SOME NEW RESULTS ON THE CONTINUED FRACTION METHOD AND ITS APPLICATIONS TO CONTINUOUS AND DISCRETE SYSTEMS

by

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ABSTRACT

This thesis deals with continued fraction methods, with emphasis on applications to both continuous and discrete systems. It contains new developments on the subject.

Realization of arbitrary digital transfer function, in one and two dimensions, is considered. Several new structures and their realizability conditions are discussed. The structures are canonic with respect to both delays and multipliers. They can also be used in realizing arbitrary transfer functions in the continuous s-domain.

An explicit recursive formula for the elements of the first column of Routh's array is presented. Closed forms for Cauer-type and Stieltjes-type expansion coefficients are derived and tabulated. These forms are used efficiently in implementing both continued fraction expansion and inversion via simple algebraic equations. The use of these forms provides insight into the expansion process.

The thesis also develops a generalized test for Hurwitz polynomials in which more information regarding the polynomial roots is obtained. Moreover, a fast Hurwitz test is proposed whereby the unwieldy operations of the conventional test are mitigated.

The algorithmic character of continued fractions is used in developing several efficient stability tests for discrete systems. Continued fractions that proceed in terms of the bilinear, the backward difference or the forward difference transform are proposed. Furthermore, the ap-

plication of continued fractions to bilinear transformation of polynomials is introduced.

Illustrative examples are provided throughout the thesis to demonstrate the validity of the proposed methods.

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Chapter I

INTRODUCTION

This thesis deals with new results on the continued fraction method and its applications to continuous and discrete systems.

The continued fraction expansion of a rational or irrational function is one of the fundamental operations in the area of circuits and systems. Since the pioneering work of Routh [1] and Hurwitz [2] in the stability analysis of continuous systems, almost one hundred years ago, the utility of the continued fraction expansion has been expanding to various aspects of analysis and design of both continuous (analog) and discrete (sampled-data or digital) systems. The history of analog network synthesis by continued fractions goes back to the mid 1920's when Cauer [3-7] showed that the driving point impedance of a passive ladder network is given by a continued fraction in which the impedance or admittance of each of the ladder branches appears explicitly. Recently, it has been shown that the passive ladder network can be transformed to active RC [8-9], switched-capacitor [10-11] or digital ladder networks [12-14] by the direct simulation of the elements of the passive prototype or by simulating its voltage-current operations. In fact, the direct application of continued fractions to discrete systems is fairly recent as it started in the early 1970's [15]. An interesting observation is the fact that unlike in the case of continuous systems, research was first oriented to the synthesis problem rather than the stability analysis problem. The application of continued fraction expansion to the latter is only six years old [16]. The continued fraction method has also been applied to synthesis of system functions in more than one variable. It is of interest to note that tangible investigation was initiated almost simultaneously for both continuous [17] and discrete [18] systems.

The theory of continued fraction expansion is a well-developed branch of mathematics [19]. There has recently been a renewed interest in the subject of continued fractions [20-22]. This is due in part to the advent of computers and the resulting importance of the algorithmic character of continued fractions [20-21]. It is also due to the close connections between continued fractions and the related topics of orthogonal polynomials, Gauss quadrature, integral representation of continued fractions... etc. [22]. According to Jones and Thron [22], Wynn in two statements attempts to sharply delineate the role of continued fractions. One of his statements is: "The theory of continued fractions has been preeminently an avenue to new and unexpected results ...".

One may recognize the operation of continued fraction expansion as Euclid's algorithm [19, 23] for finding the greatest common divisor (g.c.d.) of two integers. According to Knuth [24, p. 294], "Euclid's algorithm, which is found in Book 7, propositions 1 and 2 of his <u>Elements</u> (c.300 B.C.), and which many scholars conjecture was actually Euclid's rendition of an algorithm due to Eudoxus (c.375 B.C.), ... is the oldest nontrivial algorithm which has survived to the present day". The algorithm as presented by Euclid, computes the positive g.c.d. of two given positive integers. It is readily extended to polynomials in one variable, which of course may be regarded as generalized numbers, and further to polynomials in any number of variables [23].

A function G may be expressed by a continued fraction in the form [19]

$$G = b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \cfrac{a_n}{b_n}}}} \tag{1.1}$$

A simple terminating continued fraction expansion has each of the a_r equal to unity and the subscript n finite. In this case, the expansion requires that G be a rational function of two polynomials. However, a function that is not rational, that is a function that possesses for instance an infinite number of poles, will have a nonterminating continued fraction representation. The b_r quantities may be numbers, polynomials or rational functions. The method of finding a simple continued fraction expansion by (1.1) consists of the repetition of a two-step cycle of reciprocation (inversion) and division for every cycle but the first. The first cycle consists only of a division or is omitted completely when the expansion is carried out on 1/G rather than G.

As pointed out, a rational function may be expanded in a terminating continued fraction expansion. Conversely, a terminating continued fraction expansion could be simplified into a single fraction that is a rational function of two polynomials. This operation is of considerable practical interest in the areas of circuits and systems, both analysis and design [25-27]. The operation is known in the literature as the continued fraction inversion [28-29]. The actual operation of the continued fraction inversion involves many multiplications and is known to be more laborious and tedious [28] compared to expansion which involves long divisions.

In modern system theory, the design of a system comprises two general steps [30-32], i.e., 1) approximation and, 2) realization. The approximation step is the process of generating a transfer function satisfying a set of desired specifications, which may concern the amplitude, phase, and possibly the system time-response. The realization step is the process of converting the transfer function into a network. The continued fraction expansion and inversion operations have successfully been applied to both approximation [26, 33] and realization [15, 25]. The results are always encouraging and attractive. For instance, realization by continued fractions yields canonic [15] networks of ladder type which achieve a high degree of modularity and suitability for large-scale integration [15].

The main objective of this thesis is to introduce new results on the continued fraction method and its applications to modern systems, both continuous and discrete.

The thesis consists of six chapters, including the introduction.

In Chapter II, new canonic ladder realizations of system transfer functions in one and two variables are presented. The realizations are based on applications of a mixture of the first and second Cauer continued fraction forms. The resulting structures are shown to be implementable (computable [30]) as digital filters as they avoid the presence of delay-free loops. In the continuous domain, the new structures could be realized on an analog computer.

Chapter III presents a novel approach for the continued fraction expansion and inversion. Cauer-type forms (first, second and mixed) and Stieltjes-type forms are considered. First, a recursive formula for the elements of the first column of Routh's array is derived. Closed forms for the continued fraction expansion coefficients (also called partial quotients [4]) in terms of the rational function coefficients are then presented and tabulated. The closed forms are considered as direct representation of the divide-and-invert cycles in Euclid's algorithm, by simple algebraic equations. The forms in question are used efficiently and equally well in both continued fraction expansion and inversion. Moreover, they could be looked upon as closed forms for the elements of two-element kind (RC, RL, LC) Cauer-type passive ladder networks, the elements of lossy RLC ladder networks of the Stieltjes-type [34-37], or the multiplier elements of the ladder structures discussed in Chapter II. Finally, the closed forms for the expansion coefficients can be used in deriving a new set of necessary and sufficient conditions for a rational function to have a certain continued fraction expansion.

New applications of continued fractions to continuous systems are introduced in Chapter IV. A generalized test for Hurwitz polynomials is proposed in which further information regarding the nature of the roots, such as location and multiplicity, is obtained. The generalized test is performed using the continued fraction method exclusively. A simple aperiodicity test is described as a special case of the generalized test. Furthermore, a fast Hurwitz test is presented in which the unwieldy operations encountered in the conventional Hurwitz test are considerably mitigated.

Chapter V discusses new applications with respect to discrete systems. A simplified stability test for discrete systems is presented. The test is based on a new z-domain continued fraction expansion of the discrete reactance function. The new test yields a simplified algorithm for testing stability which compares favorably with Jury's commonly used [38-39] tabular form. Also, new z-domain continued fraction expansions are suggested. Their applications in testing stability are discussed. Furthermore, a new z-domain continued fraction inversion is proposed. Its application to bilinear transformation of polynomials is introduced.

Chapter VI contains some concluding remarks as well as suggestions for future research work. Well-chosen illustrative examples are included throughout the thesis to demonstrate the validity of proposed theory and methods.

Chapter II

NEW CANONIC REALIZATIONS OF TRANSFER FUNCTIONS USING THE CONTINUED FRACTION METHOD

2.1 INTRODUCTION

In classical network theory, the procedures for realizing transfer functions reduce to the problem of realizing an associated driving-point function [4-6]. The continued fraction method may then be applied to the realization of the driving point function. The resulting passive network is most often that of a two-element kind or at most a special type of RLC network that is realizable by an LC ladder network terminated in a resistance. Moreover, the network uses the minimum possible number of elements and requires no mutual inductance or ideal transformers [4], besides being virtually insensitive to first-order perturbations of the element values [8].

With the advent of solid-state integrated circuits,the interest of transforming the passive ladder network into an active [8-9] digital [12-14] ladder network has emerged and has been shown to be possible either by the direct simulation of the elements of the passive prototype [8, 12] or by simulating its voltage-current interrelationships [8, 13-14]. Particularly, with the rapid advances in digital integrated circuit technology, there has been an increasing interest in digital signal processing in recent years [30]. As a consequence, various schemes of realizing digital transfer functions using methods in the discrete-domain exclusively have been investigated [15, 32, 40]. The continued fraction method has been successfully used [15, 41-45] resulting in structures that are canonic, with respect to both delays and multipliers, and well-suited for implementation using large-scale integration. Essentially two approaches have been used. In the two-step procedure [44-45], the transfer function denominator is realized in the ladder form; and the transmission zeros are then obtained as a linear combination of internal variables in the ladder. This procedure is similar to that used in the continuous domain for the synthesis of singly-terminated reactive two-ports [8]. On the other hand, in the one-step procedure the continued fraction expansion is directly carried out on the transfer function realizing the function poles and zeros simultaneously [15, 41-43].

A broader knowledge of all possible network realizations of a digital transfer function may have practical as well as theoretical interests [42]. This is due in part to the practical constraints associated with the physical implementation that usually cause the network performance to deviate from the desired performance. In this respect, it has been shown [46] that the ladder realizations obtained by continued fractions perform well in the presence of arithmetical roundoff errors. It is, however, desirable to synthesize all possible realizations of a given function so that a complete error analysis of each model can be made [15].

In this chapter, several new digital network structures based on the approach of the one-step procedure are presented. Transfer functions in one and two variables are considered. Some of the structures reported in the literature [15, 41-43], that have been obtained using the same approach, are not implementable (computable) due to the presence of delay-free loops. The new structures avoid this problem. They can also be used to realize arbitrary transfer functions in the continuous (analog) domain.

First, standard forms of continued fractions, and conditions for their existence, are reviewed. The new structures are then introduced. Illustrative examples are included.

2.2 STANDARD EXPANSION FORMS FOR TRANSFER FUNCTION REALIZATION

Consider the following real transfer function in the complex variable x

$$G(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0} \tag{2.1}$$

where,

$x = s$ in the continuous domain

$x = z$ in the discrete domain

and, a_i and b_i $(i = 0, 1, 2, \dots, n)$ are real constants.

Without any loss of generality, the transfer function (2.1) will be regarded as a digital transfer function $G(z)$.

$G(z)$ can be expanded into several continued fraction forms. There are, however, three most important ones in engineering applications [28].

1. <u>The Cauer First Form</u> [15, 28]:

$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z + \cfrac{1}{\ddots + \cfrac{1}{B_n z + \cfrac{1}{A_n}}}}}} \tag{2.2}$$

where the expansion coefficients $\{A_i\}$ and $\{B_i\}$ are real constants.

The above expansion form is termed an "expansion about infinity". This is because of the fact that each term of the continued fraction is obtained by removing the asymptotic value of the function (or the remainder function) at infinity. For this expansion about infinity, the numerator and denominator polynomials are written in their usual form of descending powers of z, and poles at infinity and constants are alternately removed at each division cycle. The division in which the polynomials are written in descending powers of z is usually referred to as "forward division" [4].

2. <u>The Cauer Second Form</u> [15, 28]:

$$G(z) = A_0 + \cfrac{1}{B_1 z^{-1} + \cfrac{1}{A_1 + \cfrac{1}{B_2 z^{-1} + \cfrac{1}{\ddots + \cfrac{1}{B_n z^{-1} + \cfrac{1}{A_n}}}}}} \tag{2.3}$$

where the constants $\{A_i\}$ and $\{B_i\}$ are not the same as those in (2.2).

This expansion is termed an "expansion about zero". Here the polynomials are written in ascending powers of z, each division part of the cycle alternately removing poles at the origin and constants from the function. This division is referred to as "reverse division".

It is well known [4-6] that Cauer first and Cauer second forms are used to synthesize driving point functions of RC (RL) passive ladder networks. Furthermore, the Cauer second form plays a significant role in control system analysis [28].

3. The Stieltjes Form [15, 28, 34]:

$$G(z) = A_0 + \cfrac{1}{B_1 z + A_1 + \cfrac{1}{B_2 z + A_2 + \cfrac{1}{\ddots + \cfrac{1}{B_n z + A_n}}}} \tag{2.4a}$$

where again the constants $\{A_i\}$, $\{B_i\}$ are not the same as those in (2.2) and (2.3).

In this case, a pole at infinity and a constant are simultaneously removed at each division cycle but the first; A_0 vanishes when $a_n = 0$. The performed division is a "forward division". Another Stieltjes expansion form [15] is obtained by performing a "reverse division" as follows:

$$G(z) = A_0 + \cfrac{1}{B_1 z^{-1} + A_1 + \cfrac{1}{B_2 z^{-1} + A_2 + \cfrac{1}{\ddots + \cfrac{1}{B_n z^{-1} + A_n}}}} \tag{2.4b}$$

The Stieltjes expansion form has the application to synthesis of lossy LC Ladder networks [34-36].

2.2.1 Continued Fractions on the Reciprocal of a Transfer Function

This concept is known, in the classical network theory, by the reciprocation theorem for finding the dual of a given ladder network [4]. The process is merely carrying the continued fraction expansion on the reciprocal of the rational function. For instance, the Cauer first form performed on $1/G(z)$ takes the following form:

$$G(z) = \cfrac{1}{\cfrac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}} = \cfrac{1}{A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{\ddots + \cfrac{1}{B_n z + \cfrac{1}{A_n}}}}}} \tag{2.5}$$

where the constants $\{A_i\}$ and $\{B_i\}$ are not the same as those in (2.2).

2.2.2 Existence of a Certain Continued Fraction Form

Necessary and sufficient conditions must be satisfied for a rational function to have a certain continued fraction expansion [15, 19]. For instance, to determine the existence of Cauer first form of (2.2), G(z) is expressed by

$$G(z) = \frac{a_n}{b_n} + \frac{a'_{n-1}z^{n-1} + a'_{n-2}z^{n-2} + \ldots + a'_1 z + a'_0}{z^n + b'_{n-1}z^{n-1} + \ldots + b'_1 z + b'_0} \tag{2.6}$$

where the leading denominator coefficient has been normalized for convenience only.

From the specified G(z) as given by (2.6), the 2n x 2n matrix M is formed as follows:

$$M = \begin{bmatrix} a'_{n-1} & a'_{n-2} & a'_{n-3} & \cdots & a'_2 & a'_1 & a'_0 & 0 & 0 & \cdots \\ 1 & b'_{n-1} & b'_{n-2} & \cdots & b'_3 & b'_2 & b'_1 & b'_0 & 0 & \cdots \\ 0 & a'_{n-1} & a'_{n-2} & \cdots & a'_3 & a'_2 & a'_1 & a'_0 & 0 & \cdots \\ 0 & 1 & b'_{n-1} & \cdots & b'_4 & b'_3 & b'_2 & b'_1 & b'_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{2.7}$$

Let us denote the ith principal minor of M by D_i. For the existence of Cauer first form (2.2), $D_i \neq 0$ for i = 1, 2, 3,..., r where r = 2n-1 if $b_0 = 0$, otherwise, r = 2n [19]. For the existence of Stieltjes form (2.4a), it is only necessary to insure that the odd-numbered principal minors are nonzero, i.e., $D_i \neq 0$ for i = 1, 3, 5, ..., 2n-1. Hence, the existence of Cauer first form implies the existence of Stieltjes form (2.4a). However, the converse is not true.

On the other hand, for the existence of Cauer second form or Stieltjes form (2.4b) similar conditions must be satisfied. One has to, first, write the polynomials of $G(z)$ in ascending powers of z and then rewrite the elements of M accordingly. The existence of Cauer second form implies the existence of Stieltjes form (2.4b) whereas the converse is not true.

An alternative set of necessary and sufficient conditions have been developed by Jury [47] via the rearrangement of the elements of M in inner form. A new set of conditions are developed in Chapter III of this thesis. The new set of conditions determines the expansion coefficients at the same time.

It is worth noting that the existence of Cauer first form (or Stieltjes form (2.4a)) does not necessarily imply the existence of Cauer second form[1] (or Stieltjes form (2.4b)) as the following example illustrates.

2.2.2.1 Example 2.1

Consider the following transfer function

$$G(z) = \frac{z^3 + 4z^2 + z + 1}{z^3 + 5z^2 + 6z + 1} \tag{2.8}$$

---

[1] It is well known, however, that if $G(z)$ is interpreted as a driving point function ($z=s$), then the existence of Cauer first form implies the existence of Cauer second form and visa-versa [4].

It has been shown [14] that the Cauer first form of this function does not exist. However, Cauer second form exists and is given by

$$G(z) = 1 + \cfrac{1}{-0.2z^{-1} + \cfrac{1}{-0.8621 + \cfrac{1}{1.7521z^{-1} + \cfrac{1}{0.6621 + \cfrac{1}{-7.5521z^{-1} + \cfrac{1}{-0.6621}}}}}} . \tag{2.9}$$

2.3 REALIZATION OF A TRANSFER FUNCTION USING THE CONTINUED FRACTION EXPANSION

Some of the well-known applications of Cauer-type and Stieltjes-type forms have been mentioned in the previous section. Recently [15, 41-43], these forms have been applied to synthesis of arbitrary transfer functions. Mitra and Sherwood [15, 41], Hwang [42] and Constantinides [43] have independently presented canonic realizations based on Cauer and Stieltjes expansion forms carried out on the transfer function G(z) or its reciprocoal [41]. Their results are classified according to Mitra and Sherwood [15, 41] and are summarized in Table 2.1. It has been shown that in order to implement a function G(z), one needs building blocks that realize the following functions:

(i) For Cauer-type forms:

$$G_1(z) = \frac{1}{A + T(z)} \tag{2.10}$$

$$G_2(z) = \frac{1}{Bz + T(z)} \tag{2.11}$$

$$G_3(z) = \frac{1}{Bz^{-1} + T(z)} \tag{2.12}$$

Table 2.1

Classification of Canonic ladder realizations of G(z)

	Cauer first	Stieltjes form (2.4a)	Cauer second	Stieltjes form (2.4b)
Expansion carried out on G(z) [15]	Type IA	Type IIA	Type IB	Type IIB
Expansion carried out on 1/G(z) [41]	Type IIIA	Type IVA	Type IIIB	Type IVB

(ii) For Stieltjes-type forms:

$$G_4(z) = \frac{1}{Bz + A + T(z)} \tag{2.13}$$

$$G_5(z) = \frac{1}{Bz^{-1} + A + T(z)} \tag{2.14}$$

Building blocks that realize (2.10) are needed if Stieltjes expansion is carried out on $1/G(z)$.

Fig. 2.1 shows realizations of $G_1(z)$ through $G_5(z)$. The use of these building blocks is best illustrated by an example.

2.3.1 Example 2.2

Consider the transfer function given by (2.8). Referring to Table 2.1, Type IB ladder realization is possible by expanding the function into the Cauer second form given by (2.9). (2.9) is first written as

$$G(z) = 1 + \frac{1}{-0.2z^{-1} + T_1(z)} \tag{2.15a}$$

where

$$T_1(z) = \frac{1}{-0.8621 + T_2(z)} \tag{2.15b}$$

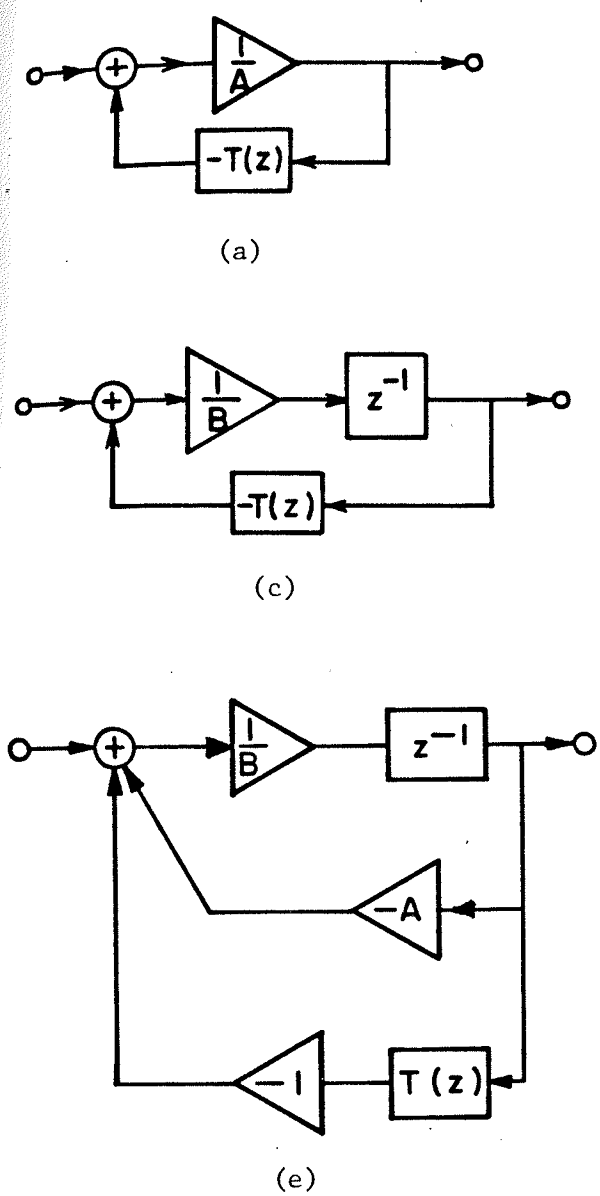

Fig. 2.1 (a) and (b) Sections realizing $G_1(z)$
(c) Section realizing $G_2(z)$
(d) Section realizing $G_3(z)$
(e) Section realizing $G_4(z)$
(f) Section realizing $G_5(z)$

The second term in (2.15a) is realizable the way indicated in Fig. 2.1(d), in which the box $1/T(z)$ is represented by $-0.862 \pm T_2(z)$ of (2.15b). Using a step-by-step procedure, the complete realization of (2.9) is obtained as shown in Fig. 2.2. It is worth noting that other realizations of $G(z)$ are possible. For instance it has been shown [41] that the Cauer first form exists if the expansion is carried out on the reciprocal of $G(z)$ given by (2.8). Hence, referring to Table 2.1, Type III A realization is possible.

It is worth noting that the canonic realization of Fig. 2.2 can be adapted for use in realizing $G(z)$ as a transfer function in the continuous complex variable s. This can simply be achieved by replacing delays (z^{-1}) by integrators (s^{-1}), adders by summing amplifiers ,and multipliers by amplifiers. In this respect, it is important to point out that all the canonic realizations indicated in Table 2.1 can be used in the synthesis of continuous transfer functions. Unfortunately, however, some of these realizations, viz. Type IIB and IVB, can not be used in the synthesis of digital transfer functions due to the presence of delay-free loops. New realizations which avoid this problem will be presented next.

2.4 NEW CANONIC REALIZATIONS OF ARBITRARY TRANSFER FUNCTIONS

2.4.1 Other Expansion Forms for Transfer Function Realization

The digital transfer function is expressed as

$$G(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0} \qquad (2.16)$$

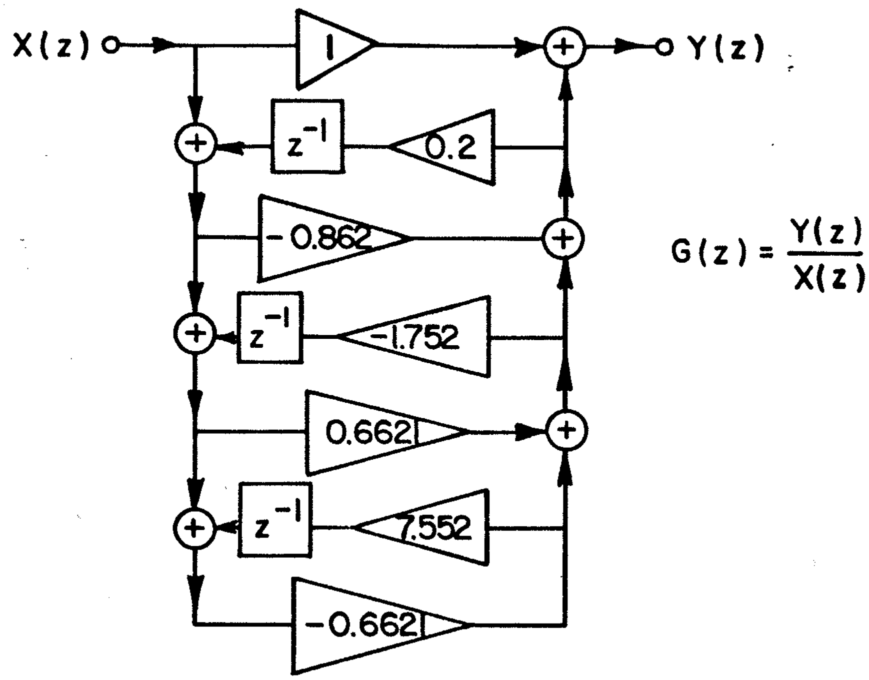

Fig. 2.2 Realization of G(z) of (2.9) in the Type IB ladder form

Assuming even n, G(z) can be expanded into one of the following forms:

Case IA:

$$G(z) = A_0 + \cfrac{1}{A_1 + B_1 z + \cfrac{1}{B_2 z^{-1} + A_2 + \cfrac{1}{\ddots + \cfrac{1}{B_n z^{-1} + A_n}}}} \tag{2.17}$$

where $\{A_i\}$ and $\{B_i\}$ are real constants. In this case a_n is allowed to be zero resulting in $A_0 = 0$. The expansion (2.17), with $A_0 = 0$, is known in the literature by the Cauer third [48] or the mixed Cauer form [26, 49]. It has successfully been applied to control system design [26, 48] and system identification [49]. In particular, in approximating a large linear system, the reduced-order model obtained by mixed Cauer form gives a satisfactory approximation for both the transient and the steady-state responses [26]. Equation (2.17) is a combination of the Cauer first and second forms, carried out on G(z) and 1/G(z), respectively. In other words, for $A_1 = B_2 = A_3 = B_4 \dots = 0$, (2.17) will be reduced to (2.2) and for $A_0 = B_1 = A_2 = B_3 \dots = 0$, (2.17) will assume Cauer second form of 1/G(z).

Case IB:

Here, the expansion is carried out on the reciprocal of G(z) as follows:

$$G(z) = \cfrac{1}{A_0 + \cfrac{1}{A_1 + B_1 z + \cfrac{1}{B_2 z^{-1} + A_2 + \cdots + \cfrac{1}{B_n z^{-1} + A_n}}}} \tag{2.18}$$

One can see that (2.18) is a combination of the Cauer first and second forms carried out on $1/G(z)$ and $G(z)$, respectively. For $A_1 = B_2 = A_3 = B_4 \ldots = 0$, (2.18) will be identical with (2.5) and for $A_0 = B_1 = A_2 = B_3 = \ldots = 0$, (2.18) will be identical with (2.3).

Case IIA:

In this case, $G(z)$ is expanded in such a way that the first division cycle, in Euclid's algorithm, is a "reverse division". The expansion takes the following form:

$$G(z) = A_0 + \cfrac{1}{B_1 z^{-1} + A_1 + \cfrac{1}{A_2 + B_2 z + \cdots + \cfrac{1}{A_n + B_n z}}} \tag{2.19}$$

If $a_0 = 0$, then $A_0 = 0$. For $A_1 = B_2 = A_3 = B_4 = \ldots = 0$, (2.19) will be identical with (2.3) and for $A_0 = B_1 = A_2 = B_3 \ldots = 0$ (2.19) will be identical with (2.5).

Case IIB:

In this case, the expansion is carried out on $1/G(z)$ in such a way that the first division cycle is a "reverse division". The expansion takes the following form

$$G(z) = \cfrac{1}{A_0 + \cfrac{1}{B_1 z^{-1} + A_1 + \cfrac{1}{A_2 + B_2 z + \cfrac{1}{\ddots + \cfrac{1}{A_n + B_n z}}}}} \tag{2.20}$$

For $A_1 = B_2 = A_3 = B_4 \dots = 0$, (2.20) will be identical with Cauer second form of $1/G(z)$, and for $A_0 = B_1 = A_2 = B_3 \dots = 0$, (2.20) will be identical with (2.2).

2.4.2 The New Canonic Structures

In order to implement any of the above four expansions, one needs building blocks that realize the two functions G_4 and $G_5(z)$ given by (2.13) and (2.14), viz. the realization of Figs. 2.1(e) and (f), respectively. Also, one may need the realization of either Fig. 2.1(a) or (b) to implement the first cycle of the expansion forms (2.18) and (2.20).

Using a step-by-step procedure, the final realizations of $G(z)$ as given by (2.17) and (2.18) are obtained as shown in Fig. 2.3 and Fig. 2.4, respectively. The realization of $G(z)$ as given by (2.19) is obtained, with $A_0 = 0$, as shown in Fig. 2.5. And finally, Fig. 2.6 shows the realization of $G(z)$ as given by (2.20). The structures are indicated

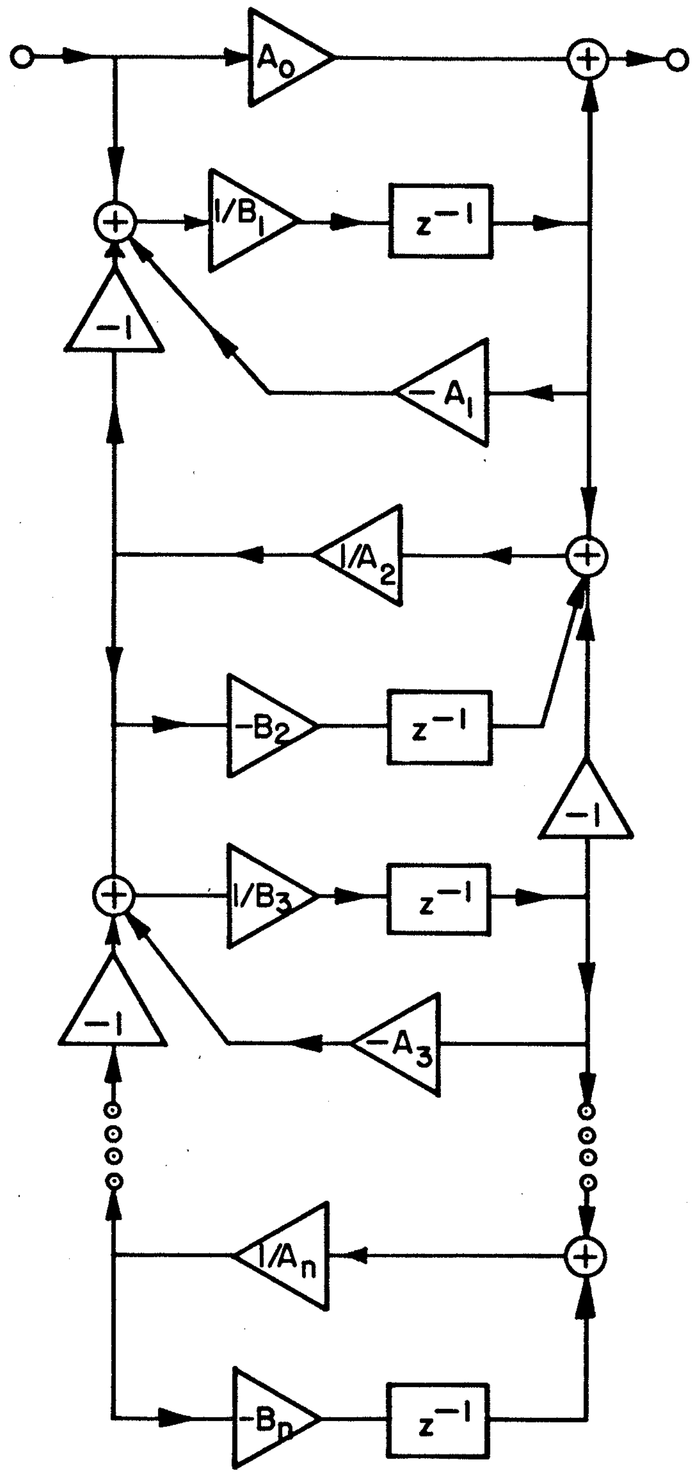

Fig. 2.3 Case IA ladder realization

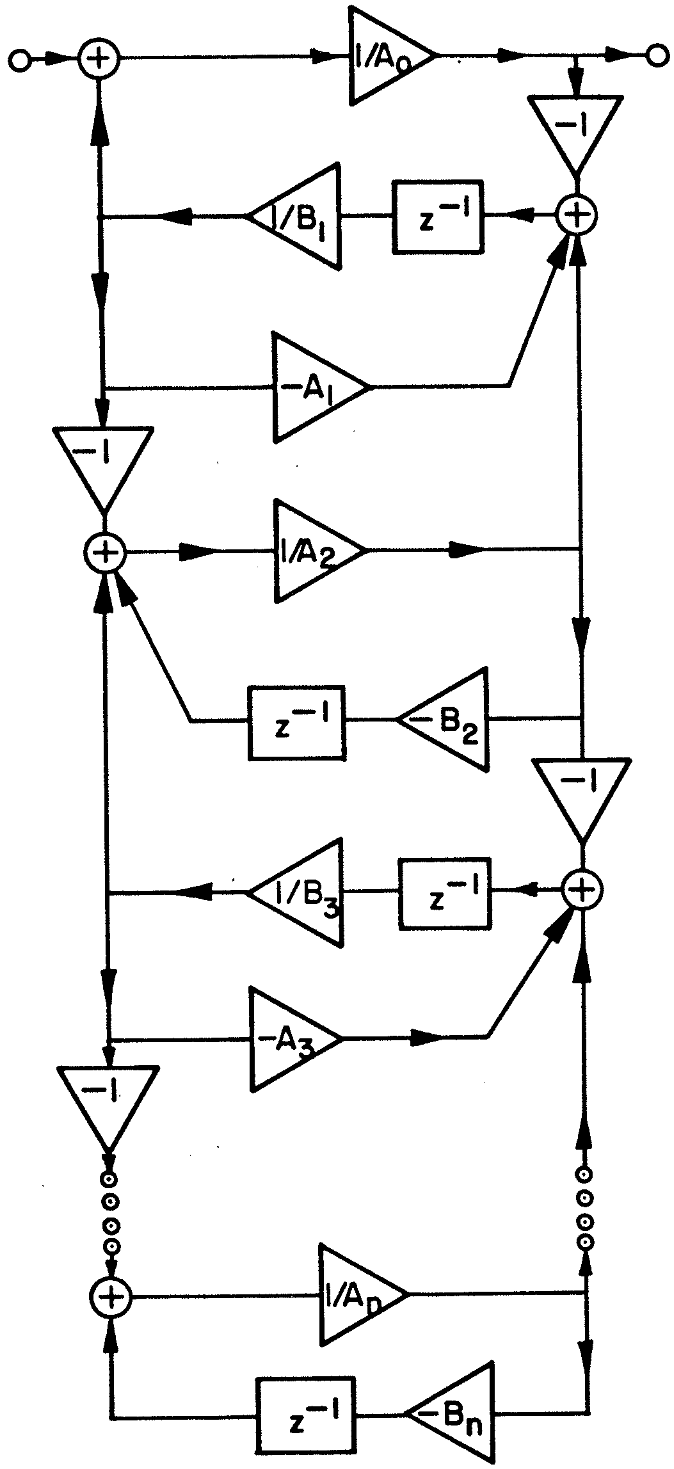

Fig. 2.4 Case IB ladder realization

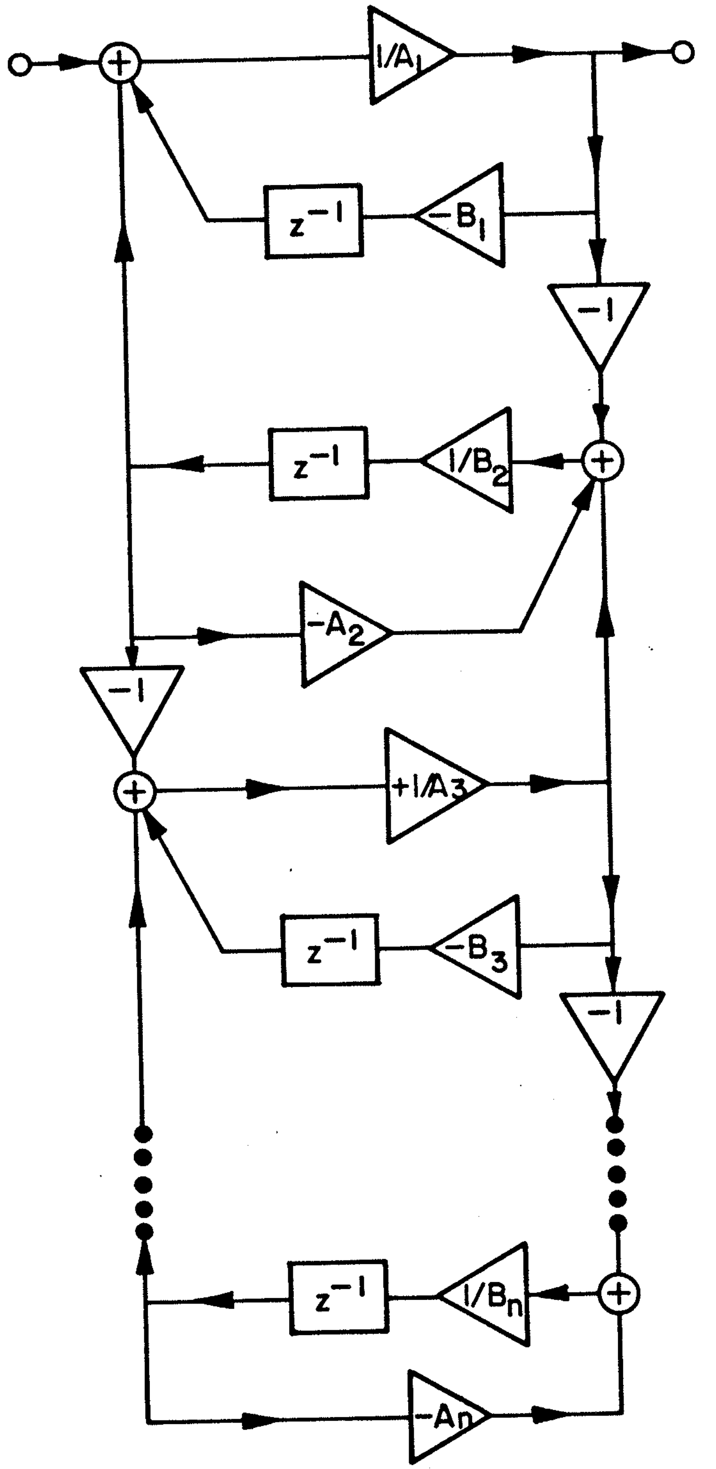

Fig. 2.5 Case IIA ladder realization (with A_0=0)

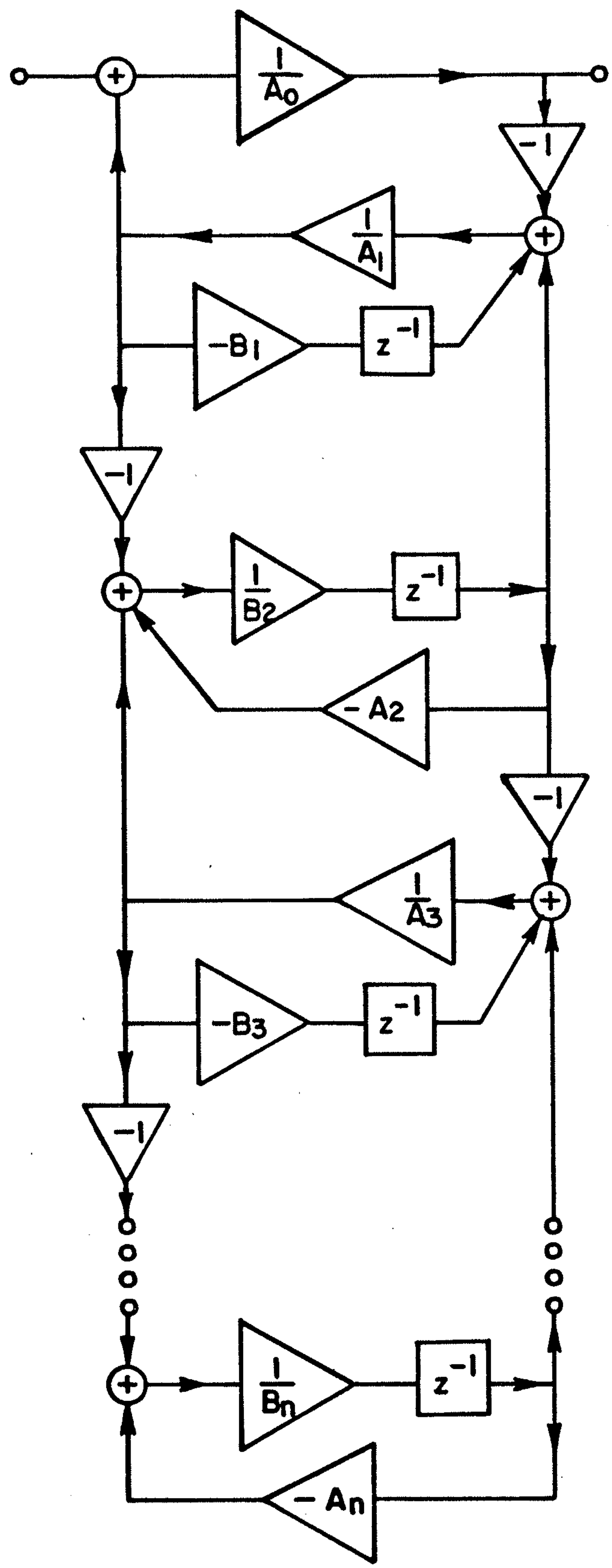

Fig. 2.6 Case IIB ladder realization

for n even. If n is odd, the procedure holds, except that the last stage in both Figs. 2.3 and 2.4 would employ the section shown in Fig. 2.1(e) and the last stage in both Figs. 2.5 and 2.6 would adopt the section shown in Fig. 2.1(f). All the structures are canonic with respect to both delays (n delays) and multipliers (2 n + 1 multipliers). One can see that no delay-free loops exist in each of the structures of Fig. 2.3, 2.4 and 2.5, hence they are implementable in the discrete z-domain. In Fig. 2.6, one may notice the presence of a delay-free loop through the multipliers $1/A_0$ and $1/A_1$. Fig. 2.6 makes use of Fig. 2.1(a) in implementing the first cycle of (2.20). Using the alternative building block of Fig. 2.1(b) will also result in a delay-free loop. The presence of a delay-free loop does not matter if the structure is to be used in realizing a continuous transfer function where the delays are replaced by integrators, the adders by summing amplifiers and the multipliers by amplifiers. However in order to implement digital transfer functions using (2.20), one must have either $A_0 = 0$ ($b_0 = 0$) or $A_1 = 0$ in (2.20). For instance, if $A_0 = 0$,(2.20) is written as

$$G(z) = B_1 z^{-1} + A_1 + \cfrac{1}{A_2 + B_2 z + \cfrac{1}{\ddots + \cfrac{1}{A_n + B_n z}}} \tag{2.21}$$

which is simply implementable as shown in Fig. 2.7. On the other hand if $A_1 = 0$, two realizations are possible as indicated in Figs. 2.8(a) and 2.8(b). In Fig. 2.8(a), the first cycle of the expansion is

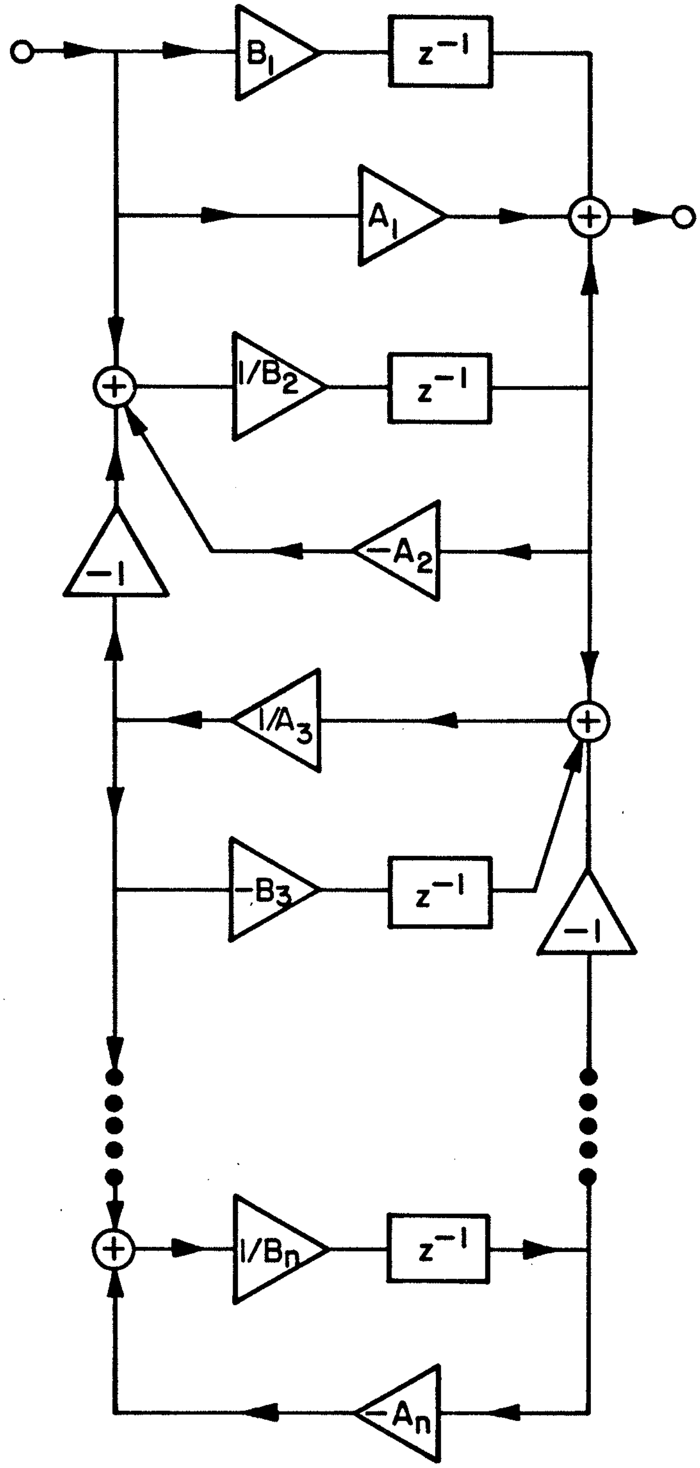

Fig. 2.7 An implementable Case IIB ladder realization obtained if $A_0=0$

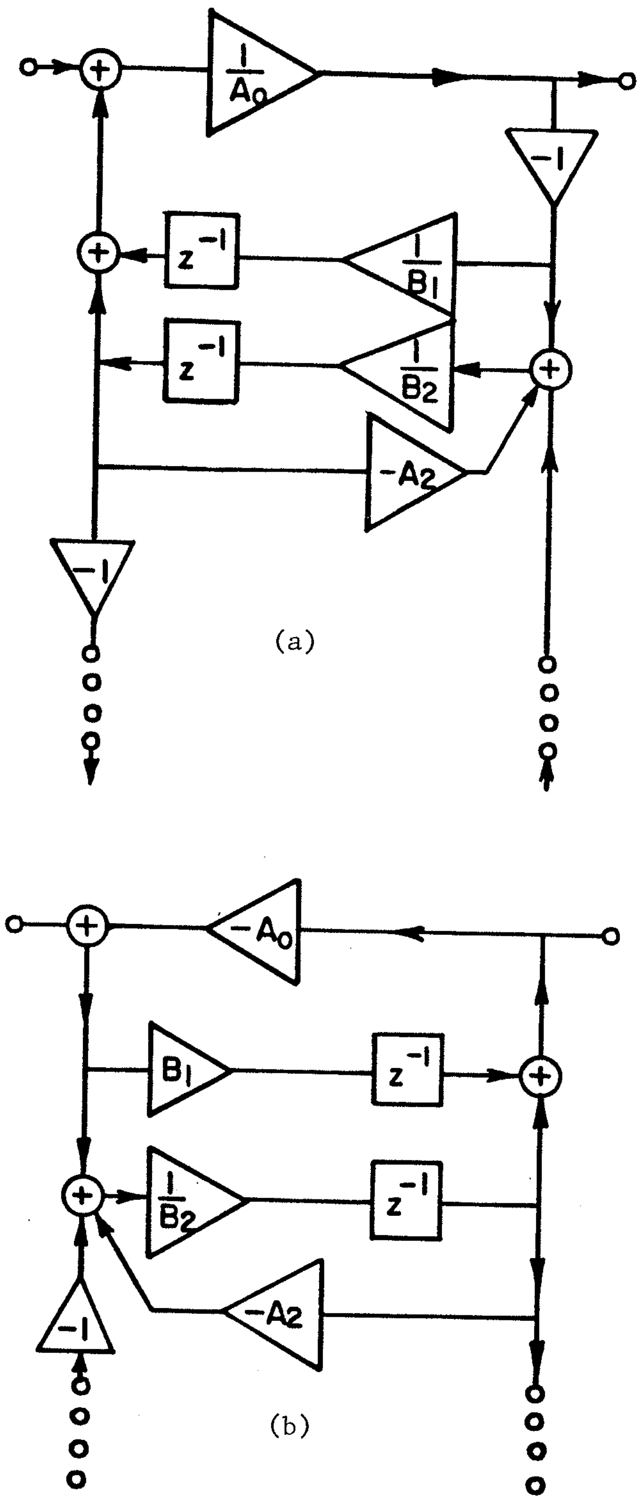

Fig. 2.8 Implementable Case IIB ladder realizations obtained if $A_1=0$
(a) The first stage is implemented according to Fig. 2.1(a)
(b) The first stage is implemented according to Fig. 2.1(b)

implemented according to Fig. 2.1(a) whereas in Fig. 2.8(b) it is implemented according to Fig. 2.1(b).

An interesting observation regarding the new realizations is the fact that although the above expansions are combinations of Cauer first and second forms, the resulting structures are mixtures of types IIA (IV A) and IIB (IV B) of Table 2.1 that are based on Stieltjes expansion forms. Furthermore, Types IA, IB, IIIA and IIIB [15, 41] ladder realizations of Table 2.1 are merely special cases of the new ladder structures. For instance, Type IA [15] ladder realization is obtainable as a special case of Case IA ladder realization of Fig. 2.3 by letting $A_1 = B_2 = A_3 = B_4 \cdots = 0$.

Finally, an important concept in digital network theory [50] is that of network transposition [50-51]. The transposition of a network is defined as the operation of reversing the direction of all of the branches in the network. The resulting network is referred to as the transpose network. In this respect, as if to underscore the uniqueness of the automatic order reduction associated with continued fractions, each of the new structures is virtually its own transpose. In other words, if the new realizations are redrawn with the multipliers of (-1) absorbed into the other multiplier values, then each new realization will be self-adjoint [51].

2.4.3 Realizability Conditions

As with continued fraction and most other realization methods there are some degenerate cases when the realizations discussed here do not exist. As pointed out in Subsection 2.2.2, necessary and sufficient conditions must be satisfied for a rational function to have a certain continued fraction expansion. To the best of the author's knowledge the conditions for the mixed-Cauer [26, 48-49] continued fraction expansion are unavailable in the literature. However, in a way analogous to the two-pair extraction method[2] [52-55] where conditions are established for each iteration [53], necessary and sufficient conditions for the existence of mixed-Cauer forms can be established for each division cycle. For instance, in the first division cycle of each of the four expansions (2.17) through (2.20) one needs for realizability that $a_n/b_n \neq a_0/b_0$. If $A_0 = 0$ in (2.19) one needs for realizability that $a_n \neq 0$ and $b_1 - A_1 a_1 \neq 0$ in the first division cycle. Similar conditions have to apply at each division cycle.

Intuitively, one may assume that if both Cauer first and second forms exist[3] for G(z) or 1/G(z), the mixed Cauer form exists also for G(z) or 1/G(z), respectively. However, one can show that if G(z) (or its reciprocal) is expandable into Cauer first only or Cauer second only, the mixed Cauer form may or may not exist.

New sets of necessary and sufficient conditions for the existence of the mixed Cauer form are given in chapter III of this thesis.

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2 Recently [55], it has been pointed out that the ladder realizations based on the two-pair extraction method are in fact obtainable from those of [15].

3 In classical network theory [7], if G(z), for z=s, is interpreted as a driving-point function then the existence of one Cauer form implies the existence of the other two. In this respect, it is noted that well known text books, written by distinct authors [4-5] overlooked or did not mention the Cauer-third form.

Generally, it may be noted that relatively strong relationships must exist between the transfer function coefficients in order for an expansion to be possible. This situation indicates that a useful design technique might be to assume that a particular expansion form is desired and then to evaluate the coefficients of G(z) to meet other design criteria subject to the expansion constraints. In this respect, the recent results of Jones and Steinhardt [46] are encouraging. They have shown that every digital transfer function represented by a directed graph realization based on the continued fraction method is a stable function. This would insure an implementation in the desired expansion form with the resulting savings in the number of delays and multipliers that must be used.

2.4.4 Illustrative Examples

2.4.4.1 Example 2.3

Consider the digital transfer function

$$G(z) = \frac{1200z^4 + 320z^3 + 270z^2 + 45z + 1}{600z^4 + 100z^3 + 129z^2 + 10z} \tag{2.22}$$

This function has been considered in [15]. It has been shown [15] that both Types IA and IIA realizations of Table 2.1 do not exist for this function. Types IB and IIB have been shown to be possible. Type IIB is not implementable due to the presence of delay-free loops. Hence, one is left with Type IB only as an implementable digital realization. In the following, however, it will be shown that two more implementable realizations for G(z) of (2.22) are possible using the proposed new ladder structures.

Case IA: Expand G(z) according to (2.17):

$$G(z) = 2 + \cfrac{1}{5z + 0.0 + \cfrac{1}{3 + 0.2z^{-1} + \cfrac{1}{-5z + 0.5435 + \cfrac{1}{0.1472 + 0.1693z^{-1}}}}} \tag{2.23}$$

Note the zero term in the second division cycle. It appears because $b_0 = 0$ in (2.22). The realization is given in Fig. 2.9.

Case IIB: since $b_0 = 0$, use the form (2.21)

$$G(z) = 0.1z^{-1} + 2 + \cfrac{1}{0.8264 + 10z + \cfrac{1}{1.9064z^{-1} + 1.9724 + \cfrac{1}{-0.0928 - 0.4447z}}} \tag{2.24}$$

which is implementable according to Fig. 2.7. The multiplier constants are (B_1, A_1, $1/B_2$, $-A_2$, $1/A_3$, $-B_3$, $1/B_4$, $-A_4$) = (0.1, 2, 0.1, -0.8264, 0.5070, -1.9064, -2.2489, 0.0928).

A common problem in the design is that of range of values for multipliers [15]. The mixed-Cauer-based structures may have relatively small ranges of multiplier values. The ratio of the largest to the smallest multiplier constant for Case IIB is

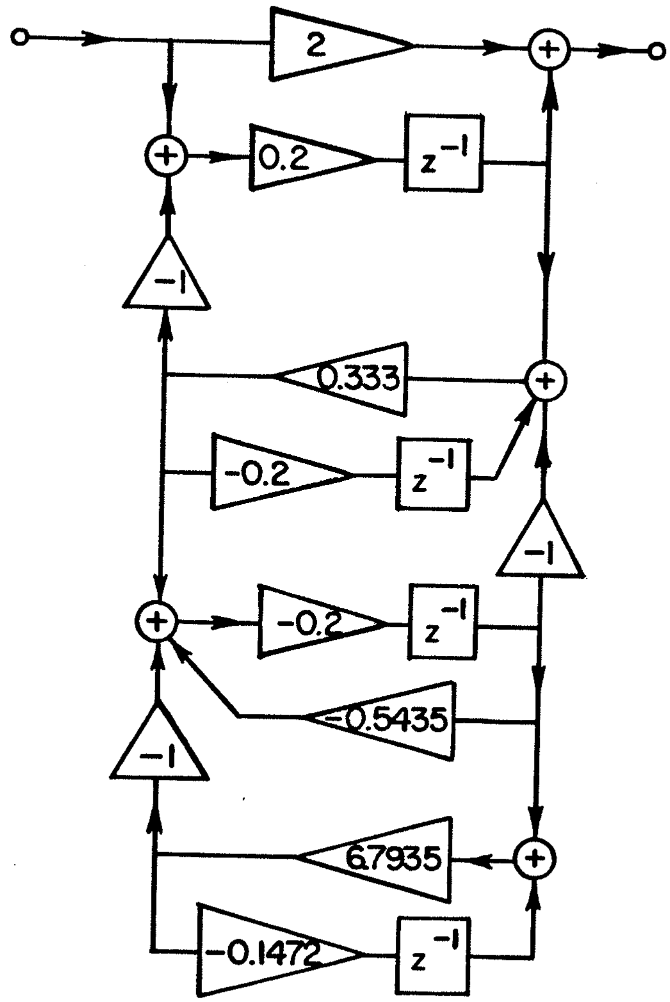

Fig. 2.9 Case IA ladder realization of G(z) of (2.22)

2.2489/0.0928 or 24.2338. The same filter function has been realized in [15] using Type IB ladder. The ratio for Type IB is 29.98/0.0120 or 2498 which is more than 100 times as high as Case IIB.

2.4.4.2 Example 2.4

Consider the following digital transfer function

$$G(z) = \frac{z^3 + 4z^2 + z + 1}{z^3 + 5z^2 + 6z + 1} \tag{2.8}$$

This function has been considered in Example 2.1 where it has been shown that Cauer first expansion does not exist whereas Cauer second does, which has the realization of Fig. 2.2.

One may notice that $a_3/b_3 = a_0/b_0 = 1$. This makes the existence of any of the expansion forms (2.17) through (2.20) impossible as pointed out earlier. To see that, let us perform the expansion (2.17) of Case IA

$$\begin{array}{r|l}
 & \quad 1 \\ \cline{2-2}
z^3 + 5z^2 + 6z + 1 & z^3 + 4z^2 + z + 1 \\
 & z^3 + 5z^2 + 6z + 1 \\ \cline{2-2}
 & \quad -z^2 - 5z
\end{array}
\qquad
\begin{array}{|l}
\quad -z \\ \hline
z^3 + 5z^2 + 6z + 1 \\
z^3 + 5z^2 \\ \hline
\qquad\quad 6z + 1
\end{array}$$

Now, if one proceeds according to (2.17), a constant term cannot be removed. However, one can remove a z^{-1} term followed by the desired constant term as follows

$$\begin{array}{r|l}
 & -\frac{1}{5}\,z^{-1} \\ \hline
-z^2 - 5z & 6z + 1 \\
 & \frac{1}{5}z + 1 \\ \hline
 & \frac{29}{5}\,z
\end{array}$$

Divide divisor and dividend by z and proceed in the same cycle.

$$\begin{array}{r|l}
 & -\frac{29}{25} \\ \hline
-z - 5 & \frac{29}{5} \\
 & \frac{29}{5} + \frac{29}{25}\,z \\ \hline
 & -\frac{29}{25}\,z
\end{array}$$

Now proceed to the next cycle as follows:

$$\begin{array}{r|l}
 & \frac{25}{29} + \frac{125}{29}\,z^{-1} \\ \hline
-\frac{29}{25}\,z & -z - 5 \\
 & -z \\ \hline
 & -5 \\
 & \underline{-5} \\
 & 0\;0
\end{array}$$

The final form of the expansion is

$$G(z) = 1 + \cfrac{1}{-z - \frac{29}{25} - \frac{1}{5}\,z^{-1} + \cfrac{1}{\frac{25}{29} + \frac{125}{29}\,z^{-1}}} \tag{2.25}$$

The realization of (2.25) is shown in Fig. 2.10 which takes the form of Case IA realization of Fig. 2.3 with a slight modification, that is a delay and a multiplier are added in the feedback path of the second stage to implement (-1/5). z^{-1} of the second cycle of (2.25). The values of the multiplier constants are (A_0, $1/B_1$, $-A_1$, $1/A_2$, $-B_2$, + 1/5) = (1, -1, 29/25, 29/25, -125/29, + 1/5). The ratio of the largest to the smallest multiplier is 21.557. Referring to Table 2.1, one can show that, out of the realizations indicated in the first row, only Type IB is possible. The expansion is given by (2.9) from which one can see that the ratio of largest to smallest multiplier values is 7.5521/0.2 or 37.7606. This ratio is approximately 1.75 times that of the modified Case IA ladder realization. Another advantage of the new structure, which is unexpected, over that of Type IB is the fact that it uses one multiplier less.

2.4.4.3 Example 2.5

In this example, it is shown that additional new structures may be generated by combining the proposed structures with those of Table 2.1. In so doing, and to avoid the presence of delay free loops,one should avoid the following situations:

(i) Two consecutive division cycles that belong to either Type IIB or IVB of Table 2.1.

(ii) Two consecutive division cycles, while proceeding in the expansion, that are a constant followed by a cycle of Stieltjes form (2.4b) or vice-versa.

(iii) Situations similar to Type IIIB when $a_n \neq 0$ [41].

Now consider the following digital transfer function

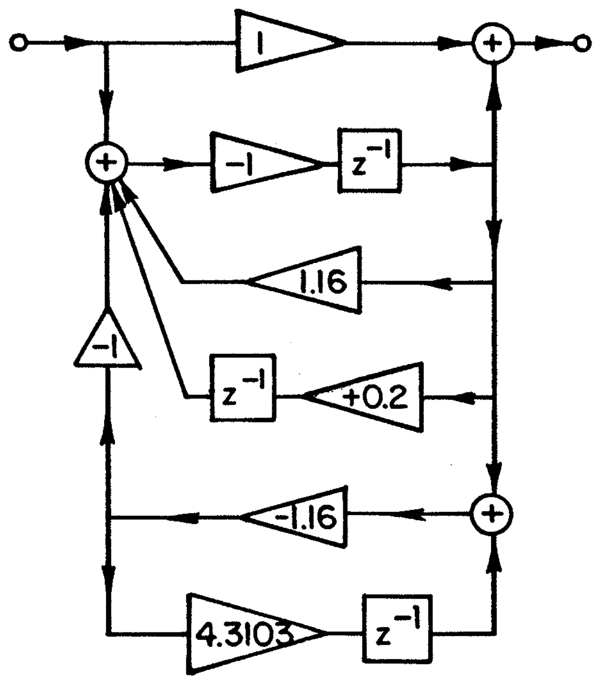

Fig. 2.10 Realization of G(z) of (2.25) in the modified Case IA ladder

$$G(z) = \frac{720z^2 + 240z + 12}{720z^3 + 600z^2 + 72z + 1} \tag{2.26}$$

which has been considered in [15].

One may carry out the first two division cycles according to Type IA of Table 2.1 as follows

$$G(z) = \cfrac{1}{z + \cfrac{1}{2 + T(z)}} \tag{2.27}$$

where

$$T(z) = \frac{120z + 10}{360z^2 + 60z + 1} \tag{2.28}$$

T(z) is then expanded according to Case IA as follows

$$T(z) = \cfrac{1}{3z + 0.1 + \cfrac{1}{0.5556z^{-1} + 6.666}} \tag{2.29}$$

The final realization of G(z) is shown in Fig. 2.11 which has no delay-free loops.

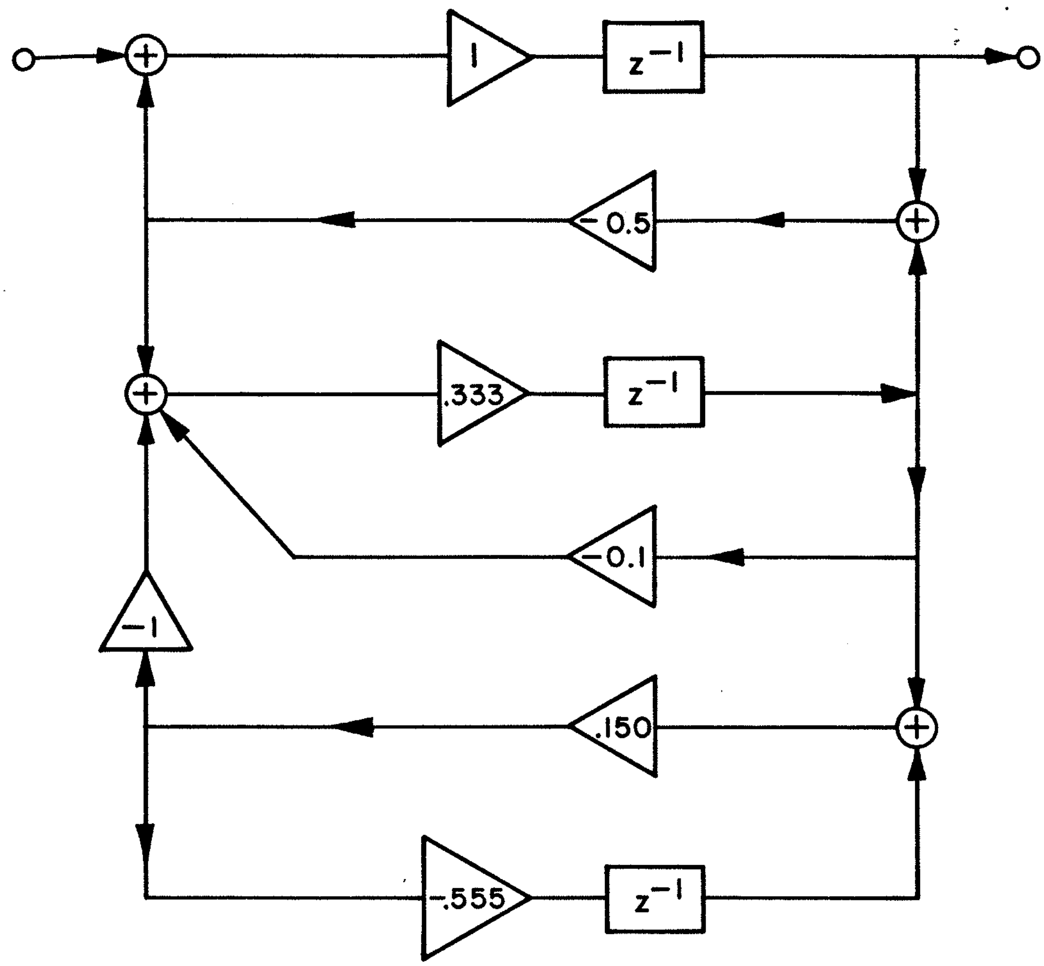

Fig. 2.11 Realization of G(z) of Example 2.5

2.5 NEW REALIZATIONS OF TWO-DIMENSIONAL TRANSFER FUNCTIONS

The digital transfer function of a two-dimensional (2-D) system can be written as a ratio of 2-D polynomials in the following manner [18, 56-57]

$$G(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} \tag{2.30}$$

where

$$A(z_1, z_2) = \sum_{i=0}^{M_a} \sum_{j=0}^{N_a} a_{ij} z_1^i z_2^j \tag{2.31a}$$

and

$$B(z_1, z_2) = \sum_{i=0}^{M_b} \sum_{j=0}^{N_b} b_{ij} z_1^i z_2^j \tag{2.31b}$$

The realization of (2.30) has the potential applications to the processing of seismic records in geophysical industry, and the enhancement of photographic data such as weather photos and medical x-rays [56].

Recent implementations [18, 58-59] have made use of the continued fraction method. In this respect, the expansion forms used are the extensions of one-dimensional expansions [15, 41-43]. For instance, the one-dimensional ladder realizations of Table 2.1 have been extended by Mitra et al. [18] to the 2-D case. Similarly, new 2-D ladder realizations can be obtained as extensions of the one-dimensional ladder structures presented earlier. For example, if one can expand the transfer function $G(z_1, z_2)$ as

$$G(z_1, z_2) = C_1 + \cfrac{1}{A_1 z_1 + C_2 + \cfrac{1}{B_1 z_2^{-1} + C_3 + \cfrac{1}{A_2 z_1 + C_4 + \cfrac{1}{\ddots}}}} \tag{2.32}$$

then, one can obtain a realizable structure as shown in Fig. 2.12 which is the extension of Case IA ladder realization of Fig. 2.3. Another structure can be obtained by simply interchanging the roles of z_1 and z_2 in (2.32). The expansion (2.32) includes Types IA and IIIB of [18] as special cases.

The operations involved in 2-D continued fractions are the usual divide-and-invert cycles of Euclid's algorithm, discussed earlier in the one-dimensional case. In the 2-D case, however, the added dimension materially increases the stringency of the conditions for the existence of a certain continued fraction form. Generally, extension of continued fractions to 2-D functions requires careful ordering of the numerator and denominator terms. For instance, in (2.32), in order to extract the constant term C_1 followed by a z_1 term one has to write the numerator and denominator in decreasing powers of z_1 as follows: (without missing terms):

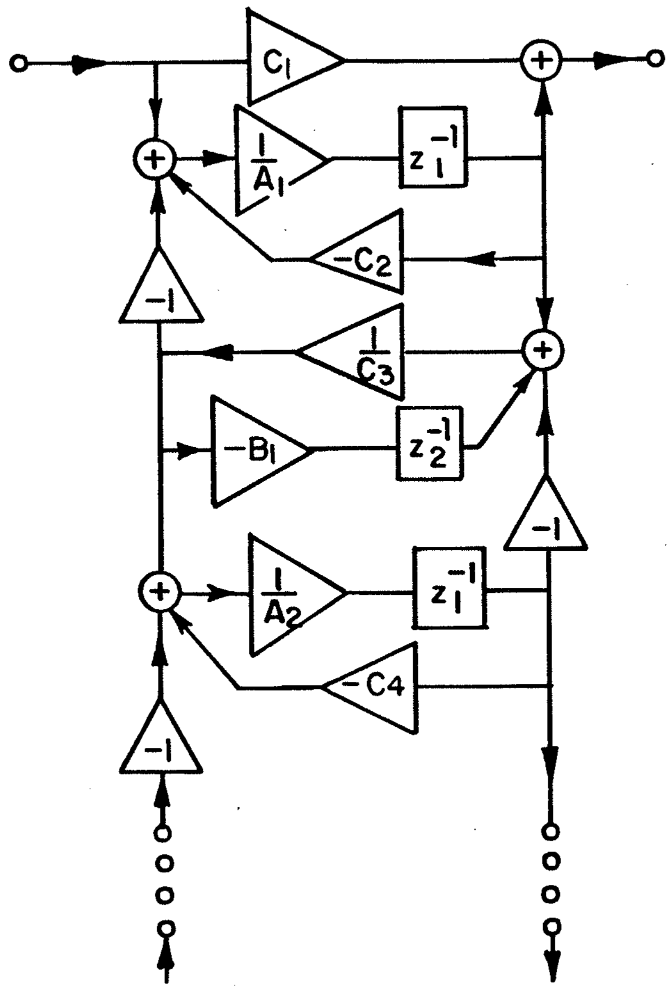

Fig. 2.12 Case IA ladder realization in two dimensions

	$z_1^m z_2^n$	$z_1^m z_2^{n-1}$	...	$z_1^m z_2$	$z_1^{m-1} z_2^n$	...	$z_1^{m-1} z_2^0$	...	$z_1^0 z_2^n$	$z_1^0 z_2^{n-1}$	...	$z_1^0 z_2^0$
Num.	a_{mn}	$a_{m,n-1}$	...	$a_{m,0}$	$a_{m-1,n}$	...	$a_{m-1,0}$	...	$a_{0,n}$	$a_{0,n-1}$	...	a_{00}
Den.	b_{mn}	$b_{m,n-1}$	...	$b_{m,0}$	$b_{m-1,n}$	...	$b_{m-1,0}$	...	$b_{0,n}$	$b_{0,n-1}$	...	b_{00}
Rem.	0	0	...	0	$c_{m-1,n}$	...	$c_{m-1,0}$	...	$c_{0,n}$	$c_{0,n-1}$	...	c_{00}

The constant coefficient, C_1 is given by a_{mn}/b_{mn}. As in the one-dimensional case, the first term c_{mn} in the remainder polynomial is zero. The next n terms must also be zero. If they are not, further steps in the expansion will not be able to reduce the degree of resulting terms. This would indicate that this expansion fails. The z_1 coefficient, A_1, in the expansion is then given by $b_{mn}/c_{m-1,n}$. Again the first n+1 terms in the next remainder must be zero for the expansion to continue. The next two coefficients C_2 and B_1, in the expansion (2.32) are obtained by reordering the remainder polynomials as increasing powers of z_2, ... etc. Thus, as pointed out in [18], the conditions for existence of a certain expansion form are established for each division cycle. The following example illustrates the advanced operations.

2.5.1 Example 2.6

Consider the following 2-D transfer function

$$G(z_1, z_2) = \frac{48z_1^2z_2 + 12z_1^2 + 144z_1z_2 + 33z_1 + 81z_2 + 15}{48z_1^2z_2 + 12z_1^2 + 120z_1z_2 + 27z_1 + 27z_2 + 2} \tag{2.33}$$

The above function has been considered elsewhere [18]. A realization has been given [18] for $G(z_1, z_2)$ as Type IA. An alternative realization is possible using a combination of Type[4] IIA [18] and Case IA of Fig. 2.12. One can expand $G(z_1, z_2)$ using the Stieltjes form in the first two cycles then proceed using the mixed Cauer form as follows:

$$G(z_1, z_2) = 1 + \cfrac{1}{2z_1 + \frac{1}{2} + \cfrac{1}{-12z_2 - \frac{26}{9} + \cfrac{1}{-\frac{81}{4}z_1^{-1} - 9}}} \tag{2.34}$$

in which the following operations are performed:

Since $G(z_1, z_2)$ is written in decreasing powers of z_1, then one can carry out the first two cycles

$$\begin{array}{r|l} & \quad 1 \\ \hline 48z_1^2z_2 + 12z_1^2 + 120z_1z_2\ldots & 48z_1^2z_2 + 12z_1^2 + 144z_1z_2\ldots \\ & 48z_1^2z_2 + 12z_1^2 + 120z_1z_2\ldots \\ \cline{2-2} & 24z_1z_2 + 6z_1 + 54z_2 + 13 \end{array}$$

$$\begin{array}{l|l} & \quad 2z_1 + \\ \hline 24z_1z_2 + 6z_1 + 54z_2 + 13 & 48z_1^2z_2 + 12z_1^2 + 120z_1z_2\ldots \\ & 48z_1^2z_2 + 12z_1^2 + 108z_1z_2\ldots \\ \cline{2-2} & 12z_1z_2 + z_1 + 27z_2 + 2 \end{array}$$

---

[4] It is assumed here that z_1 or z_2 delay elements cannot be implemented. Hence, Types IIA and IVA of [18] can be implemented with no delay-free loops using z_1^{-1} and z_2^{-1} delays adopted here. Type IIA will then be indicated by Fig. 10, in [18], in which z_1 and z_2 are replaced by z_1^{-1} and z_2^{-1}, respectively.

Now, one has to reorder the divisor and dividend polynomials in decreasing powers of z_2, remove a constant then proceed to the next cycle which is a mixed-Cauer:

$$
\begin{array}{r|l}
 & +\frac{1}{2} \\
\cline{2-2}
24z_1z_2+54z_2+6z_1+13 & 12z_1z_2+27z_2+z_1+2 \\
 & 12z_1z_2+27z_2+3z_1+\frac{13}{2} \\
\cline{2-2}
 & \qquad\qquad -2z_1-\frac{9}{2}
\end{array}
$$

$$
\begin{array}{l}
\qquad -12z_2-\frac{26}{9} \\
\hline
24z_1z_2+54z_2+6z_1+13 \\
24z_1z_2+54z_2 \\
\hline
\qquad\qquad 6z_1+13 \\
\qquad\qquad \frac{52}{9}\,z_1+13 \\
\hline
\qquad\qquad \frac{2}{9}\,z_1
\end{array}
$$

$$
\begin{array}{r|l}
 & -9-\frac{81}{4}\,z_1^{-1} \\
\cline{2-2}
\frac{2}{9}\,z_1 & -2z_1-\frac{9}{2} \\
 & -2z_1-\frac{9}{2} \\
\cline{2-2}
 & \quad 00
\end{array}
$$

Chapter III

NOVEL APPROACHES TO CONTINUED FRACTION EXPANSION AND INVERSION

3.1 INTRODUCTION

Expanding a rational function into a continued fraction and inverting a continued fraction into a rational function are two fundamentally important operations in network analysis and synthesis, both analog and digital, control systems and stability tests, etc. [4-6, 15, 25, 28, 60]. The continued fraction expansion involves many divisions whereas the inversion is related to many multiplications. The actual execution of these two operations is in general laborious and tedious, particularly the continued fraction inversion [28]. Methods have been presented by many authors [25, 28, 34, 48, 61-71] in order to avoid or simplify the actual execution of these drudgery operations. In this respect, one of the most popular methods is Routh's algorithm [72].

Routh's algorithm and continued fractions were first associated by Wall [19]. Since then, Routh's algorithm has been applied to network theory problems in general [73-75] including continued fraction expansion and inversion [28], [48], [71]. In the applications of Routh's algorithm the elements of Routh's array which are actually used are those of the first column [72].

In this chapter, a simple explicit recursive formula for the elements of the first column of Routh's array is presented. The use of this formula will eliminate the need for constructing the whole array [72]. Using this formula, closed forms for Cauer first and Cauer second expansion coefficients, in terms of the rational function coefficients, are derived and tabulated. The closed forms are considered as direct representation of the divide-and-invert cycles, in Euclid's algorithm, by simple algebraic equations. The forms in question are used very efficiently and equally well in both continued fraction expansion and inversion as they are simple and easy-to-use. Closed forms are similarly derived for mixed Cauer and Stieltjes expansion forms. It is worth noting that these forms can be interpreted as explicit expressions for the elements of two-element kind (RC, RL, LC) Cauer-type passive ladder networks, the elements of lossy RLC ladder networks of the Stieltjes-type [34-37], or the multiplier elements of the ladder structures discussed in Chapter II. Moreover, they could be used in deriving a new set of necessary and sufficient conditions for a rational function to have a certain continued fraction expansion.

3.2 ELEMENTS OF THE FIRST COLUMN OF ROUTH'S ARRAY

Routh's algorithm has a wide range of applications in network synthesis, control system analysis and stability tests [25, 60, 73]. It is a computational method that develops a sequence of computed numbers from two rows of numbers. The array is usually written in the following form:

$$\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots \end{array}$$

Computed rows:

A_{31}	A_{32}	A_{33}	A_{34}
A_{41}	A_{42}	A_{43}	...
A_{51}	A_{52}	A_{53}	...

etc.

The two rows are normally coefficients of a rational function. The third, fourth and subsequent rows are evaluated from the following relation [73].

$$A_{j,k} = A_{j-2,k+1} - \frac{A_{j-2,1}A_{j-1,k+1}}{A_{j-1,1}} \qquad j = 3,4\ldots;\ k = 1,2,\ldots \tag{3.1}$$

3.2.1 The Recursive Formula

In most of the applications of Routh's algorithm, the elements of the first column of the array are actually used. Next, a recursive formula is proposed for the direct computation of the elements of the first column without laborious generation of the whole array.

Given the following two rows

a_n	a_{n-1}	a_{n-2}	.....	a_1	a_0
b_n	b_{n-1}	b_{n-2}	.....	b_1	b_0

Define the vector A as

$$A = [A_0(n) \quad A_1(n) \quad A_2(n) \quad \ldots\ldots \quad A_{2n+1}(n)]^t$$

where $A_i(n)$, $i = 0,1,2,\ldots, 2n+1$ are the elements of the first column of Routh's array.

$A_i(n)$ is expressed in the following recursive formula:

$$A_i(n) = A_{i-2}(n-1) - \alpha_{i-1}A_{i-1}(n-1) \qquad i = 2, 3, \ldots \tag{3.2}$$

with

$$A_0(n) = a_n \qquad A_1(n) = b_n$$

and

$$\alpha_i = \frac{A_{i-1}(n)}{A_i(n)} \qquad i = 1, 2, \ldots \tag{3.3}$$

A proof for (3.2) is given in Appendix A.

Using (3.2), $A_2(n)$ is given by

$$A_2(n) = a_{n-1} - \alpha_1 b_{n-1}$$

where

$$\alpha_1 = \frac{A_0(n)}{A_1(n)}$$

and $A_3(n)$ is given by

$$A_3(n) = b_{n-1} - \alpha_2(a_{n-2} - \alpha_1 b_{n-2})$$

where

$$\alpha_2 = \frac{A_1(n)}{A_2(n)}$$

etc.

Generally, the expression of $A_i(n)$ is systematically generated from the preceding two expressions $A_{i-1}(n)$ and $A_{i-2}(n)$ by simply replacing the coefficients a_k, b_k, ... by a_{k-1}, b_{k-1}, ... respectively. It should be pointed out that since $a_m = b_m = 0$, $m < 0$ the expression of $A_i(n)$ becomes shorter after halfway, eventually yielding last entries

$$A_{2n}(n) = a_0 \qquad b_0 = 0$$

$$A_{2n+1}(n) = b_0 \qquad b_0 \neq 0$$

3.2.2 Example 3.1

The example given by Chen and Shieh [28] is used here. The following function

$$G(s) = \frac{10\ s^2 + 171\ s + 360}{s^3 + 71\ s^2 + 702\ s + 720}$$

is to be expanded using Cauer second form as follows

$$G(s) = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 s^{-1} + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_{2n} s^{-1} + \cfrac{1}{\alpha_{2n+1}}}}}}}$$

In order to expand G(s) in the above form, the generating two rows a_n, a_{n-1}, ..., a_o and b_n, b_{n-1}, ..., b_o should read

$$\begin{array}{llll} a_3 = 720 & a_2 = 702 & a_1 = 71 & a_0 = 1 \\ b_3 = 360 & b_2 = 171 & b_1 = 10 & b_0 = 0 \end{array}$$

using the recursive formula given by (3.2), one has

$$\begin{array}{ll}
 & A_0(n) = a_3 = 720 \\
\alpha_1 = 2 & \\
 & A_1(n) = b_3 = 360 \\
\alpha_2 = 1 & \\
 & A_2(n) = a_2 - \alpha_1 \underline{b_2} = 360 \\
\alpha_3 = 3 & \\
 & A_3(n) = \underline{b_2} - \alpha_2 \, \underline{(a_1 - \alpha_1 b_1)} = 120 \\
\alpha_4 = 5 & \\
 & A_4(n) = \underline{a_1 - \alpha_1 b_1} - \alpha_3 \, \underline{(b_1 - \alpha_2 a_0)} = 24 \\
\alpha_5 = 6 & \\
 & A_5(n) = \underline{b_1 - \alpha_2 a_0} - \alpha_4 \underline{a_0} = 4 \\
\alpha_6 = 4 & \\
 & A_6(n) = \underline{a_0} = 1
\end{array}$$

Note that the repetition of terms in the expressions of $A_i(n)$ could be used to save computation time.

It is worth noting that the recursive formula (3.2) is general in the sense that it can be applied to both Cauer first and Cauer second expansions. For instance, if G(s) is to be expanded in Cauer first form, namely

$$G(s) = \cfrac{1}{\alpha_1 s + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 s + \begin{matrix} \cdot \\ \cdot \end{matrix}}}}$$

the generating two rows would read

$$\begin{array}{llll} a_3 = 1 & a_2 = 71 & a_1 = 702 & a_0 = 720 \\ b_3 = 10 & b_2 = 171 & b_1 = 360 & b_0 = 0 \end{array}$$

Subsequently $A_i(n)$, i = 0, 1,...,6 are obtained as follows

$A_0(n) = 1 \quad A_1(n) = 10 \quad A_2(n) = 53.9 \quad A_3(n) = 47.44$

$A_4(n) = 407.74 \quad A_5(n) = 142.86 \quad A_6(n) = 720$

3.3 CLOSED FORMS AND TABLES FOR THE CONTINUED FRACTION EXPANSION COEFFICIENTS

3.3.1 Cauer First and Cauer Second

The expression for α_i given by (3.3) is new in the sense that it is an explicit form of the continued fraction expansion coefficients expressed in terms of preceding coefficients and the given function coefficients. As pointed out in Example 3.1, α_i given by (3.3) can be used for both Cauer first and Cauer second. An alternative expression for α_i that is more useful, particularly in continued fraction inversion, is derived next.

Consider the transfer function

$$G(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0} \tag{3.4}$$

G(x) can be expanded into one of the following Cauer first forms

Case a:

$$G(x) = \alpha_1 + \cfrac{1}{\alpha_2 x + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots}}} \tag{3.5}$$

Case b:

$$G(x) = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 x + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots}}}} \tag{3.6}$$

where, (see Appendix A)

$$\alpha_1 = \frac{A_o(n)}{A_1(n)}$$

and

$$\alpha_i = \frac{1}{\prod_{j=1}^{i-1} \alpha_j} \cdot \frac{A_o(n)}{A_i(n)} \qquad i = 2,3,\ldots,2n+1 \qquad (3.7)$$

where $A_i(n)$ is given by (3.2) with $A_o(n) = a_n$ and $A_1(n) = b_n$ for Case a, and $A_o(n) = b_n$ and $A_1(n) = a_n$ for Case b. The scheme of direct computation of α_i using the recursive closed form (3.7) for both Case a and b is found in Table 3.1. For Cauer second form, if one writes G(x) as follows:

$$G(x) = \frac{a_n + a_{n-1}x + \ldots + a_1x^{n-1} + a_ox^n}{b_n + b_{n-1}x + \ldots + b_1x^{n-1} + b_ox^n} \qquad (3.8)$$

then the scheme in Table 3.1 can readily be used when the expansion is carried out on G(x) (Case a) or its reciprocol (Case b). (3.7) is considered as direct representation of the divide-and-invert cycles, of Cauer-type continued fraction expansion, by simple algebraic equations [70]. Due to the cyclic nature of the continued fraction expansion a recursive pattern similar to (3.7), or (3.2), would appear in all types of continued fraction expansion. Mixed-Cauer and Stieltjes types are considered in the following subsections.

3.3.2 Mixed Cauer Form

For this type consider G(x) given by (3.4), with $a_n = 0$ for convenience only. Then, G(x) is expanded as follows

$$G(x) = \cfrac{1}{\alpha_1 x + \beta_1 + \cfrac{1}{\alpha_2 + \beta_2 x^{-1} + \cfrac{1}{\alpha_3 x + \beta_3 \\ \ddots}}} \qquad (3.9)$$

Table 3.1 Coefficients of continued fraction expansion of Cauer first and second forms

	Recursive Formula: $A_i(n) = A_{i-2}(n-1)-\alpha_{i-1}A_{i-1}(n-1)$		
case a	$\alpha_1 = a_n$	$\cdot\ [b_n]^{-1}$	$A_0(n) = a_n \quad A_1(n) = b_n$
	$\alpha_2 = \dfrac{a_n}{\alpha_1}$	$\cdot\ [a_{n-1}-\alpha_1 b_{n-1}]^{-1}$	
	$\alpha_3 = \dfrac{a_n}{\alpha_1\alpha_2}$	$\cdot\ [b_{n-1}-\alpha_2(a_{n-2}-\alpha_1 b_{n-2})]^{-1}$	
	$\alpha_4 = \dfrac{a_n}{\alpha_1\alpha_2\alpha_3}$	$\cdot\ [a_{n-2}-\alpha_1 b_{n-2}-\alpha_3(b_{n-2}-\alpha_2(a_{n-3}-\alpha_1 b_{n-3}))]^{-1}$	
	$\alpha_5 = \dfrac{a_n}{\alpha_1\alpha_2\alpha_3\alpha_4}$	$\cdot\ [b_{n-2}-\alpha_2(a_{n-3}-\alpha_1 b_{n-3})-\alpha_4(a_{n-3}-\alpha_1 b_{n-3}-\alpha_3(b_{n-3}-\alpha_2(a_{n-4}-\alpha_1 b_{n-4}))))]^{-1}$	
	$\vdots$		
	$\alpha_{2n} = \dfrac{a_n}{\alpha_1\alpha_2 \cdots \alpha_{2n-1}}$	$\cdot\ [a_0]^{-1}$	$b_0 = 0$
	$\alpha_{2n+1} = \dfrac{a_n}{\alpha_1\alpha_2 \cdots \alpha_{2n}}$	$\cdot\ [b_0]^{-1}$	$b_0 \neq 0$
case b	$\alpha_1 = b_n$	$\cdot\ [a_n]^{-1}$	$A_0(n) = b_n \quad A_1(n) = a_n$
	$\alpha_2 = \dfrac{b_n}{\alpha_1}$	$\cdot\ [b_{n-1}-\alpha_1 a_{n-1}]^{-1}$	
	$\alpha_3 = \dfrac{b_n}{\alpha_1\alpha_2}$	$\cdot\ [a_{n-1}-\alpha_2(b_{n-2}-\alpha_1 a_{n-2})]^{-1}$	
	$\alpha_4 = \dfrac{b_n}{\alpha_1\alpha_2\alpha_3}$	$\cdot\ [b_{n-2}-\alpha_1 a_{n-2}-\alpha_3(a_{n-2}-\alpha_2(b_{n-3}-\alpha_1 a_{n-3}))]^{-1}$	
	$\alpha_5 = \dfrac{b_n}{\alpha_1\alpha_2\alpha_3\alpha_4}$	$[a_{n-2}-\alpha_2(b_{n-3}-\alpha_1 a_{n-3})-\alpha_4(b_{n-3}-\alpha_1 a_{n-3}-\alpha_3(a_{n-3}-\alpha_2(b_{n-4}-\alpha_1 a_{n-4}))))]^{-1}$	
	$\vdots$		
	$\alpha_{2n} = \dfrac{b_n}{\alpha_1\alpha_2 \cdots \alpha_{2n-1}}$	$[b_0]^{-1}$	$a_0 = 0$
	$\alpha_{2n+1} = \dfrac{b_n}{\alpha_1\alpha_2 \cdots \alpha_{2n}}$	$[a_0]^{-1}$	$a_0 \neq 0$

It can be shown that

$$\alpha_1 = \frac{b_{n-1}}{a_n}$$

$$\alpha_i = \frac{1}{\prod_{j=1}^{i-1} \alpha_j} \cdot \frac{A_o(n)}{A_i(n)} \qquad i = 2,3,\ldots,n \tag{3.10a}$$

and,

$$\beta_1 = \frac{b_o}{a_o}$$

$$\beta_i = \frac{1}{\prod_{j=1}^{i-1} \alpha_j} \cdot \frac{A_o(n)}{A_i(n)} \qquad i = 2,3,\ldots,n \tag{3.10b}$$

where in (3.10a), $A_i(n)$ is expressed as

$$A_i(n) = A_{i-2}(n-1) - \alpha_{i-1}A_{i-1}(n-1) - \beta_{i-1}A_{i-1}(n) \tag{3.11a}$$

with,

$$A_o(n) = b_{n-1} \qquad A_1(n) = a_n$$

and in (3.10b), $A_i(n)$ is expressed as

$$A_i(n) = A_{i-2}(n+1) - \beta_{i-1}\, A_{i-1}(n+1) - \alpha_{i-1}\, A_{i-1}(n) \tag{3.11b}$$

with

$$A_o(n) = b_o \qquad A_1(n) = a_o$$

Note that the third term in both (3.11a) and (3.11b) represents the mixed nature of the expansion. The scheme of direct computation is found in Table 3.2. It should be pointed out that Shieh and Goldman [48] have developed a generalized Routh array for the expansion form (3.9). In this respect, it is worth noting that $A_i(n)$ given by (3.11a) and (3.11b) are recursive formulas for the elements A_{11}, A_{21}, $A_{31}, \ldots,$

Table 3.2 Coefficients of mixed Cauer Continued Fraction Expansion

Recursive Formulas:

$$A_i(n) = A_{i-2}(n-1) - \alpha_{i-1}A_{i-1}(n-1) - \beta_{i-1}A_{i-1}(n) \qquad (3.11a)$$

$$, A_0(n) = b_n \quad A_1(n) = a_{n-1}$$

$$A_i(n) = A_{i-2}(n+1) - \beta_{i-1}A_{i-1}(n+1) - \alpha_{i-1}A_{i-1}(n) \qquad (3.11b)$$

$$, A_0(n) = b_0 \quad A_i(n) = a_0$$

$$\alpha_1 = b_n \; [a_{n-1}]^{-1}$$

$$\beta_1 = b_0 \; [a_0]^{-1}$$

$$\alpha_2 = \frac{b_n}{\alpha_1} [b_{n-1} - \alpha_1 \; a_{n-2} - \beta_1 \; a_{n-1}]^{-1}$$

$$\beta_2 = \frac{b_0}{\beta_1} [b_1 - \beta_1 \; a_1 - \alpha_1 \; a_0]^{-1}$$

$$\alpha_3 = \frac{b_n}{\alpha_1\alpha_2} [a_{n-2} - \alpha_2 \; (b_{n-2}-\alpha_1 a_{n-3}-\beta_1 a_{n-2}) - \beta_2(b_{n-1}-\alpha_1 a_{n-2}-\beta_1 a_{n-1})]^{-1}$$

$$\beta_3 = \frac{b_0}{\beta_1\beta_2} [a_1-\beta_2(b_2-\beta_1 a_2-\alpha_1 a_1) - \alpha_2 \; (b_1-\beta_1 a_1-\alpha_1 a_0)]^{-1}$$

.

.

.

etc. and the elements $A_{1,n+1}$, $A_{2,n}$, $A_{3,n-1}$, ... etc., respectively, of the triangular pattern of the generalized Routh array developed in [48].

The mixed Cauer form (3.9) is used in Chapter II in the realization of Case IA Ladder structure. Other mixed Cauer forms are developed in Chapter II. The derivation of a computation scheme, similar to that of Table 3.2, for each of these forms becomes routine and straightforward; and hence is omitted.

3.3.3 <u>The Stieltjes Form</u>

Here, G(x) of (3.4) is expanded as follows

$$G(x) = \alpha_1 + \cfrac{1}{\alpha_2 x + \alpha_3^{-1} + \cfrac{1}{\alpha_4 x + \alpha_5^{-1} + \cfrac{1}{\ddots + \cfrac{1}{\alpha_{2n} x + \alpha_{2n+1}^{-1}}}}} \tag{3.12}$$

Cauer I - Case a ⟵|⟶ Modified version of Cauer I-Case a

In this case, one may observe that the odd-ordered α's (starting from α_3 onwards) have been expressed in the form α_i^{-1}, which has intentionally been made in order to facilitate obtaining the recursive formula. In fact, this makes one inversion cycle of this expansion equivalent to two cycles of Cauer first-Case a in Euclid's algorithm [71]. The important consequence of this is the fact that the recursive formula $A_i(n)$ for $i \leq 3$ will be the same as that given by (3.2) whereas for $i > 3$, $A_i(n)$ will be modified versions of (3.2) for both even and odd values of i. To see this one writes

$$\alpha_1 = \frac{A_o(n)}{A_1(n)} = \frac{a_n}{b_n}$$

$$\alpha_2 = \frac{a_n}{\alpha_1} [A_2(n)]^{-1}$$

$$\alpha_3 = \frac{a_n}{\alpha_1 \alpha_2} [A_3(n)]^{-1} \tag{3.13a}$$

where $A_2(n)$ and $A_3(n)$ are obtained according to (3.2). For $i > 3$ one has,

$$\alpha_i = \frac{1}{\prod_{j=1}^{(i-2)/2} \alpha_{2j}} \cdot \frac{A_1(n)}{A_i(n)} \qquad i = 4,6,8,\ldots,2n \tag{3.13b}$$

where

$$A_i(n) = A_{i-1}(n-1) - \alpha_{i-1}^{-1} A_{i-2}(n-1) \tag{3.13c}$$

with $A_2(n)$ and $A_3(n)$ as initial conditions.

Also

$$\alpha_i = \frac{1}{\prod_{j=1}^{(i-1)/2} \alpha_{2j}} \cdot \frac{A_1(n)}{A_i(n)} \qquad i = 5,7,9,\ldots,2n+1 \tag{3.13d}$$

where

$$A_i(n) = A_{i-3}(n-1) - \alpha_{i-1} A_{i-1}(n-1) \tag{3.13e}$$

with $A_2(n)$ and $A_4(n)$ as initial conditions.

The scheme of direct computation of α_i is found in Table 3.3. Note that if the expansion is carried out on the reciprocol of G(x), the recursive formulas will be the same as above with a_k and b_k interchanged (Case b in Table 3.3). Another Stieltjes form results when the expansion cycles are carried out in the "reverse division". In this case, if one writes G(x) in the form of (3.8) then the scheme in Table 3.3 can readily be used.

Table 3.3 Coefficients of Stieltjes Continued Fraction Expansion

Recursive Formulas:

$$A_i(n) = A_{i-1}(n-1) - \alpha_{i-1}^{-1} A_{i-2}(n-1) \qquad i_{even} \neq 2$$

$$A_i(n) = A_{i-3}(n-1) - \alpha_{i-1} A_{i-1}(n-1) \qquad i_{odd} \neq 1,3$$

case a

$$\alpha_1 = a_n \cdot [b_n]^{-1}$$

$$\alpha_2 = \frac{a_n}{\alpha_1} \cdot [a_{n-1} - \alpha_1 b_{n-1}]^{-1}$$

$$\alpha_3 = \frac{a_n}{\alpha_1\alpha_2} \cdot [b_{n-1} - \alpha_2(a_{n-2} - \alpha_1 b_{n-2})]^{-1}$$

$$\alpha_4 = \frac{b_n}{\alpha_2} \cdot [b_{n-2} - \alpha_2 (a_{n-3} - \alpha_1 b_{n-3}) - \alpha_3^{-1}(a_{n-2} - \alpha_1 b_{n-2})]^{-1}$$

$$\alpha_5 = \frac{b_n}{\alpha_2\alpha_4} \cdot [a_{n-2} - \alpha_1 b_{n-2} - \alpha_4(b_{n-3} - \alpha_2(a_{n-4} - \alpha_1 b_{n-4}) - \alpha_3^{-1}(a_{n-3} - \alpha_1 b_{n-3}))]^{-1}$$

$$\vdots$$

case b

$$\alpha_1 = b_n \cdot [a_n]^{-1}$$

$$\alpha_2 = \frac{b_n}{\alpha_1} \cdot [b_{n-1} - \alpha_1 a_{n-1}]^{-1}$$

$$\alpha_3 = \frac{b_n}{\alpha_1\alpha_2} \cdot [a_{n-1} - \alpha_2(b_{n-2} - \alpha_1 a_{n-2})]^{-1}$$

$$\alpha_4 = \frac{a_n}{\alpha_2} \cdot [a_{n-2} - \alpha_2(b_{n-3} - \alpha_1 a_{n-3}) - \alpha_3^{-1}(b_{n-2} - \alpha_1 a_{n-2})]^{-1}$$

$$\alpha_5 = \frac{a_n}{\alpha_2\alpha_4} \cdot [b_{n-2} - \alpha_1 a_{n-2} - \alpha_4(a_{n-3} - \alpha_2(b_{n-4} - \alpha_1 a_{n-4}) - \alpha_3^{-1}(b_{n-3} - \alpha_1 a_{n-3}))]^{-1}$$

$$\vdots$$

It should be pointed out that a modified Routh array for Stieltjes type expansion is available in the literature [34-35, 71]. In this respect, it has been shown [71] that equations (3.13c) and (3.13e) are in fact recursive formulas for the elements of the first column of the modified Routh array.

3.4 CONDITIONS FOR THE EXISTENCE OF A CERTAIN CONTINUED FRACTION EXPANSION

Necessary and sufficient conditions must be satisfied for a rational function to have a certain continued fraction expansion. As pointed out in Chapter II, conditions for the existence of different expansion types, except the mixed Cauer type, are available in the literature [19, 47]. The conditions are formulated in a matrix form where for a rational function of degree n, at least 2n-1 determinants must be evaluated [47]. It should be noted that Wall's matrix [19] is presented in minor array form whereas Jury's matrix [47] is presented in the inner form.

A new set of necessary and sufficient conditions can be obtained using the recursive formula $A_i(n)$ introduced above and its relationship with Routh's algorithm of the conventional (Cauer first and second), the generalized (mixed Cauer), or the modified (Stieltjes) array. In this respect, it is well known that in the conventional Routh's array [73] if a first-column element vanishes, strict interpretation of Routh's algorithm breaks down. As a consequence, the associated Cauer first (or second) expansion fails.[5] Therefore, one can simply see that the condi-

[5] It is assumed that the numerator and denominator of the rational function are relatively prime, ie., they contain no common factors [76].

Table 3.4

Conditions for existence of Certain Continued Fraction Expansions

Expansion	Existence Condition $A_i(n) \neq 0$
Cauer first and/or Cauer Second	$A_i(n)$ given by (3.2)
Mixed Cauer	$A_i(n)$ given by (3.11a) and (3.11b)
Stieltjes*	$A_i(n)$ given by (3.13c) (also $A_2(n) \neq 0$)

*If $A_i(n)$ given by (3.13e) vanishes, the expansion still exists [71].

tions for existence of Cauer first or Cauer second form is that $A_i(n)$ given by (3.2) must not be zero for $i = 2,3,\ldots, 2n+1$. Similar conditions for mixed Cauer and Stieltjes forms are obtained in Table 3.4.

Unlike the existing methods of checking for the existence of a certain expansion, the use of the recursive formula $A_i(n)$ has the advantage that it provides the expansion coefficients while at the same time checking existence of the expansion. Moreover, it systematizes the computations and provides a structure which gives insight into the expansion process. Hence, it allows one to easily check alternative paths of expansion should one path fail. This, for instance, has the useful application in checking all the possible canonic realizations of an arbitrary transfer function.

3.5 CONTINUED FRACTION INVERSION

3.5.1 Inversion of Cauer First and Second Expansions

The existing methods of continued fraction inversion of Cauer first and second expansions involve either state space formulation [28] or Routh's array construction [25,61-67] or both [28]. Chen [68] has offered closed forms, for inversion of Cauer second, which are found rather complicated.

The approach here is to make use of the recursive formula and Table 3.1 presented earlier. It will be shown next, by an illustrative example, that our approach is simple, straightforward and requires less work. In fact, inversion is implemented directly through the solution of very simple algebraic equations [70] using the fact that the recursive formula $A_i(n)$ takes simpler forms as i approaches $2n+1$.

3.5.1.1 Example 3.2

Given the following expansion:

$$G(s) = \cfrac{1}{2+\cfrac{1}{s^{-1}+\cfrac{1}{3+\cfrac{1}{5s^{-1}+\cfrac{1}{6+\cfrac{1}{4s^{-1}}}}}}}$$

Find the corresponding rational function.

Solution:

i. One writes

$n = 3$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 3$, $\alpha_4 = 5$, $\alpha_5 = 6$ and $\alpha_6 = 4$

ii. Using Table 3.1 - Case b, write expressions for α_1, $\alpha_2, \ldots, \alpha_6$

$$\alpha_1 = b_3[A_3]^{-1} = 2$$

$$\alpha_2 = \frac{b_3}{\alpha_1}[b_2 - \underline{\alpha_1 a_2}]^{-1} = 1$$

$$\alpha_3 = \frac{b_3}{\alpha_1\alpha_2}[\underline{a_2} - \alpha_2(\underline{b_1 - \alpha_1 a_1})]^{-1} = 3$$

$$\alpha_4 = \frac{b_3}{\alpha_1\alpha_2\alpha_3}[\underline{b_1 - \alpha_1 a_1} - \alpha_3(\underline{a_1 - \alpha_2 b_o})]^{-1} = 5$$

$$\alpha_5 = \frac{b_3}{\alpha_1\alpha_2\alpha_3\alpha_4}[\underline{a_1 - \alpha_2 b_o} - \underline{\alpha_4 b_o}]^{-1} = 6$$

$$\alpha_6 = \frac{b_3}{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}[\underline{b_o}]^{-1} = 4$$

iii. Rewrite the expressions of α_i starting from α_6 backward solving them for the rational function coefficients one by one. The rational function coefficients will then be given by:

Let $b_3 = 1$

$$b_0 = \frac{b_3}{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} = \frac{1}{720}$$

$$a_1 = \frac{b_3}{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} + \alpha_2 b_0 + \alpha_4 b_0 = \frac{1}{72}$$

$$b_1 = \frac{b_3}{\alpha_1\alpha_2\alpha_3\alpha_4} + \alpha_1 a_1 + \alpha_3(a_1 - \alpha_2 b_0) = \frac{71}{720}$$

$$a_2 = \frac{b_3}{\alpha_1\alpha_2\alpha_3} + \alpha_2(b_1 - \alpha_1 a_1) = \frac{171}{720}$$

$$b_2 = \frac{b_3}{\alpha_1\alpha_2} + \alpha_1 a_2 = \frac{702}{720}$$

$$a_3 = \frac{b_3}{\alpha_1} = \frac{1}{2}$$

G(s) will be given by:

$$G(s) = \frac{\frac{1}{2} + \frac{171}{720}s + \frac{1}{72}s^2}{1 + \frac{702}{720}s + \frac{71}{720}s^2 + \frac{1}{720}s^3}$$

3.5.2 <u>Inversion of the Mixed Cauer Expansion</u>

Shieh and Goldman [48] have proposed an inversion method that makes use of Routh's array. Here the recursive formulas (3.11a) and (3.11b) and Table 3.2 are used. Unlike Cauer first (or Cauer second) form, expressions of $A_i(n)$ do not become shorter after halfway. In spite of this, the inversion is still implemented via solution of simple algebraic equations as the following example illustrates.

3.5.2.1 Example 3.3

A continued fraction expansion is given by

$$G(s) = \cfrac{1}{s + 1 + \cfrac{1}{2 + 3s^{-1} + \cfrac{1}{3s + 2}}}$$

Find the corresponding rational function.

Solution:

i. One writes

$$n = 3, \qquad \alpha_1 = 1 \qquad \alpha_2 = 2 \qquad \alpha_3 = 3$$
$$\beta_1 = 1 \qquad \beta_2 = 3 \qquad \beta_3 = 2$$

ii. Using Table 3.2, write expressions for α_1, β_1, α_2, β_2, α_3, β_3

$$\alpha_1 = b_3\,[a_2]^{-1} = 1 \qquad \text{(Ex3-1)}$$

$$\beta_1 = b_o\,[a_o]^{-1} = 1 \qquad \text{(Ex3-2)}$$

$$\alpha_2 = \frac{b_3}{\alpha_1}\,\underline{[b_2 - \alpha_1 a_1 - \beta_1 a_2]}^{-1} = 2 \qquad \text{(Ex3-3)}$$

$$\beta_2 = \frac{b_0}{\beta_1}\,\underline{[b_1 - \beta_1\,a_1 - \alpha_1 a_o]}^{-1} = 3 \qquad \text{(Ex3-4)}$$

$$\alpha_3 = \frac{b_3}{\alpha_1\alpha_2}\,[\underline{a_1} - \alpha_2\,\underline{(b_1-\alpha_1 a_o-\beta_1 a_1)} - \beta_2\,\underline{(b_2-\alpha_1 a_1-\beta_1 a_2)}]^{-1} = 3 \qquad \text{(Ex3-5)}$$

$$\beta_3 = \frac{b_0}{\beta_1\beta_2}\,[\underline{a_1} - \beta_2\,\underline{(b_2-\beta_1 a_2-\alpha_1 a_1)} - \alpha_2\,\underline{(b_1-\beta_1 a_1-\alpha_1 a_o)}]^{-1} = 2 \qquad \text{(Ex3-6)}$$

iii. Solve the above expressions in the rational function coefficients as follows: (Note the repetition of terms in the above expressions). Assume $b_o=1$, then using (Ex3-6):

$$a_1 - \beta_2 (b_2-\beta_1 a_2-\alpha_1 a_1) - \alpha_2 (b_1-\beta_1 a_1-\alpha_1 a_o) = \frac{b_o}{\beta_1 \beta_2 \beta_3} = \frac{1}{6}$$

and from (Ex3-5), b_3 is determined by

$$b_3 = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 = 1$$

Now using (Ex3-1) and (Ex3-2), one respectively has

$$a_2 = \frac{b_3}{\alpha_1} = 1$$

and

$$a_0 = \frac{b_o}{\beta_1} = 1$$

The rest of the coefficients are determined as follows:

From (Ex3-3), one has

$$b_2 - \alpha_1 a_1 - \beta_1 a_2 = \frac{b_3}{\alpha_1 \alpha_2} = \frac{1}{2} \tag{Ex3-7}$$

and from (Ex3-4)

$$b_1 - \beta_1 a_1 - \alpha_1 a_o = \frac{b_o}{\beta_1 \beta_2} = \frac{1}{3} \tag{Ex3-8}$$

Use (Ex3-7) and (Ex3-8) in (Ex3-6)

$$a_1 = \frac{14}{6}$$

Use (Ex3-7)

$$b_2 = \frac{23}{6}$$

and from (Ex3-8)

$$b_1 = \frac{22}{6}$$

Then G(s) is given by

$$G(s) = \frac{s^2 + \frac{14}{6} s + 1}{s^3 + \frac{23}{6} s^2 + \frac{22}{6} s + 1}$$

3.5.3 Inversion of Stieltjes Type Expansion

Most of the published inversion methods [25,48,61-68,70] have been restricted to inversion of Cauer type expansions. Chen and Chang [69] have proposed a method that can be applied to inversion of Stieltjes expansion. Their method, however, involves construction and multiplication of 2n transmission matrices. Recently [71], an inversion method that makes use of the modified Routh's array has been reported. A simpler method [71] based on the recursive formulas $A_i(n)$, given by (3.13c) and (3.13e), and Table 3.3 is used in the inversion of Stieltjes expansion. In this case, however, although the recursive formula $A_i(n)$ does take simpler forms as i approaches 2n+1 one may solve simultaneous equations to get the first few function coefficients. The rest of the coefficients are obtained one by one. The following example illustrates the above fact.

3.5.3.1 Example 3.4

A Stieltjes expansion is given by

$$G(s) = \cfrac{1}{s+0.15 + \cfrac{1}{4s+1 + \cfrac{1}{5s+0.5 + \cfrac{1}{3s+1.5}}}}$$

Find the corresponding rational function.

Solution:

i. One writes

$n = 4,\quad \alpha_1 = 0,\quad \alpha_2 = 1,\quad \alpha_3 = 6.6667,\quad \alpha_4 = 4,\quad \alpha_5 = 1,\quad \alpha_6 = 5,$
$\alpha_7 = 2,\quad \alpha_8 = 3,\quad \alpha_9 = 0.6667$

ii. Using Table 3.3 - Case a, write expressions for $\alpha_1, \alpha_2, \dots \alpha_9$

$$\alpha_1 = a_4 \cdot [b_4]^{-1} = 0 \tag{Ex4-1}$$

$$\alpha_2 = b_4 \cdot [a_3]^{-1} = 1 \tag{Ex4-2}$$

$$\alpha_3 = \frac{b_4}{\alpha_2} \cdot [b_3-\alpha_2 \underline{a_2}]^{-1} = 6.6667 \tag{Ex4-3}$$

$$\alpha_4 = \frac{b_4}{\alpha_2} \cdot [b_2-\alpha_2 a_1-\alpha_3^{-1}\underline{a_2}]^{-1} = 4 \tag{Ex4-4}$$

$$\alpha_5 = \frac{b_4}{\alpha_2\alpha_4} \cdot [\underline{a_2}-\alpha_4(\underline{b_1-\alpha_2 a_0-\alpha_3^{-1}a_1})]^{-1} = 1 \tag{Ex4-5}$$

$$\alpha_6 = \frac{b_4}{\alpha_2\alpha_4} \cdot [a_1-\alpha_4(b_0-\alpha_3^{-1}a_0) - \alpha_5^{-1}(\underline{b_1-\alpha_2 a_0-\alpha_3^{-1}a_1})]^{-1} = 5 \tag{Ex4-6}$$

$$\alpha_7 = \frac{b_4}{\alpha_2\alpha_4\alpha_6} \cdot [\underline{b_1-\alpha_2 a_0-\alpha_3^{-1}a_1} - \alpha_6\ (\underline{a_0-\alpha_5^{-1}(b_0-\alpha_3^{-1}a_0)})]^{-1} = 2 \tag{Ex4-7}$$

$$\alpha_8 = \frac{b_4}{\alpha_2\alpha_4\alpha_6} \cdot [b_0-\alpha_3^{-1}a_0-\alpha_7^{-1}(\underline{a_0-\alpha_5^{-1}(b_0-\alpha_3^{-1}a_0)})]^{-1} = 3 \tag{Ex4-8}$$

$$\alpha_9 = \frac{b_4}{\alpha_2\alpha_4\alpha_6\alpha_8} \cdot [\underline{a_0-\alpha_5^{-1}(b_0-\alpha_3^{-1}a_0)}]^{-1} = 0.6667 \tag{Ex4-9}$$

iii. Solve the above expressions in the rational function coefficients as follows:

Let $b_4 = 1$

Using (Ex4-9) and (Ex4-8), one respectively has

$$a_0 - \alpha_5^{-1}(b_0 - \alpha_3^{-1}a_0) = \frac{b_4}{\alpha_2\alpha_4\alpha_6\alpha_8\alpha_9} = 0.025 \qquad \text{(Ex4-10)}$$

$$b_o - \alpha_3^{-1}a_0 = 0.025\,\alpha_7^{-1} + \frac{b_4}{\alpha_2\alpha_4\alpha_6\alpha_8} = 0.0292 \qquad \text{(Ex4-11)}$$

(Ex4-11) into (Ex4-10)

$$a_0 = 0.025 + \alpha_5^{-1}\ 0.0292 = 0.0542$$

From (Ex4-11)

$$b_0 = 0.0292 + \alpha_3^{-1}a_0 = 0.0373$$

Use (Ex4-7)

$$b_1 - \alpha_2 a_0 - \alpha_3^{-1}a_1 = 0.025\,\alpha_6 + \frac{b_4}{\alpha_2\alpha_4\alpha_6\alpha_7} = 0.15 \qquad \text{(Ex4-12)}$$

Using (Ex4-11) and (Ex4-12) into (Ex4-6), a_1 is given by

$$a_1 = 0.0292\,\alpha_4 + 0.15\,\alpha_5^{-1} + \frac{b_4}{\alpha_2\alpha_4\alpha_6} = 0.3168$$

From (Ex4-12)

$$b_1 = 0.15 + \alpha_2 a_0 + \alpha_3^{-1}a_1 = 0.2517$$

The rest of the coefficients are obtained one by one

$$a_2 = \alpha_4(b_1 - \alpha_2 a_0 - \alpha_3{}^{-1}a_1) + \frac{b_4}{\alpha_2\alpha_4\alpha_5} = 0.85$$

$$b_2 = \alpha_2 a_1 + \alpha_3{}^{-1}a_2 + \frac{b_4}{\alpha_2\alpha_4} = 0.6943$$

$$b_3 = \alpha_2 a_2 + \frac{b_4}{\alpha_2\alpha_3} = 1$$

$$a_3 = \frac{b_4}{\alpha_2} = 1$$

$$a_4 = \alpha_1 b_4 = 0$$

The transfer function is given by

$$G(s) = \frac{s^3 + 0.85s^2 + 0.3168s + 0.054}{s^4 + s^3 + 0.6943s^2 + 0.2517s + 0.0373}$$

CHAPTER IV

NEW APPLICATIONS OF THE CONTINUED FRACTION METHOD TO THE CONTINUOUS SYSTEMS

4.1 INTRODUCTION

A fundamental consideration in the design of a system, whether continuous or discrete is the stability of the system transfer function. A concept useful in this connection, in the continuous case, is that of a Hurwitz polynomial, which is defined as a polynomial having no roots in the right-half s-plane. Since the characteristic polynomial of a physical system (that is, the denominator polynomial of a system function) is a Hurwitz polynomial, the concept is widely used in engineering problems [4, 77-78].

Routh [1] devised a method for testing a polynomial for stability; the test formulated by Hurwitz [2] is essentially the same as Routh's test. The Hurwitz test [2, 77] applies the continued fraction expansion to the even and odd parts of the polynomials. In this chapter, a generalized test for Hurwitz polynomials is presented. The basic operation involved is the execution of two continued fraction expansions. The first continued fraction expansion determines whether the given polynomial is strictly Hurwitz or modified Hurwitz with m pairs of simple roots on the imaginary axis. The subsequent continued fraction detects, by inspection, the numbers of real roots and complex roots, respectively. A simple aperiodicity test for continuous systems follows as a special case. Moreover, a fast Hurwitz test is proposed. The test is based on the mixed-Cauer form whereby the number of inversion cycles in Euclid's algorithm of the continued fraction expansion, executed using Cauer first form, is reduced approximately to one half. This considerably saves the computation time of performing the test.

4.2 HURWITZ POLYNOMIALS

Consider a polynomial h(s) given by

$$h(s) = \sum_{i=o}^{n} a_i s^i \tag{4.1}$$

h(s) is said to be Hurwitz if the following conditions are satisfied [78]:

1. h(s) is real when s is real.
2. The roots of h(s) have real parts which are zero or negative.

For convenience [79], let us call the polynomial h(s) having all roots in the left-half plane the strictly Hurwitz, and the polynomial $h_m(s)$ having simple roots on the imaginary axis, in addition, the modified Hurwitz. It is well known [77-80] that when the Hurwitz test is applied on $h_m(s)$, a premature termination occurs and as a consequence, the factor m(s) in $h_m(s)$ can be isolated, corresponding to the pairs of imaginary axis roots and thus one can detect how many roots are on the imaginary axis. Furthermore by dividing $h_m(s)$ by m(s), $h_m(s)$ is reduced to a new subpolynomial that is strictly Hurwitz.

The application of the continued fraction expansion on h(s) of order n yields n positive coefficients implying that all roots of h(s) are in the left-half plane. In case of $h_m(s)$, a differentiation of m(s) is needed to carry out the continued fraction expansion to eventually obtain n positive coefficients [77].

Next, a generalized Hurwitz test is proposed in which a simple method is developed to gain further information regarding the number of real roots, complex roots and imaginary roots in a Hurwitz polynomial. The multiplicity of respective roots is also detected simultaneously. The operation involved is just one more application of the continued fraction expansion.

4.3 GENERALIZED TEST FOR HURWITZ POLYNOMIALS

4.3.1 A Simple Transformation

Let us apply a transformation

$$s \to p^2$$

That is

$$h(s) \to h(p^2) \tag{4.2}$$

As shown in Fig. 4.1, a real root $s_1 = -\sigma_1$ maps in the p-plane as a pair of imaginary axis roots p_1, $p_1^* = \pm\, j\sqrt{\sigma_1}$. A pair of complex roots $(s_2, s_2^*) = (\gamma\, e^{j\theta},\ \gamma\, e^{-j\theta})$, where $\frac{\pi}{2} < \theta < \pi$ will map in the p-plane as a set of four roots of quadrantal symmetry $(p_2, p_2^*, -p_2, -p_2^*) = (\sqrt{\gamma}\, e^{j\frac{\theta}{2}}, \sqrt{\gamma}\, e^{-j\frac{\theta}{2}}, -\sqrt{\gamma}\, e^{j\frac{\theta}{2}}, -\sqrt{\gamma}\, e^{-j\frac{\theta}{2}})$ of which the first two roots are in the shaded area of the right-half plane.

Since $h(p^2)$ is an even polynomial, one differentiates to obtain $h'(p^2)$. He then performs the continued fraction expansion on $h(p^2)/h'(p^2)$ in which the patterns will develop among the set of the expansion coefficients. Corresponding to (p_1, p_1^*), one has a couple pattern (+ +) of two successive positive entries and for $(p_2, p_2^*, -p_2, -p_2^*)$ one has a quad pattern which is the combination of two negative entries and two positive entries such as (+ + − −),(− + + −) etc. It is obvious that the number of quad patterns corresponds to the number of pairs of complex roots in h(s) whereas the number of couple patterns corresponds to the number of real roots in h(s).

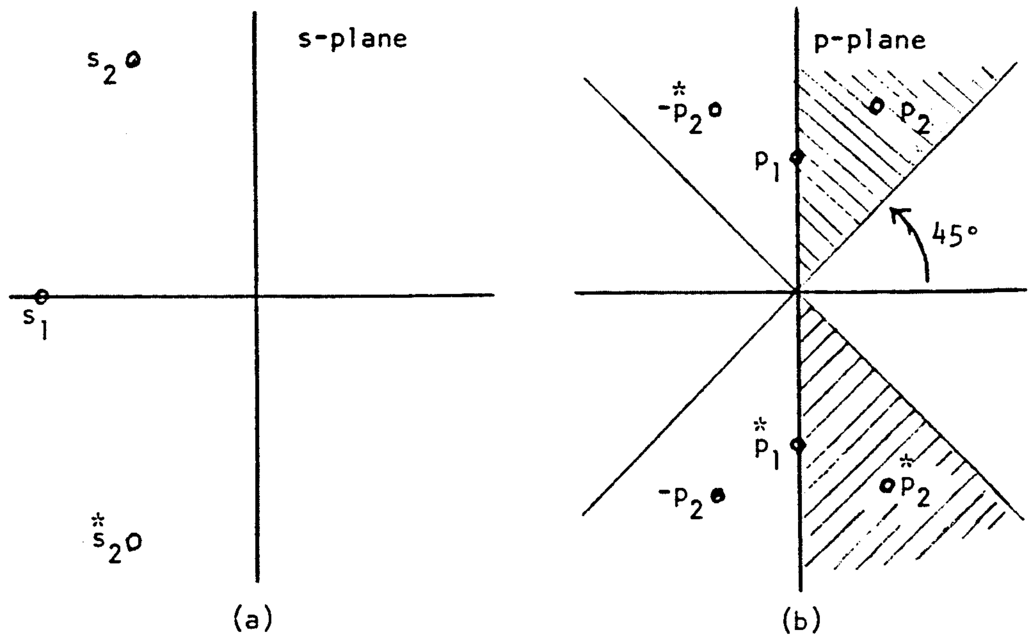

Fig. 4.1 (a) Location of roots of a Hurwitz polynomial h(s).

(b) Mapping of the s-plane roots in the p-plane.

recognize the patterns by inspection. The preceding analysis leads to the following generalized Hurwitz test.

4.3.2 Performance of the Generalized Hurwitz Test

Given a polynomial $h(s)$ of order n, one writes

$$h(s) = \sum_{i=o}^{n} a_i s^i = M(s) + N(s) \tag{4.3}$$

where $M(s)$ and $N(s)$ are the even and odd parts, respectively. Construct a function $\phi(s) = M(s)/N(s)$ for even n or $\phi(s) = N(s)/M(s)$ for odd n.

A. Case of strictly Hurwitz polynomials

1. Apply the continued fraction expansion on $\phi(s)$. No premature termination occurs and all n expansion coefficients $(\alpha_1, \alpha_2, \cdots \alpha_n)$ are positive.

$$\phi(s) = \alpha_1 s + \cfrac{1}{\alpha_2 s + \cfrac{1}{\ddots + \cfrac{1}{\alpha_n s}}} \tag{4.4}$$

2. Apply another continued fraction expansion on $h(p^2)/h'(p^2)$ to obtain a new set of 2n expansion coefficients $(\beta_1, \beta_2, \cdots, \beta_{2n})$.

$$\frac{h(p^2)}{h'(p^2)} = \beta_1 s + \cfrac{1}{\beta_2 s + \cfrac{1}{\ddots + \cfrac{1}{\beta_{2n} s}}} \tag{4.5}$$

Recognize two kinds of patterns in β's. The number of quad patterns such as (+ + − −) corresponds to the number of pairs of complex roots in $h(s)$, and the number of couple patterns (+ +) corresponds to the number of real roots in $h(s)$.

B. Case of modified Hurwitz Polynomials

1. Apply the continued fraction expansion on (s) and the premature termination occurs with the factor

$$m(s) = s^{2m} + b_{2(m-1)}s^{2(m-1)} + \dots + b_o \tag{4.6}$$

which is common to M(s) and N(s). Differentiate m(s) to obtain dm(s)/ds and carry out continued fraction expansion on m(s)/m'(s). All n expansion coefficients ($\alpha_1, \alpha_2, \dots, \alpha_n$) are positive. The number of pairs of simple roots of h(s) on the imaginary axis is m.

2. Obtain g(s) = h(s)/m(s) and apply another continued fraction expansion on $g(p^2)/g'(p^2)$ to get a new set of 2(n-m) coefficients (β_1, β_2, ..., $\beta_{2(n-m)}$). The number of quad patterns corresponds to the number of pairs of complex roots of h(s). The number of couple patterns corresponds to the number of real roots in h(s).

C. Case of multiple roots

1. Apply the continued fraction expansion as in step 1 of Case A or Case B to obtain all positive coefficients.

2. In the continued fraction expansion on $h(p^2)/h'(p^2)$, premature terminations occur but through proper differentiations, all expansion coefficients can eventually be obtained. The number of premature terminations corresponds to the multiplicity of respective roots. If n_1 quads and n_2 couples appear after vth premature termination, there are n_1 pairs of complex roots and n_2 real roots of multiplicity (v + 1), respectively, in h(s).

4.3.3. A Simple Aperiodicity Test as a Special Case

The aperiodicity is one of the significant problems in the study of systems, whether continuous or discrete [38, 47]. Generally, it arises in obtaining a response, in the time domain, that has no oscillations or that has oscillations of a finite number only. In the continuous case, this situation is represented in the frequency domain by obtaining the necessary and sufficient conditions for all the roots of the characteristic polynomial to be distinct and on the negative real axis in the s-plane [47].

There are various methods of testing whether or not a given continuous system is aperiodic. A popular method is that given by Jury [47]. His method is based on the notion of inners of a square matrix. In this method, for a system of order n, a 2n-1 by 2n-1 matrix has to be constructed and shown to be positive innerwise. A method, that is believed to be simpler, straightforward and easy to apply, can be established as a special case of the analysis provided in the preceding subsections. The method is presented in the form of a theorem.

Theorem

From (4.1), assuming $a_i > 0 \;\forall\; i$, form the following rational function:

$$\frac{h(p^2)}{h'(p^2)} = \beta_1 s + \cfrac{1}{\beta_2 s + \cfrac{1}{\ddots + \cfrac{1}{\beta_{2n} s}}}$$

If the system represented by (4.1) is aperiodic, then the above expansion exists with no premature terminations yielding positive coefficients.

The proof is readily ascertained from the analysis in Subsection 4.3.1.

4.3.4. Illustrative Examples

4.3.4.1 Example 4.1: Case of strictly Hurwitz

$$h(s) = s^4 + 4\,s^3 + 7\,s^2 + 8\,s + 4$$

$$= (s^4 + 7\,s^2 + 4) + (4\,s^3 + 8\,s) = M(s) + N(s)$$

1. Apply the continued fraction expansion on:

$$\phi(s) = \frac{s^4 + 7s^2 + 4}{4s^3 + 8s}$$

$$\begin{array}{rl} & \frac{1}{4}s \\ \cline{2-2} 4s^3 + 8s\;) & s^4 + 7s^2 + 4 \\ & s^4 + 2s^2 \\ \cline{2-2} & \qquad 5s^2 + 4 \end{array} \qquad \begin{array}{rl} & \frac{4}{5}\,s \\ \cline{2-2}) & 4s^3 + 8s \\ & 4s^3 + \frac{16}{5}\,s \\ \cline{2-2} & \qquad \frac{24}{5}\,s \end{array} \qquad \begin{array}{rl} & \frac{25}{24}s \\ \cline{2-2}) & 5s^2 + 4 \\ & 5s^2 \\ \cline{2-2} & \qquad 4 \end{array} \qquad \begin{array}{rl} & \frac{6}{5}\,s \\ \cline{2-2}) & \frac{24}{5}\,s \\ & \frac{24}{5}\,s \\ \cline{2-2} & 0 \end{array}$$

No premature termination occurs and all four expansion coefficients $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{1}{4}, \frac{4}{5}, \frac{25}{24}, \frac{6}{5})$ are positive. Thus h(s) is a strictly Hurwitz polynomial.

2. The second continued fraction expansion is now applied on

$$\frac{h(p^2)}{h'(p^2)} = \frac{p^8 + 4p^6 + 7p^4 + 8p^2 + 4}{8p^7 + 24p^5 + 28p^3 + 16p}$$

to yield $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8) = (\overset{c}{\underline{+\;+}}\;-\;\overset{q}{\underline{+\;+\;-}}\;\overset{c}{\underline{+\;+}})$

having two couples and one quad. Thus it is determined by inspection that h(s) has two real roots and a pair of complex roots.

4.3.4.2. Example 4.2: Case of modified Hurwitz

$$h(s) = s^5 + 2s^4 + 3s^3 + 3s^2 + 2s + 1 = (s^5 + 3s^3 + 2s) + (2s^4 + 3s^2 + 1) = N(s) + M(s)$$

1. Apply the continued fraction expansion on

$$\phi(s) = \frac{s^5 + 3s^3 + 2s}{2s^4 + 3s^2 + 1}$$

$$\begin{array}{rl}
 & \frac{1}{2}s \\ \hline
2s^4 + 3s^2 + 1\ ) & s^5 + 3s^3 + 2s \\
 & s^5 + \frac{3}{2}s^3 + \frac{1}{2}s \\ \hline
 & \frac{3}{2}s^3 + \frac{3}{2}s
\end{array}
\qquad
\begin{array}{rl}
 & \frac{4}{3}s \\ \hline
) & 2s^4 + 3s^2 + 1 \\
 & 2s^4 + 2s^2 \\ \hline
 & s^2 + 1
\end{array}
\qquad
\begin{array}{rl}
 & \frac{3}{2}s \\ \hline
) & \frac{3}{2}s^3 + \frac{3}{2}s \\
 & \frac{3}{2}s^3 + \frac{3}{2}s \\ \hline
 & 0
\end{array}$$

m(s) $\longrightarrow s^2 + 1$

Premature termination

Continue with m(s)/m'(s)

$$\begin{array}{rl}
 & \frac{1}{2}s \\ \hline
2s\ ) & s^2 + 1 \\
 & s^2 \\ \hline
 & 1
\end{array}
\qquad
\begin{array}{rl}
 & 2s \\ \hline
) & 2s \\
 & 2s \\ \hline
 & 0
\end{array}$$

Premature termination occurs with $m(s) = s^2 + 1$ and differentiation is required once to obtain all positive coefficients, i.e., $(\alpha_1, \alpha_2, \alpha_3, \alpha'_4, \alpha'_5) = (\frac{1}{2}, \frac{4}{3}, \frac{3}{2}, \frac{1}{2}, 2)$. Thus h(s) is a modified Hurwitz polynomial having a pair of roots on the imaginary axis.

2. $$g(s) = h(s)/m(s) = s^3 + 2s^2 + 2s + 1$$

Apply another continued fraction expansion on

$$\frac{g(p^2)}{g'(p^2)} = \frac{p^6 + 2p^4 + 2p^2 + 1}{6p^5 + 8p^3 + 4p}$$

to obtain six coefficents $(\frac{1}{6}, 9, -\frac{1}{6}, -8, \frac{1}{6}, 3) = (\underline{+\ +^{q} -\ -}\ \ \underline{+^{c}+})$. Thus h(s) has a real root and a pair of complex roots in addition to a pair of imaginary axis roots.

4.3.4.4 Example 4.3: Case of multiple roots

$$h(s) = s^5 + 3s^4 + 5s^3 + 5s^2 + 3s + 1$$

1. The continued fraction expansion on $(s^5 + 5s^3 + 3s)/(3s^4 + 5s^2 + 1)$ yields $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (+ + + + +)$ without premature termination. Thus h(s) is a strictly Hurwitz polynomial.
2. The second continued fraction expansion on $h(p^2)/h'(p^2)$ results in the following coefficients with one premature termination.

$$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \downarrow \beta'_7, \beta'_8, \beta'_9, \beta'_{10}) = (\underline{+\ +^{q} -\ -}\ \ \underline{+^{c}+} \underset{\text{premature termination}}{\downarrow} \underline{+\ +^{q'} -\ -})$$

Existence of quad patterns q' and q after and (therefore necessarily) before premature termination indicates that h(s) has a pair of complex roots of multiplicity two. A couple pattern c before premature termination, indicates that h(s) has a simple real root.

4.4. A FAST HURWITZ TEST

The conventional method [77] of Hurwitz test is based on expanding the test functions $\phi(s)$ into continued fractions of the Cauer first form. The actual execution of the divide and-invert operations associated with the continued fraction expansion is tedious and laborious [81]. To avoid drudgery, the idea [28] has been to totally avoid the actual execution of the divide-and-invert operations.

The objective here is to propose a new method of Hurwitz test. The method, however, is based on the actual execution of the continued fraction expansion though it is shown to be convenient and faster than the conventional method. The new method is presented in the form of a theorem.

Theorem

From the expression (4.3) of h(s), form a test function $\phi(s) = M(s)/N(s)$ for even n or $N(s)/M(s)$ for odd n. Carry out the continued fraction expansion on $\phi(s)$ removing the pole at infinity and also at the origin simultaneously in each cycle. That is

$$\phi(s) = \alpha_1 s + \alpha_2 s^{-1} + \cfrac{1}{\alpha_3 s + \alpha_4 s^{-1} + \cfrac{1}{\alpha_5 s + \alpha_6 s^{-1} + \cfrac{1}{\ddots}}} \tag{4.7}$$

If the coefficients α_i $(i = 1, 2, \ldots, n)$ are all positive, h(s) is a Hurwitz polynomial.

Proof

Let us first carry out the continued fraction expansion about infinity removing a pole at infinity in each cycle, that is

$$\phi(s) = \beta_1 s + \cfrac{1}{\beta_2 s + \cfrac{1}{\beta_3 s + \cfrac{1}{\ddots}}} \tag{4.8}$$

The conventional Hurwitz test states that h(s) is a Hurwitz polynomial if the coefficients β_i (i = 1, 2, ..., n) are all positive. The number of negative coefficients, if any, is equal to the number of roots in the right half plane.

As a matter of interest let us now remove the pole at the origin in each cycle:

$$\phi(s) = \gamma_1 s^{-1} + \cfrac{1}{\gamma_2 s^{-1} + \cfrac{1}{\gamma_3 s^{-1} + \cfrac{1}{\ddots}}} \tag{4.9}$$

Since $\phi(s)$ can be regarded as a lossless driving point impedance [5] $Z_{LC}(s)$, (4.8) and (4.9) correspond to the synthesis of the first Cauer network of Fig. 4.2(a) and the Cauer second network of Fig. 4.2(b), respectively. As a matter of fact, (4.7) corresponds to Fig. 4.2(c) of the band-pass type ladder network.

It is well known that all three networks are canonic and if one exists, the other two also exist. This fact, in turn, implies that α_i and γ_i are positive, or vice versa. Thus either one of the three types of continued fraction expansions may be used as the Hurwitz test.

End of Proof

It is worth noting that Cauer first form has been used in the expansion $h(p^2)/h'(p^2)$ given by (4.5). In this respect, one can easily show that, for certain h(s), the same kinds and numbers of couple and quad patterns will

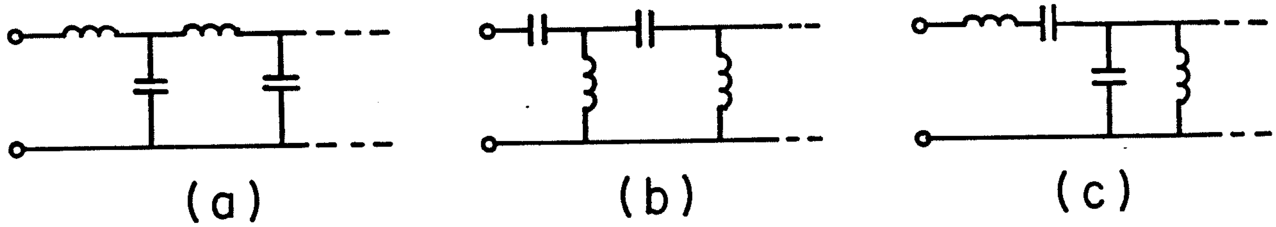

Fig. 4.2 Three Canonic realizations of $\phi(s) = z_{LC}(s)$

(a) Cauer - first type
(b) Cauer - second type
(c) Mixed - Cauer type

develop among the expansion coefficients if the expansion form given by either (4.7) or (4.9) is used.

However, the mixed form of (4.7) is proposed because the number of inversion cycles of the continued fraction expansion in the algorithm is reduced approximately to one half of the number of inversion cycles associated with either the expansion (4.8) or (4.9). As a consequence, the heavy labor is considerably mitigated. The following example illustrates the above fact.

4.4.1 <u>Illustrative Example</u>

4.4.1.1 Example 4.4: Even-ordered polynomial

$$h(s) = 6\ s^6 + 6\ s^5 + 23\ s^4 + 14\ s^3 + 22\ s^2 + 6\ s + 6$$

$$\phi(s) = \frac{6\ s^6 + 23\ s^4 + 22\ s^2 + 6}{6\ s^5 + 14\ s^3 + 6\ s}$$

i) proposed method

Carry out the continued fraction expansion on $\phi(s)$ according to (4.7)

$$\begin{array}{r|l}
 & s + s^{-1} \\
6s^5 + 14s^3 + 6s & 6s^6 + 23s^4 + 22s^2 + 6 \\
 & 6s^6 + 14s^4 + 6s^2 \\
\hline
 & 9s^4 + 16s^2 + 6 \\
 & 6s^4 + 14s^2 + 6 \\
\hline
 & 3s^4 + 2s^2
\end{array}$$

$$\begin{array}{r|l}
 & 2s + 3s^{-1} \\
3s^4 + 2s^2 & 6s^5 + 14s^3 + 6s \\
 & 6s^5 + 4s^3 \\
\hline
 & 10s^3 + 6s \\
 & 9s^3 + 6s \\
\hline
 & s^3
\end{array}$$

$$\begin{array}{r|l}
 & 3s + 2s^{-1} \\
s^3 & 3s^4 + 2s^2 \\
 & 3s^4 + 2s^2 \\
\hline
 & 0
\end{array}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (1, 1, 2, 3, 3, 2)$ and therefore h(s) is Hurwitz.

ii) conventional method:

$$\begin{array}{r|l} & s \\ \hline 6s^5 + 14s^3 + 6s & 6s^6 + 23s^4 + 22s^2 + 6 \\ & \underline{6s^6 + 14s^4 + 6s^2} \\ & \qquad 9s^4 + 16s^2 + 6 \end{array}$$

$$\begin{array}{r|l} & \frac{2}{3}s \\ \hline & 6s^5 + 14s^3 + 6s \\ & \underline{6s^5 + \frac{32}{3}s^3 + 4s} \\ & \qquad \frac{10}{3}s^3 + 2s \end{array}$$

$$\begin{array}{r|l} & \frac{27}{10}s \\ \hline & 9s^4 + 16s^2 + 6 \\ & \underline{9s^4 + \frac{54}{10}s^2} \\ & \qquad \frac{53}{5}s^2 + 6 \end{array}$$

$$\begin{array}{r|l} & \frac{50}{159}\ s \\ \hline \frac{53}{5}\ s^2 + 6 & \frac{10}{3}\ s^3 + 2s \\ & \underline{\frac{10}{3}\ s^3 + \frac{300}{159}\ s} \\ & \qquad \frac{6}{53}\ s \end{array}$$

$$\begin{array}{r|l} & \frac{2809}{30}\ s \\ \hline & \frac{53}{5}\ s^2 + 6 \\ & \underline{\frac{53}{5}\ s^2} \\ & \qquad 6 \end{array}$$

$$\begin{array}{r|l} & \frac{1}{53}\ s \\ \hline & \frac{6}{53}\ s \\ & \underline{\frac{6}{53}\ s} \\ & \quad 0 \end{array}$$

$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6,) = (1, \frac{2}{3}, \frac{27}{10}, \frac{50}{159}, \frac{2809}{30}, \frac{1}{53})$ and h(s) is Hurwitz.

It is worth noting that for odd-ordered polynomials, one has to carry out the first cycle according to the conventional test then proceed according to the proposed test.

Chapter V

NEW APPLICATIONS OF THE CONTINUED FRACTION METHOD TO THE DISCRETE SYSTEMS

5.1 INTRODUCTION

One of the main problems in the study of discrete systems is the determination of stability [38]. The problem reduces to obtaining the necessary and sufficient conditions for the roots of D(z) (discrete system characteristic equation) to lie inside the unit circle in the z-plane [82]. In this respect, the continued fraction method has recently been used in testing stability [16, 46, 83]. The idea is to carry out the continued fraction expansion on a certain test function. In [16], the test function is the bilinear equivalent of a reactance function whereas the test function in [83] is the lossless-discrete-integrator [13] (LDI) equivalent of a reactance function. In [46], the test function is the ratio of the two polynomials $z^n D(z^{-1})$ and D(z), where n is the order of D(z). This ratio has been used earlier by Jury [38] in testing stability by the "division method". His method, however, does not employ the continued fractions.

In this chapter, several additional stability tests based on the continued fractions, are presented. It is shown that the algorithmic character of continued fractions may lead to simple stability tests that compare favorably with Jury's commonly used tabular form [39, 84].

Moreover, the application of continued fractions to bilinear transformation of polynomials is introduced. The transformation is a fundamental step in designing digital [30-31, 85] or sampled-data [11] filters.

5.2 A SIMPLIFIED STABILITY TEST FOR DISCRETE SYSTEMS USING A NEW Z-DOMAIN CONTINUED FRACTION METHOD

Recently Schüssler [86] has explored some interesting properties of a polynomial D(z), where D(z) is the denominator of the transfer function of a stable discrete system.

Let D(z) be a polynomial with real coefficients of degree n given by

$$D(z) = \sum_{m=0}^{n} c_m z^m = c_n \prod_{m=1}^{n} (z-z_{\infty m}) \tag{5.1}$$

A polynomial decomposition $D(z) = F_1(z) + F_2(z)$ similar to the even and odd decomposition of a Hurwitz polynomial, in the continuous case, has been developed [86] where for D(z) of degree n, $F_1(z)$ and $F_2(z)$ are given by

$$F_1(z) = \frac{1}{2} [D(z) + z^n D(z^{-1})] \tag{5.2}$$

$$F_2(z) = \frac{1}{2} [D(z) - z^n D(z^{-1})] \tag{5.3}$$

$F_1(z)$ and $F_2(z)$ are mirror image and antimirror image polynomials, respectively.

It has been stated [86] that the following are the necessary and sufficient conditions for the zeros of D(z) to be within the unit circle.

a) The zeros of $F_1(z)$ and $F_2(z)$ are located on the unit circle

b) They are simple; and

c) They separate each other (5.4)

One can see that the properties of $F_1(z)$ and $F_2(z)$ indicated by (5.4)[6] are closely related to equivalent ones of even and odd parts of a denominator polynomial of a stable continuous system. These properties have been utilized [16] to develop stability tests using the continued fraction expansion in the z-domain. It has been shown [16] that the following expansion is always possible iff D(z) has all its zeros inside the unit circle:

$$\phi_0(z) = \frac{F_1(z)}{F_2(z)} = K_0\ f(z) + \cfrac{1}{K_1\ f(z) + \cfrac{1}{\ddots + \cfrac{1}{K_{n-1}\ f(z)}}} \tag{5.5}$$

where f(z) is defined by the z-domain bilinear function or its reciprocal, that is

$$f(z) \text{ or } 1/f(z) = \frac{z+1}{z-1} \tag{5.6}$$

and $K_i (i = 0, 1, \ldots , n-1)$ are real and positive. In fact, the function $\phi_0(z)$ is the bilinear discrete equivalent of a reactance function. The expansion given by (5.5) is the equivalent continued fraction (Cauer) form of a reactive network, or the Hurwitz test for stability of a continuous system. As pointed out earlier, in Chapter III, the divide-and-invert operations associated with the standard Cauer-type expansions are tedious and laborious. Likewise, the expansion given by

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6 Recently [87], it has been shown that conditions (5.4) are necessary and sufficient for the zeros to be either all inside or all outside the unit circle. For the zeros to be all inside the unit circle, the additional condition $|c_0/c_n|<1$ must be satisfied.

(5.5) involves drudgery and is even more laborious. To see that, consider the ith inversion cycle of the expansion (5.5), for $f(z) = (z + 1)/(z-1)$

$$K_i = \lim_{z\to 1} \left[\frac{z-1}{z+1} \cdot \phi_i(z)\right] \qquad (5.7a)$$

$$i = 0, 1, \ldots, n-1$$

$$\phi_i(z) = K_i \frac{z+1}{z-1} + \phi_{i+1}(z) \qquad (5.7b)$$

Having obtained $K_i, \phi_{i+1}(z)$ is obtained using (5.7b), then, the expansion is carried out on $1/\phi_{i+1}(z)$, etc.

Equations (5.7a) and (5.7b) describe the operations involved in one inversion cycle. Obviously, the total number of inversion cycles is equal to the order n. The operation of (5.7b) is tedious compared to the simple substitution, z=1, of (5.7a).

Next, a new z-domain continued fraction method is proposed in which the number of operations described by (5.7b) is reduced to n/2 for n even and to (n+1)/2 for n odd resulting in a considerable simplification in testing stability.

5.2.1 The New Continued Fraction Method

The new expansion method for testing stability is presented in the following propositions:

Proposition 1: For even n, the term z^2-1 is a factor of $F_2(z)$. The proof follows directly from the definition of $F_2(z)$. Note that

$$F_2(z) = \frac{1}{2} \sum_{m=0}^{n} c_m (z^m - z^{n-m}) \Big|_{z=\pm 1} = 0$$

Proposition 2: It is always possible to expand $\phi_0(z) = F_1(z)/F_2(z)$ of even degree n as:

$$\phi_0(z) = R_0 \frac{z-1}{z+1} + K_0 \frac{z+1}{z-1} + \phi_1(z)$$

where,

(1) K_0 and R_0 are real and positive

(2) $\phi_1(z)$ is of degree n-2 and has zeros at $z=\pm 1$.

The proof follows from the properties of $\phi_0(z)$ [16, 86] and proposition 1.

The above leads to the following proposition:

Proposition 3: The following expansion is always possible iff D(z), of even degree n, has all its zeros inside the unit circle:

$$\phi_0(z) = \frac{F_1(z)}{F_2(z)} = R_0 \frac{z-1}{z+1} + K_0 \frac{z+1}{z-1} + \cfrac{1}{R_1 \frac{z-1}{z+1} + K_1 \frac{z+1}{z-1} + \cfrac{1}{\ddots + \cfrac{1}{R_{\frac{n-2}{2}} \frac{z-1}{z+1} + K_{\frac{n-2}{2}} \frac{z+1}{z-1}}}} \tag{5.8}$$

where K_i and R_i are real and positive.

The following example shows the simplification associated with (5.8) in testing stability.

5.2.1.1 Example 5.1

Consider the 4th order polynomial D(z) given by

$$D(z) = 16z^4 + 24 z^3 + 18 z^2 + 6 z + 1$$

Form the following discrete reactance test function

$$\phi_0(z) = \frac{F_1(z)}{F_2(z)} = \frac{17\ z^4 + 30\ z^3 + 36\ z^2 + 30\ z + 17}{15\ z^4 + 18\ z^3 - 18\ z - 15}$$

Expand $\phi_0(z)$ according to (5.8)

$$\phi_0(z) = R_0\ \frac{z-1}{z+1} + K_0\ \frac{z+1}{z-1} + \phi_1(z) \tag{Ex1-1}$$

$$R_0 = \lim_{z\to-1} \left[\frac{z+1}{z-1} \cdot \phi_0(z)\right] = \frac{5}{24}$$

$$K_0 = \lim_{z\to1} \left[\frac{z-1}{z+1} \cdot \phi_0(z)\right] = \frac{65}{96}$$

Using (Ex1-1)

$$\phi_1\ (z) = \frac{119}{32} \cdot \frac{z^2-1}{15\ z^2 + 18\ z + 15}$$

$$\frac{1}{\phi_1(z)} = R_1\ \frac{z-1}{z+1} + K_1\ \frac{z+1}{z-1} \tag{Ex1-2}$$

$$R_1 = \lim_{z\to-1} \left[\frac{z+1}{z-1} \cdot \frac{1}{\phi_1(z)}\right] = \frac{96}{119}$$

$$K_1 = \lim_{z\to1} \left[\frac{z-1}{z+1} \cdot \frac{1}{\phi_1(z)}\right] = \frac{384}{119}$$

The above example shows that the new stability test involves basically the simple substitutions z = +1 or −1, besides the generation of the second order function $\phi_1(z)$. If $\phi_0(z)$ is to be expanded in the form (5.5), one needs to generate three functions $\phi_{i+1}(z)$ according to (5.7b). In this case the resulting K_i (i = 0, 1, 2, 3) are given by

$$(K_0, K_1, K_2, K_3) = \left(\frac{65}{96}, \frac{384}{199}, \frac{39601}{11424}, \frac{11424}{3980}\right).$$

The presented continued fraction expansion proceeds in terms of the bilinear function (z−1)/(z+1) and its reciprocal, simultaneously. Two expansion coefficients are obtained in each cycle. It is worth noting that for odd-ordered functions, one has to carry out the first cycle according to (5.5) then proceed according to (5.8).

Besides other tests such as those presented in [38], the proposed test and the test given in [16] are of interest. The latter tests have the advantage that the resulting stability constraints are simple, viz. the n expansion coefficients are real and positive, and independent on the order n of D(z) being even or odd. Moreover, it is possible that by using the signs of the expansion coefficients one can obtain information on the number of roots of D(z) outside the unit circle if unstable [16].

Steffen [88] has derived an algorithm for testing stability based on the expansion (5.5) reported in [16]. An algorithm for implementing the expansion (5.8) is developed next.

5.2.2 An Algorithm for Implementing the New Expansion

Define the rational function $\phi_i(z)$ recursively by, assuming even n

$$\phi_i(z) = R_i \frac{z-1}{z+1} + K_i \frac{z+1}{z-1} + \frac{1}{\phi_{i+1}(z)} \qquad i = 0, 1, \ldots, (n-2)/2 \tag{5.9}$$

where

$$\begin{aligned} R_i &= \lim_{z \to -1} \left[\frac{z+1}{z-1} \cdot \phi_i(z)\right] \\ K_i &= \lim_{z \to 1} \left[\frac{z-1}{z+1} \cdot \phi_i(z)\right] \end{aligned} \tag{5.10}$$

Let

$$\phi_i(z) = \frac{P_i(z)}{Q_i(z)} \tag{5.11}$$

$$P_i(z) = \sum_{m=0}^{n-j} p_m^{(i)} z^m \qquad \text{and} \qquad Q_i(z) = \sum_{m=0}^{n-j} q_m^{(i)} z^m$$

where $j = 2i$.

If one uses the substitution (5.11) in (5.9), brings the right side of the resulting equation to a common denominator, and recalls that $P_i(z)$ and $Q_i(z)$ are polynomials with no common factors he then obtains (by equating denominators)

$$P_{i+1}(z) = \frac{Q_i(z)}{z^2-1}$$

and

$$Q_{i+1}(z) = \frac{P_i(z) - [(R_i + K_i)\, z^2 + 2\,(K_i - R_i)\, z + (R_i + K_i)]\, P_{i+1}(z)}{z^2-1} \tag{5.12}$$

where according to (5.10), R_i and K_i are calculated as follows:

$$R_i = \frac{1}{4}\,\frac{P_i(-1)}{P_{i+1}(-1)} \quad \text{and} \quad K_i = \frac{1}{4}\,\frac{P_i(+1)}{P_{i+1}(+1)} \tag{5.13}$$

Equations (5.12) require the division by z^2-1. This division can be done using the following scheme. Let $A(z)$ and $B(z)$ be polynomials with the property $B(z) = (z^2-1)\,A(z)$

where

$$B(z) = \sum_{\mu=0}^{\nu+2} b_\mu\, z^\mu \quad \text{and} \quad A(z) = \sum_{\mu=0}^{\nu} a_\mu\, z^\mu \tag{5.14}$$

Knowing b_μ, one can determine the coefficients a_μ by the formulas

$$a_{2\mu} = \sum_{\lambda=\mu+1}^{\frac{\nu}{2}} b_{2\lambda} \qquad \mu = 0,\ 1,\ \ldots,\ \frac{\nu}{2}$$

and

$$a_{2\mu-1} = \sum_{\lambda=\mu+1}^{\frac{\nu}{2}} b_{2\lambda-1} \qquad \mu = 1,\ 2,\ \ldots,\ \frac{\nu}{2} \tag{5.15}$$

Hence, the coefficients of $P_{i+1}(z)$ are determined by

$$p_{2m}^{(i+1)} = \sum_{\lambda=m+1}^{\frac{n-j}{2}} q_{2\lambda}^{(i)} \qquad m = 0,\ 1,\ \ldots,\ \frac{n-j}{2} - 1$$

$$p_{2m-1}^{(i+1)} = \sum_{\lambda=m+1}^{\frac{n-j}{2}-1} q_{2\lambda-1}^{(i)} \qquad m = 1, 2, \ldots, \frac{n-j}{2} -1 \quad . \tag{5.16}$$

To evaluate the polynomials $Q_{i+1}(z)$, define the polynomials $R_{i+1}(z)$ of degree n-j by

$$R_{i+1}(z) = P_i(z) - [(R_i + K_i)\, z^2 + 2\,(K_i - R_i)\, z + (R_i + K_i)]\, P_{i+1}(z) \tag{5.17}$$

The coefficients $r_m^{(i+1)}$ of $R_{i+1}(z)$ are given by

$$r_{n-j}^{(i+1)} = p_{n-j}^{(i)} - p_{n-j-2}^{(i+1)} \cdot (R_i + K_i)$$

$$r_{n-j-1}^{(i+1)} = p_{n-j-1}^{(i)} - [p_{n-j-2}^{(i+1)} \cdot 2(K_i - R_i) + p_{n-j-3}^{(i+1)} \cdot (R_i + K_i)]$$

$$r_m^{(i+1)} = p_m^{(i)} - [(p_{m-2}^{(i+1)} + p_m^{(i+1)}) \cdot (R_i + K_i) + p_{m-1}^{(i+1)} \cdot 2(K_i - R_i)] \qquad m = 2, 3, \ldots n-j-2$$

$$r_1^{(i+1)} = p_1^{(i)} - [p_0^{(i+1)} \cdot 2(K_i - R_i) + p_1^{(i+1)} \cdot (R_i + K_i)]$$

$$r_0^{(i+1)} = p_0^{(i)} - p_0^{(i+1)} \cdot (R_i + K_i) \tag{5.18}$$

Applying (5.15) to the coefficients $r_m^{(i+1)}$, one gets the coefficients $q_m^{(i+1)}$.

One can summarize the algorithm in the following steps: Assuming $P_i(z)$ and $Q_i(z)$ to be known for a fixed value of i:

1. Determine the polynomial $P_{i+1}(z)$ using (5.16).
2. Compute the constants R_i and K_i using (5.13). Note that $R_i > 0$ and $K_i > 0$ for stability.
3. Evaluate the polynomial $R_{i+1}(z)$ using the formulas (5.18).
4. Perform the division of $R_{i+1}(z)$ by (z^2-1) using (5.15) to get the coefficients of $Q_{i+1}(z)$.

Fig. 5.1 shows the flow diagram of computation

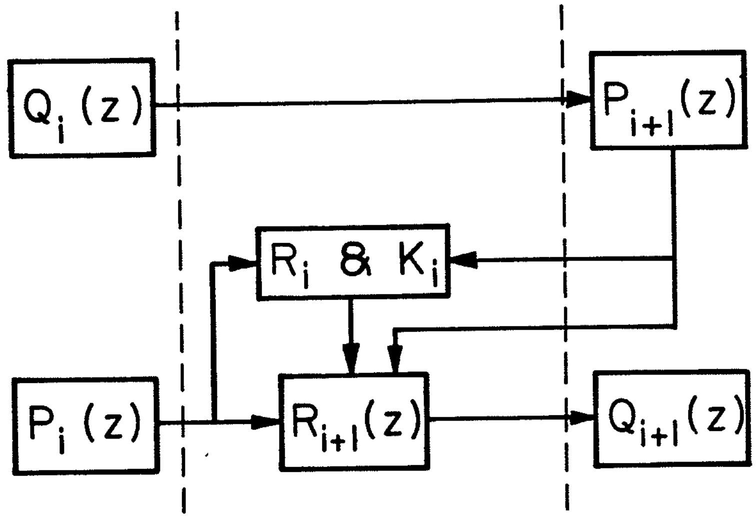

Fig. 5.1 Flow diagram for the calculation of $P_{i+1}(z)$ and $Q_{i+1}(z)$ of (5.12)

5.2.2.1 Example 5.2

Consider the 4th order polynomial D(z) of Example 5.1. The computations are summarized as shown in Table 5.1.

One may notice that the new algorithm is twice as fast as Steffen's algorithm [88] in the sense that one is able to get two expansion coefficients instead of one coefficient in each computation cycle. For odd-ordered functions, however, one has to carry out the first computation cycle according to Steffen's algorithm then proceed according to the new algorithm.

The new algorithm can be further simplified using the properties of $P_i(z)$ and $Q_i(z)$ being mirror image and antimirror image polynomials respectively. Also, one can easily show[7] that the polynomial $R_i(z)$ defined by (5.17) is a mirror image polynomial. Consequently, in the above example one actually needs to compute only the entries indicated in the dotted lines in Table 5.1. In this respect, it is worth noting that Steffen's algorithm [88] can be simplified in the same manner.

In terms of the required number of operations, the new algorithm compares favorably with the commonly used table form [39, 84] of testing stability. In the above example one may notice that multiplication is needed only in obtaining the coefficients of the polynomial $R_1(z)$ indicated in the dotted lines. The total number of multiplications and additions is found to be 5 and 20, respectively. Using the simplified table form of stability [39] one would need 10 multiplications and 20 additions. One can show that using a simplified version of Steffen's

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7 Generally [83], a polynomial $P_n(z)$ of degree n is called a mirror image polynomial iff $z^k P_n(z^{-1}) = P_n(z)$ where k = n + m and m is the number of zeros of $P_n(z)$ at z=o.

Table 5.1 Computations of K_i and R_i in Example 5.2

i	j		R_i	$P(-1)$	z^4	z^3	z^2	z^1	z^0	$P(+1)$	K_i
0	0	$P_0(z)$		10	17	30	36	30	17	130	$\frac{65}{96}$
		$Q_0(z)$	$\frac{5}{24}$		15	18	0	−18	−15		
1	2	$P_1(z)$		12			15	18	15	48	
		$R_1(z)$			$\frac{119}{32}$	0	$-\frac{119}{32}$	0	$\frac{119}{32}$		
		$Q_1(z)$	$\frac{96}{119}$				$\frac{119}{32}$	0	$-\frac{119}{32}$		$\frac{384}{119}$
2	4	$P_2(z)$		$\frac{119}{32}$					$\frac{119}{32}$	$\frac{119}{32}$	

algorithm the required number of multiplications and additions is 7 and 28, respectively. In this respect, it is important to point out that the reduction in the number of computation cycles, due to the new algorithm, to one half does not necessarily mean a similar reduction in the number of multiplications and additions. However, saving in computations using the new algorithm becomes more apparent for higher order polynomials.

5.3 OTHER Z-DOMAIN CONTINUED FRACTION EXPANSIONS

The discrete reactance function $\phi_0(z) = F_1(z)/F_2(z)$, developed by Schüssler [86], is the bilinear-transformed image of a continuous reactance function. Methods [16, 89-90] of expanding Schüssler's reactance function into continued fractions have been presented. The expansion (5.5) [16] proceeds in terms of the bilinear function $(z-1)/(z+1)$ (or its reciprocal), whereas the expansion (5.8) [89-90] proceeds in terms of the bilinear function and its reciprocal, simultaneously.

Recently, Davis [83] has presented another discrete reactance function and a continued fraction method of expanding it in terms of the well known [91] backward difference, $1-z^{-1}$, and forward difference, $z-1$, functions on an alternating basis. Davis's reactance function is related to the continuous counterpart by the lossless-Discrete-Integrator (LDI) s to z transformation that has been introduced by Bruton [13] in 1975.

Schüssler's reactance function is modified here so that it can be expanded into continued fractions that proceed in terms of either the backward difference or the forward difference transform. The modification is achieved by using the elementary synthetic division [92-93] which has also been recognized [19] as an application of Euclid's algorithm. The application of the new expansion methods to testing stability of discrete systems, in the z-domain exclusively, is discussed.

5.3.1 Expansions That Proceed in Terms of the Backward Difference Transform

It has been shown [86] that $\phi_0(z)$ is related to a continuous reactance function $\phi(s)$ by the bilinear transformation, that is

$$\phi_0(z) = \phi(s^{-1}) \Bigg|_{s^{-1} = \frac{z+1}{z-1}} = \phi(\frac{z+1}{z-1}) = \frac{F_1(z)}{F_2(z)} \tag{5.19}$$

One may be interested in obtaining the discrete function that is related to $\phi(s)$ by the backward difference (BD) transform, $s = \frac{z-1}{z}$, i.e.

$$\phi_{BD}(z) = \phi(s^{-1}) \Bigg|_{s^{-1} = \frac{z}{z-1}} = \phi(\frac{z}{z-1}) = \frac{F_{BD1}(z)}{F_{BD2}(z)} \tag{5.20}$$

In fact, $\phi(z/z-1)$ can be obtained by applying a sequence of two elementary operations to the denominator and numerator polynomials of $\phi(z+1/z-1)$. The operations are:

1. Shifting the zeros by a real constant
2. Scaling the magnitude of the zeros.

The procedure is as follows:

$$\phi(\frac{z+1}{z-1}) \xrightarrow[\text{by } -1]{\text{shift}} \phi(\frac{z}{z-2}) \xrightarrow[\text{by } 2]{\text{scale}} \phi(\frac{z}{z-1}) = \phi_{BD}(z) \tag{5.21}$$

It has been shown [86] that for the stable case $F_1(z)$ and $F_2(z)$, given by (5.2) and (5.3), respectively, have simple alternating zeros on the unit circle.

The operations described by (5.21) simply map the zeros of $F_1(z)$ and $F_2(z)$ from the unit circle into the shaded circle as shown in Fig. 5.2 with the zeros at z=+1 and z=−1 mapped into z=+1 and z=0, respectively. The resulting polynomials are $F_{BD1}(z)$ and $F_{BD2}(z)$ of (5.20). It is worth noting that $\phi_{BD}(z)$ is not a discrete reactance function [94] because of the fact that its poles and zeros do not alternate on the unit circle. For example z/(z−1) is not a discrete reactance function although it is a $\phi_{BD}(z)$ function.

The operations (5.21) are best illustrated by an example.

5.3.1.1 Example 5.3

Consider the 3rd order polynomial D(z) given by

$$D(z) = 2z^3 + 2z^2 + z$$

$$\phi_0(z) = \frac{F_1(z)}{F_2(z)} = \frac{2z^3 + 3z^2 + 3z + 2}{2z^3 + z^2 - z - 2}$$

Shift zeros of both $F_1(z)$ and $F_2(z)$ by −1. This is done using the elementary synthetic division [92]. For $F_1(z)$:

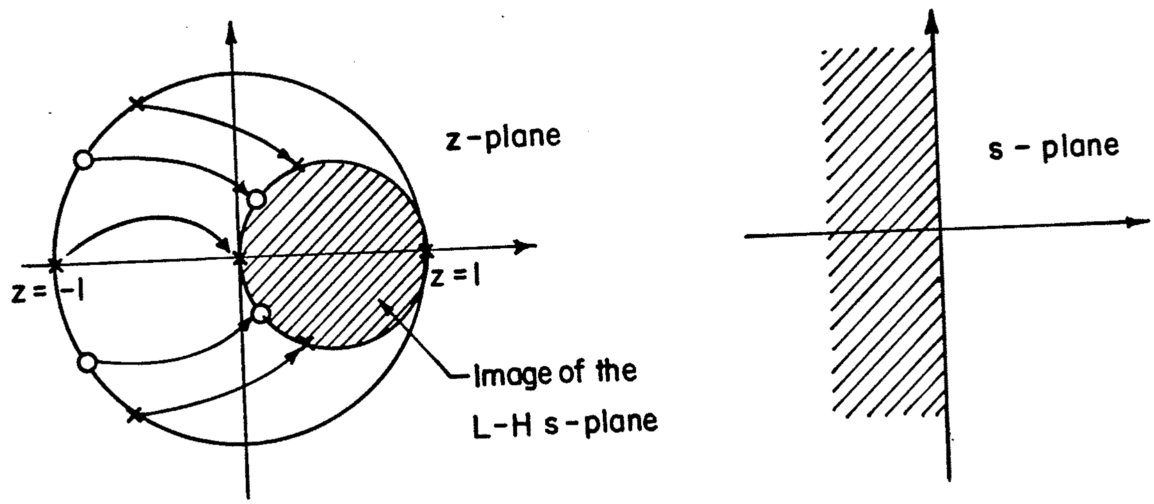

Fig. 5.2 Properties of $\phi_{BD}(z)$

$$
\begin{array}{r|rrrr}
-1 & 2 & 3 & 3 & 2 \\
 & & -2 & -1 & -2 \\
\hline
-1 & 2 & 1 & 2 & \lfloor 0 \\
 & & -2 & 1 & \\
\hline
-1 & 2 & -1 & \lfloor 3 & \\
 & & -2 & & \\
\hline
 & 2 & \lfloor -3 & & \\
 & \lfloor 2 & & & \\
\end{array}
$$

Thus,

$$F_1\left(\frac{z}{z-2}\right) = 2\, z^3 - 3\, z^2 + 3z\,.$$

Now scale the magnitude of the zeros by 2. This is done by multiplying the coefficients of z^0, z^1, z^2, --- etc, by 2^0, 2^1, 2^2, --- etc, respectively. Hence,

$$F_{BD1}(z) = 8\, z^3 - 6\, z^2 + 3\, z.$$

The above two operations are repeated on $F_2(z)$ to obtain $F_{BD2}(z)$. Note that the synthetic division does not require any multiplications [93]. $\phi_{BD}(z)$ is given by

$$\phi_{BD}(z) = \frac{8\, z^3 - 6\, z^2 + 3\, z}{8\, z^3 - 10\, z^2 + 3\, z - 1}\,.$$

The properties of $F_{BD1}(z)$ and $F_{BD2}(z)$ can now be established from the properties of $F_1(z)$ and $F_2(z)$ using the mapping illustrated in Fig. 5.2. For instance, one can show that:

1. The term $z(z-1)$ is a factor of $F_{BD2}(z)$ for even order n.

2. The terms z and z-1 are factors of $F_{BD1}(z)$ and $F_{BD2}(z)$, respectively, for odd-ordered polynomials.

Accordingly, continued fraction expansions can be derived by successive removal of poles at z=0 or z=1 or in a mixed fashion by removing the poles at z=0 and z=1, simultaneously. In other words, the following continued fraction expansions are proposed if D(z) has all its zeros inside the unit circle:

a) $$\phi_{BD}(z) = K_0 \frac{z}{z-1} + \cfrac{1}{K_1 \frac{z}{z-1} + \cfrac{1}{\ddots + \cfrac{1}{K_{n-1} \frac{z}{z-1}}}} \tag{5.22}$$

where K_i (i=1, 2, ..., n-1) are real and positive. K_0 is given by

$$K_0 = \lim_{z\to 1} \left[\frac{z-1}{z} \cdot \phi_{BD}(z)\right] \tag{5.23}$$

b) $$\phi_{BD}(z) \text{ or } 1/\phi_{BD}(z) = R_0 \frac{z-1}{z} + \cfrac{1}{R_1 \frac{z-1}{z} + \cfrac{1}{\ddots + \cfrac{1}{R_{n-1} \frac{z-1}{z}}}} \tag{5.24}$$

where R_i are real and positive. R_o is given by

$$R_0 = \lim_{z\to 0} \left[\frac{z}{z-1} \cdot \phi_{BD}(z)\right] \tag{5.25}$$

c) $$\phi_{BD}(z) = K_0 \frac{z}{z-1} + R_0 \frac{z-1}{z} + \cfrac{1}{K_1 \frac{z}{z-1} + R_1 \frac{z-1}{z} + \cfrac{1}{\ddots + \cfrac{1}{K_{\frac{n-2}{2}} \frac{z}{z-1} + R_{\frac{n-2}{2}} \frac{z-1}{z}}}} \tag{5.26}$$

where n is even, R_i and K_i are real and positive. K_0 and R_0 are determined by (5.23) and (5.25) respectively.

5.3.1.2 Example 5.4

Determine the stability of the 3rd order polynomial of Example 5.3.

Using the results of Example 5.3, one has

$$\phi_{BD}(z) = K_0 \frac{z}{z-1} + \phi_{BD1}(z)$$

$$K_0 = \lim_{z\to 1} \left[\frac{z-1}{z} \cdot \phi_{BD}(z)\right] = \frac{5}{7} > 0$$

$$\phi_{BD1}(z) = \frac{16}{7} \frac{z(z-1)}{8\ z^2-2\ z+1}$$

$\phi_{BD1}(z)$ is of even order. Proceed according to (5.26)

$$\frac{1}{\phi_{BD1}(z)} = K_1 \frac{z}{z-1} + R_1 \frac{z-1}{z}$$

$$K_1 = \lim_{z\to 1} \left[\frac{z-1}{z} \cdot \frac{1}{\phi_{BD1}(z)}\right] = \frac{49}{16} > 0$$

$$R_1 = \lim_{z\to 0} \left[\frac{z}{z-1} \cdot \frac{1}{\phi_{BD1}(z)}\right] = \frac{7}{16} > 0$$

D(z) is a characteristic polynomial of a stable system.

Note that the expansions (5.22) and (5.24) are simpler than that of (5.5). Also, the expansion (5.26) is simpler than that of (5.8). However, this is at the expense of obtaining $\phi_{BD}(z)$ from $\phi_0(z)$, which involves basically the performance of two elementary synthetic divisions that require only additions.

It is worth noting that Davies [92] has used the synthetic divisions in the bilinear transformation of polynomials. Using his results, one may suggest a stability test for the polynomial D(z) by using the synthetic division in the z-domain to obtain the continuous counterpart D(s). Hurwitz test is then applied on the even and odd parts of D(s). For discrete systems, however, it is desirable to have stability tests using methods in the z-domain exclusively [86]. In this respect, one can see that testing stability as shown in the above example is done by performing a combination of synthetic divisions and continued fractions carried out in the z-domain exclusively.

5.3.2 Expansions That Proceed in Terms of the Forward Difference Transform

A discrete function that is related to the continuous reactance function by the forward difference (FD) transform can be developed as follows:

$$\phi_0(z) \longrightarrow \phi_{BD}(z) = \phi\left(\frac{z}{z-1}\right) \xrightarrow{z\to 1/z} \phi\left(\frac{1}{1-z}\right)$$

$$\xrightarrow[\text{by}-1]{\text{scale}} \phi\left(\frac{1}{1+z}\right) \xrightarrow[\text{by}-2]{\text{shift}} \phi\left(\frac{1}{z-1}\right) = \phi_{FD}(z) \tag{5.27}$$

(5.27) maps the unit circle to the line given by z=1 as shown in Fig. 5.3. $\phi_{FD}(z)$ is

$$\phi_{FD}(z) = \frac{F_{FD1}(z)}{F_{FD2}(z)} \tag{5.28}$$

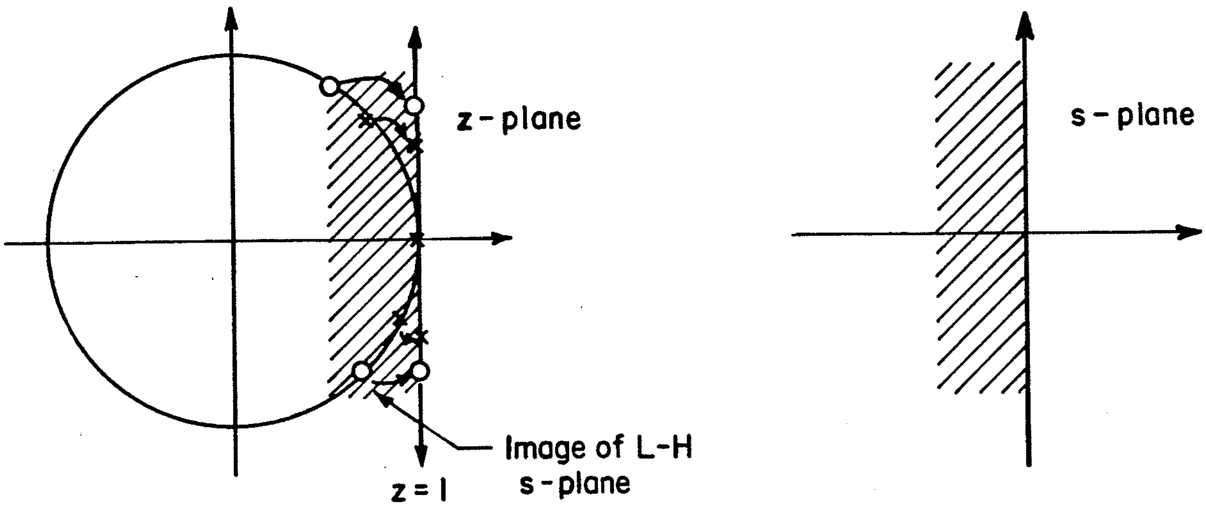

Fig. 5.3 Properties of $\phi_{FD}(z)$

The polynomials $F_{FD1}(z)$ and $F_{FD2}(z)$ have simple alternating zeros located at the line z=1. Continued fraction expansions can now be developed by studying the properties of $F_{FD1}(z)$ and $F_{FD2}(z)$. The following expansions are proposed; if D(z) has all its zeros inside the unit circle.

a) $$\phi_{FD}(z) = K_0 \frac{1}{z-1} + \cfrac{1}{K_1 \frac{1}{z-1} + \cfrac{1}{\ddots + \cfrac{1}{K_{n-1} \frac{1}{z-1}}}} \tag{5.29}$$

where, $K_i (i=0,1,\ldots,n-1)$ are real and positive. The above expansion is carried out by the successive removal of poles at z=1. K_0 is given by

$$K_0 = \lim_{z\to 1} [z-1 \cdot \phi_{FD}(z)] \tag{5.30}$$

b) $$\phi_{FD}(z) \text{ or } 1/\phi_{FD}(z) = R_0(z-1) + \cfrac{1}{R_1(z-1) + \cfrac{1}{\ddots + \cfrac{1}{R_{n-1}(z-1)}}} \tag{5.31}$$

where R_i are real and positive. The expansion is carried out by the successive removal of poles at infinity. R_0 is given by

$$R_0 = \lim_{z\to\infty} [\frac{1}{z-1} \cdot \phi_{FD}(z)] \tag{5.32}$$

c) $$\phi_{FD}(z) = K_0 \frac{1}{z-1} + R_0(z-1) \cfrac{1}{K_1 \frac{1}{z-1} + R_1(z-1) + \cfrac{1}{\ddots + \cfrac{1}{K_{\frac{n-2}{2}} \frac{1}{z-1} + R_{\frac{n-2}{2}}(z-1)}}} \qquad (5.33)$$

where n is even, K_i and R_i are real and positive.

The above expansions are simpler than the backward-difference-based expansions at the expense of extra synthetic divisions as evidenced by (5.27).

5.4 BILINEAR TRANSFORMATION OF POLYNOMIALS

A bilinear transformation of the variable in a polynomial arises in several situations in the theory of discrete systems. For instance, it is a fundamental step in designing digital or sampled-data system via a specified continuous system function [11, 30-31, 85]. In this respect, the operation is the conversion of the polynomial $D(s) = \sum_{i=0}^{n} b_i s^i$ to the polynomial $D(z) = \sum_{i=0}^{n} a_i z^i$, where $s = (z-1)/(z+1)$. Closed forms for a_k in terms of b_k have been given [38]. These forms involve unwieldly and complicated algebraic manipulation [95]. To avoid drudgery, different computation methods have been reported by many authors [92, 95-100]. Power [95-96] and Fielder [97] have presented a matrix method to obtain the coefficients of D(z). The method has been simplified by Jury [99]. Moreover, Jury and Chan [100] have given a comprehensive discussion of the matrix method. A simpler method has been described by Davies [92]. His method involves combination of elementary operations which involve basically successive application of synthetic divisions [93].

Although the abovementioned methods involve much less computation than direct substitution, it should not be assumed that these are the only alternatives; other efficient and simple methods should not be overlooked.

A new method based on continued fractions is proposed here. Unlike the existing methods, the new method systematizes the computations and avoids the need to remember or to calculate, for instance, the appropriate Q-matrix as in [99]. It also appears to be at least as simple as the method described by Davies [92]. Two basic operations are involved, viz. a continued fraction expansion in the s-domain and a continued fraction inversion in the z-domain.

5.4.1 The New Method

Given a polynomial $D(s) = M(s) + N(s)$, where $M(s)$ and $N(s)$ are the even and odd parts of $D(s)$, respectively. Form a continuous reactance function $\phi(s) = M(s)/N(s)$ or $N(s)/M(s)$ and carry out the Hurwitz test on $\phi(s)$, that is [81]

$$\phi(s) = K_0 s^{-1} + \cfrac{1}{K_1 s^{-1} + \cfrac{1}{\ddots + \cfrac{1}{K_{n-1} s^{-1}}}} \qquad (5.34)$$

$\phi(s)$ is related to the discrete counterpart, $\phi_0(z)$, according to (5.19). Hence, using the set of real constants $\{K_i\}$ obtained in (5.34), one can write the following expression for $\phi_0(z)$

$$\phi_0(z) = \frac{F_1(z)}{F_2(z)} = K_0 \frac{z+1}{z-1} + \cfrac{1}{K_1 \frac{z+1}{z-1} + \cfrac{1}{\ddots + \cfrac{1}{K_{n-1} \frac{z+1}{z-1}}}} \qquad (5.35)$$

The remaining problem is the simplification of the above expansion to yield the polynomials $F_1(z)$ and $F_2(z)$. $D(z)$ is then written as

$$D(z) = F_1(z) + F_2(z) \qquad (5.36)$$

5.4.2 An Algorithm for Implementing the z-domain Continued Fraction Inversion

It is worth noting that the new method makes use of the already available algorithms [70] of implementing the continued fraction expansion (5.34). Next, an algorithm for inverting the expansion (5.35) is developed.

The divide-and-invert cycles of (5.35) are defined recursively by

$$\phi_i(z) = K_{n-i} \frac{z+1}{z-1} + \frac{1}{\phi_{i-1}(z)} \qquad (5.37)$$

$$i = 0, 1, 2, \ldots, n$$

where[8]

$$\phi_i(z) = \frac{P_i(z)}{Q_i(z)} . \qquad (5.38)$$

and

$$P_i(z) = \sum_{\nu=0}^{i} p_\nu^{(i)} z^\nu \quad ; \quad Q_i(z) = \sum_{\nu=0}^{i} q_\nu^{(i)} z^\nu$$

with

$$P_0(z) = 1 \quad ; \quad Q_0(z) = 0$$

Using (5.37), it is seen that

$$Q_i(z) = (z-1)\, P_{i-1}(z) \qquad 0 < i \leq n \qquad (5.39)$$

---

[8] Note that in this case our $\frac{F_1(z)}{F_2(z)}$ will turn out to be $\phi_n(z)$.

Define the polynomials $R_i(z)$ as follows

$$R_i(z) = (z-1)\ Q_{i-1}(z) \qquad 0 < i \le n \tag{5.40}$$

Then,

$$P_i(z) = R_i(z) + K_{n-i}(z+1)\ P_{i-1}(z) \tag{5.41}$$

Equations (5.39) and (5.40) require the multiplication by z-1. This can be done using the following scheme. Let A(z) and B(z) be polynomials with the property

$$B(z) = (z-1)\ A(z)$$

where

$$B(z) = \sum_{\mu=0}^{m+1} b_\mu\ z^\mu \qquad \text{and} \qquad A(z) = \sum_{\mu=0}^{m} a_\mu\ z^\mu$$

Knowing the coefficients a_μ, one can determine b_μ by the formula (Note that $a_{-1} = a_{m+1} = 0$)

$$b_\mu = a_{\mu-1} - a_\mu \qquad \mu = 0, 1, \ldots, m+1$$

Therefore, the coefficients of $Q_i(z)$ and $R_i(z)$ are determined respectively by

$$q_\lambda^{(i)} = p_{\lambda-1}^{(i-1)} - p_\lambda^{(i-1)} \tag{5.42}$$

$$\lambda = 0, 1, \ldots, i$$

$$r_\lambda^{(i)} = q_{\lambda-1}^{(i-1)} - q_\lambda^{i-1} \tag{5.43}$$

The coefficients of $P_i(z)$ of (5.41) are obtained as follows

$$p_i^{(i)} = r_i^{(i)} + K_{n-i}\ p_{i-1}^{(i-1)}$$

$$p_\nu^{(i)} = r_\nu^{(i)} + K_{n-i}\,[p_\nu^{(i-1)} + p_{\nu-1}^{(i-1)}] \qquad \nu = 1, 2, \ldots i-1 \tag{5.44}$$

$$p_0^{(i)} = r_0^{(i)} + K_{n-i}\, p_0^{(i-1)}$$

For i=n, construct $\phi_n(z)$ as follows

$$\phi_n(z) = \frac{P_n(z)}{Q_n(z)} = \frac{F_1(z)}{F_2(z)} \tag{5.45}$$

Now, D(z) is obtained as $F_1(z) + F_2(z)$. Fig. 5.4 shows the flow diagram for the calculation of the polynomials $P_i(z)$ and $Q_i(z)$. The algorithm is best illustrated by an example.

5.4.3 Example 5.5

Consider the 3rd order polynomial

$$D(s) = 2\,s^3 + 6\,s^2 + 14\,s + 10 = 2(s^3 + 3\,s^2 + 7s + 5)$$

Carry out the Hurwitz test as follows

$$\phi(s) = \frac{5 + 3s^2}{7s + s^3} = \frac{5}{7}\,s^{-1} + \cfrac{1}{\frac{49}{16}\,s^{-1} + \cfrac{1}{\frac{16}{7}\,s^{-1}}}$$

Apply the bilinear transformation $s=(z-1)/(z+1)$ on $\phi(s)$. $\phi_n(z)$ is expressed by

$$\phi_n(z) = \frac{5}{7}\,\frac{z+1}{z-1} + \cfrac{1}{\frac{49}{16}\,\frac{z+1}{z-1} + \cfrac{1}{\frac{16}{7}\,\frac{z+1}{z-1}}}$$

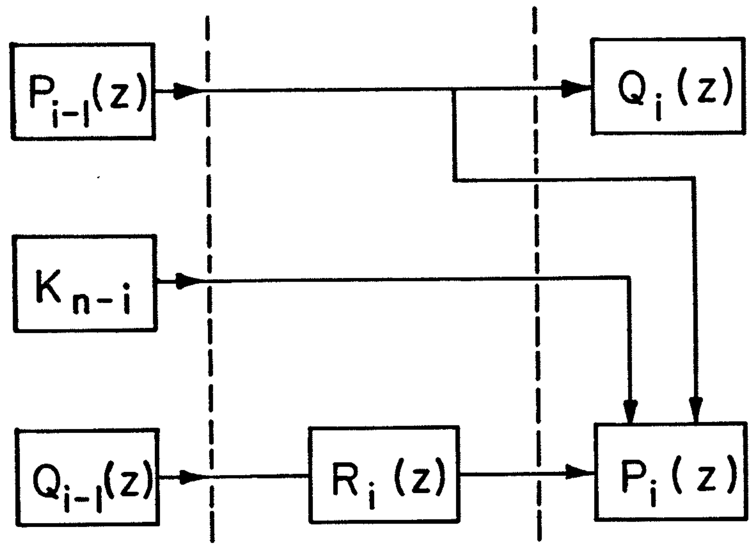

Fig. 5.4 Flow diagram for the calculation of $P_i(z)$ and $Q_i(z)$ of (5.39) and (5.41), respectively

The procedure of inverting the above expansion is illustrated in Table 5.2. From Table 5.2, one has

$$\phi_n(z) = \frac{F_1(z)}{F_2(z)} = \frac{2\ z^3 + 3\ z^2 + 3\ z + 2}{2\ z^3 + z^2 - z - 2}$$

and D(z) is given by

$$D(z) = 4\ z^3 + 4\ z^2 + 2\ z = 2\ (2\ z^3 + 2\ z^2 + z).$$

The algorithm for continued fraction inversion can be simplified by use of the property that $P_i(z)$ and $Q_i(z)$ are mirror image and antimirror image polynomials, respectively. Also, one can easily show that the polynomials $R_i(z)$ are mirror image polynomials. Finally, it is important to point out that an alternative method can be established by expanding $\phi(s)$ in a mixed Cauer form [81]. In this case, an algorithm for implementing a continued fraction inversion for the expansion (5.8) is needed. The algorithm can be derived in the light of the algorithm described above.

Table 5.2 Computation of $P_i(z)$ and $Q_i(z)$ in Example 5.5

	Z^3	Z^2	Z^1	Z^0	K_{n-i}	i
$Q_0(z)$				0		0
$P_0(z)$				1		
$Q_1(z)$			1	-1		1
$R_1(z)$			0	0		
$P_1(z)$			$\frac{16}{7}$	$\frac{16}{7}$	$\frac{16}{7}$	
$Q_2(z)$		$\frac{16}{7}$	0	$-\frac{16}{7}$		2
$R_2(z)$		1	-2	1		
$P_2(z)$		8	12	8	$\frac{49}{16}$	
$Q_3(z)$	8	4	-4	-8		3
$R_3(z)$	$\frac{16}{7}$	$-\frac{16}{7}$	$-\frac{16}{7}$	$\frac{16}{7}$		
$P_3(z)$	8	12	12	8	$\frac{5}{7}$	

Chapter VI

CONCLUSIONS AND DISCUSSIONS

The use of continued fraction expansions in realizing digital transfer functions, in one and two dimensions, is studied. Several new canonic ladder structures are developed and their realizability conditions are discussed. The structures are based on the application of a mixture of Cauer first and Cauer second continued fraction expansions. As a consequence, three new mixed Cauer forms are proposed [101]. It is shown that the resulting structures are general in the sense that published ladder realizations [15, 41-43] are special cases of the presented realizations. Furthermore, the new structures are shown to compare favorably with those of [15, 41-43] as far as the range of values of multipliers is concerned. Unlike some of the existing realizations [15, 41], the new realizations avoid the presence of delay-free loops; hence they are implementable in the discrete z-domain. Also, it is pointed out that they can be implemented in the continuous s-domain.

The approach used in obtaining the new structures is the one-step procedure [101] in which the continued fraction expansion is directly carried out on the transfer function realizing the poles and zero simultaneously. A problem with this approach is the fact that each realization has its corresponding realizability conditions which must be met in order to complete the synthesis procedure. Generally such conditions limit the synthesis procedure to certain classes of transfer functions.

It is pointed out that a useful design technique might be to assure that a particular expansion form is desired and then to evaluate the transfer function coefficients to meet other design criteria subject to the expansion constraints.

Another approach reported in the literature [44-45] is the two-step procedure in which the transfer function denominator is realized in the ladder form via the application of a Hurwitz-type expansion. The numerator is then realized as a linear combination of internal variables in the ladder. This approach is also limited to certain classes of transfer functions [44]. A useful topic for future research is to find a synthesis procedure which lessens the severity of realizability constraints and eventually leads to a synthesis method that is amenable to general stable transfer functions. It is the author's conjecture that a more sophisticated use of Davis's [83] continued fraction expansion, instead of Hurwitz expansion, in realizing the transfer function denominator would be a necessity. The continued fraction expansions introduced in Chapter V, that proceed in terms of the backward difference or the forward difference transform, can also be used [94]. The elements of the resulting ladder structures would be the discrete forward difference and/or the discrete backward difference integrators.

In a broad sense, the two-step procedure is analogous to that used in synthesis of singly-terminated reactive two-ports. A method for realizing digital transfer functions using doubly-terminated two-pairs has been reported [40]. The method [40] does not apply continued fractions. An interesting research topic is to develop a continued fraction version, of the method in [40], which would eventually be the counterpart of the well-known [8] synthesis procedure of doubly-terminated reactive two-ports.

A recurisve formula for the elements of the first column of Routh's array is presented. The use of this formula eliminates the need of constructing the whole array [72]. Consequently, explicit expressions for Cauer-type (First, Second and mixed) and Stieltjes-type expansion coefficients are systematically generated and tabulated. It is shown that these expressions are used efficiently in implementing both continued fraction expansion and inversion via solution of simple algebraic equations. The expressions represent the cyclic nature of continued fractions and provide insight into the expansion process. Therefore, they can be used in checking the existence of a certain expansion while at the same time the expansion coefficients are determined.

The continued fraction method is exclusively used in performing a proposed generalized test for Hurwitz polynomials. The basic operation involved is the execution of two continued fraction expansions. The generalized test provides further information regarding the roots' location and multiplicity. A simple aperiodicity test is obtained, as a special case of the generalized test, in which the operation involved is the execution of only one continued fraction expansion. Moreover a fast Hurwitz test is proposed in which the mixed Cauer form is used. It is demonstrated that the test is faster than the conventional Hurwitz test which uses Cauer first form.

A new z-domain continued fraction expansion of Schüssler's [86] discrete reactance function is presented. The expansion is considered as the discrete counterpart of the fast Hurwitz test [81]. The new expansion leads to a simplified algorithm for testing stability of discrete systems. The algorithm is shown to compare favorably with Jury's [39] tabular form. It is worth noting that the algorithm in question can be looked upon as the discrete counterpart of the generalized Routh's algorithm developed by Shieh and Goldman [48] for mixed Cauer form. Other z-domain continued fraction expansions, that proceed in terms of either the backward difference or the forward difference transform, are proposed. Their application in testing stability is also discussed.

The application of continued fractions to bilinear transformation of polynomials is introduced. The basic operations involved are continued fraction expansion in the s-domain and continued fraction inversion in the z-domain.

Most of the material developed in Chapters IV and V have been formulated by Jury [47] in terms of the notion of inners of a square matrix. It is believed that Jury's work [47], [38, Ch. 3], can be formulated in terms of continued fractions. In particular, modern system theory increasingly calls for the investigation of the stability of polynomials of two [47, 102] and several [47, 103] variables. It is to be hoped that methods and algorithms similar in simplicity to those developed in Chapters IV and V will be developed to facilitate the stability test of multivariable systems.

APPENDIX A

DERIVATION OF (3.2) AND (3.7)

Let G(x) be expressed as

$$G(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + a_0} \qquad \text{(A.1)}$$

G(x) is expanded into Cauer first form as follows

$$G(x) = \alpha_1 + \cfrac{1}{\alpha_2 x + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots}}} \qquad \text{(A.2)}$$

(A.2) is obtained by performing the following divide -and-invert cycles:

$$\begin{array}{r|l}
 & \alpha_1 \\
\hline
b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 & a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \\
 & a_n x^n + b_{n-1}\frac{a_n}{b_n} x^{n-1} + \ldots b_1 \frac{a_n}{b_n} x + b_0 \frac{a_n}{b_n} \\
\hline
 & (a_{n-1} - b_{n-1}\frac{a_n}{b_n}) x^{n-1} + (a_{n-2} - b_{n-2}\frac{a_n}{b_n}) x^{n-2} + \ldots
\end{array}$$

$$\begin{array}{r|l}
 & \alpha_2 x \\
\hline
(a_{n-1} - b_{n-1}\frac{a_n}{b_n}) x^{n-1} + \ldots & b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 \\
 & b_n x^n + \alpha_2 (a_{n-2} - b_{n-2}\frac{a_n}{b_n}) x^{n-1} + \ldots \\
\hline
 & [b_{n-1} - \alpha_2 (a_{n-2} - b_{n-2}\frac{a_n}{b_n})] x^{n-1} + \ldots
\end{array}$$

$$\begin{array}{r|l}
 & \alpha_3 \\
\hline
[b_{n-1} - \alpha_2 (a_{n-2} - b_{n-2}\frac{a_n}{b_n})] x^{n-1} + \ldots & (a_{n-1} - \alpha_1 b_{n-1}) x^{n-1} + \ldots
\end{array}$$

It is seen that

$$\alpha_1 = \frac{a_n}{b_n}$$

$$\alpha_2 = \frac{b_n}{a_{n-1}-\alpha_1 b_{n-1}}$$

$$\alpha_3 = \frac{a_{n-1}-\alpha_1 b_{n-1}}{b_{n-1}-\alpha_2(a_{n-2}-\alpha_1 b_{n-2})}$$

$$\vdots$$

$$\alpha_i = \frac{A_{i-1}(n)}{A_i(n)} \tag{A.3}$$

where

$$A_i(n) = A_{i-2}(n-1) - \alpha_{i-1}A_{i-1}(n-1) \qquad i = 2, 3, 4, \ldots$$

with

$$A_0(n) = a_n \quad \text{and} \quad A_1(n) = b_n \tag{A.4}$$

From equations (A.3), it is seen that α_i is the ratio of $A_{i-1}(n)$ and $A_i(n)$. Hence [73], $A_i(n)$, $i = 1, 2, \ldots, 2n + 1$, are the elements of the first column of Routh's array.

Equations (A.3) can be rewritten as follows:

$$\alpha_1 = a_n \cdot [b_n]^{-1}$$

$$\alpha_2 = \frac{a_n}{\alpha_1} \cdot [a_{n-1}-\alpha_1 b_{n-1}]^{-1}$$

$$\alpha_3 = \frac{a_n}{\alpha_1\alpha_2} \cdot [b_{n-1}-\alpha_2(a_{n-2}-\alpha_1 b_{n-2})]^{-1}$$

$$\vdots$$

$$\alpha_i = \frac{1}{\alpha_1\alpha_2\ldots\alpha_{i-1}} \cdot \frac{a_n}{A_i(n)}$$

where $A_i(n)$ is given by (A.4).

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