Linear Perturbations of Type IIB SUGRA in Flux Compactifcations

By

Bradley Stuart Cownden

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Abstract

We consider linear perturbations of the background type IIB SUGRA solutions and find the equations of motion for the moduli. In particular, we allow for spacetime fluctuations of the positions of D3-branes in the compact dimensions. We postulate an ansatz for the 5-form flux due to the motion of the D3-branes, and a corresponding first-order part of the metric. The movement of the D3-branes is then shown to affect the warp factor at linear order. Using the equations of motion for the D3-branes, the universal volume modulus, and the universal axion, we construct a second-order, effective action. Finally, based on the form of the effective action, we examine a Kähler potential for the moduli space.

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1 Introduction

1.1 String Theory

For more than a century, physicists have searched for the laws and constituents that best describe the universe at its most fundamental level. Their success can be summarized in the Standard Model (SM), which unites electromagnetism with the strong and weak nuclear forces in terms of an $SU(3) \times SU(2) \times U(1)$ gauge group. There are twelve gauge bosons that mediate the fundamental forces and act on the two primary forms of matter: quarks and leptons. The SM also predicts that the Higgs field gives mass to the fundamental particles we observe. The SM can be written in the language of quantum field theory and has proven robust over decades of experimental verification. Together, the SM and Einstein's General Relativity (GR) successfully describe the vast majority of matter and interactions in the universe through a Lagrangian containing almost 20 dimensionless parameters, each of which require experimental determination.

However, despite numerous successes (notably, the prediction of the Higgs boson and its subsequent verification at the LHC), the SM + GR description fails to address some other important areas. For instance, the fact that gravity cannot be incorporated into the SM in a renormalizable way indicates that there will be new physics to consider at very high energies. Also, there are no predictions for *why* the free parameters take the values that they do. Finally, spacetime singularities, such as those found in the centre of a black hole, are not resolvable classically. For these and additional reasons, alternative theories that incorporate the successes of the SM + GR description, but seek to resolve its unanswered questions, have been proposed.

String theory is an example of a theory that attempts to reach beyond the scope of the SM + GR description of the universe. By starting with a fundamentally different description of the universe – as strings instead of points – what arises is an elegant framing of the laws of nature. By constructing a quantum field theory from fundamental strings, we can recover the structures of GR and Quantum Field Theory (QFT) *without* the set of dimensionless constants; string theory contains no dimensionless parameters. These are the essential ingredients for any theory whose purpose is to describe the universe on both its largest and its smallest scales. There are additional features that string theory brings to light: the existence of extra dimensions, a consistent quantum theory of gravity, and the resolution of spacetime singularities, to name a few.

In order to see how the simple assertion of fundamental strings can effortlessly unite previously incompatible theories, we will summarize the introductions of [1] and [2] to the quantization of an open string in the light-cone gauge. A one-dimensional string in a D-dimensional spacetime will sweep out a two-dimensional world-sheet, which we can parameterize in terms of some τ and σ, so that the position of the world-sheet is given by $X^\mu(\tau,\sigma)$. Since these parameters are arbitrary, any physical quantity must be independent of our choice of parameters; indeed, any action must depend only on the embedding in spacetime. One of the simplest invariant actions we can write involves defining some world-sheet metric, γ_{ab} where $\{a,b\} = \{\tau,\sigma\}$, and is called the Polyakov action:

$$S_{\mathrm{P}} = -\frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d\tau d\sigma \sqrt{-\det(\gamma_{ab})}\,\gamma^{ab}\partial_a X^\mu \partial_b X_\mu\,, \tag{1.1}$$

where $\mathcal{M}$ is the string world-sheet and α' is a constant with units $[\text{length}]^2$. In fact, the constant outside the integral in (1.1) is the string tension: $T = (2\pi\alpha')^{-1}$. The high degree of symmetry in the Polyakov action, *e.g.* D-dimensional Poincaré invariance, diffeomorphism invariance, *etc.* will be transferred to the two-dimensional field theory that it describes on the string world-sheet. In this theory, $X^\mu(\tau,\sigma)$ is a massless Klein-Gordon scalar field coupled to γ_{ab}. In order to describe the amplitudes of spacetime processes, we compute the matrix elements of the quantum field theory on the world-sheet.

For illustrative purposes, we choose to quantize the system in the light-cone gauge, *i.e.* where the coordinates of the spacetime are $x^\pm = 1/\sqrt{2}(x^0 \pm x^1)$ and x^i for $i = 2,\ldots,D-1$, so that $m^2 x^+ = p^+\tau$. We then find that the equations of motion yield a wave equation for the transverse components of the open string

$$\partial_\tau^2 X^i = \left(\frac{\ell}{2\pi\alpha' p^+}\right)^2 \partial_\sigma^2 X^i\,, \tag{1.2}$$

where ℓ is the length of the string. From this point, we are able to invoke the normal principles of quantum field theory to quantize the string and so express X^i in terms of a sum of creation and annihilation operators. By requiring invariance among the transverse Lorentz generators, $M^{\mu\nu} = M^{-i}$ such that $[M^{-i}, M^{-j}] = 0$, we find that the number of dimensions, D, is fixed to $D = 26$.

We can now construct the state space via acting on the vacuum state with creation operators. While the state space of a single, open string is not a realistic model, it does serve to illustrate the main features of a string spectrum. We find that the ground state for these strings is, in fact, a scalar state with $m^2\alpha' = -1$. This unstable state is called the tachyon. The first excited state of the theory is a massless vector with $(D-2)$ possible polarizations. This state is reminiscent of the vector potential of electromagnetism, which carries with it $(4-2) = 2$ possible polarizations in four dimensions. The next set of excited states have masses of $m^2\alpha' = 1$ and are massive tensor states. If we were to repeat the quantization procedure above for closed strings instead of open strings, we would find mass*less* tensor states that would decompose into a symmetric-traceless part, an antisymmetric part, and a scalar trace part. Incredibly, the symmetric-traceless massless tensor states of closed strings are precisely the states required to represent gravitons. The antisymmetric portion of the tensor states correspond to a Kalb-Ramond field (a tensor generalization of a Maxwell gauge field), while the trace portion of the states represents a massless scalar field called the dilaton. Thus, from the first few states of the most naïve constructions of open and closed strings, we have already recovered the field content of GR as well as a SM-like theory, all from a single, one-dimensional object. From here it is not difficult to see how we could construct more physical models for the universe from fundamental strings and their interactions.

While string theory is able to resolve many issues that plague contemporary physics, it brings along with it some issues of its own. Primarily, direct observation of fundamental strings would require experiments at or near the Planck scale, and so remain entirely implausible. Indirect tests must contend with the fact that, although string theory can uniquely describe the universe, the predicted classical solutions span a vast space called the string landscape. Each configuration in the string landscape yields a different four-dimensional theory which is not necessarily stable. This, in turn, leads to universal nucleation: the tunnelling from one vacuum solution to another, producing a universe wherein new universes spontaneously form within one another and then expand in size at the speed of light.

Despite these unresolved issues, string theory has made considerable theoretical progress. The simple example of the bosonic string outlined above can be extended to a more general theory that incorporates supersymmetry (a symmetry that unifies bosons and fermions), called superstrings. Through a similar process to the one that set the critical dimension previously, in the case of superstrings, we find that $D = 10$. Additionally, the negative mass-squared tachyon states are no longer present. The types of theories that are derived from superstrings are grouped based on their inclusion of open/closed strings, the number of spacetime supersymmetry charges ($\mathcal{N}$) they carry, and whether or not the strings are oriented (*i.e.* does the direction of increasing σ play a role). The main theories – all of which contain gravity – that preserve 10-dimensional Poincaré invariance are summarized in Table 1.1.

Type	**Open/Closed**	**Oriented**	$\mathcal{N}$	**Gauge Group**
I	Open & Closed	No	1	$SO(32)$
IIA	Closed	Yes	2	$U(1)$
IIB	Closed	Yes	2	None
Heterotic	Closed	Yes	1	$E_8 \times E_8$
Heterotic	Closed	Yes	1	$SO(32)$

Table 1.1: *The primary superstring theories and their characteristics.*

Although these theories may seem distinct, additional considerations due to extra dimensions lead us to the conclusion that there exists some mapping between the supersymmetric theories. Consider that, since the extra dimensions are not readily observable, it is reasonable to postulate that they might be compact with some small radius R. One of the effects that compact dimensions have on strings is to introduce R into the mass formula for the states that move in the compact dimensions. However, the mass spectrum is invariant under the transformation

$$R \leftrightarrow \frac{\alpha'}{R}\,. \tag{1.3}$$

That is, the physics of strings in compact dimensions where $R \ll \sqrt{\alpha'}$ is *indistinguishable* from compact dimensions where $R \gg \sqrt{\alpha'}$. The coincidence of the physics when $R \leftrightarrow \alpha'/R$ is called T-duality. In addition to T-duality, there is also S-duality, which relates the physics of a system with coupling strength g to that of one with coupling strength $1/g$. Altogether, the combination of the two duality relations, allowing single or multiple dimensions to be compact, and – as we will explain below – the existence of other string-type objects, act to create a "web" on which the supersymmetric theories of Table 1.1 lie. This way, all five superstring theories can be mapped to one another. For example, Heterotic $SO(32)$ can be mapped directly to Heterotic $E_8 \times E_8$ by T-duality; however, type IIB with D3-branes is no longer purely type IIB, but rather is some closely related theory. We also find that an additional theory emerges: M-theory. M-theory, as opposed to the other superstring theories, lives in 11 dimensions and may be the most fundamental description of the other superstring theories [3]. The web of supersymmetric theories and their mappings is illustrated in Figure 1.1.

In addition to the open and closed strings already discussed, another fundamental object in string theory is the D-brane. D-branes, or Dirichlet Branes, are defined as the plane upon which open string endpoints are constrained to lie on. They are derived from the Dirichlet boundary conditions that arise in the variation of (1.1) for a relativistic, open string. The simplest kind of D-brane is one that fills all of the space it is embedded in; however, a Dp-brane is a spatial, p-dimensional hyperplane within the d-dimensional space. The motion of the endpoints of open strings is free along the Dp-brane because these points obey Neumann boundary conditions.

When we quantize open strings in the presence of a Dp-brane, we find that the world-volume of the brane supports a $U(1)$ gauge field and a massless scalar for each coordinate direction normal to the brane (these coordinates satisfy Dirichlet boundary conditions) [2]. Furthermore, stacking

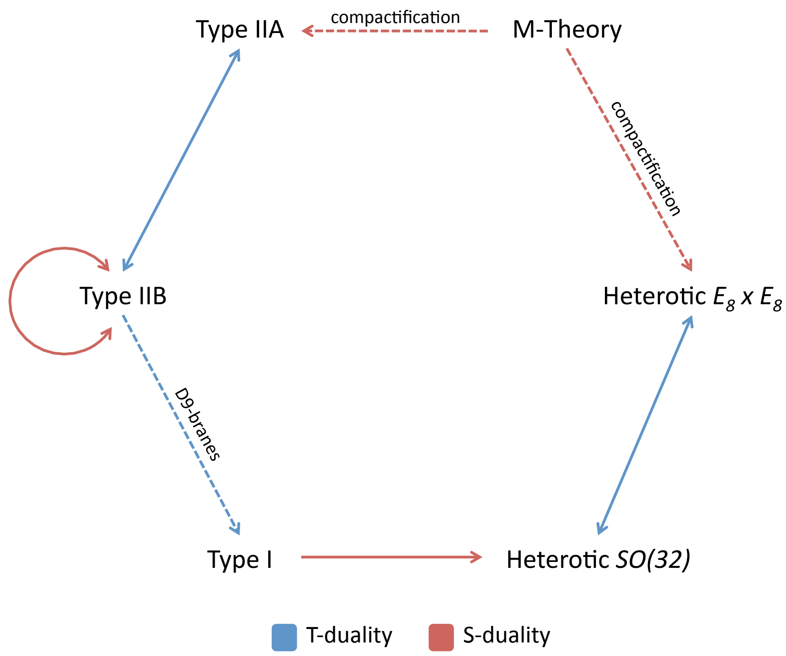


Figure 1.1: *Directional mapping between the five major $D = 10$ superstring theories and M-Theory. Dashed lines correspond to additional objects or operations required in conjunction with the corresponding duality.*

N branes on top of one another produces a $U(N)$ gauge field on the brane world-volume. This means that Dp-branes can support different types of charges for the different types of gauge fields that exist on their world-volume. In this work, we set these world-volume gauge groups to zero for simplicity.

D-branes also have tension, charge, and mass. In the case of a static Dp-brane, the tension, T_p, times the volume, V_p, gives the mass of the brane. We can therefore consider fixed branes to be very massive, *i.e.* semi-classical, objects. The tension of a Dp-brane is related to the tension of a D$(p-1)$-brane by $T_p(g) = T_{p-1}(g) \cdot 2\pi\sqrt{\alpha'}$ and is a function of the string coupling, g. By convention, we choose the string coupling such that the tension of a D1-brane is $T_1(g) \cdot 2\pi\alpha' g = 1$.

Having outlined some of the basic types and characteristics of superstrings, we now wish to focus our attention on type IIB superstring theory. In particular, we wish to see how the low-energy limit of this theory naturally leads into a theory of supergravity, *i.e.* type IIB SUGRA.

1.2 Type IIB SUGRA

Type IIB superstrings have a number of spacetime fields corresponding to string states built from an increasing number of creation operators acting on the vacuum. However, when we are considering the low-energy limit of this theory, we are only concerned with the lightest states. Since there are states with zero mass, we take these to be of primary importance and therefore consider any state with $m > 0$ to be too heavy to interact.

Recall that, as mentioned in § 1.1, all string theories posses a metric, an anti-symmetric tensor, B_{MN}, and the dilaton, ϕ, as massless scalar fields. There also exists a ten-dimensional scalar axion C_0, a 2-form[1] potential C_{MN}, and a 4-form field C_{MNPQ}. It is conventional to define the axio-dilaton $\tau = C_0 + ie^{-\phi}$ and to combine the 3-forms $F_3 = dC_2$ and $H_3 = dB_2$ into the complex 3-form $G_3 = F_3 - \tau H_3$. From these, we construct the field strength $\tilde{F}_5 = dC_4 - C_2 \wedge H_3$, which must obey the self-duality condition $\tilde{F}_5 = \star \tilde{F}_5$ [4, 5]. In doing so, we arrive that the action for type IIB superstrings in the form given by [6]:

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(\mathcal{R} - \frac{\partial_M \tau \partial^M \bar{\tau}}{2\,(\mathrm{Im}\,\tau)^2} \right) - \frac{1}{2\kappa_{10}^2} \int \Bigg[\frac{G_3 \wedge \star \bar{G}_3}{12\,\mathrm{Im}\,\tau} + \frac{1}{4} \tilde{F}_5 \wedge \star \tilde{F}_5 \\ + \frac{i}{4\,\mathrm{Im}\,\tau} C_4 \wedge G_3 \wedge \bar{G}_3 \Bigg] + \ldots + S_{\mathrm{loc}} \,, \end{aligned} \tag{1.4}$$

where $\mathcal{R}$ is the 10D Ricci scalar, and κ_{10} is the 10D Newton's constant. The terms denoted by dots are those with more than two derivatives (these do not contribute to the low-energy limit). The action due to local sources is S_{loc} and is contributed to by D3-branes. These are governed by an action that we shall refer to as the Dirac-Born-Infield (DBI) action[2]

$$S_{DBI} = \int d^{10}x \sqrt{-g} \left(-\frac{T_3}{2} \int d^4\xi \sqrt{-\gamma} \left[\gamma^{ab} P(g)_{ab} - (3-1) \right] \right) \delta^{10}(x, X(\xi)) \,, \tag{1.5}$$

[1] The language of differential forms is used extensively throughout this work. Appendices A.3 and A.4 provide the reader with some background on forms and differential geometry.

[2] In fact, this action reduces to the DBI action when electromagnetism on the brane world-volume is turned off [2].

and the Wess-Zumino coupling of the D3-brane's charge

$$S_{WZ} = -\mu_3 \int d^{10}x\sqrt{-g} \int_\xi P(C_4)\delta^{10}(x, X(\xi))\,. \tag{1.6}$$

The actions in (1.5) and (1.6) will be discussed in more detail in Chapter 2.

Unlike flat space, where field lines can continue to infinity, compact spaces must have pairs of opposing charges. Since the D3-brane has a charge μ_3 (which is known to equal the tension, T_3) that lives in the compact dimensions, there must exist some source of opposite charge where the brane field lines can end. These sources of opposite charge are Orientifold planes, or O-planes. The properties of the compact dimensions are such that O-planes are located at points that are invariant under geometric symmetry transformations [7].

Since these massive, charged D3-branes are located in the compact space, there will be warping due to their presence. The most general 10D metric that preserves 4D Poincaré invariance is

$$ds^2_{10} = e^{2A(y)}\hat{\eta}_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\tilde{g}_{mn}dy^m dy^n\,, \tag{1.7}$$

where the warp factor, A, is a function of the internal space only. We initially take $\tilde{g}_{mn}$ to be the metric of an unspecified, 6D compact manifold. Also, we adopt the convention that the spacetime coordinates are x^μ and the compact dimensions have coordinates y^m.

Let us consider a background solution to (1.4) where fluxes and local sources are set to zero. This means that only the geometry of the compact dimensions contributes. From the Bianchi identities and the (sourceless) Einstein equations, we find that the warp factor must be a constant, and so can be set to zero without loss of generality. A similar constraint for τ sets its background value to be $e^{-\phi}$. From the Einstein equations we find that $\tilde{R}_{mn} = 0$, *i.e.* the compact manifold must be Ricci-flat. The most general compact, Ricci-flat manifold is the Calabi-Yau 3-fold[3]; henceforth, $\tilde{g}_{mn}$ will be taken to be the metric of a Calabi-Yau 3-fold.

Now consider the inclusion of D3-branes, which will affect the value of the warp factor. If the i^{th} D3-brane has embedding coordinates $Y^{\not{p}}_{(i)}$ and charge T_3, then the Einstein equations yield a

[3] See Appendix A.4 for additional information regarding these types of manifolds.

Poisson equation for A [6], given by

$$\widetilde{\nabla}^2 e^{-4A} = -\frac{g_s}{2}\left|G_3^{(0)}\right|^2 - 2\kappa_{10}^2 T_3 \sum_i^N \tilde{\delta}^6(y, Y_{(i)}) - 2\kappa_{10}^2 T_3' \rho_{\rm loc}\,, \tag{1.8}$$

where $G_3^{(0)}$ is the background value for the complex 3-form, T_3' is the tension of additional (negatively) charged local sources which are *not* D3-branes, and $\rho_{\rm loc}$ is the density of these sources. The role of D3-branes in the warping of the extra dimensions was outlined first in [8] and has been explored in a variety of contexts. In this work, we need only consider the case of a single D3-brane, since, provided that the branes are sufficiently spaced apart, extending our solutions to N branes is trivial.

Like any potential, C_4 can be gauge transformed by some $d\beta_3$ without changing the form of the field strength ($\tilde{F}_5$). However, unlike in flat space, there also exist shifts that are non-trivial gauge transformations (*i.e.* cannot be written as a globally-defined $d\beta_3$ over the entire manifold) and have zero field strength. These shifts correspond to harmonic forms on the Calabi-Yau and are significant in the dimensionally-reduced theory due to their relation to the geometry of the compact space. These types of non-trivial effects translate into massless scalar fields in the dimensionally-reduced theory and are referred to as moduli. Let us now turn to the types of moduli present in type IIB SUGRA.

1.3 Moduli in Type IIB SUGRA

The moduli to contend with in type IIB SUGRA correspond to non-trivial shifts of background values – and therefore background solutions – that do not enter into the equations of motion. Quantitatively, moduli follow from the variation of some quantity $\mathcal{Q} \to \mathcal{Q} + \delta\mathcal{Q}$, that satisfies a second-order differential equation $\mathcal{O}^2\delta\mathcal{Q} = 0$. For example, the allowed deformations of the Calabi-Yau manifold, δg, are those that change the manifold's size and shape, but not its Ricci-flatness, and are governed by the Lichnerowicz equation (A.5.4). Appendix A.5 discusses the moduli space for a more general class of manifolds – Kähler manifolds – and then specializes to Calabi-Yau 3-folds.

From Yau's theorem (§ A.5), we know that the moduli space of Calabi-Yau manifolds is that of Ricci-flat Kähler metrics. This space can be separated into Kähler moduli and complex structure moduli, and is itself Kähler. Therefore, by the Kähler condition, we can describe the moduli space in terms of a Kähler potential, $\mathcal{K}$. Being able to describe the moduli space in terms of a Kähler potential is a requirement for a consistent 4D supersymmetric theory.

To see this more explicitly, we follow the discussion of [9], which considers a 10-dimensional metric that preserves 4-dimensional maximal symmetry. First, we let $u^A(x)$ parameterize the complex and Kähler moduli of the compact manifold. After dimensional reduction, the kinetic action (up to factors of the 10-dimensional Planck mass) for the moduli is

$$S_{\mathcal{K}} = -\frac{1}{\kappa_4^2}\int d^4x\sqrt{-\eta}\,\eta^{\mu\nu}\partial_\mu u^A \partial_\nu u^{\bar{B}} G_{A\bar{B}}(u)\,, \tag{1.9}$$

where η is the metric of the 4D space [9]. The Kähler metric, $G_{A\bar{B}}$, is related to the Kähler potential, $\mathcal{K}$, by $G_{A\bar{B}} = \partial_A \bar{\partial}_B \mathcal{K}$. In Chapter 6, we compare the form of the second-order effective action built explicitly with the moduli to the result from an ansatz for the Kähler potential. By matching the terms in each, we are able to write an expression for $\mathcal{K}$ and therefore produce a theory consistent with a 4D supersymmetric theory.

Besides via geometric variations of the internal manifold, moduli are also produced by the compactification of fields in the presence of extra dimensions. Kaluza and Klein first used the compactification model to unify gravity and electromagnetism (see [10] for a review of this topic). A side-effect of this is the production of zero-modes, *i.e.* moduli, in the 4D theory. To illustrate this effect, we consider the simple example of a flat, 5-dimensional spacetime with one spatial dimension y compactified into a circle of radius R. Following the example in [11], we start with the action for a real, massless scalar field

$$S = -\frac{1}{2}\int d^5x\, \partial_M \varphi \partial^M \varphi\,, \tag{1.10}$$

where φ is periodic in the compact dimension, such that $\varphi(x, y) = \varphi(x, y + 2\pi R)$, and satisfies the

5D wave equation $\Box_5 \varphi = 0$. Using the periodicity of φ in y, we can rewrite the scalar field in terms of its eigenstates in the compact dimension

$$\varphi(x,y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \varphi_n(x) e^{iny/R} \,. \tag{1.11}$$

Using this expression for φ in the equation of motion yields

$$\partial_\mu \partial^\mu \varphi_n(x) - \frac{n^2}{R^2} \varphi_n(x) = 0 \,, \tag{1.12}$$

i.e. the φ_n are spacetime fields with massess n/R. After integrating over y, the action becomes

$$S = -\frac{1}{2} \int d^4x \, \partial_\mu \varphi_0 \partial^\mu \varphi_0^* - \sum_{n=1}^{\infty} \int d^4x \left[\partial_\mu \varphi_n \partial^\mu \varphi_n^* + \frac{n^2}{R^2} \varphi_n \varphi_n^* \right] \,. \tag{1.13}$$

As we can see, this action indicates that the scalar field has acquired a mass term for $n \neq 0$. However, there is a single massless mode, φ_0, that persists. In the limit of $R \to 0$, the masses of the non-zero states become infinitely large, and can therefore be neglected. The massless scalar field, on the other hand, must be considered in more detail. It was long thought that these zero modes had to be fixed by quantum corrections to the theory. However, [6] found that some moduli could be fixed *classically* by fluxes in the extra dimensions. This process in called flux compactification, and is the focus of our next section.

In terms of the fields and geometry of type IIB SUGRA, the moduli that we are concerned with are the perturbations around the background solutions described in § 1.2. For example, the axio-dilaton, τ, has a constant background value, as determined by the 10D Einstein equations. We must consider what effects the first-order perturbation $\tau \to \tau + \delta\tau$ have on the dimensionally-reduced theory. As we will see, some moduli will be fixed by flux compactification, *e.g.* τ; however, some moduli will remain free, such as the positions of the D3-branes in the extra dimensions.

1.4 Flux Compactification

Let us first examine a simplified scenario involving the classical fixing of zero modes by flux compactification. Following the discussion of [2], we consider a 6D theory of gravity and electromagnetism where two the spatial dimensions are compact, but with a volume that can vary in spacetime. We take the metric to be of the form

$$ds_6^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + R^2(x)\bar{g}_{ab}(y)dy^a dy^b\,, \tag{1.14}$$

where a, $b = 1, 2$ and the scale factor $R(x)$ has been has been separated from the internal space metric. The volume of the compact space is then given by $R^2(x)V_2$, where V_2 is computed with $\bar{g}_{ab}$. Qualitatively, the resulting potential $V(R)$ from integration over the compact space will be proportional to the Euler number of the manifold, χ, and inversely proportional to R^4: $V(R) = -a_g\,\chi/R^4$ with $a_g > 0$. If the compact space has $\chi \geq 1$, such as for a 2-sphere, the potential is negative definite, which will cause the radius to tend to zero. In the case of a 2-torus, *i.e.* $\chi = 0$, the potential vanishes and $R(x)$ is a modulus of the theory. Furthermore, when $\chi < 0$ the potential is positive definite and the radius will tend to infinity. Integration over the compact space, *i.e.* dimensional reduction, will also affect the gravitational constant, G. Dimensional analysis tells us that the dimensionally-reduced constant, $G^{(4)}$, is equal to the higher-dimensional constant, $G^{(6)}$, divided by the volume of the compact dimensions. Since the volume can vary in time, a redefinition of $g_{\mu\nu}(x)$ is required to keep $G^{(4)}$ constant. This introduces the Weyl factor, $\Omega(x)$, into the 4D metric. The Weyl factor is the time-dependant ratio of the changing volume of the compact dimensions to the static volume:

$$e^{-2\Omega(x)} = \frac{V_2(R(x))}{V_2}\,. \tag{1.15}$$

In the presence of compact dimensions, the magnetic flux is quantized and can be utilized to stabilize the radius. If $\Phi = 2\pi n$ with $n \in \mathbb{Z}$, then we know that the B field goes like $B \sim n/R^2$.

After a similar redefinition of the 4D metric, we find that the flux contributes a term that goes like $V(R) \sim n^2/R^6$. Therefore, the total potential is

$$V(R) = -a_g \frac{\chi}{R^4} + a_f \frac{n^2}{R^6}\,, \tag{1.16}$$

where $a_f > 0$. After dimensional reduction, the action for the radius,

$$S_{R(x)} = \frac{1}{2\kappa_4^2} \int d^4x \left[\partial_\mu R(x) \partial^\mu R(x) + V(R)\right]\,, \tag{1.17}$$

will always have a (positive) non-zero potential provided there is a non-zero magnetic flux. Thus, the addition of magnetic flux has stabilized the radius by providing $V(R^*)$ with a stable equilibrium point for some $R^* > 0$.

Having demonstrated flux compactifications in a simple theory, we now consider this effect in the case of type IIB SUGRA by turning on the flux sources for G_3. The moduli we consider will now have some background value, plus a first-order perturbation. In this case, we find that the 3-form flux must satisfy the imaginary self-duality condition $\tilde{\star} G_3 = iG_3$. As shown in [12], the equation of motion for $\delta\tau$ picks up a mass term

$$\partial_\mu \partial^\mu \delta\tau = \frac{ig_s}{4} \delta g_{mn} G^m{}_{pq} G^{npq} + \frac{g_s^2}{2} \delta\tau |G_3|^2\,, \tag{1.18}$$

where g_s is the string coupling. The first term above corresponds to the mixing between the dilaton and the geometric moduli, while the second term is a mass generated by the flux. Thus, in the presence of flux, $\delta\tau$ obtains a mass and so is fixed. Via a similar process, some of the complex structure moduli described in § 1.3 obtain mass terms and are fixed in the presence of fluxes.

Another effect that quantized fluxes have is the stabilization of the Calabi-Yau against geometric singularities, which was first shown in [6] and will be summarized here. Consider a manifold with a singularity described by a deformed conifold geometry, and in the presence of fluxes. A deformed conifold is a cone over a space with topology $S^2 \times S^3$, and is described by the complex coordinates

$(w_1, \ldots, w_4)$ such that

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = z\,, \tag{1.19}$$

where z is a complex structure modulus that controls the size of S^3. When $z = 0$, the conifold is singular. However, near the conifold there are two important cycles[4]: cycle A lives on the S^3 and corresponds to taking the w^i in (1.19) to be real; cycle B is the 6D dual of A and corresponds to taking w^1, w^2, w^3 to be imaginary and w^4 to be real [13]. Over these cycles, the fluxes F_3 and H_3 are quantized so that

$$2\pi M = \frac{1}{2\pi\alpha'}\int_A F_3 \quad \text{and} \quad -2\pi K = \frac{1}{2\pi\alpha'}\int_B H_3\,, \tag{1.20}$$

where $M, K \in \mathbb{Z}^+$. Solving the equations of motion requires that z approaches a small, but non-zero value given by

$$z \sim \exp\left[-\frac{2\pi K}{g_s M}\right]\,, \tag{1.21}$$

which prevents the conifold from becoming singular. This resolution of the conifold singularity also fixes the minimum value of the warp factor to

$$e^{A_{\min}} \sim z^{1/3} \sim \exp\left[-\frac{2\pi K}{3 g_s M}\right]\,. \tag{1.22}$$

What are the implications of an exponential warp factor with a small, but finite, minimum? Primarily, strings that exist in regions of small warping can represent the Standard Model. This is because the graviton state is maximal in regions of large warping, and so strings away from these regions will support light modes [14]. Furthermore, for a theory with n compact dimensions, the effective 4D Planck scale, $M_{Pl} = 2 \times 10^{18}$ GeV, is related to the fundamental $(4+n)$ Planck scale through the geometry of the extra dimensions. Barring the case of large extra dimensions,

[4] See Appendix A.3 for additional information regarding cycles.

the fundamental Planck scale is exceedingly high. However, owing to the exponential scaling of the warp factor, we are able to establish a large mass hierarchy for very reasonable values of the flux quanta. This hierarchy may manifest itself through scattering events at seemingly low energies being able to probe the fundamental Planck scale.

Moduli, both fixed and un-fixed by compactification, have a wide range of applications in 4D effective theories. One such subject – which we will not explore in this work – is in string cosmology. Here, the free moduli are taken as candidates for the inflaton field. Before the start of inflation, the inflaton is constrained by the quantum fluctuations that dominate that epoch. As the universe expands, these fluctuations are no longer sufficient to constrain the inflaton and so it becomes perturbatively unstable. The period in our universe when inflation occurred would correspond to the movement of this modulus down some shallow potential, with inflation ceasing when the field encounters a local minimum. While some progress has been made on this subject, *e.g.* [15] and [16], no definitive model for string inflation has been discovered.

Having seen the power of flux compactification to fix moduli, we now turn to the focus of this thesis: moduli that are *not* fixed at the classical level. As mentioned in §1.3, the 4D supersymmetric theory that results from the dimensional reduction of the full, 10D one possess a quadratic action for the moduli that can be described in terms of a Kähler potential, $\mathcal{K}$. Due to its importance in the 4D theory, a direct calculation of $\mathcal{K}$ is an essential part of establishing a consistent theory. In this work, we will derive the Kähler potential in the presence of warping.

1.5 Mobile D3-Branes as Moduli

As discussed in the previous sections, there are specific moduli that are not fixed at the classical level and so will contribute to the Kähler potential. One such modulus is the universal volume modulus, which corresponds to a rescaling of the internal Calabi-Yau metric $\tilde{g} \to e^{2u(x)}\tilde{g}$ under zero warping. However, in the presence of warping there are additional factors to consider, such as non-trivial constraints. In [17], the contribution of the volume modulus to the Kähler potential in the presence of warping was explicitly calculated. We will review these results in § 5.1.

Another modulus being considered is the universal axion. The axion corresponds to a shift in the 4-form potential, C_4, by a harmonic form. Again, their role in the determination of $\mathcal{K}$ is complicated by the presence of warping. In § 5.2 we review the results of [18], which considers a scalar axion (*i.e.* shifts by a non-trivial 4-form). Using these results as a guideline, we then derive the kinetic action for a 2-form axion in § 5.3.

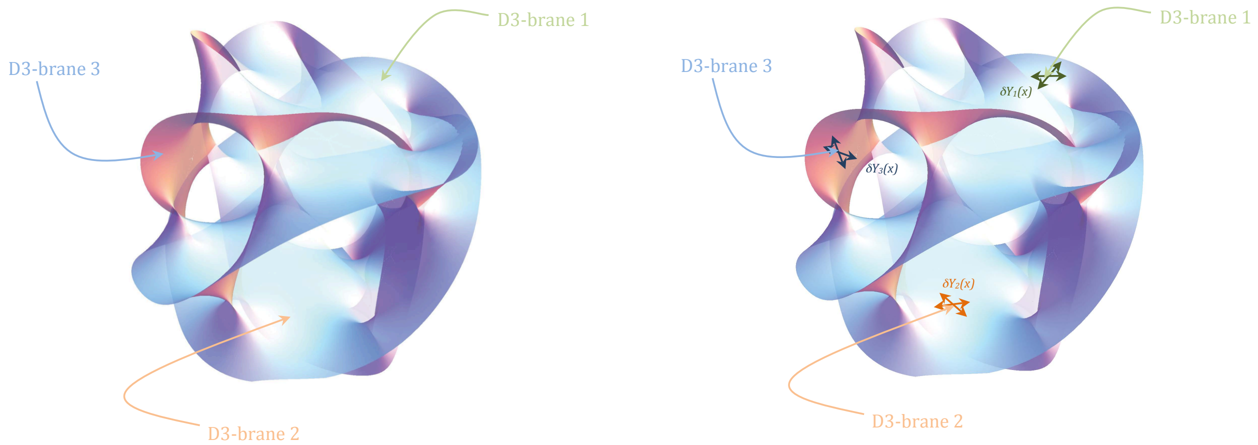
Finally, the focus of this work is on introducing mobile D3-branes as moduli into type IIB SUGRA. In order to illustrate how D3-branes can constitute a modulus in type IIB SUGRA, we can imagine a configuration of D3-branes at various positions in the compact dimensions. The fully 10D theory will be affected by the existence of the D3-branes through their influence on the warp factor, as in (1.8). In order to determine the dimensionally-reduced, 4D theory, we must integrate over the compact dimensions, and, therefore, the positions of the D3-branes. However, since these branes are taken to be static, their contributions are only through delta functions and so the theory remains independent of the positions of the branes.

In order to address the role that the positions of the D3-branes play in the 4D theory, we consider small perturbations of the D3-brane's positions by taking $Y_{(i)} \to Y_{(i)} + \delta Y_{(i)}(x)$. After solving the 10D system, the dimensional reduction of the theory to 4D will produce a kinetic action for the modulus, $\delta Y_{(i)}(x)$. This scenario is illustrated in Figure 1.2

When considering the effects due to D3-branes, the difficulties posed by warping in solving the equations of motion for the 10D system were first outlined in [19]; a general solution to these issues was framed in [20]. Attempts at deriving the Kähler potential in [21] and [22] were not successful at incorporating the influence of the warp factor and its dependence on the moduli.

A direct derivation of the impact of stringy objects on the 4D effective theory has been hampered by the difficulties resulting from including fluxes and/or non-trivial warping. The treatment of type IIB SUGRA with O3- and O7-planes in zero warping, but with fluxes, was covered in [23]. However, since warping was shown to generate the large scale hierarchies discussed in § 1.4, trivial warping solutions present little additional insight. There have also been a number of incomplete attempts to properly account for warping in the presence of fluxes, including those by [24–30].

In order to derive the quadratic action for the D3-branes, we start with two primary ingredients:



(a) *Compact dimensions with D3-branes at fixed positions,* $Y_{(i)}$.

(b) *The first-order perturbation* $Y_{(i)} \to Y_{(i)} + \delta Y_{(i)}(x)$ *introduces a modulus into the 4D theory.*

Figure 1.2: *The positions of D3-branes in the compact dimensions are moduli of the system* [manifold visualization © User: Jbourjai / Wikimedia Commons / CC-BY-SA-3.0].

ansätze for the 10D metric and the 5-form flux. We require that the ansatz for the background metric preserves 4D Poincaré variance, as in (1.7). The first-order contributions to the metric due to the modulus $\delta Y(x)$, are expressed, in part, by including a metric compensator in the off-diagonal part of the 10D metric [20]. Also, the Minkowski space metric is further rescaled by the Weyl factor, $\Omega(x)$, as we discussed in § 1.4, to maintain a time-independent, 4D Newton's constant. It is understood that both $\Omega(x)$ and the warp factor contain background *and* first-order contributions. The metric is then

$$ds_{10}^2 = e^{2\Omega} e^{2A} \hat{\eta}_{\mu\nu} dx^\mu dx^\nu + 2e^{2\Omega} e^{2A} \hat{\partial}_\mu B_m(x, y) dx^\mu dy^m + e^{-2A} \tilde{g}_{mn} dy^m dy^n \,, \tag{1.23}$$

where $A = A(x, y)$, $\Omega = \Omega(x)$, and $B_1(x, y)$ is the compensator field. Choosing the compensator to be in the off-diagonal part of the metric preserves the Ricci-flatness of the internal metric, $\tilde{g}_{mn}(y)$. Of course, we are free to choose a gauge where this term appears in the (m, n) component of the metric, however we find in practice that the off-diagonal gauge is preferable computationally.

The 4-form potential, C_4, will also require additional first-order terms that are related to the metric compensator. However, it is easier in practice to begin with an ansatz for $\tilde{F}_5$ (which must satisfy the self-duality condition). Due to this self-duality condition, half of the degrees of freedom in the flux are redundant at the level of the equations of motion. Therefore, we are free to choose whichever half of these degrees are most convenient.

We have seen that G_3 acts to stabilize the complex structure moduli; furthermore, the volume modulus, universal axion, and D3-branes are decoupled from the moduli that are fixed by G_3. Indeed, a non-zero 3-form would only serve to complicate the ansatz for the D3-brane motion. Thus, we take G_3 to be zero in this work, knowing that the complex structure moduli *could* be fixed by its presence[5].

This thesis is organized as follows: in Chapter 2, an introduction to D3-branes as electromagnetic sources is given, and the procedure for solving the type IIB SUGRA system is outlined. This includes establishing the Einstein equations and the equations of motion for the D3-brane position and its flux, $\tilde{F}_5$. Next, Chapter 3 introduces the particular ansätze used for the metric and the flux. The Einstein equations are then established for this choice of g_{MN} and $\tilde{F}_5$, and the constraint equations that descend from them are identified and solved. Then, the equations of motion for the D3-brane position and the flux are solved with the given ansätze. These equations hint that there are local terms due to gauge transformations that must be separated from $\tilde{F}_5$ and, therefore, the potential. To do so, we establish a new ansatz for the flux that is equivalent to the previous one when evaluated on-shell. This new ansatz allows the local terms to be separated out and the global degrees of freedom to be written in a closed form. Using the non-trivial Bianchi identity, local contributions to the flux are identified, up to second order. Finally, we establish that these local contributions will not enter the equations of motion at first-order and so can be dropped.

In Chapter 4, the results of the Einstein equations and equations of motion are gathered and used to construct a second-order quadratic action for the D3-brane position modulus. This action is then dimensionally-reduced to a 4D effective action for the scalar field descending from the D3-brane position. Chapter 5 reviews similar calculations of the effective action for the volume modulus and scalar axion. Then, the scalar axion is reformulated as a 2-form (for reasons that will become clear

[5] The response of G_3 to variations of the other Kähler metric moduli has been considered by the author and may be included in a later work.

later), and the effective theory for this modulus is established.

Finally, Chapter 6 establishes the quadratic action in the presence of all three moduli. Using ansätze motivated by the D3-brane flux, we are able to combine the degrees of freedom for each modulus. We then calculate the quadratic action for all the moduli. Using an ansatz for the Kähler potential, $\mathcal{K}$, that includes all moduli, we determine the effective action for the total moduli space and compare it to the result of dimensionally-reducing the quadratic action. The self-duality of $\tilde{F}_5$ means that the degrees of freedom in the flux can be chosen to be in any set of complementary components. Therefore, in Chapter 7 the opposite set of components are chosen to contain the relevant degrees of freedom. The quadratic action for this choice of components is computed and compared to the previous results.

As a final note, there is a wide variety of notation used throughout this work. Since we are most often working with objects that exist in either the extended, 4-dimensional Minkowski space or in compact, 6-dimensional Calabi-Yau space, we adopt a notation to differentiate between the two regions. The coordinates assigned to the flat, 4D space are x^μ and any mathematical objects, operators, *etc.* are denoted by hats: $\hat{\partial}$ or $\hat{\eta}$. The coordinates in the compact, 6D space are y^m and objects or operators there are denoted by tildes: $\tilde{\nabla}$ or $\tilde{g}$. When this notation is used, no warp factors are present in the object's definition. Conversely, when describing the entire 10-dimensional space, we use capital lettered coordinates X^M and any 10D object or operator will be devoid of accent. These objects will have warping implied in their definitions. In the event that a mixture of internal/external objects are used, the dimensionality of each object may be indicated by their subscript, *e.g.* the 10D exterior derivative is denoted d_{10}. When considering the embedding of the D3-branes (see Appendix A.2 for notes regarding embedding coordinates and parallel propagators on compact spaces) in the compact space, we adopt a slashed notation. For example, the first-order displacement of the D3-branes can be written in covariant form in the compact space via the use of a parallel propagator: $\tilde{g}_{mn}\Lambda^n{}_{\not{p}}\delta Y^{\not{p}}$. Whenever possible, background information for topics is provided in the Appendices, with references to this material throughout.

2 Solving Type IIB SUGRA With D3-Branes

2.1 D3-Branes as Electromagnetic Sources

As mentioned in Chapter 1, D3-branes are able to source a variety of gauge fields on their world-volumes. One such gauge field in type IIB strings is the massless Kalb-Ramond (KR) field, described by the antisymmetric field strength $B_{\mu\nu}$. In fact, the KR field has an associated conserved current on the brane world-volume whose density is tangent to the surface of the brane. Furthermore, the Kalb-Ramond field couples to the Maxwell gauge field, A_μ, that already lives on the brane. The overall field strength for the combined gauge fields is $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$, where $F_{\mu\nu}$ is the regular electromagnetic field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $\mathcal{F}$ will then follow a Maxwell-like equation, such as $d\mathcal{F}_2 = \star J_1$, where J_1 is an "electric" current density that is contributed to by Maxwell and KR charges.

However, the Kalb-Ramond field is not the only gauge field sourced by D3-branes. Other gauge fields of higher dimensions are also present, and their field strengths will also conform to equations similar to classical Maxwell equations. For example, the 5-form field strength $\tilde{F}_5$ obeys Maxwell-like condition that goes like $d_{10} \star_{10} \tilde{F}_5 = \star_{10} J_4^e$ for some 4-form "electric" current, J_4^e. Of course, since D3-branes are point sources in the compact dimensions, J_4^e will be similar to a Maxwell current due to a point charge.

A unique feature of type IIB SUGRA is the self-duality condition for the field strength $\tilde{F}_5$. This condition, written as $\star_{10}\tilde{F}_5 = \tilde{F}_5$, must be imposed by hand at the level of the equations of motion. What this tells us is that half of the degrees of freedom in $\tilde{F}_5$ are redundant; we conventionally double the contribution of whichever half of the degrees we choose to be the "actual" degrees of freedom and neglect the other terms. In terms of the gauge groups sourced by the D3-branes, the self-duality condition is equivalent to doubling either the electric or magnetic portion of the 4-form current and setting the other portion to zero.

In choosing where the relevant degrees of freedom lie in our ansatz for the field strength, we will in fact be choosing how to characterize the D3-brane. This will, in turn, effect equations such as the Bianchi identity for $\tilde{F}_5$ and may pose or resolve certain issues. However, it is important to remember that – in principle, at least – either choice is valid.

Choosing to take the D3-brane to be an electric source sets the relevant degrees of freedom to the (4,0), (3,1), and (2,2) legs of C_4[1]. Conversely, a magnetic description for the D3-branes entails setting the degrees of freedom to lie in the (0,4), (1,3), and (2,2) legs of C_4. In this chapter, we will outline the calculations required to build a second-order action for either characterization of the D3-brane charge.

2.2 Einstein Equations

Let us outline the process of solving the type IIB SUGRA system where the time-dependent positions of the D3-branes in the extra dimensions is the modulus being considered. Since the D3-branes are semi-classical, we can use the regular Einstein equations to describe the geometry of the full, ten-dimensional space. Letting $A \in [0, 9]$, the 10D Ricci tensor is given in terms of the Christoffel symbols

$$R_{MN} = \partial_A \Gamma^A_{MN} - \partial_N \Gamma^A_{AM} + \Gamma^A_{AB}\Gamma^B_{NM} - \Gamma^A_{NB}\Gamma^B_{AM}\,, \tag{2.1}$$

where each 10D Γ is given by $\Gamma^A_{BC} = 1/2\, g^{AD}(\partial_B g_{DC} + \partial_C g_{DB} - \partial_D g_{BC})$. The Ricci scalar is built from the inverse metric and (2.1): $\mathcal{R} = g^{MN} R_{MN}$. Both the Ricci tensor and scalar are used to define the Einstein tensor

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} \mathcal{R}\,. \tag{2.2}$$

[1] An overall factor of 1/2 is included in the (2,2) component, as this will also contribute to the description of the D3-brane as a magnetic source.

The energy-momentum tensor receives contributions from the 5-form flux and from the branes themselves. Just as in classical electromagnetism, the contribution from the flux , $\tilde{F}_5$, to T_{MN} is

$$T^5_{MN} = \frac{1}{4 \cdot 4!} \tilde{F}_{MPQRS} \tilde{F}_N{}^{PQRS} \,. \tag{2.3}$$

The additional contribution to T_{MN} is due to the fact that the D3-branes are a dynamical source of energy density. To determine this contribution we vary the action given in (1.5) with respect to the metric. A short calculation shows that the pullback due to the metric is

$$P(g)_{ab} = g_{MN} \partial_a X^M \partial_b X^N \,. \tag{2.4}$$

Then, noting that $\delta g_{MN}/\delta g^{PQ} = -g_{PM} g_{QN}$, the contribution from the D3-branes is

$$T^{D3}_{PQ} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{DBI}}{\delta g^{PQ}} = -\frac{T_3}{2} \int d^4\xi \sqrt{-\gamma}\, \gamma^{ab} g_{PM} g_{QN} \partial_a X^M \partial_b X^N \delta^{10}(x, X(\xi)) \,. \tag{2.5}$$

Above, and throughout our calculations, we denote the coordinates of the D3-brane world-volume by ξ, which feel the pullback γ_{ab}, which is equal to the induced metric on-shell.

The Einstein equations relate the effects of the curvature to the stress-energy from the relevant sources. Given the discussion above, we have that

$$E_{MN} = R_{MN} - \frac{1}{2} g_{MN} \mathcal{R} - (T^5_{MN} + T^{D3}_{MN}) \,. \tag{2.6}$$

Setting $E_{MN} = 0$ gives the equation for the metric of the system; however, we will use (2.6) along with an ansatz for the 10D metric to determine dynamic and constraint equations for the compensator field, $B_m(x, y)$, and the warp and Weyl factors, $A(x, y)$ and $\Omega(x)$.

2.3 Equations of Motion

The movement of the D3-branes will be affected not only by the geometry of the background, but also by the coupling of the charge to the field strength. As given by (1.5), the so-called "DBI-like" action is the geometric contribution from each brane and relies on the pullback given by (2.4),

$$S_{DBI} = -\frac{T_3}{2}\int d^{10}x\sqrt{-g}\int d^4\xi\sqrt{-\gamma}\left[\gamma^{ab}P(g)_{ab} - 2\right]\delta^{10}(x, X(\xi))\,.$$

The coupling of the charge of each D3-brane to the flux is given by the Wess-Zumino coupling, (1.6)

$$S_{WZ} = -\mu_3\int d^{10}x\sqrt{-g}\int_\xi P(C_4)\delta^{10}(x, X(\xi))\,,$$

and relies on a different pullback: the pullback due to the potential, $P(C_4)$. This pullback is a 4-form and has components

$$[P(C_4)]_{abcd} = C_{MNPQ}\partial_a X^A\partial_b X^B\partial_c X^C\partial_d X^D\,. \tag{2.7}$$

To find the equation of motion for the D3-brane position, we will vary these two actions with respect to the X^A coordinate. Since we know that the constant terms in S_{DBI} will vanish under variation, we can disregard that portion of the action in establishing $S_{D3} = S_{DBI} + S_{WZ}$. Furthermore, we will write the Wess-Zumino action in component notation by using

$$\int \beta_p \wedge \star\alpha_p = \frac{1}{p!}\int d^n x\sqrt{-g}\;\alpha_{\mu_1\ldots\mu_p}\beta^{\mu_1\ldots\mu_p}(-1)^{p(n-p)+1}\,, \tag{2.8}$$

for any n-dimensional p-forms α and β. Then,

$$\begin{aligned} S_{D3} = &-\frac{T_3}{2}\int d^{10}x\sqrt{-g}\int d^4\xi\sqrt{-\gamma}\;\gamma^{ab}P(g)_{ab}\,\delta^{10}(x, X(\xi)) \\ &+\int d^{10}x\sqrt{-g}\left(-\frac{\mu_3}{4!}\int d^4\xi\,\varepsilon^{abcd}C_{MNPQ}\partial_a X^M\partial_b X^N\partial_c X^P\partial_d X^Q\right)\delta^{10}(x, X(\xi))\,, \end{aligned} \tag{2.9}$$

where ε is the totally antisymmetric Levi-Civita tensor. After varying with respect to the 10D coordinates, we find that

$$\delta S_{D3} = \int d^{10}x\sqrt{-g}\int d^4\xi\,\delta X^M\left[T_3\partial_a(\sqrt{-\gamma}\,\gamma^{ab}g_{MN}\partial_b X^N)\right.$$
$$\left.+\frac{\mu_3}{3!}\partial_a(\varepsilon^{abcd}C_{MNPQ}\partial_b X^N\partial_c X^P\partial_d X^Q)\right]\delta^{10}(x,X(\xi))\,. \tag{2.10}$$

We choose to work in the static gauge, which aligns the world-volume coordinates with the external Minkowski space, so that the induced metric, γ_{ab}, equal to the 4D metric, $g_{\mu\nu}$. This gauge choice also collapses the spacetime portion of the 10D delta distribution $\delta^{10}(x,X(\xi))$,[2] leaving only the integral over the internal space intact.

After choosing the static gauge, we can define the equation of motion for the D3-brane, E_M, by

$$E_M = \int d^6y\sqrt{\tilde{g}}\left[T_3\hat{\partial}_\mu(\sqrt{-g_4}\,g^{\mu\nu}g_{MN}\hat{\partial}_\nu X^N)\right.$$
$$\left.+\frac{\mu_3}{3!}\hat{\partial}_a(\varepsilon^{abcd}C_{MNPQ}\hat{\partial}_b X^N\hat{\partial}_c X^P\hat{\partial}_d X^Q)\right]\tilde{\delta}^6(y,Y), \tag{2.11}$$

where the delta distribution is now a density, *i.e.* $\tilde{\delta}^6(y,Y) = \delta^6(y,Y)/\sqrt{\tilde{g}}$, which is non-zero only at the position of the D3-brane, $Y^{\not{b}}$. Furthermore, the warp factors that are in the definition of g_{mn}, the 6D metric, have been factored out so that $g_{mn} = e^{-2A}\tilde{g}_{mn}$. This way, the delta distribution density satisfies

$$\int d^6y\sqrt{\tilde{g}}\,f(y)\,\tilde{\delta}^6(y,Y) = f(Y)\,. \tag{2.12}$$

There is also the equation of motion for the flux to consider. In addition to the normal free-field action,

$$S_{\text{free}} = -\frac{1}{2\kappa^2}\int\frac{1}{4}\tilde{F}_5\wedge\star_{10}\tilde{F}_5\,, \tag{2.13}$$

the flux also has a coupling to the charged branes given by S_{WZ}. We can use (2.7) to express $P(C_4)$

[2] See Appendix A.2 for a more detailed discussion on this and related topics.

explicitly, and then use (2.8) to show that

$$S_{WZ} = \frac{T_3}{2}\int d^{10}x\sqrt{-g}\int C_4 \wedge \star_{10}\,\epsilon_\xi\,\delta^{10}(x, X(\xi))\,, \tag{2.14}$$

where $[\epsilon_\xi]_{MNPQ} \equiv 1/\sqrt{-\gamma}\,\varepsilon^{abcd}\partial_a X_M \partial_b X_N \partial_c X_P \partial_d X_Q$. A factor of 1/2 is introduced into S_{WZ} because the charge of the brane is split between an electric type and a magnetic type; therefore, the overall brane charge coupling is $\mu_3 = T_3/2$. We can vary both (2.13) and (2.14) with respect to the potential to find that

$$\delta S_{\text{free}} = \frac{1}{4\kappa^2}\int \delta C_4 \wedge d_{10} \star_{10} \tilde{F}_5\,, \tag{2.15}$$

$$\delta S_{WZ} = \frac{T_3}{2}\int d^{10}x\sqrt{-g}\int \delta C_4 \wedge \star_{10}\,\epsilon_\xi\,\delta^{10}(x, X(\xi))\,. \tag{2.16}$$

Finally, setting $\delta S_{\text{free}} + \delta S_{WZ} = 0$ for all δC_4, we can write the equation of motion for the flux,

$$E_6 = d_{10} \star_{10} \tilde{F}_5 + 2\kappa^2 T_3 \int d^4\xi\sqrt{-\gamma}\,\star_{10}\,\epsilon_\xi\,\delta^{10}(x, X(\xi))\,. \tag{2.17}$$

All the requisite equations for solving the 10D system have now been established, and hold for any particular choice of flux or metric. In the proceeding chapter, we will see that these equations give two categories of results: dynamic equations – which are only valid on-shell and will be used in Chapter 4 to construct a quadratic action – and constraint equations, which are always valid and will be used to relate the warp factor to the positions of the D3-branes.

3 Mobile D3-Branes: an Ansatz

3.1 Results of the Einstein Equations

Using the general method established in § 2.2, we can now take perturbations of the static D3-brane configuration. We must first determine the Einstein equations up to first order, taking the background warp factor to be given by the static solution Poisson equation (1.8). The full, 10D metric will be as in (1.23):

$$ds_{10}^2 = e^{2\Omega}e^{2A}\hat{\eta}_{\mu\nu}dx^\mu dx^\nu + 2e^{2\Omega}e^{2A}\hat{\partial}_\mu B_m(x,y)dx^\mu dy^m + e^{-2A}\tilde{g}_{mn}dy^m dy^n\,.$$

The computation of the Einstein Equations to first-order includes determining the Ricci tensor to the same order. Due to the presence of both background and first-order contributions, the calculation of R_{MN} is sufficiently detailed that it is included in Appendix A.1. The full expressions for the components of the Ricci tensor come from (2.1), and are

$$\begin{aligned}
R_{\mu\nu} &= \hat{\partial}_\mu\hat{\partial}_\nu(4A-2\Omega) - \hat{\eta}_{\mu\nu}\hat{\partial}^2(A+\Omega) + e^{4A}e^{2\Omega}\left(\tilde{\partial}_\ell A\hat{\partial}^2 B^\ell\hat{\eta}_{\mu\nu} - \widetilde{\nabla}^2 A\hat{\eta}_{\mu\nu} + \hat{\partial}_\mu\hat{\partial}_\nu\widetilde{\nabla}^\ell B_\ell\right)\,, && (3.1)\\
R_{\mu m} &= 2\hat{\partial}_\mu\tilde{\partial}_m A - 8\hat{\partial}_\mu A\tilde{\partial}_m A + e^{4A}e^{2\Omega}\left(\hat{\partial}_\mu\widetilde{\nabla}^\ell\widetilde{\nabla}_{[m}B_{\ell]} - \widetilde{\nabla}^2 A\hat{\partial}_\mu B_m + 4\widetilde{\nabla}^\ell A\hat{\partial}_\mu\tilde{\partial}_{[m}B_{\ell]}\right)\,, && (3.2)\\
R_{mn} &= \hat{\partial}^2\widetilde{\nabla}_{(m}B_{n)} + 4\widetilde{\nabla}_{(m}A\hat{\partial}^2 B_{n)} - \hat{\partial}^2 B^\ell\widetilde{\nabla}_\ell A\tilde{g}_{mn} + \widetilde{\nabla}^2 A\tilde{g}_{mn} + e^{-4A}e^{-2\Omega}\hat{\partial}^2 A\tilde{g}_{mn}\\
&\quad -8\tilde{\partial}_m A\tilde{\partial}_n A + \tilde{R}_{mn}\,. && (3.3)
\end{aligned}$$

Using these, we can calculate the Ricci curvature, $\mathcal{R}$, to first order

$$\mathcal{R} = 6e^{-2A}e^{-2\Omega}\hat{\partial}^2(A-\Omega) + 2e^{2A}\left(\hat{\partial}^2\widetilde{\nabla}^\ell B_\ell + \widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell + \widetilde{\nabla}^2 A - 4\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\right)\,, \qquad (3.4)$$

and the components of the Einstein tensor, as per (2.2),

$$
\begin{aligned}
G_{\mu\nu} &= (\hat{\partial}_\mu\hat{\partial}_\nu - \hat{\eta}_{\mu\nu}\hat{\partial}^2)(4A - 2\Omega + e^{4A}e^{2\Omega}\widetilde{\nabla}^\ell B_\ell) + 2e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}(2\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A - \widetilde{\nabla}^2 A)\,, &(3.5)\\
G_{\mu m} &= 2\hat{\partial}_\mu\tilde{\partial}_m A - 8\hat{\partial}_\mu A\tilde{\partial}_m A + e^{4A}e^{2\Omega}\hat{\partial}_\mu\Big(\widetilde{\nabla}^\ell\widetilde{\nabla}_{[m}B_{\ell]} + 4\widetilde{\nabla}^\ell A\tilde{\partial}_{[m}B_{\ell]}\\
&\quad +2B_m(2\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A - \widetilde{\nabla}^2 A)\Big)\,, &(3.6)\\
G_{mn} &= \hat{\partial}^2\widetilde{\nabla}_{(m}B_{n)} + 4\widetilde{\nabla}_{(m}A\hat{\partial}^2 B_{n)} - 8\tilde{\partial}_m A\tilde{\partial}_n A + \tilde{g}_{mn}(4\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A - 2\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell - \hat{\partial}^2\widetilde{\nabla}^\ell B_\ell)\\
&\quad +e^{-4A}e^{-2\Omega}\tilde{g}_{mn}\hat{\partial}^2(3\Omega - 2A)\,. &(3.7)
\end{aligned}
$$

Having determined the Einstein tensor, we must now consider the contributions to the energy-momentum tensor from both the flux and the mobile D3-brane. To do so, we introduce the ansatz for the field strength; we propose[1]

$$
\begin{aligned}
\tilde{F}_5 &= e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{-4A} - e^{-4A}e^{2\Omega}\tilde{\star}(\hat{d}B_1\wedge\tilde{d}e^{4A})\\
&\quad +e^{4\Omega}d_{10}(e^{4A}\hat{\star}\hat{d}B_1) + e^{4\Omega}\star_{10} d_{10}(e^{4A}\hat{\star}\hat{d}B_1)\,. \qquad (3.8)
\end{aligned}
$$

Recall that a valid ansatz for $\tilde{F}_5$ must obey the ten-dimensional self-duality condition, $\star_{10}\tilde{F}_5 = \tilde{F}_5$. Consider $\star_{10}$ acting on each term in $\tilde{F}_5$: the terms in the second line of (3.8) are trivially self-dual, so we need only consider those terms in the first line. Using the background part of the metric on the first term in (3.8) gives

$$
\star_{10}(e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}e^{4A}) = \tilde{\star}\tilde{d}e^{-4A}\,, \qquad (3.9)
$$

and using the off-diagonal part gives

$$
\star_{10}(e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}e^{4A}) = -e^{2\Omega}e^{-4A}\tilde{\star}(\hat{d}B_1\wedge\tilde{d}e^{4A})\,. \qquad (3.10)
$$

[1] This ansatz was established previous to this author's involvement in a work currently in progress with A. R. Frey and B. Underwood.

Similarly, the dual of the second term is

$$\star_{10}(\tilde{\star}\tilde{d}e^{-4A}) = e^{4\Omega}\hat{\epsilon} \wedge \tilde{d}e^{4A} + e^{4\Omega}\hat{\star}(\hat{d}B_1 \wedge \tilde{d}e^{4A})\,. \tag{3.11}$$

Lastly,

$$\star_{10}(-e^{-4A}e^{2\Omega}\tilde{\star}(\hat{d}B_1 \wedge \tilde{d}e^{4A})) = -e^{4\Omega}\hat{\star}(\hat{d}B_1 \wedge \tilde{d}e^{4A})\,, \tag{3.12}$$

exactly cancelling the second term in (3.11). Thus, (3.8) is a valid ansatz for the field strength.

Using (2.3), the background contributions to T^5_{MN} that come from the ansatz for $\tilde{F}_5$ are those that arise from the $e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}e^{4A}$ and $\tilde{\star}\tilde{d}e^{-4A}$ terms in (3.8), while the first-order contributions coming from the remaining parts. Taking only the background pieces, we quickly find that the background components of T^5 are

$$T^5_{\mu\nu} = -4e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\,, \tag{3.13}$$

$$T^5_{\mu m} = 0\,, \tag{3.14}$$

$$T^5_{mn} = 4\tilde{g}_{mn}\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A - 8\tilde{\partial}_m A\tilde{\partial}_n A\,. \tag{3.15}$$

The first-order contributions to T^5_{MN} come from expanding (2.3) to first order:

$$\delta T^5_{MN} = \frac{1}{96}(\tilde{F}_{MPQRS}\delta\tilde{F}_N{}^{PQRS} + \delta\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS} + 4\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQR}{}_T\delta g^{ST})\,. \tag{3.16}$$

Contributions to $\delta T^5_{\mu\nu}$ come from both the $\tilde{F}_{\mu PQRS}\delta\tilde{F}_\nu{}^{PQRS}$ and $\delta\tilde{F}_{\mu PQRS}\tilde{F}_\nu{}^{PQRS}$ terms and result in

$$\delta T^5_{\mu\nu} = 2e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell\,. \tag{3.17}$$

The off-diagonal contribution comes from all three terms in (3.16). To wit,

$$\tilde{F}_{\mu PQRS}\delta\tilde{F}_n{}^{PQRS} = -8(4!)e^{2\Omega}e^{4A}\widetilde{\nabla}^\ell A(4\hat{\partial}_\mu B_{[n}\widetilde{\nabla}_{\ell]}A - \hat{\partial}_\mu\widetilde{\nabla}_{[n}B_{\ell]})\,, \tag{3.18}$$

$$\delta\tilde{F}_{\mu PQRS}\tilde{F}_n{}^{PQRS} = 4(4!)e^{2\Omega}e^{4A}(\hat{\partial}_\mu\widetilde{\nabla}_n B_\ell\widetilde{\nabla}^\ell A - \hat{\partial}_\mu\widetilde{\nabla}_\ell B_n\widetilde{\nabla}^\ell A)\,, \tag{3.19}$$

$$\text{and}\quad \tilde{F}_{\mu PQRS}\tilde{F}_n{}^{PQR}{}_T\delta g^{ST} = -4(4!)e^{2\Omega}e^{4A}\,\widetilde{\nabla}_n A\,\hat{\partial}_\mu B_\ell\,\widetilde{\nabla}^\ell A\,, \tag{3.20}$$

such that the total $\delta T^5_{\mu m}$ is

$$\delta T^5_{\mu m} = 4e^{2\Omega}e^{4A}(\widetilde{\nabla}^\ell A\hat{\partial}_\mu\widetilde{\nabla}_{[m}B_{\ell]} - \widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\hat{\partial}_\mu B_m)\,. \tag{3.21}$$

Lastly, δT^5_{mn} contains two contributions each from both $\tilde{F}_{mPQRS}\delta\tilde{F}_n{}^{PQRS}$ and $\delta\tilde{F}_{mPQRS}\tilde{F}_n{}^{PQRS}$ terms. These are

$$\tilde{F}_{mPQRS}\delta\tilde{F}_n{}^{PQRS} = 4(4!)\big(\widetilde{\nabla}_m A\,\hat{\partial}^2 B_n + \widetilde{\nabla}_n A\,\hat{\partial}^2 B_m - \tilde{g}_{mn}\widetilde{\nabla}^\ell A\,\hat{\partial}^2 B_\ell\big)\,, \tag{3.22}$$

$$\text{and}\quad \delta\tilde{F}_{mPQRS}\tilde{F}_n{}^{PQRS} = 4(4!)\big(\hat{\partial}^2 B_m\widetilde{\nabla}_n A + \hat{\partial}^2 B_n\widetilde{\nabla}_m A - \tilde{g}_{mn}\widetilde{\nabla}^\ell A\,\hat{\partial}^2 B_\ell\big)\,. \tag{3.23}$$

Summing the contributions gives

$$\delta T^5_{mn} = 4\widetilde{\nabla}_{(m}A\hat{\partial}^2 B_{n)} - 2\tilde{g}_{mn}\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell\,. \tag{3.24}$$

Finally, the total energy-momentum tensor for the 5-form $\tilde{F}_5$ is the sum of the background and first-order pieces and has components

$$T^5_{\mu\nu} = 2e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}(\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell - 2\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A)\,, \tag{3.25}$$

$$T^5_{\mu m} = 4e^{2\Omega}e^{4A}(\widetilde{\nabla}^\ell A\hat{\partial}_\mu\widetilde{\nabla}_{[m}B_{\ell]} - \widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\hat{\partial}_\mu B_m)\,, \tag{3.26}$$

$$T^5_{mn} = 4\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\tilde{g}_{mn} - 8\tilde{\partial}_m A\tilde{\partial}_n A + 4\widetilde{\nabla}_{(m}A\hat{\partial}^2 B_{n)} - 2\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell\tilde{g}_{mn}\,. \tag{3.27}$$

As we saw previously, the charged D3-branes contribute to the stress-energy tensor via (2.5), which is

$$T^{D3}_{PQ} = -\frac{T_3}{2} \int d^4\xi \sqrt{-\gamma}\, \gamma^{ab} g_{PM} g_{QN} \partial_a X^M \partial_b X^N \delta^{10}(x, X(\xi)) \,.$$

Since we are working in the static gauge, γ^{ab} is the 4D metric, $g^{\mu\nu}$. This means that the relevant components of T^{D3}_{MN} are

$$\begin{aligned} T^{D3}_{\mu\nu} &= -T_3 e^{8A} e^{2\Omega} \hat{\eta}_{\mu\nu} \tilde{\delta}^6(y, Y) \,, && (3.28) \\ T^{D3}_{\mu m} &= -T_3 \left(e^{8A} e^{2\Omega} \hat{\partial}_\mu B_m + e^{4A} \tilde{g}_{mn} \Lambda^n_{\not{\beta}} \hat{\partial}_\mu \delta Y^{\not{\beta}} \right) \tilde{\delta}^6(y, Y) \,, && (3.29) \\ T^{D3}_{mn} &= 0 \,. && (3.30) \end{aligned}$$

N.B. The first-order part of the brane position is written out in terms of a parallel propagator, $\Lambda^n_{\not{\beta}}$, which takes vectors from the static brane to the mobile brane, located at the position $Y^{\not{\beta}}$ on the surface of the manifold. Thus, the first-order part of the 10D vector X^M is *technically* a sum over the embedding coordinates: $\Lambda^m_{\not{\beta}} \delta Y^{\not{\beta}}(x)$. However, we often suppress the summed indices in scenarios where they are not pertinent to the calculation at hand, and focus primarily on the spacetime dependence of the perturbation.

Following the form of (2.6), the contributions to the stress-energy tensor are summed and then subtracted from the Einstein tensor to give the Einstein equations. For the given ansatz, we find that the Einstein equations have components

$$\begin{aligned} E_{\mu\nu} = G_{\mu\nu} - T_{\mu\nu} &= (\hat{\partial}_\mu \hat{\partial}_\nu - \hat{\eta}_{\mu\nu} \hat{\partial}^2)(4A - 2\Omega + e^{2\Omega} e^{4A} \widetilde{\nabla}^\ell B_\ell) \\ &\quad + 2 e^{4A} e^{2\Omega} \hat{\eta}_{\mu\nu} (4 \widetilde{\nabla}^\ell A \widetilde{\nabla}_\ell A - \widetilde{\nabla}^2 A - \widetilde{\nabla}^\ell A \hat{\partial}^2 B_\ell) \\ &\quad + T_3 \kappa^2 e^{8A} e^{2\Omega} \hat{\eta}_{\mu\nu} \tilde{\delta}^6(y, Y) \,, && (3.31) \\ E_{\mu m} = G_{\mu m} - T_{\mu m} &= 2 \hat{\partial}_\mu \tilde{\partial}_m A - 8 \hat{\partial}_\mu A \tilde{\partial}_m A \\ &\quad + e^{4A} e^{2\Omega} \left(\hat{\partial}_\mu \widetilde{\nabla}^\ell \widetilde{\nabla}_{[m} B_{\ell]} + 2 \hat{\partial}_\mu B_m (4 \widetilde{\nabla}^\ell A \widetilde{\nabla}_\ell A - \widetilde{\nabla}^2 A) \right) \\ &\quad + T_3 \kappa^2 (e^{8A} e^{2\Omega} \hat{\partial}_\mu B_m + e^{4A} \tilde{g}_{mn} \Lambda^n_{\not{\beta}} \hat{\partial}_\mu \delta Y^{\not{\beta}}) \tilde{\delta}^6(y, Y) \,, && (3.32) \\ E_{mn} = G_{mn} - T_{mn} &= \hat{\partial}^2 (\widetilde{\nabla}_{(m} B_{n)} - \tilde{g}_{mn} \widetilde{\nabla}^\ell B_\ell) + e^{-4A} e^{-2\Omega} \tilde{g}_{mn} \hat{\partial}^2 (3\Omega - 2A) \,. && (3.33) \end{aligned}$$

Setting $E_{MN} = 0$, we can see that the terms present in (3.31) - (3.33) fall into two main categories: those terms that contain second-order spacetime derivatives (dynamic terms), and those that contain up to first-order spacetime derivatives (constraint terms). The constraint terms must always be set to zero, while the dynamic terms are zero when the equations of motion are satisfied, but are non-zero when working off-shell. We will use these dynamic terms in Chapter 4 to help construct the dimensionally-reduced effective theory.

The constraint equations come from $E_{\mu\nu}$ and $E_{\mu m}$ components, since E_{mn} is totally dynamic. From $E_{\mu\nu}$ we see that

$$4A - 2\Omega + e^{4A} e^{2\Omega} \widetilde{\nabla}^{\ell} B_{\ell} = 0\,. \tag{3.34}$$

N.B. Since B_1 is first-order, the above equation is actually a first-order constraint. Thus, we must take

$$A \to \delta A \quad \text{and} \quad \Omega \to \delta\Omega\,. \tag{3.35}$$

The result of (3.34) is that the first line of (3.31) disappears. The other terms in $E_{\mu\nu}$ can be rewritten using the help of the background solution for the warp factor, which, for stationary D3-branes, is

$$\widetilde{\nabla}^2 e^{-4A} = -2T_3\kappa^2 \tilde{\delta}^6(y, Y)\,. \tag{3.36}$$

Using the Poisson equation for the warp factor, (3.31) becomes

$$E_{\mu\nu} = \frac{1}{2} e^{8A} e^{2\Omega} \hat{\eta}_{\mu\nu}(-2T_3\kappa^2\tilde{\delta}^6(y,Y)) + T_3\kappa^2 e^{8A} e^{2\Omega} \hat{\eta}_{\mu\nu} \tilde{\delta}^6(y,Y) - 2e^{4A} e^{2\Omega} \hat{\eta}_{\mu\nu} \widetilde{\nabla}^{\ell} A \hat{\partial}^2 B_{\ell}\,. \tag{3.37}$$

Thus, the dynamic term in $E_{\mu\nu}$ is simply $-2e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}\widetilde{\nabla}^{\ell} A \hat{\partial}^2 B_{\ell}$. We can also use (3.36) to simplify $E_{\mu m}$. Doing so results in zero dynamic contribution but, rather, the constraint equation

$$2\hat{\partial}_{\mu}\tilde{\partial}_m A - 8\hat{\partial}_{\mu} A \tilde{\partial}_m A + e^{4A} e^{2\Omega} \hat{\partial}_{\mu} \widetilde{\nabla}^{\ell} \widetilde{\nabla}_{[m} B_{\ell]} + e^{4A} T_3 \kappa^2 \tilde{g}_{mn} \Lambda^{n}_{\not{b}} \hat{\partial}_{\mu} \delta Y^{\not{b}} \tilde{\delta}^6(y,Y) = 0\,. \tag{3.38}$$

From inspection, we can see that (3.33) is entirely dynamic.

3.2 Solving the Constraint Equations

By solving the constraint equations, we are able to relate the warping of the internal manifold to the presence of the D3-branes and their flux. The first-order constraint (3.38) from the off-diagonal components of the Einstein equations can be written as

$$-\frac{1}{2}\hat{\partial}_\mu \widetilde{\nabla}_m(\delta e^{-4A}) + \frac{1}{2}e^{2\Omega}\hat{\partial}_\mu \widetilde{\nabla}^\ell(\tilde{d}B_1)_{m\ell} + \kappa^2 T_3 \tilde{g}_{mn}\Lambda^n_{\not{p}}\hat{\partial}_\mu \delta Y^{\not{p}}\tilde{\delta}^6(y,Y) = 0\,. \tag{3.39}$$

By noting that $\tilde{\star}\tilde{d}\tilde{\star}\alpha_p = (-1)^{p(d-p)}\widetilde{\nabla}^m\alpha_{mn_2\dots n_p}$ for some p-form α_p, we can write (3.39) as

$$\tilde{d}(\delta e^{-4A}) + e^{2\Omega}\tilde{\star}\tilde{d}\tilde{\star}\tilde{d}B_1 - 2\kappa^2 T_3\,\delta Y_1\tilde{\delta}^6(y,Y) = 0\,, \tag{3.40}$$

with $\delta Y_1 = \tilde{g}_{mn}\Lambda^n_{\not{p}}\delta Y^{\not{p}}$. Taking $\tilde{\star}\tilde{d}\tilde{\star}$ of both sides causes the term proportional to B_1 to vanish and leaves

$$\tilde{\star}\tilde{d}\tilde{\star}\tilde{d}(\delta e^{-4A}) = 2\kappa^2 T_3\,\tilde{\star}\tilde{d}\tilde{\star}(\delta Y_1\tilde{\delta}^6(y,Y))\,. \tag{3.41}$$

Converting back to component notation, this is

$$\widetilde{\nabla}^2(\delta e^{-4A}) = 2\kappa^2 T_3\tilde{g}_{mn}\delta Y^{\not{p}}\widetilde{\nabla}^m(\Lambda^n_{\not{p}}\tilde{\delta}^6(y,Y))\,. \tag{3.42}$$

Since the first-order D3-brane position is only a function of spacetime, it is covariantly constant and has moved outside of the brackets. What remains inside is a particular combination of the parallel propagator and the Dirac distribution. In (A.2.2), we see that this is related to the partial derivative of $\tilde{\delta}^6(y,Y)$ with respected to the slashed coordinates. This means that

$$\widetilde{\nabla}^2(\delta e^{-4A}) = -2\kappa^2 T_3\delta Y^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y)\,, \tag{3.43}$$

which is precisely the first-order variation of the Poisson equation that determines the background value of the warp factor if $\delta = \delta_Y$, *i.e.* the variation is with respect to the embedding coordinates $Y^{\not{b}}$. This consistency means that (3.39) is, in principal, solvable.

There are two important constraints to solve: the constraint for the first-order variation of the warp factor due to the D3-branes, and the constraint from the (μ, ν) component of Einstein equations, (3.34). To solve these constraints, we adopt the following proposed solution the warp factor based on (3.43)

$$e^{-4A(x,y)} = e^{-4A_0} - 2\kappa^2 T_3 G(y, Y)\,, \tag{3.44}$$

where $G(y, Y)$ is a bi-scalar Green's function. We must also consider the first-order part of the Weyl factor. By examining its definition in terms of the warp factor, varying with respect to the D3-brane coordinates gives

$$\delta_Y e^{-2\Omega} = -2\delta_Y \Omega e^{-2\Omega_0} = \frac{1}{\tilde{V}} \int d^6y \sqrt{\tilde{g}}\, \delta_Y e^{-4A}\,. \tag{3.45}$$

Recall that the background value for the Weyl factor, $e^{-2\Omega_0}$, is a constant. Then, using (3.44), we find that

$$\delta_Y \Omega \propto \int d^6y \sqrt{\tilde{g}}\, \delta_Y G(y, Y) = \int d^6y \sqrt{\tilde{g}}\, \frac{\delta G(y, Y)}{\delta Y^{\not{b}}} \delta Y^{\not{b}} = \int d^6y \sqrt{\tilde{g}}\, \partial_{\not{b}} G(y, Y) \delta Y^{\not{b}}\,. \tag{3.46}$$

An important relationship between a bi-scalar Green's function and a bi-tensor Green's function is given by (A.2.15); namely, that the two are related via

$$\widetilde{\nabla}^{\ell} G_{\not{b}\ell}(y, Y) = -\partial_{\not{b}} G(y, Y)\,. \tag{3.47}$$

Using this, we can relate the partial derivative with respect to the embedding coordinates to the total derivative over the compact space. Since $\widetilde{\nabla}^m \delta Y^{\not{b}} = 0$, the integral over the total derivative vanishes. Thus, $\delta_Y \Omega = 0$. In Appendix A.6, we motivate this assertion without requiring an explicit solution for the warp factor, but by making certain simplifying assumptions about the compact space.

We can now we turn our attention back to the constraint given by (3.34), with $\delta_Y \Omega = 0$

$$4\delta_Y A + e^{4A} e^{2\Omega} \widetilde{\nabla}^\ell B_\ell = 0 \quad \Rightarrow \quad \delta_Y e^{-4A} = e^{2\Omega} \widetilde{\nabla}^\ell B_\ell \,. \tag{3.48}$$

This allows us to use (3.39) to determine an equation for B_1,

$$\widetilde{\nabla}_m (\delta_Y e^{-4A}) - e^{2\Omega} \widetilde{\nabla}^\ell (\tilde{d} B_1)_{m\ell} - 2\kappa^2 T_3 \tilde{g}_{mn} \Lambda^n_{\not{p}} \delta Y^{\not{p}} \tilde{\delta}^6(y, Y) = 0 \,. \tag{3.49}$$

Then, using the relation for $\delta_Y e^{-4A}$ above, we find that

$$\widetilde{\nabla}_m (e^{2\Omega} \widetilde{\nabla}^\ell B_\ell) - e^{2\Omega} \widetilde{\nabla}^\ell \left(\widetilde{\nabla}_m B_\ell - \widetilde{\nabla}_\ell B_m \right) = 2\kappa^2 T_3 \tilde{g}_{mn} \Lambda^n_{\not{p}} \delta Y^{\not{p}} \tilde{\delta}^6(y, Y) \,. \tag{3.50}$$

We can collect the first two terms and note that these are simply $\tilde{g}^{\ell n} \tilde{R}_{nm} B_\ell$, which vanishes on the Ricci-flat Calabi-Yau. What we are left with is a Poisson equation for B_1:

$$\widetilde{\nabla}^2 B_m = 2\kappa^2 T_3 e^{-2\Omega} \tilde{g}_{mn} \Lambda^n_{\not{p}} \delta Y^{\not{p}} \tilde{\delta}^6(y, Y) \,. \tag{3.51}$$

Therefore, B_1 is given by a bi-tensor Green's function,

$$B_m = 2\kappa^2 T_3 e^{-2\Omega} \delta Y^{\not{p}} G_{\not{p} m}(y, Y) \,, \tag{3.52}$$

that solves the equation

$$\widetilde{\nabla}^2 G_{\not{p} m} = \tilde{g}_{mn} \Lambda^n_{\not{p}} \tilde{\delta}^6(y, Y) \,. \tag{3.53}$$

Plugging this solution back into $\delta_Y e^{-4A} = e^{2\Omega} \widetilde{\nabla}^\ell B_\ell$, and using (A.2.15), we find that

$$\delta_Y e^{-4A} = -2\kappa^2 T_3 \delta Y^{\not{p}} \partial_{\not{p}} G(y, Y) \,, \tag{3.54}$$

which exactly matches the ansatz in (3.44) since the bi-scalar Green's function satisfies $\widetilde{\nabla}^2 G(y,Y) = \tilde{\delta}^6(y,Y)$.

3.3 Equations of Motion: Brane Position and Flux

Recall that, as discussed in § 2.1, we are able to chose to describe the D3-branes as either electric or magnetic sources, thereby setting the relevant degrees of freedom to either half of the components of C_4. Given the expression for $\tilde{F}_5$ in (3.8), we will choose the (4,0), (3,1), and (2,2) components of C_4 to correspond to the global degrees of freedom, as these can be written as $d_{10}C_4$ in closed form. With the degrees of freedom set, we can refer to § 2.3 to determine the equation of motion for the D3-brane position up to first order. Recall (2.11):

$$E_M = \int d^6y\sqrt{\tilde{g}}\left[T_3\hat{\partial}_\mu(\sqrt{-g_4}\, g^{\mu\nu}g_{MN}\hat{\partial}_\nu X^N) \right.$$
$$\left. + \frac{\mu_3}{3!}\hat{\partial}_a(\varepsilon^{abcd}C_{MNPQ}\hat{\partial}_b X^N\hat{\partial}_c X^P\hat{\partial}_d X^Q)\right]\tilde{\delta}^6(y,Y).$$

Let us first consider $M = \rho$ and examine the background contribution. Using the metric ansatz of (1.23) we have

$$E_\rho = \int d^6y\sqrt{\tilde{g}}\left[T_3\hat{\partial}_\mu(\sqrt{-g_4}\, g^{\mu\nu}g_{\rho N}\hat{\partial}_\nu X^N) \right.$$
$$\left. + \frac{\mu_3}{3!}\hat{\partial}_a(\varepsilon^{abcd}C_{\rho NPQ}\hat{\partial}_b X^N\hat{\partial}_c X^P\hat{\partial}_d X^Q)\right]\tilde{\delta}^6(y,Y)\,. \tag{3.55}$$

To all orders, only the (4,0) component of C_4 will contribute. Thus, by setting $E_\rho = 0$,

$$\int d^6y\sqrt{\tilde{g}}\left[T_3\hat{\partial}_\mu(e^{4A}e^{4\Omega}\delta^\mu_\rho) - \mu_3\hat{\partial}_\mu(e^{4A}e^{4\Omega}\delta^\mu_\rho)\right]\tilde{\delta}^6(y,Y) = 0\,. \tag{3.56}$$

Since the background solution represents static D3-branes, the left-hand side of the above equation must vanish. Indeed, this is a trivial constraint provided that $\mu_3 = T_3$, as was discussed in § 1.2 (recall that the charge coupling of the D-brane has previously been halved in the definition of the Wess-Zumino action).

Next, we consider $M = \not{b}$. In this case, we are looking for first-order contributions from the metric and the 4-form potential. The contribution from C_4 is via the (3,1) component, which is $e^{4A}e^{4\Omega}\hat{\star}\hat{d}B_1$. This will exactly cancel the contribution from the off-diagonal metric, leaving

$$E_{\not{b}} = T_3 \int d^6y \sqrt{\tilde{g}}\, e^{2\Omega} \tilde{g}_{mn} \hat{\partial}_\mu (\Lambda^m_{\not{b}} \Lambda^n_{\not{a}} \hat{\partial}^\mu Y^{\not{a}}(x)) \tilde{\delta}^6(y, Y)\,. \tag{3.57}$$

N.B. the presence of the delta distribution indicates that the equation of motion is to be evaluated at coincidence, *i.e.* when $Y^{\not{b}}(x) = y$. From Appendix A.2, we know that the parallel propagator evaluated at coincidence is $[\Lambda^m_{\not{b}}] = \delta^m_{\not{b}}$. Thus, terms such as $\hat{\partial}_\mu[\Lambda^m_{\not{b}}] = \hat{\partial}_\mu \delta^m_{\not{b}} = 0$. Therefore, the first-order equation of motion for the D3-brane position is

$$E_{\not{b}} = T_3 \int d^6y \sqrt{\tilde{g}}\, e^{2\Omega} \tilde{g}_{mn} \Lambda^m_{\not{b}} \Lambda^n_{\not{a}} \hat{\partial}^2 Y^{\not{a}} \tilde{\delta}^6(y, Y)\,. \tag{3.58}$$

We also wish to determine E_6 given the ansätze presented. Recall (2.17):

$$E_6 = d_{10} \star_{10} \tilde{F}_5 + 2\kappa^2 T_3 \int d^4\xi \sqrt{-\gamma}\, \star_{10} \epsilon_\xi\, \delta^{10}(x, X(\xi))\,.$$

The integral over the string coordinates, *i.e.* the source term, is evaluated using the definition of ϵ_ξ given in § 2.3, and becomes

$$2\kappa^2 T_3(-e^{-6A}\,\tilde{\epsilon} + e^{-6A}\hat{d}\tilde{\star}\delta Y_1)\tilde{\delta}^6(y, Y)\,, \tag{3.59}$$

where $\delta Y_1 \equiv \tilde{g}_{mn}\Lambda^n_{\not{b}} Y^{\not{b}}(x)$. This means that the equation of motion for the flux takes the form

$$d_{10} \star_{10} \tilde{F}_5 = S_6\,, \tag{3.60}$$

where S_6 is a six-dimensional point source.

By substituting (3.8) into the equation of motion for the flux, we can find a relation between the source term and the off-diagonal compensator, B_m. An important calculational tool at this level (*i.e.* the level of the equations of motion) is the 10D self-duality of the flux. This allows us to use $\star_{10}\tilde{F}_5 = \tilde{F}_5$ when determining (3.60). In calculating $d_{10}\tilde{F}_5$, we note that there is one term in particular requires extra attention: $e^{4\Omega}d_{10}\star_{10} d_{10}(e^{4A}\hat{\star}\hat{d}B_1)$. We shall calculate this term in steps; applying the innermost exterior derivative first results in

$$d_{10}(e^{4A}\hat{\star}\hat{d}B_1) = e^{4A}\hat{d}\hat{\star}\hat{d}B_1 + 4e^{4A}\tilde{d}A\,\hat{\star}\hat{d}B_1 + e^{4A}\tilde{d}\hat{\star}\hat{d}B_1\,. \tag{3.61}$$

The three terms that result from $\star_{10}$ acting on (3.61) are

$$e^{4A}\star_{10}(\hat{d}\hat{\star}\hat{d}B_1) = e^{-4A}e^{-4\Omega}\tilde{\star}\hat{\partial}^2 B_1\,, \tag{3.62}$$

$$4e^{4A}\star_{10}(\tilde{d}A\hat{\star}\hat{d}B_1) = 4e^{-2\Omega}\tilde{\star}(\hat{d}B_1\wedge\tilde{d}A)\,, \tag{3.63}$$

$$\text{and}\quad e^{4A}\star_{10}(\tilde{d}\hat{\star}\hat{d}B_1) = -e^{-2\Omega}\tilde{\star}(\tilde{d}\hat{d}B_1)\,. \tag{3.64}$$

Finally, the outermost d_{10} acting on (3.62)-(3.64) gives

$$d_{10}(e^{4A}\tilde{\star}\hat{d}B_1) = e^{-4A}d_{10}(\tilde{\star}\hat{\partial}^2 B_1) + 4e^{2\Omega}\tilde{d}\tilde{\star}(\hat{d}B_1\wedge\tilde{d}A) - e^{2\Omega}\tilde{d}(\tilde{\star}\tilde{d}\hat{d}B_1)\,. \tag{3.65}$$

Calculating the other terms in $d_{10}\tilde{F}_5$ is more straightforward, and we find that

$$d_{10}\tilde{F}_5 = d_{10}(\tilde{\star}\tilde{d}e^{-4A}) + d_{10}(e^{-4A}\tilde{\star}\hat{\partial}^2 B_1) - e^{2\Omega}\tilde{d}(\tilde{\star}\tilde{d}\hat{d}B_1)\,. \tag{3.66}$$

Thus, when $E_6 = 0$, we have

$$d_{10}\left(\tilde{\star}\tilde{d}e^{-4A} + e^{-4A}\tilde{\star}\hat{\partial}^2 B_1 - e^{2\Omega}\tilde{\star}\tilde{d}\hat{d}B_1\right) = 2\kappa^2 T_3(e^{-6A}\,\tilde{\epsilon} + e^{-6A}\tilde{\star}\hat{d}\delta Y_1)\tilde{\delta}^6(y,Y)\,. \tag{3.67}$$

We will return to this equation later to extract the dynamic contribution to the quadratic action from E_6. We can also see that the constraint equation, (3.39), is reproduced here by taking any terms with one or fewer spacetime derivatives.

3.4 Local vs. Global Degrees of Freedom

Of particular importance is the ability to separate the contributions to C_4 due to the D3-branes as point sources from those that are from global degrees of freedom. We wish to write $\tilde{F}_5$ as $d_{10}C_4' + S_5$, where the globally-defined degrees of freedom are contained in C_4'. This is because *only* the globally-defined degrees of freedom (those degrees able to be written entirely as $d_{10}C_4'$) contribute to the four-dimensional theory [18]. However, when considering the (4,0), (3,1), and (2,2) components of C_4 to be the relevant legs, the Bianchi identity is trivial; all components are globally defined.

As mentioned in § 2.1, type IIB SUGRA allows us to characterize the D3-branes as either magnetic *or* electric sources. Furthermore, both characterizations must produce the same effective theory. The effect of the Hodge star acting on any set of components is to map them to the complementary set, *e.g.* the (4,1) component of $\tilde{F}_5$ is mapped to the (0,5) component by $\star_{10}\tilde{F}_5$. At the end of the previous section, we saw that $d_{10} \star_{10} \tilde{F}_5 \neq 0$, which means that the *magnetic* characterization of the D3-branes produces a non-trivial Bianchi identity. From this, we know that – in the magnetic description – not all components are globally defined.

By drawing a comparison between the flux in the magnetic brane characterization and the field strength in classical electrodynamics, we know that the right-hand side of (3.67) must be the source term for the magnetic flux. This source term will necessarily depend explicitly on the brane position, Y. In order to write a potential that contains only globally-defined terms, we must separate out those terms in the flux that depend on Y from those that can be written as $d_{10}C_4'$.

The issue of the existence of local terms arises from the nature of the compact manifold: the manifold must be defined in coordinate patches, causing the potentials in the overlapping region to be related by a non-trivial gauge transformation. We can see this effect clearly in the case of a magnetic monopole in classical electrodynamics. For a magnetic monopole, Gauss' Law gives $\vec{\nabla} \cdot \vec{B} = 4\pi q_m \delta^3(\vec{x})$, where q_m is the "magnetic charge" of the monopole. Solving this, we find that

$$\vec{B} = \frac{q_m}{r^2}\hat{r}\,, \tag{3.68}$$

which has the vector potential

$$\vec{A} = \frac{q_m(1-\cos\theta)}{r\sin\theta}\hat{\phi}\,. \tag{3.69}$$

We can see immediately that $\vec{A}$ is singular when $\theta = \pi$. Thus, we define (3.69) in coordinate patches:

$$\begin{aligned} \vec{A}_N &= \frac{q_m(1-\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{for} \quad 0 \le \theta \le \pi/2 \\ \vec{A}_S &= -\frac{q_m(1+\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{for} \quad \pi/2 \le \theta \le \pi\,. \end{aligned} \tag{3.70}$$

This means that, in the region of overlap, the potentials differ by a gauge transformation, *i.e.*

$$\left[\vec{A}_N - \vec{A}_S\right|_{\theta=\pi/2} = \vec{\nabla}(2q_m\phi)\,, \tag{3.71}$$

that is directly related to the magnetic charge. Similarly, the SUGRA potential C_4 contains non-global contributions due to gauge transformations in the overlapping coordinate patches of the Calabi-Yau manifold.

To separate out the local contributions, it is advantageous to propose a new ansatz for the flux that maintains the relevant degrees of freedom in the (0,5), (1,4), and (2,3) components. Furthermore, since the the constraint equations of § 3.2 are true at all times, they must remain unchanged. But can a different ansatz correspond to the same on-shell solution? The answer to this is best illustrated by a simplified example. Consider a real scalar field φ subject to

$$S = -\frac{1}{2}\int d^4x\,\mathcal{L} \qquad \text{where} \quad \mathcal{L} = \partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2\,. \tag{3.72}$$

Under the field redefinition

$$\varphi \to \varphi + \lambda\varphi^2\,, \tag{3.73}$$

the $\varphi\varphi \to \varphi\varphi$ scattering amplitude remains zero [31]. Thus, the on-shell theory is unaffected by a field redefinition that is proportional to the equations of motion.

In a similar way, the flux $\tilde{F}_5$ is also defined only up to terms that disappear on-shell (and do not violate the self-duality condition). Recall that the dynamic term from (3.67) was proportional to $\tilde{\star}\hat{\partial}^2 B_1$; this means that any ansätze for $\tilde{F}_5$ that differ only by terms proportional to the dynamic equation of motion for B_1 will describe equivalent on-shell solutions. To wit, we may propose some $\tilde{F}_5^{\text{new}}$ such that

$$\tilde{F}_5^{\text{new}} = \tilde{F}_5 + e^{4A} e^{4\Omega} \hat{d}\hat{\star}\hat{d}B_1 + e^{-4A}\tilde{\star}\hat{d}\hat{\star}\hat{d}\tilde{\star}B_1 \,. \tag{3.74}$$

A short calculation shows that the last two terms above are 10D duals, and so $\tilde{F}_5^{\text{new}}$ reduces to $\tilde{F}_5$ on-shell. Explicitly, this new ansatz is

$$\tilde{F}_5^{\text{new}} = e^{4\Omega}\hat{\epsilon} \wedge \tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{4A} - e^{2\Omega}\hat{d}\tilde{\star}\tilde{d}B_1 + e^{4\Omega}\hat{\star}\hat{d}B_1 \wedge \tilde{d}e^{4A} - e^{4A}e^{4\Omega}\hat{\star}\hat{d}\tilde{d}B_1 \,. \tag{3.75}$$

In order to be a valid ansatz, (3.75) must also satisfy the 10D self-duality condition. To show this, we factor so that

$$\tilde{F}_5^{\text{new}} = e^{4\Omega}\hat{\epsilon} \wedge \tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{-4A} - e^{4\Omega}\hat{\star}\hat{d}\tilde{d}(e^{4A}B_1) - e^{2\Omega}\tilde{\star}\tilde{d}\hat{d}B_1 \,. \tag{3.76}$$

By comparing these terms with those present in (3.8) we can see that the ten-dimensional star produces the above terms, plus one positive copy and one negative copy of the term $e^{-4A}e^{2\Omega}\tilde{\star}(\hat{d}B_1 \wedge \tilde{d}e^{4A})$, which then cancels out. Thus, $\tilde{F}_5^{\text{new}}$ is 10D self-dual, as required.

The other requirement for this new ansatz is that it produce the same constraint equations as the previous one. Let us now prove that $\tilde{F}_5^{\text{new}}$ yields the same constraint equations as the previous ansatz. Once we have established that both ansätze are valid, we will then use (3.75) and separate out the non-global degrees of freedom.

3.5 Equivalence of Ansätze for $\tilde{F}_5$

In order to show that $\tilde{F}_5^{\rm new}$ produces the same constraint equations as those derived previously, we must repeat the procedure of § 3.1. The contributions to the energy-momentum tensor from (3.75) are

$$T^5_{\mu\nu} = -4e^{4A}e^{2\Omega}\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\,\hat{\eta}_{\mu\nu} \tag{3.77}$$
$$T^5_{\mu m} = -4e^{4A}e^{2\Omega}(\widetilde{\nabla}^\ell A\hat{\partial}_\mu\widetilde{\nabla}_{[m}B_{\ell]} - \widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\hat{\partial}_\mu B_m) \tag{3.78}$$
$$T^5_{mn} = 4\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A\tilde{g}_{mn} - 8\tilde{\partial}_m A\tilde{\partial}_n A\,. \tag{3.79}$$

These contributions are nearly identical to those from the previous 5-form, but terms with two spacetime derivatives acting on B_1 are now absent. Combining these results with the Einstein tensor derived previously, we find that the new components of the Einstein equations are

$$\begin{aligned} E_{\mu\nu} &= (\hat{\partial}_\mu\hat{\partial}_\nu - \hat{\eta}_{\mu\nu}\hat{\partial}^2)(4A - 2\Omega + e^{4A}e^{2\Omega}\widetilde{\nabla}^\ell B_\ell) \\ &\quad + e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}(8\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A - 2\widetilde{\nabla}^2 A - T_3\kappa^2 e^{4A}\tilde{\delta}^6(y,Y)) \end{aligned} \tag{3.80}$$
$$\begin{aligned} E_{\mu m} &= 2\hat{\partial}_\mu\tilde{\partial}_m A - 8\hat{\partial}_\mu A\tilde{\partial}_m A + e^{4A}e^{2\Omega}\Big(\hat{\partial}_\mu\widetilde{\nabla}^\ell\widetilde{\nabla}_{[m}B_{\ell]} + 2\hat{\partial}_\mu B_m(4\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell A - \widetilde{\nabla}^2 A)\Big) \\ &\quad - T_3\kappa^2 e^{4A}(e^{4A}e^{2\Omega}\hat{\partial}_\mu B_m + \hat{\partial}_\mu\delta Y^{\not{p}}\Lambda^n_{\not{p}}\tilde{g}_{mn})\tilde{\delta}^6(y,Y) \end{aligned} \tag{3.81}$$
$$\begin{aligned} E_{mn} &= \hat{\partial}^2(\widetilde{\nabla}_{(m}B_{n)} + 4\widetilde{\nabla}_{(m}A\,B_{n)} - 2\tilde{g}_{mn}\widetilde{\nabla}^\ell A\,B_\ell - \tilde{g}_{mn}\widetilde{\nabla}^\ell B_\ell) \\ &\quad + e^{-4A}e^{-2\Omega}\tilde{g}_{mn}\hat{\partial}^2(3\Omega - 2A)\,. \end{aligned} \tag{3.82}$$

From these, we are able to extract the constraint equations, which are identical to those given by the previous ansatz, *i.e.*

$$4A - 2\Omega + e^{4A}e^{2\Omega}\widetilde{\nabla}^\ell B_\ell = 0 \tag{3.83}$$
$$\text{and}\quad e^{2\Omega}\widetilde{\nabla}^\ell(\tilde{d}B_1)_{m\ell} - \widetilde{\nabla}_m(\delta e^{-4A}) = -2T_3\kappa^2\tilde{g}_{mn}\Lambda^n_{\not{p}}\delta Y^{\not{p}}\tilde{\delta}^6(y,Y)\,. \tag{3.84}$$

There is a difference, however, in the dynamic contributions from the new ansatz. In this case, only the (m,n) component of the Einstein equations will contribute to the second-order action; that is

$$\delta E_{MN} = \hat{\partial}^2(\widetilde{\nabla}_{(m}B_{n)} + 4\widetilde{\nabla}_{(m}A\,B_{n)} - 2\tilde{g}_{mn}\widetilde{\nabla}^{\ell}A\,B_{\ell} - \tilde{g}_{mn}\widetilde{\nabla}^{\ell}B_{\ell}) \\ +e^{-4A}e^{-2\Omega}\tilde{g}_{mn}\hat{\partial}^2(3\Omega - 2A)\,. \tag{3.85}$$

Finally, the dynamic contribution from the equation of motion for $\tilde{F}_5$ is now

$$\delta E_6 = -e^{4\Omega}\hat{d}\tilde{\star}\hat{d}\tilde{d}(e^{4A}B_1)\,. \tag{3.86}$$

To illuminate the local nature of some of the terms in our new ansatz, we use the solutions to the constraint equations found in § 3.2 to rewrite the (0,5) and (1,4) components – those that contain the relevant degrees of freedom – as

$$\tilde{F}_5^{\mathrm{new}} = \tilde{\star}\tilde{d}\delta e^{-4A} - e^{2\Omega}\tilde{\star}\tilde{d}\hat{d}B_1 + \ldots = -e^{2\Omega}\tilde{\star}\tilde{d}(\tilde{\star}\tilde{d}\tilde{\star}B_1) - e^{2\Omega}\tilde{\star}\tilde{d}\hat{d}B_1 + \ldots\,, \tag{3.87}$$

Where the dots represent the other components of the flux. Furthermore, recall that the solution for the compensator field, B_m, was given by

$$\widetilde{\nabla}^2 B_m = 2\kappa^2 T_3 e^{-2\Omega}\tilde{g}_{mn}\Lambda^n_{\not{p}}\delta Y^{\not{p}}\tilde{\delta}^6(y,Y) = e^{-2\Omega}\delta\tilde{Y}_m\,, \tag{3.88}$$

for an appropriately defined $\delta\tilde{Y}_m$. Next, we use the definition of the Hodge-de Rahm operator as outlined in Appendix A.3 to see that

$$\Delta B_1 = \tilde{\star}\tilde{d}\tilde{\star}\tilde{d}B_1 + \tilde{d}\tilde{\star}\tilde{d}\tilde{\star}B_1 = -\widetilde{\nabla}^2 B_1\,. \tag{3.89}$$

Substituting this into (3.87) gives

$$-e^{2\Omega}\tilde{\star}[-\widetilde{\nabla}^2 B_1 - \tilde{\star}\tilde{d}\tilde{\star}\tilde{d}B_1] - e^{2\Omega}\tilde{\star}\tilde{d}\hat{d}B_1\,. \tag{3.90}$$

Distributing the $\tilde{\star}$ into the brackets and permuting the $\hat{d}$ through the final term, we can write these degrees of freedom as

$$e^{2\Omega}\tilde{\star}\widetilde{\nabla}^2 B_1 - e^{2\Omega}(\hat{d}+\tilde{d})\tilde{\star}\tilde{d}B_1\,. \tag{3.91}$$

Thus, the relevant degrees of freedom in $\tilde{F}_5^{\text{new}}$ can be written as

$$\tilde{F}_5^{\text{new}} = d_{10}C_4' - S_5\,, \tag{3.92}$$

where the source term is such that $S_5 = \tilde{\star}\delta\tilde{Y}_1$ and the globally-defined degrees of freedom are given by

$$C_4' = -e^{2\Omega}\tilde{\star}\tilde{d}B_1\,. \tag{3.93}$$

3.6 Second-Order Local Terms in $\tilde{F}_5$

We are now able to see the local contributions to the field strength up to first order; however, our goal is to create a *second*-order action in ten dimensions. In order to properly count the degrees of freedom, we must find the local contributions to $\tilde{F}_5$ up to second order. To do so, we turn to the equation of motion for $\tilde{F}_5$ as derived in § 2.3.

We wish to consider all contributions to $\tilde{F}_5$ up to second order in $\delta Y(x)$, while also accounting for the expansion of the six-dimensional delta function near the static brane position. First, recall (2.17), which – after performing the integration over the world-volume – can be written on-shell as

$$d_{10}\tilde{F}_5 = -2\kappa^2 T_3 \star_{10} \hat{\varepsilon}^{\mu_1\ldots\mu_4}\hat{\partial}_{\mu_1}X^M\hat{\partial}_{\mu_2}X^N\hat{\partial}_{\mu_3}X^P\hat{\partial}_{\mu_4}X^Q\delta^6(y,Y)\,. \tag{3.94}$$

The contributions to $d_{10}\tilde{F}_5$ up to second order in the brane position are

$$d_{10}\tilde{F}_5 = 2\kappa^2 T_3\left(\tilde{\star} - \tilde{\star}\hat{d}Y_1 + \frac{1}{2}\tilde{\star}(\hat{d}Y_1\wedge\hat{d}Y_1)\right)\tilde{\delta}^6(y,Y)\,, \tag{3.95}$$

where $Y_m = \tilde{g}_{mn}\Lambda^n_{\not{b}}\delta Y^{\not{b}}$. However, we must also expand the delta function about the brane position, $Y^{\not{b}}$. If we allow the brane position to be $Y^{\not{b}}$, we can expand in the neighbourhood of this point in terms of Synge's world-function, σ (see Appendix A.2 for additional information regarding this). In the context of our previous calculations, we let the difference between the position of the mobile brane and the static solution be $\delta Y^{\not{b}}(x)$. When expanding in the neighbourhood of the mobile brane, the first-order brane position is $\hat{\partial}_\mu Y_m = \tilde{g}_{mn}\Lambda^n_{\underline{m}}\hat{\partial}_\mu\sigma^{\underline{m}}(x)$, where the first-order brane position is described in terms of the time-dependent, underlined coordinates. Using parallel propagators again, we can relate $\sigma^{\underline{m}}$ to $\sigma^{\not{b}}$ so that $\hat{\partial}_\mu Y_m = -\tilde{g}_{mn}\Lambda^n_{\not{b}}\hat{\partial}_\mu\sigma^{\not{b}}$.

Using the generic expansion for any tensor in the near-coincident limit (given by (A.2.11)), we can express the delta function in the neighbourhood of Y_0 in terms of derivatives with respect to the slashed coordinates as

$$\tilde{\delta}^6(y, Y) = \tilde{\delta}(y, Y_0) - \partial_{\not{b}}\tilde{\delta}^6(y, Y_0)\sigma^{\not{b}} + \frac{1}{2}\partial_{\not{a}}\partial_{\not{b}}\tilde{\delta}^6(y, Y_0)\sigma^{\not{a}}\sigma^{\not{b}} + \mathcal{O}(\sigma^3)\,. \tag{3.96}$$

We denote $\sigma_{\not{b}} = \partial_{\not{b}}\sigma$ as a vector with respect to the slashed coordinates. Using the properties of the parallel propagator, we can relate the derivative of the delta function with respect to the slashed coordinates to the covariant derivative with respect to the internal coordinates:

$$\partial_{\not{b}}\tilde{\delta}^6(y, Y) = -\widetilde{\nabla}_m(\Lambda^m_{\not{b}}\tilde{\delta}^6(y, Y))\,. \tag{3.97}$$

We are also able to note that, for some one-form α_1 on the Calabi-Yau manifold, $\tilde{\star}\tilde{d}\tilde{\star}\alpha_1 = -\widetilde{\nabla}^\ell\alpha_\ell$. Using both of these facts, the second term of (3.96) can be written as

$$\partial_{\not{b}}\left(\tilde{\delta}^6(y, Y_0)\sigma^{\not{b}}\right) = \tilde{\star}\tilde{d}\tilde{\star}\left(\tilde{\sigma}_1\tilde{\delta}^6(y, Y_0)\right)\,, \tag{3.98}$$

where $\tilde{\sigma}_m \equiv \tilde{g}_{mn}\Lambda^n_{\not{b}}\sigma^{\not{b}}$. Using this expansion for the delta function in (3.95), the terms up to second order in σ are

$$d_{10}\tilde{F}_5 = 2\kappa^2 T_3 \Big(\tilde{\star}\tilde{\delta}^6(y,Y_0) - \tilde{d}\tilde{\star}(\tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)) - \hat{d}\tilde{\star}(\tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)) - \tilde{\star}\hat{d}\tilde{\sigma}_1\tilde{\star}\tilde{d}\tilde{\star}(\tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)) \\ +\frac{1}{2}\tilde{\star}\sigma^{\not{p}}\partial_{\not{p}}\tilde{\star}\tilde{d}\tilde{\star}(\tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)) + \frac{1}{2}\tilde{\star}(\hat{d}\tilde{\sigma}_1 \wedge \hat{d}\tilde{\sigma}_1)\tilde{\delta}^6(y,Y_0)\Big)\,. \tag{3.99}$$

In order to simplify further, we wish to be able to write $d_{10}\tilde{F}_5$ in terms of ten-dimensional derivatives. To this end, we consider what terms arise from the expression $\tilde{d}\tilde{\star}(\hat{d}\tilde{\sigma}_1 \wedge \tilde{\sigma}_1\tilde{\delta}^6(y,Y_0))$. Using the associativity of the Hodge star, and the fact that the partial derivatives of the parallel propagators vanish at coincidence, *i.e.* when $Y_m = \tilde{g}_{mn}\Lambda^n_{\not{p}}Y_0^{\not{p}}$, we find that

$$\tilde{d}\tilde{\star}(\hat{d}\tilde{\sigma}_1 \wedge \tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)) = \tilde{\star}\tilde{\sigma}_1\hat{d}\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0)) + \hat{d}(\tilde{\star}\tilde{\sigma}_1)\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0))\,. \tag{3.100}$$

Likewise, we propose an additional term and find that it has two contributions:

$$\hat{d}(\tilde{\star}\tilde{\sigma}_1\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0)) = \hat{d}(\tilde{\star}\tilde{\sigma}_1)\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0) - \tilde{\star}\tilde{\sigma}_1(\hat{d}\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0))\,. \tag{3.101}$$

By adding (3.100) and (3.101), we find that

$$\hat{d}(\tilde{\star}\tilde{\sigma}_1)\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0) = \frac{1}{2}\tilde{d}\tilde{\star}(\hat{d}\tilde{\sigma}_1 \wedge \tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)) + \frac{1}{2}\hat{d}(\tilde{\star}\tilde{\sigma}_1\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0))\,. \tag{3.102}$$

Finally, by substituting this result into the last term of the first line in (3.99), we can see that

$$\begin{aligned} d_{10}\tilde{F}_5 &= 2\kappa^2 T_3\big(\tilde{\star}\tilde{\delta}^6(y,Y_0) - d_{10}[\tilde{\star}\tilde{\sigma}_1\tilde{\delta}^6(y,Y_0)] \\ &\quad +\frac{1}{2}d_{10}[\tilde{\star}\tilde{\sigma}_1\sigma^{\not{p}}\partial_{\not{p}}\tilde{\delta}^6(y,Y_0)] + \frac{1}{2}d_{10}[\tilde{\star}(\tilde{\sigma}_1 \wedge \hat{d}\tilde{\sigma}_1)\tilde{\delta}^6(y,Y_0)]\big)\,. \end{aligned} \tag{3.103}$$

This form of the equation of motion for $\tilde{F}_5$ allows us to easily separate the global and local contributions to the flux. In particular, we see that

$$\tilde{F}_5 = d_{10}C'_4 + d_{10}S^0_4 - 2\kappa^2 T_3 \tilde{\star}\tilde{\sigma}_1\tilde{\delta}^6(y,Y_0) + \kappa^2 T_3\tilde{\star}\tilde{\sigma}_1\sigma^{\not{b}}\partial_{\not{b}}\tilde{\delta}^6(y,Y_0)$$
$$+\kappa^2 T_3\tilde{\star}(\tilde{\sigma}_1\wedge\hat{d}\tilde{\sigma}_1)\tilde{\delta}^6(y,Y_0)\,, \tag{3.104}$$

where S^0_4 is the monopole source term, defined in patches such that $d^2_{10}S^0_4 = 2\kappa^2 T_3\tilde{\star}\tilde{\delta}^6(y,Y_0)$. The reason this does not vanish is because S^0_4 is not a globally defined form, and therefore the usual identity $d^2\alpha = 0$ does not apply. As a check, we can compare the terms found in (3.104) to those that would be generated by the expansion of the delta distribution in (3.92). Indeed, these two expressions agree when we remember that $\hat{\partial}_\mu\tilde{Y}_m = -\hat{\partial}_\mu\tilde{\sigma}_m$.

3.7 Effects of Local Terms In the Equations of Motion

We have already seen that the global degrees of freedom correspond to those that can be written as $\tilde{F}_5 = d_{10}C'_4$ and lie in the (0,5), (1,4), and half of the (2,3) components. However, in § 3.4 we demonstrated that the non-global term can be separated from the global terms in (3.75). It is important to determine what affect these local terms have on the equation of motion for the D3-brane position.

To investigate this further, we consider only the local terms of $\tilde{F}_5$, as given by (3.104). Using these degrees of freedom, there is no contribution from the Wess-Zumino action and so the total action for the brane position is

$$S = -\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\,\frac{1}{2}|\tilde{F}_5|^2 - \frac{T_3}{2}\int d^{10}x\sqrt{-g}\int d^4\xi\sqrt{-\gamma}(\gamma^{ab}P[g]_{ab}-2)\delta^{10}(x,X(\xi))\,. \tag{3.105}$$

We have already seen that, in the static gauge, the induced metric is equal to the 4D metric. For this calculation, it suffices to point out that $\gamma^{\mu\nu}$ depends only on the spacetime coordinates. Furthermore, we can write the pullback $P[g]_{ab}$ in terms of a set of general metrics that are functions of spacetime and the brane position. This will make the pullback's dependence on $Y^{\underline{m}}(x)$ manifest:

$$P[g]_{\mu\nu} = g_{\mu\nu}(x,Y) + 2g_{\mu\underline{n}}(x,Y)\hat{\partial}_\nu Y^{\underline{n}} + g_{\underline{mn}}(x,Y)\hat{\partial}_\mu Y^{\underline{m}}\hat{\partial}_\nu Y^{\underline{n}} \,. \tag{3.106}$$

Thus, we define $\mathcal{L}$ by

$$\begin{aligned} -\frac{1}{2\kappa^2}\int d^{10}x\mathcal{L} \equiv -\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\,\Bigg[\frac{1}{2}\frac{1}{5!}\tilde{F}_{MNPQR}\tilde{F}^{MNPQR} \\ -\frac{T_3}{2}e^{6A}\left\{\gamma^{\mu\nu}(g_{\mu\nu} + 2g_{\mu\underline{n}}\hat{\partial}_\nu Y^{\underline{n}} + g_{\underline{mn}}\hat{\partial}_\mu Y^{\underline{m}}\hat{\partial}_\nu Y^{\underline{n}}) - 2\right\}\tilde{\delta}^6(y,Y)\Bigg] \,. \end{aligned} \tag{3.107}$$

Using the Euler-Lagrange equations to compute the equation of motion requires that we take partial derivatives of $\mathcal{L}$ with respect to $Y^{\underline{m}}$ and $\hat{\partial}_\mu Y^{\underline{m}}$. Let us first examine these partial derivatives acting on the relevant components of the flux. First, we have

$$\begin{aligned} \frac{\partial \tilde{F}_{\ell_1\ldots\ell_5}}{\partial Y^{\underline{m}}} = \kappa^2 T_3\Big[2\Lambda^p_{\underline{m}}\tilde{\epsilon}_{\ell_1\ldots\ell_5 p}\tilde{\delta}^6(y,Y_0) - \Lambda^p_{\underline{m}}\tilde{\epsilon}_{\ell_1\ldots\ell_5 p}\sigma^{\not{q}}\partial_{\not{q}}\tilde{\delta}^6(y,Y_0) \\ -\Lambda^{\not{q}}_{\underline{m}}\Lambda^p_{\not{b}}\tilde{\epsilon}_{\ell_1\ldots\ell_5 p}\sigma^{\not{b}}\partial_{\not{q}}\tilde{\delta}^6(y,Y_0)\Big] \,, \end{aligned} \tag{3.108}$$

where we have noted that $\partial\sigma^{\not{b}}/\partial Y^{\underline{m}} = -\Lambda^{\not{b}}_{\underline{m}} + \mathcal{O}(2)$. The other partial derivative of F_5 to consider is

$$\frac{\partial \tilde{F}_{\ell_1\ldots\ell_4\mu}}{\partial Y^{\underline{m}}} = -\kappa^2 T_3 \Lambda^m_{\underline{m}}\Lambda^n_{\not{b}}\tilde{\epsilon}_{\ell_1\ldots\ell_4 mn}\hat{\partial}_\mu\sigma^{\not{b}}\tilde{\delta}^6(y,Y_0) \,. \tag{3.109}$$

The only non-zero derivative with respect to $\hat{\partial}_\mu Y^{\underline{m}}$ is

$$\frac{\partial \tilde{F}_{\ell_1\ldots\ell_4\mu}}{\partial(\hat{\partial}_\mu Y^{\underline{m}})} = \kappa^2 T_3 \Lambda^m_{\underline{m}}\Lambda^n_{\not{b}}\tilde{\epsilon}_{\ell_1\ldots\ell_4 mn}\sigma^{\not{b}}\tilde{\delta}^6(y,Y_0) \,. \tag{3.110}$$

Now we can state the contributions to the Euler Lagrange equations; namely,

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial Y^{\underline{m}}} = & \frac{1}{5!}\left[\frac{\partial \tilde{F}_{\ell_1\ldots\ell_5}}{\partial Y^{\underline{m}}}\tilde{F}^{\ell_1\ldots\ell_5} + \frac{\partial \tilde{F}_{\ell_1\ldots\ell_4\mu}}{\partial Y^{\underline{m}}}\tilde{F}^{\ell_1\ldots\ell_4\mu}\right] \\ & + \partial_{\underline{m}}\tilde{\delta}^6(y,Y_0)\left\{e^{6A}T_3\left[\gamma^{\mu\nu}(g_{\mu\nu} + 2g_{\mu\underline{n}}\hat{\partial}_\nu Y^{\underline{n}} + g_{\underline{pn}}\hat{\partial}_\mu Y^{\underline{p}}\hat{\partial}_\nu Y^{\underline{n}}) - 2\right]\right\} \\ & + e^{6A}T_3\gamma^{\mu\nu}(\partial_{\underline{m}}g_{\mu\nu} + 2\partial_{\underline{m}}g_{\mu\underline{n}}\hat{\partial}_\nu Y^{\underline{n}} + \partial_{\underline{m}}g_{\underline{pn}}\hat{\partial}_\mu Y^{\underline{p}}\hat{\partial}_\nu Y^{\underline{n}})\tilde{\delta}^6(y,Y_0)\,, \end{aligned} \tag{3.111}$$

and

$$\frac{1}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial(\hat{\partial}_\mu Y^{\underline{m}})} = \frac{1}{5!}\left[\frac{\partial \tilde{F}_{\ell_1\ldots\ell_4\nu}}{\partial(\hat{\partial}_\mu Y^{\underline{m}})}\tilde{F}^{\ell_1\ldots\ell_4\nu}\right] + e^{6A}T_3\gamma^{\mu\nu}(2g_{\nu\underline{m}} + 2g_{\underline{mn}}\hat{\partial}_\nu Y^{\underline{n}})\tilde{\delta}^6(y,Y_0)\,. \tag{3.112}$$

Consider the term with the curly braces of (3.111). We know that the derivative acting on the delta distribution can be converted to the y^m coordinates via

$$\partial_{\underline{m}}\tilde{\delta}^6(y,Y_0) = -\widetilde{\nabla}_\ell(\Lambda^\ell_{\underline{m}}\tilde{\delta}^6(y,Y_0))\,. \tag{3.113}$$

This means that the variation of the total action with respect to the D3-brane position is

$$\frac{\delta S}{\delta Y^{\underline{m}}} = -\frac{T_3}{2}\int d^{10}x\sqrt{-g}\,\delta Y^{\underline{m}}\Lambda^\ell_{\underline{m}}\tilde{\delta}^6(y,Y_0)\widetilde{\nabla}_\ell\{\ldots\}\,, \tag{3.114}$$

where everything within the braces depends only on x^μ and $Y^{\underline{m}}$. Thus, the we can factor out the total derivative, which means that is a boundary term that does not contribute to the equation of motion. Since we now know that there will be no contribution from that line of (3.111), we are free to drop it from our calculation.

The equation of motion for the first-order brane position is generated via the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial Y^{\underline{m}}} - \hat{\partial}_\mu\left(\frac{\partial \mathcal{L}}{\partial(\hat{\partial}_\mu Y^{\underline{m}})}\right) = 0\,, \tag{3.115}$$

and includes only terms up to first order. As discussed earlier, $\sigma^{\not{b}}$ is the brane displacement away from the static position Y_0 and is necessarily first-order. If it were not, then a shift of coordinates on the compact space would set $\sigma^{\not{b}} = 0$. This means that, since all the terms involving the flux are *at least* second order, the first-order equation of motion for the brane position must come from only terms that are defined globally.

Importantly, when we consider the static background, *i.e.* $Y^{\underline{m}} = \delta^{\not{b}}_{\underline{m}} Y^{\not{b}}_0$, the equation of motion is

$$0 = \frac{\partial \mathcal{L}}{\partial Y^{\underline{m}}} \propto \tilde{\delta}^6(y, Y_0) \left[\gamma^{\mu\nu} \partial_{\underline{m}} g_{\mu\nu} + 2\Lambda^{m}_{\underline{m}} \partial_m \tilde{F}_{\ell_1 \ldots \ell_5} \tilde{F}^{\ell_1 \ldots \ell_5}\right] . \tag{3.116}$$

The background solution is evaluated in the static gauge, and so $\gamma^{\mu\nu} = e^{-2A} e^{-2\Omega} \hat{\eta}^{\mu\nu}$ and $g_{\mu\nu} = e^{2A} e^{2\Omega} \hat{\eta}_{\mu\nu}$; furthermore, $[\tilde{F}_5]_{(0,5)} = \tilde{\star} \tilde{d} e^{-4A}$. Thus, (3.116) becomes

$$\frac{\partial \mathcal{L}}{\partial Y^{\underline{m}}} \propto \tilde{\delta}^6(y, Y_0) \left[4e^{-2A} \partial_{\underline{m}} e^{2A} + 2\Lambda^{m}_{\underline{m}} e^{4A} \partial_m e^{-4A}\right] = 0 \,. \tag{3.117}$$

This further confirms that the position of the D3-brane is a modulus of the theory, since the equation of motion is satisfied at any position. The non-trivial vanishing of the equation of motion at the level of the background is an important consistency check, as this result was established in § 3.3 when considering the opposite degrees of freedom.

Having established that local terms do not contribute to the equation of motion (and, therefore, the quadratic action), we can now turn our attention towards constructing an effective action for mobile D3-branes.

4 Effective Action for D3-Branes

4.1 Contributions to the Quadratic Action

In order to produce a properly quadratic effective action, we must first construct a ten-dimensional action comprised of the product of first-order quantities, then perform dimensional reduction over the compact dimensions. We will follow the prescription given in [18] for writing the kinetic action of a modulus and thereby produce the ten-dimensional action for the D3-brane position. In Chapter 6, we will consider all three of the volume modulus, universal axion, and brane position, and construct an effective action for that system.

To begin, we first consider a theory with fields Ψ_A and a Lagrangian density that includes, at most, two derivatives of the fields such that

$$S = \int d^n x \, \mathcal{L}(\Psi, \partial_M \Psi, \partial_M \partial_N \Psi) \,. \tag{4.1}$$

It can be shown that the second-order action is given by the first-order fluctuations contracted with the linearized equations of motion. Collecting the respective contributions to the first-order parts of the Einstein equations (δE_{MN}) and the metric (δg_{MN}) from each modulus, the first contribution to (4.1) is

$$S_R = \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \, \delta g^{MN} \delta E_{MN} \,, \tag{4.2}$$

i.e. the gravity sector quadratic action. Likewise, using the first-order parts of the equation of motion for the flux (δE_6) and the 4-form potential (δC_4) from each modulus, the next contribution to (4.1) is

$$S_5 = \frac{1}{4\kappa^2} \int \delta C_4 \wedge \delta E_6 \,, \tag{4.3}$$

i.e. the 5-form sector quadratic action. The equation of motion of the D3-brane position, E_M, also contributes to the quadratic action. We construct this action from the dynamic portions of brane equation of motion, given by δE_M, contracted with the first-order brane coordinate, δX^M, to give

$$S_{\mathrm{D3}} = \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g}\, \delta X^M \delta E_M \,, \tag{4.4}$$

i.e. the brane sector quadratic action. The total quadratic action is then the sum of all the contributions,

$$S = \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \, \left(\delta g^{MN} \delta E_{MN} + \delta X^M \delta E_M + \star_{10}(\delta C_4 \wedge \delta E_6) \right) \,. \tag{4.5}$$

We will begin assembling the total quadratic action by first constructing the action for each sector separately. Then, upon combining all sectors, we will determine the overall effective action and commence with the dimensional reduction.

4.2 Quadratic Action Sectors

In order to determine the quadratic action for the mobile D3-branes, we must decide on which legs of the potential – and, therefore, the flux – will contain the relevant degrees of freedom. In this section, we will choose the (4,0), (3,1), and (2,2) legs of the potential to represent the global degrees of freedom. For now, we use the flux ansatz given by (3.8), with

$$\tilde{F}_5 = e^{4\Omega} \hat{\epsilon} \wedge \tilde{d} e^{4A} + e^{4\Omega} d_{10}(e^{4A} \hat{\star} \hat{d} B_1) + \dots \,, \tag{4.6}$$

where the dots represent the components of the flux that do not contain "electric" degrees of freedom.

4.2.1 Gravity Sector

To construct the gravity sector action, we collect the contributions to the first-order pieces of the metric and Einstein equations. By directly varying (1.23), we can see that

$$\delta g^{\mu\nu} = \delta(e^{-2A}e^{-2\Omega}\hat{\eta}^{\mu\nu}) = -2\delta_Y A e^{-2A_0}e^{-2\Omega_0}\hat{\eta}^{\mu\nu} \tag{4.7}$$

$$\delta g^{mn} = \delta(e^{2A}\tilde{g}^{mn}) = 2\delta_Y A e^{2A_0}\tilde{g}^{mn}\,, \tag{4.8}$$

where the exponentials now only contain background parts. Recall from § 3.1 that there are no dynamic contributions from $E_{\mu m}$; therefore, we do not need to consider $\delta g^{\mu m}$. Going forward, we will always factor out the first-order contributions from the exponentials, and so – for notational simplicity – we will drop the subscripts when appropriate. Furthermore, since the first-order part of the warp factor is only due to the D3-branes here, we can drop the subscript on the delta and understand that $\delta_Y A = \delta A$.

The dynamic parts of the Einstein tensor are given by (3.31) and (3.33) and are

$$\delta E_{\mu\nu} = -2e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}\hat{\partial}^2 B_\ell \widetilde{\nabla}^\ell A\,, \tag{4.9}$$

$$\delta E_{mn} = \hat{\partial}^2\left(\widetilde{\nabla}_{(m}B_{n)} - \tilde{g}_{mn}\widetilde{\nabla}^\ell B_\ell\right) - 2e^{-4A}e^{-2\Omega}\tilde{g}_{mn}\hat{\partial}^2\delta A\,. \tag{4.10}$$

Plugging these into (4.2), the gravity sector action becomes

$$\begin{aligned} S_R &= \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\,\delta g^{MN}\delta E_{MN} = \frac{1}{4\kappa^2}\int d^4x\sqrt{-\hat{\eta}}\int d^6y\sqrt{\tilde{g}}\,e^{-2A}e^{4\Omega}\delta g^{MN}\delta E_{MN} \\ &= \frac{1}{4\kappa^2}\int d^4x\int d^6y\sqrt{\tilde{g}}\,e^{4\Omega}\left[16\delta A\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell - 10\delta A\hat{\partial}^2\widetilde{\nabla}^\ell B_\ell - 24\delta A e^{-4A}e^{-2\Omega}\hat{\partial}^2\delta A\right]. \end{aligned} \tag{4.11}$$

Note that we can write $\delta A = -1/4\, e^{4A}\delta e^{-4A}$, and – recalling the constraint equation (3.48) –

$\widetilde{\nabla}^\ell B_\ell = e^{-2\Omega}\delta e^{-4A}$. Using these substitutions, we find that

$$S_R = \frac{1}{4\kappa^2}\int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega}\left[e^{4A}\widetilde{\nabla}^\ell\{(\delta e^{-4A})\hat{\partial}^2 B_\ell\} + e^{4A}e^{-2\Omega}(\delta e^{-4A})\hat{\partial}^2(\delta e^{-4A})\right]. \quad (4.12)$$

4.2.2 5-Form Sector

To compute the quadratic action for the 5-form sector, we collect the dynamic contributions to δE_6 and the complementary first-order pieces of δC_4. For the D3-brane, these are

$$\delta E_6 = d_{10}(e^{-4A}\tilde{\star}\hat{\partial}^2 B_1) \quad \text{and} \quad \delta C_4 = e^{4\Omega}(\delta e^{-4A}\hat{\epsilon} + e^{4A}\hat{\star}\hat{d}B_1)\,. \quad (4.13)$$

Note that "complimentary" here means that the wedge product $\delta E_6 \wedge \delta C_4$ has four spacetime and six internal indices. This means that the appropriate term is

$$\delta E_6 \wedge \delta C_4 = 4\delta A e^{4A} e^{4\Omega}\hat{\epsilon} \wedge \tilde{d}(e^{-4A}\tilde{\star}\hat{\partial}^2 B_1)\,. \quad (4.14)$$

Given the formula for S_5 in (4.3), we will be integrating the above over all ten dimensions and then adding to the other contributions to the total action (4.5). Thus, it will be useful to convert the integral of the forms into component notation:

$$\int \delta E_6 \wedge \delta C_4 = \int d^4x \sqrt{-\hat{\eta}} \int d^6y \sqrt{\tilde{g}}\, 4e^{4\Omega}\delta A(\hat{\partial}^2\widetilde{\nabla}^\ell B_\ell - 4\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell)\,. \quad (4.15)$$

Finally, we again rewrite δA in terms of the direct variation of the exponential, so that

$$S_5 = \frac{1}{4\kappa^2}\int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega}\left[-e^{4A}\widetilde{\nabla}^\ell\{(\delta e^{-4A})\hat{\partial}^2 B_\ell\} - e^{4A}e^{-2\Omega}(\delta e^{-4A})\hat{\partial}^2(\delta e^{-4A})\right]. \quad (4.16)$$

4.2.3 Brane Sector

The final contribution to the total quadratic action is from the brane sector. In §2.3, we determined the equation of motion for the D3-brane position from directly varying the sum of the DBI and

WZ actions. We then defined E_A with the integral over the compact dimensions intact; however, to match the definitions in § 4.1, E_A should really be only

$$\begin{aligned} E_A = 2\kappa^2 T_3 \Big[&\hat{\partial}_\mu (e^{2A} e^{2\Omega} \hat{\eta}^{\mu\nu} g_{MN} \Lambda^M_A \Lambda^N_B \hat{\partial}_\nu X^B) \\ &+ \frac{1}{6} \hat{\partial}_a (\hat{\varepsilon}^{abcd} C_{MNPQ} \Lambda^M_A \Lambda^N_B \Lambda^P_C \Lambda^Q_D \, \hat{\partial}_b X^B \hat{\partial}_c X^C \hat{\partial}_d X^D) \Big] \, \tilde{\delta}^6(y, Y) \,. \end{aligned} \tag{4.17}$$

Using this definition, and the fact that $A = \rho$ component gave a trivial constraint, we need only consider the $A = \underline{m}$ component (as opposed to the slashed coordinates, the underlined coordinates are a more precise measure of the brane displacement):

$$\begin{aligned} E_{\underline{m}} = 2\kappa^2 T_3 \Lambda^m_{\underline{m}} \Big[&\hat{\partial}_\mu (e^{2A} e^{2\Omega} \hat{\eta}^{\mu\nu} g_{mN} \Lambda^N_B \hat{\partial}_\nu X^B) \\ &+ \frac{1}{6} \hat{\partial}_a (\hat{\varepsilon}^{abcd} C_{mbcd} \Lambda^N_B \Lambda^P_C \Lambda^Q_D \, \hat{\partial}_b X^B \hat{\partial}_c X^C \hat{\partial}_d X^D) \Big] \, \tilde{\delta}^6(y, Y) \,. \end{aligned} \tag{4.18}$$

Given that we are considering only the "electric" degrees of freedom, only the (3,1) component of C_4 will contribute to the brane equation of motion, as evidenced by the form of $\tilde{F}_5$ in (4.6). After permuting indices and expanding both contributions from the metric, we find that

$$E_{\underline{m}} = 2\kappa^2 T_3 \Lambda^m_{\underline{m}} \Big[e^{4A} e^{4\Omega} \hat{\partial}^2 B_m + e^{2\Omega} \tilde{g}_{mn} \Lambda^n_{\underline{n}} \hat{\partial}^2 Y^{\underline{n}} - (e^{4A} e^{4\Omega} \hat{\partial}^2 B_m) \Big] \, \tilde{\delta}^6(y, Y) \,. \tag{4.19}$$

Thus, the contributions from the potential and the off-diagonal metric exactly cancel one another. This is an important consistency check, since we know that the background equation of motion for the D3-branes is simply a static solution. By taking $Y^{\underline{m}} = y$, we recover this background configuration.

At last, we find that the brane sector action is

$$S_{\mathrm{D3}} = \frac{1}{4\kappa^2} \int d^4x \int d^6y \sqrt{\tilde{g}} \, 2T_3 \kappa^2 e^{2\Omega} \Lambda^m_{\underline{m}} \Lambda^n_{\underline{n}} Y^{\underline{m}} \tilde{g}_{mn} \hat{\partial}^2 Y^{\underline{n}} \, \tilde{\delta}^6(y, Y) \,. \tag{4.20}$$

4.3 Total Quadratic Action for D3-Branes

We can now use (4.5) to determine the total quadratic action for the D3-brane position as a modulus. Examining the results from Section 4.2, we can see that those terms that contain warping exactly cancel, albeit in a highly non-trivial way. Since the first-order brane position depends only on the spacetime variables, what we are left with is

$$S = -\frac{T_3}{2} \int d^4x \, e^{2\Omega} \hat{\partial}^\mu Y^{\underline{m}} \hat{\partial}_\mu Y^{\underline{n}} \int d^6y \sqrt{\tilde{g}} \, \tilde{g}_{\underline{mn}} \tilde{\delta}^6(y, Y) \, . \tag{4.21}$$

The remaining six-dimensional integral is simply the value of a background-dependent metric at the brane position, and so does not contain any degrees of freedom. Therefore, it would seem that the effect of the warping of the manifold on the D3-branes is exactly countered by the backreaction of the branes with the potential, C_4.

Having seen that mobile D3-branes remain as unfixed moduli in the four-dimensional effective theory arising from type IIB SUGRA, we now want to consider the effects of the presence of multiple moduli. We wish to see if introducing the volume modulus and axion into the 10D theory generates coupling between the moduli that will, in turn, fix some or all of them in the effective theory. To do so, we will repeat many of the procedures from the previous chapters, but will now include contributions for all the moduli. When possible, we will refer to results already established in Chapters 3 and 5.

5 Other Moduli in Warped Compactifications

Thus far, we have encountered an effective action for the spacetime scalar Y, where the global degrees of freedom are taken to be in the (4,0), (3,1), and (2,2) components of the potential. In order to construct an effective theory for multiple moduli, we wish to write similar potentials for the volume modulus and universal axion, where the relevant degrees of freedom for those scalars are in similar components of the respective potentials.

Let us first present a summary of results for the other moduli being considered, namely the universal volume modulus and the universal axion. In the first sections of this chapter we will review the existing literature for the volume modulus and the scalar version of the universal axion. We will then present an original calculation of the effective theory for the 2-form version of the axion. This description of the axion ensures that the global degrees of freedom are in the same components as those for the D3-brane and volume modulus.

5.1 The Universal Volume Modulus

Recall the universal volume modulus, $c(x)$, as a modulus of type IIB SUGRA compactifications. Let us briefly review the wavefunction for the volume modulus in the presence of warping as presented in [17, 18].

To begin, we take the metric ansatz to be

$$ds_{10}^2 = e^{2A}e^{2\Omega}\hat{\eta}_{\mu\nu}dx^\mu dx^\nu - 2e^{2A}e^{2\Omega}\hat{\partial}_\mu c(x)\tilde{\partial}_m K(y)dx^\mu dy^m + e^{-2A}\tilde{g}_{mn}dy^m dy^n\,, \tag{5.1}$$

where $K(y)$ is a compensator field and the warp factor A, and Weyl factor Ω, are functions of the volume modulus. In the context of background versus first-order solutions, we take $c(x)$ and its associated compensator field $K(y)$ to be first-order. Next, the 5-form field strength ansatz is

$$\tilde{F}_5 = e^{4\Omega}\hat{\epsilon} \wedge \tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{-4A} - e^{4\Omega}\hat{\star}\hat{d}c \wedge \tilde{d}K \wedge \tilde{d}e^{4A}\,, \tag{5.2}$$

and obeys the self-duality condition $\star_{10}\tilde{F}_5 = \tilde{F}_5$. There are two constraints that arise from the Einstein equations, namely

$$\hat{\partial}_\mu\hat{\partial}_\nu c\left(\widetilde{\nabla}^2 K - e^{-4A}\frac{\partial e^{-2\Omega}}{\partial c} + e^{-2\Omega}\frac{\partial e^{-4A}}{\partial c}\right) = 0 \tag{5.3}$$

$$\text{and}\quad \hat{\partial}_\mu c\,\tilde{\partial}_m\frac{\partial e^{-4A}}{\partial c} = 0\,. \tag{5.4}$$

Solving these constraints gives a Poisson equation for the compensator field

$$\widetilde{\nabla}^2 K(y) = e^{-4A} - e^{-2\Omega}\,. \tag{5.5}$$

The first-order components of the metric field are now given by

$$\delta g_{\mu\nu} = -e^{2A}e^{2\Omega}\left(e^{2\Omega} + \frac{1}{2}e^{4A}\right)c(x)\hat{\eta}_{\mu\nu}\,, \tag{5.6}$$

$$\delta g_{\mu m} = -e^{2A}e^{2\Omega}\hat{\partial}_\mu c(x)\tilde{\partial}_m K\,, \tag{5.7}$$

$$\delta g_{mn} = \frac{1}{2}e^{2A}c(x)\tilde{g}_{mn}\,, \tag{5.8}$$

and their inverses by

$$\delta g^{\mu\nu} = \left(e^{-2A} + \frac{1}{2}e^{2A}e^{-2\Omega}\right)c(x)\hat{\eta}^{\mu\nu}\,, \tag{5.9}$$

$$\delta g^{\mu m} = e^{2A}\hat{\partial}^\mu c(x)\tilde{\partial}^m K\,, \tag{5.10}$$

$$\delta g^{mn} = -\frac{1}{2}e^{6A}c(x)\tilde{g}^{mn}\,. \tag{5.11}$$

The dynamical equations of motion for the first-order contributions come from the (m,n) component of the Einstein equations and from the exterior derivative of the field strength, *i.e.*

$$\begin{aligned}\delta E_{mn} &= \hat{\partial}^2 c(x)\Big[\widetilde{\nabla}^2 K\tilde{g}_{mn} - \widetilde{\nabla}_m\tilde{\partial}_n K - 4\tilde{\partial}_{(m}A\tilde{\partial}_{n)}K + 2\tilde{\partial}^\ell A\tilde{\partial}_\ell K\tilde{g}_{mn} \\ &\quad +\frac{1}{2}\tilde{g}_{mn}(e^{-2\Omega} - 3e^{-4A})\Big]\,, \qquad (5.12)\\ \text{and}\quad \delta E_6 &= -e^{4\Omega}\hat{d}\hat{\star}\hat{d}c\wedge\tilde{d}K\wedge\tilde{d}e^{4A}\,. \qquad (5.13)\end{aligned}$$

Based on the form of $\tilde{F}_5$ in (5.2), the components of C_4 that carry the global degrees of freedom are

$$\delta C_4 = e^{4A}e^{4\Omega}\hat{\epsilon}\,. \qquad (5.14)$$

The reason that the (3,1) component in not included in δC_4 is because this would produce a term in $\tilde{F}_5$ that is proportional to the equation of motion for $c(x)$. As we saw with the D3-branes, when terms proportional to the equation of motion are present in the flux, it is an indication that the other legs of the potential contain a local gauge transformation. Thus, we choose to contain the global degrees of freedom within the (4,0) component of C_4, such that $[\delta C_4]_{(0,4)} = 0$.

As a check for the dimensional reduction to be done in Chapter 6, it is useful to note the result for the dimensionally reduced kinetic action, which is

$$S = -\frac{3\tilde{V}}{4\kappa^2}\int d^4x e^{4\Omega}\hat{\partial}_\mu c(x)\hat{\partial}^\mu c(x) = -\frac{3}{4\kappa_4^2}\int d^4x e^{4\Omega}\hat{\partial}_\mu c(x)\hat{\partial}^\mu c(x)\,. \qquad (5.15)$$

N.B. $\tilde{V}$ is the unwarped volume of the Calabi-Yau manifold.

5.2 The Axion As a Scalar

Let us now turn to the results of [18], where the scalar axion is the modulus being considered. Then we will consider rewriting the axion as a 2-form and derive the relevant equations of motion and constraints required to construct a dimensionally-reduced theory.

We again begin with a metric ansatz that is motivated by the gauge transformations of the nontrivial C_4 background and the necessity for a metric compensator:

$$ds_{10}^2 = e^{2A} e^{2\Omega} \hat{\eta}_{\mu\nu} dx^\mu dx^\nu + 2e^{2A} e^{2\Omega} \hat{\partial}_\mu b_0(x) B'_m(y) dx^\mu dy^m + e^{-2A} \tilde{g}_{mn} dy^m dy^n \,, \tag{5.16}$$

where b_0 and its compensator, B'_1, are first-order, and, therefore, the warp factor and Weyl factor both take their background values. The 4-form potential must depend on a different compensator due to the gauge transformation properties of C_4. We take the fluctuations of the potential (in the absence of Chern-Simons terms) to be

$$\delta C_4 = b_0(x) \wedge \omega_4 - \hat{d} b_0(x) \wedge K_3(y) \,, \tag{5.17}$$

where ω_4 is harmonic. In the traditional formulation of this modulus, we can see that the (0,4) and (3,1) legs of the potential contain the global degrees of freedom, opposite to the D3-brane and volume modulus descriptions. It is for this reason that § 5.3 reformulates the axion with the global degrees of freedom in the (4,0) and (3,1) components of the potential. The 5-form field strength follows from (5.17), and the self-duality condition $\star_{10} \tilde{F}_5 = \tilde{F}_5$, and is

$$\begin{aligned} \tilde{F}_5 = {} & e^{4\Omega} \hat{\epsilon} \wedge \tilde{d} e^{4A} + \tilde{\star} \tilde{d} e^{-4A} + \hat{d} b_0 \wedge \big(\omega_4 + \tilde{d} K_3\big) \\ & + e^{4\Omega} \hat{\star} \hat{d} b_0 \wedge B'_1 \wedge \tilde{d} e^{4A} + e^{4A} e^{2\Omega} \hat{\star} \hat{d} b_0 \wedge \tilde{\star} \big(\omega_4 + \tilde{d} K_3\big) \,. \end{aligned} \tag{5.18}$$

Using (5.18) to determine $E_6 = d_{10} \tilde{F}_5$, we obtain a constraint equation from setting the non-dynamic part to zero

$$e^{4A} \left(\tilde{\star} (\omega_4 + \tilde{d} K_3) + e^{2\Omega} \tilde{d} B'_1 \right) = \gamma_2 \,, \tag{5.19}$$

with γ_2 closed. Thus, the integrability condition is

$$e^{2\Omega} \tilde{d} \tilde{\star} \tilde{d} B'_1 = \tilde{d} e^{-4A} \wedge \tilde{\star} \gamma_2 \,, \tag{5.20}$$

which is the consistency equation necessary for (5.19) to be solvable. The dynamical equation of motion comes from the (m, n) component of the Einstein equation and the dynamic parts of $d_{10}\tilde{F}_5$:

$$\delta E_{mn} = \hat{\partial}^2 b_0 \left(\widetilde{\nabla}_{(m} B'_{n)} + 4\tilde{\partial}_{(m} A\, B'_{n)} - 2\tilde{g}_{mn} B'_\ell \tilde{\partial}^\ell A\right) \tag{5.21}$$

$$\delta E_6 = \hat{d}\left(e^{2\Omega} \hat{\star} \hat{d} b_0\right) \wedge \left[\gamma_2 - e^{2\Omega}\tilde{d}(e^{4A} B'_1)\right] . \tag{5.22}$$

To determine the dimensionally reduced kinetic action for b_0, we must evaluate

$$S = -\frac{1}{4\kappa^2}\int d^4x e^{2\Omega}\hat{\partial}_\mu b(x)\hat{\partial}^\mu b(x)\int_{CY}\omega_4 \wedge \gamma_2 . \tag{5.23}$$

We can expand ω_4 and γ_2 in terms of a basis of (1,1) harmonic 2-forms by writing $\omega_4^I = \tilde{\star}\tilde{\omega}_2^I$ and $\gamma_2^I = C^{IJ}\tilde{\omega}_2^J$. In the case of the universal axion (as opposed to generalized axions), it can be shown that $\gamma_2^{I=1} = e^{2\Omega}\tilde{J}$ and $\omega_4^{I=1} = \tilde{\star}\tilde{J}$[1]. This allows us to evaluate the integral over the Calabi-Yau

$$\int_{CY}\omega_4 \wedge \gamma_2 = \int_{CY} e^{2\Omega}\tilde{\star}\tilde{J}\wedge\tilde{J} = 3e^{2\Omega}\tilde{V} . \tag{5.24}$$

Therefore, the dimensionally reduced kinetic action for the universal axion is

$$S = -\frac{3\tilde{V}}{4\kappa^2}\int d^4x e^{4\Omega}\hat{\partial}_\mu b(x)\hat{\partial}^\mu b(x) . \tag{5.25}$$

N.B. As with the volume modulus, the axion's dimensionally-reduced quadratic action is proportional to the unwarped volume of the Calabi-Yau, $\tilde{V}$.

5.3 Reformulation of the Axion As a 2-Form

Let us now rewrite the normal scalar axion, $b_0(x)$, in terms of a 2-form, $b_2(x)$, in such a way that the relevant degrees of freedom are contained in the (4,0), (3,1), and (2,2) components of C_4. We

[1] See Appendix A.4 for more on the Kähler form, $\tilde{J}$, as well as other forms on complex manifolds.

start by defining $b_2(x)$ such that

$$\hat{d}b_2(x) \equiv e^{-2\Omega}\hat{\star}\hat{d}b_0(x)\,. \tag{5.26}$$

Thus, the metric compensator that appears in (5.16) is now

$$\delta g_{\mu m} = e^{-2\Omega}e^{2A}[\hat{\star}\hat{d}b_2]_\mu B'_m\,, \tag{5.27}$$

where $B'_1 = B'_1(y)$. With this redefinition in mind, the 5-form flux ansatz for the axion is

$$\tilde{F}_5 = e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{-4A} + \hat{d}b_2\wedge(\omega_2+\tilde{d}K_1) + e^{-4A}e^{-2\Omega}\hat{\star}\hat{d}b_2\wedge\tilde{\star}(\omega_2+\tilde{d}K_1+B'_1\wedge\tilde{d}e^{4A})\,, \tag{5.28}$$

where ω_2 is harmonic and $K_1 = K_1(y)$. The self-duality of the background contributions in $\tilde{F}_5$ follows from § 3.1. The first-order contribution from these terms are exactly canceled by the duals of the latter terms.

Using this expression for $\tilde{F}_5$, we can determine a new equation of motion from $d_{10}\tilde{F}_5 = E_6$:

$$\begin{aligned} d_{10}\tilde{F}_5 = {}& -4e^{-4A}\tilde{d}\tilde{\star}\tilde{d}A + e^{-4A}e^{-2\Omega}\big((\hat{d}\hat{\star}\hat{d}b_2 - 4\tilde{d}A\hat{\star}\hat{d}b_2)\wedge\tilde{\star}(\omega_2+\tilde{d}K_1+B'_1\wedge\tilde{d}e^{4A}) \\ & -\hat{\star}\hat{d}b_2\wedge\tilde{d}\tilde{\star}(\omega_2+\tilde{d}K_1+B'_1\wedge\tilde{d}e^{4A})\big)\,. \end{aligned} \tag{5.29}$$

$$\tag{5.30}$$

The first-order piece of (5.29) gives a dynamical equation,

$$\delta E_6 = e^{-2\Omega}e^{-4A}\hat{d}\hat{\star}\hat{d}b_2\wedge\tilde{\star}(\omega_2+\tilde{d}K_1+B'_1\wedge\tilde{d}e^{4A})\,, \tag{5.31}$$

and a constraint equation,

$$\begin{aligned} & e^{-2\Omega}\hat{\star}\hat{d}b_2\wedge\tilde{d}\big(e^{-4A}\tilde{\star}(\omega_2+\tilde{d}K_1+B'_1\wedge\tilde{d}e^{4A})\big) = 0 \\ & \qquad i.e.\quad e^{-4A}\tilde{\star}(\omega_2+\tilde{d}K_1+B'_1\wedge\tilde{d}e^{4A}) = \gamma_4\,, \end{aligned} \tag{5.32}$$

for the closed 4-form γ_4. Substituting (5.32) into (5.31) allows the pertinent dynamical equation to be written as

$$\delta E_6 = e^{-2\Omega}\hat{d}\hat{\star}\hat{d}b_2 \wedge \gamma_4\,. \tag{5.33}$$

Using (5.32) to define γ_4 also allows for (5.28) to be written as

$$\tilde{F}_5 = e^{4\Omega}\hat{\epsilon} \wedge \tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{-4A} + e^{4A}\hat{d}b_2 \wedge \tilde{\star}\gamma_4 - \hat{d}b_2 \wedge B_1' \wedge \tilde{d}e^{4A} + e^{-2\Omega}\hat{\star}\hat{d}b_2 \wedge \gamma_4\,. \tag{5.34}$$

As we have seen, this reformulation of b_0 has altered the off-diagonal metric, and potentially will also change the form of the Einstein equation. This, in turn, may affect the final determination of the quadratic action via the contraction $\delta g^{MN}\delta E_{MN}$. We note that the generalized form of the Einstein tensor from an off-diagonal modulus contribution is given in [18] and is based on the metric ansatz $\delta g_{\mu m} = e^{2A}e^{2\Omega}\hat{\partial}_\mu B_m(x,y)$. It can be shown[2] that the substitution $\hat{\partial}_\mu B_m = e^{-4\Omega}[\hat{\star}\hat{d}b_2]_\mu B_m'$ does not require a re-derivation of the Einstein tensor, and thus is a valid direct substitution. Therefore, we are able to directly quote the first order pieces of the Einstein equations and substitute the off-diagonal metric given in (5.27) where applicable. We then find that the only non-zero contribution to the Einstein tensor, G_{MN}, comes from the off-diagonal component

$$\delta G_{\mu m} = \frac{1}{2}e^{2\Omega}\widetilde{\nabla}^\ell\big(e^{4A}\tilde{d}(\hat{\partial}_\mu B)_{m\ell}\big) = -\frac{1}{4}e^{-2\Omega}[\hat{\star}\hat{d}b_2]_\mu\widetilde{\nabla}^\ell\big(e^{4A}\tilde{d}B_1'\big)_{m\ell}\,. \tag{5.35}$$

The first-order pieces of the stress tensor for the 2-form axion are determined in the usual fashion, namely via $\delta T_{MN} = \frac{1}{96}(\delta\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS} + \tilde{F}_{MPQRS}\delta\tilde{F}_N{}^{PQRS} + 4\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQR}{}_T\delta G^{ST})$. The only non-zero component is again found in the off-diagonal piece and is given by

$$\delta T_{\mu m} = -e^{-2\Omega}[\hat{\star}\hat{d}b_2]_\mu\widetilde{\nabla}^\ell A\big(2\omega_{[m\ell]} + 4\widetilde{\nabla}_{[m}K_{\ell]} + 2B'_{[m}\widetilde{\nabla}_{\ell]}e^{4A}\big)\,. \tag{5.36}$$

[2] See Appendix A.7 for this calculation.

Finally, we see that the only first-order piece of the Einstein equation for b_2 is

$$\delta E_{\mu m} = e^{-2\Omega}[\hat{\star}\hat{d}b_2]_\mu \Big(\widetilde{\nabla}^\ell(e^{4A}\tilde{d}B_1')_{[m\ell]} + \tfrac{1}{2}(\tilde{\star}\gamma_4)_{[m\ell]}\widetilde{\nabla}^\ell(e^{4A})\Big)\,. \tag{5.37}$$

N.B. Since there are no $\hat{\partial}^2$ terms present, (5.37) is a constraint equation. This constraint to gives $\tilde{\star}\gamma_4$ as a function of B_1' and A

$$(\tilde{\star}\gamma_4)_{[m\ell]}\widetilde{\nabla}^\ell(e^{4A}) = -2\widetilde{\nabla}^\ell(e^{4A}\tilde{d}B_1')_{[m\ell]}\,. \tag{5.38}$$

We can now determine the dimensionally-reduced kinetic action for the 2-form axion. Based on the form of (5.34), and our choice of the relevant degrees of freedom, the first-order part of C_4 for the axion is

$$\delta C_4 = b_2 \wedge (\omega_2 + \tilde{d}K_1)\,. \tag{5.39}$$

By recalling (5.33), we can see that the kinetic action is

$$S = \frac{1}{4\kappa^2}\int \delta C_4 \wedge \delta E_6 = \frac{1}{4\kappa^2}\int b_2 \wedge (\omega_2 + \tilde{d}K_1) \wedge e^{-2\Omega}\hat{d}\hat{\star}\hat{d}b_2 \wedge \gamma_4\,. \tag{5.40}$$

After integration by parts and noting that γ_4 is closed, the action becomes

$$S = \frac{1}{4\kappa^2}\int e^{-2\Omega} b_2 \wedge \hat{d}\hat{\star}\hat{d}b_2 \wedge \gamma_4 \wedge \omega_2\,. \tag{5.41}$$

As we saw in the previous section, in order to proceed we must expand γ_4 and ω_2 in terms of a basis of (1,1) harmonic 2-forms[3]. It follows from § 5.2, and the discussion in [18], that $\gamma_4 = e^{-2\Omega}\tilde{\star}\tilde{J}$ and $\omega_2 = \tilde{J}$. Thus, (5.41) becomes

$$S = -\frac{3\tilde{V}}{4\kappa^2}\int_{\mathcal{M}_4} e^{-4\Omega}\hat{d}b_2 \wedge \hat{\star}\hat{d}b_2\,, \tag{5.42}$$

[3] Although γ_4 is initially taken to be closed, we can consider adding an exact piece to create some new harmonic form $\gamma_4' = \gamma_4 + \tilde{d}K_3$. Taking $\gamma_4 \to \gamma_4'$ in (5.41) results in a term that goes like $\omega_2 \wedge \tilde{d}K_3$, which then vanishes after integration by parts. Thus, only the closed part of the harmonic 4-form contributes, which is simply γ_4.

where $\mathcal{M}_4$ is Minkowksi space.

As a check, we will translate this 2-form axion back to a scalar axion, and compare this result to the one derived by [18]. To do so, we define the 3-form $h_3 = \hat{d}b_2$ such that $\hat{d}h_3 = 0$. This constraint on $\hat{d}h_3$ must be introduced into the action via a scalar Lagrange multiplier that we shall call (suggestively) b'_0

$$S = -\frac{3\tilde{V}}{2\kappa^2}\int_{\mathcal{M}_4}\left(\frac{1}{2}(e^{-4\Omega}h_3 \wedge \hat{\star}h_3) - b'_0\hat{d}h_3\right). \tag{5.43}$$

The Euler-Lagrange equation for this action allows us to solve for h_3 in terms of b'_0

$$e^{-4\Omega}\hat{\star}h_3 + \hat{d}b'_0 = 0 \quad \rightarrow \quad h_3 = -e^{4\Omega}\hat{\star}\hat{d}b'_0\,. \tag{5.44}$$

Substituting this into the action and simplifying, we find

$$S = -\frac{3\tilde{V}}{4\kappa^2}\int_{\mathcal{M}_4} e^{4\Omega}\hat{d}b'_0 \wedge \hat{\star}\hat{d}b'_0 = -\frac{3\tilde{V}}{4\kappa^2}\int d^4x e^{4\Omega}\hat{\partial}_\mu b'(x)\hat{\partial}^\mu b'(x)\,. \tag{5.45}$$

As we can see, this is precisely the form of (5.25). It is important to note that there is *no relation* between the b_0 in the definition of the 2-form b_2 via (5.26) and the b'_0 in the dimensionally-reduced action above. The naming of the Lagrange multiplier introduced into (5.43) only reflects the fact that the correct scalar axion that descends from the 2-form axion is the Lagrange multiplier. By defining the 2-form axion in terms of the scalar axion in (5.26) we have, in effect, "destroyed" the initial scalar axion.

6 Effective Action for "Electric" Moduli

6.1 Electric Ansatz for All Moduli

Now that we have seen how to create the four-dimensional effective theory for the volume modulus and axion (both scalar and 2-form versions), we can turn our attention back to the 10D SUGRA system where the D3-branes *and* the other moduli are turned on. Following the procedure from Chapter 4, we will gather the necessary contributions to construct the total quadratic action.

We wish to write each of the moduli in terms of their "electric" degrees of freedom; that is, by taking the global degrees of freedom to be in the (4,0), (3,1), and (2,2) components for each of the potentials. We can accomplish this by modelling our expression for $\tilde{F}_5$ by the original D3-brane ansatz in (3.8). However, we will find that there are additional terms for the volume modulus and universal axion that are proportional to their dynamic equations of motion. As we saw in § 3.4, terms that are proportional to the equation of motion disappear on shell and so can be absorbed by a field definition.

Let us take

$$\begin{aligned}\tilde{F}_5 &= e^{4A}\hat{\epsilon}\wedge \tilde{d}e^{-4A} + \tilde{\star}\tilde{d}e^{-4A} - e^{-4A}e^{2\Omega}\tilde{\star}(\hat{d}B_1\wedge \tilde{d}e^{4A}) + e^{4\Omega}d_{10}(e^{4A}\hat{\star}\hat{d}B_1)\\ &\quad + e^{4\Omega}\star_{10} d_{10}(e^{4A}\hat{\star}\hat{d}B_1) + \hat{d}b_2\wedge(\omega_2+\tilde{d}K_1) + e^{-4A}e^{-2\Omega}\hat{\star}\hat{d}b_2\wedge\tilde{\star}(\omega_2+\tilde{d}K_1)\,, \qquad (6.1)\end{aligned}$$

where now

$$\hat{d}B_1(x,y) = -\hat{d}c(x)\wedge\tilde{d}K + e^{-4\Omega}\hat{\star}\hat{d}b_2\wedge B_1^a + e^{-2\Omega}\hat{d}Y^{\underline{m}}G_{1,\underline{m}}\,, \qquad (6.2)$$

to be the "electric" ansatz for the field strength, which now includes all the moduli. Based on this formulation, the 10D metric

$$ds_{10}^2 = e^{2A} e^{2\Omega} \hat{\eta}_{\mu\nu} dx^\mu dx^\nu + 2e^{2A} e^{2\Omega} \hat{\partial}_\mu B_m dx^\mu dy^m + e^{-2A} \tilde{g}_{mn} dy^m dy^n \,, \tag{6.3}$$

also encompasses all the moduli under consideration when B_m is given by (6.2).

As discussed previously, changing the electromagnetic character of the D3-branes means that the field strength will satisfy a set of corresponding Maxwell-like equations. In particular, (6.1) now has a trivial Bianchi identity so that $d_{10}\tilde{F}_5 = 0$, *i.e.* there are no local contributions. This also means that the Wess-Zumino action given by (1.6) is non-zero. We will see this effect when we calculate the brane sector quadratic action.

Using this electric description of the moduli will enable us to write a quadratic action that is entirely global. This means that the local terms identified in Chapter 3 will no longer exist. We will find that proper dimensional reduction is now possible and an effective theory can be constructed.

6.2 Quadratic Action Sectors Revisited

As mentioned in § 4.1, there is an additional contribution to the quadratic action: a contribution from a non-zero background axion that *does not* give a linearized term when integrated by parts and so must be inserted by hand[1]. Contrary to an infinite, 4D Minkowski space – where a non-zero axion with zero field strength vanishes – a non-trivial background value for the axion is permitted if the internal dimensions are compactified on a torus. The integral that arises from this term is evaluated at the D3-brane, and so can be written as a ten-dimensional integral with an accompanying six-dimensional delta function; we will denote this by the $D3$ subscript.

[1] This type of term is dropped in [18] in an implied integration by parts; we reintroduce it here as a correction to that work.

The total quadratic action is now comprised of four contributions:

$$S = \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\,\delta g^{MN}\delta E_{MN} + \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\,Y^{\underline{m}}E_{\underline{m}} + \frac{1}{4\kappa^2}\int \delta C_4\wedge\delta E_6$$
$$-T_3\int_{D3} b_2\wedge\omega_{\underline{mn}}\hat{d}Y^{\underline{m}}\hat{d}Y^{\underline{n}}\,. \tag{6.4}$$

In order to determine the 4D effective action from the dimensional reduction of (6.4), we must repeat the procedures of Chapters 3 and 4. Luckily, the calculation of the Einstein tensor in § 2.2 is sufficiently general that it will still apply when $\hat{\partial}_\mu B_m$ is given by (6.2), and so will not be repeated here.

6.2.1 Gravity Sector: All Moduli

Let us start by collecting the contributions to the first-order piece of the metric, δg_{MN}. Then, the dynamic parts of the Einstein equations, δE_{MN}, will be contracted with the inverse of the first-order metric.

By directly varying g^{MN}, we find that:

$$\delta g^{\mu\nu} = \delta\left(e^{-2A}e^{-2\Omega}\hat{\eta}^{\mu\nu}\right) = -2(\delta\Omega+\delta A)e^{-2A}e^{-2\Omega}\hat{\eta}^{\mu\nu} \tag{6.5}$$
$$\delta g^{mn} = \delta\left(e^{2A}\tilde{g}_{mn}\right) = 2\delta A e^{2A}\tilde{g}^{mn}\,, \tag{6.6}$$

where the terms δA and $\delta\Omega$ are the variations of the warp and Weyl factors with respect to *all* the moduli. Note that, once again, there is no contribution from $\delta E_{\mu m}$ from any of the moduli; therefore, we do not need to calculate $\delta g^{\mu m}$. Again, all exponentials will have their first-order contributions factored out, and so are written implicitly as only the background values.

Using the results from § 3.1, we find that the dynamic portions of the Einstein equations are

$$\delta E_{\mu\nu} = -2e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell \tag{6.7}$$
$$\delta E_{\mu m} = 0 \tag{6.8}$$
$$\delta E_{mn} = \hat{\partial}^2\left(\widetilde{\nabla}_{(m}B_{n)} - \tilde{g}_{mn}\widetilde{\nabla}^\ell B_\ell\right) + e^{-4A}e^{-2\Omega}\tilde{g}_{mn}\hat{\partial}^2(3\Omega-2A)\,. \tag{6.9}$$

Plugging into (4.2), we find

$$
\begin{aligned}
S_R &= \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\;\delta g^{MN}\delta E_{MN} = \frac{1}{4\kappa^2}\int d^4x\sqrt{-\hat{\eta}}\int d^6y\sqrt{\tilde{g}}\,e^{-2A}e^{4\Omega}\delta g^{MN}\delta E_{MN} \\
&= \frac{1}{4\kappa^2}\int d^4x\int d^6y\sqrt{\tilde{g}}\,e^{4\Omega}\Big[16(\delta A+\delta\Omega)\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell - 10\delta A\hat{\partial}^2\widetilde{\nabla}^\ell B_\ell \\
&\quad +12e^{-4A}e^{-2\Omega}\delta A\hat{\partial}^2(3\delta\Omega-2\delta A)\Big]\,. \qquad (6.10)
\end{aligned}
$$

6.2.2 5-Form Sector: All Moduli

To evaluate the 5-form sector action, we must determine the dynamic parts of E_6, as well as the corresponding parts of C_4. Since we know that the equation of motion for $\tilde{F}_5$ includes a non-dynamic "electric source" for the D3-branes, the only dynamic contributions come from $d_{10}\tilde{F}_5$ and are

$$
\delta E_6 = d_{10}(e^{-4A}\tilde{\star}\hat{\partial}^2 B_1) + e^{-4A}e^{-2\Omega}\hat{d}\hat{\star}\hat{d}b_2\wedge\tilde{\star}(\omega_2+\tilde{d}K_1)\,. \qquad (6.11)
$$

The global components of C_4 are those in the (4,0), (3,1) and (2,2) legs of C_4, and can written as $d_{10}C_4 = \tilde{F}_5$. These are

$$
C_4 = e^{4A}e^{4\Omega}\hat{\epsilon} + e^{4A}e^{4\Omega}\hat{\star}\hat{d}B_1 + b_2\wedge(\omega_2+\tilde{d}K_1)\,. \qquad (6.12)
$$

The relevant wedge products – those that ensure $\delta C_4\wedge\delta E_6$ has four legs in Minkowski space and six in the compact space – are

$$
\begin{aligned}
\delta E_6\wedge\delta C_4 &= 4(\delta A+\delta\Omega)e^{4A}e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}(e^{-4A}\tilde{\star}\hat{\partial}^2 B_1) + e^{4A}e^{4\Omega}\hat{\star}\hat{d}B_1\wedge\hat{d}(e^{-4A}\tilde{\star}\hat{\partial}^2 B_1) \\
&\quad +e^{-4A}e^{-2\Omega}b_2\wedge(\omega_2+\tilde{d}K_1)\wedge\hat{d}\hat{\star}\hat{d}b_2\wedge\tilde{\star}(\omega_2+\tilde{d}K_1)\,. \qquad (6.13)
\end{aligned}
$$

The last term in (6.13) is a pure axion term was shown in § 5.3 to contribute

$$
-\frac{3\tilde{V}}{4\kappa^2}\int e^{-4\Omega}\hat{\star}\hat{d}b_2\wedge\hat{d}b_2 \qquad (6.14)
$$

to the 5-form sector action. However, we must examine the result of other terms in (6.13), after being incorporated into (4.3). Translating the integrals of the forms into component notation gives

$$\int \delta e^{4A} e^{4\Omega} \hat{\epsilon} \wedge \tilde{d}(e^{-4A} \tilde{\star} \hat{\partial}^2 B_1) = \int d^4x \int d^6y \sqrt{\tilde{g}}\, 4e^{4\Omega}(\Omega + \delta A)(\hat{\partial}^2 \widetilde{\nabla}^\ell B_\ell - 4\widetilde{\nabla}^\ell A \hat{\partial}^2 B_\ell) \quad (6.15)$$

$$\text{and} \quad \int e^{4\Omega} \tilde{\star} \hat{\partial}^2 B_1 \wedge \hat{d} \hat{\star} \hat{d} B_1 = \int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega} \hat{\partial}^2 B^\ell \hat{\partial}^2 B_\ell \,. \quad (6.16)$$

Adding together the contributions, we see that the final expression for the 5-form sector action is

$$\begin{aligned} S_5 = &\frac{1}{4\kappa^2} \int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega} \left[4(\delta A + \delta\Omega)(\hat{\partial}^2 \widetilde{\nabla}^\ell B_\ell - 4\widetilde{\nabla}^\ell A \hat{\partial}^2 B_\ell) + \hat{\partial}^2 B^\ell \hat{\partial}^2 B_\ell \right] \\ &- \frac{3\tilde{V}}{4\kappa^2} \int e^{-4\Omega} \hat{\star} \hat{d} b_2 \wedge \hat{d} b_2 \,. \end{aligned} \quad (6.17)$$

We can immediately combine the terms of (6.10) and (6.17) and note that, after substantial cancellation, the quadratic terms for the volume modulus the axion are recovered (*c.f.* Chapter 5):

$$\begin{aligned} S_R + S_5 = &\frac{1}{4\kappa^2} \int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega} \left[3c(x) \hat{\partial}^2 c(x) + \hat{\partial}^2 B^\ell \hat{\partial}^2 B_\ell \right] \\ &- \frac{3}{4\kappa^2} \int e^{-4\Omega} \hat{\star} \hat{d} b_2 \wedge \hat{d} b_2 \,. \end{aligned} \quad (6.18)$$

The term proportional to $\hat{\partial}^2 B^\ell \hat{\partial}^2 B_\ell$ is a higher-derivative contribution than those considered in this work. What we see is that, even in the presence of mobile D3-branes, the sum of the gravity and 5-form sectors is identical to the static brane case. The mixing of the moduli and the effects of the branes will manifest instead in the contribution to the quadratic action from the background axion.

6.2.3 Brane Sector: All Moduli

In § 4.2, we saw that the linearized equation of motion for the brane position, $E_{\underline{m}}$, in the static gauge is

$$E_{\underline{m}} = 2\kappa^2 T_3 \Lambda^m_{\underline{m}} \Big[\hat{\partial}_\mu (e^{2A} e^{2\Omega} \hat{\eta}^{\mu\nu} g_{mN} \Lambda^N_B \hat{\partial}_\nu X^B) + \frac{1}{6} \hat{\partial}_a (\hat{\varepsilon}^{abcd} C_{mNPQ} \Lambda^N_B \Lambda^P_C \Lambda^Q_D \hat{\partial}_b X^B \hat{\partial}_c X^C \hat{\partial}_d X^D \Big] \tilde{\delta}^6(y, Y)\,. \tag{6.19}$$

Since $E_{\underline{m}}$ will be contracted with the first-order brane position, $Y^{\underline{m}}$, we take only terms in (6.19) up to first order. This means that, because of the $\hat{\partial}_a$ that acts on the (2,2) component of C_4, the background axion becomes first-order, and the resulting term in $E_{\underline{m}}$ will be second order. Furthermore, we once again find that contributions from the off-diagonal part of the metric will exactly those from the (4,1) component of C_4. The result is that the linearized equation of motion is

$$E_{\underline{m}} = 2\kappa^2 T_3 e^{2\Omega} \Lambda^m_{\underline{m}} \Lambda^n_{\underline{n}} \tilde{g}_{mn} \hat{\partial}^2 Y^{\underline{n}} \tilde{\delta}^6(y, Y_0)\,. \tag{6.20}$$

N.B. The delta function $\tilde{\delta}^6(y, Y)$ has been expanded using (3.96) and we have retained only the terms that keep $E_{\underline{m}}$ first-order. Thus, we find that the brane sector quadratic action is

$$S_{\rm D3} = \frac{1}{4\kappa^2} \int d^4x \int d^6y \sqrt{\tilde{g}}\, \Lambda^m_{\underline{m}} \Lambda^n_{\underline{n}}\, 2\kappa^2 T_3 e^{2\Omega} Y^{\underline{n}} \tilde{g}_{mn} \hat{\partial}^2 Y^{\underline{n}} \tilde{\delta}^6(y, Y_0)\,. \tag{6.21}$$

6.3 Total Quadratic Action Revisited

The total quadratic action is the sum of the gravity, 5-form, and brane sector quadratic actions. As noted above, there is significant cancellation between the gravity and 5-form sectors. The remaining contributions to the total quadratic action are

$$S = -\frac{3\tilde{V}}{4\kappa^2}\int d^4x\, e^{4\Omega}\hat{\partial}^\mu c(x)\hat{\partial}_\mu c(x) - \frac{3\tilde{V}}{4\kappa^2}\int e^{-4\Omega}\hat{d}b_2 \wedge \hat{\star}\hat{d}b_2$$
$$+\frac{1}{4\kappa^2}\int d^4x \int d^6y \sqrt{\tilde{g}}\, \Lambda^m_{\underline{m}}\Lambda^n_{\underline{n}}\, 2T_3\kappa^2 e^{2\Omega} Y^{\underline{n}}\tilde{g}_{mn}\hat{\partial}^2 Y^{\underline{n}}\tilde{\delta}^6(y, Y_0) - T_3\int_{D3} b_2 \wedge \hat{d}Y^{\underline{m}}\hat{d}Y^{\underline{n}}\omega_{\underline{mn}} \quad (6.22)$$

What remains is to convert the 2-form axion into a scalar axion following the procedure outlined in § 5.3, which also showed that $\omega_2 = \tilde{J}_2$ for the universal axion. Afterwards, we will perform the dimensional reduction required to generate the effective theory. However, we will want to compare this effective theory to the one derived from a Kähler potential. Let us now determine what kind of interactions are predicted by the standard Kähler potential presented in [22].

6.4 Kähler Metric

Consider the kinetic action for a set of moduli described by some Kähler potential,

$$S_{\mathcal{K}} = -\frac{1}{\kappa_4^2}\int d^4x\sqrt{-\eta}\,\mathcal{L}_{\mathcal{K}} = -\frac{1}{\kappa_4^2}\int d^4x\sqrt{-\eta}\,\eta^{\mu\nu}\partial_\mu u^A \partial_\nu u^{\bar{B}} G_{A\bar{B}}(u)\,, \tag{6.23}$$

where the Kähler metric is given by $G_{A\bar{B}} = \partial_A \partial_{\bar{B}}\mathcal{K}$. We take the ansatz for the Kähler potential in terms of the complex moduli basis fields ρ and ϕ to be

$$\mathcal{K} = -3\log\left[-i(\rho - \bar{\rho}) - \gamma k(\phi, \bar{\phi})\right]\,, \tag{6.24}$$

where $\rho = b_0 + i(c_0 + \frac{\gamma}{2}k(\phi, \bar{\phi}))$ and $k(\phi, \bar{\phi})$ is the Kähler potential of the internal space. The moduli kinetic action then has

$$\mathcal{L}_{\mathcal{K}} = \partial_\rho\partial_{\bar{\rho}}\mathcal{K}\,\partial\rho\partial\bar{\rho} + \partial_\rho\partial_{\bar{\phi}}\mathcal{K}\,\partial\rho\partial\bar{\phi} + \partial_{\bar{\rho}}\partial_\phi\mathcal{K}\,\partial\bar{\rho}\partial\phi + \partial_\phi\partial_{\bar{\phi}}\mathcal{K}\,\partial\phi\partial\bar{\phi}\,, \tag{6.25}$$

Using (6.24), each term is

$$\partial_\rho\partial_{\bar\rho}\mathcal{K}\,\partial\rho\partial\bar\rho = \frac{3}{4c_0^2}\Big((\partial b_0)^2 + (\partial c_0)^2 + \gamma\partial c_0(\partial_\phi k\,\partial\phi + \partial_{\bar\phi} k\,\partial\bar\phi) + \frac{\gamma^2}{4}(\partial_\phi k\,\partial\phi + \partial_{\bar\phi}k\,\partial\bar\phi)^2\Big) \tag{6.26}$$

$$\partial_\rho\partial_{\bar\phi}\mathcal{K}\,\partial\rho\partial\bar\phi = i\frac{3\gamma\partial_{\bar\phi}k}{4c_0^2}\Big(\partial b_0 + i\partial c_0 + i\frac{\gamma}{2}(\partial_\phi k\,\partial\phi + \partial_{\bar\phi}k\,\partial\bar\phi)\Big)\partial\bar\phi \tag{6.27}$$

$$\partial_{\bar\rho}\partial_\phi\mathcal{K}\,\partial\bar\rho\partial\phi = -i\frac{3\gamma\partial_\phi k}{4c_0^2}\Big(\partial b_0 - i\partial c_0 - i\frac{\gamma}{2}(\partial_{\bar\phi}k\,\partial\bar\phi + \partial_\phi k\,\partial\phi)\Big)\partial\phi \tag{6.28}$$

$$\partial_\phi\partial_{\bar\phi}\mathcal{K}\,\partial\phi\partial\bar\phi = \frac{3\gamma}{2c_0}\Big(\frac{\partial^2 k}{\partial\phi\partial\bar\phi} + \frac{\gamma\partial_\phi k\partial_{\bar\phi}k}{2c_0}\Big)\partial\phi\partial\bar\phi\,. \tag{6.29}$$

Adding everything together gives

$$\begin{aligned}\mathcal{L}_{\mathcal{K}} = {} & \frac{3}{4c_0^2}\Big((\partial b_0)^2 + (\partial c_0)^2\Big) + \frac{3\gamma}{2c_0}\partial_{\bar\phi}\partial_\phi k\,\partial\phi\partial\bar\phi - \frac{3i\gamma}{4c_0^2}\partial b_0(\partial_\phi k\,\partial\phi - \partial_{\bar\phi}k\,\partial\bar\phi)\\ & -\frac{3\gamma^2}{16c_0^2}(\partial_\phi k\partial_\phi k\,\partial\phi\partial\phi - 2\partial_\phi k\partial_{\bar\phi}k\,\partial\phi\partial\bar\phi + \partial_{\bar\phi}k\partial_{\bar\phi}k\,\partial\bar\phi\partial\bar\phi)\,.\end{aligned} \tag{6.30}$$

Based on the form of (6.30), what we anticipate seeing in the quadratic action are: separate quadratic terms for the scalar axion, volume modulus, and the D3-brane positions; a second-derivative of the internal Kähler potential; sets of derivatives (real, imaginary, and mixed) acting on k; a coupling between the first derivatives of k and the first derivative of the scalar axion. Let us now return to calculating the effective action. Once that is complete, we will compare the form of (6.30) to our result.

6.5 Comparison of the Effective & Kähler Actions

In order to more clearly see the role of k in the quadratic action, we will transform the real y^m coordinates into the z^I complex coordinates of the Calabi-Yau 3-fold. Recall[2] that the metric of the manifold (either Calabi-Yau or moduli space) is related to the Kähler form by

$$\tilde{J}_2 = \frac{1}{2}\tilde{J}_k{}^\ell\tilde{g}_{\ell m}dy^k\wedge dy^m = i\tilde{g}_{i\bar\jmath}\,dz^i\wedge dz^{\bar\jmath}\,, \tag{6.31}$$

[2] For additional information on Kähler manifolds and moduli spaces, see Appendix A.5

and the Kähler form is related to the internal Kähler potential by

$$\tilde{J}_2 = i\partial\bar{\partial}k(z,\bar{z})\,. \tag{6.32}$$

We can set the brane position $Y^{\underline{m}}$ to be the complex coordinate Z^I by converting to $\underline{m} = I = (i,\bar{\imath}) = (\{1,2,3\},\{\bar{1},\bar{2},\bar{3}\})$. Thus, the internal Kähler potential is implicitly dependent on x^μ through its dependence on the brane position (complex coordinates):

$$\hat{d}k_{\bar{\jmath}} = \partial_i k_{\bar{\jmath}}\,\hat{d}Z^i + \partial_{\bar{\imath}} k_{\bar{\jmath}}\,\hat{d}Z^{\bar{\jmath}}\,. \tag{6.33}$$

N.B. We have denoted the partial derivative of k with respect to $\bar{\jmath}$ as $k_{\bar{\jmath}} \equiv \partial_{\bar{\jmath}}k$.

Next, we can rewrite the sum over the embedding coordinates (after taking $\Lambda^m_{\underline{m}}\Lambda^n_{\underline{n}}\omega_{mn} = \omega_{\underline{mn}} = \tilde{J}_{\underline{mn}}$) as $\hat{d}Y^{\underline{m}}\hat{d}Y^{\underline{n}}\tilde{J}_2 = 2\hat{d}Z^i\hat{d}Z^{\bar{\jmath}}\,\tilde{J}_{i\bar{\jmath}}$. Then, using the anti-symmetry of the Kähler form, we can see that

$$\tilde{J}_{i\bar{\jmath}}\,\hat{d}Z^i \wedge \hat{d}Z^{\bar{\jmath}} = \frac{1}{2}(\tilde{J}_{i\bar{\jmath}}\,\hat{d}Z^i \wedge \hat{d}Z^{\bar{\jmath}} + \tilde{J}_{\bar{\jmath}i}\,\hat{d}Z^{\bar{\jmath}} \wedge \hat{d}Z^i)\,. \tag{6.34}$$

Relating the Kähler form to $k(Z,\bar{Z})$, and using (6.33), we find that

$$\hat{d}Y^{\underline{m}}\hat{d}Y^{\underline{n}}\,\omega_{\underline{mn}} = i(\hat{d}k_{\bar{\jmath}} \wedge \hat{d}Z^{\bar{\jmath}} - k_{\bar{\imath}\bar{\jmath}}\,\hat{d}Z^{\bar{\imath}} \wedge \hat{d}Z^{\bar{\jmath}} - \hat{d}k_j \wedge \hat{d}Z^j + k_{ij}\,\hat{d}Z^i \wedge \hat{d}Z^j)\,. \tag{6.35}$$

However, the terms proportional to $k_{\bar{\imath}\bar{\jmath}}$ and k_{ij} both vanish identically because the partial derivatives commute while the wedge products anti-commute. So, the overall contribution to the effective action from the brane-axion coupling is

$$-T_3\int_{D_3} b_2 \wedge \hat{d}Y^{\underline{m}}\hat{d}Y^{\underline{n}}\,\omega_{\underline{mm}} = iT_3\int \hat{d}b_2 \wedge (k_{\bar{\jmath}}\,\hat{d}Z^{\bar{\jmath}} - k_j\,\hat{d}Z^j) = iT_3\int \hat{d}b_2 \wedge k_I\hat{d}Z^I\,, \tag{6.36}$$

where we have denoted $k_I\hat{d}Z^I \equiv k_{\bar{\jmath}}\hat{d}Z^{\bar{\jmath}} - k_j\hat{d}Z^j$. Furthermore, we have evaluated the integral over the compact coordinates using

$$\int d^6y\sqrt{\tilde{g}}\,\tilde{\delta}^6(y,Y_0)=1\,, \tag{6.37}$$

such that $S_{\rm D3}$ is now only an integral over the Minkowski spacetime.

We now wish to convert the 2-form axion in (6.22) into a scalar axion. Consider the parts of the action containing the 2-form axion:

$$-\frac{3\tilde{V}}{4\kappa^2}\int e^{-4\Omega}\hat{d}b_2\wedge\hat{\star}\hat{d}b_2+iT_3\int\hat{d}b_2\wedge k_I\,\hat{d}Z^I\,. \tag{6.38}$$

By defining $h_3\equiv\hat{d}b_2$ – such that the condition $\hat{d}h_3=0$ is enforced by the Lagrange multiplier, $b_0(x)$ – (6.38) becomes

$$-\frac{3\tilde{V}}{2\kappa^2}\left[\int\frac{1}{2}e^{-4\Omega}h_3\wedge\hat{\star}h_3-\frac{2i\zeta}{3}\,h_3\wedge k_I\,\hat{d}Z^I+b_0\hat{d}h_3\right]\,, \tag{6.39}$$

where $\zeta=\kappa^2T_3/\tilde{V}$. The Euler-Lagrange equation equation that arises from (6.39) gives

$$h_3=e^{4\Omega}\left[\frac{2i\zeta}{3}k_I\,\hat{\star}\hat{d}Z^I-\hat{\star}\hat{d}b_0\right]\,. \tag{6.40}$$

Substituting back in for h_3 and simplifying, we find that

$$\begin{aligned}&-\frac{3\tilde{V}}{4\kappa^2}\int d^4xe^{4\Omega}\hat{\partial}^\mu b_0\hat{\partial}^\mu b_0+\frac{3\tilde{V}}{4\kappa^2}\left(\frac{2\zeta}{3}\right)^2\int d^4xe^{4\Omega}k_Ik_J\,\hat{\partial}^\mu Z^I\hat{\partial}_\mu Z^J\\&+i\frac{\tilde{V}\zeta}{\kappa^2}\int d^4xe^{4\Omega}k_I\,\hat{\partial}^\mu b_0\hat{\partial}_\mu Z^I\,.\end{aligned} \tag{6.41}$$

We can see that there is a regular quadratic axion term, a quadratic coordinate term, and a mixing term. This is precisely the form given by the ansatz for the Kähler potential in (6.30).

Converting the remaining term in (6.22) that contains the brane position requires again setting the brane position to be the complex coordinate Z and rewriting the Calabi-Yau metric in terms of holomorphic and antiholomorphic indices that anti-commute: $\Lambda^m_{\underline{m}}\Lambda^n_{\underline{n}}\tilde{g}_{mn}Y^{\underline{n}}\hat{\partial}^2Y^{\underline{m}}=2\tilde{g}_{i\bar{\jmath}}Z^i\hat{\partial}^2Z^{\bar{\jmath}}$. In doing so, the action becomes

$$
\begin{aligned}
S = &-\frac{3\tilde{V}}{4\kappa^2}\int d^4x e^{4\Omega}\left(\hat{\partial}^\mu b_0\hat{\partial}_\mu b_0 + \hat{\partial}^\mu c(x)\hat{\partial}_\mu c(x)\right) \\
&+\frac{\kappa^2 T_3}{3\tilde{V}}\int d^4x e^{4\Omega}\left[k_i k_j\,\hat{\partial}^\mu Z^i\hat{\partial}_\mu Z^i - 2k_i k_{\bar{\jmath}}\,\hat{\partial}^\mu Z^i\hat{\partial}_\mu Z^{\bar{\jmath}} + k_{\bar{\imath}}k_{\bar{\jmath}}\,\hat{\partial}^\mu Z^{\bar{\imath}}\hat{\partial}_\mu Z^{\bar{\jmath}}\right] \\
&+T_3\int d^4x e^{4\Omega}\left[-\frac{i}{2}\hat{\partial}_\mu b_0(k_i\hat{\partial}^\mu Z^i - k_{\bar{\imath}}\hat{\partial}^\mu Z^{\bar{\imath}}) + e^{-2\Omega}\tilde{g}_{i\bar{\jmath}}\,\hat{\partial}^\mu Z^i\hat{\partial}_\mu Z^{\bar{\jmath}}\right]\,.
\end{aligned}
\tag{6.42}
$$

N.B. All terms containing the internal Kähler potential, k, and its derivatives are implicitly evaluated at the brane position, $y = Y$. Note also that the quadratic actions for the volume modulus and scalar axion are exactly as they appeared in Chapter 5, thus providing an important consistency check.

We can now compare this quadratic action directly with that predicted in § 6.4 from the Kähler potential, and determine the constant γ. Equating the coefficients for the quadratic terms in either equation allows us to find that

$$
\gamma = \frac{2T_3\kappa^2}{3\tilde{V}} = \frac{2T_3\kappa_4^2}{3}. \tag{6.43}
$$

Furthermore, we can see that each type of coupling anticipated in § 6.4 has been realized. We can now conclusively state that the Kähler potential for the moduli is given by (6.24).

7 Effective Action for "Magnetic" Moduli

7.1 Magnetic Ansätze and Degrees of Freedom

Having established a theory for the moduli of type IIB SUGRA in Chapter 6 for the "electric" description, *i.e.* using the (4,0), (3,1), and (2,2) components of the 4-form potential, we now wish to confirm this result in the case when the opposite set of degrees of freedom is considered. As discussed in Chapter 3, the "magnetic" description of the moduli places the global degrees of freedom in the (0,4), (1,3), and (2,2) components of the potential. This raises certain issues regarding the gauge transformations of the potential in overlapping coordinate patches. However, these local contributions were separated out from the global ones in § 3.5. Having ensured that we are considering only global degrees of freedom, the effective action for the "magnetic" moduli can now be determined.

The relevant terms in C_4 for all moduli (the axion can now be written in its original scalar form) are those that are globally defined:

$$C_4 = -e^{2\Omega}\tilde{\star}\tilde{d}B_1^{D3} + b_0 \wedge \omega_4 - \hat{d}b_0 \wedge K_3 \,. \tag{7.1}$$

The other relevant equations can be expressed in terms of a total B_1, equal to the sum of the individual metric compensators, that obeys the constraint equation

$$\widetilde{\nabla}^\ell B_\ell = e^{-4A}e^{-2\Omega}(2\delta\Omega - 4\delta A)\,, \tag{7.2}$$

where the variations are with respect to all moduli, and – as before – all exponentials terms take their background values.

7.2 Quadratic Action Sector for Magnetic Moduli

The total action for all the moduli in the magnetic description will be given by (4.5), and does not require additional contributions. We can instead examine the dynamic equations for each modulus to try to write the contributions to each sector in a concise way. For example, the contribution to the Einstein equation from the volume modulus was given in (5.12) to be

$$\delta E_{mn} = \hat{\partial}^2 c(x)[2\tilde{g}_{mn}\widetilde{\nabla}^\ell A\widetilde{\nabla}_\ell K - 4\widetilde{\nabla}_{(m}A\widetilde{\nabla}_{n)}K - \widetilde{\nabla}_m\widetilde{\nabla}_n K + \tilde{g}_{mn}\widetilde{\nabla}^2 K - \frac{3}{2}\tilde{g}_{mn}e^{-4A} + \frac{1}{2}\tilde{g}_{mn}e^{-2\Omega}]\,.$$

Now, by defining $\hat{\partial}_\mu B^c_m \equiv -\hat{\partial}_\mu c\widetilde{\nabla}_m K$, and noting that $\delta^c\Omega = -1/2e^{2\Omega}c(x)$ and $\delta^c A = -1/4e^{4A}c(x)$, we are able to write δE_{mn} for the volume modulus in a form more reminiscent of the dynamic contribution to the Einstein equation from the D3-branes:

$$\begin{aligned}\delta E_{mn} &= 4\widetilde{\nabla}_{(m}A\hat{\partial}^2 B^c_{n)} + \widetilde{\nabla}_m\hat{\partial}^2 B^c_n - \tilde{g}_{mn}\widetilde{\nabla}^\ell\hat{\partial}^2 B^c_\ell - 2\tilde{g}_{mn}\widetilde{\nabla}^\ell A\hat{\partial}^2 B^c_\ell \\ &\quad + \tilde{g}_{mn}e^{-4A}e^{-2\Omega}\hat{\partial}^2(3\delta^c\Omega - 2\delta^c A)\,. \end{aligned} \tag{7.3}$$

The dynamic contribution from the scalar axion can be rewritten in a similar way by making use of the fact that $\hat{\partial}_\mu B^a_m \equiv \hat{\partial}_\mu b_0 B'_m \Rightarrow \widetilde{\nabla}^\ell B^a_\ell = 0$.

Let us now examine the contributions to each sector of the quadratic action from the moduli in terms of the total B_1, δA, and $\delta\Omega$.

7.2.1 Gravity Sector: All Moduli

When written as magnetic sources, the moduli only contribute dynamic terms to the Einstein equation through the (m, n) component. Therefore, we need only consider the corresponding first-order component of the metric; that is

$$\delta g^{mn} = \delta(e^{2A}\tilde{g}^{mn}) = 2\delta A e^{2A}\tilde{g}^{mn}\,. \tag{7.4}$$

The total Einstein equation for all moduli is

$$\delta E_{mn} = \hat{\partial}^2[4\widetilde{\nabla}_{(m}AB_{n)} - 2\tilde{g}_{mn}\widetilde{\nabla}^{\ell}AB_{\ell} + \widetilde{\nabla}_{(m}B_{n)}] + e^{-4A}e^{-2\Omega}\tilde{g}_{mn}\hat{\partial}^2(\delta\Omega + 2\delta A)\,. \tag{7.5}$$

Contracting δg^{mn} with δE_{mn} and substituting into the equation for S_R yields the gravity sector quadratic action for these moduli,

$$S_R = \frac{1}{4\kappa^2}\int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega}\left[16\delta A(e^{-4A}e^{-2\Omega}\hat{\partial}^2(\delta\Omega + \delta A) - \widetilde{\nabla}^{\ell}A\hat{\partial}^2 B_{\ell})\right]\,. \tag{7.6}$$

7.2.2 5-Form Sector for Magnetic Moduli

The method of writing the flux equation of motion for all moduli is similar to the one for writing the Einstein equation: we express the sum of the flux equations for each modulus in terms of a total B_1. It is important to note that the flux equation of motion is $E_6 = d_{10} \star_{10} \tilde{F}_5 +$ (sources), *i.e.* taking the (0,5), (1,4), and (2,3) components of $\tilde{F}_5$ to be the global degrees of freedom means the flux equation will only have global terms in the (4,2), (3,3), and (2,4) components. To wit,

$$\delta E_6 = -e^{4\Omega}\hat{d}\hat{\star}\hat{d}\tilde{d}(e^{4A}B_1) + e^{2\Omega}\hat{d}\hat{\star}\hat{d}b_0 \wedge \tilde{\star}\gamma_2\,, \tag{7.7}$$

where γ_2 is the harmonic 2-form described in § 5.2. The 5-form sector quadratic action is formed by wedging this flux equation of motion with the total 4-form potential, given by (7.1). This produces

$$\begin{aligned}\delta C_4 \wedge \delta E_6 = {}& e^{6\Omega}\tilde{\star}\tilde{d}B_1^{D3} \wedge \hat{d}\hat{\star}\hat{d}\tilde{d}(e^{4A}B_1) - e^{4\Omega}\tilde{\star}\tilde{d}B_1^{D3} \wedge \hat{d}\hat{\star}\hat{d}b_0 \wedge \gamma_2 - e^{4\Omega}b_0 \wedge \omega_4 \wedge \hat{d}\hat{\star}\hat{d}\tilde{d}(e^{4A}B_1)\\ &+e^{2\Omega}b_0 \wedge \omega_4 \wedge \hat{d}\hat{\star}\hat{d}b_0 \wedge \gamma_2\,.\end{aligned} \tag{7.8}$$

We can immediately recognize the regular quadratic term for the scalar axion as the last term above. Furthermore, we know that the last term on the first line of (7.8) will vanish after integration by parts over the compact space because ω_4 is harmonic. This leaves only the first two terms to consider. Using (A.3.12), the second term contributes

$$-\frac{1}{4\kappa^2}\int e^{4\Omega}\hat{d}\hat{\star}\hat{d}b_0 \wedge \tilde{d}B_1^{D3} \wedge \tilde{\star}\gamma_2 \tag{7.9}$$

to the 5-form sector action. Since γ_2 is harmonic, so too is $\tilde{\star}\gamma_2$. After integration by parts, the above term vanishes because $\tilde{d}\tilde{\star}\gamma_2 = 0$.

The non-vanishing terms in the 5-form sector are now

$$S_5 = -\frac{1}{4\kappa^2}\int e^{4A}e^{6\Omega}\tilde{d}\tilde{\star}\tilde{d}B_1^{D3} \wedge \hat{d}\hat{\star}\hat{d}B_1 - \frac{3\tilde{V}}{4\kappa^2}\int d^4x\, e^{4\Omega}\hat{\partial}_\mu b(x)\hat{\partial}^\mu b(x)\,. \tag{7.10}$$

In order to simplify the mixing term, it is useful to recall the solution for the metric compensator for the D3-branes, given by (3.51):

$$\widetilde{\nabla}^2 B_\ell^{D3} = 2\kappa^2 T_3 e^{-2\Omega}\tilde{g}_{\ell m}\Lambda_{\underline{n}}^{m} Y^{\underline{n}}\tilde{\delta}^6(y, Y_0)\,.$$

Using the definition of the Hodge-deRahm operator, and noting that $\Delta\alpha_1 = -\widetilde{\nabla}^2\alpha_1$ for any 1-form α_1 on the compact manifold, we find that

$$-\tilde{d}\tilde{\star}\tilde{d}B_1^{D3} = e^{-2\Omega}\tilde{\star}(\tilde{d}\delta_Y e^{-4A} - 2\kappa^2 T_3\Lambda_1 Y\tilde{\delta}^6(y, Y_0)\,, \tag{7.11}$$

where Λ_1 is a 1-form with respect to the y^m coordinates, and δ_Y denotes the variation with respect to the mobile D3-branes. Using this relation in the expression for the 5-form sector action, and converting the integral to component notation, we find that

$$\begin{aligned} S_5 = {} & \frac{1}{4\kappa^2}\int d^4x \int d^6y\sqrt{\tilde{g}}\, e^{4A}e^{4\Omega}\hat{\partial}^2 B_\ell\Big(\widetilde{\nabla}^\ell\delta_Y e^{-4A} - 2\kappa^2 T_3\Lambda_{\underline{m}}^{\ell} Y^{\underline{m}}\tilde{\delta}^6(y, Y_0)\Big) \\ & -\frac{3\tilde{V}}{4\kappa^2}\int d^4x\, e^{4\Omega}\hat{\partial}_\mu b(x)\hat{\partial}^\mu b(x)\,. \end{aligned} \tag{7.12}$$

7.2.3 Brane Sector for Magnetic Moduli

The final contribution to the total quadratic action comes from contracting the equation of motion for the brane position with the first-order brane position. Once again, we examine (2.11) to determine the first-order terms. However, unlike the result from § 6.2, our choice of which components of C_4 contain the relevant degrees of freedom means that there is no longer cancellation between the off-diagonal part of the metric and the potential. Instead, we find that

$$E_{\underline{m}} = T_3 \int d^6y \sqrt{\tilde{g}}\, \Lambda^m_{\underline{m}} (e^{4A} e^{4\Omega} \hat{\partial}^2 B_m + e^{2\Omega} \tilde{g}_{mn} \Lambda^n_{\underline{n}} \hat{\partial}^2 Y^{\underline{n}}) \tilde{\delta}^6(y, Y_0)\,. \tag{7.13}$$

This means that the brane sector quadratic action is

$$S_{\mathrm{D3}} = \frac{1}{4\kappa^2} \int d^4x \int d^6y \sqrt{\tilde{g}}\, \Lambda^m_{\underline{m}} 2T_3 \kappa^2 Y^{\underline{m}} (e^{4A} e^{4\Omega} \hat{\partial}^2 B_m + e^{2\Omega} \tilde{g}_{mn} \Lambda^n_{\underline{n}} \hat{\partial}^2 Y^{\underline{n}}) \tilde{\delta}^6(y, Y_0)\,. \tag{7.14}$$

7.3 Total Quadratic Action for Magnetic Moduli

In adding each contribution to the total quadratic action, we can see that the term in (7.14) proportional to $\hat{\partial}^2 B_m$ is exactly cancelled by the similar term in (7.12). We are then left with

$$\begin{aligned} S = \frac{1}{4\kappa^2} \int d^4x \int d^6y \sqrt{\tilde{g}}\, e^{4\Omega} \Big[& 16 e^{-4A} e^{-2\Omega} \delta A \hat{\partial}^2 (\delta A + \delta\Omega) - 16 \delta A \widetilde{\nabla}^\ell \hat{\partial}^2 B_\ell + e^{4A} \hat{\partial}^2 B_\ell \widetilde{\nabla}^\ell \delta_Y e^{-4A} \\ & + 2\kappa^2 T_3 e^{-2\Omega} \Lambda^m_{\underline{m}} \Lambda^n_{\underline{n}} \tilde{g}_{mn} Y^{\underline{m}} \hat{\partial}^2 Y^{\underline{n}} \tilde{\delta}^6(y, Y_0) \Big] - \frac{3\tilde{V}}{4\kappa^2} \int d^4x\, e^{4\Omega} \hat{\partial}_\mu b(x) \hat{\partial}^\mu b(x)\,. \end{aligned} \tag{7.15}$$

We can now write the variations of the warp and Weyl factors in terms of each of the moduli, such that

$$\delta A = -\frac{1}{4} e^{4A} (c(x) + \delta_Y e^{-4A}) \quad \text{and} \quad \delta\Omega = -\frac{1}{2} e^{2\Omega} c(x)\,. \tag{7.16}$$

By writing out the moduli explicitly like this, it is clear which terms will depend on the internal coordinates and which will simply be functions of spacetime. After using the constraint in (7.2) and substituting for δA, we find that the term $-16\delta A\widetilde{\nabla}^\ell A\hat{\partial}^2 B_\ell$ gives

$$-e^{-2\Omega}\hat{\partial}^2 c(x)(2\delta\Omega - 4\delta A) - e^{4A}\hat{\partial}^2 B_\ell \widetilde{\nabla}^\ell \delta_Y e^{-4A} - e^{4A}\delta_Y e^{-4A}\hat{\partial}^2\widetilde{\nabla}^\ell B_\ell\,. \qquad (7.17)$$

Thus, the term proportional to $e^{4A}\widetilde{\nabla}^\ell\delta_Y e^{-4A}$ that arises from the 5-form sector action is exactly cancelled.

After additional simplification, we also recover the quadratic term for the volume modulus and a coupling between the D3-brane position and volume modulus that is

$$-6e^{-2\Omega}\delta\Omega\hat{\partial}^2\delta_Y e^{-4A}\,. \qquad (7.18)$$

Since $\delta\Omega$ is a function of spacetime only, and since (3.54) tells us that $\delta_Y e^{-4A} \propto -Y^{\underline{m}}(x)\partial_{\underline{m}}G(y,Y) = Y^{\underline{m}}(x)\widetilde{\nabla}^\ell G_{\underline{m}\ell}(y,Y) = \widetilde{\nabla}^\ell(Y^{\underline{m}}(x)G_{\underline{m}\ell}(y,Y))$, (7.18) is proportional to the integral of a total derivative and therefore vanishes on the compact Calabi-Yau. Hence, the total quadratic action for the magnetic description of the moduli is

$$\begin{aligned} S = &-\frac{3\tilde{V}}{4\kappa^2}\int d^4x\int d^6y\sqrt{\tilde{g}}\,e^{4\Omega}\Big[3\hat{\partial}_\mu c(x)\hat{\partial}^\mu c(x) + 3\hat{\partial}_\mu b(x)\hat{\partial}^\mu b(x)\Big] \\ &-\frac{T_3}{2}\int d^4x\int d^6y\sqrt{\tilde{g}}\,e^{2\Omega}\Lambda^m_{\underline{m}}\Lambda^n_{\underline{n}}\tilde{g}_{mn}\hat{\partial}_\mu Y^{\underline{m}}\hat{\partial}^\mu Y^{\underline{n}}\tilde{\delta}^6(y,Y_0)\,. \end{aligned} \qquad (7.19)$$

Comparing this to the corresponding action for the electric description of the moduli in (6.22), there is complete agreement amongst the kinetic terms, however a key ingredient missing: coupling between the axion and the mobile D3-branes.

8 Discussion

We have used the conventional description of moduli in type IIB SUGRA to investigate the effects of mobile D3-branes in flux compactifications. We constructed the Einstein equations and separated from them the dynamic contributions and constraint equations. We then solved the constraint equations and expressed the warping of the compact manifold in terms of the position of a D3-brane. Owing to the non-trivial nature of the Bianchi identity for the flux, the local degrees of freedom were isolated at linear order. By using a related (but equivalent) expression for the flux ansatz, the second-order local terms were successfully written in closed form. This required adding terms to the flux ansatz that were proportional to the equations of motion, and so vanished on shell. Classically, this meant that the descriptions were equivalent. We then showed that the local terms do not contribute to the equations of motion to the order considered. Furthermore, we recovered the non-trivial cancellation in the equations of motion at the level of the background, an important consistency check with the equation of motion from the opposite characterization of the D3-branes.

When considered on its own, the effective action built from the first-order position of the D3-branes showed no dependence on the warp factor, although the cancellation of the warping was highly non-trivial. Regardless, the brane position remained a modulus of the theory. After reviewing the formulation of effective actions for similar moduli, the inclusion of additional moduli that could couple with the D3-branes was considered. A flux ansatz was then proposed that described all the moduli using similar degrees of freedom. This required establishing an equivalent theory for the universal axion in terms of a 2-form.

With these steps completed, the quadratic action for all moduli was determined. An important difference was the inclusion of a term that *was not* captured in a linearized equation of motion. The dimensional reduction of the quadratic action again displayed non-trivial cancellations of the warp factor. In fact, we saw that the effective action was precisely of the form suggested by the kinetic action for a set of complex moduli described by a Kähler potential. The brane position was shown to corresponded to the complex coordinates of the Calabi-Yau 3-fold and entered the moduli space Kähler potential through the internal Kähler potential of the manifold, and complex moduli basis fields.

Owing to the self-duality of the flux in type IIB SUGRA, choosing the global degrees of freedom to be in any set of independent components should yield the same result. To confirm this, we went on to describe the D3-branes as magnetic sources, and – after proper consideration of the local source terms – endeavoured to reproduce the effective action in (6.42). We were able to produce the correct quadratic action terms for each of the moduli individually, however the mixing terms predicted in § 6.4 did not appear. So far, attempts to cast the D3-brane as magnetic sources have been unsuccessful.

We anticipate that the required mixing term between the axion and the D3-branes should arise from terms in the D3-brane equation of motion, as presented in § 3.7. Although we asserted that $\partial\sigma^{\not{b}}/\partial Y^{\underline{m}} = -\Lambda^{\not{b}}_{\underline{m}} + \mathcal{O}(2)$, it could be that the Synge's world-function, $\sigma^{\not{b}}$, is able to take a non-zero background value. This would mean that terms that originally appeared to be second order would, in fact, contribute at first order, and thus be relevant to the quadratic action. Work continues on examining this possibility in the hopes that these issues can be resolved and the results obtained in Chapter 6 can be confirmed using the magnetic degrees of freedom.

A Appendices

A.1 The Ricci Tensor

Here we will include the details of the derivations for equations (3.1), (3.2), and (3.3) from §3.1. We start by examining the particular components of the Ricci tensor: $R_{\mu\nu}$, $R_{\mu m}$, and R_{mn}. In terms of 10-dimensional indices $\{A, B, \ldots\}$, these are

$$R_{\mu\nu} = \partial_A \Gamma^A_{\nu\mu} - \hat{\partial}_\nu \Gamma^A_{A\mu} + \Gamma^A_{AB}\Gamma^B_{\nu\mu} - \Gamma^A_{\nu B}\Gamma^B_{A\mu} \tag{A.1.1}$$

$$R_{\mu m} = \partial_A \Gamma^A_{m\mu} - \tilde{\partial}_m \Gamma^A_{A\mu} + \Gamma^A_{AB}\Gamma^B_{m\mu} - \Gamma^A_{mB}\Gamma^B_{A\mu} \tag{A.1.2}$$

$$R_{mn} = \partial_A \Gamma^A_{mn} - \tilde{\partial}_n \Gamma^A_{Am} + \Gamma^A_{AB}\Gamma^B_{nm} - \Gamma^A_{nB}\Gamma^B_{Am}\,. \tag{A.1.3}$$

Using the metric in (1.23), we can determine the relevant terms to first order. In particular, the terms to be included in $R_{\mu\nu}$, by (A.1.1), are:

$$\hat{\partial}_\lambda \Gamma^\lambda_{\nu\mu} = (2\hat{\partial}_\mu\hat{\partial}_\nu - \hat{\eta}_{\mu\nu}\hat{\partial}^2)(A+\Omega) + e^{4A}e^{2\Omega}\hat{\partial}^2 B^\ell \tilde{\partial}_\ell A \hat{\eta}_{\mu\nu} \tag{A.1.4}$$

$$\tilde{\partial}_\ell \Gamma^\ell_{\nu\mu} = e^{4A}e^{2\Omega}(4\tilde{\partial}_\ell A \hat{\partial}_\nu\hat{\partial}_\mu B^\ell + \hat{\partial}_\nu\hat{\partial}_\mu\tilde{\partial}_\ell B^\ell - 4\tilde{\partial}_\ell A\tilde{\partial}^\ell A\hat{\eta}_{\mu\nu} - \tilde{\partial}^2 A\hat{\eta}_{\mu\nu}) \tag{A.1.5}$$

$$\hat{\partial}_\nu \Gamma^\lambda_{\lambda\mu} = 4\hat{\partial}_\nu\hat{\partial}_\mu(A+\Omega) + e^{4A}e^{2\Omega}\tilde{\partial}_\ell A\hat{\partial}_\nu\hat{\partial}_\mu B^\ell \tag{A.1.6}$$

$$\hat{\partial}_\nu \Gamma^\ell_{\ell\mu} = -6\hat{\partial}_\nu\hat{\partial}_\mu A - e^{4A}e^{2\Omega}\tilde{\partial}_\ell A\hat{\partial}_\nu\hat{\partial}_\mu B^\ell \tag{A.1.7}$$

$$\Gamma^\rho_{\rho\ell}\Gamma^\ell_{\nu\mu} = 4e^{4A}e^{2\Omega}(\tilde{\partial}_\ell A\hat{\partial}_\nu\hat{\partial}_\mu B^\ell - \tilde{\partial}_\ell A\tilde{\partial}^\ell A\hat{\eta}_{\mu\nu}) \tag{A.1.8}$$

$$\Gamma^m_{m\ell}\Gamma^\ell_{\nu\mu} = e^{4A}e^{2\Omega}(\tilde{\Gamma}^m_{m\ell}\hat{\partial}_\nu\hat{\partial}_\mu B^\ell - \tilde{\Gamma}^m_{m\ell}\tilde{\partial}^\ell A\hat{\eta}_{\mu\nu} - 6\tilde{\partial}_\ell A\hat{\partial}_\nu\hat{\partial}_\mu B^\ell + 6\tilde{\partial}_\ell A\tilde{\partial}^\ell A\hat{\eta}_{\mu\nu}) \tag{A.1.9}$$

$$\Gamma^\lambda_{\nu\ell}\Gamma^\ell_{\lambda\mu} = e^{4A}e^{2\Omega}(\tilde{\partial}_\ell A\hat{\partial}_\nu\hat{\partial}_\mu B^\ell - \tilde{\partial}_\ell A\tilde{\partial}^\ell A\hat{\eta}_{\mu\nu}) \tag{A.1.10}$$

$$\Gamma^m_{\nu\rho}\Gamma^\rho_{m\mu} = e^{4A}e^{2\Omega}(\tilde{\partial}_m A\hat{\partial}_\nu\hat{\partial}_\mu B^m - \tilde{\partial}_m A\tilde{\partial}^m A\hat{\eta}_{\mu\nu})\,. \tag{A.1.11}$$

N.B. Terms higher than first-order have been excluded; the Christoffel symbols on the Calabi-Yau, $\tilde{\Gamma}$, contribute at the background level. Plugging (A.1.4) - (A.1.11) into (A.1.1) gives $R_{\mu\nu}$ as in (3.1).

Next, we calculate $R_{\mu m}$. It is convenient to define the first-order term $\Phi^m_{\mu n} \equiv e^{4A}e^{2\Omega}\tilde{g}^{\ell m}(\tilde{\partial}_\ell A\hat{\partial}_\mu B_n + \hat{\partial}_\mu\tilde{\partial}_{[\ell}B_{n]})$, since it is common to many of the following calculations. Thus, the contributions to $R_{\mu m}$ from (A.1.2) are:

$$\hat{\partial}_\nu\Gamma^\nu_{\mu m} = \hat{\partial}_\mu\tilde{\partial}_m A \tag{A.1.12}$$

$$\tilde{\partial}_\ell\Gamma^\ell_{\mu m} = -\tilde{\partial}_m\hat{\partial}_\mu A - \tilde{\partial}_\ell\Phi^\ell_{\mu m} \tag{A.1.13}$$

$$\tilde{\partial}_m\Gamma^\nu_{\mu\nu} = 4\tilde{\partial}_m\hat{\partial}_\mu A + e^{4A}e^{2\Omega}\left(4\tilde{\partial}_m A\tilde{\partial}_\ell A\hat{\partial}_\mu B^\ell + \tilde{\partial}_m(\tilde{\partial}_\ell A\hat{\partial}_\mu B^\ell)\right) \tag{A.1.14}$$

$$\tilde{\partial}_m\Gamma^\ell_{\ell\mu} = -6\tilde{\partial}_m\hat{\partial}_\mu A - e^{4A}e^{2\Omega}\left(4\tilde{\partial}_m A\tilde{\partial}_\ell A\hat{\partial}_\mu B^\ell + \tilde{\partial}_m(\tilde{\partial}_\ell A\hat{\partial}_\mu B^\ell)\right) . \tag{A.1.15}$$

It is convenient to note the sum of (A.1.14) and (A.1.15) at this point, as it will reduce potential confusion later on

$$\tilde{\partial}_m\Gamma^\nu_{\nu\mu} + \tilde{\partial}_m\Gamma^\ell_{\ell\mu} = -2\tilde{\partial}_m\hat{\partial}_\mu A\,. \tag{A.1.16}$$

There are additional terms that benefit from summing before plugging into (A.1.2), and they will be calculated as such in the following. Continuing with the contributions to $R_{\mu m}$, we find that

$$(\Gamma^\nu_{\nu\lambda} + \Gamma^\ell_{\ell\lambda})\Gamma^\lambda_{m\mu} = \tilde{\partial}_m A\hat{\partial}_\mu(4A - 2\Omega) \tag{A.1.17}$$

$$(\Gamma^\nu_{\nu n} + \Gamma^\ell_{\ell n})\Gamma^n_{m\mu} = 2\tilde{\partial}_m A\hat{\partial}_\mu A + 2\tilde{\partial}_n A\Phi^n_{\mu m} - \tilde{\Gamma}^\ell_{\ell n}\Phi^n_{\mu m} \tag{A.1.18}$$

$$\Gamma^\nu_{m\lambda}\Gamma^\lambda_{\nu\mu} = 4\tilde{\partial}_m A\hat{\partial}_\mu(A + \Omega) + e^{4A}e^{2\Omega}\tilde{\partial}_m A\tilde{\partial}^\ell A\hat{\partial}_\mu B_\ell \tag{A.1.19}$$

$$\begin{aligned}\Gamma^\nu_{m\ell}\Gamma^\ell_{\nu\mu} = &-e^{4A}e^{2\Omega}\tilde{\partial}^\ell A[\hat{\partial}_\mu\widetilde{\nabla}_{(m}B_{\ell)} + 4\tilde{\partial}_{(m}A\hat{\partial}_{|\mu|}B_{\ell)} - \tilde{g}_{m\ell}\hat{\partial}^\nu B_k\tilde{\partial}^k A \\ &+ e^{-4A}e^{-2\Omega}\hat{\partial}_\mu A\tilde{g}_{m\ell}]\end{aligned} \tag{A.1.20}$$

$$\Gamma^n_{m\lambda}\Gamma^\lambda_{n\mu} = -\tilde{\partial}_m A\hat{\partial}_\mu A - \tilde{\partial}_n A\Phi^n_{\mu m} \tag{A.1.21}$$

$$\Gamma^n_{m\ell}\Gamma^\ell_{n\mu} = -\tilde{\Gamma}^n_{m\ell}\Phi^\ell_{\mu n} + 6\hat{\partial}_\mu A\tilde{\partial}_m A + \tilde{\partial}_\ell A\Phi^\ell_{\mu m} + \tilde{\partial}_m A\Phi^\ell_{\mu\ell} - \tilde{\partial}^n A\tilde{g}_{m\ell}\Phi^\ell_{\mu n}\,. \tag{A.1.22}$$

Plugging (A.1.12) - (A.1.22) into (A.1.2) gives $R_{\mu m}$ as in (3.2).

Finally, the terms that contribute to R_{mn} from (A.1.3) are:

$$\hat{\partial}_\mu \Gamma^\mu_{mn} = \hat{\partial}^2 \widetilde{\nabla}_{(m} B_{n)} + 2\tilde{\partial}_{(m} A \hat{\partial}^2 B_{n)} - \hat{\partial}^2 B_\ell \tilde{\partial}^\ell A \tilde{g}_{mn} \tag{A.1.23}$$

$$\tilde{\partial}_\ell \Gamma^\ell_{mn} = \tilde{\partial}_\ell \tilde{\Gamma}^\ell_{mn} - 2\tilde{\partial}_m \tilde{\partial}_n A + \tilde{\partial}_\ell (\tilde{g}_{mn} \tilde{\partial}^\ell A) \tag{A.1.24}$$

$$\tilde{\partial}_n \Gamma^\mu_{\mu m} = 4\tilde{\partial}_n \tilde{\partial}_m A \tag{A.1.25}$$

$$\tilde{\partial}_n \Gamma^\ell_{\ell m} = \tilde{\partial}_n \tilde{\Gamma}^\ell_{\ell m} - 6\tilde{\partial}_n \tilde{\partial}_m A \tag{A.1.26}$$

$$\begin{aligned}(\Gamma^\mu_{\mu k} + \Gamma^\ell_{\ell k})\Gamma^k_{nm} = {} & \tilde{\Gamma}^\ell_{\ell k} \tilde{\Gamma}^k_{nm} - \tilde{\partial}_m A \tilde{\Gamma}^\ell_{\ell n} - \tilde{\partial}_n A \tilde{\Gamma}^\ell_{\ell m} + \tilde{g}_{nm} \tilde{\partial}^k A \tilde{\Gamma}^\ell_{\ell k} - 2\tilde{\partial}_k A \tilde{\Gamma}^\ell_{\ell k} + 4\tilde{\partial}_n A \tilde{\partial}_m A \\ & - 2\tilde{\partial}_\ell A \tilde{\partial}^\ell A \tilde{g}_{mn}\end{aligned} \tag{A.1.27}$$

$$\Gamma^\mu_{n\nu} \Gamma^\nu_{\ell m} = 4\tilde{\partial}_n A \tilde{\partial}_m A \tag{A.1.28}$$

$$\begin{aligned}\Gamma^\ell_{nk} \Gamma^k_{\ell m} = {} & \tilde{\Gamma}^\ell_{nk} \tilde{\Gamma}^k_{\ell m} - 2(\tilde{\partial}_{(m} A \tilde{\Gamma}^\ell_{n)\ell} + \tilde{\partial}_\ell A \tilde{\Gamma}^\ell_{nm}) + \tilde{\partial}^k A (\tilde{g}_{\ell m} \tilde{\Gamma}^\ell_{nk} + \tilde{g}_{\ell n} \tilde{\Gamma}^\ell_{mk}) \\ & + 8\tilde{\partial}_m A \tilde{\partial}_n A - 2\tilde{\partial}^\ell A \tilde{\partial}_\ell A \tilde{g}_{mn} \,.\end{aligned} \tag{A.1.29}$$

Likewise, plugging (A.1.23) - (A.1.29) into (A.1.3) yields R_{mn} as in (3.3). The the Ricci tensor on the Calabi-Yau, $\tilde{R}_{mn}$, is defined in terms of the Christoffel symbols there: $\tilde{R}_{mn} = \tilde{\partial}_\ell \tilde{\Gamma}^\ell_{mn} - \tilde{\partial}_n \tilde{\Gamma}^\ell_{\ell m} + \tilde{\Gamma}^\ell_{\ell k} \tilde{\Gamma}^k_{nm} - \tilde{\Gamma}^\ell_{nk} \tilde{\Gamma}^k_{\ell k} = 0$.

A.2 Delta Distributions, Parallel Propagators, and Synge's World-Function in Curved Space

An essential part of solving the constraint equations in Chapter 3 is understanding the role that the geometry of the compact manifold plays in the equations. Our naïve treatment of vectors, delta functions, and Green's functions fails to capture some of the subtleties involved in extra dimensions. For this reason, we will review the portions of [32] that provide the mathematical groundwork for a proper treatment of bi-tensors, *i.e.* tensorial functions of two points in spacetime.

First, we introduce the invariant, n-dimensional Dirac distribution $\delta^n(x, x')$ via the definition

$$\int_{\mathcal{M}} d^n x \sqrt{-g}\, f(x) \delta^n(x, x') = f(x') \,. \tag{A.2.1}$$

The Dirac distribution is symmetric in its arguments, so that taking $\mathcal{M} \to \mathcal{M}'$ and $f(x) \to f(x')$ in (A.2.1) results in $f(x') \to f(x)$.

Next, we consider some parallel propagator, $\Lambda^{\alpha}{}_{\alpha'}(x, x')$. $\Lambda^{\alpha}{}_{\alpha'}$ is used to parallel transport any vector from x to x' along the geodesic, β, that links these two points. The inverse operation, parallel transporting a vector from x' to x along β, is performed by $\Lambda^{\alpha'}{}_{\alpha}$. The combination of the invariant Dirac distribution and $\Lambda^{\alpha}{}_{\alpha'}$ can be shown to satisfy the identities:

$$\nabla_{\alpha}(\Lambda^{\alpha}{}_{\beta'}(x, x')\delta^n(x, x')) = -\partial_{\beta'}\delta^n(x, x') \tag{A.2.2}$$

$$\nabla_{\beta'}(\Lambda^{\alpha}{}_{\beta'}(x, x')\delta^n(x, x')) = -\partial_{\alpha}\delta^n(x, x') \,. \tag{A.2.3}$$

Synge's world-function is a scalar function of two points on a curved manifold and is defined by

$$\sigma(x, x') = \frac{1}{2}(\lambda_1 - \lambda_0)\int_{\lambda_0}^{\lambda_1} g_{\mu\nu}(z)t^{\mu}t^{\nu}d\lambda \,, \tag{A.2.4}$$

where the tangent vector, $t^{\mu} = dz^{\mu}/d\lambda$, obeys the geodesic equation

$$\frac{Dt^{\mu}}{d\lambda} = 0 \,. \tag{A.2.5}$$

We take $z^{\mu}(\lambda)$ to be the set of relations that describes the unique geodesic, β, connecting the two points. We can see that (A.2.4) describes σ to numerically be half the squared geodesic distance between x and x'.

Synge's world-function can be used to create a dual vector with respect to either sets of coordinates by taking its partial derivative, $\sigma_{\alpha} \equiv \partial\sigma/\partial x^{\alpha}$. By taking repeated derivatives, we can create a bi-tensor of any rank. However, the most useful quantity for our purposes is the single derivative, σ^{α}, since this corresponds to a vector with respect to x^{α} that is tangent to β. We can then use parallel propagators to evaluate this vector at the x' position. For example, we can express σ^{α} in terms of $\sigma^{\alpha'}$ via

$$\sigma^{\alpha} = -\Lambda^{\alpha}_{\alpha'}\sigma^{\alpha'} \,. \tag{A.2.6}$$

This scenario is illustrated in the context of the first-order perturbation of the D3-brane postion by Figure A.1.

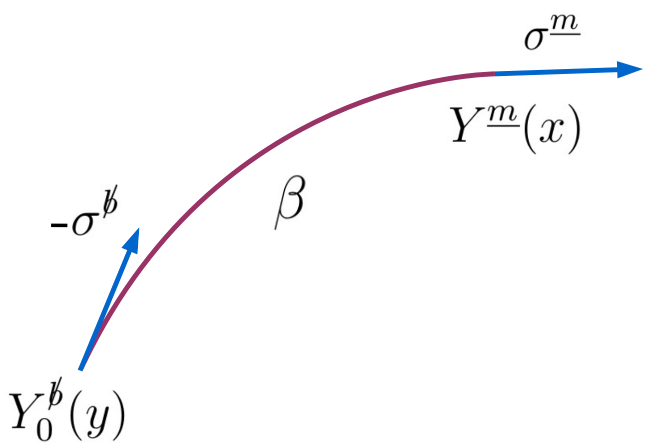


Figure A.1: *The geodesic β connects the stationary brane position, $Y_0^{\not{p}}(y)$, to the perturbed positon, $Y^{\underline{m}}(x)$. Derivatives of σ produce dual vectors with respect to one set of coordiantes.*

In the coincidence limit of $x \to x'$, we find that the parallel propagator collapses to a Dirac delta,

$$[\Lambda^{\alpha}_{\alpha'}] = \delta^{\alpha}_{\alpha'} \,, \tag{A.2.7}$$

its first derivatives vanish,

$$[\nabla_{\gamma}\Lambda^{\alpha}_{\alpha'}] = [\nabla_{\gamma'}\Lambda^{\alpha}_{\alpha'}] = 0 \,, \tag{A.2.8}$$

and its second derivatives evaluate to the Ricci tensor,

$$\left[\nabla_\delta \nabla_\gamma \Lambda^\alpha_{\beta'}\right] = \left[\nabla_\delta \nabla_{\gamma'} \Lambda^\alpha_{\beta'}\right] = -\frac{1}{2} R^{\alpha'}{}_{\beta'\gamma'\delta'} \,, \tag{A.2.9}$$

$$\left[\nabla_{\delta'} \nabla_\gamma \Lambda^\alpha_{\beta'}\right] = \left[\nabla_{\delta'} \nabla_{\gamma'} \Lambda^\alpha_{\beta'}\right] = +\frac{1}{2} R^{\alpha'}{}_{\beta'\gamma'\delta'} \,. \tag{A.2.10}$$

Finally, in the near-coincidence limit, we are able to expand any tensor at x in terms of Synge's world-function and the tensor's value at x' via

$$A_{\alpha\beta}(x) = \Lambda^\alpha_{\alpha'} \Lambda^\beta_{\beta'} (A_{\alpha'\beta'}(x') - \nabla_{\gamma'} A_{\alpha'\beta'}(x') \sigma^{\gamma'} + \frac{1}{2} \nabla_{\delta'} \nabla_{\gamma'} A_{\alpha'\beta'}(x') \sigma^{\gamma'} \sigma^{\delta'} + \ldots) \,, \tag{A.2.11}$$

where the dots represent terms of higher order in σ.

Consider some electromagnetic vector potential, A_α, in a curved, 4D spacetime. In the Lorenz gauge, A_α satisfies the wave equation

$$\Box A^\alpha - R^\alpha{}_\beta A^\beta = -4\pi j^\alpha \,, \tag{A.2.12}$$

where $R^\alpha{}_\beta$ is the Ricci tensor and j^α is the current density. We can write the generic solution to (A.2.12) as

$$A^\alpha(x) = \int d^4x' \sqrt{-g'}\, G^\alpha{}_{\beta'}(x, x') j^{\beta'}(x') \,, \tag{A.2.13}$$

where the bi-tensor Green's function $G^\alpha{}_{\beta'}(x, x')$ satisfies

$$\Box G^\alpha{}_{\beta'}(x, x') - R^\alpha{}_\beta G^\beta{}_{\beta'}(x, x') = -4\pi \Lambda^\alpha{}_{\beta'}(x, x') \delta^4(x, x') \,. \tag{A.2.14}$$

The conventional solutions to (A.2.14) involve defining the Green's function in two causal regions: the retarded bi-tensor Green's function, $G_+{}^\alpha{}_{\beta'}(x, x')$, and the advanced bi-tensor Green's function,

$G_-{}^{\alpha}{}_{\beta'}(x,x')$. In the case of a Ricci-flat manifold, the Green's functions satisfy

$$\nabla_\alpha G_\pm{}^{\alpha}{}_{\beta'}(x,x') = -\partial_{\beta'} G_\pm(x,x')\,, \tag{A.2.15}$$

where $G_\pm(x,x')$ are the associated scalar Green's functions. We will find that this relation is crucial to solving the constraint equations for the D3-brane system.

A.3 Introduction to Differential Forms

Differential forms are a powerful notational tool used to provide coordinate-independent descriptions of tensors with many indices on higher dimensional manifolds. By generalizing concepts such as scalars and vector fields to zero- and one-forms, respectively, we can use form notation to perform calculations that would be otherwise hampered by component considerations. Of particular use are the concepts of exterior differentiation and Hodge duals, which naturally encode the metric. Furthermore, the skew symmetric properties of forms can be used to produce the simplifications that are used throughout this work.

To compile a list of useful identities, we will follow the developments outlined in [9] and [33]. Then, we provide an example of the form notation in the context of four-dimensional electromagnetism.

We define a p-form α_p as a totally skew-symmetric covariant tensor of rank p. Thus, a zero-form is a scalar, while a one-form is a vector. The wedge product of a p-form α_p and a q-form β_q is a $(p+q)$-form given by

$$\left(\alpha_p \wedge \beta_q\right)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\alpha_{[\mu_1\ldots\mu_p}\beta_{\mu_{p+1}\ldots\mu_{p+q}]}\,. \tag{A.3.1}$$

The wedge product has the property that

$$\alpha_p \wedge \beta_p = (-1)^{pq}\beta_q \wedge \alpha_p\,. \tag{A.3.2}$$

The space of p-forms at x is a vector space and has a basis

$$\{dx^{m_1} \wedge dx^{m_2} \wedge \ldots \wedge dx^{m_p}\}, \qquad m_1 < m_2 < \ldots < m_p\,. \tag{A.3.3}$$

Thus, the components of a p-form, α_p are given by

$$\alpha_p = \frac{1}{p!}\,\alpha_{m_1 m_2 \ldots m_p} dx^{m_1} \wedge dx^{m_2} \wedge \ldots \wedge dx^{m_p}\,, \tag{A.3.4}$$

with $\alpha_{m_1 m_2 \ldots m_p}$ skew symmetric.

Exterior differentiation on a form is essentially taking the curl of a skew symmetric tensor and maps a p-form to a $(p+1)$-form

$$\big(d\alpha_p\big)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1}\alpha_{\mu_2 \ldots \mu_{p+1}]}\,. \tag{A.3.5}$$

Crucially, due to the skew symmetry, $d\alpha_p$ is covariant because the Christoffel symbols cancel. This means that the exterior derivative of a form is covariant *regardless* of the metric. Another important consequence of the skew symmetry is that

$$d\big(d\alpha_p\big) = 0\,. \tag{A.3.6}$$

By defining $\mathcal{M}_p$ to be a p-dimensional manifold with a boundary $\partial\mathcal{M}_p$, we can restate Stokes' theorem for a $(p-1)$-form in a coordinate-invariant way, *i.e.* that

$$\int_{\mathcal{M}_p} d\alpha_{p-1} = \int_{\partial\mathcal{M}_p} \alpha_{p-1}\,. \tag{A.3.7}$$

The Hodge star $\star$ in an d-dimensional space is a map from p-forms to $(n-p)$-forms and is defined by its action on the basis elements

$$\star(dx^{m_1} \wedge \ldots \wedge dx^{m_p}) = \frac{1}{(n-p)!}\sqrt{g}\,g^{m_1 k_1} \ldots g^{m_p k_p}\varepsilon_{k_1 \ldots k_p k_{p+1} \ldots k_n} dx^{k_{p+1}} \wedge \ldots \wedge dx^{k_n}\,, \tag{A.3.8}$$

where ε is the totally antisymmetric Levi-Civita tensor density. An alternative definition (which is used throughout this work) includes the factor of the square root of the determinant of the metric in the definition of the tensor ϵ,

$$\epsilon^{0\ldots n} = -1/\sqrt{g}\,. \tag{A.3.9}$$

N.B. This is for a space with a Minkowski signature metric; the internal Calabi-Yau has a Euclidean signature metric and so $\tilde{\epsilon}^{\,4\ldots 9} = 1/\sqrt{\tilde{g}}$. The Hodge dual also has a simplifying identity when acting on a p-form:

$$\star\star\ \alpha_p = (-1)^{p(n-p)+1}\alpha_p \quad \text{Minkowski signature} \tag{A.3.10}$$

$$\star\star\ \alpha_p = (-1)^{p(n-p)}\alpha_p \quad \text{Euclidean signature}\,. \tag{A.3.11}$$

Using the Hodge star we can define an inner product on the space of real forms

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p) = \int \alpha_p \wedge \star\beta_p = \frac{1}{p!}\int \alpha_{\mu_1\ldots\mu_p}\beta^{\mu_1\ldots\mu_p}\sqrt{g}\,dx^1\wedge\ldots\wedge dx^n\,. \tag{A.3.12}$$

We are also able to define the adjoint of the exterior derivative, $d^\dagger$, that maps p-forms to $(p-1)$-forms and obeys

$$(\alpha_p, d\beta_{p-1}) = (d^\dagger\alpha_p, \beta_{p-1})\,. \tag{A.3.13}$$

Explicitly, (A.3.13) tells us that

$$(\alpha, d\beta) = \int d\beta\wedge\star\alpha = \int \big(d(\beta\wedge\star\alpha) - (-1)^{p-1}\beta\wedge d\star\alpha\big)\,. \tag{A.3.14}$$

On a compact manifold, $\partial\mathcal{M} = 0$ and so the first term vanishes. We are left with

$$(\alpha, d\beta) = (-1)^p\int \beta\wedge d\star\alpha = (-1)^{p+p(n-p)}\int \beta\wedge\star(\star\, d\star\alpha)\,. \tag{A.3.15}$$

We recover (A.3.13) by defining $d^\dagger \equiv (-1)^{p+p(n-p)} \star d \star$ so that

$$(\alpha, d\beta) = \int \beta \wedge \star d^\dagger \alpha = (d^\dagger \alpha, \beta)\,. \tag{A.3.16}$$

As with the exterior derivative, the successive applications of the adjoint on a form gives zero:

$$d^\dagger d^\dagger \alpha_p = \star d \star \star d \star \alpha_p = (-1)^{p(n-p)} \star d^2 \star \alpha_p = 0\,. \tag{A.3.17}$$

The natural second-order differential operator in form notation is the Hodge-deRahm operator that takes p-forms to p-forms

$$\Delta \equiv d\, d^\dagger + d^\dagger d\,. \tag{A.3.18}$$

Using Δ we can classify different types of forms on a manifold. One special type of form is a harmonic form, which is annihilated by Δ: $\Delta\omega = 0$. Thus, if ω is harmonic,

$$\begin{aligned} (\omega, \Delta\omega) &= 0 \\ (\omega, (d\, d^\dagger + d^\dagger d)\, \omega) &= 0 \\ (d^\dagger \omega, d^\dagger \omega) + (d\omega, d\omega) &= 0\,. \end{aligned} \tag{A.3.19}$$

Since the inner product is positive semi-definite, then both $d\omega = 0$ and $d^\dagger \omega = 0$. It can also be shown that the reciprocal is true; thus,

$$\Delta\omega = 0 \qquad \Longleftrightarrow \qquad d\omega = 0 \text{ and } d^\dagger \omega = 0\,. \tag{A.3.20}$$

Consider the smooth, connected manifold $\mathcal{M}$. We can define the p-chain α_p as the sum

$$\alpha_p = \sum_i c_i N_i\,, \tag{A.3.21}$$

where $c_i \in \mathbb{C}$ and N_i is a smooth, p-dimensional, oriented sub-manifold of $\mathcal{M}$. Various types of chains can be considered for the various values of c_i. Now consider acting on the p-chain α_p with the ∂ operator

$$\partial\alpha_p = \sum_i c_i \partial N_i \,, \tag{A.3.22}$$

where ∂N_i is the oriented boundary of N_i. Thus, $\partial\alpha_p$ is a $(p-1)$-chain. In order to explore the relationship between forms – which relate to elements of volume – and chains – which relate to boundaries – let C denote some small p-cube of volume

$$C \in \mathbb{C}^n : \{a^1 \leq x^1 \leq b^1, \ldots, a^p \leq x^p \leq b^p, x^{p+1} = c^{p+1}, \ldots, x^n = c^n\} \,, \tag{A.3.23}$$

and let η_c be some $(n-p)$-form given by

$$\begin{aligned} \eta_c = \theta(x^1 - a^1)\theta(b^1 - x^1) \ldots \; & \theta(x^p - a^p)\theta(b^p - x^p) \\ & \times\delta(x^{p+1} - c^{p+1}) \ldots \delta(x^n - c^n) dx^{p+1} \wedge \ldots \wedge dx^n \,, \end{aligned} \tag{A.3.24}$$

where $\theta(x-a)$ is the heavy-side step function. Then, if ω_p is an arbitrary p-form, we find that

$$\int_{\mathcal{M}} \omega_p \wedge \eta_c = \int_C \omega_{12\ldots p}\, dx^1 \wedge \ldots \wedge dx^p = \int_C \omega \,. \tag{A.3.25}$$

This can be extended to any volume by taking an appropriate linear combination of cubes. Thus, we define the inner product of a p-form, ω_p, with a p-chain, C_p, by

$$\pi(\omega, C) = \int_C \omega \,. \tag{A.3.26}$$

Using the form-version of Stokes' theorem given by (A.3.7), we can see that

$$(-1)^{p+1} \int_{\mathcal{M}} \omega \wedge d\eta_c = \int_C d\omega = \int_{\partial C} \omega \,, \tag{A.3.27}$$

i.e. $(-1)^{p+1} d\eta_c \sim \partial C$ for any chain C. We can therefore define a cycle as a chain with no boundary; any two cycles C and C' are considered to be equivalent if they differ by a boundary, *e.g.* $C'_p \sim C_p + \partial C_{p+1}$. This equivalence in cycles is illustrated in Figure A.2 for the case of the torus T_2. The number of independent cycles on a manifold is equal to the simplical homology of the manifold, given by the ratio of the set of cycles over the set of chains that comprise boundaries. Homologies and cohomologies are addressed in detail in § A.4.

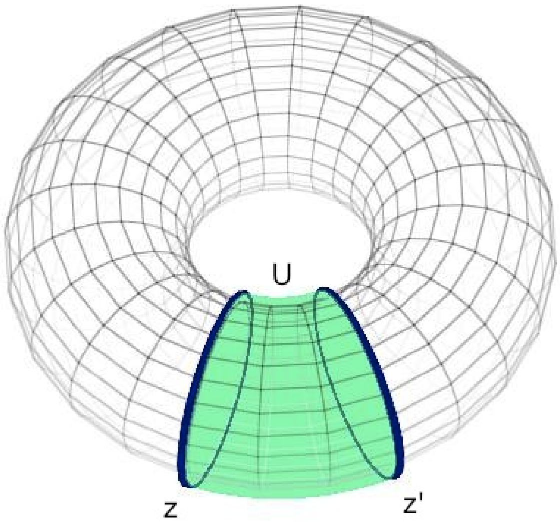


Figure A.2: *Curves z and z' are cycles because they have no boundary and are not themselves boundaries. These two cycles are equivalent because they differ only by the boundary of U.*

A.3.1 Example: Electromagnetism

As an example of the use of forms in an already familiar problem, we review the procedure outlined in [34] for the formulation of classical electromagnetism in terms of a one-form gauge field A_1 and a 2-form field strength $F_2 = dA_1$. Maxwell's equations in the presence of sources are

$$dF_2 = \star J_1^m \quad \text{and} \quad d \star F_2 = \star J_1^e \,, \tag{A.3.28}$$

where J_1^e is the electric current density 4-vector and J_1^m is the magnetic one. For a point-like electric source with charge e and density $\rho_e = e\delta^3(x)$, we can write the charge in terms of the field strength:

$$e = \int_{S^2} \star F_2 \,. \tag{A.3.29}$$

N.B. The closure of $\mathbb{R}^3$ is the two-sphere S^2. In the sourceless case, (A.3.28) becomes

$$dF_2 = 0 \quad \text{and} \quad d \star F_2 = 0 \,. \tag{A.3.30}$$

Here, the self-duality condition $F_2 = \star F_2$ is manifest; furthermore, this condition is still required for higher-dimensional analogues.

A.4 Forms on Complex Manifolds

A standard model for the internal dimensions in a 10D SUGRA theory is the Calabi-Yau 3-fold. This choice is set by the Einstein equations for the background metric, and results in some crucial restrictions on the types of forms permitted in the extra dimensions. Since the resultant 4D theory is determined via dimensional reduction, we postulate that the exact form of the 6D manifold should not affect the final result. We will briefly review some important properties that are used throughout this work by following the summaries provided in [11], [35], and [9].

Following the review of form notation above, we first examine some properties of forms on compact manifolds. For a compact manifold without boundary and positive definite metric, Hodge's theorem states that any p-form can be uniquely decomposed into harmonic, exact, and co-exact parts, *i.e.*

$$\omega = \alpha + d\beta + d^\dagger\gamma \,. \tag{A.4.1}$$

Note that a closed form is one whose exterior derivative is zero, while an exact form is one that can be written in terms of the exterior derivative of another form. Thus, all exact forms are closed.

In general, we are able to write closed forms as an exact form within any local coordinate patch. However, this may not be the case globally.

One way we classify different manifolds is in terms of topological invariants, such as cohomology classes. For some manifold $\mathcal{M}$, we define the deRham cohomology class in terms of the set of closed p-forms $\mathcal{Z}^p$ and the set of exact p-forms $\mathcal{B}^p$ to be

$$\mathcal{H}^p = \frac{\mathcal{Z}^p}{\mathcal{B}^p} \, . \tag{A.4.2}$$

Thus, $\mathcal{H}^p$ is the set of closed p-forms where any two elements are considered equivalent if they differ by an exact form: $\alpha_p \sim \alpha_p + d\beta_{p-1}$. Importantly, each cohomology class contains precisely *one* harmonic form, although this form will depend on the metric of $\mathcal{M}$.

The cohomology class also determines the number of linearly independent harmonic p-forms by the Betti numbers, $b_p = \dim(\mathcal{H}^p)$. We can then build the Euler characteristic χ of an n-dimensional manifold by the alternating sum of the Betti numbers

$$\chi = \sum_{p=0}^{n} (-1)^p b_p \, . \tag{A.4.3}$$

We will see that Calabi-Yau manifolds are a subset of a much more general class of manifolds and are identified by their values of χ and b_p.

Let us specify $\mathcal{M}$ to be a complex manifold with local coordinates z^μ. We define the mixed tensor (also known as the complex structure) $J_m{}^n$ as

$$J_m{}^n = i dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i dz^{\bar{\mu}} \otimes \frac{\partial}{\partial z^{\bar{\mu}}} \, , \tag{A.4.4}$$

where the complex coordinates z^μ have $\mu = \{1 \ldots n\}$ and $z^{\bar{\mu}} = (\bar{z^\mu})$. $J_m{}^n$ is a real tensor with the identity

$$J_m{}^n J_n{}^p = -\delta_m{}^p \, . \tag{A.4.5}$$

The complex structure corresponds to variations of the components of the metric of mixed type, *i.e.* one real and one imaginary index. Another important structure corresponds to variations of the components of the metric of pure type, *i.e.* all real or all imaginary indices, and is denoted Ω. In the case of a Calabi-Yau manifold (see below), Ω is a nonzero, holomorphic (3,0)-form[1] given by

$$\Omega = \frac{1}{3!}\Omega_{\mu\nu\rho}(x)dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \, , \tag{A.4.6}$$

where each component is a function of position. Furthermore, Ω is harmonic and covariantly constant. We discuss the relationship between variations of the metric and the moduli space of the manifold in Appendix A.5.

On a hermitian manifold there exists a unique connection that allows for the covariant derivatives of both the metric and the complex structure to vanish. Furthermore, the torsion $\Gamma_{[mn]}{}^{r}$ is pure (all real or all imaginary) in its lower indices. It can be shown that the existence plus the uniqueness conditions for this connection imply that

$$\Gamma_{\mu\nu}{}^{\kappa} = g^{\kappa\bar{\rho}}\partial_{\mu}g_{\nu\bar{\rho}} \, . \tag{A.4.7}$$

Thus, only a few terms in the Riemann tensor are non-vanishing:

$$R_{\mu\bar{\nu}\bar{\rho}}{}^{\bar{\sigma}} = -R_{\bar{\nu}\mu\bar{\rho}}{}^{\bar{\sigma}} = \partial_{\mu}\Gamma_{\bar{\nu}\bar{\rho}}{}^{\bar{\sigma}} \, . \tag{A.4.8}$$

Since the Riemann tensor is already antisymmetric, it is straightforward to define a Ricci-form $\mathcal{R}$ as

$$\mathcal{R} = iR_{\mu\bar{\nu}\bar{\rho}}{}^{\bar{\rho}} \, dz^{\mu} \wedge dz^{\bar{\nu}} \, . \tag{A.4.9}$$

[1] Previously, the notation (m, n) would have referred to m 4D indices and n 6D indices; in the context of complex manifolds this refers to m holomorphic and n anti-holomorphic indices.

Using (A.4.7) and noting that $\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$, we see that

$$\mathcal{R} = i\partial\bar{\partial}\log\sqrt{g}\,, \tag{A.4.10}$$

and that Ricci-form is closed, *i.e.* $d\mathcal{R} = 0$. This defines a cohomology class via

$$c_1 = \frac{1}{2\pi}\mathcal{R} \tag{A.4.11}$$

known as the first Chern class, which is invariant under smooth changes to the complex structure of the manifold.

A Calabi-Yau manifold is a compact Kähler manifold with $c_1 = 0$. A Kähler manifold is a type of Hermitian manifold whose associated complex structure is closed. A Calabi-Yau manifold with nonzero Euler number, χ, has Betti numbers $b_{1,0} = b_{0,1} = 1/2\, b_1 = 0$. Since b_1 is a topological invariant, we are permitted to establish results in the case of a Ricci-flat metric that will apply to all other Calabi-Yau manifolds with $\chi \neq 0$. The Ricci-flatness of the Calabi-Yau motivates the inclusion of compensator fields in our metric ansatz so that this condition is met.

A.5 Kähler Manifolds and Their Moduli

Recall the definition of a Calabi-Yau manifold as a compact Kähler manifold with vanishing first Chern class and closed Kähler form, J. Following the definition of the complex structure in (A.4.4), we define the Kähler form of a compact manifold to be

$$J = \frac{1}{2}J_k{}^{\ell}g_{\ell m}dx^k \wedge dx^m = ig_{\mu\bar{\nu}}dx^{\mu} \wedge dx^{\bar{\nu}}\,. \tag{A.5.1}$$

Furthermore, any metric that has a closed fundamental form is called a Kähler metric. Since J is closed, we find immediately that $dJ = 0 \Rightarrow \partial J = \bar{\partial}J = 0$, and, using (A.5.1), we arrive at the

Kähler condition:

$$\partial_\lambda g_{\mu\bar{\nu}} = \partial_\mu g_{\lambda\bar{\nu}} \quad \text{and} \quad \bar{\partial}_\lambda g_{\mu\bar{\nu}} = \bar{\partial}_\nu g_{\mu\bar{\lambda}}\,. \tag{A.5.2}$$

The Kähler condition implies that the metric (of Kähler type) can be expressed in terms of a local Kähler potential, $\mathcal{K}$; and thus, so too can the Kähler form

$$g_{\mu\bar{\nu}} = \partial_\mu \bar{\partial}_\nu \mathcal{K} \Rightarrow J = i\partial\bar{\partial}\mathcal{K}\,. \tag{A.5.3}$$

Yau's theorem states that for, a Calabi-Yau manifold $\mathcal{M}$ with Kähler form J_0, there exists a unique, Ricci-flat Kähler metric for $\mathcal{M}$ whose associated Kähler form J is in the same cohomology class as J_0. This means that the parameter space of a Calabi-Yau manifold is equivalent to the parameter space of Ricci-flat metrics. By perturbing the parameter space of Ricci-flat metrics from $g_{mn} \to g_{mn} + \delta g_{mn}$ such that $R_{mn}(g) = 0$ and $R_{mn}(g+\delta g) = 0$, we obtain the Lichnerowicz equation:

$$\nabla^\ell \nabla_\ell \delta g_{mn} + 2R_m{}^p{}_n{}^q \delta g_{pq} = 0\,. \tag{A.5.4}$$

The zero modes of (A.5.4) of pure type, $\delta g_{\mu\nu}$ and $\delta g_{\bar{\mu}\bar{\nu}}$, and those of mixed type, $\delta g_{\bar{\mu}\nu}$, solve the equation separately. Thus, there are two main types of metric variations to consider: those of mixed type, which are associated with a real, harmonic (1,1)-form, and those of pure type, which are associated with a complex, harmonic (2,1)-form. Due to Yau's theorem, there is a one-to-one correspondence between the metric variations and the elements of the cohomology class of the same type.

As described in Appendix A.4, the dimension of a cohomology class is defined as the Betti number, b. Similarily, modes of mixed type correspond to variations of the Kähler class and are defined by Hodge numbers $h_{1,1} = \dim(\mathcal{H}^{1,1})$ real parameters; modes of pure type correspond to variations of the complex structure and yield $h_{2,1} = \dim(\mathcal{H}^{2,1})$ complex parameters.

Let us first consider the space of (2,1)-forms. Let each χ_α define a (2,1) cohomology class via $\chi_\alpha = 1/2\,\chi_{\alpha\kappa\lambda\bar{\mu}} dx^\kappa \wedge dx^\lambda \wedge dx^{\bar{\mu}}$, and let these $\mathcal{H}^{2,1}$ be indexed by $\alpha = \{1, 2, \ldots, h_{2,1}\}$. Each deformation within a cohomology class is then given by

$$\chi_{\alpha\kappa\lambda\bar{\mu}} = -\frac{1}{2}\Omega_{\kappa\lambda}{}^{\bar{\nu}}\frac{\partial \delta g_{\bar{\mu}\bar{\nu}}}{\partial z^\alpha}\,, \tag{A.5.5}$$

where the z^α are the parameters for the complex structure. *N.B.* $\Omega_{\mu\nu}{}^{\bar{\lambda}} = \Omega_{\mu\nu\sigma}g^{\sigma\bar{\lambda}}$. We can then express the variations of the metric of pure type by the inverse of the previous relation

$$\delta g_{\bar{\mu}\bar{\nu}} = -\frac{1}{||\Omega||^2}\bar{\Omega}_{\bar{\mu}}{}^{\rho\sigma}\chi_{\alpha\rho\sigma\bar{\nu}}\delta z^\alpha\,. \tag{A.5.6}$$

In order to describe the moduli space, we can consider endowing the space with some metric G that has both pure and mixed indices. This makes the correspondence between the variations of the metric and the elements of the cohomology classes manifest. Using (A.5.6), we can show that the mixed metric in the moduli space, $G_{\alpha\bar{\beta}}$, can be given by

$$G_{\alpha\bar{\beta}} = -\frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial z^{\bar{\beta}}}\log\left(i\int_{\mathcal{M}}\Omega\wedge\bar{\Omega}\right)\,, \tag{A.5.7}$$

i.e., the moduli space is itself Kähler. The moduli space, by (A.5.3), therefore has its own Kähler potential that is given by

$$k = -\log\left(i\int_{\mathcal{M}}\Omega\wedge\bar{\Omega}\right)\,. \tag{A.5.8}$$

Next, we examine the space of (1,1)-forms. Note that, for the real (1,1)-forms ϱ, ϑ, the inner product on the moduli space is given by

$$G(\varrho,\vartheta) = \frac{1}{2V}\int_{\mathcal{M}} d^6x\sqrt{g}\,\varrho_{\mu\bar{\nu}}\vartheta_{\rho\bar{\sigma}}g^{\mu\bar{\sigma}}g^{\rho\bar{\nu}} = \frac{1}{2V}\int \varrho\wedge\star\vartheta\,. \tag{A.5.9}$$

It can be shown that, by defining the triple intersection κ as

$$\kappa(\varrho,\vartheta,\tau) \equiv \int \varrho\wedge\vartheta\wedge\tau\,, \tag{A.5.10}$$

then it follows from the identity

$$V = \frac{1}{3!}\kappa(J,J,J) \tag{A.5.11}$$

and a relation for $\star\vartheta$ in terms of J and κ, that the inner product can be written as

$$G(\varrho,\vartheta) = -3\left[\frac{\kappa(\varrho,\vartheta,J)}{\kappa(J,J,J)} - \frac{3}{2}\frac{\kappa(\varrho,J,J)\kappa(\vartheta,J,J)}{\kappa^2(J,J,J)}\right]. \tag{A.5.12}$$

In terms of the cohomology class $\mathcal{H}^2(\mathcal{M},\mathbb{Z})$, parameterized by ω^A, $A=\{1,\ldots,b_{1,1}\}$, we can see that the metric for the space of (1,1)-forms is given by

$$G_{A\bar{B}} = -\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^{\bar{B}}}\log\kappa(J,J,J)\,, \tag{A.5.13}$$

i.e. the Kähler potential for this space is the logarithm of the volume of the Calabi-Yau manifold.

A.6 Calculation of $\delta\Omega$

We wish to determine how the first-order part of the Weyl factor, $e^{2\Omega}$, contributes to the constraint equation

$$4\delta A - 2\delta\Omega + e^{4A}e^{2\Omega}\widetilde{\nabla}^\ell B_\ell = 0\,. \tag{A.6.1}$$

To do so, we first recall that $e^{2\Omega} = \tilde{V}_{CY}/\tilde{V}_W$, where the unwarped volume of the Calabi-Yau is $\tilde{V}_{CY}$ and is given by

$$\tilde{V}_{CY} = \int d^6y\sqrt{\tilde{g}}\,, \tag{A.6.2}$$

while the warped volume of the Calabi-Yau, which depends on the background solution for $A_0(y)$, is

$$\tilde{V}_W = \int d^6y \sqrt{\tilde{g}}\, e^{-4A_0(y)} \,. \tag{A.6.3}$$

Thus, the first-order Weyl factor can be expressed as

$$e^{-2\Omega} = \frac{1}{\tilde{V}_{CY}} \int d^6y \sqrt{\tilde{g}}\, e^{-4A} \,. \tag{A.6.4}$$

Furthermore, the background Einstein equation leads to a constraint that gives the warp factor in terms of the positions of the D3-branes via

$$\widetilde{\nabla}^2 e^{-4A_0} = -2 \sum_i T_3 \kappa^2 \tilde{\delta}^6(y, Y_i) \,, \tag{A.6.5}$$

The solution to (A.6.5) in flat space is a 6-dimensional Green's function that goes as r^{-4}. Thus, in order to determine the first-order part of the Weyl factor, we consider perturbing the solution of (A.6.5) for two D3-branes by moving the position of one of the branes by some $\delta\vec{y}$ and substituting this into (A.6.4):

$$-2e^{-2\Omega}\delta\Omega = \frac{1}{\tilde{V}_{CY}} \int d^6y \sqrt{\tilde{g}} \left(\frac{1}{|\vec{y} - \vec{y}_0 - \delta\vec{y}|^4} - \frac{1}{|\vec{y} - \vec{y}_0|^4} \right) . \tag{A.6.6}$$

Without specifying more about the manifold, this is as far as we can proceed; however, it is instructive to consider a flat, finite-radius manifold. Using spherical symmetry, and taking $|\delta\vec{y}| \ll |\vec{y} - \vec{y}_0|$ so that $\epsilon = |\delta\vec{y}|/|\vec{y} - \vec{y}_0| \ll 1$, (A.6.6) can be expanded to linear order in ϵ to give

$$\int d^6y \left(\frac{1}{|\vec{y} - \vec{y}_0 - \delta\vec{y}|^4} - \frac{1}{|\vec{y} - \vec{y}_0|^4} \right) \approx \int d^6y \frac{4\epsilon \cos\theta}{|\vec{y} - \vec{y}_0|^4} \,. \tag{A.6.7}$$

Provided that this space has a finite radius, *i.e.* $|\vec{y} - \vec{y}_0| \leq R$, (A.6.6) vanishes.

Although this example is instructive, we have made a number of simplifying assumptions in order to evaluate the integral over the Calabi-Yau. In §3.2 we are able to see that, in the case of the warp factor being described by a bi-scalar Green's function, we find the same conclusion without having to make any assumptions regarding the internal manifold.

A.7 Einstein Equations for the 2-Form Axion

We wish to determine if the reformulation of the universal axion as a 2-form includes any new terms into the Einstein equations. If so, there will be additional considerations for the 2-form axion. We begin by specifying the metric ansatz for $b_2(x)$,

$$ds_{10}^2 = e^{2A}e^{2\Omega}\hat{\eta}_{\mu\nu}dx^\mu dx^\nu + 2e^{2A}e^{-2\Omega}[\hat{\star}\hat{d}b_2]B'_m(y)dx^\mu dy^m + e^{-2A}\tilde{g}_{mn}dy^m dy^n\,, \tag{A.7.1}$$

where B'_1 is co-closed, *i.e.* $\widetilde{\nabla}^\ell B'_\ell = 0$. For simplicity, we denote $[\hat{\star}\hat{d}b_2]_\mu$ by t_μ so that $\partial^\mu t_\mu = \partial^\mu[\hat{\star}\hat{d}b_2]_\mu = -\hat{\star}\hat{d}\hat{\star}(\hat{\star}\hat{d}b_2) = -\hat{\star}\hat{d}\hat{d}b_2 = 0$. Using (A.1.1), the terms (up to first order) that contribute to $R_{\mu\nu}$ are

$$\hat{\partial}_\lambda\Gamma^\lambda_{\nu\mu} = (2\hat{\partial}_\mu\hat{\partial}_\nu - \hat{\eta}_{\mu\nu}\hat{\partial}^2)(A+\Omega) \tag{A.7.2}$$

$$\tilde{\partial}_\ell\Gamma^\ell_{\nu\mu} = e^{4A}e^{-2\Omega}\hat{\partial}_{(\mu}t_{\nu)}(4\tilde{\partial}_\ell AB'^\ell + \tilde{\partial}_\ell B'^\ell) - e^{4A}e^{2\Omega}\hat{\eta}_{\mu\nu}(4\tilde{\partial}_\ell A\tilde{\partial}^\ell A + \widetilde{\nabla}^2 A) \tag{A.7.3}$$

$$\hat{\partial}_\nu\Gamma^\lambda_{\lambda\mu} = 4\hat{\partial}_\nu\hat{\partial}_\mu(A+\Omega) + e^{4A}e^{-2\Omega}\hat{\partial}_\nu t_\mu\tilde{\ell}AB'^\ell \tag{A.7.4}$$

$$\hat{\partial}_\nu\Gamma^\ell_{\ell\mu} = -e^{4A}e^{-2\Omega}\hat{\partial}_\nu t_\mu\tilde{\partial}^\ell AB'_\ell - 6\hat{\partial}_\nu\hat{\partial}_\mu A \tag{A.7.5}$$

$$(\Gamma^\lambda_{\lambda n} + \Gamma^m_{mn})\Gamma^n_{\nu\mu} = e^{4A}(\tilde{\Gamma}^m_{mn} - 2\tilde{\partial}_n A)(e^{-2\Omega}B'^n\hat{\partial}_{(\nu}t_{\mu)} - e^{2\Omega}\tilde{\partial}^n A\hat{\eta}_{\nu\mu}) \tag{A.7.6}$$

$$\Gamma^\rho_{\mu m}\Gamma^m_{\nu\rho} = e^{4A}e^{-2\Omega}\tilde{\partial}^m AB'_m\hat{\partial}_\nu t_\mu - e^{4A}e^{2\Omega}\hat{\eta}_{\nu\mu}\tilde{\partial}_m A\tilde{\partial}^m A \tag{A.7.7}$$

$$\Gamma^\lambda_{\mu n}\Gamma^n_{\lambda\nu} = e^{4A}e^{-2\Omega}\tilde{\partial}^n AB'_n\hat{\partial}_\mu t_\nu - e^{4A}e^{2\Omega}\hat{\eta}_{\nu\mu}\tilde{\partial}_n A\tilde{\partial}^n A\,. \tag{A.7.8}$$

The result of these contributions is that

$$R_{\mu\nu} = \hat{\partial}_\mu\hat{\partial}_\nu(4A - 2\Omega) - \hat{\eta}_{\mu\nu}\hat{\partial}^2(A+\Omega) - e^{4A}e^{2\Omega}\widetilde{\nabla}^2 A\,. \tag{A.7.9}$$

The more difficult of the three components of R_{MN} for the 2-form axion is the off-diagonal component. To ease the calculation, we will again define some $\Psi^m_{\mu n} \equiv e^{4A}e^{-2\Omega}t_\mu \tilde{g}^{\ell m}(\tilde{\partial}_\ell A B'_n + \tilde{\partial}_{[\ell}B'_{n]})$ and compute the sums of certain combinations of Christoffel symbols whenever it is convenient to do so. The contributions to $R_{\mu m}$ are given by (A.1.2) and include

$$\hat{\partial}_\lambda \Gamma^\lambda_{m\mu} = \hat{\partial}_\mu \tilde{\partial}_m A \tag{A.7.10}$$

$$\tilde{\partial}_\ell \Gamma^\ell_{m\mu} = -\hat{\partial}_\mu \tilde{\partial}_m A - \tilde{\partial}_\ell \Psi^\ell_{\mu m} \tag{A.7.11}$$

$$\tilde{\partial}_m \Gamma^\ell_{\ell\mu} = -6\hat{\partial}_\mu \tilde{\partial}_m A - \tilde{\partial}_m \Psi^\ell_{\mu\ell} \tag{A.7.12}$$

$$\tilde{\partial}_m \Gamma^\nu_{\nu\mu} = 4\hat{\partial}_\mu \tilde{\partial}_m A + e^{4A}e^{-2\Omega}t_\mu \left(4\tilde{\partial}_m A \tilde{\partial}_\ell A B'^\ell + \tilde{\partial}_m(\tilde{\partial}_\ell A B'^\ell)\right) \tag{A.7.13}$$

$$(\Gamma^\rho_{\rho\lambda} + \Gamma^q_{q\lambda})\Gamma^\lambda_{m\mu} = 4\tilde{\partial}_m A \hat{\partial}_\mu (A+\Omega) + e^{4A}e^{-2\Omega}t_\mu \tilde{\partial}_m A \tilde{\partial}^\ell A B'_\ell - \tilde{\partial}_m A \Psi^\ell_{\mu\ell} - 6\tilde{\partial}_m A \hat{\partial}_\mu A \tag{A.7.14}$$

$$(\Gamma^\nu_{\nu\ell} + \Gamma^q_{q\ell})\Gamma^\ell_{m\mu} = 2\tilde{\partial}_\ell A \Psi^\ell_{\mu m} + 2\tilde{\partial}_m A \hat{\partial}_\mu A - \tilde{\Gamma}^q_{q\ell}\Psi^\ell_{\mu m} - \tilde{\Gamma}^\ell_{\ell m}\hat{\partial}_\mu A \tag{A.7.15}$$

$$\Gamma^\rho_{m\lambda}\Gamma^\lambda_{\rho\mu} = 4\tilde{\partial}_m A \hat{\partial}_\mu (A+\Omega) + e^{4A}e^{-2\Omega}t_\mu \tilde{\partial}_m A \tilde{\partial}^\ell A B'_\ell \tag{A.7.16}$$

$$\Gamma^\rho_{m\ell}\Gamma^\ell_{\rho\mu} = -\hat{\partial}_\mu A \tilde{\partial}_m A - e^{4A}e^{-2\Omega}t_\mu \tilde{\partial}^\ell A \left(\widetilde{\nabla}_{(m}B'_{\ell)} + 4\tilde{\partial}_{(m}AB'_{\ell)} - \tilde{g}_{\ell m}\tilde{\partial}^q A B'_q\right) \tag{A.7.17}$$

$$\Gamma^q_{m\lambda}\Gamma^\lambda_{q\mu} = -\tilde{\partial}_\ell A \Psi^\ell_{\mu m} - \tilde{\partial}_m A \hat{\partial}_\mu A \tag{A.7.18}$$

$$\begin{aligned}\Gamma^q_{m\ell}\Gamma^\ell_{q\mu} = {} & \tilde{\partial}_\ell A \Psi^\ell_{\mu m} + \tilde{\partial}_m A \Psi^\ell_{\mu\ell} - \tilde{g}_{m\ell}\tilde{\partial}^q A \Psi^\ell_{\mu q} + 6\tilde{\partial}_m A \hat{\partial}_\mu A - \tilde{\Gamma}^q_{m\ell}\Psi^\ell_{\mu q} \\ & -\tilde{\Gamma}^\ell_{m\ell}\hat{\partial}_\mu A\,.\end{aligned} \tag{A.7.19}$$

Combining these and using the definition of Ψ results in

$$R_{\mu m} = 2\hat{\partial}_\mu \tilde{\partial}_m A - 8\hat{\partial}_\mu A \tilde{\partial}_m A - e^{4A}e^{-2\Omega}t_\mu (\widetilde{\nabla}^2 B'_m + \widetilde{\nabla}^2 A B'_m + 4\widetilde{\nabla}^\ell A \tilde{\partial}_{[\ell}B'_{m]})\,. \tag{A.7.20}$$

The final component is R_{mn}, which is made up of

$$\hat{\partial}_\mu \Gamma^\mu_{mn} = e^{-4A} e^{-2\Omega} \tilde{g}_{mn} \hat{\partial}^2 A \tag{A.7.21}$$

$$\tilde{\partial}_\ell \Gamma^\ell_{mn} = \tilde{\partial}_\ell (\tilde{g}_{mn} \tilde{\partial}^\ell A) - 2\tilde{\partial}_m \tilde{\partial}_n A + \tilde{\partial}_\ell \tilde{\Gamma}^\ell_{mn} \tag{A.7.22}$$

$$\tilde{\partial}_n \Gamma^\mu_{\mu m} = 4\tilde{\partial}_n \tilde{\partial}_m A \tag{A.7.23}$$

$$\tilde{\partial}_n \Gamma^\ell_{\ell m} = -6\tilde{\partial}_n \tilde{\partial}_m A + \tilde{\partial}_n \tilde{\Gamma}^\ell_{\ell m} \tag{A.7.24}$$

$$\begin{aligned}(\Gamma^\mu_{\mu q} + \Gamma^p_{pq})\Gamma^q_{nm} &= \tilde{\partial}^q A \tilde{g}_{nm} \tilde{\Gamma}^\ell_{\ell q} - 2\tilde{g}_{nm} \tilde{\partial}^\ell A \tilde{\partial}_\ell A + 4\tilde{\partial}_m A \tilde{\partial}_n A - 2\tilde{\partial}_\ell A \tilde{\Gamma}^\ell_{nm} - 2\tilde{\partial}_{(m} A \tilde{\Gamma}^\ell_{n)\ell} \\ &\quad + \tilde{\Gamma}^q_{nm} \tilde{\Gamma}^\ell_{\ell q}\end{aligned} \tag{A.7.25}$$

$$\Gamma^\mu_{n\nu} \Gamma^\nu_{\mu m} = 4\tilde{\partial}_n A \tilde{\partial}_m A \tag{A.7.26}$$

$$\begin{aligned}\Gamma^p_{nq} \Gamma^q_{pm} &= 8\tilde{\partial}_m A \tilde{\partial}_n A - 2\tilde{\partial}_{(m} A \tilde{\Gamma}^\ell_{n)\ell} + 2\tilde{\partial}_\ell A \tilde{\Gamma}^\ell_{nm} + 2\tilde{\partial}^p A \tilde{g}_{\ell(m} \tilde{\Gamma}^\ell_{n)p} - 2\tilde{g}_{mn} \tilde{\partial}^\ell A \tilde{\partial}_\ell A \\ &\quad + \tilde{\Gamma}^\ell_{np} \tilde{\Gamma}^p_{\ell m}\,,\end{aligned} \tag{A.7.27}$$

and is given by (A.1.3). We therefore find that

$$R_{mn} = -8\tilde{\partial}_m A \tilde{\partial}_n A + e^{-4A} e^{-2\Omega} \tilde{g}_{mn} \hat{\partial}^2 A + \tilde{g}_{mn} \widetilde{\nabla}^2 A + \tilde{R}_{mn}\,, \tag{A.7.28}$$

where $\tilde{R}_{mn}$ is the Ricci tensor of the internal space; for the Calabi-Yau $\tilde{R}_{mn} = 0$.

Using (A.7.9), (A.7.20), and (A.7.28), the Ricci scalar is

$$\mathcal{R} = 6e^{-2A} e^{-2\Omega} \hat{\partial}^2 (A - \Omega) + 2e^{2A} (\widetilde{\nabla}^2 A - 4\widetilde{\nabla}^\ell A \widetilde{\nabla}_\ell A)\,, \tag{A.7.29}$$

which means that the components of the Einstein tensor, given by $G_{MN} = R_{MN} - 1/2 g_{MN} \mathcal{R}$, are

$$G_{\mu\nu} = (\hat{\partial}_\mu \hat{\partial}_\nu - \hat{\eta}_{\mu\nu} \hat{\partial}^2)(4A - 2\Omega) + 2e^{4A} e^{2\Omega} \hat{\eta}_{\mu\nu} (2\widetilde{\nabla}^\ell A \widetilde{\nabla}_\ell A - \widetilde{\nabla}^2 A) \tag{A.7.30}$$

$$\begin{aligned}G_{\mu m} &= 2\hat{\partial}_\mu \tilde{\partial}_m A - 8\hat{\partial}_\mu A \tilde{\partial}_m A - e^{4A} e^{-2\Omega} t_\mu [\widetilde{\nabla}^2 B'_m + 2\widetilde{\nabla}^2 A B'_m \\ &\quad + 4\widetilde{\nabla}^\ell A (\tilde{\partial}_{[\ell} B'_{m]} - \widetilde{\nabla}^\ell A B'_m)]\end{aligned} \tag{A.7.31}$$

$$G_{mn} = -8\tilde{\partial}_m A \tilde{\partial}_n A + 4\widetilde{\nabla}^\ell A \widetilde{\nabla}_\ell A \tilde{g}_{mn} + e^{-4A} e^{-2\Omega} \tilde{g}_{mn} \hat{\partial}^2 (3\Omega - 2A)\,. \tag{A.7.32}$$

We can compare this Einstein tensor to the one calculated for the D3-brane in § 3.1 under the substitution $\hat{\partial}_\mu B_m \to e^{-4\Omega}[\hat{\star}\hat{d}b_2]_\mu B'_m$. Indeed, there is complete agreement between (A.7.30)-(A.7.32) and (3.5)-(3.7). Furthermore, the stress tensor, T_{MN}, does not require taking derivatives, so it too will remain valid under direct substitution. Thus, reformulation of the scalar axion into a 2-form axion does not require a re-derivation of the Einstein equations.

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