam. This study guide can include anything the student chooses such as definitions, rules, diagrams, and example problems. For a unit like matrices, students are introduced to the mechanics of how matrix operations work and then are exposed to exactly three applications: matrix multiplication for inventory problems, transition experiments, and networks. Within each of these applications, there is only one type of problem that can be asked. Further, it is made quite clear and stated explicitly which of the three applications are required for any given problem. Here, a conscientious student would study by looking at numerous examples of each problem type until they could recreate the steps and calculator key presses to arrive at the correct answer. Students who are not particularly fond of memorizing could even write down the examples and steps in their study guides. The average results on the provincial standards test for the matrices component were above 80%. Whatever teaching method is being used for the matrices unit appears to be working in producing correct answers and high marks from students. However, there are many more things to understand about matrices than transitions and networks. Matrices can help us analyze systems of linear equations as well as lead us into thinking about advanced topics such as graph theory. There are special properties of symmetrical matrices, or the identity matrix that are worth discussion. There must be a way to teach the matrices unit in a way for students to achieve success on the final exam but to also gain the long term benefits of conceptual understanding. However, with the standards test including only the arithmetic of matrices and a predictable set of problems, teaching for conceptual understanding is not teaching to the test.

As mentioned earlier, the average for the Probability component of the standards exam has been consistently lower than the other units in the course. The unit can be taught, and is presented in the curriculum's specific learning outcomes, as a set of rules and formulas that help students arrive at the correct answer when applied to the appropriate problem. This is an example of what Skemp (2006) referred to as instrumental learning. There are three possible reasons that instrumental understanding is so prominent in learning Mathematics: it is easier to understand, the rewards (right answers) are more apparent and immediate, and because less understanding is required. Consider a student who has just learned about the addition rule of probability and is working on an assignment. It will be obvious to the student that they should use the tool which he/she just learned. This removes the challenge of having to decipher when the addition rule is appropriate, stripping the task of its complexity and any real value; the student can then find the probability of each of events A and B and then add them together. This is much easier than having to understand how the mutual exclusivity of events A and B will change the answer, or having to choose what events *A* and *B* might be in the first place.

Students have more trouble with probability because, unlike some other units, problems from the probability unit do not lend themselves to a procedural approach. In most branches of mathematics, particularly at the high school level, counter intuitive results are not encountered until concepts reach a high level of abstraction. This is not the case for probability where probabilistic reasoning differs from logical reasoning in its counter intuitive nature (Batanero, 2005). Probability problems tend to be difficult to recognize as a particular type of problem, and each one is likely to be unlike any sample questions the student has seen previously. Small changes to a situation dramatically change the results and sometimes even the rule to be used in finding the right answer. It is in these situations where a more conceptual understanding of probability would be beneficial.

The official educational policy statement on what learning mathematics should look like in the province is the Manitoba mathematics curriculum (MECY, 2009). While it can be easy for teachers and students to look at this document and focus on the specific learning outcomes that are presented as indicators of learning, there is a significant section in the curriculum that speaks to a need for more than instrumental understanding. The term 'practice' is used in the curriculum document in place of what many students would call 'problem solving'. The word 'practice' here is used to describe activities in which students have already been given ways to solve the problem and are being asked to apply the appropriate template. Problem solving then is described as an activity that is engaging, open-ended, and requires conceptual understanding. Curriculum writers (MECY, 2009) suggest that “a true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement” (p. 10). Through solving problems, learners build on their conceptual understanding and engagement; two fundamental parts to molding students’ willingness to persevere in future problem-solving tasks (MECY, 2009). This strategy would enable students to not only perform better on exams, but would also provide a foundation for learning more complex probability concepts.

Although the study uses a specific tool, the tree diagram, as the central point in learning probability, it is the notion of moving the learning of mathematics away from memorization and algorithmic computations to a conceptual understanding that is central to the study as a whole. The tree diagram as a learning tool facilitated and supported students in succeeding at constructing that conceptual understanding.

The benefits of this shift have been introduced in this chapter of the paper. In a society that is growing quickly in areas of automated computation, the most notable benefits of moving mathematics education towards conceptual understanding is that it will foster learners who are more adaptable to new tasks. Technological advances have made it no longer necessary for humans to calculate but to think creatively to solve problems. The purpose of this study, and the use of the tree diagram, was to gain a richer understanding of how a tool that inherently emphasizes the big picture of a concept impacts student learning and the students' perception of their own learning.

It is important to learn mathematics in a way that is adaptable to new situations. This idea can be stated by both teachers and students as a learning goal—that the purpose of learning each small piece of a puzzle is to be able to eventually piece it back together and apply the whole to new problems. However, the approach to learning math in schools is not always successful in achieving this goal. An argument can be made as to how this approach is ineffective and that piecing individual components together does not lead to an understanding of the whole. This is the reason that the probability unit was singled out for this study. It is an area that lends itself to be broken down into individual parts but at the same time resists fitting into a procedural frame for solving problems. Word problems in the probability unit vary in their mathematical structure as well as their contexts.

Developing an ability to solve these problems requires a conceptual understanding among the topics' multiple concepts.

In order to notice the complex nature of how learners used tree diagrams to make sense of probability, I used a blend of qualitative methodologies to guide the data collection and analysis. In the next chapter, I provide the reasoning and support for the use of various methodologies including practitioner action research, phenomenology, and narrative inquiry. I also provide details of the specific types of data that were collected and how they were designed to make the students' learning experiences visible.

Chapter 3 – Research Design and Procedures

In this chapter, I provide an overview of the different methodologies that informed my research design. While the methodological foundations provide the broader orientations for doing educational research, I also detail the research methods and procedures used in the study. The methods used are individual processes adapted from a variety of methodologies. It is important to provide reasons supporting the choices made with regard to these adaptations and how they fit the intentions and context of the study.

3.1 Methodology

The guiding principle in the decision making with regard to methodology points back to the research questions stated in the first chapter. The goal of addressing these questions guided my decisions about the most appropriate methodologies to use.

1. How can a mathematical tool such as a *tree diagram* assist in moving a student's mathematical learning from computing algorithms to conceptual understanding?
2. Further, what role can the tree diagram have in enabling students to *show* their learning and conceptual understanding?
3. How does an emphasis on conceptual understanding as opposed to a more instrumental approach impact student perception of what it means to understand mathematics?

The unit of study here is the *individual* student and not the group of students as a collective. The study explores *how* students learn probability; that is to gain insight into the cognitive processes that are being accessed, the approaches to new situations, and why particular problems are more difficult than others. The study does not address any quantifiable measurements in terms of assessment scores, averages, or comparisons to other students. Since there is an emphasis on the qualities of learning probability and mathematics with the data primarily being collected as explanations of this learning, a *qualitative* approach to the study was used. However, further refinement was required to guide my decision making about the types of data I would collect and how I would analyze the data to inform my research questions. While this led to a blended adaptation of several methodologies, I consider the study primarily as an act of qualitative practitioner research.

Qualitative practitioner research. Practitioner research is rooted in the belief that teachers as practitioners are legitimate knowers who have gained important and valuable perspectives about the situations in which they practice (Borko, Whitcomb, & Byrnes, 2008). For this reason, practitioner research is embedded into the context of the practitioner as both the researcher and teacher—a central decision maker in the experiences studied. That is, the research is being done *by* a teacher *about* his practice to inform his practice. While these traits are present in other research methodologies such as interpretive research, there are two central features of practitioner research: the role of the researcher and the purpose for the research. In practitioner research, there is a unique dual role of the researcher as both the researcher and the practitioner. Since the purposes of these studies are embedded into the practitioner's perspective, they are quite effective in uncovering complexities in the local context. However, this does not diminish their usefulness across contexts. (Lytle & Cochran-Smith, 1992; Zeichner, 1999).

The primary reason that this study was based on practitioner research principles was to address my positioning as a teacher-researcher. My stance within the classroom for this study was a significant component to understanding the students as data sources. As the teacher, I was guided by a purpose of noticing student learning with the intention of better causing it. As the researcher, I positioned myself to gain an understanding of the experiences of the students as they were learning and to do so from their perspectives. Borko et. al (2008) identify an "aim to understand human activity in situ and from the perspective of participants" (p. 1029) to be a significant part of practitioner research.

The principles of practitioner research also guided the purpose of this study. The story of my experiences teaching the Grade 12 Applied Mathematics course over several years highlighted my cognitive dissonance between how I wanted students to understand probability and their actual learning experiences. The noticing of this tension led to my desire to gain an understanding of how students could learn to think about probability. This type of cognitive dissonance is a typical focus for practitioner research studies. Borko et. al (2008) state that the focus of practitioner research is to "address questions and issues that arise from discrepancies between what the practitioner intends and what actually occurs" (p. 1030). This focus on problems and issues that arise directly from within the practitioner's practice is enabled by the nature of practitioner research. Borko et. al (2008) assert that "in all variants of practitioner research, the researcher's professional context is the site for inquiry, and problems and issues within professional practice are the focus of investigation" (p. 1030).

In order to enact this focus, a degree of intentionality and planning are incorporated into practitioner research. All versions of practitioner research also share the features of intentionality and systematicity (Cochran-Smith & Lytle, 1993). Intentionality ensures that the research processes used are planned and deliberate while systematicity encompasses the way that the data is gathered, organized, and analyzed. Section 3.5 of this chapter outlines the detailed instructional materials, activities, and time line for the research process. In teaching the Grade 12 Applied Mathematics course over the years, I have gradually developed the unit plan for teaching probability. However, it was necessary to further adapt and refine the plan in order to optimize the time available for the study. A shift in focus and intention of the lessons, activities, and assessment enabled me to notice the students' learning experiences from my role as the researcher. The systematic organization of the data collection for a practitioner research study lent itself well to the teaching and learning structure of the unit. The types of data collected played significant roles to inform both of my roles as the teacher and researcher.

Data analysis for practitioner research studies draw on a connection between the structured analyses of the participants' learning with the practitioner's perspective.

> The interweaving of systematic analyses of candidates' learning (or other educational outcomes) with the teacher educator's intentions, decisions, interpretations, and reflections enables practitioner researchers to construct detailed accounts of teaching and learning that allow them to derive insights and conclusions not available to outside researchers. (Borko et. al, 2008, p. 1031)

These insights and conclusions that are only available to the researcher as practitioner were central to the analysis in this study. While the variety of data collected from the participants throughout the learning process all contributed to an understanding of the participants' experiences, it was my interpretations of the data as they were being collected that led to deeper insights and conclusions.

Phenomenology. Elements of my research methods have been informed by phenomenology, primarily in my intention of understanding the essence of what it means to think probabilistically and how that thinking can be affected and supported by visual learning tools.

Phenomenological studies aim to describe the meaning of the lived experiences of a phenomenon for several individuals (Creswell, 2007). In doing so, these studies focus on the commonality between the participants' experiences of the phenomenon. This premise is founded in the belief that a phenomenon can be reduced to a universal essence. The data collection, analysis, and interpretation in a phenomenological study then aim to build a composite description of the essence of the lived experience. A significant philosophical assumption across most perspectives of phenomenological studies is that the lived experiences of persons are conscious ones (Van Manen, 1990). This philosophical perspective implies an inextricable connection between the reality of an object to an individual's consciousness of it. This notion has important implications to how the data in a phenomenological study are analyzed and interpreted.

In framing the intention of this study, I viewed the experience of thinking probabilistically as the phenomenon of interest. I aimed to construct an understanding of the essence of how students experience the learning of probability. In constructing this understanding, it was important to see more than just the students' actions and behaviours but their reflections and thought processes before, during, and after the learning had taken place. Phenomenological interpretation enables the deep structures of experiences to be explored. Van Manen (1979) asserts that "the challenge for phenomenology is to make available, through a reflective use of method and descriptions, "opportunities for seeing" through the surface structure of everyday life the ground structures of common education phenomena and experiences" (pp. 9-10). The deep structure here that may be common across the students is the challenges they experience while thinking about uncertainty—that is to think probabilistically. Giving the students opportunities to reflect on their learning experiences may reveal a common belief that there should be a single right approach to all problems. Throughout the learning of probability with tree diagrams and the big ideas of probability, students experienced questions that did not emphasize the final answer, but rather the impact of change and the nature of uncertainty. The student process of interpreting problems to be solved was emphasized rather than the calculation of an answer to the problem. Throughout this experience, my intent was to notice the shift in perspective about learning mathematics and how it impacted the students' abilities and attitudes to learning.

There is a major procedural step in conducting phenomenological research that is paralleled in this study. This connection provided the general orientation that informed the way I thought about the research questions. The procedural step is framed by two broad, general questions: "What have you experienced in terms of the phenomenon?" and "What contexts or situations have typically influenced or affected your experience of the phenomenon" (Moustakas, 1994, p. 58)? The first question was critical in focusing my attention on collecting data that would enable me to build rich descriptions of the common experiences of thinking probabilistically across all of the participants. The second question focused my attention to how the context of using a visual tool such as the tree diagram to think about probability influenced the participants' ability to develop that thinking. A phenomenological approach was very effective in framing the intent behind the research questions for this study. In addition to this approach, the various forms of data collection in this study were also informed by narrative inquiry.

Narrative inquiry. Narrative inquiry is the study of experience as story (Connelly & Clandinin, 2006). Here, the term 'story' refers to the personal meaning-making that happens when individuals interpret their experiences of the world. “To use narrative inquiry methodology is to adopt a particular view of experience as phenomenon under study” (Connelly & Clandinin, 2006, p. 375). In order to think narratively about a phenomenon, in this case about learners developing probabilistic thinking, it is necessary to consider how the inquiry can be framed to move from the creation of field texts to the composition and sharing of interim research texts. This task can be challenging due to the non-temporal nature of living and thinking within a story. It is necessary then that the process of narrative inquiry be a recursive one (Connelly & Clandinin, 2006). The research process moves from the field to field texts and back again when the experiences are relived in the sharing of the interim text.

Field texts in a narrative inquiry can include a variety of data types from transcripts of conversations, to field notes, and other texts created by the researcher to represent aspects of the lived experience. For this study, I interpreted the various types of data collected throughout the students' learning experiences by reading and responding to the students through interactive writing or to myself through field notes. Interactive writing is a form of data that I have used as a teacher even outside the scope of this study. Later in this chapter, I provide support for its use in the classroom for learning as well as in this study for research. Interactive writing is also viewed as a valuable process for teaching meta-cognition and can play a significant role in data generation and interpretation in a narrative inquiry (McFeetors, 2008).

The interactions that I had with the data as they were collected helped me interpret the participants' experiences while the story of the learning was happening. When the unit ended, I constructed narrative texts from all data gathered to be shared with the participants during their closure interviews—an important component to a narrative inquiry.

3.2 Data Analysis

The goal of data analysis for this study was to create a coherent narrative of each of the five selected students' experiences in learning probability through the use of the tree diagram. Once the five students were selected, the original products of learning as well as the interactive writing from those students were analyzed for this purpose. I had analyzed both types of data, the products of learning and the interactive writing, during the instruction, with this first analysis being more as a teacher than as a researcher. Specifically, the interactive writing was analyzed as I constructed responses to each student on the same day that the students had written to me (Mason & McFeetors, 2002). After the instruction was completed, both forms of data were analyzed again to construct interim texts for each interview participant.

The initial products of learning identified the starting point for the students and their approach to learning a new concept. When introduced to a new concept or visual representation of a familiar concept, what are the goals of the learner? Whether the tree diagram worked or did not work for the student in trying to solve various problems could highlight *how* the student was using the tree. As the students progressed with their learning and the analysis progressed from the products of learning to the interactive writing, more evidence was provided to highlight how the student saw math. What could be considered a success for the student with regard to this tree diagram that was being used?

The data analysis did not emphasize the content being learned but rather focused on how the student learned. For example, consider the topic of mutual exclusivity. This concept can be described, taught, and learned in a number of ways from direct instruction, to the use of visual representations like a Venn diagram or the tree diagram used in this study. The area of interest for analysis was not only how the tree diagram could make the learning of mutual exclusivity more meaningful to the student, but also how these enabled the student to become aware of their use of the tree diagram to learn the concept. In making the student aware of how they learned, the idea was to enable them to show their learning to me for assessment purposes and to provide a foundation for learning more complex probability topics. This is where the careful consideration of prompts and responses to the students' interactive writing was valuable as they highlighted these learning processes.

Unlike the products of learning and interactive writing, the closure interviews generated data that was collected after the learning had occurred. The interviews gave the learners chances to not only reflect on how they learned but also to have the story of how they learned told back to them. This was an opportunity for these learners to think about the journey they took in learning probability with tree diagrams and what that journey could mean for learning mathematics in general. In other words, the intention of the closure interviews was both pedagogic and inquiry-oriented in their intention, as was appropriate for the relationship of teacher-researcher with the students (Clandinin & Connelly, 2000).

3.3 Interpretation

Before considering what was involved in the interpretation of the data for this study, a distinction is needed between data analysis and data interpretation.

> Interpretation, by definition, involves going beyond descriptive data. Interpretation means attaching significance to what was found, making sense of the findings, offering explanations, drawing conclusions, extrapolating lessons, making inferences, considering meanings, and

otherwise imposing order on an unruly but surely patterned world. (Patton, 2002, p. 570)

This distinction is important to this study because the various types of data collected each offered different opportunities to see the qualities of the participants' learning experiences. While the data analysis provided opportunities for me to read, examine, and think about the data I collected, the interpretation enabled me to make sense of the ideas that emerged from the data.

From the beginning, the focus of this project had not been on the teaching techniques used in teaching probability, but rather the perceptions of the students as they lived through an experience of learning probability from a specific perspective. This perspective was defined as having an emphasis on understanding the big ideas of a concept and their relation to other mathematical concepts as opposed to an emphasis on procedures, algorithms, and formulas. In order to notice these experiences and any implications that they may have on learning mathematics in general, a phenomenological approach was taken in interpreting the data to answer the research questions.

After determining who the participants would be, the learning products and interactive writing for those particular students was viewed again and a narrative text was constructed. This was the focal point of the interpretive process as both the participant and I made sense of the two week experience. In interpreting the interviews from their transcripts, careful attention was given to the verbs used by the participant as they described the actions taken to participate in the unit. Were they focused on doing well as students or were they more focused on causing learning? How did the participants describe their actions and intentions as participants of the unit? First, using the three data types collected throughout the two-week period of the probability unit, a narrative was constructed to examine the lived experience of the student as they progressed through the learning. Connections were then made across the chosen students to identify any common themes in learning probability. Finally, the experiences of the students, along with the connections across students and the nature of learning probability were examined to explore possible implications for learning mathematics in general.

Narratives were constructed from the evidence of the students' experiences as they lived through the learning of probability as well as their reflections of those experiences. These narratives were not intended to represent a general experience. Rather, writing a narrative was the process I used to interpret from a big-idea perspective each student's responses, interactive writing, and closure interviews to bring into view the essence of how that student had learned mathematics.

It was interesting to examine the deep structures found in both learning probability specifically and learning mathematics generally. That is, the study aimed to make visible the pedagogic value in approaching the learning of mathematics with an emphasis on big ideas and conceptual understanding. This examination applied to various stages of learning from introduction, to showing understanding, to getting stuck, and finally applying understanding to new concepts.

The conceptual understanding valued here was noticed in several ways. Student responses about the thinking they did when faced with a new problem or when the student was unclear about what to do to solve a problem was especially valuable in evaluating the students' understanding.

> Students show understanding of a specific item of mathematical knowledge when, faced with situations of cognitive imbalance, they decide to voluntarily tackle, work out and give satisfactory answers that are appropriate for the situation and that involve the use of such knowledge. (Romero, 2006, p. 12)

Also of interest was the extent to which understanding prior mathematics was vital to learning new mathematical concepts.

The final part of the interpretation process was using the outcomes of the phenomenological interpretation outlined in this section to address the three research questions of this study.

3.4 Research Context

It is important, particularly with qualitative studies to understand the context in which the research is being done and how it relates to the data collection and analysis. The context here includes both the researcher and the environment of the research. Factors to consider include but are not limited to the socio-economic background, culture, age, and geographical positioning of both the school and the students.

The high school involved in the study is located in an affluent neighborhood in Winnipeg. Due to a variety of programs offered, the school has a reputation for attracting students from outside of the catchment area creating a more diverse student body. The three program streams offered at the school are English, French Immersion, and International Baccalaureate. The course that has been selected for the purpose of this study is Grade 12 Applied Mathematics and is a part of the English program. The new curriculum for Grade 12 Applied Mathematics includes the following units: probability, financial mathematics, design and measurement, sinusoidal functions, polynomial functions, exponential functions, and logical reasoning. While the unit on matrices provided a good example in chapter 2 of how a mathematical concept could be taught instrumentally, it has since been removed from the Grade 12 Applied Mathematics curriculum.

The course is one of three mathematics programs offered at the grade twelve level; the other two programs are Essentials and Pre-Calculus. All grade twelve mathematics courses offered in the province of Manitoba are required to include a provincial standards test as an integral part of the course assessment and comprise 30% of the students' final mark. All other test, exam, and assessment distribution are decided by the classroom teacher.

3.5 Data Collection Process

The three research questions represent three levels of the study. The first question was intended to focus the exploration on the students' experiences as they learned the details of the topic of probability. These details, as identified in the previous chapter, include calculating the final probabilities of situations involving uncertainty, using counting principles such as the fundamental counting principle, combinations, and permutations to determine the total number of possibilities for different events.

The probability unit. It is important that students are able to show proficiency in calculating probabilities, combinations, and permutations as it is a component of the Manitoba curriculum as well as a significant part of the students' assessment in class and as well as on the standards test. The Manitoba Grade 12 Applied Mathematics Curriculum document identifies six specific outcomes for the probability unit (MECY, 2009, p. 37).

Table 1
Specific Outcomes for Probability

12A.P.1:	Interpret and assess the validity of odds and probability statements
12A.P.2:	Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events.
12A.P.3:	Solve problems that involve the probability of independent and dependent events.
12A.P.4:	Solve problems that involve the fundamental counting principle.
12A.P.5:	Solve problems that involve permutations.
12A.P.6:	Solve problems that involve combinations.

Five of these six outcomes begin with the verb *solve* in relation to problem solving. It is important to note that the notion of *problem solving* as it appears in the curriculum document is not being used in the same way as on the standards test. The test authors interpret the activity of problem solving as the ability to apply previous knowledge to a question with the goal of determining a single final answer. How this manifests on the standards test are questions that ask the student to provide either a listing of a sample space, a final probability of various probability experiments, or a number of possible outcomes. While there are single marks awarded for appropriate work (which includes applying a correct formula or showing understanding of mutual exclusivity and independence when appropriate), there are also single marks awarded only for the correct final answer based on the students work.

The following table shows an organization timeline of anticipated topics for the probability unit.

Table 2
Sequence of Lesson Plans for Teaching of Probability Unit

Lesson 01: Theoretical and experimental probability
Lesson 02: Organized counting and the FCP
Lesson 03: Combinations and permutations
Lesson 04: Introduction to tree diagrams and sample space
Lesson 05: Problem solving with tree diagrams
Lesson 06: Problem solving with tree diagrams
Lesson 07: The multiplication and addition rules with tree diagrams
Lesson 08: Mutually exclusive events with tree diagrams
Lesson 09: Independent and dependent events with tree diagrams
Lesson 10: Unit test

The unit is structured over 9 lessons with a time frame of two weeks and is completed with the writing of a unit test. An introduction to probability is given through simple experiments involving dice or cards before the tree diagram is introduced (lesson 04) as a tool for calculating, learning, and showing the learning of thinking probabilistically. From this point forward, the tree diagram is used as the foundation and representation of understanding in all key concepts of the unit from counting sample spaces to mutual exclusivity and dependent events.

Products of learning. Homework, assignments, student work, or projects are terms that have been used to describe the products of learning that teachers collect from students with the intent of making visible the learning that has occurred. When the activity of completing these products of learning is rich, complex, and non-algorithmic, they can provide opportunities for student-centered classroom discussions and continued analysis (Ely & Cohen, 2010). When students record their understanding by creating products of learning, they inform my decision making as both a teacher and a researcher.

Products of learning are collected on a regular basis throughout the teaching of the unit—four times over the course of 10 days of lessons. These products consisted of students' written responses to a variety of types of questions: calculation questions that showed students how tools could be used to not only calculate answers but to broaden their conceptual understanding, short response questions that were designed to cause reflection on this conceptual understanding, and questions that provided students with opportunities to be more aware of their thought processes as they built an understanding of various concepts.

The purpose of the products of learning within the context of this unit and this research project were to provide students with opportunities to engage with probability through the use of tree diagrams. Questions were less meta-cognitive in nature and focused on the use of the tree diagram in multiple ways: to calculate probabilities, to express probabilistic thinking, and to extend learning by modeling variations of a problem. These products of learning were collected and viewed through several passes. The first pass was intended to view the data through a teaching lens. As a teacher, the information found here assisted me in making decisions from day to day. Misunderstandings shown by the students were addressed in subsequent lessons and future products of learning were adjusted accordingly. The products of learning also provided evidence the students' ability to perform calculations appropriately and arrive at suitable probabilities. During this first pass, I recorded field notes based on student responses. These notes focused on areas that students had shown either an understanding or a misunderstanding of concepts. Among calculation problems, written responses were provided for questions that aimed to access broader topics in probability. These questions required the learner to have a more conceptual understanding of the topic. While problems that drew on simple calculations were represented within the products of learning, I have not included them in this section because their intended purposes were less divergent.

Before the use of tree diagrams can be useful to the learning of probability, they must be constructed appropriately. This construction is not trivial and can be a confusing task for many students. Common errors include creating a separate tree for each event of an experiment and including different events of the experiment within the same level of a tree. There are questions in the assignments that require students to construct tree diagrams for various scenarios. Question 2a from Assignment 3 asks for the construction of a tree diagram to represent a scenario where a student guesses on three multiple choice questions (See Appendix C for the complete contents of Assignment 3). A problem such as this one was intended to guide students in thinking about the experiment without worrying too much about the probabilities of each outcome. Being able to count the sample space in an organized way is a critical component to thinking probabilistically about an experiment. Using the tree to create the sample space is an opportunity for students to be organized in their counting methods. This allowed them to consider the different sample space sizes if changes were made to the experiment. As a teacher, I read the responses to this question to gauge the students' understanding of probability experiments. I looked specifically at their ability to distinguish between the different events occurring in one experiment. Examples varied from being very obviously broken into parts (toss a coin and then throw a die) to more subtle (the results of a slot machine). Responses here indicated the students' readiness to move onto more advanced thinking in probability or whether more thinking should be done about sample spaces and counting.

One of the benefits of using a tree diagram is that the leaf nodes of every tree diagram lay out all of the possible outcomes of an experiment. If an experiment has too many outcomes, the tree can be reorganized into more general groupings that suit the needs of the learner or at least can be used as a visualization tool without drawing it out. Now that the sample space is made apparent, the learner can begin to think about the probability of various outcomes.

In some experiments, each outcome has the same probability while in others, some are weighted considerably more likely than other outcomes. Regardless of how the outcomes are weighted, it is important to understand that the sample space, and therefore the tree diagram, is including *every* possible outcome. Thus, when the outcomes from one branch sum to 1, this indicates that every possible outcome for that specific part of the experiment is included. At the end, the leaf nodes sum to 1 to indicate that every possible outcome is included in the tree diagram. These ideas can be accessed by the following questions on a product of learning.

> *Explain why the probabilities of each of the final outcomes sum to 1.*
>
> *Explain why the probabilities of the branches from any particular stem sum to 1.*

Responses to these questions above require explanations about the reason the outcomes sum to 1. First, learners must develop an understanding of what a probability of 1 indicates and then show their understanding of why the outcomes sum to that total.

To build on the notion that the tree represents all possible outcomes, the intention in the following two questions was to use it as a tool to show an understanding of complementary events.

> *If you have determined that the probability of event* A *occurring is* P(A)*, use the tree to show how you can determine* P(Not A).
>
> *By how many outcomes will this tree be different if you remove one of the branches in the first level? By how much does this impact the probability of the desired event?*

Once a student has indicated which outcomes they consider a *success*, it should not be a big step to identify which outcomes are considered *failures*. However, some interesting questions could be asked about complementary events occurring in the middle of the tree rather than the end. While complementary events of the final outcome of an experiment can be found by looking at all of the leaf nodes that are not the desired outcome, the complementary events of an event that occurs in the middle must also consider the branches that stem from the undesired paths. I made field notes based on my observations of whether students could differentiate between these two questions.

Similarly, the last question from the three questions above was intended to access the students understanding of how the outcome of one part of the whole experiment impacts the rest of the experiment. This is affected by several factors such as the number of outcomes in the subsequent parts and the number of subsequent parts (levels of the tree) that follow the removed branch.

Further, here are some additional questions that were included in guiding the students through the creation of the products of learning:

Explain why this tree diagram does not represent the experiment appropriately? [Show a tree diagram where each of the events in the experiment are drawn on the same level]

Can you explain why multiplying along a path in the tree leads to the probability of that outcome?

How can you determine by looking at a tree diagram if the events in the experiment are dependent or independent?

How can you determine by looking at a tree diagram if two events are mutually exclusive or not?

Is a tree suitable for representing a permutation? A combination? Explain

These questions were designed to provide opportunities for the students to interact and engage with tree diagrams in order to develop probabilistic thinking. Trees would be used as a primary visual representation for thinking about how different events relate to each other and the impact those relationships have on the final probabilities of desired outcomes.

During the second pass, the products of learning were viewed from a researcher’s perspective, along with the second form of data, the students' interactive writing. Products of learning that showed student work—particularly written work that asked the student to explain their thinking—would be valuable in making apparent the thought processes involved in solving the problems. D'Ambrosio, Kastberg, & dos Santos (2010) emphasize this point: “The analysis of student written work is a way for teachers to understand how students approach a task, how they interpret it, and what knowledge they draw on in answering the questions or in solving the task” (p. 494).

The notes that I made initially were closely examined to draw on these aspects of the students learning and woven together to form a narrative of the learner’s development of understanding. When using student work for research purposes, it is important to first hypothesize about the concepts in the activity that may cause problems for students (Zimmerman, 2006). My own experiences in teaching probability were used to draw on the concepts that were often confusing to students. I took field notes to identify these problem areas so that they could be revisited when the learner reflected on the journey taken to an eventual understanding of the concept.

While these products of learning provide a picture of the learner’s approach to learning the details of probability, a more in-depth approach would be required to understand the processes with which the learner was thinking while he/she was attempting the probability problems. For this purpose, the second data collection mechanism was used: interactive writing.

Interactive writing. Interactive writing in a classroom environment involves prompts from the teacher in the form of a question to which all of the students are given time in class to write a response. The written responses are read and the teacher writes a short, but meaningful response to each student’s writing. In order to be meaningful, teacher responses are to address the individuality of each student as well as suggest actions for that particular student to improve his/her learning.

The question prompts that are used are often determined by the teacher during a lesson. Prompts can vary considerably and illuminate anything from study habits to explanations of mathematical concepts. For the purposes of teaching the probability unit, and for this study, the students were asked to engage in interactive writing four times in the span of ten lessons. Each time the students wrote, I responded before the beginning of the next lesson. This response not only served as acknowledgment to the student but as a primary interpretation for research purposes of the information that the student had provided.

While interactive writing can be used for many different purposes from getting to know the students as learners to helping them develop understanding of concepts, the intent of interactive writing within this study was to access each student's experiences while learning probability through its bigger ideas and concepts; namely their journey in getting better at thinking probabilistically. As a researcher, I was not as interested in seeing how well a student formed an emerging understanding of a concept like mutually exclusive events; the interest in this topic would be reserved for me as a teacher trying to assist the student in developing that understanding. Rather, I was interested in understanding the nature of each student's experience as they learned to think probabilistically through tree diagrams. In order to make visible the nature of the students' learning experiences, interactive writing questions were designed to encourage students to revisit learning moments they've had that day; particularly successful ones. While interactive writing prompts were created during the lesson to cater to specific aspects of the lesson that students found challenging, I also had an idea going into each lesson of what big ideas should be included.

Table 3
Interactive Writing Questions

What example would you use from today's class to explain independent and dependent events?
What was a good moment you had in today's work?
When you were working with a classmate; what was a moment when you/your partner benefited from working together?
Describe the learning moment you had in today's class when you understood the concept. How did you know you understand it? What did you do to cause that understanding?

(table continues)

Which question helped you understand the big idea of the day?
After today's lesson, do you feel prepared to attempt any problem on the final exam? If not, what do you think you need to do to feel prepared?
Explain why this tree diagram does not represent the experiment appropriately?
Can you explain why multiplying along a path in the tree leads to the probability of that outcome?
How can you determine by looking at a tree diagram if the events in the experiment are dependent or independent?
How can you determine by looking at a tree diagram if two events are mutually exclusive or not?
Is a tree suitable for representing a permutation? A combination?

In questions like the first one, I was not looking for whether the student understood the concept of independent and dependent events (that was the focal point of products of learning as data). Rather, as a means to understand their experiences in learning, the interactive writing was to prompt them to think a second time about their learning. It is through the in-depth nature of the explanations in the writing that enabled me to gain a better understanding of each student's learning.

It is important to acknowledge that interactive writing in a meaningful way does not happen by accident. Careful planning and development of teacher-student relationships are important in engaging in interactive writing that is valuable to both teacher and student. According to Albert, Mayotte, and Sohn (2002), "interactive observational assessment is a pedagogical approach that invites engagement between teacher and student through written dialogue to help students develop their understanding of mathematics" (p. 396). For this reason, the original plan for the research project was to teach the probability unit during the last 1/3 of the course after the students would have had opportunities to participate in interactive writing for the first 2/3 of the course. This plan was to ensure that the student responses could move past the initial reasons for writing into more personal and authentic purposes. Mason & McFeetors (2002) further identify as many as four reasons that occur in sequence from the student's perspective to engage in interactive writing: writing to respond, writing to report, writing to reflect, and finally writing to relate. They state, “as they participate further in that process, students usually find themselves with fewer teacher-centered reasons for what and how they write” (p. 533).

It was important for research purposes that toward the end of the unit students engaged in interactive writing to reflect and relate. Writing to reflect is when a student is able to take a concept about which they are uncertain and make it the object of careful consideration (Mason & McFeetors, 2002). This process was important as they would be asked to explain their thoughts on thinking about the bigger ideas in learning probability.

However, after the plan for building pedagogical relationships and trust through interactive writing was established, a significant challenge was presented in the research process. Due to constraints from the ethics review board, I was unable to teach the unit to my own class. Instead, I taught the two week probability unit to a group of students from another teacher's class. These students were unfamiliar with me as a teacher and were not familiar with learning activities such as interactive writing. Although it was challenging to get a lot of students to write to reflect, there was still a lot of valuable data that provided insight to their thinking. Another challenge of doing the research project with an unfamiliar class was the inability to build rich pedagogical relationships as a two week period was too short for this. These challenges are identified now to give context to the way that the data was actually collected. I address how I account for the challenges in later chapters.

However, the idea of relational learning—the trust that students develop for teachers as they share their thinking—is a powerful notion. In doing this research project, I noticed the areas in both the teaching and research that would have benefited greatly from more time and opportunities to build relationships with the students. The interest that the students showed for the research project and willingness to participate was a good foundation for our relationship that led to the quality of data that I was able to collect, but more time to build quality relationships would have made the data that much richer.

Closure interviews. Based on the evidence provided in the products of learning and interactive writing, five students were chosen as a purposeful sample. Purposeful sampling here meant that I selected individuals from the sample space of data who purposefully informed an understanding of the research problem (Creswell, 2007). Since the sample space of a class of thirty students is relatively small, a purposeful sample from that space will be even smaller. Thus, a deliberate stratified purposeful sampling method was used; this included subgroups in the sample to facilitate connections and comparisons to be made across the groups. Two students were chosen for their demonstration of a strong conceptual understanding of probability, two students were chosen for their current state of developing a conceptual understanding, and the fifth student was chosen for being stuck and frustrated with learning probability altogether.

There are several important reasons that a closure interview with each of the five chosen students was selected as the final part of the data collection process. While collecting products of learning and engaging in interactive writing gathered evidence of the students learning while it is happening, an interview gave the students a chance to reflect on their learning after the unit was complete. When the students reflected and retold the key moments of their learning of probability, they had a chance to make meaning of their learning experiences. The interviews were conducted in the math office at school at times that worked around the participants' availability. The conversations were audio-recorded with my laptop and lasted between forty five and sixty minutes. After the interviews were complete, I transcribed the recordings which prepared me to analyze the data.

The questions that I asked in the interviews focused on guiding the students in recounting their experiences during the two week period of the unit. This provided the opportunity for the students to think about and tell the story of their learning.

> Telling stories is essentially a meaning-making process. When people tell stories, they select details of their experience from their stream of consciousness. … It is this process of selecting constitutive details of experience, reflecting on them, giving them order, and thereby making sense of them that makes telling stories a meaning-making experience. (Seidman, 2006, p. 7)

This meaning-making experience was used to highlight key aspects of the lived experience of learning for both the researcher and the student. These aspects included but were not limited to the thinking occurring when approaching a new problem, the thinking occurring when a student was stuck on a problem, and *what* was remembered conceptually about a topic once the learning of it has passed and *how* it was remembered. The interview process gave insight into the student's probabilistic thinking and how it had changed over the course of the learning; particularly counting methods and organization, intuition with regards to uncertainty, and the impact that a change to a situation involving chance could have on the final probability of an outcome. A large part of this study was to enable students to be more aware of how they learned mathematics with the hope that this awareness could lead the students to become better at learning. Using interviews as a data collection process fit these purposes and the one articulated by Seidman (2006) below.

> The purpose of in-depth interviewing is not to get answers to questions, not to test hypotheses, and not to 'evaluate' as the term is normally used...At the root of in-depth interviewing is an interest in understanding the lived experience of other people and the meaning they make of that experience. (p. 9)

Before the interviews took place and after five participants had been selected, I had planned to use the products of learning and record of interactive writing for those participants to construct narrative texts for each of the individuals' lived experiences. This narrative text was to be used to detail the journey that the students had taken in engaging with tree diagrams to learn probability. I had planned to share the narrative texts with the participants during their interviews as a starting point in examining the lived experience of the participants as they progressed through the learning. However, there were significant challenges with the writing of the narrative texts in part due to the lack of development of pedagogical relationships between me and the students. Also, the development of some of the participants did not naturally form a narrative. These challenges are outlined in further detail in later chapters. In the cases where I was not able to form a full narrative, I still used the data to create a portrait of those participants. These portraits highlighted aspects of the participants as learners that were valuable in contributing to the themes of this project and informing my research questions.

The interviews were transcribed in full to depict my interactions with the participants. The interactive nature of an interview along with the chance for detailed explanations from the students enabled me to explore the nature of the students' experiences as they described *how* they learned probability. While the interview conversations flowed naturally with follow-up questions and responses, the originally planned interview questions included:

Table 4
Closure Interview Questions

I've written this description of your journey through learning probability based on your assignments and interactive writing; can you please read it and describe a moment that tree diagrams worked for you?
I noticed from our interactive writing on this day, that you used the phrase <insert a student response that shows reflection of a successful moment>; Does that phrase apply to your approach to the whole unit?
How long after you were introduced to the tree diagram did you feel you were prepared to try a question on your own? If you saw a probability question like the ones in our lesson; could you use a tree diagram to think it through? If you saw a probability question that you've never seen before; could you use a tree diagram to help you?
What do you do when you have a situation where drawing a tree is impractical?

(table continues)

Do you still visualize it or use it in any way? Do you find that the tree diagram has made it easier for you to understand probability? Do you think that formulas help you understand mathematics? Does it help to see how formulas are derived? Here are some of the formulas we've looked at; do you like formulas? How do formulas work for you in math? How did you see this unit? In order to use the trees effectively, was it more important to understand the math or to calculate the correct probabilities? On the unit test, which question did you feel good about? What did you do to make you feel ready for this? How did that work for you? Why did you choose this number of levels for the tree? The number of branches for each level? (After formally introducing the multiplication rule / addition rule) Do you find the rules easier to use than the tree when simply wanting to calculate the final answer? Do you prefer using the tree or the rules in general? Why? Can you explain why the multiplication rule works? Addition rule? How is this question different from the previous (given any two probability questions)?

While it was important to start the interview questions by allowing the students to show their understanding of various probability concepts (consequently showing them what it means to *understand*), the purpose of the closure interviews were to guide the learners to reflect on key learning moments. With regards to the third research question, the interviews were also used to see the perceptions of students on the learning they did during this unit and how it may be extended to learning mathematics in general. Questions to follow up the conceptual questions included:

Table 5
More Closure Interview Questions

Describe a concept that was originally confusing to you but that you eventually figured out. How did you figure out that concept? What are some of the key learning moments that you had during this unit?

(table continues)

How do you think you can recreate some of those moments in the other units of this course?

The following figure shows the anticipated time line of data collection points throughout the teaching sequence of the unit.

Table 6
Outline for Data Collection

Lesson 01: Theoretical and experimental probability
Product of Learning 1 – Introduction
Interactive writing 1
Lesson 02: Organized counting and the FCP
Interactive writing 2
Lesson 03: Combinations and permutations
Product of Learning 2 - Counting
Lesson 04: Introduction to tree diagrams and sample space
Lesson 05: Problem solving with tree diagrams
Interactive writing 3
Lesson 06: Problem solving with tree diagrams
Product of Learning 3 – Sample Space
Lesson 07: The multiplication and addition rules with tree diagrams
Lesson 08: Mutually exclusive events with tree diagrams
Product of Learning 4 – Problem solving
Lesson 09: Independent and dependent events with tree diagrams
Interactive writing 4
Lesson 10: Unit test **Closure Interviews with 5 students**

The data collection process included four products of learning from the entire class, four opportunities for the students to write interactively as well as read my responses, and the creation of narrative texts to be used in closure interviews with four students at the end of the unit.

In this chapter, I have outlined how the data were collected, analyzed, and interpreted. In the following chapter, I describe the details of how the data collection actually unfolded. I will also show how each of the data types contributed to my understanding of student learning.

Chapter 4 – Implementing the Action Strategy

In this chapter, I detail the two week implementation of the action strategy. The chapter is structured as a day by day account of the teaching and learning of the unit. While I describe the data that was collected along the way, the focus of the chapter is to reflect on the pedagogy that was happening. Significant moments in the learning are highlighted and revisited in later chapters. The end of the chapter contains an overview of each of the data types collected and how they were used in the data analysis.

4.1 Story of the Unit

The teaching and learning of the unit unfolded according to the following time-line:

Friday, May 16, 2014. I introduced myself to the class and explained the intent and process of the research project I was planning to complete with them. The classroom teacher had previously prepared them for this introduction and the students were eager to ask questions about the project. Many of the questions reflected an interest in a Master's degree in general and the requirements of completing a thesis. For a group of students whose longest paper has probably never exceeded 10 pages, the idea of writing a document that can be in the 100's of pages was very exciting. The majority of the students in the class were unfamiliar to me; I had taught five of the students previously, but only one of them in a mathematics setting. I emphasized that the research project is not about my teaching methods or classroom environment but rather the learning experiences that they'll have as they learn probability.

I also used this meeting to explain and distribute the consent and assent forms. The students were instructed to read and return all completed forms to the classroom teacher, regardless if the response was affirmative or negative for participating in the research.

Tuesday, May 20, 2014. The first lesson of the unit took place on this day. Students were given an overview of probability, the difference between theoretical and experimental probability, and the notion of thinking probabilistically and the different implications of uncertainty. I also introduced the idea of constructing tree diagrams and how to use them to think about simple probability experiments. This introduction was a significant indicator to the learners that there would be an intervention in this research project—a different way of learning a new math concept.

The big idea here was that thinking probabilistically is rooted in counting; particularly in the organized counting of outcomes. The likelihood of an event is the number of outcomes of that event in comparison to the total number of outcomes for the experiment. At this point, the students had some success with making connections between counting and probability for simple scenarios like getting a number greater than 3 when rolling a die.

I then introduced students to the use of tree diagrams to represent various probability experiments. Drawing a tree to represent a situation like tossing a coin twice strengthened the students' connection to counting and probability. *Using tree diagrams to build connections between different concepts in probability* is a theme that is explored further in the next chapter. As we explored the example together, many students were able to count the number of outcomes of getting *at least* one heads.

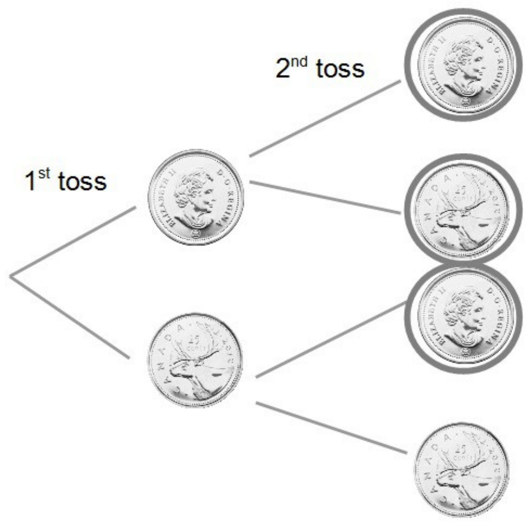


Figure 4.1. The probability of getting at least one heads is 3/4. Each of the three circled outcomes is reached from a path that contains one or two heads.

This example with coins worked well in serving our next class conversation about outcomes that are *not* equally likely. I illustrated this by getting the students to think about the coin being weighted; that is, considering a scenario where the likelihood of tossing a heads was far greater than the likelihood of tossing a tails (I used a 75% likelihood for heads and a 25% likelihood for tails). At this point, many students realized that counting outcomes had its limitations; that it only works when outcomes are equally likely. The probability of getting at least one heads would be a lot more than just counting 3 out of 4 outcomes if the coin was weighted to favour heads. I could tell from the students' questions and remarks that the tree was useful in making this connection; one student even commented on possibly altering the tree to show 3 branches that leads to heads and 1 that leads to tails and re-counting the outcomes.

I gave Assignment 1 to the students and gave them approximately ten minutes to work on it (See Appendix A for the complete contents of Assignment 1). This assignment was a short exercise designed to be started in class and completed at home if necessary. There are two questions on the assignment: one that requires the students to draw a tree diagram and another in which drawing a tree diagram would not be feasible. The tree diagram question is positioned first and was attempted by the majority of students during class with myself present. This question asked students to draw a tree diagram that represented the simple scenario of rolling one die. Many students were able to draw the simple tree diagram that represented the scenario and did not have difficulty answering the questions that guided their thinking about the different probabilities of various outcomes that were possible.

Although I was not able to be with the students while they attempted the second question of the assignment since it was completed at home, I was hoping to see the thinking that was happening in their responses. The question involves a scenario of drawing from a standard deck of fifty-two cards. There were no students who attempted to draw a tree with fifty-two branches but many responses described how certain scenarios were more likely than others because they had more *leaf nodes* or *outcomes*. Students were using the idea of a tree diagram to visualize different outcomes and how probable they were in relation to each other.

> Question 2:
>
> *A card is drawn from a regular deck of 52 cards.*
>
> *a. What is the probability that a Jack is drawn?*
>
> *b. What is the probability that a red Jack is drawn?*
>
> *Which of the previous events (a or b) are more likely to happen? Explain how you know.*
>
> In response, John wrote:
>
> *The probability of any Jack is drawn because [there are] more leaf nodes available. 4 Jacks and only 2 red jacks.*

This example shows how John makes reference to a tree diagram that he did not actually draw. Responses like this illustrate the potential of using visualizations not only as actual organizers of data, but abstract constructs for the way learners can think about ideas.

With ten minutes remaining in this period, I stopped the students from working on the assignment and transitioned them into engaging with me through the first interactive writing. I asked the students to think about the following questions and respond before the end of the period:

Describe a moment in today's lesson when you felt you understood the concept.

Describe a moment in today's lesson when you felt confused. What did you do to get unconfused?

I could see by walking around the room and chatting with students as they wrote their responses that this was a new and unfamiliar exercise for them. Not only were many students unaccustomed to writing in a math course, they were especially unfamiliar with writing about their learning in a math course. Since the students were not expecting to be reflecting on how they experienced the lesson, many had difficulty expressing thoughts about their learning. Many responses either voiced no moments of confusion or only indicated confusing moments that were caused by factors like the LCD projector malfunctioning or being behind in the notes from being absent. However, these responses led me to come back to the students both as a whole class and in smaller groups to probe further about their learning experiences. Some students began to see the value of reflecting about their learning as it made them more aware of how to become better at it. The idea of learners shaping their identities which resulted in being able to expand the way they learned was a key moment for some students—this idea will be revisited in later chapters.

Wednesday, May 21, 2014. I guided the students through several situations involving counting. I emphasized the difference between permutations and combinations. Tree diagrams were used to visualize these differences by looking at the leaf nodes. Leaf nodes that indicated outcomes for a combination would show that the outcomes were indeed equivalent and should not be counted more than once. For students, the way they were thinking about each problem required them to keep the problem within context. This process shifted their conversations away from what to *do* next towards what they *understood* about the problem.

The example used during class was for the upcoming student council election where the student body voted to elect two co-presidents from a pool of three candidates.

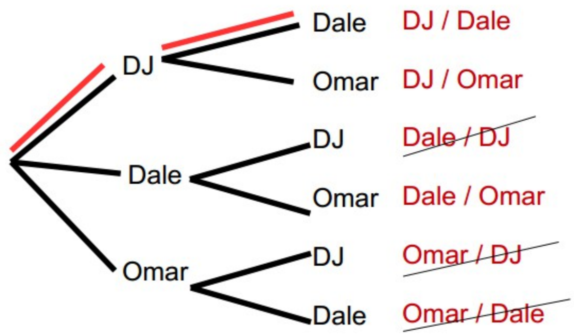


Figure 4.2. A tree diagram showing duplicate outcomes in a combination.

In considering the context of this situation, the outcome for Dale/DJ can be crossed out because it results in the same outcome as selecting DJ/Dale as the schools co-presidents; the order in which the candidates were selected did not result in two different outcomes and should be counted once.

Tree diagrams such as this one were used later as a format for students to show their understanding of probability and organized counting. At this time in the unit, tree diagrams served to help form that understanding. I used this figure because I found in my experience teaching this unit in prior years that using formal descriptions for two types of context situations, permutations and combinations, was not meaningful for many students. The students' confusion in distinguishing between these two types of counting was made apparent again in their responses to counting problems in the assignments as well as the interactive writing.

My informal interactions informed me that the language used in these scenarios such as Question 2 quoted above created confusion for some students. In trying to understand the difference between permutations and combinations, students would say that for combinations, the “order does not matter" intending to imply that the order in which selections are made did not cause new distinct outcomes, so such outcomes should not be counted twice. However, some students incorrectly believed the phrase "order does not matter" implied that the order of the selection of choices impacted the outcome in a literal sense.

For example, in a problem that asked for the number of different open-face sandwiches that could be made from selecting 3 toppings out of 8 possible topping choices, many students determined this to be a permutation. Their reasoning was that the order of which the toppings were placed did indeed lead to a different sandwich. While this may be true to a person who is particular about how ingredients are placed in a sandwich, it is reasonable to say that most people would consider a BLT sandwich any sandwich that contains bacon, lettuce, and tomato without regard for the order in which the ingredients are placed. The discussions highlighted some confusion among students with regard to the language of probability

Visual representations such as the tree diagram, tables, and Venn diagrams helped students negotiate this disequilibrium. As the teacher, I was able to use these tools to get students to think about what should and should not be counted as the same outcome. Their interactions with me during class showed that they appreciated these mechanisms as a way to negotiate their understanding of the difference between permutations and combinations.

During the last 10 minutes of this period, I gave the second interactive writing task to the students and asked them to think and respond before the end of the period. The exercise involved the following questions:

> *Given a counting problem with more than one part, come up with a strategy for yourself that will determine if that problem is a permutation or a combination. Explain how your strategy would work on the following examples (and determine if each is a permutation or combination—you don't have to*

actually do the counting)

a) How many different sandwiches can you make at Subway with 3 different veggies given you can choose from 20.

b) How many ways can you line up 4 friends to take a picture?

c) How many possibilities are there for a 6/49 lottery? (6 winning numbers are chosen between 1-49; a winning ticket matches all 6 numbers).

To students, this interactive writing task was considerably different from the previous one. My intention was to give students an opportunity to engage with the mathematical ideas thoughtfully. Instead of just doing mathematics to find a right answer as many of the students were accustomed to, here they were encouraged to write more and explore their thinking process along the way. Many students found this change in expectation to be frustrating: they encountered difficulty coming up with their own strategies for differentiating between permutations and combinations. Students had difficulty thinking generally about permutations and combinations. Rather, many of them related their strategies to the example of sandwiches in the question or previous examples they had seen in class.

Here are some student strategies for differentiating between permutations and combinations:

Carl: I'd relate it to things we've learned such as the locker combos or the food combos.

Gina: You can't put the sauce on, then the veggies, and then the bread. The order overall matters although it doesn't matter what order you put the veggies on.

Bob: If you put the sandwiches together with the same toppings no matter the order they will taste the same.

Carl identifies two types of examples to help him determine what a combination is: a *correct* use of the word combo to indicate a group of food items, and an *incorrect* use of the same word to indicate a lock code. It can be useful for learners who are trying to define a term or concept to think not only about what that concept *is*, but also what it *is not*. Gina makes a commendable attempt to think about the context of the scenario. This works while she is talking about sandwiches, and by all accounts she could be correct in saying that the veggies need to be placed first and thus implying that the order does matter. However, her approach may lead to challenges in being able to generalize what combinations are. Bob has an understanding that the number of possible sandwiches would be a combination but he did not expand that idea to communicate what combinations are.

This was a key moment for the students as learners. For some of the students, it was the first realization that this different way of learning through tree diagrams would not be easy. For students who may have believed that this new visual tool would be a simple way to find a correct answer, this realization caused some frustration. Other students welcomed the difficult thinking that they had to do with tree diagrams because they recognized that the topic of probability was complex and difficult—they realized that the tree diagrams gave them some traction with the complex thinking required to understand the concepts they were learning. This role of tree diagrams as providing learners with opportunities to think richly will be revisited in later chapters.

Thursday, May 22, 2014. The concept of counting with permutations and combinations was developed further with the students on this day. This teaching was done through the use of guided examples and explanations of the differences between permutations and combinations. Counting concepts were expanded to include the fundamental counting principle as well as details of the calculations of permutations and combinations using factorial notation and the TI-83 calculator. It will be interesting to revisit this theme of student experiences of this lesson through the interactive writing and products of learning. The amount of steps, rules, numbers, and calculations are a drastic contrast to the experiences provided earlier through the visual representations of probability experiments.

Throughout the research period, there were several significant constraints including time restrictions, teaching a class of students that I did not know well, noticing the learning as both a teacher and a researcher, and preparing students for the upcoming provincial standards test. At this point of the unit, I had begun to realize that these constraints were causing me to approach the teaching of the unit differently than I would in a normal situation. I was spending too much time at the front of the room talking. I wanted to talk less moving forward so decided to allocate more time for students to try problems on their own as I walked around the room observing and guiding them.

During the second half of the period, I distributed Assignment 2 to the students and gave them the rest of the period to work on it (See Appendix B for the complete contents of Assignment 2). Like the first assignment, the length of Assignment 2 was short. My intent was to provide students with a sense of how much variety there can be in different counting scenarios and how that variety creates difficulty in problem solving. On this assignment, there are 6 questions that are all different in their context.

I took this opportunity to engage in conversations with various groups of students while they were thinking about the problems. I spoke to students individually as well as in small groups of two or three students. These conversations contributed positively to the classroom experience in several ways. Students continued to develop pedagogical relationships with me as the teacher and became more comfortable in sharing their thoughts as well as their frustrations about their learning. I was able to be effective as a learning coach by noticing things that were causing those frustrations and to guide their learning. For example, one group of students felt frustrated that they had to draw a tree diagram for an experiment that they considered too large to draw. I asked them to imagine the tree instead and to consider what would happen to the sample space if one of the options was taken away. This redirected their thinking towards how different events in the experiment related to each other.

There was enough time in the period for most of the students to at least attempt all of the questions. Students spent the majority of the time thinking about the problems and discussing with each other as well as myself rather than calculating; the actual calculations did not take much time at all once students had a good understanding of each problem. Many students found it challenging that there did not appear to be any template for the problems; each problem felt like one they had never seen before.

Friday, May 23, 2014. With numerous activities going on around the school, about half of the students in the class were not in attendance on this date. We took a step back and explored how counting fits into thinking probabilistically. I showed the students some ways that tree diagrams can be used to assist in counting experiments and understanding sample spaces.

I engaged the students in interactive writing during the last ten minutes of the period; students were asked to think and respond before they left. The interactive writing posed the following questions:

> *From the following list of topics we've learned so far: sample spaces, combinations and permutations, fundamental counting principle, tree diagrams, and the "probability formula"*
>
> *a) Choose one of the topics that you understand well and describe how you know you understand it.*
>
> *b) Choose one of the topics that you are still confused about and describe why you think you are having trouble with it.*

I determined from speaking to the students during the ten minutes that they were writing that many of them did not understand the intent of the questions. The intent of the questions here was to direct the students in thinking about their own meta-cognitive processes as they learn a new and complex concept in mathematics. Rather than reflect on their thinking, many students directed their attention to the act of doing mathematical calculations as a way to gauge their understanding. In my responses to their writing which were returned to them the next day, I encouraged students to extend their thinking to the implications that their actions had on their thinking. For example, several students identified that they would further their understanding through practice or study. I would ask further questions about what that practice or studying might look like for them.

I included quotation marks around the term “probability formula” to draw the students' attention to the notion that the uses of formulas have not been the norm in our learning of probability to this point in the unit. The term 'formula' is a prominent one in learning school mathematics. I identified through conversations with the students that many of them considered the learning of a formula and using it appropriately as the end goal and indicator of understanding a mathematical concept. However, the use of the “probability formula” here proved not to be useful, even to the students who were eager for it. Its usefulness was limited because the variables in the formula are vague and the problems in probability are unique and authentic.

Monday, May 26, 2014. With some experiences with counting and sample spaces, the students were able to shift attention back to thinking about probability as a whole by exploring some different probability experiments. Building up students' experiences with complex thinking by maintaining the complexity in a problem enabled students to attempt difficult problems. In this period, we used trees to represent experiments that had weighted probability distributions; that is, the outcomes on any given level of the tree may have a different likelihood of occurring. For example, instead of the equally distributed likelihood of getting a heads or tails on a coin toss or number on a die roll, a weighted likelihood would be used to represent the chance of rain for tomorrow at 20% (making the chance of no rain being 80%). Students were able to engage with these kinds of experiments during the lesson because it was easy to authentically place them into context.

At the end of the period, I assigned the third product of learning to the students (See Appendix C for the complete contents of Assignment 3). There was no time dedicated to completing this assignment so the students were encouraged to try the questions at home and give them back to me within the following two days. I was sure to let the students know that I could be reached in a variety of ways if they had questions about the assignment; to come see me during lunch, before or after class, or by email and through the blog.

From my experiences in teaching this course before, I knew that the part of this assignment that students found most challenging is the creation of the actual tree diagrams. When faced with scenarios that involve more than one part, many students are tempted to either create more than one tree or to use the branches of one level of the tree to incorrectly represent different parts of the scenario. In order to address some of these misconceptions, I spent the last few minutes of the period drawing the students attention to different thought processes involved in creating a tree diagram. I encouraged them to ask questions about the different paths of the trees as they created them and also to consider if the leaf nodes represented all of the possible outcomes of the scenario.

Tuesday, May 27, 2014. The students and I spent this period trying some of the questions together from previous standards test for probability. This practice built some confidence in the students' ability to read, understand, and even correctly answer some of the problems on a standards test but also drew their attention to some areas that they still did not understand. It was interesting to explore the students' perceptions of being able to discuss mathematical problems that they are attempting but do not fully understand. Students appeared to enjoy this activity as they could see that I was randomly pulling questions from past exams. There was a sense of building confidence as they were able to at least discuss and think about most of the problems. Students also appreciated the opportunity to take a step towards preparing for the final exam of the course.

Wednesday, May 28, 2014. After seeing some terms on the standards test such as *dependent events* that the students were unfamiliar with, we spent some time discussing those terms with the tree diagram as a visual learning tool. While the purpose of math education is to enable learners to develop rich understandings of concepts like probability, the provincial standards test is a significant part of the course for students; it is required and constitutes 30% of their final mark. Providing students with opportunities to familiarize themselves with terminology is an important step in preparing for the test. As a teacher, I have seen that it is more effective for students to navigate the learning of terminology through building conceptual understanding and context rather than by formal definitions.

The tree diagram was a successful tool in developing understanding of a concept like *dependent events*. It provided a visual representation that showed context by making apparent the connections of dependency to other concepts in probability. I directed the students to create a tree diagram for a scenario that involved picking two objects out of a bag first by replacing the first pick and then by not replacing the first pick. The bag had two types of objects: 5 balls and 4 cubes.

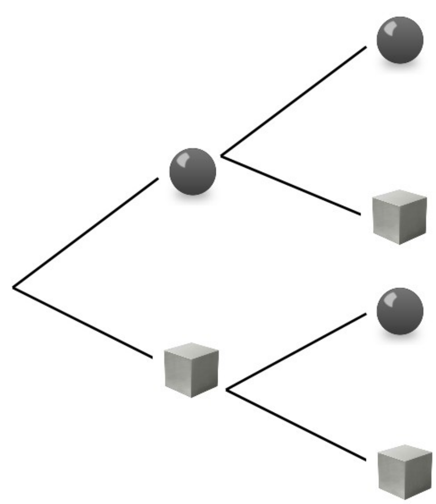


Figure 4.3. A tree diagram showing an experiment of picking two objects.

Like many times before, students wrote the likelihood of taking each branch of the tree. They had to consider what impact replacing the first marble or not had on the probability of the second pick. At this point, many students noticed that replacing the first marble gave them a clean slate for the second pick; the probabilities for outcomes of the second pick were the same regardless of which node they were branching from.

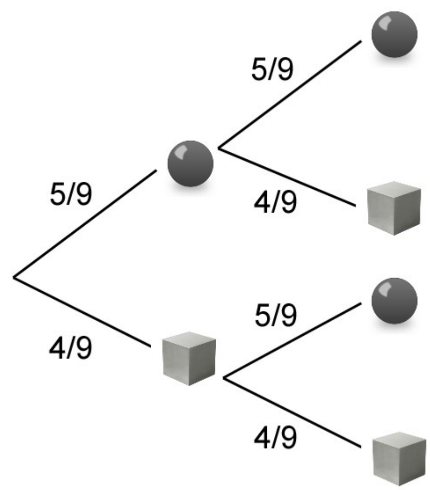


Figure 4.4. A tree diagram indicating the probabilities of taking each branch.

Some students thought aloud about what the probabilities would be along the branches for the second pick without replacing the first. Many students were able to identify that not replacing the first pick not only impacted the whole pool of marbles to pick from but specifically decreased the available marbles of the colour that was picked first and not replaced.

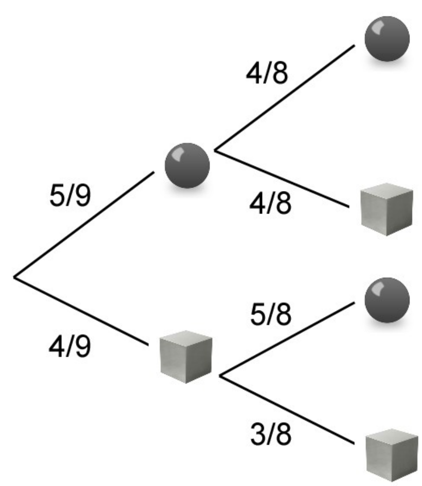


Figure 4.5. A tree diagram showing an experiment of picking objects without replacing them.

Through further discussion between the students, they concluded that they could easily spot a tree diagram that showed dependent events compared to independent events; they could compare the probabilities on the second level to check if they were different from each other. The discussion made evident an emerging and rich understanding of *dependent events* among the students.

At the end of the discussion, I revealed that there is actually a formula called the *multiplication rule of probability* that is used to calculate the probabilities of two or more events occurring. This is a calculation that students had been doing by multiplying along the paths of the tree diagram.

> *The multiplication rule of probability for independent events:* P(A and B) = P(A) x P(B)
>
> *The multiplication rule of probability for dependent events:* P(A and B) = P(A) x P(B | A)
>
> *Where the "|" symbol identifies a "such that" scenario. Where we multiply the* P(A) *with the* P(B) *such that A has occurred.*

This simple formula contrasted the complex discussion by confining the idea to clearly defined set of rules used to calculate the probability of dependent events. As I explained these formulas to the students, some asked why it was necessary to have two versions. Upon discussion, many students agreed that the two versions were unnecessary; the $P(B|A)$ would yield the same probability even if event B was not dependent on the results of event A.

The students saw that the formula was only reiterating calculations that they had already been using with the tree diagram. This was another key moment during the students' experience in this different way of learning that will be revisited in later chapters. Prior to this experience, many students would be eager for a math formula that helps them easily find an answer. However, this time with learning probability, many students did not prefer to use the formula as it did not help them think about whether two sets of desired outcomes were independent or dependent; some stated that using the formula without first thinking about the context of the scenario would be difficult. They did see value in the formula as a way of validating what already made sense to them. Students voiced that it made more sense to multiply along the path to get the probability of the final outcome than to use a formula because they got to see how that path fit with the rest of the experiment and that they could see the total probability decreasing as they multiplied along.

The data from this lesson illustrates how the tree diagram can bring attention to connections between several probability concepts from dependent and independent events to the multiplication rule. These connections are made by showing how dependent or independent events impact the visual pathways of the tree diagram. The understanding gained by exploring these connections is relational (Skemp, 2006). In chapter 2, the term conceptual understanding was described as encompassing other forms of understanding, including relational understanding, that contribute to the knowledge of the meaning of a concept. As learners developed their relational understanding of dependent events from this lesson, this understanding contributed to their broader conceptual understanding of probability. This conceptual understanding was evident in the way that students talked about the multiplication rule, a procedural skill, and how it connected to the concept of dependent events while they discussed methods for calculating the desired probability. Hiebert (1996) notes that "when students develop methods for constructing new procedures they are integrating their conceptual knowledge with their procedural skill" (p. 8).

At the end of the lesson, I gave Assignment 4 to the students (See Appendix D for the complete contents of Assignment 4). This assignment was relatively short and consisted of three problems. Students had sufficient time to complete most if not all of the assignment during class. Some students worked in small groups of three or four while others worked individually. As I walked around the room, I talked to students about their thinking as they tried to solve the problems. I encouraged them to think about what aspects of the tree diagram changed or stayed the same when representing dependent events compared to independent exclusive events.

Thursday, May 29, 2014. This lesson was used to further explore concepts with the tree as a visual learning tool. The students and I continued our conversation around the idea of dependent and independent events from the previous lesson. I gave the students opportunities to identify and explain the difference between these two types of events by working through example scenarios together. Many of these scenarios had already been introduced at the beginning of the unit before I had formally introduced the concept of dependent events. Previously, students had correctly generated tree diagrams by authentically considering the context of each scenario. After learning about concepts such as dependency and mutually exclusive events, students had a chance to revisit scenarios that they were already familiar with and see how these concepts fit into their understanding of probability. This was the final lesson in the teaching of the probability unit.

I gave the final interactive writing task during the last ten minutes of the period; students were asked to think and respond before they left. Given a typical probability experiment, the students were asked to explain how they could use a tree diagram to show understanding of mutually exclusivity and dependent events:

> *Explain how you know that these events are mutually exclusive.*
>
> *Explain how you know by looking at the tree diagram that these events are dependent.*

Several students were able to articulate their understanding of these concepts in probability by using the tree diagram to assist in their explanations. They noticed that when circling leaf nodes for outcomes they were interested in, two events that included common leaf nodes being circled indicated that the two events were mutually exclusive. Students also noticed that they could easily spot when two events were dependent by looking at the weighted branches of the tree; if the second level of branches differed from node to node, then the outcomes of the second event were affected by the first event implying that it is dependent on the first.

4.2 Secondary Use of Data: From Teaching to Researching

In addition to the products of learning and interactive writing, I recorded field notes and observations about various learning moments. These notes included my responses to the student writing during the interactive writing process, notes to myself in response to the quality of student learning shown through the products of learning, as well as records of informal conversations and questions that students asked me during and after class. In addition to being an important part of the teaching and learning process, these notes became a valuable part of the research data as well.

All of the data collected was used as part of the teaching and learning process of the unit. Student writing, misconceptions shown through the assignments, or questions asked in class were all used by me as the teacher in order to make decisions about guiding the students' learning. Once the teaching and learning process was complete, I was able to take a second look at the data from the perspective of the researcher. However, I first completed the process for obtaining consent and assent for using the data for research.

The process for collecting consent and assent forms was facilitated by the classroom teacher. The process took place over the entire time of the ten day teaching block. At the end of the teaching block, the forms were returned to me with the following results: Of a group of twenty-seven students, four students who were under the age of 18 returned assent forms along with parental consent forms agreeing to participate in the study and eleven students over the age of 18 returned consent forms agreeing to participate in the study. There were twelve students who did not return consent forms; the data collected from these students is not included in any way in the research project.

The research project includes data collected from fifteen students. From the subset of fifteen students, I was able to make informed decisions regarding the purposeful sampling for the interview process. I selected five students who had diverse experiences throughout the learning of the unit. The students were at varying stages of moving from learning mathematics through an instrumental, algorithmic approach towards an approach centred on conceptual understanding. Some students I selected for the interviews were emerging not only as learners of math but also were developing their identities of perceptions of how they learn. Other students I wanted to speak with were having difficulties with the challenge of transitioning to learning some complex ideas for understanding; they were eager for formulas, steps, and finding the right answer. One student in particular, Trevor, had an enlightening experience during the short period of time that we spent together. Not only did he make great strides in learning the mathematics of probability, but also gained incredible insight into how he learns and how to get better at learning.

Being a beginning researcher, this was my first experience with collecting research data of this magnitude over a short period of time. The experience was intense and at times overwhelming but also very exciting. As a teacher, my senses for gathering information about the learners were focused on the purpose for making decisions to extend their learning. As a researcher, I found that a new dimension was added to my senses for gathering information; from the comments made between students, details in student body language and engagement, and mannerisms in asking and answering questions. All of these details added to a picture of the experiences of the students as they were learning math. The overwhelming part of the research stemmed from the desire to somehow capture all of these details.

There was one lesson in particular when I overheard two contrasting comments from two different students. While I would have loved to put the classroom on pause at each of these moments and extend the conversations with these students and take extensive notes about their learning moments, I was unable to do that as the teacher. However, these two moments did serve to be valuable in providing me with some insight to the overall experience of some students throughout this process.

One comment was something like “I feel like I'm learning!” and the other was quite simply “I hate this unit". These two bold statements made me curious. At the very least, they indicated to me that what I was doing with these students during this ten day unit was an intervention. The learning that was happening was a different kind of learning from what the students were accustomed to. These differences were a result of more than just the differing teaching styles between me and the classroom teacher; I was able to see that the students were being thoughtful about the way they think about and learn math. For many, it was a new experience.

As a researcher, I would have loved to have an opportunity to sit down with those students and ask some questions about their statements: What does it *feel* like to learn? How is this feeling different from the previous feelings you've had that caused you to make the statement? What would you attribute to the cause of this new feeling? What do you hate about the unit? While I still feel a considerable distance away from becoming an experienced researcher, I hope that I was able to capitalize on some of the opportunities later in the research process to gain insight to some of these questions.

4.3 Products of Learning

As illustrated in the story of the unit in section 4.1, the products of learning were given to the students as assignments approximately every other lesson, for a total of 4 instances. In each instance, students were normally given some time in class to try the assignment but usually not enough time to complete them. Mathematical problems on the assignments were varied and authentic with little to no repetition. Many of the problems included opportunities for students to make decisions and see how those decisions impacted the probabilities of different outcomes in the scenario.

Throughout the data collection process, I viewed the products of learning first and foremost as a teacher. I read through and analyzed students' responses in order to guide the decisions I was making as a teacher; decisions on how to plan the next lesson. There were many instances of student work and written responses that showed misunderstandings of concepts, a reliance on algorithms and formulas, as well as a resistance to this particular way of learning through big ideas. All of these instances provided meaningful insight into what the students needed to guide and shape their learning.

As mentioned in the previous section, on May 21 the class explored the concept of counting with permutations and combinations. This proved to be a difficult concept for many students to grasp as the subtle difference between permutations and combinations not only required different methods for calculating, but also a different way of conceptualizing and visualizing.

On the *Product of Learning 2 – Counting*, one student named Carlo answered the following question:

Question: *In a class of 27 students:*

a. In how many ways can the class elect a president, vice-president, secretary, and treasurer

Carlo's answer: ***27 x 26 x 25 x 24 = 421 200 possible outcomes***

P VP S T

Question: *b. In how many ways can the teacher select 3 students to receive a scholarship, each worth the same amount?*

Carlo's answer: ***27 x 26 x 25 = 17550 possible outcomes***

Here, the student does not see a difference between parts a and b of the question and continues to apply the fundamental counting principle to both situations. As a teacher, my planning for the next experience with permutations and combinations would involve trying to draw the learners' attention to the differences between the two ways of counting. In this example, the student had viewed the words around the problem as trivial and ignored the context of the situation. A more meaningful approach to the problem would be to consider what affect the fact that the three scholarships are equal in amount would have on the context of the situation.

While certainly useful in guiding my teaching decisions, the products of learning offered considerable value to the study as a research project as well. At its core, this research project is about the experiences of students as they learn mathematics through an approach that is centred on conceptual understanding, using a visual representation to develop the thinking required for that understanding. This particular type of data enabled me to see some of that thinking as a part of the research.

While Carlo missed the difference between the permutation and combination in this question, several students took another approach to the question. Sonia, Alex, and Bob all explicitly stated that in part a, the 'order mattered' and in part b, the 'order did not matter'.

Their solutions then looked like the following:

Question: *In a class of 27 students:*

a. In how many ways can the class elect a president, vice-president, secretary, and treasurer

A typical student answer: ***nPr = 27P4 = 421,200 ways*** ***order matters***

Question: *b. In how many ways can the teacher select 3 students to receive a scholarship, each worth the same amount?*

A typical student answer: ***nCr = 27C3 = 2925 ways*** ***order doesn't matter***

Here, *nPr* and *nCr* refer to functions on the TI-83 graphing calculators that calculate permutations and combinations. Being correct and straightforward, these responses do not offer much insight into the students' understanding of permutations and combinations; only that these students were able to differentiate between two situations that required the two different functions. These students have shown that they have at least an instrumental understanding (Skemp, 2006) of when to proceed with a permutation function and when to proceed with a combination function. As a teacher planning for their learning, this becomes problematic, particularly because the *nPr* and *nCr* functions' usefulness is limited as soon as there is any degree of variation to the problem being solved. While the responses provided in the previous example did not provide much insight to the rich thinking that the students may or may not have been exploring as they solved the problem, they certainly did provide a starting point to a conversation.

Another element of the teaching and learning process that proved to be a valuable piece to the research data were the teaching notes that I wrote to myself. These notes sometimes led to informal conversations with students about their learning experiences and at other times served as a piece to the bigger picture of a student's narrative through the learning of the unit.

An example of a teaching note that I wrote to myself about the above example was: *Speak to the student about what 'order' means; and ask "how do you decide when it matters?"* This note would remind me to speak to the students who approached the problem by considering whether 'order' was important or not. While selecting three students for an equal scholarship would be considered a combination where the order of selecting the students does not matter, it certainly could be reasonable to describe it as a scenario where the order *does* matter. Perhaps the student who was picked first would feel better than the one who was picked last. Some learners experienced difficulty distinguishing between the ambiguous uses of the word 'order' here.

In resolving this ambiguity, it was interesting to talk with students after they provided responses such as "order matters" to determine what they meant. A particular conversation with Bob made me aware that his thinking of order led back to the idea of counting. To make the distinction between permutations and combinations, Bob asked himself whether he would count an outcome differently if he had changed the order of the selection around. While this approach was more effective for interpreting some examples than for others, it certainly showed me what it looked like for a student to explore a complex concept in probability—a valuable insight during the research project.

4.4 Interactive Writing

Interactive writing throughout the teaching and learning of this unit was valuable in three distinct and equally significant ways. The questions that I posed for students to initially respond to could be organized into two categories: questions asking to show their understanding of math concepts being learned in class and questions asking students to reflect on their experiences while learning.

Writing about math concepts. First, responses to the *math concepts* questions provided insight to the extent of which students were thinking probabilistically. These questions would typically require students to explain the reasoning behind their responses to probability problems. The responses were particularly useful to me (both as a teacher and as a researcher) when the students responded incorrectly to the probability problem. Students' explanations when they get a problem incorrect can show their thought processes in approaching math problems. With their responses, I can identify whether they are trying to understand the context of the problem and thinking probabilistically or if they are trying simply to connect the problem to something they have seen in class.

During one of the interactive writing opportunities, students were asked to respond to the following question:

> *Consider the following experiment: There is a bag containing 3 blue marbles and 4 red marbles. A marble is picked and not returned to the bag, and then a second marble is picked.*
> *Draw a tree diagram to represent this experiment including the probabilities of taking each branch.*
> *Explain how you know by looking at the tree that these events are dependent.*

A student, Evan, responded by drawing the following tree diagram and wrote the following response:

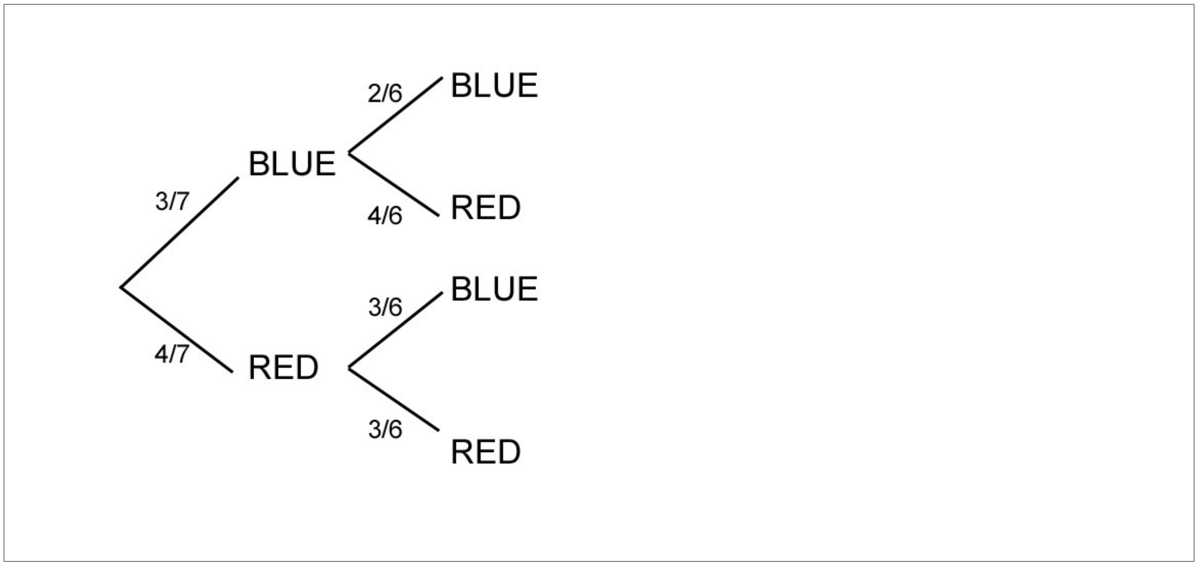
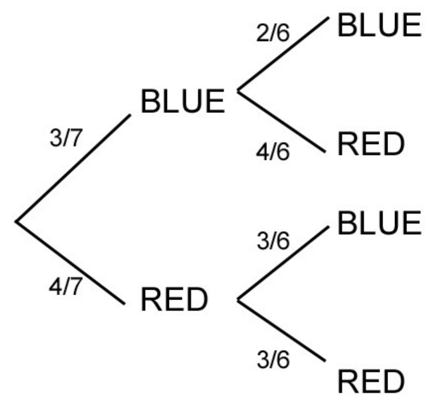


Figure 4.6. Evan's use of the tree diagram to think about dependent events.

Here, Evan has incorrectly stated that the events are dependent because the probability of choosing a BLUE on the second pick has changed from the original 3/7 chance of picking a BLUE on the first pick. While it is useful to acknowledge that Evan does not understand how the tree diagrams show that the events are dependent, his response does not reveal much about *why* he does not understand.

A response from another student, Bryan, to the same interactive writing question showed a more subtle misunderstanding of dependent events. Bryan wrote:

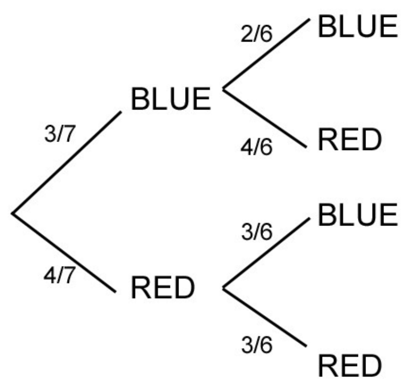


This is dependent because the second part of the tree depends upon what colour of marble you pull out.

Figure 4.7. Bryan's use of the tree diagram to think about dependent events.

While Bryan could actually understand the meaning of dependent events, his response is doing little more than restating a definition. Responses like this to interactive writing tasks do not say enough about a student's thinking but are useful as talking points with the students either through writing back and forth or through informal conversations. In this case, I wrote back to Bryan to encourage him to write a little more about what he meant in saying that the “second part depends on the first”. It was through the *interactive* part of the writing that I understood Bryan shared the same misunderstanding as Evan about looking at change in probabilities from 3/7 to 2/6 indicating dependence.

While Evan and Bryan required my responses to push them further in making their understanding (or misunderstanding) of the math concepts more apparent, other students like Bob used the opportunities to show their understanding not only to the teacher but to themselves. Here, Bob responds to the same question:



By keeping one marble you are decreasing the number of marbles in the bag for the second pick. If you initially pick a red there is less red for the second pick. If you pick a blue there is less blue for the second pull. There is a change in probability for each pull.

Figure 4.8. Bob's use of the tree diagram to think about dependent events.

Opportunities like this for students to explain their learning, not just show their steps, encouraged them to think richly about how the concepts they were learning are connected. Interactive writing made these opportunities for thinking visible to me as their teacher, and as the researcher.

Writing about learning experiences. The second way in which the interactive writing was valuable as research data was through the questions which asked the students to reflect on their learning. For many students, writing about their learning in a math course was a new and uncomfortable experiences.

Here are two questions that both access student reflections about being confused; they were asked at two different points of the two week research period:

Describe a moment in today's lesson when you felt confused. What did you do to get unconfused?

Choose one of the topics that you are still confused about and describe why you think you are having trouble with it.

The following responses show that many students do not equate being confused with misunderstanding or as having anything to do with their own thinking. Rather, they look to external factors that distracted their attention during class. These comments suggest that they think that the learning happens during the reception of the teacher-led portion of the lesson, rather than during their engagement with the practice questions.

> *Student*: Rather confused when the projector stopped working multiple times. Other than that followed lesson pretty well.
>
> *Student*: I was confused because I missed the notes from last class.
>
> *Student*: Fundamental Counting Principle. It just seems to confuse me and I need to practice more.

For their next steps towards becoming unconfused, many responses indicated typical behaviours associated with being a good student: practising more, reading over the notes, and paying attention in class. This information is useful as research data as it allows me to understand to what extent students are aware of how they learn and how to become better at it.

Some students began to show that they could take advantage of the interactive writing opportunities to understand more about how they learn. Harry, for instance, expressed how he found the examples clear during class but had difficulties doing problems on his own: “combinations and permutations, it is hard to tell the difference between them. In the notes the difference is clear but in a real problem there is some confusion knowing which one to use.” While this student is still unclear about why what he calls “real problems” are more difficult than the examples he's seen in class, his response here shows some thinking and moving towards a better understanding of his learning. He is learning something that does not typically yield to attempts to understand it just by receiving the teacher's general explanations and notes. That the “real problems” are confusing to him is natural, not a consequence of weak teaching or weak learning. Harry will need to understand this big idea about the nature of math as a learnable thing before he gets engaged in the complex math he is likely to encounter at university.

Responding to interactive writing as an initial interpretation. The third way in which interactive writing process was useful to me as a researcher was through the act of responding to the students' writing. Responding acted as an immediate opportunity to interpret the data; it illuminated how students make sense about their learning, why they get confused, and what they do when they get confused. My responses normally provided some guidance for how they could proceed as learners but also served as an early indication to myself of the themes that were emerging about the research.

On a question that required students to count the number of possible committees that could be formed from a group of 5 women and 7 men given various restrictions, many of them were not able to think about the context of the scenario. In learning about permutations and combinations, students learned to recognize clues that hinted whether a situation was a permutation or a combination and they applied the appropriate counting formula. This normally consisted of multiplying partial factorials for the given spots available. For example, a typical response that many students gave for the question was:

12 x 11 x 10 x 9 = 11880 possible committees without any restrictions

This response is incorrect because the student misunderstood the scenario as a permutation, or did not think about the scenario at all and applied a formula that they recognized.

In another part of the question that asked the number of committees possible with exactly 3 women and 1 man, almost all of the students did not know how to proceed in the question. This research study is not aiming to prove that a particular method of teaching, such as using the tree diagram, will be a sure way to enable students with the understanding needed to solve these kinds of problems, but this research data shows a lot about what students think about when they attempt these problems. The study is about becoming a more effective teacher by studying the learning of students. In particular, data show the impact of a student's eagerness for a formula on the student's ability to think richly about an authentic problem.

4.5 Closure Interviews

Preparing for the closure interviews was a daunting task. The only experience I had previously with interviewing for research was for a project on a much smaller scale as a part of a course on qualitative research. What I felt most nervous about was whether I would be able to capture the richness of the experiences that the students had over the research period. I could sense this richness through seeing the students learn each day, by conversing with them, and by building the trust relationship between them as the teacher. I was concerned that I would not be able to capture that richness by not asking the right questions or missing opportunities to follow up with good questions; many adults, let alone adolescents, have a difficult time expressing ideas about their thinking and learning.

The first of the five interviews presented significant challenges and many of my concerns materialized. However, I was relieved and grateful that I was able to adjust my interview strategy before the start of the next interview. By the end of the last interview, I had collected a lot of valuable data. A careful consideration of the interview transcripts was the final interpretive act. All data sets were then considered to identify the major themes emerging from the study.

Narrative texts. My plan before beginning the closure interviews was to review the entire collection of data from each of the interview participants. This collection included the students' products of learning, interactive writing, the teaching notes I made about the students, and the final unit test. From this larger set of data, I planned to write a narrative text; a story of change that the students experienced on the journey of the learning of probability. I would then bring the narrative text to the interview and use it as a focus for the conversation, asking the participants to reflect back on significant parts of the journey.

This plan presented significant challenges, particularly in creating the narrative texts. There was a mismatch between the planned approach and the actual opportunity for rich conversations. This mismatch was not due to the lack of change in the short time period of the research. Rather, creating a rich narrative text depends on building rich relationships between teacher and student. I did not understand the participants as richly as I would have if I had taught them from the beginning of the semester. From their perspective, they had not had an opportunity to develop a trusting relationship with me that would enable them to fearlessly share their partial understandings and their personal thinking. It turned out that the best closure interview, the one in which the participant opened up the most and was most thoughtful about their responses, was with Trevor—one of the few students from the class that I had taught from previous years.

Even with the challenges, I did attempt to create narrative texts for each of the interview participants (See Appendix E and Appendix F for the narrative texts I created and used for Sheila and Trevor respectively). I used the data available to paint verbal pictures of who the participants were as both students and learners. Although I was unable to show the participants the change that they had experienced and ask them to reflect on it in rich ways, there were certainly still many valuable gems of data that emerged. These gems served as strong talking points during the interviews. The process of creating narrative texts to prepare for the interviews was not a failed endeavour; it simply did not unfold as originally planned.

Process. I conducted the closure interviews over the span of one week following the teaching of the unit and prior to the final exam. I typically interviewed one participant per day. Immediately following the first interview, I sensed that my approach needed adjustments. I transcribed the interview right away and collaborated with my thesis advisor to plan for next interviews appropriately. The issue with the first interview was that I, as the interviewer, talked far more than the interviewee. I was excited to talk about the learning that I saw through the teaching experience as well as the data. The interview showed my excitement through my description of the data but included mostly one word or very brief responses from the participant.

Lam: And what I noticed was uh...I gave this question to you shortly after we talked about tree diagrams during the first class. Just 'this is how you make a tree' and 'this is what it can be used for' and what you said was 'it wasn't until' or... 'once we applied the formula' then you kind of saw how they worked together. Right? Because the formula I'm talking about is the P(E) where E is an event that we want.

Sheila: yeah

Collaboration with my thesis advisor between the first interview and the other four was useful in shifting my position as an interviewer. It helped me realize as a beginner researcher how important it is to listen and use what the participants are saying to respond and open up opportunities to get them to say more. The other interviews went much better in this regard, there were several occasions where the students would begin to talk about their learning and I was able to ask follow-up questions that not only got them to think more but to express it as well.

Lam: I'll take you to something that you said helped you understand... I mean, it was just the first lesson and we had just started talking about tree diagrams but you said "I felt engaged and there were examples"

Kevin: Yeah.

Lam: So examples are something that you just mentioned are something that help you understand. How do you think that is? How do you think examples help you understand?

Kevin: Examples are.. well, that's what we're learning. You can give us the concept of what

> we're supposed to be learning but then as soon as you give us examples and work us through the problems that's when we actually start using it. So, it's a lot of like using stuff is what is more important.

In this example, I helped Kevin think about how he learns math by prompting him to say more about something that he had said in an interactive writing response. Students often use words to describe their learning without giving much thought to what those words mean or at least how they can be interpreted.

The closure interviews provided many opportunities like this that were of value to the research but also created opportunities for the participants to revisit and think more deeply about their learning experiences. At times, they would provide a more complete description of what they had already said and in the process gain a better understanding of their perspectives. On other occasions, I would challenge their beliefs by giving prompts or asking questions that would encourage them to consider alternate perspectives.

Throughout the five closure interviews, there were a variety of questions, prompts, and responses. Although some interviews and responses were more revealing in showing student thinking and learning, all of them offered some insight to the participants' experiences.

Chapter 5 – Key Findings

In this chapter I present the findings that resulted from the analysis and interpretation of the data sets gathered in this classroom inquiry. The chapter begins by examining the smaller pieces of data in the interactive writing and field notes that I had taken throughout the teaching and learning of the unit. Data samples and connections to moments in chapter 4 will highlight how these smaller pieces of data were valuable in informing both my teaching and research. The chapter then leads into an analysis of the most significant form of data—the closure interviews. This form of data showcased the students' voices in most depth and provided incredible insight into their experiences in learning math not only during the two-week research period but throughout their lives as well. During the descriptions of each participant's interview, any themes that naturally emerged from the analysis are described and then summarized at the end of the chapter.

5.1 Data Flow

I have been using interactive writing and field notes for the past five years to inform my teaching. I have found that they have been useful in giving me insight into students' thoughts on a variety of topics from how they study, to when they are learning, and how they are learning. I often use these insights to make decisions on how to proceed with the structuring of the activities, conversations, and experiences in the classroom. However, this is a long process; it is often not until the later part of a course that the students have built a trust in the process and me as the facilitator. They begin by approaching the interactive writing as they do many activities in school—as a task to complete for the teacher. As they begin to see how their writing in addition to my responses can help them gain awareness in their learning, they begin to write more authentically in their intentions to reflect on their learning (Mason & McFeetors, 2002).

In this research project, I was not able to build relationships with the students over the course of a semester. Also, for many of the students, writing in a math course was something that they had never experienced before. For these reasons, the types of responses may not have contributed as much as they potentially could have in terms of informing my teaching decisions. However, there were several key instances and responses in the interactive writing that acted as catalysts to good conversations with students. There were other moments that helped me adjust what I was doing for the next lesson. All of these small parts of the data collected from the interactive writing and field notes of the conversations that followed from them contributed to my readiness for the final form of data collection—the closure interviews.

There were instances where seemingly shallow responses to an interactive question prompted a good conversation. In response to a question to describe a moment in the lesson that the student felt confused, the student responded, “I did not find a moment where I was confused possibly because we are learning only the basics to probability.” This was a common response to the questions as students were just starting to become comfortable with me as a teacher and the interactive writing as a process. Many students were still uncomfortable with the math learning that was going on as well. However, after reading responses like this, I came back to some of the learners individually and the class as a whole to dig deeper. I discussed with this student what they considered were the *basics* of probability and we talked about possible reasons that they did not feel confused. Through those conversations, the student identified that being able to do the assignments by connecting to the lesson was why they did not feel confused. This served as a valuable talking point to revisit when the same student did encounter topics that were more complex. The student became confused when they could not connect assignment problems to the concepts and examples presented during the lesson. This discomfort often came through in my observations as I circulated the room while the students worked on the assignments, which were documented in field notes.

For example, there was a common error when drawing the tree diagrams—students would draw two separate tree diagrams to represent an experiment with two different parts. Then they would get stuck because calculating the probability of each event individually did not assist them in thinking about the probability of the entire experiment that included all of the events together. This caused frustration for many students as they expressed not wanting to draw the tree because they were getting it wrong. These types of responses prompted me to convince some students to keep on thinking about the tree diagrams—I guided them through the creation of the diagrams so that they could see the value of the trees in the opportunities for thinking about the problems. I was assisting students in getting through the transition to a better way of learning about probability.

A valuable aspect of the interactive writing and field notes data was the way they set up for the closure interviews. My impressions from these two smaller types of data primed my sense of possibilities with the interviews by providing potential topics to discuss. For example, in the interactive writing, there were many students who pointed to *example* problems as the most effective way of learning. I wanted to know more about this idea—particularly what the students found valuable about seeing examples and if it assisted them in thinking about the concepts they were learning in addition to helping them see how a familiar problem could be approached. This topic became a common talking point across a few of the closure interviews.

5.2 Closure Interviews

The closure interviews were the most significant form of data in this research project. Conversations with these students immediately after they had experienced the learning of probability revealed many things about how they learned and how they saw themselves as learners.

Some of the students who were interviewed showed change in the short period of time that I worked with them. Other students showed an emergence or at least a willingness to think about changing. That is, the students began to see that the possibility of changing the way they learned math could benefit them. These were subtle but significant shifts in the way these students saw themselves as learners—to see themselves as students who could engage in the sort of change that they may not have considered prior to this experience. Some of the students also showed a resistance to change. Speaking to these students provided valuable insight to the thinking of a resistant learner and how to move them towards a direction of rich thinking. The following section provides an overview of each of the closure interviews that I conducted.

Kevin. Kevin was a student with whom I was able to build a relationship with over the short period of time I spent with the class. Without generalizing his unique experiences and personality, I could see that he exemplified a type of student that I had encountered many times before during my career as a math teacher. Kevin's approach to learning math was fixed firmly around the examples of math problems that he could find. It did not matter if these examples came from the notes, a textbook, the teacher, or the internet, they were all equally valuable in shaping his ability to solve math problems. Kevin's math learning was based on action. In replying to a question about how he learned math, he responded, "Listen, take good notes, go home after every class, and I study and you know, look at my notes and do the homework. Then on the tests I make study sheets before and flash cards and then study those."

As Kevin spoke more about the actions he took in his approach to learning, it became evident that his idea of what it means to learn and understand really referred to an ability to *do* and *calculate*. The reason that he looked to examples to build this ability was because he was looking for a template to the solutions for problems he was doing. This approach was so entrenched in his identity as a learner and his perception of mathematics that he really believed that the learning of mathematics really only could begin when an example was given and the concepts being learned could be applied.

> Examples are... well, that's what we're learning. You can give us the concept of what we're supposed to learn but then as soon as you give us examples and work us through the problems that's when we actually start using it. Using stuff is what is more important. (Kevin)

This statement is quite explicit and points to what really matters in learning math. If Kevin was not able to do the math or at least see it being done by someone else, then there was nothing else for him to think about richly or make connections with. This is an approach that I have seen repeatedly in students as they develop identities as math learners that are anchored in survival adaptations. Approaching problem solving by looking at examples or by decoding key words does not work for authentic problems that are rich and complex. Clement (2005) elaborates on the drawbacks of this approach by stating that "when problems involve several relationships between and among quantities—the pitfalls with the key-word approach become readily apparent" (p. 2).

When I asked Kevin how he would approach a problem that he had never encountered before and that he could not match up with an existing example that he had seen previously, he was really stumped. He thought a bit about other resources that he could access for examples such as the teacher or other students who had taken the same course before finally concluding that he would have to somehow find an example similar to the problem he was trying to solve.

It seems that the notion of diving into a problem and thinking about it richly and within the context that it is presented is not yet seen by Kevin as a possible approach to problem solving. When our conversation steered towards a hypothetical situation in the future in which Kevin could find himself with math problems in university and no access to a teacher or other students, he still looked towards an action-based plan to seeking out examples. He said "I would just google whatever the unit is or whatever the question is... I look for the explanation of it and then examples of it and then questions that I can try on my own."

Kevin, and students like him, would benefit from opportunities to engage with the learning of math not only through examples but through ideas. Visual representations like the tree diagram could provide such opportunities in learning probability. While there are certainly common elements in the way probability questions present themselves, the richness in probabilistic thinking lies in the connections between the ideas. When asked whether his approach to learning math had to change in the learning of this particular unit, he replied:

> I did need to approach it a lot differently because we weren't taking notes by hand, you were giving us the notes. I actually like that approach; it's a lot better because we can actually have our attention fully on you and not taking notes. (Kevin)

Although Kevin recognized that there was a difference in the learning of this unit, he attributed the difference to the teaching style of the teacher and the activities that were going on in class rather than his actual learning. However, in our conversation, Kevin did express using the tree diagram in a way that helped him when he was confused about a problem. He said, "The split tree diagrams where you had to do like 20% and 80% here and then it split off. That was a little difficult at the start but after we did a lot of examples I think that cleared it up quite fast." Here, he uses the tree but not as a tool to illuminate some of the ideas that are confusing him but rather, as another example that he can apply as a template to fit with his current problem. While there is a glimpse of opportunity for Kevin to do some rich thinking about probability, he is still at the beginning of that journey. He is a student and a learner who is aware and explicit about how he learns and what it means to learn. To Kevin, learning math is best accomplished by looking at examples for a set of criteria to apply to the next problem. While he enjoys different experiences in the classroom, he has not yet seen the value of engaging with ideas in math that is not immediately practical in answering the questions provided.

Georgia. Georgia, like Kevin, looked to examples of math problems to build her understanding. However, Georgia showed that she was on the cusp of using those examples and visualizations like the tree diagram to think in more complex ways about the math concepts she was learning and how they connected to each other.

I first noticed Georgia in the hallways working on applied math not only before and after class, but even in the morning before school and at lunch time. She was a quiet person who rarely engaged in conversations or asked questions during class. As both a teacher and a researcher, I was very curious to talk with Georgia about her study habits and her experiences as she learned this unit in particular. She spent a lot of time engaging with the mathematics on her own. I wanted to gain insight to what she did when she was confused? How did she see herself as a learner? As I spoke with her during our closure interview, I learned that similar to Kevin, Georgia looked first to examples of math problems to build her understanding. When I asked what she does when she is first learning a concept that she does not understand, Georgia responded, “Let's say I don't understand a lesson we just did. I would just look for examples on Google.”

As I learned more about Georgia as a learner, however, I realized that she used examples to learn in a very different way than Kevin. While Kevin looked to examples with the intent of gaining a template for the steps to find the answer to a problem, Georgia used them to see how some of the details fit into the context of more general concepts. When I asked her how examples help her understand a concept in math she stated, “It just shows you like how to put it into context basically and just how to use the formula.”

Although there is a desire inherent in her comments to determine a formula for calculating the correct answer, Georgia also placed value on thinking about the context of a problem. As we explored this idea further, she showed a good understanding of herself as a learner. She said, “If I don't understand why I'm doing something then I don't remember it. I need like, logic to it... I like Applied better because it makes more sense than Precal."

Like many students who enrol in the Applied Mathematics course, Georgia did so as a result of not having positive experiences in Pre-Calculus Mathematics. As students go through this transition, some take the opportunity to reflect on the differences between the two programs. Here, Georgia understands quite well that she appreciates the opportunities in Applied Mathematics to see more contexts that enable her to make sense of the concepts being learned. I pointed out that there are many things that can be learned in math without context and without understanding them but that those things mostly involve performing a calculation or following steps to an algorithm. Georgia responded, "I don't like that. I like to know why. Because then if I can't figure something out I can at least reason and think about it logically."

At this point, I was quite interested in what role visual representations like the tree diagram had in Georgia's process of making sense of the things she was learning. In her interactive writing, she wrote that "tree diagrams help provide a simple and clear way to solve a question." When a student uses words such as 'simple' and 'clear' in describing her learning, I am always interested in hearing more about the thinking associated with feeling that way. During our closure interview, I asked Georgia to say more about that particular interactive writing response. While pointing at a tree diagram on one of her assignments, Georgia said, "When you're doing math it can get complicated but with this we can just see 'ok, if you want this for this and yes for that' then the answer is right in front of you." She was pointing to how she created the tree and filled out the probabilities of taking each branch.

While Georgia sees herself as someone who learns well by looking at examples, she is on the cusp of changing what she does with those examples. She describes the process of creating this visual representation not as a means to finding the answer, but as a valuable process that enables her to think about the outcomes and how changes in one event can affect the probability of the entire experiment; she is thinking probabilistically.

Veronica. I wanted to conduct a closure interview with Veronica because I noticed intent in her learning focused on grades, similar to other students I have taught in the past. While Kevin and Georgia both used examples to learn, they associated positive experiences with successfully completing a question or getting the right answer. Veronica prioritized receiving good marks above getting a question right and certainly above understanding the big ideas behind the question. I saw a lot of reluctance early on in the unit as Veronica had difficulty letting go of tasks that she self-identified as necessary to being a good student: taking notes, completing assignments, and following steps.

Many students I have met over my teaching career work very hard at being good students. They do their homework, they do extra practice questions, and they spend many hours memorizing facts that enable them to have success in school. These strategies work for many students, until they do not. Veronica was such a student; she spent much of her childhood practicing and doing math and achieving success in school.

> Learning math was pretty easy, you know. Addition, subtraction, that came really simple. Um, when it came to learning multiplication, my mom would write out like, everything and I would memorize. I learned that if I practised like, math, then I got better at it. Like, you give me questions and I would do them over and over and over. (Veronica)

Like Georgia, Veronica uses the word 'simple' here to describe her learning. In this context, the word is being used to indicate that the ability to do addition and subtraction was *easy*. Once Veronica was able to manage the addition of each of the ten digits, she could fit that ability into the algorithm of adding any number of digits together. This approach to learning math worked for Veronica until she encountered math that she called too 'complex'.

> Grade 10 was pretty easy too, actually. Wasn't too complex yet. I didn't find it hard. I got away with... sometimes, I wouldn't do my practice homework and I would ace it anyway. It was easy. But then in grade 11 Pre-Calculus, that's when it starting getting really hard. That's when I realized I couldn't get away [without practicing]. I was introduced to a lot more new stuff and it was more complex. Like, I think it just got more into detail. (Veronica)

Veronica equates a concept being complex as having lots of details. She began to feel the limitations of memorizing and practising as the algorithms became overwhelming to remember. This lack of success did considerable damage to her identity as a math learner as she was not only experiencing frustration with not being able to do the math but saw her success in the classroom and grades steadily declining. She claimed, "When I did pretty poorly in grade 11 and grade 12 Pre-Calculus, it took a lot out of... like, I mean I was feeling very down." Over the years, Veronica had come to see learning math as an emotional identity-building activity. Although she was no longer experiencing what she considers a successful experience in learning math and not achieving the grades she expected, she did not identify her approach to learning as the reason for her difficulties. She thought her difficulties came from a lack of practising and that she could no longer get away with that.

Throughout the short period of time that I spent with Veronica and throughout our conversation, I learned that she was not as simple a learner as she portrayed. Like Georgia, Veronica was on the verge of becoming more aware of herself as a learner. While she attributed her past success to being able to carry out steps and procedures, she was willing to try a new approach to learning. To Veronica, any approach to learning that brings into light the process of the learning would be new; to this point, any learning was an invisible by-product of her efforts.

I was eager to learn what the experience of reluctantly trying something new was like for Veronica. I asked her if she needed to change her approach to learning this unit. She responded, "This unit? Is uh... it's easy if you get it, sort of. Yeah, it's cool! The topic is interesting but it is sometimes hard to think it through because I didn't understand the question." Veronica definitely sees the challenge in learning for conceptual understanding. Problems are authentic and connected in rich and complex ways. This approach did not match with Veronica's familiarity with seeking steps and procedures in notes and examples. She claimed, "I look for steps and procedures [in the notes]. That's what I usually look at the most when I'm studying for math... I'm just going over procedures... for different scenarios."

Veronica still put forth efforts to try to think about the concepts being learned. Here, she is discussing with me about a question that she got wrong on the test.

> I spent a lot of time on that question. At first, I was going to do it by drawing it all on one level. But then I said, 'wait, that's not going to work'. Then I remembered back to one question where you showed us four levels... and I just kind of, you know, did that! You know... like if the car starts, then that's the path you take. (Veronica)

I saw in her assignments and unit test that she tries to use the tree diagram to get started on a problem but is not ready to fully commit to thinking about concepts. Veronica would start a math problem by creating a tree diagram and by doing so would make some conceptual connections, but shortly after, she would be eager to look for a formula that she thinks would apply. She stated, "So I went down to that branch where the car won't start and then I multiplied them...because I remembered the formula." In most cases, Veronica would use the wrong formula and get the calculation wrong. Veronica has not yet made the connection between making meaning of the problems she is solving and the calculations involved in solving them.

Sheila. The student who I had overheard during one of the lessons that she 'hated this unit' was Sheila. Through a closure interview with Sheila, I wanted to explore the reasons for those feelings as well as gain understanding about the experiences of someone who resisted being introspective to her own math learning.

Sheila proved to be a challenging person to engage with in conversations about learning. She was guarded not only in sharing her personal inner discourse with me, but also from herself. In many instances, she showed that she chose to ignore her own cognition; learning math was just not an area that she spent much time thinking about. Between many one word responses and awkward silences, Sheila would share little in the way of her thought processes. When asked to describe what she does when she is confused about a concept she is learning, she commented:

> Ummm, well I try a few different ways even if I think right away that they're going to be wrong. Because when I see the right way to do something I always say 'I never would have thought of that', 'It's so out of the box.' So I will try to do different ways as well. Ummm, it's almost like, even if I think it's wrong or I think it's silly or... sometimes, I'll think it's too easy so I don't go that way. Yeah. Even if it seems way too easy... or too hard. Or I usually look at the mark value. Like, if it's 1 mark and I keep going and going and going then like obviously I'm not getting it. (Sheila)

These quotes contain seemingly disconnected trains of thought that give little insight into Sheila's thought processes as she learns math. She approaches many problems without a clear strategy or intent of moving her understanding forward. These statements also show that while she bases a lot of her decisions on intuition; she is unsure of the basis for her intuition.

Sheila's experiences with the tree diagram, while different, did not convince her that learning math could be a positive experience.

> With the tree, I could see that I could get a 1 and a 2 and a 3... and I can picture it, almost. Like, rolling the dice. And then, by, ummm... applying the formula to it... then I was able to think to myself that... well it takes too much time to do all of that so obviously there must be a quicker

way. (Sheila)

Sheila is hesitant to engage with the opportunities for probabilistic thinking that the tree provides. She indicates that the thinking about context really gets in the way of just doing the question and calculating the right answer. It is this reluctance to think in rich ways that creates difficulties for Sheila when she encounters what she calls 'word problems'.

Something that I found remarkable is how two different learners can have such different experiences as they participate in the same lesson. The *probabilistic thinking* that got in Sheila's way was the same thinking that the next student, Trevor, found incredibly useful and actually pivotal in his transition into thinking about math conceptually. Trevor saw the tree diagrams as a way to push his thinking about math problems to the next level and to reflect on that thinking in order to get better at it.

Trevor. Trevor was one of the few students in this class that I had taught prior to the research project. He had been a student in one of my grade 10 computer science classes two years earlier. I remember that he was a confident student and we were able to build a good relationship during our time together. A two week period of time is too short to foster a meaningful pedagogical relationship. This realization leads me to believe that Trevor was more enthusiastic and explicit in his responses due to our existing pedagogical relationship. It was the details in his descriptions of his learning experiences that set his closure interview apart from the other participants.

I chose to interview Trevor for two reasons. First, I could see from his assignments and tests that he understood the unit on probability very well. Second, it appeared that he had made quite a transition in the way he learned math as he explored this unit. When I first began interacting with Trevor during this unit, he exuded the same confidence that I remembered. He appeared to be a student who was accustomed to experiencing success in learning math. However, as I interacted with him more, I learned that he was a student who equated being successful at math with finding the subject simple and easy to understand. He spent the majority of his school life identifying himself this way as a math learner.

Trevor's positive identity was challenged when he reached high school and encountered math concepts that were intimidating and difficult for him to understand. He had placed value and pride in previously finding the concepts up to grade 8 simple and clear. When Trevor encountered math that he saw as challenging and complicated, he looked to reasons outside of the math learning and the math itself. To him, learning mathematics was constant and did not require change; it was external factors such as the teacher that created the difficulties of learning. When asked to describe his prior experiences with learning math, Trevor said, “Math has been very simple and very easy to understand up until grade 9 in which I had a teacher that could not teach me the way I am able to learn.” In this quote, Trevor describes the learning of math to be 'simple' and 'easy'. He is using these terms to imply that there should be little ambiguity to doing math and that there is always a right answer that he is trying to uncover. Even his use of the word 'understand' here was used in a way to not really imply the actual understanding of math concepts, but rather the doing of math tasks.

Trevor had a clear idea of himself as a math learner and when the learning environment fit that profile, then he was able to be successful. He valued concrete values and applications over what he described as *obscure* concepts. Making connections to numbers that made sense to Trevor was important to his ability to understand the math being learned. Even when learning an abstract concept such as the standard form representation of a periodic function, Trevor would be deliberate in using specific and concrete numbers to form the basis of his understanding.

After facing challenges in grade 9 with learning math, Trevor felt a resurgence in grade 10 in which he again felt that the math became simple, clear, and easy to understand. However, he also again looked to external factors as the cause for this resurgence rather than his doing anything different as a learner. As a part of once again finding math simple, it was important to Trevor that he was able to learn without putting forth much effort. Having to exert himself in learning math was linked in his mind to not being successful and not being able to understand. He said, "My teacher really was eccentric about teaching and wanting her class to learn. From that I learned without trying and a lot of the concepts back in grade 9 actually made sense to me." Trevor is placing importance on the learning of math being easy and again using terms like "made sense" loosely to imply that he was able to do the math and still did not like the messiness of understanding it in rich and complex ways.

During the learning of the probability unit in grade 12 Applied Mathematics in which the research data was collected, Trevor immediately identified the difference in the way that he learned. He was being encouraged to think about math in new ways. While Trevor was accustomed to learning math by starting with concrete examples and then move towards visual representations, tree diagrams provided opportunities for him to not only start with the visual representation but also to use that representation as an anchor in his conceptual understanding of probability.

> My approach to learning for this unit was quite different because for me it is usually the numbers first before the visual representations whereas with the trees... the trees actually did help me in a way that I wasn't use do so it was kind of odd for me but at the same time I could understand it very simply... it's almost as if I was going backwards starting with the visualizations and then into numbers. (Trevor)

What I found most fascinating was that Trevor was still using the same language to describe his learning but using it in different ways. Here, he describes how tree diagrams enabled him to understand probability concepts 'very simply'. He had previously used 'simple' to describe the ability to do problems and follow mathematical steps with ease; here, he is referring to something much different. Tree diagrams were not only enabling him to solve for unknown probabilities, but they were encouraging him to think about the context that the problems presented.

It is apparent that Trevor was quite aware of the changes in his approach to learning math during this unit but he also recognized that the changes worked for him. Even though he felt quite *odd* starting with visualization and working *backwards*, he saw that visualization enabled him to see how the tree diagram represented the conceptualization and relationships among the different events. He was able to see this concept without having to start with specific examples or concrete numbers. Understanding the concept of probability and how events related to each other enabled Trevor to approach a variety of contextual problems by thinking about the big idea of how any event fits in relation to an experiment's sample space. He then used this conceptual understanding to apply contexts and solve problems.

By making this breakthrough from the abstract towards the particular, Trevor showed that visual representations such as the tree diagram can assist in the understanding of math concepts by showing relationships of big ideas rather than building through concrete examples and numbers. This breakthrough becomes particularly meaningful because it comes from a student who did not initially identify himself as being able to learn complex ideas.

Trevor saw the value of the approach he took in learning probability. During the unit test, he encountered a problem that he found difficult; it was one of the few questions that he did not answer correctly on the test. Upon reflection on his approach to that question, Trevor approached the problem again through the conceptual understanding rather than the actions that he took. That is, he focused on how he did not understand the context of the problem rather than how he simply did not know how to proceed with solving the problem.

> When I read that there were 3 women and 1 man in the problem I automatically assumed that the order mattered but then when I read more I realized it could actually be a combination and

> could be in any order. So I thought that adding the chances of each one together would give me the right answer which it did not. (Trevor)

Even though he did not solve the problem correctly or even fully understand how to do the problem afterwards, Trevor's approach to thinking about the problem showed a significant transition in how he is able to learn mathematics. He saw the value of this process of change because he had success learning probability through the big ideas and representing relationships of ideas through a visual diagram. Trevor transitioned into seeing the value of basing his understanding of a new concept in its basic premises. He has used this transition to build his capacity as a math learner by having his experiences during this unit complementing his previous experiences. By being aware of how he learned and how he changed, Trevor was able to connect to his previous experiences as a learner and see them more richly. Trevor has since decided to try Pre-Calculus math again at the high school level where he currently has a mark of 99%. This is an indication for himself that he can find success in repositioning himself as someone who is able to learn through abstract concepts and big ideas.

5.3 Pedagogical Insights

The themes that emerged as I analyzed the data were admittedly dispersed as scattered elements of insight into the learning experiences of the students during the research. Some insights appeared once in particular student experiences and others appeared more than once across several students. While the scattered elements may not have formed a narrative of a specific journey in learning, they are certainly connected as facets to the complex nature of the change these students experienced while becoming more aware of their own learning. In all of the data collected, from the products of learning and interactive writing to the most significant closure interviews, the following three insights emerged:

Mathematical formulas can be effective in summarizing relationships effectively but are insufficient in enabling students to build rich understanding of concepts, particularly when the formulas are presented as a procedure to determine an answer to a clearly defined problem. The idea that conceptual understanding can be developed by learning formulas in a way that enables learners to make connections between the formula and the context of the situation it represents is not a new one nor is it limited to just probability. “If students are asked frequently to formulate mathematical models for situations and to interpret results of algebraic calculations, they develop greater understanding of and skill in those processes” (Huntley et. al, 2000, p. 344).

Tree diagrams played a significant role as an intervention to the way students view themselves as math learners. While many students use visual tools such as graphs and diagrams to complement their learning, using a visual tool as the central focus of their learning was a new experience. The rich tasks designed around a visual representation effectively shifted the experiences in the classroom towards conceptual understanding. The impact of a different strategy in learning probability, particularly the tree diagram, can be significant to the learners’ development of probability concepts (Aspinwall & Shaw, 2000). Students were able to develop probabilistic thinking, were more engaged, and more aware of their learning.

Students are able to *expand their repertoire of processes for learning math* when they become more aware of how they are learning. The authors of the MECY (2006) assert that when students develop an awareness of how they learn, they can use that awareness to adjust and advance their learning. This change in perception of how a student sees themselves as able to learn math is connected to engaging activities in the classroom. Learning through an experiential foundation leads to a more permanent, adaptable, and dynamic understanding of the math concept being learned. The following chapter will focus on these pedagogical insights in moving towards shedding light on the research questions.

Chapter 6 – Pedagogical Insights and Conclusions

In this chapter, I explore the three pedagogical insights that emerged from chapter 5: *math formulas*, *the role of tree diagrams*, and *student identities as math learners*. I use samples from the various types of data to illustrate different facets of the themes as well as refer back to significant moments during the data collection that added to my understanding of how these students were experiencing the learning of probability through tree diagrams. I also explore any implications that those experiences had towards learning math in general and to my research questions.

6.1 Pedagogical Insights

Math formulas. The insight that I thought I understood the most going into the research was the notion that using mathematical formulas as the focal point for learning math concepts was ineffective, and perhaps even damaging to the development of conceptual understanding. A large part of the planning for my courses and for this research project was based on that notion. I wanted to see if the data that I collected would support what I experienced with learners over the years as a classroom teacher. While approaching the research with the preconceived anticipation of proving that teaching to a formula was ineffective and was limited in scope, there was unexpected richness that emerged around this insight. While the data did support that teaching mathematical formulas as a calculation procedure was ineffective in enabling students to build a rich understanding of concepts, the data also showed that formulas could be useful in summarizing relationships and connections between concepts. The formulas provided the learners another way of making sense of the concepts they were learning that added to the complexity of the process.

It is tempting for students to view mathematical formulas simply as tools that are useful for calculating an answer to a math problem. The question for students then becomes how to determine which formula is appropriate for a given math problem and then where to put the numbers in the right positions to perform the calculation. When formulas are used this way, there is little space to think about the relationship between the different variables in the formula. Students continue to view formulas as tools created by mathematicians. The origins of where formulas come from are typically dismissed as involving concepts that are far too complex for learners to understand.

In actuality, formulas are not rigid tools that only serve the one purpose of calculating answers. Many of them originate from relationships between variables that are connected in interesting ways. When these rich connections are not explored, I believe students struggle to notice how formulas can assist in conceptual understanding. This is one of the reasons that many of my students do not like what they call “word problems”. Here, Sheila describes what makes her confused in math class.

> It was just really clear when you were explaining the tree diagrams to us and the formulas and then once… See the thing that personally just always screws me up is WORD problems [emphasized]. Like I just... I'm good with equations and formulas and stuff like that but as soon as it's a word problem, then it's sometimes hard for me to see it. (Sheila)

When Sheila talks about being good with equations and formulas, she is talking about the comfort of being able to look at a formula and recognize how to proceed. She is able to base her decisions on examples that she has seen before. She dislikes word problems because they often require a deeper understanding which is different from procedural knowledge, especially authentic probability problems. When asked further to explain her thought processes as she worked through the probability formulas, she described looking for key words.

> Umm. I really talked to myself about the addition rule and the multiplication rule and I asked myself am I looking at 'and' or 'or' and I knew what 'or' meant... to add them. ummm.
> Yeah and then... I knew that. You [the teacher] made a point of saying that 'make sure you multiply first' when you're looking at 'and'. (Sheila)

This shows how teaching and learning about formulas simply as procedures can be detrimental to the learners ability to develop a rich understanding of the concepts around the formula. Even in an environment of building experiential foundations that I was trying to create, students like Sheila showed that they were still eager for formulas and to look for key words or hints on how to proceed without having to think about the math. Realistically, I did not expect to change everyone's approach to learning and thinking about math in the two weeks of teaching that I had with them. While many students showed the same resistance to thinking about math as Sheila did, some others did show growth in becoming more aware of how they learned.

When I present formulas in my class in general, and in this unit particularly, they are never presented as the focal point of the learning. Rather, they are introduced after learners have had an opportunity to create a more experiential foundation of the concept being learned. Then, formulas are introduced as a topic for further discussion about the math concepts and how they connect to each other. This probability unit, and as an extension—counting, include the possible use of several formulas which appear in Table 7 below.

Table 7
Formulas Used in Probability

The “probability formula”	The probability of event E occuring: P(E) = # ways E can happen / # total possible outcomes
The fundamental counting principle	The total number of ways that events A, B, C, ... can happen is: m x n x p... where m, n, p is the number of ways that A, B, and C can occur respectively. (table continues)

The multiplication rule of probability	The probability of events A and B occurring are the probability of A occurring times the probability of B occurring: P(A and B) = P(A) P(B) and for dependent events, the probability of A occurring times the probability of B occurring given that A has occurred: P(A and B) = P(A) P(B|A)
The addition rule of probability	The probability of events A or B occurring are given by the probability of A occurring plus the probability of B occurring: P(A or B) = P(A) + P(B) and for non-mutually exclusive events, minus the probability that both events occur: P(A or B) = P(A) + P(B) – P(A and B)

While these formulas can certainly be used to calculate the probability of various experiments, it is clear to many students that they are ineffective for authentic problems that are even just slightly more complex than the most basic probability experiments.

Consider the "probability formula" for example, in a basic experiment such as tossing a coin—almost every student can answer immediately that there is 1/2 chance of getting a heads. They can even connect this result to the formula by identifying the 1 as the number of ways they could achieve the desired outcome of heads to the two possible outcomes of the experiment. However, if the students' default approach to problems is to think about how to fill up each of the variables in the formula, then they have difficulties with thinking about different variations of the same experiment. What happens if the coin is weighted and the two outcomes are not equally likely? What happens if you toss the coin more than once and are interested in getting two heads in a row? Both the number of ways that these events can happen as well as the total possible outcomes are much less obvious.

Similarly, the *fundamental counting principle* causes considerable confusion for students who attempt to apply it without thinking of the context of the problem. Many counting experiments contain variations and restrictions that affect the final number of possible outcomes. Students who think about how they can make the problem fit the formula often encounter challenges. Here, Kevin expresses frustration with not knowing how to use the fundamental counting principle:

> Like the uhh... these kinds of questions... like you can have a lineup of people? That one was actually a challenging unit for me... probably one I had most difficulty with because I didn't understand the whole multiplication where you had like 4x3x2x1 and then only 4 people could go here then if one was there there was 3 here... that was the most difficult one because of all that stuff but... (Kevin)

Kevin did not know how to approach a counting problem. He had set up blank spaces for each part of the experiment but did not know how to proceed. His thinking revolved around what the values of m, n, and p might be. His confusion and the confusion of many other students are caused in part by the lack of context of the way the formula is used. Students are stuck with a formula that provides little to think about other than the number of ways that each event can occur.

After putting in a number 4 for the number of ways to choose the first person, many students ask the question: “but who was chosen first?” Using the fundamental counting principle here creates a disconnect between the act of counting and the complex nature of the experiment being counted. When probed further, Kevin is able to identify how he would like to proceed with understanding the problem:

> See what I would have done was the 27 x 27 x 27. But then what I didn't understand at the start was that if you have 27, one has to sit there so one person is gone so you would have 26 there.

> And then one person gone it would be 25 and then 24 because I didn't understand the whole line up of one person sitting there and staying. So that's what I didn't understand a lot of that so I would really emphasize that. Like I would give the example where you could visualize it because a lot of this was difficult to see but we did a lot of examples where we could visualize. Like the seating of the photographs and stuff like that was very useful. (Kevin)

Kevin is expressing a desire to think more about how each part of the experiment connects to the other. Unlike Sheila, Kevin has shown a willingness to look at formulas in richer ways than just a tool or procedure for calculating. He is in a transition towards being able to think about formulas in order to help him understand difficult problems.

When formulas are presented as a part of a bigger picture of conceptual understanding, they can serve as a powerful way of summarizing that understanding. An example of this can be shown through the addition and multiplication rules of probability. Tree diagrams are visual representations of probability experiments. Once a tree diagram is created, students can then focus their attention on different aspects of the tree diagram to ask different questions and explore various answers. In the case of the addition rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This formula includes rather abstract elements. The *P(A and B)* can be quite difficult to think about for complex scenarios. This difficulty is compounded if the student does not have a good understanding of mutually exclusive events.

However, when a student has been able to create a good understanding of concepts like mutual exclusivity and how it affects the likelihood of either one of events *A* or *B* occurring, then the formula can add to that understanding in meaningful ways. The subtraction of the *P(A and B)* can be visualized using the tree diagram as learners see what exactly it is that they are taking away: the extra duplicate outcomes that they have counted. Further to this formula, the learners can see when subtracting the *P(A and B)* would *not* impact the overall probability of *P(A or B)*—when there is no overlap between outcomes *A* and *B* occurring. Here, the formula is helping to illustrate an important part of mutual exclusivity and summarizes the idea well: when events *A* and *B* are mutually exclusive, subtracting their intersection is equivalent to subtracting nothing.

Some students were able to show this use of the formula during the unit test. One of the questions asked students to describe any events that were mutually exclusive and to explain why the events were mutually exclusive. One student responded, "Drawing a card from a deck and getting Aces & Kings are mutually exclusive events as there can't be a King and Ace on the same card. You get 0 if you take away the Kings *and* Ace." It was interesting to see students use the formula to support their ideas; to many students, this was not a far stretch from their experiences with the way and reason that formulas have been used in mathematics.

The role of tree diagrams. When I first taught this course in my first year of teaching, I remember spending a day or two exploring tree diagrams as a small part of the probability unit. I presented it as a possible visual representation of probability experiments but not as a focus of the learners thinking about the experiment. As I taught the course again in each of the subsequent semesters and saw the value that tree diagrams provided to the students' learning, the diagrams took on a more prominent role in the unit. Now, when I teach the probability unit, tree diagrams are introduced on the very first day of the unit and are a focal point for learning throughout all of the different concepts in the unit.

This process of the tree diagram takeover occurred because I made two significant realizations. First, I could see that students who were learning through the probability formulas were neither experiencing success in solving problems nor were they understanding probability concepts. Second, I could see that the tree diagrams provided incredible value in building rich understandings of probability. Using them as a focal point brought these values to the forefront of both the learner's consciousness and my own understanding of their learning.

The most obvious value that the tree diagram had was that it served as a thinking tool—that is, it both encouraged and enabled probabilistic thinking in students. The tree diagram achieved this by simultaneously showing the *big picture* of probability scenarios while maintaining their *complexity* and still providing opportunities to focus on details of smaller parts of the scenarios for *calculations*.

Big picture. Many students identified that the most immediate reward of using a tree diagram to approach probability problems was that it helped *simplify* the problem for them. In this context, what students meant by *simple* was that they were able to get a sense of all of the possible outcomes, namely the sample space, of an experiment without being overwhelmed by different cases to consider. This applied even to scenarios that would require far too many outcomes to draw a tree diagram. In those cases, the tree diagram would help build a mental picture of the sample space.

Once students felt like they had a grasp of the sample space of the scenario, they felt confident that they could determine the probability of any desired event—if not by counting the outcomes they wanted in comparison to the total outcomes possible, then by multiplying the probabilities along the branches of the tree. One student described the use of tree diagrams the following way in one of the interactive writing entries, "I understand the tree diagrams well because I find they are a clear and simple way to solve a question." When asked to explain further about this statement, the student said, "Because when you're doing math it can get complicated but with this we can just see 'ok, if you want this for this and yes for that' then the answer is right in front of you."

Students often equate something *simple* with something that does not have many things to remember. For this reason, many formulas are not considered simple to students; they have to remember the formula itself, which scenarios it applies to, and what values go into each of the variables, as well as variations to the formula and how it is used.

What many of the students who participated in this project are not aware of is that while they feel that tree diagrams simplify things for them, they are actually enabling them to see the probability scenarios in *complex* ways. The reason that students used the word simple to describe tree diagrams was because there were not a lot of things to remember. Instead, creating them and working with them required thinking about the context of the scenario and how it could be translated into a visual representation rather than trying to make the scenario fit a generalized formula. This is not an easy task and does lead to a lot of questions and mistakes made by students. However, when students get stuck on the tree diagram, the questions they ask and the concepts they think about require them to think probabilistically. These questions often expand the students' thinking to imagine the impact of changes to the sample space through subtle changes to the probability experiments. Thinking about these types of questions is the essence of what it means to think probabilistically. When stuck on a problem, students are equipped not only to think about examples they have seen or formulas that they have worked with, but they are able to think specifically about the problem's context in conceptual ways.

Details for calculations. While the tree diagram may have created a framework for students to think probabilistically; many of them may not have been more aware of this thinking. However, another result of using the tree diagram that may have been more obvious to students was the ability to use it as a tool to calculate answers—in this case, probabilities to different desired events. The unit test included a question that many students found challenging. The probability scenario described two people trying to start two different cars on a cold winter day. Probabilities were given for each of the cars successfully starting, both unrelated to each other. Students were asked to determine what the likelihood is that both of the cars would start. Here is Kevin's reflection about that question:

> I just did like what you taught us to do really. It's just write out all your stuff; you don't want to leave anything out. I didn't write the tree because you know you told us on the last day that trees aren't necessary—they're a great step—but I didn't find it was necessary for this because I could find out all of the examples of what could happen. (Kevin)

When asked to explain how the student knew to multiply the two probabilities, he responded, "Uh because... [pause] because it's like both will start so Colette's car is .45 and Gabrielles car is .62 so I know the chances of them both starting would be .45 x .62 because both events have to happen."

While Kevin shows that he is still thinking a lot about examples he has seen, he is also showing how he uses the tree to understand this scenario. Saying that both of the events have to happen shows an understanding of why the probability of a desired outcome decreases when a second event occurring is required. Not only does the desired outcome include Colette's car starting which is only 45% likely, after that happens, Gabrielle's car needs to start which would be 62% of the 45% times that Colette's car started. The probability of both cars starting is calculated by $(0.45)(0.62) = 27.9\%$.

Kevin shows that the tree diagram helped him see the outcome that he wanted out of the four possible outcomes. Seeing the path along the branch that he wanted to take also helped him think about how to calculate the final probability of the outcome he wanted. He did all of this without referencing or thinking about the multiplication rule of probability. If any of the probabilities changed, or even the desired outcome changed to wanting both cars not to start, then Kevin would be able to use his understanding to adapt to those changes in the scenario when calculating his answer.

There was a second part to that question that was more difficult. Students were asked to determine the probability of only one car starting. Students find questions like this very challenging because there is more than one case or outcome to consider. They are interested in a car starting but are not picky about which car that is. This example with the two cars is relatively small compared to possible scenarios that would become even more complex without a rich understanding of which outcomes are desired. In this case, and for even larger scaled problems, tree diagrams can be useful in organizing the sample space so that students can think about which outcomes they want. In this case, as they look at each of the 4 possible paths, they can see and consider which ones they want—the ones that have exactly one of the cars starting.

The same student, Kevin, was able to get part b of the question correct. Here, he is reflecting on his approach to the question:

> This was actually… Like I pretty much did this part from example A because I already had all of the given material that I needed. Like one car will start so it would be either Colette's and Gabrielle's wouldn't so it would be .45 and .38 and then if Gabrielle's started then Colette's wouldn't so the chances that she wouldn't was 100-48 which would be 62 percent or 0.62 times .55 which were the chances that Gabrielle's would start. (Kevin)

One of the most common errors that students make when solving probability problems is to either add probabilities when they should multiply or multiplying probabilities when they should be added. I asked Kevin how he knew to add the two outcomes he wanted rather than multiply like he did in the previous part to this same question. He responded:

> Uh... because it was the chances that only one car would start.. you had to use both examples like because in the end it would be the chances that both of them would start... it wasn't just if hers will start but both of them. (Kevin)

While he could have certainly articulated his ideas more clearly, I could see that Kevin was developing an understanding of when the probability of a desired outcome is expanded due to accepting a wider range of possibilities (like in this case with accepting either one of the cars not starting). Similarly, he was developing an understanding of when probabilities would be restricted due to a more limited definition of what a successful outcome would be.

Consider changing the question even further to ask the student to determine the probability of *at least* one car starting. A seemingly subtle change can be overlooked if a student is not able to see the big picture of the scenario while still maintaining the complexity of the relationship of the whole system. In using the tree diagram, this subtle change simply directs the students' attention to look at more branches that could possibly satisfy their desired outcome.

Maintaining complexity. As a teacher, while it is a benefit to provide students with a tool to perform calculations effectively, it is certainly not the greatest benefit. The greatest benefit of using the tree diagram is that it maintains and embraces the complexity of what it means to think probabilistically. From my experiences in using the tree diagram, I view its ideal use for students is as an organizational cognitive framework. This is extremely valuable because authentic probability situations would not actually typically lend themselves to tree diagrams. There would be too many outcomes or too many levels to fit on a page. It is important to consider why teaching and learning through visual representations like a tree diagram may still be useful if it is not practical to draw them for every problem.

I asked Trevor about how he remembers working through the same question on the unit test—cars starting on a cold winter day. I let him know that he did very well on the question without even drawing a tree diagram and asked him to reflect on his approach:

> Alright, so when I started the car starting question... when it initally says to create a sample space, I thought to go and define what each of my outcomes is represented as so I started off with 'does the car start or not start' for the first person Colette and then I did the same thing for Gabrielle if her car started or not started. And then from that there were only two outcomes for each person so I just basically wrote down the tree without ever having to draw it because I just

> saw the outcomes as almost like multiple straight lines instead of it branching out ... I just started at the end and worked my way back. (Trevor)

Trevor has realized that the actual structure of a tree diagram is not the important part of the exercise; nor is the shape of the tree crucial to understanding probability or to finding the right answer to this question. What he has realized is that it is powerful to be able to organize the way he is thinking about this mathematical scenario. He is not only listing off possible outcomes, but he is being organized in the way he is thinking about them—this leads to his confidence that he is considering all of the outcomes he wants.

Trevor continues to describe how he knew to multiply along the branches as opposed to adding them.

> I've been working with probabilities for quite a while. And I knew to multiply because it's two different chances out of 100 I guess is the way I saw it. It has to be... there's only a 100 total that you can go out of so I knew the multiply because that's the chance of one event and the other. (Trevor)

This response is unique in comparison with other students. A common response would be similar to what Kevin expressed about considering that the second event occurring would comprise of that percentage of the probability of the first event occurring. For example, 62% of the 45% would indicate that there is a likelihood of 27.9% for both cars to start. However, this quote shoes that Trevor is thinking about how he might have two very likely events (over 50%) and if he wanted both to happen and he indeed added them together, he would get a final probability that was greater than 100% which is impossible. Not only is Trevor using the tree diagram to think about this problem, he is using it to explain his thinking. Also, he is using the concept of complement events (thinking about why he *shouldn't* add the probabilities rather than why he *should* add them). A potent and rich way to think about how an idea is defined is to not only think about what that idea *is* but to spend as much time thinking about what it *is not*. This is another core aspect of thinking probabilistically.

Student identities as math learners. Enabling students to shape, and in many cases *reshape*, their identities as math learners was an underlying theme of the entire research project. A few unexpected findings were made apparent from the data.

Looking back to the beginning of this project when I began writing my proposal, I remember believing that if I could help learners overcome some of their issues with understanding mathematics, that they would enjoy learning math more. After talking to some of the students during and after the learning and research was complete, I was able to reflect on their experiences with learning math during this unit and in general. This shifted my perspective on student identity and math learning; the conversations illuminated the notion that some students not only did not like math, but they did not *want* to like it. This disposition could still be a defense against the negative experiences that the learner has associated with mathematics, a lack of interest in thinking mathematically, or another issue altogether.

Differences are to be expected in any diverse group of students. While a few students were eager to reflect on and discuss their experiences with learning math, many other students were not aware of how they learned mathematics. Some students were just aware that they have had a difficult time making sense of mathematics but attributed these difficulties to exterior factors such as the teacher or the math concepts themselves. Other students had given up on thinking of themselves as math learners altogether. The next two sections outline the differences in identities that I noticed in the students followed by how the students were able to expand their repertoires for learning math by becoming more aware of their identities.

Differences in identities. Naturally, students create identities of themselves as learners from a very young age. Experiences in school, particularly in math, shape these identities in both positive and negative ways (Ashcraft, 2002). By the time a student reaches the grade 12 Applied Mathematics course, each student is at a remarkably different stage in building their identity. This difference is not only marked by the stage of their understanding of mathematics or their confidence in learning mathematics, but is rooted in how they see themselves as learners of mathematics. That is, how do they see themselves as learners who are able to get better at mathematics. This difference is complex because it is connected to many things. In the short time I spent with this group of students, I was able to see a wide variety of learners at different stages of building their identities. The students showed a wide range of varying awareness of themselves as learners. Some students were quite aware of how they learned mathematics and were able to expand on that awareness while other students had very little insight to how they learned. Many of these students expressed through the interactive writing activities that many of their past frustrations, confusion, and difficulty with learning math came from outside sources.

When asked to describe a moment in class that made them confused, more than one referred to a minor glitch with the LCD projector or an action such as missing notes.

> *Student*: Rather confused when the projector stopped working multiple times. Other than that followed lesson pretty well.
>
> *Student*: I was confused because I missed the notes from last class.
>
> *Student*: Fundamental Counting Principle. It just seems to confuse me and I need to practice more.

It is a fair statement to make that these two incidents would affect the learning in a class, but they are certainly not the essence of the learning. These responses came naturally to the open ended nature of the question being asked; they showed what stage a student was at in building their identities as able learners of math. In this case, the students were quite a distance from being aware of how to get better at learning. The one student who was able to identify the topic that caused the confusion still only looked to the action of practice as a way of learning better.

Other students responded to the same question in ways that showed more thoughtfulness about why learning some concepts were difficult. One student said, "Combinations and permutations, it is hard to tell the difference between them. In the notes the difference is clear but in a real problem there is some confusion knowing which one to use." This student was able to identify the reason for his confusion. In addition to recognizing the specific topic, he expressed the cause of the confusion was not being able to distinguish between the two types of counting scenarios. The space that this student is in lends itself to being guided towards a conceptual understanding of a topic such as combinations and permutations. The complexity of the identity that this student is showing in comparison to any other student who might be confused by the same topic is illuminated by the question: what type of experience would best assist this student in making that move towards understanding. It takes not only the teachers insight to this student's thinking, but the student's own self-awareness of their understanding and how to best move that understanding forward.

Expanding repertoires for learning. Within the complex differences of the learners, one insight consistently emerged: learners who became more aware of how they learned mathematics were able to expand their repertoires of processes for learning. To illustrate, when Trevor described the confusion he experienced while approaching a counting problem on the unit test, he spoke of how he thought to himself during the confusion:

> When I read that it was 3 women and 1 man I automatically assumed that the order mattered but then as it was a combination I thought well it could be in any order. So I thought that adding the chances of each one together would give me the right answer which it did not.

(Trevor)

Trevor is showing that even at the moment when he is talking to me in the interview, he does not have a good understanding of the problem. He describes taking a guess at how to proceed but is also able to recognize that it did not lead to the correct answer. The ability to recognize that the answer is not correct shows conceptual understanding of the problem. Further, what is impressive about Trevor's reflection is that he is thinking mathematically and probabilistically about the question—asking what he needs to understand more, not just what he can do to get to the right answer.

During the closure interview with Trevor, he presented as a student who was quite aware of his learning. This awareness enabled Trevor to expand his repertoire for learning math. Here, he describes his experience learning probability with a visual representation as the main focus:

> For my approach to learning it was quite different because for me it is usually the numbers first before the visual representations whereas with the trees... it wasn't difficult for me to understand as from a young age I've always enjoyed gambling. I mean taking chances on doing something. So the trees actually did help me in a way that I wasn't used to so it was kind of odd for me but at the same time I could understand it very simply because I was used to a lot of events involving probabilities and chances. But yeah, the way I've been taught recently—it works—for the initial understanding and then going further from that. It's almost as if I was going backwards starting with the visualizations and then into numbers. (Trevor)

For Trevor, this two week period of learning probability in a different way was not just a novelty for which the excitement would wear off over time. Instead, his experience during those two weeks shifted his positioning in both his understanding of mathematics and his identity as a math learner. This was possible because he has had previous experiences in the years prior to grade 12 to think about how he learned math. These opportunities for reflection positioned Trevor to be in a place of readiness for moving forward in his identity building.

Trevor moved forward in his identity building by adding to his repertoire of ways to learn math. Trevor made a remarkable transition from self-identifying as someone who could do math but someone who had difficulty thinking about it more conceptually to a learner who was able to use abstract visual representations to think richly about mathematical concepts. In admitting that using tree diagrams 'work' in building an initial understanding, but also admitting that the feeling is uncomfortable and definitely new, Trevor is acknowledging the powerful potential of approaching learning a new concept in math not through examples but through concepts.

6.2 Answering My Research Questions

In this section, I formulate answers to my three research questions. My first question *was how can a mathematical tool such as a tree diagram assist in moving a student's mathematical learning from computing algorithms to conceptual understanding?* An important shift in my own positioning was to move away from completely disregarding the usefulness of computing algorithms. Learning is complex and taking an instrumental approach to developing a strategy for calculating answers has a place in the big picture of understanding. The tree diagram showed that it can assist learners in doing just that. However, the potential of the tree diagram was not just in finding the right answer, but to be able to make sense of it.

I posed this research question to evaluate what about tree diagrams I considered to be so valuable in learning probability. The research question asks how mathematical tools *such as* the tree diagram can move learners towards conceptual understanding. This question implies that there are characteristics of the tree diagram that hold true for other mathematical tools as well.

Visual tools can be valuable learning tools by providing structure for rich thinking (Simmt et. al, 2012). The tree diagram helped with the cognition of learners as they worked through probability problems. The structure of the trees supported understanding through the flexibility they had to adapt for contextual complexity. Rather than just give students something to *do*, the trees enabled students to approach authentic problems with something to *think* about by providing a framework for that thinking. This holds true for other visual diagrams such as the Venn diagram for learning set theory, and network graphs for matrices (Zimmermann & Cunningham, 1991). In creating each of these diagrams for their respective problems, learners would be required to think about the context of the different scenarios. The diagrams would provide many opportunities to think—particularly about complex parts of the problem that may be causing confusion. When a learner is confused about a set theory problem, creating or analyzing a Venn diagram leads to questions about what is distinct or common between the different sets—questions that are at the core of the concept of sets.

By taking the default position of creating a visual representation, a tree diagram, for every new problem they approach, students in this project were guided towards looking at the big picture of the problem. This means that students were able to see different parts of each problem simultaneously and how they connected to each other. They were able to see how changes to one part of the problem effectively changed other parts of the problem and the final outcomes. Seeing the big picture in an organized way provided an entry point for every student in a non-intimidating manner. Even students who did not know how to approach a problem had something to think about that did not require memorization or mastery of an algorithm—all they needed was an ability to think about possible outcomes of a probability experiment. They could experience some success even if they may have missed some outcomes or organized them incorrectly. Regardless of how far along a learner was in developing conceptual understanding of probability, they would be thinking probabilistically along the way.

On two different levels, the use of visual diagrams effectively moved students from the use of algorithms towards conceptual understanding in solving authentic problems. First, the diagram structure set up opportunities for students to see the big picture of a problem while maintaining its complexity. Second, they provided a means to calculate the answer to the problem in a way that still involved thinking about the details of the problem rather than an algorithm that was disconnected to the context of the problem. Students generally appreciated this move towards conceptual understanding—feeling that the approach was simple, yet challenging in rich ways.

When I first asked my second question, *further, what role can the tree diagram have in enabling students to show their learning and conceptual understanding,* I was aiming to determine how useful tree diagrams were as assessment tools after the learning has happened. Since completing the research, I realized that the usefulness of tree diagram as an assessment tool cannot be separated from its effectiveness as a learning tool; I have shown that tree diagrams are effective because they are valuable *during* the learning. However, this insight does not imply that tree diagrams cannot show how the quality of understanding a learner has about probability—they can actually be very valuable devices for showing what a learner is thinking.

Even during the first stage of creating a tree diagram, a lot can be seen of the learners' conceptual understanding. Tree diagrams can be created to incorrectly represent a probability experiment. This visual is an opportunity to see if a student has a misunderstanding of the scenario, the possible outcomes, or the likelihood of any part of the experiment happening. Some common errors in creating tree diagrams include: separating parts of the experiment into different trees, showing levels and paths of the tree that do not represent an outcome, or incorrectly showing the weighting of branches. These types of errors provide insight to what the student is thinking.

Another useful aspect of the tree diagram as assessment is by serving as a conversation point. Tree diagrams are created as visual representations of experiments or scenarios; they do not have to be connected to a specific question about that experiment. Once a diagram is created, many questions can be asked about the experiment with the tree as a focal point showing the depth and quality of a learners understanding. Asking about sample spaces, dependence, mutual exclusivity, or the probabilities of any of the outcomes provide opportunities for learners to talk about and show their understanding.

Further, having the tree diagram as a central point of a problem provides an opportunity to think about how changes to the problem impact the overall probability of any of the outcomes. Learners can show their understanding by talking or writing about how the diagram would be affected by adding or removing an outcome to a part of the experiment. Thinking about the changes to the size of the sample space is integral to understanding the likelihood of any desired event. Through talking and writing about different aspects of the tree diagram, learners are constantly showing themselves and the teacher different aspects of the quality of their learning and understanding.

While my third research question, *how does an emphasis on conceptual understanding as opposed to a more instrumental approach impact student perception of what it means to understand mathematics,* aims at addressing the general implications of learning math for conceptual understanding, it is closely related to the theme of student identity. One of the most significant indicators that I use to assess my success as a math teacher comes from a simple question: do you enjoy learning math any more than when you started this course? It may appear to be trivial to give so much weight to a simple question, but the importance of the identity that a learner brings to a new learning experience as well as the identity they build during that experience cannot be understated. Recognizing that learners are very different from each other is the first step in understanding the role of identity-making in shaping student perception of themselves as learners. This project illuminated how learning through conceptual understanding can affect that identity building.

Sheila and Trevor were students who both started the two-week unit on probability with previous experiences with math learning that shaped their identities in significant ways. Sheila began the unit with the belief that math was not a subject that she liked, nor one that she wanted to like. Her identity was shaped by the negative experiences she has had in learning math; this resulted in her not being able to see herself as an able learner. Although the two-week unit on probability appeared to be different, she did not trust that being pulled back into engaging with a subject that she has had such negative experiences with would be worth her time and energy. The experience during the research showed her that it was possible for her to understand math but that it would take time. In addition to taking time to build a richer understanding of math, it would take even more time to approach problems in meaningful and authentic ways, as opposed to approaching them with an instrumental perspective. Realistically, the decision on whether Sheila would invest this time or not into learning math could not be swayed during a two week period. However, even putting her, and other students like her, on a path to becoming aware of that decision can be considered a success. They are seeing the possibility of shifting their perceptions of themselves as a type of person who is able to engage with math in different ways.

Trevor also had experiences that shaped his identity significantly. While he also experienced some challenges throughout his life as a math learner, Trevor's learning journey typically led to success with school mathematics. He attributed his challenges to exterior factors like the teacher's teaching style rather than internal factors like how he learns. Like Sheila, he had a predominantly instrumental approach to learning math. The experience he had learning probability through conceptual understanding played a significant role in shifting his identity as a math learner. Prior to the research project, he did not see himself as learning math through abstract ideas, but rather by seeing examples of how algorithms and procedures can be executed. Throughout the learning of probability, he was able to become more aware of what it could be like to learn math differently. While he certainly felt the discomfort of being confused, he appreciated the value of building a richer understanding of the concepts he was learning—an appreciation that shaped his identity in a significant way over the long term. He has continued to study mathematics after his experiences with this project and is achieving both academic success as well as enjoyment of learning math that he has never experienced before. While it is not reasonable to give credit to his short experience with learning probability with tree diagrams, it is clear that the experience did play a role in shaping his identity.

Incidentally, Trevor did visit me to have a conversation a year after the two-week period we spent together. He wanted to share that he decided to return to high school to take some higher level math courses to qualify for the university program that he wishes to enter. He related that the experiences that he had during those two weeks and the conversations we had afterwards did play a role in his decision making. Shortly after our closure interview, he had committed to not only come back to school for those courses but to make the year a *learning* year, not just an *achieving* year. That is, he was committed to and achieved a deeper enjoyment from engaging with higher level topics in rich ways. I could sense the satisfaction that he felt as he described not only excelling in the courses as he achieved marks in the high 90s but also how much more he enjoyed learning and encountering difficult concepts.

6.3 Positioning and Next Steps

Going into this research project, I had a strong inclination as a teacher that tree diagrams could play a significant role in the thinking, learning, and developing understanding of probability. The project itself had a powerful influence on deepening my understanding of this role as well as bringing to light other insights that affected my positioning and readiness to answer my research questions. In answering the research questions, I was able to reflect on what the insights mean to me as a teacher as well as what they could potentially mean for learning and teaching math in high school.

Nearing the end of the writing of this thesis while I was preparing for the final chapters, my thesis advisor asked me if I was able to identify a single theme as the *most* important to me. I thought about it and determined that the most important theme—the reason that provided a foundation for the work I have done over the years—was to show the incredible value of providing rich and active learning experiences in the math classroom. The intricate details of how such a big idea could or should come to fruition in a classroom are all linked in a complex system. Learners bring complex identities to the classroom, teachers have to address competing demands from various stakeholders, and active learning for understanding can be very challenging for both teachers and learners—at times uncomfortable and messy.

Data showed that the range of positioning of learners within mathematics is alive and well. Students see themselves relating to learning math in many different ways, but any move towards becoming more aware of that relationship is a step towards getting better at learning. By the time some students reach grade 12, they are too far along in their resentment and mistrust of math that an introduction to learning math in a way that can engage them in thinking is not enough to change their perception of both themselves and math as a subject that they can learn.

Current research in mathematics education state that there is remarkable value to learning actively through rich tasks and for conceptual understanding (Munter, 2014; Schoenfeld, 1992). Understanding and implementing better pedagogy for teaching math is not enough to solve all of the problems of school math. There is no simple cure to the complex system of relationships between learners and school math. However, the potential of providing experiences that shape student identities in positive ways so that they can continue to learn math and think mathematically is a powerful idea.

In reflecting on the details of the project, there are some areas that worked well and other areas that did not happen as planned. Some of these challenges arose from the necessity of conducting the research over a short period of time and with an unfamiliar class of students. This experience made me curious to continue learning about the role of pedagogical relationships and how relational learning can impact the shaping of students' identities. However, a benefit of conducting the research over a short period of time was that it required me to notice and listen to the student learning intensely as both the teacher and researcher. This listening was valuable in contributing to my decision making as a teacher. Noticing how the students were learning informed me of what was happening, whether students were learning well, how I knew they were learning, and how to better cause that learning. This positioning as an inquiring teacher will most certainly be a habit for me as I continue to teach.

The next step for me and others as math teachers is not to point to curriculum redesign or even restructuring of classroom activities to be more contextual and engaging. These things can come later. The most important first next step is to look at ourselves as learners and to understand the profound impact that our development as professionals can have on the students that come into our classrooms each year. In developing the themes of this thesis, there is a robust theme that has developed concurrently as an element of the other themes—and that is my professional learning through inquiry as a result of this project. I address this theme separately in the next chapter.

Chapter 7 – Professional Learning Through Inquiry

7.1 Writing Chapter 7

The idea for writing this additional chapter began in December of 2014—6 months after completing the data collection. I was preparing to teach the probability unit for the first time since conducting the research for this project the previous semester. It was a coincidence that I was preparing to teach the unit again just as I was preparing to write the findings of my thesis. I noticed that I was looking forward to teaching the unit more than I usually do. There was a different kind of anticipation during the planning; I felt better prepared to not only create rich learning experiences for a new set of students but also better prepared to listen to their experiences and to notice their learning.

I realized that this anticipation was a small part of a larger phenomenon—that my experiences throughout the process of completing a master's thesis had become a powerful form of professional development. This was actually an unexpected theme that emerged. When I began this thesis, I believed that I had something to offer to the math teachers at my school, my school division, and to the research community at large with regards to teaching mathematics for conceptual understanding. I felt like I was out to provide evidence that the benefits of teaching and learning mathematics this way could make significant changes and improvements to school mathematics. Since completing the research, I have taught the probability unit to four different sets of students. As I am nearing the end of the research and writing process, this unexpected theme made it apparent that what I *gained* as a professional from the process was just as powerful.

7.2 Professional Learning from a Master's Thesis

I applied for and was accepted into the Master of Education program at the University of Manitoba in 2011—two years into my career as a high school math teacher. I entered the program that early into my career because my experiences during my first year of teaching had drawn my attention to many questions about teaching and learning. At the end of my bachelor of education and into the beginning of my first year of teaching, I spent the entire summer preparing for the courses I was to teach the following year. That planning was focused almost entirely on creating lesson plans based on the outcomes of the curriculum documents. As the year went on, my lesson plans worked well in the sense that they helped me stay organized. The administrators at the school all gave me very positive assessments and feedback—they spoke highly of my organization and content knowledge. While these could be considered positive steps in an early career, something felt wrong—my experiences as a teacher were very positive but I could see that the learning experiences of the students was not as positive. The students did well on the assignments and tests but I could tell through conversations with them that they did not have rich conceptual understandings of the math they were learning—they were often eager to learn the formulas and most conversations centred on how to calculate the correct answers.

Entering the master's program and starting the coursework immediately provided me with things to think about as I explored reasons for the learning challenges of the students in my class. Possible reasons were made apparent through journals, books, and interacting with other graduate students and professors. The notion of building experiential, rich, and complex learning experiences for conceptual understanding contrasted my teacher-centred focus on breaking down complex ideas into small parts and teaching algorithms and procedures for an instrumental understanding of mathematics. This contrast led to my topic for this research project for which I developed the research questions.

When I read the questions now, three years after originally posing them, I remember the excitement I felt when I had written them. I was excited to begin my development and growth in learning to build rich learning experiences for students. That excitement turned into confidence as I gained experience teaching the course again and again—each time making significant improvements in creating richer learning experiences. The ideas I was developing through reading and writing for the thesis proposal appeared to address the challenges that the learners in my class had experienced. Students were more authentically engaged and asked more meaningful questions in our conversations. Rather than look for formulas and how to calculate answers, students were more eager to engage in discussions, ask questions, and explore concepts. The excitement I felt that had turned into confidence then turned into a strong belief that the transition I was making as a teacher was benefiting the students in profound ways—this led to a feeling of deep satisfaction and fulfilment.

As I wrote my thesis proposal and continued to teach, I saw from my experience that teaching and learning for conceptual understanding was at the core of the improved engagement with and learning of mathematics in my classroom. I chose to use probability as the unit of study for my thesis because it was the first topic in mathematics that drew my attention to the problems that learners were facing when they learned it as a set of procedures. I wanted to provide data to support the idea that teaching probability with tree diagrams was a better way to foster and enable conceptual understanding. While my desire to collect data did yield a lot of evidence to further support my beliefs in learning for conceptual understanding, it also led to another form of professional development. While I was conducting the research, I expected to see valuable data; what I did not expect was the amount of learning that I would experience during the data collection, analysis, and writing process. In other words, while I expected to learn from the data as a researcher, the impact that the project had on my learning as a teacher was more significant than I had anticipated.

As stated earlier, my position first shifted from a teacher-centred approach and lesson planning toward a student-centred approach of designing learning experiences. This was a dramatic improvement that can be credited to the coursework of the master's program as well as the reading I did during the writing of the proposal. My belief that I have improved as a teacher is a profound one that leads to some significant implications—particularly regarding how that improvement is made visible. It can be argued that an improvement in my teaching should imply an improvement in student achievement. However, even student achievement can be measured in many ways—results on a standards test, a shift in attitude towards learning math, or an ability to learn future course work. In the context of this study, it is unclear if the students in the class performed better on the standards test because of their learning experiences with tree diagrams. It is also unclear how their learning experiences with tree diagrams will impact their learning in the future. What is clear from this study is that my professional judgement of what counts as rich mathematical learning as well as my judgement of the quality of that learning in students has been enhanced. Based on this judgement, my students' achievement in mathematical learning has improved. The enhancements in my professional judgement of both math learning and how to notice that learning has been central to my journey in developing as a professional. At the heart of this journey is my emergence from knowing as a teacher to that of a teacher-researcher.

The researcher role that I developed enabled me to truly listen to students as they made sense of difficult math problems. Teachers are not always able to have in-depth conversations with students—particularly in June after the learning has occurred. These conversations along with the revisiting of teaching tools and assignments provided many valuable insights to the processes that students went through as they grappled with learning math concepts. Assignments and interactive writing are both mechanisms that I have used for years prior to this research project. While they informed my teaching decisions in the classroom, it was particularly interesting to look at them from a different stance—that of a researcher. I was able to draw different value and information about student learning from this analysis than the teacher role in me would. While as a teacher I would likely be focused on viewing this data in order to make teaching decisions for the next day in class, the shift towards research provided a broader view of the connections between teaching and learning in the class.

Among the various types of quality learning that I experienced in working through a master's thesis, it was the teaching of the probability unit to a new set of students that ran concurrently with my data analysis that really made apparent to me the potential of using a master's thesis as professional development. However, as a beginner researcher, it would not have been possible for me to gain as much value from the process of analyzing data without the guidance of my thesis advisor.

Thesis advisor as coach. Another element of the process of this research project that was pivotal to my professional learning was access to my thesis advisor as a coach. This was my first research project of this magnitude and the advice and guidance of my advisor has been extremely valuable. The value of this guidance was particularly apparent during the data analysis and interpretation stages of the research project. If it were not for my advisor as a coach, I would not have been able to trust my instincts and how they could help me interpret the complex data. I was able to connect what I was able to capture on paper with what I could recall from the living experience of collecting the data—a connection that enabled me to capture and create a rich portrayal of the data. In the context of writing a master's thesis, the development of an ability to analyze data to make connections is valuable. However, in the context of shaping the journey of a teacher striving for excellence—being able to listen to students as they are learning is essential.

Collaboration with my thesis advisor as coach played a significant role in the emergence of my professional judgement. Collaboration in this sense was not limited to me as the learner receiving information but rather the building of a pedagogical relationship with the advisor. It is in the nature of a trusting relationship that both parties can gain an understanding of how to learn and grow. The growth that I experienced from this collaboration will impact how I position myself for future collaborations.

7.3 Teacher Inquiry

The analyzing and interpreting of data from students who were learning probability while concurrently listening to students struggling to learn probability has proven to be a potentially powerful form of teacher inquiry. I certainly credit the experiences that I had with working on this master's thesis as significantly moving me towards becoming an inquiring teacher. Inquiring teachers are described as those who base their decision making on questions about their teaching—questions framed around what the teachers are doing in their classrooms and how they know it is working (Liljedahl, 2014). Inquiring teachers want evidence of success to come from their own experiences and classrooms so they can better make decisions to improve those successes.

This process of listening and noticing the learning as it is happening was illuminated by my positioning during the research as a teacher-researcher. This positioning was defined by the principles of the practitioner-research methodology that informed my study. As I was noticing the attributes of the students' learning, I began to notice my own learning as a teacher show similar attributes. Through engaging students in problem solving, I noticed the cyclical relationship between how rich tasks with the tree diagram enabled the students to think probabilistically. In turn, thinking probabilistically opened up opportunities to engage in more rich tasks. As a teacher-researcher, engaging in the rich tasks of noticing how students learn gave me insight to the complex structures of that learning. Having a better understanding of how students learn enabled me to position myself to be a better listener.

My stance as the teacher-researcher in this study guided me in challenges that I encountered as both the teacher and the researcher. During the first of the closure interviews, even after the teaching and learning of the unit had ended, I had not transitioned to my role as the researcher. The turns I took in that conversation were very long and came from my role as a teacher. I learned from that experience that allowing myself to listen is an important part of being a researcher—especially during conversations that are intended to honour the interviewees' perspectives.

While the master's thesis has illuminated many relevant themes and issues around the teaching and learning of mathematics in school, it is worthwhile to pay some attention to the value it brings to professional learning as well. When I started this thesis, I originally intended to prove that learning for conceptual understanding was superior to learning through procedures and formulas. The thesis did not succeed in proving this point because the connections and contrast between conceptual understanding and procedural understanding are not simple and clear, but are connected in complex ways. The thesis did succeed in showing several powerful ideas: that listening to students to make decisions can be the foundation of causing better learning and that listening to students thoughtfully can be a powerful form of professional inquiry and development. The story of my professional learning journey encapsulates this idea—that there is a complex relationship between developing an understanding of what it means to learn math and the professional judgement of the quality of that learning.

References

Albert, L. R., & Mayotte, G., & Sohn, S. C. (2002). Making observations interactive. *Mathematics Teaching in the Middle School*. *7*(7), 396-401.

Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science*, *11*(5), 181-185.

Aspinwall, L., & Shaw, K. L. (2000). Enriching students' mathematical intuitions with probability games and tree diagrams. *Mathematics Teaching in the Middle School*, *6*(4) 214-220.

Batanero, C., & Sanchez, E. (2005). What is the nature of high school students' conceptions and misconceptions about probability? In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 241-266). New York: Springer.

Borko, H., Whitcomb, J., & Byrnes, K. (2008). Genres of research in teacher education. In M. Cochran-Smith, S. Feiman-Nemser, & D. J. McIntyre (Eds.), *Handbook of research on teacher education* (3rd ed.)*, Enduring questions in changing contexts* (pp. 1017-1049). New York: Routledge.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem solving framework. *Educational Studies in Mathematics, 58*, 45-75.

Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. San Francisco, CA: Jossey-Bass.

Clandinin, D. J., Huber, J., Huber, M., Murphy, S., Murray Orr, A., Pearce, M., & Steeves, P. (2006). *Composing diverse identities: Narrative inquiries into the interwoven lives of children and teachers.* New York: Routledge.

Clement, L. L., & Bernhard, J. Z. (2005). A problem-solving alternative to using key words. *Mathematics Teaching in the Middle School, 10*(7), 360-365.

Cochran-Smith, M., & Lytle, S. (1993). *Inside/outside: Teacher research and knowledge.* New York: Teachers College Press.

Creswell, J. (2007) *Qualitative inquiry & research design* (2nd ed.). London: SAGE.

D'Ambrosio, B., Kastberg, S., & dos Santos, J. R. (2010). Learning from student approaches to algebraic proofs. *Mathematics Teacher*, *103*(7), 489-495.

Dick, J., & Childrey, M. (2012). Enhancing understanding of transformation matrices. *Mathematics Teacher, 105*(8), 622-26.

Ely, R. E., & Cohen, J. S. (2010). Put the right spin on student work. *Mathematics Teaching in the Middle School, 16*(4), 208-215.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12-21.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Huntley, M. A., Sasmussen, C. L., Villarubi, R. S., Sangtong, J., & Fey, J. T. (2000). Effects of standards-based mathematics education: A study of the Core-Plus Mathematics Project algebra and functions strand. *Journal for Research in Mathematics Education. 31*(3), 328-361.

Johnson, M. L. (1983). Writing in mathematics classes: A valuable tool for learning. *The Mathematics Teacher, 76*(2), 117-119.

Liljedahl, P. (2014). Approaching professional learning: What teachers want. *The Mathematics*

Enthusiast, 11(1), 109-122.

Lockwood, E. (2012). Counting using sets of outcomes. *Mathematics Teaching in the Middle School, 18*(3), 132-135.

Lytle, S., & Cochran-Smith, M. (1992). Teacher research as a way of knowing. *Harvard Educational Review, 62*(4), 447-475.

Manitoba Education, Citizenship, and Youth (MECY). (2006). *Rethinking classroom assessment with purpose in mind.* Winnipeg, MB: MECY.

Manitoba Education, Citizenship, and Youth (MECY). (2009). *Grades 9 to 12 mathematics: Manitoba curriculum framework of outcomes*. Winnipeg, MB: MECY.

Mason, R. T., & McFeetors, P. J. (2002). Interactive writing in mathematics class: Getting started. *Mathematics Teacher, 95*(7), 532-536.

McFeetors, P. J. (2008). Using student data for teacher reflection. *Canadian Journal for New Scholars in Education, 1*(1), 1-17.

Moore, D. S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: New Approaches to Numeracy* (pp. 95-138). Washington, DC: Mathematical Sciences Education Board and National Research Board.

Moustakas, C. (1994). *Phenomenological research methods*. Thousand Oaks, CA: SAGE.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584-635.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Patton, M. Q. (2002). *Qualitative research & evaluation methods* (3rd ed.). London: SAGE.

Polya, G. (1945). *How to solve it*. Princeton, NY: Princeton University Press.

Romero, J. G., & Marí, J. L. G. (2006). Assessing understanding in mathematics: Steps towards an operative model. *For the Learning of Mathematics, 26*(2), 10-15.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Seidman, I. (2006) *Interviewing as qualitative research: A guide for researchers in education and the social sciences* (3rd ed.). New York: Teachers College.

Simmt, E., Sookochoff, S., McFeetors, P. J., Mason, R. T. (2012). Curriculum development to promote visualization and mathematical reasoning. In S. P. Norris (Ed.), *Reading for evidence and interpreting visualizations in mathematics and science education* (pp. 147-163). Rotterdam, Netherlands: Sense.

Skemp, R. R. (2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School, 12*(2), 88-95.

Stanic, G., & Kilpatrick, J. (1989). Historical perspectives on problem solving in the mathematics curriculum. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1-22). Reston, VA: National Council of Teachers of Mathematics.

Stein, C. C. (2007). Let's talk: Promoting mathematical discourse in the classroom. *Mathematics Teacher, 101*(4), 285-289.

van Manen, M. (1979). The phenomenology of pedagogic observation. *Canadian Journal of Education*, *4*(1), 5-16.

van Manen, M. (1990). *Researching lived experience: Human science for an action sensitive pedagogy*. London, ON: The University of Western Ontario.

Willingham, D. T. (2010). Is it true that some people just can't do math? *American Educator,* 14-19, 39.

Zimmerman, G. (2006). Probability simulation. In D. S. Mewborn (ed.), *Teachers engaged in research* (pp. 19-37). Charlotte, NC: Information Age Publishing.

Zimmerman, W., & Cunningham, S. (1991). Editors' introduction: What is mathematical visualization. In W. Zimmerman, & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1-8). Washington, DC: Mathematical Association of America.

Zeichner, K. (1999). The new scholarship of teacher education. *Educational Researcher, 28*(9), 4-15.

Appendix A

Assignment 1 – Introduction to Probability

1. A six-sided die is rolled.

a. Draw a simple tree diagram to show this experiment.

b. What is the probability that an odd number shows?

c. What is the probability that a number less than 3 shows?

d. What is the probability that a number greater than 4 does not show?

e. Describe your own event E that could happen from this experiment.

E:

f. Use the tree diagram to show that:

If the probability of E happening is P(E), then the probability of E *not* happening is given by 1 – P(E).

note: By "use the tree diagram to show", it is meant that you should describe which leaf nodes of the tree you would count to determine P(E) and P(Not E)

2. A card is drawn from a regular deck of 52 cards.

a. What is the probability that a Jack is drawn?

b. What is the probability that a red Jack is drawn?

c. Which of the previous events (from parts a and b) are *more* likely to happen? Explain how you know.

d. Describe an event with the same experiment (one card drawn from a regular deck of 52 cards) that is even more likely than both events a and b. Show that your event is more likely by determining its probability.

Appendix B

Assignment 2 – Counting

1. A nickel and a dime are tossed on a table. In how many ways can they fall?

2. How many numbers of three different digits less than 500 can be formed from the digits 1, 2, 3, 4, 5, 6, and 7?

3. If there are six doors in a building:

in how many ways can a student enter ***one*** and:

a. Leave by a different door?

b. Leave by the same door?

c. Leave by any door?

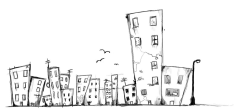
4. In a class of 27 students:

a. In how many ways can the class elect a president, vice-president, secretary, and treasurer

b. In how many ways can the teacher select 3 students to receive a scholarship, each worth the same amount?

5. There are three main roads between the cities A and B, and two between B and C. In how many ways can a person drive from A to C and return, going through B on both trips, without driving on the same

road twice?



A

B

C

6. How many “words” of four different letters can be made from the letters a, e, i, o, r, s, and t?

Appendix C

Assignment 3 – Probability with Sample Spaces

1. A dice is rolled and a coin is tossed.

a. Draw a tree diagram for this experiment ***and*** list the sample space

What is the probability that

b. A *four* and a *heads* appears?

c. An *even number* and a *heads* appears? Explain why this event has a higher probability than part b.

2. Bob is taking a multiple choice test and has no idea of the answers to the last three questions. He decides to guess at the answers (For each question, he has 1/4 chance of guessing the correct answer)

a. If each level of the tree represents each of the 3 questions, draw a tree diagram for this experiment.

b. If *c* stands for correct answer and *w* stands for wrong answer, list all of the possible outcomes of his 3 guesses (*ex. correct, wrong, wrong* is one possible outcome)

c. Find the probability that he guesses all three answers correctly.

3. A slot machine has 3 wheels and each wheel has a 0, 1, and 2 painted on it. When a button is pushed, each wheel will display a number. Each wheel operates independently. The complete number shown is the amount of money won. Thus, if the wheels together show 1, 2, 0, then $120 is won.

a. List all the possible outcomes for this machine.

b. Find the probability of winning $111.

c. Find the probability of winning nothing.

d. Find the probability of winning at least one dollar.

e. Find the probability of winning at least $100

Appendix D

Assignment 4 – Problems

1. An experiment consists of randomly drawing a card from a deck of 20 cards numbered 1 to 20. What is the probability of drawing a card that is an even number *or* a multiple of 5? Show your work.

2. Determine if each of the following events are mutually exclusive. Explain your reasoning.

a. The Miami Heat and the San Antonio Spurs winning the NBA championship this year.

b. Choosing a Red card or a Black card from a standard deck of 52 cards

c. Choosing a Red card or a Face card (J, Q, K, A) from a standard deck of 52 cards

3. According to the Tourist Information Centre in Churchill, tourists experience sunny weather 55% of the time while visiting Churchill in October. The statistics also indicate that, if it is sunny, the probability of seeing a polar bear is 40%. If it is not sunny, the probability of seeing a polar bear is 30%.

a. Create a diagram for this situation to show the sample space.

b. Calculate the probability of each possible outcome.

c. What is the probability that a tourist will see a polar bear? Show your work.

Appendix E

Narrative Text for Sheila

Draw different representations of periodic functions: formula, graph, explanation. Which is your go-to? Which helps you understand the experiment the best?
In the beginning of the unit, and shortly after being introduced to the tree diagram as a tool for representing a probability experiment; Sheila identifies that it was not until the formula was applied to the tree diagram that she understood how to use either. The formula here refers to P(E) = #ways E can happen / #total number of outcomes.

Was it the ability to see the visual size of the desired outcomes circled in relationship to the total number of outcomes there were that helped make the connection between tree and formula?
When asked to come up with a strategy for determining the difference between perms and combs, Sheila answered that tree diagrams can be used but did not say more about how. Either she said this only because we had learned about trees the previous class or she understands more but has not communicated it; find out.

You value clear examples and instruction; what do you do when you encounter something that you are unclear about? Example: interactive writing about perms and combs.

How could tree diagrams be used to represent the *difference* between perms and combs?
Sheila continues to apply her strategy without much success; separate trees are made for each part of the experiment. Understandably, the idea that tree diagrams link different parts of the experiment has created an understanding that if one part has no effect or even no relation, then they should not be linked in the same tree. There is a certain amount of linearity and sequence in the tree that is perhaps unintentional and even detrimental.

While using a tree to describe independent and dependent events, Sheila indicates that the probability of getting a blue changes from the first level to the second level; this is indeed a misunderstanding of what dependent events cause. Sheila has yet to make the connection between the concept of dependent events and how it would be visualized with the tree.

In assignment 3, Sheila is able to use tree diagrams to determine sample spaces (rolling a die and flipping a coin, and guessing on 3 multiple choice questions).

In assignment 4, Sheila is able to set up a tree diagram and then apply the multiplication formula and addition rule to solve several problems using these concepts.

During the unit test, Sheila mistakenly set up the tree with branches for different parts of the experiment (Gabrielle and Colette's car not starting). It appears that while she had some success with setting up the tree diagram, it may have been because the examples were similar to the ones I showed in class (seeing a bigfoot depending on weather). The problem with the car starting may have thrown off Sheila because the two parts of the experiment are not sequential or related.

Find out if that's the case.

Appendix F

Narrative Text for Trevor

Trevor is showing very good ability to complete the assignments. He also shows extreme confidence about being able to learn math. Throughout his interactive writing, he does not indicate moments of confusion. He even describes in his own words as 'understanding perfectly' the concepts in probability.

What were his experiences in learning math in the past? Access how he has developed this confidence. What does he mean about finding math easy?

What did you find easy about learning probability? Why?

He does admit to a moment of being confused on one of the counting problems but immediately states that he was able to figure it out. What did he do to figure it out? What does figuring it out mean to him?

Aha. He did have some trouble with answering the final question on the unit test. Get him to describe what that moment felt like—why does he think he got that question wrong?

Can you think of a moment that you were confused? What was hard about it?

The last question on the unit test for example.

As early as assignment 2, Trevor is drawing tree diagrams in the margins of the assignment even if they are not required as a part of the question. Why is he doing this and how is it helping him think about the problem? He sees tree diagrams as being useful in doing questions at least... did he find it useful during the learning?

How was your overall experience over the past ten days learning probability?

What is next for Trevor? Has he noticed any changes in the way he learns math or the possibilities for how to learn math?

Will you be studying math any time in the near future?
How do you prefer to learn it?

How does he see himself as a math learner and did that change from this experience?
Draw different representations of periodic functions: formula, graph, explanation. Which is your go-to? Which helps you understand the experiment the best?

What part of your notes do you find most useful? Definitions, steps, formulas, diagrams, examples?