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THREE DIMENSIONAL ANALYSIS OF HARMONIC WAVES IN A PERIODICALLY LAMINATED MEDIUM

BY

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NOMENCLATURE

$\bar{A}$, $\bar{B}$ $\bar{U}$, $\bar{V}$, $\bar{W}$	Amplitude of propagating wave
$[A_1]$, $[B_1]$	Stiffness and Mass matrix of the laminated media
$C^{(i)}_{\ell,n}$	Material constants of the $(i)^{th}$ lamina (i) = f : reinforcing layer (i) = m : matrix layer ℓ = 1, 2, 3...6 n = 1, 2, 3...6
c	Phase Velocity
d	Thickness of a unit cell
d_1, d_f, $2h^{(1)}$	Thickness of reinforced layer
d_2, d_m, $2h^{(2)}$	Thickness of matrix layer
$\bar{d}$	Ratio of thicknesses of reinforced layer and the matrix layer
f_m (η_i)	Interpolation function, m = 1, 2, 3, 4
I, (i)	Superscript (i) and I represent the same quantity, i.e., the $(i)^{th}$ or I^{th} lamina
$[k_i]$, $[m_i]$	Stiffness and mass matrix of the $(i)^{th}$ lamina
$\bar{k}$, ξ	Dimensionless wave number
k, k_x, k_y, k_z	Wave number and the x- , y- and z-component of the wave number
$\{r_i\}$	Matrix of nodal quantities for the $(i)^{th}$ lamina
$\{R\}$	Matrix of nodal quantities for the entire media
$[S^a]$, $[S^p]$	Impedance matrix
t, T	Time, superscript denotes transpose
$T^{(i)}_{kin}$, $V^{(i)}_{Pot}$	Kinetic energy and potential energy of the $(i)^{th}$ lamina (i) = f : reinforced layer (i) = m : matrix layer

U_I, V_I, W_I	x- , y- and z-component of the global displacement of the $(i)^{th}$ lamina
$\bar{U}_I$, $\bar{V}_I$, $\bar{W}_I$	Complex conjugate of U_I, V_I, W_I
$U_o^{ft,mk}$, $V_o^{ft,mk}$	Local displacement of mid plane of the k^{th} pair of
$W_o^{fk,mk}$	reinforcing and matrix layer.
u_i, v_i, w_i	nodal x- , y- and z-displacement of the i^{th} node
α, ϕ	Horizontal and vertical angle
β	Nondimensional phase velocity
$\gamma^{(i)}$, $\varepsilon^{(i)}$	Shear strain and axial strain of the $(i)^{th}$ lamina (i) = f reinforced layer (i) = m matrix layer
γ	Ratio of shear modulus
η	Nondimensional ratio of thicknesses of reinforced layer and the total thickness of the unit cell
η_i	Generalized coordinate of the $(i)^{th}$ lamina
$\bar{\eta}$	Nondimensional y-component of wave number
θ	Ratio of mass densities
λ_1, λ_2, λ_3	Lagrangian multiplier
Λ	Wavelength
μ	Shear Modulus, Lame's constant
ν	Poission's ratio
$\rho^{(i)}$	Density of $(i)^{th}$lamina (i) = f : reinforcing layer (i) = m ; matrix layer
$\sigma^{(i)}$	Stresses of the $(i)^{th}$ lamina
χ_i	Nodal shear stress parallel to x-y plane at i^{th} node

σ_i	Nodal normal stress parallel to y-axis at i^{th} node
τ_i	Nodal shear stress parallel to y-z plane at i^{th} node
ω	Frequency
Ω	Normalized nondimensional frequency

CHAPTER 1

INTRODUCTION

1.1 LITERATURE REVIEW

The increasing use of composite materials like fibre-reinforced composite and graphite-epoxy composite in a variety of structural applications has generated extensive research efforts in the area of dynamic behaviour of periodically laminated media. In the past two decades, various approximate theories and exact solutions for wave propagation in the laminated media have been proposed.

The early approximate theory was made by specifying the behaviour of the plate through its thickness. The most common of which is the Love-Kirchhoff hypothesis [1, 2, 3]. But, when the material properties differ appreciably from layer to layer and/or when a high degree of orthotropy exists in one or more layers, the validity of these theories was found to be questionable [4]. To improve these approximate theories, the effect of transverse shear deformation has been included. This approach produced satisfactory results for the static analysis of thick laminates [5]. Later, Nelson and Lorch [6] introduced a refined theory that includes transverse shear, transverse normal and quadratic terms in the kinematic assumption.

Postma [7], Rytov [8],and White and Angona [9] then developed the effective modulus theory on the basis of both static and dynamic consideration. This is the approximate elasticity solutions in which attempts were made to bridge the gap between the approximate plate theories and the general theory of elasticity formulations, by smoothing and averaging procedures on special laminates with periodic structure through thickness. However, this method does not account for the effects of the geometric dispersion. Therefore, an effective stiffness method was proposed by Sun, Achenbach and Herrmann [10, 11, 12]. This method was then developed in more detail by other investigators [13, 14, 15].

Other approximate approaches that have been used extensively are the mixture theory [16, 17, 18], the theory of interacting continuum [19, 20], and the new quotient method [21, 22]. Recently, Mengi et al [23] used higher order plate theory together with a smoothing operation to study the dispersion characteristic of two-layer periodic laminated media.

In addition to the approximate analysis mentioned above, exact solutions for wave propagating in laminated media have also been presented in the literature. Most actual analyses are based on two-dimensional equations where each laminate is isotropic. The antiplane [24-26] and plane strain [27] problems have been dealt with by imposing the displacement and stress continuity conditions at the interfaces. Recently, Delph et al.[28-30] have done extensive work in using Floquet's theory in conjunction with the elasticity solution to analyse the dispersion characteristic for harmonic wave propagation through laminated media. Dispersion relations for laminated composites in a three-dimensional setting have been obtained by Kulkarmi and Pagano [31], where the solution for vibration frequencies for cylindrical bending has been presented. Yamada and Nemat-Nasser [32] have also examined the dispersive effects in layered orthotropic elastic composites, where the direction of the corresponding harmonic waves makes an arbitrary angle with respect to the layers.

In most of the analysis mentioned above, the laminae were assumed to be isotropic. The most recent work was done by Shah and Datta [33] in which the finite element method is presented for studying harmonic wave propagation in a periodically laminated medium, where each lamina may have anisotropic properties. However, in their work, antiplane and plane strain motions are dealt with separately. The method used is discussed in detail in Ref. [33].

1.2 PRESENT SCOPE

In this thesis, the finite element method or stiffness method proposed by Shah and Datta [33], for the antiplane and plane strain motions, is extended to three-dimensional problems. The method and formulation technique used is discussed in Chapter 2. In order to assess the accuracy of this method, numerical results are first compared with the results obtained by the effective modulus [7] and effective stiffness methods [11]. But first of all, the effective modulus and effective stiffness method have to be extended to handle a three dimensional analysis for anisotropic materials. These will be discussed and formulated in Chapter 3. Numerical results are then presented for fiber-reinforced composite and graphite-epoxy composite with different layer thicknesses. The computer programs used to assist the numerical computation are listed in Appendix H, I and J. The first computer program is written for the three dimensional analysis of wave propagation through laminated layers using the effective modulus method. The second is for an effective stiffness analysis while Appendix J is the computer program written to solve the same problem using the finite element method.

CHAPTER 2

HARMONIC WAVES IN A PERIODICALLY LAMINATED MEDIUM: FINITE ELEMENT METHODS (F.E.M.)

2.1 INTRODUCTION

In this chapter, the behaviour of harmonic waves propagating in a periodically layered, infinite, elastic body is examined. In most previous studies on the similar subject mentioned in Chapter 1, each lamina was assumed to be isotropic. However, in order to be more general, the present analysis will deal with three dimensional, anisotropic lamina. In this method, an interpolation function is assumed for each lamina which is characterized by a discrete number of generalized coordinates at the interfaces. These generalized coordinates are the interface displacements and stresses, thus ensuring the continuity of these quantities across the boundary planes. By applying Hamilton's principle [34] and using Floquet's theory [35], the dispersion equation is obtained. The solutions of this equation yield the frequency-wave-number relationships. In the following, the method will be discussed in detail.

2.2. EQUATION OF LINEAR ELASTICITY

In the following, harmonic waves propagating through a periodically layered, elastic body of unbounded extent is considered. For the purpose of this discussion, a two-layered periodically laminated elastic body, as shown in Fig.2.1, is considered though the method presented can be applied to any number of periodicity. Any two adjacent laminae in the body then compose a unit cell. Both laminae in the unit cell are assumed to be homogeneous, orthorhombic and perfectly bonded to contiguous layers. The two laminae of a typical unit cell, C_n, as shown in Fig. 2.1, have elastic constants ($C_{11}^{(i)}$, $C_{12}^{(i)}$, $C_{13}^{(i)}$, $C_{22}^{(i)}$, $C_{23}^{(i)}$, $C_{33}^{(i)}$, $C_{44}^{(i)}$, $C_{55}^{(i)}$, $C_{66}^{(i)}$); ($C_{11}^{(i+1)}$, $C_{12}^{(i+1)}$, $C_{13}^{(i+1)}$, $C_{22}^{(i+1)}$, $C_{23}^{(i+1)}$, $C_{33}^{(i+1)}$, $C_{44}^{(i+1)}$, $C_{55}^{(i+1)}$, $C_{66}^{(i+1)}$), thickness ($2\,h^{(i)}$, $2\,h^{(i+1)}$) and densities ($\rho^{(i)}$, $\rho^{(i+1)}$), respectively. For a particular lamina, the relevant stress-strain relations are

$$\begin{Bmatrix} \sigma_{xx}^{(i)} \\ \sigma_{yy}^{(i)} \\ \sigma_{zz}^{(i)} \\ \sigma_{yz}^{(i)} \\ \sigma_{xz}^{(i)} \\ \sigma_{xy}^{(i)} \end{Bmatrix} = \begin{bmatrix} C_{11}^{(i)} & C_{12}^{(i)} & C_{13}^{(i)} & 0 & 0 & 0 \\ C_{12}^{(i)} & C_{22}^{(i)} & C_{23}^{(i)} & 0 & 0 & 0 \\ C_{13}^{(i)} & C_{23}^{(i)} & C_{33}^{(i)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^{(i)} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^{(i)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(i)} \\ \varepsilon_{yy}^{(i)} \\ \varepsilon_{zz}^{(i)} \\ \gamma_{yz}^{(i)} \\ \gamma_{xz}^{(i)} \\ \gamma_{xy}^{(i)} \end{Bmatrix} \tag{2.1}$$

where $\sigma_{mn}^{(i)}$ represent the stress and $\varepsilon_{mn}^{(i)}$, $\gamma_{mn}^{(i)}$ represent the strain of the I^{th} (superscript $^{(i)}$) lamina.

Let U_I (x_i, y_i, z_i, t), V_I (x_i, y_i, z_i, t) and W_I (x_i, y_i, z_i, t) be the Cartesian components of the displacement in the x, y and z directions respectively, for the I^{th} lamina. The strain components in the $(i)^{th}$ lamina can then be expressed as

$$\varepsilon_{xx}^{(i)} = \frac{\partial U_I}{\partial x_i} \qquad \varepsilon_{yy}^{(i)} = \frac{\partial V_I}{\partial y_i} \qquad \varepsilon_{zz}^{(i)} = \frac{\partial W_I}{\partial z_i}$$

$$\gamma_{yz}^{(i)} = \left(\frac{\partial W_I}{\partial y_i} + \frac{\partial V_I}{\partial z_i}\right) \qquad \gamma_{xz}^{(i)} = \left(\frac{\partial W_I}{\partial x_i} + \frac{\partial U_I}{\partial z_i}\right) \qquad \gamma_{xy}^{(i)} = \left(\frac{\partial V_I}{\partial x_i} + \frac{\partial U_I}{\partial y_i}\right) \tag{2.2}$$

Substituting Eqs. 2.2 into Eqs. 2.1, the stress-displacement relationships may be obtained. Substitution of these stress-displacement relationships into the stress equations of motion leads to Navier's equation of motion [36]. The solution to these Navier equations give the displacement and stresses in each lamina. At present, no analytical solutions to these Navier's equations are available. Thus, in this thesis, a finite element method is proposed to solve the problem.

2.3 TRACTION AND DISPLACEMENT CONTINUITY

For a two-layered periodically laminated elastic body, the continuities of traction and displacement at the interface between two consecutive lamina (1) and (2), and (2) and (3) are:

$$
\begin{aligned}
U_2 \; (x_2, -h^{(2)}, z_2, t) &= U_1 \; (x_1, h^{(1)}, z_1, t) \\
V_2 \; (x_2, -h^{(2)}, z_2, t) &= V_1 \; (x_1, h^{(1)}, z_1, t) \\
W_2 \; (x_2, -h^{(2)}, z_2, t) &= W_1 \; (x_1, h^{(1)}, z_1, t) \\
\sigma_{yx}^{(2)}(x_2, -h^{(2)}, z_2, t) &= \sigma_{yx}^{(1)}(x_1, h^{(1)}, z_1, t) \\
\sigma_{yy}^{(2)}(x_2, -h^{(2)}, z_2, t) &= \sigma_{yy}^{(1)}(x_1, h^{(1)}, z_1, t) \\
\sigma_{yz}^{(2)}(x_2, -h^{(2)}, z_2, t) &= \sigma_{yz}^{(1)}(x_1, h^{(1)}, z_1, t)
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
U_2 \; (x_2, h^{(2)}, z_2, t) &= U_3 \; (x_3, -h^{(1)}, z_3, t) \\
V_2 \; (x_2, h^{(2)}, z_2, t) &= V_3 \; (x_3, -h^{(1)}, z_3, t) \\
W_2 \; (x_2, h^{(2)}, z_2, t) &= W_3 \; (x_3, -h^{(1)}, z_3, t) \\
\sigma_{yx}^{(2)}(x_2, h^{(2)}, z_2, t) &= \sigma_{yx}^{(3)}(x_3, - h^{(1)}, z_3, t)
\end{aligned}
$$

$$\sigma_{yy}^{(2)} (x_2, h^{(2)}, z_2, t) = \sigma_{yy}^{(3)} (x_3, -h^{(1)}, z_3, t)$$
$$\sigma_{yz}^{(2)} (x_2, h^{(2)}, z_2, t) = \sigma_{yz}^{(3)} (x_3, -h^{(1)}, z_3, t) \tag{2.4}$$

Using Floquet's theory [35], Eqs. 2.4 can be rewritten as

$$U_2 \ (x_2. h^{(2)}, z_2, t) = U_1 \ (x_1, -h^{(1)}, z_1, t) \ e^{ik_y d}$$
$$V_2 \ (x_2, h^{(2)}, z_2, t) = V_1 \ (x_1, -h^{(1)}, z_1, t) \ e^{ik_y d}$$
$$W_2 \ (x_2, h^{(2)}, z_2, t) = W_1 \ (x_1, -h^{(1)}, z_1, t) \ e^{ik_y d}$$
$$\sigma_{yx}^{(2)} (x_2, h^{(2)}, z_2, t) = \sigma_{yx}^{(1)} (x_1, -h^{(1)}, z_1, t) \ e^{ik_y d}$$
$$\sigma_{yy}^{(2)} (x_2, h^{(2)}, z_2, t) = \sigma_{yy}^{(1)} (x_1, -h^{(1)}, z_1, t) \ e^{ik_y d}$$
$$\sigma_{yz}^{(2)} (x_2, h^{(2)}, z_2, t) = \sigma_{yz}^{(1)} (x_1, -h^{(1)}, z_1, t) \ e^{ik_y d} \tag{2.5}$$

where k_y is the Floquet wave number in the y-direction and

$$d = 2(h^{(1)} + h^{(2)})$$

By writing the equations in this form, the stresses and displacements at the interface of two consecutive laminae are automatically satisfied.

The dispersion equation for harmonic-wave propagation is obtained by using Eqs. 2.3 and 2.5. For two dimensional isotropic laminates, Delph, Herrmann, and Kaul [29, 30] used exact solutions to Navier's equations of motion together with continuity conditions and Floquet's theory equation (2.3) and (2.5) to obtain the exact dispersion relationship. For two dimensional anisotropic laminates, Shah and Datta [33] proposed an approximate method by expressing the displacement components in each lamina in terms of interpolation functions and then employed Eqs. 2.3 and 2.5 to obtain the approximate dispersion relationship. For three dimensional analysis, Yamada and Nemat-Nasser [32] proposed an approximate analysis and an approximate method called the new quotient method for waves propagating through a layered orthotropic elastic composite. In this thesis, the approach by Shah and Datta [33] has been adopted and extended to three dimensional analysis. This method is chosen because of its simplicity and accuracy in the two-dimensional analysis.

2.4 DISPLACEMENT EQUATIONS

In this three dimensional analysis to obtain the approximate dispersion equations, the displacements have to be defined first. The displacement components in each lamina are expressed in terms of interpolation functions as

$$\begin{Bmatrix} U_I \\ V_I \\ W_I \end{Bmatrix} = \bar{E} \begin{bmatrix} u_i & u_j & \dfrac{\chi_i}{C_{66}^{(i)}} - \dfrac{\partial v_i}{\partial x} & \dfrac{\chi_j}{C_{66}^{(i)}} - \dfrac{\partial v_j}{\partial x} \\ v_i & v_j & \dfrac{1}{C_{22}^{(i)}}\{\sigma_i - C_{12}^{(i)}\dfrac{\partial u_i}{\partial x} - C_{23}^{(i)}\dfrac{\partial w_i}{\partial z}\} & \dfrac{1}{C_{22}^{(i)}}\{\sigma_j - C_{12}^{(i)}\dfrac{\partial u_j}{\partial x} - C_{23}^{(i)}\dfrac{\partial w_j}{\partial z}\} \\ w_i & w_j & \dfrac{\tau_i}{C_{44}^{(i)}} - \dfrac{\partial v_i}{\partial z} & \dfrac{\tau_j}{C_{44}^{(i)}} - \dfrac{\partial v_j}{\partial z} \end{bmatrix} \begin{Bmatrix} f_1(\eta_i) \\ f_2(\eta_i) \\ f_3(\eta_i) \\ f_4(\eta_i) \end{Bmatrix} \tag{2.6}$$

where $j = i + 1$, $\bar{E} = e^{i(k_x x + k_z z - \omega t)}$

$$f_1(\eta_i) = \frac{1}{4}\,(2 - 3\,\eta_i + \eta_i{}^3)$$

$$f_2(\eta_i) = \frac{1}{4}\,(2 + 3\,\eta_i - \eta_i{}^3)$$

$$f_3(\eta_i) = \frac{h^{(i)}}{4}\,(1 - \eta_i - \eta_i{}^2 + \eta_i{}^3)$$

$f_4(\eta_i) = \dfrac{h^{(i)}}{4}(-1 - \eta_i + \eta_i^2 + \eta_i{}^3)$ in which $\eta_i = y_i/h^{(i)}$ is the natural coordinate

u_i, v_i, w_i, χ_i, σ_i and τ_i are the i^{th} nodal value given by $\{u_i, \chi_i, v_i, \sigma_i, w_i, \tau_i\}^T = \{U_I, \sigma_{yx}^{(i)}, V_I, \sigma_{yy}^{(i)}, W_i, \sigma_{yz}^{(i)}\}^T_{\,y_i = -h_i^{(i)}}$

Similarly, $\{u_j, \chi_j, v_j, \sigma_j, w_j, \tau_j\}^T = \{U_I, \sigma_{yx}^{(i)}, V_I, \sigma_{yy}^{(i)}, W_I, \sigma_{yz}^{(i)}\}_{y_i=h^{(i)}}$

k_x, k_z are the wave numbers in the x and z directions respectively where $k_x = \frac{2\pi}{\Lambda_x}$, $k_z = \frac{2\pi}{\Lambda_z}$. Λ_x, Λ_z, are the wavelengths in the x and z directions, respectively, and ω is the circular frequency. Superscript T denotes a transpose.

The interpolation function $f_m(\eta_i)$ is chosen in order to satisfy the stress and displacement continuity at the interface.

The complete derivation of Eqs. 2.6 is given in Appendix A. In Eqs. 2.6, if the z-component of the displacement is suppressed, the problem is reduced to a two dimensional plane-strain problem similar to that outlined in [33].

2.5 FORMULATION OF ELEMENTAL STIFFNESS AND MASS MATRICES

The potential energy and kinetic energy [34] for the $(i)^{th}$ lamina are

$$V_{pot}^{(i)} = \frac{1}{2}\int_0^{\Lambda_z}\int_0^{\Lambda_x}\int_{-h^{(i)}}^{h^{(i)}} \{\sigma_{xx}\bar{\varepsilon}_{xx} + \sigma_{yy}\bar{\varepsilon}_{yy} + \sigma_{zz}\bar{\varepsilon}_{zz} + \sigma_{yz}\bar{\gamma}_{yz} + \sigma_{xz}\bar{\gamma}_{xz} + \sigma_{xy}\bar{\gamma}_{xy}\}\,dydxdz$$

$$T_{kin}^{(i)} = \frac{\omega^2}{2}\int_0^{\Lambda_z}\int_0^{\Lambda_x}\int_{-h^{(i)}}^{h^{(i)}} \rho^{(i)}\{U_I\bar{U}_I + V_I\bar{V}_I + W_I\bar{W}_I\}\ dydxdz \tag{2.7}$$

where a bar over a quantity designates the complex conjugate.

Using the assumed displacement (2.6) and substituting it into the potential and kinetic energy expression, equations (2.7), lead to the forms given in matrix notation as

$$V_{pot}^{(i)} = \frac{1}{2}\{\bar{r}_i\}^T\ [k_i]\ \{r_i\}$$

$$T_{kin}^{(i)} = \frac{\omega^2}{2}\{\bar{r}_i\}^T\ [m_i]\ \{r_i\} \tag{2.8}$$

where $\{r_i\}^T = \{u_i, \chi_i, v_i, \sigma_i, w_i, \tau_i, u_j, \chi_j, v_j, \sigma_j, w_j, \tau_j\}^T$ and $[k_i]$ and $[m_i]$ are the stiffness and mass matrices of the (i)th lamina.

The long hand expression for $(V_{pot}^{(i)} - T_{kin}^{(i)})$ can be obtained by substituting the stress-strain relationship, which are functions of the nodal values, into Eqs. 2.7. This expression is shown as:

$$
\begin{aligned}
V_{pot}^{(i)} - T_{kin}^{(i)} = \frac{\Lambda_x \Lambda_z}{2} \Big\{ & C_{11}^{(i)} k_x^2 h^{(i)} \int_{-1}^{1} U_I \bar{U}_I \, d\eta_i + C_{12}^{(i)} \, ik_x \int_{-1}^{1} (U_I \bar{V}_I' - V_I' \bar{U}_I) d\eta_i \\
& + C_{13}^{(i)} h^{(i)} k_x k_z \int_{-1}^{1} (W_I \bar{U}_I + U_I \bar{W}_I) d\eta_i + \frac{C_{22}^{(i)}}{h^{(i)}} \int_{-1}^{1} V_I' \bar{V}_I' d\eta_i \\
& + C_{23}^{(i)} ik_z \int_{-1}^{1} (W_I \bar{V}_I' - V_I' \bar{W}_I) d\eta_i \quad + C_{33}^{(i)} k_z^2 h^{(i)} \int_{-1}^{1} W_I \bar{W}_I d\eta_i \\
& + \frac{C_{44}^{(i)}}{h^{(i)}} \int_{-1}^{1} W_I' \bar{W}_I' d\eta_i \quad + C_{44}^{(i)} ik_z \int_{-1}^{1} (V_I \bar{W}_I' - W_I' \bar{V}_I) d\eta_i \\
& + C_{44}^{(i)} k_z^2 h^{(i)} \int_{-1}^{1} V_I \bar{V}_I d\eta_i \quad + C_{55}^{(i)} h^{(i)} k_x^2 \int_{-1}^{1} W_I \bar{W}_I d\eta_i \\
& + C_{55}^{(i)} h^{(i)} k_x k_z \int_{-1}^{1} (W_I \bar{U}_I + U_I \bar{W}_I) d\eta_i \quad + C_{55}^{(i)} k_z^2 \, h^{(i)} \int_{-1}^{1} U_I \bar{U}_I d\eta_i \\
& + C_{66}^{(i)} h^{(i)} k_x^2 \int_{-1}^{1} V_I \bar{V}_I d\eta_i \quad + C_{66}^{(i)} ik_x \int_{-1}^{1} (V_I \bar{U}_I' - U_I' \bar{V}_I) d\eta_i \\
& + \frac{C_{66}^{(i)}}{h^{(i)}} \int_{-1}^{1} U_I' \bar{U}_I' d\eta_i \quad - \rho_i h^{(i)} \omega^2 \int_{-1}^{1} U_I \bar{U}_I d\eta_i \\
& - \rho_i h^{(i)} \omega^2 \int_{-1}^{1} V_I \bar{V}_I d\eta_i - \rho_i h^{(i)} \omega^2 \int_{-1}^{1} W_I \bar{W}_I d\eta_i
\end{aligned}
\tag{2.10}
$$

in which η_i is the natural coordinate.

$$i = \sqrt{-1}$$

and the prime denotes the differentiation with respect to η_i. The evaluations of the integrals of Eq. 2.10 are shown in Appendix B and Appendix C.

Making use of Eqs. 2.8, the expression (2.10) can be rewritten as

$$\begin{aligned}(V_{pot}^{(i)} - T_{kin}^{(i)}) &= \frac{1}{2} \{\bar{r}_i\}^T [[k_i] - \omega^2[m_i]] \{r_i\} \\ &= \frac{1}{2} \{\bar{r}_i\}^T [S^a] \{r_i\} \qquad (2.11)\end{aligned}$$

where the impedance matrix $[S^a]$ is Hermitian.

2.6 GENERATION OF EIGENVALUE AND EIGENVECTOR PROBLEMS

Consider an m-layer periodicity. Floquet's relationship (2.5) can be written as

$$\begin{aligned}&\{u_{m+1}, \chi_{m+1}, v_{m+1}, \sigma_{m+1}, w_{m+1}, \tau_{m+1}, u_{m+2}, \chi_{m+2}, v_{m+2}, \tau_{m+2}, w_{m+2}, \tau_{m+2}\}^T \\ &= \{u_1, \chi_1, v_1, \sigma_1, w_1, \tau_1, u_2, \chi_2, v_2, \sigma_2, w_2, \tau_2\}^T e^{ik_y d}\end{aligned}$$

$$\text{where } d = \sum_{i=1}^{m} 2\, h^{(i)} \qquad (2.12)$$

By applying Hamilton's Principle to (2.11) and then utilizing Floquet's relation (2.12), the equilibrium equations for nodes 1 to m can be written. After rearranging terms, these equations yield the dispersion relation as an algebraic eigenvector problem,

$$[A_3]\ \{R_3\} = \omega^2\ [B_3]\ \{R_3\} \qquad (2.13)$$

where $\{R_3\}^T = \{u_1, \chi_1, v_1, \sigma_1, w_1, \tau_1, u_2, \chi_2 \ldots\ldots w_m, \tau_m\}^T$

and $[A_3]$ and $[B_3]$ are 6 m x 6 m matrices with complex-valued polynomial elements which are functions of material and geometric properties, wave number k_x, k_y and k_z and frequency ω.

The roots of the dispersion relation

$$\det \left| [A_3] - \omega^2 [B_3] \right| \tag{2.14}$$

define a surface in frequency-wave-number space which is generally discontinuous at $k_y = \frac{n\pi}{d}$, n = 1, 2... The assembly process of Eq. 2.14 is outlined in Appendix D using the form of an impedance matrix. A detailed discussion of the disperson surface has been reported by Delph, Harrmann, and Kaul [28, 30]. It has also been shown by Shah and Datta [33] that, for a two dimensional analysis, the numerical results obtained by this finite element method are in close agreement with the exact solution for isotropic laminates. In this study numerical results are presented for three dimensional analysis on boron-aluminium composites and graphite-epoxy composites. Attention is focused on the lowest three branches (SH, SV, and P) of the dispersion curve for real wave number k ($k = \frac{2\pi}{\Lambda}$). The reason for concentrating on the lowest 3 branches is that for a structural dynamic problem, it is the lower frequencies that cause the most severe problems. SH is the lowest out of plane mode, SV is the lowest transverse mode and P is the lowest longitudinal mode propagating in any arbitrary direction through the lamina. SH, SV and P modes of the dispersion curve are defined in the following section.

2.7 ANTIPLANE AND PLANE STRAIN MOTIONS

The derivation of the dispersion relation in this Chapter is for the general three-dimensional case. However, with a simple modification, the dispersion relations for antiplane and plane strain motions can be obtained. The method of obtaining these dispersion relations will be discused in the following.

2.7.1 ANTIPLANE STRAIN

For the antiplane motion, the equation of displacement is given as

$$W_I(x_i, y_i, 0, t) = \{f_1(\eta_i)\, w_i + f_2(\eta_i)\, w_j + f_3(\eta_i)\frac{\tau_i}{C_{44}^{(i)}} + f_4(\eta_i)\frac{\tau_j}{C_{44}^{(i)}}\}\, e^{i(k_x x - \omega t)} \tag{2.15a}$$

where $f_m(\eta_i)$, k_x, ω are defined in Eq. 2.6

$$w_i = W_I(x_i, -h^{(i)}, 0); \quad w_j = W_I\,(x_i, h^{(i)}, 0) \tag{2.15b}$$

$\tau_i = \sigma_{yz}^{(i)}(x_i, -h^{(i)}, 0)$; $\tau_j = \sigma_{yz}^{(i)}\,(x_i, h^{(i)}, 0)$ at a particular time t.

Eq. 2.15b is obtained by substituting the conditions $U_I = V_I = \sigma_{yx}^{(i)} = \sigma_{yy}^{(i)} = 0.0$ into Eq. 2.6. Then, the potential energy and kinetic energy for the (i)th lamina are obtained by integrating over both the lamina thickness and the wavelength Λ.

$$V_{pot}^{(i)} = \frac{1}{2}\int_o^{\Lambda}\int_{-h^{(i)}}^{h^{(i)}} [C_{44}^{(i)}\, \gamma_{yz}^{(i)}\, \bar{\gamma}_{yz}^{(i)} + C_{55}^{(i)}\, \gamma_{xz}^{(i)}\, \bar{\gamma}_{xz}^{(i)}]\, dx_i dy_i$$

$$T_{kin}^{(i)} = \frac{\omega^2}{2}\int_o^{\Lambda}\int_{-h^{(i)}}^{h^{(i)}} \rho^{(i)}\, W_I \bar{W}_I\, dx_i dy_i \tag{2.16}$$

After differentiating the assumed displacement field, its substitution into the potential and kinetic energy expression leads to forms given in matrix notation by

$$V_{pot}^{(i)} = \frac{1}{2}\,\{\bar{r}_1\}^T\,[k_1]\,\{r_1\}$$

$$T_{kin}^{(i)} = \frac{1}{2}\,\{\bar{r}_1\}^T\,[m_1]\,\{r_1\} \tag{2.17}$$

where $\{r_1\}^T = \{w_i, \tau_i, w_j, \tau_j\}^T$

$[k_1]$ and $[m_1]$ are the stiffness and mass matrices of the lamina.

If m-layer periodicity is considered, Floquet's relation Eq. 2.5 can be written as,

$$\{w_{m+1}\ \tau_{m+1}\ w_{m+2}\ \tau_{m+2}\}^T = \{w_1\ \tau_1\ w_2\ \tau_2\}^T\ e^{ik_y d} \tag{2.18}$$

where $d = \sum_{i=1}^{m} 2\ h^{(i)}$

By applying Hamilton's Principle and using Floquet's relation, the equilibrium equations for the nodes can be written. These equations yield the dispersion relation as an algebraic eigenvalue problem as in Eq. 2.13. However, for this case,

$$\{R_1\} = \{w_1,\ \tau_1,\ \ldots.\ w_m,\ \tau_m\}^T$$

and [A], and [B] are 2 m x 2 m matrices.

The lowest frequency obtained by solving the dispersion relation

$$\det \left| [A_1] - \omega^2 [B_1] \right| = 0$$

is the frequency of the lowest SH mode of the propagating waves.

2.7.2 Plane Strain

Similarly, the dispersion relations for plane strain motion can be obtained by substituting the conditions

$W_I = \sigma_{yz}^{(i)} = 0$ into Eq. 2.6

It is given in the form

$$\begin{Bmatrix} U_I \\ \\ V_I \end{Bmatrix} = e^{i(k_x x - \omega t)} \begin{bmatrix} u_i & u_j & \dfrac{\chi_i}{C_{66}^{(i)}} - \dfrac{\partial v_i}{\partial x_i} & \dfrac{\chi_j}{C_{66}^{(i)}} - \dfrac{\partial v_j}{\partial x_i} \\ v_i & v_j & -\dfrac{C_{12}^{(i)}}{C_{22}^{(i)}} \dfrac{\partial u_i}{\partial x_i} + \dfrac{\sigma_i}{C_{22}^{(i)}} & \dfrac{\sigma_j}{C_{22}^{(i)}} - \dfrac{C_{12}^{(i)}}{C_{22}^{(i)}} \dfrac{\partial u_j}{\partial x_i} \end{bmatrix} \begin{Bmatrix} f_1(\eta_i) \\ f_2(\eta_i) \\ f_3(\eta_i) \\ f_4(\eta_i) \end{Bmatrix}$$

where u_i, v_i, χ_i, σ_i are defined in Eq. 2.6

The potential energy and kinetic energy for $(i)^{th}$ lamina become

$$V_{pot}^{(i)} = \frac{1}{2} \int_{o}^{\Lambda} \int_{-h^{(i)}}^{h^{(i)}} [\sigma_{xx}\, \bar{\varepsilon}_{xx}^{(i)} + \sigma_{yy}\, \bar{\varepsilon}_{yy}^{(i)} + \sigma_{xy}\, \bar{\gamma}_{xy}^{(i)}]\; dx_i\; dy_i$$

$$T_{kin}^{(i)} = \frac{\omega^2}{2} \int_{o}^{\Lambda} \int_{-h^{(i)}}^{h^{(i)}} \rho^{(i)}\; [U_I \bar{U}_I + V_I \bar{V}_I]\; dx_i\; dy_i \qquad (2.20)$$

Following the same step as in 2.7.1, after differentiating the assumed displacement field and substituting it into the potential and kinematic energy expression, the form obtained is

$$V_{pot}^{(i)} = \frac{1}{2} \{\bar{r}_2\}^T [k_2] \{r_2\}$$

$$T_{kin}^{(i)} = \frac{1}{2} \{\bar{r}_2\}^T [m_2] \{r_2\} \qquad (2.21)$$

where $\{r_2\}^T = \{u_i, \chi_i, v_i, \sigma_i, u_j, \chi_j, v_j, \sigma_j\}^T$

The $[k_2]$ and $[m_2]$ are the stiffness and mass matrices.

Similarly, by applying Hamilton's Principle and Floquet's relation to a m-layer periodicity lamina, the dispersion relation can be obtained as

$$[A_2]\; \{R_2\} = \omega^2\; [B_2]\; \{R_2\} \qquad (2.22)$$

where $\{R_2\}^T = \{u_1, \chi_1 \; .... \; v_m, \sigma_m\}^T$

and $[A_2]$ and $[B_2]$ are 4 m x 4 m matrices.

When $k_x = 0$, the dispersion equation (2.22) yields 2(2 m x 2 m) equations. One of these equations is the dispersion equation for longitiudinal (P) waves propagating in the direction of propagating waves, while the other is for shear (SV) waves propagating normal to the direction of propagating waves.

The dispersion relation for plane strain and antiplane strain motion, derived above by inputing the proper conditions into Eq. 2.6, is the same as that derived by Shah and Datta [33].

CHAPTER 3

OTHER THEORIES ON WAVE PROPAGATION IN A STRATIFIED MEDIUM

3.1 INTRODUCTION

The practical importance of laminated and fiber reinforced composites has stimulated many analytical studies of these materials. For this kind of laminated medium, Postma [7], Rytov [8] and white and Angona [9] have computed the effective elastic constants on the basis of both static and dynamic consideration. However, it was shown in [11] that this effective modulus theory has limited applicability for wave propagation in practical laminates where ratio of layer stiffness is high. (Shear modulus of reinforced layer/shear modulus of matrix layer = 50) Sun, Achenbach and Herrmann [11, 12, 13] then proposed the effective stiffness theory. These two theories are based on a two-dimensional analysis performed on a stratified medium consisting of alternating plane, parallel layers of two homogeneous isotropic materials.

In this chapter, these two theories will be extended to accommodate a three dimensional analysis on the medium that consists of either isotropic or anisotropic materials. The governing dispersion equations for these two theories will be derived based on Postma's [7] presentation and Sun et al's [11] continuum theory. The results obtained by using these theories will be compared with those produced from the finite element method.

3.2 EFFECTIVE MODULUS METHOD

The customary approach in constructing a theory to describe the mechanical behaviour of a laminated composite consists of replacing the composite by a homogeneous and isotropic medium whose material constants are determined in terms of the geometry and the material properties of the constituents of the composite. Theories of this type are termed "effective modulus theories". In the following discussion, Postma's [7] approach is followed closely but his approach is extended to a three dimensional case with either isotropic or anisotropic laminae.

3.2.1 Stress-Strain Relationship

Fig. 3.1 shows a typical laminated medium. For the purpose of this discussion, a layered structure consisting of alternating plane, parallel layers of materials which can be regarded as anisotropic, is considered. Supposing Hooke's law is valid, it can be expressed in matrix form as

$$\{\sigma\} = [C]\ \{\gamma\} \tag{3.1}$$

where

$$\{\sigma\}^T = < \sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{yz}\ \sigma_{xz}\ \sigma_{xy} >$$

$$\{\gamma\}^T = < \varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ \gamma_{yz}\ \gamma_{xz}\ \gamma_{xy} >$$

σ_{xx} means the normal component of the traction across a surface element perpendicular to the x-axis etc.

σ_{yz} = the tangential component parallel to the y-axis of the traction across a surface element perpendicular to the z-axix, etc.

and ε_{xx} = linear dilatation of line elements in the direction of the x-axis in the unstrained state

γ_{yz} = decrease in angle between two line elements which are parallel to the y and z axes in the unstrained state

[C] is the coefficient matrix consisting of material constants.

It is clear that Eq. 3.1 is the same as Eq. 2.1. If U, V and W are the cartesian componets of the displacement, the strain-displacement relationship shown in Eq. 2.2 also holds for this case.

Consider now a stratified medium consisting of a large number of alternating plane, parallel layers of two homogeneous anisotropic materials for which their elastic constants are ($C_{11}^{(1)}$, $C_{12}^{(1)}$, $C_{13}^{(1)}$, $C_{22}^{(1)}$, $C_{23}^{(1)}$, $C_{33}^{(1)}$, $C_{44}^{(1)}$, $C_{55}^{(1)}$, $C_{66}^{(1)}$); ($C_{11}^{(2)}$, $C_{12}^{(2)}$, $C_{13}^{(2)}$, $C_{22}^{(2)}$, $C_{23}^{(2)}$, $C_{33}^{(2)}$, $C_{44}^{(2)}$, $C_{55}^{(2)}$, $C_{66}^{(2)}$) respectively. The layer thickness and density for the first material is d_1 and $\rho^{(1)}$; for the second it is d_2 and $\rho^{(2)}$. Comparing d_1 and d_2 with that defined in Chapter 2, $d_1 = 2\ h^{(1)}$; $d_2 = 2\ h^{(2)}$. With the coordinate axes shown in Fig. 3.1 and consider an elementary rectangular parallelopiped with faces parallel to the coordinate planes, let the height of the parallelopipe be $n(d_1 + d_2)$ where n is an integer, and let the length and width be L and B respectively. To simplify the problem, the following calculations are based on a unit cell, that is $n = 1$.

On the face perpendicular to the y-axis, a traction σ_{yy} such that there are no tangential components σ_{xy} and σ_{zy} is applied. Similarly, on the face perpendicular to the x-axis, only the normal tractions $\sigma_{xx}^{(1)}$ on the layer d_1 and a normal traction $\sigma_{xx}^{(2)}$ on the layer d_2 are present. On the face perpendicular to the z-axis, there will be normal tractions $\sigma_{zz}^{(1)}$ and $\sigma_{zz}^{(2)}$. The normal tractions $\sigma_{xx}^{(1)}$, $\sigma_{xx}^{(2)}$, $\sigma_{zz}^{(1)}$, $\sigma_{zz}^{(2)}$ are such that $\varepsilon_{xx}^{(1)} = \varepsilon_{xx}^{(2)} = \varepsilon_{xx}$ and $\varepsilon_{zz}^{(1)} = \varepsilon_{zz}^{(2)} = \varepsilon_{zz}$ where $\varepsilon_{xx}^{(1)}$ is the linear dilatation of a line element in the direction of the x-axis in the layers d_1, etc. This restriction holds in order to insure the continuity of the displacement as a function of the radius vector. The linear dilatation of a linear element parallel to the y-axis in the d_1 and d_2 layers are

$\varepsilon_{yy}^{(1)}$ and $\varepsilon_{yy}^{(2)}$ respectively. In general, $\varepsilon_{yy}^{(1)}$ is not equal to $\varepsilon_{yy}^{(2)}$.

For the first anisotropic medium, Hooke's law gives

$$\begin{aligned}
\sigma_{xx}^{(1)} &= C_{11}^{(1)}\,\varepsilon_{xx} + C_{12}^{(1)}\,\varepsilon_{yy}^{(1)} + C_{13}^{(1)}\,\varepsilon_{zz}\\
\sigma_{yy}^{(1)} &= C_{12}^{(1)}\,\varepsilon_{xx} + C_{22}^{(1)}\,\varepsilon_{yy}^{(1)} + C_{23}^{(1)}\,\varepsilon_{zz}\\
\sigma_{zz}^{(1)} &= C_{13}^{(1)}\,\varepsilon_{xx} + C_{23}^{(1)}\,\varepsilon_{yy}^{(1)} + C_{33}^{(1)}\,\varepsilon_{zz}
\end{aligned} \tag{3.2a}$$

Hooke's law for the second anisotropic layer is obtained by replacing the superscript $^{(1)}$ of the above equation by a superscript $^{(2)}$. This set of equations will be designated as (3.2b). If the weighed average tractions on the faces perpendicular to the x-axis and z-axis are considered, they become

$$\begin{aligned}
\sigma_{xx} &= \frac{\sigma_{xx}^{(1)} d_1 + \sigma_{xx}^{(2)} d_2}{d_1 + d_2}\\
\sigma_{zz} &= \frac{\sigma_{zz}^{(1)} d_1 + \sigma_{zz}^{(2)} d_2}{d_1 + d_2}
\end{aligned} \tag{3.3}$$

While the traction on the face perpendicular to the y-axis remains as σ_{yy}.

By substituting Eqs. 3.2a and 3.2b into Eq. 3.3, the equation become:

$$\begin{aligned}
(d_1+d_2)\sigma_{xx} &= [d_1C_{11}^{(1)}+d_2C_{11}^{(2)}]\varepsilon_{xx} + d_1C_{12}^{(1)}\varepsilon_{yy}^{(1)} + d_2C_{12}^{(2)}\varepsilon_{yy}^{(2)} + [d_1C_{13}^{(1)}+d_2C_{13}^{(2)}]\varepsilon_{zz}\\
(d_1+d_2)\sigma_{yy} &= [d_1C_{12}^{(1)}+d_2C_{12}^{(2)}]\varepsilon_{xx} + d_1C_{22}^{(1)}\varepsilon_{yy}^{(1)} + d_2C_{22}^{(2)}\varepsilon_{yy}^{(2)} + [d_1C_{23}^{(1)}+d_2C_{23}^{(2)}]\varepsilon_{zz}\\
(d_1+d_2)\sigma_{zz} &= [d_1C_{13}^{(1)}+d_2C_{13}^{(2)}]\varepsilon_{xx} + d_1C_{23}^{(1)}\varepsilon_{yy}^{(1)} + d_2C_{23}^{(2)}\varepsilon_{yy}^{(2)} + [d_1C_{23}^{(1)}+d_2C_{33}^{(2)}]\varepsilon_{zz}
\end{aligned} \tag{3.4}$$

If ε_{yy} is defined by

$$(d_1 + d_2)\,\varepsilon_{yy} = d_1\,\varepsilon_{yy}^{(1)} + d_2\,\varepsilon_{yy}^{(2)} \tag{3.5}$$

then ε_{yy} is the overall dilatation of a linear element parallel to the y-axis, which contains an equal number of sections through the layers d_1 and d_2.

By substituting Eqs. 3.2a and 3.2b into Eq. 3.5, $\varepsilon_{yy}^{(1)}$ and $\varepsilon_{yy}^{(2)}$ can be expressed as:

$$\varepsilon_{yy}^{(1)} = \frac{[c_{12}^{(2)} - c_{12}^{(1)}]\varepsilon_{xx} + (\bar{d} + 1)c_{22}^{(2)}\ \varepsilon_{yy} + [c_{23}^{(2)} - c_{23}^{(1)}]\varepsilon_{zz}}{[c_{22}^{(1)} + c_{22}^{(2)}\ \bar{d}]}$$

$$\varepsilon_{yy}^{(2)} = \frac{-\bar{d}[c_{12}^{(2)} - c_{12}^{(1)}]\varepsilon_{xx} + (\bar{d} + 1)c_{22}^{(1)}\ \varepsilon_{yy} - \bar{d}[c_{23}^{(2)} - c_{23}^{(1)}]\varepsilon_{zz}}{[c_{22}^{(1)} + c_{22}^{(2)}\ \bar{d}]}$$

in which $\bar{d}$ is defined as $\frac{d_1}{d_2}$ (3.6)

In order to get the relation between the normal stresses and the strains, Eq. 3.6 is substituted into Eq. 3.4 to obtain

$$\begin{aligned} D\ \sigma_{xx} = {} & \{[\bar{d}c_{11}^{(1)} + c_{11}^{(2)}]\ [c_{22}^{(1)} + \bar{d}c_{22}^{(2)}] - \bar{d}\ [c_{12}^{(1)} - c_{12}^{(2)}]^2\}\varepsilon_{xx} \\ & + (\bar{d} + 1)\ [\bar{d}\ c_{12}^{(1)}\ c_{22}^{(2)} + c_{12}^{(2)}\ c_{22}^{(1)}]\varepsilon_{yy} \\ & + \bar{d}\ [c_{12}^{(1)} - c_{12}^{(2)}]\ [c_{23}^{(2)} - c_{23}^{(1)}] + [c_{22}^{(1)} + \bar{d}\ c_{22}^{(2)}][\bar{d}\ c_{13}^{(1)} + c_{13}^{(2)}]\varepsilon_{zz} \end{aligned} \tag{3.7a}$$

$$\begin{aligned} D\ \sigma_{yy} = {} & \{[c_{22}^{(1)}+\bar{d}c_{22}^{(2)}][\bar{d}c_{12}^{(1)}+c_{12}^{(2)}] + \bar{d}[c_{12}^{(2)}-c_{12}^{(1)}][c_{22}^{(1)}-c_{22}^{(2)}]\}\varepsilon_{xx} \\ & + (\bar{d} + 1)^2\ c_{22}^{(1)}\ c_{22}^{(2)}\ \varepsilon_{yy} \\ & + \{\bar{d}\ [c_{22}^{(1)}-c_{22}^{(2)}][c_{23}^{(2)}-c_{23}^{(1)}] + [\bar{d}c_{23}^{(1)}+c_{23}^{(2)}][c_{22}^{(1)}+\bar{d}c_{22}^{(2)}]\}\varepsilon_{zz} \end{aligned} \tag{3.7b}$$

$$\begin{aligned} D\ \sigma_{zz} = {} & \{[\bar{d}c_{13}^{(1)}+c_{13}^{(2)}][c_{22}^{(1)}+\bar{d}c_{22}^{(2)}] + \bar{d}[c_{12}^{(2)}-c_{12}^{(1)}][c_{23}^{(1)}-c_{23}^{(2)}]\}\varepsilon_{xx} \\ & + \{(\bar{d}+1)\ [\bar{d}c_{23}^{(1)}c_{22}^{(2)} + c_{23}^{(2)}c_{22}^{(1)}]\}\varepsilon_{yy} \\ & + \{[\bar{d}c_{33}^{(1)}+c_{33}^{(2)}][c_{22}^{(1)}+\bar{d}c_{22}^{(2)}] - \bar{d}[c_{23}^{(2)}-c_{23}^{(1)}]^2\}\varepsilon_{zz} \end{aligned}$$

where $D = (\bar{d} + 1)\ [C_{22}^{(1)} + \bar{d}\ C_{22}^{(2)}]$ (3.7c)

In the same way, if a tangential traction σ_{xy} is applied to the faces perpendicular to the y-axis as shown in Fig. 3.2, the following is obtained:

$$(d_1 + d_2)\ \gamma_{xy} = d_1\ \gamma_{xy}^{(1)} + d_2\ \gamma_{xy}^{(2)}$$

since $$\sigma_{xy}^{(1)} = C_{66}^{(1)}\ \gamma_{xy}^{(1)} = C_{66}^{(2)} \gamma_{xy}^{(2)}$$

Therefore, σ_{xy} can be expressed as

$$\sigma_{xy} = \frac{(\bar{d} + 1)\ C_{66}^{(1)}\ C_{66}^{(2)}}{\bar{d}\ C_{66}^{(2)} + C_{66}^{(1)}}\ \gamma_{xy} \tag{3.8a}$$

Similarly, for the traction σ_{zy},

$$\sigma_{zy} = \frac{(\bar{d} + 1)\ C_{44}^{(1)}\ C_{44}^{(2)}}{\bar{d}\ C_{44}^{(2)} + C_{44}^{(1)}}\ \gamma_{zy} \tag{3.8b}$$

Finally, by applying a trangential force $\sigma_{zx}^{(1)}\ Ld_1$ to the face perpendicular to the z-axis of the d_1 layer and a tangential force $\sigma_{zx}^{(2)}\ Ld_2$ to the corresponding face of the d_2 layers and based on the fact that $\gamma_{zx}^{(1)}$ must be equal to $\gamma_{zx}^{(2)}$ in order to insure the continuity of the displacement, it is found that $\sigma_{zx}^{(1)} = C_{55}^{(1)}\ \gamma_{zx}$; $\sigma_{zx}^{(2)} = C_{55}^{(2)}\ \gamma_{zx}$. By taking σ_{zx} as the average tangential traction on the parallelopiped, σ_{zx} can be expressed as

$$\sigma_{zx} = \frac{C_{55}^{(1)}\ \bar{d} + C_{55}^{(2)}}{(\bar{d} + 1)}\ \gamma_{zx} \tag{3.8c}$$

3.2.2 Effective Elastic Constants

By comparing these results with Eq. 3.1, the static effective elastic constants of the layered composite are found to be:

$$\bar{c}_{11} = \frac{[\bar{d}c_{11}^{(1)}+c_{11}^{(2)}]\ [c_{22}^{(1)}+\bar{d}c_{22}^{(2)}] - \bar{d}[c_{12}^{(1)} - c_{12}^{(2)}]^2}{(1 + \bar{d})\ [c_{22}^{(1)} + \bar{d}\ c_{22}^{(2)}]}$$

$$\bar{c}_{12} = \frac{[c_{22}^{(1)}c_{12}^{(2)}+\bar{d}c_{12}^{(1)}c_{22}^{(2)}]}{c_{22}^{(1)} + \bar{d}\ c_{22}^{(2)}}$$

$$\bar{c}_{13} = \frac{[\bar{d}c_{13}^{(1)}+c_{13}^{(2)}]\ [c_{22}^{(1)}+\bar{d}c_{22}^{(2)}] + \bar{d}\ [c_{12}^{(1)}-c_{12}^{(2)}]\ [c_{23}^{(2)}-c_{23}^{(1)}]}{(1 + \bar{d})\ [c_{22}^{(1)} + \bar{d}\ c_{22}^{(2)}]}$$

$$\bar{c}_{22} = \frac{(1 + \bar{d})\ c_{22}^{(1)}\ c_{22}^{(2)}}{[c_{22}^{(1)} + \bar{d}\ c_{22}^{(2)}]}$$

$$\bar{c}_{23} = \frac{\bar{d}\ c_{23}^{(1)}\ c_{22}^{(2)} + c_{23}^{(2)}\ c_{22}^{(1)}}{[c_{22}^{(1)} + \bar{d}\ c_{23}^{(2)}]}$$

$$\bar{c}_{33} = \frac{[\bar{d}c_{33}^{(1)}+c_{33}^{(2)}]\ [c_{22}^{(1)}+\bar{d}c_{22}^{(2)}] - \bar{d}\ [c_{23}^{(1)}-c_{23}^{(2)}]^2}{(1 + \bar{d})\ [c_{22}^{(1)} + \bar{d}\ c_{22}^{(2)}]}$$

$$\bar{c}_{44} = \frac{(1+ \bar{d})\ c_{44}^{(1)}\ c_{44}^{(2)}}{\bar{d}\ c_{44}^{(2)} + c_{44}^{(1)}}$$

$$\bar{c}_{55} = \frac{c_{55}^{(1)}\ \bar{d} + c_{55}^{(2)}}{(\bar{d} + 1)}$$

$$\bar{c}_{66} = \frac{(1 + \bar{d})\ c_{66}^{(1)}\ c_{66}^{(2)}}{\bar{d}\ c_{66}^{(2)} + c_{66}^{(1)}} \tag{3.9}$$

For plane strain problems, Eqs. 3.9 reduce to the form presented by Shah and Datta [37] which is similar to that listed in [7]. This supports the derivation of the effective constants for three dimensional analysis.

3.2.3. Displacement Equations of Motion

The equations of motion for the layered medium are defined as:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 U}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 W}{\partial t^2} \tag{3.10}$$

where U, V, and W are the displacement components in the x, y and z direction respectively.

$$\rho = \frac{\rho^{(1)}\,\bar{d} + \rho^{(2)}}{1 + \bar{d}}$$ is the effective mass density

t is the time

From Eqs. 3.1, 3.7, 3.8 and 3.10, the governing field equations of motions are:

$$(\bar{C}_{11}\frac{\partial^2}{\partial x^2}+\bar{C}_{66}\frac{\partial^2}{\partial y^2}+ \bar{C}_{55}\frac{\partial^2}{\partial z^2})\ U + (\bar{C}_{12}+\bar{C}_{66})\frac{\partial^2 V}{\partial x\partial y} + (\bar{C}_{13}+\bar{C}_{55})\frac{\partial^2 W}{\partial x\partial z} = \rho\ \ddot{U}$$

$$(\bar{C}_{66}+\bar{C}_{12})\frac{\partial^2 U}{\partial x\partial y}+(\bar{C}_{66}\frac{\partial}{\partial x^2}+\bar{C}_{22}\frac{\partial^2}{\partial y^2} + \bar{C}_{44}\frac{\partial^2}{\partial z^2})V + (\bar{C}_{23}+\bar{C}_{44})\frac{\partial^2 W}{\partial y\partial z} = \rho\ \ddot{V}$$

$$(\bar{C}_{55}+\bar{C}_{13})\frac{\partial^2 y}{\partial x\partial z} + (\bar{C}_{44}+\bar{C}_{23})\frac{\partial^2 V}{\partial y\partial z} + (\bar{C}_{55}\frac{\partial^2}{\partial x^2}+\bar{C}_{44}\frac{\partial^2}{\partial y^2}+\bar{C}_{33}\frac{\partial^2}{\partial z^2})W = \rho\ \ddot{W}$$

where $\ddot{U} = \dfrac{\partial^2 U}{\partial t^2}$ etc

(3.11)

If the displacement equations are defined as:

$$U = \bar{U}\ e^{ik_x x + ik_y y + ik_z z - i\omega t}$$
$$V = \bar{V}\ e^{ik_x x + ik_y y + ik_z z - i\omega t}$$
$$W = \bar{W}\ e^{ik_x x + ik_y y + ik_z z - i\omega t} \qquad (3.12)$$

where k_x, k_y and k_z are the wave numbers

$\bar{U}$, $\bar{V}$ and $\bar{W}$ are amplitudes

ω is the circular frequency of the propagating wave,

and substituting these into Eq. 3.11, the governing field equations will result in a standard eigenvalue problem which can easily be solved. In matrix form, the equations are:

$$\begin{bmatrix} \bar{C}_{11}k_x^2 + \bar{C}_{66}k_y^2 + \bar{C}_{55}k_z^2 & (\bar{C}_{12} + \bar{C}_{66})k_x k_y & (\bar{C}_{13} + \bar{C}_{55})\ k_x k_z \\ (\bar{C}_{66} + \bar{C}_{12})k_x k_y & \bar{C}_{66}k_x^2 + \bar{C}_{22}k_y^2 + \bar{C}_{44}k_z^2 & (\bar{C}_{23} + \bar{C}_{44}\ k_y k_z) \\ (\bar{C}_{55} + \bar{C}_{13})k_x k_z & (\bar{C}_{23} + \bar{C}_{44})k_y k_z & \bar{C}_{55}k_x^2 + \bar{C}_{44}k_y^2 + \bar{C}_{33}k_z^2 \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{Bmatrix} =$$

$$+\ \omega^2 \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{Bmatrix} \qquad (3.13)$$

In compact form, it is expressed as:

$$[A_1]\ \{x_i\} = \omega^2\ [B_1]\ \{x_i\}$$

where $\{x_i\}^T = \{\bar{U}\ \bar{V}\ \bar{W}\}^T$ (3.14a)

The dispersion relationship similar to Eq. 2.14 is given by

$$\det\ [[A_1] - \omega^2\ [B_1]] = 0 \qquad (3.14b)$$

3.2.4 Antiplane and Plane Strain Motion

Similar to section 2.7, the equations for antiplane and plane strain motions can be obtained by substituting the governing conditions into Eq. 3.13.

For antipane strain motion,

$$\bar{U} = \bar{V} = 0$$

Then, the $[A_1]$ and $[B_1]$ matrices of Eq. 3.14 reduces to m x m matrix for a m-layer periodicity medium

For plane strain motion,

$$\bar{W} = 0$$

Then, the $[A_1]$ and $[B_1]$ matrices of Eq. 3.14 are 2 m x 2 m matrices for an m-layer periodicity medium.

The definition of SH, SV and P mode of propagating wave is similar to section 2.7.

3.3. EFFECTIVE STIFFNESS METHOD

Herrmann and Achenbach [10] have proposed a conceptually different approach to constructing continuum models for the dynamic analysis of directionally reinforced composites. Instead of introducing a representative homogeneous medium by means of "effective moduli", representative elastic moduli are used for the matrix, and the elastic and geometric properties of the reinforcing elements are combined into effective stiffnesses. Certain assumptions are made regarding the deformation of the reinforcing elements. With these assumptions and by applying a smoothing operation, approximate kinetic and strain energy densities for the composite material are obtained. Hamilton's principle is then applied to yield the displacement equations of motion. The displacements for both the reinforcing layers and the matrix layers are expressed as linear expansion about the mid-planes of the layers. In this manner, effective stiffnesses are introduced for both the matrix layers and the reinforcing layers. This theory is distinguished from the effective modulus theory primarily because bending, shear and extensional stiffnesses of the reinforcing elements enter the strain energy density of the laminated medium. In the following discussion, the theory proposed by Herrmann et al [10, 11] will be followed closely but is extended to three-dimensional analysis and includes anisotropic materials.

3.3.1 Kinematics and Smoothing Operation

A stratified medium consisting of a large number of alternating plane, parallel layers of two different materials is considered (Fig. 3.3). For a general case, the following derivation of the equations of motion is based on anisotropic material. Similarly, the material constants and the thicknesses of the stiff reinforcing layers and the soft matrix layers are denoted by (C_{11}^f, C_{12}^f, C_{13}^f, C_{22}^f, C_{23}^f, C_{33}^f, C_{44}^f, C_{55}^f, C_{66}^f, d_f) and (C_{11}^m, C_{12}^m, C_{13}^m, C_{22}^m, C_{23}^m, C_{33}^m, C_{44}^m, C_{55}^m, C_{66}^m, d_m) respectively.

Compared to the notation used in Chapter 2, $C_{ij}^f = C_{ij}^{(i)}$, $d_f = 2\ h^{(i)}$, $C_{ij}^m = C_{ij}^{(i+1)}$, $d_m = 2\ h^{(i+1)}$. The k^{th} pair of reinforcing and matrix layers whose midplane positions are defined by y^{fk} and y^{mk} respectively (Fig. 3.3) are considered. The displacement at the midplane of the reinforcing layer is denoted by

U_o^{fk} (x, y^{fk}, z, t) for component in x direction

V_o^{fk} (x, y^{fk}, z, t) for component in y direction

W_o^{fk} (x, y^{fk}, z, t) for component in z direction

For the displacement at the midplane of the matrix layer, the superscript ^{f} is changed to ^{m}.

At this point, two local coordinate systems are introduced. These two local coordinate systems (x, $\bar{y}^f$, z) and (x, $\bar{y}^m$, z) are redefined with axes parallel to the x, y and z axes but with origins in the mid-planes of the reinforcing layer and the matrix layer respectively (Fig. 3.4).

The displacements in the k^{th} reinforcing layer (U^{fk}, V^{fk}, W^{fk}) and the displacements in the k^{th} matrix layer (U^{mk}, V^{mk}, W^{mk}) are expanded in an infinite series of Legendre polynominals [37]. For the k^{th} reinforcing layer, the displacements are written as:

$$U^{fk} = \sum_{n=o}^{\alpha} p_n \left(\frac{\bar{y}^f}{d_f}\right) U_n^{fk} \ (x, y^{fk}, z, t)$$

$$V^{fk} = \sum_{n=o}^{\alpha} p_n \left(\frac{\bar{y}^f}{d_f}\right) V_n^{fk} \ (x, y^{fk}, z, t)$$

$$W^{fk} = \sum_{n=o}^{\alpha} p_n \left(\frac{\bar{y}^f}{d_f}\right) W_n^{fk} \ (x, y^{fk}, z, t) \qquad (3.15)$$

where $p_n(\frac{\bar{y}^f}{d_f})$ are the Legrendre polynomials:

$$P_o(\frac{\bar{y}^f}{d_f}) = 1, \ P_1(\frac{\bar{y}^f}{d_f}) = \frac{\bar{y}^f}{d_f}, \ p_2(\frac{\bar{y}^f}{d_f}) = [3(\frac{\bar{y}^f}{d_f})^2 - 1]/2 \text{ etc.} \qquad (3.16)$$

n is the number of terms in the polynomial

A similar expansion can be written for the matrix layer by changing superscript ^{f} to ^{m}. Conceptually, a displacement approximation can achieve greater accuracy if a sufficient number of terms in the series is retained. However, with many terms, the manipulation becomes tedius and the approximation does not result in the desired simplication. In this analysis it is, therefore, stipulated that the thicknesses of the layers are sufficiently small as compared to any characteristic length of the deformation such that only linear terms in Eq. 3.16 need be retained. Thus the displacements are:

$$U^{fk} = U_o^{fk}\ (x,\ y^{fk},\ z,\ t) + \bar{y}^f\ \psi_{2x}^{fk}\ (x,\ y^{fk},\ z,\ t)$$
$$V^{fk} = V_o^{fk}\ (x,\ y^{fk},\ z,\ t) + \bar{y}^f\ \psi_{2y}^{fk}\ (x,\ y^{fk},\ z,\ t)$$
$$W^{fk} = W_o^{fk}\ (x,\ y^{fk},\ z,\ t) + \bar{y}^f\ \psi_{2z}^{fk}\ (x,\ y^{fk},\ z,\ t) \tag{3.17}$$

and

$$U^{mk} = U_o^{mk}\ (x,\ y^{mk},\ z,\ t) + \bar{y}^m\ \psi_{2x}^{mk}\ (x,\ y^{mk},\ z,\ t)$$
$$V^{mk} = V_o^{mk}\ (x,\ y^{mk},\ z,\ t) + \bar{y}^m\ \psi_{2y}^{mk}\ (x,\ y^{mk},\ z,\ t)$$
$$W^{mk} = W_o^{mk}\ (x,\ y^{mk},\ z,\ t) + \bar{y}^m\ \psi_{2z}^{mk}\ (x,\ y^{mk},\ z,\ t) \tag{3.18}$$

In Eqs. 3.17

U_o^{fk}, V_o^{fk} and W_o^{fk} represents the displacements at the median plane of the k^{th} reinforcing layer

ψ_{2x}^{fk} and ψ_{2z}^{fk} are the antisymmetric thickness shear deformations

ψ_{2y}^{fk} represents the symmetric thickness stretch deformation

Based on compatibility, that is, the displacements at the interface of the k^{th} reinforcing and matrix layers are continuous, the following relation is obtained:

$$U_o^{mk}\ (x,\ y^{mk},\ z,\ t) - U_o^{fk}\ (x,\ y^{fk},\ z,\ t)$$
$$= \frac{1}{2}\ d_f\ \psi_{2x}^{fk}\ (x,\ y^{fk},\ z,\ t) + \frac{1}{2}\ d_m\ \psi_{2x}^{mk}\ (x,\ y^{mk},\ z,\ t) \tag{3.19}$$

for compatibility along the y and z axes, U_o^{mk}, U_o^{fk}, ψ_{2x}^{fk}, ψ_{2x}^{mk}, are changed into V_o^{mk}, V_o^{fk}, ψ_{2y}^{fk}, ψ_{2y}^{mk} and W_o^{mk}, W_o^{fk}, ψ_{2z}^{fk}, ψ_{2z}^{mk} respectively.

From the displacement equations, the strain energy stored in elements of unit surface area of the k^{th} pair of reinforcing and matrix layers can be computed.

In an anisotropic body, the strain energy density, in the condensed form, can be written as:

$$V_{pot} = \frac{1}{2} \int_{vol} \{\sigma_{ij}\}^T\ \{\varepsilon_{ij}\} dV \qquad i, j = x,\ y,\ \text{or}\ z \tag{3.20}$$

where ε_{ij} is the strain tensor and summation implied.

The strain-displacement relationship, Eqs. 2.2, still holds for this case.

For the k^{th} reinforcing layer, the components of the strain tensor can be approximated by substituting the assumed displacement components, Eqs. 3.17 into Eqs. 2.2. It should be noted that the differentiation in the y-direction should be with respect to the local coordinate of the reinforcing layer $\bar{y}^f$. With this operation, the components of the strain tensor are found to be:

$$\varepsilon_{xx}^{fk} = \frac{\partial}{\partial x} U_o^{fk} + \bar{y}^f \frac{\partial}{\partial x} \psi_{2x}^{fk} \tag{3.21a}$$

$$\varepsilon_{yy}^{fk} = \frac{\partial V}{\partial \bar{y}^f} + \psi_{2y}^{fk} \tag{3.31b}$$

$$\varepsilon_{zz}^{fk} = \frac{\partial}{\partial z} W_o^{fk} + \bar{y}^f \frac{\partial}{\partial z} \psi_{2z}^{fk} \tag{3.21c}$$

$$\varepsilon_{xy}^{fk} = \varepsilon_{yx}^{fk} = \frac{1}{2} (\psi_{2x}^{fk} + \frac{\partial}{\partial x} V_o^{fk} + \bar{y}^f \frac{\partial}{\partial x} \psi_{2y}^{fk}) \tag{3.21d}$$

$$\varepsilon_{zy}^{fk} = \varepsilon_{yz}^{fk} = \frac{1}{2} (\psi_{2z}^{fk} + \frac{\partial}{\partial z} V_o^{fk} + \bar{y}^f \frac{\partial}{\partial z} \psi_{2y}^{fk}) \tag{3.21e}$$

$$\varepsilon_{xz}^{fk} = \varepsilon_{zx}^{fk} = \frac{1}{2} (\frac{\partial}{\partial z} U_o^{fk} + \bar{y}^f \frac{\partial}{\partial z} \psi_{2x}^{fk} + \frac{\partial}{\partial x} W_o^{fk} + \bar{y}^f \frac{\partial}{\partial x} \psi_{2z}^{fk}) \tag{3.21f}$$

The corresponding expressions for the strain components in the k^{th} matrix layer can be obtained by changing the superscript f in Eqs. 3.21a-3.21f to m. For easier and clearer presentation, the following notation is adopted.

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}$$

$$\partial_{13} = \frac{\partial}{\partial x \partial z}, \quad \partial_{23} = \frac{\partial}{\partial y \partial z}, \text{ etc.}$$

Substituting relation (2.1) and expression (3.21a-f) into the strain energy equation, (Eq. 3.20) and integrating over the thickness d_f, the strain energy stored per unit surface area of the k^{th} reinforcing layer is found to be:

$$V_{pot}^{fk} \text{ / unit surface area} = \frac{1}{2} \int_{-d_f/2}^{d_f/2} \{\sigma_{ij}\}^T \{\varepsilon_{ij}\} d\bar{y}^f$$

$$= \frac{1}{2} C_{11}^{fk} \{d_f(\partial_1 U_o^{fk})^2 + \frac{1}{12} d_f^3 (\partial_1 \psi_{2x}^{fk})^2\} + C_{12}^{fk} d_f \psi_{2y}^{fk} (\partial_1 U_o^{fk})$$

$$+ C_{13}^{fk} \{d_f(\partial_1 U_o^{fk})(\partial_3 W_o^{fk}) + \frac{1}{12} d_f^3 (\partial_1 \psi_{2x}^{fk})(\partial_3 \psi_{2z}^{fk})\}$$

$$+ \frac{1}{2} C_{22}^{fk} d_f (\psi_{2y}^{fk})^2 + C_{23}^{f} d_f \psi_{2y}^{fk} (\partial_3 W_o^{fk})$$

$$+ \frac{1}{2} C_{33}^{fk} \{d_f(\partial_3 W_o^{fk})^2 + \frac{1}{12} d_f^3 (\partial_3 \psi_{2z}^{fk})^2\}$$

$$+ \frac{1}{2} C_{44}^{fk} \{d_f(\psi_{2z}^{fk} + \partial_3 V_o^{fk})^2 + \frac{1}{12} d_f^3 (\partial_3 \psi_{2y}^{fk})^2\}$$

$$+ \frac{1}{2} C_{55}^{fk} \{d_f (\partial_3 U_o^{fk} + \partial_1 W_o^{fk})^2 + \frac{1}{12} d_f^3 (\partial_3 \psi_{2x}^{fk} + \partial_1 \psi_{2z}^{fk})^2\}$$

$$+ \frac{1}{2} C_{66}^{fk} \{d_f (\psi_{2x}^{fk} + \partial_1 V_o^{fk})^2 + \frac{1}{12} d_f^3 (\partial_1 \psi_{2y}^{fk})^2\}] \qquad (3.22)$$

Similar computation can be carried out to derive the expression for the strain energy stored in an element of unit surface area of the kth matrix layer. This expression, V_{pot}^{mk}, can be written by replacing in Eq. 3.22 subscripts and supercripts f by subscripts and superscripts m.

After the formulation of the strain energy density, the next step will be the formulation of the kinetic energy density. The kinetic energy of a continuum is defined by

$$T_{kin} = \frac{1}{2} \int_{vol} \rho\{\dot{U}^2 + \dot{V}^2 + \dot{W}^2\} dV_{vol} \qquad (3.23)$$

Therefore, the kinetic energy per unit surface area of the kth reinforcing layer is obtained as:

$$T_{kin/unit\ area}^{fk} = \frac{1}{2} \rho_f d_f (\dot{U}_o^{fk})^2 + \frac{1}{24} d_f^3 \rho_f (\dot{\psi}_{2x}^{fk})^2$$

$$+ \frac{1}{2} \rho_f d_f (\dot{V}_o^{fk})^2 + \frac{1}{24} d_f^3 \rho_f (\dot{\psi}_{2y}^{fk})^2$$

$$+ \frac{1}{2} \rho_f d_f (\dot{W}_o^{fk})^2 + \frac{1}{24} d_f^3 \rho_f (\dot{\psi}_{2z}^{fk})^2 \qquad (3.24)$$

where ρ_f is the mass density of the reinforcing material.

If $$\eta = \frac{d_f}{(d_f + d_m)} \quad \text{and} \quad I_f = \frac{1}{12} d_f^2 \, \eta \, \rho_f, \tag{3.25}$$

Eq. 3.24 can be expressed as

$$\begin{aligned} T_{kin}^{fk} = \frac{1}{2} (d_f + d_m) \{ & \eta\rho_f (\dot{U}_o^{fk})^2 + I_f \, (\dot{\psi}_{2x}^{fk})^2 \\ & + \eta\rho_f \, (\dot{V}_o^{fk})^2 + I_f \, (\dot{\psi}_{2y}^{fk})^2 \\ & + \eta\rho_f \, (\dot{W}_o^{ft})^2 + I_f \, (\dot{\psi}_{2z}^{fk})^2 \} \end{aligned} \tag{3.26}$$

For the k^{th} matrix layer, the kinetic energy per unit surface area is obtained by replacing in Eq. 3.24 superscripts and subscripts f by m. This equation can then be expressed in a similar form as Eq. 3.26

$$\begin{aligned} T_{kin}^{mk} = \frac{1}{2} (d_f + d_m) \{ & (1-\eta) \, \rho_m \, (\dot{U}_o^{mk})^2 + I_m \, (\dot{\psi}_{2x}^{mk})^2 \\ & + (1-\eta) \, \rho_m \, (\dot{V}_o^{mk})^2 + I_m \, (\dot{\psi}_{2y}^{mk})^2 \\ & + (1-\eta) \, \rho_m \, (\dot{W}_o^{mk})^2 + I_m \, (\dot{\psi}_{2z}^{mk})^2 \} \end{aligned} \tag{3.27}$$

where $I_m = \frac{1}{12} d_m^2 \, (1-\eta) \, \rho_m$

Assuming that within a certain height H in the y-direction, there are n reinforcing layers and n matrix layers, then the total strain and kinetic energies stored in a rectangular parallelpiped of sides H, unity and unity, of the laminated medium are given by

$$(V_{pot})_H = \sum_{i=1}^{n} (V_{pot}^{fk} + V_{pot}^{mk})_i \tag{3.28}$$

and $$(T_{kin})_H = \sum_{i=1}^{n} (T_{kin}^{fk} + T_{kin}^{mk})_i \quad \text{respectively} \tag{3.29}$$

Based on the basic premise of the effective stiffness theory, the summation over 2n discrete points y^{fk} and y^{mk} may be approximated by a weighted integration over y thus,

$$(V_{pot})_H = \sum_{i=1}^{n} (V_{pot}^{fk} + V_{pot}^{mk})_i \simeq \int_H \frac{1}{d_f + d_m} \, (V_{pot}^f + V_{pot}^m) \, dy \tag{3.30}$$

$$(T_{kin})_H = \sum_{i=1}^{n} (T_{kin}^{fk} + T_{kin}^{mk})_i \simeq \int_H \frac{1}{d_f+d_m} (T_{kin}^f + T_{kin}^m)\, dy \qquad (3.31)$$

The superscript k has been removed on the right-hand sides of Eqs.3.30 and 3.31 to indicate that V_{pot}^f, V_{pot}^m, T_{kin}^f and T_{kin}^m are now defined for all y. By means of this "smoothing operation" the layered medium has been replaced by a homogeneous continuum whose strain and kinetic energy densities are functions of x, y, z and t, i.e.

$$V_{pot} = \frac{1}{d_f+d_m} [V_{pot}^f\ (x, y, z, t) + V_{pot}^m\ (x, y, z, t)] \qquad (3.32)$$

$$T_{kin} = \frac{1}{d_f+d_m} [T_{kin}^f(x, y, z, t) + T_{kin}^m\ (x, y, z, t)] \qquad (3.33)$$

The field variables (U_o^{fk}, V_o^{fk}, W_o^{fk}, U_o^{mk}, V_o^{mk}, W_o^{mk}, ψ_{2x}^{fk}, ψ_{2x}^{mk}, ψ_{2y}^{fk}, ψ_{2y}^{mk}, ψ_{2z}^{fk}, ψ_{2z}^{mk}) which are thus far defined only on certain discrete parallel planes, $y = y^{fk}$ and $y = y^{mk}$, have now become a function of y, and hence, the superscript k can be deleted.

Now, the state of deformation in the laminated medium can be described by twelve field variables (U_o^f, V_o^f, W_o^f, U_o^m, V_o^m, W_o^m, ψ_{2x}^f, ψ_{2x}^m, ψ_{2y}^f, ψ_{2y}^m, ψ_{2z}^f, ψ_{2z}^m) which are a function of x, y, z and t. However, by observing that U_o^f, V_o^f, W_o^f, and U_o^m, V_o^m, W_o^m should be considered as representing the same quantity, namely, the "gross displacements", the number of twelve field variables is reduced to nine. The gross displacements are then referred to as U, V and W. (ψ_{2x}^f, ψ_{2x}^m, ψ_{2y}^f, ψ_{2y}^m, ψ_{2z}^f, ψ_{2z}^m) describe the "local deformations" in the reinforcing layers and matrix layers. Therefore, they do not represent the same magnitude. The local deformations and the gradients of the gross displacements are related, however, by conditions of continuity at the interfaces of the layers. In view of Eq. 3.19, the gross displacements and the local deformations can be written as

$$U(x,y^{mk},z,t) - U(x,y^{fk},z,t) = \frac{1}{2} d_f\psi^f_{2x}(x,y^{fk},z,t) + \frac{1}{2} d_m\psi^m_{2x}(x,y^{mk},z,t)$$

$$V(x,y^{mk},z,t) - V(x,y^{fk},z,t) = \frac{1}{2} d_f\psi^f_{2y}(x,y^{fk},z,t) + \frac{1}{2} d_m\psi^m_{2y}(x,y^{mk},z,t)$$

$$W(x,y^{mk},z,t) - W(x,y^{fk},z,t) = \frac{1}{2} d_f\psi^f_{2z}(x,y^{fk},z,t) + \frac{1}{2} d_m\psi^m_{2z}(x,y^{mk},z,t) \qquad (3.34)$$

Noting that $y^{mk} = y^{fk} + \frac{1}{2}(d_m+d_f)$ (3.35)

and assuming that the thicknesses of the layers are sufficiently small, the difference relation, Eq. 3.34, can be replaced by a differential relation between the local deformations and the gradients of the gross displacements:

i.e. $$\frac{\partial}{\partial y} U(x,y,z,t) = \eta\, \psi^f_{2x}(x,y,z,t) + (1-\eta)\, \psi^m_{2x}(x,y,z,t)$$

$$\frac{\partial}{\partial y} V(x,y,z,t) = \eta\, \psi^f_{2y}(x,y,z,t) + (1-\eta)\, \psi^m_{2y}(x,y,z,t)$$

$$\frac{\partial}{\partial y} W(x,y,z,t) = \eta\, \psi^f_{2z}(x,y,z,t) + (1-\eta)\, \psi^m_{2z}(x,y,z,t) \qquad (3.36)$$

The continuity condition Eq. 3.19 has thus been generalized to hold at any point in the continuum. In deriving Eq. 3.36, the differences between the local deformation at y^{fk} and y^{mk} are neglected because the local deformations (ψ^f_{2x}, ψ^m_{2x}, ψ^f_{2y}, ψ^m_{2y}, ψ^f_{2z}, ψ^m_{2z}) had originally been defined in the domains of the reinforcing layers and the matrix layers.

In view of Eqs. 3.22 and 3.32, the approximate strain energy function of the laminated medium may now be written in terms of the derivatives of the gross displacements and of the local deformations and their derivatives. It is expressed as

$$\begin{aligned} V_{pot} = {} & \frac{1}{2}\{\eta\, C^f_{11}+(1-\eta)C^m_{11}\}(\partial_1 U)^2 + \frac{1}{24}\eta\, d_f^2 C^f_{11}(\partial_1\psi^f_{2x})^2 + \eta\, C^f_{12}\psi^f_{22}(\partial_1 U) \\ & + \{\eta\, C^f_{13}+(1-\eta)C^m_{13}\}(\partial_1 U)(\partial_3 W) + \frac{1}{12}\eta\, d_f^2 C^f_{13}(\partial_1\psi^f_{2x})(\partial_3\psi^f_{2z}) \\ & + \frac{1}{2}\eta\, C^f_{22}(\psi^f_{22})^2 + \eta\, C^f_{23}\psi^f_{22}(\partial_3 W) + \frac{1}{2}\{\eta C^f_{33}+(1-\eta)C^m_{33}\}(\partial_3 W)^2 \\ & + \frac{1}{24}\eta\, d_f^2 C^f_{33}(\partial_3\psi^f_{2z})^2 + \frac{1}{2}\eta\, C^f_{44}\{\partial_3 V+\psi^f_{2z}\}^2 + \frac{1}{24}\eta d_f^2 C^f_{44}(\partial_3\psi^f_{2y})^2 \end{aligned}$$

$$+ \frac{1}{2} \{\eta C^f_{55}+(1-\eta) C^m_{55}\} [\partial_3 U+\partial_1 W]^2 + \frac{1}{24} \eta d^2_f C^f_{55} [\partial_3 \psi^f_{2x}+\partial_1 \psi^f_{2z}]^2$$

$$+ \frac{1}{2} \eta C^f_{66} [\partial_1 V+\psi^f_{2x}]^2 + \frac{1}{24} \eta d^2_f C^f_{66} (\partial_1 \psi^f_{2y}]^2$$

$$+ \frac{1}{24} (1-\eta) d^2_m C^m_{11} (\partial_1 \psi^m_{2x})^2 + (1-\eta) C^m_{12} \psi^m_{22} (\partial_1 U)$$

$$+ \frac{1}{12} (1-\eta) d^2_m C^m_{13} (\partial_1 \psi^m_{2x})(\partial_3 \psi^m_{2z}) + \frac{1}{2} (1-\eta) C^m_{22} (\psi^m_{2y})^2$$

$$+ (1-\eta) C^m_{22} \psi^m_{2z} (\partial_3 W) + \frac{1}{24} (1-\eta) d^2_m C^m_{33} (\partial_3 \psi^m_{2z})^2$$

$$+ \frac{1}{2} (1-\eta) C^m_{44} [\partial_3 V+\psi^m_{2z}]^2 + \frac{1}{24} (1-\eta) d^2_m C^m_{44} (\partial_3 \psi^m_{2y})^2$$

$$+ \frac{1}{24} (1-\eta) d^2_m C^m_{55} [\partial_3 \psi^m_{2x}+\partial_1 \psi^m_{2z}]^2 + \frac{1}{2} (1-\eta) C^m_{66} [\partial_1 V+\psi^m_{2x}]^2$$

$$+ \frac{1}{24} (1-\eta) d^2_m C^m_{66} (\partial_1 \psi^m_{2y})^2 \tag{3.37}$$

Similarly, the approximate kinetic energy can be obtained from Eqs. 3.26, 3.27 and 3.37 as

$$T_{kin} = \frac{1}{2} \rho_c \dot{U}^2 + \frac{1}{2} I_f (\dot{\psi}^f_{2x})^2 + \frac{1}{2} I_m (\dot{\psi}^m_{2x})^2$$

$$+ \frac{1}{2} \rho_c \dot{V}^2 + \frac{1}{2} I_f (\dot{\psi}^f_{2y})^2 + \frac{1}{2} I_m (\dot{\psi}^m_{2y})^2$$

$$+ \frac{1}{2} \rho_c \dot{W}^2 + \frac{1}{2} I_f (\dot{\psi}^f_{2z})^2 + \frac{1}{2} I_m (\dot{\psi}^m_{2z})^2$$

in which

$$\rho_c = \eta \rho_f + (1-\eta) \rho_m$$

$$I_f = \frac{1}{12} d^2_f \eta \rho_f$$

$$I_m = \frac{1}{12} d^2_m (1-\eta) \rho_m \tag{3.38}$$

3.3.2 Displacement Equations of Motion

Thus far, only the strain and kinetic energy densities of the medium have been derived. The next step will be the derivation of the displacement equations of motion. Consider a fixed regular region R of the laminated medium. Hamilton's principle for independent variations of the dependent field quantities in R at time t_o and t_1 may be written as

$$\delta \int_{t_o}^{t_1} (T^R_{kin} - V^R_{pot})\, dt + \int_{t_o}^{t_1} \delta W_1 dt = 0 \tag{3.39}$$

where δW_1 represents the variation of the work done by body forces and T^R_{kin} and V^R_{pot} are the total kinetic and strain energies.

i.e. $V^R_{pot} = \int_R V_{pot} dR$

$T^R_{kin} = \int_R T_{kin} dR$ in which dR denotes the scalar volume element.

In the absence of body forces, the variational problem then reduces to finding the Euler equation for

$$\delta \int_{t_o}^{t_1} \int_R (T_{kin} - V_{pot}) dt\, dR = 0 \tag{3.40}$$

Examination of Eq. 3.40 reveal that there are nine field variables namely $U, V, W, \psi^f_{2x}, \psi^m_{2x}, \psi^f_{2y}, \psi^m_{2y}, \psi^f_{2z}, \psi^m_{2z}$. These nine field variables can be further reduced to six by eliminating $\psi^m_{2x}, \psi^m_{2y}, \psi^m_{2z}$ by means of continuity equations 3.34. However, in order to be consistent with reference [10], a more elegant method which is to introduce the continuity conditions as subsidiary conditions through Lagrangian multiplier [34] is followed. The variational problem may be redefined as

$$\delta \int_{t_o}^{t_1} \int_R (T_{kin} - V_{pot} - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3) dt\, dR = 0 \tag{3.41}$$

where the Lagrangian multipliers λ_1, λ_2 and λ_3 are functions of x, y, z and t and from Eq. 3.36,

$$S_1 = \frac{\partial}{\partial y} U - \eta\, \psi^f_{2x} - (1-\eta)\, \psi^m_{2x}$$

$$S_2 = \frac{\partial}{\partial y} V - \eta\, \psi^f_{2y} - (1-\eta)\, \psi^m_{2y}$$

$$S_3 \quad \frac{\partial}{\partial y} W - \eta\, \psi^f_{2z} - (1-\eta)\, \psi^m_{2z} \tag{3.42}$$

Since $(T_{kin} - V_{pot} - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3)$ depends only on the dependent field variables and their first order derivatives, the system of Euler equations may be written as:

$$\sum_{r=1}^{4} \frac{\partial}{\partial q_r} \left[\frac{\partial(T_k - V_p - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3)}{\partial(\partial f_s/\partial q_r)}\right] - \frac{\partial(T_k - V_p - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3)}{\partial f_s} = 0 \qquad (3.43)$$

In Eq. 3.43, f_s represent the twelve dependent variables (U, V, W, ψ^f_{2x}, ψ^m_{2x}, ψ^f_{2y}, ψ^m_{2y}, ψ^f_{2z}, ψ^m_{2z}, λ_1, λ_2 and λ_3); q_r is the spatial variables x, y, z and the time, variable t.

Substitution of the strain energy density, Eq. 3.37, and the kinetic energy density, Eq. 3.38, into Eq. 3.43 will yield a system of twelve equations. A typical computation is shown in Appendix E.

Defining the quantities

$$Q_{\ell n} = \eta\, C^f_{\ell n} + (1-\eta)\, C^m_{\ell n}, \qquad n = 1, 2 \ldots 6$$
$$\ell = 1, 2 \ldots 6$$

and using the notation

$$\partial_{ij}\, U_k = \partial^2 U_k / \partial x_i \partial x_j$$

where $\partial x_1 = \partial x, \quad \partial x_2 = \partial y, \quad \partial x_3 = \partial z$

$U_1 = U \qquad U_2 = V \qquad U_3 = W,$

the twelve equations are expressed as

$$Q_{11}\partial_{11}U + Q_{55}\partial_{33}U + [Q_{55} + Q_{13}]\,\partial_{13}W + \eta\, C^f_{12}\partial_1\psi^f_{2y} + (1-\eta)C^m_{12}\partial_1\psi^m_{2y} + \partial_2\lambda_1 = \rho_c\, \ddot{U} \qquad (3.44)$$

$$Q_{66}\partial_{11}V + Q_{44}\partial_{33}V + \eta C^f_{66}\partial_1\psi^f_{2x} + \eta C^f_{44}\partial_3\psi^f_{2z} + (1-\eta)\,[C^m_{66}\partial_1\psi^m_{2x} + C^m_{44}\partial_3\psi^m_{2z}] + \partial_2\lambda_2 = \rho_c\, \ddot{V} \qquad (3.45)$$

$$Q_{33}\partial_{33}W + Q_{55}\partial_{11}W + [Q_{55} + Q_{13}]\,\partial_{13}U + \eta\, C^f_{23}\partial_3\psi^f_{2y} + (1-\eta)C^m_{23}\partial_3\psi^m_{2y} + \partial_2\lambda_3 = \rho_c\, \ddot{W} \qquad (3.46)$$

$$\frac{1}{12}\,\eta\, d_f^2\, [C_{11}^f \partial_{11}\psi_{2x}^f + C_{13}^f \partial_{13}\psi_{2z}^f + C_{55}^f \partial_{13}\psi_{2z}^f + C_{55}^f \partial_{33}\psi_{2x}^f]$$

$$- \eta\, C_{66}^f \partial_1 V - \eta\, C_{66}^f \psi_{2x}^f + \eta\, \lambda_1 = I_f\, \ddot{\psi}_{2x}^f \qquad (3.47)$$

$$\frac{1}{12}(1-\eta) d_m^2 [C_{11}^m \partial_{11}\psi_{2x}^m + C_{13}^m \partial_{13}\psi_{2z}^m + C_{55}^m \partial_{13}\psi_{2z}^m + C_{55}^m \partial_{33}\psi_{2x}^m]$$

$$- (1-\eta) C_{66}^m \partial_1 V - (1-\eta) C_{66}^m \psi_{2x}^m + (1-\eta)\lambda_1 = I_m\, \ddot{\psi}_{2x}^m \qquad (3.48)$$

$$\frac{1}{12}\,\eta\, d_f^2\, [C_{33}^f \partial_{33}\psi_{2z}^f + C_{13}^f \partial_{13}\psi_{2x}^f + C_{55}^f \partial_{13}\psi_{2x}^f + C_{55}^f \partial_{11}\psi_{2z}^f]$$

$$- \eta\, C_{44}^f \partial_3 V - \eta\, C_{44}^f \psi_{2z}^f + \eta\, \lambda_3 = I_f\, \ddot{\psi}_{2z}^f \qquad (3.49)$$

$$\frac{1}{12}(1-\eta) d_m^2\, [C_{33}^m \partial_{33}\psi_{2z}^f + C_{13}^m \partial_{13}\psi_{2x}^m + C_{55}^m \partial_{13}\psi_{2x}^m + C_{55}^m \partial_{11}\psi_{2z}^f]$$

$$- (1-\eta) C_{44}^m \partial_3 V - (1-\eta) C_{44}^m \psi_{2z}^m + (1-\eta)\, \lambda_3 = I_m\, \ddot{\psi}_{2z}^m \qquad (3.50$$

$$\frac{1}{12}\,\eta\, d_f^2\, [C_{66}^f \partial_{11}\psi_{2y}^f + C_{44}^f \partial_{33}\psi_{2y}^f] - \eta\, C_{12}^f \partial_1 U - \eta\, C_{23}^f \partial_3 W$$

$$- \eta\, C_{22}^f \psi_{2y}^f + \eta\, \lambda_2 = I_f\, \ddot{\psi}_{2y}^f \qquad (3.51)$$

$$\frac{1}{12}\,(1-\eta)\; d_m^2\; [C_{66}^m \partial_{11}\psi_{2y}^m + C_{44}^m \partial_{33}\psi_{2y}^m] - (1-\eta)\; C_{12}^m \partial_1 U - (1-\eta)\; C_{23}^m \partial_3 W$$

$$- (1-\eta)\; C_{22}^m \psi_{2y}^m + (1-\eta)\; \lambda_2 = I_m\, \ddot{\psi}_{2y}^m \qquad (3.52)$$

$$\partial_2 U - \eta\, \psi_{2x}^f - (1-\eta)\, \psi_{2x}^m = 0 \qquad (3.53)$$

$$\partial_2 V - \eta\, \psi_{2y}^f - (1-\eta)\, \psi_{2y}^m = 0 \qquad (3.54)$$

$$\partial_2 W - \eta\, \psi_{2z}^f - (1-\eta)\, \psi_{2z}^m = 0 \qquad (3.55)$$

3.3.3 Propagation of Plane Harmonic Waves

The displacement equations of motion, Eqs. 3.44-3.55 can be used to study the propagation of plane harmonic waves in an arbitrary direction. Assuming that the variables of Eq. 3.44-3.55 have the form,

$$\bar{A}\, e^{ik_x x + ik_y y + ik_z z - ikct} \qquad (3.56)$$

where $\bar{A}$ is a constant amplitude

k_x, k_y, k_z are the wave numbers in the x, y and z direction respectively,

c is the phase velocity of the propagating wave and

$\omega = kc$ = angular frequencies

Eqs. 3.44 to 3.55 can be expressed as eigenvalue-eigenvector problems.

$$[A]\ \{x\} - \omega^2\ [B]\ \{x\} = 0 \qquad (3.57)$$

where $\{x\}^T = \langle U, V, W, \psi^f_{2x}, \psi^m_{2x}, \psi^f_{2z}, \psi^m_{2z}, \psi^f_{2y}, \psi^m_{2y}, \lambda_1, \lambda_2, \lambda_3 \rangle$

Eq. 3.57 can be expressed as

$$[S^P]\ \{x\} = [[A] - \omega^2[B]]\ \{x\} = 0$$

The explicit form of $[S^P]$ for this problem is given in Appendix F. Similarly, the roots of this dispersion relation

$$\det\ [[A] - \omega^2[B]]$$

define a surface in phase-velocity vs wave-number space. A detailed discussion of these dispersion curves has been reported by Sun, Achenbach and Herrmann [11]. Their results using this effective stiffness method will be compared with those obtained using finite element method in Chapter 4.

3.3.4 Antiplane and Plane Strain Motion

Case 1: Governing equation of propagation of longitudinal waves (P waves) in the direction of the layering on x-y plane.

For waves of this type, the field variables are of the form

$$(U, \psi^f_{2y}, \psi^m_{2y}, \lambda_2) = (\bar{A}, \bar{A}^f_{2y}, \bar{A}^m_{2y}, \bar{B}_2)\ \exp\ [ik(x-ct)] \qquad (3.58)$$

All other field variables vanish identically. The substitution of this governing condition into the displacement equations of motion, Eqs. 3.44-3.55 yields a system of four homogeneous equations for $\bar{A}_1$, $\bar{A}^f_{2y}$, $\bar{A}^m_{2y}$ and

$\bar{B}_2$. The dispersion equation is obtained by requiring that the determinant of the coefficients vanishes. The non-dimensional form of the dispersion equation obtained is given as:

$$[(1-\eta) + \eta\theta]^2 \xi^2 \beta^4 - \{[(1-\eta) + \eta\theta]\,[(1-\eta) + \eta\,\delta^f_{66} + \eta\,\delta^f_{11} + (1-\eta)\,\delta^m_{11}]\xi^2$$
$$+ 12\eta\,[(1-\eta) + \eta\theta]\,[\delta^f_{22} + \eta\,\delta^m_{22}/(1-\eta)]\}\beta^2$$
$$+\{[\eta\,\delta^f_{11} + (1-\eta)\delta^m_{11}]\,[(1-\eta) + \eta\,\delta^f_{66}]\xi^2$$
$$+ 12\eta\,[\delta^f_{22} + \eta\delta^m_{22}/(1-\eta)]\,[\eta\,\delta^f_{11} + (1-\eta)\,\delta^m_{11}] - 12\eta^2\,[\delta^f_{12} - \delta^m_{12}]^2\} = 0 \tag{3.59}$$

where $\beta = c/(C^m_{66}/\rho_m)^{1/2}$ is the dimensionless phase velocity

$\xi = kd_f$ is the dimensionless wave number

$\theta = \rho_f/\rho_m$ is the ratio of mass densities

and $\delta^f_{ij} = C^f_{ij}/C^m_{66}$

$\delta^m_{ij} = C^m_{ij}/C^m_{66}$

If the materials of both the reinforced and matrix layers are isotropic, Eq. 3.59 degenerate to Eq. 63 of Ref. [11] in which $\delta^f_{66} = \gamma = \mu_f/\mu_m$. γ is the ratio of shear moduli of the reinforced layer, μ_f and the matrix layer, μ_m.

Case 2: Governing equation of propagation of SV waves in the direction of the layering on x-y plane.

For a shear wave, the nonvanishing field variables are V, ψ^f_{2x}, ψ^m_{2x} and λ_1. That is

$$(V, \psi^f_{2x}, \psi^m_{2x}, \lambda_1) = (\bar{A}_2, \bar{A}^f_{21}, \bar{A}^m_{21}, \bar{B}_1) \exp[ik(x - ct)] \tag{3.60}$$

Following the previously described procedure, the dispersion equation corresponding to the system of solutions (3.60) in nondimensional form is given as:

$$[(1-\eta) + \eta\theta]^2\xi^2\beta^4 - \{[(1-\eta) + \eta\theta]\ [(1-\eta) + \eta\delta^f_{66} + \eta\delta^f_{11} + (1-\eta)\delta^m_{11}]\xi^2$$
$$+ 12\ [(1-\eta) + \eta\theta]\ [(1-\eta)\delta^f_{66} + \eta]\ \eta/(1-\eta)\}\beta^2$$
$$+ \{[(1-\eta) + \eta\delta^f_{66}]\ [\eta\delta^f_{11} + (1-\eta)\delta^m_{1}]\xi^2$$
$$+ 12\eta\ \delta^f_{66}/(1-\eta)\} = 0 \tag{3.61}$$

Eq. 3.61 wll degenerate to Eq. 72 of Ref. [11] if the materials of the reinforced and matrix layers are isotropic.

Case 3: Governing equation of propagation of SH waves in the direction of the layering.

For SH waves, the displacements are parallel to the z-direction while the waves propagate in the x-direction. The nonvanishing field variables are of the form

$$(W, \psi^f_{2z}, \psi^m_{2z}, \lambda_3) = (\bar{A}_3, \bar{A}^f_{23}, \bar{A}^m_{23}, \bar{B}_3)\ \exp\ [ik(x-ct)] \tag{3.62}$$

Substituting of Eq. 3.62 into Eqs. 3.44 - 3.55 will yield two uncoupled systems of equations, governing symmetric and antisymmetric motion, respectively. For symmetric motion, the field quantity is represented by

$$W = \bar{A}_3\ \exp\ [ik\ (x-ct)] \tag{3.63}$$

for which the constant phase velocity is obtained as

$$\beta = \{\frac{\eta\delta^f_{55} + (1-\eta)\delta^m_{55}}{[(1-\eta) + \eta\theta]}\}^{1/2} \tag{3.64}$$

The antisymmetric system is represented by the following solutions:

$$(\psi^f_{2z}, \psi^m_{2z}, \lambda_3) = (\bar{A}^f_{23}, \bar{A}^m_{23}, \bar{B}_3)\ \exp\ [ik\ (x-ct)] \tag{3.65}$$

Then, the nondimensional form of the dispersion equation obtained is given as:

$$[(1-\eta)+\eta\theta]\xi^2\beta^2 - \{[(1-\eta)\delta^m_{55} + \eta\delta^f_{55}]\xi^2 + 12\eta\ [\delta^f_{44} + \eta\delta^m_{44}/(1-\eta)] = 0 \tag{3.66}$$

Eq. 3.64 and Eq. 3.66 are the same as Eq. 75 and Eq. 77 respectively of Ref. [11] if the material properties of the reinforced and matrix layers are isotropic.

For all the cases mentioned above, the dispersion equation can be expressed in matrix form as in Eq. 3.57. But [A] and [B] are reduced to 4n x 4n for a n-layer periodicity.

For a wave propagating normal to the layering, the dispersion equation can be obtained in a similar procedure by substituting the governing displacement conditions into Eq. 3.44-3.55. It will not be discussed here. Ref. [11] provides a good discussion on the procedure.

CHAPTER 4

NUMERICAL RESULTS AND DISCUSSION

4.1 INTRODUCTION

In this Chapter, the numerical results of several examples are presented and discussed. For a two dimensional analysis of harmonic waves propagating in a periodically layered, isotropic, infinite, elastic body, Shah and Datta [33] have shown that the present finite element method gives excellent results when compared with the exact analysis proposed by Delph, Herrmann and Kaul [28, 29, 30]. For a three dimensional analysis on the same problem but using anisotropic material, Ref. [32] would be a good reference to determine the accuracy of the present analytical method outlined in Chapter 2. However, at time of writing this thesis, the detailed numerical results of Ref. [32] were not received. Therefore, comparison with their work cannot be carried out. In order to gauge the accuracy and range of applicability of this present approach, a two-layer periodic isotropic laminated medium is considered. The numerical results obtained by using this proposed finite element method are compared with those obtained by using the effective modulus method [Chapter3.2] and the effective stiffness method [Chapter 3.3].

The second example presented considers a wave propagating through a two-layer, infinite elastic body with anisotropic material properties. The composite chosen is that of boron-fiber reinforced layers sandwiched between thin layers of aluminum. The numerical results are first compared with that obtained by extended effective modulus method and the extended effective stiffness method. Then the behaviour of the dispersion curves for waves propagating through this media are discussed individually.

In the third example, the material considered is graphite-epoxy composite. There are two types of graphite-epoxy composites considered. A formula, outlined in Appendix G, is used to obtain their respective material constants.

For all the examples mentioned above, only a two-layer periodic medium is considered. In order to obtain better results, each layer within the unit cell is subdivided into "lamina" at mid-plane (Fig. 2.1), thus creating a four-layer periodicity. Higher layered periodicity can be created by further subdividing the lamina. However, it is shown in [33] that the numerical results for lower modes of wave propagation do not appreciably change by increasing the number of lamina. Therefore, only the results of four layered periodicity are presented here.

The numerical results presented in this Chapter are obtained from FORTRAN programs written for the extended effective modulus method, extended effective stiffness method and the finite element method. These programs are included in Appendices H, I and J.

4.2 ISOTROPIC LAMINATED MEDIUM

The dispersion curves for waves propagating through an isotropic laminated medium using the effective modulus method, effective stiffness method and the elasticity theory have been presented by Sun, Anchebach and Herrmann [11, 12]. However, their presentations are based on two dimensional analysis. For an isotropic material, the numerical results obtained by using a three dimensional analysis and those obtained using a two dimensional analysis should be identical.

In this example, the problem is analyzed using the same material properties and parameters given in Ref. [11]. The material constants are tabulated in Table 4.1. The non-dimensional parameters used are defined as:

$\gamma = \dfrac{\mu_1}{\mu_2}$ ratio of shear moduli

μ_1 = shear modulus of reinforced layer

μ_2 = shear modulus of matrix layer

$\beta = \dfrac{c}{(\mu_2/\rho^{(2)})^{\frac{1}{2}}}$ dimensionless phase velocity

c = phase velocity of propagating wave

$\rho^{(2)}$ = density of matrix layer

$\xi = k\ (2\ h^{(1)})$ dimensionless wave number

$2\ h^{(i)}$ = thickness of reinforced layer

k = wave number

$= \dfrac{2\pi}{\Lambda}$

Λ = wave length of propagating wave

$\theta = \rho^{(1)}/\rho^{(2)}$ ratio of mass densities

$\rho^{(1)}$ = density of reinforced layer

To be consistent with the results presented in Ref. [11], the numerical computations are carried out for three values of γ, namely, γ = 100, 50 and 10, and for $\eta = 0.8$, $\theta = 3$. η is shown in Eq. 3.25. The Poisson ratio of the reinforced layer and the matrix layer are 0.3 and 0.35, respectively.

Based on these material properties and parameters and using the effective modulus and effective stiffness method, the curves presented in Ref. [11] are reproduced. The results obtained by using the finite element method based on the same material properties and parameters are then plotted on the graphs. These graphs are shown in Fig. 4.1 to Fig. 4.4. It is found that the results using the finite element method coincide with the results obtained by using theory of elasticity for γ = 10, 50 and 100. This proves the validity of the proposed finite element method and with this observation, the proposed method can be applied to the analysis of harmonic waves propagating in anisotropic composites, with confidence.

4.3 FIBER-REINFORCED BORON-ALUMINIUM COMPOSITE

Datta and Ledbetter [39] have computed the elastic constants for the fiber-reinforced boron-aluminium composite. This composite is a heat resistance material and is used mainly for aerospace structures such as space shuttles. Shah and Datta then used the same elastic constants and geometric parameters to perform a plane-strain analysis to derive the dispersion characteristics of harmonic waves propagating in the composite [33]. For analyticl purpose, the composite is modelled as a two-layer medium. The first layer is the boron-fiber reinforced layer with anisotropic material properties while the second layer is the thin layer of aluminium. In the following discussion, the method outlined in Chapter 2 is used to determine the dispersion characteristics in this two-layer medium. The material constants for the fiber reinforced layer are slightly different (about 6% smaller) than those presented in Ref. [39], and are listed in Table 4.2. The reason for the slight difference in values is that more experimental works have been carried out to determine the properties of the composite. The new set of material constants, provided by Datta, are presumably a better representation of the composite. The parameters used in the discussion are:

$\Omega = \dfrac{\omega}{\omega_s}$ — Dimensionless frequency of waves

$$\omega_s = \frac{\pi}{2h^{(2)}} \cdot \sqrt{\frac{\mu_2}{\rho^{(2)}}}$$

ω = frequency of waves

$\bar{\eta} = \dfrac{2h^{(2)}}{\pi} k_y$ — Dimensionless y-component of wave number

k_y = y-component of wave number

$\bar{k} = \dfrac{2h^{(2)}}{\pi} k$ — Dimensionless wave number

k = wave number = $\dfrac{2\pi}{\Lambda}$

$\bar{d} = \dfrac{2h^{(1)}}{2h^{(2)}}$ — Ratio of thickness of reinforced layer and the thickness of matrix layer

α — Horizontal angle made by propagating wave on x-z plane with respect to x-axis

ϕ — Vertical angle made by propagating wave with respect to x-z plane

For comparison with the effective modulus and the effective stiffness method, a β vs ξ plot for waves propagating parallel and normal to the layering is first considered. $\bar{d}$ is taken to be 12.

For waves propagating parallel to the layering and along the global x-axis ($\alpha = 0°$, $\phi = 0°$), the results are presented in Fig. 4.5. It is shown that for small wave numbers ($\xi < 2$), the quasi-longitudinal mode (P-wave) is in agreement with the effective modulus and the effective stiffness method. As the wave number increases, the results obtained using finite element method departs substantially from those obtained by using the other two methods. This dispersive behaviour is similar to that observed for the isotropic case presented in Fig. 4.3. It has been summarized in Ref. [24] that for large γ (γ = 50, 100), there should be a rapid decrease of the phase veolocity of the lowest longitudinal mode (P-wave). This is because there is a shift of the participation in the motion of both reinforcing and matrix layers to the matrix layer only. For both isotropic and anisotropic materials, only the finite element method shows this behaviour (Fig. 4.3 and 4.5). As for the quasi-transverse mode (SV-wave), there is no remarkable difference between the three methods.

For waves propagating normal to the layering, ($\alpha = 0°$, $\phi = 90°$), the results are presented in Fig. 4.6. In this figure, there is a remarkable variation between the three methods, expecially for higher wave number. For isotropic material, Ref [11] shows that the effective stiffness method and effective modulus method is comparable to the exact method only for small value of ξ ($\xi < 1$) for both P-waves and SV-waves. The results are shown in Fig. 4.4. However, the results obtained using finite element method still coincides with that obtained from the theory of elasticity. For plane-strain analysis of a laminated boron-aluminium composite, Ref. [37] concluded that the dispersive behaviour of wave propagating normal to the layering agrees with experimental observations. This varifies the validity of the finite element method. However, due to the lack of numerical results in the three dimensional analysis of harmonic waves propagating through anisotropic media, Fig. 4.6 can only be shown but can not be comapred with those obtained using the theory of elasticity.

For the rest of the graphical presentation in this section, the numerical results will be presented using the material properties in Table 4.2 but the frequency of the propagating wave is determined, instead of the phase velocity. Ω, the dimensionless frequency, is obtained by normalizing the roots of the dispersion relation in Eq. 2.13.

Figs. 4.7 to 4.18 show the dispersion curves for the lowest longitudinal (P), lowest transverse (SV) and the lowest symmetric SH mode propagating in the x-y, y-z and x-z plane. It can be concluded that for small magnitude of $\bar{k}$ where $\bar{k}$ is the nondimensional wave number, say $\bar{k} \leq 0.07$, the frequency, Ω, for the three modes is directly proportional to $\bar{k}$. This agrees with the actual behaviour as stated in Ref. [40]. One disadvantage of this Ω vs $\bar{k}$ plot is that for $\bar{k} = 0$, the phase velocity of the waves cannot be determined. Comparing the 3-sets of graphs, namely Figs. 4.7-4.10 for waves on the x-y plane, Figs. 4.11-4.14 for waves on the y-z plane and Figs. 4.15-4.18 for waves on the x-z planes, it is observed that the waves are less dispersed when they travel along the x-z plane. This is due to the fact that they are travelling along the matrix layer which is the homogeneous isotropic layer. Greater dispersion is observed for the waves travelling along the x-y and y-z planes because for these two planes, the reinforcing layer and the matrix layer are involved. The dispersion is the greatest for waves propagating normal to the layering ($\alpha = 0°$, $\phi = 90°$ or $\alpha = 90°$, $\phi = 90°$) due to the maximum participation of both layers in vibrating the particles.

Figs. 4.19-4.27 are plots showing the variation of frequencies with respect to the directions of progagation on the plane for a particular $\bar{k}$. Similar observations, as described above, are obtained.

Instead of examining the variation in frequencies with respect to angular changes in the direction of propagating waves, the computer program listed in Appendix J is capable of handling the analysis by treating the y-component of the nondimensional wave number, $\bar{\eta}$ as a known quantity. These Ω vs. $\bar{\eta}$ plots are presented in Figs. 4.28 to 4.32. The SH mode can be easily determined from the output. However, the longitudinal and transverse modes cannot be distinguished. Therefore, the lowest five branches, regardless of modes, are plotted. It is observed that for large values of $\bar{k}$ (0.5, 1.0) there is very little variation in the frequencies as $\bar{\eta}$ changes correspondingly. This is due to the fact that, for small $\bar{\eta}$ and large magnitudes in $\bar{k}$, the waves are travelling along the matrix layer only.

Similar plots are carried out for the Ω vs $\bar{k}$ plot for a different $\bar{\eta}$ and at a horizontal angle of 0° and 45°. The results of the lowest 5 branches are shown in Figs. 4.33 to 4.38. Only the lowest out-of plane, (SH) mode can be identified.

4.4 GRAPHITE-EPOXY FIBER REINFORCED COMPOSITE

In this section, the behaviour of dispersion curves for waves propagating through two types of graphite-epoxy composites will be considered. Graphite-epoxy composite has high strength and high electrical resistance and it can be used as thermal insulation in nuclear power plants. Since the material constants change when the percentage of fibre in the composite differ, a formula is provided by Datta [42], and is listed in Appendix G, to calculate the material constants for the two types of composites used in this analysis. The value of $\bar{c}$ in the formula is used as a control. If $\bar{c}$ is different, other material constants for that type of graphite epoxy composite can be calculated. The following discussion will be based on two different values of $\bar{c}$, namely, when $\bar{c} = 0.3$ and $\bar{c} = 0.668$. Similar to the second example, the composite is modelled to be a two layer medium. The first layer is the graphite fiber reinforced layer while the second layer is the thin layer of epoxy.

4.4.1 Graphite-Epoxy Composite (i)

For $\bar{c} = 0.3$, the material constants are calculated according to the formula in Appendix G. For convenience, these values have been tabulated in Table 4.3. In this section, two different ratios of thicknesses between the reinforcing layer and the matrix layer are considered. The two ratios are

$$\bar{d} = 4.0 \text{ and } \bar{d} = 9.0$$

The parameters used in Section 4.3 are used here for graphical presentation. For $\bar{d} = 4.0$, the frequencies are proportional to $\bar{k}$ from 0 to as far as 0.09 which is equivalent to a wave length of 44.4 $h^{(2)}$ (Fig. 4.39-4.42). In these figures, attempts to differentiate the propagating waves into SH, SV or P modes are not carried out. Therefore, only the lowest 3 branches, irrespective of modes, are plotted. For the waves travelling on each plane, only one representative direction of propagation will be considered. For example, when waves are propagating on the x-y plane, the direction of propagation is taken to be 45° with respect to the x-axis [$\alpha = 0°$, $\phi = 45°$]. This is shown in Fig. 4.39. In Fig. 4.42, the arbitrary direction of the propagating waves through the medium is chosen to be at $\alpha = 45°$, $\phi = 45°$.

Fig. 4.43 ($\alpha = 0°$), Fig. 4.44 ($\alpha = 45°$) and Fig. 4.45 ($\alpha = 90°$) shows the lowest 3 branches plot for the variation of frequencies with respect to the value of $\bar{\eta}$ for a constant value of $\bar{k}$. It is shown in these three graphs that for $\bar{k} = 1.0$, the frequencies of the lowest three branches are independent of $\bar{\eta}$. This is due to the fact that the waves are propagating along the epoxy layer which is the isotropic layer.

Fig. 4.46 ($\alpha = 45°$) shows the characteristic variation of Ω with respect to $\bar{k}$ for this graphite-epoxy composite.

For $\bar{d} = 9.0$, Figs. 4.47 to 4.50 show that Ω is proportional to $\bar{k}$ if $\bar{k}$ is smaller than 0.04. This shows that when the thickness of the reinforcing layer increases, greater dispersion of the propagating waves is observed.

Figs. 4.51 to 4.54 are similar plots to Figs. 4.43 to 4.46 for $\bar{d} = 9$. Similar behaviour is observed.

4.4.2 Graphite-Epoxy Composite (ii)

For $\bar{c} = 0.668$, the material constants are calculated and tabulated in Table 4.4. Two ratios of thicknesses $\bar{d} = 4.0$ and $\bar{d} = 9.0$ are considered. The parameters used in the graphical presentations are similar to that of Section 4.4.1. Figs. 4.55 to 4.70 plotted using the same parameters as in Figs. 4.39 to 4.54 correspondingly. The general behaviour of the dispersion curves are the same as in Section 4.4.1. The only difference is that the waves are travelling with a higher frequency.

4.5 COMPARISON OF THE THREE METHODS:

(Effective Modulus, Effective Stiffness and Finite Element Method.)

In analyzing for the first branch of the frequency spectrum using three layered tranversely isotropic sphere, Fyre [41] compared his results using the finite element method, with that obtained by the Shell theory. He showed that for a soft middle layer, the shell theory produce much higher values than the finite element method. To verify this point, the same material properties he used have been reproduced here. They are listed in Table 4.5. Three types of matrix layer will be used. The first type is a very soft layer whose material constants are about ten times smaller than the reinforcing layer. It is designated as 2-i. The material constants, for the second type designated as 2-ii, are about half of that of the reinforcing layer. For the third type, 2-iii, it is of the same material as the reinforcing layer. Only two-layer periodic medium similar to those in Chapter 2 and Chapter 3 is considered here.

$\bar{d}$ is taken to be 0.6 and the periodicity is equal to 4 for all three cases. The results of the dispersion relation using the effective modulus [Chapter 3.2], effective stiffness [Chapter 3.3] and finite element [Chapter 2] method are obtained with the aid of the computer programs in Appendix H, I and J. The phase velocity, β, vs wave number, ξ, plots of the results are shown in Figs. 4.71 and 4.72.

Fig. 4.71 shows the SV mode for waves propagating along the x-axis. It is shown that when using matrix layer 2-i, there is a large discrepency between the three methods. The effective stiffness method produces a higher value than the other two methods, even for a small wave number. This discrepency decreases when matrix layer 2-ii is used. For matrix layer 2-iii, there is no difference in the results obtained by the three methods. This is because for the effective modulus and effective stiffness method, the dispersion relations are formulated by applying an averaging or smoothing operation through the media. This operation has been discussed in Chapter 3. As the difference between the layer stiffnesses increases, it is understandable that the first two approximate methods will yield a result that is farther away from the exact value. Finite element method is subdividing the media into smaller laminae without averaging the material properties of the laminae. Interpolation functions (Eq. 2.6) are used to ensure compatibility of the laminae. Therefore, it is reasonable to assume that it should yield a better approximation to the exact value. In Ref. [33], it is shown that for isotropic material, the present finite element method is in agreement with the results obtained using the theory of elasticity. Therefore, it can be concluded that the finite element method should be used instead of the other two methods if the matrix layer is too soft when compared with the reinforced layer ($\gamma > 10$).

Fig. 4.72 shows a similar plot but for the SH mode. The same conclusion can be deduced.

4.6 CONCLUSION

A finite element method has been presented in this thesis for the study of harmonic waves propagating in a layered composite medium. This approximate method utilizes the interpolation functions, satisfying the continuity of displacements and tractions at the interfaces of a periodically laminated composite medium, and Floquet's theory. The formulation of the dispersion relations is based on a three-dimensional analysis of the harmonic waves propagating in a periodically layered, anisotropic, infinite body. However, the method can be readily applied to any two-dimensional analysis. Two-dimensional analysis is applicable only to harmonic waves travelling in isotropic or transversely anisotropic media where the plane-strain and antiplane strain motions are not coupled. However, the composites for practical applications can seldom be modified to be transversely anisotropic materials. Therefore, the proposed three-dimensional analysis should be used. To assess the accuracy of this proposed method, comparison to the effective modulus and the stiffness method were carried out. It is shown that the finite element yields better approximations than the other two methods (Fig. 4.1 to Fig. 4.6). For two-dimensional analysis on isotropic materials, the present method yields excellent results when compared with that obtained by using the theory of elasticity (Fig. 4.1 to Fig. 4.4).

The dispersive hehaviour of harmonic waves propagating through fiber-reinforced boron-aluminium composite and graphite epoxy composite are also presented (Fig. 4.7 to 4.70). Both composites are modelled as a layer of anisotropic reinforced material and a layer of isotropic matrix material. Ref. [37] has shown that for the plane-strain analysis of a wave propagating in fiber-reinforced boron-aluminium composite, the analytical results obtained from the present finite element formulation agree with the experimental observations.

At present an analytical solution for waves propagating through an anisotropic laminate is not available. Therefore, an approximate method (finite element method) to solve such problems is proposed. Due to the lack of numerical results in the same research area in literature, it is hoped that the results presented in this thesis can serve as a bench mark for further approximate theories.

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A P P E N D I X A

DERIVATION OF THE INTERPOLATION FUNCTION

APPENDIX A

By using third degree approximation and interpolation, the functions U_I, V_I and W_I can be derived. This is done in order to satisfy the stress and strain continuity.

Using the interpolation functions, U_I, V_I and W_I can be expressed as:

$$U_I = f_1(\eta_i)u_1 + f_2(\eta_i)u_2 + f_3(\eta_i)u_3 + f_4(\eta_i)u_4$$

$$V_I = f_1(\eta_i)v_1 + f_2(\eta_i)v_2 + f_3(\eta_i)v_3 + f_4(\eta_i)v_4$$

$$W_I = f_1(\eta_i)w_1 + f_2(\eta_i)w_2 + f_3(\eta_i)w_3 + f_4(\eta_i)w_4 \quad \text{(A-1)}$$

where $f_1(\eta_i) = \frac{1}{4}(2-3\eta_i+\eta_i^3)$

$$f_2(\eta_i) = \frac{1}{4}(2-3\eta_i-\eta_i^3)$$

$$f_3(\eta_i) = \frac{h^{(i)}}{4}(1-\eta_i-\eta_i^2+\eta_i^3)$$

$$f_4(\eta_i) = \frac{h^{(i)}}{4}(-1-\eta_i+\eta_i^2+\eta_i^3)$$

in which η_i is the vertical local coordinates with the property

$$-h^{(i)} \leqslant y_{(i)} \leqslant h^{(i)}; \; -1 \leqslant \eta_i \leqslant 1. \text{ or } \eta_i = \frac{y_i}{h^{(i)}}.$$

u, v and w are function of x and z and are the nodal displacements of the i, i+1 node [Fig. 2.1].

The deriviation of $f_1(\eta_i)$, $f_2(\eta_i)$, $f_3(\eta_i)$, and $f_4(\eta_i)$, is given in mos finite element textbooks. It is also derived in Ref. [43].

For a particular lamina, the relevant stress-strain relations are

$$\begin{Bmatrix} \sigma_{xx}^{(i)} \\ \sigma_{yy}^{(i)} \\ \sigma_{zz}^{(i)} \end{Bmatrix} = \begin{bmatrix} C_{11}^{(i)} & C_{12}^{(i)} & C_{13}^{(i)} \\ C_{12}^{(i)} & C_{22}^{(i)} & C_{23}^{(i)} \\ C_{13}^{(i)} & C_{23}^{(i)} & C_{33}^{(i)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(i)} \\ \varepsilon_{yy}^{(i)} \\ \varepsilon_{zz}^{(i)} \end{Bmatrix}$$

$$\sigma_{yz}^{(i)} = C_{44}^{(i)} \gamma_{yz}^{(i)}$$

$$\sigma_{xz}^{(i)} = C_{55}^{(i)} \gamma_{xz}^{(i)}$$

$$\sigma_{xy}^{(i)} = C_{66}^{(i)} \gamma_{xy}^{(i)}$$

where $C_{ij}^{(i)}$ is the elastic constant of the $(i)^{th}$ layer

$\varepsilon_{kk}^{(i)}$ is the normal strain in k-k plane

$\gamma_{mn}^{(i)}$ is the shear strain in m-n plane.

It is known from the theory of linear elasticity that

$$\varepsilon_{xx}^{(i)} = \frac{\partial U_I}{\partial x} \quad ; \quad \gamma_{xy}^{(i)} = \frac{\partial V_I}{\partial x} + \frac{\partial U_I}{\partial y}$$

$$\varepsilon_{yy}^{(i)} = \frac{\partial V_I}{\partial y} \quad ; \quad \gamma_{yz}^{(i)} = \frac{\partial W_I}{\partial y} + \frac{\partial V_I}{\partial z}$$

$$\varepsilon_{zz}^{(i)} = \frac{\partial W_I}{\partial z} \quad ; \quad \gamma_{zx}^{(i)} = \frac{\partial U_I}{\partial z} + \frac{\partial W_I}{\partial x}$$

let $\sigma_{yy}^{(i)}\Big|_{\eta=-1} = \sigma_i$;

$\sigma_{yx}^{(i)}\Big|_{\eta=-1} = \chi_i$;

and $\sigma_{yz}^{(i)}\Big|_{\eta=-1} = \tau_i$

Then $\sigma_{yy}^{(i)} = C_{12}^{(i)} \frac{\partial U_I}{\partial x} + C_{22}^{(i)} \frac{\partial V_I}{\partial y} + C_{23}^{(i)} \frac{\partial W_I}{\partial z}$

$$\sigma_{yx}^{(i)} = C_{66}^{(i)} \left(\frac{\partial U_I}{\partial z} + \frac{\partial W_I}{\partial x}\right)$$

$$\sigma_{yz}^{(i)} = C_{44}^{(i)} \left(\frac{\partial W_I}{\partial y} + \frac{\partial V_I}{\partial z}\right)$$

In order to satisfy the stress continuity at the interface between two consecutive layers, the evaluation of σ_{yx}, σ_{yz} and σ_{yy} at the boundary nodes is required. With reference to Fig. 2.1, the boundary nodes are at $\eta_i = -1$ and $\eta_i = +1$.

For the I (superscript (i)) lamina and the i node

$$\sigma_{yx}^{(i)} \Big|_{\eta_i = -1} = C_{66}^{(i)} \left[f_3'(-1)\, u_3 + \frac{\partial v_1}{\partial x}\right]$$

$$= C_{66}^{(i)} \left[u_3 + \frac{\partial v_1}{\partial x}\right] = \chi_i$$

thus $$u_3 = \left(\frac{\chi_i}{C_{66}^{(i)}} - \frac{\partial v_1}{\partial x}\right)$$

$$\sigma_{yx}^{(i)} \Big|_{\eta_i = +1} = C_{66}^{(i)} \left[u_4 + \frac{\partial v_2}{\partial x}\right] = \chi_{i+1}$$

thus $$u_4 = \left(\frac{\chi_{i+1}}{C_{66}^{(i)}} - \frac{\partial v_2}{\partial x}\right)$$

Similarly,

$$\sigma_{yy}^{(i)} \Big|_{\eta_i = -1} = C_{12}^{(i)} \frac{\partial u_1}{\partial x} + C_{22}^{(i)} v_3 + C_{23}^{(i)} \frac{\partial w_1}{\partial z} = \sigma_i$$

thus, $$v_3 = \frac{1}{C_{22}^{(i)}} \left[\sigma_i - C_{12}^{(i)} \frac{\partial u_1}{\partial x} - C_{23}^{(i)} \frac{\partial w_1}{\partial z}\right]$$

$$\sigma_{yy}^{(i)} \Big|_{\eta_i=+1} = C_{12}^{(i)} \frac{\partial u_2}{\partial x} + C_{22}^{(i)} v_4 + C_{23}^{(i)} \frac{\partial w_2}{\partial z} = \sigma_{i+1}$$

$$v_4 = \frac{1}{C_{22}^{(i)}} [\sigma_{i+1} - C_{12}^{(i)} \frac{\partial u_2}{\partial x} - C_{23}^{(i)} \frac{\partial w_2}{\partial z}]$$

and
$$\sigma_{yz}^{(i)} \Big|_{\eta_i=-1} = C_{44}^{(i)} (w_3 + \frac{\partial v_1}{\partial z}) = \tau_i$$

thus, $w_3 = (\frac{\tau_i}{C_{44}^{(i)}} - \frac{\partial v_1}{\partial z})$

$$\sigma_{yz}^{(i)} \Big|_{\eta_i=+1} = C_{44}^{(i)} (w_4 + \frac{\partial v_2}{\partial z}) = \tau_{i+1}$$

$$w_4 = (\frac{\tau_{i+1}}{C_{44}^{(i)}} - \frac{\partial v_2}{\partial z})$$

Realizing that subscript 2 above is refering to the i+1 node and by letting $i+1 = j$, the components of displacement equations become

$$U_I = f_1(\eta_i) u_i + f_2(\eta_i)u_j + f_3(\eta_i)[\frac{\chi_i}{C_{66}^{(i)}} - \frac{\partial v_i}{\partial x}] \qquad + f_4(\eta_i)[\frac{\chi_j}{C_{66}^{(i)}} - \frac{\partial v_j}{\partial x}]$$

$$V_I = f_1(\eta_i) v_i + f_2(\eta_i) v_j + f_3(\eta_i)[\frac{1}{C_{22}^{(i)}}(\sigma_i - C_{12}^{(i)} \frac{\partial u_i}{\partial x} - C_{23}^{(i)} \frac{\partial w_i}{\partial z})] + f_4(\eta_i)[\frac{1}{C_{22}^{(i)}}(\sigma_j - C_{12}^{(i)} \frac{\partial u_j}{\partial x} - C_{23}^{(i)} \frac{\partial w_j}{\partial z})]$$

$$W_I = f_1(\eta_i) w_i + f_2(\eta_i)w_j + f_3(\eta_i)[\frac{\tau_i}{C_{44}^{(i)}} - \frac{\partial v_i}{\partial z}] \qquad + f_4(\eta_i)[\frac{\tau_{i+1}}{C_{44}^{(i)}} - \frac{\partial v_j}{\partial z}]$$

A P P E N D I X B

INTEGRATION OF INTERPOLATION FUNCTION

APPENDIX B

In the derivation of the potential and kinetic energy (see Appendix C) several integrations involving the interpolation functions are considered. These integrals will be summarized below.

(A) The evaluation of the integral $\int_{-1}^{1} \{f\} \{f\}^T \, d\eta$ can be given by the symmetric matrix [A], as shown:

$$
\begin{aligned}
A_{11} &= 78/105 \quad ; & A_{12} &= 9/35 \\
A_{13} &= 22h/105 \quad ; & A_{14} &= -13h/105 \\
A_{22} &= 78/105 \quad ; & A_{23} &= 13h/105 \\
A_{24} &= -22h/105 \quad ; & A_{33} &= 8h^2/105 \\
A_{34} &= -6h^2/105 \quad ; & A_{44} &= 8h^2/105
\end{aligned}
$$

(B) The evaluation of the integral $\int_{-1}^{1} \{f'\} \{f'\}^T \, d\eta$ can be given by the symmetric matrix [B] as shown:

$$
\begin{aligned}
B_{11} &= 3/5 \quad ; & B_{12} &= -3/5 \\
B_{13} &= h/10 \quad ; & B_{14} &= h/10 \\
B_{22} &= 3/5 \quad ; & B_{23} &= -h/10 \\
B_{24} &= -h/10 \quad ; & B_{33} &= 4h^2/15 \\
B_{34} &= -h^2/15 \quad ; & B_{44} &= 4h^2/15
\end{aligned}
$$

(C) The evaluation of the integral $\int_{-1}^{1} \{f\} \{f'\}^T \, d\eta$ can be given by the skew-symmetric matrix [D] as shown:

$$
\begin{aligned}
D_{11} &= -1/2 \quad ; & D_{12} &= 1/2 \\
D_{13} &= h/5 \quad ; & D_{14} &= -h/5 \\
D_{22} &= 1/2 \quad ; & D_{23} &= -h/5 \\
D_{24} &= h/5 \quad ; & D_{33} &= 0 \\
D_{34} &= -h^2/15 \quad ; & D_{44} &= 0
\end{aligned}
$$

A P P E N D I X   C

EVALUATION OF THE INTEGRAL IN ENERGY EQUATION

(EQ. 2.10 OF CHAPTER 2)

APPENDIX C

(A) The evaluation of the integral $\int_{-1}^{1} U \bar{U} \, d\eta_i$ can be gine by the matrix [U] whose coefficients are:

$U_{1,1} = A^{*}_{11}$; $U_{1,2} = A_{13}/C_{66}$

$U_{1,3} = -ik_x A_{13}$; $U_{1,7} = A_{12}$

$U_{1,8} = A_{14}/C_{66}$; $U_{1,9} = -ik_x A_{14}$

$U_{2,2} = A_{33}/C^{2}_{66}$; $U_{2,3} = -ik_x A_{33}/C^{2}_{66}$

$U_{2,7} = A_{23}/C_{66}$; $U_{2,8} = A_{34}/C^{2}_{66}$

$U_{2,9} = -\ ik_x A_{34}/C_{66}$;

$U_{3,3} = k^{2}_{x}\ A_{33}$; $U_{3,7} = ik_x A_{23}$

$U_{3,8} = ik_x A_{34}/C_{66}$; $U_{3,9} = k^{2}_{x} A_{34}$

$U_{7,7} = A_{22}$; $U_{7,8} = A_{24}/C_{66}$

$U_{7,9} = -ik_x A_{24}$

$U_{8,8} = A_{44}/C^{2}_{66}$; $U_{8,9} = -ik_x A_{44}/C_{66}$

$U_{9,9} = k^{2}_{x}\ A_{44}$

Notes: For the above coefficients of [U]

i) $i = \sqrt{-1}$

ii) [U] is a Hermitian matrix

iii) the rest of the coefficients that are not listed above are all equal to zero.

The notes applied to all other integrals unless specified otherwise.

(B) For the evaulation of the integral $\int_{-1}^{1} U' \bar{U}' \, d\eta_i$ change $[A]_{ij}$ of U to $[B]_{ij}$.

---

*Matrix [A], [B] and [D] are given in Appendix B.

(C) The evaluation of the integral $\int_{-1}^{1} V\ \bar{V}\ d\eta_i$, can be given by the matrix [V] where

$$
\begin{array}{llll}
V_{1,1} = k_x^2 A_{33}\ C_{12}^2/C_{22}^2 & ; & V_{1,3} = ik_x A_{13}\ C_{12}/C_{22} \\
V_{1,4} = ik_x A_{33}\ C_{12}/C_{22}^2 & ; & V_{1,5} = k_x k_z A_{33}\ C_{12}C_{23}/C_{22}^2 \\
V_{1,7} = k_x^2 A_{34}\ C_{12}^2/C_{22}^2 & ; & V_{1,9} = ik_x A_{23}\ C_{12}/C_{22} \\
V_{1,10} = ik_x A_{34}\ C_{12}/C_{22}^2 & ; & V_{1,11} = k_x k_z A_{34}\ C_{12}C_{23}/C_{22}^2 \\
V_{3,3} = A_{11} & ; & V_{3,4} = A_{13}/C_{22} \\
V_{3,5} = -ik_z A_{13}\ C_{23}/C_{22} & ; & V_{3,7} = -ik_x A_{14}\ C_{12}/C_{22} \\
V_{3,9} = A_{12} & ; & V_{3,10} = A_{14}/C_{22} \\
V_{3,11} = -ik_z A_{14}\ C_{23}/C_{22} & ; & \\
V_{4,4} = A_{33}/C_{22}^2 & ; & V_{4,5} = -ik_z A_{33}\ C_{23}/C_{22}^2 \\
V_{4,7} = -ik_x A_{34}\ C_{12}/C_{22}^2 & ; & V_{4,9} = A_{23}/C_{22} \\
V_{4,10} = A_{34}/C_{22}^2 & ; & V_{4,11} = -ik_z A_{34}\ C_{23}/C_{22}^2 \\
V_{5,5} = k_z^2 A_{33}\ C_{23}^2/C_{22}^2 & ; & V_{5,7} = k_x k_z A_{34}\ C_{12}C_{23}/C_{22}^2 \\
V_{5,9} = ik_z A_{23}\ C_{23}/C_{22} & ; & V_{5,10} = ik_z A_{34}\ C_{23}/C_{22}^2 \\
V_{5,11} = k_z^2 A_{34}\ C_{23}^2/C_{22}^2 & ; & \\
V_{7,7} = k_x^2 A_{44}\ C_{12}^2/C_{22}^2 & ; & V_{7,9} = ik_x A_{24}\ C_{12}/C_{22} \\
V_{7,10} = ik_x A_{44}\ C_{12}/C_{22}^2 & ; & V_{7,11} = k_x k_z A_{44}\ C_{12}C_{23}/C_{22}^2
\end{array}
$$

$V_{9,9} = A_{22}$; $V_{9,10} = A_{24}/C_{22}$

$V_{9,11} = -ik_z A_{24}\ C_{23}/C_{22}$;

$V_{10,10} = A_{44}/C_{22}^2$; $V_{10,11} = -ik_z A_{44}\ C_{23}/C_{22}^2$

$V_{11,11} = k_z^2 A_{44}\ C_{23}^2/C_{22}^2$

(D) For the evaluation of the integral $\int_{-1}^{1} V'\ \bar{V}'\ d\eta_i$, change $[A]_{ij}$ of $[V]$ to $[B]_{ij}$

(E) The evaluation of the integral $\int_{-1}^{1} W\ \bar{W}\ d\eta_i$, can be given by the matrix $[W]$ where

$W_{3,3} = k_z^2\ A_{33}$; $W_{3,5} = ik_z A_{13}$

$W_{3,6} = ik_z A_{33}/C_{44}$; $W_{3,9} = k_z^2 A_{34}$

$W_{3,11} = ik_z A_{23}$; $W_{3,12} = ik_z A_{34}/C_{44}$

$W_{5,5} = A_{11}$; $W_{5,6} = A_{13}/C_{44}$

$W_{5,9} = -ik_z A_{14}$; $A_{5,11} = A_{12}$

$W_{5,12} = A_{14}/C_{44}$;

$W_{6,6} = A_{33}/C_{44}^2$; $W_{6,9} = -ik_z\ A_{34}/C_{44}$

$W_{6,11} = A_{23}/C_{44}$; $W_{6,12} = A_{34}/C_{44}^2$

$W_{9,9} = k_z^2 A_{44}$; $W_{9,11} = ik_z A_{24}$

$W_{9,12} = ik_z A_{44}/C_{44}$;

$W_{11,11} = A_{22}$; $W_{11,12} = A_{24}/C_{44}$

$W_{12,12} = A_{44}/C_{44}^2$

(F) For the evaluation of the integral $\int_{-1}^{1} W' \bar{W}' \, d\eta_i$, change $[A]_{ij}$ of [W] to $[B]_{ij}$.

(G) The evaluation of $\int_{-1}^{1} (U \bar{V}' - V'\bar{U}) \, d\eta_i$ can be given by the matrix [UV] as shown

$$UV_{1,1} = 2\, ik_x D_{13} C_{12}/C_{22}$$

$$UV_{1,2} = ik_x D_{33} C_{12} T_1/(C_{22} C_{66})$$

$$UV_{1,3} = (k_x^2 D_{33} C_{12}/C_{22} - D_{11}) T_1$$

$$UV_{1,4} = -D_{13}/C_{22}$$

$$UV_{1,5} = ik_z D_{13} C_{23}/C_{22}$$

$$UV_{1,7} = ik_x D_{23} C_{12}/C_{22} + ik_x D_{14} C_{12}/C_{22}$$

$$UV_{1,8} = -ik_x D_{34} C_{12}/(C_{22} C_{66})$$

$$UV_{1,9} = -k_x^2 D_{34} C_{12}/C_{22} - D_{12}$$

$$UV_{1,10} = -D_{14}/C_{22}$$

$$UV_{1,11} = ik_z D_{14} C_{23}/C_{22}$$

$$UV_{2,3} = D_{13}/C_{66}$$

$$UV_{2,4} = -D_{33} T_1/(C_{22} C_{66})$$

$$UV_{2,5} = ik_z D_{33} C_{23} T_1/(C_{22} C_{66})$$

$$UV_{2,7} = ik_x D_{34} C_{12}/(C_{22} C_{66})$$

$$UV_{2,9} = D_{23}/C_{66}$$

$$UV_{2,10} = -D_{34}/(C_{22} C_{66})$$

$$UV_{2,11} = ik_z D_{34} C_{23}/(C_{22} C_{66})$$

$$UV_{3,3} = 2\, ik_x\, D_{13}$$

$$UV_{3,4} = -ik_x\, D_{33} T_1 / C_{22}$$

$$UV_{3,5} = -k_x k_z\, D_{33}\, C_{23} T_1 / C_{22}$$

$$UV_{3,7} = -\, D_{12} - k_x^2\, D_{34}\, C_{12}/C_{22}$$

$$UV_{3,8} = -D_{14}/C_{66}$$

$$UV_{3,9} = ik_x\, D_{14} + ik_x\, D_{23}$$

$$UV_{3,10} = -ik_x\, D_{34}/C_{22}$$

$$UV_{3,11} = -k_x k_z\, D_{34}\, C_{23}/C_{22}$$

$$UV_{4,7} = D_{23}/C_{22}$$

$$UV_{4,8} = -D_{34}/(C_{22}\, C_{66})$$

$$UV_{4,9} = ik_x\, D_{34}/C_{22}$$

$$UV_{5,7} = ik_z\, D_{23}\, C_{23}/C_{22}$$

$$UV_{5,8} = -ik_z\, D_{34}\, C_{23}/(C_{22} C_{66})$$

$$UV_{5,9} = -k_x k_z\, D_{34}\, C_{23}/C_{22}$$

$$UV_{7,7} = 2\, ik_x\, D_{24}\, C_{12}/C_{22}$$

$$UV_{7,8} = ik_x\, D_{44}\, C_{12}\, T_1\, /(C_{22}\, C_{66})$$

$$UV_{7,9} = (k_x^2\, D_{44}\, C_{12}/C_{22} - D_{22})\, T_1$$

$$UV_{7,10} = -D_{24}/C_{22}$$

$$UV_{7,11} = ik_z\ D_{24}\ C_{23}/C_{22}$$

$$UV_{8,9} = D_{24}/C_{66}$$

$$UV_{8,10} = -D_{44}\ T_1\ /(C_{22}C_{66})$$

$$UV_{8,11} = ik_z\ D_{44}\ C_{23}\ T_1\ /(C_{22}C_{66})$$

$$UV_{9,9} = 2\ ik_x\ D_{24}$$

$$UV_{9,10} = -ik_x\ D_{44}\ T_1\ /C_{22}$$

$$UV_{9,11} = -k_x k_z\ D_{44}\ C_{23}\ T_1\ /C_{22}$$

where $T_1 = 1.0$

and $[UV]_{ji}$ is the negative of the conjugate of $[UV]_{ij}$

(H) For the evaluation of $\int_{-1}^{1} (V\bar{U}' - U'\bar{V}) d\eta_i$, set $T_1 = -1.0$ for [UV].

(I) The evaluation of $\int_{-1}^{1} (W\bar{U} + U\bar{W})\ d\eta_i$ can be given by the matrix [UW] as:

$$UW_{1,3} = -ik_z\ A_{13} \quad ; \quad UW_{1,5} = A_{11}$$

$$UW_{1,6} = A_{13}/C_{44} \quad ; \quad UW_{1,9} = -ik_z A_{14}$$

$$UW_{1,11} = A_{12} \quad ; \quad UW_{1,12} = A_{14}/C_{44}$$

$$UW_{2,3} = -\ ik_z A_{33}/C_{66} \quad ; \quad UW_{2,5} = A_{13}/C_{66}$$

$$UW_{2,6} = A_{33}/(C_{44}C_{66}) \quad ; \quad UW_{2,9} = -ik_z A_{34}/C_{66}$$

$$UW_{2,11} = A_{23}/C_{66} \quad ; \quad UW_{2,12} = A_{34}/(C_{44}C_{66})$$

$$UW_{3,3} = 2\ k_x k_z A_{33} \quad ; \quad UW_{3,5} = ik_x A_{13}$$

$$UW_{3,6} = ik_x A_{33}/C_{44} \quad ; \quad UW_{3,7} = ik_z A_{23}$$

$$UW_{3,8} = ik_z A_{34}/C_{66} \quad ; \quad UW_{3,9} = k_x k_z A_{34}$$

$UW_{3,11} = ik_xA_{23}$; $UW_{3,12} = ik_xA_{34}/C_{44}$

$UW_{5,7} = A_{12}$; $UW_{5,8} = A_{14}/C_{66}$

$UW_{5,9} = -ik_xA_{14}$;

$UW_{6,7} = A_{23}/C_{44}$; $UW_{6,8} = A_{34}/(C_{44}C_{66})$

$UW_{6,9} = -ik_xA_{34}/C_{44}$;

$UW_{7,9} = -ik_zA_{24}$; $UW_{7,11} = A_{22}$

$UW_{7,12} = A_{24}/C_{44}$

$UW_{8,9} = -ik_zA_{44}/C_{66}$; $UW_{8,11} = A_{24}/C_{66}$

$UW_{8,12} = A_{44}/(C_{44}C_{66})$

$UW_{9,9} = 2k_xk_zA_{44}$; $UW_{9,11} = ik_xA_{24}$

$UW_{9,12} = ik_xA_{44}/C_{44}$

(J) The evaluation of $\int_{-1}^{1} (W\,\bar{V}' - V'\bar{W})\,d\eta_i$ can be given by the matrix [WV] as shown

$WV_{1,3} = k_xk_zD_{33}C_{12}T_1\ /C_{22}$;

$WV_{1,5} = ik_xD_{13}C_{12}/C_{22}$; $WV_{1,6} = ik_xD_{33}C_{21}T_1/(C_{22}C_{44})$

$WV_{1,9} = -k_xk_zD_{34}C_{12}/C_{22}$; $WV_{1,11} = ik_xD_{23}C_{12}/C_{22}$

$WV_{1,12} = -ik_xD_{34}C_{12}/(C_{22}C_{44})$;

$WV_{3,3} = 2ik_zD_{13}$; $WV_{3,4} = ik_zD_{33}T_1/C_{22}$

$WV_{3,5} = (D_{11} - k_z^2D_{33}C_{23}/C_{22})\ T_1$

$WV_{3,6} = -D_{13}/C_{44}$

$WV_{3,7} = -k_x k_z D_{34} C_{12}/C_{22}$; $WV_{3,9} = ik_z D_{14} + ik_z D_{23}$

$WV_{3,10} = -ik_z D_{34}/C_{22}$; $WV_{3,11} = -D_{12} - k_z^2 D_{34} C_{23}/C_{22}$

$WV_{3,12} = -D_{14}/C_{44}$

$WV_{4,5} = D_{13}/C_{22}$; $WV_{4,6} = D_{33} T_1/(C_{22} C_{44})$

$WV_{4,9} = ik_z D_{34}/C_{22}$; $WV_{4,11} = D_{23}/C_{22}$

$WV_{4,12} = -D_{34}/(C_{22} C_{44})$

$WV_{5,5} = 2ik_z D_{13} C_{23}/C_{22}$

$WV_{5,6} = ik_z D_{33} C_{23} T_1/(C_{22} C_{44})$

$WV_{5,7} = ik_x D_{14} C_{12}/C_{22}$; $WV_{5,9} = D_{12} - k_z^2 D_{34} C_{23}/C_{22}$

$WV_{5,10} = -D_{14}/C_{22}$

$WV_{5,11} = ik_z D_{23} C_{23}/C_{22} + ik_z D_{14} C_{23}/C_{22}$

$WV_{5,12} = -ik_z D_{34} C_{23}/(C_{22} C_{44})$

$WV_{6,7} = ik_x D_{34} C_{12}/(C_{22} C_{44})$; $WV_{6,9} = D_{23}/C_{44}$

$WV_{6,10} = -D_{34}/(C_{22} C_{44})$; $WV_{6,11} = ik_z D_{34} C_{23}/(C_{22} C_{44})$

$WV_{7,9} = k_x k_z D_{44} C_{12} T_1/C_{22}$

$WV_{7,11} = ik_x D_{24} C_{12}/C_{22}$; $WV_{7,12} = ik_x D_{44} C_{12} T_1/(C_{22} C_{44})$

$WV_{9,9} = 2ik_z D_{24}$; $WV_{9,10} = -ik_z D_{44} T_1/C_{22}$

$WV_{9,11} = (D_{22} - k_z^2 D_{44} C_{23}/C_{22}) T_1$

$WV_{9,12} = -D_{24}/C_{44}$

$WV_{10,11} = D_{24}/C_{22}$; $WV_{10,12} = +D_{44}T_1/(C_{22}C_{44})$

$WV_{11,11} = 2ik_x D_{24}C_{23}/C_{22}$

$WV_{11,12} = ik_z D_{44}C_{23}T_1/(C_{22}C_{44})$

where $T_1 = 1.0$

and $[WV]_{ji}$ is the negative of the conjugate of $[WV]_{ij}$.

(K) For the evaluation of $\int_{-1}^{1}(V\bar{W}' - W\bar{V}')\,d\eta_i$, set $T_1 = -1.0$ for [WV].

A P P E N D I X D

ASSEMBLY PROCESS OF STIFFNESS MATRIX

APPENDIX D

For a structural problem of dynamic nature,

$$\{P\} = [k]\,\{U\} - \omega^2\,[m]\,\{U\} \tag{D-1}$$

where $\{P\}$ = member forces

$[k]$ = elemental stiffness matrix

ω = circular frequency

$[m]$ = elemental mass matrix

$\{U\}$ = vector with unknown quantities

Equation D.1 can be written as

$$\{P\} = \{[k] - \omega^2\,[m]\}\,\{U\} \tag{D-2}$$

Let $[S]^{(q)}_{ij} = [k]^{(q)}_{ij} - \omega^2\,[m]^{(q)}_{ij}$ where superscript denotes layer number

$[S]^{(q)}_{ij}$ is called the impedance matrix.

By assembling these elemental matrices $[S]^{(q)}_{ij}$, the following is obtained:

$$\begin{bmatrix} S^{(1)}_{11} & S^{(1)}_{12} & & & & \\ S^{(1)}_{21} & S^{(1)}_{22} + S^{(2)}_{11} & S^{(2)}_{12} & & & \\ & S^{(2)}_{21} & S^{(2)}_{22} + S^{(3)}_{11} & S^{(3)}_{12} & & \\ & & S^{(3)}_{21} & S^{(3)}_{22} + S^{(4)}_{11} & S^{(4)}_{12} & \\ & & \vdots & S^{(4)}_{21} & S^{(4)}_{22} & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \end{Bmatrix}$$

If the equilibrium of joint 2 and 3 is considered,

$$S^{(1)}_{21}\,u_1 + [S^{(1)}_{22} + S^{(2)}_{11}]u_2 + S^{(2)}_{12}\,u_3 = 0 \tag{D-3}$$

and

$$S^{(2)}_{21}\,u_2 + [S^{(2)}_{22} + S^{(3)}_{11}]u_3 + S^{(3)}_{12}\,u_4 = 0 \tag{D-4}$$

are obtained.

Using Floquet's Theory, we obtain $u_3 = u_1 * E$

$$u_4 = u_2 * E$$

where $E = e^{ik_y d}$ is a constant.

Substituting these values of u_3 and u_4 into Eqs. D-3 and D-4, we obtain:

$$S_{21}^{(1)} u_1 + [S_{22}^{(1)} + S_{11}^{(2)}]u_2 + S_{12}^{(2)}(E\ u_1) = 0 \qquad \text{(D-5)}$$

$$S_{21}^{(2)} u_2 + [S_{22}^{(2)} + S_{11}^{(3)}](Eu_1) + S_{12}^{(3)}(E\ u_2) = 0 \qquad \text{(D-6)}$$

After rearranging Eqs. D-5 and D-6 can be written as

$$[S_{21}^{(1)} + ES_{12}^{(2)}]u_1 + [S_{22}^{(1)} + S_{11}^{(2)}]u_2 = 0$$

$$E\ [S_{22}^{(2)} + S_{11}^{(3)}]u_1 + [S_{21}^{(2)} + ES_{12}^{(3)}]\ u_2 = 0$$

Thus, by this assembly process, the same number of equations as the number of unknowns can be obtained.

A P P E N D I X   E

EVALUATION OF EULER EQUATION

(EQ. 3.43 OF CHAPTER 3)

APPENDIX E

In the effective stiffness method, the 12 dependent variables can be derived by coordinating a system of Euler equations which were written as

$$\sum_{r=1}^{4} \frac{\partial}{\partial q_r}\left[\frac{\partial(T_k - V_p - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3)}{\partial(\partial f_s/\partial q_r)}\right] - \frac{\partial(T_k - V_p - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3)}{\partial f_s} = 0 \qquad \text{(E-1)}$$

where f_s represent the twelve dependent variables

$(U, V, W, \psi^f_{2x}, \psi^m_{2x}, \psi^f_{2y}, \psi^m_{2y}, \psi^f_{2z}, \psi^m_{2z}, \lambda_1, \lambda_2, \lambda_3)$

and q_r are the spatial variables x, y, z, and time t

(I) For $f_s = U$, the Euler Equation become:

$$\frac{\partial}{\partial x}\left[\frac{\partial F}{\partial(\partial U/\partial x)}\right] - \frac{\partial F}{\partial U} + \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial(\partial U/\partial y)}\right] - \frac{\partial F}{\partial U} + \frac{\partial}{\partial z}\left[\frac{\partial F}{\partial(\partial U/\partial z)}\right] - \frac{\partial F}{\partial U} + \frac{\partial}{\partial t}\left[\frac{\partial F}{\partial(\partial U/\partial t)}\right] - \frac{\partial F}{\partial U} = 0$$

where $F = T_k - V_p - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3$

$\frac{\partial F}{\partial U} = 0$, using the notation $\frac{\partial}{\partial x} = \partial_1$, $\frac{\partial}{\partial y} = \partial_2$, $\frac{\partial}{\partial z} = \partial_3$, the equation become:

$$\partial_1\left[\frac{\partial F}{\partial(\partial_1 U)}\right] + \partial_2\left[\frac{\partial F}{\partial(\partial_2 U)}\right] + \partial_3\left[\frac{\partial F}{\partial(\partial_3 U)}\right] + \frac{\partial}{\partial t}\left[\frac{\partial F}{\partial \dot{U}}\right] = 0$$

$$\partial_1 \left\{ \begin{aligned} &\frac{\partial T_k}{\partial(\partial_1 U)} = 0 \\ &\frac{\partial V_p}{\partial(\partial_1 U)} = \{\eta C^f_{11} + (1-\eta) C^m_{11}\}(\partial_1 U) + \eta C^f_{12}\psi^f_{2y} + \{\eta C^f_{13} + (1-\eta) C^m_{13}\}\partial_3 W \\ &\qquad\qquad + (1-\eta)\, C^m_{12}\psi^m_{2y} \\ &\frac{\partial \lambda_1 S_1}{\partial(\partial_1 U)} = \frac{\partial \lambda_2 S_2}{\partial(\partial_1 U)} = \frac{\partial \lambda_3 S_3}{\partial(\partial_1 U)} = 0 \\ &\frac{\partial F}{\partial U} = 0 \end{aligned} \right.$$

$$\partial_2 \begin{cases} \dfrac{\partial T_k}{\partial(\partial_2 U)} = 0 \\ \dfrac{\partial V_f}{\partial(\partial_2 U)} = 0 \\ \dfrac{\partial \lambda_1 s_1}{\partial(\partial_2 U)} = \lambda_1, \ \dfrac{\partial \lambda_2 s_2}{\partial(\partial_2 U)} = \dfrac{\partial \lambda_3 s_3}{\partial(\partial_2 U)} = 0 \end{cases}$$

$$\partial_3 \begin{cases} \dfrac{\partial T_k}{\partial(\partial_3 U)} = 0 \\ \dfrac{\partial V_p}{\partial(\partial_3 U)} = \{\eta C^f_{55}+(1-\eta)C^m_{55}\} \ \{\partial_3 U+\partial_1 W\} \\ \dfrac{\partial \lambda_1 s_1}{\partial(\partial_3 U)} = \dfrac{\partial \lambda_2 s_2}{\partial(\partial_3 U)} = \dfrac{\partial \lambda_3 s_3}{\partial(\partial_3 U)} = 0 \end{cases}$$

$$\frac{\partial}{\partial t} \begin{cases} \dfrac{\partial T_k}{\partial \dot{U}} = \rho_c \ \dot{U} \\ \dfrac{\partial V_p}{\partial \dot{U}} = \dfrac{\partial \lambda_1 s_1}{\partial \dot{U}} = \dfrac{\partial \lambda_2 s_2}{\partial \dot{U}} = \dfrac{\partial \lambda_3 s_3}{\partial \dot{U}} = 0 \end{cases}$$

Then, the Euler equation becomes

$$-\{\eta C^f_{11}+(1-\eta)C^m_{11}\} \ \partial_{11}U-\eta C^f_{12}\partial_1\psi^f_{2y} - \{\eta C^f_{13}+(1-\eta)C^m_{13}\} \ \partial_{13}W-(1-\eta)C^m_{12}\partial_1\psi^m_{2y}$$
$$-\partial_2\lambda_1 - \{\eta C^f_{55}+(1-\eta)C^m_{55}\} \ \{\partial_{33}U+\partial_{31}W\} + \rho_c\ddot{U} = 0$$

or

$$\{\eta C^f_{11}+(1-\eta)C^m_{11}\} \ \partial_{11}U + \{\eta C^f_{55}+(1-\eta)C^m_{55}\} \ \partial_{33}U + \{[\eta C^f_{13}+(1-\eta)C^m_{13}]$$
$$+[\eta C^f_{55}+(1-\eta)C^m_{55}]\} \ \partial_{13}W + \eta C^f_{12}\partial_1\psi^f_{2y} + (1-\eta)C^m_{12}\partial_1\psi^m_{2y}$$
$$+\partial_2\lambda_1 = \rho_c \ \ddot{U} \tag{E-2}$$

The other 11 equations can be derived in a similar way.

A P P E N D I X   F

ELEMENTS OF THE IMPEDANCE MATRIX

APPENDIX F

Omitting the superscript (i) for the (i)th lamina and defining

$$Q_n = \eta\, C^f_{nn} + (1 - \eta)\, C^m_{nn} \qquad \text{where } n = 1, 2\ldots 6$$

$$Q_7 = \eta\, C^f_{13} + (1 - \eta)\, C^m_{13}$$

$[S^p]$ is a 12 x 12 Hermitian matrix whose non-zero elements are given by:

$$S^p_{1,1} = -k_x^2\, Q_1 - k_z^2\, Q_5 - \rho_c\, \omega^2$$

$$S^p_{1,3} = -k_x k_z\, Q_5\, Q_7$$

$$S^p_{1,8} = ik_x \eta\, C^f_{12}$$

$$S^p_{1,9} = ik_x\, (1 - \eta)\, C^m_{12}$$

$$S^p_{1,10} = S^p_{2,11} = S^p_{3,12} = -ik_y$$

$$S^p_{2,2} = -k_x^2\, Q_6 - k_z^2\, Q_4 - \rho_c\, \omega^2$$

$$S^p_{2,4} = ik_x \eta\, C^f_{66}$$

$$S^p_{2,5} = ik_x\, (1 - \eta)\, C^m_{66}$$

$$S^p_{2,6} = ik_z\, \eta\, C^f_{44}$$

$$S^p_{2,7} = ik_z\, (1 - \eta)\, C^m_{44}$$

$$S^p_{3,3} = -k_x^2\, Q_5 - k_z^2\, Q_3 - \rho_c\, \omega^2$$

$$S^p_{3,8} = ik_z\, \eta\, C^f_{23}$$

$$S^p_{3,9} = ik_z\, (1 - \eta)\, C^m_{23}$$

$$S^p_{4,4} = -\frac{1}{12}\eta\, d_f^2\, [k_x^2 C^f_{11} + k_z^2 C^f_{55}] - \eta\, C^f_{66} - \rho_c\, \omega^2$$

$$S^p_{4,6} = -\frac{1}{12}\eta\, d_f^2\, k_x k_z\, [C^f_{13} + C^f_{55}]$$

$$S^p_{4,10} = S^p_{6,12} = S^p_{8,11} = -\eta$$

$$S^p_{5,5} = -\frac{1}{12}(1-\eta)\, d_m^2\, [k_x^2\, C^m_{11} + k_z^2\, C^m_{55}] - (1-\eta)\, C^m_{66} - \rho_c\, \omega^2$$

$$S^p_{5,7} = -\frac{1}{12}(1-\eta)\, d_m^2\, k_x k_z\, [C^m_{13} + C^m_{55}]$$

$$S^p_{5,10} = S^p_{7,12} = S^p_{9,11} = -(1-\eta)$$

$$S^p_{6,6} = -\frac{1}{12}\eta\, d_f^2\, [k_x^2\, C^f_{55} + k_z^2\, C^f_{33}] - \eta\, C^f_{44} - \rho_c\, \omega^2$$

$$S^p_{7,7} = -\frac{1}{12}(1-\eta)\, d_m^2\, [k_x^2\, C^m_{55} + k_z^2\, C^m_{33}] - (1-\eta)\, C^m_{44} - \rho_c\, \omega^2$$

$$S^p_{8,8} = -\frac{1}{12}\eta\, d_f^2\, [k_x^2\, C^f_{66} + k_z^2\, C^f_{44}] - \eta\, C^f_{22} - \rho_c\, \omega^2$$

$$S^p_{9,9} = -\frac{1}{12}(1-\eta)\, d_m^2\, [k_x^2\, C^m_{66} + k_z^2\, C^m_{44}] - (1-\eta)\, C^m_{22} - \rho_c\, \omega^2$$

f denotes the reinforced layer

m is the matrix layer

C_{ij} is the material constants of the layer

k_x, k_y, k_z are the components of wave number in the x, y and z-direction

ω is the angular frequency

η is defined in Eq. 3.25 ; $\eta = \dfrac{d_f}{d_f + d_m}$

ρ_c is defined in Eq. 3.38 ; $\rho_c = \eta\rho_f + (1-\eta)\,\rho_m$

A P P E N D I X   G

FORMULA TO CALCULATE THE MATERIAL CONSTANTS OF GRAPHITE-EPOXY FIBER-REINFORCED COMPOSITE

APPENDIX G

The material constants of graphite-epoxy fiber-reinforced composite are given as:

$$C_{11} = E_L + 2\,\nu_{LT}\,C_{13}$$

$$C_{13} = C_{12} = 2\,\nu_{LT}\,K_T$$

$$C_{22} = C_{33} = K_T + \mu_{TT}$$

$$C_{23} = K_T - \mu_{TT}$$

$$C_{44} = \mu_{TT} = \frac{1}{2}\,(C_{33} - C_{23})$$

$$C_{55} = C_{66} = \mu_{LT}$$

where

E_L = lateral modulus of elasticity

ν_{LT} = Poission's ratio

K_T = Bulk's modulus

μ_{TT} = transverse shear modulus

μ_{LT} = lateral shear modulus

The constants E_L, K_T, μ_{LT}, μ_{TT} and ν_{LT} are determined from the properties of graphite and epoxy.

Epoxy matrix is isotropic with properties denoted by subscript m. Its material properties are given as:

$$E_m = 5.35 * 10^9 \;\; N/m^2$$

$$K_m = 6.06 * 10^9 \;\; N/m^2$$

$$\mu_m = 1.95 * 10^9 \;\; N/m^2$$

$$\nu_m = 0.353$$

The graphite fibers layer are transversely isotropic with properties given by the primed quantities.

$$E'_L = 2.32 * 10^{11} \quad N/m^2$$

$$K'_T = 15.0 * 10^{9} \quad N/m^2$$

$$\mu'_{LT} = 24.0 * 10^{9} \quad N/m^2$$

$$\mu'_{TT} = 5.02 * 10^{9} \quad N/m^2$$

$$\nu'_{LT} = 0.290$$

With these properties of graphite and epoxy, E_L, K_T, μ_{LT}, μ_{TT} and ν_{LT} can be obtained using the following relations.

$$D_1 = \frac{1 - \bar{c}}{K'} + \frac{\bar{c}}{K_m} + \frac{1}{\mu_m}$$

$$E_L = E_m (1 - \bar{c}) + \bar{c}\, E'_L + \frac{4\bar{c}(1-\bar{c})\ (\nu'_{LT} - \nu_m)^2}{D_1}$$

$$\nu_{LT} = \nu_m (1-\bar{c}) + \bar{c}\, \nu'_{LT} + \frac{\bar{c}(1-\bar{c})\ (\nu'_{LT} - \nu_m)\ (\frac{1}{K_m} - \frac{1}{K'_T})}{D_1}$$

$$K_T = K_m + \frac{\bar{c}(K_m + \mu_m)\ (K'_T - K_m)}{(1-\bar{c})\ K'_T + \bar{c}K_m + \mu_m}$$

$$\frac{\mu_{TT}}{\mu_m} = 1 + \frac{2\bar{c}\ (K_m + \mu_m)\ (\mu'_{TT} - \mu_m)}{K_m\mu_m + (K_m + 2\mu_m)\ [\bar{c}\mu_m + (1-\bar{c})\ \mu'_{TT}]}$$

$$n = \mu'_{LT} / \mu_m$$

$$\frac{\mu_{LT}}{\mu_m} = \frac{1 + \bar{c}(n-1)/(n+1)}{1 - \bar{c}(n-1)/(n+1)}$$

In the above relations, $\bar{c}$ is a constant value that can be chosen to determine the various material constants for different types of graphite epoxy fiber reinforced composite. $\bar{c}$ is chosen to be 0.3 and 0.668 for the two cases of presentation mentioned in Chapter 4. The computed material constants are shown in Table 4.3 and Table 4.4 respectively.

A P P E N D I X H

FORTRAN PROGRAM FOR THE EFFECTIVE MODULUS METHOD

```
C
C      **************************************************************
C      *                                                            *
C      *               EFFECTIVE MODULIS METHOD                     *
C      *                    PROGRAMMED BY                           *
C      *                   JOHNNY K.T. YEO                          *
C      *             THE UNIVERSITY OF MANITOBA                     *
C      *                   DECEMBER, 1982                           *
C      *                                                            *
C      **************************************************************
C
C NOTES :
C          1) ALL INPUTS ARE FORMAT-FREE.
C          2) NLAYER,IPRINT,NKBAR,NPHI,NALFA ARE INTEGER VALUES
C          3) H,C,RHO,KBAR ARE REAL
C
C ..................................................................
C
C                       INPUT DESCRIPTION
C
C ..................................................................
C
C     A.  START CARD - ONE CARD FOR NLAYER AND IPRINT.
C                     * NLAYER = NUMBER OF LAYERS
C                     * IPRINT = NUMBER OF SETS OF VECTORS AS OUTPUT
C
C     B.  MATERIAL PROPERTIES CARD - NLAYER OF CARDS FOR :
C                       H(I),C11(I),C12(I),C13(I),C22(I),C23(I),
C                       C33(I),C44(I),C55(I),C66(I),RHO(I)
C                     * H(I)    = LAYER THICKNESS
C                     * CJK(I)  = MATERIAL CONSTANTS OF THE LAYER
C                     * RHO(I)  = DENSITY OF THE MATERIAL
C                     *      I  = 1 TO NLAYER
C
C     C.  BASIC CONTROL CARD - ONE CARD FOR NKBAR.
C                     * NKBAR   = NUMBER OF WAVE NUMBER TO BE EVALUATED
C
C     D.  WAVE NUMBER CARD - AS MANY CARDS AS REQUIRED FOR KBAR.
C                     * KBAR(J)= VALUE OF WAVE NUMBER
C                     *      J = 1 TO NKBAR
C
C     E.  ANGLE CONTROL CARD - ONE CARD FOR NPHI
C                     * NPHI    = NUMBER OF VERTICAL ANGLES
C
C     F.  VERTICAL ANGLE CARD - AS MANY CARDS AS REQUIRED FOR PHI.
C                     * PHI(K)  = VERTICAL ANGLES IN DEGREE.
C                     *     K   = I TO NPHI
C
C     G.  ANGLE CONTROL CARD - ONE CARD FOR NALFA
C                     * NALFA   = NUMBER OF HORIZONTAL ANGLES
C
C     H.  HORIZONTAL ANGLES CARD - AS MANY CARDS AS REQUIRED FOR ALFA.
C                     * ALFA(L) = HORIZONTAL ANGLE IN DEGREE
C                     *      L  = 1 TO NALFA
C
C
C ..................................................................
C
C                       OUTPUT DESCRIPTION
C
C ..................................................................
C
C     ZETA  = DIMENSIONLESS WAVE NUMBER
C     OM(M) = PHASE VELOCITY OF PROPAGATING WAVE
```

```
C
C ..............................................................
C
C                         SAMPLE DATA
C
C ..............................................................
C
C  CARD A : 2   1
C  CARD B :
C       8.0 2.56  0.583 0.583 1.797 0.745 1.797 0.526 0.559 0.559 2.534
C       0.5 1.107 0.573 0.573 1.107 0.573 1.107 0.267 0.267 0.267 2.702
C  CARD C : 1
C  CARD D : 0.0
C  CARD E : 1
C  CARD F : 45
C  CARD G : 1
C  CARD H : 45
C
C ..............................................................
C
C
C
C **** DECLARATION    ****
C
      INTEGER L1,L2,L3,N1,N2,N3
      REAL*8  PI,H(2),C11(2),C12(2),C13(2),C22(2),C23(2),
     .        C33(2),C44(2),C55(2),C66(2),RHO(2),ZETA(20)
      REAL*8 KHBAR,XK,XK2,ZK,ZK2,YK,YK2,KBAR(20),PHI(10),ALFA(10)
      COMPLEX*16  A(3,3),B(3,3),EIGA(3),EIGB(3),OM(3),
     .            OMSQ(3),Z(3,3),WK(3,6),CIMGG,ZERO
C
      ZERO =(0.0,0.0)
      CIMGG=(0.0,1.0)
      PI   =4.*ATAN(1.0)
C
C ***** MAIN LINE PROGRAM *****
C
      CALL TRAPS(99999, 99999, 99999, 99999, 99999)
C
C THIS SUBROUTINE WILL TRAPS ANY NUMBER APPROACHING ZERO
C
C
C       READ IN NUMBER OF LAYERS
C
      READ, NLAYER ,IPRINT
C
C       READ IN PROPERTIES OF EACH LAYER
C
      READ, (H(I),C11(I),C12(I),C13(I),C22(I),C23(I),C33(I),
     .       C44(I),C55(I),C66(I),RHO(I),I=1,NLAYER)
      PRINT 10, NLAYER
 10   FORMAT ('1'////,40X,'NEMBER OF LAYERS =',3X,I2/////,1X,
     .        'LAYER PROPERTIES *'//)
      DO 20 I=1,NLAYER
         PRINT 30, I,H(I),C11(I),C12(I),C13(I),C22(I)
         PRINT 31,C23(I),C33(I),C44(I),C55(I),C66(I),RHO(I)
 30      FORMAT('-','LAYER=',I2,2X,'H =',G16.9,'C11=',G16.9,
     .          'C12=',G16.9,'C13=',G16.9,'C22=',G16.9)
 31      FORMAT('0',9X,'C23=',G16.9,'C33=',G16.9,'C44=',G16.9,
     .          'C55=',G16.9,'C66=',G16.9,'RHO=',G16.9)
 20   CONTINUE
C
      DBAR = H(1)/H(2)
      C11BAR=((DBAR*C11(1)+C11(2))*(C22(1)+DBAR*C22(2))
```

```
     .         -DBAR*(C12(1)-C12(2))**2)/((1+DBAR)*(C22(1)
     .         +DBAR*C22(2)))
      C12BAR=(C22(1)*C12(2)+DBAR*C12(1)*C22(2))/
     .         (C22(1)+DBAR*C22(2))
      C13BAR=((DBAR*C13(1)+C13(2))*(C22(1)+DBAR*C22(2))
     .         +(C23(2)-C23(1))*DBAR*(C12(1)-C12(2)))
     .         /((1+DBAR)*(C22(1)+DBAR*C22(2)))
C
      C22BAR=((1+DBAR)*C22(1)*C22(2))/((C22(1)+DBAR*C22(2)))
      C23BAR=(DBAR*C23(1)*C22(2)+C23(2)*C22(1))/(C22(1)+DBAR*C22(2))
C
      C33BAR=((DBAR*C33(1)+C33(2))*(C22(1)+DBAR*C22(2))
     .         -DBAR*(C23(1)-C23(2))**2)/
     .         ((1+DBAR)*(C22(1)+DBAR*C22(2)))
C
      C44BAR=(1+DBAR)*C44(1)*C44(2)/(DBAR*C44(2)+C44(1))
      C55BAR=(DBAR*C55(1)+C55(2))/(DBAR+1)
      C66BAR=((1+DBAR)*C66(1)*C66(2))/(DBAR*C66(2)+C66(1))
C
      RHOBAR= (RHO(2)+DBAR*RHO(1))/(1.+DBAR)
C
C
      NUM =1
      PRINT 40
 40   FORMAT ('0'///,5X,'THE EFFECTIVE MODULUS OF THE TWO LAYERS IS '/)
      PRINT 30,NUM,H(1)+H(2),C11BAR,C12BAR,C13BAR,C22BAR
      PRINT 31, C23BAR,C33BAR,C44BAR,C55BAR,C66BAR,RHOBAR
C
C
      READ, NKBAR
      DO 50 L1=1,NKBAR
         READ, KBAR(L1)
         ZETA(L1) = H(1)*KBAR(L1)
 50   CONTINUE
C
C ***** READ PHI IN DEGREE *****
C
      READ, NPHI
      DO 60 L2=1,NPHI
          READ, PHI(L2)
 60   CONTINUE
C
C ***** READ ALFA IN DEGREE *****
C
      READ, NALFA
      DO 65 L3=1,NALFA
          READ, ALFA(L3)
 65   CONTINUE
C
C
      DO 70 N1=1,NKBAR
         PRINT 75, KBAR(N1), ZETA(N1)
 75      FORMAT ('-',////,10X,'KBAR =',F12.8,5X,'ZETA=',F12.8//)
         DO 80 N2=1,NPHI
            ANGLE = PHI(N2)*PI/180.
            KHBAR = KBAR(N1)*COS(ANGLE)
            YK    = KBAR(N1)*SIN(ANGLE)
            DO 90 N3=1,NALFA
               ROT =ALFA(N3)*PI/180
               XK  =KHBAR*COS(ROT)
               ZK  = KHBAR*SIN(ROT)
               PRINT 100,PHI(N2),YK,ALFA(N3),XK,ZK
 100           FORMAT (' ',3X,'PHI =',F8.5,3X,'KAPAY =',F10.7,3X,
     .                 'ALFA =',F8.5,3X,'KAPAX =',F10.7,3X,
```

```
     .                      'KAPAZ =',F10.7)
                DO 11 JJ=1,3
                   DO 22 KK=1,3
                      A(JJ,KK)=ZERO
                      B(JJ,KK)=ZERO
 22                CONTINUE
 11             CONTINUE
C
                XK2 = XK*XK
                YK2 = YK*YK
                ZK2 = ZK*ZK
C
C ****** FORMULATION OF THE A MATRIX ******
C
                A(1,1) =-C11BAR*XK2-C66BAR*YK2-C55BAR*ZK2
                A(1,2) =-XK*YK*(C12BAR+C66BAR)
                A(1,3) =-XK*ZK*(C13BAR+C55BAR)
C
                A(2,1) = A(1,2)
                A(2,2) =-C66BAR*XK2-C22BAR*YK2-C44BAR*ZK2
                A(2,3) =-YK*ZK*(C23BAR+C44BAR)
C
                A(3,1) = A(1,3)
                A(3,2) = A(2,3)
                A(3,3) =-C55BAR*XK2-C44BAR*YK2-C33BAR*ZK2
C
C ****** FORMULATION OF B MATRIX
C
                B(1,1) =-RHOBAR
                B(2,2) =-RHOBAR
                B(3,3) =-RHOBAR
C
C
                IJOB=1
                IA  =3
                IB  =3
                IZ  =3
                N   =3
                CALL EIGZC(A,IA,B,IB,N,IJOB,EIGA,EIGB,Z,IZ,WK
     .                      ,INFER,IER)
                DO 120 M=1,3
                   OMSQ(M)=EIGA(M)/EIGB(M)
                   OM(M)   =CDSQRT(OMSQ(M))/(C44(2)/RHO(2))**0.5
                   OM(M)   =OM(M)/KBAR(N1)
                   PRINT 125, M, OM(M)
 125               FORMAT ('0',1X,I2,5X,2G16.9)
 120            CONTINUE
                IF (IPRINT .GT. 0) THEN DO
                   IV=1
                   PRINT 140, IV
 140               FORMAT ('0'//,1X,'M',10X,'THE CORRESPONDING Z MATRIX'
     .                      ' OF CELL ',I1,1X,'IS :'//)
                   DO 150 K=1,3
                      PRINT 160, K, (Z(L,K),L=1,3)
 160                  FORMAT ('0',I2,1X,3(2F10.6,1X))
 150               CONTINUE
                   IPRINT=IPRINT-1
                END IF
C
 90          CONTINUE
 80       CONTINUE
 70    CONTINUE
       STOP
       END
```

A P P E N D I X   I

FORTRAN PROGRAM FOR THE EFFECTIVE STIFFNESS METHOD

```
C
C     ****************************************************************
C     *                                                              *
C     *                  EFFECTIVE STIFFNESS METHOD                  *
C     *                        PROGRAMMED BY                         *
C     *                       JOHNNY K.T. YEO                        *
C     *                    UNIVERSITY OF MANTIBA                     *
C     *                                                              *
C     ****************************************************************
C
C NOTES :
C         1) ALL INPUTS ARE FORMAT-FREE.
C         2) NLAYER,IPRINT,NKBAR,NPHI,NALFA ARE INTEGER VALUES
C         3) H,C,RHO,KBAR ARE REAL
C
C ..................................................................
C
C                      INPUT DESCRIPTION
C
C ..................................................................
C
C     A.  START CARD - ONE CARD FOR NLAYER AND IPRINT.
C                   * NLAYER = NUMBER OF LAYERS
C                   * IPRINT = NUMBER OF SETS OF VECTORS AS OUTPUT
C
C     B.  MATERIAL PROPERTIES CARD - NLAYER OF CARDS FOR :
C                    H(I),C11(I),C12(I),C13(I),C22(I),C23(I),
C                    C33(I),C44(I),C55(I),C66(I),RHO(I)
C                   * H(I)   = LAYER THICKNESS
C                   * CJK(I) = MATERIAL CONSTANTS OF THE LAYER
C                   * RHO(I) = DENSITY OF THE MATERIAL
C                   *     I  = 1 TO NLAYER
C
C     C.  BASIC CONTROL CARD - ONE CARD FOR NKBAR.
C                   * NKBAR  = NUMBER OF WAVE NUMBER TO BE EVALUATED
C
C     D.  WAVE NUMBER CARD - AS MANY CARDS AS REQUIRED FOR KBAR.
C                   * KBAR(J)= VALUE OF WAVE NUMBER
C                   *      J = 1 TO NKBAR
C
C     E.  ANGLE CONTROL CARD - ONE CARD FOR NPHI
C                   * NPHI   = NUMBER OF VERTICAL ANGLES
C
C     F.  VERTICAL ANGLE CARD - AS MANY CARDS AS REQUIRED FOR PHI.
C                   * PHI(K) = VERTICAL ANGLES IN DEGREE.
C                   *     K  = I TO NPHI
C
C     G.  ANGLE CONTROL CARD - ONE CARD FOR NALFA
C                   * NALFA  = NUMBER OF HORIZONTAL ANGLES
C
C     H.  HORIZONTAL ANGLES CARD - AS MANY CARDS AS REQUIRED FOR ALFA.
C                   * ALFA(L) = HORIZONTAL ANGLE IN DEGREE
C                   *      L  = 1 TO NALFA
C
C
C ..................................................................
C
C                      OUTPUT DESCRIPTION
C
C ..................................................................
C
C     ZETA  = DIMENSIONLESS WAVE NUMBER
C     OM(M) = PHASE VELOCITY OF PROPAGATING WAVE
C
```

```
C ..........................................................................
C
C                           SAMPLE DATA
C
C ..........................................................................
C
C  CARD A : 2   1
C  CARD B :
C        8.0 2.56  0.583 0.583 1.797 0.745 1.797 0.526 0.559 0.559 2.534
C        0.5 1.107 0.573 0.573 1.107 0.573 1.107 0.267 0.267 0.267 2.702
C  CARD C : 1
C  CARD D : 0.0
C  CARD E : 1
C  CARD F : 45
C  CARD G : 1
C  CARD H : 45
C
C ..........................................................................
C
C
C
C **** DECLARATION    ****
C
      INTEGER L1,L2,L3,N1,N2,N3
      REAL*8  PI,H(2),C11(2),C12(2),C13(2),C22(2),C23(2),
     .        C33(2),C44(2),C55(2),C66(2),RHO(2)
      REAL*8  ETA,DF,DM,RHOF,RHOM,RHOC,IFF,IMM,KHBAR,
     .        XK,XK2,ZK,ZK2,YK,KBAR(20),PHI(10),ALFA(10),ZETA(20)
      COMPLEX*16  A(12,12),B(12,12),EIGA(12),EIGB(12),OM(12),
     .            OMSQ(12),Z(12,12),WK(12,24),CIMGG,ZERO
C
      ZERO =(0.0,0.0)
      CIMGG=(0.0,1.0)
      PI   =4.*ATAN(1.0)
C
C ***** MAIN LINE PROGRAM *****
C
      CALL TRAPS(99999, 99999, 99999, 99999, 99999)
C
C THIS SUBROUTINE WILL TRAPS ANY NUMBER APPROACHING ZERO
C
C
C      READ IN NUMBER OF LAYERS
C
      READ, NLAYER ,IPRINT
C
C      READ IN PROPERTIES OF EACH LAYER
C
      READ, (H(I),C11(I),C12(I),C13(I),C22(I),C23(I),C33(I),
     .      C44(I),C55(I),C66(I),RHO(I),I=1,NLAYER)
      PRINT 10, NLAYER
 10   FORMAT ('1'////,40X,'NEMBER OF LAYERS =',3X,I2/////,1X,
     .         'LAYER PROPERTIES *'//)
      DO 20 I=1,NLAYER
         PRINT 30, I,H(I),C11(I),C12(I),C13(I),C22(I)
         PRINT 31,C23(I),C33(I),C44(I),C55(I),C66(I),RHO(I)
 30      FORMAT('-','LAYER=',I2,2X,'H =',G16.9,'C11=',G16.9,
     .            'C12=',G16.9, C13=',G16.9,'C22=',G16.9)
 31      FORMAT('0',9X,'C23=',G16.9,'C33=',G16.9,'C44=',G16.9,
     .            'C55=',G16.9,'C66=',G16.9,'RHO=',G16.9)
 20   CONTINUE
C
      READ, NKBAR
      DO 50 L1=1,NKBAR
```

```
              READ, KBAR(L1)
              ZETA(L1) = H(1)*KBAR(L1)
   50    CONTINUE
C
C ***** READ PHI IN DEGREE *****
C
         READ, NPHI
         DO 60 L2=1,NPHI
            READ, PHI(L2)
   60    CONTINUE
C
C ***** READ ALFA IN DEGREE *****
C
         READ, NALFA
         DO 65 L3=1,NALFA
            READ, ALFA(L3)
   65    CONTINUE
C
C
         DO 70 N1=1,NKBAR
            PRINT 75, KBAR(N1), ZETA(N1)
   75       FORMAT ('-',////,10X,'KBAR =',F12.8,5X,'ZETA =',F12.8//)
            DO 80 N2=1,NPHI
               ANGLE = PHI(N2)*PI/180.
               KHBAR = KBAR(N1)*COS(ANGLE)
               YK    = KBAR(N1)*SIN(ANGLE)
               DO 90 N3=1,NALFA
                  ROT =ALFA(N3)*PI/180
                  XK  =KHBAR*COS(ROT)
                  ZK  = KHBAR*SIN(ROT)
                  PRINT 100,PHI(N2),YK,ALFA(N3),XK,ZK
  100             FORMAT (' ',3X,'PHI =',F8.5,3X,'KAPAY =',F10.7,3X,
     .                     'ALFA =',F8.5,3X,'KAPAX =',F10.7,3X,
     .                     'KAPAZ =',F10.7)
                  DO 11 JJ=1,12
                     DO 22 KK=1,12
                        A(JJ,KK)=ZERO
                        B(JJ,KK)=ZERO
   22                CONTINUE
   11             CONTINUE
C
C
                  DF = H(1)
                  DM = H(2)
                  ETA= DF/(DF+DM)
                  RHOF = RHO(1)
                  RHOM = RHO(2)
                  RHOC = ETA*RHOF + (1-ETA)*RHOM
                  IFF  = DF*DF*RHOF*ETA/12.0
                  IMM  = DM*DM*RHOM*(1-ETA)/12.0
                  XK2  = XK*XK
                  ZK2  = ZK*ZK
C
C   ****** FORMATION OF A MATRIX ******
C
                  A( 1, 1)=-XK2*(ETA*C11(1)+(1-ETA)*C11(2))
     .                     -ZK2*(ETA*C55(1)+(1-ETA)*C55(2))
                  A( 1, 3)=-XK*ZK*((ETA*C55(1)+(1-ETA)*C55(2))
     .                            +(ETA*C13(1)+(1-ETA)*C13(2)))
                  A( 1, 8)= XK*ETA*C12(1)*CIMGG
                  A( 1, 9)= XK*(1.-ETA)*C12(2)*CIMGG
                  A( 1,10)=-YK*CIMGG
C
                  A( 2, 2)=-XK2*(ETA*C66(1)+(1.-ETA)*C66(2))
```

```
     .              -ZK2*(ETA*C44(1)+(1.-ETA)*C44(2))
      A( 2, 4)= XK*ETA*C66(1)*CIMGG
      A( 2, 5)= XK*(1.-ETA)*C66(2)*CIMGG
      A( 2, 6)= ZK*ETA*C44(1)*CIMGG
      A( 2, 7)= ZK*(1.-ETA)*C44(2)*CIMGG
      A( 2,11)=-YK*CIMGG
C
      A( 3, 1)= A( 1, 3)
      A( 3, 3)=-XK2*(ETA*C55(1)+(1.-ETA)*C55(2))
     .         -ZK2*(ETA*C33(1)+(1.-ETA)*C33(2))
      A( 3, 8)= ZK*ETA*C23(1)*CIMGG
      A( 3, 9)= ZK*(1.-ETA)*C23(2)*CIMGG
      A( 3,12)=-YK*CIMGG
C
      A( 4, 2)=-A( 2, 4)
      A( 4, 4)=-XK2*ETA*DF*DF*C11(1)/12.0
     .         -ZK2*ETA*DF*DF*C55(1)/12.0 - ETA*C66(1)
      A( 4, 6)=-XK*ZK*(ETA*DF*DF*C13(1)/12.0
     .                + ETA*DF*DF*C55(1)/12.0)
      A( 4,10)=-ETA
C
      A( 5, 2)=-A( 2, 5)
      A( 5, 5)=-XK2*(1.-ETA)*DM*DM*C11(2)/12.0
     .         -ZK2*(1.-ETA)*DM*DM*C55(2)/12.0-(1.-ETA)*C66(2)
      A( 5, 7)=-XK*ZK*(1.-ETA)*DM*DM*(C13(2)+C55(2))/12.0
      A( 5,10)=-(1.-ETA)
C
      A( 6, 2)=-A( 2, 6)
      A( 6, 4)= A( 4, 6)
      A( 6, 6)=-XK2*ETA*DF*DF*C55(1)/12.0
     .         -ZK2*ETA*DF*DF*C33(1)/12.0 - ETA*C44(1)
      A( 6,12)=-ETA
C
      A( 7, 2)=-A( 2, 7)
      A( 7, 5)= A( 5, 7)
      A( 7, 7)=-XK2*(1.-ETA)*DM*DM*C55(2)/12.0
     .         -ZK2*(1.-ETA)*DM*DM*C33(2)/12.-(1.-ETA)*C44(2)
      A( 7,12)=-(1.-ETA)
C
      A( 8, 1)=-A( 1, 8)
      A( 8, 3)=-A( 3, 8)
      A( 8, 8)=-XK2*ETA*DF*DF*C66(1)/12.
     .         -ZK2*ETA*DF*DF*C44(1)/12. - ETA*C22(1)
      A( 8,11)=-ETA
C
      A( 9, 1)=-A( 1, 9)
      A( 9, 3)=-A( 3, 9)
      A( 9, 9)=-XK2*(1.-ETA)*DM*DM*C66(2)/12.
     .         -ZK2*(1.-ETA)*DM*DM*C44(2)/12.-(1.-ETA)*C22(2)
      A( 9,11)=-(1.-ETA)
C
      A(10, 1)=-A( 1,10)
      A(10, 4)= A( 4,10)
      A(10, 5)= A( 5,10)
C
      A(11, 2)=-A( 2,11)
      A(11, 8)= A( 8,11)
      A(11, 9)= A( 9,11)
C
      A(12, 3)=-A( 3,12)
      A(12, 6)= A( 6,12)
      A(12, 7)= A( 7,12)
C
C ****** FORMULATION OF THE B MATRIX ******
```

```
C
            B( 1,  1)=-RHOC
            B( 2,  2)=-RHOC
            B( 3,  3)=-RHOC
            B( 4,  4)=-IFF
            B( 5,  5)=-IMM
            B( 6,  6)=-IFF
            B( 7,  7)=-IMM
            B( 8,  8)=-IFF
            B( 9,  9)=-IMM
C
C
C
            IJOB=1
            IA  =12
            IB  =12
            IZ  =12
            N   =12
            CALL EIGZC(A,IA,B,IB,N,IJOB,EIGA,EIGB,Z,IZ,WK
     .                  ,INFER,IER)
            DO 120 M=1,12
               OMSQ(M)=EIGA(M)/EIGB(M)
               OM(M)   =CDSQRT(OMSQ(M))/((C44(2)/RHOM)**0.5)
               OM(M)   =OM(M)/KBAR(N1)
               PRINT 125, M, OM(M)
 125           FORMAT ('0',1X,I2,5X,2G16.9)
 120        CONTINUE
            IF (IPRINT .GT. 0) THEN DO
               IV=1
               PRINT 140, IV
 140           FORMAT ('0'//,1X,'M',10X,'THE CORRESPONDING Z MATRIX'
     .                  ' OF CELL ',I1,1X,'IS :'//)
               DO 150 K=1,12
                  PRINT 160, K, (Z(L,K),L=1,12)
 160              FORMAT ('0',I2,1X,6(2F10.6,1X)/,4X,6(2F10.6,1X))
 150           CONTINUE
               IPRINT=IPRINT-1
            END IF
C
  90        CONTINUE
  80     CONTINUE
  70  CONTINUE
      STOP
      END
$ENTRY
```

A P P E N D I X J

FORTRAN PROGRAM FOR THE FINITE ELEMENT METHOD

There are two main line computer programs written for the finite element method outlined in Chapter 2. The first program read the value of $\bar{k}$ travelling in any arbitrary direction that make a horizontal angle, α, with the x-axis and a vertical angle, ϕ, with respect to the x-z plane. The components of $\bar{k}$ are calculated by taking sine and cosine of the angles. The other main line program will read in $\bar{k}$ and the y-component, $\bar{\eta}$ and angle, α, it makes with respect to the x-axis in (x-z) plane. Only the second program will be listed in this Appendix. However, it does not cause much difficulties in converting one program to the other. The procedure of modification will be to read the vertical angle, ϕ, as input and y-component of $\bar{k}$ is calculated by taking the sine of angle ϕ.

```
C
C      ***********************************************************
C      *                                                         *
C      *        HARMONIC WAVE PROPAGATION ANALYSIS PROGRAM       *
C      *                          BY                             *
C      *                   JOHNNY K.T. YEO                       *
C      *               THE UNIVERSITY OF MANITOBA                *
C      *                   DECEMBER, 1982                        *
C      ***********************************************************
C
C NOTES :
C          1) ALL INPUTS ARE FORMAT-FREE EXCEPT FOR ETA.
C          2) NP,IPRINT,NKHBAR,NALFA,NETA ARE INTEGER VALUES
C          3) H,C,RHO,KHBAR ARE REAL
C
C ..........................................................................
C
C                         INPUT DESCRIPTION
C
C ..........................................................................
C
C     A.  START CARD - ONE CARD FOR NP AND IPRINT.
C                      * NP     = NUMBER OF PERIODICITY
C                      * IPRINT = NUMBER OF SETS OF VECTORS AS OUTPUT
C     B.  MATERIAL PROPERTIES CARD - (NP+1) OF CARDS FOR :
C                        H(I),C11(I),C12(I),C13(I),C22(I),C23(I),
C                        C33(I),C44(I),C55(I),C66(I),RHO(I)
C                      * H(I)    = LAYER THICKNESS
C                      * C(I)    = MATERIAL CONSTANTS OF THE LAYER
C                      * RHO(I)  = DENSITY OF THE MATERIAL
C                      *      I  = 1 TO (NP+1)
C     C.  BASIC CONTROL CARD - ONE CARD FOR NKHBAR.
C                      * NKHBAR = NUMBER OF HORIZONTAL WAVE NUMBER TO BE
C                                 EVALUATED
C     D.  WAVE NUMBER CARD - AS MANY CARDS AS REQUIRED FOR KHBAR.
C                      * KBAR(J)= VALUE OF WAVE NUMBER
C                              J = 1 TO NKHBAR
C     E.  ANGLE CONTROL CARD - ONE CARD FOR NALFA
C                      * NALFA  = NUMBER OF HORIZONTAL ANGLES
C     F.  HORIZONTAL ANGLE CARD - AS MANY CARDS AS REQUIRED FOR ALFA.
C                      * ALFA(K)= HORIZONTAL ANGLES IN DEGREE.
C                      *      K = 1 TO NALFA
C     G.  VERTICAL COMPONENT CONTROL CARD - ONE CARD FOR NETA
C                      * NETA   = NUMBER OF VERTICAL COMPONENT OF
C                                 WAVE NUMBER
C     H.  VERTICAL COMPONENT CARD - AS MANY CARDS AS REQUIRED FOR ETA.
C                      * ETA(L) = VERTICAL DIMENSIONLESS WAVE NUMBER
C                            L  = 1 TO NETA
C
C                              FORMAT (2F10.7)
C
C
C ..........................................................................
C
C                         OUTPUT DESCRIPTION
```

```
C .....................................................................
C
C     ZETA  = DIMENSIONLESS WAVE NUMBER
C     OM(M) = ANGULAR FREQUENCY OF PROPAGATING WAVE
C
C .....................................................................
C
C                         SAMPLE DATA
C
C .....................................................................
C
C  CARD A : 4    1
C  CARD B :
C       4.0 2.56  0.583 0.583 1.797 0.745 1.797 0.526 0.559 0.559 2.534
C       4.0 2.56  0.583 0.583 1.797 0.745 1.797 0.526 0.559 0.559 2.534
C       0.5 1.107 0.573 0.573 1.107 0.573 1.107 0.267 0.267 0.267 2.702
C       0.5 1.107 0.573 0.573 1.107 0.573 1.107 0.267 0.267 0.267 2.702
C       4.0 2.56  0.583 0.583 1.797 0.745 1.797 0.526 0.559 0.559 2.534
C  CARD C : 1
C  CARD D : 2.0
C  CARD E : 1
C  CARD F : 45
C  CARD G : 1
C  CARD H :    2    (FORMAT 2F10.7)
C
C .....................................................................
C
C
C
C
C      **********   DECLARATION    ***********
C
      INTEGER NP1, NP2, IC2
      REAL *8    H(5),C11(5),C12(5),C13(5),C22(5),C23(5),C33(5),
     .           C44(5),C55(5),C66(5),RHO(5),ALFA(15),DEPTH
      REAL *8    A(4,4),B(4,4),D(4,4),KHBAR(45),XK,ZK,PI,KAPPA
      COMPLEX *16     AK(36,36), BK(36,36), CK(36,36), DK(36,36),
     &                AM(36,36), BM(36,36), DZ(36,36), EIGA(36),
     &                EIGB(36), OMSQ(36), OM(36), ETA(50),
     &                WK(24,48), E, YK, Z(36,36), CIMGG, ZERO
C
C
      COMMON/BLK1/ NP1, NP2, IC2
      COMMON/BLK2/CIMGG, ZERO
      COMMON/BLK3/ PI
C
C
      ZERO=(0.0 ,0.0)
      CIMGG=(0.0, 1.0)
      PI=4. * ATAN(1.0)
C
C
C      **********   MAIN LINE PROGRAM   ************
C
      CALL TRAPS(99999, 99999, 99999, 99999, 99999)
C
C THIS SUBROUTINE WILL TRAPS ANY NUMBER APPROACHING ZERO
C
C  **** READ IN NUMBER OF PERIODICITY  ****
C
C  **** IPRINT IS THE NUMBER OF EIGENVECTORS TO BE PRINTED ****
C
      READ, NP , IPRINT
      NP1= NP + 1
```

```
      NP2= NP + 2
      IC2= 6 * NP2
C
C READ IN PROPERTIES OF NP + 1 LAYERS.
C
      READ, (H(I),C11(I),C12(I),C13(I),C22(I),C23(I),C33(I),
     .       C44(I),C55(I),C66(I),RHO(I),I=1,NP1)
      PRINT 30, NP
   30 FORMAT('1'////' ', 40X, 'PERIODICITY =', 3X, I2/////
     *          ' ', 1X, 'LAYER PROPERTIES :'//)
C
C
      X1 = RHO(NP) / C55(NP)
      XMULT= SQRT(X1)
      DO 40 I=1, NP1
      PRINT 50, I,H(I),C11(I),C12(I),C13(I),C22(I)
      PRINT 51, C23(I),C33(I),C44(I),C55(I),C66(I),RHO(I)
   50 FORMAT('-','LAYER=',I2,2X,'H =',G16.9,
     .          'C11=',G16.9,'C12=',G16.9,'C13=',G16.9,
     .          'C22=',G16.9)
   51 FORMAT('0',9X,'C23=',G16.9,'C33=',G16.9,'C44=',G16.9,
     .          'C55=',G16.9,'C66=',G16.9,'RHO=',G16.9)
   40 CONTINUE
C
      ICOUNT = 6 * NP
      DEPTH  = 0.0
      DO 10 I=1, NP
         DEPTH = DEPTH + H(I)
   10 CONTINUE
C
C READ IN KAPPABAR
C
      READ, NKHBAR
      DO 35 J=1, NKHBAR
         READ, KHBAR(J)
   35 CONTINUE
C
C READ ALFA IN DEGREE
C
      READ, NALFA
      DO 37 J=1, NALFA
         READ, ALFA(J)
   37 CONTINUE
C
C READ IN ETA (KAPPAY)
C
      READ, NETA
      DO 52 J=1, NETA
         READ 55, ETA(J)
   55    FORMAT (2F10.7)
   52 CONTINUE
C
C     READ, NCELL
C
      CALL INFORM(A, B, D)
C
C
C
      DO 60 L1=1, NKHBAR
         KAPPA = KHBAR(L1) * PI / 2.
C
         DO 70 L2=1, NALFA
            ANGLE = ALFA(L2) * PI / 180.
            XK    = KHBAR(L1)*COS(ANGLE)*PI/2.
```

```
                ZK     = KHBAR(L1)*SIN(ANGLE)*PI/2.
                PRINT 66,KAPPA, ALFA(L2), XK, ZK ,KHBAR(L1)
      66        FORMAT ('-'//'-',5X,'KAPPA=',F10.5,3X,'ANGLE =',F10.5,
     .                 3X,'KAPAX=',F10.5,3X,'KAPAZ=',F10.5,3X,
     .                 '( KAPPABAR=',F10.5,3X,')')
C
             DO 11 JJ=1, IC2
               DO 22 KK=1, IC2
                 AK(JJ, KK) = ZERO
                 BK(JJ, KK) = ZERO
                 CK(JJ, KK) = ZERO
                 DK(JJ, KK) = ZERO
                 AM(JJ, KK) = ZERO
                 BM(JJ, KK) = ZERO
      22       CONTINUE
      11    CONTINUE
C
C        **** FIRST FORM THE MATRICES INVOLVING UUBAR ****
C
          CALL UUBAR(NP,H,C11,C55,C66,RHO,XK,ZK,A,1,AK)
C
C
          CALL UUBAR(NP,H,C11,C55,C66,RHO,XK,ZK,A,2,BK)
C
          CALL ADDING(NP,AK,BK)
C
C
          CALL UUBAR(NP,H,C11,C55,C66,RHO,XK,ZK,B,3,BK)
C
          CALL ADDING(NP, AK, BK)
C
C
          CALL UUBAR(NP,H,C11,C55,C66,RHO,XK,ZK,A,4,AM)
C
C
C
C *****  THEN FORM THE MATRICES INVOLVING VVBAR  *****
C
          CALL VVBAR(NP,H,C12,C22,C23,C44,C66,RHO,XK,ZK,B,1,BK)
C
          CALL ADDING(NP, AK, BK)
C
          CALL VVBAR(NP,H,C12,C22,C23,C44,C66,RHO,XK,ZK,A,2,BK)
C
          CALL ADDING(NP, AK, BK)
C
          CALL VVBAR(NP,H,C12,C22,C23,C44,C66,RHO,XK,ZK,A,3,BK)
C
          CALL ADDING(NP, AK, BK)
C
          CALL VVBAR(NP,H,C12,C22,C23,C44,C66,RHO,XK,ZK,A,4,BM)
C
          CALL ADDING(NP, AM, BM)
C
C         ****  FORM THE MATRICES INVOLVING WWBAR  ****
C
          CALL WWBAR(NP,H,C33,C44,C55,RHO,XK,ZK,A,1,BK)
C
          CALL ADDING(NP, AK, BK)
C
          CALL WWBAR(NP,H,C33,C44,C55,RHO,XK,ZK,B,2,BK)
C
          CALL ADDING(NP, AK, BK)
C
```

```
      CALL WWBAR(NP,H,C33,C44,C55,RHO,XK,ZK,A,3,BK)
C
      CALL ADDING(NP, AK, BK)
C
      CALL WWBAR(NP,H,C33,C44,C55,RHO,XK,ZK,A,4,BM)
C
      CALL ADDING(NP, AM, BM)
C
C     ****  FORM THE MATRICES INVOLVING UVBAR DOING :        ****
C     ****        (UVBAR' - V'UBAR) & (VUBAR' - U'VBAR)      ****
C
      CALL UVBAR(NP,H,C12,C22,C23,C66,XK,ZK,D,1,BK)
C
      CALL ADDING(NP, AK, BK)
C
      CALL UVBAR(NP,H,C12,C22,C23,C66,XK,ZK,D,2,BK)
C
      CALL ADDING(NP, AK, BK)
C
C     ****  FORM THE MATRICES INVOLVING WUBAR DOING :        ****
C     ****        (WUBAR + UWBAR)                            ****
C
      CALL WUBAR(NP,H,C13,C44,C55,C66,XK,ZK,A,1,BK)
C
      CALL ADDING(NP, AK, BK)
C
      CALL WUBAR(NP,H,C13,C44,C55,C66,XK,ZK,A,2,BK)
C
      CALL ADDING(NP, AK, BK)
C
C     ****  FORM THE MATRICES INVOLVING WVBAR DOING          ****
C     ****        (WVBAR' - V'WBAR) & (VWBAR' - W'VBAR)      ****
C
      CALL WVBAR(NP,H,C12,C22,C23,C44,XK,ZK,D,1,BK)
C
      CALL ADDING(NP, AK, BK)
C
      CALL WVBAR(NP,H,C12,C22,C23,C44,XK,ZK,D,2,BK)
C
      CALL ADDING(NP, AK, BK)
C
C
C  *******   TO SHIFT UP AK AND AM BY 6 ROWS      *********
C
      NP6 =6 * NP
      NPSQ=6 * NP2
C
      DO 33 I=1, NP6
         IR = I + 6
         DO 44 J=1, NPSQ
            AK(I, J) = AK(IR, J)
            AM(I, J) = AM(IR, J)
   44    CONTINUE
   33 CONTINUE
C
C
C  *** THE FOLLOWING WILL FORM THE 'D' MATRIX (WHICH IS PART OF THE
C      THE ASSEMBLY MATRIX AND TO BE MODIFIED) .  THE D MATRIX OF
C      THE K MATRIX IS STORED IN BK, WHILE THAT OF THE M MATRIX IS
C      STORED IN BM.  ***
C
C
      CALL FORMDD (NP, AK, AM, BK, BM)
C
```

```
C
            DO 77 J=1, NETA
               YK = PI / 2. * ETA(J)
               E  = CDEXP(YK * 2. *  DEPTH * CIMGG)
               DO 80 K=1, ICOUNT
                  DO 90 L=1, ICOUNT
                     CK(K, L) = AK(K, L) +  E * BK(K, L)
                     DK(K, L) = - ( AM(K, L) +  E * BM(K, L) )
   90             CONTINUE
   80          CONTINUE
C
C
               IJOB =  1
               IA   = IC2
               IB   = IC2
               IZ   = IC2
               N    = ICOUNT
               CALL EIGZC (CK, IA, DK, IB, N, IJOB, EIGA, EIGB, Z,
     *                     IZ, WK, INFER, IER)
C
C
               PRINT 100, ETA(J)
  100          FORMAT('-', 44X, 'ETA =', 3X, 2G16.9/)
               PRINT 105
  105          FORMAT('0', T4, 'M', T25, 'OMEGA(M)', T65, 'EIGA(M)',
     *                T109, 'EIGB(M)'/)
C
               DO 110 M=1, ICOUNT
                  OMSQ(M) = EIGA(M) / EIGB(M)
                  OM(M)   = CDSQRT( OMSQ(M) ) * 2.DO / PI
                  OM(M)   = OM(M) * XMULT
                  PRINT 120, M, OM(M), EIGA(M), EIGB(M)
  120             FORMAT('0', 1X, I2, 6(5X, G16.9))
  110          CONTINUE
C
               IF (IPRINT .GT. 0) THEN DO
C                 DO 503 IN=1,ICOUNT
C                    DO 504 JN=1,ICOUNT
C                       DZ(IN,JN)=Z(IN,JN)
C504              CONTINUE
C503           CONTINUE
C
C           ** DZ IS USED TO CALCULATE THE MODULI **
C
               IV=1
               PRINT 140, IV
 140           FORMAT ('-'//' ',1X,'M',10X,'THE CORRESPONDING Z MATRIX OF'
     *                 ' CELL ',I1,1X,'IS :'//)
               DO 151 K=1, ICOUNT
                  PRINT 161, K, (Z(L,K),L=1,ICOUNT)
 161              FORMAT ('0',I2,1X,6(2F10.6,1X)/,4X,6(2F10.6,1X)/,
     .                    4X,6(2F10.6,1X)/,4X,6(2F10.6,1X)//)
 151           CONTINUE
C
C THE FOLLOWING SHOULD BE INCLUDED IF MORE THAN 1 UNIT CELL IS USED.
C NCELL IS THE NUMBER OF UNIT CELL USED
C
C
C              DO 141 IV=2,NCELL
C                 PRINT 140, IV
C                 DO 142 JJ=1,ICOUNT
C                    DO 143 KK=1,ICOUNT
C                       Z(JJ,KK)=Z(JJ,KK)*E
C143                CONTINUE
```

```
C                 PRINT 161, JJ, (Z(LL,JJ),LL=1,ICOUNT)
C142            CONTINUE
C141         CONTINUE
C
C
C
C   THE ABOVE PROCEDURE EVALUATES ALL THE Z MATRICES OF NCELL
C
C   THE SAME PROCEDURE IS EMPLOYED TO EVALUATE THE CORRESPONDING
C   MODULI OF THESE Z MATRICES
C
C      IV=1
C      PRINT 170, IV
C170    FORMAT('-'//' ',1X,'M',10X,'THE CORRESPONDING MODULI OF Z'
C     .      ,' MATRIX OF CELL ',I1,1X,'IS :'//)
C      DO 180 K=1,ICOUNT
C         PRINT 190,K,(CDABS(DZ(L,K)),L=1,ICOUNT)
C190       FORMAT('0',I2,5X,6(F10.6,2X)/,8X,6(F10.6,2X)/,
C     .                 8X,6(F10.6,2X)/,8X,6(F10.6,2X)//)
C     .                 8X,6(F10.6,2X)/,8X,6(F10.6,2X))
C180    CONTINUE
C      DO 1300 IV=2,NCELL
C         PRINT 170, IV
C         DO 1400 JJ=1,ICOUNT
C            DO 1500 KK=1,ICOUNT
C               DZ(JJ,KK)=DZ(JJ,KK)*E
C1500         CONTINUE
C            PRINT 190,JJ,(CDABS(DZ(LL,JJ)),LL=1,ICOUNT)
C1400      CONTINUE
C1300   CONTINUE
C506   CONTINUE
C
C
                  IPRINT=IPRINT-1
               END IF
   77          CONTINUE
   70       CONTINUE
   60 CONTINUE
      STOP
      END
C
C
C
C*****************************************************************
C
C                  SUBROUTINE INFORM                             *
C                                                                *
C*****************************************************************
C
C
      SUBROUTINE INFORM(A, B, D)
C
C
      REAL*8  A(4,4), B(4,4), D(4,4)
C
C     *******    TO FORM THE MATRICES A, B AND D     *******
C
      A(1, 1) = 78. / 105
      A(2, 1) = A(1, 2) = 9. / 35.
      A(3, 1) = A(1, 3) = 22. / 105.
      A(4, 1) = A(1, 4) = - 13. / 105.
C
      A(2, 2) = 78. / 105.
      A(3, 2) = A(2, 3) = 13. / 105.
```

```
      A(4, 2) = A(2, 4) = - 22. / 105.
C
      A(3, 3) = 8. / 105.
      A(4, 3) = A(3, 4) = - 6. / 105.
C
      A(4, 4) = 8. / 105.
C
C
      B(1, 1) = 3. / 5.
      B(2, 1) = B(1, 2) = - 3. / 5.
      B(3, 1) = B(1, 3) = 1. / 10.
      B(4, 1) = B(1, 4) = 1. / 10.
C
      B(2, 2) = 3. / 5.
      B(3, 2) = B(2, 3) = - 1. / 10.
      B(4, 2) = B(2, 4) = - 1. / 10.
C
      B(3, 3) = 4. / 15.
      B(4, 3) = B(3, 4) = - 1. / 15.
C
      B(4, 4) = 4. / 15.
C
C
      D(1, 1) = - 1. / 2.
      D(1, 2) = 1. / 2.
      D(1, 3) = 1. / 5.
      D(1, 4) = - 1. / 5.
C
      D(2, 1) = - D(1, 2)
      D(2, 2) = 1. / 2.
      D(2, 3) = - 1. / 5.
      D(2, 4) = 1. / 5.
C
      D(3, 1) = - D(1, 3)
      D(3, 2) = - D(2, 3)
      D(3, 3) = 0.0D0
      D(3, 4) = - 1. / 15.
C
      D(4, 1) = - D(1, 4)
      D(4, 2) = - D(2, 4)
      D(4, 3) = - D(3, 4)
      D(4, 4) = 0.0D0
C
      RETURN;END
C
C
C
C
C
C**********************************************************
C                                                         *
C                 SUBROUTINE  UUBAR                       *
C                                                         *
C**********************************************************
C
C
      SUBROUTINE UUBAR (NP,H,C11,C55,C66,RHO,XK ZK,AB,N,AKNAM)
C
C
      COMMON/BLK1/NP1, NP2, IC2
      COMMON/BLK2/CIMGG, ZERO
      REAL *8          H(NP1),C11(NP1),C55(NP1),C66(NP1),RHO(NP1),
     *                 AB(4, 4), XK, XK2, ZK, ZK2
      COMPLEX *16      AKNAM(IC2,IC2), U(12,12), CONST ,CIMGG, ZERO
```

```
C
      XK2 = XK * XK
      ZK2 = ZK * ZK
      DO 1000 JJ=1, IC2
         DO 1050 KK=1, IC2
            AKNAM(JJ, KK) = ZERO
 1050    CONTINUE
 1000 CONTINUE
C
      DO 991 L1=1, 12
         DO 992 L2=1, 12
            U(L1,L2) = ZERO
  992    CONTINUE
  991 CONTINUE
      DO 10 I=1, NP1
C
         U(1, 1) = AB(1, 1)
         U(1, 2) = AB(1, 3) * H(I) / C66(I)
         U(2, 1) =  U(1, 2)
         U(1, 3) = - AB(1, 3) * H(I) * XK * CIMGG
         U(3, 1) = -  U(1, 3)
         U(1, 4) =  U(4, 1) = ZERO
         U(1, 7) = AB(1, 2)
         U(7, 1) =  U(1, 7)
         U(1, 8) = AB(1, 4) * H(I) / C66(I)
         U(8, 1) =  U(1, 8)
         U(1, 9) = - AB(1, 4) * H(I) * XK * CIMGG
         U(9, 1) = -  U(1, 9)
         U(1,10) =  U(10,1) = ZERO
C
         U(2, 2) = AB(3, 3) * H(I) * H(I) / C66(I) / C66(I)
         U(2, 3) =-AB(3, 3)*H(I)*H(I)*XK*CIMGG/C66(I)
         U(3, 2) =- U(2, 3)
         U(2, 7) = AB(2, 3) * H(I) / C66(I)
         U(7, 2) =  U(2, 7)
         U(2, 8) = AB(3, 4) * H(I) * H(I) / C66(I) / C66(I)
         U(8, 2) = U(2, 8)
         U(2, 9) = - AB(3, 4) * H(I) * H(I) * XK * CIMGG / C66(I)
         U(9, 2) = - U(2, 9)
C
         U(3, 3) = AB(3, 3) * H(I) * H(I) * XK2
         U(3, 4) =  U(4, 3) =  ZERO
         U(3, 7) = AB(2, 3) * H(I) * XK * CIMGG
         U(7, 3) = - U(3, 7)
         U(3, 8) = AB(3, 4) * H(I) * H(I) * XK * CIMGG / C66(I)
         U(8, 3) = - U(3, 8)
         U(3, 9) = AB(3, 4) * H(I) * H(I) * XK2
         U(9, 3) =  U(3, 9)
C
C
         U(7, 7) = AB(2, 2)
         U(7, 8) = AB(2, 4) * H(I) / C66(I)
         U(8, 7) =  U(7, 8)
         U(7, 9) = - AB(2, 4) * H(I) * XK * CIMGG
         U(9, 7) = -  U(7, 9)
C
         U(8, 8) = AB(4, 4) * H(I) * H(I) / C66(I) / C66(I)
         U(8, 9) =-AB(4, 4)*H(I)*H(I)*XK*CIMGG/C66(I)
         U(9, 8) =- U(8, 9)
C
         U(9, 9) = AB(4, 4) * H(I) * H(I) * XK2
C
C        PRINT, ((U(L1,L2), L1=1,12), L2=1,12)
C
```

```
C
         IF ( N .EQ. 1 ) THEN DO
            CONST = C11(I) * XK2 * H(I)
         ELSE DO
            IF ( N .EQ. 2 ) THEN DO
               CONST = C55(I) * H(I) *ZK2
            ELSE DO
               IF (N .EQ. 3) THEN DO
                  CONST = C66(I) / H(I)
               ELSE DO
                  CONST =-RHO(I) * H(I)
               END IF
            END IF
         END IF
C
C
         ITIME = I - 1
         DO 20 J=1, 12
            IR = J + 6 * ITIME
            DO 30 K=1, 12
               IC = K + 6 * ITIME
               AKNAM(IR, IC) = AKNAM(IR, IC) + CONST * U(J, K)
   30       CONTINUE
   20    CONTINUE
   10 CONTINUE
C
      RETURN;END
C
C
C
C
C**********************************************************
C                                                         *
C            SUBROUTINE VVBAR                             *
C                                                         *
C**********************************************************
C
C
      SUBROUTINE VVBAR (NP,H,C12,C22,C23,C44,C66,RHO,XK,ZK,AB,N,BKNBM)
C
      COMMON/BLK1/ NP1, NP2, IC2
      COMMON/BLK2/ CIMGG, ZERO
      REAL *8       H(NP1),C12(NP1),C22(NP1),C23(NP1),C44(NP1),
     *              C66(NP1), RHO(NP1), XK, XK2, ZK, ZK2, AB(4, 4)
      COMPLEX *16   BKNBM(IC2,IC2), V(12,12), CONST, CIMGG, ZERO
C
      XK2 = XK * XK
      ZK2 = ZK * ZK
C
      DO 1000 JJ=1, IC2
         DO 1050 KK=1, IC2
            BKNBM(JJ, KK) = ZERO
 1050    CONTINUE
 1000 CONTINUE
C
      DO 991 L1=1, 12
         DO 992 L2=1, 12
            V(L1,L2) = ZERO
  992    CONTINUE
  991 CONTINUE
C
      DO 10 I=1, NP1
C
         V(1, 1) = AB(3, 3) * H(I) * H(I) * XK2 * C12(I) * C12(I)
```

```
     *                 / C22(I) / C22(I)
         V(1, 2) = V(2, 1) = ZERO
         V(1, 3) = AB(1, 3) * H(I) * XK * C12(I) * CIMGG / C22(I)
         V(3, 1) = - V(1, 3)
         V(1, 4) =AB(3, 3)*H(I)*H(I)*C12(I)*XK*CIMGG/C22(I)/C22(I)
         V(4, 1) =-V(1, 4)
         V(1, 5) = AB(3,3)*ZK*XK*H(I)*H(I)*C12(I)*C23(I)/C22(I)/C22(I)
         V(5, 1) = V(1, 5)
         V(1, 7) = AB(3, 4) * H(I) * H(I) * XK2 * C12(I) * C12(I)
     *                 / C22(I) / C22(I)
         V(7, 1) = V(1, 7)
         V(1, 8) = V(8, 1) = ZERO
         V(1, 9) = AB(2, 3) * H(I) * XK * C12(I) * CIMGG / C22(I)
         V(9, 1) = - V(1, 9)
         V(1,10) = AB(3, 4) * H(I) * H(I) * XK * C12(I) * CIMGG
     *                 / C22(I) / C22(I)
         V(10,1) = -V(1,10)
         V(1,11) = AB(3, 4)*ZK*XK*H(I)*H(I)*C12(I)*C23(I)/C22(I)/C22(I)
         V(11,1) = V(1,11)
C
C
         V(3, 3) = AB(1, 1)
         V(3, 4) = AB(1, 3) * H(I) / C22(I)
         V(4, 3) = V(3, 4)
         V(3, 5) =-AB(1, 3)*ZK*H(I)*C23(I)*CIMGG/C22(I)
         V(5, 3) =-V(3, 5)
         V(3, 7) = - AB(1, 4) * H(I) * XK * C12(I) * CIMGG / C22(I)
         V(7, 3) = - V(3, 7)
         V(3, 8) = V(8, 3) = ZERO
         V(3, 9) = AB(1, 2)
         V(9, 3) =  V(3, 9)
         V(3,10) = AB(1, 4) * H(I) / C22(I)
         V(10,3) =  V(3,10)
         V(3,11) =-AB(1, 4)*ZK*H(I)*C23(I)*CIMGG/C22(I)
         V(11,3) =-V(3,11)
C
         V(4, 4) = AB(3, 3) * H(I) * H(I) / C22(I) / C22(I)
         V(4, 5) =-AB(3,3)*H(I)*H(I)*ZK*C23(I)*CIMGG/C22(I)/C22(I)
         V(5, 4) =-V(4, 5)
         V(4, 7) = - AB(3, 4) * H(I) * H(I) * XK * C12(I) * CIMGG
     *                 / C22(I) / C22(I)
         V(7, 4) = - V(4, 7)
         V(4, 8) = V(8, 4) = ZERO
         V(4, 9) = AB(2, 3) * H(I) / C22(I)
         V(9, 4) = V(4, 9)
         V(4,10) = AB(3, 4) * H(I) * H(I) / C22(I) / C22(I)
         V(10,4) = V(4,10)
         V(4,11) =-AB(3,4)*H(I)*H(I)*ZK*C23(I)*CIMGG/C22(I)/C22(I)
         V(11,4) =-V(4,11)
C
         V(5, 5) = AB(3,3)*H(I)*H(I)*ZK2*C23(I)*C23(I)/C22(I)/C22(I)
         V(5, 7) = AB(3,4)*H(I)*H(I)*ZK*XK*C12(I)*C23(I)
     .                 /C22(I)/C22(I)
         V(7, 5) = V(5, 7)
         V(5, 9) = AB(2,3)*H(I)*ZK*C23(I)*CIMGG/C22(I)
         V(9, 5) =-V(5, 9)
         V(5,10) = AB(3,4)*H(I)*H(I)*ZK*C23(I)*CIMGG/C22(I)/C22(I)
         V(10,5) =-V(5,10)
         V(5,11) = AB(3,4)*H(I)*H(I)*ZK2*C23(I)*C23(I)/C22(I)/C22(I)
         V(11,5) = V(5, 11)
C
         V(7, 7) = AB(4, 4) * H(I) * H(I) * XK2 * C12(I) * C12(I)
     *                 / C22(I) / C22(I)
         V(7, 8) = V(8, 7) = ZERO
```

```
         V(7, 9) = AB(2, 4) * H(I) * XK * C12(I) * CIMGG / C22(I)
         V(9, 7) = - V(7, 9)
         V(7,10) = AB(4, 4)*H(I)*H(I)*XK*CIMGG*C12(I)/C22(I)/C22(I)
         V(10,7) =-V(7,10)
         V(7,11) = AB(4,4)*H(I)*H(I)*ZK*XK*C12(I)*C23(I)/C22(I)/C22(I)
         V(11,7) = V(7,11)
C
C
         V(9, 9) = AB(2, 2)
         V(9,10) = AB(2, 4) * H(I) / C22(I)
         V(10,9) = V(9,10)
         V(9,11) =-AB(2,4)*H(I)*ZK*C23(I)*CIMGG/C22(I)
         V(11,9) =-V(9,11)
C
         V(10,10) = AB(4, 4) * H(I) * H(I) / C22(I) / C22(I)
         V(10,11)=-AB(4,4)*H(I)*H(I)*ZK*C23(I)*CIMGG/C22(I)/C22(I)
         V(11,10)=-V(10,11)
         V(11,11)= AB(4,4)*H(I)*H(I)*ZK2*C23(I)*C23(I)/C22(I)/C22(I)
C
C
         IF ( N .EQ. 1 ) THEN DO
            CONST = C22(I) / H(I)
         ELSE DO
            IF ( N .EQ. 2 ) THEN DO
               CONST = C44(I) * ZK2 * H(I)
            ELSE DO
               IF (N .EQ. 3) THEN DO
                  CONST=C66(I)*XK2*H(I)
               ELSE DO
                  CONST=-RHO(I)*H(I)
               END IF
            END IF
         END IF
C
         ITIME = I - 1
         DO 20 J=1, 12
            IR = J + 6 * ITIME
            DO 30 K=1, 12
               IC = K + 6 * ITIME
               BKNBM(IR, IC) = BKNBM(IR, IC) + CONST * V(J, K)
   30       CONTINUE
   20    CONTINUE
   10 CONTINUE
C
      RETURN;END
C
C****************************************************************
C
C                  SUBROUTINE WWBAR                              *
C                                                                *
C****************************************************************
C
C
      SUBROUTINE WWBAR(NP,H,C33,C44,C55,RHO,XK,ZK,AB,N,CKNCM)
C
C
      COMMON/BLK1/NP1,NP2,IC2
      COMMON/BLK2/CIMGG,ZERO
      REAL*8    H(NP1),C33(NP1),C44(NP1),C55(NP1),RHO(NP1),
     @          AB(4,4),XK,XK2,ZK,ZK2
      COMPLEX*16  CKNCM(IC2,IC2),W(12,12),CONST,CIMGG,ZERO
C
      XK2=XK*XK
      ZK2=ZK*ZK
```

```
      DO 1000 JJ=1,IC2
         DO 1050 KK=1,IC2
            CKNCM(JJ,KK) = ZERO
 1050    CONTINUE
 1000 CONTINUE
C
      DO 991 L1=1, 12
         DO 992 L2=1, 12
            W(L1,L2) = ZERO
  992    CONTINUE
  991 CONTINUE
      DO 10 I=1,NP1
         W(3 ,3)= AB(3,3)*ZK2*H(I)*H(I)
         W(3 ,5) =AB(1,3)*ZK*H(I)*CIMGG
         W(5 ,3)=-W(3 ,5)
         W(3, 6)= AB(3,3)*H(I)*H(I)*ZK*CIMGG/C44(I)
         W(6, 3)=-W(3, 6)
         W(3, 9)= AB(3,4)*H(I)*H(I)*ZK2
         W(9, 3)= W(3, 9)
         W(3,11)= AB(2,3)*H(I)*ZK*CIMGG
         W(11,3)=-W(3,11)
         W(3,12)= AB(3,4)*H(I)*H(I)*ZK*CIMGG/C44(I)
         W(12,3)=-W(3,12)
C
         W(5, 5)= AB(1,1)
         W(5, 6)= AB(1,3)*H(I)/C44(I)
         W(6, 5)= W(5, 6)
         W(5, 9)=-AB(1,4)*H(I)*ZK*CIMGG
         W(9, 5)=-W(5, 9)
         W(5,11)= AB(1,2)
         W(11,5)= W(5,11)
         W(5,12)= AB(1,4)*H(I)/C44(I)
         W(12,5)= W(5,12)
C
         W(6, 6)= AB(3,3)*H(I)*H(I)/C44(I)/C44(I)
         W(6, 9)=-AB(3,4)*H(I)*H(I)*ZK*CIMGG/C44(I)
         W(9, 6)=-W(6, 9)
         W(6,11)= AB(2,3)*H(I)/C44(I)
         W(11,6)= W(6,11)
         W(6,12)= AB(3,4)*H(I)*H(I)/C44(I)/C44(I)
         W(12,6)= W(6,12)
C
         W(9, 9)= AB(4,4)*H(I)*H(I)*ZK2
         W(9,11)= AB(2,4)*H(I)*ZK*CIMGG
         W(11,9)=-W(9,11)
         W(9,12)= AB(4,4)*H(I)*H(I)*ZK*CIMGG/C44(I)
         W(12,9)=-W(9,12)
C
         W(11,11)= AB(2,2)
         W(11,12)= AB(2,4)*H(I)/C44(I)
         W(12,11)= W(11,12)
C
         W(12,12)= AB(4,4)*H(I)*H(I)/C44(I)/C44(I)
C
         IF (N .EQ. 1) THEN DO
            CONST = C33(I)*ZK2*H(I)
         ELSE DO
            IF (N .EQ. 2) THEN DO
               CONST = C44(I)/H(I)
            ELSE DO
               IF (N .EQ. 3) THEN DO
                  CONST = C55(I)*XK2*H(I)
               ELSE DO
                  CONST =-RHO(I)*H(I)
```

```
                END IF
             END IF
          END IF
C
          ITIME = I - 1
          DO 20 J=1, 12
             IR = J + 6 * ITIME
             DO 30 K=1, 12
                IC = K + 6 * ITIME
                CKNCM(IR, IC) = CKNCM(IR, IC) + CONST * W(J, K)
   30        CONTINUE
   20     CONTINUE
   10 CONTINUE
C
C
       RETURN;END
C
C
C
C****************************************************************
C                                                               *
C                 SUBROUTINE  UVBAR                             *
C                                                               *
C****************************************************************
C
C
       SUBROUTINE UVBAR (NP,H,C12,C22,C23,C66,XK,ZK,D,N,DK12)
C
       COMMON/BLK1/ NP1, NP2, IC2
       COMMON/BLK2/ CIMGG, ZERO
       REAL *8      H(NP1),C12(NP1),C22(NP1),C23(NP1),C66(NP1),
      *             D(4, 4),CHSIGN, XK, XK2, ZK, ZK2
       COMPLEX *16 DK12(IC2,IC2), UV(12,12), CONST, CIMGG, ZERO
C
       DO 1000 JJ=1, IC2
          DO 1050 KK=1, IC2
             DK12(JJ, KK) = ZERO
 1050     CONTINUE
 1000 CONTINUE
C
       DO 991 L1=1, 12
          DO 992 L2=1, 12
             UV(L1,L2) = ZERO
 992      CONTINUE
 991   CONTINUE
C
       XK2 = XK * XK
       ZK2 = ZK * ZK
       IF ( N .EQ. 1 ) THEN DO
          CHSIGN = 1.0D0
       ELSE DO
          CHSIGN = - 1.0D0
       END IF
C
       DO 10 I=1, NP1
C
          UV(1, 1) = 2. * D(1, 3) * H(I) * XK * C12(I) * CIMGG / C22(I)
          UV(1, 2) = UV(2, 1) = ZERO
          UV(1, 3) = - D(1, 1) * CHSIGN
          UV(3, 1) = - UV(1, 3)
          UV(1, 4) = - D(1, 3) * H(I) / C22(I)
          UV(4, 1) = - UV(1, 4)
          UV(1, 5) =  D(1, 3)*ZK*C23(I)*H(I)*CIMGG/C22(I)
          UV(5, 1) = UV(1, 5)
```

```
      UV(1, 7) = D(2, 3) * H(I) * XK * C12(I) * CIMGG / C22(I)
     *           + D(1, 4) * H(I) * XK * C12(I) * CIMGG / C22(I)
      UV(7, 1) = UV(1, 7)
      UV(1, 8) = - D(3, 4) * H(I) * H(I) * XK * C12(I) * CIMGG
     *           / C22(I) / C66(I)
      UV(8, 1) = UV(1, 8)
      UV(1, 9) = ( - D(3, 4) * H(I) * H(I) * XK2 *C12(I) / C22(I) )
     *           - D(1, 2)
      UV(9, 1) = - UV(1, 9)
      UV(1,10) = - D(1, 4) * H(I) / C22(I)
      UV(10,1) = - UV(1,10)
      UV(1,11) = UV(11,1) = D(1, 4)*H(I)*ZK*C23(I)*CIMGG/C22(I)
C
      UV(2, 2) = ZERO
      UV(2, 3) = D(1, 3) * H(I) / C66(I)
      UV(3, 2) = - UV(2, 3)
      UV(2, 7) = D(3, 4) * H(I) * H(I) * XK * C12(I) * CIMGG
     *           / C22(I) / C66(I)
      UV(7, 2) = UV(2, 7)
      UV(2, 8) = UV(8, 2) = ZERO
      UV(2, 9) = D(2, 3) * H(I) / C66(I)
      UV(9, 2) = - UV(2, 9)
      UV(2,10) = - D(3,4) * H(I) * H(I) / C22(I) / C66(I)
      UV(10,2) = - UV(2,10)
      UV(2,11) = UV(11,2) = D(3,4)*H(I)*H(I)*ZK*C23(I)*CIMGG/
     .                      C22(I)/C66(I)
C
      UV(3, 3) = 2. * D(1, 3) * H(I) * XK * CIMGG
      UV(3, 4) = UV(4, 3) = ZERO
      UV(3, 7) = ( - D(1, 2) ) - D(3, 4) * H(I) * H(I) * XK2
     *                        * C12(I) / C22(I)
      UV(7, 3) = - UV(3, 7)
      UV(3, 8) = - D(1, 4) * H(I) / C66(I)
      UV(8, 3) = - UV(3, 8)
      UV(3, 9) = D(1, 4) * H(I) * XK * CIMGG
     *         + D(2, 3) * H(I) * XK * CIMGG
      UV(9, 3) = UV(3, 9)
      UV(3,10) = - D(3, 4) * H(I) * H(I) * XK * CIMGG / C22(I)
      UV(10,3) = UV(3,10)
      UV(3,11) = -D(3, 4)*H(I)*H(I)*ZK*XK*C23(I)/C22(I)
      UV(11,3) = -UV(3,11)
C
      UV(4, 4) = ZERO
      UV(4, 7) = D(2, 3) * H(I) / C22(I)
      UV(7, 4) = - UV(4, 7)
      UV(4, 8) = - D(3, 4) * H(I) * H(I) / C22(I) / C66(I)
      UV(8, 4) = - UV(4, 8)
      UV(4, 9) = D(3, 4) * H(I) * H(I) * XK * CIMGG / C22(I)
      UV(9, 4) = UV(4, 9)
C
      UV(5, 7) = D(2,3)*H(I)*ZK*C23(I)*CIMGG/C22(I)
      UV(7, 5) = UV(5, 7)
      UV(5, 8) =-D(3,4)*H(I)*H(I)*ZK*C23(I)*CIMGG/C22(I)/C66(I)
      UV(8, 5) = UV(5, 8)
      UV(5, 9) =-D(3,4)*H(I)*H(I)*XK*ZK*C23(I)/C22(I)
      UV(9, 5) =-UV(5, 9)
C
      UV(7, 7) = 2. * D(2, 4) * H(I) * XK * CIMGG * C12(I) / C22(I)
      UV(7, 9) = - D(2, 2) *CHSIGN
      UV(9, 7) = - UV(7, 9)
      UV(7,10) = - D(2, 4) * H(I) / C22(I)
      UV(10,7) = - UV(7,10)
      UV(7,11) =UV(11,7) =D(2,4)*H(I)*ZK*C23(I)*CIMGG/C22(I)
C
```

```
          UV(8, 8) = ZERO
          UV(8, 9) = D(2, 4) * H(I) / C66(I)
          UV(9, 8) = - UV(8, 9)
          UV(9, 9) = 2. *D(2, 4) * H(I) * XK * CIMGG
C
C
          IF (N .EQ. 1) THEN DO
             CONST = C12(I) *XK * CIMGG
          ELSE DO
             CONST = C66(I) *XK *CIMGG
          END IF
C
          ITIME = I - 1
          DO 20 J=1, 12
             IR = J + 6 * ITIME
             DO 30 K=1, 12
                IC = K + 6 * ITIME
                DK12(IR, IC) = DK12(IR, IC) + CONST * UV(J, K)
   30        CONTINUE
   20     CONTINUE
   10 CONTINUE
C
      RETURN;END
C
C
C
C***************************************************************
C                                                              *
C              SUBROUTINE WUBAR  DOING WU +  UW                *
C                                                              *
C***************************************************************
C
C
      SUBROUTINE WUBAR(NP,H,C13,C44,C55,C66,XK,ZK,AB,N,EK12)
C
C
      COMMON/BLK1/ NP1, NP2, IC2
      COMMON/BLK2/ CIMGG, ZERO
      REAL *8         H(NP1),C13(NP1),C44(NP1),C55(NP1),C66(NP1),
     *                XK, XK2, ZK, ZK2, AB(4, 4)
      COMPLEX *16     EK12(IC2,IC2),WU(12,12), CONST, CIMGG, ZERO
C
      XK2 = XK * XK
      ZK2 = ZK * ZK
C
      DO 1000 JJ=1, IC2
         DO 1050 KK=1, IC2
            EK12(JJ, KK) = ZERO
 1050    CONTINUE
 1000 CONTINUE
C
      DO 991 L1=1, 12
         DO 992 L2=1, 12
            WU(L1,L2) = ZERO
 992     CONTINUE
 991  CONTINUE
C
      DO 10 I=1, NP1
C
         WU(1, 3)=-AB(1,3)*H(I)*ZK*CIMGG
         WU(3, 1)=-WU(1, 3)
         WU(1, 5)= AB(1,1)
         WU(5, 1)= WU(1, 5)
         WU(1, 6)= AB(1,3)*H(I)/C44(I)
```

```
      WU(6, 1)= WU(1, 6)
      WU(1, 9)=-AB(1,4)*H(I)*ZK*CIMGG
      WU(9, 1)=-WU(1, 9)
      WU(1,11)= AB(1,2)
      WU(11,1)= WU(1,11)
      WU(1,12)= AB(1,4)*H(I)/C44(I)
      WU(12,1)= WU(1,12)
C
      WU(2, 3)=-AB(3,3)*H(I)*H(I)*ZK*CIMGG/C66(I)
      WU(3, 2)=-WU(2, 3)
      WU(2, 5)= AB(1,3)*H(I)/C66(I)
      WU(5, 2)= WU(2, 5)
      WU(2, 6)= AB(3,3)*H(I)*H(I)/C44(I)/C66(I)
      WU(6, 2)= WU(2, 6)
      WU(2, 9)=-AB(3,4)*H(I)*H(I)*ZK*CIMGG/C66(I)
      WU(9, 2)=-WU(2, 9)
      WU(2,11)= AB(2,3)*H(I)/C66(I)
      WU(11,2)= WU(2,11)
      WU(2,12)= AB(3,4)*H(I)*H(I)/C44(I)/C66(I)
      WU(12,2)= WU(2,12)
C
      WU(3, 3)= 2.*AB(3,3)*H(I)*H(I)*ZK*XK
      WU(3, 5)= AB(1,3)*H(I)*XK*CIMGG
      WU(5, 3)=-WU(3, 5)
      WU(3, 6)= AB(3,3)*H(I)*H(I)*XK*CIMGG/C44(I)
      WU(6, 3)=-WU(3, 6)
      WU(3, 7)= AB(2,3)*H(I)*ZK*CIMGG
      WU(7, 3)=-WU(3, 7)
      WU(3, 8)= AB(3,4)*H(I)*H(I)*ZK*CIMGG/C66(I)
      WU(8, 3)=-WU(3, 8)
      WU(3, 9)= 2.*AB(3,4)*H(I)*H(I)*ZK*XK
      WU(9, 3)= WU(3, 9)
      WU(3,11)= AB(2,3)*H(I)*XK*CIMGG
      WU(11,3)=-WU(3,11)
      WU(3,12)= AB(3,4)*H(I)*H(I)*XK*CIMGG/C44(I)
      WU(12,3)=-WU(3,12)
C
      WU(5, 7)= AB(1, 2)
      WU(7, 5)= WU(5, 7)
      WU(5, 8)= AB(1,4)*H(I)/C66(I)
      WU(8, 5)= WU(5, 8)
      WU(5, 9)=-AB(1,4)*H(I)*XK*CIMGG
      WU(9, 5)=-WU(5, 9)
C
      WU(6, 7)= AB(2,3)*H(I)/C44(I)
      WU(7, 6)= WU(6, 7)
      WU(6, 8)= AB(3,4)*H(I)*H(I)/C44(I)/C66(I)
      WU(8, 6)= WU(6, 8)
      WU(6, 9)=-AB(3,4)*H(I)*H(I)*XK*CIMGG/C44(I)
      WU(9, 6)=-WU(6, 9)
C
      WU(7, 9)=-AB(2,4)*H(I)*ZK*CIMGG
      WU(9, 7)=-WU(7, 9)
      WU(7,11)= AB(2,2)
      WU(11,7)= WU(7,11)
      WU(7,12)= AB(2,4)*H(I)/C44(I)
      WU(12,7)= WU(7,12)
C
      WU(8, 9)=-AB(4,4)*H(I)*H(I)*ZK*CIMGG/C66(I)
      WU(9, 8)=-WU(8, 9)
      WU(8,11)= AB(2,4)*H(I)/C66(I)
      WU(11,8)= WU(8,11)
      WU(8,12)= AB(4,4)*H(I)*H(I)/C44(I)/C66(I)
      WU(12,8)= WU(8,12)
```

```
C
        WU(9, 9)= 2.*AB(4,4)*H(I)*H(I)*ZK*XK
        WU(9,11)=AB(2,4)*XK*H(I)*CIMGG
        WU(11,9)=-WU(9,11)
        WU(9,12)= AB(4,4)*H(I)*H(I)*XK*CIMGG/C44(I)
        WU(12,9)=-WU(9,12)
C
        IF (N .EQ. 1) THEN DO
           CONST = C13(I)*H(I)*XK*ZK
        ELSE DO
           CONST = C55(I)*H(I)*XK*ZK
        END IF
C
C
        ITIME = I - 1
        DO 20 J=1, 12
           IR = J + 6 * ITIME
           DO 30 K=1, 12
              IC = K + 6 * ITIME
              EK12(IR, IC) = EK12(IR, IC) + CONST *WU(J, K)
   30      CONTINUE
   20   CONTINUE
   10 CONTINUE
C
C
      RETURN;END
C
C
C***************************************************************
C                                                              *
C               SUBROUTINE WVBAR DOING WVBAR'-V'WBAR AND       *
C                                      VWBAR'-W'VBAR           *
C                                                              *
C***************************************************************
C
C
      SUBROUTINE WVBAR(NP,H,C12,C22,C23,C44,XK,ZK,D,N,FK12)
C
C
      COMMON/BLK1/NP1,NP2,IC2
      COMMON/BLK2/CIMGG,ZERO
      REAL*8   H(NP1),C12(NP1),C22(NP1),C23(NP1),C44(NP1),
     @         D(4,4),XK,XK2,ZK,ZK2
      COMPLEX*16  FK12(IC2,IC2),WV(12,12),CONST,CIMGG,ZERO
C
      XK2=XK*XK
      ZK2=ZK*ZK
      DO 1000 JJ=1,IC2
         DO 1050 KK=1,IC2
            FK12(JJ,KK) = ZERO
 1050    CONTINUE
 1000 CONTINUE
C
      DO 991 L1=1, 12
         DO 992 L2=1, 12
            WV(L1,L2) = ZERO
  992    CONTINUE
  991 CONTINUE
C
      IF (N .EQ. 1) THEN DO
         CHSIGN = 1.0D0
      ELSE DO
         CHSIGN =-1.0D0
      END IF
```

```
C
      DO 10 I=1,NP1
         WV(1, 5)= D(1,3)*H(I)*XK*C12(I)*CIMGG/C22(I)
         WV(5, 1)= WV(1, 5)
         WV(1, 9)=-D(3,4)*H(I)*H(I)*XK*ZK*C12(I)/C22(I)
         WV(9, 1)=-WV(1, 9)
         WV(1,11)= D(2,3)*H(I)*XK*C12(I)*CIMGG/C22(I)
         WV(11,1)= WV(1,11)
         WV(1,12)= D(3,4)*H(I)*H(I)*XK*C12(I)*CIMGG/C22(I)/C44(I)
         WV(12,1)= WV(1,12)
C
         WV(3, 3)=2.*D(1,3)*H(I)*ZK*CIMGG
         WV(3, 5)= D(1,1)*CHSIGN
         WV(5, 3)=-WV(3, 5)
         WV(3, 6)=-D(1,3)*H(I)/C44(I)
         WV(6, 3)=-WV(3, 6)
         WV(3, 7)=-D(3,4)*H(I)*H(I)*ZK*XK*C12(I)/C22(I)
         WV(7, 3)=-WV(3, 7)
         WV(3, 9)= D(1,4)*H(I)*ZK*CIMGG
     .                            +D(2,3)*H(I)*ZK*CIMGG
         WV(9, 3)= WV(3, 9)
         WV(3,10)=-D(3,4)*H(I)*H(I)*ZK*CIMGG/C22(I)
         WV(10,3)= WV(3,10)
         WV(3,11)=(-D(1,2))-D(3,4)*H(I)*H(I)*ZK2*C23(I)/C22(I)
         WV(11,3)=-WV(3,11)
         WV(3,12)=-D(1,4)*H(I)/C44(I)
         WV(12,3)=-WV(3,12)
C
         WV(4, 5)= D(1,3)*H(I)/C22(I)
         WV(5, 4)=-WV(4, 5)
         WV(4, 9)= D(3,4)*H(I)*H(I)*ZK*CIMGG/C22(I)
         WV(9, 4)= WV(4, 9)
         WV(4,11)= D(2,3)*H(I)/C22(I)
         WV(11,4)=-WV(4,11)
         WV(4,12)=-D(3,4)*H(I)*H(I)/C22(I)/C44(I)
         WV(12,4)=-WV(4,12)
C
         WV(5, 5)= 2.*D(1,3)*ZK*H(I)*CIMGG*C23(I)/C22(I)
         WV(5, 7)= D(1,4)*H(I)*XK*C12(I)*CIMGG/C22(I)
         WV(7, 5)= WV(5, 7)
         WV(5, 9)= (-D(1,2))-D(3,4)*H(I)*H(I)*ZK2*C23(I)/C22(I)
         WV(9, 5)=-WV(5, 9)
         WV(5,10)=-D(1,4)*H(I)/C22(I)
         WV(10,5)=-WV(5,10)
         WV(5,11)= D(1,4)*H(I)*ZK*CIMGG*C23(I)/C22(I)
     .                            +D(2,3)*H(I)*ZK*CIMGG*C23(I)/C22(I)
         WV(11,5)= WV(5,11)
         WV(5,12)=-D(3,4)*ZK*H(I)*H(I)*C23(I)*CIMGG/C22(I)/C44(I)
         WV(12,5)= WV(5,12)
C
         WV(6, 7)= D(3,4)*H(I)*H(I)*XK*C12(I)*CIMGG/C22(I)/C44(I)
         WV(7, 6)= WV(6, 7)
         WV(6, 9)= D(2,3)*H(I)/C44(I)
         WV(9, 6)=-WV(6, 9)
         WV(6,10)=-D(3,4)*H(I)*H(I)/C22(I)/C44(I)
         WV(10,6)=-WV(6,10)
         WV(6,11)= D(3,4)*H(I)*H(I)*ZK*C23(I)*CIMGG/C22(I)/C44(I)
         WV(11,6)= WV(6,11)
C
         WV(7,11)= D(2,4)*H(I)*XK*C12(I)*CIMGG/C22(I)
         WV(11,7)= WV(7,11)
C
         WV(9, 9)= 2.*D(2,4)*H(I)*ZK*CIMGG
         WV(9,11)= D(2,2)*CHSIGN
```

```
            WV(11,9)=-WV(9,11)
            WV(9,12)=-D(2,4)*H(I)/C44(I)
            WV(12,9)=-WV(9,12)
C
            WV(10,11)= D(2,4)*H(I)/C22(I)
            WV(11,10)=-WV(10,11)
C
            WV(11,11)= 2.*D(2,4)*H(I)*ZK*C23(I)*CIMGG/C22(I)
C
            IF (N .EQ. 1) THEN DO
               CONST = C23(I) * ZK* CIMGG
            ELSE DO
               CONST = C44(I) * ZK * CIMGG
            END IF
C
            ITIME = I - 1
            DO 20 J=1, 12
               IR = J + 6 * ITIME
               DO 30 K=1, 12
                  IC = K + 6 * ITIME
                  FK12(IR, IC) = FK12(IR, IC) + CONST *WV(J, K)
      30       CONTINUE
      20    CONTINUE
      10 CONTINUE
C
C
         RETURN;END
C
C
C*******************************************************
C                                                      *
C                   SUBROUTINE ADDING                  *
C                                                      *
C*******************************************************
C
C
         SUBROUTINE ADDING (NP, SK, MK)
C
         COMMON/BLK1/NP1, NP2, IC2
         COMMON/BLK2/ CIMGG, ZERO
         COMPLEX *16   SK(IC2,IC2), MK(IC2,IC2), CIMGG, ZERO
C
         NP6 = 6 * NP
         NPSQ = 6 * ( NP + 2 )
         NPSQM6 = NPSQ - 6
C
C      FIRST, ADD UP ALL THE MATRICES AND STORE THE RESULT IN AK
C
         DO 10 I=7, NPSQM6
            DO 20 J=1, NPSQ
               SK(I, J) = SK(I, J) + MK(I, J)
      20    CONTINUE
      10 CONTINUE
C
         RETURN;END
C
C
C
C
C
C*******************************************************
C                                                      *
C                   SUBROUTINE FORMDD                  *
C                                                      *
```

```
C*******************************************************
C
C
      SUBROUTINE FORMDD (NP, AK, AM, BK, BM)
C
      COMMON/BLK1/ NP1, NP2, IC2
      COMMON/BLK2/ CIMGG, ZERO
      COMPLEX *16   AK(IC2,IC2), AM(IC2,IC2), BK(IC2,IC2), BM(IC2,IC2)
     *              ,CIMGG, ZERO
C
      DO 1000 JJ=1, IC2
         DO 1050 KK=1, IC2
            BK(JJ, KK) = ZERO
            BM(JJ, KK) = ZERO
 1050    CONTINUE
 1000 CONTINUE
C
      NP6 = 6 * NP
      DO 10 I=1, NP6
         DO 20 J=1, 12
            IC = J + NP6
            BK(I, J) = AK(I, IC)
            BM(I, J) = AM(I, IC)
   20    CONTINUE
   10 CONTINUE
C
      RETURN;END
C
$ENTRY
```

T A B L E S

PERIODICITY = 4 , $\bar{d}$ = 4

	Thickness $(2h^{(i)})$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}	Density $\rho^{(i)}$
γ = 10	4.0	35	15	15	35	15	35	10	10	10	3.0
	1.0	4.333	2.333	2.333	4.333	2.333	4.333	1.0	1.0	1.0	1.0
γ = 50	4.0	175	75	75	175	75	175	50	50	50	3.0
	1.0	4.333	2.333	2.333	4.333	2.333	4.333	1.0	1.0	1.0	1.0
γ = 100	4.0	350	150	150	350	150	350	100	100	100	3.0
	1.0	4.333	2.333	2.333	4.333	2.333	4.333	1.0	1.0	1.0	1.0

Table 4.1: Material constants of composites for different isotropic lamina.

NOTE: To achieve a periodicity of 4, the properties of 5 layers are required as input. The first and second layers are subdivided into two layers each. The fifth layer is the same as the first sub-divided layer.

PERIODICITY = 4 , $\bar{d}$ = 12

$C_{ij} = C_{ij} * 10^{11} N/m^2$

ρ is in g/cm

Layer	Thickness $h^{(i)}$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}	Density
1	6.0	2.6907	0.5850	0.5850	1.8860	0.7634	1.8860	0.5613	0.6019	0.6019	2.5200
2	6.0	2.6907	0.5850	0.5850	1.8860	0.7634	1.8860	0.5613	0.6019	0.6019	2.5200
3	0.5	1.1070	0.5730	0.5730	1.1070	0.5730	1.1070	0.2670	0.2670	0.2670	2.7020
4	0.5	1.1070	0.5730	0.5730	1.1070	0.5730	1.1070	0.2670	0.2670	0.2670	2.7020
5	6.0	2.6907	0.5850	0.5850	1.8860	0.7634	1.8860	0.5613	0.6019	0.6019	2.5200

Table 4.2: Material constants for fiber-reinforced Boron-Aluminium Composite.

NOTE: 5 layer as input

PERIODICITY = 4 , $\bar{d}$ = 4 Units: C_{ij} : * $10^{11} N/m^2$

ρ: g/cm

Layer	Thickness $h^{(i)}$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}	Density
1	2.0	0.7669	0.0503	0.0503	0.1007	0.0507	0.1007	0.0250	0.0328	0.0328	1.200
2	2.0	0.7669	0.0503	0.0503	0.1007	0.0507	0.1007	0.0250	0.0328	0.0328	1.200
3	0.5	0.0865	0.0475	0.0475	0.0865	0.0475	0.0865	0.0195	0.0195	0.0195	1.800
4	0.5	0.0865	0.0475	0.0475	0.0865	0.0475	0.0865	0.0195	0.0195	0.0195	1.800
5	2.0	0.7669	0.0503	0.0503	0.1007	0.0507	0.1007	0.0250	0.0328	0.0328	1.200

Table 4.3: Material constants for graphite-epoxy composite (i).

PERIODICITY = 4 , $\bar{d}$ = 9

Units: C_{ij} : $10^{11} N/m^2$
ρ: g/cm

Layer	Thickness $h^{(i)}$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}	Density
1	4.50	1.6073	0.0644	0.0644	0.1392	0.0692	0.1392	0.0350	0.0707	0.0707	1.20
2	4.50	1.6073	0.0644	0.0644	0.1392	0.0692	0.1392	0.0350	0.0707	0.0707	1.20
3	0.50	0.0865	0.0475	0.0475	0.0865	0.0475	0.0865	0.0195	0.0195	0.0195	1.80
4	0.50	0.0865	0.0475	0.0475	0.0865	0.0475	0.0865	0.0195	0.0195	0.0195	1.80
5	4.50	1.6073	0.0644	0.0644	0.1392	0.0692	0.1392	0.0350	0.0707	0.0707	1.20

Table 4.4: Material constants for graphite-epoxy composite (ii).

PERIODICITY = 4 : $\bar{d} = 0.6$

Layer		Thickness $2h^{(i)}$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}	DENSITY
1		0.6	7.000	0.300	3.000	0.700	3.000	7.000	2.000	2.000	2.000	1.000
2–	i	1.0	0.700	0.300	0.300	0.700	0.300	0.700	0.200	0.020	0.020	1.000
	ii	1.0	3.850	0.300	1.650	0.700	1.650	3.850	1.100	1.010	1.010	1.000
	iii	1.0	7.000	0.300	3.000	0.700	3.000	7.000	2.000	2.000	2.000	1.000

Table 4.5: Material constants of transversely isotropic laminae

NOTE: Same as the note in Table 4.1. Layer 2 denotes the three types of matrix layers to be considered. Only one of these should be used at one consideration.

F I G U R E S

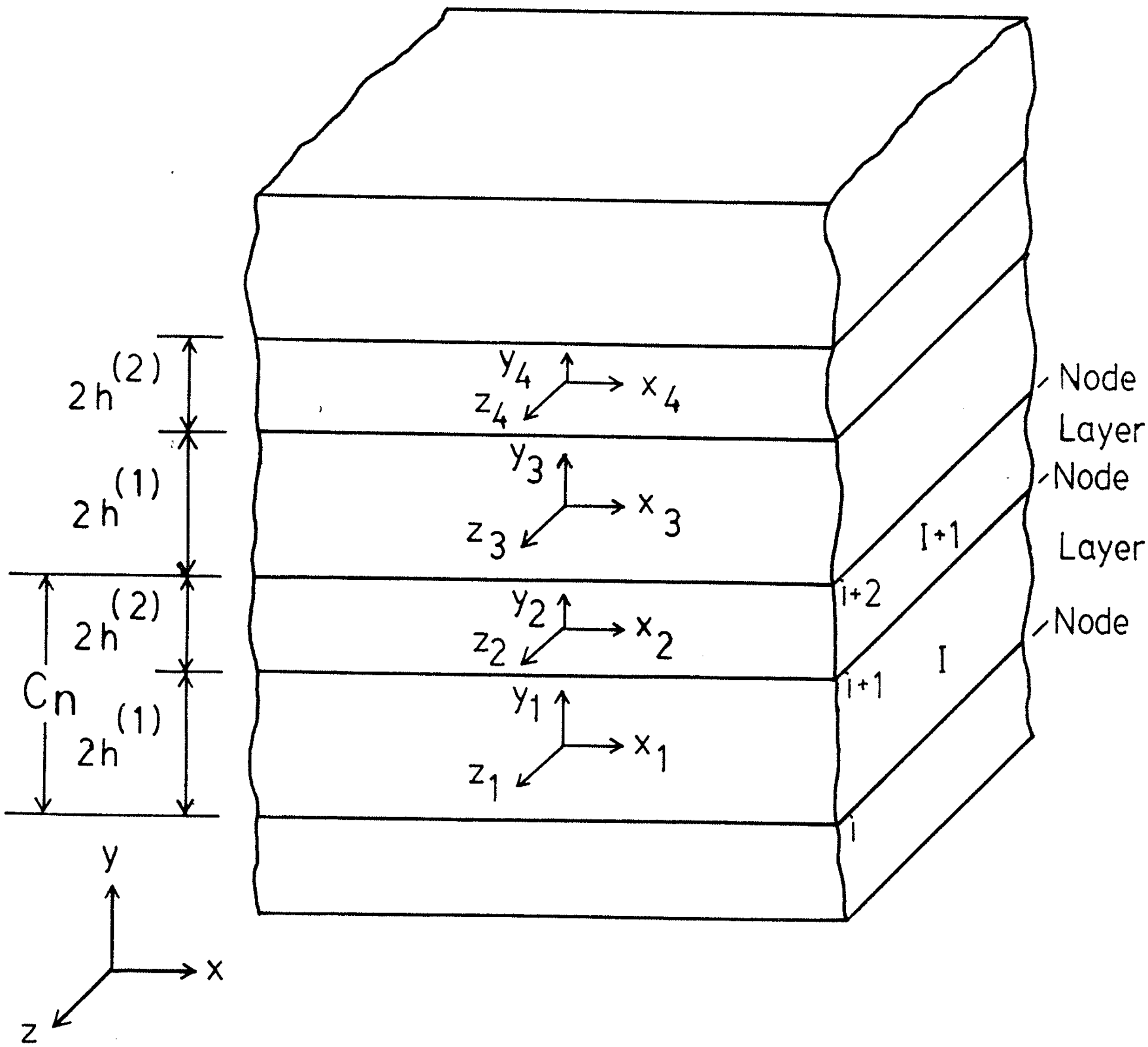


Fig. 2.1 Geometry of periodically laminated infinite medium.

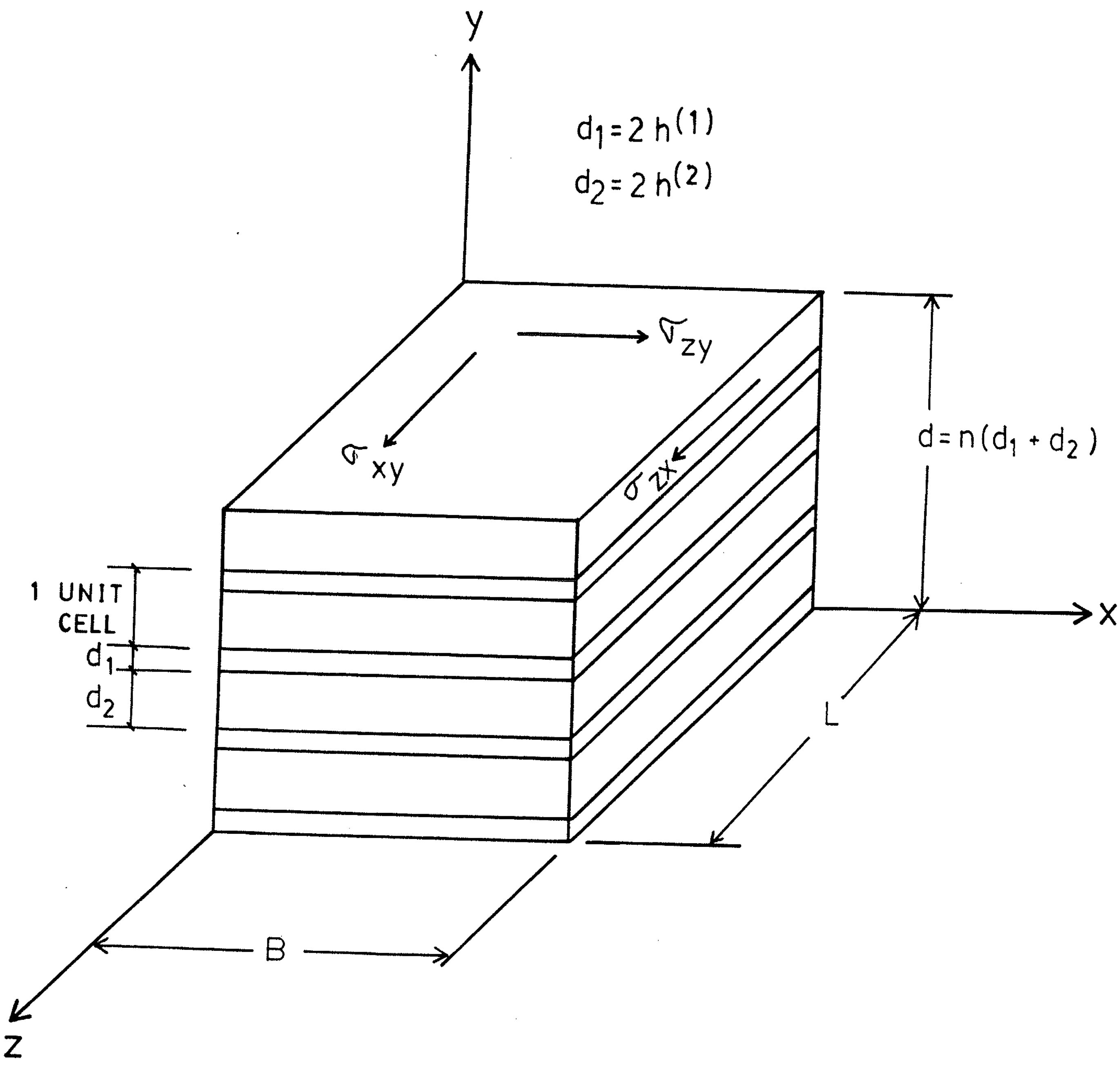


Fig. 3.1 Typical finite laminated medium showing shear stresses acting on the surface of the medium.

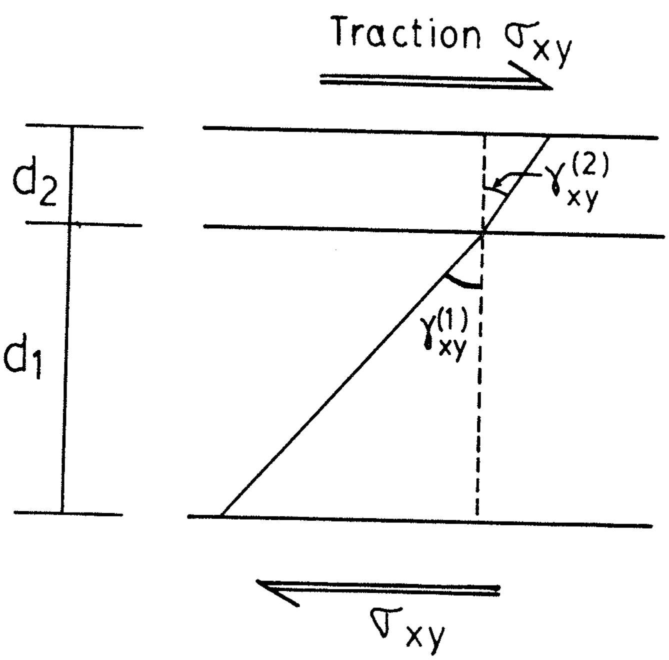


Fig. 3.2 Laminated layers subjected to tangential traction.

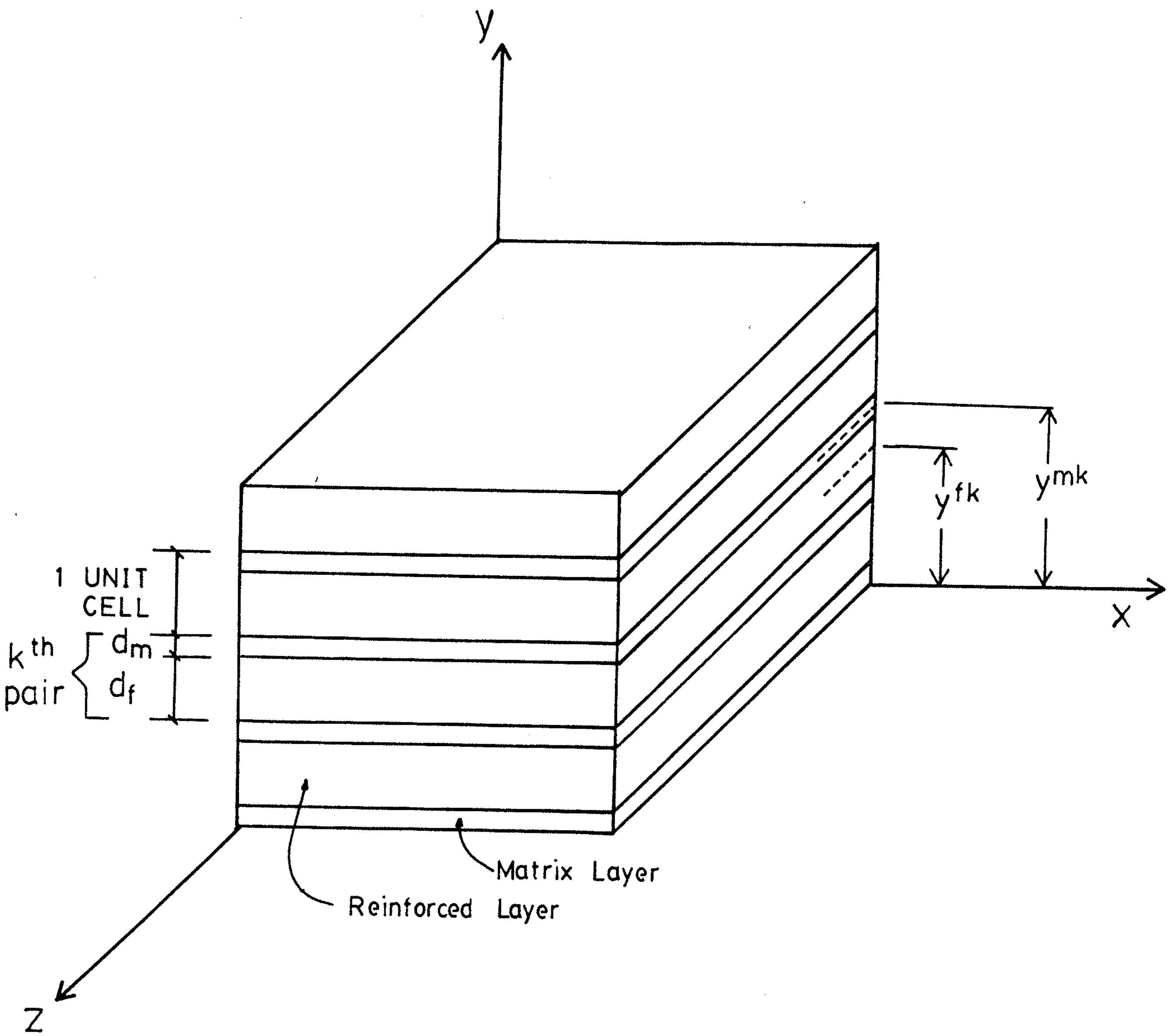


Fig. 3.3 Laminated medium used for the Effective Stiffness Method showing the layer properties and local coordinates.

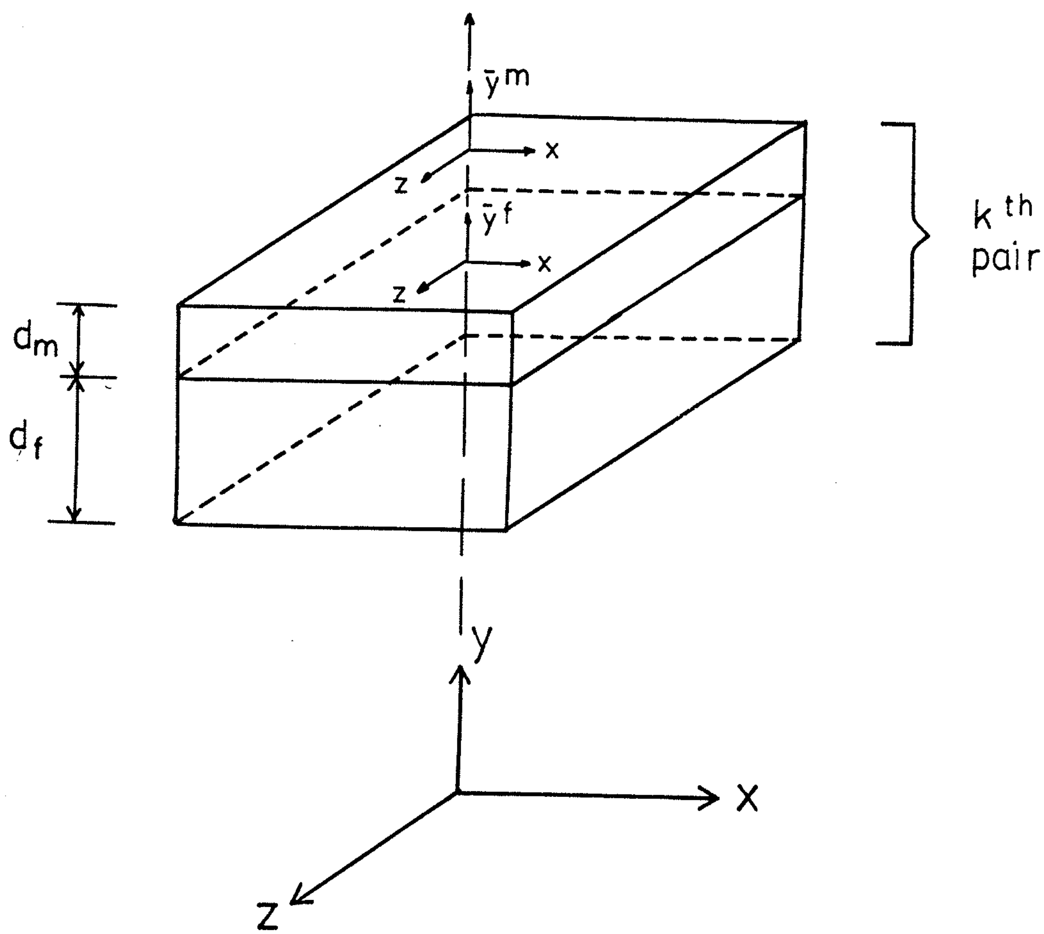


Fig. 3.4 Pair of reinforcing and matrix layers.

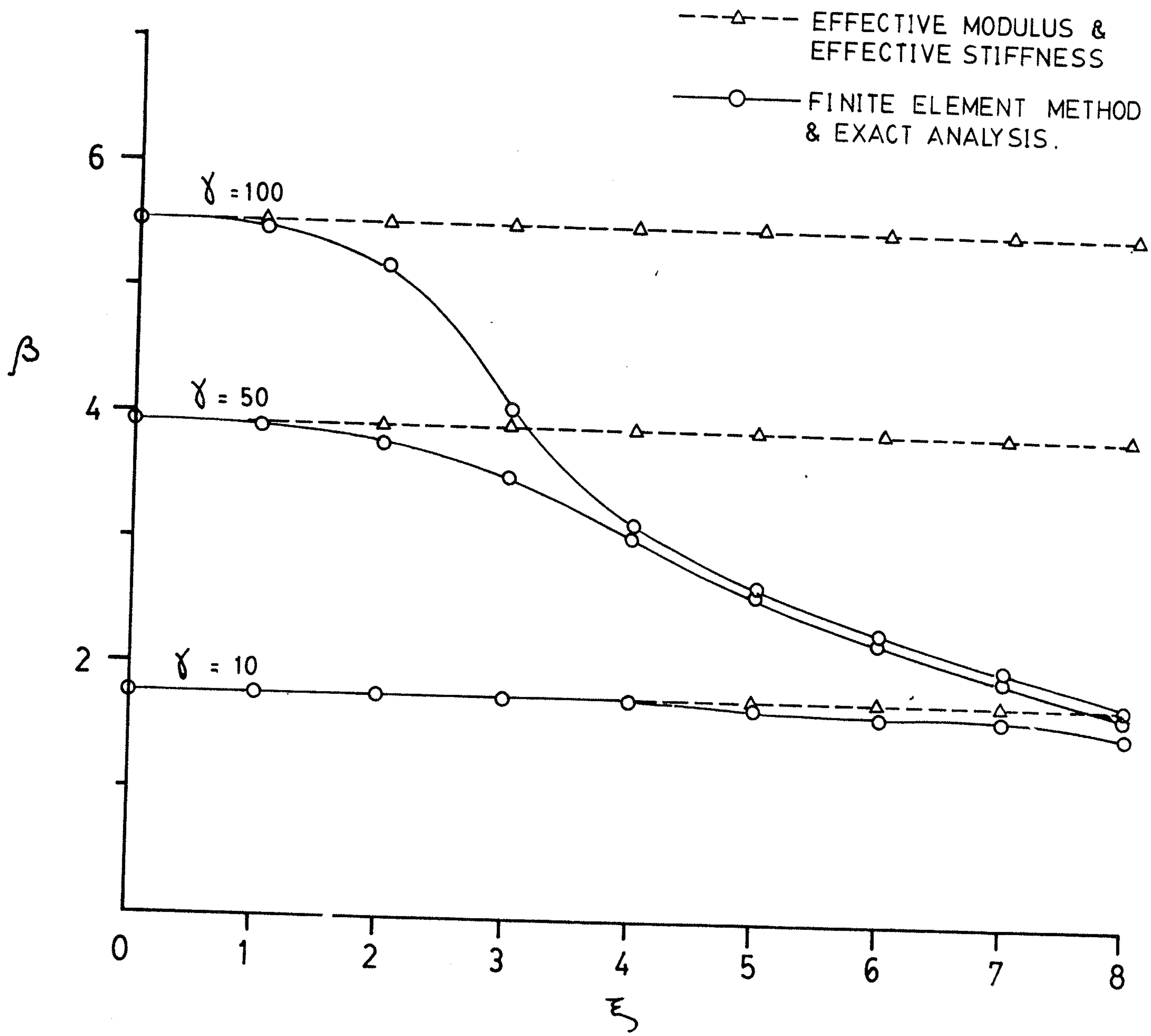


Fig. 4.1 Lowest symmetric SH mode propagating in the direction of the layering for isotropic material.

Fig. 4.2 Phase velocity vs. wave number plot for the lowest anti-symmetric mode (SV) propagating in the direction of the layering.

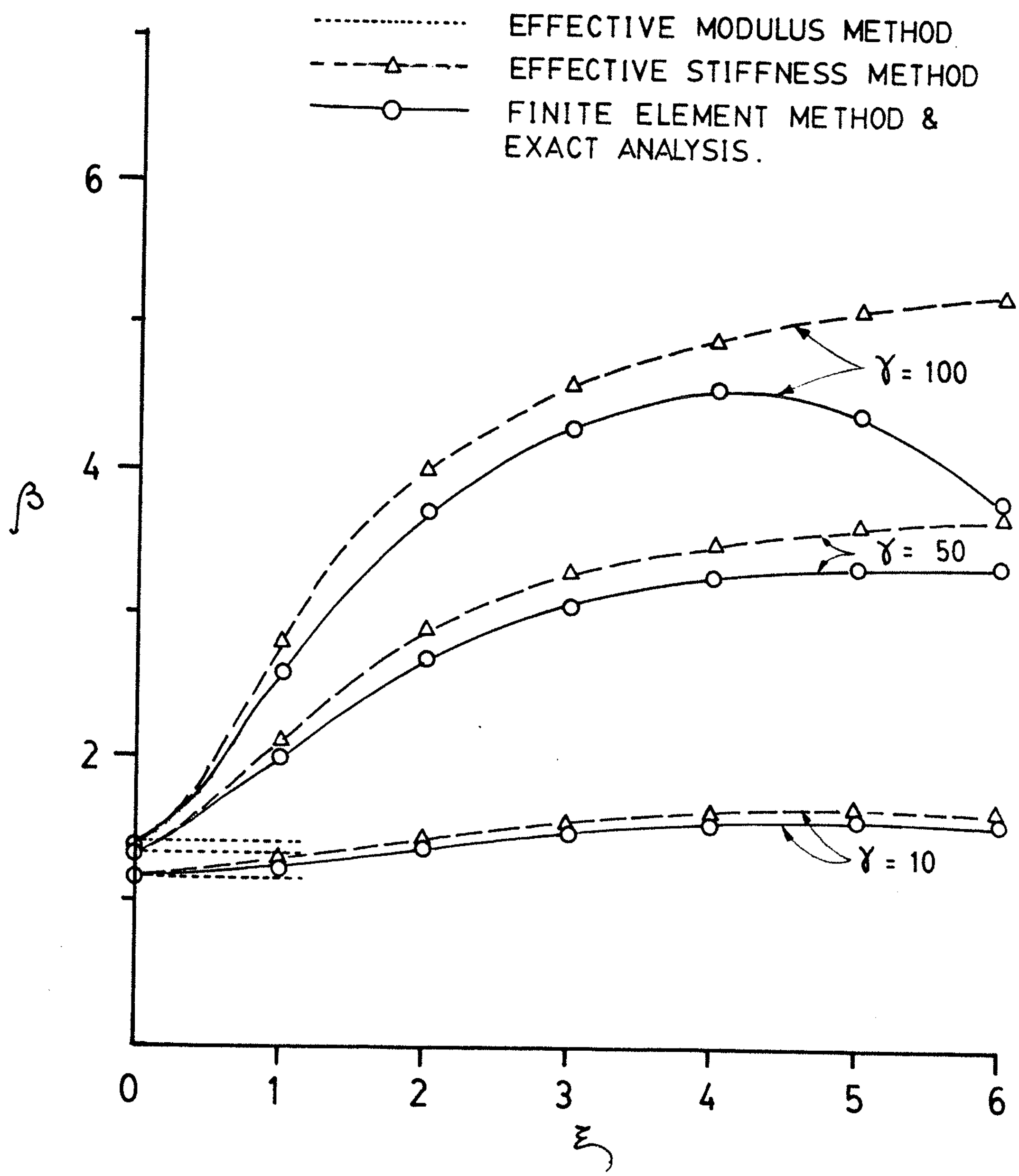


Fig. 4.3 Lowest symmetric P mode propagating in the direction of the layering.

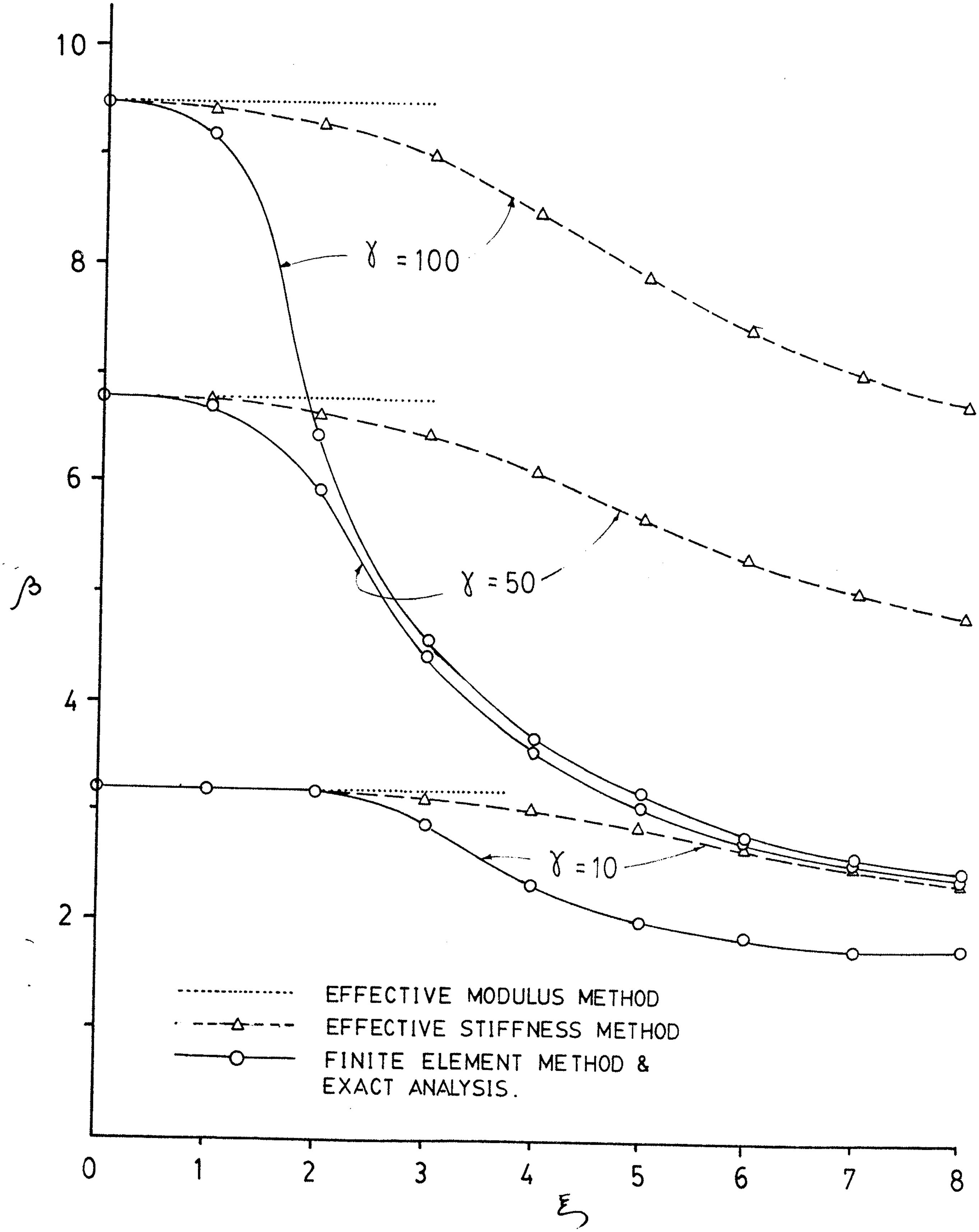


Fig. 4.4 Lowest transverse (SV) and longitudinal (P) mode propagating normal to the layering.

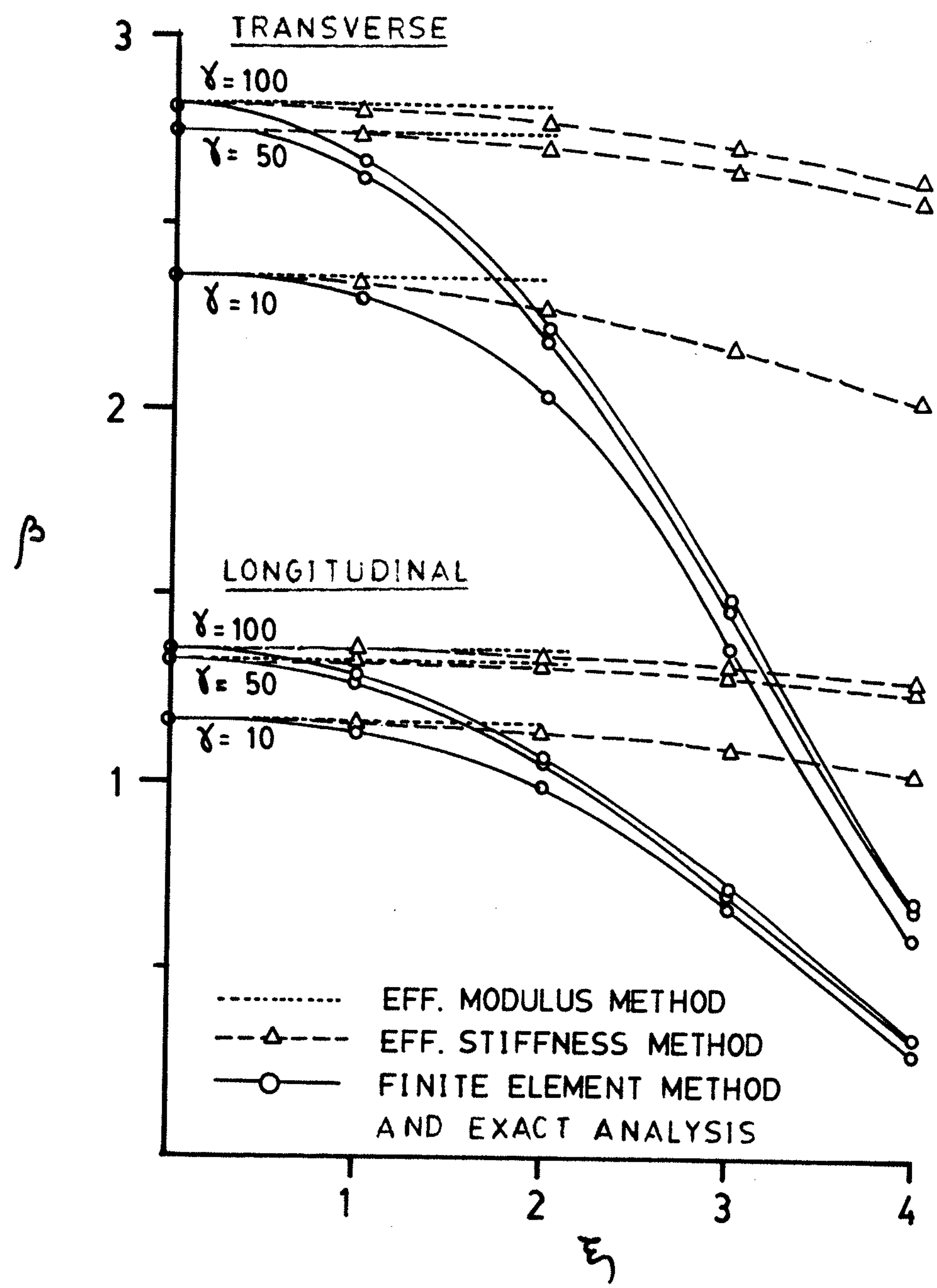


Fig. 4.5 Lowest transverse (SV) and longitudinal (P) mode propagating in the direction of the layering for anisotropic material.

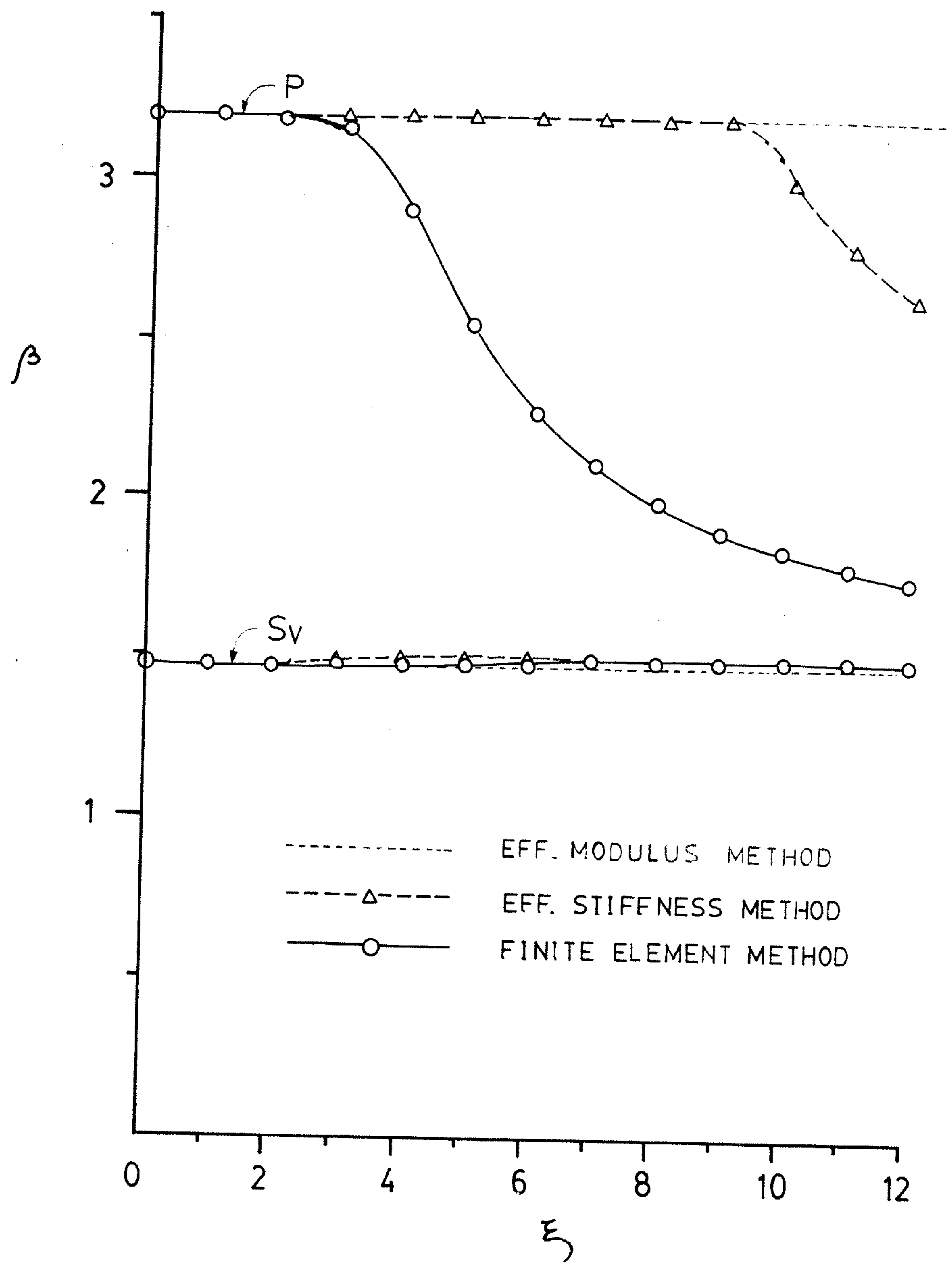


Fig. 4.6 Lowest SH, SV and P mode propagating normal to the layering for anisotropic material, $\alpha = 0°$, $\phi = 90°$.

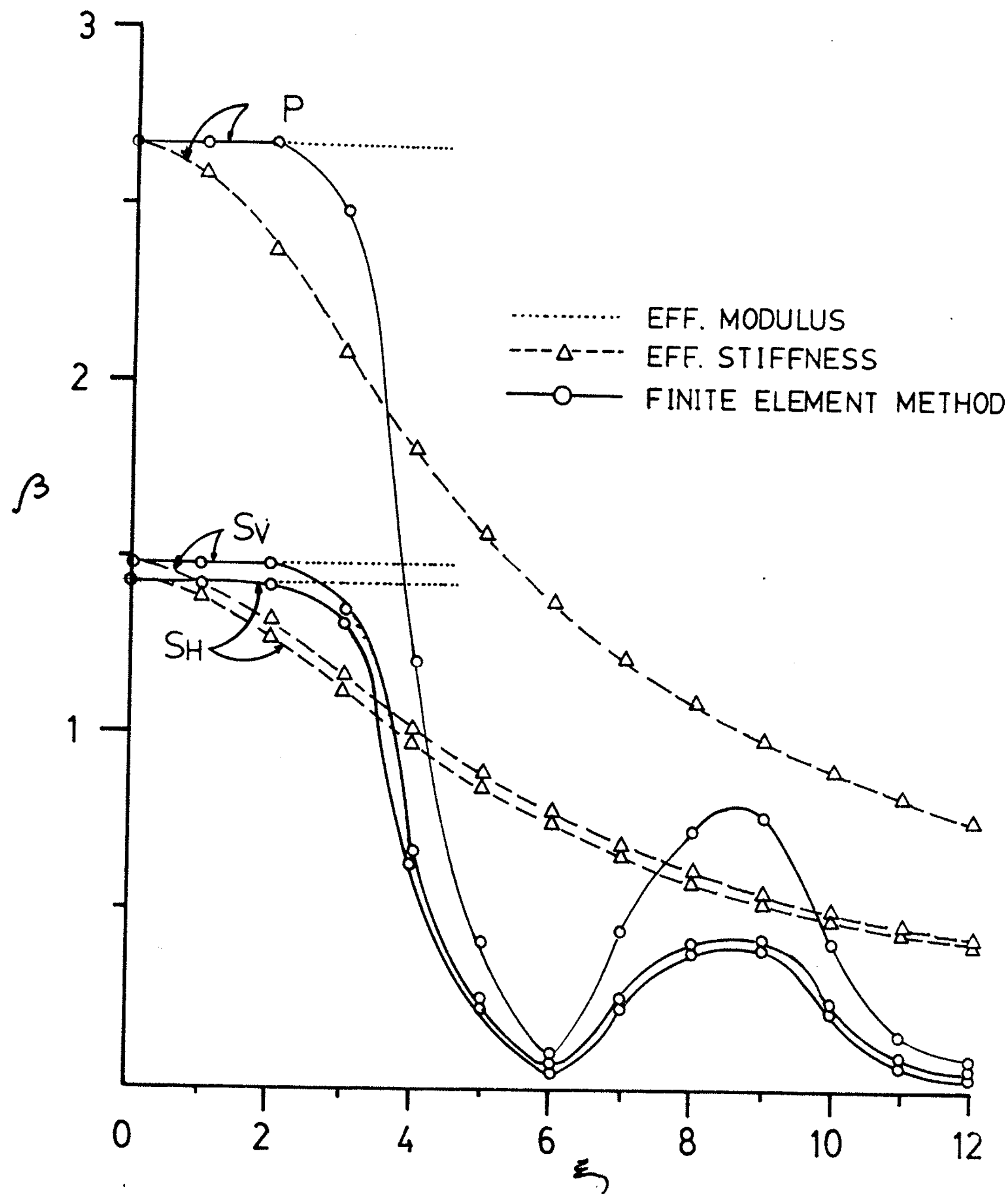


Fig. 4.7 Lowest SH, SV and P mode propagating along x-y plane, ϕ = 0°.

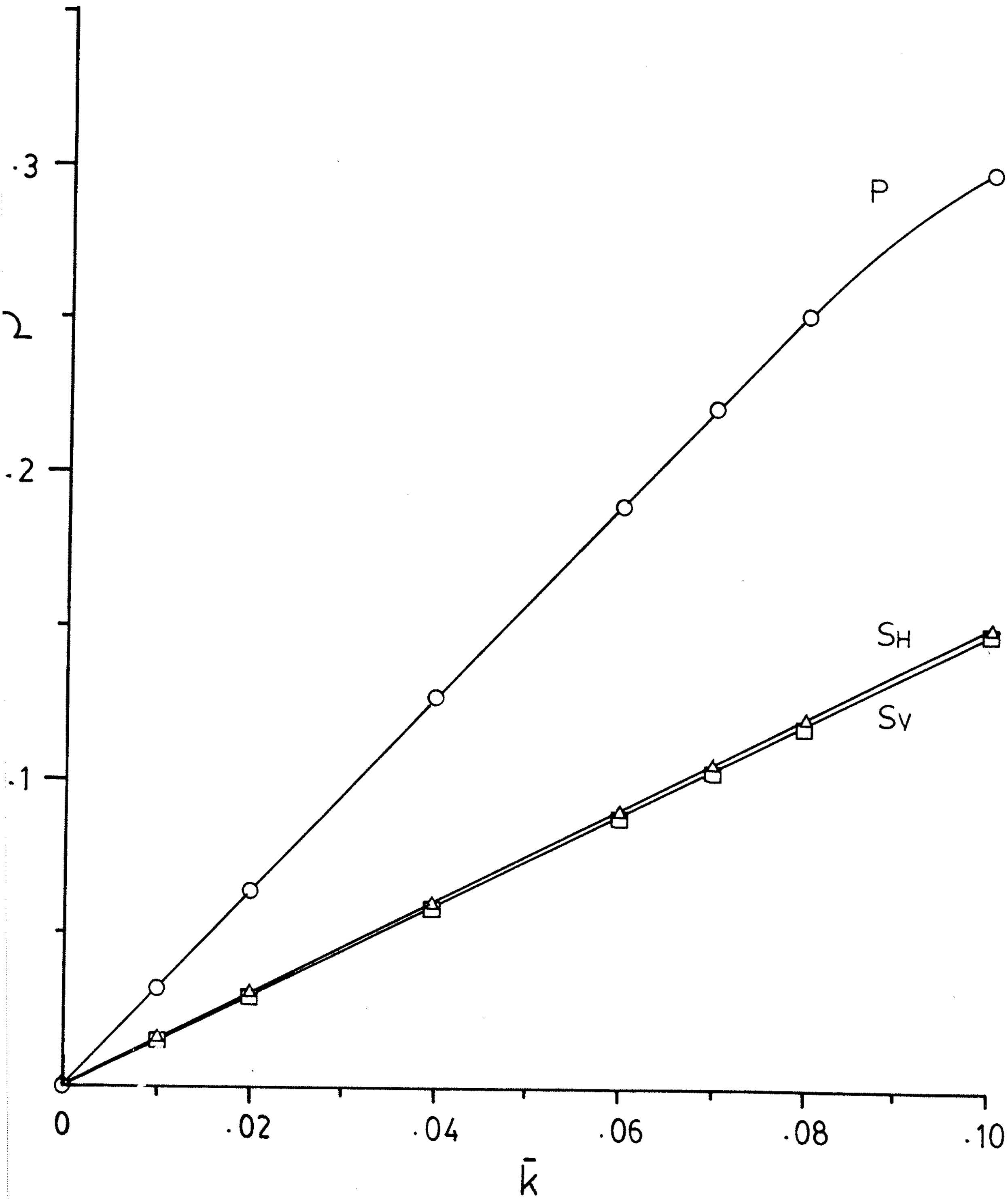


Fig. 4.8 Lowest SH, SV and P mode propagating along x-y plane, ϕ = 30°.

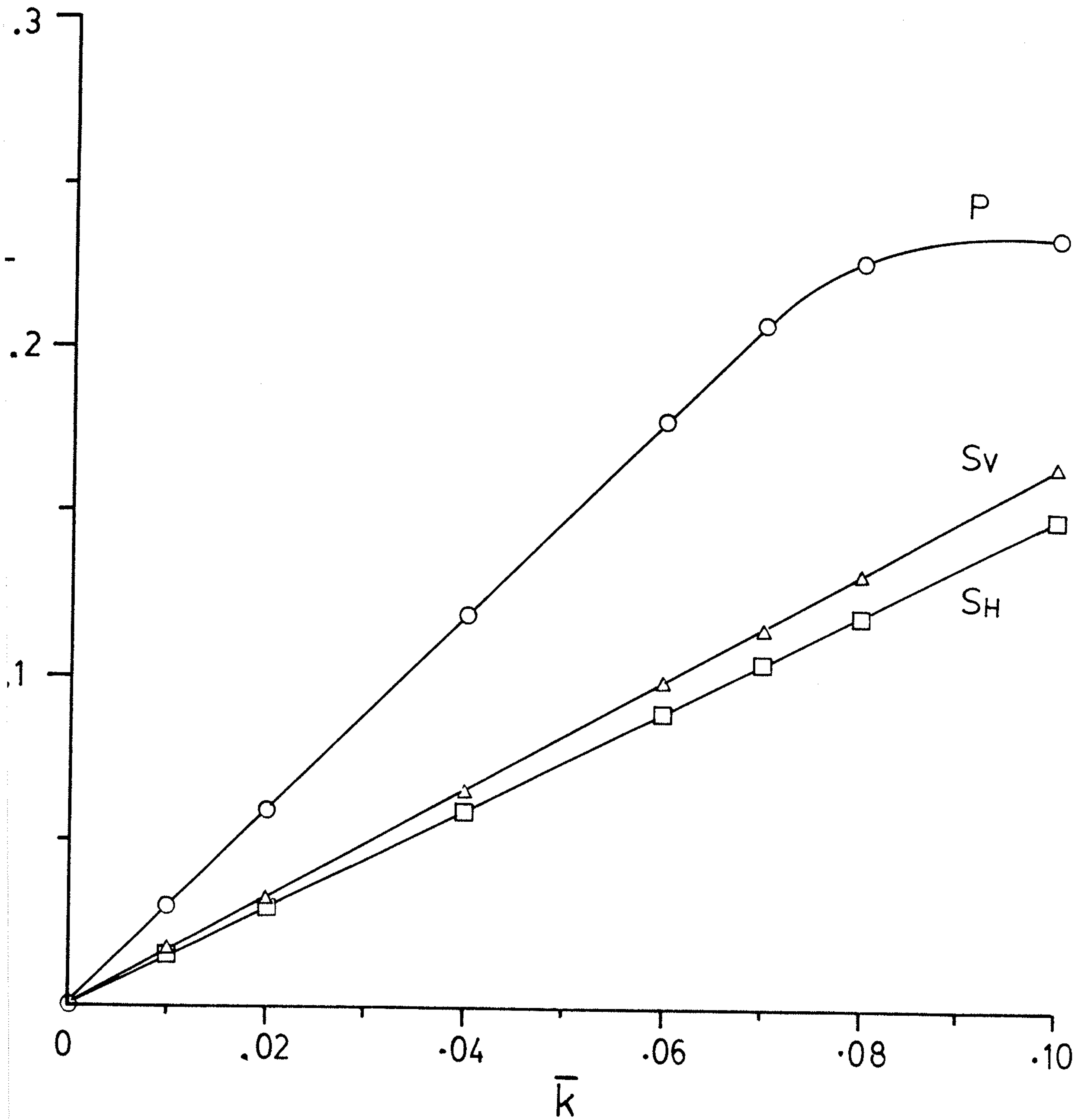


Fig. 4.9 Lowest SH, SV and P mode propagating along x-y plane, ϕ = 60°.

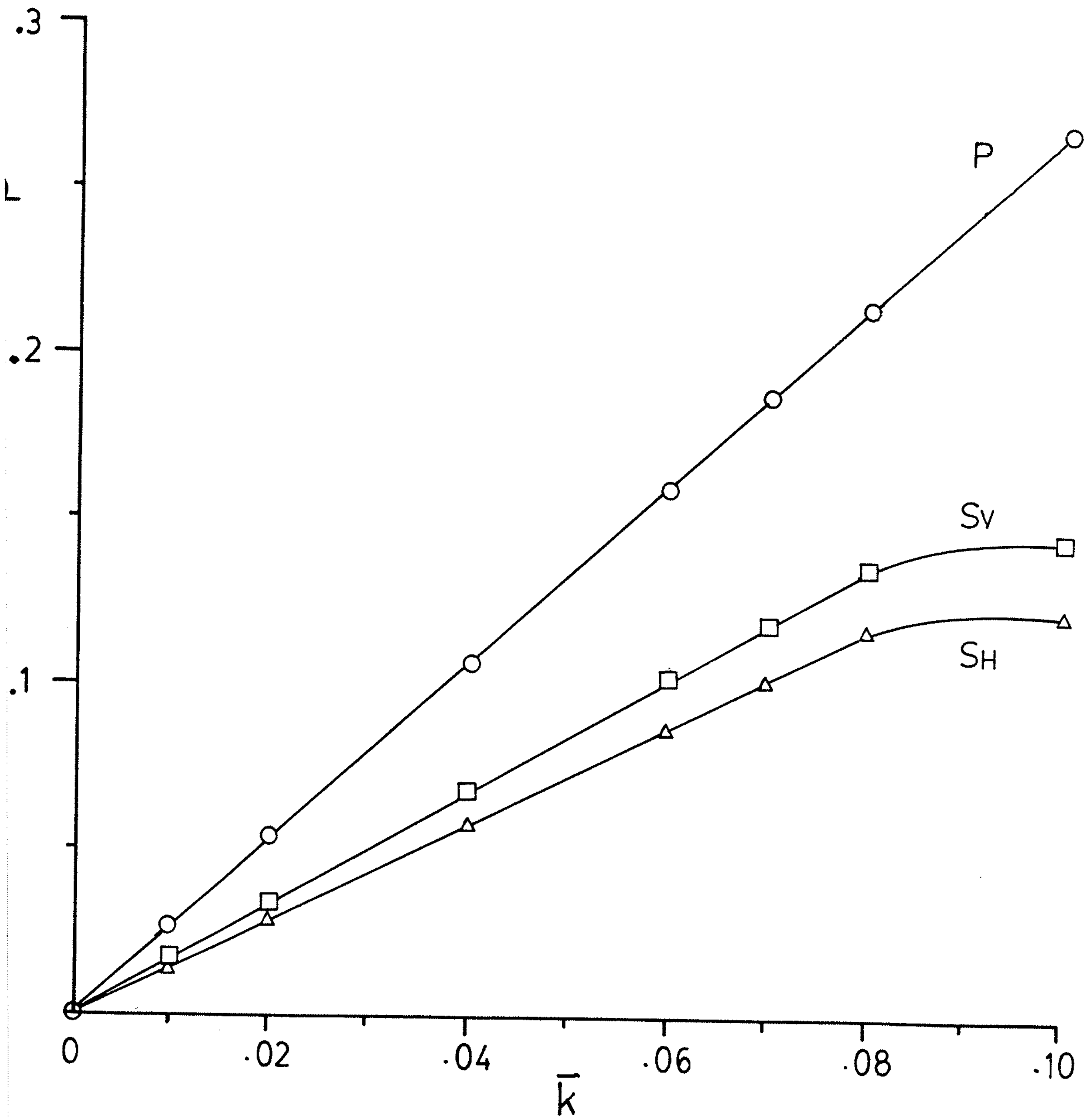


Fig. 4.10 Lowest SH, SV and P mode propagating along x-y plane, ϕ = 90°.

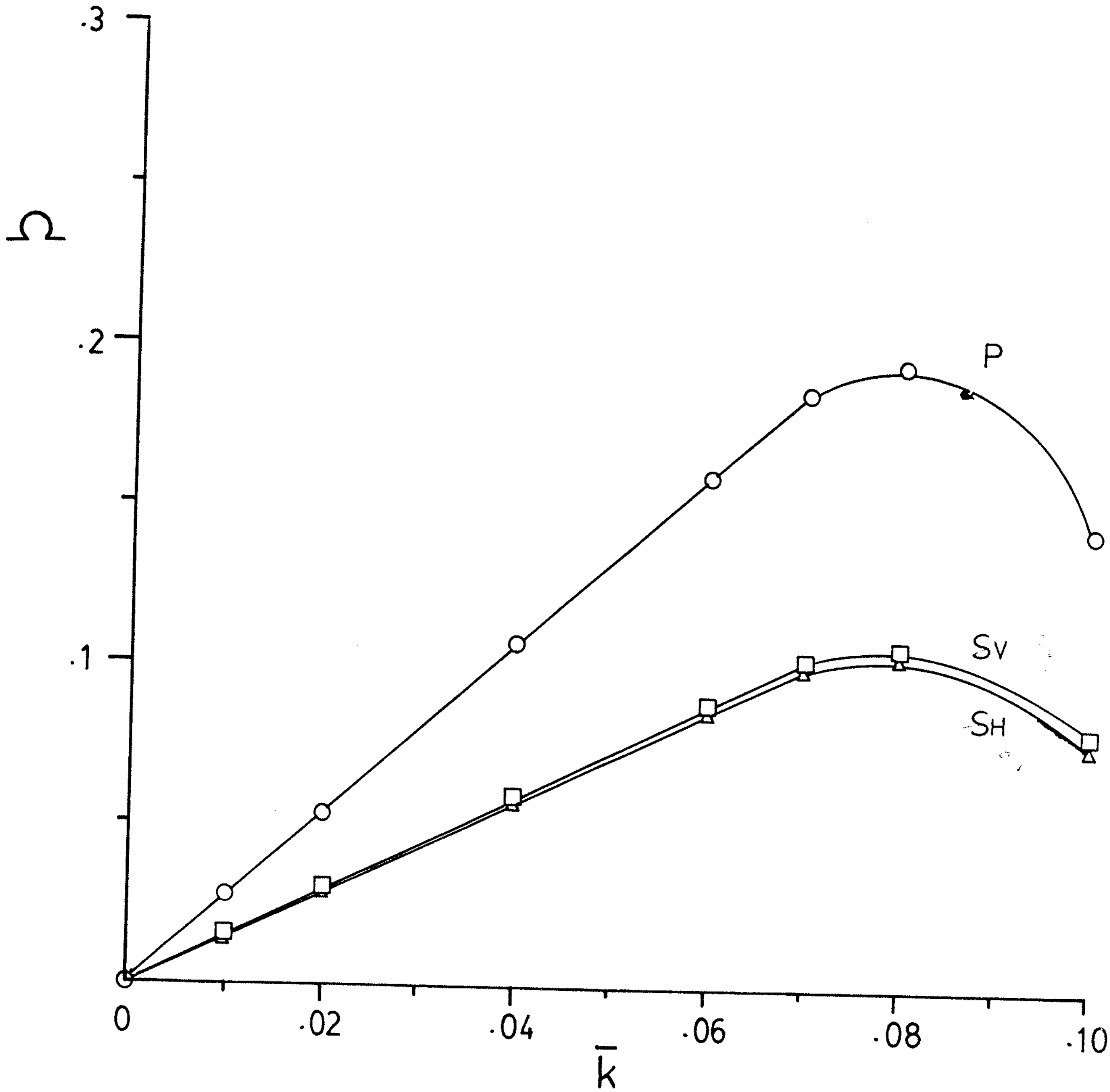


Fig. 4.11 Lowest SH, SV and P mode propagating along y-z plane, $\phi = 0°$, $\alpha = 90°$.

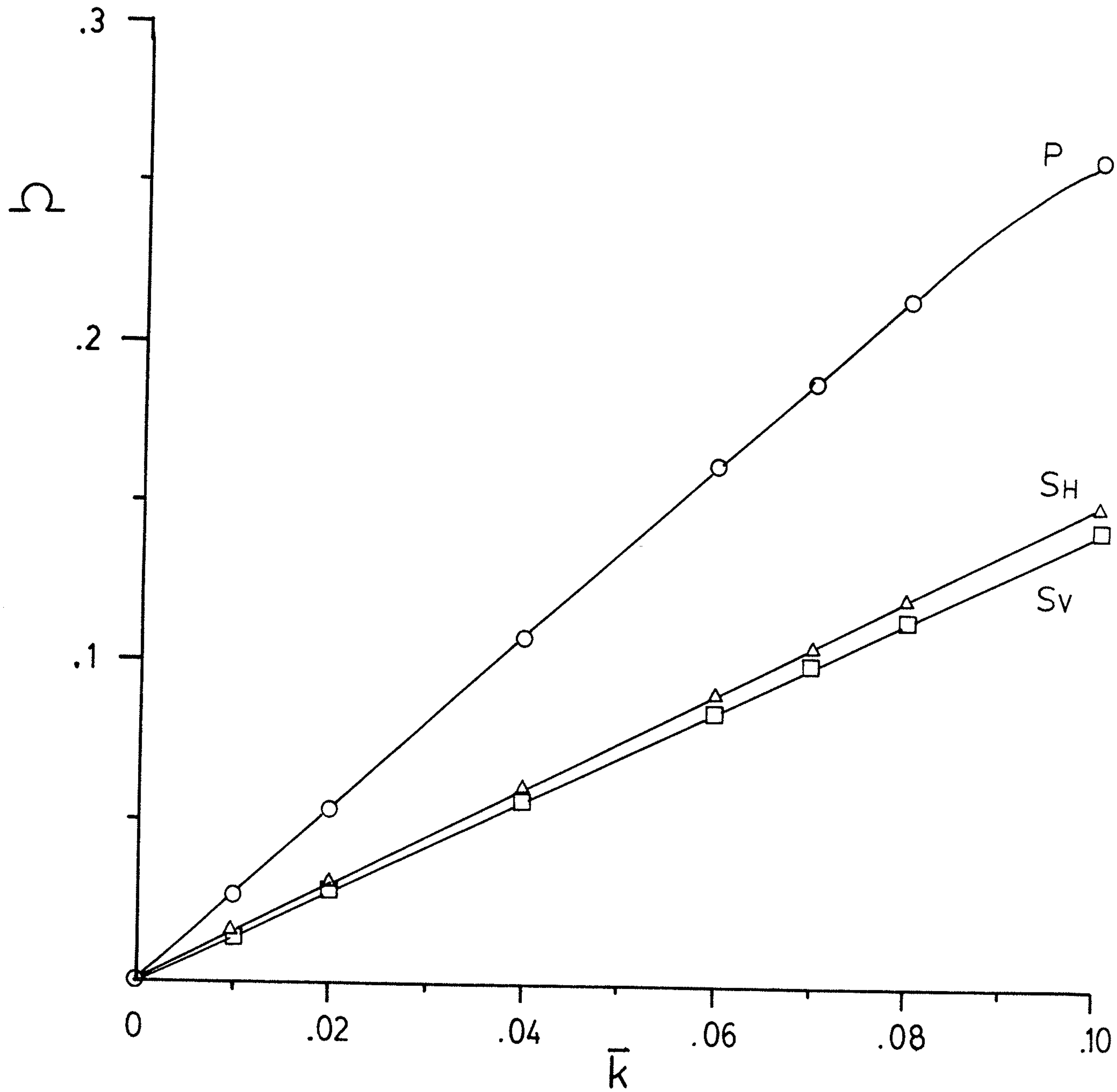


Fig. 4.12 Lowest SH, SV and P mode propagating along y-z plane, ϕ = 30°.

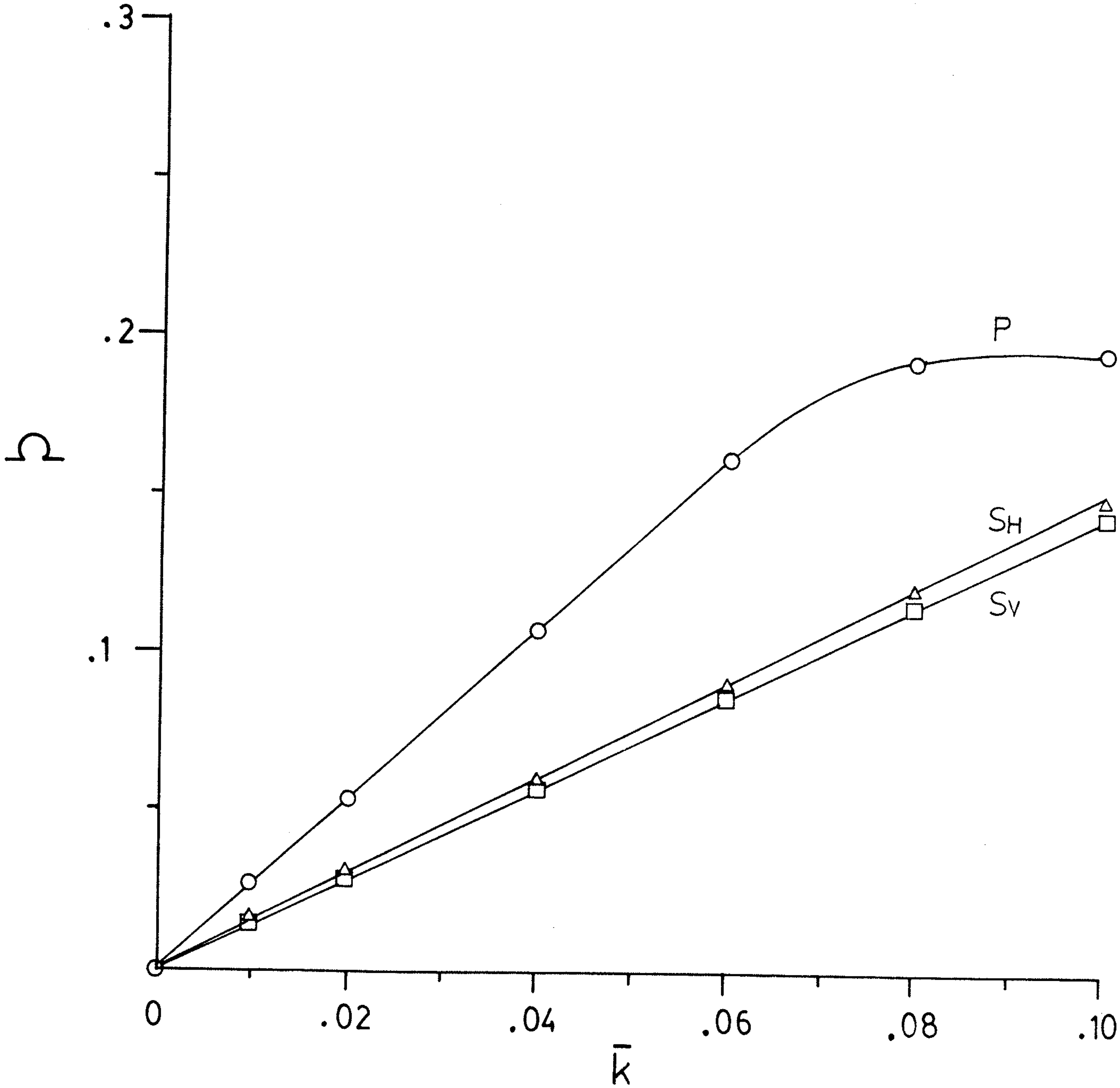


Fig. 4.13 Lowest SH, SV and P mode propagating along y-z plane, ϕ = 60°.

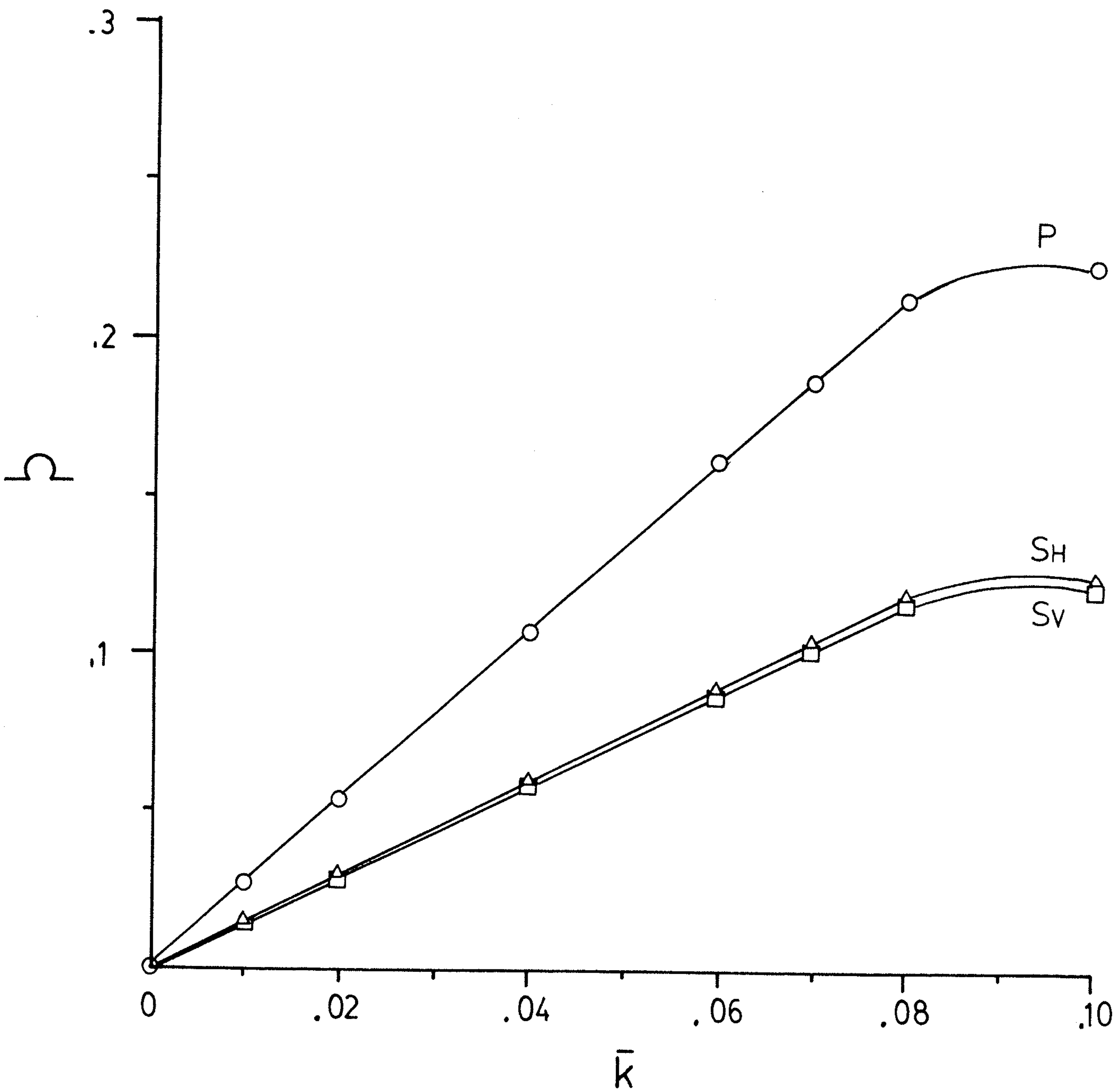


Fig. 4.14 Lowest SH, SV and P mode propagating along y-z plane, ϕ = 90°. (This figure is the same as Fig. 4.10)

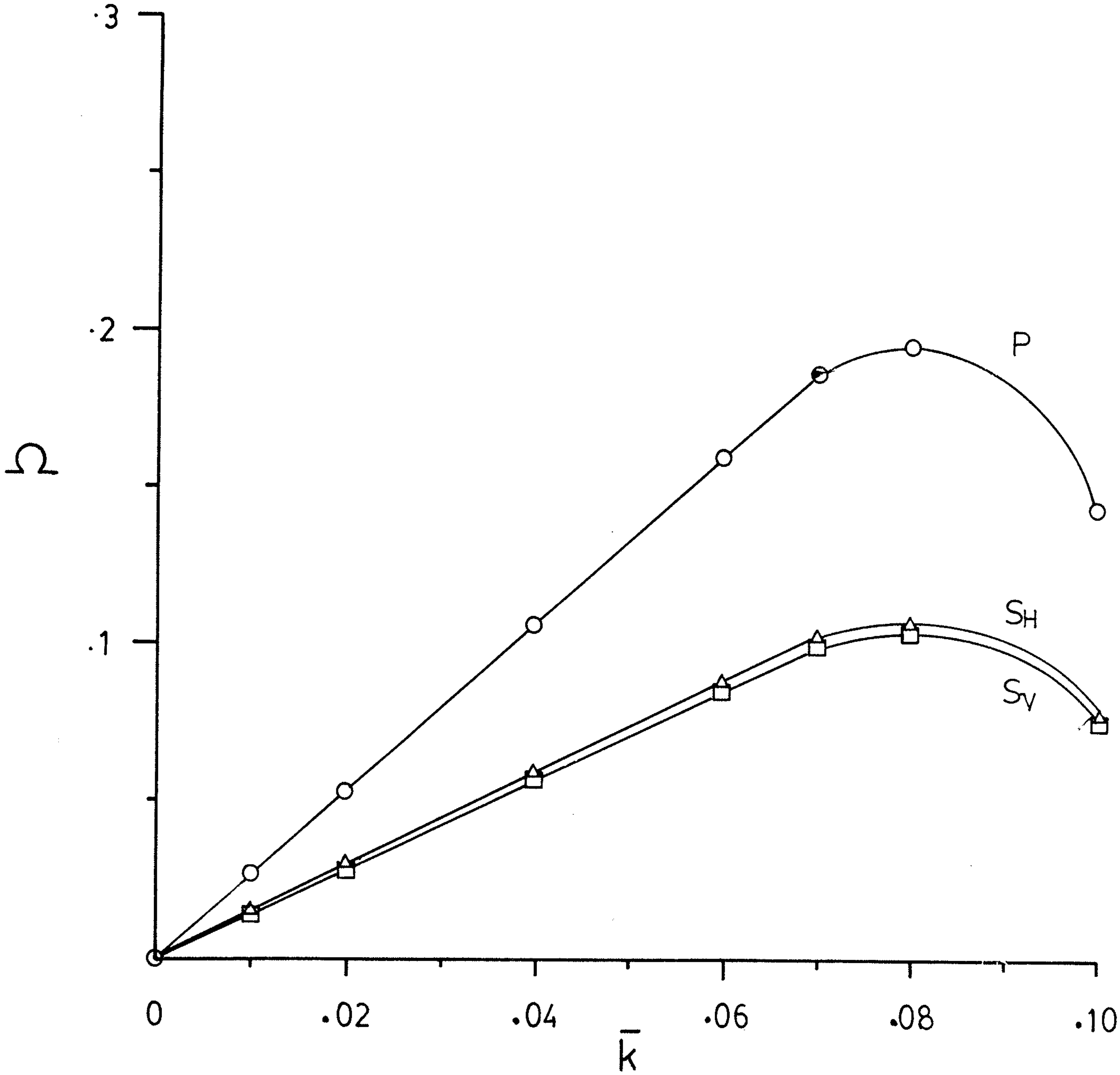


Fig. 4.15 Lowest SH, SV and P mode propagating along x-z plane, $\alpha = 0°$. (This figure is the same as Fig. 4.7)

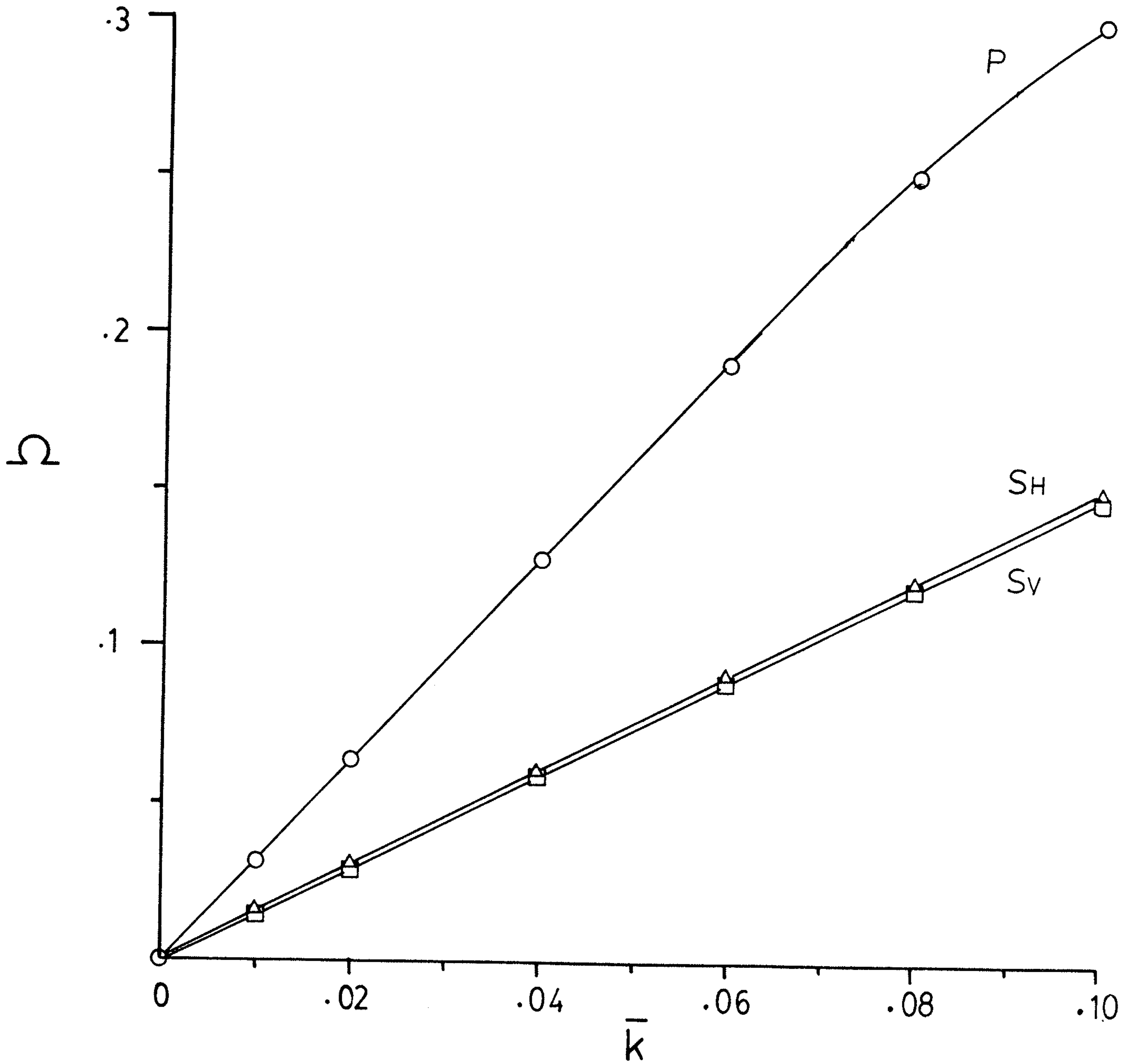


Fig. 4.16 Lowest SH, SV and P mode propagating along x-z plane, α = 30°.

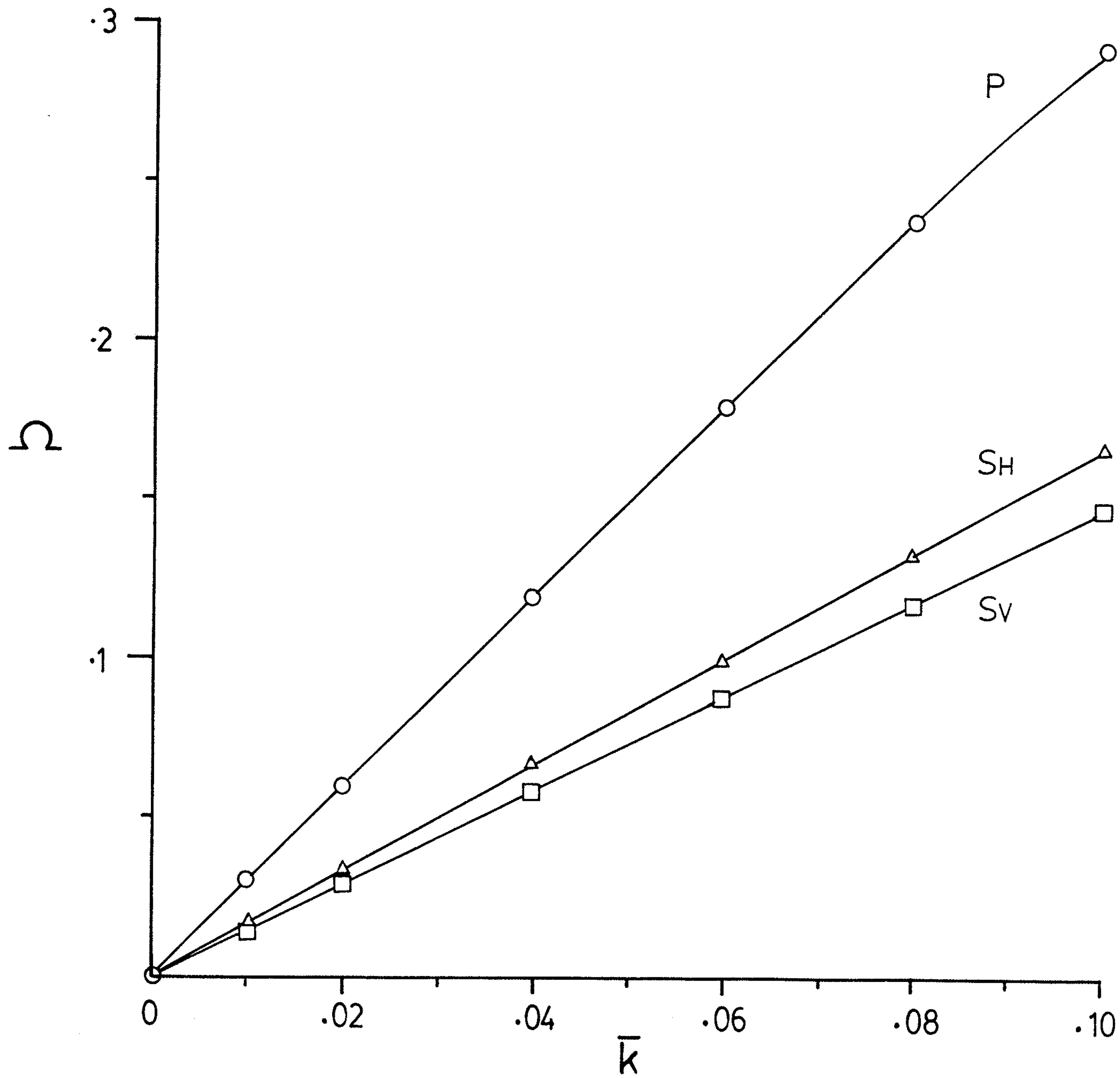


Fig. 4.17 Lowest SH, SV and P mode propagating along x-z plane, α = 60°.

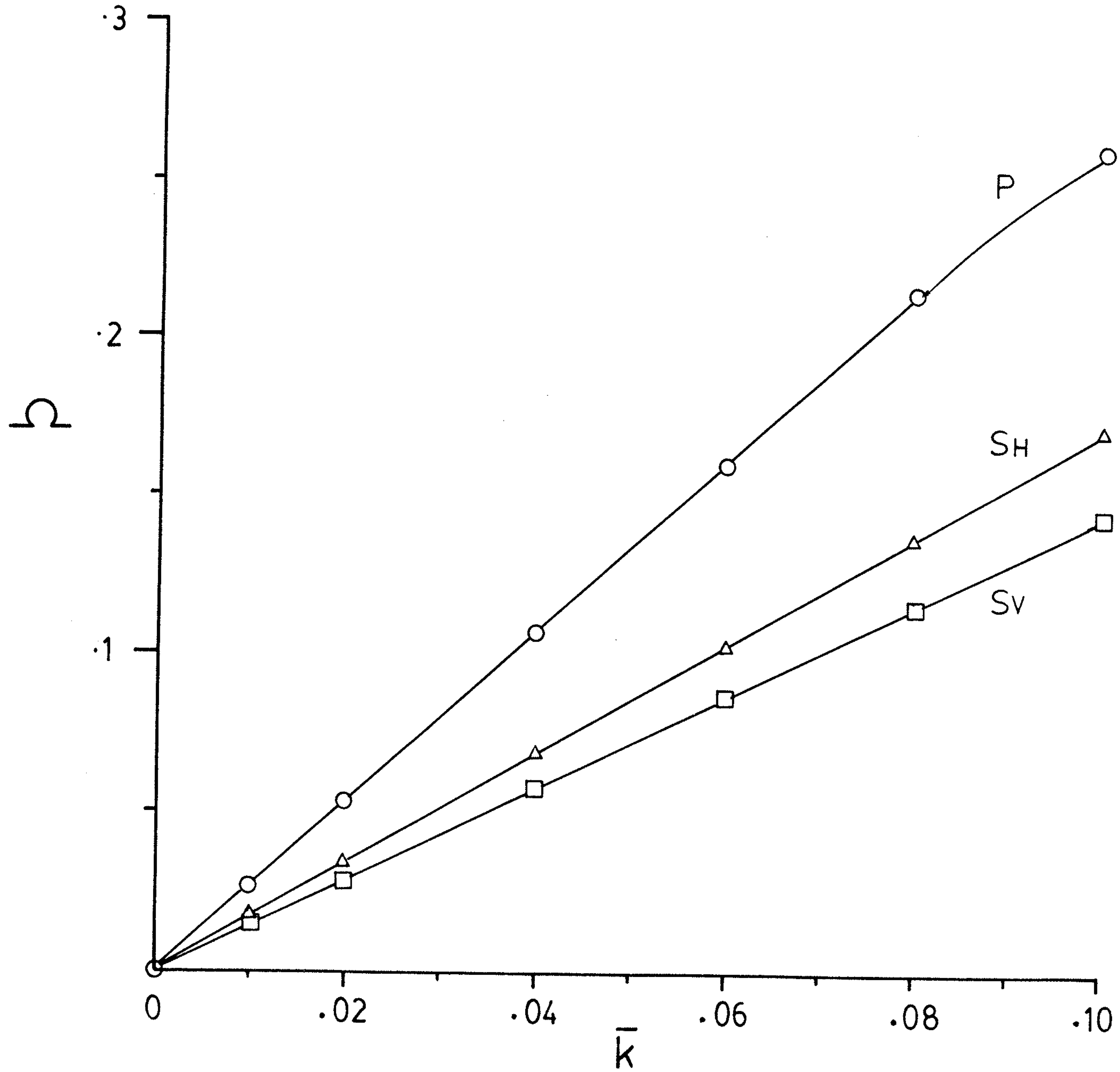


Fig. 4.18 Lowest SH, SV and P mode propagating along x-z plane, α = 90°. (This figure is the same as Fig. 4.11)

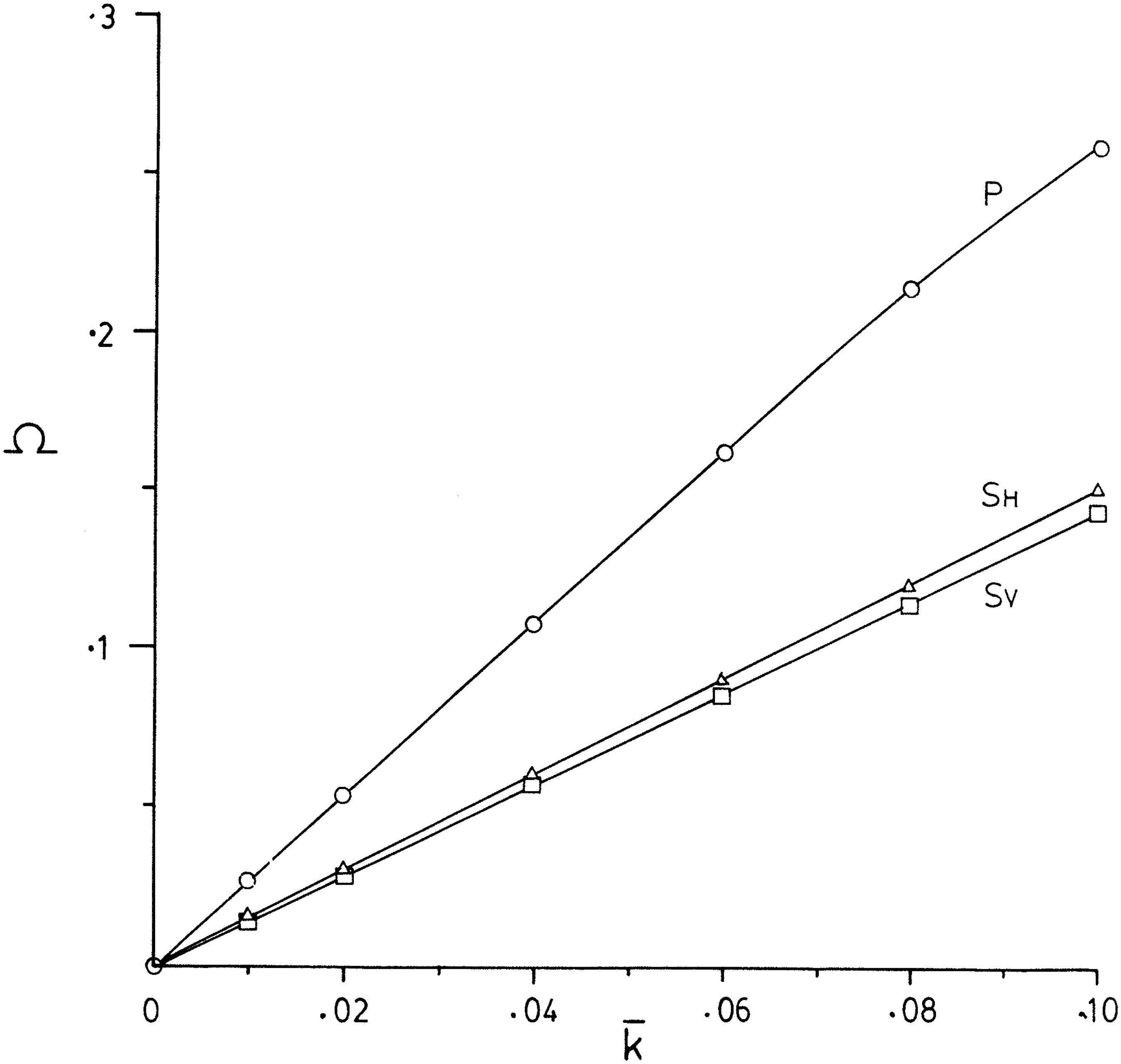


Fig. 4.19 Frequency vs. vertical angle plot for SH mode propagating on x-y plane.

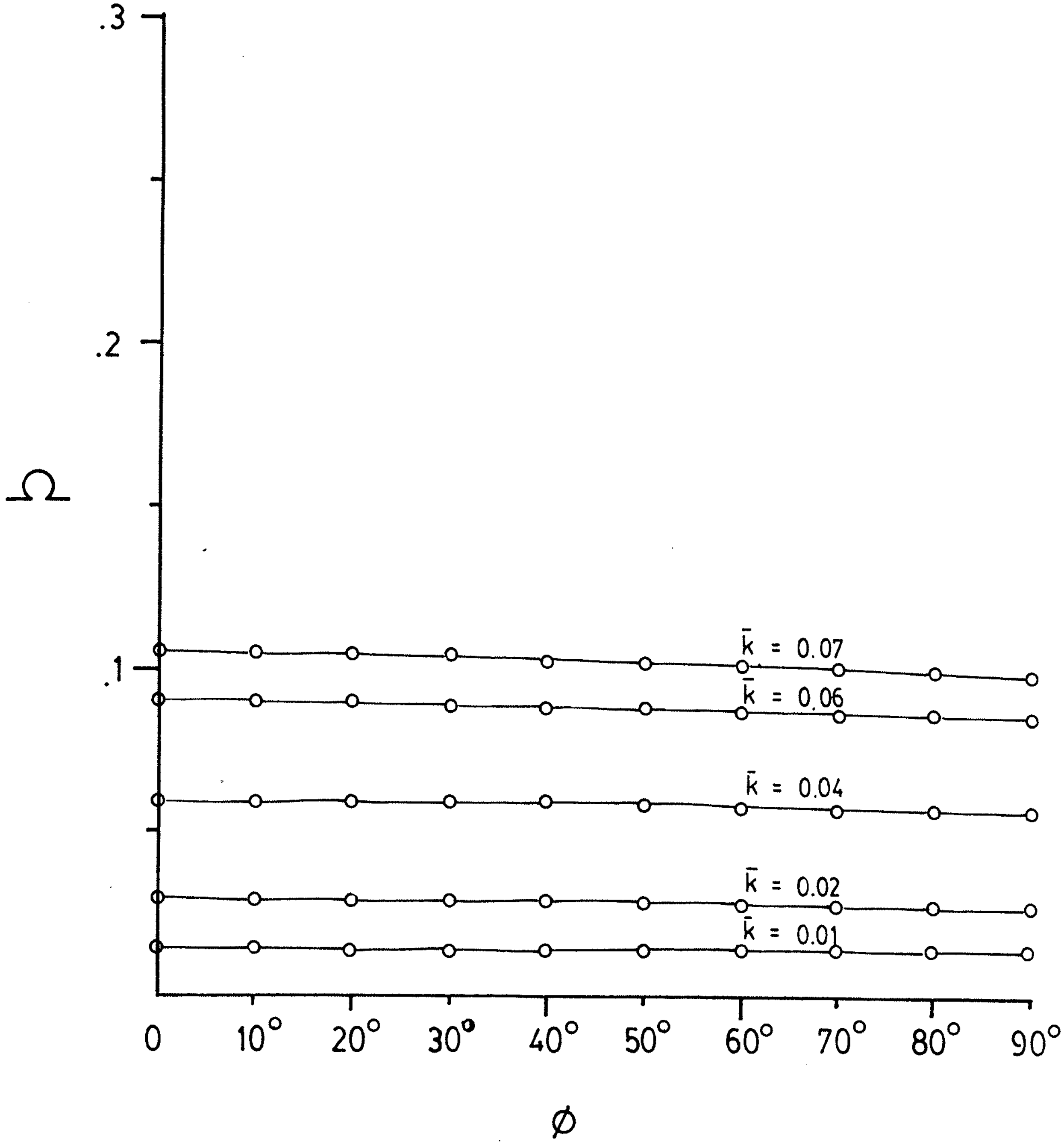


Fig. 4.20 Frequency vs. vertical angle plot for SV mode propagating on x-y plane.

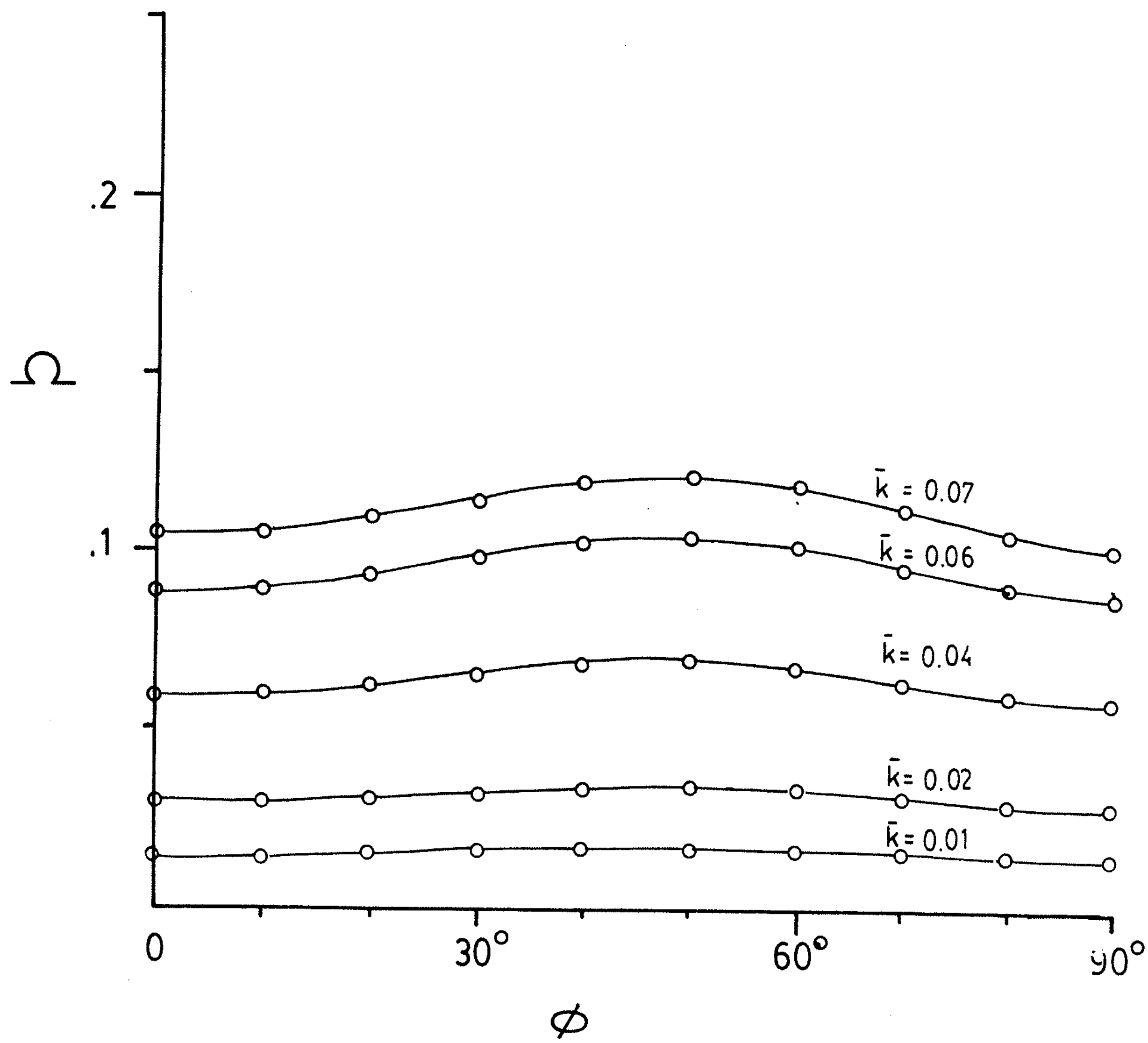


Fig. 4.21 Frequency vs. vertical angle plot for P mode propagating on x-y plane.

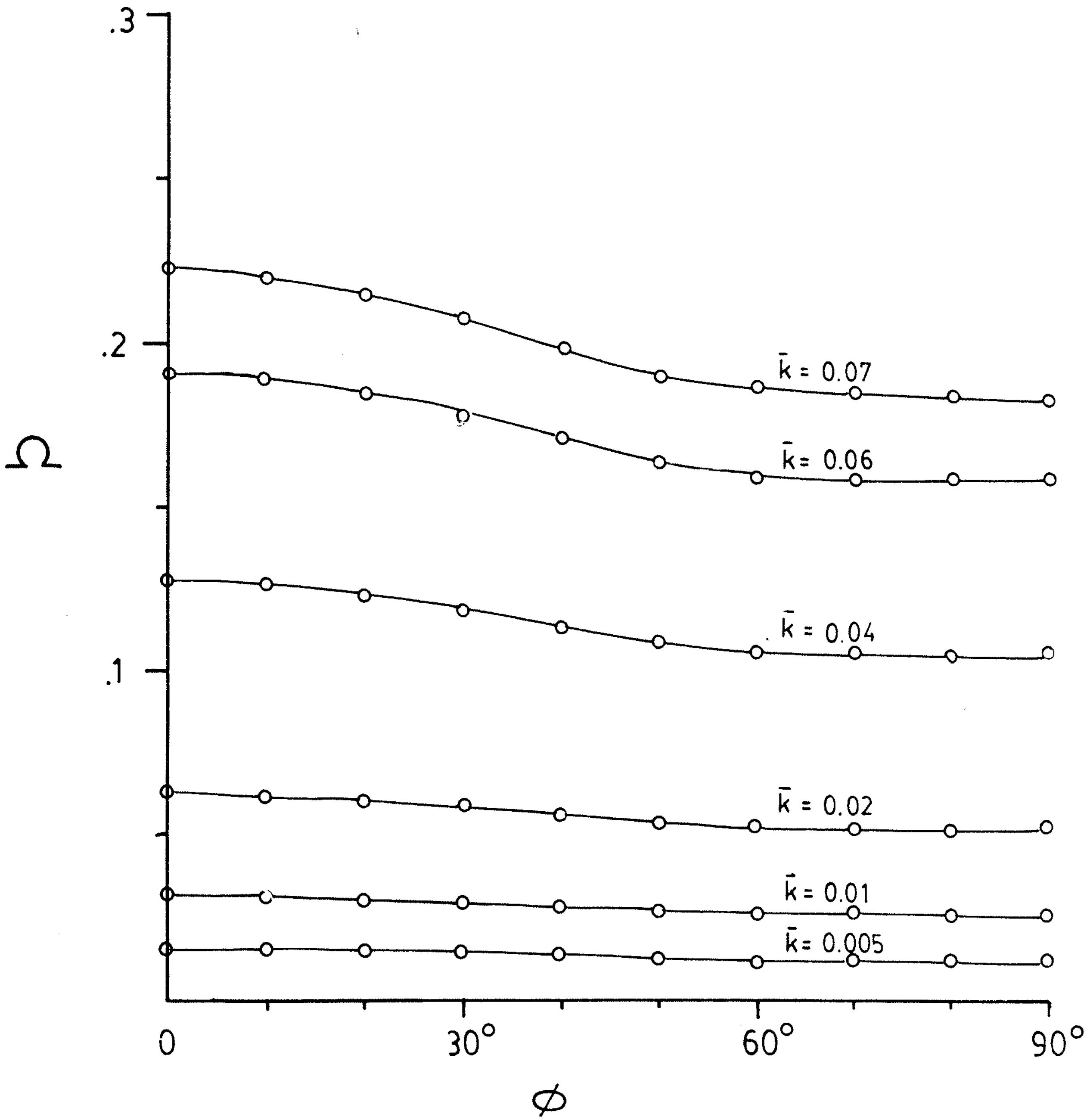


Fig. 4.22 Frequency vs. vertical angle plot for lowest SH mode propagating on y-z plane.

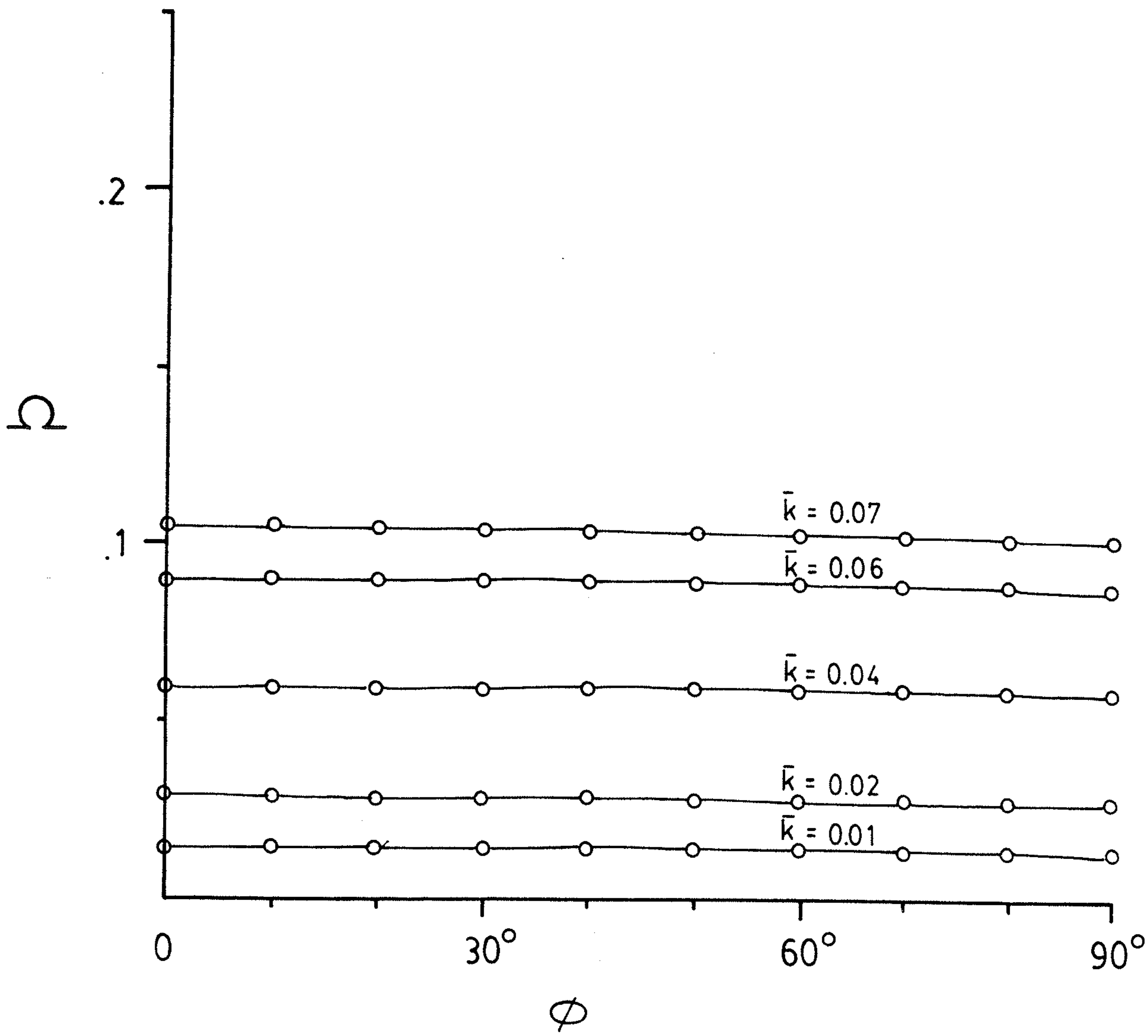


Fig. 4.23 Frequency vs. vertical angle plot for lowest SV mode propagating on y-z plane.

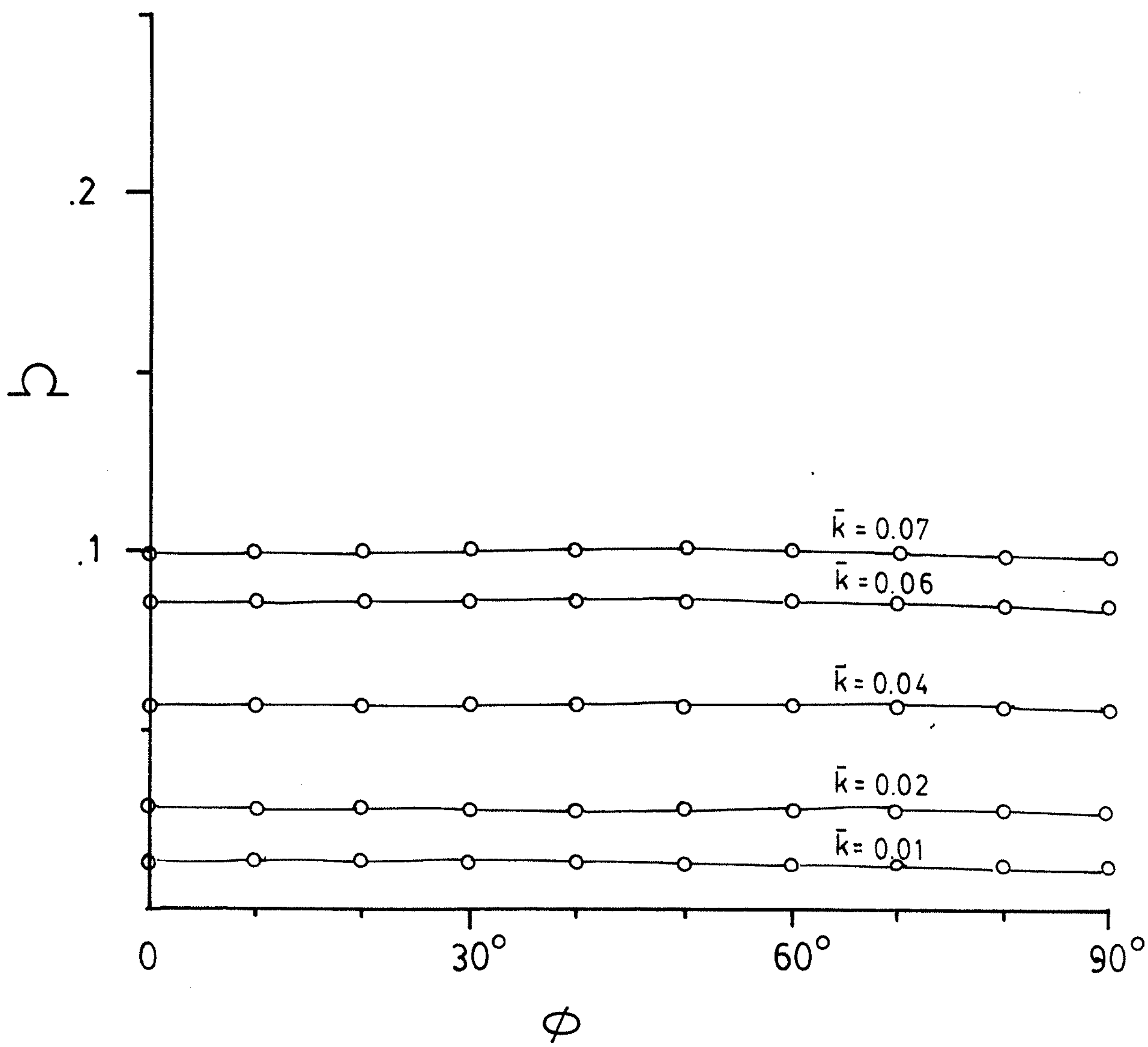


Fig. 4.24 Frequency vs. vertical angle plot for lowest P mode propagating on y-z plane.

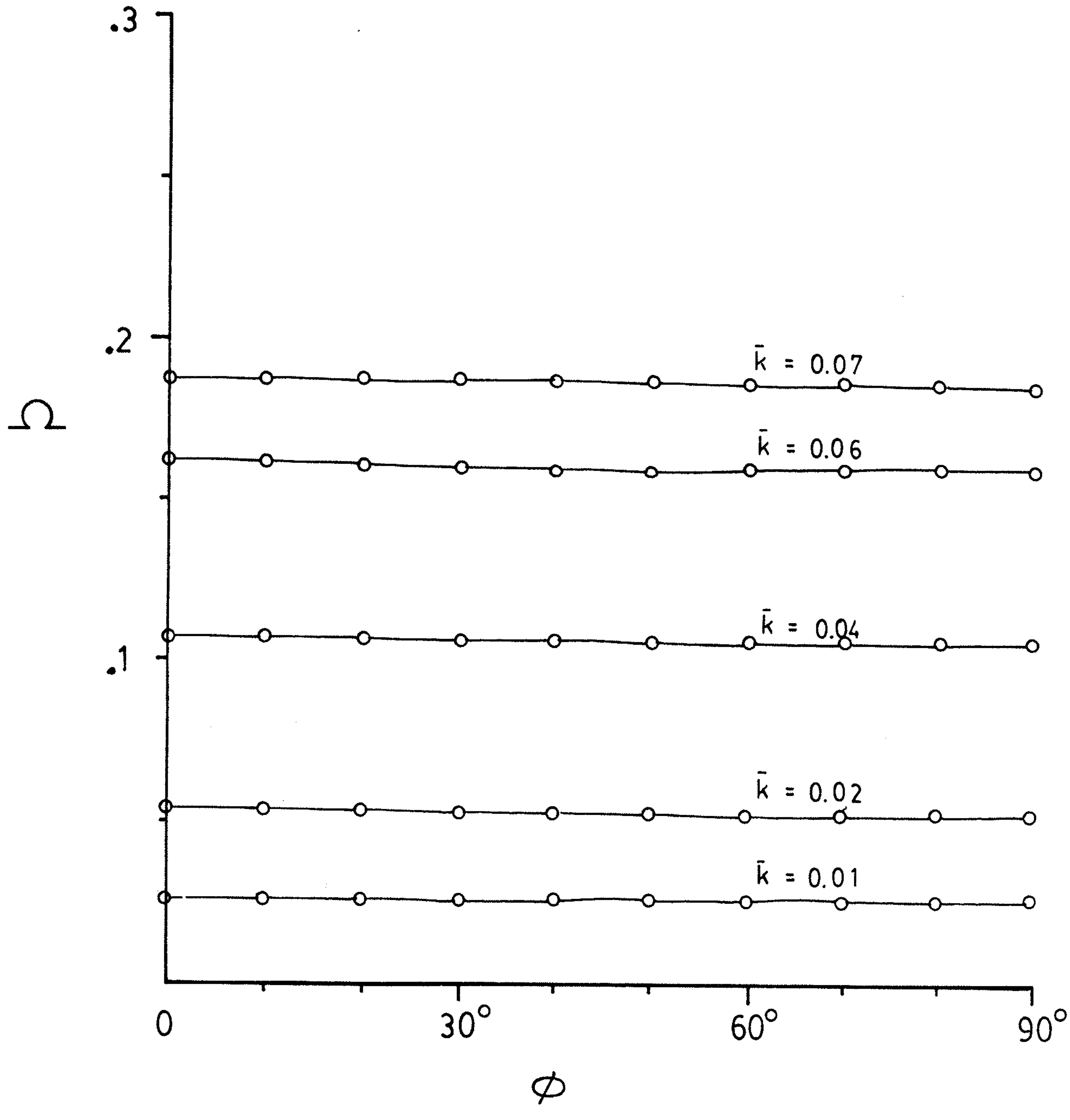


Fig. 4.25 Frequency vs. horizontal plane plot for lowest SH mode propagating on x-z plane.

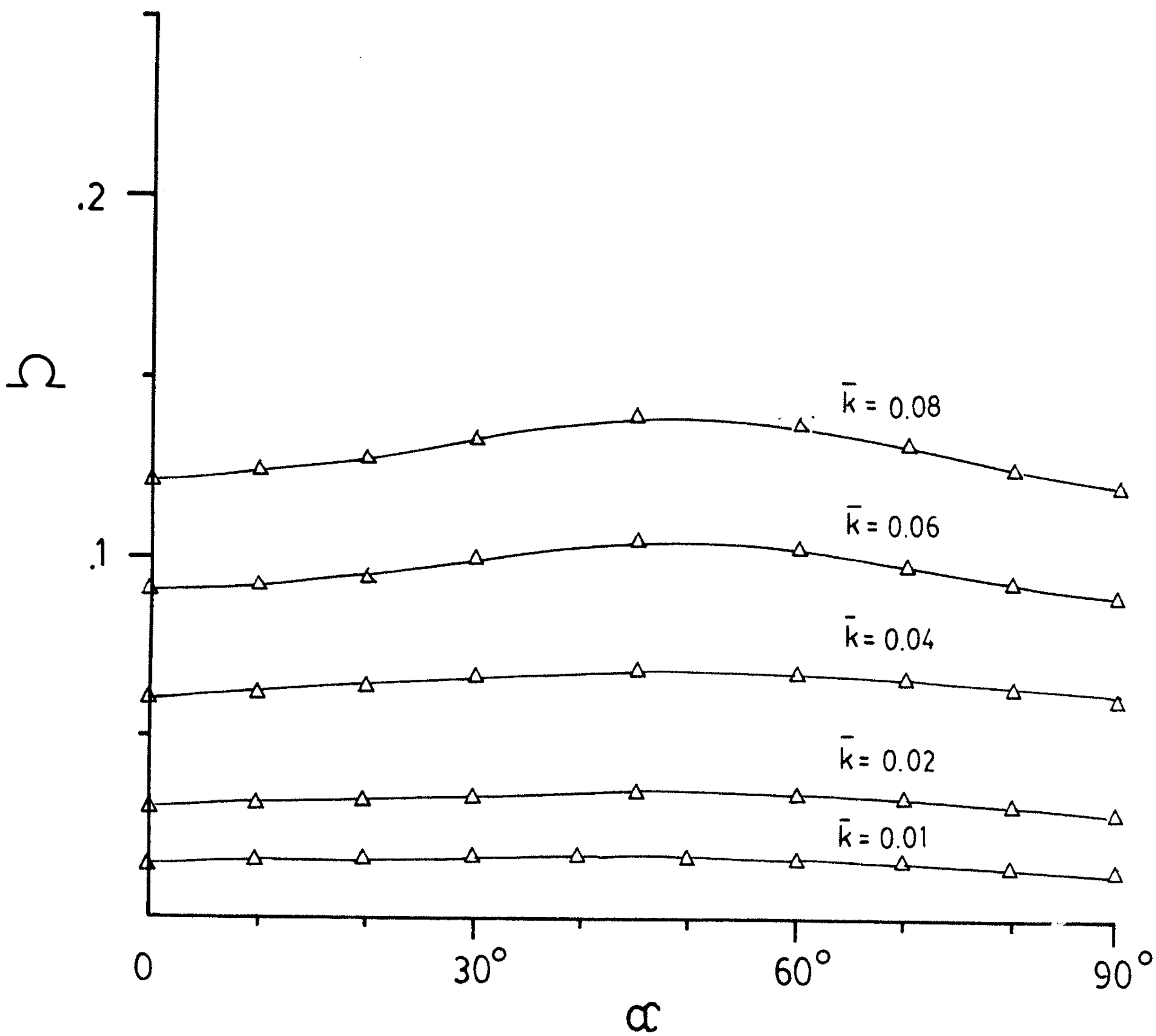


Fig. 4.26 Frequency vs. horizontal angle plot for lowest SV mode propagating on x-z plane.

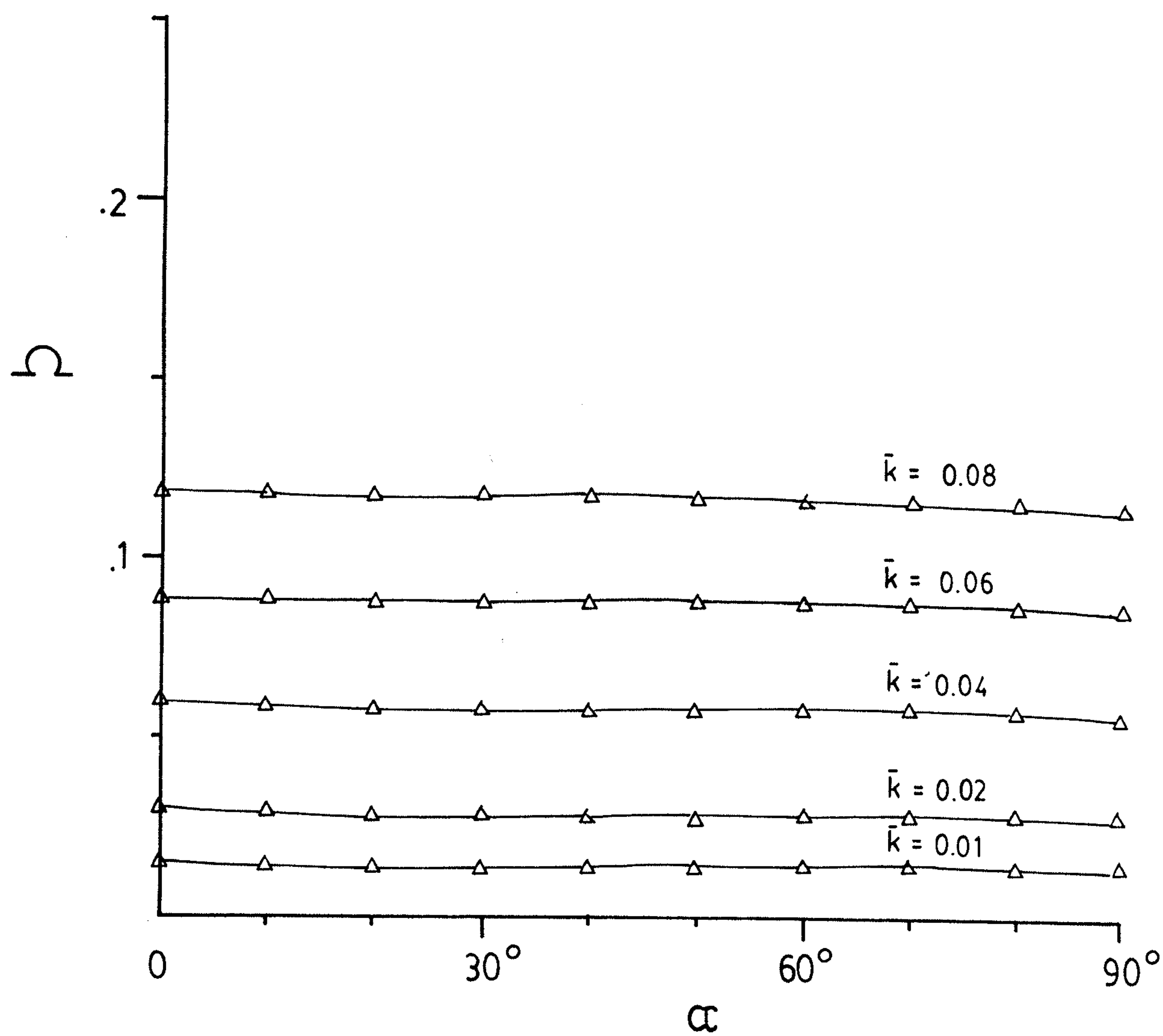
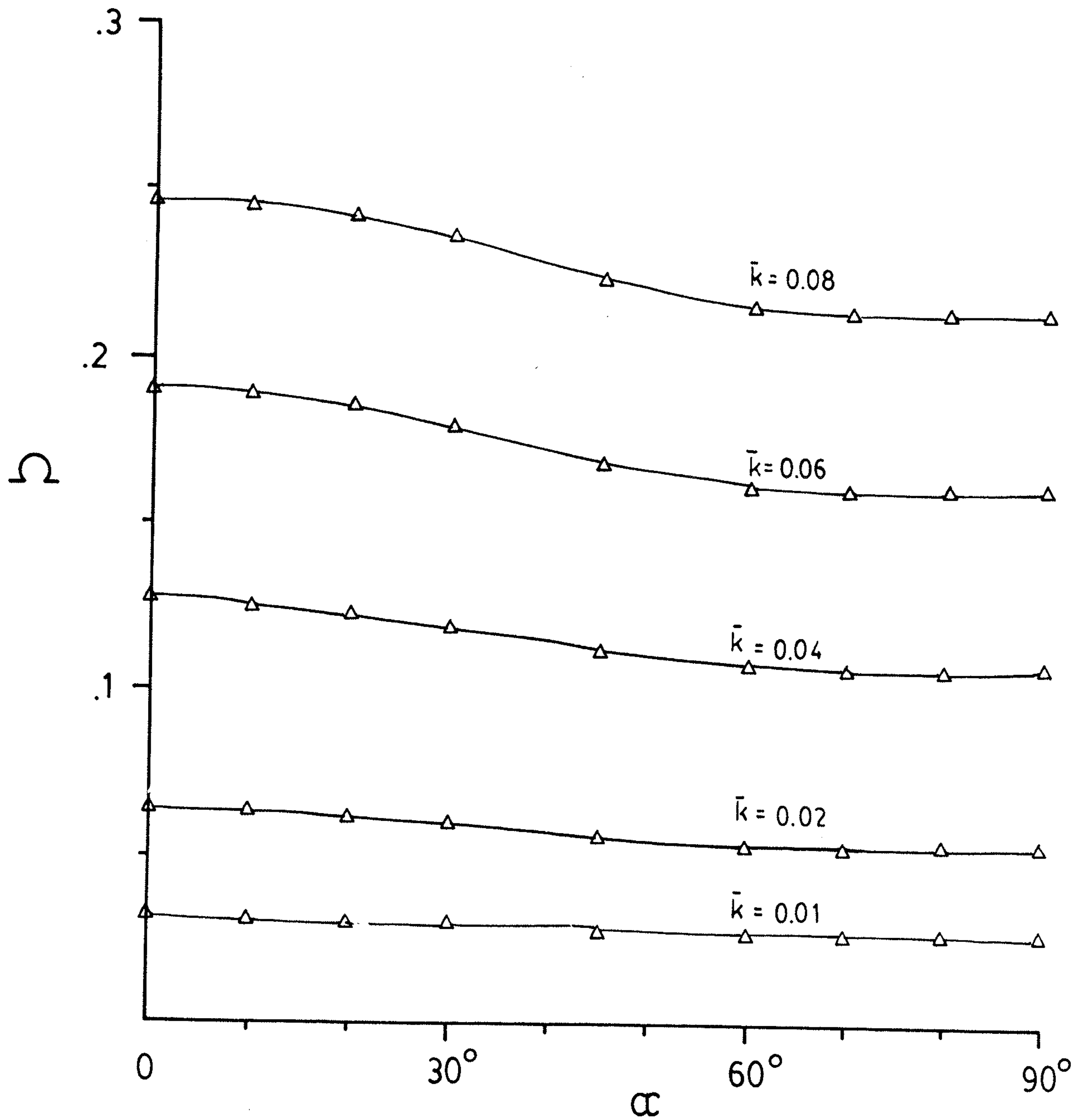


Fig. 4.27 Frequency vs. horizontal angle plot for lowest P mode propagating on x-z plane.



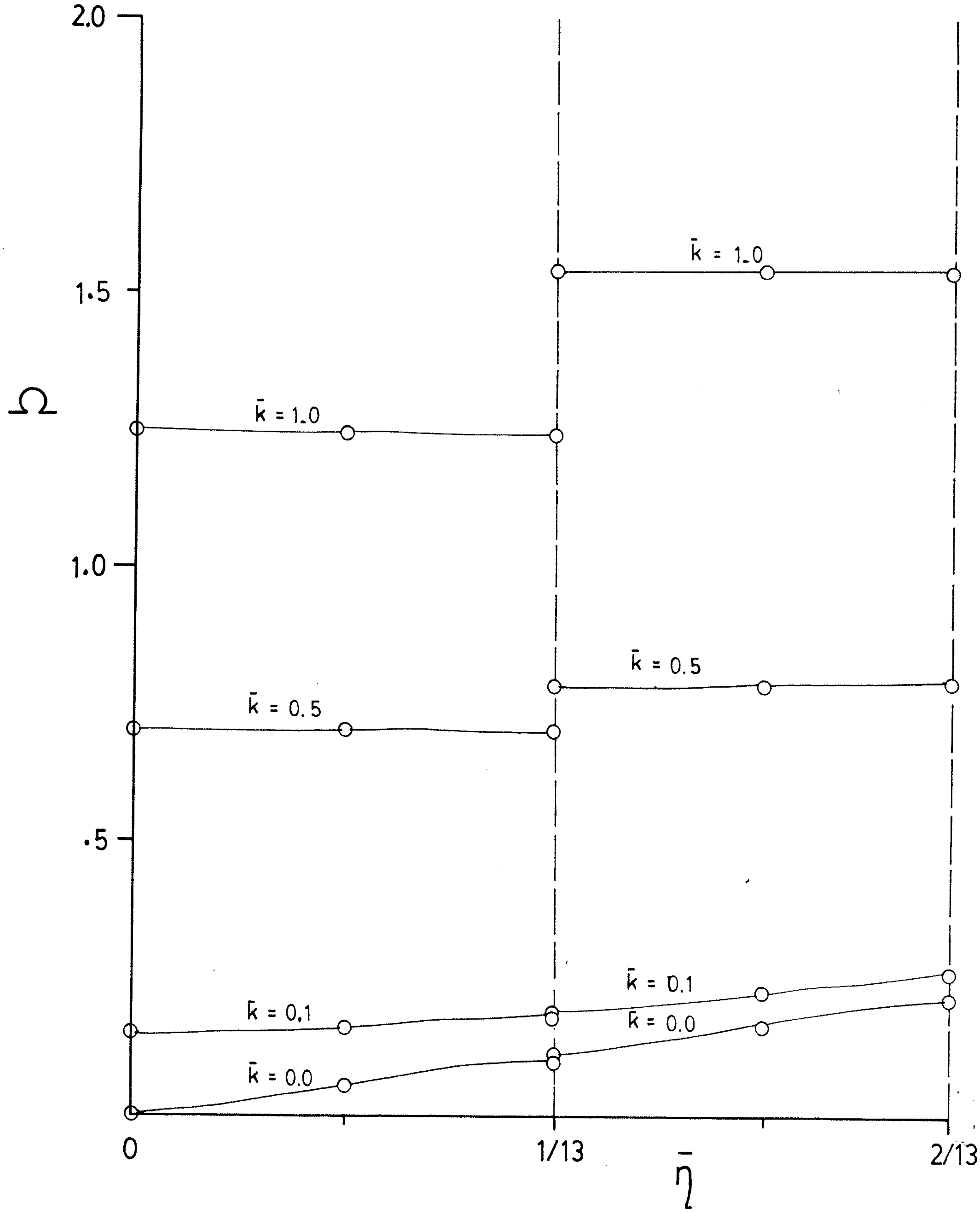


Fig. 4.28 Curves of constant $\bar{k}$ for lowest SH mode, $\alpha = 0°$.

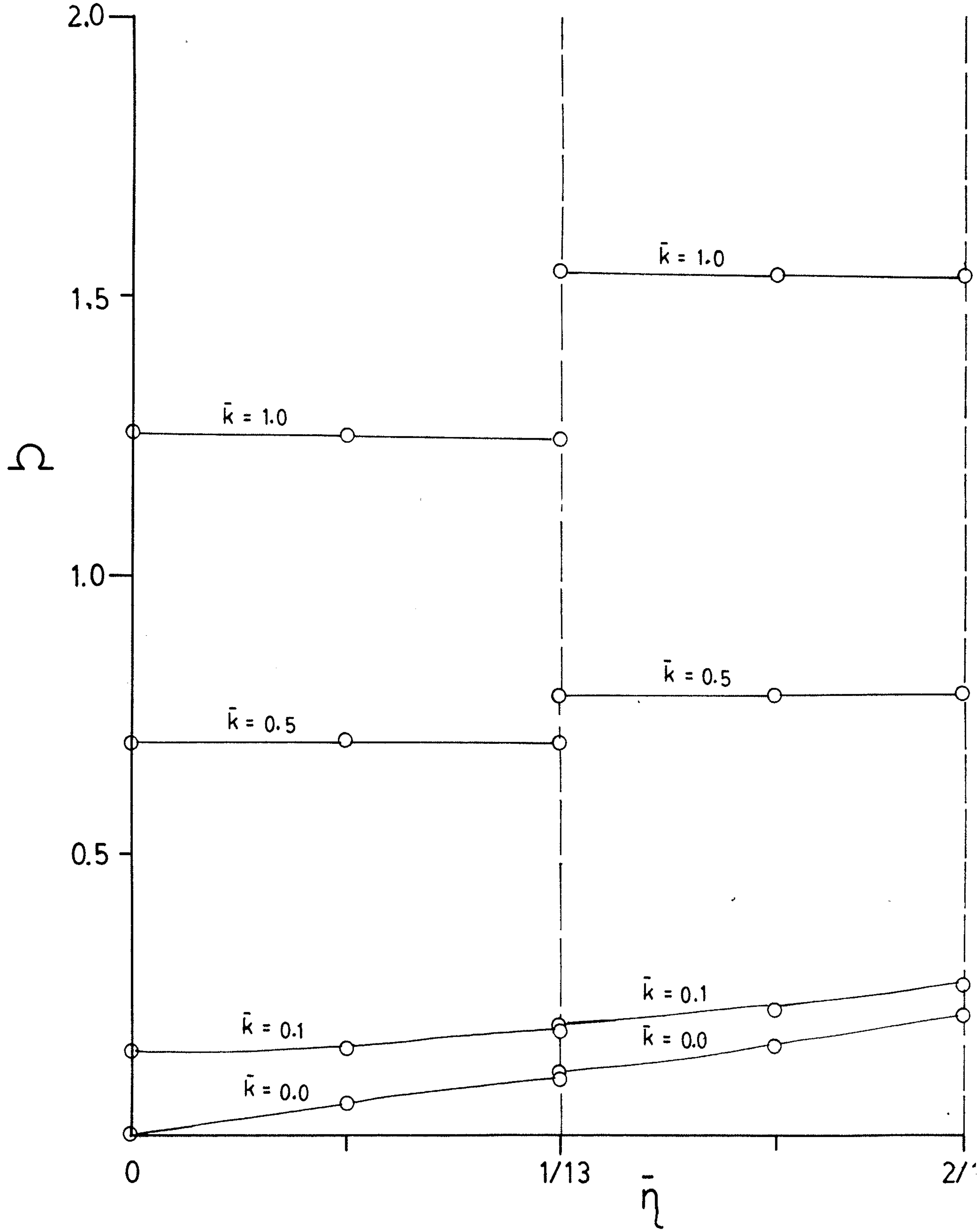


Fig. 4.29 Curves of constant $\bar{k}$ for lowest SH mode, α = 90°.

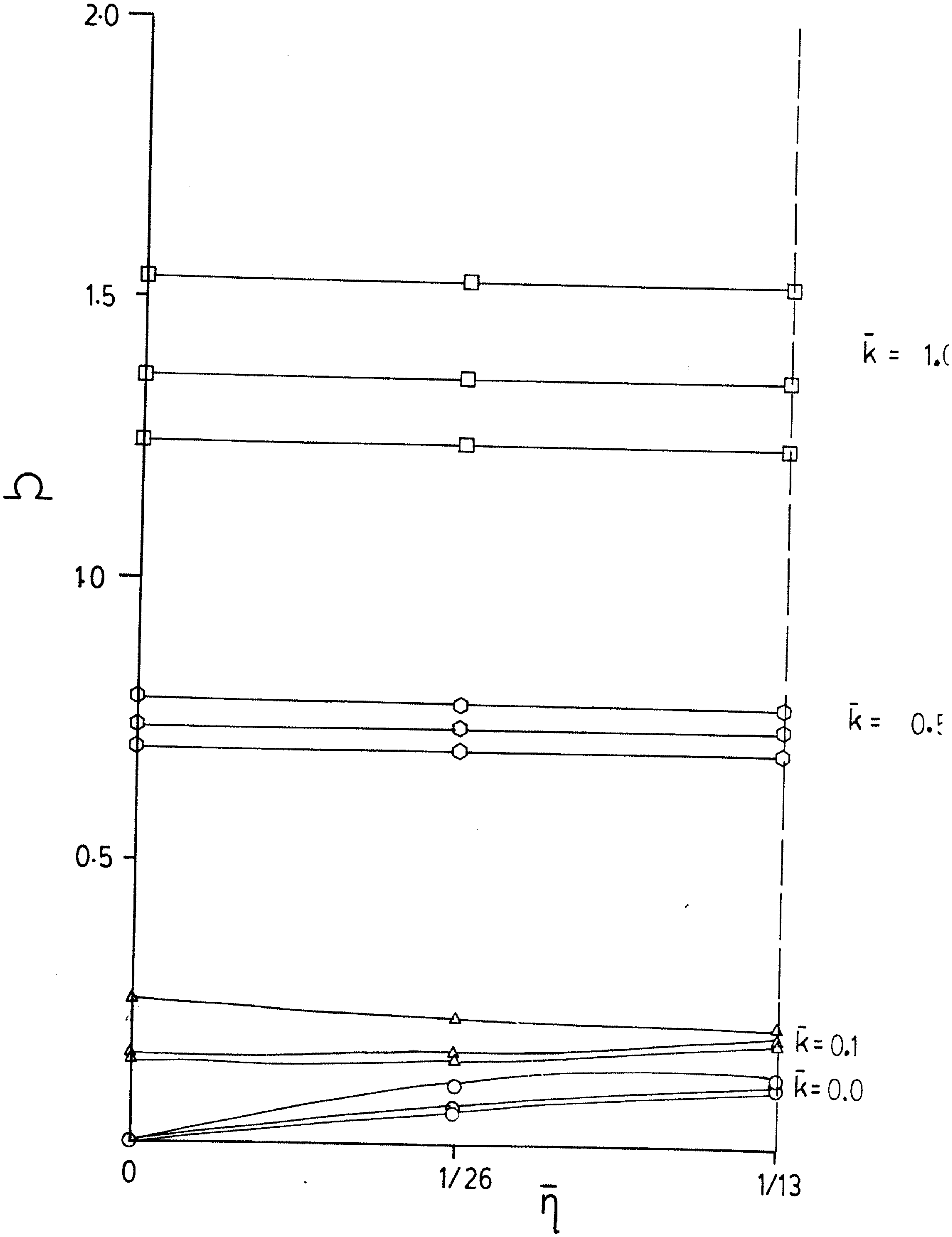


Fig. 4.30 Curves of constant $\bar{k}$ on x-y plane, α = 0°.

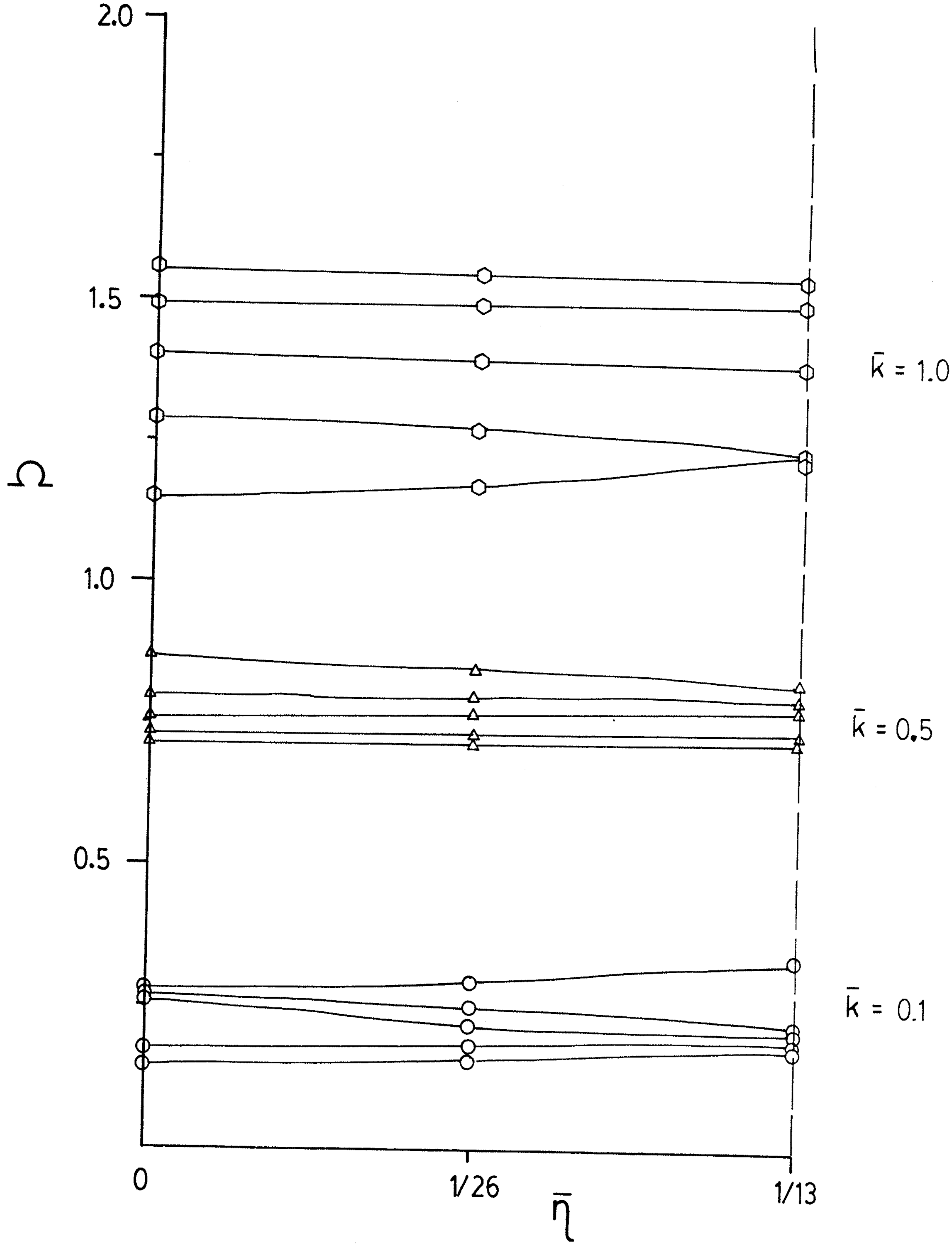


Fig. 4.31 Lowest 5 branches for curves of constant $\bar{k}$ on x-y plane, α = 45°.

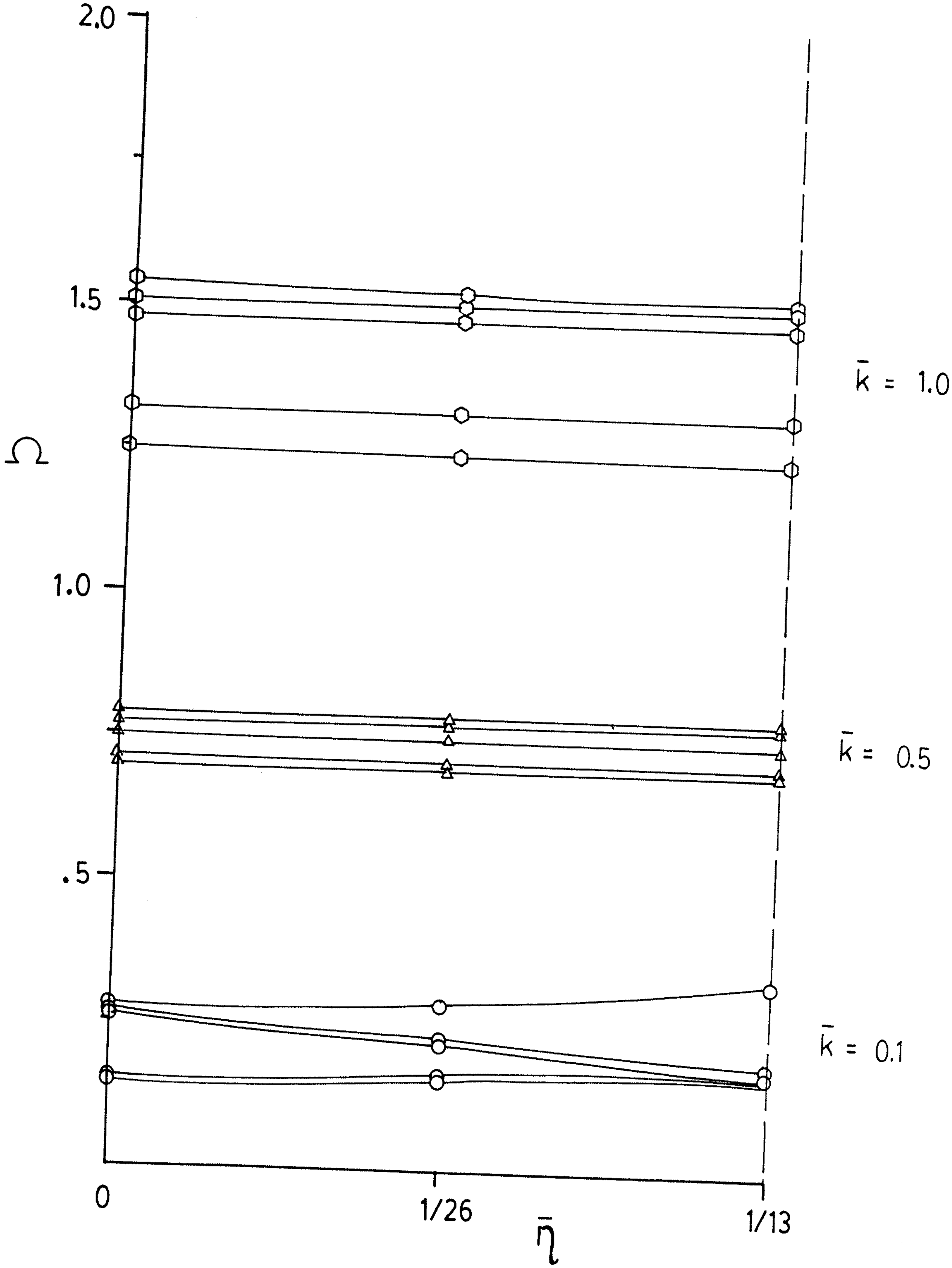


Fig. 4.32 Lowest 5 branches for curves of constant $\bar{k}$ on y-z plane, α = 90°.

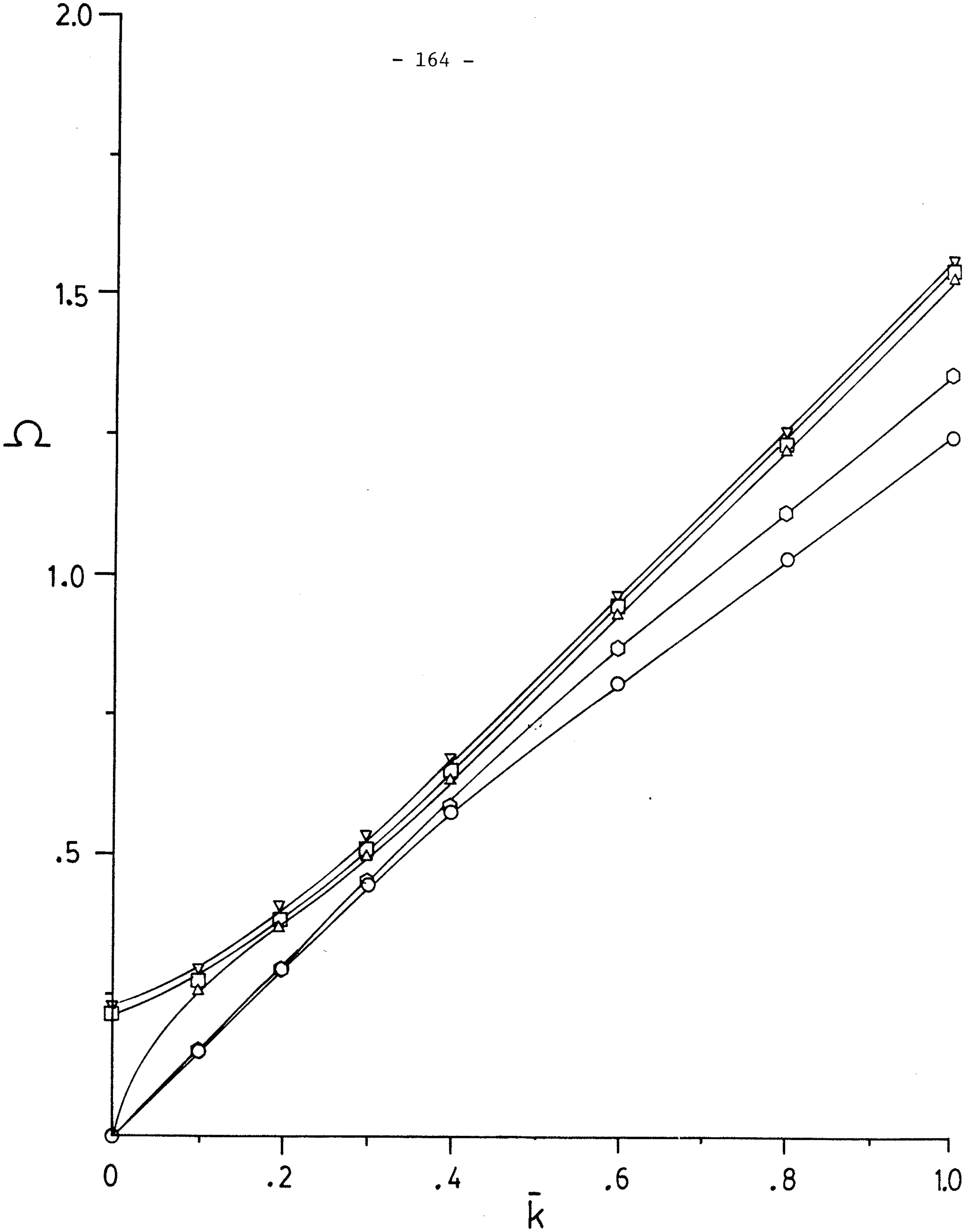


Fig. 4.33 Lowest 5 branches for $\alpha = 0°$, $\bar{\eta} = 0.0$.

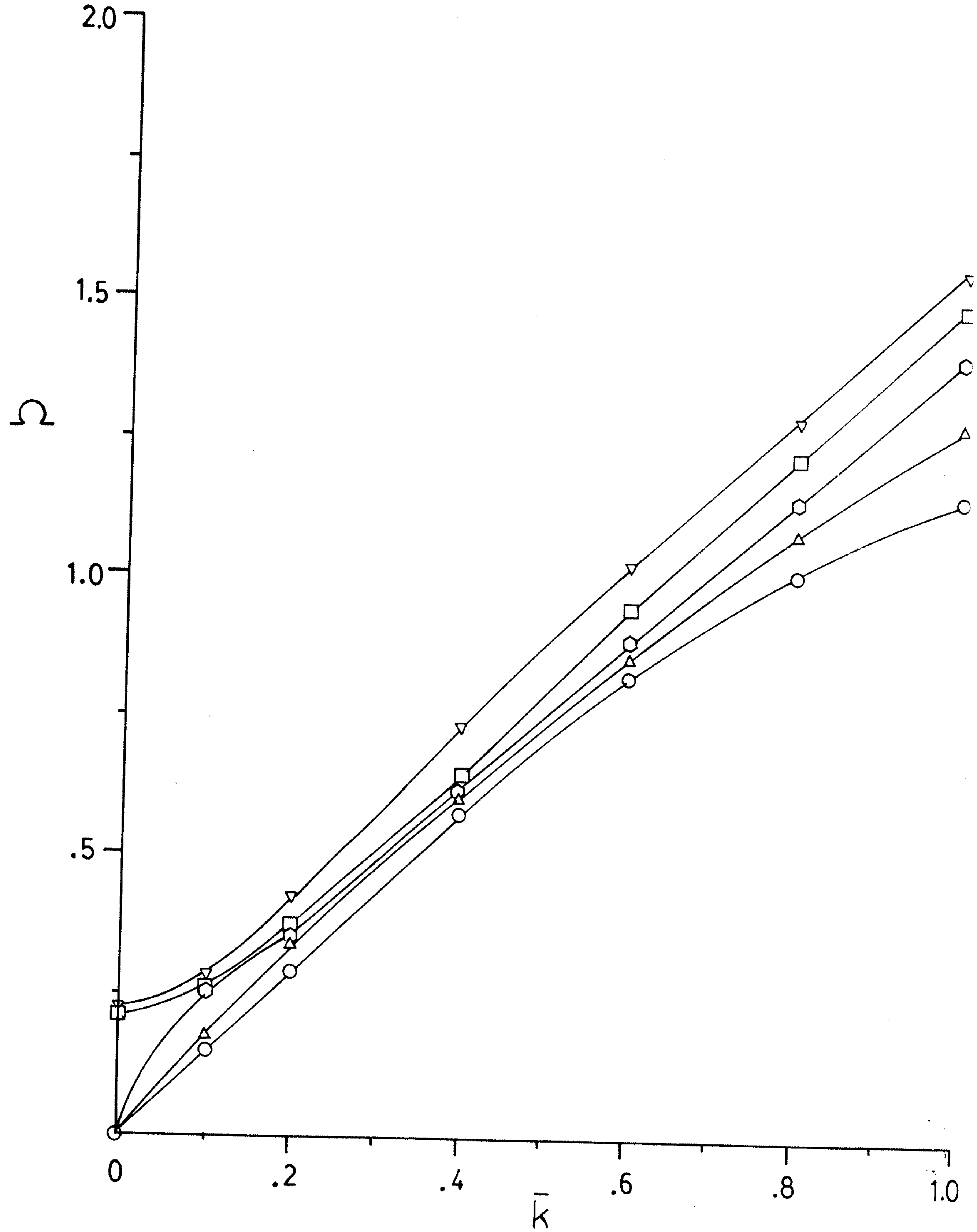


Fig. 4.34 Lowest 5 branches for $\alpha = 45°$, $\bar{\eta} = 0.0$.

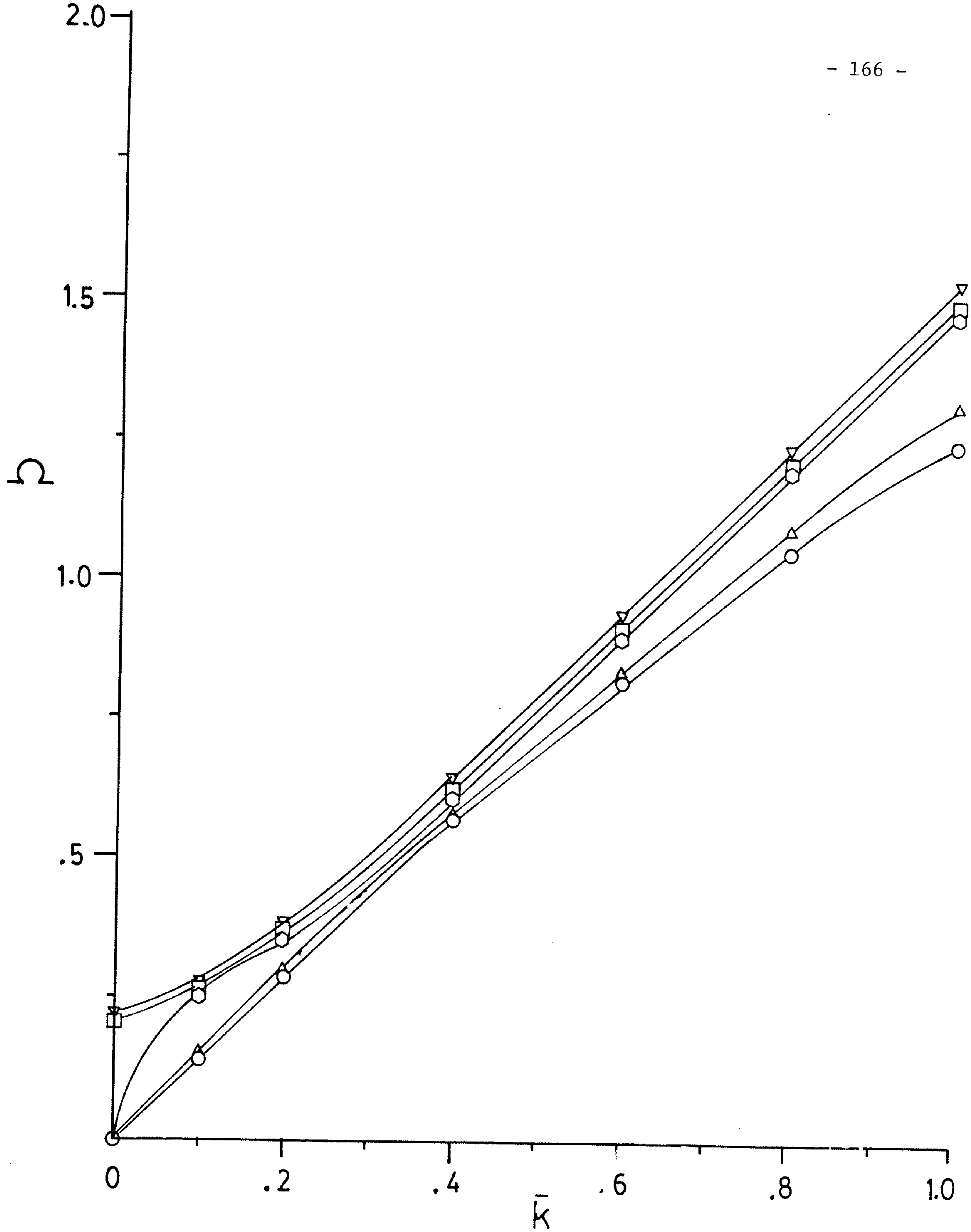
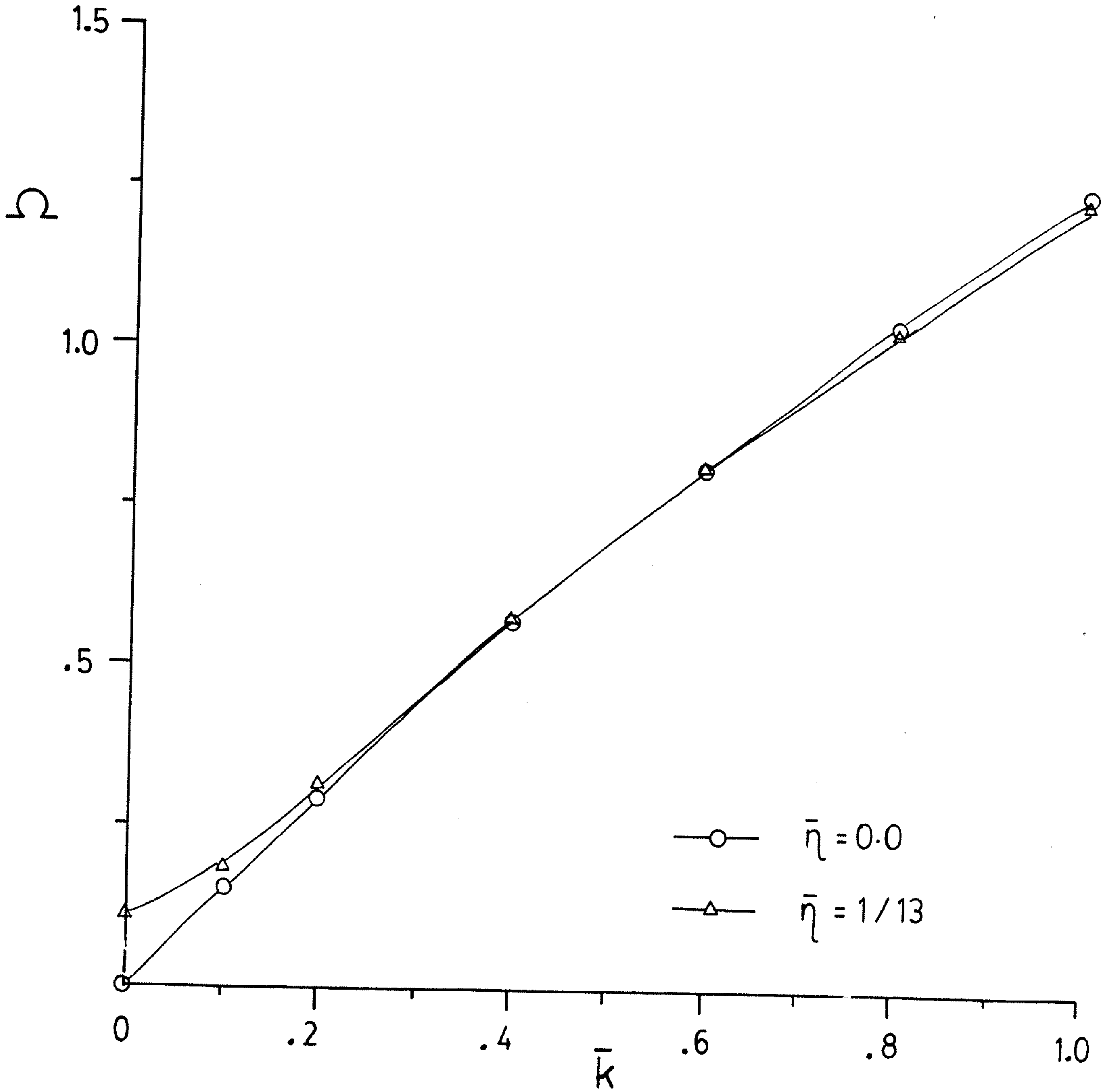


Fig. 4.35 Lowest 5 branches for $\alpha = 90°$, $\bar{\eta} = 0.0$.

Fig. 4.36 Frequency vs. wave number plot for lowest SH mode, η = 0.0 and $\bar{\eta}$ = 1/13.



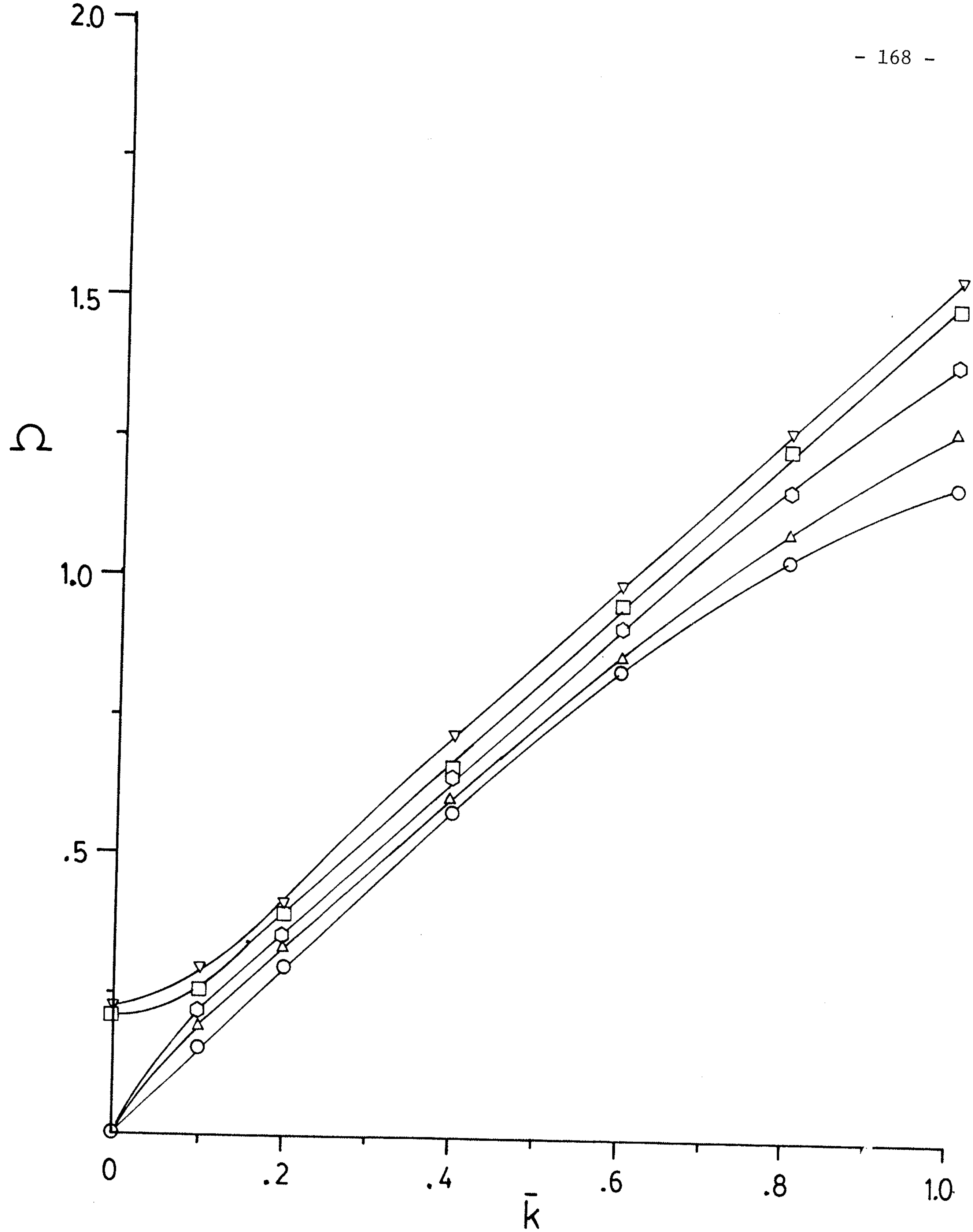


Fig. 4.37 Lowest 5 branches for $\alpha = 45°$, $\bar{\eta} = 1/26$.

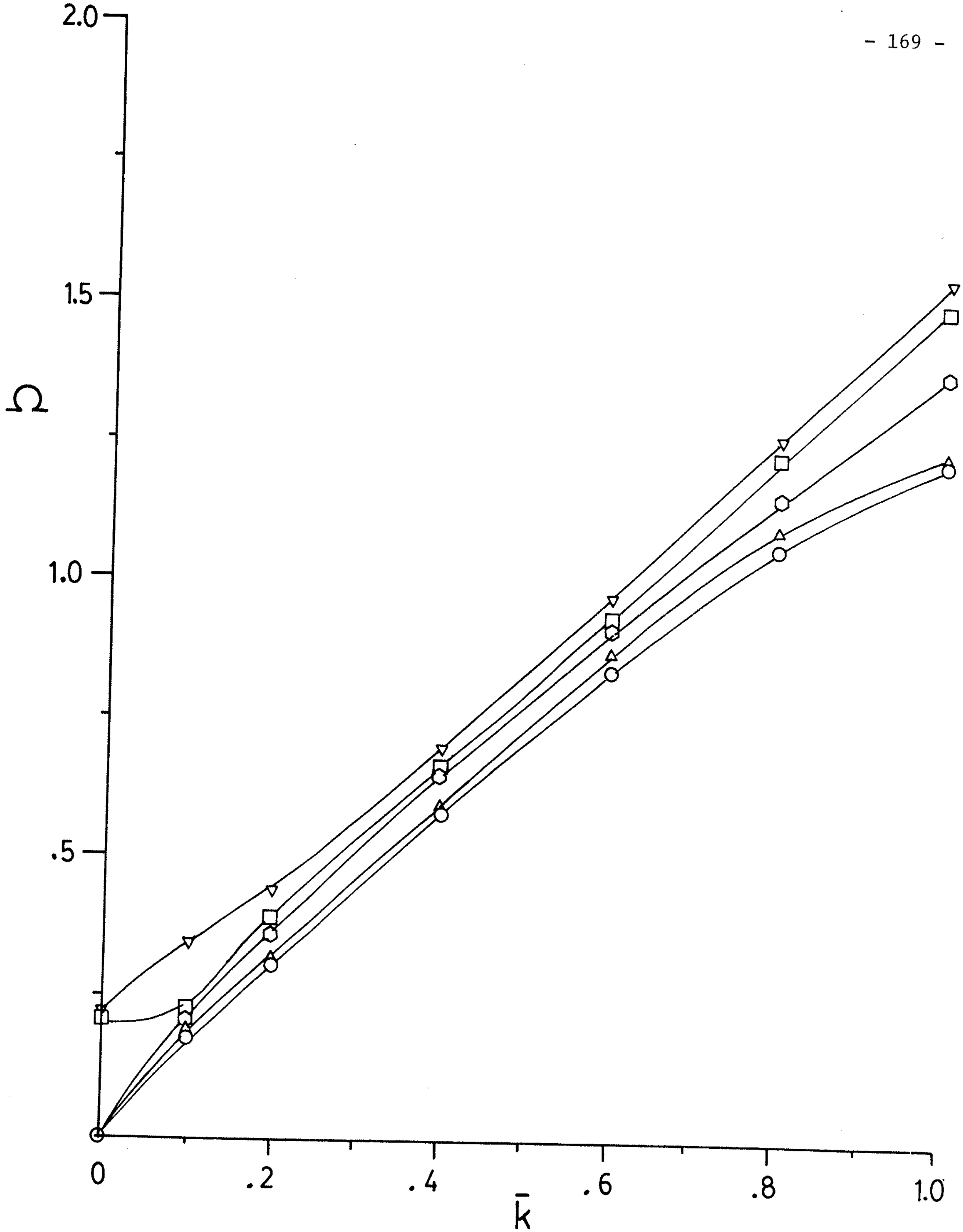
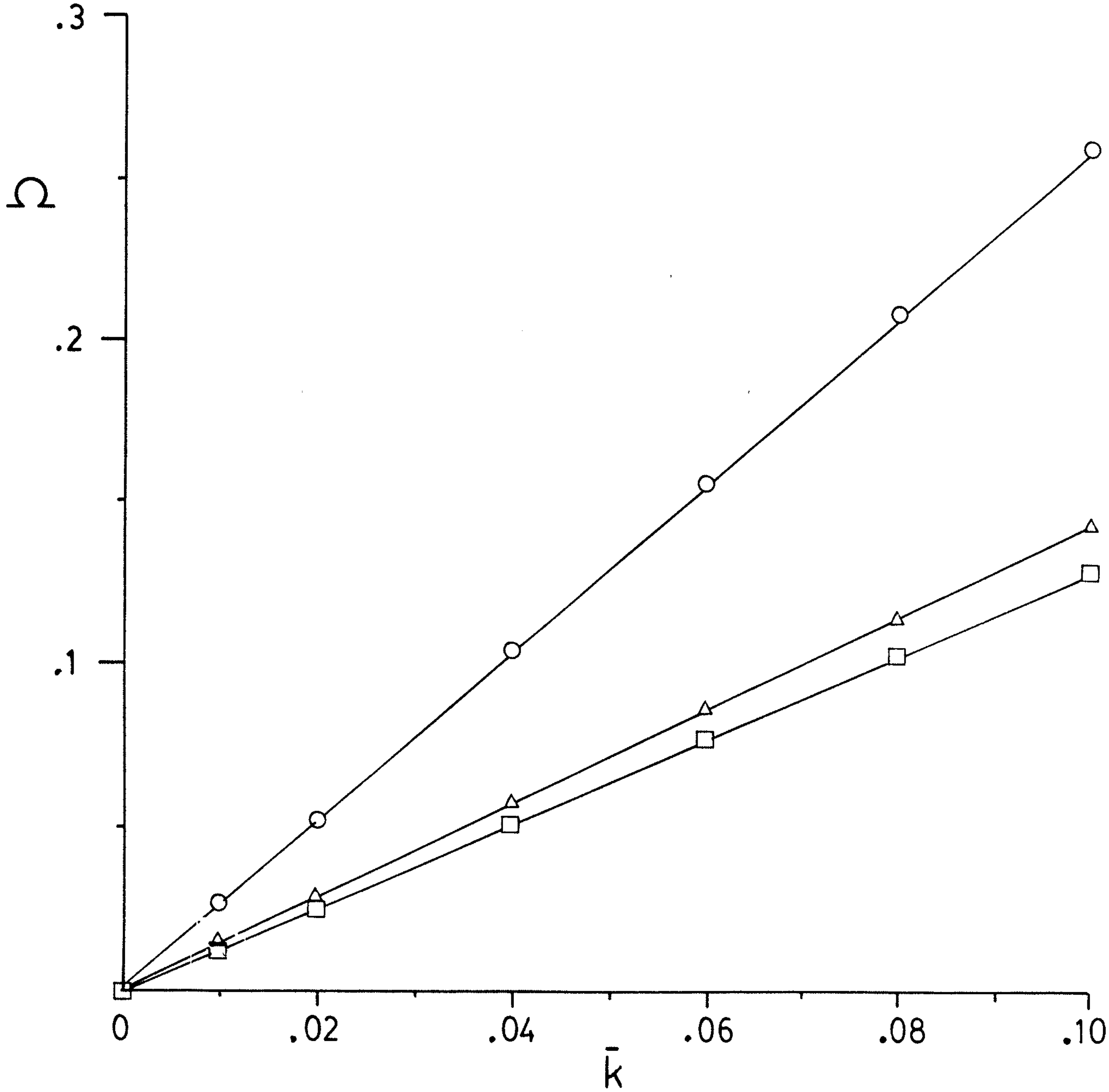


Fig. 4.38 Lowest 5 branches for $\alpha = 45°$, $\bar{\eta} = 1/13$.

Fig. 4.39 Lowest 3 branches for wave propagating on x-y plane, $\alpha = 0°$, $\phi = 45°$.



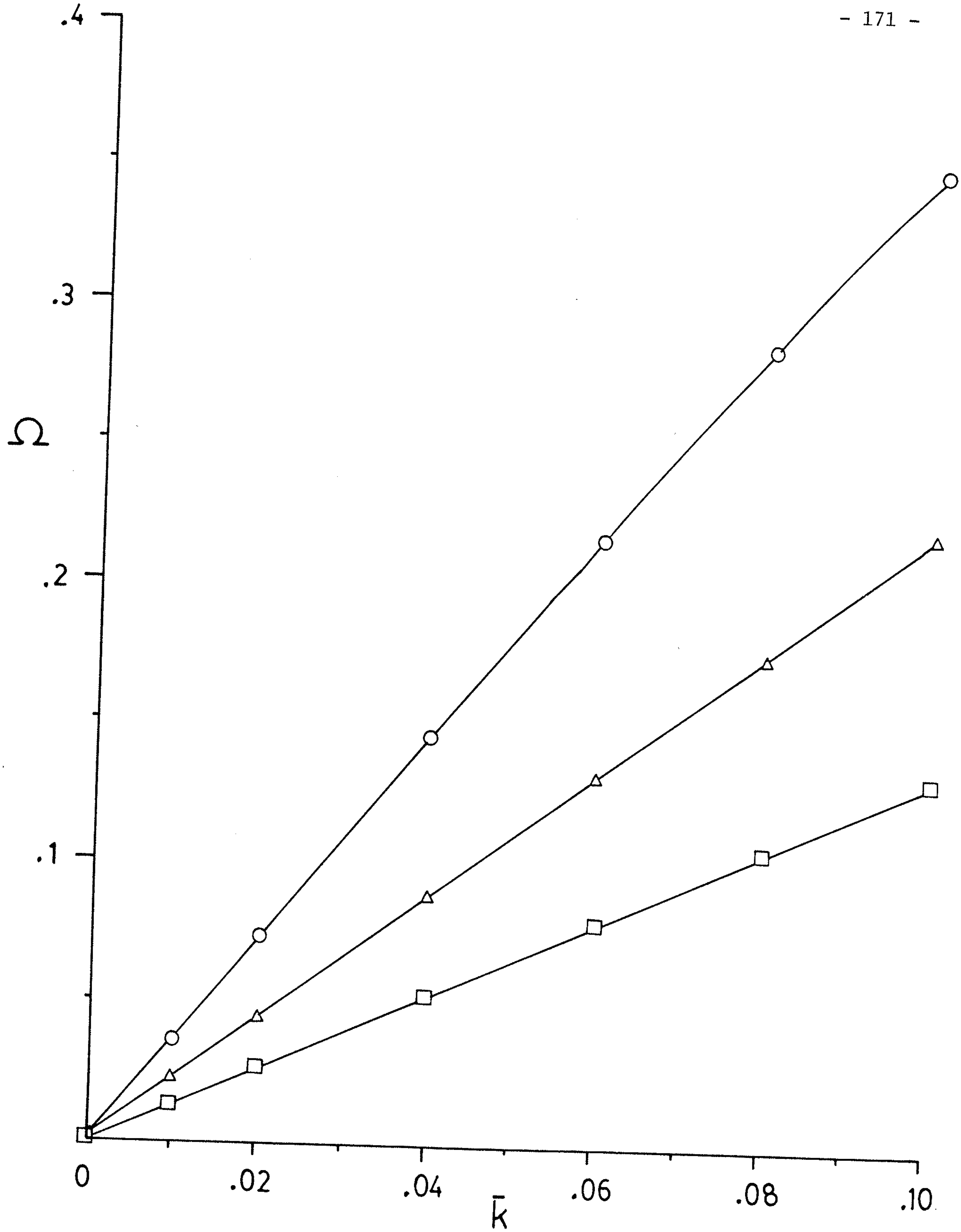


Fig. 4.40 Lowest 3 branches for wave propagating on y-z plane, $\alpha = 90°$, $\phi = 45°$.

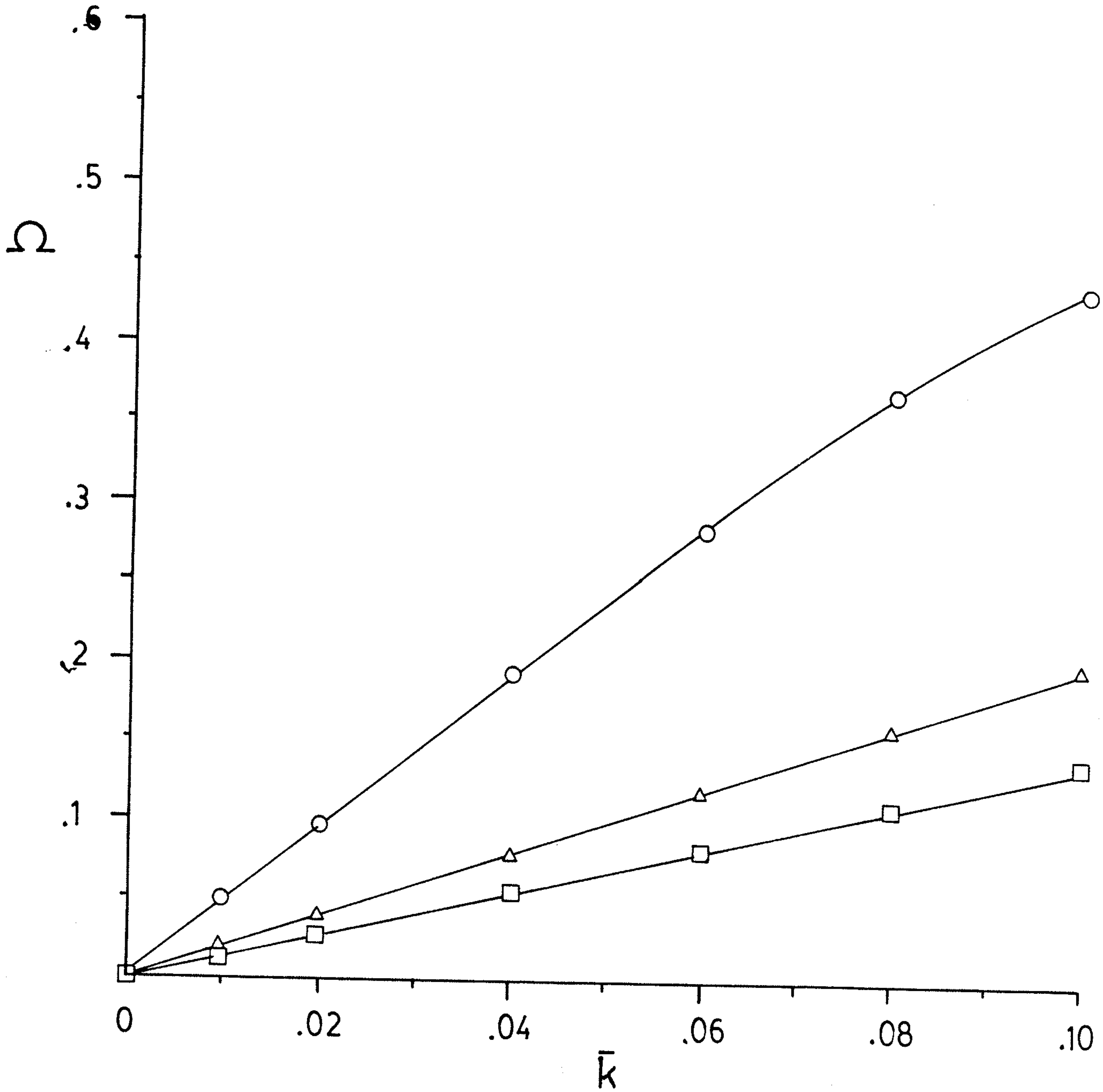


Fig. 4.41 Lowest 3 branches for wave propagating on x-z plane, α = 45°, ϕ = 0°.

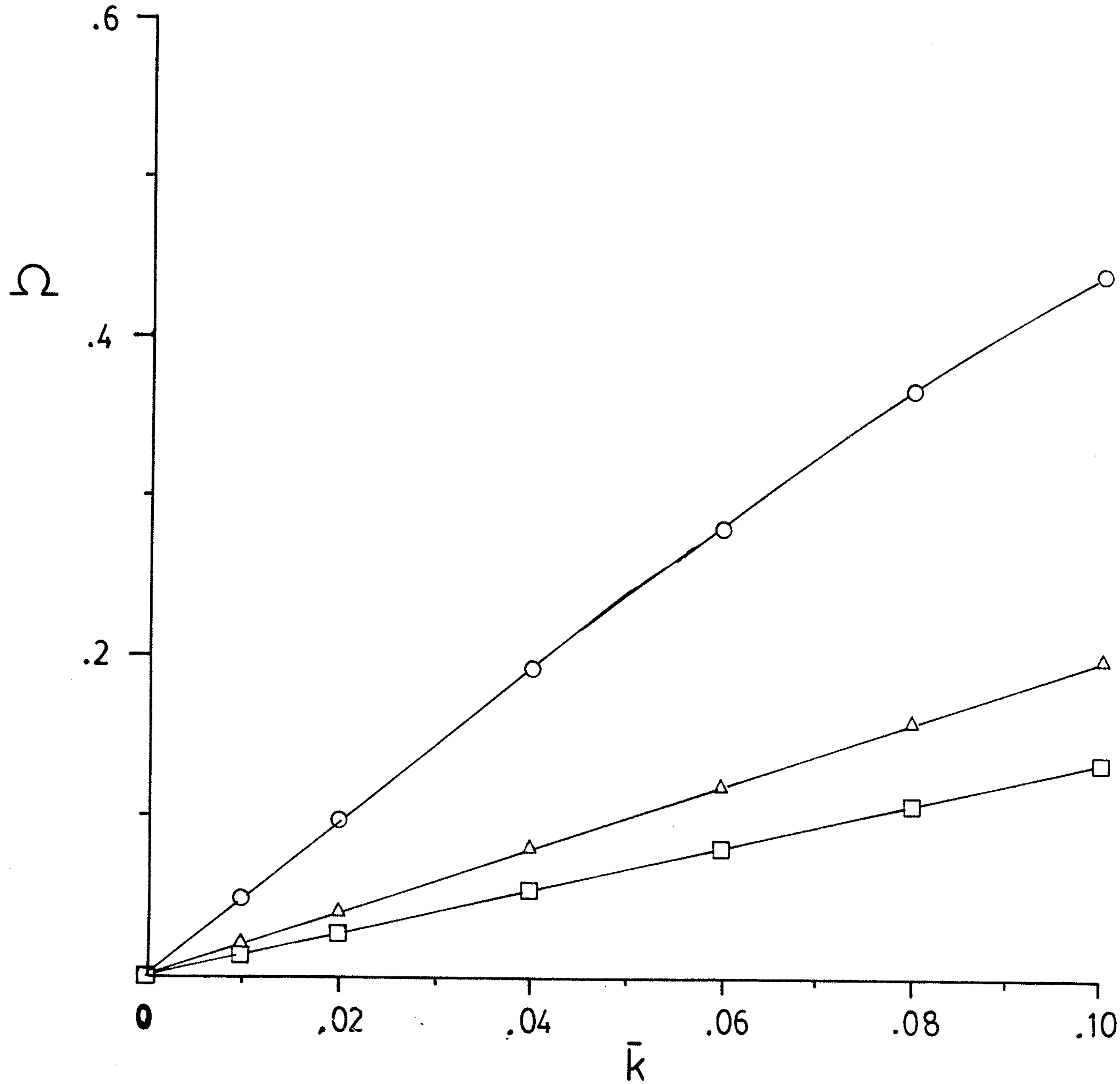


Fig. 4.42 Lowest 3 branches for wave propagating at $\alpha = 45°$, $\phi = 45°$.

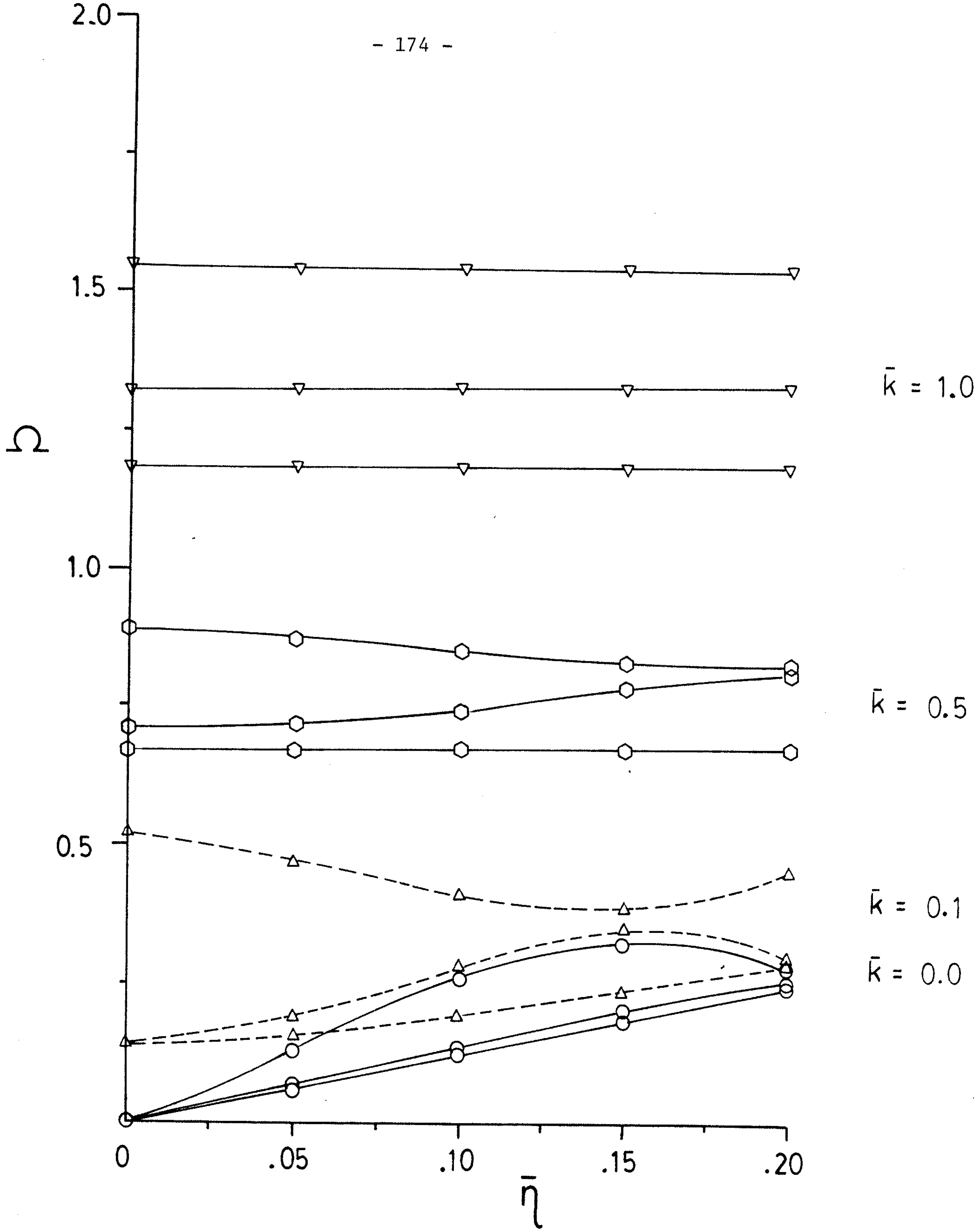


Fig. 4.43 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 0°$.

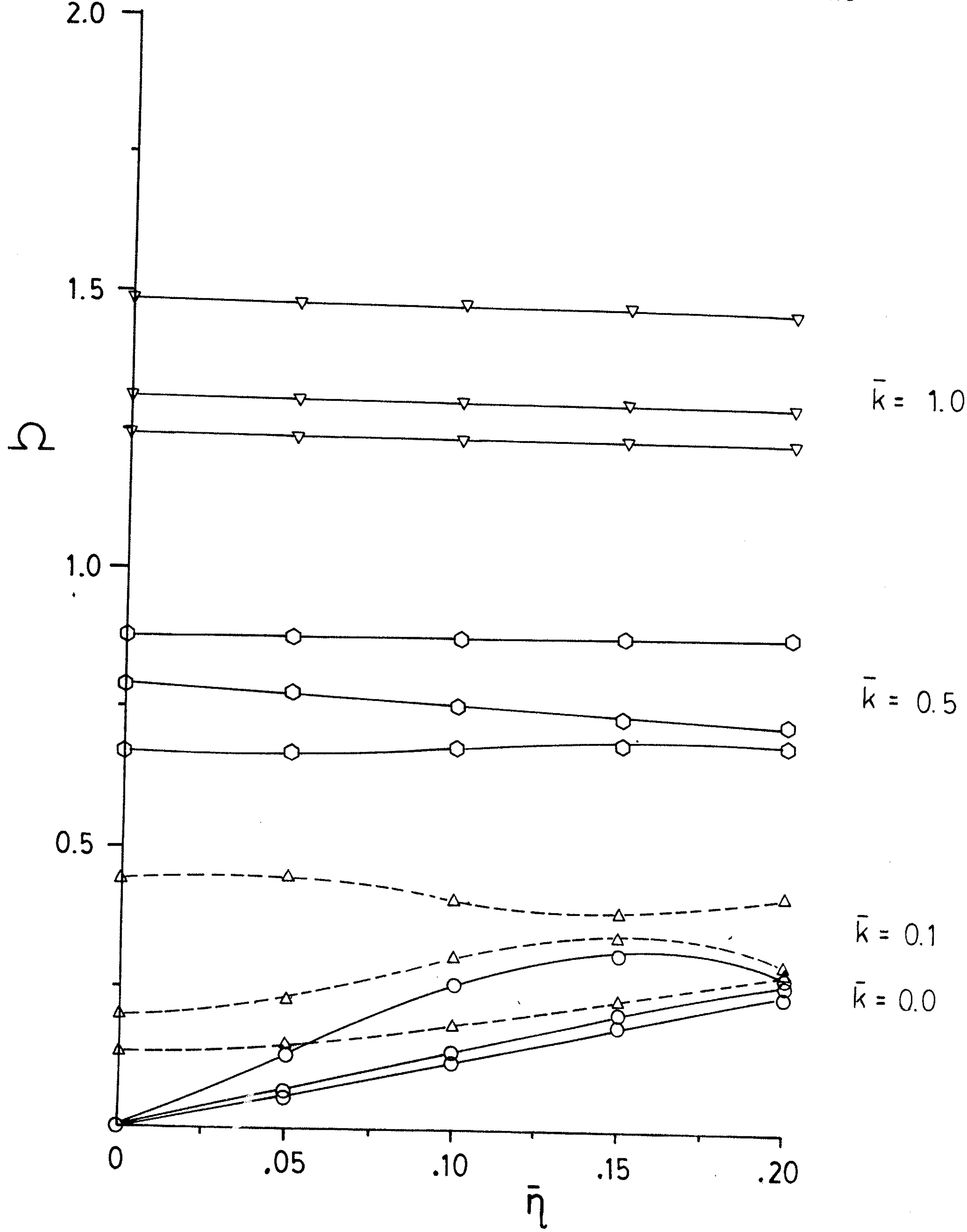


Fig. 4.44 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 45°$.

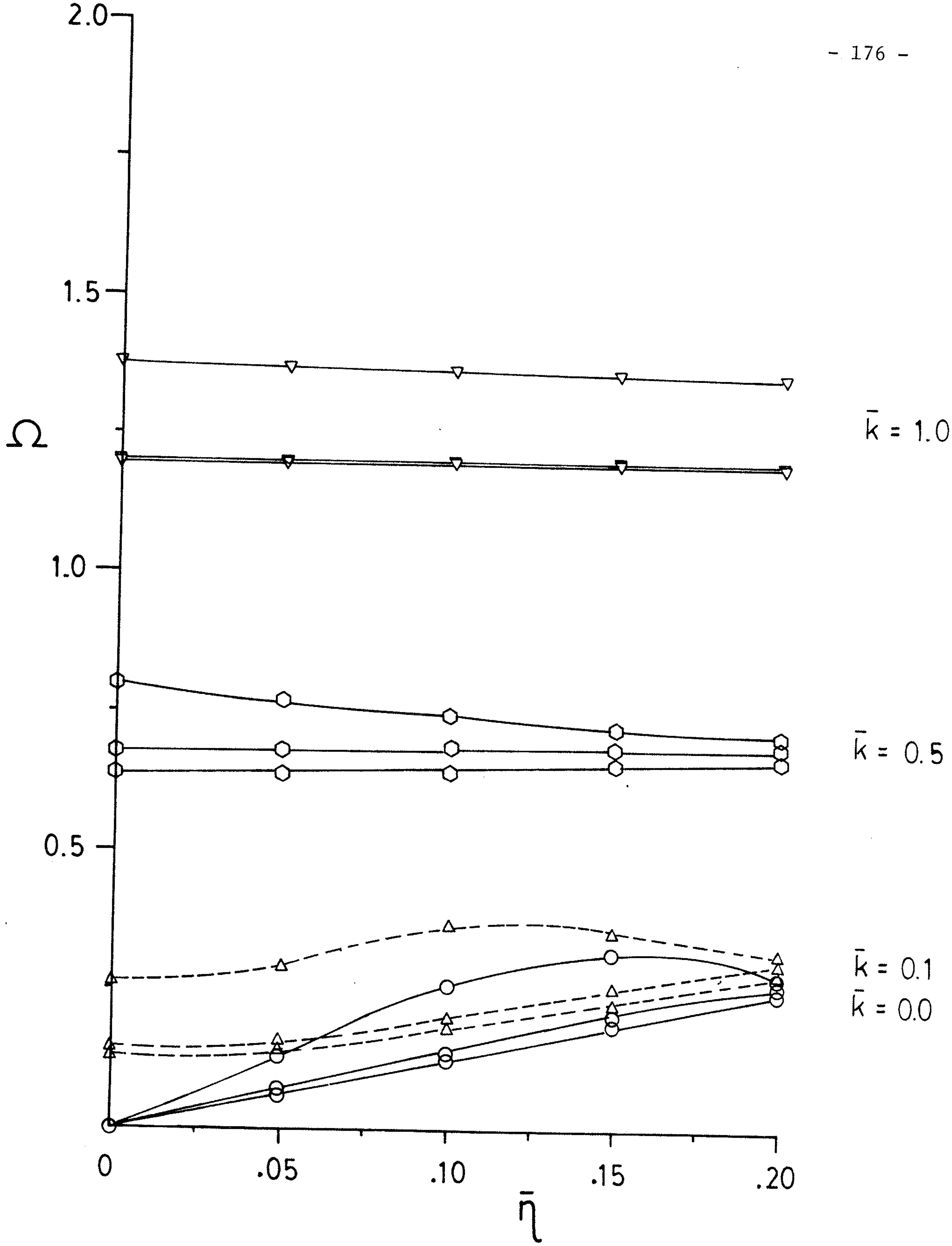


Fig. 4.45 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 90°$.

Fig. 4.46   Lowest branch for constant $\bar{\eta}$, $\alpha = 45°$.

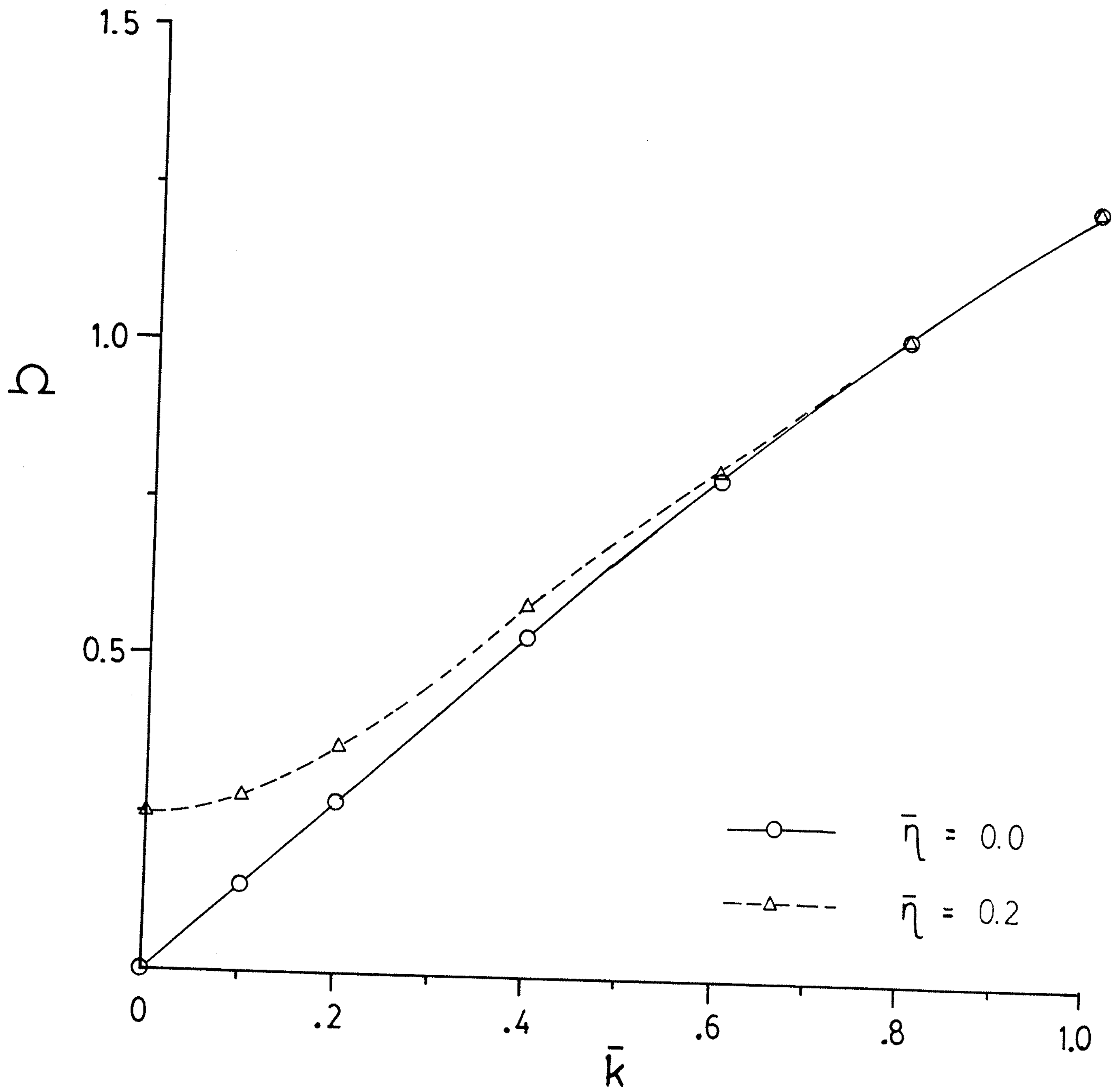


Fig. 4.47 Lowest 3 branches for wave propagating on x-y plane, $\alpha = 0°$, $\phi = 45°$; $\bar{d} = 9$.

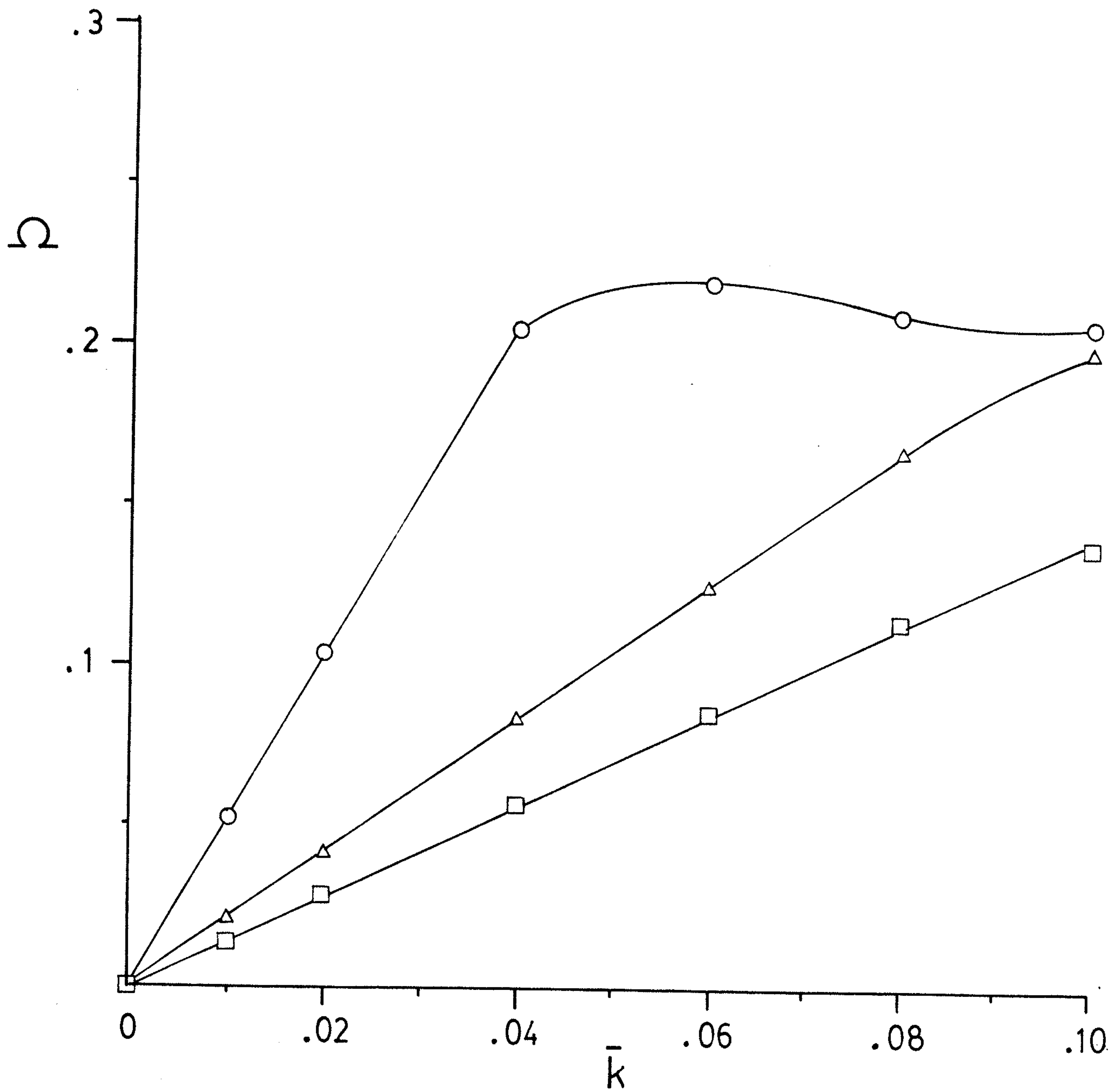
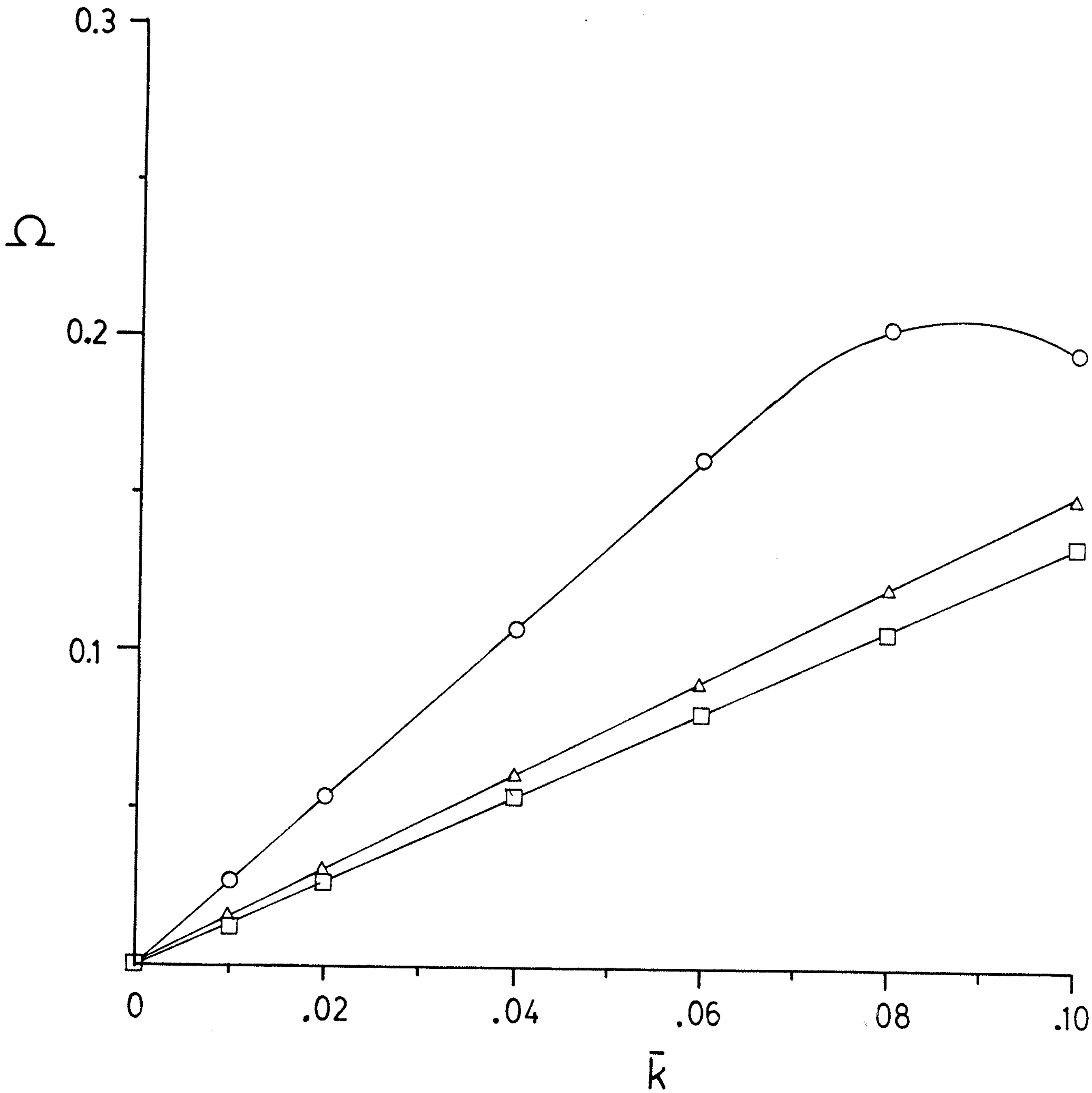


Fig. 4.48 Lowest 3 branches for wave propagating on y-z plane, $\alpha = 90°$, $\phi = 45°$; $\bar{d} = 9$.



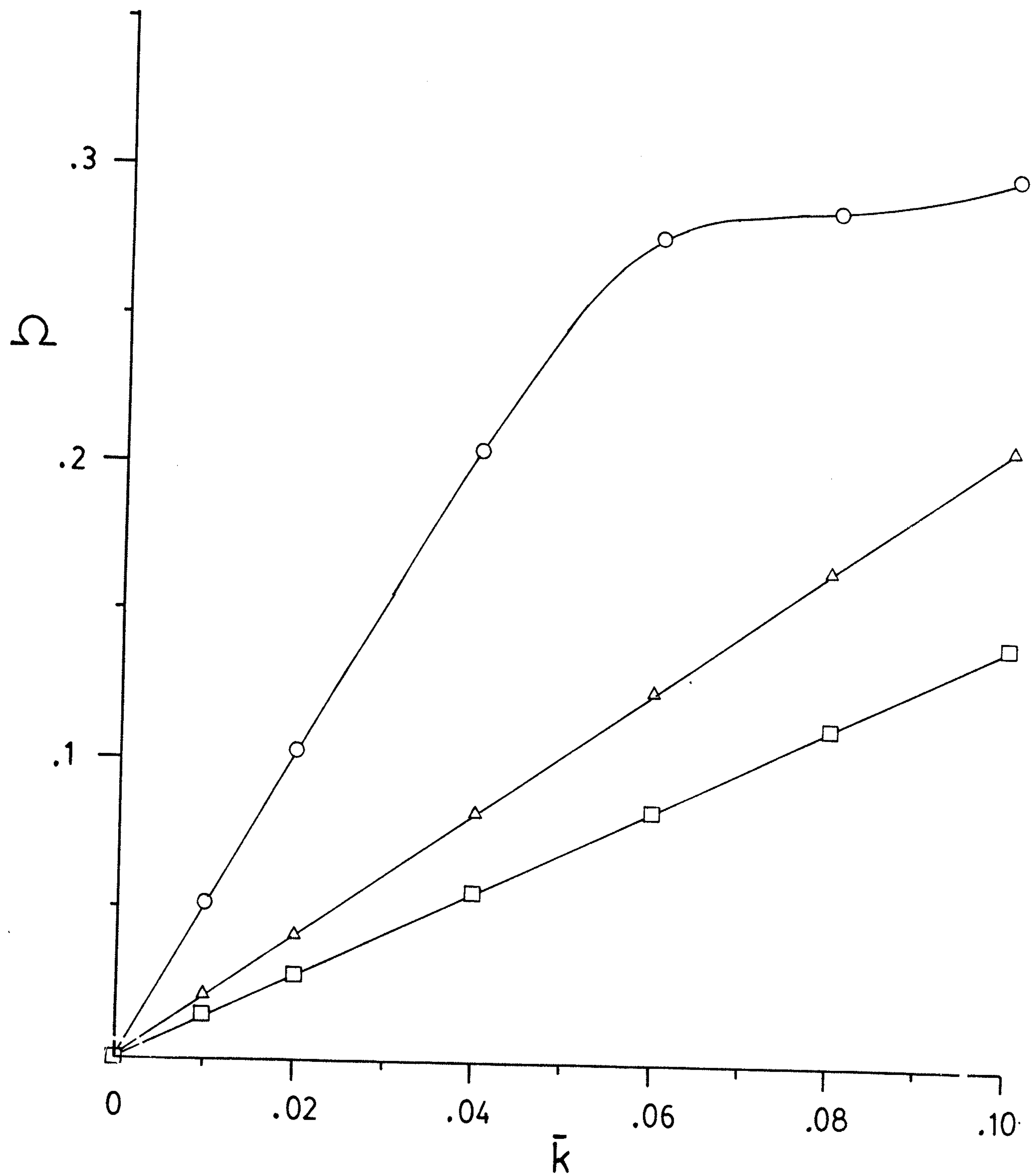
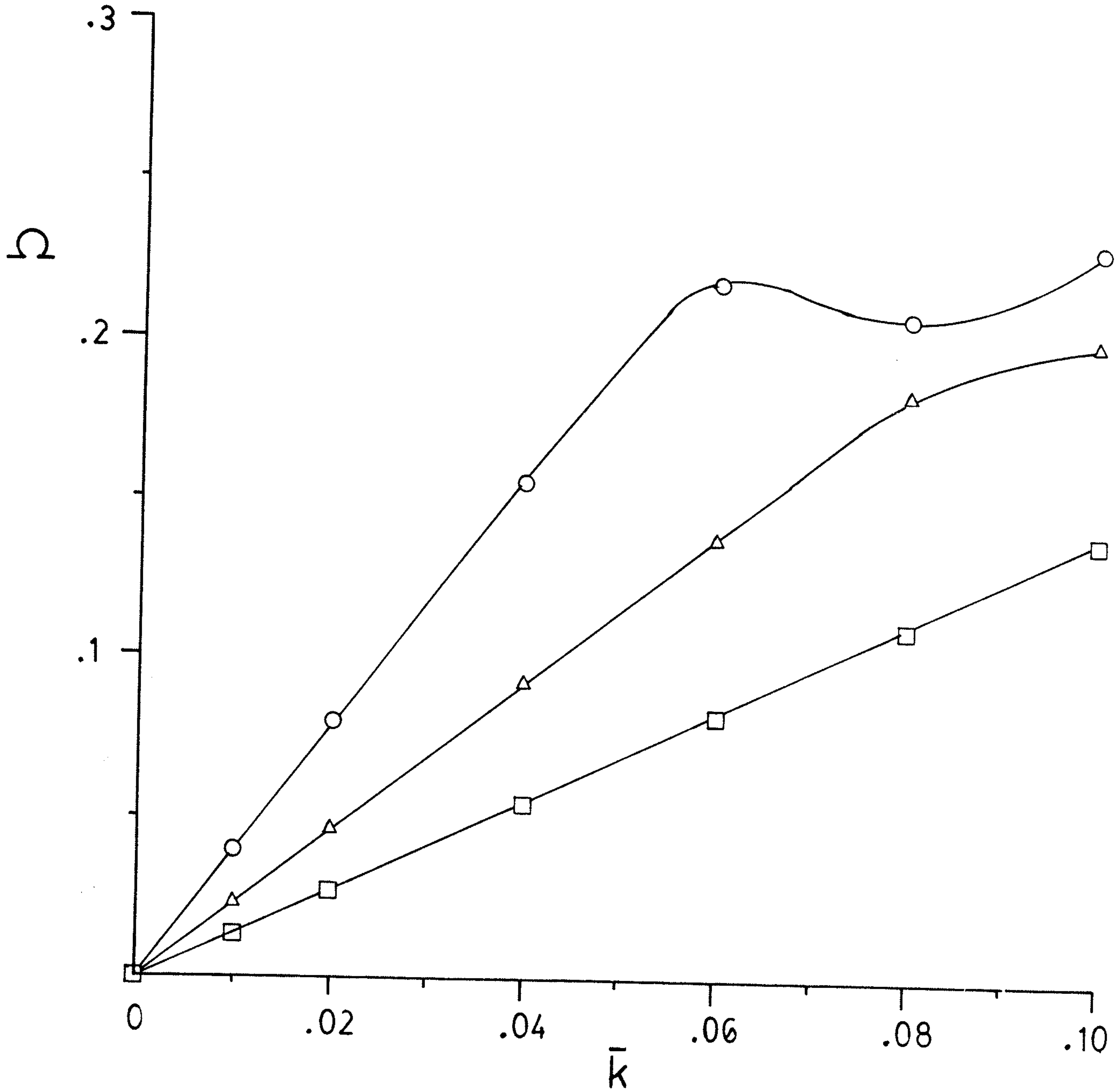


Fig. 4.49 Lowest 3 branches for wave propagating on x-z plane, $\alpha = 45°$, $\phi = 0°$; $\bar{d} = 9$.

Fig. 4.50 Lowest 3 branches for wave propagating at $\alpha = 45°$, $\phi = 45°$; $\bar{d} = 9$.



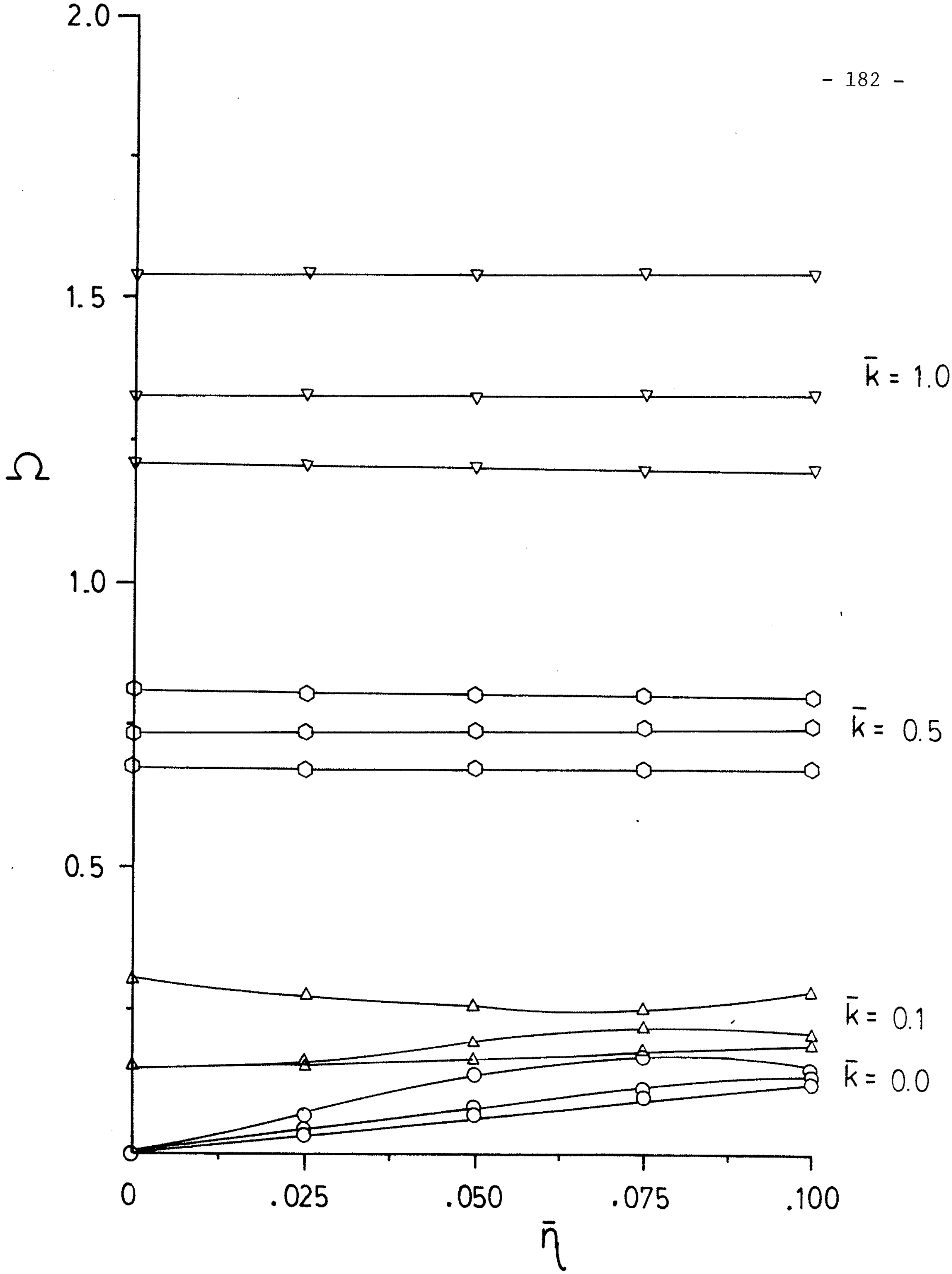


Fig. 4.51 Lowest 3 branches for curves of constant $\bar{k}$, α = 0.°; $\bar{d}$ = 9.

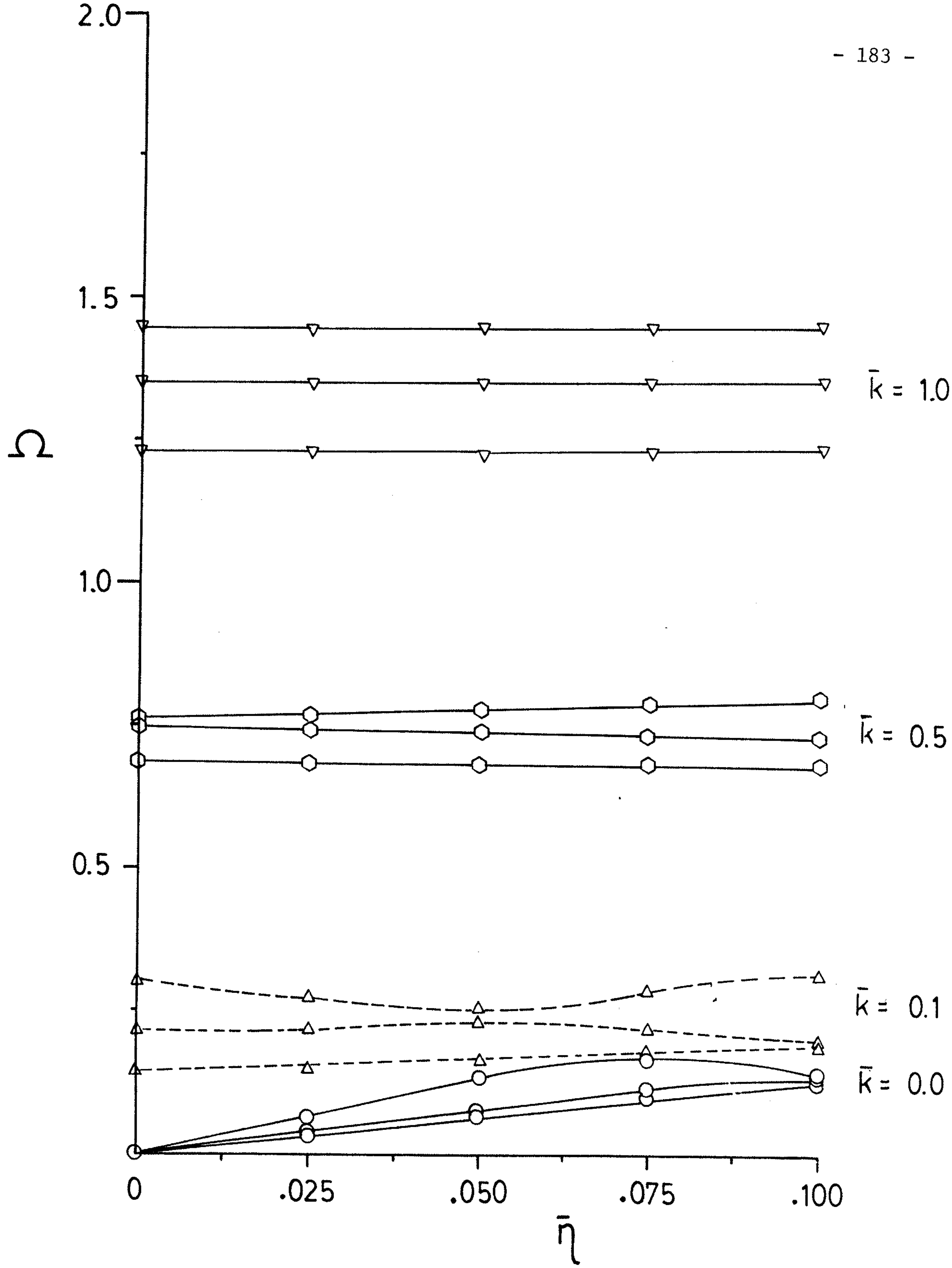


Fig. 4.52 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 45°$; $\bar{d} = 9$.

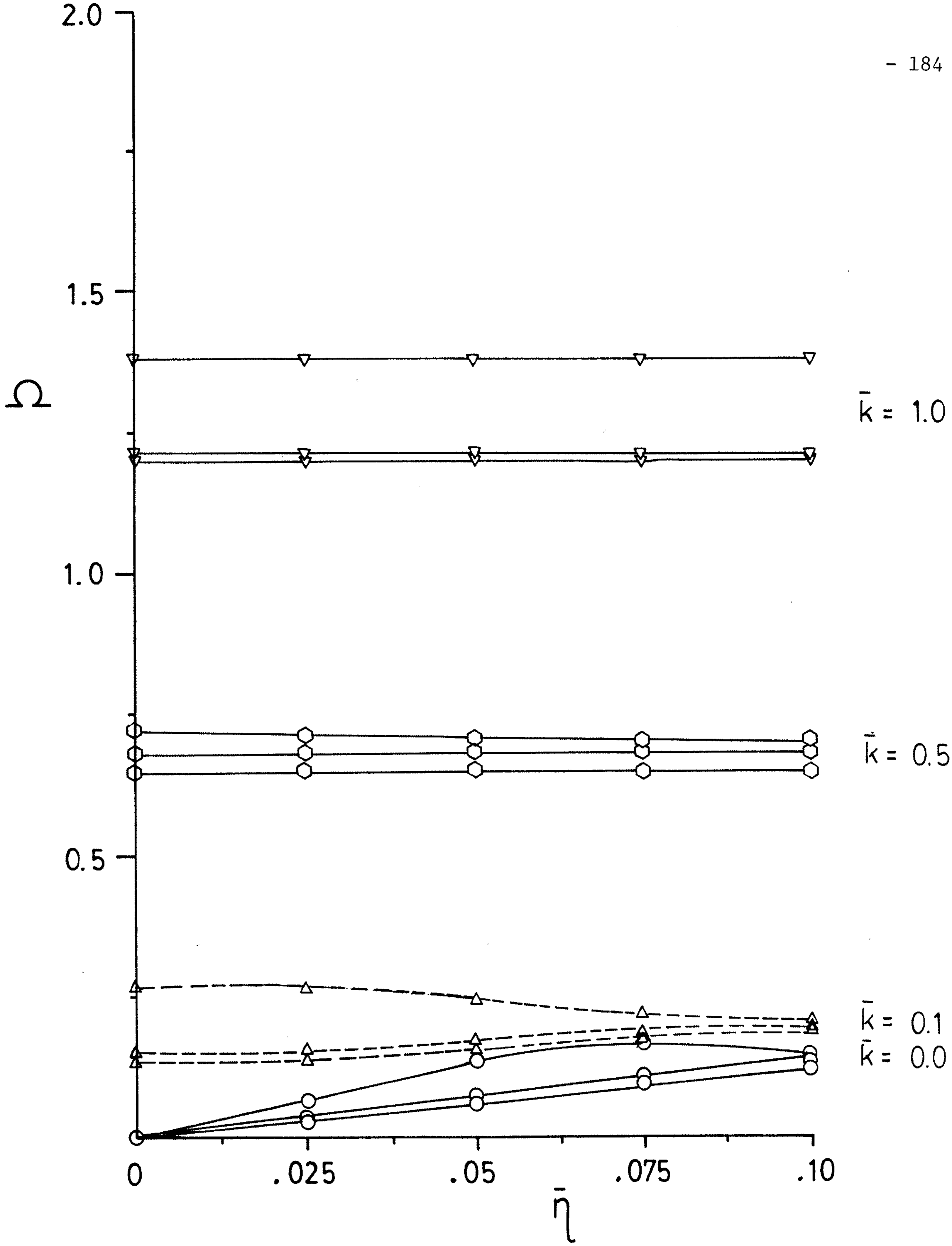


Fig. 4.53 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 90°$; $\bar{d} = 9$.

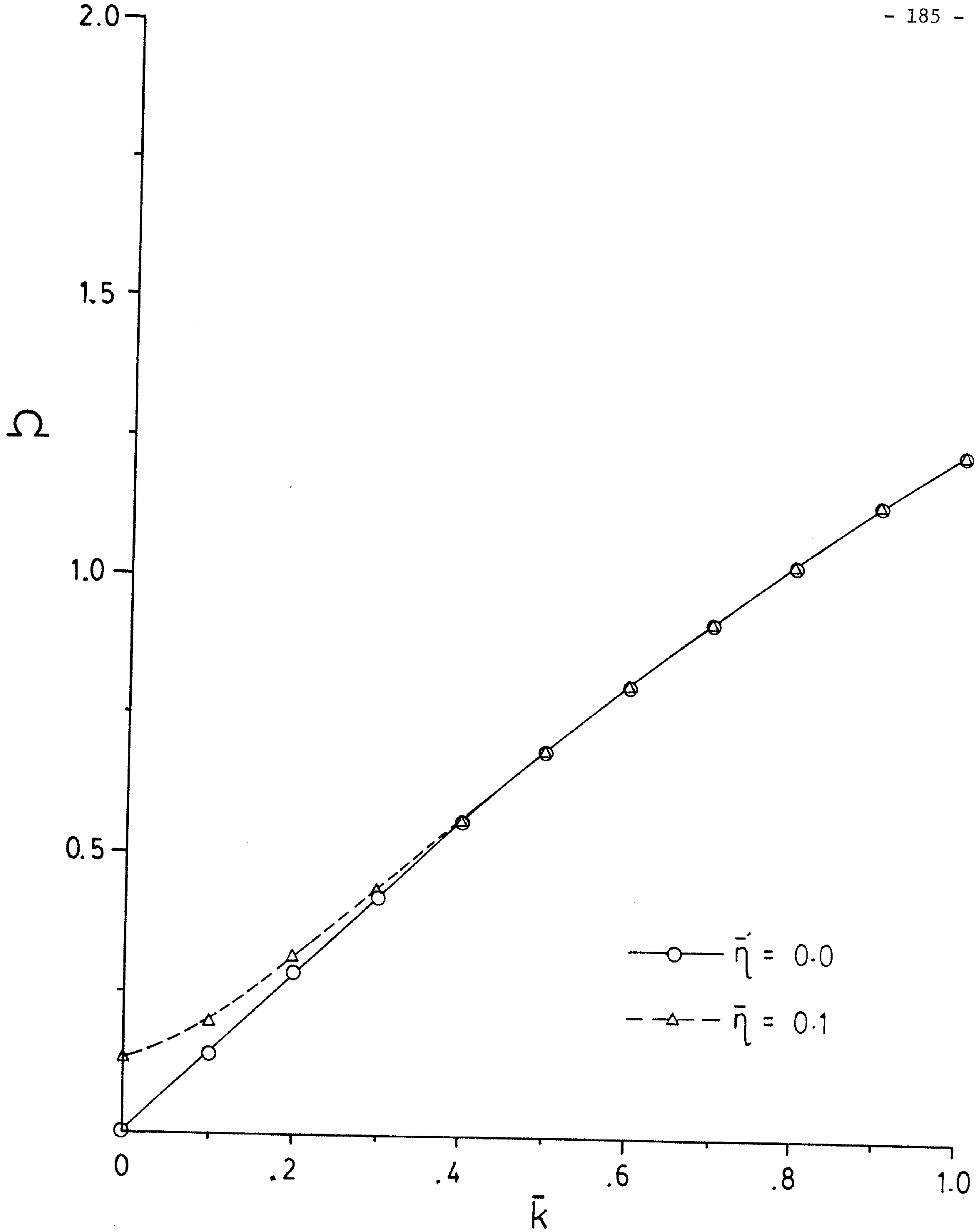


Fig. 4.54 Lowest branch for constant $\bar{\eta}$, α = 45°; $\bar{d}$ = 9.

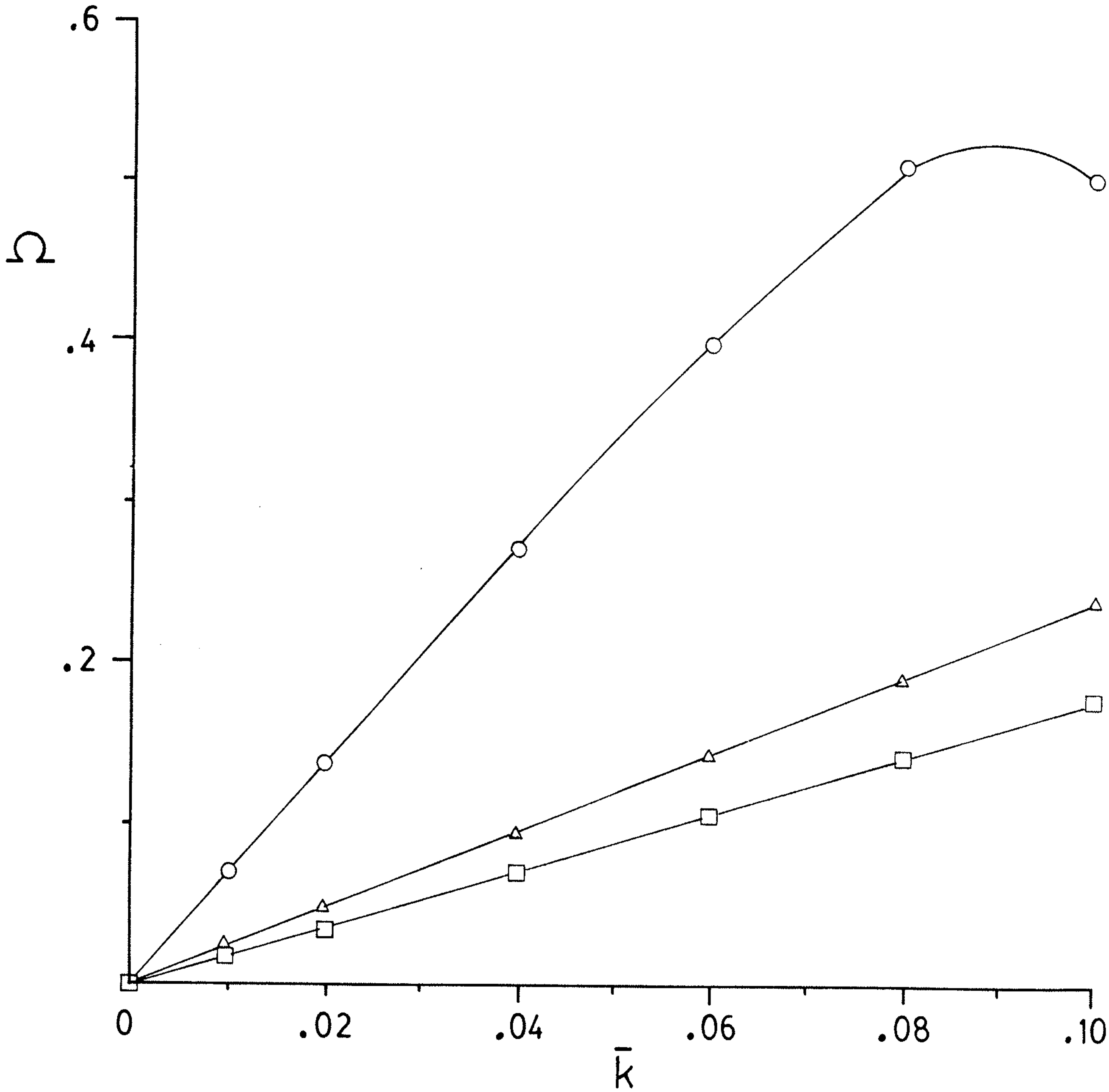


Fig. 4.55 Lowest 3 branches for wave propagating on x-y plane, $\alpha = 0°$, $\phi = 45°$.

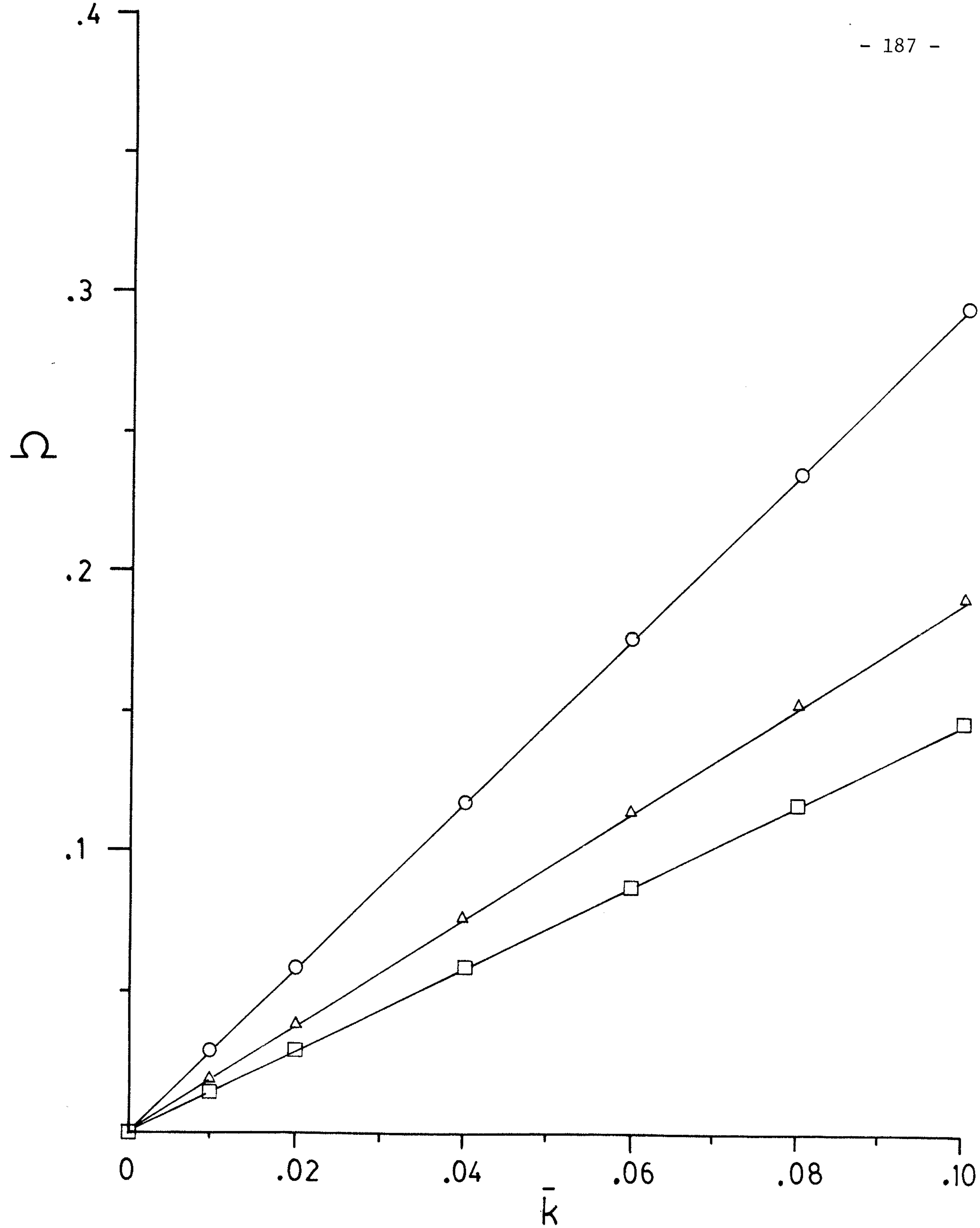


Fig. 4.56 Lowest 3 branches for wave propagating on y-z plane, α = 90°, $\phi = 0°$.

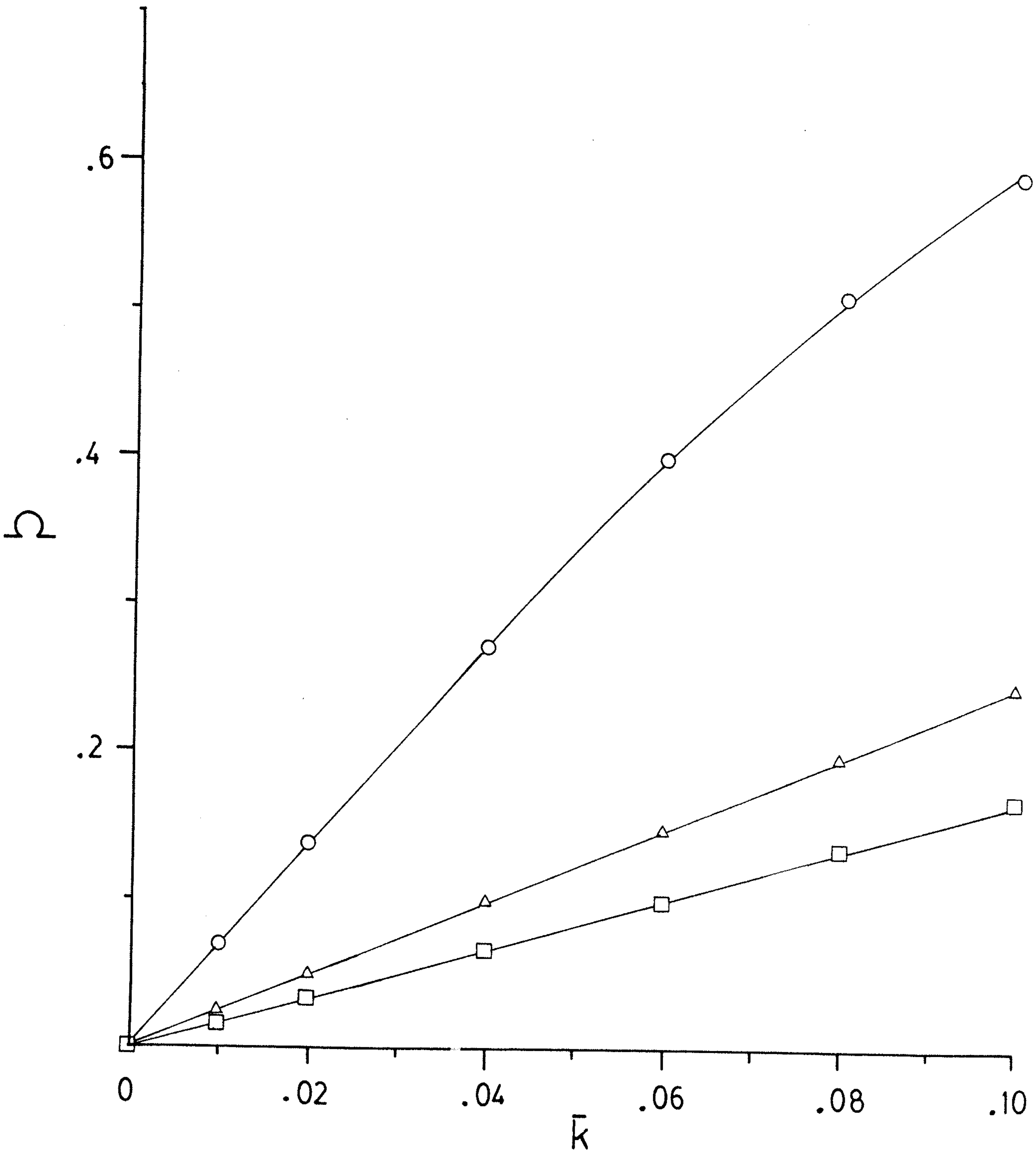


Fig. 4.57 Lowest 3 branches for wave propagating on x-z plane, $\alpha = 45°$, $\phi = 0°$.

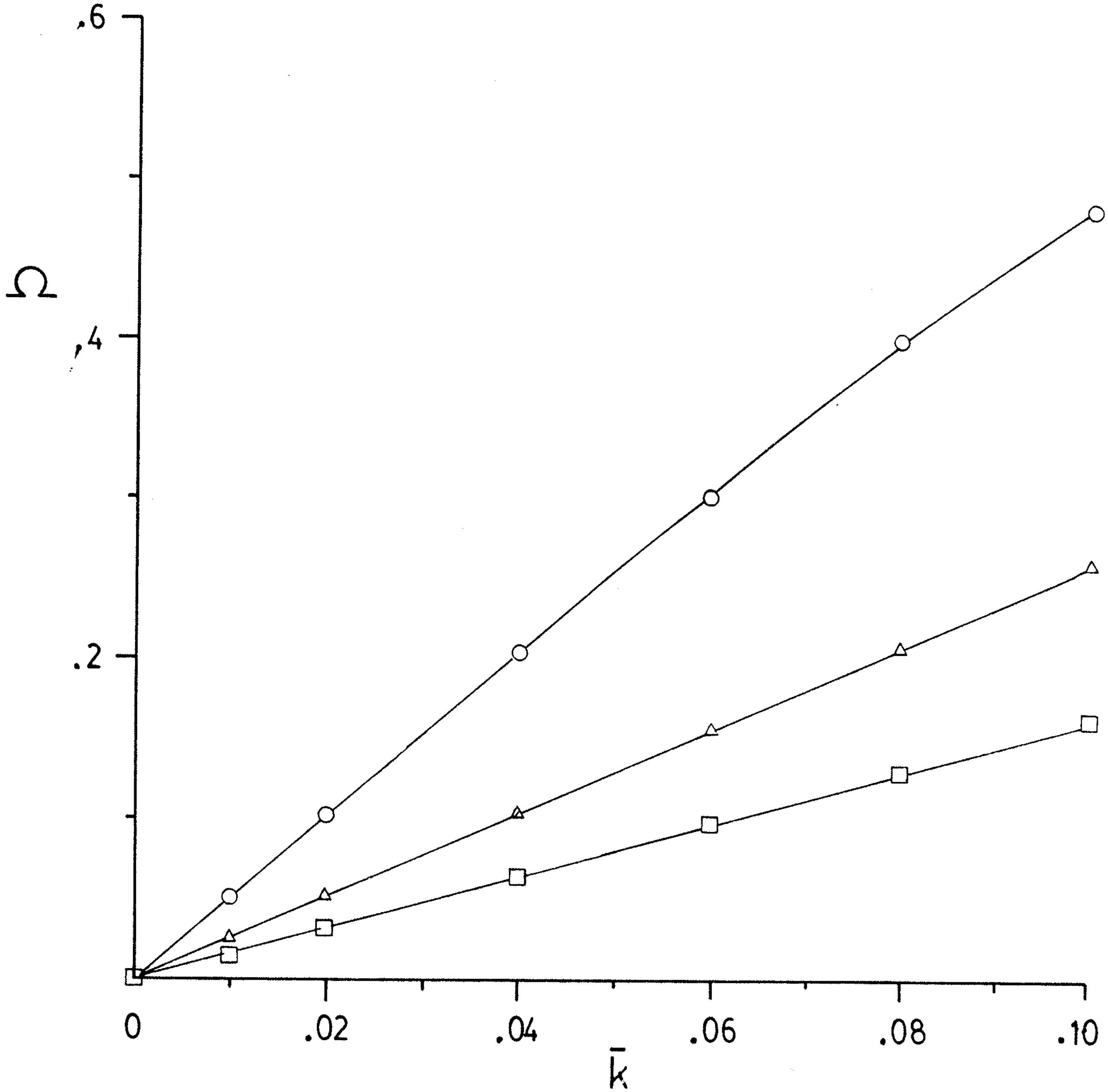


Fig. 4.58 Lowest 3 branches for wave propagating at $\alpha = 45°$, $\phi = 45°$.

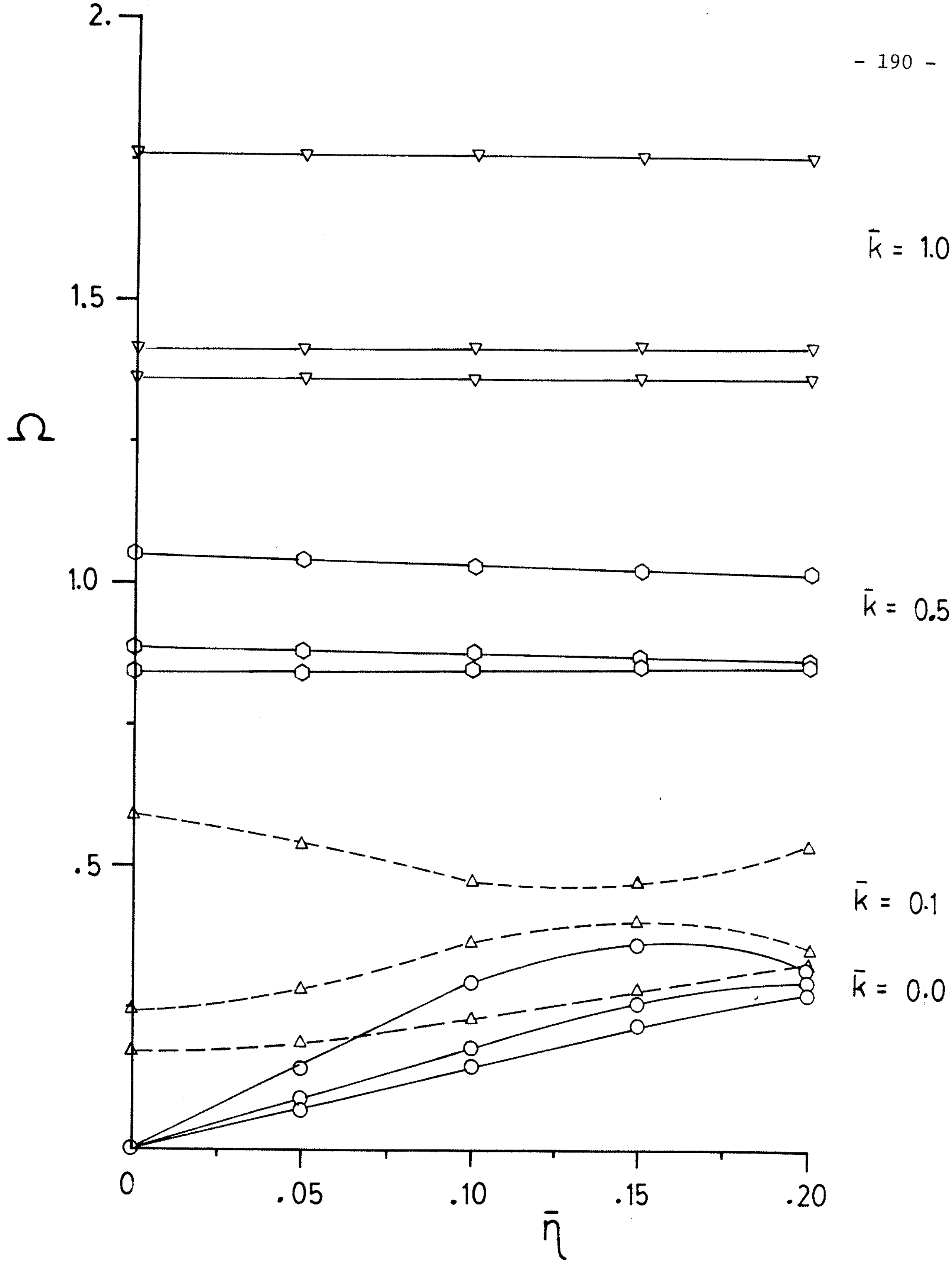


Fig. 4.59 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 0°$.

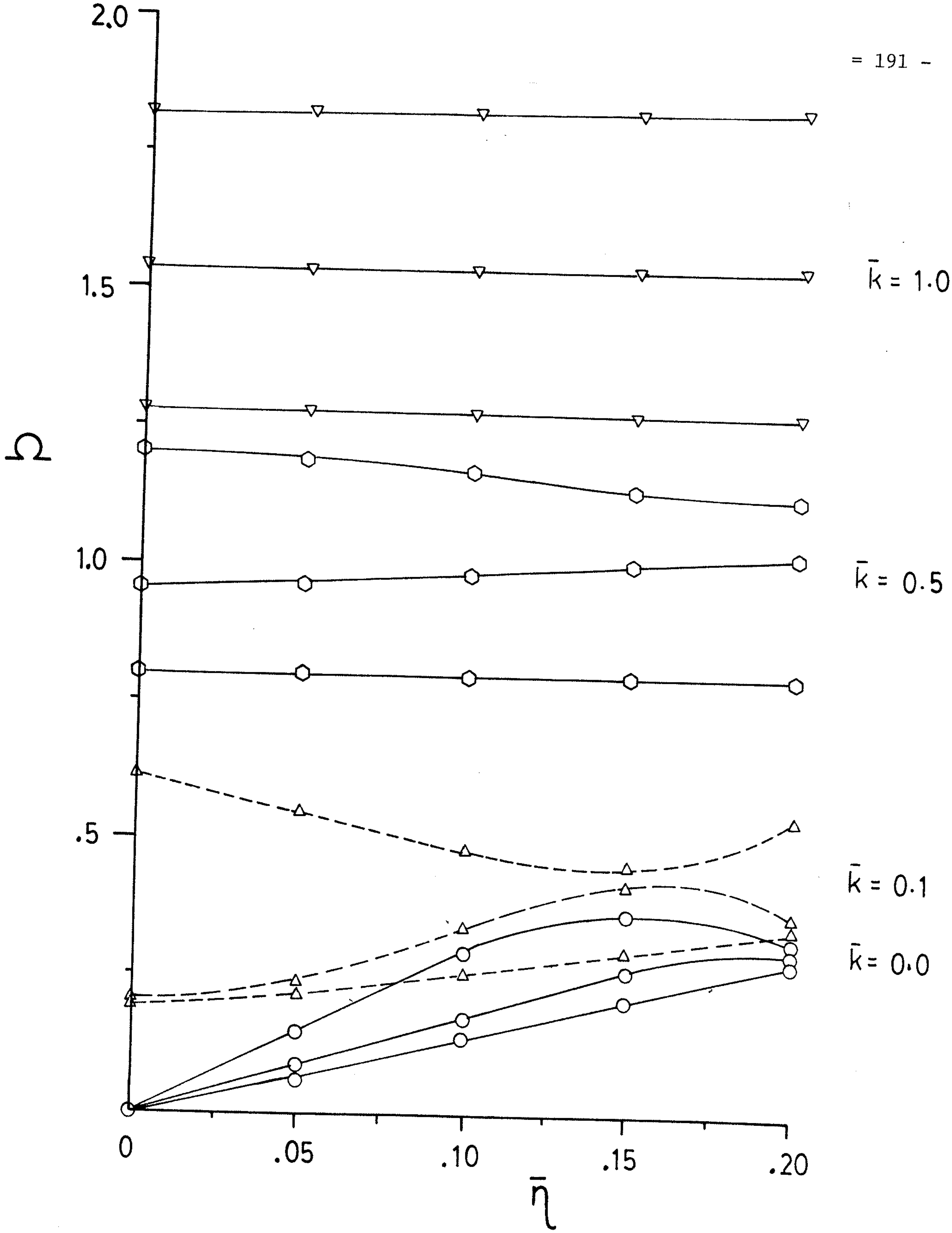


Fig. 4.60 Lowest 3 branches for curves of constant $\bar{k}$, α = 45°.

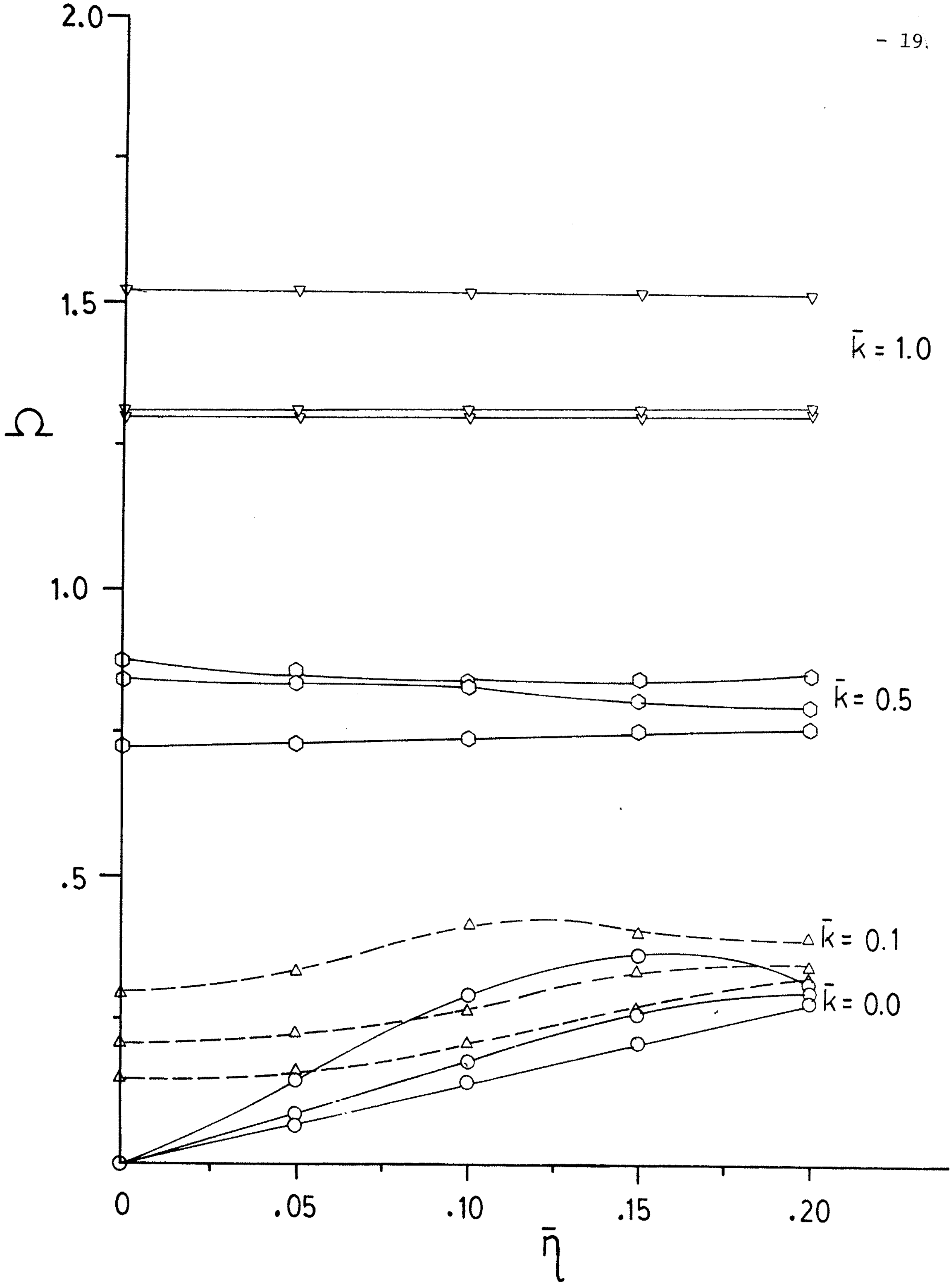


Fig. 4.61 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 90°$.

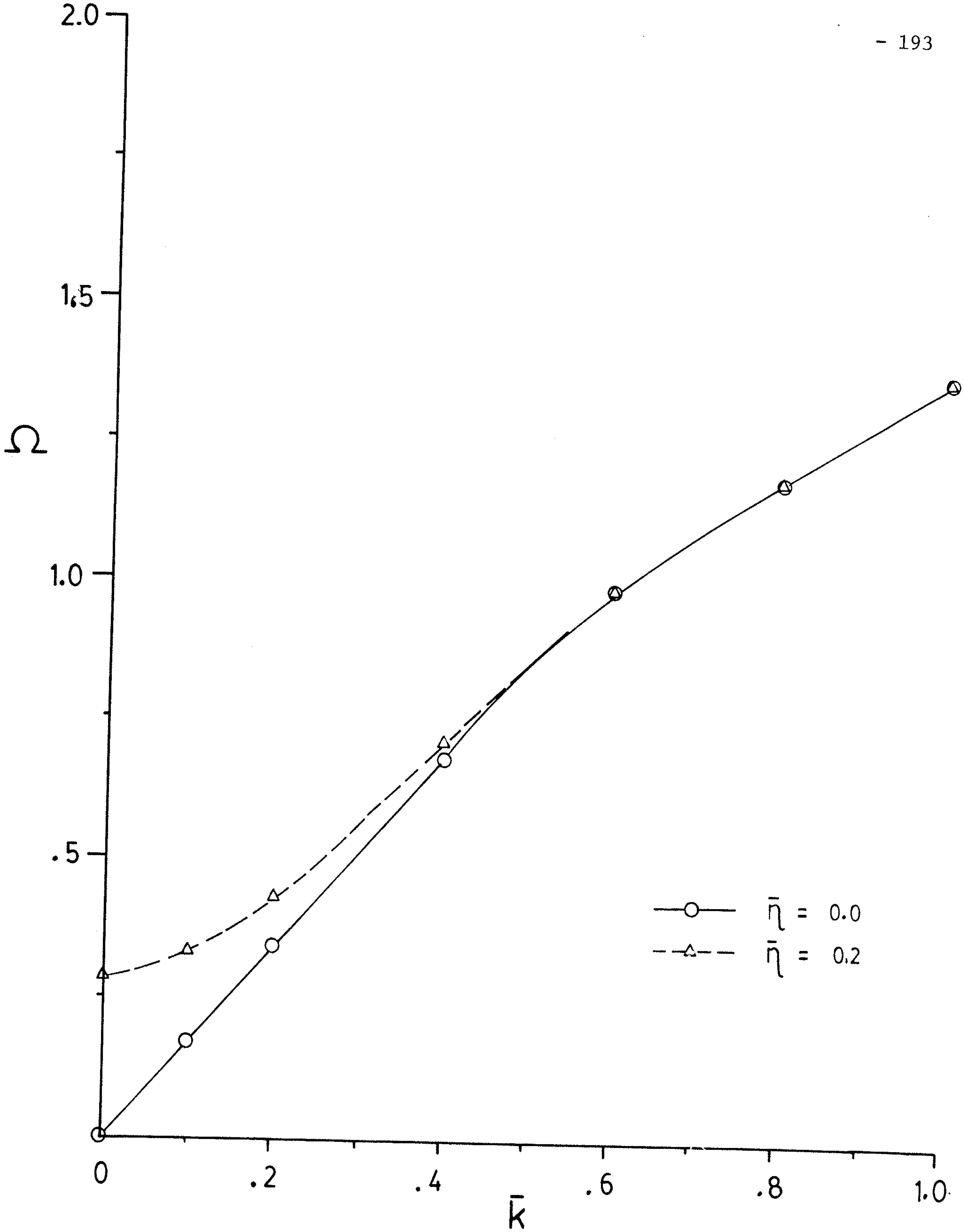


Fig. 4.62. Lowest branch for constant $\bar{\eta}$, α = 45°.

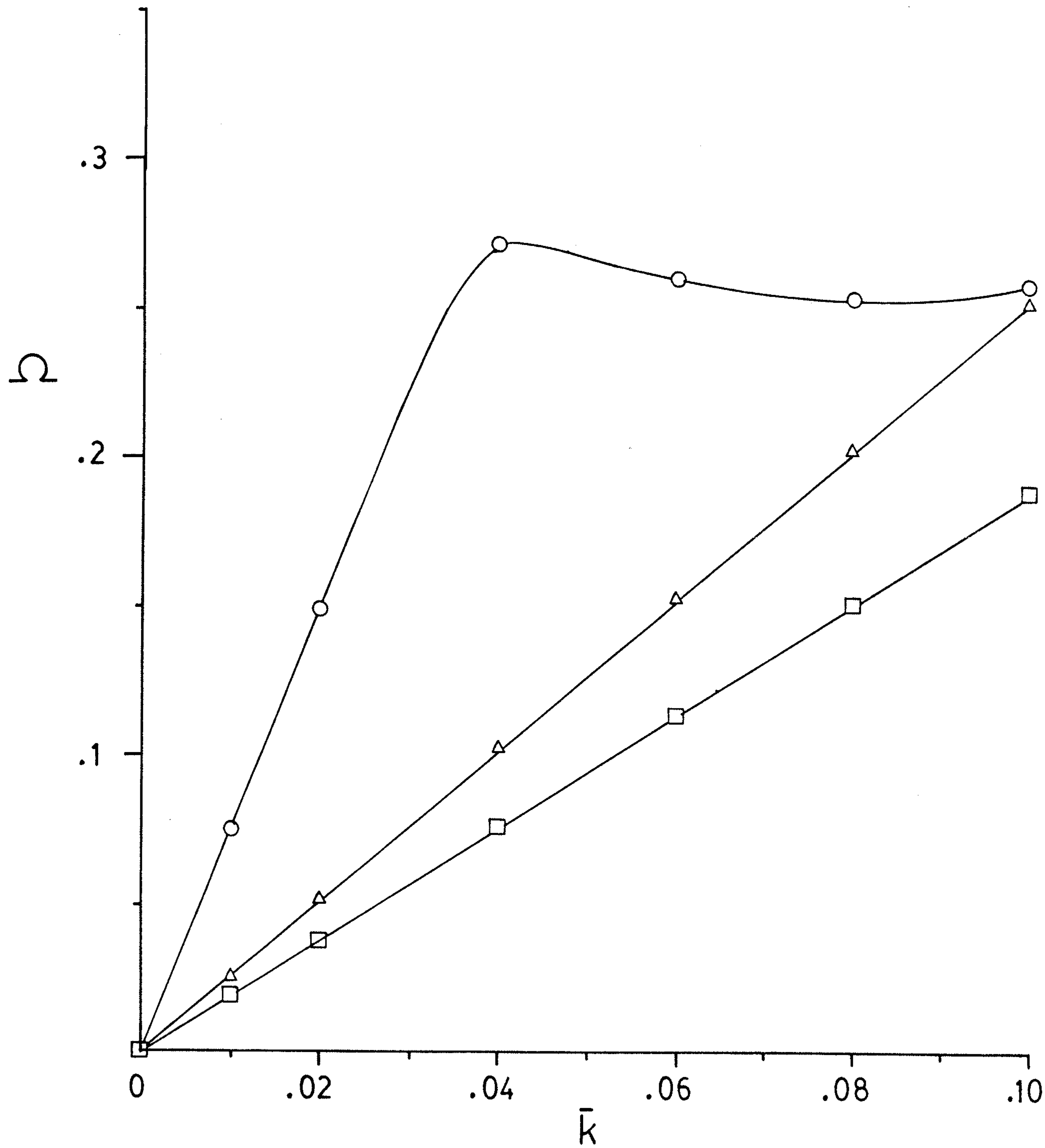
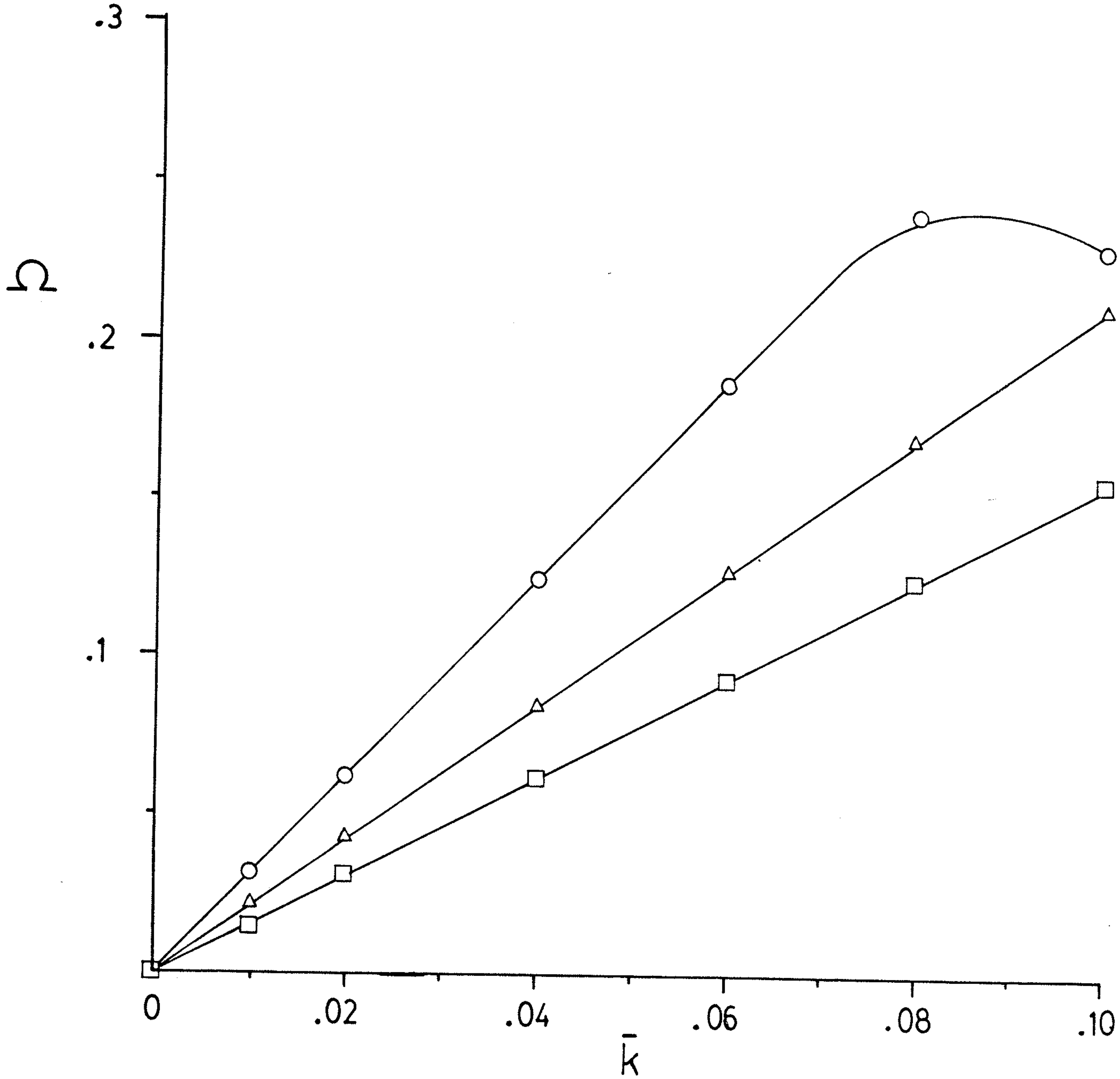


Fig. 4.63 Lowest 3 branches for wave propagating on x-y plane, $\alpha = 0°$, $\phi = 42°$; $\bar{d} = 9$.

Fig. 4.64 Lowest 3 branches for wave propagating at y-z plane, $\alpha = 90°$, $\phi = 45°$; $\bar{d} = 9$.



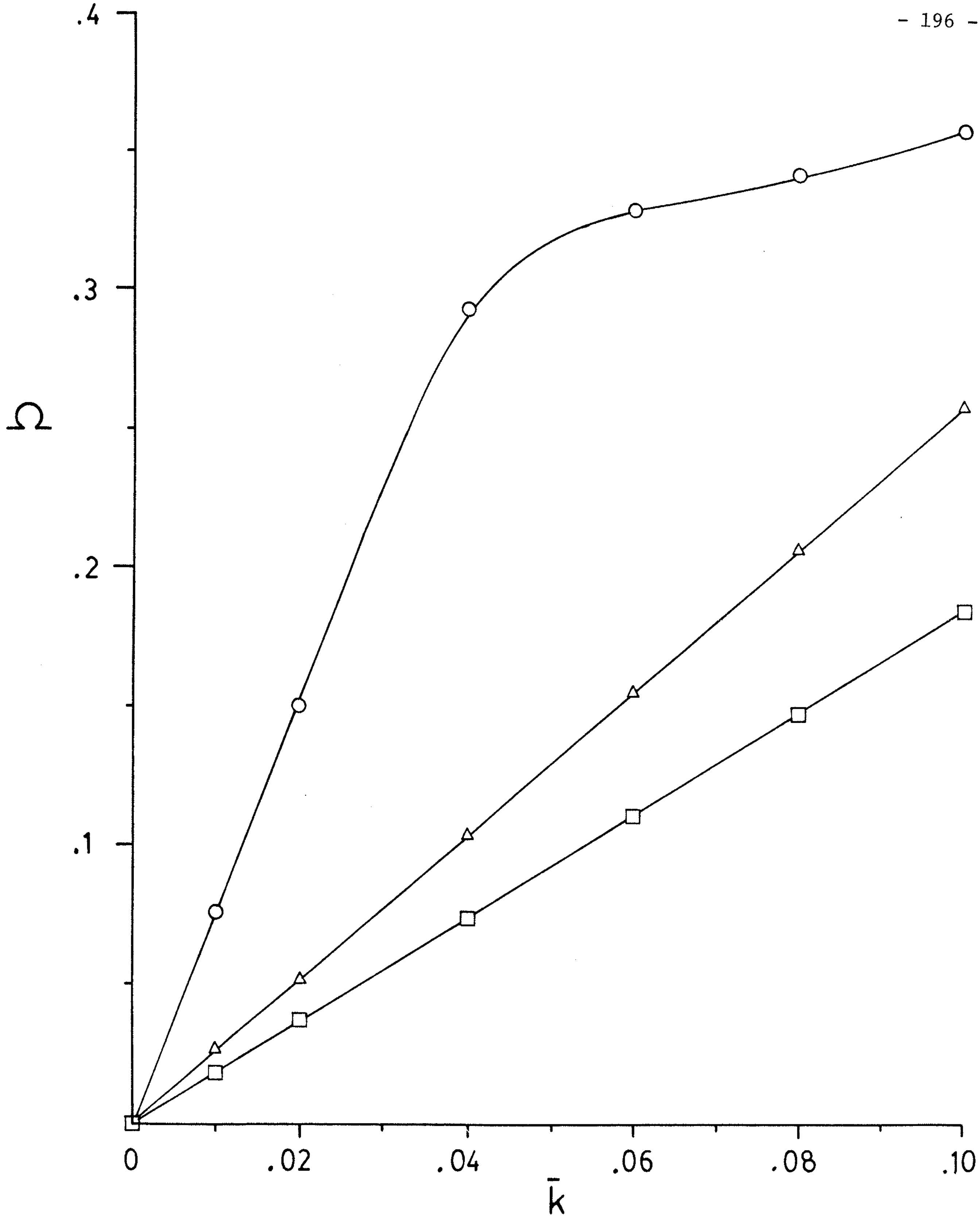


Fig. 4.65 Lowest 3 branches for wave propagating at x-z plane, $\alpha = 45°$, $\phi = 0°$; $\bar{d} = 9$.

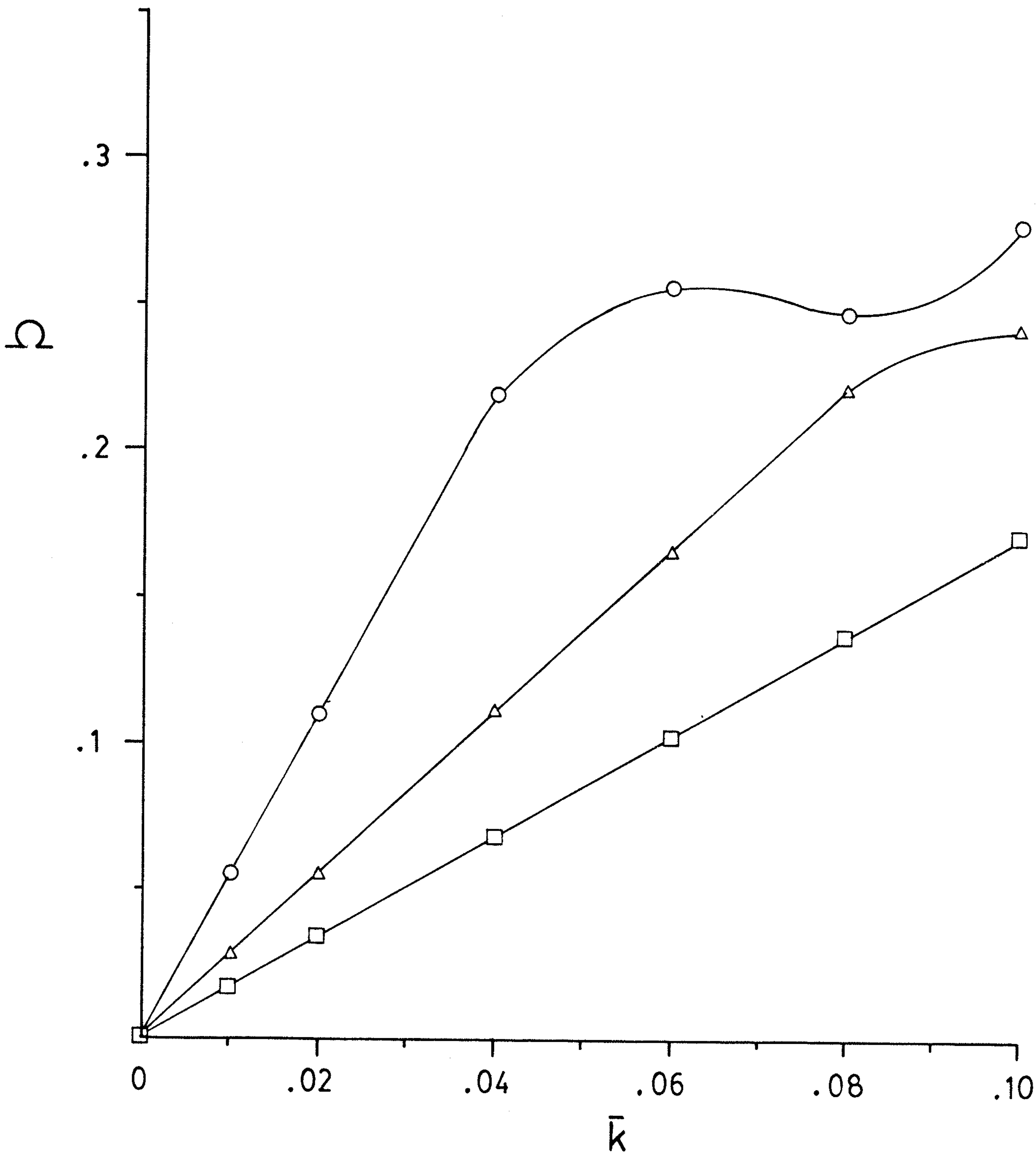


Fig. 4.66 Lowest 3 branches for wave propagating at $\alpha = 45°$, $\phi = 45°$; $\bar{d} = 9$.

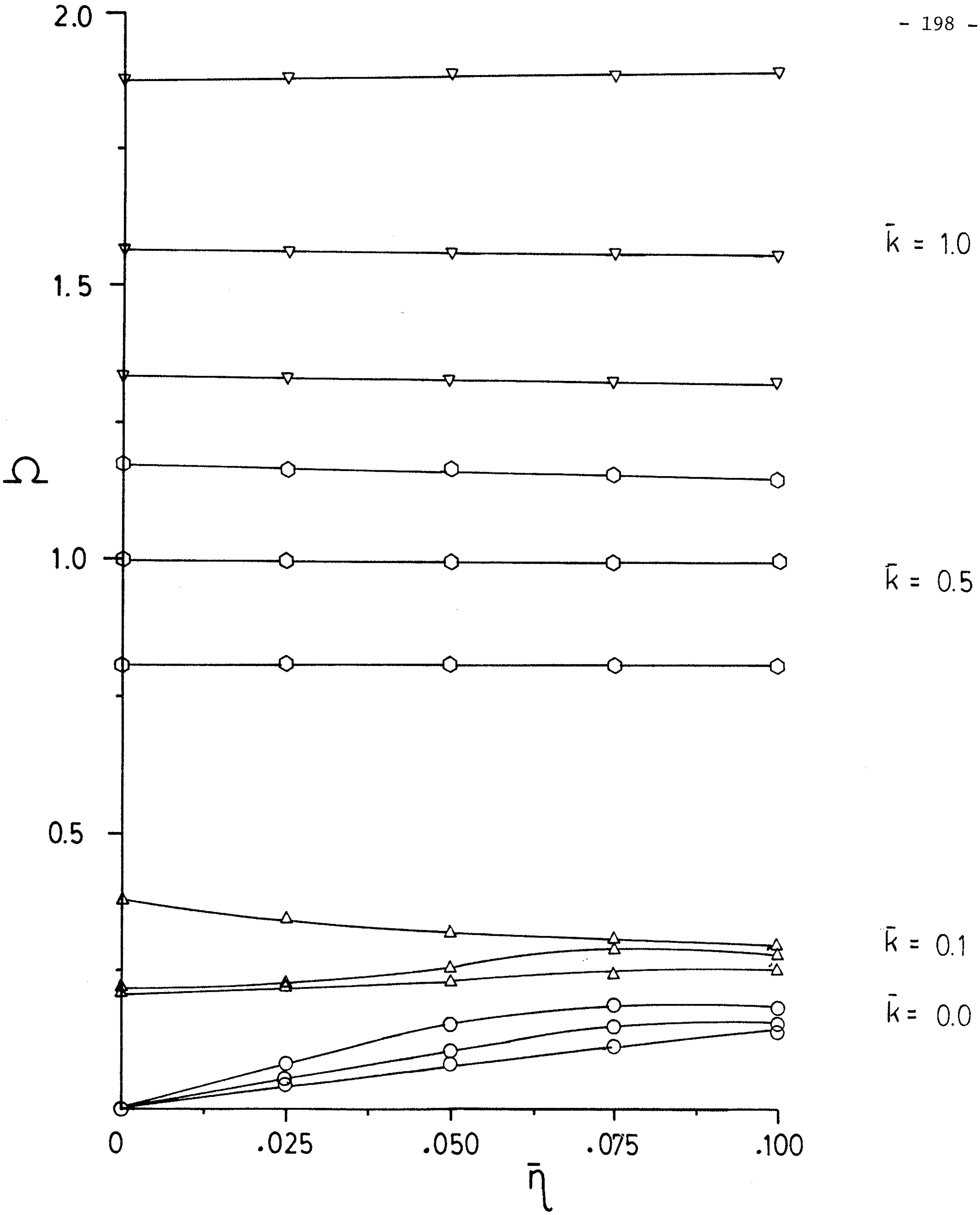


Fig. 4.67 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 0°$; $\bar{d} = 9$.

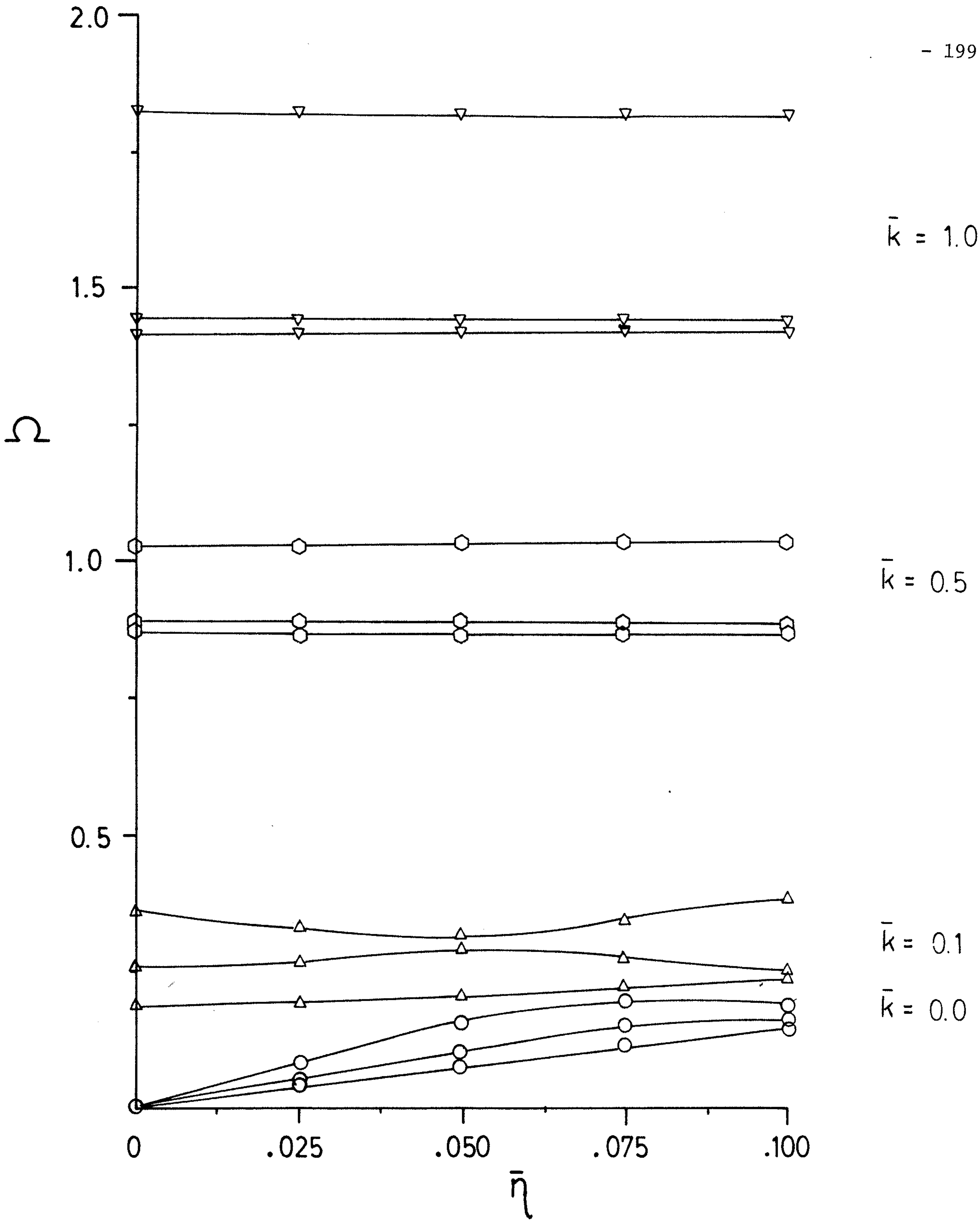


Fig. 4.68 Lowest 3 branches for curves of constant $\bar{k}$, $\alpha = 45°$; $\bar{d} = 9$.

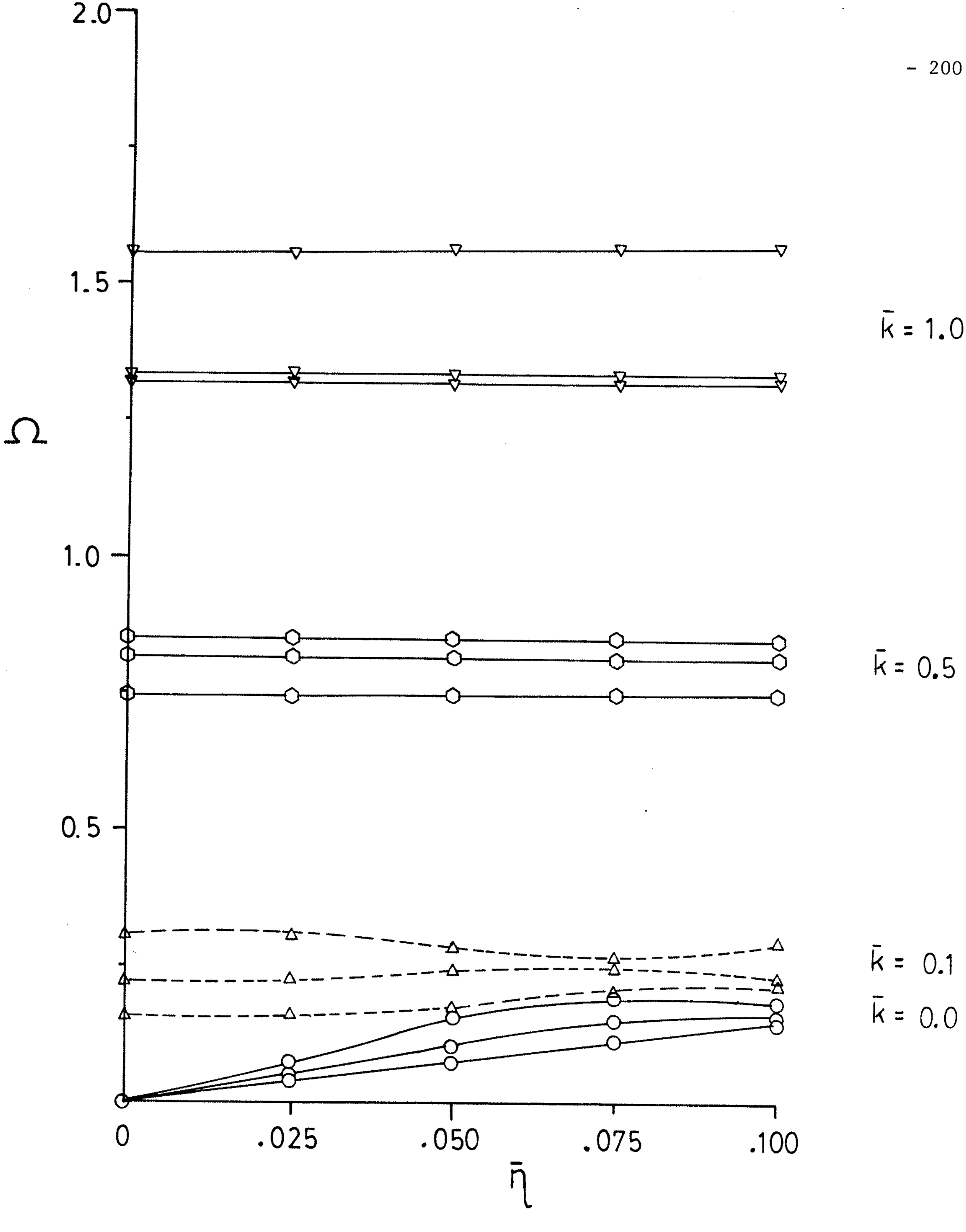


Fig. 4.69 Lowest 3 branches for curves of constant $\bar{k}$, α = 90°; $\bar{d}$ = 9.

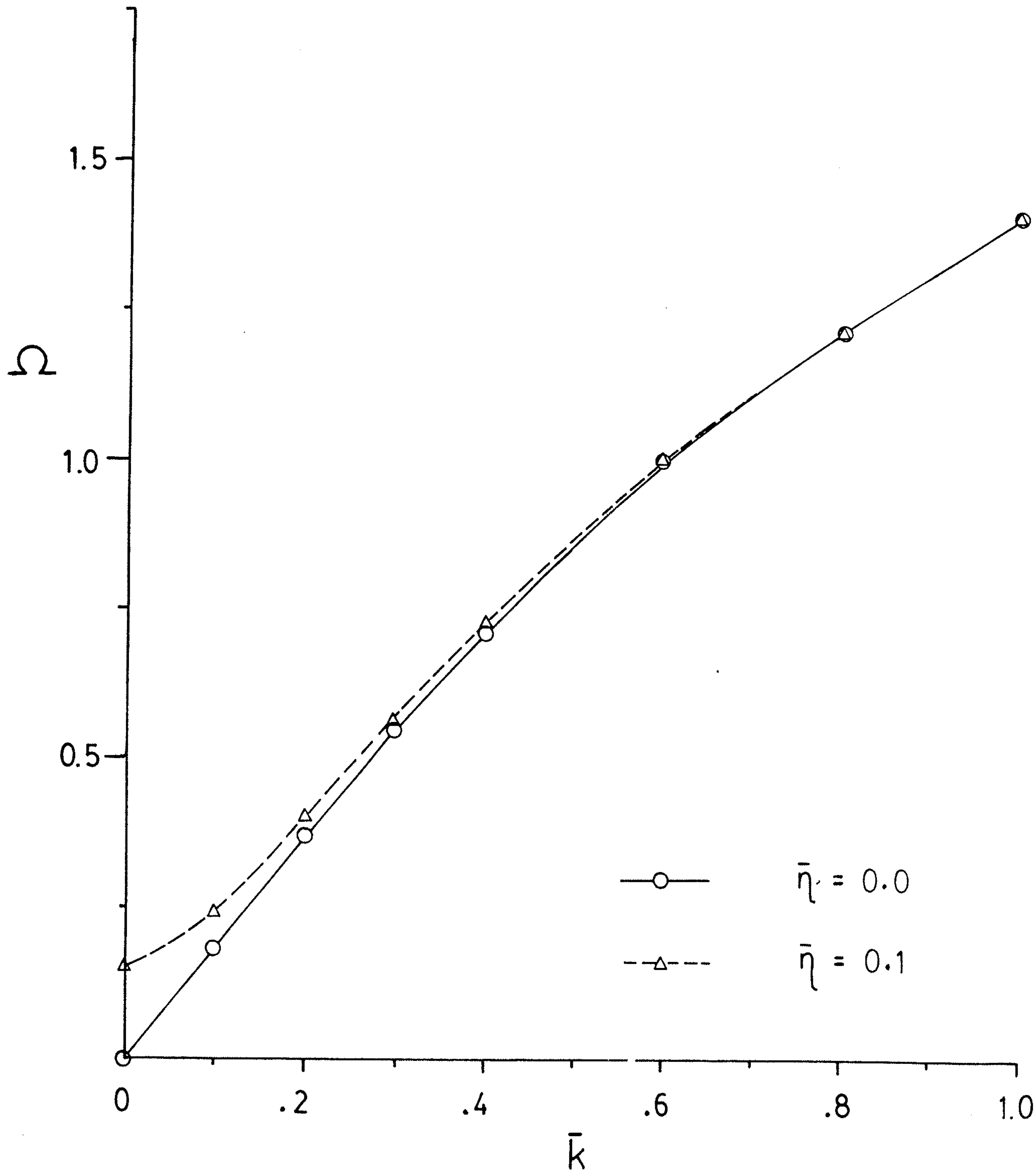


Fig. 4.70 Lowest branch for constant $\bar{\eta}$, $\alpha = 45°$; $\bar{d} = 9$.

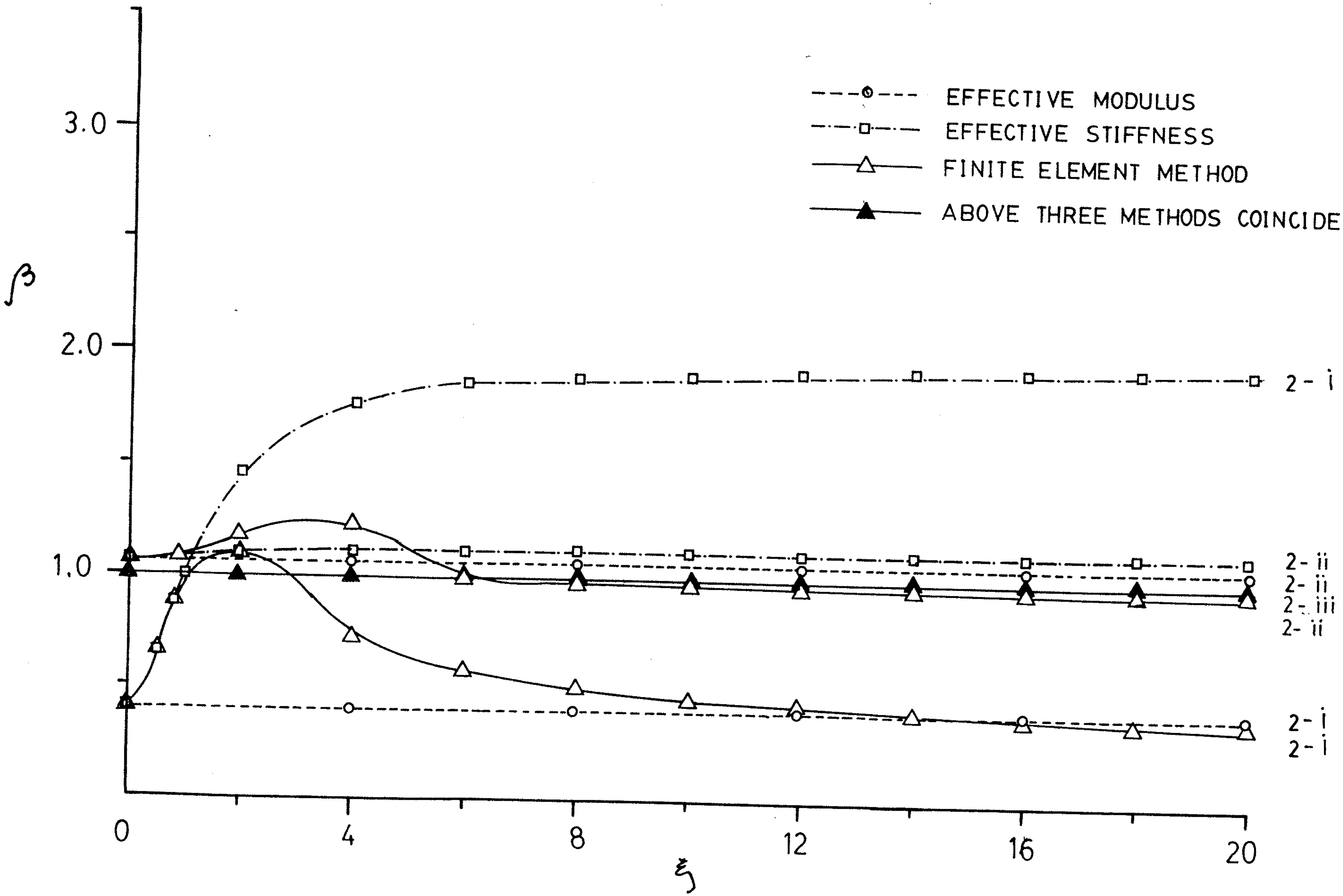


Fig. 4.71 Lowest SV mode for wave propagating through 3 isotropic media with different thicknesses.

Fig. 4.72 Lowest SH mode for wave propagating through 3 isotropic media with different thicknesses.