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Research Article

On Locally Uniformly Differentiable Functions on a Complete Non-Archimedean Ordered Field Extension of the Real Numbers

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We study the properties of locally uniformly differentiable functions on $\mathcal{N}$, a non-Archimedean field extension of the real numbers that is real closed and Cauchy complete in the topology induced by the order. In particular, we show that locally uniformly differentiable functions are C^1, they include all polynomial functions, and they are closed under addition, multiplication, and composition. Then we formulate and prove a version of the inverse function theorem as well as a local intermediate value theorem for these functions.

1. Introduction

We start this section by reviewing some basic terminology and facts about non-Archimedean fields. So let F be an ordered non-Archimedean field extension of $\mathbb{R}$. We introduce the following terminology.

Definition 1.1 ($\sim$, $\approx$, $\ll$, S_F, λ). For $x, y \in F^* := F \setminus \{0\}$, we say $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$, where $|\cdot|$ denotes the usual absolute value on F:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \tag{1.1}$$

For nonnegative $x, y \in F$, one says that x is infinitely smaller than y and write $x \ll y$ if $nx < y$ for all $n \in \mathbb{N}$, and we say that x is infinitely small if $x \ll 1$ and x is finite if $x \sim 1$; finally, we say that x is approximately equal to y and write $x \approx y$ if $x \sim y$ and $|x-y| \ll |x|$. We also set $\lambda(x) = [x]$, the class of x under the equivalence relation $\sim$.

The set of equivalence classes S_F (under the relation $\sim$) is naturally endowed with an addition via $[x]+[y] = [x \cdot y]$ and an order via $[x] < [y]$ if $|y| \ll |x|$ (or $|x| \gg |y|$), both of which are readily checked to be well defined. It follows that $(S_F, +, <)$ is an ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is [1], the class of 1. It follows from the previous part that the projection λ from F^* to S_F is a valuation.

The theorem of Hahn [2] provides a complete classification of non-Archimedean extensions of $\mathbb{R}$ in terms of their skeleton groups. In fact, invoking the axiom of choice, it is shown that the elements of any such field F can be written as formal power series over its skeleton group S_F with real coefficients, and the set of appearing exponents forms a well-ordered subset of S_F.

From general properties of formal power series fields [3, 4], it follows that if S_F is divisible, then F is real closed; that is, every positive element of F is a square in F and every polynomial of odd degree over F has at least one root in F. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [5], and for an overview of the related valuation theory, the book by Krull [6]. A thorough and complete treatment of ordered structures can also be found in [7].

Throughout this paper, we will denote by $\mathcal{N}$ any totally ordered non-Archimedean field extension of $\mathbb{R}$ that is complete in the order topology and whose skeleton group $S_{\mathcal{N}}$ is Archimedean, that is, a subgroup of $\mathbb{R}$. The coefficient of the qth power in the Hahn representation of a given x will be denoted by $x[q]$, and the number d will be defined by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$. It is easy to check that, for $q \in S_{\mathcal{N}}$, $0 < d^q \ll 1$ if and only if $q > 0$, and $d^q \gg 1$ if and only if $q < 0$; moreover, $x \approx x[\lambda(x)]d^{\lambda(x)}$ for all $x \neq 0$.

The smallest such field $\mathcal{N}$ is the field L of the formal Laurent series whose skeleton group is $S_L = \mathbb{Z}$, and the smallest such field that is also real closed is the Levi-Civita field $\mathcal{R}$, first introduced in [8, 9]. In this latter case $S_{\mathcal{R}} = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, the set of exponents in the Hahn representation of x is a left-finite subset of $\mathbb{Q}$; that is, below any rational bound r there are only finitely many exponents. The Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ (when viewed as maps from $S_{\mathcal{R}} = \mathbb{Q}$ into $\mathbb{R}$) are left-finite, it is possible to represent these numbers on a computer [1]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [10], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. For a review of the Levi-Civita field $\mathcal{R}$, see [11, 12] and the references therein.

In the wider context of valuation theory, it is interesting to note that the topology induced by the order on $\mathcal{N}$ is the same as that introduced via the valuation λ, as shown in Remark 1.2. It follows therefore that the field $\mathcal{N}$ is just a special case of the class of fields discussed in [13].

Remark 1.2. The mapping $\Lambda : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$, given by $\Lambda(x,y) = \exp(-\lambda(x-y))$, is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the order topology (we will use τ_v to denote either one of the two topologies in the rest of the paper). If A is an open set in the order topology and $a \in A$, then there exists $r > 0$ in $\mathcal{N}$ such that, for all $x \in \mathcal{N}$, $|x-a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$; then we also have that, for all

$x \in \mathcal{N}$, $\Lambda(x, a) < l \Rightarrow x \in A$, and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously.

In this paper, we will study the properties of locally uniformly differentiable functions on $\mathcal{N}$, thus expanding on the work done in [14]. In particular, we will show that this class of functions is closed under addition, multiplication, and composition of functions. Then we will state and prove a more general version of the inverse function theorem than that proved in [14], as well as a local intermediate value theorem for $\mathcal{N}$-valued locally uniformly differentiable functions on $\mathcal{N}$. The stronger condition (local uniform differentiability) on the function than that of the real case is needed for the proof of both theorems because of the total disconnectedness of the field $\mathcal{N}$ in the order topology.

2. Preliminaries

In this section we review some of the topological properties of the field $\mathcal{N}$ which helps the reader understand the differences between $\mathcal{N}$ and $\mathbb{R}$. We begin this with the following definition.

Definition 2.1. Let $A \subset \mathcal{N}$. Then we say that A is compact in $(\mathcal{N}, \tau_v)$ if every open cover of A in $(\mathcal{N}, \tau_v)$ has a finite subcover.

Remark 2.2. Since τ_v is induced by a metric on $\mathcal{N}$, namely, the valuation metric Λ mentioned in the Introduction, it follows by the Borel-Lebesgue Theorem (see, e.g., [15, Section 9.2]) that A is compact in $(\mathcal{N}, \tau_v)$ if and only if A is sequentially compact.

Theorem 2.3. *$(\mathcal{N}, \tau_v)$ is a totally disconnected topological space. It is Hausdorff and nowhere locally compact. There are no countable bases. The topology induced to $\mathbb{R}$ is the discrete topology.*

Proof. Let $A \subset \mathcal{N}$ contain more than one point, and let $a \neq b$ in A be given. Without loss of generality, we may assume that $a < b$. Let

$$G_1 = \{x \in \mathcal{N} : |x - a| \ll b - a\}, \qquad G_2 = \mathcal{N} \setminus G_1. \tag{2.1}$$

Then G_1 and G_2 are disjoint and open in $(\mathcal{N}, \tau_v)$, $a \in G_1 \cap A$ and $b \in G_2 \cap A$, and $A \subset G_1 \cup G_2 = \mathcal{N}$. This shows that any subset of $(\mathcal{N}, \tau_v)$ containing more than one point is disconnected, and hence $(\mathcal{N}, \tau_v)$ is totally disconnected. It follows that $(\mathcal{N}, \tau_v)$ is Hausdorff. That $(\mathcal{N}, \tau_v)$ is Hausdorff also follows from the fact that it is a metric space ([16, p. 66, Problem 7(a)]).

To prove that $(\mathcal{N}, \tau_v)$ is nowhere locally compact, let $x \in \mathcal{N}$ be given and let U be a neighborhood of x. We show that the closure $\overline{U}$ of U is not compact. Let $\epsilon > 0$ in $\mathcal{N}$ be such that $(x - \epsilon, x + \epsilon) \subset U$ and consider the sets

$$\begin{gathered} M_{-1} = \{y \in \mathcal{N} : y < x \text{ or } y - x \gg d \cdot \epsilon\}, \\ M_n = (x + (n-1)d \cdot \epsilon, x + (n+1)d \cdot \epsilon) \quad \text{ for } n = 0, 1, 2, \ldots, \end{gathered} \tag{2.2}$$

where d is the infinitely small positive number defined in the introduction. Then it is easy to check that M_n is open in $(\mathcal{N}, \tau_v)$ for all $n \geq -1$, and $\bigcup_{n=-1}^{\infty} M_n = \mathcal{N}$; in particular, $\overline{U} \subset \bigcup_{n=-1}^{\infty} M_n$. But it is impossible to select finitely many of the M_n's to cover $\overline{U}$ because each of the infinitely many elements $x + nd \cdot \epsilon$ of $\overline{U}$, $n = -1, 0, 1, 2, \ldots$, is contained only in the set M_n.

There cannot be any countable bases because the uncountably many open sets $M_X = (X - d, X + d)$, with $X \in \mathbb{R}$, are disjoint. The open sets induced on $\mathbb{R}$ by the sets M_X are just the singletons $\{X\}$. Thus, in the induced topology, all sets are open and the induced topology is therefore discrete. □

As an immediate consequence of the fact that $(\mathcal{N}, \tau_v)$ is totally disconnected, it follows that, for any $x_0 \in \mathcal{N}$, the connected component of x_0 is $\{x_0\}$. Moreover, there are sets that are both open and closed, as we will show hereinafter.

Definition 2.4. Let $\Omega \subset \mathcal{N}$. Then we say that Ω is clopen in $(\mathcal{N}, \tau_v)$ if it is both open and closed.

Proposition 2.5. *For any $x_0 \in \mathcal{N}$ and for any $a > 0$ in $\mathcal{N}$, the set $\Omega = \{x : |x - x_0| \ll a\}$ is clopen in $(\mathcal{N}, \tau_v)$.*

Proof. Let $x \in \Omega$ be given. For all $y \in (x - ad, x + ad)$, we have that

$$\left|y - x_0\right| \leq \left|y - x\right| + |x - x_0| < ad + |x - x_0| \ll a. \tag{2.3}$$

Thus, $(x - ad, x + ad) \subset \Omega$ and hence Ω is open.

Now let $x \in \mathcal{N} \setminus \Omega$. Then for all $y \in (x - ad, x + ad)$, we have that

$$\left|y - x_0\right| \geq \left|\left|y - x\right| - |x - x_0|\right| \approx |x - x_0| \not\ll a; \tag{2.4}$$

so $y \in \mathcal{N} \setminus \Omega$. Thus, $(x - ad, x + ad) \subset \mathcal{N} \setminus \Omega$ and hence $\mathcal{N} \setminus \Omega$ is open. That is, Ω is closed. □

Similarly we can show that the sets $\{x : |x - x_0| \sim a\}$ and $\{x : |x - x_0| \approx a\}$ are clopen for any $x_0 \in \mathcal{N}$ and any $a > 0$ in $\mathcal{N}$.

Proposition 2.6. *Let $x_0 \in \mathcal{N}$ be given and let Ω be a neighborhood of x_0. Then there is a clopen set L such that $x_0 \in L \subset \Omega$.*

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be such that $(x_0 - \epsilon, x_0 + \epsilon) \subset \Omega$. Let $L = \{x \in \mathcal{N} : |x - x_0| \ll \epsilon\}$. Then L is clopen by Proposition 2.5 and $x_0 \in L \subset (x_0 - \epsilon, x_0 + \epsilon) \subset \Omega$. □

It follows that the clopen sets form a base for the order topology. Moreover, the quasi-component of any $x_0 \in \mathcal{N}$ is $\{x_0\}$.

As an immediate consequence of the fact that $(\mathcal{N}, \tau_v)$ is nowhere locally compact, we obtain the following result.

Corollary 2.7. *For all $a < b$ in $\mathcal{N}$, none of the intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ are compact in $(\mathcal{N}, \tau_v)$.*

Since τ_v is induced on $\mathcal{N}$ by the order, we define boundedness of a set in $(\mathcal{N}, \tau_v)$ as follows.

Definition 2.8. Let $A \subset \mathcal{N}$. Then we say that A is bounded in $(\mathcal{N}, \tau_v)$ if there exists $M > 0$ in $\mathcal{N}$ such that $|x| \leq M$ for all $x \in A$.

Proposition 2.9. *Let A be compact in $(\mathcal{N}, \tau_v)$. Then A is closed and bounded in $(\mathcal{N}, \tau_v)$. Moreover, A has an empty interior in $(\mathcal{N}, \tau_v)$; that is,*

$$\operatorname{int}(A) := \{a \in A : \exists r > 0 \text{ in } \mathcal{N} \ni (a - r, a + r) \subset A\} = \emptyset. \tag{2.5}$$

Proof. That A is closed in $(\mathcal{N}, \tau_v)$ follows from the fact that $(\mathcal{N}, \tau_v)$ is a Hausdorff topological space and A is compact in $(\mathcal{N}, \tau_v)$ (see [17, p. 36]).

Now we show that A is bounded in $(\mathcal{N}, \tau_v)$. For each $n \in \mathbb{N}$, let $G_n = (-d^{-n}, d^{-n})$. Then, for each $n \in \mathbb{N}$, G_n is open in $(\mathcal{N}, \tau_v)$. Moreover, since the skeleton group of $\mathcal{N}$ is Archimedean it follows that $A \subset \bigcup_{n \in \mathbb{N}} G_n = \mathcal{N}$. Since A is compact in $(\mathcal{N}, \tau_v)$, we can choose a finite subcover; thus, there is $m \in \mathbb{N}$ and there exist $j_1 < j_2 < \cdots < j_m$ in $\mathbb{N}$ such that

$$A \subset \bigcup_{l=1}^{m} G_{j_l} = G_{j_m} = \left(-d^{-j_m}, d^{-j_m}\right). \tag{2.6}$$

It follows that $|x| < d^{-j_m}$ for all $x \in A$, and hence A is bounded in $(\mathcal{N}, \tau_v)$.

Finally, we show that $\operatorname{int}(A) = \emptyset$. Assume not. Then there exist $a < b$ in A such that $[a, b] \subset A$. Since $[a, b]$ is a closed subset of the compact set A, it follows that $[a, b]$ is compact in $(\mathcal{N}, \tau_v)$, which contradicts Corollary 2.7. □

The following examples show that there are (countably infinite) closed and bounded sets that are not compact while there are uncountable sets that are compact in $(\mathcal{N}, \tau_v)$.

Example 2.10. Let $A = [0, 1] \cap \mathbb{Q}$. Then clearly, A is countably infinite and bounded in $(\mathcal{N}, \tau_v)$. We show that A is closed in $(\mathcal{N}, \tau_v)$. Let $x \in \mathcal{N} \setminus A$ be given and let $G_0 = (x - d, x + d)$. If $G_0 \cap A \neq \emptyset$, then there exists $q \in A$ such that $G_0 \cap A = \{q\}$. Let $r = |q - x|$ and let $G = (x - r, x + r)$. Then G is open in $(\mathcal{N}, \tau_v)$ and $G \cap A = \emptyset$. Thus, $\mathcal{N} \setminus A$ is open, and hence A is closed in $(\mathcal{N}, \tau_v)$.

Next we show that A is not compact in $(\mathcal{N}, \tau_v)$. For each $q \in A$, let $G_q = (q - d, q + d)$. Then G_q is open in $(\mathcal{N}, \tau_v)$ for each q and $A \subset \bigcup_{q \in A} G_q$, but we cannot select a finite subcover since each $t \in A$ is contained only in G_t.

Example 2.11. Let $C_{\mathcal{N}}$ denote the Cantor-like set constructed in the same way as the standard real Cantor set C; but instead of deleting the middle third, we delete from the middle an open interval $(1 - 2d)$ times the size of each of the closed subintervals of $[0, 1]$ at each step of the construction. Then $C_{\mathcal{N}}$ is compact in $(\mathcal{N}, \tau_v)$.

It turns out that if we view $\mathcal{N}$ as an infinite dimensional vector space over $\mathbb{R}$, then τ_v is not a vector topology; that is, $(\mathcal{N}, \tau_v)$ is not a linear topological space.

Theorem 2.12. *τ_v is not a vector topology.*

Proof. Assume to the contrary that $(\mathcal{N}, \tau_v)$ is a vector topology. Then, by continuity of scalar multiplication, there exists an open set $O_{\mathbb{R}} \subset \mathbb{R}$ and there exists an open set $O_{\mathcal{N}} \subset \mathcal{N}$ such that

$\alpha x \in (1-d, 1+d)$ for all $\alpha \in O_{\mathbb{R}}$ and for all $x \in O_{\mathcal{N}}$. Let $\alpha_0 \in O_{\mathbb{R}}$ and $x_0 \in O_{\mathcal{N}}$ be given. Since $O_{\mathbb{R}}$ is open, there exists $r > 0$ in $\mathbb{R}$ such that $(\alpha_0 - 2r, \alpha_0 + 2r) \subset O_{\mathbb{R}}$. Hence

$$\alpha_0 x_0 \in (1-d, 1+d), \qquad (\alpha_0 + r)x_0 \in (1-d, 1+d). \tag{2.7}$$

Thus,

$$r|x_0| = |(\alpha_0 + r)x_0 - \alpha_0 x_0| < 2d, \tag{2.8}$$

which contradicts the fact that $r|x_0| \gg 2d$, since both r and $|x_0|$ are finite and d is infinitely small. □

Since any normed vector space, with the metric topology induced by its norm, is a linear topological space ([18, Proposition III.1.3]), we readily infer from Theorem 2.12 that there can be no norm on $\mathcal{N}$ that would induce the same topology as τ_v on $\mathcal{N}$.

We finish this section with the following criterion for convergence for an infinite series, which does not hold for real numbers series.

Proposition 2.13. *For each $n \in \mathbb{N}$, let x_n be an element of $\mathcal{N}$. Then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence (x_n) converges to zero.*

Proof. Assume that $\sum_{n=1}^{\infty} x_n$ converges, and let (y_n) denote the sequence of partial sums of the series: $y_n = \sum_{i=1}^{n} x_i$. Thus (y_n) converges and hence it is Cauchy. Now let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $N \in \mathbb{N}$ such that for each $n, m > N$ we have that $|y_n - y_m| < \epsilon$. It follows that $|x_{n+1}| = |y_{n+1} - y_n| < \epsilon$ for all $n > N$ or, equivalently, $|x_n| < \epsilon$ for all $n > N + 1$. Hence the sequence (x_n) converges to zero.

Now assume that the sequence (x_n) converges to zero. Let (y_n) be the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$ and let $\epsilon > 0$ be given in $\mathcal{N}$. Then there is an $N \in \mathbb{N}$ such that $|x_n| < d\epsilon$ for all $n > N$. Thus, for all $n > m > N$, we have that $|y_n - y_m| = |\sum_{i=m+1}^{n} x_i| < (m-n)d\epsilon \ll \epsilon$. Thus (y_n) is Cauchy, and hence $\sum_{n=1}^{\infty} x_n$ converges since $(\mathcal{N}, \tau_v)$ is complete. □

3. Local Uniform Continuity

In this section we introduce the concept of local uniform continuity of a function from a subset of $\mathcal{N}$ to $\mathcal{N}$ and study properties of such functions that will be relevant to our discussion of locally uniformly differentiable functions later in Section 4. We start with the following definitions.

Definition 3.1. Let $A \subset \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. Then one says that f is continuous at x_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in A$ and $|x - x_0| < \delta$.

Definition 3.2. Let $A \subset \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. Then one says that f locally uniformly continuous at x_0 if there is a neighborhood Ω of x_0 in A such that f is uniformly continuous on Ω. That is, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ whenever $x, y \in \Omega$ and $|y - x| < \delta$.

Exactly as in real calculus, one can easily show that if $f, g : A \to \mathcal{N}$ are (locally uniformly) continuous at $x_0 \in A$ and if $\alpha \in \mathcal{N}$, then $f + \alpha g$ is (locally uniformly) continuous at x_0. Moreover, if $A, B \subset \mathcal{N}$ and if $f : A \to B$ is (locally uniformly) continuous at $x_0 \in A$ and $g : B \to \mathcal{N}$ is (locally uniformly) continuous at $f(x_0)$, then $g \circ f : A \to \mathcal{N}$ is (locally uniformly) continuous at x_0.

Lemma 3.3. *Let $x_0 \in \mathcal{N}$ be given and let Ω be a neighborhood of x_0. Then there exist sequences (x_n) and (y_n) as well as mutually disjoint clopen sets U_n and U_0 and a continuous function f such that*

(1) *the set $\{x_n, y_n : n \in \mathbb{N}\}$ has no limit point;*

(2) $\lim_{n\to\infty}|x_n - y_n| = 0$*;*

(3) $x_0 \notin \{x_n, y_n : n \in \mathbb{N}\}$*;*

(4) $x_n \in U_n$ *for each* $n \in \mathbb{N} \cup \{0\}$*;*

(5) $y_n \notin U_m$ *for any* $n, m \in \mathbb{N}$*;*

(6) $f(\cup_{n\in\mathbb{N}} U_n) = \{1\}$, $f(\mathcal{N} \setminus \cup_{n\in\mathbb{N}} U_n) = \{0\}$.

Proof. Since $\mathcal{N}$ is not compact, there is a sequence $(x_n) \subset \Omega$ that has no limit point in $\mathcal{N}$. Without loss of generality, we can take $x_0 \notin \{x_n : n \in \mathbb{N}\}$ since $\{x_n : n \in \mathbb{N}\} \setminus \{x_0\}$ will still have no limit point. For each $n \in \mathbb{N}$, let $y_n \in \mathcal{N}$ be such that $|y_n - x_n| < d^n$ and $y_n \neq x_m$ for any $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. Since (d^n) is a null sequence in $\mathcal{N}$, it follows that $|y_n - x_n| \to 0$ as $n \to \infty$. Assume that $\{x_n, y_n : n \in \mathbb{N}\}$ has a limit point in $\mathcal{N}$ and let c be such a limit point. Since (x_n) has no limit point, there is an $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \cap \{x_n : n \in \mathbb{N}\} = \emptyset$. There exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|y_n - x_n| < \epsilon/2$. Since c is the limit point of $\{x_n, y_n : n \in \mathbb{N}\}$, there must be $M > N$ such that $y_M \in (c - \epsilon/2, c + \epsilon/2)$. But then $|x_M - c| \leq |x_M - y_M| + |y_M - c| < \epsilon/2 + \epsilon/2 = \epsilon$. This is a contradiction. Hence $\{x_n, y_n : n \in \mathbb{N}\}$ has no limit point.

Since $\{x_n, y_n : n \in \mathbb{N}\}$ has no limit point, there exist U_n' and U_0' such that $x_0 \in U_0'$, $\{x_n, y_n : n \in \mathbb{N}\} \cap U_n' = \{x_n\}$ and $\{x_n, y_n : n \in \mathbb{N}\} \cap U_0' = \emptyset$. For each $n \in \mathbb{N} \cup \{0\}$ let $\epsilon_n > 0$ in $\mathcal{N}$ be such that $\epsilon_n \leq d^n$ and $(x_n - \epsilon_n, x_n + \epsilon_n) \subset U_n'$. Then $(x_0 - \epsilon_0, x_0 + \epsilon_0)$ and all of $(x_n - \epsilon_n/2, x_n + \epsilon_n/2)$, $n \in \mathbb{N}$, are mutually disjoint open sets. By Proposition 2.6, there are clopen neighborhoods U_n of x_n such that $U_n \subset (x_n - \epsilon_n/2, x_n + \epsilon_n/2)$ and a clopen neighborhood U_0 of x_0 such that $U_0 \subset (x_0 - \epsilon_0/2, x_0 + \epsilon_0/2)$. $\bigcup_{n=1}^{\infty} U_n$ is open as it is the union of open sets, but it is also closed as we will show hereinafter. Let $x \in \mathrm{cl}(\bigcup_{n=1}^{\infty} U_n)$ be given. Then there exists a sequence $(z_m) \subset \bigcup_{n=1}^{\infty} U_n$ such that $z_n \to x$. (x_n) has no limit points, so x is separated from (x_n). Therefore, there exists $N \in \mathbb{N}$ such that $(x - d^N, x + d^N) \cap \{x_n : n \in \mathbb{N}\} = \emptyset$. Moreover, there exists $M \in \mathbb{N}$ such that for every $m \geq M$, $|z_m - x| < (1/2)d^N$. It follows that $|z_m - x_k| \geq ||x_k - x| - |z_m - x|| > d^N - (1/2)d^N = (1/2)d^N \gg d^k$ for all $k > N$ and for all $m \geq M$. $U_k \subset (x_k - d^k, x_k + d^k)$, for each $k \in \mathbb{N}$; it follows that $\{z_m : m \geq M\} \cap \bigcup_{k=N+1}^{\infty} U_k = \emptyset$. That is, $\{z_m : m \geq M\} \subset \bigcup_{k=1}^{N} U_k$ which is a finite union of closed sets and hence is itself closed. Since $z_m \to x$ and $\{z_m : m \geq M\} \subset \bigcup_{k=1}^{N} U_k$ which is closed, it follows that $x \in \bigcup_{k=1}^{N} U_k \subset \bigcup_{k=1}^{\infty} U_k$. Thus, $\bigcup_{k=1}^{\infty} \mathrm{U}_k = \mathrm{cl}(\bigcup_{k=1}^{\infty} U_k)$ and hence $\bigcup_{k=1}^{\infty} U_k$ is closed. Define f on $\mathcal{N}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \displaystyle\bigcup_{n=1}^{\infty} U_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Then f is continuous on $\mathcal{N}$ because $\bigcup_{n=1}^{\infty} U_n$ is clopen. $\square$

In the real case, any function that is continuous on a neighborhood is also locally uniformly continuous on that neighborhood. This property does not hold in non-Archimedean fields.

Theorem 3.4. *Let $x_0 \in \mathcal{N}$ be given and let Ω be a neighborhood of x_0. Then there is a continuous function $f : \Omega \to \mathcal{N}$ that is not locally uniformly continuous at x_0.*

Proof. Let $\Omega_1 = \Omega \cap (x_0 - d, x_0 + d)$ and apply Lemma 3.3 to x_0 and Ω_1 to get f_1, U_1, $(x_{1,n})$, and $(y_{1,n})$ which correspond to f, U_0, (x_n), and (y_n), respectively, in that lemma. Let $\Omega_2 = U_1 \cap (x_0 - d^2, x_0 + d^2)$ and apply Lemma 3.3 to x_0 and Ω_2 to get f_2, U_2, $(x_{2,n})$, and $(y_{2,n})$. Continuing inductively, we can apply Lemma 3.3 to x_0 and $\Omega_{k+1} = U_k \cap (x_0 - d^{k+1}, x_0 + d^{k+1})$ in order to get U_{k+1}, f_{k+1}, $(x_{k+1,n})$, and $(y_{k+1,n})$. The resulting f_k's satisfy $f_k(y_{l,n}) = 0$ for every $k, l, n \in \mathbb{N}$ and $f_k(x_{l,n}) = \delta_{k,l}$ (the Kronecker delta) for every $k, l, n \in \mathbb{N}$. Let $f = \sum_{k=1}^{\infty} d^k f_k$, which converges (pointwise), by Proposition 2.13, since $|d^k f_k(x)| \le d^k \to 0$ as $k \to \infty$ for all $x \in \Omega$.

To show that f is continuous on Ω, let $t \in \Omega$ be given. Let $\epsilon > 0$ in $\mathcal{N}$ be given and let $N \in \mathbb{N}$ be such that $d^N \ll \epsilon$. For each $n < N$, let $\delta_n > 0$ be such that $|f_n(x) - f_n(t)| < d^N$ whenever $x \in \Omega$ and $|x - t| < \delta_n$, which is possible since each f_n is continuous at t. Let $\delta = \min\{\delta_n : n < N\}$. Then for all $x \in \Omega$ satisfying $|x - t| < \delta$, we have that

$$\begin{aligned}
\left|f(x) - f(t)\right| &= \left|\sum_{n=1}^{\infty} d^n \left(f_n(x) - f_n(t)\right)\right| \le \left|\sum_{n=1}^{N} d^n \left(f_n(x) - f_n(t)\right)\right| \\
&\quad + \left|\sum_{n=N+1}^{\infty} d^n f_n(x)\right| + \left|\sum_{n=N+1}^{\infty} d^n f_n(t)\right| \le \sum_{n=1}^{N} d^n \left|d^N\right| \\
&\quad + \left|\sum_{n=N+1}^{\infty} d^n\right| + \left|\sum_{n=N+1}^{\infty} d^n\right| < d^N + d^N + d^N \ll \epsilon.
\end{aligned} \tag{3.2}$$

Thus, f is continuous at t, for all $t \in \Omega$, and hence f is continuous on Ω.

Now we show that f is not locally uniformly continuous at x_0. Let Δ be a neighborhood of x_0, let $M \in \mathbb{N}$ be such that $(x_0 - d^M, x_0 + d^M) \subset \Delta$, and let $\epsilon = (1/2)d^M$. So $\Omega_M \subset (x_0 - d^M, x_0 + d^M) \subset \Delta$. It follows that $x_{M,n} \in \Omega_M \subset \Delta$ and $y_{M,n} \in \Omega_M \subset \Delta$ are such that $|y_{M,n} - x_{M,n}| \to 0$ as $n \to \infty$, but

$$\left|f(y_{M,n}) - f(x_{M,n})\right| = \left|\sum_{k=1}^{\infty} d^k \left(f_k(y_{M,n}) - f_k(x_{M,n})\right)\right| = d^M > \epsilon. \tag{3.3}$$

Therefore f is not locally uniformly continuous at x_0. □

One can in fact show that there are continuous functions which are not locally uniformly continuous without using the property of total disconnectedness of $\mathcal{N}$. By using the method prescribed previously, the derivatives of the constructed functions are calculated easily, and this will prove useful when dealing with local uniform differentiability in Section 4.

Example 3.5. In the following, we will provide an explicit example of a function which is continuous but not locally uniformly continuous. Imitating the proof of Theorem 3.4, we let

$$\begin{aligned} x_{m,n} &= nd^m, \\ y_{m,n} &= nd^m + d^{n+m}, \\ U_{m,n} &= \left\{ x \in \mathcal{N} : |x - nd^m| \underset{\sim}{\ll} d^{m+m+1} \right\}, \\ U_{m,o} &= \left\{ x \in \mathcal{N} : |x| \underset{\sim}{\ll} d^{m+1} \right\}, \end{aligned} \tag{3.4}$$

and let

$$f(x) = \begin{cases} d^m & \text{if } x \in U_{m,n} \text{ for some } m, n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then f is well defined on $\mathcal{N}$ since the $U_{m,n}$'s are mutually disjoint sets. We will show that f is continuous on $\mathcal{N}$ but f is not locally uniformly continuous at 0. Let $t \in \mathcal{N}$ be given. We will distinguish the following three cases.

Case 1 ($t < 0$). In this case, the function f is constant on the open set $(-\infty, 0)$ containing t, and hence f is continuous at t.

Case 2 ($t > 0$). Let $M \in \mathbb{N}$ be such that $d^{M+1} \ll t \underset{\sim}{\ll} d^M$. Then $t \in U_{M-1,0}$ but $t \notin U_{M,0}$. Let $\Delta = U_{M-1,0} \setminus U_{M,0}$. Then Δ is clopen, and $\bigcup_{n=1}^{\infty} U_{M,n} \subset \Delta$ is also clopen. $f(\bigcup_{n=1}^{\infty} U_{M,n}) = \{d^M\}$ and $f(\Delta \setminus \bigcup_{n=1}^{\infty} U_{M,n}) = \{0\}$. Therefore f is continuous on Δ since f is constant on disjoint open sets that cover Δ. Hence f is continuous at $t \in \Delta$.

Case 3 ($t = 0$). Let $\epsilon > 0$ in $\mathcal{N}$ be given and let $M \in \mathbb{N}$ be such that $d^M \ll \epsilon$. Let $\delta = d^{M+1}$. Then, for $|x| < \delta$, we have that $x \notin U_{k,n}$ for all $k < M$ and every $n \in \mathbb{N}$. So $|f(x) - f(0)| \leq d^M \ll \epsilon$. This shows that f is continuous at $t = 0$.

Thus, f is continuous at t for all $t \in \mathcal{N}$ and hence f is continuous on $\mathcal{N}$. To see that f is not locally uniformly continuous at 0, let Ω be any neighborhood of 0. Let $M \in \mathbb{N}$ be such that $(-d^M, d^M) \subset \Omega$ and let $\epsilon = (1/2)d^M$. Then the sequences $(x_{M,n})$ and $(y_{M,n})$ in Ω defined previously are such that $|y_{M,n} - x_{M,n}| = d^{n+M} \to 0$ as $n \to \infty$, but $|f(y_{M,n}) - f(x_{M,n})| = d^M > \epsilon$. Thus, for every neighborhood Ω of 0, there exists $\epsilon_0 > 0$ such that for each $\delta > 0$, there are $x, y \in \Omega$ such that $|y - x| < \delta$ while $|f(y) - f(x)| \geq \epsilon_0$, which shows that f is not locally uniformly continuous at 0.

4. Local Uniform Differentiability

Definition 4.1. Let $A \subset \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. Then we say that f is locally uniformly differentiable (LUD) at x_0 if there is a neighborhood Ω of x_0 in A such that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x) - f'(x)(y - x)| < \epsilon|y - x|$ whenever $x, y \in \Omega$ and$|y - x| < \delta$.

Definition 4.2. Let $A \subset \mathcal{N}$ and let $f : A \to \mathcal{N}$. Then we say that f is locally uniformly differentiable (LUD) on A if f is locally uniformly differentiable at x for all $x \in A$.

Definition 4.3. Let $A \subset \mathcal{N}$, let $f : A \to \mathcal{N}$, and let $k \in \mathbb{N}$ be given. Then we say that f is LUD^k if kth derivative of f, $f^{(k)}$, exists and $f^{(l)}$ is LUD on A for each $l \in \{0, 1, \dots, k-1\}$.

The following two results follow readily from Definition 4.3.

Proposition 4.4. *Let $A \subset \mathcal{N}$ and let $f : A \to \mathcal{N}$ be LUD^k. Then $f^{(n)}$ is LUD^l on A for all $l \geq 1$ and $n \geq 0$ satisfying $l + n \leq k$.*

Proposition 4.5. *Let $l \in \mathbb{N}$ be given, let $A \subset \mathcal{N}$, and let $f : A \to \mathcal{N}$ be such that f is LUD^l and $f^{(l)}$ is LUD on A. Then f is LUD^{l+1} on A.*

A noteworthy property of local uniform differentiability is that it is not inherited by the function from its derivatives, nor passed from the function onto its derivatives. Indeed, the following is an explicit example of a function whose derivative is everywhere zero, but it is not itself locally uniformly differentiable.

Example 4.6. Let $x_{m,n}$, $y_{m,n}$, and $U_{m,n}$ be as in Example 3.5. Define $g : \mathcal{N} \to \mathcal{N}$ by

$$g(x) = \begin{cases} (n-1)^2 d^{2m} & \text{if } x \in U_{m,n} \text{ for some } m, n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then g is well defined since the $U_{m,n}$ are mutually disjoint. Let $x \in \mathcal{N}$ be given. If $x \in U_{m,n}$ for some $m, n \in \mathbb{N}$, then $x \approx nd^m$ and hence $|g(x)| = (n-1)^2 d^{2m} < |x|^2$. Also, if $x \notin U_{m,n}$ for any $m, n \in \mathbb{N}$, then $|g(x)| = 0 < |x|^2$. Therefore, $|g(x)| < |x|^2$ for all $x \in \mathcal{N}$.

Note that g is locally constant on $\mathcal{N} \setminus \{0\}$ and hence $g' = 0$ on $\mathcal{N} \setminus \{0\}$. We will show that g is differentiable at 0 with $g'(0) = 0$ too. So let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta = \epsilon$. Then for $0 < |x| < \delta$, we have that

$$\left| \frac{g(x) - g(0)}{x} \right| = \left| \frac{g(x)}{x} \right| < |x| < \delta = \epsilon. \tag{4.2}$$

This shows that g is differentiable at 0 with $g'(0) = 0$. Altogether, $g' = 0$ on $\mathcal{N}$. Therefore g is C^∞ on $\mathcal{N}$ with $g^{(k)} = 0$ for all $k \in \mathbb{N}$.

Now we show that g is not LUD at 0. Consider the sequences $(x_{n,n})_{n \in \mathbb{N}}$ and $(y_{n,n})_{n \in \mathbb{N}}$, where $x_{n,n} = nd^n$ and $y_{n,n} = nd^n + d^{2n}$. Both of these sequences converge to zero. Moreover,

$$|y_{n,n} - x_{n,n}| = d^{2n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{4.3}$$

but

$$|g(y_{n,n}) - g(x_{n,n})| = (n-1)^2 d^{2n} \geq d^{2n} = |y_{n,n} - x_{n,n}| \quad \text{for } n \geq 2. \tag{4.4}$$

Thus, for any neighborhood Ω of 0, $\epsilon_0 = 1$, and for any $\delta > 0$, there are $y, x \in \Omega$ such that $|y - x| < \delta$, but $|g(y) - g(x) - g'(x)(y - x)| \geq |y - x|$. This shows that g is not LUD at 0.

Example 4.6 shows that the property LUD is not necessarily inherited from the derivatives of the function, since $g' = 0$ is LUD$^\infty$. Here is another example that shows that the LUD property is not passed on from a function to its derivatives.

Example 4.7. Let $f : \mathcal{N} \to \mathcal{N}$ be given by

$$f(x) = \begin{cases} xd^{\lambda(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{4.5}$$

We will show that f is LUD on $\mathcal{N}$ with derivative

$$f'(x) = g(x) = \begin{cases} d^{\lambda(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{4.6}$$

and then we will show that f' is not LUD by showing that it is not LUD at 0. So let $x_0 \neq 0$ be given. Let $\Omega = \{x \in \mathcal{N} : x \sim x_0\}$ which is an open (clopen) neighborhood of x_0. Then for all $x, y \in \Omega$, we have that

$$\begin{aligned} \left| f(y) - f(x) - g(x)(y - x) \right| &= \left| yd^{\lambda(y)} - xd^{\lambda(x)} - d^{\lambda(x)}(y - x) \right| \\ &= \left| yd^{\lambda(x_0)} - xd^{\lambda(x_0)} - d^{\lambda(x_0)}(y - x) \right| = 0. \end{aligned} \tag{4.7}$$

This shows that f is locally uniformly differentiable at x_0 with derivative $f'(x_0) = g(x_0) = d^{\lambda(x_0)}$.

For $x_0 = 0$, let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta = \epsilon d$. Then for all $x \neq y$ in $(-\delta, \delta)$, we will show that

$$\left| f(y) - f(x) - g(x)(y - x) \right| < \epsilon \left| y - x \right|. \tag{4.8}$$

Case 1 $(x = 0)$. Then $f(x) = g(x) = 0$ and $y \neq 0$. It follows that

$$\left| f(y) - f(x) - g(x)(y - x) \right| = \left| f(y) \right| = d^{\lambda(y)} |y| = d^{\lambda(y)} \left| y - x \right| < \epsilon \left| y - x \right|. \tag{4.9}$$

because $d^{\lambda(y)} \sim |y| < \delta \ll \epsilon$.

Case 2 $(y = 0)$. Then $f(y) = 0$ and $x \neq 0$. It follows that

$$\begin{aligned} \left| f(y) - f(x) - g(x)(y - x) \right| &= \left| -f(x) + g(x)x \right| = \left| -d^{\lambda(x)}x + d^{\lambda(x)}x \right| \\ &= 0 < \epsilon \left| y - x \right|. \end{aligned} \tag{4.10}$$

Case 3 ($x \neq 0 \neq y$). Then

$$\begin{aligned}
\left|f(y) - f(x) - g(x)(y - x)\right| &= \left|d^{\lambda(y)} y - d^{\lambda(x)} x - d^{\lambda(x)}(y - x)\right| \\
&= \left|d^{\lambda(y)} - d^{\lambda(x)}\right|\left|y\right| < d^{-1}\left|y - x\right|\left|y\right| \\
&< d^{-1}\delta\left|y - x\right| = \epsilon\left|y - x\right|.
\end{aligned} \tag{4.11}$$

Thus, f is locally uniformly differentiable at 0 with derivative $f'(0) = g(0) = 0$. Altogether, it follows that f is LUD on $\mathcal{N}$ with derivative

$$f'(x) = g(x) = \begin{cases} d^{\lambda(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{4.12}$$

Next we show that f' is not differentiable at 0. Take the sequence $(x_n) = d^n$ and the sequence $(y_n) = 2d^n$. Then both sequences converge to 0. But

$$\begin{aligned}
\lim_{n \to \infty} \frac{f'(x_n) - f'(0)}{x_n} &= \lim_{n \to \infty} \frac{d^n}{d^n} = 1, \\
\lim_{n \to \infty} \frac{f'(y_n) - f'(0)}{y_n} &= \lim_{n \to \infty} \frac{d^n}{2d^n} = \frac{1}{2}.
\end{aligned} \tag{4.13}$$

If $f'(x)$ were differentiable at 0, then both limits in (4.13) would be equal to $f''(0)$. Since the two limits are different, we conclude that f' is not differentiable at 0, and hence f' is not LUD on $\mathcal{N}$.

The previous two examples illustrate that the most natural definition for LUD^k is given in Definition 4.3.

Proposition 4.8. *Let $f : A \to \mathcal{N}$ be LUD at $x_0 \in A$. Then f is C^1 at x_0.*

Proof. Let Ω be a neighborhood of x_0 in A such that f is uniformly differentiable on Ω and let $\delta_0 > 0$ be such that $(x_0 - \delta_0, x_0 + \delta_0) \subset \Omega$. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there is $\delta > 0$, $\delta \leq \delta_0$, such that for all $x, y \in \Omega$ satisfying $|y - x| < \delta$ we have that

$$\left|f(y) - f(x) - f'(x)(y - x)\right| < \frac{\epsilon}{2}\left|y - x\right|. \tag{4.14}$$

It follows that for $0 < |x - x_0| < \delta$,

$$\left|f'(x) - f'(x_0)\right| \leq \left|\frac{f(x_0) - f(x)}{x_0 - x} - f'(x)\right| + \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad . \tag{4.15}$$

□

Remark 4.9. Proposition 4.8 shows that the class of locally uniformly differentiable functions is a subset of the class of C^1 functions. However, this is still large enough to include all polynomial functions as Corollary 4.14 will show.

Proposition 4.10. *Let $f : A \to \mathcal{N}$ be locally uniformly differentiable at $x_0 \in A$. Then f is locally uniformly continuous at x_0.*

Proof. Let Δ be a neighborhood of x_0 in A such that f is uniformly differentiable on Δ. By Proposition 4.8, f' is continuous at x_0. Let $\Omega \subset \Delta$ be a neighborhood of x_0 such that for every $x \in \Omega$, $|f'(x)| < 1 + |f'(x_0)|$. Since f is uniformly differentiable on Δ, there exists $\delta_1 > 0$ such that for all $x, y \in \Omega \subset \Delta$ satisfying $|y - x| < \delta_1$, we have that

$$\left|f(y) - f(x) - f'(x)(y - x)\right| < \left|y - x\right|. \tag{4.16}$$

Let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta = \min\{\delta_1, \epsilon/(2 + |f'(x_0)|)\}$. Then for all $x, y \in \Omega \subset \Delta$ satisfying $|y - x| < \delta$, we have that

$$\begin{aligned} \left|f(y) - f(x)\right| &< \left|y - x\right| + \left|f'(x)\right|\left|y - x\right| < \left|y - x\right|\left(2 + \left|f'(x_0)\right|\right) \\ &< \delta\left(2 + \left|f'(x_0)\right|\right) < \epsilon. \end{aligned} \tag{4.17}$$

□

Proposition 4.11. *Let $A \subset \mathcal{N}$, let $x_0 \in A$ be given, let $\alpha \in \mathcal{N}$ be given, and let $f, g : A \to \mathcal{N}$ be LUD at x_0. Then $f + \alpha g$ is LUD at x_0, with derivative*

$$(f + \alpha g)'(x_0) = f'(x_0) + \alpha g'(x_0). \tag{4.18}$$

Proof. Without loss of generality, we may assume that $\alpha \neq 0$. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that $(x_0 - \delta, x_0 + \delta) \subset A$,

$$\begin{aligned} \left|f(y) - f(x) - f'(x)(y - x)\right| &< \frac{\epsilon}{2}\left|y - x\right|, \\ \left|g(y) - g(x) - g'(x)(y - x)\right| &< \frac{\epsilon}{2|\alpha|}\left|y - x\right|, \end{aligned} \tag{4.19}$$

whenever $x, y \in (x_0 - \delta, x_0 + \delta)$. It follows that, for all $x, y \in (x_0 - \delta, x_0 + \delta)$, we have that

$$\begin{aligned} &\left|(f + \alpha g)(y) - (f + \alpha g)(x) - f'(x)(y - x) - \alpha g'(x)(y - x)\right| \\ &\quad = \left|\left[f(y) - f(x) - f'(x)(y - x)\right] + \alpha\left[g(y) - g(x) - g'(x)(y - x)\right]\right| \\ &\quad \le \left|f(y) - f(x) - f'(x)(y - x)\right| + |\alpha|\left|g(y) - g(x) - g'(x)(y - x)\right| \\ &\quad < \frac{\epsilon}{2}\left|y - x\right| + |\alpha|\frac{\epsilon}{2|\alpha|}\left|y - x\right| = \epsilon\left|y - x\right|. \end{aligned} \tag{4.20}$$

□

Proposition 4.12 (Chain Rule). *Let $A, B \subset \mathcal{N}$, let $x_0 \in A$ be given, and let $g : A \to B$ and $f : B \to \mathcal{N}$ be such that g is LUD at x_0 and f is LUD at $g(x_0)$. Then $f \circ g$ is LUD at x_0 with derivative $(f \circ g)'(x_0) = (f' \circ g(x_0)) \cdot g'(x_0)$.*

Proof. Let Δ be a neighborhood of $g(x_0)$ in B such that f is uniformly differentiable on Δ, and let Ω be a neighborhood of x_0 in A such that $g(\Omega) \subset \Delta$ and g is uniformly differentiable and uniformly continuous on Ω (Proposition 4.10). The condition that $g(\Omega) \subset \Delta$ is always possible because g is continuous. Since g is continuous at x_0 and f' is continuous at $g(x_0)$ (Proposition 4.8), it follows that $f' \circ g$ is continuous at x_0. So there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $(x_0 - \delta_1, x_0 + \delta_1) \subset \Omega$ and for all $x \in (x_0 - \delta_1, x_0 + \delta_1)$ we have that

$$\left|f' \circ g(x)\right| < \left|f' \circ g(x_0)\right| + 1. \tag{4.21}$$

Also, since g' is continuous at x_0, there is a $\delta_2 > 0$ in $\mathcal{N}$ such that $(x_0 - \delta_2, x_0 + \delta_2) \subset \Omega$, and for all $x \in (x_0 - \delta_2, x_0 + \delta_2)$, we have that

$$\left|g'(x)\right| < \left|g'(x_0)\right| + \frac{1}{2}. \tag{4.22}$$

Since g is uniformly differentiable on Ω, there is a $\delta_3 > 0$ in $\mathcal{N}$ such that $(x_0 - \delta_3, x_0 + \delta_3) \subset \Omega$, and for all $x, y \in (x_0 - \delta_3, x_0 + \delta_3)$ satisfying $|y - x| < \delta_3$, we have that

$$\left|g(y) - g(x)\right| < \left(\left|g'(x)\right| + \frac{1}{2}\right)\left|y - x\right| \leq \left(\left|g'(x_0)\right| + 1\right)\left|y - x\right|. \tag{4.23}$$

Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then, since f is uniformly differentiable on Δ, there exists an $\eta > 0$ such that whenever $u, v \in \Delta$ and $|u - v| < \eta$, it follows that

$$\left|f(v) - f(u) - f'(u)(v - u)\right| < \frac{\epsilon}{2\left(\left|g'(x_0)\right| + 1\right)}|v - u|. \tag{4.24}$$

Since g is uniformly differentiable on Ω, there exists $\delta_4 > 0$ in $\mathcal{N}$ such that $(x_0 - \delta_4, x_0 + \delta_4) \subset \Omega$, and for all $x, y \in (x_0 - \delta_4, x_0 + \delta_4)$ satisfying $|y - x| < \delta_4$, we have that

$$\left|g(y) - g(x) - g'(x)(y - x)\right| < \frac{\epsilon}{2\left(\left|f' \circ g(x_0)\right| + 1\right)}\left|y - x\right|. \tag{4.25}$$

Finally, g is uniformly continuous on Ω. Thus, there exists $\delta_5 > 0$ in $\mathcal{N}$ such that $(x_0 - \delta_5, x_0 + \delta_5) \subset \Omega$, and for all $x, y \in (x_0 - \delta_5, x_0 + \delta_5)$ satisfying $|y - x| < \delta_5$, we have that

$$\left|g(y) - g(x)\right| < \eta. \tag{4.26}$$

Let

$$\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}. \tag{4.27}$$

Then $(x_0-\delta, x_0+\delta) \subset \Omega$, and for all $x, y \in (x_0-\delta/2, x_0+\delta/2)$, we have that $|y-x|<\delta$ and $|g(y)-g(x)|<\eta$; $g(y), g(x) \in \Delta$ and hence

$$\begin{aligned}
&\left|f\circ g(y) - f\circ g(x) - f'\circ g(x) g'(x)(y-x)\right| \\
&\quad \le \left|f\circ g(y) - f\circ g(x) - f'\circ g(x)(g(y)-g(x))\right| \\
&\qquad + \left|f'\circ g(x) g(y) - f'\circ g(x) g(x) - f'\circ g(x) g'(x)(y-x)\right| \\
&\quad = \left|f(g(y)) - f(g(x)) - f'(g(x))(g(y)-g(x))\right| \\
&\qquad + \left|f'\circ g(x)\right|\left|g(y)-g(x)-g'(x)(y-x)\right| \\
&\quad < \frac{\epsilon}{2\left(\left|g'(x_0)\right|+1\right)}\left|g(y)-g(x)\right| + \frac{\epsilon}{2\left(\left|f'\circ g(x_0)\right|+1\right)}\left(\left|f'\circ g(x_0)\right|+1\right)\left|y-x\right| \\
&\quad < \frac{\epsilon}{2}|y-x| + \frac{\epsilon}{2}|y-x| = \epsilon|y-x|.
\end{aligned} \tag{4.28}$$

□

Proposition 4.13 (Product Rule). *Let $A \subset \mathcal{N}$, let $x_0 \in A$ be given, and let $f, g : A \to \mathcal{N}$ be LUD at x_0. Then fg is LUD at x_0 with $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.*

Proof. Let Ω be a neighborhood of x_0 in A such that f and g are uniformly differentiable on Ω and g is uniformly continuous on Ω (Proposition 4.10). Since f, g, and f' are continuous at x_0, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $(x_0-\delta_1, x_0+\delta_1) \subset \Omega$, and for all $x \in (x_0-\delta_1, x_0+\delta_1)$, we have that

$$\left|f(x)\right| < \left|f(x_0)\right| + 1, \qquad \left|g(x)\right| < \left|g(x_0)\right| + 1, \qquad \left|f'(x)\right| < \left|f'(x_0)\right| + 1. \tag{4.29}$$

Let $\epsilon > 0$ in $\mathcal{N}$ be given. Since f and g are uniformly differentiable on Ω, and g is uniformly continuous on Ω, there exists $\delta > 0$ such that whenever $|y-x|<\delta$ and $x, y \in \Omega$, we have that

$$\begin{aligned}
\left|g(y)-g(x)\right| &< \frac{\epsilon}{3\left(\left|f'(x_0)\right|+1\right)}, \\
\left|f(y)-f(x)-f'(x)(y-x)\right| &< \frac{\epsilon}{3\left(\left|g(x_0)\right|+1\right)}\left|y-x\right|, \\
\left|g(y)-g(x)-g'(x)(y-x)\right| &< \frac{\epsilon}{3\left(\left|f(x_0)\right|+1\right)}\left|y-x\right|.
\end{aligned} \tag{4.30}$$

Let $x, y \in (x_0-\delta_1, x_0+\delta_1)$ be such that $|y-x|<\delta$. Then,

$$\begin{aligned}
&\left|f(y)g(y) - f(x)g(x) - \left(f'(x)g(x) + f(x)g'(x)\right)(y-x)\right| \\
&\quad \le \left|f(y)g(y) - f(x)g(y) - f'(x)g(x)(y-x)\right|
\end{aligned}$$

$$\begin{aligned}
&+ \left|f(x)g(y) - f(x)g(x) - f(x)g'(x)(y-x)\right| \\
&\leq \left|f(y)g(y) - f(x)g(y) - f'(x)g(y)(y-x)\right| \\
&\quad + \left|f'(x)(y-x)(g(y) - g(x))\right| \\
&\quad + \left|f(x)g(y) - f(x)g(x) - f(x)g'(x)(y-x)\right| \\
&= \left|g(y)\right|\left|f(y) - f(x) - f'(x)(y-x)\right| \\
&\quad + \left|f'(x)\right|\left|g(y) - g(x)\right|\left|y-x\right| \\
&\quad + \left|f(x)\right|\left|g(y) - g(x) - g'(x)(y-x)\right| \\
&< \left(\left|g(x_0)\right| + 1\right)\frac{\epsilon}{3\left(\left|g(x_0)\right| + 1\right)}\left|y-x\right| \\
&\quad + \left(\left|f'(x_0)\right| + 1\right)\frac{\epsilon}{3\left(\left|f'(x_0)\right| + 1\right)}\left|y-x\right| \\
&\quad + \left(\left|f(x_0)\right| + 1\right)\frac{\epsilon}{3\left(\left|f(x_0)\right| + 1\right)}\left|y-x\right| \\
&= \left(\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\right)\left|y-x\right| = \epsilon\left|y-x\right|.
\end{aligned} \tag{4.31}$$

□

Since the function $f(x) = x$ is LUD on $\mathcal{N}$, as can be easily checked, it follows from Propositions 4.11 and 4.13 that any polynomial function is LUD on $\mathcal{N}$.

Corollary 4.14. *Let $f : \mathcal{N} \to \mathcal{N}$ be a polynomial function. Then f is LUD on $\mathcal{N}$.*

Since the derivative of a polynomial function is again a polynomial function, we readily obtains the following result.

Corollary 4.15. *Let $f : \mathcal{N} \to \mathcal{N}$ be a polynomial function. Then f is LUD$^\infty$ on $\mathcal{N}$.*

Proposition 4.16. *The function $h : \mathcal{N} \setminus \{0\} \to \mathcal{N} \setminus \{0\}$ defined as $h(x) = 1/x$ is LUD$^\infty$.*

Proof. First we note that h is infinitely often differentiable on $\mathcal{N} \setminus \{0\}$, with derivatives

$$h^{(l)}(x) = \frac{(-1)^l l!}{x^{l+1}} \text{ for each } l \in \mathbb{N}. \tag{4.32}$$

Now we prove that h is LUD on $\mathcal{N} \setminus \{0\}$. So let $x_0 \in \mathcal{N} \setminus \{0\}$ and let $\epsilon > 0$ in $\mathcal{N}$ be given. Let

$$\Omega = \left\{x \in \mathcal{N} : |x - x_0| < \min\left\{|x_0|d, \frac{\epsilon d}{|x_0|^3}\right\}\right\}. \tag{4.33}$$

Then Ω is a neighborhood of x_0 and $x \approx x_0$ for all $x \in \Omega$. Moreover, for all $x, y \in \Omega$, we have that

$$\begin{aligned}\left|h(y) - h(x) + \frac{1}{x^2}(y-x)\right| &= \left|\frac{1}{y} - \frac{1}{x} + \frac{1}{x^2}(y-x)\right| = \left|\frac{1}{x^2} - \frac{1}{xy}\right|\left|y-x\right| \\ &= \frac{\left|y-x\right|}{x^2\left|y\right|}\left|y-x\right| < \epsilon\left|y-x\right|,\end{aligned} \tag{4.34}$$

since

$$x^2\left|y\right| \approx |x_0|^3, \qquad \left|y-x\right| \le \left|y-x_0\right| + |x-x_0| < \frac{2\epsilon d}{|x_0|^3} \ll \frac{\epsilon}{|x_0|^3}, \tag{4.35}$$

so that

$$\frac{\left|y-x\right|}{x^2\left|y\right|} < \epsilon. \tag{4.36}$$

Thus, for all $x_0 \in \mathcal{N} \setminus \{0\}$, h is LUD at x_0, and hence h is LUD on $\mathcal{N} \setminus \{0\}$.

Applying Propositions 4.11 and 4.13, it then follows that $h^{(l)}(x) = (-1)^l l!/x^{l+1}$ is LUD on $\mathcal{N} \setminus \{0\}$ for all $l \in \mathbb{N}$, and hence h is LUD$^\infty$ on $\mathcal{N} \setminus \{0\}$. □

5. Inverse Function Theorem

The following version of the inverse function theorem for LUD functions was proven in [14].

Theorem 5.1 (inverse function theorem). *Let $A \subset \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be locally uniformly differentiable on A, and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then there is a neighborhood Ω of x_0 in A and a function $g : f(\Omega) \to \mathcal{N}$, such that*

(i) $g = f|_\Omega^{-1}$;

(ii) $f|_\Omega$ *is injective;*

(iii) $f(\Omega)$ *is open;*

(iv) g *is locally uniformly differentiable on* $f(\Omega)$, *with* $g' = 1/f' \circ g$.

In this paper, we state and prove a more general version of the inverse function theorem for functions from (a subset of) $\mathcal{N}$ to $\mathcal{N}$.

Theorem 5.2 (General Inverse Function Theorem for One-Variable Functions). *Let $A \subset \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be LUD^l on A, and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then there is a neighborhood Ω of x_0 in A and a function $g : f(\Omega) \to \mathcal{N}$, such that*

(i) $g = f|_\Omega^{-1}$;

(ii) $f|_\Omega$ *is injective;*

(iii) $f(\Omega)$ *is open;*

(iv) *g is LUD^l on $f(\Omega)$;*

(v) $g' = 1/f' \circ g$.

Proof. (i), (ii), (iii), and (v) are proven in [14]. To prove (iv), first recall that the function $h : \mathcal{N} \setminus \{0\} \to \mathcal{N} \setminus \{0\}$ given by $h(x) = 1/x$ is LUD$^\infty$ on $\mathcal{N} \setminus \{0\}$ (by Proposition 4.16). Let $y \in f(\Omega)$ be given; then there exists $\xi \in \Omega$ such that $y = f(\xi)$. We show by induction on n that g is LUD^n at $f(\xi)$ for all $n \le l$. We know that g is LUD^1 at $f(\xi)$ from [14]. Now assume that g is LUD^k $(k < l)$ at $f(\xi)$. Then $g^{(k)} = (g')^{(k-1)} = (h \circ f' \circ g)^{(k-1)}$ (by (v)). We have that h is LUD^k on $\mathcal{N} \setminus \{0\}$ (by Proposition 4.16), f' is LUD^k at ξ (by Proposition 4.4), and g is LUD^k at $f(\xi)$. Thus, by the Chain Rule (Proposition 4.12), it follows that $h \circ f' \circ g$ is LUD^k at $f(\xi)$. It follows that $g^{(k)} = (h \circ f' \circ g)^{(k-1)}$ is LUD at $f(\xi)$ (by Proposition 4.4) and hence g is LUD^{k+1} at $f(\xi)$ (by Proposition 4.5). This completes the induction and shows that g is LUD^l at $f(\xi)$.

□

6. Intermediate Value Theorem

The intermediate value theorem is an important key result in real analysis. However, while all continuous real-valued functions on $\mathbb{R}$ have the intermediate value property, this is not the case for $\mathcal{N}$-valued functions on $\mathcal{N}$. In fact, since $\mathcal{N}$ is not connected, any function which takes on two distinct constant values on a separation of the field will be continuous but will not attain any value between the constants. The following example illustrates this.

Example 6.1. Let $f : [0,1] \to \mathcal{N}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \sim 1, \\ 0 & \text{if } x \ll 1. \end{cases} \tag{6.1}$$

Then f is LUD$^\infty$ on $[0,1]$ as f is locally constant everywhere. But $f(x)$ does not attain on $[0,1]$ any values between $f(0) = 0$ and $f(1) = 1$. So even the property of LUD$^\infty$ is not strong enough to ensure an intermediate value property for the function.

The next question is whether any kind of local intermediate value property can be assured. That is, can we find sufficient conditions for a function to have the intermediate value property on some neighborhood of a point? The answer is yes as we will see in Theorem 6.3, but first we present the following example which shows that even the LUD$^\infty$ property is not quite sufficient to ensure the local intermediate value property, and it demonstrates the need for the added hypothesis to Theorem 6.3.

Example 6.2. Let $f : \mathcal{N} \to \mathcal{N}$ be given by

$$f(x) = \begin{cases} d^{2\lambda(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{6.2}$$

We will show that f is LUD$^\infty$ on $\mathcal{N}$. First, note that f is locally constant everywhere but at 0. Hence f is trivially LUD$^\infty$ on $\mathcal{N} \setminus \{0\}$ with $f' = 0$. It remains to show that f is LUD at 0 with $f'(0) = 0$. Let $\Omega = (-1, 1)$ and let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta = \epsilon d$. Let $x, y \in \Omega$ be such that $|y - x| < \delta$. Without loss of generality, we may assume that $|x| < |y|$. We distinguish two possible cases.

Case 1 ($|y| \sim |x|$). Then $|f(y) - f(x)| = 0 < \epsilon |y - x|$.

Case 2 ($|y| \gg |x|$). Then

$$\left|f(y) - f(x)\right| \approx d^{2\lambda(y)} \sim \left|y\right|^2 < \delta\left|y\right| \ll \epsilon\left|y - x\right|, \tag{6.3}$$

since $\delta \ll \epsilon$ and $|y| \approx |y - x|$, and this shows that $|f(y) - f(x)| \ll \epsilon |y - x|$. So, f is LUD$^{\infty}$ on $\mathcal{N}$ (with all derivatives equal to 0 everywhere); however, clearly f does not have the intermediate value property in any neighborhood of 0.

Theorem 6.3 (Local intermediate value theorem). *Let $A \subset \mathcal{N}$ and let $f : A \to \mathcal{N}$ be LUD at $x_0 \in A$ with $f'(x_0) \neq 0$. Then there is a neighborhood Ω of x_0 such that for any $a < b$ in $f(\Omega)$ and for any $c \in (a, b)$, there is an $x \in \Omega$ such that $f(x) = c$. Moreover, x is strictly between $f^{(-1)}(a)$ and $f^{(-1)}(b)$.*

Proof. Without loss of generality, we may assume that $f'(x_0) > 0$, since if $f'(x_0) < 0$, we could then apply this proof to $(-f)$ and get the desired result. Since f is LUD at x_0, there exists a neighborhood Δ of x_0 in A such that f is uniformly differentiable on Δ. Since f' is continuous at x_0 (by Proposition 4.8), there is $\delta_1 > 0$ such that $(x_0 - \delta_1, x_0 + \delta_1) \subset \Delta$, and for any $x \in (x_0 - \delta_1, x_0 + \delta_1)$, we have that

$$f'(x) > \frac{2}{3} f'(x_0). \tag{6.4}$$

Since f is uniformly differentiable on Δ, there exists $\delta < \delta_1$ such that for all $x, y \in (x_0 - \delta_1, x_0 + \delta_1)$ satisfying $|y - x| < \delta$ we have that

$$\left|f(y) - f(x) - f'(x)(y - x)\right| < \frac{f'(x_0)}{3}\left|y - x\right|. \tag{6.5}$$

It follows that for $y, x \in (x_0 - \delta/2, x_0 + \delta/2)$ we have that

$$\frac{f(y) - f(x)}{y - x} > f'(x) - \frac{f'(x_0)}{3} > \frac{1}{3} f'(x_0) > 0. \tag{6.6}$$

Hence f is strictly increasing on $\Omega_1 = (x_0 - \delta/2, x_0 + \delta/2)$. Applying Theorem 5.2 to f yields a neighborhood $\Omega_2 \subset \Omega_1$ of x_0 such that $f(\Omega_2)$ is open. Let $\epsilon > 0$ be such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subset f(\Omega_2)$ and let $\Omega = f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ which is an open neighborhood of x_0. Now let $a, b \in f(\Omega)$ be such that $a < b$, and let $c \in (a, b)$ be given. Then $c \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \subset f(\Omega_2)$ since $a, b \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ and $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ is a convex set. So there is $x \in \Omega_2$ such that $f(x) = c$. It follows that $x \in \Omega$ because $f(x) = c \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$. It is also true that $a < c = f(x) < b$; and since f is increasing on Ω, it follows that $f^{-1}(a) < x < f^{-1}(b)$. □

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