GENERALIZATION OF WHITTAKER-HENDERSON GRADUATION USING ℓ_p-NORMS

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TABLE OF CONTENTS

0. ABSTRACT

Given a vector of ungraduated values $\underset{\sim}{u}'' = (u_1'', \ldots, u_n'')^T$ and a constant $\lambda \geq 0$, the Whittaker-Henderson graduation method finds the optimal values $\underset{\sim}{u}^\lambda = (u_1^\lambda, \ldots, u_n^\lambda)^T$, called the graduated values, which minimize $F(\underset{\sim}{u}) + \lambda S(\underset{\sim}{u})$ over all $\underset{\sim}{u} = (u_1, \ldots, u_n)^T$, where F is a measure of the lack of fit of $\underset{\sim}{u}$ to $\underset{\sim}{u}''$ and S is a measure of the lack of smoothness of the values in $\underset{\sim}{u}$. This paper gives a generalization of the Whittaker-Henderson graduation method. F and S are defined in terms of ℓ_p-norms and ℓ_q-norms respectively, where $1 \leq p, q \leq \infty$. Methods of finding graduated values in each case are investigated and some numerical examples are given. Sets of graduated values thus obtained are compared. Monotone properties of the graduated values are established.

1. INTRODUCTION: CLASSIFICATION

Given a vector of ungraduated values $\underset{\sim}{u}'' = (u_1'', \ldots, u_n'')^T$, and a constant $\lambda \geq 0$, the well-known Whittaker-Henderson Type B method finds the graduated values $u^\lambda = (u_1^\lambda, \ldots, u_n^\lambda)^T$ which minimize $F(\underset{\sim}{u}) + \lambda S(\underset{\sim}{u})$ over all $\underset{\sim}{u} \equiv (u_1, \ldots, u_n)^T$. F and S are defined as follows:

$$F(\underset{\sim}{u}) \equiv \sum_{x=1}^{n} w_x (u_x - u_x'')^2$$

and

$$S(\underset{\sim}{u}) \equiv \sum_{x=1}^{n-z} (\Delta^z u_x)^2$$

where $w_x > 0$ are weights assigned to u_x'' and $\Delta^z u_x$ are the z-th differences of u_x.

The formula for the graduated values is obtained elegantly by Greville [3], using linear algebra, and by Shiu [10] using advanced calculus.

Schuette [8] uses the measures $F(\underset{\sim}{u}) \equiv \sum_{x=1}^{n} w_x |u_x'' - u_x|$ and $S(\underset{\sim}{u}) \equiv \sum_{x=1}^{n-z} |\Delta^z u_x|$ and shows that $\underset{\sim}{u}^\lambda$ can be obtained by formulating the problem as a linear programming problem. In his discussion of Schuette's paper [4], Greville suggested "It would be most interesting and worthwhile if someone would perform the same task for the Chebyshev norm that Schuette has done for the ℓ_1-norm".

Some other suggestions are given in the discussion of Schuette's paper [9], for example, to use the ℓ_1-norm for fit and the Chebyshev norm for smoothness, or use the ℓ_1-norm for fit and the ℓ_2-norm for smoothness,...,etc.

This paper gives a generalization of the Whittaker-Henderson method by using ℓ_p-norms (the definition and properties of norms are given in Chapter 1). We use

$$F_p(\underset{\sim}{u}) \equiv \sum_{x=1}^{n} w_x |u_x - u_x''|^p \quad \text{and} \quad S_q(\underset{\sim}{u}) \equiv \sum_{x=1}^{n-z} |\Delta^z u_x|^q$$

where $1 \leq p,\ q < \infty$. In case p and q equal infinity, we define

$$F_\infty(\underset{\sim}{u}) \equiv \max_{1 \leq x \leq n} |u_x'' - u_x| \quad \text{and} \quad S_\infty(\underset{\sim}{u}) \equiv \max_{1 \leq x \leq n-z} |\Delta^z u_x|.$$

We classify the above cases (for different p and q) into categories which can be explained by the following diagram:

	$q = 1$	$1 < q < \infty$	$q = \infty$
$p = 1$	corner	rim	corner
$1 < p < \infty$	rim	interior	rim
$p = \infty$	corner	rim	corner

Proof of existence of an optimal solution, or graduated values, and, in some cases, uniqueness of the optimal solution are given in Chapter 1. It is shown that the optimal solution is unique when $1 < p < \infty$.

The methods of finding graduated values in each case are investigated in Chapters 4 to 6. The "corner" cases are discussed in Chapter 4. The graduated values are obtained by means of linear programming, which perhaps is the most widely used optimization model in operations research. Chapter 5 considers the "interior" cases. We differentiate $F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$ and obtain the graduated values by solving the equation $F_p'(\underset{\sim}{u}) + \lambda S_q'(\underset{\sim}{u}) = \underset{\sim}{0}$. The remaining cases, we call them "rim" cases, ~~which~~ are more complicated. They are discussed in Chapter 6.

Chapter 7 gives some numerical examples for each case and the resulting graduated values are compared.

Monotone properties of the optimal solutions are obtained in Chapter 3. We show that $F_p(\underset{\sim}{u}^\lambda) + \lambda S_q(\underset{\sim}{u}^\lambda)$ and $F_p(\underset{\sim}{u}^\lambda)$ are nondecreasing functions of λ and that $S_q(\underset{\sim}{u}^\lambda)$ is a nonincreasing function of λ. (The special case when $p = q = 2$ has been treated by Chan, Chan and Mead [1].)

Although in actuarial applications the graduated values are normally required to be positive, we allow them to be negative in this paper. It turns out that, in most cases, the graduated values will be positive if the ungraduated values are positive, even though we do not impose non-negativity constraints.

2. NORMS, EXISTENCE AND UNIQUENESS OF SOLUTION

We generalize the Whittaker-Henderson graduation method as follows:

$$\min_{\underset{\sim}{u}} F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u}) \qquad \text{(WH,p,q)}$$

where

$$F_p(\underset{\sim}{u}) \equiv \sum_{x=1}^{n} w_x |u_x'' - u_x|^p$$

and

$$S_q(\underset{\sim}{u}) \equiv \sum_{x=1}^{n-z} |\Delta^z u_x|^q$$

with $w_x > 0$ and $1 \le p,\ q < \infty$.

Before we define $F_\infty(\underset{\sim}{u})$ and $S_\infty(\underset{\sim}{u})$, we first give the definitions and some properties of norms.

Given a vector $\underset{\sim}{y} = (y_1, \ldots, y_n)$ in R^n, the ℓ_p-*norm* of $\underset{\sim}{y}$ is defined as $||\underset{\sim}{y}||_p = \left(\sum_{x=1}^{n} |y_x|^p\right)^{1/p}$ where $1 \le p < \infty$.

In case $p = \infty$, the ℓ_∞-norm of $\underset{\sim}{y}$ is defined as $||\underset{\sim}{y}||_\infty = \lim_{p\to\infty} ||\underset{\sim}{y}||_p$. It is intuitively clear that the following property holds:

<u>LEMMA 1.1</u> $\qquad ||\underset{\sim}{y}||_\infty = \max_{1\le x\le n} |y_x|$

<u>Proof</u>: It suffices to show that $||\underset{\sim}{y}||_\infty \ge \max_{1\le x\le n} |y_x|$ and $||y||_\infty \le \max_{1\le x\le n} |y_x|$. Let k be the co-ordinate for which $|y_k|$ attains the value $\max_{1\le x\le n} |y_x|$. We have

$$||\underset{\sim}{y}||_p = \left(\sum_{x=1}^{n} |y_x|^p\right)^{1/p} \geq (|y_k|^p)^{1/p} = |y_k| = \max_{1\leq x\leq n} |y_x|,$$

for each p. Therefore,

$$\max_{1\leq x\leq n} |y_x| \leq ||\underset{\sim}{y}||_\infty. \qquad (2a)$$

Conversely,

$$||\underset{\sim}{y}||_p = \left(\sum_{x=1}^{n} |y_x|^p\right)^{1/p} \leq (n|y_k|^p)^{1/p} = n^{1/p}|y_k|$$

$$= n^{1/p} \max_{1\leq x\leq n} |y_x|.$$

The inequality still holds if we take limits on both sides:

$$||\underset{\sim}{y}||_\infty = \lim_{p\to\infty} ||\underset{\sim}{y}||_p \leq \lim_{p\to\infty} n^{1/p} \max_{1\leq x\leq n} |y_x| = \max_{1\leq x\leq n} |y_x|. \qquad (2b)$$

Therefore, the equality follows immediately from (2a) and (2b).

More generally, we can define the *weighted* ℓ_p*-norm* of $\underset{\sim}{y}$, with weights $w_x > 0$, as $\left(\sum_{x=1}^{n} w_x|y_x|^p\right)^{1/p}$ where $1 \leq p < \infty$. Therefore, $F_p(\underset{\sim}{u})$ and $S_q(\underset{\sim}{u})$, as defined above, can be considered as the p-th power of the weighted ℓ_p-norm of $\underset{\sim}{u}''-\underset{\sim}{u}$ and the q-th power of the ℓ_p-norm of $\Delta^z\underset{\sim}{u}$ respectively.

The weighted ℓ_∞-norm of $\underset{\sim}{y}$ is defined as

$$\lim_{p\to\infty}\left(\sum_{x=1}^{n} w_x|y_x|^p\right)^{1/p},$$

for which a modification of Lemma 1.1 is valid:

$$\lim_{p\to\infty}\left(\sum_{x=1}^{n} w_x|y_x|^p\right)^{1/p} = \max_{1\leq x\leq n} |y_x|.$$

The proof of this equality is quite similar to the proof of Lemma 1.1: Let k be the co-ordinate for which

$$|y_k| = \max_{1 \le x \le n} |y_x|$$

and j be the co-ordinate for which

$$w_j = \max_{1 \le x \le n} w_x.$$

Since

$$\left(\sum_{x=1}^{n} w_x |y_x|^p\right)^{1/p} \ge (w_k |y_k|^p)^{1/p} = w_k^{1/p} |y_k|,$$

taking limits on both sides, we get

$$\lim_{p\to\infty}\left(\sum_{x=1}^{n} w_x |y_x|^p\right)^{1/p} \ge |y_k| = \max_{1 \le x \le n} |y_x|.$$

Conversely,

$$\left(\sum_{x=1}^{n} w_x |y_x|^p\right)^{1/p} \le (nw_j |y_k|^p)^{1/p} = (nw_j)^{1/p} |y_k|.$$

Taking limits on both sides, we get

$$\lim_{p\to\infty}\left(\sum_{x=1}^{n} w_x |y_x|^p\right)^{1/p} \le |y_k| = \max_{1 \le x \le n} |y_x|.$$

Therefore, we define $F_\infty(\underset{\sim}{u})$ and $S_\infty(\underset{\sim}{u})$ as follows:

$$F_\infty(\underset{\sim}{u}) \equiv \max_{1 \le x \le n} |u_x - u''_x|$$

and

$$S_\infty(\underset{\sim}{u}) = \max_{1 \le x \le n-z} |\Delta^z u_x|.$$

$F_\infty(\underset{\sim}{u})$ is the ℓ_∞-norm of $\underset{\sim}{u}''-\underset{\sim}{u}$ and $S_\infty(\underset{\sim}{u})$ is the ℓ_∞-norm of $\Delta^z\underset{\sim}{u}$. Note that the weights disappear in the term $F_\infty(\underset{\sim}{u})$.

We are now going to show the existence and, for the case that $1 < p < \infty$, the uniqueness of the optimal solution of (WH,p,q).

It is obvious that $F_p(\underset{\sim}{u})$ and $S_q(\underset{\sim}{u})$ are continuous functions of $\underset{\sim}{u}$. Therefore $F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$ is also continuous. We show the existence of an optimal solution by the continuity of $F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$ and the following theorem:

<u>THEOREM</u> Any continuous function defined on a closed bounded subset in R^n attains its minimum (and maximum) values on that closed bounded set.

The proof of this theorem can be found in almost any text on mathematical analysis (e.g. [5, p.101]).

Let $\lambda S_q(\underset{\sim}{u}'') = c > 0$ (in case $c = 0$, $\underset{\sim}{u}''$ is obviously the optimal solution). Define

$$D_r = \{\underset{\sim}{y} \in R^n: |y_x - u''_x| \le r,\ x = 1,\ldots,n\}.$$

We can find $r > 0$ such that any $\underset{\sim}{u}$ outside the subset D_r is not an optimal solution, that is $F_p(\underset{\sim}{u}) > c$, which implies that

$$F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u}) > c = F_p(\underset{\sim}{u}'') + \lambda S(\underset{\sim}{u}'').$$

We can take $r = \left(\dfrac{c}{\min\limits_x w_x}\right)^{1/p}$ if $1 \le p < \infty$. In case $p = \infty$, we take $r = c$.

Therefore, the existence of an optimal solution follows from the continuity of $F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$ and from D_r being a closed bounded subset.

We will now show that, if $1 < p < \infty$, the optimal solution is unique. We prove this by using the strict convexity of the function $F_p(\underset{\sim}{u})$ when $1 < p < \infty$.

Recall that a function $G: R^n \to R$ is convex if

$$G(\beta\underset{\sim}{y}+(1-\beta)\underset{\sim}{y}^*) \leq \beta G(\underset{\sim}{y}) + (1-\beta)G(\underset{\sim}{y}^*)$$

for all $\underset{\sim}{y}$, $\underset{\sim}{y}^*$ in R^n and $\beta \in [0,1]$. G is strictly convex if

$$G(\beta\underset{\sim}{y}+(1-\beta)\underset{\sim}{y}^*) < \beta G(\underset{\sim}{y}) + (1-\beta)G(\underset{\sim}{y}^*)$$

for all $\underset{\sim}{y}$, $\underset{\sim}{y}^*$ in R^n with $\underset{\sim}{y} \neq \underset{\sim}{y}^*$ and $\beta \in (0,1)$. Also, we have the following properties: if G, H are convex functions and $\beta \geq 0$, then βG and G+H are also convex. If, in addition, one of G and H is strictly convex, then so is G+H.

When p and q equal 1 or infinity, it can be easily seen from the above definitions that $F_p(\underset{\sim}{u})$ and $S_q(\underset{\sim}{u})$ are convex functions of u.

When $1 < p < \infty$, we can show that $F_p(\underset{\sim}{u})$ is strictly convex and $S_q(\underset{\sim}{u})$ is convex. For the proof we need the following lemma:

<u>LEMMA 1.2</u> Let $G(y) = |y|^p$, $p > 1$. Then $G'(y) = \mathrm{sgn}(y)p|y|^{p-1}$ for any $y \in R$ and, except when $y = 0$

and $p < 2$, $G''(y) = p(p-1)|y|^{p-2}$.

The function sgn(y) is defined as

$$\mathrm{sgn}(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0. \end{cases}$$

This can easily be proved by considering the cases $y > 0$, $y = 0$ and $y < 0$ [2, pp.113].

Note that $G''(y) \geq 0$, (the = sign applying only when $y = 0$) and $G'(y)$ is strictly increasing. Using facts in elementary mathematical analysis [2, pp.113], we can conclude that $G(y) = |y|^p$ is strictly convex.

THEOREM 1.1 $F_p(\underset{\sim}{u})$ is strictly convex when $1 < p < \infty$ and $S_q(\underset{\sim}{u})$ is convex for all $1 \leq q \leq \infty$. (Hence, $F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$ is strictly convex.)

Proof: Let $\underset{\sim}{u}$, $\underset{\sim}{u}^* \in R^n$ and $\beta \in [0,1]$. Since $G(y) = |y|^q$ is convex for $1 < q < \infty$ (in fact, it is strictly convex), we have

$$\begin{aligned} S_q(\beta\underset{\sim}{u}+(1-\beta)\underset{\sim}{u}^*) &= \sum_{x=1}^{n-z} |\Delta^z(\beta(u_x)) + (1-\beta)(u_x^*)|^q \\ &= \sum_{x=1}^{n-z} |\beta(\Delta^z u_x) + (1-\beta)(\Delta^z u_x^*)|^q \\ &\leq \sum_{x=1}^{n-z} \beta|\Delta^z u_x|^q + \sum_{x=1}^{n-z} (1-\beta)|\Delta^z u_x^*|^q \\ &= \beta S_q(\underset{\sim}{u}) + (1-\beta)S_q(\underset{\sim}{u}^*). \end{aligned}$$

In case q equals 1 or infinity, it can be easily seen that $S_q(\underset{\sim}{u})$ is convex.

Consider $\underset{\sim}{u}$, $\underset{\sim}{u}^* \in R^n$ with $\underset{\sim}{u} \neq \underset{\sim}{u}^*$ and $\beta \in (0,1)$. Since $G(y) = |y|^p$ is strictly convex and $\underset{\sim}{u} \neq \underset{\sim}{u}^*$, that is, $u_x \neq u_x^*$ for at least some x, we have

$$F_p(\beta\underset{\sim}{u}+(1-\beta)\underset{\sim}{u}^*) = \sum_{x=1}^{n} w_x|\beta u_x+(1-\beta)u_x^*-u_x''|^p$$

$$= \sum_{x=1}^{n} w_x|\beta(u_x-u_x'')+(1-\beta)(u_x^*-u_x'')|^p$$

$$< \beta\sum_{x=1}^{n} w_x|u_x-u_x''|^p+\sum_{x=1}^{n} w_x(1-\beta)|u_x^*-u_x''|^p$$

$$= \beta F_p(\underset{\sim}{u}) + (1-\beta)F_p(\underset{\sim}{u}^*).$$

By Theorem 1.1, we obtain the following:

THEOREM 1.2 The optimal solution of (WH,p,q) is unique when $1 < p < \infty$.

Proof: We prove this by contradiction. Let $\underset{\sim}{u}$, $\underset{\sim}{u}^*$ be two optimal solutions with $\underset{\sim}{u} \neq \underset{\sim}{u}^*$ and

$$F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u}) = M = F_p(\underset{\sim}{u}^*) + \lambda S_q(\underset{\sim}{u}^*).$$

Consider the point $\frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{u}^*$.

$$F_p(\tfrac{1}{2}\underset{\sim}{u} + \tfrac{1}{2}\underset{\sim}{u}^*) < \tfrac{1}{2}F_p(\underset{\sim}{u}) + \tfrac{1}{2}F_p(\underset{\sim}{u}^*)$$

by the strict convexity of $F_p(\underset{\sim}{u})$. Therefore,

$$F_p(\tfrac{1}{2}\underset{\sim}{u} + \tfrac{1}{2}\underset{\sim}{u}^*) + \lambda S_q(\tfrac{1}{2}\underset{\sim}{u} + \tfrac{1}{2}\underset{\sim}{u}^*)$$

$$< \tfrac{1}{2}[F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})] + \tfrac{1}{2}[F_p(\underset{\sim}{u}^*) + \lambda S_q(\underset{\sim}{u}^*)]$$

$$= \frac{M}{2} + \frac{M}{2} = M.$$

This contradicts the fact that

$$\min_{\underset{\sim}{u}} F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u}) = M.$$

3. MONOTONE PROPERTIES

Optimal solutions of $\underset{\sim}{u}^\lambda$ of the problem

$$\min_{\underset{\sim}{u}} F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u}) \qquad \text{(WH,p,q)}$$

have the *monotonic properties*:

$F_p(\underset{\sim}{u}^\lambda) + \lambda S_q(\underset{\sim}{u}^\lambda)$ is a nondecreasing function of λ,

$F_p(\underset{\sim}{u}^\lambda)$ is a nondecreasing function of λ,

$S_q(\underset{\sim}{u}^\lambda)$ is a nonincreasing function of λ.

If $S_q(\underset{\sim}{u}^\lambda) > 0$, then $F_p(\underset{\sim}{u}^\lambda) + \lambda S_q(\underset{\sim}{u}^\lambda)$ is an increasing function of λ. If, in addition, $1 < p, q < \infty$, then $F_p(\underset{\sim}{u}^\lambda)$ is an increasing function of λ and $S_q(\underset{\sim}{u}^\lambda)$ is a decreasing function of λ.

These monotone properties can be used to check if errors were made in the calculations of the $\underset{\sim}{u}^\lambda$ when several λ values are used.

Although these properties may be intuitively clear, we prove Theorems 3.1 and 3.2:

THEOREM 3.1 Let $\lambda > \lambda^* \geq 0$ and $\underset{\sim}{u}^\lambda$ and $\underset{\sim}{u}^{\lambda^*}$ denote the corresponding optimal solution of (WH,p,q). Then

$$F_p(\underset{\sim}{u}^\lambda) + \lambda S_q(\underset{\sim}{u}^\lambda) \geq F_p(\underset{\sim}{u}^{\lambda^*}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda^*}),$$

$$F_p(\underset{\sim}{u}^\lambda) \geq F_p(\underset{\sim}{u}^{\lambda^*}) \quad \text{and} \quad S_q(\underset{\sim}{u}^\lambda) \leq S_q(\underset{\sim}{u}^{\lambda^*}).$$

Proof: Since $\lambda > \lambda^*$, we obtain the following inequalities:

$$F_p(\underset{\sim}{u}^{\lambda^*}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda^*}) \leq F_p(\underset{\sim}{u}^{\lambda}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda})$$

$$\leq F_p(\underset{\sim}{u}^{\lambda}) + \lambda S_q(\underset{\sim}{u}^{\lambda}). \qquad (3.1a)$$

The first inequality comes from the fact that $\underset{\sim}{u}^{\lambda^*}$ is the optimal solution of (WH,p,q) corresponding to λ^*. The second inequality is obvious since $\lambda > \lambda^*$.

Similarly, we obtain

$$F_p(\underset{\sim}{u}^{\lambda}) + \lambda S_q(\underset{\sim}{u}^{\lambda}) \leq F_p(\underset{\sim}{u}^{\lambda^*}) + \lambda S_q(\underset{\sim}{u}^{\lambda^*}). \qquad (3.1b)$$

Adding the first inequality in (3.1a) and the inequality (3.1b), we obtain

$$0 \leq (\lambda-\lambda^*)[S_q(\underset{\sim}{u}^{\lambda^*}) - S_q(\underset{\sim}{u}^{\lambda})]. \qquad (3.1c)$$

That is, $S_q(\underset{\sim}{u}^{\lambda}) \leq S_q(\underset{\sim}{u}^{\lambda^*})$ since $\lambda > \lambda^*$. This, and the first inequality of (3.1a), imply that $F_p(\underset{\sim}{u}^{\lambda^*}) \leq F_p(\underset{\sim}{u}^{\lambda})$ and the proof is complete.

<u>THEOREM 3.2</u> Let $\lambda > \lambda^* \geq 0$ and $\underset{\sim}{u}^{\lambda}$ and $\underset{\sim}{u}^{\lambda^*}$ denote the corresponding solutions of (WH,p,q).

(a) If $S_q(\underset{\sim}{u}^{\lambda}) > 0$, then

$$F_p(\underset{\sim}{u}^{\lambda}) + \lambda S_q(\underset{\sim}{u}^{\lambda}) > F_p(\underset{\sim}{u}^{\lambda^*}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda^*}).$$

(b) If, in addition, $1 < p,\ q < \infty$, then

$$F_p(\underset{\sim}{u}^{\lambda}) > F_p(\underset{\sim}{u}^{\lambda^*}) \quad \text{and} \quad S_q(\underset{\sim}{u}^{\lambda}) < S_q(\underset{\sim}{u}^{\lambda^*}).$$

<u>Proof</u>: (a) The assumption $S_q(\underset{\sim}{u}^{\lambda}) > 0$ implies that the second inequality in (3.1a) is a strict inequality. Therefore, $F_p(\underset{\sim}{u}^{\lambda}) + \lambda S_q(\underset{\sim}{u}^{\lambda}) > F_p(\underset{\sim}{u}^{\lambda^*}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda^*})$ by (3.1a).

(b) This can be obtained if we can show $\underset{\sim}{u}^{\lambda} \neq \underset{\sim}{u}^{\lambda^*}$ and the first inequality in (3.1a) is strict. We first show $\underset{\sim}{u}^{\lambda} \neq \underset{\sim}{u}^{\lambda^*}$, which can be seen easily: From the result obtained in Chapter 5, we have

$$F_p'(\underset{\sim}{u}^{\lambda}) + \lambda S_q'(\underset{\sim}{u}^{\lambda}) = \underset{\sim}{0} = F_p'(\underset{\sim}{u}^{\lambda^*}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda^*}).$$

Suppose $\underset{\sim}{u}^{\lambda} = \underset{\sim}{u}^{\lambda^*}$; we have

$$\lambda S_q'(\underset{\sim}{u}^{\lambda}) = \lambda^* S_q'(\underset{\sim}{u}^{\lambda^*}) = \lambda^* S_q'(\underset{\sim}{u}^{\lambda})$$

and thus $S_q'(\underset{\sim}{u}^{\lambda}) = \underset{\sim}{0}$ since $\lambda \neq \lambda^*$.

If we denote the differencing matrix of order z by K (see Chapter 4 for definition), we have

$$S_q'(\underset{\sim}{u}^{\lambda}) = [q|\Delta^z u_1^{\lambda}|^{q-1} \operatorname{sgn}(\Delta^z u_1^{\lambda}), \ldots, q|\Delta^z u_{n-z}^{\lambda}|^{q-1} \operatorname{sgn}(\Delta^z u_{n-z}^{\lambda})]K,$$

(see Chapter 5, Theorem 5.1). Since K is of rank n-z (see Chapter 4),

$$|\Delta^z u_x^{\lambda}| \operatorname{sgn}(\Delta^z u_x^{\lambda}) = 0$$

for all $x = 1, \ldots, n-z$. This will imply $S_q(\underset{\sim}{u}^{\lambda}) = \underset{\sim}{0}$ which contradicts our assumption.

Now, we will use an indirect proof to show that the first inequality in (3.1a) is strict. Assume that

$$F_p(\underset{\sim}{u}^{\lambda^*}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda^*}) = F_p(\underset{\sim}{u}^{\lambda}) + \lambda^* S_q(\underset{\sim}{u}^{\lambda}).$$

Then, since $\underset{\sim}{u}^{\lambda} \neq \underset{\sim}{u}^{\lambda^*}$, there would not be a unique solution to the minimization of $F_p(\underset{\sim}{u}) + \lambda^* S_q(\underset{\sim}{u})$, which is a contradiction.

Therefore, we obtain $S_q(\underset{\sim}{u}^{\lambda}) < S_q(\underset{\sim}{u}^{\lambda^*})$. This, together with the first strict inequality in (3.1a), implies $F_p(\underset{\sim}{u}^{\lambda^*}) < F_p(\underset{\sim}{u}^{\lambda})$. Therefore the proof is complete.

IV. THE "CORNER" CASES ($p = 1$ or ∞, $q = 1$ or ∞)

In this chapter, we find the optimal solutions of

$$\min_{\underset{\sim}{u}} F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$$

by using linear programming for the cases when p and q are equal to 1 or infinity.

Case (i): $p = q = 1$

This case is proposed and solved by Schuette [8]. He solved the problem

$$\min_{\underset{\sim}{u}}[F_1(\underset{\sim}{u}) + \lambda S_1(\underset{\sim}{u})] \qquad \text{(WH,1,1)}$$

by formulating it as a linear programming problem:

$$\min\left[\sum_{x=1}^{n} w_x(P_x+N_x) + \lambda \sum_{x=1}^{n-z}(R_x+T_x)\right]$$

subject to the constraints

$$\Delta^z(P_x-N_x) + R_x - T_x = \Delta^z u_x'', \qquad x = 1,2,\ldots,n-z,$$

and

$$P_x \geq 0, \quad N_x \geq 0, \quad R_x \geq 0, \quad T_x \geq 0.$$

(P_x and N_x represent the positive and negative parts of u_x-u_x'' respectively, that is, $P_x = u_x-u_x''$ if $u_x-u_x'' \geq 0$ and $P_x = 0$ if $u_x-u_x'' < 0$; $N_x = -|u_x-u_x''|$ if $u_x-u_x'' < 0$ and $N_x = 0$ if $u_x-u_x'' \geq 0$. Similarly, R_x and T_x represent the positive and negative parts of $\Delta^z u_x$.)

<u>Case (ii)</u>: $p = q = \infty$.

This case is suggested by Greville in the discussion of Schuette's paper [4].

We solve the problem:

$$\underset{\underset{\sim}{u}}{\text{Min}}\ F_\infty(\underset{\sim}{u}) + \lambda S_\infty(\underset{\sim}{u}) \qquad (WH,\infty,\infty)$$

where

$$F_\infty(\underset{\sim}{u}) = \max_{1\le x\le n} w_x |u''_x - u_x| \quad \text{and} \quad S_\infty(\underset{\sim}{u}) = \max_{1\le x\le n-z} |\Delta^z u_x|$$

by formulating it as a linear programming problem:

$$\underset{\underset{\sim}{u},f,s}{\text{Min}}\ [f + \lambda s] \qquad (LP1)$$

subject to the constraints

$$\left.\begin{array}{l} u_x - u''_x \le f \\ -u_x + u''_x \le f \end{array}\right\} \quad x = 1,\ldots,n \qquad (4.1a)$$

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{n} u_i k_{xi} \le s \\ \\ \displaystyle -\sum_{i=1}^{n} u_i k_{xi} \le s \end{array}\right\} \quad x = 1,2,\ldots,n-z. \qquad (4.1b)$$

k_{xi} are the coefficients of the $(n-z)\times n$ difference matrix K of order z, and K can be expressed as:

$$\begin{array}{c} \overbrace{\hphantom{(-1)^z \quad (-1)^{z-1}\binom{z}{1} \quad (-1)^{z-2}\binom{z}{2} \quad \cdots \quad -\binom{z}{z-1} \quad 1}}^{z+1 \text{ columns}} \qquad \overbrace{\hphantom{0 \cdots 0}}^{n-2z-1 \text{ columns}} \\ \begin{bmatrix} (-1)^z & (-1)^{z-1}\binom{z}{1} & (-1)^{z-2}\binom{z}{2} & \cdots & -\binom{z}{z-1} & 1 & & 0 & \cdots & 0 \\ 0 & (-1)^z & (-1)^{z-1}\binom{z}{1} & \cdots & \cdots & -\binom{z}{z-1} & & 1 & \cdots & 0 \\ \vdots & & & & & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & & -(-1)^z & \cdots & -\binom{z}{z-1} & 1 \end{bmatrix} \\ \underbrace{\hphantom{0 \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad 0}}_{n-2z-1 \text{ columns}} \qquad \underbrace{\hphantom{-(-1)^z \cdots -\binom{z}{z-1} \quad 1}}_{z+1 \text{ columns}} \end{array}$$

The vector $\Delta^z \underset{\sim}{u} = (\Delta^z u_1, \ldots, \Delta^z u_{n-z})$ can be obtained from $\underset{\sim}{u}$ by a matrix multiplication: $\Delta^z \underset{\sim}{u} = K\underset{\sim}{u}$, which can be seen from the expression for K.

<u>THEOREM 4.1</u> The linear programming problem (LP1) is equivalent to the (WH,∞,∞) problem.

<u>Proof</u>: Let $\underset{\sim}{u}^\lambda$ be the optimal solution of the (WH,∞,∞) problem and $(\underset{\sim}{u}^*, f^*, s^*)$ the optimal solution of the linear programming problem (LP1). We need to show that

$$F_\infty(\underset{\sim}{u}^\lambda) + \lambda S_\infty(\underset{\sim}{u}^\lambda) = f^* + \lambda s^*.$$

Since

$$F_\infty(\underset{\sim}{u}^\lambda) = \max_{1 \le x \le n} |u_x'' - u_x^\lambda| \quad \text{and} \quad S_\infty(\underset{\sim}{u}^\lambda) = \max_{1 \le x \le n-z} |\Delta^z u_x^\lambda|,$$

$(\underset{\sim}{u}^\lambda, F_\infty(\underset{\sim}{u}^\lambda), S_\infty(\underset{\sim}{u}^\lambda))$ satisfies the constraints (4.1a) and (4.1b). Therefore,

$$F_\infty(\underset{\sim}{u}^\lambda) + \lambda S_\infty(\underset{\sim}{u}^\lambda) \ge f^* + \lambda s^*$$

since $(\underset{\sim}{u}^*, f^*, s^*)$ is the optimal solution of (LP1). Conversely, we have

$$F_\infty(\underset{\sim}{u}^\lambda) + \lambda S_\infty(\underset{\sim}{u}^\lambda) \le F_\infty(\underset{\sim}{u}^*) + \lambda S_\infty(\underset{\sim}{u}^*)$$

$$= \max_{1 \le x \le n} |u_x'' - u_x^*| + \lambda \max_{1 \le x \le n-z} |\Delta^z u_x^*| \le f^* + \lambda s^*.$$

The first inequality comes from the fact that the minimization of the (WH,∞,∞) ~~problem~~ expression extends over all $\underset{\sim}{u}$.

Case (iii). $p = 1,\ q = \infty$.

We formulate

$$\min_{\underset{\sim}{u}} \left[\sum_{x=1}^{n} w_x |u''_x - u_x| + \lambda \max_{1 \le x \le n-z} |\Delta^z u_x| \right] \qquad (\mathrm{WH},1,\infty)$$

as the following linear programming problem:

$$\mathrm{Min} \sum_{x=1}^{n} w_x (P_x + N_x) + \lambda s \qquad (\mathrm{LP2})$$

subject to the constraints

$$\left.\begin{aligned} \sum_{i=1}^{n} u_i k_{xi} &\le s \\ -\sum_{i=1}^{n} u_i k_{xi} &\le s \end{aligned}\right\} \quad x = 1,2,\ldots,n-z$$

$$P_x + N_x = u_x - u''_x \qquad x = 1,2,\ldots,n$$

and

$$P_x \ge 0, \quad N_x \ge 0, \quad S \ge 0$$

where k_{xi} is the element in the x-th row and the i-th column of the difference matrix K of z-th order.

This formulation is in fact a combination of those in case (i) and case (ii) and the proof that $(\mathrm{WH},1,\infty)$ and (LP2) are equivalent proceeds along the same lines.

Case (iv): $p = \infty,\ q = 1$.

This case is quite similar to case (iii). We formulate

$$\min_{\underset{\sim}{u}} \left[\max_{1 \le x \le n} |u''_x - u_x| + \lambda \sum_{x=1}^{n-z} |\Delta^z u_x| \right] \qquad (\mathrm{WH},\infty,1)$$

as the following linear programming problem:

$$\operatorname{Min}\left[f + \lambda \sum_{x=1}^{n-z} (R_x+T_x)\right] \qquad \text{(LP3)}$$

subject to the constraints

$$\left.\begin{aligned} u_x - u''_x &\leq f \\ -u_x + u''_x &\leq f \end{aligned}\right\} \quad x = 1,2,\ldots,n$$

$$R_x - T_x = \Delta^z u_x \qquad x = 1,2,\ldots,n-z$$

and

$$R_x \geq 0, \quad T_x \geq 0, \quad f \geq 0$$

It can be easily seen that $(WH,\infty,1)$ and (LP3) are equivalent.

V. THE "INTERIOR" CASES ($1 < p < \infty$, $1 < q < \infty$)

Consider the following problem

$$\underset{\tilde{u}}{\text{Min}} \sum_{x=1}^{n} w_x |u_x'' - u_x|^p + \lambda \sum_{x=1}^{n-z} |\Delta^z u_x|^q \qquad (\text{WH},p,q)$$

where $1 < p,\ q < \infty$.

We differentiate $F_p(\tilde{u}) + \lambda S_q(\tilde{u})$ with respect to $\tilde{u}$ and obtain the optimal solution of (WH,p,q) by solving the equations $F_p'(\tilde{u}) + \lambda S_q'(\tilde{u}) = \tilde{0}$.

Let $\tilde{u}^\lambda$ be the optimal solution of (WH,p,q). Then

$$F_p'(\tilde{u}^\lambda) + \lambda S_q'(\tilde{u}^\lambda) = \tilde{0}.$$

Conversely,

$$F_p'(\tilde{u}^\lambda) + \lambda S_q'(\tilde{u}^\lambda) = \tilde{0}$$

is also a sufficient condition for the optimality of $\tilde{u}^\lambda$. This is a property of convex functions and the details can be found in [2, pp.116]. Hence, the solution of the equation $F_p'(\tilde{u}) + \lambda S_q'(\tilde{u}) = \tilde{0}$ gives the unique optimal solution to the (WH,p,q) problem.

Before we compute the derivative $F_p'(\tilde{u}) + \lambda S_q'(\tilde{u})$, we need some preliminaries.

Let $A: R^r \to R^s$ be a function. Define

$$A'(\tilde{y}) = \begin{bmatrix} \frac{\partial A_1}{\partial y_1} & \cdots & \frac{\partial A_1}{\partial y_r} \\ \vdots & & \vdots \\ \frac{\partial A_s}{\partial y_1} & \cdots & \frac{\partial A_s}{\partial y_r} \end{bmatrix}$$

where

$$A(\underset{\sim}{y}) = (A_1(\underset{\sim}{y}),\ldots,A_s(\underset{\sim}{y})).$$

If A is a linear transformation and thus can be represented by a matrix M, i.e., $A(\underset{\sim}{y}) = M\underset{\sim}{y}$, where

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1r} \\ \vdots & & \vdots \\ m_{s1} & \cdots & m_{sr} \end{bmatrix},$$

then

$$A'(\underset{\sim}{y}) = M.$$

If $A\colon R^r \to R^s$ and $B\colon R^s \to R^t$, and $C = B(A)$ is the composite function: $R^r \to R^t$, the Chain Rule [2, pp.122] can be generalized as

$$C'(\underset{\sim}{y}) = B'(A(\underset{\sim}{y}))A'(\underset{\sim}{y}).$$

<u>THEOREM 5.1</u> $F_p(\underset{\sim}{u})$ and $S_q(\underset{\sim}{u})$ are differentiable at every $\underset{\sim}{u}$ in R^n and their derivatives are

$$F_p'(\underset{\sim}{u}) = \left[\frac{\partial F_p}{\partial u_1}, \ldots, \frac{\partial F_p}{\partial u_n}\right]$$

$$= p[|u_1-u_1''|^{p-1}\operatorname{sgn}(u_1-u_1''),\ldots,|u_n-u_n''|^{p-1}\operatorname{sgn}(u_n-u_n'')]W$$

$$S_q'(\underset{\sim}{u}) = \left[\frac{\partial S_q}{\partial u_1}, \ldots, \frac{\partial S_q}{\partial u_n}\right]$$

$$= q[|\Delta^z u_1|^{q-1}\operatorname{sgn}(\Delta^z u_1),\ldots,|\Delta^z u_{n-z}|^{q-1}\operatorname{sgn}(\Delta^z u_{n-z})]K,$$

where W is the n×n diagonal matrix with elements $w_1,\ldots,w_n$.

Proof: Let

$$A(\underset{\sim}{u}) = \begin{bmatrix} |u_1-u_1''|^p \\ \vdots \\ |u_n-u_n''|^p \end{bmatrix} \quad \text{and} \quad B(\underset{\sim}{y}) = \sum_{x=1}^{n} w_x y_x;$$

then $F_p(\underset{\sim}{u})$ can be written as $(B(A))(\underset{\sim}{u})$. Hence,

$$A'(\underset{\sim}{u}) = \begin{bmatrix} p|u_1-u_1''|^{p-1}\mathrm{sgn}(u_1-u_1'') & & 0 \\ & \ddots & \\ 0 & & p|u_n-u_n''|^{p-1}\mathrm{sgn}(u_n-u_n'') \end{bmatrix}$$

and

$$B'(\underset{\sim}{y}) = [w_1,\ldots,w_n].$$

So, by the Chain Rule:

$$F_p'(\underset{\sim}{u}) = B'(A(\underset{\sim}{u}))A'(\underset{\sim}{u})$$

$$= [p|u_1-u_1''|^{p-1}\mathrm{sgn}(u_1-u_1''),\ldots,p|u_n-u_n''|^{p-1}\mathrm{sgn}(u_n-u_n'')]W.$$

Let

$$B(\underset{\sim}{y}) = \sum_{x=1}^{n-z} |y_x|^q$$

with $y \in R^{n-z}$. Then $S(\underset{\sim}{u})$ can be expressed as $B(K(\underset{\sim}{u}))$ and

$$B'(\underset{\sim}{y}) = [q|\Delta^z u_1|^{q-1}\mathrm{sgn}(\Delta^z u_1),\ldots,q|\Delta^z u_{n-z}|^{q-1}\mathrm{sgn}(\Delta^z u_{n-z})]K.$$

Having computed the derivative $F_p'(\underset{\sim}{u}) + \lambda S_q'(\underset{\sim}{u})$, we solve the equation $F_p'(\underset{\sim}{u}) + \lambda S_q'(\underset{\sim}{u}) = \underset{\sim}{0}$ by using the Newton-Raphson algorithm [7]. In order to do this, however, we need the second derivative of $F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u})$.

THEOREM 5.2 $F_p(\underset{\sim}{u})$ and $S_q(\underset{\sim}{u})$ are twice differentiable at $\underset{\sim}{u}$ (except when $u_x = u''_x$ or $\Delta^z u_x = 0$ for some x and $p < 2$ or $q < 2$.) Their second derivatives (if they exist) are

$$F'_p(\underset{\sim}{u}) = \left[\frac{\partial^2 F_p}{\partial u_i \partial u_j}\right]$$

$$= p(p-1)\begin{bmatrix} |u_1-u''_1|^{p-2} & & 0 \\ & \ddots & \\ 0 & & |u_n-u''_n|^{p-2} \end{bmatrix} W,$$

$$S''_q(\underset{\sim}{u}) = \left[\frac{\partial^2 S_q}{\partial u_i \partial u_j}\right]$$

$$= q(q-1)K^T \begin{bmatrix} |\Delta^z u_1|^{q-2} & & 0 \\ & \ddots & \\ 0 & & |\Delta^z u_{n-z}|^{q-2} \end{bmatrix} K.$$

Proof: The matrix $(F'_p(\underset{\sim}{u}))^T$ can be expressed as $p(W(A))(\underset{\sim}{u})$, where

$$A(\underset{\sim}{u}) = \begin{bmatrix} |u_1-u''_1|^{p-1}\operatorname{sgn}(u_1-u''_1) \\ \vdots \\ |u_n-u''_n|^{p-1}\operatorname{sgn}(u_n-u''_n) \end{bmatrix}.$$

Then, by the Chain Rule,

$$F''_p(\underset{\sim}{u}) = p(p-1)WA'(\underset{\sim}{u}) = p(p-1)A'(\underset{\sim}{u})W$$

where

$$A'(\underset{\sim}{u}) = \begin{bmatrix} |u_1-u_1''|^{p-2} & & & \\ & \ddots & & 0 \\ & 0 & \ddots & \\ & & & |u_n-u_n''|^{p-2} \end{bmatrix}.$$

The expression for $S_q''(\underset{\sim}{u})$ can be similarly derived.

Using the Newton-Raphson algorithm [7], we obtain the following:

<u>THEOREM 5.3</u> Let $\underset{\sim}{u}^o$ be an initial trial solution and $\underset{\sim}{u}^k$ denote the "solution" after k iterations, so that

$$\underset{\sim}{u}^{k+1} = \underset{\sim}{u}^k - [F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)]^{-1}[F_p'(\underset{\sim}{u}^k) + \lambda S_q'(\underset{\sim}{u}^k)].$$

If $\underset{\sim}{u}^\lambda = \lim_{k\to\infty} \underset{\sim}{u}^k$ exists, then $\underset{\sim}{u}^\lambda$ is the unique solution of the equation

$$F_p'(\underset{\sim}{u}) + S_q'(\underset{\sim}{u}^\lambda) = \underset{\sim}{0},$$

provided

$$F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)$$

and

$$[F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)]^{-1}$$

exist for all k. That is, u^λ is the unique optimal solution of (WH,p,q) with $1 < p,\ q < \infty$.

Notice that $S_q''(\underset{\sim}{u}^k)$ is a symmetric matrix; therefore, it is nonnegative definite. $F_p''(\underset{\sim}{u}^k)$ is also nonnegative since it is a diagonal matrix. Therefore, $F_p''(\underset{\sim}{u}^k) + \lambda''S_q''(\underset{\sim}{u}^k)$ is nonnegative definite. Moreover, $F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)$ is nonsingular if $F_p''(\underset{\sim}{u}^k)$ is nonsingular, which is the case for $u_x^k \neq u_x''$ for all $x = 1,\ldots,n$. (If $u_x^k = u_x''$ for some x, we can always change u_x^k to $u_x''+\varepsilon$ with $\varepsilon \neq 0$ so as to have $F_p''(\underset{\sim}{u})$ nonsingular.)

Greville's [3] well-known graduated values for the case $p = q = 2$ can be obtained immediately from Theorem 5.3. In this case,

$$F_2'(\underset{\sim}{u}) + \lambda S_2'(\underset{\sim}{u}) = 2W(\underset{\sim}{u}-\underset{\sim}{u}'') + 2\lambda K^T K\underset{\sim}{u},$$

and

$$F_2''(\underset{\sim}{u}) + \lambda S_2''(\underset{\sim}{u}) = 2W + 2\lambda K^T K,$$

which is positive definite and hence nonsingular. Then, for any $\underset{\sim}{u}^o$,

$$\begin{aligned}
\underset{\sim}{u}' &= \underset{\sim}{u}^o - [F_2''(\underset{\sim}{u}^o) + \lambda S_2''(\underset{\sim}{u}^o)]^{-1}[F_2'(\underset{\sim}{u}^o) + \lambda S_2'(\underset{\sim}{u}^o)] \\
&= \underset{\sim}{u}^o - [2W + 2\lambda K^T K]^{-1}[2W(\underset{\sim}{u}^o-\underset{\sim}{u}'') - 2\lambda K^T K(\underset{\sim}{u}^o)] \\
&= \underset{\sim}{u}^o - [W + \lambda K^T K]^{-1}[(W + \lambda K^T K)(\underset{\sim}{u}^o) - W\underset{\sim}{u}''] \\
&= \underset{\sim}{u}^o - \underset{\sim}{u}^o + (W + \lambda K^T K)^{-1} W\underset{\sim}{u}'' \\
&= (W + \lambda K^T K)^{-1} W\underset{\sim}{u}'',
\end{aligned}$$

which is independent of $\underset{\sim}{u}^o$.

We should point out that $F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)$ is usually a huge matrix and should not be inverted. Instead, we can find $\underset{\sim}{u}^{k+1}$ by using the Choleski square-root method [3] to solve the equations

$$[F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)][\underset{\sim}{u}^{k+1}]$$

$$= [F_p''(\underset{\sim}{u}^k) + \lambda S_q''(\underset{\sim}{u}^k)][\underset{\sim}{u}^k] - [F_p'(\underset{\sim}{u}^k) + \lambda S_q'(\underset{\sim}{u}^k)].$$

An APL program for the computation by the Newton-Raphson procedure is provided on the next page.

```
      ∇GRAD[⎕]∇
      ∇ GRAD IV
[1]   CR←1
[2]   →0×⍳0=CR
[3]   FD←(P×W+.×(×IV−UV)×(|IV−UV)*P−1)+L×Q×(⍉K)+.×(×K+.×IV)×(|K+.×IV)*Q−1
[4]   A← 19 19 ⍴,(⍉(|IV−UV)*P−2), 19 19 ⍴0
[5]   B←J⍴,(⍉(|K+.×IV)*Q−2),J⍴0
[6]   SD←((P−1)×P×W+.×A)+(Q−1)×Q×L×(⍉K)+.×B+.×K
[7]   D←(⌹SD)+.×FD
[8]   AGV←IV←IV+0.1×UV=IV←IV−D
[9]   F←+/W+.×(|AGV−UV)*P
[10]  S←+/(|K+.×AGV)*Q
[11]  M←F+L×S
[12]  CR←(⌈/|D)≥0.00001
[13]  →2
      ∇


      P←2
      Q←4
      L←1
      J←16 16
      K←16 19⍴¯1 3 ¯3 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
      W←19 19⍴,(19 1⍴3 5 8 10 15 20 23 20 15 13 11 10 9 9 7 5 5 3 1),19 19⍴0
      UV←34 24 31 40 30 49 48 48 67 58 67 75 76 76 102 100 101 115 134
      IV←36 21 32 38 32 46 49 46 69 55 68 73 78 73 103 98 103 112 135


      GRAD IV

      AGV
30.53879314 28.73168986 30.50783422 34.13565801 38.06206907 43.61998412
      48.28064435 53.07742402 58.74084863 62.80011241 67.0156958 71.79349972
      76.76184235 83.65185761 91.8353774 98.96527157 106.2047082 115.5273646
      128.347123
```

K	Difference matrix of order z
W	n×n matrix with w_x as diagonal elements
L	Coefficient λ
IV	Intial trial solution
UV	Ungraduated values
AGV	Approximated graduated values

VI. THE "RIM" CASES ($p = 1$ or ∞, $1 < q < \infty$; $1 < p < \infty$, $q = 1$ or ∞)

The remaining cases for the solution of

$$\min_{\underset{\sim}{u}} F_p(\underset{\sim}{u}) + \lambda S_q(\underset{\sim}{u}) \qquad \text{(WH,p,q)}$$

are more complicated and will be discussed in this chapter. They are:

Case (i) $\quad 1 < p < \infty,\ q = 1,$

Case (ii) $\quad 1 < p < \infty,\ q = \infty,$

Case (iii) $\quad p = 1,\ 1 < q < \infty,$

Case (iv) $\quad p = \infty,\ 1 < q < \infty.$

We formulate these (WH,p,q) problems as follows:

Case (i): $1 < p < \infty,\ q = 1.$

$$\text{Min}\left[\sum_{x=1}^{n} w_x |u_x - u_x''|^p + \lambda \sum_{x=1}^{n-z} (R_x + T_x)\right]$$

subject to

$$\Delta^z u_x = R_x - T_x \qquad x = 1,\ldots,n-z$$

$$R_x \geq 0$$

$$T_x \geq 0.$$

Case (ii): $1 < p < \infty,\ q = \infty.$

$$\min_{\underset{\sim}{u},s}\left(\sum_{x=1}^{n} w_x |u_x - u_x''|^p + \lambda s\right)$$

subject to

$$\left.\begin{aligned} \Delta^z u_x &\le s \\ -\Delta^z u_x &\le s \end{aligned}\right\} \qquad x = 1,2,\ldots,n-z$$

Case (iii): $p = 1,\ 1 < q < \infty$.

$$\text{Min}\left[\sum_{x=1}^{n} w_x (P_x + N_x) + \lambda \sum_{x=1}^{n-z} |\Delta^z u_x|^q\right]$$

subject to

$$P_x - N_x = u_x - u''_x, \qquad x = 1,2,\ldots,n$$

$$P_x \ge 0$$

$$N_x \ge 0.$$

Case (iv): $p = \infty,\ 1 < q < \infty$.

$$\underset{\underset{\sim}{u},f}{\text{Min}}\left[f + \lambda \underset{x=1}{\overset{n-z}{\Delta}} |\Delta^z u_x|^q\right]$$

subject to

$$\left.\begin{aligned} u_x - u''_x &\le f \\ -u_x + u''_x &\le f \end{aligned}\right\} \qquad x = 1,\ldots,n.$$

From the above four formulations we see that we have in fact combinations of the methods used in Chapter 4 and Chapter 5. We can use the method of Lagrange Multipliers to find the optimal solutions for the above cases. In practice, however, these calculations are quite complicated since the constraints are inequalities rather than equalities. Therefore, we have considered only examples for which $p = 2$ in cases (i) and (ii) and $q = 2$ in cases (iii) and (iv), which allows us to compute the optimal solutions by using much simpler quadratic programming methods.

VII. EXAMPLES

Numerical examples are given in this chapter. The 19 ungraduated values and weights given by Miller [6] are graduated using $z = 3$.

Tables 1 - 5 give the graduated values of the special cases $p = q = 1$, $p = q = 2$, $p = q = 3$, $p = q = 5$ and $p = q = \infty$, respectively. Some patterns of the graduated values can be detected from observation of the results for the cases $p = q = 2$, $p = q = 3$ and $p = q = 5$.

Graduated values of the other cases are also given. Tables 6 - 9 show the graduated values of the cases ($p = 1$, $q = \infty$), ($p = \infty$, $q = 1$), ($p = 2$, $q = 4$) and ($p = 4$, $q = 2$), respectively.

The graduated values of the "rim" cases are also shown. Tables 10 and 11 give the graduated values of the cases ($p = 2$, $q = 1$) and ($p = 2$, $q = \infty$), respectively. These graduated values were computed by means of quadratic programming methods.

~~Figure~~ as is 1 compares values with the graduated values for the cases $p = q = 1$, $p = q = 2$ and $p = q = \infty$ when $z = 3$ and $\lambda = 3$. ~~Figure~~ as is 2 compares the ungraduated values for $(p,q) = (1,\infty)$, $(p,q) = (\infty,1)$, $(p,q) = (2,4)$ and $(p,q) = (4,2)$ when $z = 3$ and $\lambda = 1$.

Different values of λ are chosen in each case so that a large range of the values $F_p(\underset{\sim}{u}^\lambda)$ and $S_q(\underset{\sim}{u}^\lambda)$ is covered. Notice that the Monotone properties of the optimal solutions (discussed in Chapter 3) are satisfied in the above cases.

TABLE 1

Graduated Values when p = 1, q = 1 and z = 3

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 6$	$\lambda = 10$
			Graduated Values u^λ_x				
1	34	3	34.00	34.00	34.00	15.90	22.32
2	24	5	24.00	24.00	29.00	24.00	26.68
3	31	8	31.00	31.00	31.00	31.00	31.00
4	40	10	40.00	37.50	40.00	36.90	35.29
5	30	15	30.00	43.50	46.00	41.70	39.56
6	49	20	49.00	49.00	49.00	45.40	43.79
7	48	23	48.00	48.00	48.00	48.00	48.00
8	48	20	48.00	48.00	48.00	51.46	52.18
9	67	15	67.00	51.67	51.67	55.78	56.73
10	58	13	58.00	58.00	58.00	60.96	61.68
11	67	11	67.00	67.00	67.00	67.00	67.00
12	75	10	75.00	75.00	73.00	72.01	72.71
13	76	9	76.00	76.00	76.00	76.00	78.80
14	76	9	76.00	81.92	82.14	81.26	85.27
15	102	7	102.00	92.75	91.43	87.79	92.13
16	100	5	100.00	100.00	100.00	95.59	99.37
17	101	5	101.00	103.67	107.86	104.66	107.00
18	115	3	112.33	115.00	115.00	115.00	115.00
19	134	1	134.00	134.00	121.43	126.61	123.39
	Fit	$F_1(\underset{\sim}{u}^\lambda)$	8.00	588.83	691.14	833.20	872.63
	Smoothness	$S_1(\underset{\sim}{u}^\lambda)$	415.33	76.00	35.33	6.14	0.41
	$F_1(\underset{\sim}{u}^\lambda) + \lambda S_1(\underset{\sim}{u}^\lambda)$		423.33	740.83	797.14	870.07	876.76

TABLE 2

Graduated Values when p = 2, q = 2 and z = 3

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 6$	$\lambda = 10$
			Graduated Values u_x^λ				
1	34	3	31.65	31.17	30.94	30.58	30.30
2	24	5	27.57	28.31	28.61	28.96	29.12
3	31	8	30.98	30.76	30.68	30.64	30.69
4	40	10	34.86	34.28	34.08	33.91	33.88
5	30	15	35.95	36.93	37.33	37.76	37.93
6	49	20	45.40	44.66	44.30	43.85	43.62
7	48	23	48.16	48.21	48.25	48.30	48.33
8	48	20	51.38	52.10	52.44	52.87	53.09
9	67	15	61.04	59.98	59.53	58.99	58.73
10	58	13	62.19	62.68	62.83	62.90	62.88
11	67	11	66.86	67.00	67.05	67.10	67.11
12	75	10	72.65	72.06	71.86	71.72	71.73
13	76	9	75.63	75.98	76.21	76.58	76.81
14	76	9	81.75	82.60	82.94	83.30	83.44
15	102	7	94.76	93.53	92.93	92.10	91.66
16	100	5	100.69	100.11	99.80	99.37	99.13
17	101	5	104.18	105.08	105.55	106.20	106.53
18	115	3	114.00	114.55	114.89	115.40	115.68
19	134	1	132.07	130.36	129.38	127.98	127.25
	Fit	$F_2(\underset{\sim}{u}^\lambda)$	2903.96	3979.98	4501.05	5164.62	5490.81
	Smoothness	$S_2(\underset{\sim}{u}^\lambda)$	1235.52	452.15	236.62	73.20	29.96
	$F_2(\underset{\sim}{u}^\lambda) + \lambda S_2(\underset{\sim}{u}^\lambda)$		4139.48	4884.27	5210.90	5603.82	5790.43

TABLE 3

Graduated Valeus when $p = 3$, $q = 3$ and $z = 3$

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 6$	$\lambda = 10$
			Graduated Values u^{λ}_x				
1	34	3	30.91	30.71	30.60	30.42	30.29
2	24	5	28.00	28.24	28.36	28.53	28.64
3	31	8	30.97	30.73	30.63	30.51	30.45
4	40	10	34.46	34.14	34.00	33.81	33.71
5	30	15	36.14	36.53	36.72	36.98	37.13
6	49	20	44.31	43.98	43.81	43.57	43.43
7	48	23	48.43	48.43	48.44	48.46	48.47
8	48	20	52.64	52.98	53.16	53.41	53.57
9	67	15	60.95	60.56	60.37	60.09	59.92
10	58	13	62.82	63.07	63.18	63.31	63.38
11	67	11	66.39	66.46	66.51	66.60	66.72
12	75	10	71.39	71.09	70.96	70.85	70.83
13	76	9	74.72	74.97	75.13	75.44	75.68
14	76	9	82.24	82.74	82.99	83.33	83.52
15	102	7	94.92	94.34	94.03	93.56	93.25
16	100	5	101.62	101.22	101.00	100.69	100.51
17	101	5	105.46	105.93	106.19	106.60	106.88
18	115	3	113.64	114.22	114.57	115.15	115.51
19	134	1	130.07	129.36	129.03	128.55	128.22
	Fit	$F_3(\underset{\sim}{u}^{\lambda})$	20104.45	24600.43	27071.49	30874.67	33295.07
	Smoothness	$S_3(\underset{\sim}{u}^{\lambda})$	5845.65	2593.11	1575.18	652.77	335.08
	$F_3(\underset{\sim}{u}^{\lambda}) + \lambda S_3(\underset{\sim}{u}^{\lambda})$		25950.09	29786.64	31797.02	34791.32	36645.88

TABLE 4

Graduated Values when p = 5, q = 5 and z = 3

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 6$	$\lambda = 10$
			Graduated Values u_x^λ				
1	34	3	30.12	30.03	29.99	29.92	29.89
2	24	5	28.49	28.60	28.65	28.72	28.76
3	31	8	31.60	31.45	31.36	31.22	31.12
4	40	10	34.33	34.19	34.11	33.99	33.92
5	30	15	36.10	36.27	36.36	36.50	36.59
6	49	20	43.63	43.47	43.39	43.26	43.17
7	48	23	48.56	48.57	48.57	48.54	48.51
8	48	20	53.33	53.48	53.56	53.70	53.79
9	67	15	60.99	60.82	60.73	60.60	60.52
10	58	13	63.27	63.42	63.49	63.58	63.61
11	67	11	66.63	66.90	66.97	66.94	66.84
12	75	10	70.19	70.02	69.94	69.83	69.78
13	76	9	73.31	73.42	73.51	73.74	73.95
14	76	9	82.47	82.70	82.83	83.03	83.17
15	102	7	95.14	94.88	94.73	94.49	94.31
16	100	5	102.57	102.30	102.15	101.93	101.79
17	101	5	106.41	106.65	106.78	107.01	107.17
18	115	3	113.45	113.90	114.15	114.56	114.83
19	134	1	128.85	128.61	128.47	128.25	128.09
	Fit	$F_5(\underset{\sim}{u}^\lambda)$	805003	937766	1015917	1148647	1244947
	Smoothness	$S_5(\underset{\sim}{u}^\lambda)$	189895	94220.74	62095.73	30156.85	17579.92
	$F_5(\underset{\sim}{u}^\lambda) + \lambda S_5(\underset{\sim}{u}^\lambda)$		994899	1126207	1202204	1329589	1420746

TABLE 5

Graduated Values when $p = \infty$, $q = \infty$ and $z = 3$

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 6$	$\lambda = 10$
			Graduated Values u^{λ}_x				
1	34	1	24.94	24.81	24.56	24.39	24.39
2	24	1	27.07	27.41	27.68	27.90	27.91
3	31	1	30.31	30.79	31.26	31.64	31.64
4	40	1	34.40	34.78	35.21	35.55	35.56
5	30	1	39.06	39.19	39.44	39.61	39.61
6	49	1	43.99	43.86	43.86	43.79	43.79
7	48	1	48.93	48.60	48.38	48.12	48.12
8	48	1	53.58	53.26	52.92	52.64	52.64
9	67	1	57.94	57.81	57.56	57.39	57.39
10	58	1	62.28	62.44	62.40	62.39	62.39
11	67	1	66.89	67.32	67.51	67.68	67.68
12	75	1	72.03	72.62	73.00	73.29	73.29
13	76	1	77.99	78.52	78.94	79.26	79.26
14	76	1	85.06	85.19	85.44	85.61	85.61
15	102	1	92.94	92.81	92.56	92.39	92.39
16	100	1	101.37	101.20	100.24	99.62	99.62
17	101	1	110.06	110.19	108.37	107.34	107.34
18	115	1	118.73	119.60	116.87	115.59	115.59
19	134	1	127.66	129.62	125.66	124.39	124.39
	Fit	$F_\infty(\underset{\sim}{u}^\lambda)$	9.06	9.19	9.43	9.61	9.61
	Smoothness	$S_\infty(\underset{\sim}{u}^\lambda)$	0.28	0.17	0.09	0.03	0.03
	$F_\infty(\underset{\sim}{u}^\lambda) + \lambda S_\infty(\underset{\sim}{u}^\lambda)$		9.34	9.54	9.70	9.81	9.95

TABLE 6

Graduated Values when $p = 1$, $q = \infty$ and $z = 3$

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 5$	$\lambda = 10$	$\lambda = 15$	$\lambda = 20$
			Graduated Values u^{λ}_x				
1	34	3	34.00	34.00	34.00	34.00	34.00
2	24	5	24.00	24.00	24.00	24.00	26.63
3	31	8	31.00	31.00	31.00	31.00	31.00
4	40	10	40.00	40.00	40.00	40.00	39.23
5	30	15	30.00	30.00	30.67	41.33	43.45
6	49	20	49.00	49.00	49.00	49.00	49.00
7	48	23	48.00	48.00	48.00	48.00	48.00
8	48	20	48.00	48.00	48.00	48.00	48.00
9	67	15	67.00	67.00	67.00	56.33	53.96
10	58	13	58.00	58.00	58.00	58.00	58.00
11	67	11	67.00	67.00	67.00	67.00	67.00
12	75	10	75.00	75.00	75.00	75.00	75.00
13	76	9	76.00	76.00	76.00	76.00	76.00
14	76	9	76.00	76.00	76.00	76.00	77.88
15	102	7	102.00	100.33	99.67	89.00	88.50
16	100	5	100.00	100.00	100.00	100.00	100.00
17	101	5	101.00	101.00	101.00	103.67	106.20
18	115	3	115.00	115.00	115.00	115.00	115.00
19	134	1	134.00	134.00	134.00	134.00	134.00
	Fit	$F_1(\underset{\sim}{u}^{\lambda})$	0.00	11.66	26.33	434.33	555.66
	Smoothness	$S_{\infty}(\underset{\sim}{u}^{\lambda})$	54.00	49.00	47.00	15.00	7.88
	$F_1(\underset{\sim}{u}^{\lambda}) + \lambda S_{\infty}(\underset{\sim}{u}^{\lambda})$		54.00	256.66	496.33	659.33	713.16

TABLE 7

Graduated Values when $p = \infty$, $\underline{\sigma} = 1$ and $z = 3$

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
			Graduated Values u^{λ}_x				
1	34	1	26.85	24.52	24.52	25.35	25.35
2	24	1	27.69	27.97	27.97	28.57	28.57
3	31	1	29.69	31.42	31.62	32.01	32.01
4	40	1	32.85	35.19	35.45	35.67	35.67
5	30	1	37.15	39.24	39.48	39.57	39.57
6	49	1	42.62	43.59	43.70	43.69	43.69
7	48	1	49.23	48.23	48.12	48.04	48.04
8	48	1	54.98	52.95	52.72	52.62	52.62
9	67	1	59.85	57.76	57.52	57.43	57.43
10	58	1	63.84	62.65	62.51	62.46	62.46
11	67	1	66.96	67.62	67.69	67.73	67.73
12	75	1	69.21	72.67	73.07	73.22	73.22
13	76	1	74.61	78.55	79.00	79.17	79.17
14	76	1	83.15	85.24	85.48	85.57	85.57
15	102	1	94.85	92.76	92.52	92.43	92.43
16	100	1	103.18	101.09	100.11	99.75	99.74
17	101	1	108.15	110.03	108.26	107.52	107.52
18	115	1	116.04	120.02	116.96	115.75	115.75
19	134	1	126.85	130.85	126.22	124.43	124.48
	Fit	$F_{\infty}(\underset{\sim}{u}^{\lambda})$	7.15	9.24	9.48	9.57	9.57
	Smoothness	$S_1(\underset{\sim}{u}^{\lambda})$	18.82	0.94	0.36	0.23	0.23
	$F_{\infty}(\underset{\sim}{u}^{\lambda}) + \lambda S_1(\underset{\sim}{u}^{\lambda})$		9.04	9.52	9.66	9.80	10.03

TABLE 8

Graduated Values when $p = 2$, $\sigma = 4$ and $z = 3$

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
			Graduated Values u_x^λ				
1	34	3	31.38	30.95	30.78	30.54	30.31
2	24	5	28.19	28.48	28.59	28.73	28.85
3	31	8	30.33	30.39	30.44	30.50	30.58
4	40	10	34.34	34.21	34.18	34.14	34.11
5	30	15	37.72	37.92	37.99	38.06	38.12
6	49	20	44.06	43.83	43.73	43.62	43.52
7	48	23	48.23	48.26	48.27	48.28	48.29
8	48	20	52.67	52.90	52.98	53.08	53.15
9	67	15	59.20	58.94	58.85	58.74	58.65
10	58	13	62.68	62.76	62.78	62.80	62.81
11	67	11	66.99	67.00	67.00	67.01	67.04
12	75	10	72.08	71.90	71.85	71.79	71.76
13	76	9	76.34	76.56	76.65	76.76	76.85
14	76	9	83.30	83.52	83.59	83.65	83.69
15	102	7	92.63	92.21	92.04	91.84	91.66
16	100	5	99.40	99.14	99.06	89.97	98.90
17	101	5	105.51	105.87	106.02	106.20	106.37
18	115	3	114.88	115.23	115.37	115.53	115.66
19	134	1	130.17	129.22	128.83	128.35	127.93
	Fit	$F_2(\underset{\sim}{u}^\lambda)$	4746.86	5116.96	5257.08	5419.55	5554.51
	Smoothness	$S_4(\underset{\sim}{u}^\lambda)$	3072.96	772.34	404.56	167.69	69.29
	$F_2(\underset{\sim}{u}^\lambda) + \lambda S_4(\underset{\sim}{u}^\lambda)$		5054.16	5348.66	5459.36	5587.24	5693.10

TABLE 9

Graduated Values when p = 4, q = 2 and z = 3

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 10$	$\lambda = 100$	$\lambda = 200$	$\lambda = 300$
			Graduated Values u_x^{λ}				
1	34	3	32.66	31.36	29.94	29.70	29.61
2	24	5	25.48	27.03	28.75	29.05	29.16
3	31	8	32.13	32.39	31.59	31.28	31.11
4	40	10	38.02	36.39	34.42	33.97	33.77
5	30	15	31.98	33.71	35.88	36.41	36.66
6	49	20	47.38	45.94	44.09	43.62	43.40
7	48	23	48.39	48.48	48.48	48.48	48.48
8	48	20	49.60	51.04	52.89	53.36	53.60
9	67	15	65.04	63.33	61.19	60.68	60.43
10	58	13	59.80	61.29	63.07	63.41	63.52
11	67	11	65.96	65.79	66.38	66.51	66.56
12	75	10	73.85	72.37	70.62	70.34	70.29
13	76	9	74.63	73.99	74.26	74.68	74.98
14	76	9	78.10	79.82	82.03	82.66	82.98
15	102	7	99.70	97.85	95.42	94.65	94.22
16	100	5	101.75	102.70	102.33	101.89	101.62
17	101	5	102.30	103.74	105.96	106.69	107.09
18	115	3	113.63	112.83	114.00	114.83	115.30
19	134	1	134.79	131.88	128.97	128.23	127.83
	Fit	$F_4(\underset{\sim}{u}^{\lambda})$	1564.00	18156.22	112334	160539	189039
	Smoothness	$S_2(\underset{\sim}{u}^{\lambda})$	7795.14	3514.73	642.76	297.33	179.29
	$F_4(\underset{\sim}{u}^{\lambda}) + \lambda S_2(\underset{\sim}{u}^{\lambda})$		9359.14	53303.87	176610	220005	242824

TABLE 10

Graduated Values when p = 2, q = 1 and z = 3

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 5$	$\lambda = 10$	$\lambda = 15$	$\lambda = 20$
			Graduated Values u^λ_x				
1	34	3	33.83	33.17	32.33	31.50	30.67
2	24	5	24.20	25.00	26.00	27.00	28.18
3	31	8	31.06	31.31	31.63	31.94	31.91
4	40	10	39.70	38.50	37.00	35.50	34.27
5	30	15	30.27	31.33	32.66	34.00	35.27
6	49	20	48.85	48.25	47.50	46.75	46.00
7	48	23	48.00	48.00	48.00	48.00	48.00
8	48	20	48.15	48.75	49.50	50.25	51.00
9	67	15	66.73	65.67	64.33	63.00	61.67
10	58	13	58.23	59.15	60.31	61.46	62.67
11	67	11	67.00	67.00	67.00	67.00	66.80
12	75	10	74.80	74.00	73.00	72.00	71.22
13	76	9	76.00	76.00	76.00	76.00	75.92
14	76	9	76.33	77.67	79.33	81.00	82.45
15	102	7	101.57	99.86	97.71	95.57	94.24
16	100	5	100.00	100.43	101.10	101.77	101.37
17	101	5	101.50	102.20	102.70	103.20	103.85
18	115	3	114.33	113.83	113.83	113.83	114.21
19	134	1	135.50	135.33	134.50	133.66	132.45
	Fit	$F_2(\underset{\sim}{u}^\lambda)$	12.51	204.25	787.46	1759.44	2935.42
	Smoothness	$S_1(\underset{\sim}{u}^\lambda)$	51.72	42.14	33.52	25.75	18.46
	$F_2(\underset{\sim}{u}^\lambda) + \lambda S_1(\underset{\sim}{u}^\lambda)$		64.23	414.95	1122.66	2146.19	3304.62

TABLE 11

Graduated Values when $p = 2$, $q = \infty$ and $z = 3$

x	Ungraduated Values u''_x	Weights w_x	$\lambda = 1$	$\lambda = 10$	$\lambda = 100$	$\lambda = 200$	$\lambda = 300$
			Graduated Values u_x^λ				
1	34	3	34.00	34.00	34.00	32.87	30.79
2	24	5	24.00	24.00	24.00	26.04	28.26
3	31	8	31.00	31.00	31.19	30.49	30.43
4	40	10	40.00	39.81	37.69	35.69	34.60
5	30	15	30.00	30.38	34.02	36.84	38.05
6	49	20	49.00	48.71	46.14	44.48	43.49
7	48	23	48.00	48.07	48.10	48.08	48.21
8	48	20	48.00	48.04	50.59	52.30	53.16
9	67	15	67.00	66.94	63.18	60.33	58.86
10	58	13	58.00	58.02	59.91	61.64	62.60
11	67	11	67.00	67.00	66.74	66.76	67.09
12	75	10	75.00	75.00	75.00	73.67	71.84
13	76	9	75.94	75.69	74.80	75.27	76.51
14	76	9	76.17	76.93	79.60	82.11	83.83
15	102	7	101.79	100.80	97.37	94.21	92.19
16	100	5	100.10	100.56	102.16	101.04	98.90
17	101	5	101.00	101.00	101.00	103.92	105.71
18	115	3	115.00	115.00	115.00	113.38	115.36
19	134	1	135.00	135.00	135.00	135.00	130.54
	Fit	$F_2(\underset{\sim}{u}^\lambda)$	1.65	25.54	1165.08	3733.37	5254.74
	Smoothness	$S_\infty(\underset{\sim}{u}^\lambda)$	52.70	46.74	25.96	10.54	2.71
	$F_2(\underset{\sim}{u}^\lambda) + \lambda S_\infty(\underset{\sim}{u}^\lambda)$		54.35	493.24	3761.08	5483.37	6076.74

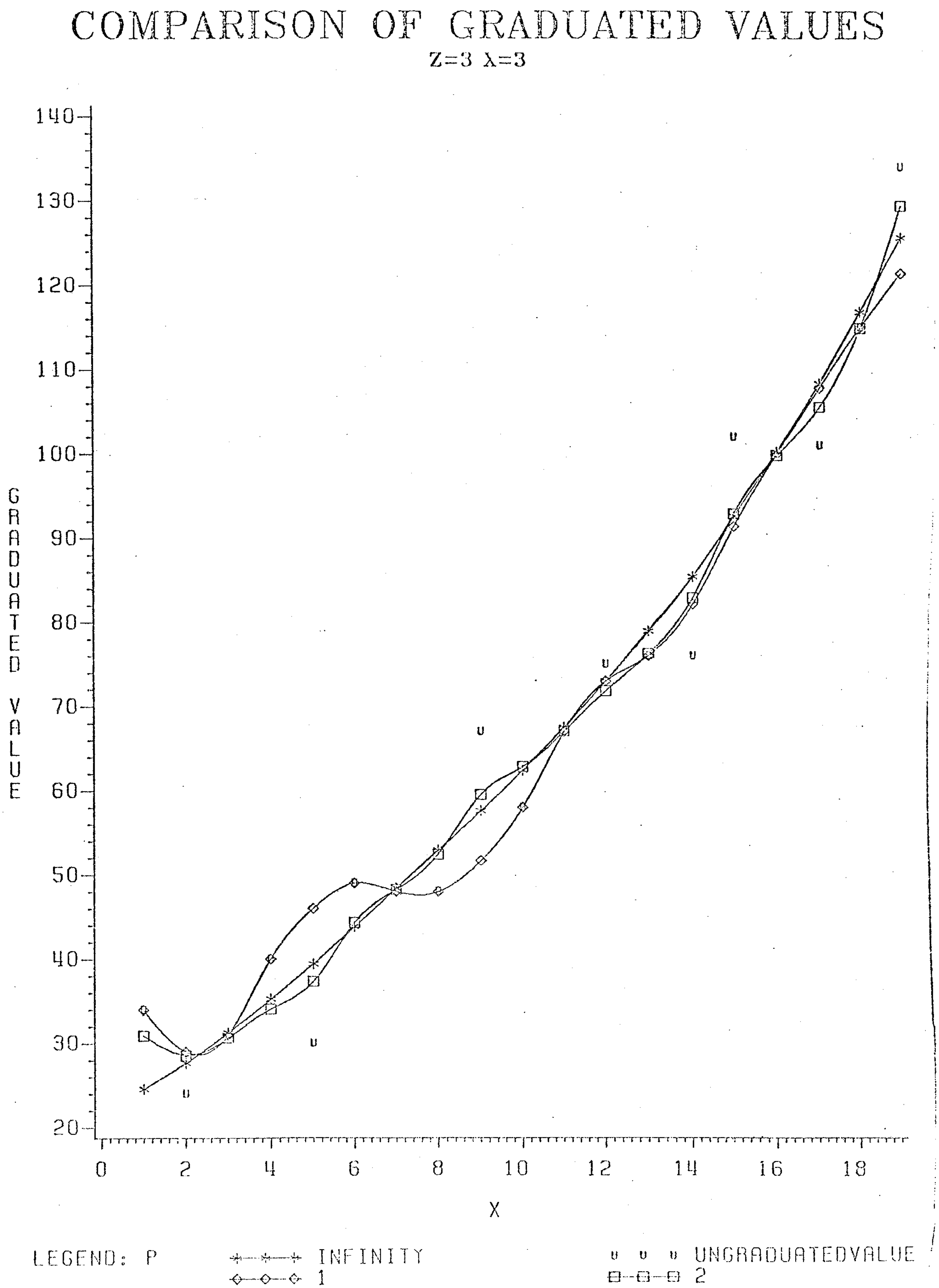
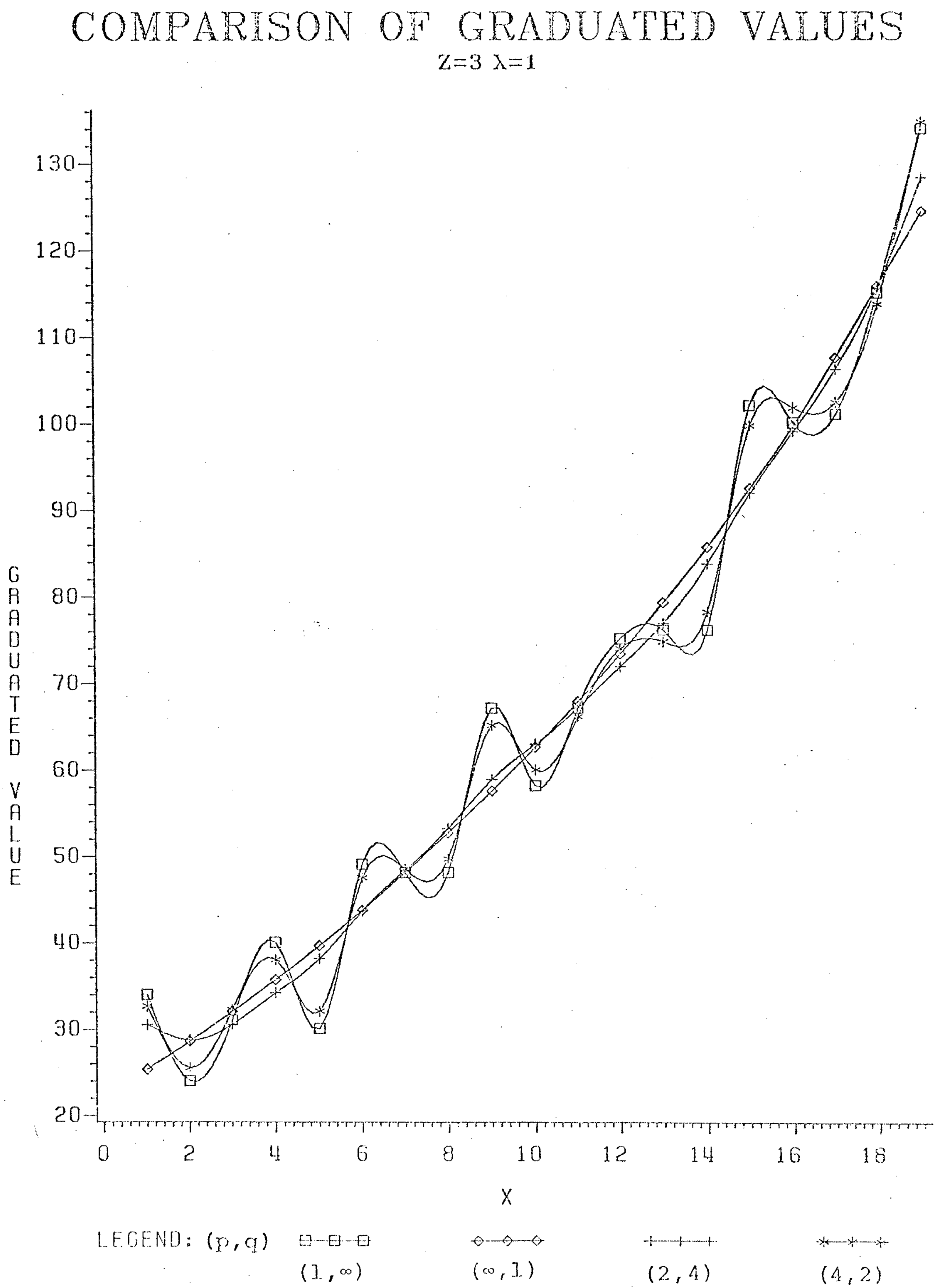

COMPARISON OF GRADUATED VALUES
Z=3 λ=3
GRADUATED VALUE
X
LEGEND: P
INFINITY
1
UNGRADUATEDVALUE
2

Figure 2


COMPARISON OF GRADUATED VALUES
Z=3 λ=1
GRADUATED VALUE
130
120
110
100
90
80
70
60
50
40
30
20
0
2
4
6
8
10
12
14
16
18
X
LEGEND: (p,q)
(1,∞)
(∞,1)
(2,4)
(4,2)

VIII. CONCLUDING REMARKS

Notice that there is inconsistency between the definition of the F- and S-functions as given in Chapter 1 for finite and for infinite values of p and q: we used the ℓ_∞- norms of $\underset{\sim}{u} - \underset{\sim}{u}''$ and $\Delta^z \underset{\sim}{u}$ for p and q infinite, but the p^{th} resp. q^{th} *powers* of the ℓ_p- and ℓ_q- norms for finite p and q.

This finds its origin in the fact that Whittaker used the sums of squares, i.e. the p^{th} power of the ℓ_p-norms for $p = 2$; here we have continued using the p^{th} power of the ℓ_p-norms as a direct extension from $p = 2$ to all values ≥ 1; since this is obviously impossible for $p = \infty$, we have used the ℓ_p-norm itself for infinite p.

A more elegant approach might have resulted from alternative definitions of F_p and S_q consistent with those of F_∞ and S_∞; F_p would be defined as the (weighted) ℓ_p-norm of $\underset{\sim}{u} - \underset{\sim}{u}''$, and S_q as the ℓ_q-norm of $\Delta^z \underset{\sim}{u}$ (rather than the p^{th} or q^{th} powers of these norms):

$$\left.\begin{aligned} "F_p" &= \left(\sum_{x=1}^{n} w_x |u_x - u''_x|^p \right)^{1/p}, \\ "S_q" &= \left(\sum_{x=1}^{n-z} |\Delta^z u_x|^q \right)^{1/q}. \end{aligned}\right. \qquad (8.1)$$

If we introduce the notation

$$\left.\begin{aligned} "F_\infty" &= \lim_{p\to\infty} "F_p", \\ "S_\infty" &= \lim_{q\to\infty} "S_q", \end{aligned}\right\}$$

we see that we need only (8.1) to define F_p and S_q consistently for all p, q ($1 \leq p, q \leq \infty$).

Note (see Chapter II, pp. 5-7) that the new definition does not change anything for infinite p and q:

$$"F_\infty(\underset{\sim}{u})" = F_\infty(\underset{\sim}{u}) = \max_{1\leq x\leq n} |\underset{\sim}{u} - \underset{\sim}{u}"|,$$

$$"S_\infty(\underset{\sim}{u})" = S_\infty(\underset{\sim}{u}) = \max_{1\leq x\leq n-z} |\Delta^{z}\underset{\sim}{u}_x|$$

as before.

A practical advantage of the new definition is that the functions F_p and S_q would be of the same order of magnitude for all p and q instead of, as before, growing exponentially with increasing p and q.

The theoretical implications of this more consistent and, therefore, more elegant approach have, however, not been explored here. But, they will be part of a further study.

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