MAXIMUM ENTROPY MODELS
FOR
BRAND SWITCHING BEHAVIOR
AND
PARTY SWITCHING BEHAVIOR

by

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A Thesis

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ABSTRACT

Brand switching is a complex phenomenon about which usually only partial information is available. Herniter (1973) considered the situation when the only information available was the proportion of market shares of different brands. In his model using the Principle of Maximum Entropy on Shannon's measure of entropy (1948),Herniter estimated various statistics like proportion of customers loyal to each brand, proportion of customers wavering between two specific brands, the probability of switching from one specified brand to another, the probability of repeat purchase of a brand and so on. His model is quite powerful since it is completely determined by specifying only the proportion of market shares of various brands. All other brand choice statistics such as repeat rates and switch rates are derived from the model. He makes two main assumptions. The first is that the consumer choice probabilities are all continuous random variables with density functions to be determined. The second assumption is that for determining the density functions and the consumer choice probabilities , the entropy function of the system must be maximized. The results of Herniter were criticized by Bass (1974) on the grounds that the entropy

estimates of brand switching depend on the market shares only and they are independent of the product category. This implies that Herniter's model has weakness of not reflecting the variations in product category characteristic thus giving the same brand loyalty rates for different products like cigarettes, cereals,soft drinks and gasoline etc. , having the same market share distributions.

One objective of the present thesis is to meet this objection by using Renyi's (1961) measure of entropy instead of Shannon's (1948). This measure contains a parameter β which can be different for different products or for different communities. The present work reveals one important weakness in Renyi's measure viz. it is not valid for large values of β for all proportions m of market shares. However, the valid bounds for m are calculated .

When the number of brands n is large, Herniter's model becomes very complicated, and the more general model obtained by using Renyi's measure is still more complicated. For this case simpler models have been developed. The results depend on the structure of the models and the information available about the system. A very important finding was that the results of a maximum entropy model depend on the assumed structure.

All the results obtained for brand switching have been applied for party switching in elections. Most of the treatment is similar. The results of political party systems have been applied to U.S. Presidential elections.

A number of problems for further research are also proposed.

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Chapter I

INTRODUCTION

It is commonly believed by management scientists that quantitative techniques can be integrated usefully into nontrivial decision processes. Quantitative approaches to decision making involve the construction and manipulation of mathematical models of the decision environment. Incorporating quantitative analysis as one input to the decision process can be of considerable aid in making better decisions. Although applications of quantitative models have been described by some authors as disappointing to some degree, the impact of quantitative decision models in the last three decades has been remarkable. The fact remains that the quantitative analysis of managerial, business, marketing, economic and social problems has become entrenched and is maturing.

In the present work, an analysis of some quantitative models in the fields of Brand Switching behavior and Party Switching behavior are studied. To put things in proper perspective, the present thesis is divided into a number of chapters. The present chapter introduces an overview of the problem, various definitions and basic concepts, work done by other researchers and a summary of the thesis.

1.1 AN OVERVIEW OF THE PROBLEM

In brand switching , buyers switch brands and one is interested in knowing the following:

1. The probability that a buyer will buy one specific brand exclusively;
2. the probability that a buyer will buy any one of two or more specified brands;
3. the probability that a buyer will switch from a given brand to another given brand;
4. the probability that the buyer will repeat the purchase of a brand;
5. the probability that a buyer will purchase brand j when he is known to have purchased ith brand earlier (transition probabilities).

Herniter [8] developed a hetrogeneous multinomial preference model to estimate the above brand selection statistics. He used the Maximum Entropy Principle and the conventional measure of entropy due to Shannon [21], viz :

$$S = -K \sum_{i=1}^{n} p_i \log p_i \qquad (1.1.1)$$

where p_i is the probability that the system is in state i .

The major weakness in Herniter's model is that it does not reflect the variations in product category characteristics thus giving the same brand switching statistics for different products having the same market share distributions. It is the purpose of this study to remove this weakness by using Renyi's measure of entropy [20] instead of

Shannon's measure. Also this generalises the results proved by Herniter [8]. Renyi's measure contains a parameter β which can be chosen to fit the product. Renyi's measure of entropy is given as follows:

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i} \qquad (1.1.2)$$

In order to accomplish this, various definitions and basic concepts must be grasped and therefore are introduced as follows.

1.2 DEFINITIONS AND BASIC CONCEPTS

BUYERS. Buyers are people who purchase. They search, purchase, use and evaluate products, services and ideas, which they expect to satisfy their needs.

NOTE. Sometimes for convenience 'customers' or 'consumers' for 'Buyers' has been used.

BUYING BEHAVIOR. The term buying behavior can be defined as the process that buyers display in searching for, purchasing, using, and evaluating products, services and ideas.

NOTE. Because in the market the buyer has many choices for satisfying his needs, and because he does not have perfect information about all these choices; his buying behavior can not be predicted with certainty. Hence in the present study, buying behavior is described by a stochastic variable.

A STUDY OF BUYING BEHAVIOR

<u>BUYER'S POINT OF VIEW</u>. It is important to study buyer behavior so that greater insight can be gained into buyer related decisions: What they buy, why they buy, and how they buy.

<u>MARKETER'S POINT OF VIEW</u>. It is important to be well versed in the field of buyer behavior so that meaningful contributions can be made to the development of marketing strategy. A careful monitoring of buyer behavior in the market place enables the marketer to measure the success or failure of a specific marketing strategy. Knowledge of consumer behavior is also used to segment the markets. Without doubt, marketers who understand buyer behavior have a competitive advantage in the market place.

<u>BRAND LOYALTY</u>. The concept of 'loyalty' used here is of a 'repeat purchasing' type and is probably fairly standard, if slightly superficial, in the analysis of consumer goods markets. Loyalty means consistent purchase of one brand of a product category. If a buyer buys a specific brand consistently no matter what happens in the market (such as price changes, out of stock conditions, competitive activity, etc.) then the buyer is said to be loyal to that brand. For example consider the buyer who buys a given brand at least once in a certain period of time. In a second period (which may or may not be successive) a certain number of the initial buyers will buy the same brand again. The buyers, who buy the same brand in both periods of time, are called "loyal" buyers.

INDIFFERENCE AMONG BRANDS. A buyer may have preference for two or more brands. If the buyer has preference for two brands say brand 1 and brand 2, so that he or she may buy either brand, he or she is said to be indifferent between those two brands. Similarly, if the buyer has preferences for brands 1, 2 and 3 so that he or she may buy either brand 1 or brand 2 or brand 3, he or she is said to be indifferent among these three brands. Such a situation is also referred to as 'divided loyalty'.

VOTER: Voters are people who are eligible and willing to vote for a party. They analyse, vote and evaluate parties which they expect to satisfy them.

VOTING BEHAVIOR. The term voting behavior can be defined as the process that voters display in analysing, voting, and evaluating parties which they expect will satisfy them.

NOTE. Because in the election the voter usually has at least two choices for satisfying his needs and because he does not have perfect information about all of the choices, therefore, his voting behavior cannot be predicted with certainty. Hence in the present study a random variable is associated with the voting behavior.

A STUDY OF VOTING BEHAVIOR

Voters Point of View. It is important to study voter behavior so that greater insight can be gained into voter's decisions regarding

(i) which party do they vote for and,

(ii) why do they vote for a particular party?

POLITICAL PARTIES POINT OF VIEW: To operate successfully parties must have a thorough understanding of what makes voters vote a particular party. They have to know how they vote. Voting behavior is simply a subset of the larger field of human behavior. Parties which do understand voter behavior have a great competitive advantage over others in the elections.

Differences in people are what makes life interesting. For example, some people may favor a party that will decrease tax rates, while others may prefer one that will try to develop more industry or take care of unemployment. Inspite of this apparent diversity in human behavior, people act very much alike when it comes to voting. Basically most people have similar motives, but they express these motives in different ways. For this reason, an understanding of motivation is very important when it comes to study voting behavior.

Party Loyalty. This means consistent voting for one party in elections. If a voter votes for a specific party consistently no matter what happens in the election (such as change of candidate, competitive activity) then the voter is said to be loyal to that party.

Indifference Among Parties. A voter may have preference for two or more parties. If the voter has preference for two parties say party 1 and party 2, so that he or she may vote either party, then he or she is said to be indifferent to those two parties. Similarly, if the voter has preferences for parties 1, 2 and 3 then he or she is said to be indifferent among the three parties and so on. Such a situation is referred to as 'divided loyalty' also.

CATEGORIES. A buyer may have a preference for one brand exclusively (loyal to a brand), or he may have a preference for two brands, three brands and so on. There are 2^N-1 such categories of preference combinations in an N brand market. The category "preference for no brand" is excluded, since buyers must purchase some product.

MEASURE OF PREFERENCE. It is assumed that the probability of purchasing a brand is numerically equal to the preference for the brand. In other words it is assumed that each buyer has a set of preferences for the brands given by a random variable α, which can take values α_i $(0 \leqslant \alpha_i \leqslant 1)$ with corresponding probabilities $p(\alpha i)$ $(=p_i$ say); where i = 1, 2, ...; n (say). The probabilities p_i's are not known and are to be estimated later using the Maximum Entropy Principle.

State of the System. A system state is the status of the system at a point in time. In brand switching models the state of the system is defined by the category, the value of α, and the brand purchased. Similarly, in vote switching models the state of the system is defined by the category, the value of α, and the party preferred by a voter in election.

Markov Chain: Consider an experiment which may be an election or the purchase of a particular brand. If there is only a finite number, n, of possible states and the probability of being in a certain state j depends only upon the immediately preceding state j-1 of the experiment, then the experiment is called a Markov Chain.

MARKOV MATRIX OR THE FIRST ORDER TRANSITION MATRIX. Associated with a Markov Chain having n posible states is a nxn matrix P= (p_{ij}) in which p_{ij} represents the conditional probability of purchasing (voting) brand (party) i if brand (party) j was purchased (voted) on the previous occasion.

Mathematically if X_t defines the state of the system at time t, then the transition probability of going from state i to j can be defined as

$$p_{ij} = P(x_1 = j \backslash x_0 = i)$$

$$= P(x_1 = j, x_0 = i) / P(x_0 = i)$$

where $P(x_1 = j, x_0 = i)$ is the joint probability of switching brands i and j, defined below.

JOINT PROBABILITIES. These are the set of probabilities of buying (voting) the same brand (party) twice in succession or of switching between brands (parties).

In what follows the entropy concept and its measures will be introduced. By using the concept known as the Maximum

Entropy Principle, transition probability and joint probability matrices will be derived for brand switching and vote switching models.

Entropy Concept. A very useful contribution of information theory to the social and behavioral sciences is the concept of entropy. It is a measure of the degree of disorder, uncertainty or randomness of a probabilistic system. Jaynes [11] while interpreting entropy writes:

> "Just as in applied statistics the crux of a problem is often the devising of some method of sampling that avoids bias, our problem is that of finding a probability assignment which avoids bias, while agreeing with whatever information is given. The great advance provided by information theory lies in the discovery that there is a unique unambiguous criterion for the 'amount of uncertainty' represented by a discrete probability distribution, which agrees with our intuitive notions that a broad distribution represents more uncertainty than does a sharply peaked one, and satisfies all other conditions which make it reasonable."

Application of entropy concept is useful when very little information is available and one wants to be as uncertain as possible. In the study of brand switching behavior, the only available data are the market shares, and one wants to obtain estimates of brand selection statistics with this limited information. The only question is what measure of entropy to use.

In brand switching , consider a discrete random variable α representing the number of purchase occasions on which a particular buyer buys a given brand. Then the variable α can take discrete values α_i $(0 \leq \alpha_i \leq 1)$, i = 1,

..., $,\alpha$. The objective is to identify the probability function $p(\alpha i)$ $(= p_i)$ with the maximum entropy measure subject to given conditions.

Conventional measure of entropy of a system consisting of n states was given by Shannon as follows:

$$S = -K \sum_{i=1}^{n} p_i \log p_i \tag{1.2.1}$$

Where i, is the state of the system defined by the category, the value of α, and the brand purchased, categories are defined on the basis of customer's loyalty to a particular brand or indifference among brands, p_i is the probability that the customer is in state i, k is an arbitrary constant and the base of the logarithm is arbitrary. Therefore the relative value rather than the absolute value of the entropy is of importance. Here natural logarithms will be used. Letting k = 1;

$$S = - \sum_{i=1}^{n} p_i \ln p_i \tag{1.2.2}$$

This measure of entropy was used by Herniter [8] to obtain brand statistics but the weakness of this measure is that it does not take into account the effect of different products or communities.

To overcome this weakness another measure known as Renyi's measure of entropy is used. This measure was given by Renyi's [20] as :

$$S = \frac{1}{1-\beta} \ln \frac{\sum_i p_i^{\beta}}{\sum_i p_i} \qquad (1.2.3)$$

This entropy contains a parameter, β which can be different for different products or for different communities. By using Renyi's entropy measure instead of Shannon's measure, brand statistics or party statistics for different products or communities can be obtained.

Renyi's measure (1.1.3) approaches Shannon's (1.1.2) measure of entropy as $\beta \to 1$. This important fact motivates us in undertaking the present study.

Furthermore, Shannon's measure of entropy is applicable only when $\sum_i p_i = 1$ that is when a customer must buy one of the brands under consideration whereas the Renyi's measure can be used even when $\sum_i p_i \leq 1$ which means Renyi's entropy can be used even when the customer may not choose any of the brands under consideration and may choose some other brand.

MAXIMUM ENTROPY PRINCIPLE (MEP). This principle states that by maximizing entropy subject to the given constraints (in the present study given market shares along with normalization conditions on density function will yield the constraint on the problem) one obtains the unbiased encoding of one's information concerning the system.

The MEP, first given explicitely by Jaynes [11] in the context of statistical mechanics, has been applied, during the last twenty-five years to a variety of fields, including non-equilibrium thermodynamics, characterisation of statistical distributions, statistical estimation theory, urban and regional planning, spatial structure, transportation, economics, insurance, data base analysis, spectral analysis, pattern recognition, image reconstruction, marketing, engineering, and psychology. The long experience of other disciplines with entropy makes it an attractive concept for behavioral and political scientists.

The MEP is based on the ancient rules of wisdom that one should acknowledge the limitations of ones knowledge, that one should use only the knowledge one has and completely avoid using any knowledge which is not available, and that one should speak the truth and nothing but the truth.

The MEP is an excellent principle of reasoning which gives the most unbiased, most objective, most uncomitted prediction subject to the given constraints. The results will be only as good as the constraints imposed. If the information given by the constraints is insufficient or irrelevant or inconsistent, we cannot expect the predictions to be reasonable. In fact, when the predictions are inconsistent or against experience or observation, we should examine the constraints and the underlying structure of the model. The results obtained by applying MEP are dependent on the underlying structure of the model of reality. If the model does not adequately reflect reality then MEP results will also not adequately apply to reality. The argument that is being built up is that MEP is a good principle and if applied to correctly structured problems it will yield feasible results.

In marketing or in an election, one obtains that probability distribution which has maximum entropy subject to the normalization constraints and the contraints on market shares or party shares. This probability distribution along with the constraints of the system completely determines the system statistics with respect to brand or party selection.

Once the system statistics are determined with respect to brand or party selection, the joint probabilities of purchasing the same brand twice or the joint probability of voting the same party twice in succession or of switching can be determined.

To obtain the first order transition matrix, conditional probabilities of purchasing (or voting) brand (or party) i on the nth occasion if brand (or party) j was purchased (or voted) on (n-1)-th occasion are obtained. These can be calculated from joint probability by using the definition of conditional probability.

1.3 WORK DONE BY OTHERS

Models for brand switching have been given by Hendry Corporation, Herniter, Bass, Morrison and Kulwani, Kapur and Bector, and others. Herniter [8], and Kapur and Bector [13] make full use of the Principle of Maximum Entropy of Jaynes [11]; Hendry [4] and Morrison and Kulwani [19] make only partial use of this principle, while others (for example Bass [1]) make use of behavioristic assumptions only.

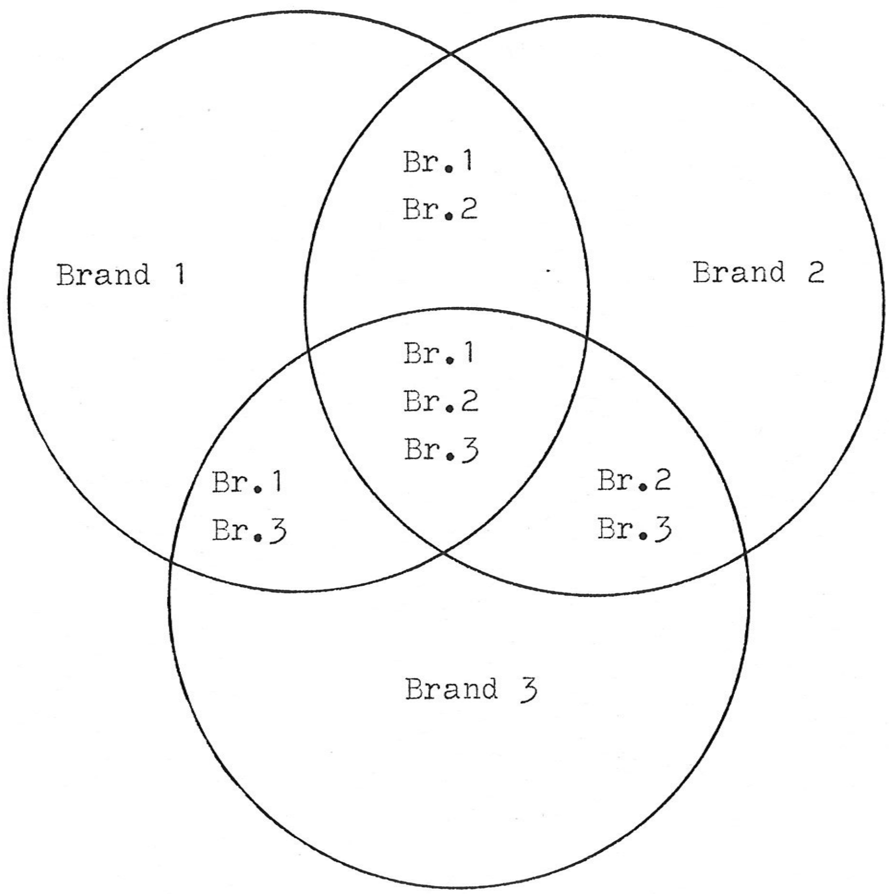
In the present study Herniter's [8] and Bass's [1] models have been used. These two models, their results, assumptions and limitations are given below.

HERNITER'S MODEL

For three-brand market Herniter considered seven categories:

1. loyal to brand 1
2. loyal to brand 2
3. loyal to brand 3
4. Indifference between brands 1 and 2
5. Indifference between Brands 1 and 3
6. Indifference between brands 2 and 3
7. Indifference among brands 1, 2 and 3. (See Figure 1-1)

Herniter assumed that each buyer has a set of preferences for the brands in the market, and there is a distribution of preferences over the population, also he assumed that the probability of purchasing a brand is numerically equal to the preference for the brand.



BRAND PREFERENCE CATEGORIES

THREE-BRAND MARKET

FIGURE 1-1

Let

1. $\alpha (0 \leq \alpha \leq 1)$ be the parameter which defines preference of a buyer for brand 1 over brand 2, or of brand 1 over brand 3 or of brand 2 over brand 3.

2. g_1, g_2, g_3, be the probabilitieis of a buyer being loyal to brands 1, 2, 3 respectively.

3. g_4, g_5, g_6 be the probabilities of his wavering between brands (1, 2), (1, 3) and (2, 3) respectively.

4. g_7 be the probability of his wavering among all the three brands 1, 2, 3.

5. $f_4(\alpha)$, $f_5(\alpha)$, $f_6(\alpha)$ be the density functions of α, which gives the preference structure in categories 4,5 and 6 respectively.

6. $f_7(\alpha, \beta)$ be the joint density functions of two random variables α, β which gives the preference structure in cateogry 7.

Then the entropy S is given by [8]:

$$S = -\sum_{i=1}^{7} g_i \ln g_i - \sum_{i=4}^{6} g_i \int_0^1 f_i(\alpha)\left[\ln f_i(\alpha) + \alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\right] d\alpha$$

$$-g_7 \int_0^1 \int_0^{1-\alpha} f_7(\alpha,\beta)\left[\ln f_7(\alpha,\beta) + \alpha\ln\alpha + \beta\ln\beta + (1-\alpha-\beta)\ln(1-\alpha-\beta)\right] d\alpha\, d\beta \quad \ldots (1.3.1)$$

Let m_1, m_2, m_3 be the market shares of the three brands, then:

$$g_1 + g_4\int_0^1 \alpha f_4(\alpha)\,d\alpha + g_5\int_0^1 \alpha f_5(\alpha)\,d\alpha + g_7\int_0^1\int_0^{1-\beta} \alpha f_7(\alpha,\beta)\,d\alpha\, d\beta \ldots (1.3.2)$$

$$g_2 + g_4\int_0^1(1-\alpha)f_4(\alpha)d\alpha + g_6\int_0^1\alpha f_6(\alpha)d\alpha + g_7\int_0^1\int_0^{1-\beta}\beta f_7(\alpha,\beta)d\alpha\, d\beta = m_2 \quad \ldots (1.3.3)$$

$$g_3 + g_5\int_0^1(1-\alpha)f_5(\alpha)d\alpha + g_6\int_0^1(1-\alpha)f_6(\alpha)d\alpha + g_7\int_0^1\int_0^{1-\beta}(1-\alpha-\beta)f_7(\alpha,\beta)d\alpha\, d\beta = m_3 \quad \ldots (1.3.4)$$

Normalisation conditions are:

$$\sum_{i=1}^{7} g_i = 1 \qquad (1.3.5)$$

$$\int_0^1 f_i(\alpha)d\alpha = 1 \quad , \quad i = 4, 5, 6 \qquad (1.3.6)$$

$$\int_0^1\int_0^{1-\alpha} f_7(\alpha,\beta)d\alpha\, d\beta = 1 \qquad (1.3.7)$$

Herniter [8] obtains the maximum entropy solution of the above problem which yields values for g's and f's and from these he deduces the repeat purchase probabilities as well as the transition probabilities; to answer the following questions:

1. The probability that a buyer will buy one specific brand exclusively;
2. The probability that a buyer will buy any one of two or more specified brands;
3. The probability that a buyer will switch from a given brand to another given brand;
4. The probability that the buyer will repeat the purchase of a brand;
5. The probability that a buyer will purchase brand j when he is known to have purchased ith brand earlier.

Herniter's model answers all these questions under the following assumptions:

1. Each buyer has a set of preferences for the brands in the market and there is a distribution of the preferences over population;
2. Brand selection is made under "stable" market conditions;
3. The effects of advertising, price and purchase timing have not been considered;
4. The total population of buyers is assumed to be known and unchanging.

Herniter's model has the following major weakness:

It gives the same answer independently of the product of which the brands are being considered and independently of the conservative or innovative attitude of the community whose purchase behavior is being investigated. Bass [1, p-4] also criticised Herniter's model:

> "The major weakness of the maximum entropy approach to the estimation of brand switching is in its inflexibility. The entropy estimates of brand switching depend only on the market shares and are independent of product category. Surely, it is reasonable to expect that brand loyalty, as reflected in repeat purchase probabilities, would vary with the product category. Cigarettes, cereals, soft drinks and gasoline should not all be expected to have the same brand loyalty rates, holding constant the market share distribution."

This weakness has been removed by using Renyi's measure of entropy instead of Shannon's measure. Renyi's measure contains a parameter β which can be chosen to fit the product or the community or both. In Herniter's model for n brands there are 2^n-1 categories, 2^n-1 g's and $2^{n+1}-n-1$ density function so that in the MEP one has to determine 2^n-1 constants and 2^n-n-1 density functions i.e. a total of $2^{n+1}-n-2$ unknowns. If n is large, this can be a huge number, e.g. if n=10, the number is 2036 and if n=20, it will be 20970130. The structure given by Herniter becomes extremely complicated when the number of brands is more than four.

Bass's Model.

In order to simplify the model, Bass [1] considered only n+1 categories by lumping together all waverers into one group called the Stochastic Preference Group [SPG]. If g_0 is the probability of a customer belonging to the SPG, n+1 probabilities are to be determined. Bass also introduced conditional probabilities $\theta_1, \theta_2, \ldots \theta_n$, where θ_i is the conditional probability of a buyer in the SPG purchasing the ith brand. Therefore in this case 2n+1 probabilities

$$g_1, g_2, \ldots g_n ; \theta_1, \theta_2, \ldots, \theta_n ; g_0 \qquad (1.3.8)$$

are determined.

Effectively we have to determine only 2n-1 probabilities because

$$g_0 = 1 - \sum_{i=1}^{n} g_i \quad , \quad \sum_{i=1}^{n} \theta_i = 1$$

This structure is much simpler than Herniter's, since there are at most 2n+1 unknowns to be determined. For 10 brands, we have to determine 21 unknown and for 20 brands, we have to determine 41 unknowns and these unknowns are constants rather than functions. Bass modified the Negative Binomial distribution [NBD] of Ehrenberg [3] and made some other behavioristic assumptions to get his Partial Negative Binomial Distribution (PNBD) model which may better be designated as the Modified Negative Binomial Distribution (MNBD) model. For the unknowns in (1.3.8) in terms of the market shares he obtained the following results:

$$\theta_i = \frac{\sqrt{m_i}}{\sum_{i=1}^{n} \sqrt{m_i}} \tag{1.3.10}$$

$$g_i = \sum_{i=1}^{n} m_i(1-\theta_i) = 1 - \frac{\sum_{i=1}^{n} m_i^{3/2}}{\sum_{i=1}^{n} m_i^{1/2}} \tag{1.3.11}$$

$$g_i = m_i - g_i\theta_i = m_i - \left(1 - \frac{\sum_i m_i^{3/2}}{\sum_i m_i^{1/2}}\right) \frac{m_i^{1/2}}{\sum m_i^{1/2}} \tag{1.3.12}$$

Bass also tried to consider the fact that there can be more brand switching in some product classes than in others. He introduced a "Brand loyalty factor", given by

$$\frac{\lambda - 1}{\lambda}$$

into the theory. This factor will have different values for different product classes.

Bass prepared tables for brand switching by taking the value of λ as 1.66 and compared them with Herniter's tables. The comparison showed that the switching for the two models is very similar and in some cases identical.

Bass [1], in his model, made the following behavioristic assumptions.

1. If x_i is a random variable defined by the number of customers in the SPG buying brand i and k_i is some minimum number choosing brand i, then the conditional distribution of a random variable $\gamma i = x_i - K i$ may be approximated by a Poisson distribution with a mean μi;

2. The mean preferences vary over the population of the SPG and therefore there exists a probability distribution $h(\mu)$ over mean preferences, which may be approximated by Gamma distribution.

In the present work another model has been also developed in which the structure of Bass's model is kept the same but MEP using Shannon's entropy is applied in place of behavioristic assumptions.

1.4 SUMMARY OF THESIS

In Chapter 2 Herniter's [8] entropy model for brand purchase behavior has been generalised through the use of Renyi's [20] measure of entropy which is a more general concept than Shannon's [21] used by Herniter and which includes Shannon's measure as a limiting case. The generalised model is more flexible than Herniter's model since it can give different marketing statistics for different products and it can give these statistics even when only a partial list of the brands is considered or when the customer may not buy any of the brands under consideration and may choose some other brand.

Chapter 3 presents another generalised entropy model for brand purchase behavior. An important result proved in this chapter is that, if none of the market shares is prescribed, the entropy will be maximum when all the market shares are equal. Another important result is that, if some of the market shares are prescribed, the entropy will be maximum when the remaining market shares are equal. The maximization of entropy is also studied when some of the probabilities of specified brands being selected are prescribed.

In chapter 4, it is shown that even when one makes use of the Principle of Maximum Entropy, the solution of a maximum entropy model for switching of brands in a market depends on the structure of the underlying model i.e. on the basic scheme of classification used and the assumptions made. This important fact is brought out in this chapter by proposing three new models of switching of brands. In the first model the MEP by using Shannon's measure of entropy is applied to the structure of Bass [1], in place of behavioristic assumptions. This enables us to determine the following interesting results:

1. The probability of a customer belonging to SPG is 1/2
2. The probability of a customer being loyal to the ith brand is proportional to ith brand market share.

3. The conditional probability of a customer of the SPG being loyal to the ith brand is equal to its market share.

Furthermore it is also shown that the same results are obtained when Renyi's measure of entropy is used. In our second model, results similar to Herniter's [8] three brand, seven categories case are derived by introducing the conditional probabilities of choosing a brand. The results obtained from this model show that one fourth of the customers are loyal to a single brand. Also out of the proportion m_1, purchasing the first brand, 1/4 m_1, are indifferent between brands 1 and 2, 1/4 m_1, are indifferent between brands 1 and 3 and 1/4 m_1, are indifferent among all the three brands.

In Model 3, the structure of model 2 is used, but, instead of market shares being prescribed, the loyalty of customers for one or more than one brand is given. The main results of this model are that, when the market shares are all equal, about 8% of the customers are loyal to each brand, while, in the case of Herniter's model, the percentage is about 11%.

In Chapter 5, under appropriate assumptions, results similar to that developed in chapter 2, 3 and 4 are discussed for voting behavior in an election. A few additional results are also derived. In case 1, the most likely voting share of each party is obtained when the proportion of wavering and loyal votes is given. Using the information both before and after the election, the behavior of switching of wavering voters is estimated in second case. Cases 3 and 4 deal with obtaining party statistics when some of the party shares are known and when information about the proportion of loyal voters only is available. In Case 5 the percentage of voters from loyalists and waverers have been obtained when no information is available about the system.

Chapter 6 presents a study of the effect of β on the entropy and various probabilities. It is also shown that Renyi's measure of entropy fails for some values of β greater than 1 and that it cannot be used for all values of m for a given β.

A value of β from U.S. Presidential elections in 1972 has been obtained and from this the behavior of voters in 1976 elections has been estimated.

In chapter 7 concluding remarks and suggestions for further work are given.

Chapter II

GENERALISATION OF HERNITER'S MODEL FOR BRAND SWITCHING BEHAVIOR - I

2.1 INTRODUCTION

Bass [1] has made a strong case for stochastic or probabilistic models (as against deterministic models) of consumer behavior. A number of such stochastic models are now available [1, 4-6, 10, 14-18].

Herniter [8] developed a heterogeneous multinomial preference probabilistic model of consumer purchase behavior for frequently purchased low-cost items. The model is quite forceful since it is completely determined by specifying only the market shares of the various brands. All other brand selection statistics such as repeat rates and switch rates, are derived from the model. He makes two main assumptions. The first is that the probabilities with which the consumer chooses various brands are all continuous random variables with density functions to be determined. The second assumption is that for determining the density funtions and the probabilities of the consumer choosing the various brands, the entropy function of the system must be maximized. Herniter [8] used the conventional measure of entropy due to Shannon [21], viz:

$$S = -K\sum_{i=1}^{n} p_i \ln p_i \tag{2.1.1}$$

where, p_i is the probability of the system being in state i and k is an arbitrary constant.

Renyi [20] derived another important measure of entropy of order β given as follows:

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i} \tag{2.1.2}$$

It will be interesting to see how far the results of Herniter [8] are modified if Renyi's measure of entropy is used. All results of Herniter should follow as particular cases, as in (2.1.2) approaches unity.

In the present generalised model, the market statistics will depend on the parameter β and this parameter can depend on the product category. This also give a greater flexibility in fitting observed empirical data to theoretical predictions of the model. The model will also be applicable when the market shares of some of the relatively minor brands may either (i) be unknown or (ii) ignored.

2.2 THE TWO-BRANDS ONE-CATEGORY CASE

Consider the case of a two-brand (brands 1 and 2) market and a customer whose preference for brand 1 is α. Assuming that the probability of purchasing a brand is numerically equal to the preference for the brand, then the probabilities of his purchasing brands 1 and 2 are respectively α and $1-\alpha$. There is a distribution of α over the population, $f(\alpha)$. The state of this system is defined by the buyer's parameter value, α, and the brand purchased. There are an infinite number of states and therefore, the state of the system is considered as a continuous rather than a discrete variable.

Let b=i be the event that brand i is purchased, then Renyi's measure of entropy for the system is given by

$$S = \int_0^1 \frac{1}{1-\beta} \ln \left[\frac{p^\beta(b=1,\alpha) + p^\beta(b=2,\alpha)}{p(b=1,\alpha) + p(b=2,\alpha)} \right] f(\alpha)\,d\alpha \qquad (2.2.1)$$

$$= \int_0^1 \frac{1}{1-\beta} \ln \left[\frac{[p(b=1|\alpha)f(\alpha)]^\beta + [p(b=2|\alpha)f(\alpha)]^\beta}{p(b=1|\alpha)f(\alpha) + p(b=2|\alpha)f(\alpha)} \right] f(\alpha)\,d\alpha \qquad (2.2.2)$$

since $p(b=1|\alpha) = \alpha$ and $p(b=2|\alpha) = 1-\alpha$, the entropy is

$$S = \frac{1}{1-\beta}\int_0^1 \ln\left[\frac{(\alpha f(\alpha))^\beta + ((1-\alpha)f(\alpha))^\beta}{\alpha f(\alpha) + (1-\alpha)f(\alpha)}\right] f(\alpha)\,d\alpha \quad \ldots (2.2.3)$$

$$= \frac{1}{1-\beta}\int_0^1 \left[(\beta-1)\ln f(\alpha) + \ln(\alpha^\beta + (1-\alpha)^\beta)\right] f(\alpha)\,d\alpha \quad \ldots (2.2.4)$$

The problem now becomes one of determining the distribution $f(\alpha)$ which maximizes the entropy (2.2.4) subject to the condition

$$\int_0^1 f(\alpha)\,d\alpha = 1 \qquad (2.2.5)$$

To find the maximum entropy of the system, the method of Lagrange multipliers as applied to the calculus of variations is used. Defining L as the Lagrangian, we have

$$L = \frac{1}{1-\beta}\int_0^1 (\beta-1)\ln f(\alpha) + \ln(\alpha^\beta + (1-\alpha)^\beta)\, f(\alpha)\,d\alpha$$
$$- (\lambda-1)\left[\int_0^1 (f(\alpha)\,d\alpha - 1)\right] \qquad (2.2.6)$$

The necessary condition for entropy to be maximum is

$$\delta L = 0 \qquad (2.2.7)$$

Applying the above condition, the following equation is obtained

$$\ln f(\alpha) - \frac{1}{1-\beta}\ln(\alpha^\beta + (1-\alpha)^\beta) + \lambda = 0 \qquad (2.2.8)$$

This gives

$$f(\alpha) = e^{-\lambda}\left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{1/1-\beta} \qquad (2.2.9)$$

where λ is determined as a function of β from the equations (2.2.5) and (2.2.9), i.e.

$$\int_0^1 \left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} d\alpha = e^{\lambda} \qquad (2.2.10)$$

Numerical integration gives Table 2-1 and Fig. 2-1.

It is easily seen that

$$\lim_{\beta\to 1}\left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \qquad (2.2.11)$$

and from (2.2.10)

$$\lim_{\beta\to 1} \lambda(\beta) = .5165$$

which agrees with the results of Herniter [8]. Also substituting $\ln(\alpha^{\beta}+(1-\alpha)^{\beta})$ from (2.2.8) in (2.2.4), we get

$$S = \frac{1}{1-\beta}\int_0^1 \left[(\beta-1)\ln f(\alpha) + (\ln f(\alpha)-\lambda)(1-\beta)\right] d\alpha \qquad (2.2.13)$$

Now using (2.2.5) it is seen that the maximum entropy is $S=\lambda(\beta)$

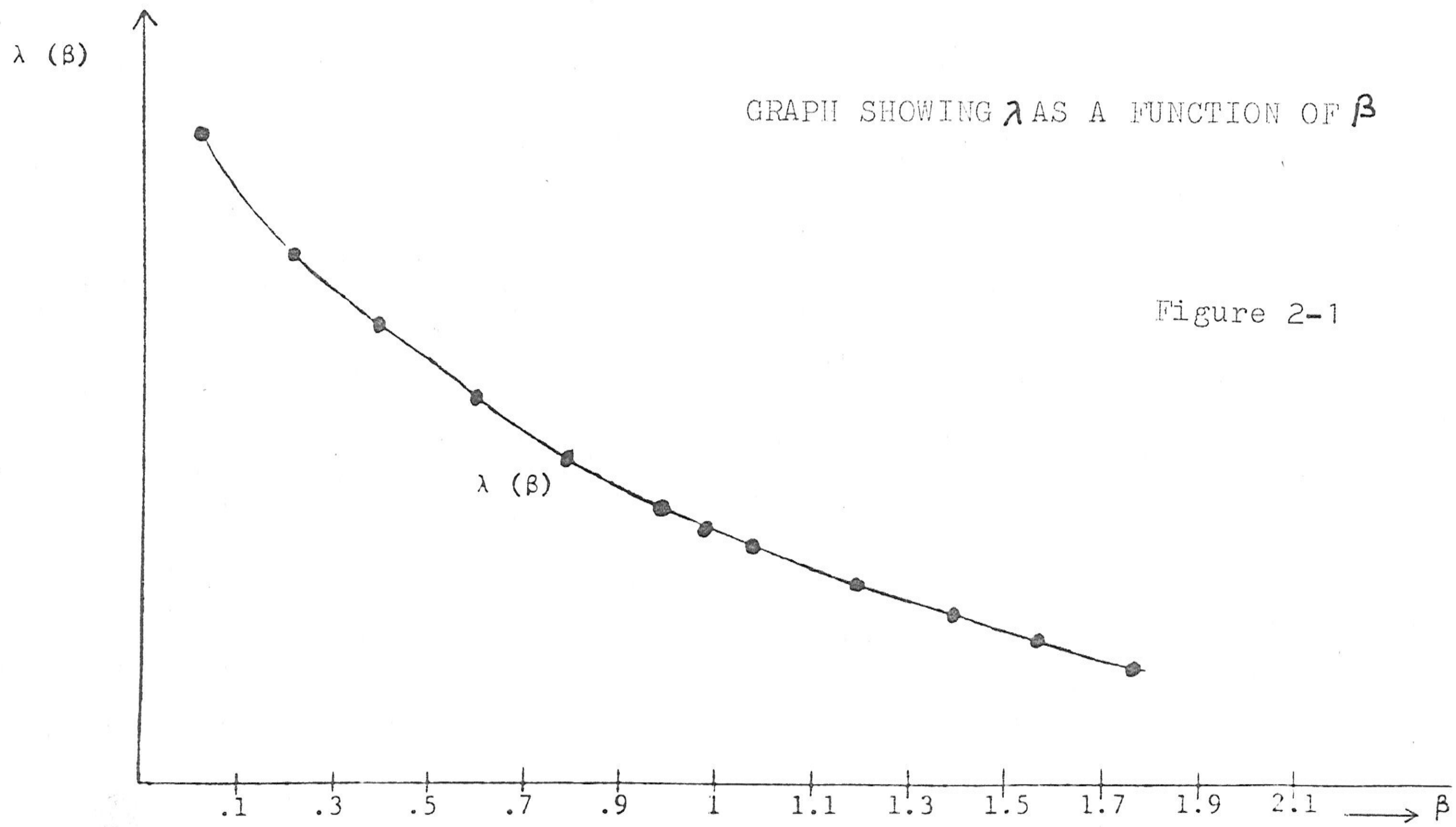
From equation (2.2.11) or Table 2-1 it is evident that the maximum entropy is a monotonic function of β.

Now from (2.2.10), it is easily seen that,

$$f(\alpha) = f(1-\alpha)\;;\; f(1/2) = 2f(0) = 2f(1) = 2e^{-\lambda} \qquad (2.2.14)$$

TABLE 2-1

β	.1	.3	.5	.7	.9	1.0	1.1	1.3	1.5	1.7	1.9	2.5
λ	.6644	.6167	.5796	.5503	.5266	.5165	.5073	.4920	.4776	.4660	.4561	.4330
$e^{-\lambda}$	.5146	.5397	.5601	.5768	.5906	.5966	.6021	.6114	.6203	.6275	.6338	.6486



GRAPH SHOWING λ AS A FUNCTION OF β

Figure 2-1

This yields that the density function $f(\alpha)$ is symmetric about $\alpha = 1/2$

Let m_1 and m_2 be the expected market shares of brands 1 and 2 respectively, then m_1 and m_2 are given by

$$m_1 = \int_0^1 \alpha f(\alpha) d\alpha \; ; \; m_2 = \int_0^1 (1-\alpha) f(\alpha)\, d\alpha \qquad (2.2.15)$$

Using (2.2.14) we have from (2.2.15)

$$m_1 = \int_0^1 \alpha\, f(\alpha) d\alpha = \int_0^1 (1-\alpha) f(1-\alpha)\, d\alpha = \int_0^1 (1-\alpha) f(\alpha) d\alpha = m_2 \; \ldots\ldots\ldots \; (2.2.16)$$

so that $m_1 = m_2 = .5$

Thus for the one category assumption, at maximum value of entropy the market shares of both the brands are equal, whatever be the value of β (including $\beta = 1$) discussed by Herniter [8]. The joint probability of purchasing brand 1 twice is

$$\int_0^1 \alpha^2 f(\alpha) d\alpha = \frac{\int_0^1 \alpha^2 \left\{\alpha^\beta + (1-\alpha)^\beta\right\}^{\frac{1}{1-\beta}} d\alpha}{\int_0^1 \left\{\alpha^\beta + (1-\alpha)^\beta\right\}^{\frac{1}{1-\beta}} d\alpha} = K(\beta) \;\; (\text{say}) \quad \ldots\ldots (2.2.18)$$

The joint probability of purchasing brand 2 twice is

$$\int_0^1 (1-\alpha)^2 f(\alpha) d\alpha = \int_0^1 f(\alpha) d\alpha - 2\int_0^1 \alpha f(\alpha) d\alpha + \int_0^1 \alpha^2 f(\alpha)\, d\alpha$$

$$= 1 - 2 \cdot \frac{1}{2} + K(\beta) = K(\beta) \qquad (2.2.19)$$

The probability of switching from one brand to another is

$$\int_0^1 \alpha(1-\alpha) f(\alpha)\, d\alpha = \int_0^1 \alpha f(\alpha) d\alpha - \int_0^1 \alpha^2 f(\alpha) d\alpha$$

$$= \frac{1}{2} - K(\beta) \tag{2.2.20}$$

so that the joint probability matrix A is (writing k for k(β)
) given by

$$A = \begin{pmatrix} K & .5-K \\ .5-K & K \end{pmatrix} \tag{2.2.21}$$

The first order transition probability matrix or Markov matrix B gives the set of conditional probabilities of purchasing brand i on the nth occassion given that brand j was purchased on the (n-1)-th occasion. Therefore,

$$B = \begin{pmatrix} \overline{\alpha^2}/\bar{\alpha} & \overline{\alpha(1-\alpha)}/\bar{\alpha} \\ \overline{\alpha(1-\alpha)}/\overline{1-\alpha} & \overline{(1-\alpha)^2}/\overline{1-\alpha} \end{pmatrix} \tag{2.2.22}$$

$$= \begin{pmatrix} 2K & 1-2K \\ 1-2K & 2K \end{pmatrix} \tag{2.2.23}$$

From (2.2.18) it is seen that k(β) gives the second moment μ_2' of the distribution f(α) about the origin, so that the variance of the distribution is given by

$$\sigma^2 = \text{variance} = \mu_2' - (\mu_1')^2 = K - .25 \tag{2.2.24}$$

Numerical integration gives Table 2-2

TABLE 2-2

β	$K(\beta)$	$\frac{1}{2}-K(\beta)$	σ^2	σ	$\sigma/\mu_1'^2$
.1	.3308	.1692	.0808	.2843	1.1372
.3	.3269	.1731	.0769	.2773	1.1092
.5	.3242	.1758	.0742	.2724	1.0896
.7	.3223	.1777	.0723	.2689	1.0755
.9	.3210	.1790	.0705	.2665	1.0660
1.0	.3205	.1795	.0705	.2655	1.0620
1.1	.3200	.1800	.0701	.2645	1.0580
1.3	.3194	.1806	.0696	.2696	1.0544
1.5	.3190	.1810	.0690	.2626	1.0504
1.7	.3186	.1814	.0686	.2619	1.0476
1.9	.3184	.1816	.0684	.2615	1.0460
2.5	.3180	.1820	.0680	.2608	1.0432

2.3 THE TWO-BRAND THREE-CATEGORY CASE

Now consider a two-brand market with three categories.

i) Customers who have a preference for brand 1 only.

ii) Customers who have a preference for brand 2 only.

iii) Customers who have a preference for both brands.

Let g_i be the probability of customer being in category i (i = 1,2,3), α the parameter which defines the customer's preference for brand 1 if the customer is in category 3, and f(α) the density function for α, $0 \leq \alpha \leq 1$, then the entropy is

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{3} g_i^{\beta}}{\sum_{i=1}^{3} g_i} + g_3 \int_0^1 \left[\frac{1}{1-\beta} \ln \frac{\{\alpha f(\alpha)\}^{\beta} + \{(1-\alpha) f(\alpha)\}^{\beta}}{\alpha f(\alpha) + (1-\alpha) f(\alpha)} \right] f(\alpha)\, d\alpha \qquad \ldots . \ (2.3.1)$$

The normalisation conditions are:

$$\sum_{i=1}^{3} g_i = 1 \quad , \quad \int_0^1 f(\alpha)\, d\alpha = 1 \qquad (2.3.2)$$

Let m_j be the expected market share of brand j(j = 1,2), then since the expected market share is simply the probability that a consumer selected at random will purchase the brand, the market share constraints are

$$g_1 + g_3 \int_0^1 \alpha f(\alpha)\, d\alpha = m_1 \ ; \qquad (2.3.3)$$

$$g_2 + g_3 \int_0^1 (1-\alpha) f(\alpha)\, d\alpha = m_2 \ . \qquad (2.3.4)$$

Entropy S in (2.3.1) is maximized subject to the normalization conditions in (2.3.2) and expected market share constraints in (2.3.3). Therefore, maximize the Lagrangian

$$L = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{3} g_i^{\beta}}{\sum_{i=1}^{3} g_i} + g_3 \int_0^1 \left[\frac{1}{1-\beta} \ln\left((\alpha^{\beta} + (1-\alpha)^{\beta}) (f(\alpha))^{\beta-1} \right) \right] f(\alpha)\, d\alpha$$

$$- (\eta - 1) \left(\sum_{i=1}^{3} g_i - 1 \right) - \lambda \left[\int_0^1 f(\alpha)\, d\alpha - 1 \right]$$

$$- \zeta_1 \left[g_1 + g_3 \int_0^1 \alpha f(\alpha)\, d\alpha - m_1 \right]$$

$$- \zeta_2 \left[g_2 + g_3 \int_0^1 (1-\alpha) f(\alpha)\, d\alpha - m_2 \right] \qquad \ldots\ldots (2.3.5)$$

for variations of g_1, g_2, g_3 and α.

Using calculus of variations on (2.3.5), we get

$$\frac{1}{1-\beta} \left[\frac{\beta g_1^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} - \frac{1}{\sum_{i=1}^{3} g_i} \right] - (\eta - 1) - \zeta_1 = 0 \quad \ldots (2.3.6)$$

$$\frac{1}{1-\beta} \left[\frac{\beta g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} - \frac{1}{\sum_{i=1}^{3} g_i} \right] - (\eta - 1) - \zeta_2 = 0 \quad \ldots (2.3.7)$$

$$\frac{1}{1-\beta} \left[\frac{\beta g_3^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} - \frac{1}{\sum_{i=1}^{3} g_i} \right] - (\eta - 1)$$

$$+\int_0^1 \left[\frac{1}{1-\beta}\ln(\alpha^\beta + (1-\alpha)^\beta) - \ln f(\alpha)\right] f(\alpha)\,d\alpha$$

$$-\zeta_1 \int_0^1 \alpha f(\alpha)\,d\alpha - \zeta_2 \int_0^1 (1-\alpha) f(\alpha)\,d\alpha = 0 \qquad (2.3.8)$$

$$\frac{g_3}{1-\beta}\left[(\beta-1) + \ln\{(f(\alpha))^{\beta-1}(\alpha^\beta + (1-\alpha)^\beta)\}\right]$$

$$-\zeta_1 g_3 \alpha - \zeta_2 g_3 (1-\alpha) - \lambda = 0 \qquad (2.3.9)$$

Equations (2.3.2) - (2.3.4) and (2.3.6) - (2.3.9) give us eight equations to determine eight unknowns g_1, g_2, g_3, $f(\alpha)$; η; ζ_1, ζ_2.

From (2.3.9) we get

$$f(\alpha) = e^{-\frac{\lambda}{g_3} - 1 - \zeta_2}\, e^{-(\zeta_1 - \zeta_2)\alpha}\left[\alpha^\beta + (1-\alpha)^\beta\right]^{\frac{1}{1-\beta}} \qquad (2.3.10)$$

Multiplying (2.3.9) by $f(\alpha)$ and then integrating it for α from 0 to 1 and using (2.3.8), we get

$$\frac{\beta}{1-\beta}\,\frac{g_3^{\beta-1}}{\sum_{i=1}^{3} g_i^\beta} - \frac{1}{1-\beta} - (\eta - 1) + \mu = 0\,;\quad \mu = \frac{\lambda}{g_3} + 1 \quad \ldots\ldots (2.3.11)$$

Substituting (2.3.6) from (2.3.7) yields

$$\zeta_1 - \zeta_2 = \frac{\beta}{1-\beta}\,\frac{g_1^{\beta-1} - g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^\beta} \qquad (2.3.12)$$

Subtracting (2.3.7) from (2.3.11) yields

$$\mu + \zeta_2 = \frac{\beta}{1-\beta}\,\frac{g_2^{\beta-1} - g_3^{\beta-1}}{\sum_{i=1}^{3} g_i^\beta} \qquad (2.3.13)$$

Subtracting for $f(\alpha)$ from (2.3.10) in (2.3.2) and (2.3.3), we get respectively,

$$\int_0^1 e^{-(\delta_1-\delta_2)\alpha}\left[\alpha^{\beta}+(1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} d\alpha = e^{\mu+\delta_2} \quad \ldots\ (2.3.14)$$

$$g_1 + g_3\, e^{-(\mu+\delta_2)}\int_0^1 e^{-(\delta_1-\delta_2)\alpha}\,\alpha\left[\alpha^{\beta}+(1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} d\alpha = m_1 \ldots (2.3.15)$$

$$g_2 + g_3\, e^{-(\mu+\delta_2)}\int_0^1 e^{-(\delta_1-\delta_2)\alpha}(1-\alpha)\left[\alpha^{\beta}+(1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} d\alpha = m_2 \ldots (2.3.16)$$

Substituting for $\mu+\delta_2$ and $\delta_1-\delta_2$ from (2.3.13) and (2.3.12) in (2.3.14) and (2.3.15), and using (2.3.2) along with the Newton Raphson iterative method, three unknown g_1, g_2 and g_3 can be determined.

Once $f(\alpha)$ is determined $\bar{\alpha}$ and $\bar{\alpha}^2$, can also be determined. This will then yield the following probability matrix A and Markovian matrix B:

$$A = \begin{pmatrix} g_1 + g_3\bar{\alpha}^2 & g_3\overline{\alpha(1-\alpha)} \\ g_3\overline{\alpha(1-\alpha)} & g_2 + g_3\overline{(1-\alpha)}^2 \end{pmatrix}$$

$$B = \begin{pmatrix} \dfrac{g_1 + g_3\bar{\alpha}^2}{g_1 + g_3\bar{\alpha}} & \dfrac{g_3\overline{\alpha(1-\alpha)}}{g_1 + g_3\bar{\alpha}} \\[2ex] \dfrac{g_3\overline{\alpha(1-\alpha)}}{g_2 + g_3\overline{(1-\alpha)}} & \dfrac{g_2 + g_3\overline{(1-\alpha)}^2}{g_2 + g_3\overline{(1-\alpha)}} \end{pmatrix}$$

Let $\beta \to 1$, then equation (2.3.10) yields

$$f(\alpha) = e^{-(\mu+\zeta_2)} \, e^{-(\zeta_1-\zeta_2)\alpha} \, \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} \tag{2.3.18}$$

equation (2.3.12) and (2.3.13) give

$$\zeta_1 - \zeta_2 = \ln \frac{g_2}{g_1} \quad , \quad \mu + \zeta_2 = \ln \frac{g_3}{g_2} \tag{2.3.19}$$

(2.3.18) and (2.3.19) imply

$$f(\alpha) = \frac{g_2}{g_3} \left(\frac{g_1}{g_2}\right)^{\alpha} \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)}$$

$$= \frac{g_1^{\alpha} \, g_2^{1-\alpha}}{g_3} \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} \tag{2.3.20}$$

which is same form as in Herniter [6, equation 18].

Using (2.3.10) in (2.3.1), the optimal entropy is given by,

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{3} g_i^{\beta}}{\sum_{i=1}^{3} g_i} + g_3 \int_0^1 f(\alpha)[(\mu+\zeta_2) + (\zeta_1-\zeta_2)\alpha] \, d\alpha$$

$$= \frac{1}{1-\beta} \ln (g_1^{\beta} + g_2^{\beta} + g_3^{\beta}) + g_3(\mu+\zeta_2) + g_3(\zeta_1-\zeta_2)\alpha \tag{2.3.21}$$

(2.3.3), (2.3.12) and (2.3.13) gives

$$S = \frac{1}{1-\beta} \ln (g_1^{\beta} + g_2^{\beta} + g_3^{\beta}) + \frac{\beta}{1-\beta} g_3 - \frac{g_2^{\beta-1} - g_3^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}}$$

$$+ \frac{\beta}{1-\beta} \, \frac{g_1^{\beta-1} - g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} (m_1 - g_1) \tag{2.3.22}$$

$$= \frac{m_1}{1-\beta}\left[\ln(g_1^\beta + g_2^\beta + g_3^\beta) + \beta\left(\frac{g_1^{\beta-1}}{\sum_{i=1}^{3} g_i^\beta} - 1\right)\right]$$

$$+ \frac{m_2}{1-\beta}\left[\ln(g_1^\beta + g_2^\beta + g_3^\beta) + \beta\left(\frac{g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^\beta} - 1\right)\right] \qquad (2.3.23)$$

Now taking the limit as $\beta \to 1$,

For different values of β Tables 2-3 to 2-5 are obtained.

Special Case: Let

$$g_1 = g_2 = g, \quad g_3 = 1 - 2g$$

Equations (2.2.5) and (2.3.12) - (2.3.15) give

$$\delta_1 - \delta_2 = 0, \quad \mu + \delta_2 = \lambda(\beta), \quad g_1 + .5\, g_3 = g_2 + .5\, g_3$$

$$= m_1 = m_2 = 0.5 \qquad (2.3.24)$$

$$\lambda(\beta) = \frac{\beta}{1-\beta} \; \frac{g_1^{\beta-1} - (1-2g)^{\beta-1}}{2g^\beta + (1-2g)^\beta} \qquad (2.3.25)$$

which enables us to solve for g and therefore for g_1, g_2, g_3 for any given β.

Table 2-3

Theoretical Results for a Two-Brand Market Entropy Model

		β =0.25				β =0.50				β =0.75			
m_1	m_2	g_1	g_2	g_3	S	g_1	g_2	g_3	S	g_1	g_2	g_3	S
.1	.9	.0270	.7556	.2174	1.0351	.0306	.7386	.2308	0.8990	.0267	.7142	.2590	0.8000
.2	.8	.0392	.5421	.4186	1.2198	.0583	.5641	.3776	1.1271	.0650	.5647	.3702	1.0612
.3	.7	.0508	.3482	.6008	1.3527	.0903	.4122	.4975	1.2664	.1132	.4398	.4469	1.2153
.4	.6	.0704	.1996	.7298	1.4412	.1339	.2889	.5772	1.3460	.1728	.3342	.4929	1.2997
.5	.5	.1123	.1123	.7754	1.4733	.1974	.1974	.6052	1.3722	.2459	.2459	.5082	1.3267

Table 2-4

Theoretical Results For a Two-Brand Market Entropy Model

		β = 1.0				β = 1.25				β = 1.50			
m_1	m_2	g_1	g_2	g_3	S	g_1	g_2	g_3	S	g_1	g_2	g_3	S
.1	.9	.0187	.6895	.2918	0.7325	.0079	.6666	.3254	0.6913	.0000*	.6383*	.3617*	0.6949*
.2	.8	.0658	.5583	.3759	1.0105	.0632	.5501	.3867	0.9713	.0585	.5421	.3994	0.9415
.3	.7	.1259	.4512	.4229	1.1787	.1332	.4555	.4113	1.1500	.1375	.4566	.4059	1.1269
.4	.6	.1952	.3569	.4479	1.2719	.2085	.3691	.4224	1.2525	.2175	.3760	.4065	1.2378
.5	.5	.2720	.2720	.4560	1.3019	.2871	.2871	.4256	1.2861	.2968	.2968	.4064	1.2751

Corresponding to values of $\beta \geqslant 1.5$ and $m_1 < .1074$, Newton-Raphson Method yields negative values of probabilities, which is not realistic because probabilities are not supposed to be negative. Therefore corresponding to $\beta = 1.5$, the values with astriks of g's and S corresponds to $m_1 = .1074$.

Table 2-5

Theoretical Results For a Two-Brand Market Entropy Model

m_1	m_2	=1.75				=2.00				=2.25			
		g_1	g_2	g_3	S	g_1	g_2	g_3	S	g_1	g_2	g_3	S
		.0000*	.5884*	.4116*	0.7721*	.0000*	.5625*	.4374*	0.8074*	.0000*	.5475*	.4525*	0.8255*
.2	.8	.0523	.5350	.4127	0.9190	.0452	.5292	.4255	0.9025	.0379	.5246	.4375	0.8905
.3	.7	.1400	.4562	.4037	1.1066	.1414	.4552	.4034	1.0898	.1421	.4539	.4040	1.0753
.4	.6	.2238	.3800	.3962	1.2259	.2286	.3823	.3890	1.2160	.2324	.3837	.3839	1.2074
.5	.5	.3033	.3033	.3934	1.2669	.3080	.3080	.3840	1.2605	.3114	.3114	.3771	1.2554

Corresponding to values of $\beta \geqslant 1.75$ and $m_1 < .1378$, Newton-Raphson Method yields negative values of probabilities,which is not realistic because probabilities are not supposed to be negative. Similar statements hold for $\beta \geqslant 2.00$, $m_1 < .1559$ and $\beta \geqslant 2.25$, $m_1 < .1674$.

Therefore corresponding to the values of β of 1.75, 2.0 and 2.25, the values with astriks of g's and S corresponds to $m_1 = .1378$, .1559 and .1674 respectively.

2.4 THREE-BRAND SEVEN CATEGORY CASE

In this case the first three categories contain customers who have preferences for only one brand, the next three categories contain customers who have preferences for only two brands and the last category contains customers who have preferences for all three brands.

The model structure of the three-brand case, as in Herniter [8], is thus

category	Preference Distribution	Brands Involved	Preferences for Brands 1	2	3
1	--	1	-	-	-
2	--	2	-	-	-
3	--	3	-	-	-
4	f	1,2	α	$1-\alpha$	
5	f	1,3	α	-	$1-\alpha$
6	f	2,3	-	α	$1-\alpha$
7	f	1,2,3	α	γ	$1-\alpha-\gamma$

The entropy of the system is

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{7} g_i^{\beta}}{\sum_{i=1}^{7} g_i} + \sum_{i=4}^{6} g_i \int_0^1 f_i(\alpha)\left[\frac{1}{1-\beta} \ln\left\{(f_i(\alpha))^{\beta-1}(\alpha^{\beta}+(1-\alpha)^{\beta})\right\}\right] d\alpha$$

$$+ g_7 \int_0^1 d\gamma \int_0^{1-\gamma} d\alpha\, f_7(\alpha,\gamma)\left[\frac{1}{1-\beta} \ln\left\{(f_7(\alpha,\gamma))^{\beta-1}(\alpha^{\beta}+\gamma^{\beta}+(1-\alpha-\gamma)^{\beta})\right\}\right]$$

$$\ldots\ldots (2.3.27)$$

The problem now is to maximize S subject to the following constraints.

$$\sum g_i = 1 \;, \quad \int_0^1 f_j(\alpha)\, d\alpha = 1 \qquad j = 4, 5, 6 \qquad (2.3.28)$$

$$\int_0^1 d\gamma \int_0^{1-\gamma} f_7(\alpha,\gamma)\, d\alpha = 1 \qquad (2.3.29)$$

$$K_1 = g_1 + g_4 \int_0^1 \alpha\, f_4(\alpha)\, d\alpha + g_5 \int_0^1 \alpha\, f_5(\alpha)\, d\alpha$$

$$+ g_7 \int_0^1 d\gamma \int_0^{1-\gamma} \alpha\, f_7(\alpha,\gamma)\, d\alpha = m_1 \qquad (2.3.30)$$

$$K_2 = g_2 + g_4 \int_0^1 (1-\alpha) f_4(\alpha)\, d\alpha + g_6 \int_0^1 \alpha\, f_6(\alpha)\, d\alpha$$

$$+ g_7 \int_0^1 d\gamma \int_0^{1-\gamma} f_7(\alpha,\gamma)\, d\alpha = m_2 \qquad (2.3.31)$$

$$K_3 = g_3 + g_5\int_0^1 (1-\alpha) f_5(\alpha)\, d\alpha + g_6\int_0^1 (1-\alpha) f_6(\alpha)\, d\alpha$$

$$+ g_7\int_0^1 d\gamma \int_0^{1-\gamma} (1-\alpha-\gamma)\, f_7(\alpha,\gamma) = m_3 \quad \ldots (2.3.32)$$

Now the following Lagrangian is to be maximized.

$$L = S - (\eta-1)\left[\sum_{i=1}^{7} g_i - 1\right] - \sum_{i=4}^{6} \lambda_i \left[\int_0^1 f_i(\alpha)\, d\alpha - 1\right]$$

$$- \lambda_7\left[\int_0^1 \int_0^{1-\gamma} f_7(\alpha,\gamma)\, d\alpha\, d\gamma - 1\right] - \zeta_1 (K_1 - m_1)$$

$$- \zeta_2 (K_2 - m_2) - \zeta_3 (K_3 - m_3) \qquad (2.3.33)$$

By using calculus of variations, following results are obtained.

$$f_4(\alpha) = \left\{ \exp \frac{\beta}{1-\beta} \frac{g_4^{\beta-1} - \alpha g_1^{\beta-1} - (1-\alpha) g_2^{\beta-1}}{g_1^{\beta} + g_2^{\beta} + g_4^{\beta}} \right\} \left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} \quad \ldots (2.3.34)$$

$$f_5(\alpha) = \left\{ \exp \frac{\beta}{1-\beta} \frac{g_5^{\beta-1} - \alpha g_1^{\beta-1} - (1-\alpha) g_3^{\beta-1}}{g_1^{\beta} + g_3^{\beta} + g_5^{\beta}} \right\} \left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} \qquad (2.3.35)$$

$$f_6(\alpha) = \left\{ \exp \frac{\beta}{1-\beta} \frac{g_6^{\beta-1} - \alpha g_2^{\beta-1} - (1-\alpha) g_3^{\beta-1}}{g_2^{\beta} + g_3^{\beta} + g_6^{\beta}} \right\} \left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} \qquad (2.3.36)$$

$$f_7(\alpha,\gamma) = \left\{\exp \frac{\beta}{1-\beta} \frac{g_7^{\beta-1} - \alpha g_1^{\beta-1} - \gamma g_2^{\beta-1} - (1-\alpha-\gamma) g_3^{\beta-1}}{g_1^{\beta-1} + g_2^{\beta-1} + g_3^{\beta-1} + g_7^{\beta-1}}\right\}$$

$$\cdot \left[\alpha^{\beta} + \gamma^{\beta} + (1-\alpha-\gamma)^{\beta}\right]^{\frac{1}{1-\beta}} \qquad (2.3.37)$$

By using the normalising equations (2.3.28) - (2.3.29) four relations among g's are obtained. Equations (2.3.30) - (2.3.32) give three relations among g's so that all seven g's can be determined.

2.5 MAXIMIZATION OF ENTROPY WHEN $\sum_{i=1}^{n} g_i < 1$

Now consider the case when a customer may not choose any of the brands under consideration and may choose some other brand so that

$$\sum_{i=1}^{n} m_i < 1 \qquad \text{and} \qquad \sum_{i=1}^{n} g_i < 1$$

In this case the use of Renyi's entropy is essential. Our analysis will be same as before except that the following constraints will not be used.

$$\sum_{i=1}^{n} g_i = 1$$

Thus in Sections 2.3 and 2.4 the results can be obtained by putting $\eta = 1$.

For example, for Section 2.3, we get

$$f(\alpha) = \left[\exp \frac{\beta}{1-\beta} \frac{g_3^{\beta-1} - \alpha g_1^{\beta-1} - (1-\alpha) g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}}\right]\left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}}$$

$$= \left[\exp \frac{\beta}{1-\beta} \frac{(g_3^{\beta-1} - g_2^{\beta-1}) - (g_2^{\beta-1} - g_1^{\beta})\alpha}{\sum_{i=1}^{3} g_i^{\beta}}\right]\left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} \quad \ldots\ (2.5.1)$$

and equations (2.3.12) - (2.3.15) are obtained. This way four equations for g_1, g_2, g_3 are obtained and those values of $\zeta_1 - \zeta_2$ are chosen which give consistent positive values for g_1, g_2, g_3 whose sum is less than unity.

The method can obviously be used for any number of brands.

Suppose now that there are more than three brands of a product, but <u>most</u> of the customers purchase three brands only, though <u>some</u> may purchase other brands. In this case

$$m_1 + m_2 + m_3 = m < 1 \ ,$$

There are three alternatives;

i) Lump together all other brands into a fourth brand and call its share as m_4 so that

$$m_1 + m_2 + m_3 + m_4 = 1 \ ,$$

and find the probabilities for the four-brands fifteen-categories case that results from doing so.

ii) Confine to the market for three brands only and neglect all the customers who purchase other brands. The market shares can be taken as

$$M_1 = \frac{m_1}{m} , \quad M_2 = \frac{m_2}{m} , \quad M_3 = \frac{m_3}{m} ,$$

such that $M_1 + M_2 + M_3 = 1$,

Now find the probabilities for the three-brand seven-categories case.

iii) Ignore the constraint $g_1 + g_2 + g_3 = 1$, maximize Renyi's entropy keeping other constraints intact and find the probabilities for the seven categories.

The three alternatives will obviously give, in general, different results. The last one seems to be the most satisfactory since this model is essentially non-linear.

Chapter III

GENERALISATION OF HERNITER'S MODEL FOR BRAND SWITCHING BEHAVIOR - II

3.1 INTRODUCTION

In the present chapter, maximization of entropy is considered, when

a) no information is available about the market shares so that there are no constraints on the entropy function except the normalising constraints, or

b) information is available about market shares of some brands only, or

c) some of the g's are prescribed.

It is proved that,

i) if none of the market shares is prescribed, then the entropy will be maximum when all the market shares are equal and

ii) if some of the market shares are prescribed, then the entropy will be maximum when the remaining market shares are equal, and

iii) when some of the g's are prescribed market shares are not necessarily equal.

3.2 MAXIMIZATION OF ENTROPY WHEN MARKET SHARES ARE NOT PRESCRIBED

The three brands case. Let g_1, g_2, g_3 be the probabilities of brands 1, 2, 3 being purchased alone, g_4, g_5, g_6 be the probabilities of a customer purchasing brands (1,2) (1,3) (2,3) respectively and let g_7 be the probability of his buying all the three brands. Let $f_4(\alpha)$ be the density function for his preference for 1 when he purchases (1,2). Similarly let $f_5(\alpha)$ and $f_6(\alpha)$ denote the corresponding density functions when he purchases (1,3) and (2,3) respectively, and let $f_7(\alpha,\gamma)$ denote the density function for his preferences α and γ for 1 and 2 when he purchases (1, 2, 3)

The entropy of the system is then given by

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{7} g_i^{\beta}}{\sum_{i=1}^{7} g_i} + \sum_{i=4}^{6} g_i \int_0^1 \left[\frac{1}{1-\beta} \ln\left((f_i(\alpha))^{\beta-1}\left(\alpha^{\beta}+(1-\alpha)^{\beta}\right)\right)\right] f_i \, d\alpha$$

$$+ g_7 \int_0^1 d\gamma \int_0^{1-\gamma} d\alpha \, f_7(\alpha,\gamma)\left[\frac{1}{1-\beta} \ln\left((f_7(\alpha,\gamma))^{\beta-1}\left(\alpha^{\beta}+\gamma^{\beta}+(1-\alpha-\gamma)^{\beta}\right)\right)\right] \quad \ldots (3.2.1)$$

This has to be maximized subject to the constraints

$$\sum_{i=1}^{7} g_i = 1 \quad , \quad \int_0^1 f_i(\alpha)\, d\alpha = 1 \quad ; \; i = 4, 5, 6 \qquad (3.2.2)$$

$$\int_0^1 d\gamma \int_0^{1-\gamma} f_7(\alpha,\gamma)\, d\alpha = 1 \qquad (3.2.3)$$

Therefore the Lagrangian

$$L = S - (\zeta - 1)\left[\sum_{i=1}^{7} g_i - 1\right] - \sum_{i=4}^{6} \lambda_i \left[\int_0^1 f_i(\alpha)\,d\alpha - 1\right]$$

$$- \lambda_7 \left[\int_0^1 d\gamma \int_0^{1-\gamma} f_7(\alpha,\gamma)\,d\alpha\,d\gamma - 1\right] \qquad (3.2.4)$$

is maximized for variations in g's and f's

Using Lagrange's method of calculus of variations,

$$\frac{1}{1-\beta}\left[\frac{\beta g_i^{\beta-1}}{\sum_{i=1}^{7} g_i^{\beta}} - \frac{1}{\sum_{i=1}^{7} g_i}\right] - (\zeta - 1) = 0 \; ; \quad i = 1, 2, 3 \qquad (3.2.5)$$

$$\frac{1}{1-\beta}\left[\frac{\beta g_i^{\beta-1}}{\sum_{i=1}^{7} g_i^{\beta}} - \frac{1}{\sum_{i=1}^{7} g_i}\right] - (\zeta - 1)$$

$$- \int_0^1 \left[\ln f_i(\alpha) - \ln(\alpha^{\beta} + (1-\alpha)^{\beta})^{1/(1-\beta)}\right] f_i(\alpha)\,d\alpha = 0$$

$$i = 4, 5, 6 \qquad (3.2.6)$$

$$\frac{1}{1-\beta}\left[\frac{\beta g_i^{\beta-1}}{\sum_{i=1}^{7} g_i^{\beta}} - \frac{1}{\sum_{i=1}^{7} g_i}\right] - (\zeta - 1)$$

$$- \int_0^1 d\gamma \int_0^{1-\gamma} d\alpha\, f_7(\alpha,\gamma)\left[\ln f_7(\alpha,\gamma) - \ln(\alpha^{\beta} + \gamma^{\beta} + (1-\alpha-\gamma)^{\beta})^{\frac{1}{1-\beta}}\right] = 0$$

$$(3.2.7)$$

$$-g_i\left[\ln f_i(\alpha)-\ln\left[\alpha^{\beta}+(1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}}\right]-g_i-\lambda_i=0\ ; \quad i=4,5,6 \qquad (3.2.8)$$

$$-g_7\left[\ln f_7(\alpha,\gamma)-\ln\left[\alpha^{\beta}+\gamma^{\beta}+(1-\alpha-\gamma)^{\beta}\right]^{\frac{1}{1-\beta}}\right]-g_7-\lambda_7=0 \qquad (3.2.9)$$

We easily get from these

$$g_1=g_2=g_3=G_1\ (\text{say})\ ;\ g_4=g_5=g_6=G_2(\text{Say});\ g_7=G_3(\text{say}) \qquad \ldots\ldots\ (3.2.10)$$

$$f_4(\alpha)=f_5(\alpha)=f_6(\alpha)=e^{-\mu}\left[\alpha^{\beta}+(1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} \qquad (3.2.11)$$

$$f_7(\alpha,\gamma)=e^{-\nu}\left[\alpha^{\beta}+\gamma^{\beta}+(1-\alpha-\gamma)^{\beta}\right]^{\frac{1}{1-\beta}} \qquad (3.2.12)$$

The constants μ and ν can be found as functions of β by using the normalising conditions (3.2.2) and (3.2.3) which also give

$$3G_1+3G_2+G_3=1 \qquad (3.2.13)$$

Again, using (3.2.5) - (3.2.12),

$$\frac{1}{1-\beta}\left[\frac{\beta G_1^{\beta-1}}{3G_1^{\beta}+3G_2^{\beta}+G_3^{\beta}}-1\right]=\zeta-1 \qquad (3.2.14)$$

$$\frac{1}{1-\beta}\left[\frac{\beta G_2^{\beta-1}}{3G_1^{\beta}+3G_2^{\beta}+G_3^{\beta}}-1\right]=\zeta-1-\mu(\beta) \qquad (3.2.15)$$

$$\frac{1}{1-\beta}\left[\frac{\beta G_3^{\beta-1}}{3G_1^{\beta}+3G_2^{\beta}+G_3^{\beta}}-1\right]=\zeta-1-\nu(\beta) \qquad (3.2.16)$$

which gives

$$\frac{\beta}{1-\beta}\ \frac{G_1^{\beta-1}-G_2^{\beta-1}}{3G_1^{\beta}+3G_2^{\beta}+G_3^{\beta}}=\mu(\beta) \qquad (3.2.17)$$

$$\frac{\beta}{1-\beta}\ \frac{G_1^{\beta-1}-G_3^{\beta-1}}{3G_1^{\beta}+3G_2^{\beta}+G_3^{\beta}}=\nu(\beta) \qquad (3.2.18)$$

From (3.2.13), (3,2.17) and (3.2.18), G_1, G_2, G_3, can be obtained.

Thus finally all seven g's and all four f's are known.

From the symmetry of the distributions (3.2.11) and (3.2.12) it follows that

$$\int_0^1 \alpha f_i(\alpha)\,d\alpha = 1/2 \qquad i=4,5,6 \qquad (3.2.19)$$

and

$$\int_0^1 d\gamma \int_0^{1-\gamma} \alpha f_7(\alpha,\gamma)\,d\alpha = 1/3 \qquad (3.2.20)$$

The market shares are then given by

$$m_1 = g_1 + g_4\int_0^1 \alpha f_4(\alpha)d\alpha + g_5\int_0^1 \alpha f_5(\alpha)d\alpha + g_7\int_0^1 d\gamma \int_0^{1-\gamma} \alpha f_7(\alpha,\gamma)d\alpha$$

$$= g_1 + 1/2\,(g_4 + g_5) + 1/3\, g_7 \qquad (3.2.21)$$

Similarly

$$m_2 = g_2 + 1/2\,(g_4 + g_6) + 1/3\, g_7 \qquad (3.2.22)$$

and

$$m_3 = g_3 + 1/2\,(g_5 + g_6) + 1/3\, g_7 \qquad (3.2.23)$$

From (3.2.10), (3.2.21), (3.2.22), (3.2.23)

$$m_1 = m_2 = m_3 = \frac{1}{3} \qquad (3.2.24)$$

so that, if the entropy is maximized subject to only the normalising constraints, then the three market shares should be equal.

n-brand case. The above discussion can be easily generalised to show that, if there are n brands and market shares are not prescribed so that the entropy is maximized only under the normalising constranints, then the market shares must all be equal. Further, the probability that a customer purchases brand i alone is independent of i and the probability that he purchases brands i,j alone is independent of which pair (i,j) has been taken and so on. Thus there are only n distinct probabilities $G_1, G_2, \ldots, G_n$ where G_k is the probability of a customer choosing k of the brands and

$$ {}^{n}C_1 G_1 + {}^{n}C_2 G_2 + {}^{n}C_3 G_3 + \ldots\ldots\ldots + {}^{n}C_n G_n = 1 \qquad (3.2.25) $$

So that

$$ G_1 < \frac{1}{n}, \quad G_2 < \frac{2}{n(n-1)}, \quad \ldots\ldots, \quad G_n < 1 \qquad (3.2.26) $$

Equation (3.2.25) can also be written as

$$ G_1 + \frac{n-1}{2} G_2 + \frac{(n-1)(n-2)}{6} G_3 + \ldots\ldots\ldots + \frac{1}{n} G_n = \frac{1}{n} \quad (3.2.27) $$

which can be interpreted as giving the market share of each brand

Also the (n-1) density functions are obtained

$$ F_2(\alpha) = \exp\left[\frac{\beta}{1-\beta} \frac{G_2^{\beta-1} - G_1^{\beta-1}}{\sum_{i=1}^{n} {}^{n}C_i G_i^{\beta}} \right] \left[\alpha^{\beta} + (1-\alpha)^{\beta} \right]^{\frac{1}{1-\beta}} \qquad (3.2.28) $$

$$ F_3(\alpha,\gamma) = \exp\left[\frac{\beta}{1-\beta} \frac{G_3^{\beta-1} - G_1^{\beta-1}}{\sum_{i=1}^{n} {}^{n}C_i G_i^{\beta}} \right] \left[\alpha^{\beta} + \gamma^{\beta} + (1-\alpha-\gamma) \right]^{\frac{1}{1-\beta}} \qquad (3.2.29) $$

. . . .

By using the (n-1) normalising conditions

$$\int_0^1 F_2(\alpha)\,d\alpha = 1\,, \quad \int_0^1 d\gamma \int_0^{1-\gamma} F_3(\alpha,\gamma)\,d\alpha = 1\,, \ldots\ldots \tag{3.2.30}$$

(n-1) equations of the form

$$\frac{\beta}{1-\beta}\,\frac{G_2^{\beta-1} - G_1^{\beta-1}}{\sum_{i=1}^{n} n_{c_i} G_i^{\beta}} = \nu_1\,, \quad \frac{\beta}{1-\beta}\,\frac{G_3^{\beta-1} - G_1^{\beta-1}}{\sum_{i=1}^{n} n_{c_i} G_i^{\beta}} = \nu_2 \ldots\ldots \tag{3.2.31}$$

where $\nu_1, \nu_2,$ .....are known. The (n-1) equations (3.2.31) are obtained, together with equation (3.2.25) give us sufficient equations to determine G_1, G_2,...G_n. Thus all G's and all F's are completely determined.

The numerical work consists of evaluating (n-1) multiple integrals and solving n simultaneous algebraic equations.

If $\beta \rightarrow 1$, (n-1) integrals to be evaluated are:

$$\int_0^1 \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\,d\alpha\,, \quad \int_0^1 d\gamma \int_0^{1-\gamma} \alpha^{-\alpha}\gamma^{-\gamma}(1-\alpha-\gamma)^{-(1-\alpha-\gamma)}\,d\alpha\,, \ldots\ldots \tag{3.2.32}$$

and the n equations to be solved are:

$$\ln\frac{G_1}{G_2} = \nu_1\,, \quad \ln\frac{G_1}{G_3} = \nu_2\,, \ldots\ldots\ln\frac{G_1}{G_n} = \nu_{n-1}$$

$$\sum_{i=1}^{n} n_{c_i} G_i = 1 \tag{3.2.33}$$

3.3 THE CASE WHEN SOME OF THE MARKET SHARES ARE PRESCRIBED:

In the three-brand case of section 3.2, let the market share of the first brand be given so that,

$$m_1 = g_1 + g_4\int_0^1 \alpha f_4(\alpha)\,d\alpha + g_5\int_0^1 \alpha f_5(\alpha)\,d\alpha + g_7\int_0^1 d\gamma \int_0^{1-\gamma} \alpha f_7(\alpha,\gamma)\,d\alpha \quad \ldots\ldots \quad (3.3.1)$$

The new Lagrangian is

$$L' = L - \zeta\Big[g_1 + g_4\int_0^1 \alpha f_4(\alpha)\,d\alpha + g_5\int_0^1 \alpha f_5(\alpha)\,d\alpha + g_7\int_0^1 d\gamma \int_0^{1-\gamma} \alpha f_7(\alpha,\gamma)\,d\alpha - m_1\Big] \quad (3.3.2)$$

Some of the equations (3.2.5) - (3.2.9) will now contain additional terms with coefficient ζ. Careful examination of the modified system of equations easily gives

$$g_2 = g_3\ , \quad g_4 = g_5\ , \quad f_4(\alpha) = f_5(\alpha), \quad (3.3.3)$$

leading to $m_2 = m_3$, (3.3.4)

so that, when entropy is maximized, the market shares of the other two brands are equal.

Similarly, if k of the market shares are given as m_1, m_2, ..., m_k, then, in equilibrium, each of the others must be

$$\frac{1 - m_1 - m_2 - \cdots\cdots - m_k}{n - k} \quad (3.3.5)$$

Thus the case that only some market shares are prescribed can be treated as a special case of the general model when all market shares are prescribed and the calculations are considerably simplified since the number of distinct g's and f's is considerably reduced

3.4 THE CASE WHEN SOME OF THE g'S ARE PRESCRIBED

Suppose that in the three brand case, g_1, g_2, g_3 are prescribed; then equation (3.2.5) is no longer available. The equation $g_4 = g_5 = g_6 = G_2$ is valid and so are equations (3.2.11) and (3.2.12). Equation (3.2.13) becomes

$$g_1 + g_2 + g_3 + 3G_2 + G_3 = 1 \qquad (3.4.1)$$

From (3.2.18) and (3.4.1), G_2 and G_3 can be obtained. Thus all g's and all f's are known. From equations (3.2.21) - (3.2.23), we find

$$m_1 - m_2 = g_1 - g_2, \quad m_2 - m_3 = g_2 - g_3, \quad m_3 - m_1 = g_3 - g_1 \quad \cdots (3.4.2)$$

so that the market shares are not necessary equal. In fact

$$g_1 > g_2 > g_3 \implies m_1 > m_2 > m_3 \qquad (3.4.3)$$

and if the market shares are equal , then g_1, g_2 and g_3 would have to be equal.

For the n-brand model, if g_1, g_2, g_3 ....,g_n are all prescribed, then (3.4.1) becomes;

$$g_1 + g_2 + \cdots + g_n + {}^nC_2 G_2 + {}^nC_3 G_3 + \cdots + {}^nC_n G_n = 1 \qquad (3.4.4)$$

and (3.4.2) is generalised to

$$m_1 - g_1 = m_2 - g_2 = \cdots\cdots = m_n - g_n \qquad (3.4.5)$$

In the three brand case, if all the g's are prescribed, the three equations (3.2.5) - (3.2.7) are not available, but the density functions are still given by (3.2.11) and (3.2.12). This is true for the n-brand case also. If all the g's are prescribed, the density functions are still given by

$$F_2(\alpha) = e^{-\mu_1}\left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} \quad ; \quad F_3(\alpha,\gamma) = e^{-\mu_2}\left[\alpha^{\beta} + \gamma^{\beta} + (1-\alpha-\gamma)^{\beta}\right]^{\frac{1}{1-\beta}} \quad \ldots (3.4.6)$$

$$F_4(\alpha,\gamma,\delta) = e^{-\mu_3}\left[\alpha^{\beta} + \gamma^{\beta} + \delta^{\beta} + (1-\alpha-\gamma-\delta)^{\beta}\right]^{\frac{1}{1-\beta}}, \ldots \quad (3.4.7)$$

Chapter IV

OTHER MAXIMUM ENTROPY MODELS FOR BRAND SWITCHING

4.1 INTRODUCTION

The models in this chapter emphasize the importance of the underlying structure. This also emphasizes that even with the maximum entropy principle, there has to be a trade-off between computational complexity and the assumptions made in the structure of the underlying model.

4.2 MAXIMUM - ENTROPY SOLUTION FOR BASS'S MODEL

In the present case the structure discussed by Bass [1] is considered but in place of behavioristic assumptions MEP with Shannon's measure of entropy is used. This enables us to determine the following results:

i) the Probability g_o of a customer belonging to the SPG is 1/2,

ii) the Probability of a customer being loyal to the ith brand is proportional to the ith brand market share,

iii) the conditional probability of a customer of the SPG being loyal to the ith brand is equal to its market share.

It is also shown that the same results are obtained when in place of Shannon's measure, Renyi's measure of entropy is used.

In Bass's structure there are 2n classes; n classes of those who purchase brands 1,2,...,n exclusively and n classes of those who belong to the SPG but purchase brands 1,2,...,n. Shannon's entropy S is now given by:

$$S = -\sum_{i=1}^{n} g_i \ln g_i - \sum_{i=1}^{n} g_0\theta_i \ln g_0\theta_i$$

$$= -\sum_{i=1}^{n} g_i \ln g_i - g_0 \ln g_0 - g_0 \sum_{i=1}^{n} \theta_i \ln \theta_i \qquad (4.2.1)$$

where

$$g_0 = 1 - \sum_{i=1}^{n} g_i \quad \text{and} \quad \sum_{i=1}^{n} \theta_i = 1 \qquad (4.2.2)$$

maximizing (4.2.1) subject to

$$m_i = g_i + g_0\theta_i \qquad (4.2.3)$$

gives:

$$-1 - \ln g_i + (1 + \ln g_i) + \sum_{i=1}^{n} \theta_i \ln \theta_i + \sum_{i=1}^{n} \lambda_i \theta_i - \lambda_i = 0 \qquad (4.2.4)$$

and

$$-g_0(1 + \ln \theta_i) - \lambda_i g_0 = 0 \qquad (4.2.5)$$

and Eliminating λ_i between (4.2.4) and (4.2.5) gives:

$$\ln \frac{g_0}{g_i} + \ln \theta_i = 0 \quad \text{or} \quad \frac{g_i}{g_0} = \theta_i \qquad (4.2.6)$$

$$\frac{g_1}{\theta_1} = \frac{g_2}{\theta_2} = \ldots = \frac{g_n}{\theta_n} = \frac{\sum_{i=1}^{n} g_i}{\sum_{i=1}^{n} \theta_i} = 1 - g_0 = g_0 \qquad (4.2.7)$$

So that

$$g_i = \frac{m_i}{2}, \quad \theta_i = m_i, \quad g_0 = \frac{1}{2} \qquad (4.2.8)$$

Results are now derived by using Renyi's entropy as follows:

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{n} g_i^{\beta} + \sum_{i=1}^{n} g_0^{\beta} \theta_i^{\beta}}{\sum_{i=1}^{n} g_i + \sum_{i=1}^{n} g_0 \theta_i} \qquad (4.2.9)$$

maximizing (4.2.9) subject to (4.2.3):

$$\frac{1}{1-\beta}\left[\frac{\beta g_0^{\beta} \theta_i^{\beta-1}}{\sum g_i^{\beta} + \sum g_0^{\beta} \theta_i^{\beta}} - \frac{g_0}{\sum g_i + \sum g_0 \theta_i}\right] - \lambda_i g_0 = 0 \qquad (4.2.10)$$

or

$$\lambda_i = \frac{\beta}{1-\beta} \frac{g_0^{\beta-1} \theta_i^{\beta-1}}{\sum g_i^{\beta} + \sum g_0^{\beta} \theta_i^{\beta}} - \frac{1}{(1-\beta)(\sum g_i + \sum g_0 \theta_i)} \qquad (4.2.11)$$

and

$$\frac{1}{1-\beta} \frac{[\beta g_i^{\beta-1} - \beta g_0^{\beta-1} \sum_{i=1}^{n} \theta_i^{\beta}]}{\sum g_i^{\beta} + \sum g_0^{\beta} \theta_i^{\beta}} - \frac{1}{1-\beta} \frac{1-1}{\sum g_i + \sum g_0 \theta_i} - \lambda_i + \sum \lambda_i \theta_i = 0 \qquad (4.2.12)$$

or

$$\frac{1}{1-\beta} \frac{[\beta g_i^{\beta-1} - \beta g_0^{\beta-1} \sum_{i=1}^{n} \theta_i^{\beta}]}{\sum g_i^{\beta} + \sum g_0^{\beta} \theta_i^{\beta}} - \lambda_i + \sum \lambda_i \theta_i = 0 \qquad (4.2.13)$$

Eliminating λ_i between (4.2.11) and (4.2.13):

$$g_i = g_0 \theta_i \qquad (4.2.14)$$

From (4.2.14) and (4.2.3):

$$g_i = \frac{m_i}{2} \quad , \quad \theta_i = m_i \quad , \quad g_0 = \frac{1}{2} \qquad (4.2.15)$$

which is the same as (4.2.8), obtained earlier by using Shannon's entropy.

This solution is much simpler than that of Bass [1] and, using (4.2.15), the following conclusions can be drawn:

i) The probability g_0 of a customer belonging to SPG is 1/2.

ii) The probability of a customer being loyal to ith brand is proportional to the ith brand market share.

iii) The conditional probability of a customer of the SPG being loyal to the ith brand is equal to its market share.

These results are not highly plausible because they show that independently of the number of brands, fifty percent of the customers are waverers. In practice, the percentage of waverers is expected to increase with the number of brands. However this is a joint consequence of Bass's structure and the MEP. Since the MEP is a principle of logical reasoning and gives valid results if the constraints and structures are validly chosen, this only shows that the basic structure of Bass is not sufficient to give valid results. In fact

Bass himself supplemented this basic structure by at least half a dozen ad-hoc, empirically justifiable and behavioristic assumptions to get results which are close enough to Herniter's results.

Herniter's structure is sufficiently detailed and it gives, with the MEP, conclusions consistent with commonsense expectations. Bass's structure is not detailed enough to give, combined with the MEP, conclusions consistent with commonsense expectations and therefore some additional constraints are necessary. These are provided by assumptions made by Bass.

4.3 BASS'S MODIFIED STRUCTURE WHEN MARKET SHARES ARE PRESCRIBED

Since Bass's structure is not sufficiently detailed it is reconsidered by introducing a few variations and additions to it. For three brands, the seven categories of Herniter are kept and, in each category of waverers, the conditional probability of choosing a brand is introduced . Thus, the probabilities are

$$f_1 = g_1 \;,\quad f_2 = g_2 \;,\quad f_3 = g_3 \;;$$

$$f_4 = g_4\theta_1 \;,\quad f_5 = g_4(1-\theta_1) \;;$$

$$f_6 = g_5\theta_2 \;,\quad f_7 = g_5(1-\theta_2) \;;$$

$$f_7 = g_6\theta_3 \;,\quad f_9 = g_6(1-\theta_3) \;;$$

$$f_{10} = g_7\theta_4 \;, \; f_{11} = g_7\theta_5 \;, \; f_{12} = g_7(1-\theta_4-\theta_5) \qquad (4.3.1)$$

Shannon's entropy, S is now given by

$$S = -\sum_{i=1}^{12} f_i \ln f_i \qquad (4.3.2)$$

Maximizing S subject to:

$$f_1 + f_4 + f_6 + f_{10} = m_1 \qquad (4.3.3)$$

$$f_2 + f_5 + f_8 + f_{11} = m_2 \qquad (4.3.4)$$

$$f_3 + f_7 + f_9 + f_{12} = m_3 \qquad (4.3.5)$$

gives:

$$f_1 = f_4 = f_6 = f_{10} = \frac{m_1}{4} \qquad (4.3.6)$$

$$f_2 = f_5 = f_8 = f_{11} = \frac{m_2}{4} \qquad (4.3.7)$$

$$f_3 = f_7 = f_9 = f_{12} = \frac{m_3}{4} \qquad (4.3.8)$$

using (4.3.1):

$$g_1 = \frac{m_1}{4} \;, \quad g_2 = \frac{m_2}{4} \;, \; g_3 = \frac{m_3}{4} \;, g_4 = \frac{m_1+m_2}{4} \ldots (4.3.9)$$

$$g_5 = \frac{m_1+m_3}{4} \;, \; g_6 = \frac{1}{4}(m_2+m_3) \;, \quad g_7 = \frac{1}{4}(m_1+m_2+m_3)$$

$$\cdots (4.3.10)$$

$$\theta_1 = \frac{m_1}{m_1+m_2}, \quad \theta_2 = \frac{m_1}{m_1+m_3}, \quad \theta_3 = \frac{m_2}{m_2+m_3} \tag{4.3.11}$$

$$\theta_4 = m_1/m_1+m_2+m_3, \theta_5 = m_2/m_1+m_2+m_3 \tag{4.3.12}$$

It can be easily shown that by maximizing Renyi's entropy subject to (4.3.3.) - (4.3.5), same results (4.3.9) - (4.3.12) will be obtained as obtained by using Shannon's entropy.

The results of this model show that

i) they are independent of whether Shannon's measure or Renyi's measure of entropy is assumed,

ii) one fourth of the customers are loyal to a single brand,

iii) out of the proportion m_1, purchasing the first brand 1/4 m_1 are loyal to brand 1 and 2, 1/4 m_1 are loyal to brands 1 and 3 and 1/4 m_1 are jointly loyal to brands 1,2 and 3.

Similarly, for n brands, the number of unknowns is

$$^nC_1 + 2\,^nC_2 + 3\,^nC_3 + \ldots\ldots + n.\,^nC_n = n.2^{n-1} \tag{4.3.14}$$

and one can show that

i) $m_i/2^{n-1}$ of the customers are loyal to the ith brand only.

ii) $m_i/2^{n-1}$ of the customers are loyal to the ith and one more brand, and

iii) $m_i/2^{n-1}$ of the customers are loyal to the ith and two more brands and so on.

The number of unknowns $n\ 2^{n-1}$ is larger than the number of unknowns $2^{n+1}-n-2$ in Herniter's model, but all these are constants, while in Herniter's model, 2^n-n-1 are functions and as seen above, the calculations are much simpler and consistent with commonsense expectations.

4.4 BASS'S MODIFIED STRUCTURE WHEN g'S ARE PRESCRIBED

The structure is same as in Model 2, but instead of market shares being prescribed, the g's are given to us. These for example can be determined by Gallup polls of customers. These polls can determine what proportion of customers is loyal to one brand only; what proportion is loyal to two specified brands and so on.

θ's and m's are determined by maximizing (4.3.2). This gives

$$\theta_1 = \theta_2 = \theta_3 = 1/2, \quad \theta_4 = \theta_5 = 1-\theta_4-\theta_5 = 1/3 \tag{4.4.1}$$

$$m_1 = g_1 + \frac{1}{2}(g_4 + g_5) + \frac{1}{3} g_7 \tag{4.4.2}$$

$$m_2 = g_2 + \frac{1}{2}(g_4 + g_6) + \frac{1}{3} g_7 \tag{4.4.3}$$

$$m_3 = g_3 + \frac{1}{2}(g_5 + g_6) + \frac{1}{3} g_7 \tag{4.4.4}$$

The maximum entropy is given by

$$S_{max.} = -\sum_{i=1}^{7} g_i \ln g_i + (g_4 + g_5 + g_6)\ln 2 + g_7 \ln 3 \qquad (4.4.5)$$

This is a concave function of g 's and is maximum when

$$g_1 = g_2 = g_3 = \frac{1}{12}\ ;\ g_4 = g_5 = g_6 = \frac{1}{6}\ ,\ g_7 = \frac{1}{4} \qquad (4.4.6)$$

According to this model, when the market shares are equal, about 8% customers are loyal to each brand, while in the case of Herniter's model, this proportion is about 11%.

This concludes that although MEP is an excellent principle of reasoning and it gives the most unbiased, most objective, most uncommitted prediction subject to the given constraints the deductions will be only as good as the constraints imposed. If the information given by the constraints is insufficient or irrelevant or inconsistent one cannot expect the predictions to be reasonable. In fact when the predictions are inconsistent or against experience or observation, one should examine the constraints and the underlying structure of the model carefully.

Chapter V

MAXIMUM ENTROPY MODELS FOR PARTY SWITCHING

5.1 INTRODUCTION

A number of models for party switching in an election are discussed. All these models use the Principle of Maximum Entropy for estimating the probability distribution. The problem of party switching in elections is similar to that of Brand Switching

Voters can be identified with buyers, parties with brands and party share with market share and so on. After an election, the proportion of votes received by each political party is known. As such the results of Herniter's [8] model and its generalisations can be used to answer the following estimates:

- i) the proportion of voters loyal to each party,
- ii) the proportion of voters who waver among a specified number of parties,
- iii) when a voter wavers among a specified number of parties, the most likely distribution of the preference random variables,
- iv) if the party shares for all parties are specified, the probability of a voter voting for the same party or switching to another party.

We also consider the following additional problems.

i) Suppose a Gallup poll giving the proportion of wavering and loyal voters is conducted before an election. From these proportions estimate the most likely share of each party after the election.

ii) Using the information both before and after the election, estimate the switching behavior of wavering voters.

iii) Also, what results can be derived about party statistics if,

a) some of the party shares are known; or

b) information about the proportion of loyal voters only is available; or

c) there is no information available about the voting shares.

In the present work, Herniter's model described below is modified to answer the above problems.

5.2 HERNITER'S MODEL FOR THREE POLITICAL PARTIES

Let

i) g_1, g_2, g_3 be the probabilities of a voter being loyal to parties 1, 2, 3 respectively;

ii) g_4, g_5, g_6 be the probabilities of a voter wavering between parties 1,2; 1,3; 2,3 respectively;

iii) g_7 be the probability of a voter wavering between 1, 2 and 3;

iv) $f_4(\alpha)$, $f_5(\alpha)$, $f_6(\alpha)$ be the density functions of a random variable α $(0 \leq \alpha \leq 1)$ which gives the preference for 1 over 2 or of 1 over 3 or of 2 over 3,

v) $f_7(\alpha, \beta)$ be the joint density function of two random variables α, β .

Then the entropy S is given by [8]:

$$S = -\sum_{i=1}^{7} g_i \ln g_i - \sum_{i=4}^{6} g_i \int_0^1 f_i(\alpha)\left[\ln f_i(\alpha) + \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\right] d\alpha$$

$$- g_7 \int_0^1 \int_0^{1-\alpha} f_7(\alpha,\beta)\left[\ln f_7(\alpha,\beta) + \alpha \ln \alpha + \beta \ln \beta\right.$$

$$\left. + (1-\alpha-\beta)\ln(1-\alpha-\beta)\right] d\alpha\, d\beta \qquad (5.2.1)$$

The constraints are

$$\sum_{i=1}^{7} g_i = 1 \qquad (5.2.2)$$

$$\int_0^1 f_i(\alpha)\, d\alpha = 1 \quad ; \quad i = 4, 5, 6 \qquad (5.2.3)$$

$$\int_0^1 \int_0^{1-\alpha} f_7(\alpha,\beta)\, d\alpha\, d\beta = 1 \qquad (5.2.4)$$

$$g_1 + g_4\int_0^1 \alpha f_4(\alpha)d\alpha + g_5\int_0^1 \alpha f_5(\alpha)d\alpha + g_7\int_0^1\int_0^{1-\alpha} \alpha f_7(\alpha,\beta)d\alpha d\beta = m_1 \quad \ldots(5.2.5)$$

$$g_2 + g_4\int_0^1 (1-\alpha)f_4(\alpha)d\alpha + g_6\int_0^1 \alpha f_6(\alpha)d\alpha + g_7\int_0^1\int_0^{1-\alpha} \beta f_7(\alpha,\beta)d\alpha d\beta = m_2 \quad (5.2.6)$$

$$g_3 + g_5\int_0^1 (1-\alpha)f_5(\alpha)d\alpha + g_6\int_0^1 (1-\alpha)f_6(\alpha)d\alpha + g_7\int_0^1\int_0^{1-\alpha} (1-\alpha-\beta)f_7(\alpha,\beta)d\alpha d\beta = m_3 \quad \ldots(5.2.7)$$

Herniter assumed that m_1, m_2, m_3 are fixed and he maximized S subject to the eight constraints (5.2.2) - (5.2.7) to determine the seven g's and four density functions f's. He also calculated the nine probabilities p_{ij} of transition from ith party to jth party:

$$P_{11} = g_1 + g_4\int_0^1 \alpha^2 f_4(\alpha)\,d\alpha + g_5\int_0^1 \alpha^2 f_5(\alpha)\,d\alpha$$
$$+ g_7\int_0^1\int_0^{1-\alpha} \alpha^2 f_7(\alpha,\beta)\,d\alpha\,d\beta\,, \qquad (5.2.8)$$

$$P_{22} = g_2 + g_4\int_0^1 (1-\alpha)^2 f_4(\alpha)\,d\alpha + g_6\int_0^1 \alpha^2 f_6(\alpha)\,d\alpha$$
$$+ g_7\int_0^1\int_0^{1-\alpha} \beta^2 f_7(\alpha,\beta)\,d\alpha\,d\beta\,, \qquad (5.2.9)$$

$$P_{33} = g_3 + g_5\int_0^1 (1-\alpha)^2 f_5(\alpha)\,d\alpha + g_6\int_0^1 (1-\alpha)^2 f_6(\alpha)\,d\alpha$$
$$+ g_7\int_0^1\int_0^{1-\alpha} (1-\alpha-\beta)^2 f_7(\alpha,\beta)\,d\alpha\,d\beta, \qquad (5.2.10)$$

$$P_{21} = P_{12} = g_4\int_0^1 \alpha(1-\alpha) f_4(\alpha)\,d\alpha$$
$$+ g_7\int_0^1\int_0^{1-\alpha} \alpha\beta f_7(\alpha,\beta)\,d\alpha\,d\beta\,, \qquad (5.2.11)$$

$$P_{13} = P_{31} = g_5\int_0^1 \alpha(1-\alpha) f_5(\alpha)\,d\alpha$$
$$+ g_7\int_0^1\int_0^{1-\alpha} \alpha(1-\alpha-\beta) f_7(\alpha,\beta)\,d\alpha\,d\beta, \qquad (5.2.12)$$

$$P_{32} = P_{23} = g_6\int_0^1 \alpha(1-\alpha) f_6(\alpha)\,d\alpha$$
$$+ g_7\int_0^1\int_0^{1-\alpha} \beta(1-\alpha-\beta) f_7(\alpha,\beta)\,d\alpha\,d\beta. \qquad (5.2.13)$$

The following conclusions can be drawn from the tables he prepared:

i) Maximum switching occurs when the three parties have equal shares of votes.

ii) Switching is least when one party is dominant.

iii) Total switching also appears to be directly related to the entropy of the system. When the entropy increases, switching increases and vice versa, however, the change in entropy is greater than the change in switching.

iv) There is equilibrium in switching between pairs of parties i.e. the fraction of votes which switches from party i to party j is equal to the fraction which switches from party j to party i.

5.3 THE CASE WHEN g'S ARE PRESCRIBED BUT m'S ARE NOT

Below the most likely voting share of each party is estimated, when through some gallup polls or otherwise g's are known but m's are not. S is maximized subject to constraints (5.2.3) and (5.2.4). This gives:

$$g_i\left[\ln f_i(\alpha) + \alpha\ln\alpha + (1-\alpha)\ln(1-\alpha) + 1\right] - \lambda_i = 0 \qquad (5.3.1)$$

$$i = 4, 5, 6$$

$$g_7\left[\ln f_7(\alpha,\beta) + \alpha\ln\alpha + \beta\ln\beta + (1-\alpha-\beta)\ln(1-\alpha-\beta) + 1\right] + \mu = 0$$

$$\ldots\ldots \quad (5.3.2)$$

Again, making use of (5.2.3) and (5.2.4):

$$f_i(\alpha) = A_i \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \qquad i = 4, 5, 6 \qquad (5.3.3)$$

where

$$A_i \int_0^1 \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\, d\alpha = 1 \qquad (5.3.4)$$

so that $A_i = 0.5981$, $\quad i = 4, 5, 6 \qquad (5.3.5)$

and

$$f_7(\alpha, \beta) = B\alpha^{-\alpha}\beta^{-\beta}(1-\alpha-\beta)^{-(1-\alpha-\beta)}, \qquad (5.3.6)$$

where

$$B\int_0^1\int_0^{1-\alpha} \alpha^{-\alpha}\beta^{-\beta}(1-\alpha-\beta)^{-(1-\alpha-\beta)}\, d\alpha\, d\beta = 1, \qquad (5.3.7)$$

so that $B=0.858 \qquad (5.3.8)$

Now, making use of the symmetry in the density function;

$$\int_0^1 \alpha f_i(\alpha)\, d\alpha = \int_0^1 (1-\alpha) f_i(\alpha)\, d\alpha = \frac{1}{2} \qquad (5.3.9)$$

$$\int_0^1\int_0^{1-\alpha} \alpha f_7(\alpha, \beta)\, d\alpha\, d\beta = \int_0^1\int_0^{1-\alpha} \beta f_7(\alpha, \beta)\, d\alpha\, d\beta$$

$$= \int_0^1\int_0^{1-\alpha} (1-\alpha-\beta) f_7(\alpha, \beta)\, d\alpha\, d\beta$$

$$= \frac{1}{3} \qquad (5.3.10)$$

so that the market shares of the three parties are:

$$m_1 = g_1 + \frac{1}{2}(g_4 + g_5) + \frac{1}{3}g_7 \qquad (5.3.11)$$

$$m_2 = g_2 + \frac{1}{2}(g_4 + g_6) + \frac{1}{3}g_7 \qquad (5.3.12)$$

$$m_3 = g_3 + \frac{1}{2}(g_5 + g_6) + \frac{1}{3}g_7 \qquad (5.3.13)$$

This shows that the entropy is maximum when the wavering votes are allocated equally among the parties concerned. In fact in the absence of any other information this is the most objective allocation that can be made.

Substituting (5.3.3) and (5.3.6) in (5.2.1), the maximum entropy is obtained as

$$S_{max.} = -\sum_{i=1}^{7} g_i \ln g_i - \ln A \sum_{i=4}^{6} g_i - g_7 \ln B \qquad (5.3.14)$$

This is a concave function of g's and is maximum subject to

$$\sum_{i=1}^{7} g_i = 1 \quad , \quad \text{when}$$

$$g_1 = g_2 = g_3 = a \qquad (5.3.15)$$

$$g_4 = g_5 = g_6 = \frac{a}{A} \qquad (5.3.16)$$

$$g_7 = \frac{a}{B} \qquad (5.3.17)$$

so that

$$a\left(3+\frac{3}{A}+\frac{1}{B}\right) = 1 \qquad (5.3.18)$$

Using (5.3.5) and (5.3.8), we get

$$a = .109 \qquad (5.3.19)$$

This gives

$$g_1 = g_2 = g_3 = .109 \qquad (5.3.20)$$

$$g_4 = g_5 = g_6 = .182 \qquad (5.3.21)$$

and

$$g_7 = .127 \qquad (5.3.22)$$

Thus, if both g's and f's are allowed to vary, the maximum entropy occurs when the party shares of the three parties are equal. In this case about 11% are loyal to each party, about 18% waver between pairs of parties and the remaining 13% are undecided among the three parties. Finally each party gets about 11% of the loyalish votes, about 18% of the votes of those who were undecided between this party and some other party and about 4% from those who were undecided among all the three parties, making a final total of 33 1/3% for that party.

5.4 THE CASE WHEN g'S AND m'S BOTH ARE PRESCRIBED

Now both g's and m's are given and f's are to be estimated i.e., the estimation of, how wavering voters have switched on the basis of both pre-election polls and election results, is to be done. Maximization of S subject to constraints (5.2.3) - (5.2.7) gives:

$$f_4(\alpha) = a_4 \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} e^{-(\mu_4-\mu_5)\alpha} \quad (5.4.1)$$

$$f_5(\alpha) = a_5 \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} e^{-(\mu_4-\mu_6)\alpha} \quad (5.4.2)$$

$$f_6(\alpha) = a_6 \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} e^{-(\mu_5-\mu_6)\alpha} \quad (5.4.3)$$

$$f_7(\alpha,\beta) = a_7 \alpha^{-\alpha} \beta^{-\beta} (1-\alpha-\beta)^{-(1-\alpha-\beta)} e^{-\mu_4\alpha-\mu_5\beta-\mu_6(1-\alpha-\beta)} \quad \ldots.(5.4.4)$$

where the seven constants a_4, a_5, a_6, a_7, μ_4, μ_5, μ_6 have to be determined by using the seven constraints (5.2.2) - (5.2.6).

when g's are given by (5.3.20) - (5.3.22) and $m_1 = m_2 = m_3$, one obtains $\mu_4 = \mu_5 = \mu_6$. In general:

$$\int_0^1 \alpha f_4(\alpha)\, d\alpha = a_4 \int_0^1 \alpha^{1-\alpha} (1-\alpha)^{-(1-\alpha)} e^{-(\mu_4-\mu_5)\alpha}\, d\alpha \quad \ldots.(5.4.5)$$

and

$$\int_0^1 (1-\alpha) f_4(\alpha) d\alpha = a_4 \int_0^1 \alpha^{-\alpha} (1-\alpha)^{\alpha} e^{-(\mu_4-\mu_5)\alpha} d\alpha$$

$$= a_4 \int_0^1 (1-\alpha)^{-(1-\alpha)} \alpha^{1-\alpha} e^{-(\mu_4-\mu_5)(1-\alpha)} d\alpha \qquad (5.4.6)$$

Which shows that, in the general case, wavering votes are not shared equally among the parties concerned.

If the votes in the categories 4, 5, 6 and 7 are shared in the ratio a: 1-a, b:1-b, c:1-c, d:e:1-d-e then

$$g_1 + g_4 a + g_5 b + g_7 d = m_1 \qquad (5.4.7)$$

$$g_2 + g_4(1-a) + g_6 c + g_7 e = m_2 \qquad (5.4.8)$$

$$g_3 + g_5(1-b) + g_6(1-c) + g_7(1-d-e) = m_3 \qquad (5.4.9)$$

since

$$\sum_{i=1}^{3} m_i = 1$$

Only two of the equations (5.4.7) - (5.4.9) are independent. Given $g_1, g_2, \ldots, g_7$ and m_1, m_2, m_3, these equations give, in general, an infinite number of solutions for the five constants a, b, c, d, e and the maximum entropy principle enables us to decide which of this infinite number of solutions is to be used.

5.5 THE CASE WHEN SOME OF THE PARTY SHARES ARE PRESCRIBED

Below it will be proved that, if only some of the party shares are known, then the entropy will be maximum when the remaining party shares are equal.

In the three party case, assume that the party share of the first party is given so that:

$$m_1 = g_1 + g_4 \int_0^1 \alpha_4 f_4(\alpha) d\alpha + g_5 \int_0^1 \alpha f_5(\alpha) d\alpha + g_7 \int_0^1 d\gamma \int_0^{1-\gamma} \alpha f_7(\alpha, \gamma) d\alpha \qquad (5.5.1)$$

Now the entropy (5.2.1) will be maximized subject to (5.2.2) - (5.2.4) and (5.5.1).

An examination of the modified system of equations gives

$$g_2 = g_3 , \qquad g_4 = g_5 \qquad f_4(\alpha) = f_5(\alpha)$$

leading to $m_2 = m_3$

so that when entropy is maximized, the party shares of the other <u>two parties</u> are equal.

5.6 THE CASE WHEN SOME OF THE g'S ARE PRESCRIBED

In this case g_1, g_2, g_3 are given and other g's and f's can be varied to maximize S. This gives

$$g_4 = g_5 = g_6 = \frac{a}{A} , \qquad g_7 = \frac{a}{B} \qquad (5.6.1)$$

and

$$g_1 + g_2 + g_3 + \frac{3a}{A} + \frac{a}{B} = 1 , \qquad (5.6.2)$$

so that

$$a = (1 - g_1 - g_2 - g_3)\, AB/(3B+A)$$

$$= (1 - g_1 - g_2 - g_3)\cdot 1618 \qquad (5.6.3)$$

and then all the probabilities are modified due to the knowledge of g_1, g_2, g_3. The density function do not change and the changes in the transition probabilities can be easily found.

5.7 THE CASE WHEN g'S AND m'S ARE NOT PRESCRIBED

If no information is available about the system S is maximized subject to constraints (5.2.2) - (5.2.4) to get equations (5.3.3) - (5.3.8).

Also (5.3.20) - (5.3.22) gives:

$$g_1 = g_2 = g_3 = a = .109 \qquad (5.7.1)$$

$$g_4 = g_5 = g_6 = \frac{a}{A} = .182 \qquad (5.7.2)$$

$$g_7 = \frac{a}{B} = .127 \qquad (5.7.3)$$

An extension of the above results to the case of four parties gives:

$$g_1 = g_2 = g_3 = g_4 = .052 \tag{5.7.4}$$

$$g_5 = g_6 = g_7 = g_8 = g_9 = g_{10} = .087 \tag{5.7.5}$$

$$g_{11} = g_{12} = g_{13} = g_{14} = .061 \tag{5.7.6}$$

$$g_{15} = .026 \tag{5.7.7}$$

$$f_i(\alpha) = A\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \qquad i = 5, 6, \ldots, 10 \tag{5.7.8}$$

$$f_j(\alpha, \beta) = B\alpha^{-\alpha}\beta^{-\beta}(1-\alpha-\beta)^{-(1-\alpha-\beta)} \qquad j = 11, 12, 13, 14 \tag{5.7.9}$$

$$f_{15}(\alpha, \beta, \gamma) = C\alpha^{-\alpha}\beta^{-\beta}\gamma^{-\gamma}(1-\alpha-\beta-\gamma)^{-(1-\alpha-\beta-\gamma)} \qquad C = 2 \tag{5.7.10}$$

$$S_{max} = 2.957 \tag{5.7.11}$$

Thus in this case each party gets 5.2% votes from loyalists, 13.05% from those who want to choose between this party and one other and only .65% from those who are undecided among all the four parties. These percentages can be compared with 10.9%, 18.2% and 4.25% in the case of three parties. The maximum entropy 2.957 can also be compared with the value of maximum entropy 2.219 in the three party case. In fact the maximum entropy (uncertainty) goes on increasing as the number of parties goes on increasing.

Chapter VI

A STUDY OF EFFECT OF β ON SWITCHING PROBABILITIES

6.1 INTRODUCTION

In the present Chapter the effect of β on brand switching or party switching is studied by solving numerically some non-linear equations derived in chapter 2. Some important conclusions arrived at are demonstrated through tables and graphs.

In Chapter 2 following was obtained:

From (2.3.2)

$$\sum_{i=1}^{3} g_i = 1 \tag{6.1.1}$$

and

$$\int_0^1 f(\alpha)\, d\alpha = 1 \tag{6.1.2}$$

From (2.3.12)

$$J_1 - J_2 = \frac{\beta}{1-\beta} \, \frac{g_1^{\beta-1} - g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} \tag{6.1.3}$$

From (2.3.13)

$$\mu + \xi_2 = \frac{\beta}{1-\beta} \frac{g_2^{\beta-1} - g_3^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} \qquad (6.1.4)$$

From (2.3.14) and (2.3.15)

$$\int_0^1 e^{-(\xi_1-\xi_2)\alpha} \left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} d\alpha = e^{\mu+\xi_2} \qquad (6.1.5)$$

$$g_1 + g_3 e^{-(\mu+\xi_2)} \int_0^1 e^{-(\xi_1-\xi_2)} \alpha \left[\alpha^{\beta} + (1-\alpha)^{\beta}\right]^{\frac{1}{1-\beta}} d\alpha = m_1 \qquad (6.1.6)$$

Using the above equations and applying the Newton-Raphson iterative method, three unknowns g_1, g_2 and g_3 can be determined for a particular value of β.

Further, the entropy can be obtained by substituting the values of g_1, g_2, g_3 and corresponding β in the following equation derived in Chapter 2:

$$S = \frac{1}{1-\beta} \ln\left(g_1^{\beta} + g_2^{\beta} + g_3^{\beta}\right) + \frac{\beta}{1-\beta} g_3 - \frac{g_2^{\beta-1} - g_3^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}}$$

$$+ \frac{\beta}{1-\beta} \frac{g_1^{\beta-1} - g_2^{\beta-1}}{\sum_{i=1}^{3} g_i^{\beta}} (m_1 - g_1) \qquad (6.1.7)$$

For different values of β Tables 6-1 and 6-2 have been obtained.

Table 6-1

Entropy Versus β For Different Values of m_1

β	Entropy S for				
	$m_1=.1$	$m_1=.2$	$m_1=.3$	$m_1=.4$	$m_1=.5$
0.25	1.0351	1.2998	1.3527	1.4412	1.4733
0.50	0.8990	1.1271	1.2664	1.3460	1.3722
0.75	0.8000	1.0611	1.2153	1.2997	1.3267
1.00	0.7325	1.0105	1.1787	1.2719	1.3019
1.25	0.6912	0.9713	1.1500	1.2525	1.2860
1.50	*0.6949 (.1074)	0.9415	1.1264	1.2378	1.2751
1.75	*0.7721 (.1378)	0.9190	1.1066	1.2258	1.2668
2.00	*0.8074 (.1559)	0.9025	1.0897	1.2159	1.2605
2.25	*0.8255 (.1674)	0.8905	1.0753	1.2074	1.2554

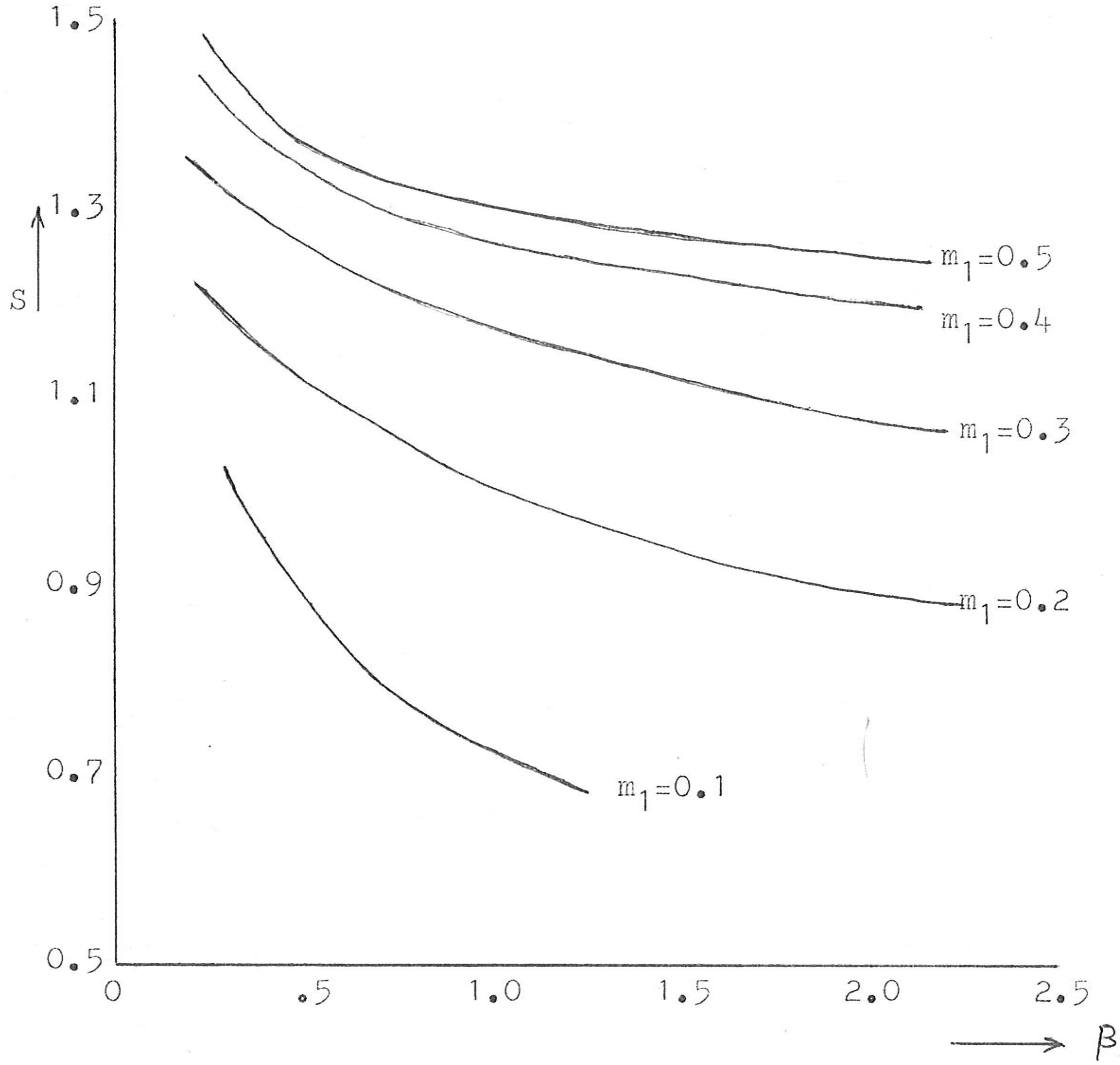
Corresponding to values of $\beta \geqslant 1.50$ and $m_1 < .1074$, Newton-Raphson Method yields negative values of probabilities, which is not realistic because probabilities are not supposed to be negative. Similar statements hold for (i) $\beta \geqslant 1.75$, $m_1 < .1378$ (ii) $\beta \geqslant 2.00$, $m_1 < .1559$ (iii) $\beta \geqslant 2.25$, $m_1 < .1674$. Therefore corresponding to the values given in β column bracked values of m_1 are choosen and corresponding entropies are calculated.

Table 6-2

g_2 Versus β For Different Values of m_1

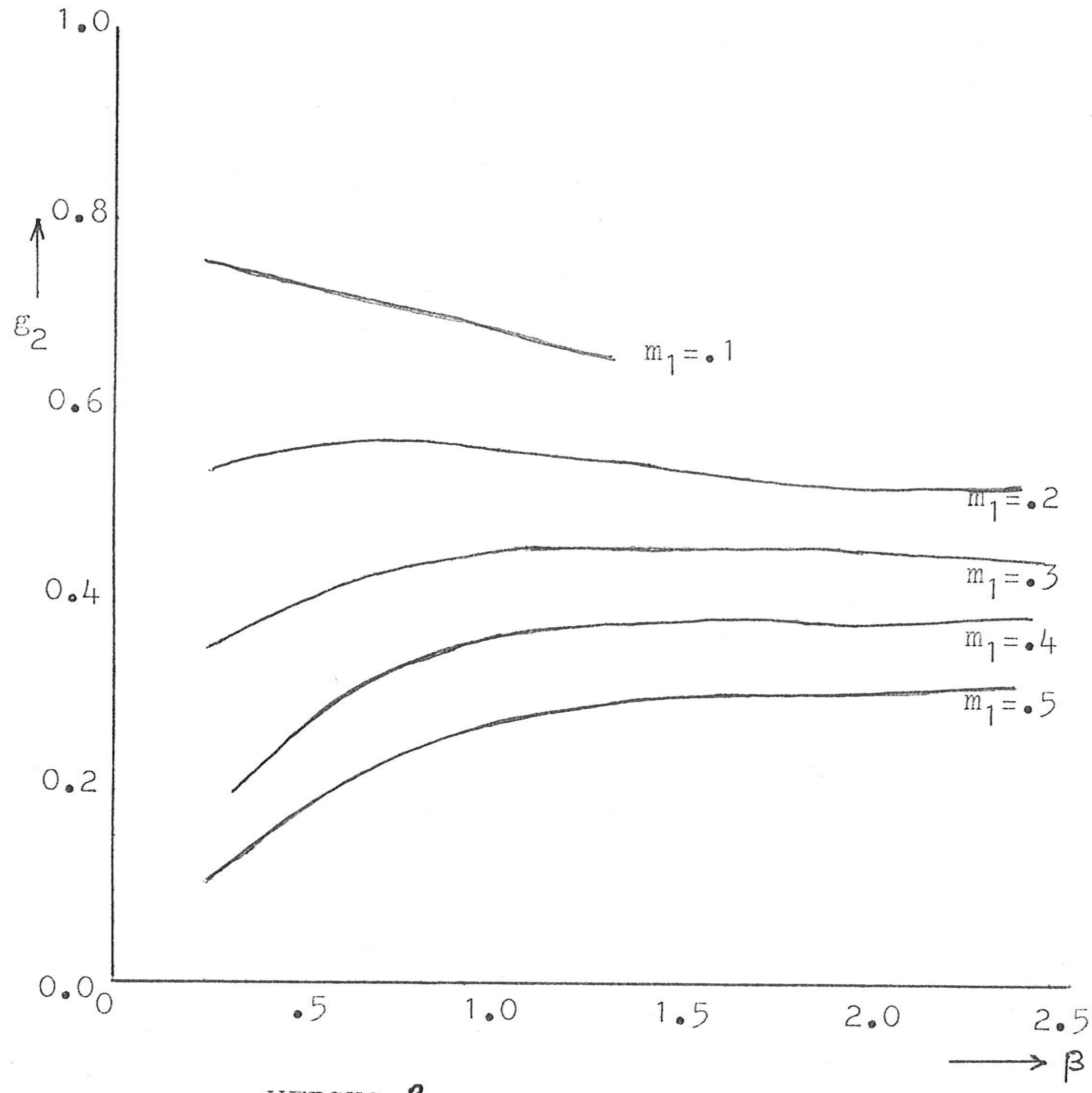
β	Probability g_2 for				
	$m_1=.1$	$m_1=.2$	$m_1=.3$	$m_1=.4$	$m_1=.5$
0.25	.7556	.5421	.3482	.1996	.1122
0.50	.7386	.5641	.4112	.2889	.1974
0.75	.7142	.5647	.4308	.3342	.2458
1.00	.6895	.5583	.4512	.3569	.2720
1.25	.6666	.5501	.4555	.3691	.2871
1.50	*.6383 (.1074)	.5421	.4566	.3760	.2968
1.75	*.5883 (.1378)	.5350	.4562	.3799	.3032
2.00	*.5625 (.1559)	.5292	.4551	.3823	.3079
2.25	*.5474 (.1674)	.5246	.4539	.3836	.3114

Corresponding to values of $\beta \geqslant 1.50$ and $m_1 < .1074$, Newton-Raphson Method yield negative values of probabilities, which is not realistic because probabilities are not supposed to be negative. Similar statements hold for (i) $\beta \geqslant 1.75$, $m_1 < .1378$ (ii) $\beta \geqslant 2.00$, $m_1 < .1559$ (iii) $\beta \geqslant 2.25$, $m_1 < .1674$. Therefore corresponding to the values given in column bracked values of m_1 are choosen and corresponding g_2's are calculated.



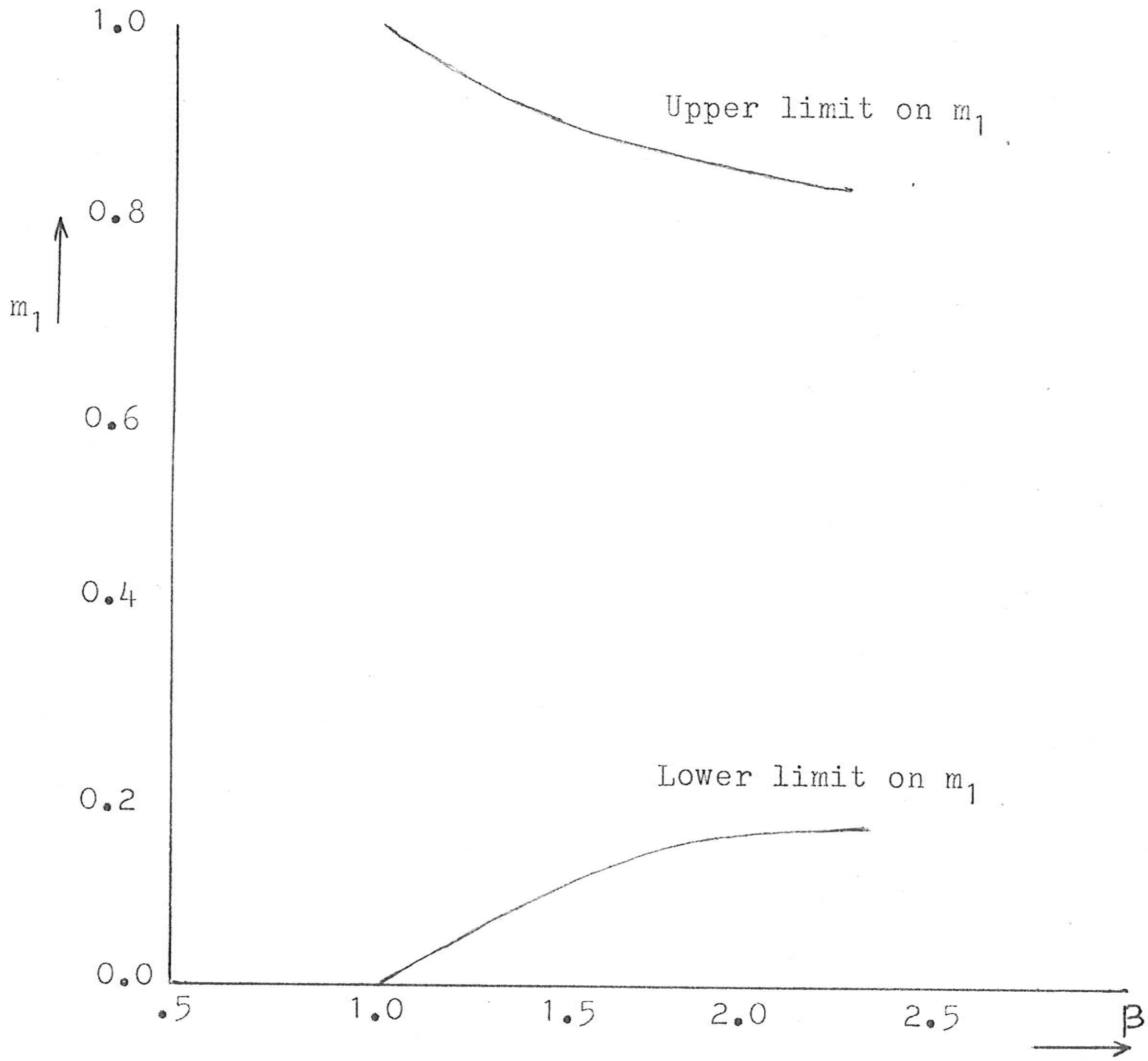
ENTROPY VERSUS β CURVES FOR DIFFERENT VALUES OF m_1

Figure 6-1



g_2 VERSUS β CURVES FOR DIFFERENT VALUES OF m_1

Figure 6-2



UPPER AND LOWER LIMITS OF VALIDITY

Figure 6-3

6.2 ENTROPY VERSUS β

The graph of S against β shows that:

i) for a given m, the entropy decreases as β increases; the rate of decrease is large in the beginning and decreases as β increases.

ii) $\frac{dS}{d\beta}$ is negative. Its magnitude decreases as m increases.

iii) For a given β, the entropy increases as m increases, though the rate of increase itself slows down.

iv) For a given entropy, β increases with m.

6.3 g_2 VERSUS β

The graph of g_2 against β shows that

i) For a given β, as m_1 increases, g_2 decreases; this is expected.

ii) For a given m_1, as β increases, g_2 first increases and then decreases.

6.4 VALIDITY OF RENYI'S ENTROPY

It is found that for $\beta > 1$, during the process of numerical computations some of the probabilities become negative; it is, therefore, not possible to work with them, as a negative number raised to some power may result in an imaginary value.

Table 6-3 gives the range of m_1 from which positive probabilities for various values of β are obtained.

Table 6-3

Value of β	Lower limit on m_1	Upper limit on m_1
1.25	.0540	.9460
1.50	.1074	.8926
1.75	.1378	.8622
2.00	.1559	.8441
2.25	.1674	.8326

The range of validity becomes narrower as β increases. Further for $\beta \leq 1$, the model is valid for m_1 between 0 and 1.

Failure of Renyi's measure of entropy for some values of $\beta > 1$ is expected, since it is a concave function of β if $\beta < 1$ but not necessarily concave if $\beta > 1$.

Thus while Renyi's measure has the advantage that there is a parameter which can be adjusted, the disadvantage is that it cannot be used for all values of m for a given β .

In fact, it shows that for a specific situation for which $\beta=1.5$, m_1 will not be outside the range of .1074 to .8926. In other words, if m_1 is found to be less than .1 or more than .9, then, for that product, β has to be chosen less than 1.5.

6.5 EMPIRICAL TESTS-U.S. PRESIDENTIAL ELECTIONS (1972 AND 1976)

In this section empirical tests of the generalised entropy model are presented. Using gallup poll results and the election results of 1972 [The Gallup Poll, Public Opinion] ,a value of the parameter β is obtained.By using this value of β and the election results of 1976, the percentage of loyal and wavering voters between two parties are obtained.

While considering the results of gallup polls the following assumptions have been made.

i) A third party whose party share was very small was kept in the wavering category of voters.

ii) 16% of the voters changed their loyalty to a party when they went to voting booth in 1972 elections and, 20% changed in 1976 elections.

iii) The data is taken from the survey that was done one month before the elections.

NOTE. To achieve better accuracy the data should be taken from a number of previous surveys and g's be obtained by using some kind of a smoothing technique on the data. Since complete data of all gallup poll surveys was not available, therefore, in the present work the results of only those surveys have been taken which were done one month before the elections.

Based on above assumption Table 6-4 gives both theoretical and observed data obtained from the gallup surveys. Different values of β were tried to fit the data of 1972. The best value of β obtained is 1.1 . With this value of β, the percentage of loyal and wavering voters are obtained for 1976 also.

Table 6-4

Year	Observed data obtained from gallup poll surveys					Theoretical data computed from entropy model for $\beta = 1.1$				
	m_1	m_2	g_1	g_2	g_3	m_1	m_2	g_1	g_2	g_3
1972	.62	.38	.365	.164	.471	.62	.38	.380	.186	.433
1976	.49	.51	.262	.285	.453	.49	.51	.270	.287	.441

In Table 6-4 a comparison of the theoretical results obtained by using the entropy model for β =1.1 with the observed data obtained from the gallup surveys shows that for the year 1976 values of g_1 , g_2 and g_3 are quite close.

Chapter VII

CONCLUDING REMARKS AND SUGGESTIONS

The present thesis has generalised Herniter's [8] model for brand switching by using Renyi's [20] measure of entropy in place of Shannon's measure. The new model gives all the results of Herniter's model in a limiting case when the parameter $\beta \to 1$. However the generalisation is achieved at the cost of increasing computational complexity and extensive numerical work had to be done to find results for different values of β. This numerical work incidentally also gave the limits on the values of β for which the new model is valid.

Another measure of entropy given by Kapur [12] is as follows:

$$S = \frac{1}{1-\beta} \ln \frac{\sum_{i=1}^{n} p_i^{\beta+\gamma-1}}{\sum_{i=1}^{n} p_i^{\gamma-1}}$$

This measure is "of order β and type γ and approaches Renyi's measure [20] as $\gamma \to 1$, and it approaches Shannon's measure [21] as both β and γ approaches unity. Use of Kapur's measure will give still more general and accurate results as now there are two parameters β and γ which may be chosen to represent two different characteristics of a product or community e.g. one parameter can be assigned to different products and the other parameter can be assigned to different consumer groups.

There is however need for a measure of entropy which will not lead to negative probabilities. Since a measure has been recently (1983) proposed by Kapur as

$$-\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{\alpha}\sum_{i=1}^{n}(1+\alpha p_i)\ln(1+\alpha p_i)$$

It can be easily verified that it's a concave function and approaches Shannon's measure as $\alpha \to 0$. It is proposed to extend all the results of the present thesis to this new measure and to compare the results obtained for this new model with the result obtained from the model using Renyi's measure of entropy. This will of course also involve numerical computations.

In Chapter 4, Shannon's measure of entropy was used to show that the solution of a maximum entropy model for switching of brands in a market depends on the structure of the underlying model. However in Model 1 of this Chapter Renyi's measure was also used and it was shown that the results obtained were independent of the measure chosen. It is proposed to apply Renyi's and Kapur's measures to all the models and compare with the one's results obtained by using Shannon's measure.

The results of brand-switching models have been extended to party switching in elections. Some data for voting have been analysed. It would be certainly interesting to get more data both about elections and about brand switching to test the validity of the various models.

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Appendix A

COMPUTER OUTPUT FOR TWO PARTY (BRAND) THREE CATEGORIES ENTROPY MODEL FOR DIFFERENT VALUES OF β

$\beta = 0.6500$

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2300	0.2300	0.5399	1.3414
0.4900	0.5100	0.2222	0.2381	0.5397	1.3411
0.4800	0.5200	0.2145	0.2463	0.5391	1.3403
0.4700	0.5300	0.2071	0.2548	0.5381	1.3390
0.4600	0.5400	0.1998	0.2635	0.5368	1.3372
0.4500	0.5500	0.1927	0.2723	0.5350	1.3349
0.4400	0.5600	0.1858	0.2814	0.5328	1.3320
0.4300	0.5700	0.1790	0.2907	0.5303	1.3286
0.4200	0.5800	0.1725	0.3002	0.5273	1.3246
0.4100	0.5900	0.1661	0.3099	0.5240	1.3201
0.4000	0.6000	0.1598	0.3199	0.5203	1.3151
0.3900	0.6100	0.1538	0.3300	0.5162	1.3095
0.3800	0.6200	0.1479	0.3404	0.5117	1.3034
0.3700	0.6300	0.1421	0.3510	0.5069	1.2967
0.3600	0.6400	0.1365	0.3618	0.5017	1.2894
0.3500	0.6500	0.1310	0.3729	0.4961	1.2816
0.3400	0.6600	0.1257	0.3842	0.4902	1.2732
0.3300	0.6700	0.1205	0.3957	0.4839	1.2642

0.3200	0.6800	0.1154	0.4074	0.4772	1.2545
0.3100	0.6900	0.1105	0.4193	0.4702	1.2443
0.3000	0.7000	0.1056	0.4315	0.4628	1.2334
0.2900	0.7100	0.1009	0.4439	0.4551	1.2219
0.2800	0.7200	0.0964	0.4566	0.4471	1.2097
0.2700	0.7300	0.0919	0.4694	0.4387	1.1969
0.2600	0.7400	0.0875	0.4825	0.4300	1.1833
0.2500	0.7500	0.0833	0.4958	0.4209	1.1690

β = 0.7000

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2385	0.2385	0.5231	1.3336
0.4900	0.5100	0.2305	0.2466	0.5229	1.3334
0.4800	0.5200	0.2228	0.2548	0.5224	1.3326
0.4700	0.5300	0.2152	0.2633	0.5215	1.3313
0.4600	0.5400	0.2078	0.2719	0.5203	1.3294
0.4500	0.5500	0.2006	0.2807	0.5187	1.3270
0.4400	0.5600	0.1935	0.2897	0.5168	1.3241
0.4300	0.5700	0.1866	0.2989	0.5146	1.3206
0.4200	0.5800	0.1798	0.3083	0.5120	1.3166
0.4100	0.5900	0.1732	0.3178	0.5090	1.3121
0.4000	0.6000	0.1667	0.3276	0.5057	1.3070
0.3900	0.6100	0.1604	0.3375	0.5021	1.3014
0.3800	0.6200	0.1542	0.3477	0.4981	1.2951
0.3700	0.6300	0.1482	0.3580	0.4938	1.2884
0.3600	0.6400	0.1423	0.3685	0.4892	1.2810
0.3500	0.6500	0.1365	0.3793	0.4842	1.2730
0.3400	0.6600	0.1309	0.3902	0.4789	1.2645
0.3300	0.6700	0.1254	0.4014	0.4732	1.2553
0.3200	0.6800	0.1200	0.4127	0.4672	1.2456
0.3100	0.6900	0.1148	0.4243	0.4609	1.2351
0.3000	0.7000	0.1097	0.4360	0.4543	1.2241
0.2900	0.7100	0.1046	0.4480	0.4474	1.2124
0.2800	0.7200	0.0997	0.4602	0.4401	1.1999
0.2700	0.7300	0.0950	0.4726	0.4325	1.1868
0.2600	0.7400	0.0903	0.4852	0.4245	1.1730
0.2500	0.7500	0.0857	0.4980	0.4163	1.1584

$\beta =$ 0.7500

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2459	0.2459	0.5082	1.3268
0.4900	0.5100	0.2379	0.2540	0.5081	1.3265
0.4800	0.5200	0.2301	0.2623	0.5076	1.3257
0.4700	0.5300	0.2224	0.2707	0.5069	1.3244
0.4600	0.5400	0.2149	0.2793	0.5058	1.3225
0.4500	0.5500	0.2076	0.2880	0.5044	1.3201
0.4400	0.5600	0.2003	0.2970	0.5027	1.3171
0.4300	0.5700	0.1932	0.3060	0.5007	1.3136
0.4200	0.5800	0.1863	0.3153	0.4984	1.3095
0.4100	0.5900	0.1795	0.3247	0.4958	1.3049
0.4000	0.6000	0.1728	0.3343	0.4929	1.2997
0.3900	0.6100	0.1663	0.3440	0.4897	1.2940
0.3800	0.6200	0.1599	0.3539	0.4862	1.2877
0.3700	0.6300	0.1536	0.3640	0.4823	1.2808
0.3600	0.6400	0.1475	0.3743	0.4782	1.2733
0.3500	0.6500	0.1415	0.3848	0.4738	1.2652
0.3400	0.6600	0.1356	0.3954	0.4690	1.2565
0.3300	0.6700	0.1298	0.4062	0.4640	1.2472
0.3200	0.6800	0.1242	0.4172	0.4586	1.2372
0.3100	0.6900	0.1186	0.4284	0.4529	1.2266
0.3000	0.7000	0.1132	0.4398	0.4470	1.2154
0.2900	0.7100	0.1079	0.4514	0.4407	1.2034
0.2800	0.7200	0.1027	0.4632	0.4341	1.1908
0.2700	0.7300	0.0976	0.4751	0.4272	1.1774
0.2600	0.7400	0.0927	0.4873	0.4200	1.1633
0.2500	0.7500	0.0878	0.4997	0.4125	1.1484

β = 0.8000

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2524	0.2524	0.4952	1.3207
0.4900	0.5100	0.2444	0.2606	0.4950	1.3204
0.4800	0.5200	0.2366	0.2688	0.4946	1.3196
0.4700	0.5300	0.2288	0.2772	0.4939	1.3182
0.4600	0.5400	0.2212	0.2858	0.4930	1.3163
0.4500	0.5500	0.2137	0.2945	0.4918	1.3139
0.4400	0.5600	0.2064	0.3033	0.4903	1.3109
0.4300	0.5700	0.1992	0.3123	0.4885	1.3073
0.4200	0.5800	0.1921	0.3214	0.4865	1.3031
0.4100	0.5900	0.1851	0.3307	0.4842	1.2984
0.4000	0.6000	0.1783	0.3401	0.4816	1.2932
0.3900	0.6100	0.1716	0.3497	0.4788	1.2873
0.3800	0.6200	0.1650	0.3594	0.4756	1.2809
0.3700	0.6300	0.1585	0.3693	0.4722	1.2739
0.3600	0.6400	0.1521	0.3793	0.4686	1.2662
0.3500	0.6500	0.1459	0.3895	0.4646	1.2580
0.3400	0.6600	0.1397	0.3999	0.4604	1.2491
0.3300	0.6700	0.1337	0.4104	0.4559	1.2396
0.3200	0.6800	0.1278	0.4211	0.4511	1.2295
0.3100	0.6900	0.1220	0.4320	0.4460	1.2187
0.3000	0.7000	0.1164	0.4430	0.4406	1.2072
0.2900	0.7100	0.1108	0.4542	0.4350	1.1950
0.2800	0.7200	0.1053	0.4656	0.4291	1.1821
0.2700	0.7300	0.1000	0.4772	0.4228	1.1685
0.2600	0.7400	0.0948	0.4889	0.4163	1.1541
0.2500	0.7500	0.0896	0.5009	0.4095	1.1389

β = 0.8500

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2582	0.2582	0.4836	1.3153
0.4900	0.5100	0.2502	0.2663	0.4835	1.3150
0.4800	0.5200	0.2423	0.2746	0.4831	1.3141
0.4700	0.5300	0.2345	0.2830	0.4825	1.3128
0.4600	0.5400	0.2268	0.2915	0.4817	1.3108
0.4500	0.5500	0.2193	0.3001	0.4806	1.3083
0.4400	0.5600	0.2118	0.3089	0.4793	1.3052
0.4300	0.5700	0.2045	0.3178	0.4778	1.3016
0.4200	0.5800	0.1973	0.3268	0.4760	1.2973
0.4100	0.5900	0.1902	0.3359	0.4739	1.2926
0.4000	0.6000	0.1832	0.3452	0.4717	1.2872
0.3900	0.6100	0.1763	0.3546	0.4691	1.2812
0.3800	0.6200	0.1695	0.3641	0.4664	1.2746
0.3700	0.6300	0.1628	0.3738	0.4634	1.2675
0.3600	0.6400	0.1563	0.3836	0.4601	1.2597
0.3500	0.6500	0.1498	0.3936	0.4566	1.2513
0.3400	0.6600	0.1435	0.4037	0.4528	1.2423
0.3300	0.6700	0.1372	0.4140	0.4488	1.2326
0.3200	0.6800	0.1311	0.4244	0.4445	1.2222
0.3100	0.6900	0.1251	0.4349	0.4400	1.2112
0.3000	0.7000	0.1192	0.4456	0.4352	1.1995
0.2900	0.7100	0.1134	0.4565	0.4301	1.1871
0.2800	0.7200	0.1077	0.4676	0.4248	1.1739
0.2700	0.7300	0.1021	0.4788	0.4192	1.1600
0.2600	0.7400	0.0966	0.4902	0.4133	1.1453
0.2500	0.7500	0.0912	0.5018	0.4071	1.1299

$\beta =$ 0.9000

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2633	0.2633	0.4733	1.3104
0.4900	0.5100	0.2553	0.2715	0.4732	1.3101
0.4800	0.5200	0.2474	0.2797	0.4729	1.3092
0.4700	0.5300	0.2395	0.2881	0.4724	1.3078
0.4600	0.5400	0.2318	0.2965	0.4716	1.3058
0.4500	0.5500	0.2242	0.3051	0.4707	1.3032
0.4400	0.5600	0.2167	0.3138	0.4696	1.3001
0.4300	0.5700	0.2092	0.3226	0.4682	1.2964
0.4200	0.5800	0.2019	0.3315	0.4666	1.2921
0.4100	0.5900	0.1947	0.3405	0.4648	1.2872
0.4000	0.6000	0.1875	0.3496	0.4628	1.2817
0.3900	0.6100	0.1805	0.3589	0.4606	1.2756
0.3800	0.6200	0.1736	0.3683	0.4582	1.2689
0.3700	0.6300	0.1667	0.3778	0.4555	1.2616
0.3600	0.6400	0.1600	0.3874	0.4526	1.2536
0.3500	0.6500	0.1534	0.3971	0.4495	1.2451
0.3400	0.6600	0.1468	0.4070	0.4462	1.2358
0.3300	0.6700	0.1404	0.4170	0.4426	1.2260
0.3200	0.6800	0.1340	0.4271	0.4388	1.2154
0.3100	0.6900	0.1278	0.4374	0.4348	1.2042
0.3000	0.7000	0.1217	0.4479	0.4305	1.1922
0.2900	0.7100	0.1156	0.4584	0.4259	1.1796
0.2800	0.7200	0.1097	0.4692	0.4211	1.1662
0.2700	0.7300	0.1039	0.4801	0.4161	1.1520
0.2600	0.7400	0.0981	0.4911	0.4108	1.1370
0.2500	0.7500	0.0925	0.5023	0.4052	1.1213

$\beta =$ 0.9500

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2679	0.2679	0.4642	1.3059
0.4900	0.5100	0.2599	0.2761	0.4641	1.3056
0.4800	0.5200	0.2519	0.2843	0.4638	1.3048
0.4700	0.5300	0.2441	0.2926	0.4633	1.3033
0.4600	0.5400	0.2363	0.3010	0.4627	1.3013
0.4500	0.5500	0.2286	0.3095	0.4619	1.2986
0.4400	0.5600	0.2210	0.3181	0.4609	1.2954
0.4300	0.5700	0.2135	0.3268	0.4597	1.2916
0.4200	0.5800	0.2061	0.3356	0.4583	1.2872
0.4100	0.5900	0.1987	0.3445	0.4567	1.2822
0.4000	0.6000	0.1915	0.3535	0.4550	1.2766
0.3900	0.6100	0.1843	0.3627	0.4530	1.2704
0.3800	0.6200	0.1772	0.3719	0.4509	1.2635
0.3700	0.6300	0.1702	0.3812	0.4486	1.2561
0.3600	0.6400	0.1634	0.3906	0.4460	1.2480
0.3500	0.6500	0.1565	0.4002	0.4433	1.2392
0.3400	0.6600	0.1498	0.4098	0.4403	1.2298
0.3300	0.6700	0.1432	0.4196	0.4372	1.2197
0.3200	0.6800	0.1367	0.4295	0.4338	1.2090
0.3100	0.6900	0.1302	0.4395	0.4302	1.1975
0.3000	0.7000	0.1239	0.4497	0.4264	1.1853
0.2900	0.7100	0.1176	0.4600	0.4224	1.1724
0.2800	0.7200	0.1115	0.4704	0.4181	1.1588
0.2700	0.7300	0.1054	0.4810	0.4135	1.1444
0.2600	0.7400	0.0995	0.4918	0.4088	1.1291
0.2500	0.7500	0.0936	0.5027	0.4037	1.1131

$\beta = 1.0000$

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2720	0.2720	0.4560	1.3019
0.4900	0.5100	0.2640	0.2801	0.4559	1.3016
0.4800	0.5200	0.2560	0.2884	0.4556	1.3007
0.4700	0.5300	0.2481	0.2966	0.4553	1.2992
0.4600	0.5400	0.2403	0.3050	0.4547	1.2971
0.4500	0.5500	0.2326	0.3135	0.4540	1.2944
0.4400	0.5600	0.2249	0.3220	0.4531	1.2911
0.4300	0.5700	0.2173	0.3306	0.4521	1.2873
0.4200	0.5800	0.2098	0.3393	0.4509	1.2828
0.4100	0.5900	0.2024	0.3481	0.4495	1.2776
0.4000	0.6000	0.1950	0.3570	0.4480	1.2719
0.3900	0.6100	0.1877	0.3660	0.4463	1.2655
0.3800	0.6200	0.1805	0.3750	0.4444	1.2585
0.3700	0.6300	0.1734	0.3842	0.4424	1.2509
0.3600	0.6400	0.1664	0.3934	0.4402	1.2426
0.3500	0.6500	0.1594	0.4028	0.4378	1.2337
0.3400	0.6600	0.1526	0.4123	0.4352	1.2241
0.3300	0.6700	0.1458	0.4218	0.4324	1.2138
0.3200	0.6800	0.1390	0.4315	0.4294	1.2028
0.3100	0.6900	0.1324	0.4413	0.4263	1.1912
0.3000	0.7000	0.1259	0.4512	0.4229	1.1788
0.2900	0.7100	0.1194	0.4613	0.4193	1.1656
0.2800	0.7200	0.1131	0.4714	0.4155	1.1517
0.2700	0.7300	0.1068	0.4817	0.4114	1.1371
0.2600	0.7400	0.1006	0.4922	0.4072	1.1216
0.2500	0.7500	0.0946	0.5028	0.4026	1.1053

β = 1.0500

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2757	0.2757	0.4486	1.2982
0.4900	0.5100	0.2676	0.2838	0.4485	1.2979
0.4800	0.5200	0.2597	0.2920	0.4483	1.2970
0.4700	0.5300	0.2518	0.3002	0.4480	1.2954
0.4600	0.5400	0.2439	0.3086	0.4475	1.2933
0.4500	0.5500	0.2361	0.3170	0.4469	1.2906
0.4400	0.5600	0.2284	0.3254	0.4462	1.2872
0.4300	0.5700	0.2208	0.3340	0.4453	1.2832
0.4200	0.5800	0.2132	0.3426	0.4442	1.2786
0.4100	0.5900	0.2057	0.3513	0.4431	1.2734
0.4000	0.6000	0.1982	0.3600	0.4417	1.2675
0.3900	0.6100	0.1909	0.3689	0.4403	1.2610
0.3800	0.6200	0.1835	0.3778	0.4387	1.2539
0.3700	0.6300	0.1763	0.3868	0.4369	1.2461
0.3600	0.6400	0.1691	0.3959	0.4350	1.2376
0.3500	0.6500	0.1620	0.4051	0.4329	1.2285
0.3400	0.6600	0.1550	0.4144	0.4306	1.2187
0.3300	0.6700	0.1481	0.4238	0.4282	1.2082
0.3200	0.6800	0.1412	0.4332	0.4256	1.1970
0.3100	0.6900	0.1344	0.4428	0.4228	1.1851
0.3000	0.7000	0.1277	0.4525	0.4198	1.1725
0.2900	0.7100	0.1210	0.4623	0.4167	1.1591
0.2800	0.7200	0.1145	0.4722	0.4133	1.1450
0.2700	0.7300	0.1080	0.4823	0.4097	1.1301
0.2600	0.7400	0.1016	0.4924	0.4059	1.1144
0.2500	0.7500	0.0954	0.5027	0.4019	1.0978

β = 1.1000

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2790	0.2790	0.4420	1.2948
0.4900	0.5100	0.2710	0.2871	0.4419	1.2945
0.4800	0.5200	0.2630	0.2953	0.4418	1.2936
0.4700	0.5300	0.2551	0.3035	0.4415	1.2920
0.4600	0.5400	0.2472	0.3118	0.4411	1.2898
0.4500	0.5500	0.2394	0.3201	0.4405	1.2870
0.4400	0.5600	0.2316	0.3285	0.4399	1.2836
0.4300	0.5700	0.2239	0.3369	0.4391	1.2795
0.4200	0.5800	0.2163	0.3455	0.4383	1.2748
0.4100	0.5900	0.2087	0.3541	0.4373	1.2694
0.4000	0.6000	0.2012	0.3627	0.4361	1.2634
0.3900	0.6100	0.1937	0.3714	0.4349	1.2568
0.3800	0.6200	0.1863	0.3802	0.4335	1.2495
0.3700	0.6300	0.1789	0.3891	0.4320	1.2415
0.3600	0.6400	0.1716	0.3981	0.4303	1.2329
0.3500	0.6500	0.1644	0.4071	0.4285	1.2236
0.3400	0.6600	0.1572	0.4162	0.4266	1.2136
0.3300	0.6700	0.1502	0.4254	0.4245	1.2029
0.3200	0.6800	0.1431	0.4347	0.4222	1.1915
0.3100	0.6900	0.1362	0.4440	0.4198	1.1794
0.3000	0.7000	0.1293	0.4535	0.4172	1.1665
0.2900	0.7100	0.1225	0.4631	0.4144	1.1529
0.2800	0.7200	0.1157	0.4728	0.4115	1.1386
0.2700	0.7300	0.1091	0.4826	0.4084	1.1234
0.2600	0.7400	0.1025	0.4925	0.4050	1.1075
0.2500	0.7500	0.0960	0.5026	0.4014	1.0907

$\beta =$ 1.1500

m_1	m_2	g_1	g_2	g_3	S
0.5000	0.5000	0.2820	0.2820	0.4360	1.2917
0.4900	0.5100	0.2740	0.2901	0.4360	1.2914
0.4800	0.5200	0.2660	0.2982	0.4358	1.2904
0.4700	0.5300	0.2581	0.3064	0.4356	1.2888
0.4600	0.5400	0.2502	0.3146	0.4352	1.2866
0.4500	0.5500	0.2423	0.3229	0.4348	1.2837
0.4400	0.5600	0.2345	0.3312	0.4343	1.2802
0.4300	0.5700	0.2268	0.3396	0.4336	1.2760
0.4200	0.5800	0.2191	0.3480	0.4329	1.2712
0.4100	0.5900	0.2114	0.3565	0.4320	1.2657
0.4000	0.6000	0.2038	0.3651	0.4311	1.2596
0.3900	0.6100	0.1963	0.3737	0.4300	1.2528
0.3800	0.6200	0.1888	0.3824	0.4288	1.2453
0.3700	0.6300	0.1813	0.3911	0.4276	1.2372
0.3600	0.6400	0.1739	0.3999	0.4261	1.2284
0.3500	0.6500	0.1666	0.4088	0.4246	1.2189
0.3400	0.6600	0.1593	0.4178	0.4230	1.2087
0.3300	0.6700	0.1521	0.4268	0.4212	1.1978
0.3200	0.6800	0.1449	0.4359	0.4192	1.1862
0.3100	0.6900	0.1378	0.4451	0.4172	1.1739
0.3000	0.7000	0.1307	0.4544	0.4149	1.1608
0.2900	0.7100	0.1237	0.4637	0.4125	1.1470
0.2800	0.7200	0.1168	0.4732	0.4100	1.1324
0.2700	0.7300	0.1100	0.4828	0.4073	1.1170
0.2600	0.7400	0.1032	0.4925	0.4043	1.1009
0.2500	0.7500	0.0965	0.5023	0.4012	1.0839