ESSAYS ON THE ECONOMICS OF UNCERTAINTY: AN INVESTIGATION INTO THE IMPACTS OF INCREASING UNCERTAINTY FOR FIVE RELATED CLASSES OF MODELS

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In Partial Fulfillment

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by

Robert Alexander Sproule

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DEDICATION

This dissertation is dedicated to my parents who on occasion have expressed an appreciation of formal thought. They are M.O. Sproule (nee Kerrigan) who at one time played J.S. Bach's Preludio XXI without flaw, and Group Captain J.A. Sproule, D.F.C. (R.C.A.F, retired) who on several occasions has professed an affinity for the mathematics of navigation, spherical trigonometry.

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Secondly, I wish to acknowledge the assistance of Charlene Hill, Department of Economics, University of Alberta. Her skill with the Philips MICOM word processor has made my task significantly lighter.

Finally, I wish to acknowledge the assistance of my wife, Joan Sigrid Johnson. Her support (which came in many forms) proved absolutely necessary to the completion of this endeavor.

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ABSTRACT

This dissertation provides an investigation into the impact of increasing uncertainty for five related classes of models. These classes of models are related in the sense that all are choice theoretical models, their objective function contains two or three arguments, and the decision-maker has two choice variables. These classes of models are seen as different because of differences in the definition of the arguments in the objective functions, or the definitions of the choice variables.

In our determination of the impact of increasing uncertainty for any one class of model, results from one or more of several other models are employed. These results are presented in Sandmo (1970 and 1971), Block and Heineke (1973), Ishii (1977), Hanson and Menezes (1978), and Tressler and Menezes (1980). These results are derived from models which contain one or two arguments, and contain one choice variable. By comparison with the five related classes of models which are the object of concern in this dissertation, these models are similar. In fact, these models, and their related results, may be seen as continuents of the five classes of models to be investigated. It is because of this that these models and their results prove useful in our determination of the impact of increasing uncertainty. The contributions of this dissertation by Chapter are as follows.

Chapter III extends Cowell's (1981) analysis of the "safe work, risky work, and leisure" decision by defining the restrictions on the utility function which are needed to render unambiguous the effects of a "Sandmo-type" increase in uncertainty of the wage-rate paid to risky work on the optimal allocation of labor supply. In this investigation, results from Block and Heineke (1973), and Tressler and Menezes (1980) are employed.

In a two-period model, Sandmo (1969) considers the comparative-static effects of a mean-preserving increase in the uncertainty of the return on the risky asset on the optimal allocation of initial wealth to first-period consumption, the safe asset, and the risky asset. Sandmo concludes that these effects are indeterminate without further restrictions on the utility function. Chapters IV and V propose two alternative solutions to this problem. The solution offered in Chapter IV is based on results from Sandmo (1970) and Hanson and Menezes (1978). The solution offered in Chapter V is based on results from Block and Heineke (1973), and Tressler and Menezes (1980).

In a choice-theoretic framework of criminal activity, Block and Heineke (1975a) consider the comparative-static effect of a mean-preserving increase in the uncer-

tainty of the rate of capture or arrest. They determine that two restrictions on the utility function are required to obtain an unambiguous result. The purpose of Chapter VI is twofold. Using the work of Block and Heineke (1973) and Tressler and Menezes (1980), this Chapter assesses the robustness of the result obtained by Block and Heineke, and secondly extends the inquiry of Block and Heineke by determining four other comparative-static effects related to the same problem.

In empirical research into the developments of tax evasion, one of several regularities has emerged: information about the tax system in a contributing factor to both the decision to evade taxes, and the amount of income underreported. Chapter VII provides a choice theoretical model of tax evasion and labor supply whose intended purpose is to provide a formal analysis of these phenomena. Using the Isachsen and Strom (1980) model of tax evasion and labor supply, this Chapter assesses the total and marginal effects of imperfect information about the parameters of the tax system. Our determination of the marginal effects is enabled by the use of restrictions on the utility function proposed by Block and Heineke (1973) and Tressler and Menezes (1980).

Chapter VIII extends the literature on the economics of labor supply under uncertainty by adapting the Leuthold (1968) and Kosters (1966) models of the family labor supply decision to a stochastic environment. The total and marginal effects of uncertainty are determined for both model-types. In the determination of the marginal effects, the results of Block and Heineke (1973) and Tressler and Menezes (1980) are employed.

Chapter IX provides a solution to a problem posed by Block and Heineke (1972 and 1975b). The problem is the identification of plausible restrictions on the utility function which permit the signing of the comparative-static effects of a "Sandmo-type" increase in interest-rate uncertainty on the optimal levels of labor supply and savings. Our investigation is enabled by the work of Sandmo (1968), Mirman (1971), Block and Heineke (1973), Levhari and Weiss (1974), and Hanson and Menezes (1978).

Chapter X presents a model of the owner-managed firm under output-price uncertainty. The results obtained here are twofold. The first is the effect of risk preference on the optimal behavior of the OMF. The second is the comparative-static effects of both an additive shift, and a multiplicative shift, in the distribution (or a "Sandmo-type" increase in the uncertainty) on the optimal level of output and the optimal level of owner-manager effort. In the determination of these comparative static effects, results from Sandmo (1971), Block and Heineke (1973), and Ishii (1977) are employed.

CHAPTER I

INTRODUCTION

With the appearance of Kenneth Arrow's Aspects of the Theory of Risk-Bearing in 1966, the literature on the economics of uncertainty has grown in exponential proportions. So great has been this growth that of late numerous surveys of the literature have been apparently warranted, and prepared.[1] The question we well might ask is: What has motivated these efforts?

An insightful answer to this question is found in a recent introductory paper by Yasuhiro Sakai (1977a). Sakai writes:

> "The theory of economic science consists to a large extent of theories about how individuals try to find the 'best' or 'optimal' decision among possibly an infinite number of alternatives, subject to a certain environment over which they have no control. It is more realistic and reasonable to assume that full information of the environment is not available to decision makers, presumably because its random nature *per se*, or because of human ignorance, or because of both...
>
> (It) is true that the old, deterministic approach is apart from economic reality because of relying upon the assumption of complete information and perfect foresight, and is less satisfactory than the new, stochastic approach. But, if we reckon the former approach merely as an old, useless piece in the 'economics museum', we would commit a serious mistake of overestimating the power and scope of the latter. The time-honored, sound and solid basis that is characteristic of the traditional approach

> should be reexamined and retained in a broader perspective. A more balanced view of this point would be that the stochastic approach aims at a generalization of the deterministic approach in the sense that the latter deals with a special case in which everything is 100 percent sure.
>
> (In) introducing uncertainty factors into economic models, we might face the unwanted danger of establishing 'uncertain economics' rather than uncertainty economics. If we regard everything as being completely uncertain and incomprehensible to us, we would fail with 100 percent probability to establish any sort of constructive proposition-and economic science would be labelled as the dismal science...
>
> Summing up, the significance of uncertainty economics resides in the fact that, being more general than certainty economics, it succeeds in offering a better explanation of well known results in economic theory and also giving a new insight into some hitherto unknown results that would not be otherwise obtainable" (pp. 51-52).

Not only does Sakai's observation explain the growth of the literature, but it also provides clear justification for the main objectives for this dissertation. These objectives are two-fold. The first is the adaptation of several existing deterministic (or non-stochastic) models to a stochastic environment. Our particular concern here is the definition of the assumption-set needed to sign the impact of increasing uncertainty on the optimal choices of the decision maker in such models.

The second is the definition of the assumption-set needed to sign the impact of increasing uncertainty on the optimal choices of the decision maker for several existing stochastic models. In all such cases considered, we solve a problem which was previously posed, and in most cases left unsolved.

All in all, this dissertation represents the investigation into the impacts of increasing uncertainty on five related classes of models. In order that we may define the domain of inquiry in greater detail and with more clarity of vision, an overview to choice models in the economics of uncertainty is required. This overview is found in the following taxonomy.

I-1 A Taxonomy of Partial, Partial Equilibrium Models

In the economics of uncertainty, there is a class of models which Michael Rothschild (1973, p. 1288) terms partial, partial equilibrium (PPE) models. He characterizes this class of models as those which explain how economic agents should react to a variation in the economic environment. But, he adds, these models do not explain where this variation comes from or what, if anything, preserves it. Although effort has been expended on the development of partial equilibrium models (e.g., Appelbaum and Lim (1982), and Tressler and Menezes (1983)) and general equilibrium models (e.g., Sakai (1978)), most of the literature on the economics of uncertainty is comprised of PPE models.

These so-called PPE models may be categorized according to the number of arguments in the objective function, and according to the number of choice variables available to the decision-maker. A simple taxonomy which captures these facts is presented in Table 1.1. For each

TABLE 1.1

A TAXONOMY OF PARTIAL, PARTIAL EQUILIBRIUM MODELS SHOWING EXAMPLES

Number of Choice Variables	Number of Arguments in the Objective Function		
	1	2	3
1	- Rothschild and Stiglitz (1970,1971) - Diamond and Stiglitz (1974) - Sandmo (1971) - Kraus (1979) - Katz (1981) (1x1)	- Sandmo (1970) - Block and Heineke (1973) - Hanson and Menezes (1978) - Tressler and Menezes (1980) (2x1)	- (3x1)
2	- Batra and Ullah (1974) - Holthausen (1976,1979) - Feder <u>et.al.</u> (1980) - Kon (1983) (1x2)	- Sandmo (1969) - Block and Heineke (1975a) - Levhari and Weiss (1974) - Cowell (1981) (2x2)	- Block and Heineke (1972, 1975b) (3x2)

entry in this taxonomy, examples of such models are offered. An example of one model having one argument in the objective function, and one choice variable, is Sandmo's (1971) model of the competitive firm under output-price uncertainty. The objective function for Sandmo's model is the expected utility of the firm's profit, and the single choice variable is the level of output.

For ease of reference, the following labelling system is used to identify model classes in the above taxonomy. A model with i number of arguments in the objective function, and j number of choice variables, is said to be a model having a dimension of ixj. Thus, for example, the Sandmo (1971) model may be termed a model of dimension 1x1.

This taxonomy is useful for one reason. It shows at a glance the subset of the PPE models which are of central concern to this dissertation: <u>This thesis will investigate the impact of increasing uncertainty on optimal behavior in several models of dimension 2x2 or 3x2</u>.

Before more can be said about our methodology, or contributions to the literature, two statements are required. The first is a definition of the solution to the general comparative-static problem for models of dimension 2x2. The second is a terse review of approaches which have been offered in the literature to the determination of the comparative-static effects of increasing uncertainty for models of dimension 2x2 and 3x2.

I-2 A General Definition of the Comparative-Static Effects of Increasing Uncertainty for Models of Dimension 2x2

Consider the random variable, z. Suppose a decision-maker formulates a subjective probability density function (SPDF) for z, i.e., formulates $f(z,v,e)$ where v is a parameter of $f(\cdot)$ which defines the mean of the family of distributions having the same variance, and e is a parameter of $f(\cdot)$ which defines the variance of the family of distributions having the same mean.

Consider next an expected utility function (EUF), $V(\cdot)=EU(\cdot)$ defined over (ϕ,γ)-space such that $U'>0$ and $U''<0$. Suppose that ϕ and γ are functions of two choice variables x_j for $j=1,2$, i.e., $\phi=\phi(x_1,x_2)$ and $\gamma=\gamma(x_1,x_2,z)$ such that $\phi'<0$, $\phi''=0$, $\gamma'>0$, and $\gamma''=0$. The problem for the decision-maker is the choice of a particular pair, (x_1^*,x_2^*), which maximizes the EUF, i.e., the optimization problem for models of dimension 2x2 is:

$$\max_{x_1,x_2} V(x_1,x_2,e) = \int U(\phi(x_1,x_2),\gamma(x_1,x_2,z))f(z,v,e)dz \qquad (1)$$

The first order conditions for a maximum are $V_1=V_2=0$. The second order conditions are $V_{11}<0$, $V_{22}<0$, and $[V_{11}-V_{22}-(V_{12})^2] = |H|>0$. Thus provided that the first and second order conditions hold, there exists a unique pair, (x_1^*,x_2^*), which satisfies Equation (1).

Of central concern is the comparative-static effect of a mean-preserving increase in the uncertainty of z, (i.e., the effect of an increase in e) on the optimal-pair.

In formal terms, the comparative-static problem for this 2x2 model is:[2]

$$dx_1^*/de = \{V_{23} \cdot V_{12} - V_{13} \cdot V_{22}\} / |H| \qquad (2)$$

$$dx_2^*/de = \{V_{13} \cdot V_{12} - V_{23} \cdot V_{11}\} / |H| \qquad (3)$$

Clearly, the unambiguous signing of Equations (2) and (3) requires that two conditions be met:[3]

<u>Condition 1</u>: V_{13}, V_{23}, and V_{12} be of unambiguous sign.

<u>Condition 2</u>: Either: (a) sign (V_{13}) = sign $(-V_{23})$ for $V_{12}<0$, or (b) sign (V_{13}) = sign (V_{23}) for $V_{12}>0$.

Clearly, there is an <u>asymmetrical</u> relationship between Conditions 1 and 2. That is, if Condition 1 fails, then Condition 2 by necessity must fail. However, Condition 2 might fail regardless of whether Condition 1 is met or not.

I-3 <u>Four Previously Proposed Solutions to this Comparative-Static Problem</u>

The literature offers four solutions to the comparative-static problem for 2x2 and 3x2 models. The first solution is one of simply acknowledging that any one or more of V_{13}, V_{23}, or V_{12} can <u>not</u> be signed unambiguously, and therefore the comparative-static results defined by Equations (2) and (3) are ambiguous in sign. Examples of this fact are offered by Sandmo (1969, p. 598), and Block and Heineke (1972, p. 525; 1975b, p. 529).

The second approach is to impose strong restrictions on the expected utility function in order to obtain unambiguous results. By assuming separability of the utility function in their analysis of the effects of increasing uncertainty, Levhari and Weiss (1974, pp. 956-59) provide an example of this approach.

A third approach is to restrict the dimension of the decision being considered. For example, by assuming $\Delta x_2=0$, one might inspect the effect, $dx_1/de \big|_{\Delta x_2=0}$. Alternatively, rather than inspect the effects of e on an interior solution (i.e., the effects of e on (x_1^*, x_2^*) for $x_1>0$ and $x_2>0$), the problem may be (in some sense) simplified by considering the effects of e on a corner solution (e.g., the effects of e on $(x_1^*, 0)$ for $x_1>0$ and $x_2=0$, or the alternative). An example of the former is provided by Holthausen (1979, p. 991),[4] and an example of the latter is provided by Cowell (1981, pp. 369-71).[5]

A fourth approach to the problem is to transform the objective function by rewriting it in its income or wealth equivalent form. This approach serves to simplify the problem by reducing the number of the arguments in the utility function to one, income or wealth. An example of the use of this approach is provided by Block and Heineke (1975a, pp. 319-23).

I-4 The Presently Proposed Solution

The solution to this comparative-static problem offered in this dissertation is unlike any of the previously proposed solutions, i.e., those outlined in Section I-3. Our solution issues from, and is contained in, the following eight considerations.

(a) If the second order conditions hold, non-ambiguity of the signs of the effects represented by Equations (2) and (3) is guaranteed if both Conditions 1 and 2 (defined in Section I-2) are satisfied.

(b) In satisfying Conditions 1 and 2, V_{13} and V_{23} must be signed unambiguously. These partial derivatives may be signed by reference to simpler models of dimension 2x1 (or 1x1). In particular, the optimization problem defined by Equation (1) allows for the joint-determination of (x_1, x_2). However, the optimization problem defined by Equation (1) may be viewed as being comprised of two independent models of dimension 2x1, i.e.,

$$\max_{x_1} V(x_1, \bar{x}_2, e) = \int U(\phi(x_1, \bar{x}_2), \gamma(x_1, \bar{x}_2, z)) f(z, v, e) dz \quad (4)$$

for x_2 held constant (i.e., for $\bar{x}_2 \varepsilon S_2$ where $S_2 = \{\bar{x}_2 \mid x_2^* - h < \bar{x}_2 < x_2^* + h \text{ and } h > 0\}$), and:

$$\max_{x_2} V(\bar{x}_1, x_2, e) = \int U(\phi(\bar{x}_1, x_2), \gamma(\bar{x}_1, x_2, z)) f(z, v, e) dz \quad (5)$$

for x_1 held constant (i.e., for $\bar{x}_1 \varepsilon S_1$ where $S_1 = \{\bar{x}_1 \mid x_1^* - h < \bar{x}_1 < x_1^* + h \text{ and } h > 0\}$). The first and second order conditions for Equations (4) and (5) are $V_j = 0$ and $V_{jj} < 0$ for $j = 1, 2$. Thus if these conditions hold, there exists a unique x_j, x_j^0, which satisfies (4) when $j=1$ and (5) when $j=2$.

(c) The comparative-static problem associated with models of dimension 2x1 is:

$$dx_j^0/de = -V_{j3}/V_{jj} \tag{6}$$

for $x_{j+s} \varepsilon S_{j+s}$ (i.e., $dx_{j+s}=0$) where s=1 if j=1, and s=-1 if j=2. Since $V_{jj}<0$ for a maximum,

$$\text{sign } (dx_j^0/de) = \text{sign } (V_{j3}) \tag{7}$$

for $x_{j+s} \varepsilon S_{j+s}$. Now V_{j3} for j=1,2 in Equation (6) is the same as V_{13} or V_{23} in Equations (2) and (3) with the only exception that x_{j+s} is determined exogenously rather than endogenously.[6,7]

(d) In view of Condition 1 (defined in Section I-2), it is clear that models of dimension 2x1 (or even 1x1) may provide useful comparative-static information in the determination of the comparative-static effects for models of dimension 2x2 (or as discussed later, 3x2), i.e., information on the sign of V_{j3}.

(e) The use of information on the comparative-static results for models of dimension 2x1 is simplified if (where necessary) a restriction on the second derivatives of $\gamma(x_1,x_2,z)$ is employed. This restriction is stated above, i.e., $\gamma''=0$.

(f) To utilize such information, the chronological development of the literature may be important. An example will serve to make this point. As we shall discuss in subsequent Chapters, one problem analogous to Equation (1) is the Sandmo (1969) portfolio problem. As discussed above, Sandmo concludes that the comparative-static effects of increasing uncertainty are indeterminate. As we will demonstrate, his conclusion is due in large part to his inability to sign V_{j3} for j=1,2. Using the presently proposed solution to the comparative-static problem, Sandmo's problem could not have been solved without the determination of the comparative-static effects of increasing uncertainty associated with two related 2x1 models. That is, from our perspective, the solution to Sandmo's problem had to await the appearance of one of two pairs of papers: either Sandmo (1970) and Hanson and Menezes (1978), or Block and Heineke (1973) and Tressler and Menezes (1980).

(g) In the problems to be considered, the approach used in signing V_{12} is case-specific and as such this topic is left for later discussion.

(h) A final consideration centres on the measurement of e. There are two measures of e, a "Sandmo-measure" and a "Rothschild-Stiglitz-measure."[8] It is important to recognize that the former is a special case of the latter. This information is vital for the integration of the information on the sign(s) of V_{j3} into the 2x2 or 3x2 models. To make this clearer, consider the following example. Suppose that V_{13} is based on one measurement of e and V_{23} is based on the other. In an evaluation of Equations (2) and (3), this fact will necessitate that the exogenous variable, e, be explicitly defined as the less general measure of uncertainty, i.e., as the "Sandmo-measure."

I-5 An Outline of this Dissertation

This dissertation is organized in the following manner. In Chapter II, a review of topics in the economics of uncertainty which are central to this thesis are presented. Of major importance here is an overview to four papers. These are Block and Heineke (1973), Tressler and Menezes (1980), Sandmo (1970), and Hanson and Menezes (1978). In terms of the taxonomy presented above, these four papers utilize models of dimension 2x1. In terms of the information which they render (i.e., in signing V_{j3} for j=1,2), these four papers provide the basis of the analysis of the comparative-static effects of increasing uncertainty in several 2x2 and 3x2 models presented in Chapters III through IX. The remainder of Chapter II may be viewed as a collection of topics in the economics of uncertainty which are (in the view of the author) a prerequisite to a comprehension of these four papers.

Chapter III is motivated by Cowell (1981). The Chapter centres on a particular model of dimension 2x2. This model is termed the "safe work, risky work, and leisure" model. In this Chapter, the restrictions on the EUF needed to sign the effects of increasing wage-rate uncertainty on the optimal level of labor supply are determined. In our analysis, information from two 2x1 dimension models is employed. This information is provided by Block and Heineke (1973) and Tressler and Menezes (1980). The model in Chapter III is representative of our <u>first class</u> of models.

Chapter IV represents the first of two approaches to solving what we term "Sandmo's (1969) portfolio problem". According to our taxonomy contained in Table 1.1, the Sandmo problem concerns a second model of dimension 2x2. In specifying the restrictions on the EUF needed to solve this problem, information is drawn from two papers which employ similar models of dimension 2x1. These two papers are Sandmo (1970), and Hanson and Menezes (1978). The unique nature of this particular model considered in this Chapter permits us to term it the second class of models.

Chapter V represents the second of two approaches to defining a solution to the "Sandmo (1969) portfolio problem". The differences between Chapters IV and V are twofold: (i) the specification of the objective function, and (ii) the source of information from 2x1 models. The two papers used here as sources are Block and Heineke (1973), and Tressler and Menezes (1980).

As is the case with Chapter V, Chapters VI and VII may be viewed as simple extensions to Chapter III. Therefore, while a new class of models is not considered in Chapters VI and VII, new applications are. In particular, in Chapter VI, the comparative-static effects of the Block and Heineke (1975a) model of criminal choice under increasing uncertainty is assessed and extended. In Chapter VII, the Isachsen and Strom (1980) model of tax evasion and labor supply is adapted to a stochastic environment.

Chapter VIII provides another example of the adaption of non-stochastic models to a stochastic environment. The non-stochastic models in question are Leuthold's (1968) model and Koster's (1966 and 1969) model of the family labor supply decision. In the adapation of the Koster's model, this Chapter considers another unique class of models: one whose dimension is 3x2. This Chapter contains therefore our assessment of a <u>third class</u> of model. The results of Block and Heineke (1973), and Tressler and Menezes (1980) are employed here again.

Chapter IX solves a comparative-static problem posed on two occasions by Block and Heineke (1972 and 1975b). This problem is motivated by a model whose dimension is 3x2. Because this model is unrelated to the Kosters model of family labor supply, the model provided by Block and Heineke represents the <u>fourth class</u> of model to be considered. The works of Sandmo (1970) and Hanson and Menezes (1978) are employed here.

Chapter X contains an investigation into the impact of increasing uncertainty for a <u>fifth</u>, and <u>final</u>, <u>class</u> of model. Motivated by a well-defined set of non-stochastic models of the same genre, Chapter X considers the impacts of output-price uncertainty on the operations of the owner-managed firm. In this Chapter, the works of Sandmo (1971), Block and Heineke (1973), and Ishii (1977) are employed.

To sum up, the models in Chapters III, V, VI, and VII may be viewed as identical in their formal properties.

For this reason, these models are said to be members of the first-class of models. The model in Chapter IV is unique, and therefore it is viewed as the sole member of the second class. Chapters VIII and IV consider the effects for two different models having dimensions of 3x2. These two models are termed members of the third- and fourth-class of models. Finally Chapter X considers a unique type of 2x2 model. This model is termed a member of the fifth class. A summary of the relegation of models (related to each Chapter) to one of five classes is presented in Table 1.2.

TABLE 1.2

CLASSES OF MODELS BY CHAPTER AND DIMENSION

Class	Chapter	Dimension
1	III, V, VI, VII	2x2
2	IV	2x2
3	VIII	3x2
4	IX	3x2
5	X	2x2

I-6 The Contributions of this Dissertation

The contributions of this dissertation are of two sorts: (a) the determination of the impact of increasing uncertainty in existing models under uncertainty, and (b) the adaptation of existing models under certainty to a stochastic environment, and the determination of the impact of increasing uncertainty on the properties of these adapted models. In brief overview, our contributions are:

(a) <u>The Determination of the Impact of Increasing Uncertainty in Existing Models Under Uncertainty</u>: There are four such contributions:

(i) In Chapter III, Cowell's (1981) model of "riskless and risky activity" is extended. This extension is the investigation into the restrictions on the EUF needed for the determination of the effects of increasing uncertainty of the wage-rate paid the risky activity.

(ii) In Chapters IV and V, two alternative solutions are offered to a problem posed by Sandmo (1969). The problem is the determination of the effects of increasing uncertainty of the return on the risky asset in a two-period, two-asset model.

(iii) In Chapter VI, a result of the Block and Heineke (1975a) model of criminal choice under uncertainty is assessed for robustness, and the predictions of the same model extended.

(iv) In Chapter IX, a problem formulated on two occasions by Block and Heineke (1972 and 1975b) is addressed. In this Chapter, restrictions on the EUF which render the comparative-static effects of increasing interest rate uncertainty unambiguous are offered.

(b) The Adaptation of Models Under Certainty to a Stochastic Environment: There are three such contributions:

(i) In Chapter VII, the Isachsen and Strom (1980) model of tax evasion and labor supply is modified to enable an assessment of the impact of imperfect information about the tax system.

(ii) In Chapter VIII, the Leuthold (1968) and Kosters (1966 and 1969) models of the family labor supply decision are adapted to enable an investigation of the effects of stochastic wage and non-wage income.

(iii) In Chapter X, a model of the owner-managed firm (OMF) under output-price uncertainty is presented. This Chapter is motivated in part by a well-defined literature on the model of the OMF under certainty.

Footnotes to Chapter I

[1] Such surveys include McCall (1971), Ahsan and Ullah (1978a), Hadar and Russell (1978), Gilbert et al. (1978), Hey (1979 and 1981), Hirshleifer and Riley (1979), Lippman and McCall (1981a), Sinn (1982), and Winkler (1982).

[2] For a complete overview to the comparative-static problem, see Chiang (1984, Chapters 8-12), or Silberberg (1978, Chapter 9).

[3] The second order conditions require that $V_{11}<0$, $V_{22}<0$ and $|H|>0$. Therefore, the signing of these three terms are non-issues.

[4] Holthausen (1979) does not employ a model of dimension 2x2 or 3x2, but rather employs one of dimension 1x2. Therefore, Holthausen's approach is suggestive of an approach to the determination of the impact for 2x2 or 3x2

models, rather than an <u>actual</u> approach.

[5]Cowell considers three cases in which h_0 denotes riskless activity and h_1 denotes risky activity. These are: (i) $h_0>0$ and $h_1=0$, (ii) $h_0=0$ and $h_1>0$, and (iii) $h_0>0$, and $h_1>0$. Cowell considers the conditions for signing the comparative-static effect of mean-preserving increase in wage-rate uncertainty in case (ii) only. See p. 370.

[6]Our methodology is not unique to the literature on the economics of uncertainty. It is however unique to the determination of the effects of <u>increasing uncertainty</u> in models of dimension 2x2. Holthausen (1976, pp. 100-1) uses the same methodology in the determination of the effects of <u>increasing risk aversion</u>.

[7]A simple definition of the relationship between Equations (2) and (3) on one hand, and Equation (6) on the other, is as follows:

(i) For $x_2 = \bar{x}_2$ (or $dx_2=0$), then $V_{12}=V_{23}=0$ <u>and</u>:

$$dx_1/de = [V_{23}\cdot V_{12}-V_{13}\cdot V_{22}]/[V_{11}\cdot V_{22}-(V_{12})^2]$$
$$= -V_{13}\cdot V_{22}/V_{11}\cdot V_{22}$$
$$= -V_{13}/V_{11}$$

(ii) For $x_1 = \bar{x}_1$ (or $dx_1=0$), then $V_{12}=V_{13}=0$ <u>and</u>:

$$dx_2/de = [V_{13}\cdot V_{12}-V_{23}\cdot V_{11}]/[V_{11}\cdot V_{22}-(V_{12})^2]$$
$$= -V_{23}\cdot V_{11}/V_{11}\cdot V_{22}$$
$$= -V_{23}/V_{22}$$

[8]There are in fact three measures of e. See Section II-3 for details.

CHAPTER II

A REVIEW OF SELECTED TOPICS FROM THE LITERATURE ON THE ECONOMICS OF UNCERTAINTY

In this Chapter, a selection of topics which forms the analytical basis of this dissertation is reviewed. These topics include the expected utility function, an array of possible types of measures of risk preference, two types of mean preserving increases in uncertainty (or alternatively, risk), and an overview to four key papers (i.e., Block and Heineke (1973), Tressler and Menezes (1980), Sandmo (1970), and Hansen and Menezes (1978)).

II-1 An Overview of the Expected Utility Hypothesis

The initial definition of the expected utility hypothesis (EUH) is due to Daniel Bernoulli (1738). Significant, pioneering advances on the explication of the nature of the EUH, and its applications, were made by Frank Ramsey (1931), and John von Neumann and Oskar Morgenstern (1944). Recent surveys of the EUH are offered by Fishburn (1981), Schoemaker (1982), and Sinn (1982).

In the tradition of Green (1976), Blackorby *et al.* (1977), and Varian (1978), our objective here is to provide a thumbnail sketch of the EUH. To achieve this, it is useful to think of the EUH as having four sets of formal concepts or theorems. These are: (i) the concept of a "lottery", (ii) the axioms which define a "lottery-space," (iii) a theorem which defines the expected utility function (EUF), and (iv) a theorem which defines the uniqueness of the EUF under functional transformations. In our explication of these concepts and theorems, it is useful to define the simplest case. This we achieve by assuming that there are "two states of the world," these being "winning" and "losing." It should be kept in mind that our results based on a two-state-world generalize to an n-state-world.

A lottery may be defined by considering an individual decision-maker with initial wealth of W_2. If the state of winning obtains, the individual's wealth increases to W_3. If the state of losing obtains, the individual's wealth decreases to W_1. It is clear that $W_1 < W_2 < W_3$. The individual's subjective probability of the first state occuring is (1-P), and his subjective probability of the second state occuring is P. A "lottery" is defined by the sum $PW_1+(1-P)W_3$. <u>The expected value of the lottery</u> is $\bar{W} = PW_1+(1-P)W_3$.

Depending on the relationship between initial wealth, the subjective probabilities, and the payoffs in both states of nature, a lottery is said to be one of three types. A lottery is said to be <u>favorable</u> if:

$$W_2 < PW_1 + (1-P)W_3 \qquad (1)$$

<u>fair</u> if:

$$W_2 = PW_1 + (1-P)W_3 \qquad (2)$$

and <u>unfavorable</u> if:

$$W_2 > PW_1 + (1-P)W_3 \tag{3}$$

The definition of a lottery enables the definition of the concept, "lottery-space." After Herstein and Milnor (1953) and Varian (1978, pp. 104-5), the lottery-space, R, is assumed to satisfy the following properties:

<u>L1</u>: $1 \cdot W_1 + (1-1)W_3 = W_1$

<u>L2</u>: $PW_1 + (1-P)W_3 = (1-P)W_3 + PW_1$

<u>L3</u>: $\bar{P}(PW_1 + (1-P)W_3) + (1-\bar{P})W_3$

$$= \bar{P}PW_1 + (1-\bar{P}P)W_3$$

These properties are to be interpreted as follows. L1 asserts that getting a prize with the probability of one is the same as getting the prize for certain. L2 asserts that order of the prizes in the lottery is irrelevant (i.e., the relationship is symmetrical). Finally, L3 asserts that complex lotteries can be reduced to simpler lotteries (i.e., the perception of a lottery depends only on the <u>net</u> probabilities of receiving various prizes).

This concept of lottery-space, R, serves as the back-drop to the decision-maker's problem in that lotteries serve as the object of "desire". In this lottery-space, the decision-maker's preferences are assumed to be complete, reflexive, and transitive.

A cardinal measure of this individual's preference ordering of lotteries is the EUF. To establish the existence of the EUF defined on R, four additional assumptions are required. After Varian (1978, pp. 105-7), these are:

C1: If $S_1 = (P\varepsilon[0,1] \mid PW_1 + (1-P)W_3 \geq W_2)$ and $S_2 = (P\varepsilon[0,1] \mid PW_1 + (1-P)W_3 \leq W_2)$, then S_1 and S_2 are closed sets for all W_1, W_2, $W_3 \varepsilon R$, where "$\leq$" is the operator, "weak preference."

C2: If $W_1 \doteq W_3$, then $PW_1 + (1-P)W_2 \doteq PW_3 + (1-P)W_2$, where "$\doteq$" is the operator, "indifferent to."

C3: There exist some "best" lottery with prize W_3^+, and some "worst" lottery with prize W_3^-. Thus for any $W_3 \varepsilon R$, $W_3^- < W_3 < W_3^+$.

C4: The lottery $PW_2^- + (1-P)W_3^+$ is preferred to $\bar{P}W_2^- + (1-\bar{P})W_3^+$ if and only if $\bar{P} > P$.

C1 is the assumption of continuity. C2 states that lotteries with indifferent prizes are assumed to be treated indifferently. C3 defines the "worst" and the "best" sets. Finally, C4 defines the relationship between lotteries with different probabilities and the preference relation.

Assumptions L1 through L3 inclusive, and C1 through C4 inclusive, are sufficient to establish the existence of the EUF. In particular, consider the following Theorem.

Theorem 1: If $(R, \geq)$ satisfy the above seven axioms, there exists a utility function defined in R such that the utility of a lottery equals the expected utiilty of its prizes, i.e.,

$$U(PW_1 + (1-P)W_3) = PU(W_1) + (1-P)U(W_3)$$

Proof: Varian (1978, pp. 106-7) Q.E.D.

This theorem which defines the EUF has been termed the expected utility hypothesis.

The properties of the EUH includes a theorem on the EUF under functional transformations. In particular,

Theorem 2: An EUF is unique up to an affine transformation.[1]

Proof: Varian (1978, p. 108) Q.E.D.

As a preference functional for ordering uncertain prospects, the EUF is not without its competitors. In a recent survey of the literature on decision-making under uncertainty, Sinn (1982, pp. 41-122) presents a taxonomy of candidates. In addition to the EUF, Sinn surveys members of what he terms the "two-parametric substitutive" criterion, and the "lexicographic" criterion. The most commonly known member of the former class is the Markowitz-Tobin "mean-variance" criterion,[2] and the most commonly known member of the latter class is Roy's "safety-first" principle.[3]

Investigations into the relationship between the EUF and its competitors have been undertaken. Feldstein (1969), for example, demonstrates that the EUF may be reconciled with the mean-variance criterion if one of two (or both) conditions hold: (i) the functional form of the EUF is quadratic, or (ii) the distribution of each security price in the Markowitz-Tobin portfolio is Gaussian.

As a consequence of Feldstein's work, the mean-variance criterion is viewed by most as an approximation to a more general model, the EUH. From this perspective, much has been written on the performance of the mean-variance criterion as an approximation of the EUF. Taking the position that the mean-variance criterion performs poorly, Borch

(1969, 1976, and 1976) argues that the mean-variance criterion contains a hidden counterintuitive and counterfactual assumption about risk preference - the assumption that the decision-maker exhibits a "risk loving" attitude. (Parenthetically, the reader should note that the concept of risk preference will be defined formally in the next Section.) Tsiang (1972 and 1974) argues that the performance of the mean-variance criterion depends on the circumstances under which it is applied. He states that if two restrictions are imposed on its use, the mean-variance criterion performs well as an approximation of the EUF. These restrictions are: (i) restrictions on the functional form of the EUF (such as non-satiation, risk aversion, decreasing absolute risk aversion, and increasing relative risk aversion), and (ii) a restriction on the size of the risk relative to the expected magnitude of wealth.

II-2 An Overview of Measurements of Risk Preference

In our thumbnail sketch of the EUH, a literature review pointed up the fact that central to a reconciliation of the EUF and the mean-variance criterion is the concept of risk preference. In this Section, we review a variety of measures of risk preference. These include: risk aversion, risk neutrality, risk loving, absolute risk aversion, relative risk aversion, partial relative risk aversion, the preference intensity function (PIF), constant absolute risk aversion (CARA), decreasing absolute risk aversion (DARA), Leland's Principle of Decreasing Risk Aversion to Concentration (PDRAC), and (what is termed in this thesis) the Hansen-Menezes-Tressler Condition (HMTC).

(a) Types of Attitudes Towards Risk - Risk Aversion, Risk Neutrality, and Risk Loving:[4] In general, the concept of attitude towards risk concerns the relationship between the utility of the decision-maker towards certain wealth and his expected utility of a fair lottery. In particular, let the utility of certain wealth, W_2, be $U(W_2)$, and let the expected utility of the fair lottery,

$$W_2 = PW_1 + (1-P)W_3$$

be:

$$E(U(W_2)) = PU(W_1) + (1-P)U(W_3) \tag{4}$$

where $U'(\cdot)>0$. Denoting C as the risk premium, we may define the concept of risk preference in terms of the relationship between the utility of certain wealth and the expected utility of a lottery, i.e.,

$$U(W_2-C) = PU(W_1)+(1-P)U(W_3) \tag{5}$$

where the sign of C defines risk preference. In particular, the individual is said to be risk averse, risk neutral, or risk loving as $C>0$, $C=0$, or $C<0$.

The attitude towards risk (i.e., the sign of C) is also related to the curvature of the EUF. To demonstrate this, consider the following Theorem:

Theorem 3: Let z be an additive random component of certain wealth, W_2. If z is distributed as $N(0,\sigma^2)$, then:

$$\text{sign } (C) = \text{sign } (-U_{WW})$$

<u>Proof</u>: By definition,

$$U(W_2-C) = E[U(W_2+z)]$$

Taking a Taylor Series expansion of both sides, we obtain:

$$\text{LHS} = U(W_2-C) = U(W_2)-U_W(W_2)\cdot C + r_1$$

and

$$\text{RHS} = E[U(W_2+z)] = U(W_2)+U_W(W_2)\cdot E(z) + \frac{U_{WW}(W_2)}{2} \cdot E(z^2) + r_2$$

where r_1 and r_2 are the remainders. Since $E(z)=0$, we obtain:

$$U_W(W_2)C = -U_{WW}(W_2)\cdot \sigma^2/2$$

or:

$$C = [-U_{WW}/U_W]\cdot \sigma^2/2$$

Since $U_W(W_2)>0$ and $\sigma^2>0$, it follows that:

$$\text{sign }(C) = \text{sign }(-U_{WW}(W_2)) \qquad \text{Q.E.D.}$$

A plausible behavioral hypothesis is that a decision-maker under uncertainty exhibits risk aversion, i.e., his EUF is concave in wealth. In measuring risk aversion, the use of the curvature of the EUF alone is problematic. The problem stems from the fact that <u>the curvature of the EUF is not invariant under an affine transformation</u>, and yet as determined in Theorem 2 the EUF is. Obviously, <u>a better measure of risk-aversion is one which is invariant</u>. Several measures of risk aversion which satisfy this restriction have been proposed. Three such candidates are:

(i) the Arrow-Pratt measure of <u>absolute</u> risk aversion denoted by:[5]

$$R_A(W) = -U_{WW}(W)/U_W(W) \tag{6}$$

(ii) the Arrow-Pratt measure of relative risk aversion denoted by:[6]

$$R_R(W) = -W \cdot U_{WW}(W)/U_W(W) \tag{7}$$

and

(iii) the Menezes-Hanson measure of partial relative risk aversion denoted by:[7]

$$R_P(W;Z) = -WU_{WW}(W+Z)/U_W(W+Z) \tag{8}$$

where W is initial wealth, and Z is a random variable.

(b) The Preference Intensity Function: In addition to the above methods, another method of measuring risk preference is the preference intensity function (PIF) which is due to Friedman and Savage (1948), and Stone (1970). To define the PIF, consider again Equation (5) in which C=0. From this, it is clear that:

$$U(\bar{W}) \gtrless EU(W) \text{ as } R_A(W) \gtrless 0 \tag{9}$$

where $\bar{W}$ denotes certain wealth. Manipulation of Equation (9) permits us to define the PIF as:

$$\theta = U(\bar{W}) - E(U(W)) \tag{10}$$

such that:

$$\theta \gtrless 0 \text{ as } R_A(W) \gtrless 0 \tag{11}$$

The PIF is not invariant under a linear transformation of the utility function. To demonstrate this, consider:

Theorem 4: If the utility function U(W) is transformed as $U^* = aU(W)+b$, then $\theta^* = a\theta$.

Proof: $\theta^* = U^*(\bar{W}) - E(U^*(\bar{W}))$

$$= U^*(\bar{W}) - U^*(\bar{W}-C)$$

$$= aU(\bar{W})+b - [aU(\bar{W}-C)+b]$$

$$= a[U(\bar{W}) - U(\bar{W}-C)]$$

$$= a\theta$$

Q.E.D.

(c) Constant and Decreasing Absolute Risk Aversion: Two behavioral hypotheses which are commonly invoked in the literature are constant absolute risk aversion (CARA), and decreasing absolute risk aversion (DARA). The history of the use of DARA as a behavioral hypothesis is longer and more replete than the history of the use of CARA. DARA has been used or defended by Daniel Bernoulli (1738),[8] Irving Fisher (1906, p. 277), and Arrow (1971, p. 35) who states that DARA is "supported by everyday observation" and is "intuitively appealing."[9] Examples of the use of CARA include Arrow (1971, pp. 90-120), and Block and Heineke (1973).

One property of a utility function which has one argument and which exhibit DARA is that its third derivative is positive (i.e., $U_{WWW}>0$). To establish this, consider:

Theorem 5: If $U(W)$ is DARA, then $U_{WWW} > 0$.

Proof: If U is DARA, then

$$dR_A(W)/dW = -[U_{WWW}U_W-(U_{WW})^2]/(U_W)^2 < 0$$

which implies:

$$U_{WWW}/U_W - (U_{WW})^2/U_W > 0$$

which implies:

$$U_{WWW}/U_W > (U_{WW})^2/U_W > 0$$

which implies $U_{WWW}>0$ since $U_W>0$. Q.E.D.

(d) Leland's PDRAC: Leland (1968) posits a plausible behavioral response to risk known as the Principle of Decreasing Risk Aversion to Concentration (PDRAC). This Principle is stated as follows: "we become less averse to risk in a (commodity) as that (commodity) becomes increasingly prominent in a constant utility bundle" (p. 269). Use or mention of Leland's PDRAC in the literature on the economics of uncertainty appears in Epstein (1975, p. 883), Menezes and Auten (1978), Whitmore and Findlay (1978, p. 216), and Lippman and McCall (1981a, p. 239).

An implication of the PDRAC is this. Suppose an individual is confronted by two situations, situations A and B, in which two goods, C_1 and C_2, are offered with certainty. Suppose the quantities of C_1 and C_2 offered in Situations A and B are as presented in Table 2.1. Suppose too that the quantities of C_1 and C_2 offered in A and B are such that the individual is indifferent between Situation A and B, i.e.,

$$U(C_1^A, C_2^A) = U(C_1^B, C_2^B) \qquad (12)$$

Suppose too that C_1 and C_2 in Situations A and B satisfy $C_2^A < C_2^B$.

Now suppose C_2^A and C_2^B become prospects (i.e., $E(C_2^A)=10$ and $E(C_2^B)=30$) such that the variances of both prospects are the same (i.e., $\mathrm{Var}(C_2^A) = \mathrm{Var}(C_2^B)$). The expected utility of the individual in both situations can be determined as follows. Using a second-order Taylor Series expansion, we can express the expected utility associated

TABLE 2.1

AN EXAMPLE RELATED TO LELAND'S PDRAC

Situation	Commodity Bundle	
	C_1	C_2
A	30	10
B	10	30

with Situation A as a function of the statistical parameters of prospect C_2^A. In particular, expanding $U(C_1^A, C_2^A)$ about the expected value of C_2^A, $E(C_2^A) = \bar{C}_2^A$, we get:

$$U(C_1^A, C_2^A) = U(C_1^A, \bar{C}_2^A) + U_2(C_1^A, \bar{C}_2^A)(C_2^A - \bar{C}_2^A)$$
$$+ U_{22}(C_1^A, \bar{C}_2^A)(C_2^A - \bar{C}_2^A)^2 + R$$

where $U_2 = \partial U/\partial C_2$ and $U_{22} = \partial^2 U/\partial C_2{}^2$. Taking the expected value of this expansion, assuming $E(R)=0$, and noting that $E(C_2^A - \bar{C}_2^A)=0$, we get:

$$EU(C_1^A, C_2^A) = U(C_1^A, \bar{C}_2^A) + U_{22}(C_1^A, \bar{C}_2^A)\sigma^2 \qquad (13)$$

By identical reasoning, we obtain for Situation B:

$$EU(C_1^B, C_2^B) = U(C_1^B, \bar{C}_2^B) + U_{22}(C_1^B, \bar{C}_2^B)\sigma^2 \qquad (14)$$

Recalling Equation (12), and subtracting (14) from (13), we obtain:

$$EU(C_1^A, C_2^A) - EU(C_1^B, C_2^B)$$
$$= \sigma^2 [U_{22}(C_1^A, \bar{C}_2^A) - U_{22}(C_1^B, \bar{C}_2^B)] \qquad (15)$$

Now Leland's PDRAC asserts that if $C_2^A < C_2^B$, then $R_A^A > R_A^B$ or $U_{22}^A < U_{22}^B$. Therefore, if $C_2^A < C_2^B$, and if the PDRAC holds, then the R.H.S. of (15) is negative, i.e.,

$$U^A_{22} < U^B_{22} \tag{16}$$

where $U^A_{22}=U_{22}(C^A_1,\bar{C}^A_2)$ and likewise U^B_{22}. And therefore:

$$EU(C^A_1,C^A_2) < EU(C^B_1,C^B_2) \tag{17}$$

In words then, if $C^A_2<C^B_2$, then Leland's PDRAC implies (17).

The implication of the PDRAC may be generalized in discrete form as:

$$\left.\frac{\Delta U_{22}}{\Delta C_2}\right|_{\Delta U=0} = \frac{U^B_{22} - U^A_{22}}{C^B_2 - C^A_2} > 0 \tag{18}$$

by virtue of Equation (16) and the assumption that $C^A_2<C^B_2$. In continuous form, <u>the implication of the PDRAC may be expressed as</u>:

$$\left.\frac{dU_{22}}{dC_2}\right|_{dU=0} = U_{222} - U_{221}\left[\frac{U_2}{U_1}\right] > 0 \tag{18'}$$

In the literature on the economics of uncertainty, the issue of the plausibility of the PDRAC as a <u>behavioral</u> assumption has been addressed. Menezes and Auten (1978) show the PDRAC to be equivalent to Irving Fisher's insight that income risk reduces the rate of time preference.

(e) <u>The HMTC</u>: Another behavioral hypothesis regarding risk preference is <u>what is termed here as</u> the Hanson-Menezes-Tressler Condition (HMTC). The naming of this term is motivated by the use of a particular hypothesis in Hanson and Menezes (1978) and Tressler and Menezes (1980).

In particular, the HMTC follows from the following problem. Let w be a random variable. Consider an expected utility function, $EU(L,wy)$, where $Y=wy$ and $L=y$, and $U_L<0$, $U_{LL}<0$, $U_Y>0$, and $U_{YY}<0$. Next consider two risky bundles, (L_1,wy_1) and (L_2,wy_2), such that $EU(L_1,wy_1)=EU(L_2,wy_2)=EU^0$ and $L_1<L_2$ and $y_1<y_2$ where $L=y$. Next consider w^* which has the same mean as w, but a greater dispersion, i.e., the p.d.f. of w^*, $f^*(w)$, is obtained from $f(w)$ by a shift satisfying Equations (4) and (5) in Diamond and Stiglitz (1974).[10] This new random variable, w^*, provides a new pair of risky bundles, (L_1,w^*y_1) and (L_2,w^*y_2).

Underlying the HMTC is a particular preference relation. This particular preference relation is Assumption [A] in both Hanson and Menezes (1978, p. 656), and Tressler-Menezes (1980, p. 432), i.e., $EU(L_1,w^*y_1)>EU(L_2,w^*y_2)$ because (L_2,w^*y_2) has a greater dispersion than (L_1,w^*y_1), i.e., because $y_1<y_2$, $E(w)=E(w^*)$, and $Var(w^*)>Var(w)$.

The HMTC is found in three steps. Firstly, taking a Taylor Series expansion of $EU(L_1,w^*y_1)$ about wy_1, and ignoring the third-degree and higher-order terms, yields:

$$\begin{aligned} EU(L_1,w^*y_1) &= EU(L_1,wy_1) + U^{1}_{22}E(w^*y_1-E(wy_1))^2 \\ &= EU(L_1,wy_1) + U^{1}_{22}y_1^2E(w^*-E(w))^2 \\ &= EU(L_1,wy_1) + U^{1}_{22}(L_1)^2\sigma^2 \qquad (19) \end{aligned}$$

where $\sigma^2=Var(w^*)$. Secondly taking a Taylor Series expansion of $EU(L_2,w^*y_2)$ about wy_2, and ignoring the third-degree and higher-order terms, yields:

$$EU(L_2, w^*y_2) = EU(L_2, wy_2) + U_{22}^2(L_2)^2\sigma^2 \qquad (20)$$

Finally, subtracting (20) from (19) yields:

$$EU(L_1, w^*y_1) - EU(L_2, w^*y_2) = \sigma^2[U_{22}^1(L_1)^2 - U_{22}^2(L_2)^2] \qquad (21)$$

since $EU(L_1, wy_1) = EU(L_2, wy_2)$, i.e., since $\Delta U = 0$.

The HMTC can be inferred from Equation (21). In particular, Assumption [A] postulates that the L.H.S. of (21) is positive, and therefore the R.H.S. of (21) is also positive, i.e., since $\sigma^2 > 0$, then:

$$\Delta(U_{YY}L^2) > 0 \qquad (22)$$

for $\Delta L = L_1 - L_2 < 0$. Therefore,

$$\left.\frac{\Delta U_{YY}L^2}{\Delta L}\right|_{\Delta U=0} \approx \left.\frac{dU_{YY}L^2}{dL}\right|_{dU=0} = L^2[dU_{YY}/dY\big|_{dU=0} \cdot dY/dL] + 2LU_{YY} < 0 \qquad (23)$$

Equation (23) (or Equations like it) is what is referred to in this dissertation as the HMTC.

Because of their importance to our subsequent analysis, three issues regarding the HMTC should be addressed. The astute reader may have noticed a striking similarity between Equations (18') and (23). Because of this similarity, the <u>first issue</u> to be addressed is the question: how does Leland's PDRAC compare to the HMTC? The difference is made discernible if we rewrite Leland's PDRAC in the context of "HMTC problem" discussed above. That is, the PDRAC asserts:

$$dU_{YY}/dY\big|_{dU=0} > 0 \qquad (24)$$

<u>Mutual compatibility</u> of the PDRAC and the HMTC requires therefore that:

$$0 < L^2 [dU_{YY}/dY\Big|_{dU=0} \cdot dY/dL] < -2LU_{YY}$$

Another point of comparison between the PDRAC and the HMTC is the difference in the prospects being considered in either case. In particular, the PDRAC considers two prospects of equal variance (i.e., see Equation (15)), whereas the HMTC considers two prospects of unequal variance. This last statement is evident from Equation (21) in that for $\Delta L=\Delta y\neq 0$, $\Delta Var(w^{*}y_{i}) \neq 0$. The reader should note that a more detailed comparison of these two preference relations may be found in a comparison of Hanson and Menezes (1978, pp. 654-6), and Menezes and Auten (1978, pp. 253-4).

In many of the subsequent chapters of this dissertation, the behavioral hypothesis of constant risk aversion to concentration (CRAC) will be used in some sense in conjunction with the HMTC. Thus, the second issue which should be addressed concerns the relationship between CRAC and the HMTC. This relationship is determined easily. From Equation (24), it is clear that CRAC implies $dU_{YY}/dY=0$ for $dU=0$. This implication reduces Equation (23) to:

$$dU_{YY}L^2/dL\Big|_{dU=0} = 2LU_{YY} < 0 \qquad (23')$$

Equation (23') leads to a very important point which is central to much of this thesis: from Equation (23') it is clear that absolute risk aversion (i.e., $U_{YY}<0$) and CRAC together imply the HMTC.

There remains the final issue of not only the plausibility of the HMTC, but the plausibility of the HMTC in combination with the assumption of CRAC. In a recent article, Hansen and Menezes (1978) address the issue of the plausibility of the HMTC by showing that the HMTC "confirms Marshall's conjecture that capital risk affects saving adversely" (p. 668). The use of CRAC would modify the Menezes and Auten (1978) finding (regarding Leland's PDRAC and Irvin Fisher's comment on time preference) already discussed above. That is, the paper by Menezes and Auten (1978) implies that the hypothesis of CRAC is equivalent to the hypothesis that time preference is invariant with respect to income risk. Since we have just argued that risk aversion and CRAC implies the HMTC, it follows that another argument in favor of the plausibility of the HMTC rests on the union of the argument in favor of CRAC (which we have just discussed), and the argument in favor of absolute risk aversion (which has been discussed in Section II-2(a)).

II-3 Two Measurements of An Increase in Risk, and Their Impacts on Optimal Behavior

In the economics of uncertainty, several measurements of an increase in risk have been defined. These include: (i) a "Sandmo increase in risk," (ii) a "Rothschild-Stiglitz increase in risk," and (iii) an "Eeckhoudt-Hansen increase in risk."[11] In this Section, we undertake the following. Firstly, we define two measures of an increase in risk, the "Sandmo-type" and the "Rothschild-Stiglitz-type". Secondly, we show that the "Sandmo-type" is a special case of the "Rothschild-Stiglitz-type." Finally, we provide a preview of comparative-static relationship between the exogenous variable, an "increase in risk", and an endogenous variable, the "optimal level of a choice-variable."

(b) A "Rothschild-Stiglitz Increase in Risk": In the following definition of the "Rothschild-Stiglitz increase in risk", we begin with first principles. In particular, we consider a family of distribution functions, $F(x,e)$, where x denotes a non-degenerate random variable, and e denotes a shift-parameter which simultaneously preserves mean of all members in the family. Consider any two members in this family, e.g., $F(x,e_1)$ and $F(x,e_2)$ for $e_1<e_2$. A "Rothschild-Stiglitz increase in risk" is defined by two conditions. These are:

$$\int_0^1 [F(x,e_1) - F(x,e_2)]dx = \int_0^1 F_e(x,e)dx = 0 \qquad (25)$$

and:

$$T(x^*,e) = \int_0^{x^*} [F(x,e_2) - F(x,e_1)]dx$$

$$= \int_0^{x^*} f_e(x,e)dx \geq 0 \qquad (26)$$

for $x \varepsilon [0,1]$. The first condition, Equation (25) requires that $F(x,e_1)$ and $F(x,e_2)$ have the same mean. Equation (26) requires that the two probability distribution functions satisfy the "single crossing property."[12] A diagrammatical representation of Equations (25) and (26) is presented in Figure 2.1.

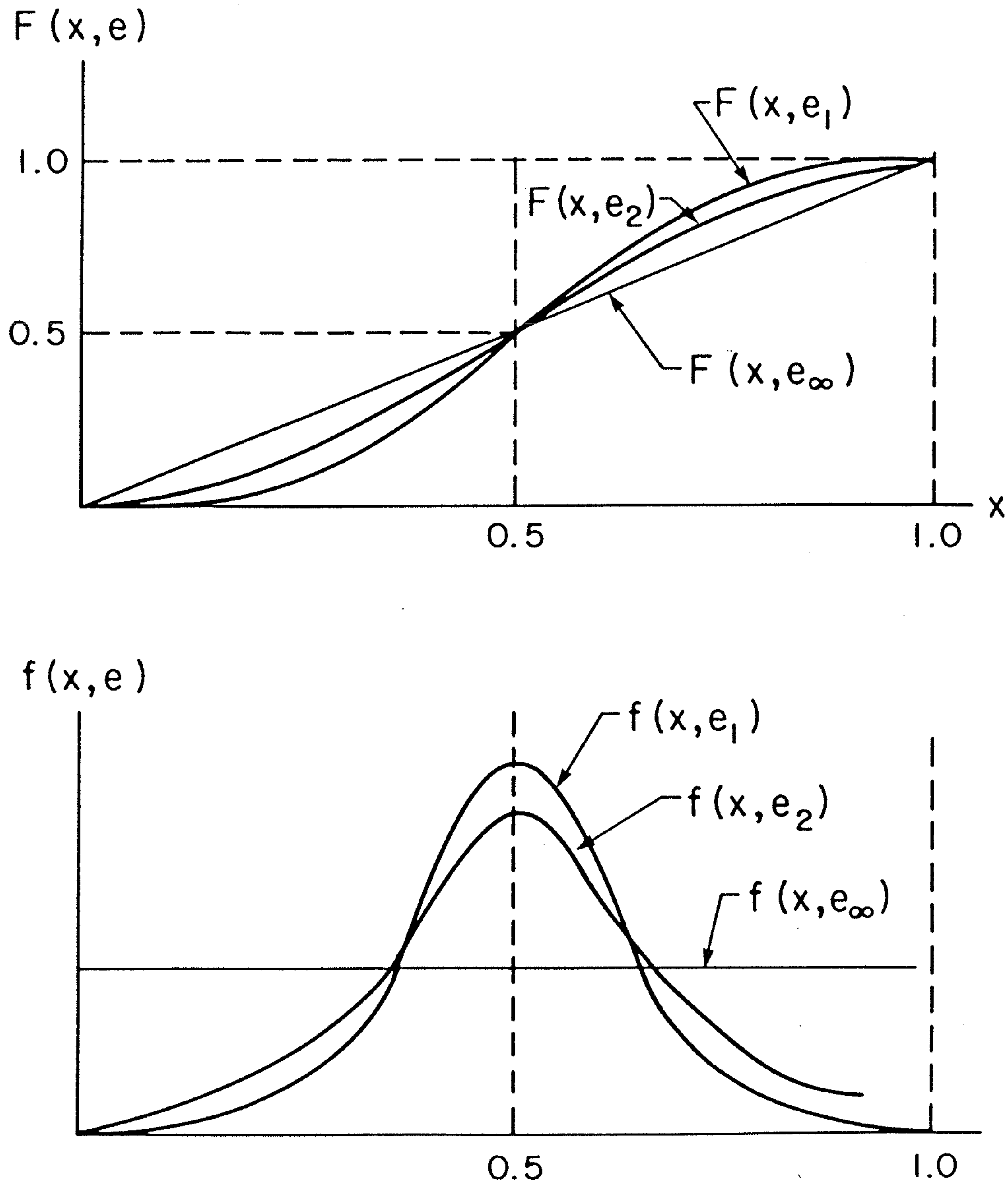

Figure 2.1.--A Diagrammatical Representation of a "Rothschild-Stiglitz" Increase in Risk

The plausibility of Equations (25) and (26) as measures which define an increase in risk is established by Rothschild and Stiglitz (1970). They show that this definition is equivalent to two other conditions: (i) that all risk averters dislike increased risk, and (ii) that an increase in risk is the addition of noise to the random variable. For purposes of subsequent discussion it is useful for us to show here the relationship between the concept, a "Rothschild-Stiglitz increase in risk", and the statement that all risk averters dislike increased risk. This we do in a sequence of three theorems. In particular, Theorem 6 defines the basis of Equation (25). Theorem 7 defines the concept of First-order Stochastic Dominance. Finally, Theorem 8 defines the concept of Second-order Stochastic Dominance, and it establishes the Rothschild-Stiglitz claim - that all risk averters dislike increased risk as defined by Equations (25) and (26).[13]

Theorem 6: For any two cumulative distribution functions $F(x,e_1)$ and $F(x,e_2)$ with identical means such that $x \varepsilon [0,1]$ and $e_1 < e_2$, then:

$$\int_0^1 [F(x,e_2) - F(x,e_1)]dx = 0$$

Proof: The equality of means implies:

$$\int_0^1 xdF(x,e_1) = \int_0^1 xdF(x,e_2)$$

or:

$$\int_0^1 xd[F(x,e_2) - F(x,e_1)] = 0 \tag{27}$$

Using integration by parts, (27) becomes:

$$\int_0^1 [F(x,e_2) - F(x,e_1)]dx = 0 \qquad \text{Q.E.D.}$$

<u>Theorem 7 [First-Order Stochastic Dominance]</u>: If $U_x(x)>0$ and $F(x,e_2)\geq F(x,e_1)$, then the expected utility of x under $f(x,e_2)$ is less than or equal to the expected utility under $f(x,e_1)$, i.e.,

$$EU(x \mid f(x,e_2)) \leq EU(x \mid f(x,e_1)) \tag{28}$$

<u>Proof</u>: Using integration by parts, it follows that:

$$EU(x \mid f(x,e_1)) - EU(x \mid f(x,e_2))$$

$$= \int_0^1 U(x)[f(x,e_1) - f(x,e_2)]dx$$

$$= U(x)[F(x,e_1) - F(x,e_2)] \Big|_0^1 \tag{29}$$

$$- \int_0^1 U_x(x)[F(x,e_1) - F(x,e_2)]dx$$

$$= - \int_0^1 U_x(x)[F(x,e_1) - F(x,e_2)]dx \tag{30}$$

Since $F(x,e_2)\geq F(x,e_1)$ and $U_x>0$, Equation (30) is positive.

Q.E.D.

<u>Theorem 8 [Second-Order Stochastic Dominance]</u>: If $U_x(x)>0$ and $U_{xx}(x)<0$, and

$$\int [F(x,e_2) - F(x,e_1)]dx \geq 0$$

then $EU(x \mid f(x,e_2)) \leq EU(x \mid f(x,e_1))$

<u>Proof</u>: By Theorem 7,

$$EU(x \mid f(x,e_1)) - EU(x \mid f(x,e_2))$$

$$= - \int U_x(x)[F(x,e1) - F(x,e2)]dx > 0 \tag{31}$$

Integrating by parts, (31) becomes

$$U_x(x^*) \int_0^{x^*} [F(z,e_2) - F(z,e_1)]dz$$

$$- \int_0^{x^*} U_{xx}(x) \int_0^{x} [F(z,e_2) - F(z,e_1)]dzdx \qquad (32)$$

Since $U_x>0$, $U_{xx}<0$, and $\int[F(x,e_2) - F(x,e_1)]dx \geq 0$, (32) is non-negative, i.e.,

$$EU(x \mid f(x,e_1)) - EU(x \mid f(x,e_2)) \geq 0 \qquad \text{Q.E.D.}$$

(b) A "Sandmo Increase in Risk": A special case of the "Rothschild-Stiglitz increase in risk" is the "Sandmo increase in risk". This "special case"-status stems from the fact that a specific distribution function (i.e., $F((x-w(e))/e)$ is used in the place of the general function, $F(x,e)$. The comparability of the "Sandmo distribution function" and the general distribution function is obtained by initializing the values of e and w(e). In particular, $x=ex+w(e)$ for $e=1$ and $w(1)=0$.

Three theorems on the "Sandmo increase in risk" are offered below. In the first, we demonstrate that the parameter e directly affects the variance of x. In the second, we show that w(e) is a scaling factor. Finally, we show that the "Sandmo distribution function," $F((x-w(e))/e)$ satisfies Equations (25) and (26), that is, we show that the "Sandmo increase in risk" is a special case of the "Rothschild-Stiglitz increase in risk".

Theorem 9: Consider $x=ex+w(e)$ for $e=1$ and $w(1)=0$. Then:

$$Var(x) = e^2 Var(x)\Big|_{e=1}$$

<u>Proof</u>: $\mathrm{Var}(x) = E[x-E(x)]^2$

$$= E[ex+w(e) - E(ex+w(e))]^2 \Big|_{e=1}$$

$$= e^2 \; E[x-E(x)]^2 \Big|_{e=1}$$

$$= e^2 \; \mathrm{Var}(x) \Big|_{e=1}$$

Q.E.D.

Theorem 9 demonstrates the role which e plays in the distribution function, F(x,e). Clearly, e is a "variance-shifting" parameter, i.e.,

$$\frac{d\mathrm{Var}(x)}{de} = 2e\mathrm{Var}(x) \Big|_{e=1} > 0 \qquad (33)$$

An increase in e above its initial value of unity increases not only the variance (as indicated by Theorem 9), but also increases the expected value of x. In order to preserve the mean in a "Sandmo increase in risk" (i.e., in order to satisfy Equation (25), an increase in e should be matched by a decline in w(e), and this decline in w(e) should ensure that the mean of x is unchanged (i.e., dE(x)=0). This next Theorem presents an implication of this requirement.

<u>Theorem 10</u>: Consider x=ex+w(e) for e=1 and w(1)=0. Then for a constant mean,

$$E(x) = -dw/de$$

<u>Proof</u>: For the Equation, x=ex+w(e), we take the expectation of both sides. This gives:

$$E(x) = eE(x) + w(e)$$

Taking the differential (and holding the mean constant), gives:

$$E(x)de + dw(e) = 0$$

which implies:

$$E(x) = -\, dw/de \qquad \text{Q.E.D.}$$

A "Sandmo increase in risk" is characterized by $de>0$ and $dw/de=-E(x)<0$. In the next Theorem we demonstrate that a "Sandmo increase in risk" is a special case of a "Rothschild-Stiglitz increase in risk."

<u>Theorem 11</u>: The random variable, $x=ex+w(e)$ for $e=1$, $w(1)=0$, satisfies Equations (25) and (26), i.e.,

$$\text{(i)} \quad \int_0^1 F_e(x,e)dx = 0$$

$$\text{(ii)} \quad \int_0^{x^*} F_e(x,e)dx \geq 0 \text{ for } x^* \varepsilon [0,1]$$

<u>Proof</u>: The random variable may be written as $x=[x-w(e)]/e$. The distribution function for this random variable is $F(x,e)=F((x-w(e))/e)$. Thus

$$F_e(x,e) = F_x(\cdot)[(-\frac{dw}{de}\cdot e - (x-w(e)))/e^2]$$

$$= F_x(\cdot)[E(x) - x]\Big|_{e=1,w(1)=0}$$

by virtue of Theorem 10. Taking the integral of this product, we get:

$$\int_0^{x^*} F_e(x,e)dx = \int_0^{x^*} F_x(\cdot)[E(x) - x]dx\Big|_{e=1,w(1)=0}.$$

Now: (i) for $x^*=1$, $\int_0^1 F_x(\cdot)[E(x)-x]dx = 0$, and

(ii) for $x^* \varepsilon [0,1]$, $\int_0^{x^*} F_x(\cdot)[E(x)-x]dx \geq 0$ Q.E.D.

(c) <u>The Comparative-Static Effect of a Mean-Preserving Increase in Risk - An Analysis of Three Cases</u>: For purposes of comparative-static analysis, the contribution of the two preceding subsections is the provision of a <u>definiton</u> of an exogenous variable. A major topic in the economics of uncertainty is not merely the presentation of definitions, but the determination of the <u>sufficient condition(s)</u> for signing the effect of a mean-preserving increase in uncertainty (i.e., the exogenous variable) on the optimal level of a control variable (i.e., the endogenous variable).

In the discussion which follows, we show that <u>the nature of this (these) sufficient condition(s) depends upon the nature of the objective function</u>. To demonstrate this, we consider three cases:

I. $$\max_{a(e)} V(a;x,e) = \int U(a,x)dF(x,e) \qquad (34)$$

II. $$\max_{a(e)} V(Z(a,x);x,e) = \int U(Z(a,x))dF(x,e) \qquad (35)$$

(a) where Z is <u>nonlinear</u> in a and x, and

(b) where Z is <u>linear</u> in a and x.

<u>Case I</u>: The sufficient condition for signing the effect of a "Rothschild-Stiglitz increase in risk" on the optimal level of the choice variable, a*(e), is provided by Rothschild and Stiglitz (1971, p. 67), and again by Diamond and Stiglitz (1974, p. 340). To explicate this sufficient condition, we present the Diamond-Stiglitz Theorem.

<u>Theorem 12 [Diamond-Stiglitz (1974)]</u>: Let a*(e) be the level of the control variable which maximizes $\int U(a,x)$.

$dF(x,e)$. If increases in e represent mean-preserving increases in risk (i.e., satisfy Equations (25) and (26)), then a* increases (decreases) with e if U_a is a strictly convex (concave) function of x, i.e., if $U_{axx}>0$ (<0).

Proof: $a^*(e)$ is defined implicitly by the first order condition for expected utility maximization, i.e.,

$$\int U_a(a,x)F_x(x,e)dx = 0$$

Implicitly differentiating, we have,

$$da^*/de = -[\int U_a F_{xe} dx]/[\int U_{aa} F_x dx]$$

Since the denominator is negative,

$$\text{sign } [da^*/de] = \text{sign } [\int U_a F_{xe} dx]$$

Noting that $F_e(0,e) = F_e(1,e) = T(0,e) = T(1,e) = 0$, and applying integration by parts twice, we have

$$\int U_a F_{xe} dx = - \int U_{ax} F_e dx = \int U_{axx} T(x,e) dx$$

where T is defined in Equation (26). By assumption T is non-negative so that:

$$\text{sign } [da^*/de] = \text{sign } [U_{axx}(a,x)]$$

assuming that U_{axx} is uniformly signed for all x. Q.E.D.

Case II(a): In a recent paper by Kraus (1979) which was subsequently extended by Katz (1981), the sufficient condition provided by Rothschild and Stiglitz (i.e., $U_{axx}>0$ (<0)) is shown to be insufficient in problems of the form:

$$\max_{a(e)} V(a;x,e,Z) = \int U(Z(a,x))dF(x,e) \qquad (36)$$

where Z is non-linear in a and x. In particular, the sign of U_{axx} in general is indeterminate in that:

$$U_{axx} = U_Z Z_{axx} + 2U_{ZZ} Z_{ax} Z_x$$
$$+ Z_a(U_{ZZ} Z_{aa} + U_{ZZZ} Z_x^2) \quad (37)$$

cannot be signed *a priori*.

Case II(b): In the third case, consider the problem defined by Equation (36) such that Z(a,x) is linear in a and x. In particular, assume Z is of the form:

$$Z = V(a)x + Y(a)$$

where V(a) and Y(a) are both positively valued functions. The first and second order conditions for a maximum are:

$$V_a = E[U_Z[V'(a)x + Y'(a)]] \quad (38)$$

and:

$$V_{aa} = E[U_{ZZ}[V'(a)x + Y'(a)]^2]$$
$$+ E[U_Z[V''(a)x + Y''(a)]] < 0 \quad (39)$$

Thus, if Equations (38) and (39) hold, there exists a unique value of a, a*, which satisfies Equation (36).

In this case, we determine the sufficient condition for signing the effect of a "Sandmo increase in risk" on a*(e). To do this, we substitute ex+w(e) for x in Equation (38), and take the total differential. Setting all increments except those for a* and e equal to zero, we obtain:

$$da^*/de \Big|_{e=1,w(1)=0} = -V_{ae}/V_{aa} \quad (40)$$

Since the denominator of Equation (40) is negative, it follows that:

$$\text{sign } (da^*/de) = \text{sign } (V_{ae}) \quad (41)$$

Now the sign of V_{ae} can be determined by using what Batra (1975) terms the "covariance method." In particular,[14]

$$V_{ae} = E[U_Z[[x-E(x)]V'(a)]]$$
$$+ E\{[U_{ZZ}][x-E(x)][V'(a)x + Y'(a)] \cdot V(a)\}$$
$$= E(xS) - E(x) \cdot E(S)$$
$$= Cov(x,S) \tag{42}$$

where:

$$S = U_Z[V'(a)] + U_{ZZ}V(a)[V'(a)x + Y'(a)] \tag{43}$$

Thus from Equations (41) and (42), it follows that:

$$\text{sign } (da^*/de) = \text{sign } (Cov(x,S)) \tag{44}$$

The sign of Cov(x,S) is identical to the sign of $\partial S/\partial x$, i.e.,

$$\partial S/\partial x = 2U_{ZZ}V'V + U_{ZZZ}(V'(a)x + Y'(a))V^2 \tag{45}$$
$$= U_{axx} \tag{46}$$

Therefore:

$$\text{sign } (da^*/de) = \text{sign } (U_{axx}) \tag{47}$$

II-4 A Summary of Four Papers: Block and Heineke (1973), Tressler and Menezes (1980), Sandmo (1970), and Hanson and Menezes (1978)

Four papers are central to this dissertation. The first is authored by Block and Heineke (1973). In their paper, Block-Heineke (B-H) address two problems: the determination of the sufficient conditions for signing the effect on the optimal level of labor supply of a mean-preserving increase in the uncertainty of (i) non-wage (or autonomous) income, and (ii) the wage-rate. With regards to the first problem, and using a "Sandmo increase in risk", B-H show that four sufficient conditions are needed. These are:

(i) an EUF which exhibits absolute risk aversion, (ii) an EUF which exhibits CARA in labor supply, (iii) an EUF which exhibits DARA in income, and (iv) an EUF which treats labor supply as an inferior good. These conditions permit B-H to assert that a "Sandmo increase" in the uncertainty of autonomous income increases the optimal level of labor supply.

With regards to the second problem, B-H were unable to specify a set of sufficient conditions. A few years later, Tressler and Menezes (1980) proposed a sufficient condition for signing the effect in the second problem. A sufficient condition which Tressler-Menezes (T-M) identify is the HMTC. Using a "Rothschild-Stiglitz increase in risk," T-M show that a mean-preserving increase in the uncertainty of the wage-rate reduces the optimal level of labor supply provided that the HMTC holds.

There is a symmetrical relationship between Block and Heineke (1973) and Tressler and Menezes (1980) on one hand, and Sandmo (1970), and Hanson and Menezes (1978) on the other. Like Block and Heineke, Sandmo addresses two problems. Sandmo solves the first. The second problem is solved by Hanson and Menezes (1978).

Since a grasp of the mechanics of the proofs used in these four papers is required at subsequent, later stages of this dissertation, a formal yet brief explication of these four papers is presented here. We begin with the first B-H problem, and the B-H result.

(a) The First B-H (1973) Problem: The first problem which

B-H consider is of the form:

$$\max_{L(e)} V(L,e) = \int_{-A}^{\infty} U(L,wL+Y^\circ)g(Y^\circ,e)dY^\circ \tag{48}$$

where $\int U(\cdot)g(Y^\circ)dY^\circ$ is the EUF with arguments in labor supply, L, and income, $Y=wL+Y^\circ$. Income has two components: autonomous income, Y°, and wage income, wL, where w represents the hourly wage-rate. In this problem, Y° is treated as a stochastic variable, and accordingly $g(Y^\circ)$ is the subjective probability density function. In both the first and the second problem, B-H assume the EUF exhibits the usual properties (i.e., $U_Y>0$, $U_{YY}<0$, $U_L<0$, and $U_{LL}<0$).

The first and second order conditions associated with Equation (48) are:

$$V_L = E(U_L) + wE(U_Y) = 0 \tag{49}$$

and:

$$V_{LL} = E(U_{LL}) + 2wE(U_{LY}) + w^2E(U_{YY}) < 0 \tag{50}$$

Thus, if Equations (49) and (50) hold, there exists a unique level of labor supply, L*, which satisfies Equation (48).

To determine the effect of a "Sandmo increase in risk" as measured by e on L*, B-H introduce three additional, plausible assumptions. Referring later to these as the B-H assumption-set, the additional assumptions are: (i) labor as an inferior good (i.e., $dL/dY^\circ\big|_{Y=E(Y)} < 0$), (ii) CARA in labor (i.e., $\partial R_A(L,Y(L))/\partial L = 0$), and (iii) DARA in income (i.e., $\partial R_A(L,Y(L))/\partial Y < 0$).

This assumption-set is deemed plausible by B-H in the following manner. Firstly, the assumption DARA in Y represents the "Arrow-Pratt" hypothesis. As B-H (1973) note, DARA in Y implies that "the individual becomes increasingly willing to accept a wage of a given size as his income increases, in the sense that odds demanded diminish. Arrow (1965) argues that both intuition and fact support this hypothesis" (p. 378).

Secondly, CARA in L implies that the Arrow-Pratt measure of absolute risk aversion is invariant in L. That is, CARA in L "implies that the odds demanded by the individual for taking a risky action are not affected by how much he 'works'" (p. 378).

Finally, the assumption that labor is an inferior good is supported by empirical evidence. For example, casual inspection of historical data reveals that the increase in the standard of living since the turn of the century is associated with a decline in the length of the average work week. For a general discussion of this phenomenon, see Gunderson (1980, pp. 83-98).

In determining the effect of a "Sandmo increase in risk," B-H follow Sandmo's (1969, 1970, and 1971) procedure, and let $Y^\circ = eY^\circ + w(e)$ for $e=1$ and $w(1)=0$. As discussed in Theorem 10, a defining characteristic of a "Sandmo increase in risk" is that:

$$dw/de = -E(Y^\circ) \tag{51}$$

In addition to Equation (51), we require the results of four lemmas and one theorem. These are:

Lemma 1: $dY/de = Y° - E(Y°)$

Proof: At $e=1$ and $w(1)=0$, the income constraint may be written as:

$$Y = wL + eY° + w(e)$$

Differentiating with respect to e, and using Equation (51), we obtain:

$$dY/de = Y° - E(Y°) \qquad \text{Q.E.D.}$$

Lemma 2: $\frac{dL^*}{de} = Cov(Y°, U_{LY} + wU_{YY})/V_{LL}$

Proof: Letting $Y° = eY° + w(e)$ at $e=1$ and $w(1)=0$ in the first order condition, and taking the implicit derivative, we obtain:

$$V_{LL}dL^* + [E(U_{LY}\frac{dY}{de}) + E(wU_{YY}\frac{dY}{de})]de$$

$$= V_{LL}dL^* + [E(U_{LY}(Y°-E(Y°)))+E(wU_{YY}(Y°-E(Y°)))]de$$

$$= V_{LL}dL^* + [E((U_{LY}+wU_{YY})Y°)-E(U_{LY}+wU_{YY})E(Y°)]de$$

$$= V_{LL}dL^* + Cov(Y°, U_{LY} + wU_{YY})\ de = 0$$

by virtue of Lemma 1 and the definition of the covariance function. Rearranging terms, we obtain:

$$dL^*/de = -Cov(Y°,U_{LY} + wU_{YY})/V_{LL} \qquad \text{Q.E.D.}$$

Lemma 3: If the EUF exhibits CARA in labor, and DARA in income, then

$$\partial R_A(L,Y(L))/\partial L < 0$$

Proof: Differentiating $R(L,Y(L))$ with respect to L, we obtain:

$$dR(\cdot)/dL = \partial R(\cdot)/\partial Y \cdot \partial Y/\partial L + \partial R(\cdot)/\partial L \qquad (52)$$

If the EUF exhibits CARA in L (i.e., $\partial R(\cdot)/\partial L=0$) and if the

EUF exhibits DARA in Y (i.e., $\partial R(\cdot)/\partial Y<0$) then Equation (52) becomes:

$$dR(\cdot)/dL = w\cdot\partial R(\cdot)/\partial Y < 0 \qquad \text{Q.E.D.}$$

Lemma 4: If labor is an inferior good, then $E(U_{LY}+wU_{YY})<0$.

Proof: From Equation (49), it follows that

$$dL/dY^{\circ} = -E(U_{LY} + wU_{YY})/V_{LL}$$

Since $V_{LL}<0$ [Equation (50)], and since $dL/dY^{\circ}<0$ by assumption, $E(U_{LY}+wU_{YY})<0$. Q.E.D.

Theorem 13: If the EUF exhibits CARA in labor, and DARA in income, then $dL^*/de>0$.

Proof: By Lemma 2,

$$dL^*/de = -Cov(Y^{\circ}, U_{LY} + wU_{YY})/V_{LL}$$

Since $V_{LL}<0$, it is clear that:

$$\text{sign } (dL^*/de) = \text{sign } (Cov(Y^{\circ}, U_{LY}+wU_{YY})) \qquad (53)$$

Now the sign of the covariance function can be determined by reference to a third variable, Y. If the derivatives of the arguments of the covariance function with respect to Y are of like signs, then the sign of the covariance function is positive. Otherwise, the sign of the function is non-positive. From the income constraint, it is clear that $dY^{\circ}/dY>0$. Secondly, by Lemma 3, we may write:

$$\partial(-U_{YY}/U_Y)\partial L = w\cdot\partial(-U_{YY}/U_Y)/\partial Y < 0$$

By Young's Theorem and by rearranging terms,[15] this may be written as:

$$\frac{\partial}{\partial Y}[-(U_{LY} + wU_{YY})/U_Y] < 0$$

which implies:

$$\frac{\partial}{\partial Y}\ [(U_{LY} + wU_{YY})/U_Y]$$

$$= [\frac{\partial}{\partial Y}(U_{LY}+wU_{YY})\cdot U_Y-(U_{LY}+wU_{YY})\cdot U_{YY}]/U_Y^2$$

$$= \frac{1}{U_Y}\ [\frac{\partial}{\partial Y}\ (U_{LY}+wU_{YY}) + (U_{LY}+wU_{YY})R_A(\cdot)] > 0$$

Since $U_Y>0$, then

$$\frac{\partial}{\partial Y}\ (U_{LY}+wU_{YY}) > -R_A(\cdot)(U_{LY}+wU_{YY})$$

Since $R_A(\cdot)>0$, and $(U_{LY}+wU_{YY})<0$ [Lemma 4], then

$$\frac{\partial}{\partial Y}\ (U_{LY}+wU_{YY}) > 0 \tag{54}$$

By $dY°/dY>0$ and Equation (54), Cov(,)>0. Since Cov(,)>0, then $dL^*/de>0$ by virtue of Equation (53). Q.E.D.

(b) The Second B-H (1973) Problem: The second problem which B-H consider is of the form:

$$\max_{L(e)} V(L,e) = \int_0^b U(L,wL)g(w,e)dw \tag{55}$$

The variables, and the general properties of the EUF, are the same as in the first problem with the exception that in the second problem the wage-rate (and not autonomous income) is treated as the stochastic variable.

The first and second order conditions for maximum are:

$$V_L = E(U_L) + E(wU_Y) = 0 \tag{56}$$

and:

$$V_{LL} = E(U_{LL}) + 2E(wU_{LY}) + E(w^2U_{YY}) < 0 \tag{57}$$

Thus, if Equations (56) and (57) hold, there exists a unique level of labor supply, L*, which satisfies Equation (55).

Using the same approach employed in their solution to the first problem, and using a "Sandmo-type" increase in uncertainty, B-H were unable to determine unambiguously the direction of the effect of a mean-preserving increase in the uncertainty of the wage-rate on the optimal level of labor supply. A solution to this problem was offered recently by Tressler and Menezes (1980). Rejecting the B-H assumption-set, Tressler and Menezes (T-M) show that the HMTC is sufficient to sign unambiguously the comparative-static effect. Provided the HMTC holds, T-M show that a "Rothschild-Stiglitz-type" increase in wage-rate uncertainty decreases the optimal level of labor supply, i.e., $dL^*/de<0$.[16]

The proof of this comparative-static result may be characterized as having the following components:

Step 1: Define the marginal rate of transformation (MRT) between hours of leisure foregone and income. Analytically, the MRT is written as $R(\cdot,\cdot)$.

Step 2: Show that:

$$\text{sign}\left(\frac{dL^*}{de}\right) = \text{sign}\left(-\frac{dR}{de}\right) \tag{58}$$

and this is achieved in three steps:

$$\begin{aligned}\text{sign}\left(\frac{dL^*}{de}\right) &= \text{sign}\,(V_{Lww}) \\ &= \int (U_L + wU_Y)g_e(w,e)dw \\ &= \text{sign}\left(-\frac{dR}{de}\right)\end{aligned} \tag{58'}$$

Step 3: Show that the HMTC implies that sign (-dR/de) is negative.

In the remainder of this Section, we sketch the proofs used by T-M in each of these three steps.

Step 1: In the objective function, EU(L,wL), L has two meanings. As the first argument, L represents the number of hours of leisure foregone. As a component of the second argument, L represents the number of hours worked to provide income. This dual meaning of L is the motivation for the construction of a concept, the MRT between hours of leisure foregone and income.

To distinguish between the two meanings of L, T-M relabel the second meaning as z. Under this relabelling, the objective function becomes:

$$EU(L,wz) = \int_0^b U(L,wz)g(w,e)dw \qquad (59)$$

subject to $L=z$, $w\varepsilon[0,b]$, and $L\varepsilon[0,T]$. Also under this relabelling, the first and second order conditions (i.e., Equations (56) and (57) are unchanged. Finally, under this relabelling, the necessary condition may be written as:

$$R(L_0,wz_0) = -\frac{EU_L(L_0,wz_0)}{E(wU_Y(L_0,wz_0))} = 1 \qquad (60)$$

where $L_0=z_0$ represents the utility maximizing level of labor supply, and $R(L_0,z_0)$ represents the MRT between hours of leisure foregone and income at the optimum value, $L_0=z_0$.

Connected with the optimum in (L,z)-space is a locus of points of constant expected utility (i.e., an iso-utility curve or indifference curve). In particular, let I_0 denote the set of risky labor-income bundles, (L,wz), which have the same expected utility as (L_0, wz_0), i.e., let

$$I_0 = ((L,z) \,|\, EU(L,wz) = EU(L_0, wz_0)) \tag{61}$$

The properties of I_0 include that it is increasing and strictly convex in (L,z)-space. In particular, for EU(L,wz)=k, it follows that

$$\frac{dz}{dL} = -\frac{EU_L(L,wz)}{E(wU_Y(L,wz))} > 0 \tag{62}$$

and:

$$\frac{d^2z}{dL^2} = -\frac{[E(U_{LL})+2E(wU_{LY})+E(w^2U_{YY})]}{E(wU_Y)} > 0 \tag{63}$$

by virtue of Equation (57). Note that in Equation (62), the RHS equals R(L,wz) by virtue of Equation (60).

The results of Equations (60), (61), and (62) are presented graphically in Figure 2.2. In the diagram note that at the optimum, I_0 is tangent to the line z=L, and that I_0 is increasing and strictly convex in (L,z)-space. For later discussion, it will be useful to remember that slope of I_0 equals the MRT_{zL}.

<u>Step 2</u>: By virtue of Rothschild and Stiglitz (1971, p. 67) and Diamond and Stiglitz (1974, p. 340), it is clear that:

$$\text{sign}\left(\frac{dL^*}{de}\right) = \text{sign}\left(V_{Lww}\right) \tag{64}$$

Recall Theorem 12. Differentiation will show that:

$$V_{Lww} = 2E(U_{YY})L + E(wU_{YYY})\cdot L^2 + E(U_{LYY})\cdot L^2 \tag{65}$$

This result (i.e., Equation (65)) is needed in the next

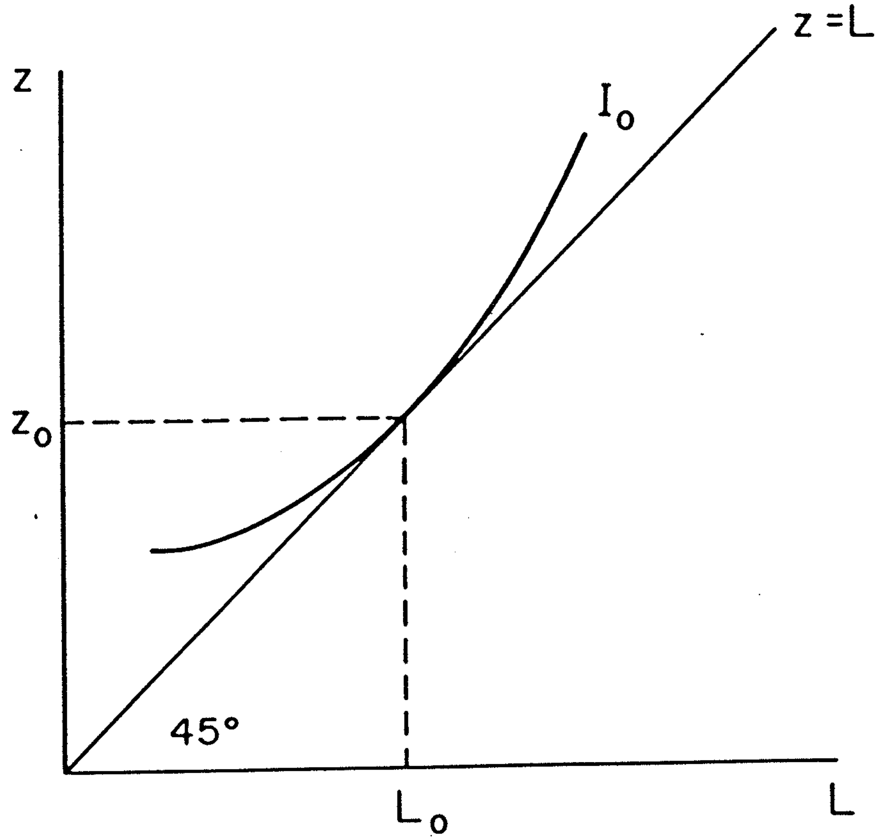

Figure 2.2.--A Diagrammatical Representation of the Optimal Level of Labor Supply

theorem.

Theorem 14: $V_{Lww} = \int (U_L+wU_Y)g_e(w,e)dw$

Proof: Integrating $\int (U_L+wU_Y)g_e(w,e)dw$ by parts twice, and using Equations (25) and (26), we obtain the result, Equation (65). Q.E.D.

Theorem 15: sign $(\int (U_L+wU_Y)g_e(w,e)dw)$ = sign $(-dR/de)$ in a small neighborhood of (L_0,z_0).

Proof: By virtue of Equation (62), let $R(L,wz)=-I_1/I_2$ where $I_1=\int U_Lg(w,e)dw<0$ and $I_2=\int wU_Y\, g(w,e)dw>0$. Now:

$$\frac{dR}{de} = -[\frac{\partial I_1}{de} \cdot I_2 - I_1 \frac{\partial I_2}{\partial e}]/(I_2)^2 \qquad (66)$$

where:

$$\frac{\partial I_1}{\partial e} = \int U_L g_e(w,e)dw$$

and:

$$\frac{\partial I_2}{\partial e} = \int wU_Y g_e(w,e)dw \qquad (67)$$

At the optimum, (L_0,z_0), it is clear that:

$$\int U_L g(w,e)dw = -\int wU_Y g(w,e)dw \qquad (68)$$

by virtue of Equation (56). Therefore, at the optimum, Equation (66) becomes:

$$\left.\frac{dR}{de}\right|_{(L_0,z_0)} = -\int (U_L+wU_Y)g_e(w,e)dw/I_2 \qquad (69)$$

Since $I_2>0$, Equation (69) implies:

$$\text{sign } (\int (U_L+wU_Y)g_e(w,e)dw) = \text{sign } (-\frac{dR}{de})$$

in a small neighborhood of (L_0,z_0). Q.E.D.

In summary, the results of Step 2 (i.e., sign (dL*/de)=sign(-dR/de)) are provided by Theorems 12, 14, and 15.

Step 3: In Step 3, we show that the HMTC implies that sign (-dR/de) is negative. T-M demonstrate this in the following manner. Let (L_1, z_1) and (L_2, z_2) be any two points on the iso-utility curve, I_0 (recall Equation (61)), such that $L_1 < L_2$ and $z_1 < z_2$. By definition of I_0, the individual is indifferent between the two risky bundles, i.e.,

$$EU(L_1, wz_1) = EU(L_2, wz_2) \tag{70}$$

In both bundles, L is known with certainty. However, w is not known with certainty, and therefore, $Y_1 = wz_1$ and $Y_2 = wz_2$ represent random or stochastic variables.

Consider a new stochastic wage rate, w*, whose subjective probability density function is g(w*,e*). This function is chosen so that the relationship between g(w,e) and g(w*,e*) ensures that: (i) E(w)=E(w*), and (ii) e<e* (i.e., the dispersion of g(w,e) is less than the dispersion of g(w*,e*). Stated differently, g(w*,e*) is chosen such that g(w,e) and g(w*,e*) satisfy Equations (25) and (26). Also under w*, we obtain a second pair of risky bundles. These are: (L_1, w^*z_1) and (L_2, wz_2). A useful related Theorem is:

Theorem 16: $\text{sign } \left(\frac{dR}{de}\right) = \text{sign } (R^*(L, w^*z) - R(L, wz))$

Proof: $\frac{dR}{de} \approx \frac{\Delta R}{\Delta e} = \frac{R^*(L, w^*z) - R(L, wz)}{e^* - e}$

Since $e^* > e$, then $\frac{dR}{de} \gtrless 0$ as $R^*(L, w^*z) \gtrless R(L, wz)$. Q.E.D.

In sum, we have progressed to the stage where the sign $(dL^*/de)=\text{sign}(-dR/de)=\text{sign}[-(R^*(L,w^*z)-R(L,wz))]$ by virtue of Theorems 12, 14, 15, and 16. <u>What we wish to show in the remaining paragraphs is that the HMTC implies $R^*(L,w^*z)>R(L,wz)$, and therefore $dL^*/de<0$.</u> To do this, the T-M proof requires that we show that three statements are mutually implied. These are:

<u>S1</u>: $R^*(L,w^*z) > R(L,wz)$ for all (L,z)

<u>S2</u>: $V^1(L_1,z_1,w,w^*) < V^2(L_2,z_2,w,w^*)$ for all (L_1,z_1) and (L_2,z_2) such that $EU(L_1,wz_1)=EU(L_2,wz_2)=EU^0$ and such that $L_1<L_2$ where $V^1(\cdot)$ and $V^2(\cdot)$ are PIFs.

<u>S3 [The HMTC]</u>: $EU(L_1,w^*z_1)>EU(L_2,w^*z_2)$ for all (L_1,z_1) and (L_2,z_2) such that $EU(L_1,wz_1)=EU(L_2,wz_2)=EU^0$ and such that $L_1<L_2$.

A comment on (S3) should be offered. The inequality in (S3) is analogous to Assumption [A] in Section II-2 (e). Recall too that Equation (23) is an implication of Assumption [A].

<u>Theorem 17</u>: (S3) implies (S2).

<u>Proof</u>: Recall (L_1,z_1) and (L_2,z_2) are chosen to ensure $EU(L_1,wz_1)=EU(L_2,wz_2)$, i.e., both bundles are on I^0. Assume w undergoes a "Rothschild-Stiglitz increase in risk" so that the new corresponding random wage-rate is w*. Using the concept of a PIF (see Section II-2(b)), we may write:

$$v_1(L_1,z_1,w) = U(L_1,\bar{w}z_1) - EU(L_1,wz_1)$$

$$v_1^*(L_1,z_1,w^*) = U(L_1,\bar{w}z_1) - EU(L_1,w^*z_1)$$

Subtracting v_1 from v_1^*, we obtain:

$$V^1(L_1,z_1,w,w^*) = EU(L_1,wz_1) - EU(L_1,w^*z_1)$$

By similar reasoning, we obtain:

$$V^2(L_2,z_2,w,w^*) = EU(L_2,wz_2) - EU(L_2,w^*z_2)$$

Now since $EU(L_1,wz_1)=EU(L_2,wz_2)=EU^0$, subtracting V^1 from V^2 gives:

$$\Delta V\Big|_{EU=EU^0} = [V^2(\bullet) - V^1(\bullet)]\Big|_{EU=EU^0}$$

$$= [EU(L_1,w^*z_1)-EU(L_2,w^*z_2)] \qquad (71)$$

By virtue of the HMTC, the RHS of Equation (71) is positive, and therefore (S3) implies (S2). Q.E.D.

Theorem 18: (S2) implies (S1).

Proof: Taking the derivative of $V(\bullet)$ with respect to z yields:

$$\frac{dV}{dz} = [EU_L(L,wz) - EU_L(L,w^*z)]\frac{dL}{dz}$$

$$+ [E(wU_Y(L,wz)] - E(w^*U_Y(L,w^*,z)] \qquad (72)$$

Along the iso-utility curve, $EU^0=EU(L,wz)$, Equation (62) holds. Substituting Equation (62) into Equation (72), we get:

$$\frac{dV}{dz}\Bigg|_{EU=EU^0} = \frac{E[wU_Y(L,wz)]}{EU_L(L,wz)} \bullet EU_L(L,w^*z)$$

$$- E[w^*U_Y(L,w^*z)]$$

$$= -\frac{E[wU_Y(L,wz)]}{EU_L(L,wz)} \bullet E[w^*U_Y(L,w^*z)]$$

$$\bullet [R^*(L,w^*z) - R(L,wz)] \qquad (73)$$

Since $U_Y>0$ and $U_L<0$, it is clear from Equation (73) that:

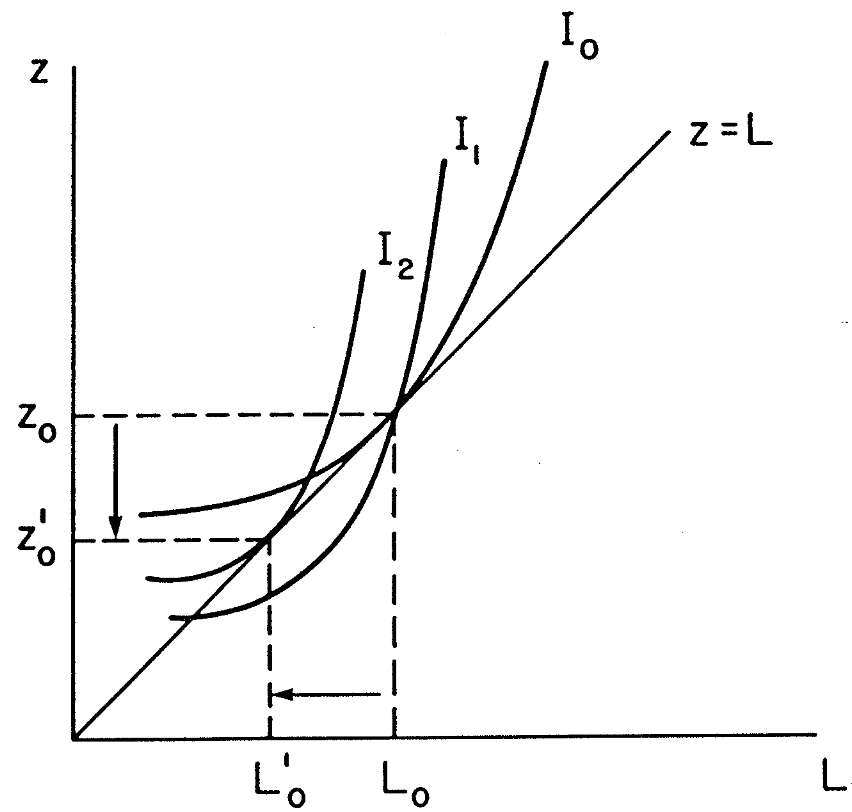

Figure 2.3.--A Diagrammatical Representation of the Impact of a "Rothschild-Stiglitz-Type" Increase in Wage-Rate Uncertainty on the Optimal Level of Labor Supply.

$$\text{sign } [\Delta V/\Delta z \Big|_{EU=EU^0}] = \text{sign } [R^*(L,w^*z) - R(L,wz)] \quad (74)$$

Since $\Delta z=z_2-z_1>0$

$$\text{sign } [\Delta V \Big|_{EU=EU^0}] = \text{sign } [R^*(L,w^*z) - R(L,wz)] \quad \text{Q.E.D.}$$

The T-M (1980) solution to the second problem posed by B-H (1973) may be presented diagrammatically. As displayed in Figure 2.3, the initial equilibrium occurs at the point of tangency of I_0 and the identity $z=L$. T-M (1980) then show that $dR/de>0$. Graphically, this means that an increase in e causes I_0 to rotate counterclockwise as is presented in Figure 2.3. Associated with an increase in e is a new family of indifference curves with one member which is tangent to $z=L$. The point of tangency between this member of the new family of indifference curves occurs at a lower level of L than the old. Therefore, in Figure 2.3, we observe $dL^*/de<0$.

(c) <u>The First Sandmo (1970) Problem</u>: The two remaining papers of import are Sandmo (1970) and Hansen and Menezes (1978). In his paper, Sandmo addresses <u>two</u> problems. The first he terms the problem of "income risk," and the second he terms the problem of "capital risk." It is the problem of "income risk" which we term the first Sandmo (1970) problem and "capital risk" the second Sandmo (1970) problem.

The first Sandmo (1970, pp. 355-6) problem may be characterized as follows. Consider an individual whose welfare is measured by a utility function with arguments in first-period consumption, C_1, and expected second-period consumption, $E(C_2)$. As usual, suppose that this function is continuous, and three times differentiable. Suppose too that this function has positive first-derivatives, and negative second-derivatives. The first Sandmo (1970) problem concerns the optimal appropriation of initial wealth, W_1, in the first-period between first-period consumption and savings which pays a rate of interest i. The level of first-period consumption is defined by:

$$C_1 = W_1 - S \tag{75}$$

In the second period, the individual receives stochastic autonomous income, Y_2. The expected (or average) value of Y_2 is $E(Y_2)=\int Y_2 g(Y_2,e)dY_2$ where $g(Y_2,e)$ is the subjective probability density function of Y_2, and e represents the "Rothschild-Stiglitz-measure" of uncertainty. The expected level of second-period consumption is

$$E(C_2) = E(Y_2) + (1+i)S \tag{76}$$

Using (75), Equation (76) may be rewritten as:

$$E(C_2) = E(Y_2) + (1+i)[W_1-C_1] \tag{77}$$

In formal terms, the first Sandmo (1970) problem is:

$$\max_{C_1} V(C_1,S,e)$$

$$= \int U(C_1,Y_2 + (1+i)(W_1-C_1))g(Y_2,e)dY_2 \tag{78}$$

The first and second order conditions for this problem are:

$$V_{C_1} = E(U_1 - (1+i)U_2) = 0 \tag{79}$$

$$V_{C_1C_1} = E(U_{11} - w(1+i)U_{12} + (1+i)^2U_{22}) < 0 \tag{80}$$

Thus, if (79) and (80) hold, there exists a unique value of C_1, C_1^*, which satisfies (78).

Sandmo determines the comparative-static effect of a "Sandmo-type" increase in the uncertainty of i, i.e., he determines the sign (dC_1^*/de). He shows that "decreasing temporal risk aversion" is sufficient to assert $dC_1^*/de<0$.[17]

(d) <u>The Second Sandmo (1970) Problem</u>: The first and the second problems are identical except in the second problem, initial wealth is allocated between first-period consumption and a capital investment, K, and K pays a stochastic rate of return of x. Therefore, the level of first-period consumption is defined by:

$$C_1 = W_1 - K \tag{81}$$

and the expected level of second-period consumption is defined by:

$$E(C_2) = E(1+x)\cdot K \tag{82}$$

where $E(1+x)=1+E(x)=1+\int x\cdot g(x,e)dx$ where $g(x,e)$ is the probability density function for x. In formal terms, the second Sandmo (1970) problem is:

$$\max_{C_1} V(C_1,K,e)$$

$$= \int U(C_1,(1+x)(W_1-C_1))g(x,e)dx \tag{83}$$

The first and second order conditions for this problem are:

$$V_{C_1} = E(U_1 - (1+x)U_2) = 0 \tag{84}$$

$$V_{C_1C_1} = E(U_{11} - 2(1+x)U_{12} + (1+x)^2U_{22}) < 0 \tag{85}$$

Thus, if (84) and (85) hold, there exists a unique value of C_1, C_1^*, which satisfies (83).

Sandmo could not determine the comparative-static effect of a "Sandmo-type" increase in the uncertainty of x, i.e., he could not determine the sign (dC_1^*/de). Recently, Hanson and Menezes (1978) have shown that the HMTC is sufficient to assert $dC_1^*/de>0$.

Footnotes to Chapter II

[1]An affine function is one which is both concave and convex. A linear function is an affine function.

[2]Key references for this criterion are Markowitz (1952), and Tobin (1958).

[3]A key reference here is Roy (1952).

[4]In this Section, we limit our discussion to univariate measures of risk preference. New directions of research into risk preference include research into multivariate measures. See for example Duncan (1977) and Karni (1979).

[5]See Arrow (1970) and Pratt (1964).

[6]See Arrow (1970) and Pratt (1964).

[7]See Menezes and Hanson (1970).

[8]For an English translation of Bernoulli's article, see Sommer (1954).

[9]Two recent investigations of DARA include Machina (1982), and Dybvig and Lippman (1983).

[10]A formal definition of the mean-preserving increase in the dispersion of w is provided in Section II-3. Note that Diamond and Stiglitz (1974) Equations (4) and (5) are our Equations (25) and (26) in Section II-3.

[11]The original articles in which these concepts are defined are Sandmo (1970), Rothschild and Stiglitz (1971), and Eeckhoudt and Hansen (1980).

[12]For a discussion of (25) and (26), see Diamond and Stiglitz (1974, pp. 338-40).

[13]Two comments should be made here. Firstly, see Nagatani (1978, pp. 71-76) and Fishburn and Vickson (1978) for a more complete treatment of the material which we cover in Theorems 6, 7 and 8. Secondly, in the following three

Lemmas, the concept of the Riemann-Stieltjes Integral is used. Although this integral and its properties are defined elsewhere (e.g., in Haaser and Sullivan (1971)), it seems appropriate to show its relationship to the Riemann integral, and finally define for it integration by parts.

Definition: Let F(x) and G(x) be two real valued functions defined on the real interval $x \varepsilon [a,b]$. Subdivide the interval into n equal parts according to $a=a_0<a_2<a_2<\ldots<a_n=b$. Let $x_i \varepsilon [a_{i-1},a_i]$. Consider the sum:

$$S = \sum_{i=1}^{n} F(x)[G(a_i) - G(a_{i-1})]$$

The Reimann-Stieltjes Integral is defined as:

$$\lim_{\Delta a_i \to 0} S = \int_a^b F(x)dG(x) = I$$

In words, then, F(x) is said to be Reimann-Stieltjes integrable relative to G(x). If F(x)=1, then I defines the Riemann-Stieltjes Integral of G(x) over [a,b].

The relationship between the Riemann and Riemann-Stieltjes Integrals can be shown in the following Lemma.

Lemma: Let F(x) and G(x) be defined on [a,b]. Suppose that $G'(x)=g(x)$ exists on [a,b] and that the functions F(x), G(x), and g(x) are integrable there.

Then:

$$\int_a^b F(x)dG(x) = \int_a^b F(x)g(x)dx$$

Furthermore, if G(x)=1, then

$$\int_a^b dG(x) = \int_a^b g(x)dx$$

Proof: See Haaser and Sullivan (1971, pp. 255-56).

Lemma [Integration by Parts]: If F(x) is integrable with respect to G(x), then G(x) is integrable with respect to F(x), and

$$\int_a^b G(x)dF(x) = [F(x)\cdot G(x)]\Big|_a^b - \int_a^b F(x)dG(x)$$

Proof: See Haaser and Sullivan (1971, pp. 254-55).

[14] The covariance function is defined as follows:

Lemma: Let X and Y denote two random variables that have the joint probability density function, f(x,y). Let the means of X and Y be E(X) and E(Y). Then the covariance of X and Y, $Cov(X,Y)=E(X-E(X))(Y-E(Y))$, is equal to $E(XY) - E(X)E(Y)$.

Proof:
$$\begin{aligned} Cov(X,Y) &= E(X - E(X))(Y - E(Y)) \\ &= E(XY-XE(Y) - YE(X) + E(X)E(Y)) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$
Q.E.D.

[15] Young's Theorem states that for any differentiable function of the form, $z=f(x,y)$, $f_{xy}=f_{yx}$. See Chiang (1984, p. 313).

[16] For a discussion about the plausibility of the HMTC, see Section II-2(e) above.

[17] Using our notation, decreasing temporal risk aversion is defined as:

$$\partial[(U_{12} - (1+i)U_{22})/U_2]/\partial C_2 < 0$$

CHAPTER III

SAFE WORK, RISKY WORK, AND LEISURE UNDER INCREASING WAGE-RATE UNCERTAINTY

The present Chapter presents an investigation into the impact of an increase in uncertainty for what we termed the first class of models. The particular problem addressed here is suggested in Cowell (1981). The solution to this problem serves as the yardstick (or a point of comparison) for the solution of similar problems - problems posed in four other Chapters, namely Chapters IV through VII inclusive.

III-1 An Introduction to the Problem

In a recent contribution to what might be termed the literature on "multiple job holdings under uncertainty", Cowell (1981) presents a first definition of the "safe work, risky work, and leisure" decision.[1] Cowell's paper is motivated by several casual observations - observations which lead him to conclude that: "workers with multiple labor activities and risky rewards are quite commonly encountered, and economic analysis has not yet been adequately developed to deal with this problem" (p. 365).[2]

One topic which does not receive complete treatment in Cowell's paper is the explication of the conditions for the signing of the effects of increasing wage-rate uncertainty on the optimal allocation of time between safe work, risky work, and leisure. In particular, the matter of the effect of increasing wage-rate uncertainty is given attention only in Cowell's investigation of the "risky work and leisure" decision - a decision in which the allocation of time to risky work and leisure is considered as a special case of the decision in which the levels of safe work, risky work, and leisure are jointly chosen.[3] Within this confine, Cowell is able to state the sufficient conditions under which a "Sandmo-type" increase in wage-rate risk affects unambiguously the optimal level of risky work. However essentially the same problem (that is, the problem concerning the effect of a mean-preserving increase in wage-rate uncertainty on the optimal level of risky work) has been already solved by Tressler and Menezes (1980) with the exception that they use a "Rothschild-Stiglitz increase in risk".[4]

The purpose of the present Chapter is twofold. The first is to extend the scope of Cowell's investigation by defining sufficient conditions (i.e., a set of restrictions on the expected utility function) which permit the determination of the effects of a marginal increase in wage-rate uncertainty on the optimal allocation of time among safe work, risky work, and leisure. The second (which is not unrelated to the first) is to unify the literature on _multiple_ job holdings under uncertainty _à la_ Cowell (1981), and the literature on _single_ job holdings under uncertainty _à la_

Block and Heineke (1973) and Tressler and Menezes (1980).

This Chapter is organized as follows. In Section III-2, the "safe work, risky work, and leisure" problem is defined, and the conditions for its solution are presented. In Section III-3, the restrictions needed to sign the effects of a "Sandmo-type" increase in wage-rate uncertainty are stated, and the implications of these restrictions are discussed. Section III-4 presents the optimal labor supply responses to a "Sandmo-type" increase in wage-rate uncertainty provided that the assumption-set defined in Section III-3 holds. Summary remarks are offered in Section III-5.

III-2 The "Safe Work, Risky Work, and Leisure" Problem

Consider an individual who holds two jobs: the first job having a certain wage-rate, W_1; and the second having an uncertain wage-rate, W_2. The expected (or average) wage-rate for W_2 is $E(W_2) = \int W_2 \cdot g(W_2, e)dW_2$ where $g(W_2, e)$ is the subjective probability density function (SPDF) of W_2, and e represents the "Rothschild-Stiglitz-measure" of uncertainty. Thus for working L_1 hours at the first job, the individual is paid wage-income of W_1L_1. Similarly, for working L_2 hours at the second job, the individual expects to be paid wage-income of $E(W_2) \cdot L_2$.

Suppose the individual has a utility function with leisure, S, and expected wage-income, $E(Y) = W_1L_1+E(W_2)L_2$, as arguments, i.e., $U=U(S,E(Y))$. Under the expected utility hypothesis (EUH), there exists a function, $EU(S,Y)$, such that $U(S,E(Y)) \gtreqless EU(S,Y))$ for $R_A(S,Y)=(-U_{YY}/U_Y) \gtreqless 0$ where $R_A(S,Y)$ is the Arrow-Pratt measure of absolute risk aversion. Suppose this expected utility function (EUF) is continuous, and three-times differentiable. Suppose too that this EUF has positive first derivatives, and negative second derivatives, i.e., the EUF exhibits absolute risk aversion or $R_A(S,Y)>0$. The time allocation problem is solved by choosing L_1 and L_2 so as to maximize the EUF. In particular, under the EUH, the problem for the individual is to:

$$\max_{L_1,L_2} V(S,L_1,L_2,e) = \int_{R_o} U(T-L_1-L_2,W_1L_1+W_2L_2)g(W_2,e)dW_2 \qquad (1)$$

such that $W_1 \varepsilon R_o$ where R_o is the domain of W_2. The first order conditions (FOC) for an interior maximum are:

$$V_{L_1} = -E(U_S) + E(W_1U_Y) = 0$$

$$V_{L_2} = -E(U_S) + E(W_2U_Y) = 0 \qquad (2)$$

The second order conditions (SOC) for an interior maximum are:

$$V_{L_1L_1} < 0; \; V_{L_2L_2} < 0;$$

$$(V_{L_1L_1} \cdot V_{L_2L_2} - (V_{L_1L_2})^2) = |H| > 0 \qquad (3)$$

where:

$$V_{L_iL_j} = E(U_{SS}) - E((W_i+W_j)U_{SY}) + E(W_iW_jU_{YY}) \qquad (4)$$

for i=1,2 and j=1,2. Thus, if (2) and (3) hold, there exists some unique combination, (S^*,L_1^*,L_2^*), which satisfies Equation (1).

One implication of risk-aversion is that at the optimum $E(W_2)-W_1>0$. In particular,

<u>Theorem 1</u>: If $R_A(S,Y)>0$, then $E(W_2)-W_1>0$.

Proof: From the first order conditions,[5]

$$W_1 E(U_Y) = E(W_2 \cdot U_Y)$$

$$= E(W_2)\cdot E(U_Y) + Cov(W_2 \cdot U_Y) \qquad (5)$$

Dividing by $E(U_Y)$ yields:

$$E(W_2)-W_1 = -Cov(W_2, U_Y)/E(U_Y) \qquad (6)$$

since $dY/dW_2>0$, and $dU_Y/dY<0$ as $R_A(S,Y)>0$, then $Cov(W_2,U_Y)<0$ as $R_A(S,Y)>0$. Thus $E(W_2)-W_1>0$ as $R_A(S,Y)>0$. Q.E.D.

In keeping with the literature on the economics of uncertainty, the R.H.S. of Equation (6) is termed the cost of risk bearing (CRB). In the context of a model of firm under output-price uncertainty, the CRB is interpreted by Pleeter and Horowitz (1974) as "...the adjustment made to production cost to represent the utility or disutility that results from the variance of profits" (p. 188). In the same context, Sakai (1978) interprets the CRB as "...the per unit (psychological) cost of risk bearing as associated with price uncertainty" (p. 290).[6]

A second implication of risk aversion is that the optimal amount of time allocated to safe work exceeds the optimal amount of time allocated to risky work (i.e., $L_1^*>L_2^*$). In particular,

Lemma 1: $Cov(W_2,U_Y) = Cov(W_2-W_1,U_Y)$

Proof: $Cov(W_2-W_1,U_Y) = E((W_2-W_1)U_Y)-E(W_2-W_1)\cdot E(U_Y)$

$$= E(W_2\cdot U_Y)-E(W_2)\cdot E(U_Y)$$

$$= Cov(W_2,U_Y)$$

Q.E.D.

Lemma 2: If $R_A(S,Y)>0$, then sign $[Cov(W_2-W_1,U_Y)]$ $= \text{sign } (L_2^*-L_1^*)$.

Proof: $d[W_2-W_1]/dY = dW_2/dY-dW_1/dY = 1/L_2^*-1/L_1^*$. Sign $[Cov(W_2-W_1,U_Y)] = \text{sign } [d(W_2-W_1)/dY]\cdot\text{sign } [dU_Y/dY]$. For $R_A(S,Y)>0$, $dU_Y/dY<0$. Therefore, sign $[Cov(W_2-W_1,U_Y)]$ $=\text{sign } [-d(W_2-W_1)/dY] = \text{sign } [-1/L_2^*+1/L_1^*] = \text{sign } [L_2^*-L_1^*]$.

Q.E.D.

Theorem 2: If $R_A(S,Y)>0$, then $L_1^*>L_2^*$.

Proof: By Lemmae 1 and 2, sign $[Cov(W_2,U_Y)] = \text{sign } [Cov(W_2-W_1,U_Y)] = \text{sign } [L_2^*-L_1^*]$. By Theorem 1, $Cov(W_2,U_Y)<0$ which implies $L_1^*>L_2^*$. Q.E.D.

III-3 The Restrictions and Their Implications

To enable signing of the comparative-static effects of a mean-preserving increase in the uncertainty of W_2 on (S^*,L_1^*,L_2^*), two restrictions are imposed on the EUF. These are:

Assumption 1: Safe work is an inferior good in the sense that:[7]

$$-1/W_1 < dL_1^*/dY\Big|_{Y=E(Y)} < 0 \tag{7}$$

Assumption 2: The EUF exhibits constant risk aversion to concentration (CRAC),[8] i.e.,

$$dU_{YY}/dY\Big|_{dU=0} = U_{YYY}-U_{YYS}[U_Y/U_S] = 0 \tag{8}$$

The implications of these two assumptions are as follows:

Lemma 3: If Assumption 2 holds, and if the EUF exhibits $R_A(S,Y)>0$, then the EUF satisfies the Hanson-Menezes-Tressler Condition (HMTC).

Proof: The HMTC requires that

$$dU_{YY}L_2^2/dL_2\Big|_{dU=0} = L_2^2[dU_{YY}/dY\Big|_{dU=0} \cdot dY/dL_2] + 2L_2U_{YY} < 0 \quad (9)$$

Clearly, Assumption 2 and $R_A(S,Y)>0$ imply the HMTC. Q.E.D.

The results of Tressler-Menezes (T-M) may be restated in the context of Equation (1) and our assumption-set. In particular, let L_1 be fixed in an arbitrary h-neighborhood of L_1* (i.e., let $\bar{L}_1 \varepsilon R_1$ where $R_1 = \{\bar{L}_1 \mid L_1^* - h < \bar{L}_1 < L_1^* + h$ and $h>0\}$). Consider the problem:

$$\max_{L_2} V(S,\bar{L}_1,L_2,e) = \int U(T-\bar{L}_1-L_2,\ W_1\bar{L}_1 + W_2L_2)g(W_2,e)dW_2 \quad (10)$$

for $\bar{L}_1 \varepsilon R_1$. Let L_2^0 be a solution to Equation (10).

Lemma 4: If Assumption 2 holds, and if the EUF exhibits $R_A(S,Y)>0$, then $V_{L_2e}<0$.

Proof: Implicit differentiation of the second FOC yields $dL_2^0/de = -V_{L_2e}/V_{L_2L_2}$. Since $V_{L_2L_2}<0$ [Equation 3],

$$\text{sign } [dL_2^0/de] = \text{sign } [V_{L_2e}]$$

If Assumption 2 holds, and if $R_A(S,Y)>0$, then the HMTC holds [Lemma 3], and if the HMTC holds, T-M (1980) show $dL_2^0/de<0$. Q.E.D.

The next three lemmas are needed to restate one result reported by Block-Heineke (B-H) in the context of Equation 1 and our assumption-set. In particular, let L_2 be fixed in an arbitrary h-neighborhood of L_2* (i.e., Let $\bar{L}_2 \in R_2$ where $R_2 = \{\bar{L}_2 \mid L_2^* - h < \bar{L}_2 < L_2^* + h$ and $h > 0\}$). Consider the problem:

$$\max_{L_1} V(S, L_1, \bar{L}_2, e) = \int U(T - L_1 - \bar{L}_2,\ W_1 L_1 + W_2 \bar{L}_2) g(W_2, e) dW_2 \tag{11}$$

for $\bar{L}_2 \in R_2$. Let L_1^0 be a solution to Equation (11).

Lemma 5: If Assumption 2 holds, then $dU_{SY}/dY\big|_{dU=0} = 0$.

Proof: Assumption 2 implies:

$$U_{YYY} - U_{YYS}[U_Y/U_S] = 0 \tag{12}$$

which in turn implies:

$$U_{YYS} - U_{YYY}[U_S/U_Y] = 0 \tag{13}$$

which is precisely $dU_{YY}/dS\big|_{dU=0} = 0$. Now for three-times differentiable functions, Young's Theorem implies

$$dU_{YY}/dS\big|_{dU=0} = dU_{SY}/dY\big|_{dU=0} = 0$$ Q.E.D.

Lemma 6: If Assumption 2 holds, then $\mathrm{Cov}(W_2, U_{SY} - W_1 U_{YY}) = 0$.

Proof: Sign $[\mathrm{Cov}(W_2, U_{SY} - W_1 U_{YY})]$ = sign $[dW_2/dY]$ $\cdot$ sign $[d(U_{SY} - W_1 U_{YY})/dY]$. If Assumption 2 holds, $dU_{YY}/dY\big|_{dU=0} = 0$ and $dU_{SY}/dY\big|_{dU=0} = 0$ [Lemma 5].

Therefore $d(U_{SY} - W_1 U_{YY})/dY\big|_{dU=0} = 0$, and $\mathrm{Cov}(W_2, U_{SY} - W_1 U_{YY}) = 0$ Q.E.D.

Lemma 7: If Assumption 2 holds, then $V_{L_1 e} = 0$.

Proof: Let $W_2 \bar{L}_2 = eW_2 \bar{L}_2 + r(e)$ for $e = 1$ and $r(1) = 0$. Taking the expectations of both sides of the equation yields:

$$E(W_2)\bar{L}_2 = eE(W_2)\bar{L}_2 + r(e).$$

Taking the total derivative of this, and holding the mean constant yields:

$$E(W_2)\bar{L}_2 \cdot de + dr = 0$$

This implies that $dr/de = -E(W_2)\bar{L}_2$. Now consider the income constraint,

$$Y = W_1 L_1 + eW_2\bar{L}_2 + r(e)$$

for e=1 and r(1)=0. It is clear that:

$$dY/de = W_2\bar{L}_2 + dr/de$$

$$= [W_2 - E(W_2)]\bar{L}_2 \qquad (14)$$

Now substitute $eW_2\bar{L}_2 + r(e)$ for $W_2\bar{L}_2$ in the first FOC such that e=1 and r(1)=0. Taking the implicit derivative of this revised FOC yields:

$$V_{L_1 L_1} \cdot dL_1^0 - [E(U_{SY} - W_1 U_{YY}) \cdot dY/de]de$$

$$= V_{L_1 L_1} \cdot dL_1^0 - [E(U_{SY} - W_1 U_{YY})(W_2 - E(W_2))\bar{L}_2]de$$

$$= V_{L_1 L_1} dL_1^0 - [E((U_{SY} - W_1 U_{YY})W_2)$$

$$- E((U_{SY} - W_1 U_{YY})E(W_2))\bar{L}_2 de$$

$$= V_{L_1 L_1} dL_1^0 - Cov(W_2, U_{SY} - W_1 U_{YY})\bar{L}_2 de = 0$$

Therefore:

$$dL_1^0/de = \bar{L}_2 \cdot Cov(W_2, U_{SY} - W_1 U_{YY})/V_{L_1 L_1}$$

$$= -V_{L_1 e}/V_{L_1 L_1}$$

If $V_{L_1 L_1} < 0$ [Equation (3)], then

$$\text{sign } [dL_1^U/de] = \text{sign } [V_{L_1 e}]$$

$$= \text{sign } [-\mathrm{Cov}(W_2, U_{SY}-W_1U_{YY})]$$

If Assumption 2 holds, then $\mathrm{Cov}(W_2, U_{SY}-W_1U_{YY})=0$ [Lemma 6], and therefore $V_{L_1 e}=0$. Q.E.D.

The final result which is required is to show that Assumptions 1 and 2 together imply $V_{L_1L_2}<0$. This is achieved in the following three lemmas:

<u>Lemma 8</u>: If Assumption 1 holds, then $E(W_1U_{YY}-U_{SY})<0$.

<u>Proof</u>: Implicit differentiation of the first FOC yields:

$$dL_1^*/dY = -E(W_1U_{YY}-U_{SY})/V_{L_1L_1}$$

Since $V_{L_1L_1}<0$ [Equation (3)], and since $dL_1^*/dY<0$ [Assumption 1], therefore $E(W_1U_{YY}-U_{SY})<0$. Q.E.D.

<u>Lemma 9</u>: If Assumption 1 holds, then $E(U_{SS}-W_1U_{SY})<0$.

<u>Proof</u>: By Assumption 1, $0>dL_1^*/dY\big|_{Y=E(Y)}>-1/W_1$ which implies

$$0 > W_1 \cdot dL_1^*/dY > -1, \text{ and implies:}$$

$$0 > W_1[-E(W_1U_{YY}-U_{SY})/V_{L_1L_1}] > -1$$

and:

$$E(U_{SS} - 2W_1U_{SY}+W_1^2U_{YY}) < E(W_1^2U_{YY}-W_1U_{SY})$$

Therefore $E(U_{SS}-W_1U_{SY})<0$. Q.E.D.

<u>Lemma 10</u>: If Assumptions 1 and 2 hold, then $V_{L_1L_2}<0$.

<u>Proof</u>: From Equation (4),

$$V_{L_1L_2} = E(U_{SS}) - E((W_1+W_2)U_{SY}) + E(W_1W_2U_{YY})$$

$$= E(U_{SS}-W_1U_{SY}) + E(W_2) \cdot E(W_1 \cdot U_{YY}-U_{SY})$$

$$- \mathrm{Cov}(W_2,U_{SY}-W_1U_{YY})$$

$$= E(U_{SS}-W_1 U_{SY}) + E(W_2) \cdot E(W_1 U_{YY} U_{SY})$$

by virtue of Assumption 2 [Lemma 6]. Since $E(W_1 U_{YY}-U_{SY})<0$ and $E(U_{SS}-W_1 U_{SY})<0$ by virtue of Assumption 1 [Lemmas 8 and 9], therefore $V_{L_1 L_2}<0$. Q.E.D.

III-4 The Optimal Labor Supply Responses to Increasing Wage-Rate Uncertainty

To determine the effects of a mean-preserving increase in the uncertainty of W_2 on the allocation of time among safe work, risky work, and leisure, consider the following theorem:

Theorem 3: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the wage-rate paid for risky work increases the optimal amount of time devoted to safe work, decreases the optimal amount of time devoted to risky work, and ambiguously affects the optimal amount of time devoted to leisure.

Proof: Take the total differential of (2), and set the increments of all variables except L_1, L_2, and e equal to zero. Using Cramer's Rule, we may write:

$$dL_1^*/de = \{V_{L_2 e} \cdot V_{L_1 L_2} - V_{L_1 e} \cdot V_{L_2 L_2}\}/\left|H\right| \qquad (15)$$

and

$$dL_2^*/de = \{V_{L_1 e} \cdot V_{L_1 L_2} - V_{L_2 e} \cdot V_{L_1 L_1}\}/\left|H\right| \qquad (16)$$

If Assumption 2 holds, then $V_{L_2 e}<0$ [Lemma 4]. If Assumption 2 holds, then $V_{L_1 e}=0$ [Lemma 7]. If Assumptions 1 and 2 hold, then $V_{L_1 L_2}<0$ [Lemma 10]. Finally $V_{L_1 L_1}<0$, $V_{L_2 L_2}<0$, and $\left|H\right|$ by assumption [Equation (3)]. In sum, if Assumptions 1 through 5 hold, $dL_1^*/de>0$ and $dL_2^*/de<0$. At the optimum, the time constraint is:

$$S^*(e) + L_1^*(e) + L_2^*(e) - T = 0 \tag{17}$$

where T represents the total time available for any individual. Differentiating (17) with respect to e, we obtain:

$$dS^*/de = -[dL_1^*/de + dL_2^*/de] \tag{18}$$

It is clear that $dS^*/de \gtrless 0$ if $dL_2^*/de \gtrless dL_1^*/de$. Q.E.D.

III-5 Summary Remarks

In this Chapter, the analysis offered by Cowell (1981) has been extended. In particular, in the context of the "safe work, risky work, leisure" decision, a set of sufficient conditions or restrictions on the utility function which enable the signing of the comparative-static effects of a "Sandmo-type" increase in wage-rate uncertainty were determined. These restrictions represent some of the restrictions used by Block and Heineke (1973) and Tressler and Menezes (1980). Under these restrictions, the effects of a "Sandmo-type" increase in wage-rate uncertainty on the optimal allocation are: (i) to decrease the optimal amount of time devoted to risky work, (ii) to increase the optimal amount of time devoted to safe work, and (iii) to ambiguously affect the optimal amount of time devoted to leisure.

Footnotes to Chapter III

[1]Examples of the literature on multiple job holdings under certainty include Shishko and Rostker (1976) and O'Connell (1979).

[2]The motivation underlying Professor Cowell's paper is two-dimensional. Not only is the paper concerned with the modelling of the "safe work, risky work, and leisure" decision *per se*, but it is concerned with the (tax) policy implications associated with this model. In this Chapter, Cowell's latter motivation is deliberately ignored.

[3]Cowell considers three cases in which h_0 denotes riskless activity and h_1 denotes risky activity. These are: (i) $h_0>0$ and $h_1=0$, (ii) $h_0=0$ and $h_1>0$, and (iii) $h_0>$ and $h_1>0$. Two comments about cases (ii) and (iii) are warranted. Firstly, Cowell considers the sufficient conditions for signing the comparative-static effect of mean-preserving increase in wage-rate uncertainty in case (ii) only. See p. 370. Secondly, in case (iii), he considers the "safe work, risky work, and leisure" decision as being comprised of two separable decisions: the "labor supply" problem and the "time portfolio" problem. See p. 371.

The present Chapter is distinct from the Cowell paper in two respects. First, we do not treat the "safe work, risky work, and leisure" problem (i.e., case (iii)) as being separable. Second, this Chapter determines the sufficient conditions for signing the comparative-static effects of a mean-preserving increase in wage-rate uncertainty for case (iii).

[4]In due fairness to Professor Cowell, he informs me that: (i) the Tressler-Menezes (1980) paper did not appear until his paper had been accepted in *Economica*, and (ii) his "paper (does) not claim generality, but (examines) two possibly interesting subcases of the intractable general formulation".

[5]For a definition of the covariance function, see Footnote 14, Chapter II.

[6]Two comments about the CRB. Firstly, Blair and Lusky (1977, p. 236) speculate that Penner (1967) was the first to use the CRB, and this was in the context of the theory of the firm under output-price uncertainty. Secondly, Tressler (1980, pp. 41-43) provides a detailed comparison of the statement by Pleeter and Horowtiz (1974) and Sakai (1978).

[7]A sufficient condition that this Assumption hold is that the marginal utility of leisure is non-decreasing in income (i.e., $E(U_{SY})\geq 0$). In particular, $E(U_{SY})\geq 0$ implies

$E(W_1 U_{YY} - U_{SY}) < 0$ and therefore $dL_1^*/dY < 0$. Likewise, $E(U_{SY}) \geq 0$ implies $E(U_{SS} - W_1 U_{SY}) = V_{L_1 L_1} - E(W_1^2 U_{YY} - W_1 U_{SY}) < 0$ which implies:

$$- 1/W_1 < -E[W_1 U_{YY} - U_{SY}]/V_{L_1 L_1} = dL_1^*/dY$$

[8] Assumption 2 is motivated by Leland's PDRAC. By adapting Assumption 2, <u>we are assuming the following preference relation under uncertainty</u>: an individual is indifferent between consumption bundles which contain different amounts of an uncertain commodity. See Section II-2(d) for the basis of this analysis.

CHAPTER IV

THE PORTFOLIO EFFECTS OF INCREASING ASSET-RETURN UNCERTAINTY IN A TWO-ASSET, TWO-PERIOD MODEL: A PRELIMINARY ASSESSMENT*

The present Chapter presents an investigation into the impact of uncertainty for what we termed the <u>second class</u> of models. The particular problem addressed here is suggested by Sandmo (1969). The solution to this problem offered here is viewed as <u>tentative</u> in that a rather strong assumption-set is required. An alternative approach which (in our view) employs a weaker assumption-set is proposed in Chapter V.

IV-1 <u>An Introduction to the Problem</u>

Sandmo (1969) considers the problem of devising a two-period consumption plan under uncertainty in which there are three defining characteristics. The first is that the utility function contains as arguments first-period consumption and second-period consumption.[1] The second is that (in the first-period) initial wealth is allocated among first-period consumption and two assets; an asset providing a certain rate of return in the second-period (i.e., a "safe asset"), and an asset providing an uncertain rate of return in the second-period (i.e., a "risky asset"). The final defining characteristic is that Sandmo selects as choice variables the level of first-period consumption and the level of the risky asset.[2] In an analysis of the qualitative effects of a mean-preserving increase in the uncertainty of the return provided the risky asset, Sandmo concludes that the effects are ambiguous without further restrictions on the utility function.[3]

The purpose of this Chapter is two-fold. The first is to propose a preliminary approach to the signing of these effects in the Sandmo (1969) portfolio problem. In particular, using the work of Sandmo (1970), and Hanson and Menezes (1978), a set of four restrictions on the utility function is proposed. The second purpose (which is not unrelated to the first) is to unify the literature on <u>two</u>-asset, two-period models <u>a la</u> Sandmo (1969), and the literature on <u>one</u>-asset, two-period models <u>a la</u> Sandmo (1970) and Hanson and Menezes (1978).

This Chapter is organized as follows. In Section IV-2, Sandmo's (1969) portfolio problem, and the conditions for its solution, are presented. In Section IV-3, the restrictions on the utility function needed to sign the effects of a "Sandmo-type" increase in the uncertainty of the return to the risky asset are stated, the implications of these restrictions are discussed, and the optimal portfolio responses to a "Sandmo-type" increase are presented. Summary remarks are offered in Section IV-4.

IV-2 The Sandmo (1969) Portfolio Problem

Consider an individual whose welfare is measured by a utility function with arguments in first-period consumption, C_1, and expected second-period consumption, $E(C_2)$. As usual, suppose that this function is continuous, and three times differentiable. Suppose too that this function has positive first-derivatives, and negative second-derivatives. The Sandmo (1969) portfolio problem concerns the optimal appropriation of initial wealth, W_1, in the first-period amongst first-period consumption; a safe asset, A_1, which pays i_1 with certainty; and, a risky asset, A_2, which pays an uncertain return, i_2. The expected (or average) return of A_2 is $E(i_2)=\int i_2 f(i_2,e)di_2$ where $f(i_2,e)$ is the subjective probability density function (SPDF) of i_2 and e represents the "Rothschild-Stiglitz-measure" of uncertainty. The level of first-period consumption is defined by:

$$C_1 = W_1 - A_1 - A_2 \tag{1}$$

and the expected level of second-period consumption is defined by:[4]

$$E(C_2) = r_1 A_1 + E(r_2)A_2 \tag{2}$$

where $r_1=(1+i_1)$ and $E(r_2)=\int r_2 g(r_2,e)dr_2=1+E(i_2)=$ $1+\int i_2 f(i_2,e)di_2$. In formal terms, the Sandmo (1969) portfolio problem is:

$$\max_{C_1,A_2} V(C_1,A_1,A_2,e)$$

$$= \int_{R_0} U(C_1,r_1[W_1-C_1]+[r_2-r_1]A_2)g(r_2,e)dr_2 \tag{3}$$

such that $r_1 \varepsilon R_0$ where R_0 is the domain of r_2 and where

$r_2 - r_1 = i_2 - i_1$. The first order conditions (FOC) for an interior maximum are:

$$V_{C_1} = E(U_1) - r_1 E(U_2) = 0$$
$$V_{A_2} = E((r_2 - r_1)U_2) = 0 \tag{4}$$

The second order conditions (SOC) for an interior maximum are:

$$V_{C_1C_1} < 0; \quad V_{A_2A_2} < 0; \quad \text{and}$$
$$(V_{C_1C_1} \cdot V_{A_2A_2} - (V_{C_1A_2})^2) = |H| > 0 \tag{5}$$

where:

$$V_{C_1C_1} = E(U_{11}) - 2r_1 E(U_{12}) + r_1^2 E(U_{22})$$
$$V_{A_2A_2} = E((r_2 - r_1)^2 U_{22})$$
$$V_{C_1A_2} = -E[(r_2 - r_1)(r_1 U_{22} - U_{12})] \tag{6}$$

Thus, if Equations (4) and (5) hold, there exists some unique combination, (C_1^*, A_1^*, A_2^*), which satisfies Equation (3).

IV-3 The Optimal Portfolio Responses to the Risk-Averse Individual to Increasing Uncertainty of the Return on the Risky Asset

To enable signing of the comparative-static effects of a mean-preserving increase in the uncertainty of r_2 on the original portfolio selection of the risk-averse individual, four restrictions are imposed on the EUF. These are:

Assumption 1: First-period consumption is a normal good, i.e.,

$$dC_1^*/dW_1 > 0 \tag{7}$$

Assumption 2: The EUF exhibits decreasing temporal risk aversion (DTRA), i.e.,[5]

$$\partial[(U_{12} - r_1 U_{22})/U_2]/\partial C_2 < 0 \tag{8}$$

Assumption 3: The EUF satisfies the Hanson-Menezes-Tressler Condition (HMTC), i.e.,

$$d[U_{22}A_2^2]/dA_2\Big|_{dU=0} = A_2^2[(r_2-r_1)U_{222}$$

$$- U_{221}[(r_2-r_1)U_2/(\dot{U}_1-r_1U_2)]]+2A_2U_{22} < 0 \quad (9)$$

Assumption 4: The risky asset is a normal good, i.e.,

$$dA_2^*/dW_1 > 0 \quad (10)$$

Three comments about these four restrictions are in order. Firstly, Sandmo (1970) shows that Assumptions 1 and 2 are sufficient to assert that an increase in "income risk" decreases the optimal level of first-period consumption. Sandmo's result can be cast in terms of Equation (3). In particular, consider the problem:

$$\max_{C_1} V(C_1,A_2,\bar{A}_2,e)$$

$$= \int U(C_1,r_1[W_1-C_1] + [r_2-r_1]\bar{A}_2)g(r_2,e)dr_2 \quad (11)$$

for $\bar{A}_2 \varepsilon R_2$ where $R_2 = \{\bar{A}_2 \,|\, A_2^*-h<\bar{A}_2<A_2^*+h \text{ and } h>0\}$. Let C_1^0 be the solution to this problem.

Lemma 1: If Assumptions 1 and 2 hold, then $V_{C_1e}<0$.

Proof: By implicit differentiation of the first FOC, $dC_1^0/de=-V_{C_1e}/V_{C_1C_1}$. By Equation (5), $V_{C_1C_1}<0$. Therefore:

$$\text{sign } (dC_1^0/de) = \text{sign } (V_{C_1e}) \quad (12)$$

If Assumptions 1 and 2 hold, Sandmo (1970, PP. 355-6) shows that $dC_1^0/de<0$ where e is the "Sandmo-measure of uncertainty". Q.E.D.

Secondly, Hanson and Menezes (1978) show that Assumption 3 is sufficient to assert that an increase in "capital risk" decreases the optimal level of savings. The Hanson-Menezes (H-M) result can be cast in terms of Equation (3). In particular, consider the problem:

$$\max V(C_1, \bar{A}_1, A_2, e) = \int U(C_1, r_1[W_1 - C_1] + [r_2 - r_1]A_2) g(r_2, e) dr_2 \qquad (13)$$

for $\bar{A}_1 \varepsilon R_1$ where $R_1 = \{\bar{A}_1 \mid A_1^* - h < \bar{A}_1 < A_1^* + h \text{ and } h > 0\}$. Let A_2^0 be a solution to this problem.

<u>Lemma 2</u>: If Assumption 3 holds, then $V_{A_2 e} < 0$.

<u>Proof</u>: By implicit differentiation of the second FOC, $dA_2^0/de = -V_{A_2 e}/V_{A_2 A_2}$. By Equation (5), $V_{A_2 A_2} < 0$. Therefore

$$\text{sign } (dA_2^0/de) = \text{sign } (V_{A_2 e}). \qquad (14)$$

If Assumption 3 holds, H-M (1978) show that $dA_2^0/de < 0$ where e is the "Sandmo measure of uncertainty".[6] Q.E.D.

A final comment concerns an implication of Assumptions 1 and 4. In particular,

$$V_{C_1 A_2} > 0. \qquad (15)$$

To verify this consider:

<u>Lemma 3</u>: If Assumptions 1 and 4 hold, then $V_{C_1 A_2} > 0$.

<u>Proof</u>: Implicit differentiation of the first FOC yields, $dC_1^*/dA_2^* = -V_{C_1 A_2}/V_{A_2 A_2}$. Since $V_{A_2 A_2} < 0$ [Equation (5)],

$$\text{sign } [dC_1^*/dA_2^*] = \text{sign } [V_{C_1 A_2}]. \qquad (16)$$

Now $dA_2^*/dW_1 = dA_2^*/dC_1^* \cdot dC_1^*/dW_1$. Since $dC_1^*/dW_1 > 0$ [Assumption 1], and $dA_2^*/dW_1 > 0$ [Assumption 4], therefore $dC_1^*/dA_2^* > 0$ and $V_{C_1 A_2} > 0$. Q.E.D.

Finally, to determine the effects of a mean-preserving increase in the uncertainty of r_2 on the allocation of initial wealth amongst the safe asset, the risky asset, and first-period consumption, consider the following theorem:

<u>Theorem 2</u>: If Assumptions 1 and 4 hold, then a "Sandmo-type" increase in the uncertainty of the rate of return on the risky asset decreases the optimal level of first-period consumption, decreases the optimal holding of the risky asset, and increases the optimal holding of the safe asset.

<u>Proof</u>: Take the total differential of (4), and set the increments of all the variables except C_1, A_2 and e to zero. Using Cramer's Rule, we may write:

$$dC_1^*/de = [V_{A_2 e} \cdot V_{C_1 A_2} - V_{C_1 e} \cdot V_{A_2 A_2}]/\left|H\right| \tag{17}$$

$$dA_2^*/de = [V_{C_1 e} \cdot V_{C_1 A_2} - V_{A_2 e} \cdot V_{C_1 C_1}]/\left|H\right| \tag{18}$$

If Assumptions 1 and 2 hold, then $V_{C_1 e}<0$ [Lemma 1]. If Assumption 3 holds, then $V_{A_2 e}<0$ [Lemma 2]. If Assumptions 1 and 4 hold, then $V_{C_1 A_2}>0$ [Lemma 3]. Finally, $V_{C_1 C_1}<0$, $V_{A_2 A_2}<0$, and $\left|H\right|>0$ by assumption [Equation (5)]. In sum, if Assumptions 1 through 4 hold, $dC_1^*/de<0$ and $dA_2^*/de<0$. Using Equation (1), the effect, dA_1^*/de, can be determined. At the optimum, Equation (1) is:

$$A_1^*(e) = W_1 - [C_1^*(e)+A_2^*(e)] \tag{19}$$

Differentiating (19) with respect to e,

$$dA_1^*/de = -[dC_1^*/de+dA_2^*/de]. \tag{20}$$

It is clear from the above that $dA_1^*/de>0$. Q.E.D.

IV-4 Summary Remarks

This Chapter offered a solution to a comparative-static problem posed by Sandmo (1969). In particular, in the context of Sandmo's two-asset, two-period model, this paper presented a set of restrictions on the utility function which enable the signing of the comparative-static effects of a "Sandmo-type" increase in the uncertainty of the return provided the risky asset. It was shown that if these restrictions hold, the effects of a "Sandmo-type" increase on the optimal allocation of initial wealth are: (i) to increase the optimal level of the safe asset, (ii) to decrease the optimal level of the risky asset, and (iii) to decrease the optimal level of first-period consumption.

Footnotes to Chapter IV

*This chapter is forthcoming in Rivista Internationale di Scienze Economiche e Commerciali.

[1] For a review of the literature on two-period models, see Sandmo (1974), Hey (1979, pp. 72-79; 1981, pp. 104-12), and Lippman and McCall (1981, pp. 238-42).

[2] Sandmo (1969, pp. 589-91).

[3] His analysis of the comparative-static effects is in three parts. Firstly, in the case of non-negative levels of the risky asset (i.e., for lenders), he states: "an increase in dispersion has the same effect on consumption as a decrease in the expected yield on the risky asset" (p. 598). Secondly, in the case of non-positive levels of the risky asset (i.e., for borrowers), he states: "an increase in dispersion has the same effect on consumption as an increase in the expected yield on the risky asset" (p. 598). Finally, the effect of a change in the expected yield on the risky asset on consumption is in general indeterminate (i.e., see Equation (22), p. 596). In particular Sandmo states: "Equation (22) implies that the effect on consump-

tion of an increase in the expected yield on the risky asset is indeterminate for all (levels of the risky asset); there are always conflicting tendencies of the substitution and income effects. A fortiori, <u>this will also be the case for increases in risk</u> (p. 598).

[4] For sheer simplicity, autonomously-determined second-period income is assumed to be zero.

[5] Sandmo (1970, p. 356).

[6] In Footnote 18, H-M (1978) claim their result also holds for a "Rothschild-Stiglitz increase in risk."

CHAPTER V

THE PORTFOLIO EFFECTS OF INCREASING ASSET-RETURN UNCERTAINTY IN A TWO-ASSET, TWO-PERIOD MODEL: AN ALTERNATIVE APPROACH

The purpose of the present Chapter is to offer an alternative approach to the problem posed in Chapter IV, the Sandmo (1969) portfolio problem. As already noted, a weakness of the approach offered in Chapter IV is the use of a strong assumption-set. This Chapter provides an approach which employs a weaker assumption-set.

The present Chapter should be viewed as the first of three applications of the results generated in Chapter III. That is, in addition to this Chapter and Chapter III, Chapters VI and VII are to be viewed also as members of the first class of models.

This Chapter is organized as follows. In Section V-1, a reformulated version of Sandmo's (1969) portfolio problem, and the conditions for its solution, are presented. In Section V-2, the restrictions on the utility function needed to sign the effects of a "Sandmo-type" increase in the uncertainty of the return to the risky asset are stated, and the implications of these restrictions are discussed. Section V-3 presents the optimal portfolio responses to a "Sandmo-type" increase provided that the assumption-set defined in Section V-2 holds. Summary remarks are offered in Section V-4.

V-1 A Reformulation of the Sandmo (1969) Portfolio Problem

Consider an individual whose welfare is measured by a utility function with arguments in first-period consumption, C_1, and expected second-period consumption, $E(C_2)$. As usual, suppose that this function is continuous, and three-times differentiable. Suppose too that this function has positive first derivatives, and negative second-derivatives. The Sandmo (1969) portfolio problem concerns the optimal appropriation of initial wealth, W_1, in the first-period amongst first-period consumption; a safe asset, A_1, which pays i_1 with certainty; and, a risky asset, A_2, which pays an uncertain return, i_2. The expected (or average) return of A_2 is $E(i_2)=\int i_2 f(i_2,e)di_2$ where $f(i_2,e)$ is the subjective probability density function (SPDF) of i_2, and e represents the "Rothschild-Stiglitz-measure" of uncertainty. The level of first-period consumption is defined by:

$$C_1 = W_1 - A_1 - A_2 \tag{1}$$

and the expected level of second-period consumption is defined by:[1]

$$E(C_2) = r_1 A_1 + E(r_2)A_2 \tag{2}$$

where $r_1=(1+i_1)$ and $E(r_2)=\int r_2 g(r_2,e)dr_2=1+E(i_2)=1+\int i_2 f(i_2,e)di_2$. In formal terms, the reformulated version of the Sandmo (1969) portfolio problem is:[2]

$$\max_{A_1, A_2} V(C_1, A_1, A_2, e) = \int_{R_0} U(W_1 - A_1 - A_2, r_1 A_1 + r_2 A_2) g(r_2, e) dr_2 \quad (3)$$

such that $r_1 \varepsilon R_0$ where R_0 is the domain of r_2. The first order conditions (FOC) for an interior maximum are:

$$V_{A_1} = -E(U_1) + r_1 E(U_2) = 0$$

$$V_{A_2} = -E(U_1) + E(r_2 U_2) = 0 \quad (4)$$

The second order conditions (SOC) for an interior maximum are:

$$V_{A_1 A_1} < 0; \; V_{A_2 A_2} < 0; \text{ and}$$

$$(V_{A_1 A_1} \cdot V_{A_2 A_2} - (V_{A_1 A_2})^2) = \left| H \right| > 0 \quad (5)$$

where:

$$V_{A_i A_j} = E(U_{11}) - E((r_i + r_j) U_{12}) + E(r_i r_j U_{22}) \quad (6)$$

for i=1,2 and j=1,2. Thus, if Equations (4) and (5) hold, there exists some unique combination (C_1^*, A_1^*, A_2^*), which satisfies Equation (3).

An implication of $E(U_2)>0$ and $E(U_{22})<0$ is absolute risk aversion (i.e., $R_A(C_1, C_2) = (-U_{22}/U_2) > 0$). One implication of risk aversion is that at the optimum $E(r_2 - r_1) > 0$. In particular,

<u>Theorem 1</u>: If $R_A(C_1, C_2) > 0$, then $E(r_2) - r_1 > 0$.

<u>Proof</u>: From Equation (4),

$$r_1 E(U_2) = E(r_2 U_2)$$

$$= E(r_2) \cdot E(U_2) + \text{Cov}(r_2, U_2)$$

Dividing by $E(U_2)$ yields

$$E(r_2) - r_1 = -\text{Cov}(r_2, U_2)/E(U_2) \quad (7)$$

Since $dC_2/dr_2 > 0$, and $dU_2/dC_2 < 0$ as $R_A(C1, C2) > 0$, then

$Cov(r_2,U_2)<0$ as $R_A(C_1,C_2)>0$. Thus $E(r_2)>r_1$ as $R_A(C_1,C_2)>0$.

Q.E.D.

With regards to Theorem 1, two comments are in order. Firstly, as noted in Chapter III, terms such as the term on the R.H.S. of Equation (7), i.e., $-Cov(r_2,U_2)/E(U_2)$, are referred to as the CRB. Secondly, in the case of the R.H.S. of Equation (7), CRB measures the amount by which the return on the risky asset (relative to the return on the safe asset) must compensate a risk-averse individual for holding a positive level of the risky asset.

Another implication of risk aversion is that the optimal amount of the safe asset exceeds the optimal amount of the risky asset (i.e., $A_1^*>A_2^*$). In particular,

Theorem 2: If $R_A(C_1,C_2)>0$, then $A_1^*>A_2^*$.

Proof: $Cov(r_2,U_2)=Cov(r_2-r_1,U_2)$ [Lemma 1, Chapter III]. If $R_A(C_1,C_2)>0$, then sign $[Cov(r_2,U_2)]=$sign $[Cov(r_2-r_1,U_2)]=$ sign$[A_2^*-A_1^*]$[Lemma 2, Chapter III]. Since $R_A(C_1,C_2)>0$ implies $Cov(r_2,U_2)<0$ [Theorem 1], $R_A(C_1,C_2)>0$ implies $A_1^*>A_2^*$.

Q.E.D.

V-2 The Restrictions and their Implications

To enable signing of the comparative-static effects of a mean-preserving increase in the uncertainty of r_2 on (C_1^*,A_1^*,A_2^*), two restrictions are imposed on the expected utility function (EUF). These are:

Assumption 1: The safe asset is an inferior good, i.e.,

$$-1/r_1 < dA_1/dC_2\Big|_{C_2=E(C_2)} < 0 \tag{8}$$

Assumption 2: The EUF exhibits constant risk aversion to concentration (CRAC), i.e.,

$$dU_{22}/dC_2\Big|_{dU=0} = U_{222}-U_{221}[U_2/U_1] = 0 \tag{9}$$

The implications of these two assumptions parallel Lemmas 1 through 8 inclusive, in Chapter III. In particular:

Lemma 1: If Assumption 2 holds, and if the EUF exhibits $R_A(C_1,C_2)>0$, then the EUF satisfies the Hanson-Menezes-Tressler Condition (HMTC), i.e., $dU_{22}A_2^2/dA_2\Big|_{dU=0} < 0$

Proof: Lemma 3, Chapter III.

Lemma 2: If Assumption 2 holds, and if the EUF exhibits $R_A(C_1,C_2)>0$, then $V_{A_2e}<0$.

Proof: Lemma 4, Chapter III.

Lemma 3: If Assumption 2 holds, then $dU_{12}/dC_2\Big|_{dU=0}=0$.

Proof: Lemma 5, Chapter III.

Lemma 4: If Assumption 2 holds, then $Cov(r_2,U_{12}-r_1U_{22})=0$.

Proof: Lemma 6, Chapter III.

Lemma 5: If Assumption 2 holds, then $V_{A_1e}=0$.

Proof: Lemma 7, Chapter III.

Lemma 6: If Assumption 1 holds, then $E(r_1U_{22}-U_{12})<0$.

Proof: Lemma 8, Chapter III.

Lemma 7: If Assumption 1 holds, then $E(U_{11}-r_1U_{12})<0$.

Proof: Lemma 9, Chapter III.

Lemma 8: If Assumptions 1 and 2 hold, then $V_{A_1A_2}<0$.

Proof: Lemma 10, Chapter III.

V-3 The Optimal Portfolio Responses to Increasing Uncertainty of the Return on the Risky Asset

To determine the effects of a mean-preserving increase in the uncertainty of r_2 on the allocation of initial wealth amongst the safe asset, the risky asset, and first-period consumption, consider the following theorem.

Theorem 2: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the rate of return on the risky asset increases the optimal holding of the safe asset, decreases the optimal holding of the risky asset, and ambiguously affects the optimal level of first-period consumption.

Proof: Take the total derivative of (4), and set the increments of all the variables except A_1, A_2, e to zero. Using Cramer's Rule, we may write:

$$dA_1^*/de = [V_{A_2e} \cdot V_{A_1A_2} - V_{A_1e} \cdot V_{A_2A_2}]/|H| \tag{10}$$

$$dA_2^*/de = [V_{A_1e} \cdot V_{A_1A_2} - V_{A_2e} \cdot V_{A_1A_1}]/|H| \tag{11}$$

If Assumption 2 holds, then $V_{A_1e}=0$ [Lemma 5]. If Assumption 2 holds, then $V_{A_2e} < 0$ [Lemma 2]. If Assumptions 1 and 2 hold, then $V_{A_1A_2}<0$ [Lemma 8]. Finally, $V_{A_1A_1}<0$, $V_{A_1A_2}<0$, and $|H|>0$ by assumption [Equation (5)]. In sum, if Assumptions 1 through 5 hold, $dA_1^*/de>0$ and $dA_2^*/de<0$.

At the optimum, Equation (1) is:

$$C_1^*(e) = W_1 - A_1^*(e) - A_2^*(e) \tag{12}$$

Differentiating (12) with respect to e,

$$dC_1^*/de = -[dA_1^*/de + dA_2^*/de] \tag{13}$$

From the above it is clear that:

$$dC_1^*/de \gtrless 0 \text{ iff } dA_1^*/de \lessgtr dA_2^*/de \qquad \text{Q.E.D.}$$

V-4 Summary Remarks

This Chapter offered an alternative solution to a comparative-static problem posed by Sandmo (1969). In particular, in the context of Sandmo's two-asset, two-period model, this Chapter presented a set of restrictions on the utility function which enable the signing of the comparative-static effects of a "Sandmo-type" increase in the uncertainty of the return provided the risky asset. It was shown that if these restrictions hold, the effects of a "Sandmo-type" increase on the optimal allocation of initial wealth are: (i) to increase the optimal level of the safe asset, (ii) to decrease the optimal level of the risky asset, and (iii) to ambiguously affect the optimal level of first-period consumption.

Footnotes to Chapter V

[1] As in Chapter IV, autonomously-determined future income is assumed to be zero for reasons of simplicity.

[2] Equation (3) represents a reformulation of the portfolio problem posed by Sandmo (1969, p. 590), i.e., a reformulation of Equation (3) in Chapter IV. In particular, from Chapter IV, the original formulation of the Sandmo (1969) problem is:

$$\max_{C_1, A_2} V(C_1, A_1, A_2, e) =$$

$$\int U(C_1, r_1[W_1 - C_1] + [r_2 - r_1]A_2)g(r_2, e)dr_2 \qquad (3')$$

Since W_1 is fixed, it is clear from (1) that the choice of any pair of C_1, A_1 and A_2 determines the third variable; that is, Equations (3) and (3') are identical.

[3]See Footnote 7, Chapter III.

CHAPTER VI

A LABOR THEORETIC ANALYSIS OF CRIMINAL CHOICE UNDER INCREASING UNCERTAINTY*

Two further applications of the results generated in Chapter III are found in this Chapter. The first is an assessment of the robustness of the comparative-static effect of increasing uncertainty reported by Block and Heineke (1975a) in their paper on a labor theoretic analysis of criminal choice. The second application is the extension of this same analysis offered by Block and Heineke.

VI-1 An Introduction to the Problem

In a recent paper, Block and Heineke (1975a) observe that: "(the) relation between the offense decision and the degree of certainty with which the penalty is administered has been debated endlessly by criminologists. Well over a century and a half ago, Sir Samuel Romilly, in a series of debates with William Paley, held that not only did certainty of punishment deter criminal activities, but also that certainty of punishment was more crucial than severity. 'So evident is that truth of this maxim that if it were possible that punishment could be reduced to an absolute certainty, a very slight penalty would be sufficient to deter almost every species of crime'" (pp. 321-2). Then using a choice model of time spent in legal activity and illegal activity, Block-Heineke (B-H) show that Romilly's hypothesis can be supported provided two assumptions hold; that is, that "increases in the certainty of punishment...discourage criminal activity" provided that psychic costs are independent of wealth and that stochastic returns are constant (pp. 322-3).

The purpose of the present Chapter is two-fold. The first is to assess the robustness of B-H's evaluation of the Romilly hypothesis by employing (like B-H) a choice-theoretic framework. To do this, the works of Block and Heineke (1973) and Tressler and Menezes (1980) are utilized. Using these same works, the second purpose of this Chapter is to enlarge the B-H (1975a) analysis of the labor supply effects of increased uncertainty so as to encompass the effects associated with increases in the uncertainty of the wage-rate paid to legal work, the rate of which offenses are committed, the rate of return to illegal activity, and the level of fine per offense.

The present Chapter is organized as follows. In Section VI-2, the B-H (1975a) model of criminal choice is defined, the first and second order conditions for its solution is presented, and the general comparative-static problem associated with the problem is defined. In Section VI-3, the case of stochastic wages paid to legal activity is considered. The results determined here are two-fold: the total effect, and the marginal effects, of uncertainty. The latter is determined using Block and Heineke (1973), and

Tressler and Menezes (1980). In Section VI-4, four cases are considered: the cases of a stochastic rate of committing offenses, a stochastic rate of return on illegal activity, a stochastic rate of capture, and finally a stochastic fine per offense. In all four cases the total effect, and marginal effects, of uncertainty are determined. Again, the latter is determined using Block and Heineke (1973), and Tressler and Menezes (1980). In the case of stochastic rate of capture, our results are compared with that reported by Block and Heineke (1975a, pp. 321-3). Summary remarks are offered in Section VI-5.

VI-2 The Block-Heineke (1975a) Model of Criminal Choice[1]

Consider an individual who divides his time three ways, L_1 hours to legal activity, L_2 hours to illegal activity, and $S=T-L_1-L_2$ hours to leisure where T represents the total time available to any individual. For working L_1 hours at legal activity, this individual receives as wage income W_1L_1 where W_1 is the wage-rate paid.

Associated with illegal activity, there are three states of nature, each of which has a well-defined payoff. The first state is not committing an offense. The payoff in this state is zero; i.e.,

$$[1-\bar{W}_2]\cdot 0 = 0 \qquad (1)$$

where $\bar{W}_2$ is the non-stochastic rate at which offenses are committed per unit of L_2.[2] The second state is committing an offense and not being captured or arrested. The payoff in this state is:

$$W_3 \cdot (1-W_4) \cdot \bar{W}_2 \quad (2)$$

where W_3 is the non-stochastic rate of return on illegal activity, and W_4 is the rate of capture or arrest such that $0<W_4<1$. The third state is committing the offense and being captured or arrested. The payoff in this state is:

$$(W_3-W_5) \cdot W_4 \cdot \bar{W}_2 \quad (3)$$

where W_5 is the fine per offense. The expected return to illegal activity over all three states is the sum of the payoffs in each state, i.e.,

$$E(R) = [W_3-W_4 \cdot W_5] \cdot \bar{W}_2>0 \quad (4)$$

In sum, for spending L_2 hours in illegal activity, the individual expects to be remunerated in the amount $E(R) \cdot L_2$.

Assume that any one of the five parameters, W_i for $i=1,5$, may not be known with certainty. The expected value of W_i is $E(W_i) = \int W_i g^i(W_i;\ e_i)dW_i$ where $g^i(W_i;\ e_i)$ is the subjective probability density function (SPDF) of W_i, and e_i represents the "Rothschild-Stiglitz-measure" of uncertainty.

Suppose the individual has a utility function with leisure, S, and expected income, $E(Y) = E(W_1) \cdot L_1+E(R)L_2$, as arguments, i.e., $U(S,E(Y))$. Under the EUH, there exists an EUF, i.e., $EU(S,Y)=\int U(S,Y)g^i(W_i,e_i)dW_i$. Suppose this EUF is continuous, and three-times differentiable. Suppose too that this EUF has positive first derivatives, and negative second derivatives. The time allocation problem is solved by choosing L_1 and L_2 so as to maximize the utility function. In particular, under the EUH, the problem for the individual is to:[3]

$$\max_{L_1, L_2} V(S, L_1, L_2, e_i)$$

$$= \int_{R_i} U(T-L_1-L_2, Z_1L_1+Z_2L_2)g^i(W_i, e_i)dW_i \qquad (5)$$

such that $W_i \varepsilon R_i$ for $i=1,5$, where R_i is the domain of W_i, and where $Z_1=W_1$ and $Z_2=E(R)=(W_3-W_4 \cdot W_5)\bar{W}_2$. The first order conditions (FOC) for an interior maximum are:

$$V_{L_1} = -E(U_S) + E(Z_1U_Y) = 0 \qquad (6)$$

$$V_{L_2} = -E(U_S) + E(Z_2U_Y) = 0 \qquad (7)$$

The second order conditions (SOC) for an interior maximum are:

$$V_{L_1L_1}<0; \; V_{L_2L_2}<0; \text{ and}$$

$$(V_{L_1L_1} \cdot V_{L_2L_2} - (V_{L_1L_2})^2) = |H|>0 \qquad (8)$$

where:

$$V_{L_kL_\ell} = E(U_{SS})-E((Z_k+Z_\ell)U_{SY})+E(Z_kZ_\ell U_{YY}) \qquad (8a)$$

for $k=1,2$ and $\ell=1,2$. Thus if Equations (6), (7), and (8) hold, there exists some unique combination, (S^*, L_1^*, L_2^*), which satisfies Equation (5).

Associated with the optimal triplet (S^*, L_1^*, L_2^*) is the general comparative-static problem. In particular, consider the effects of a change in any exogenous variable, e, on the optimal triplet. These effects are determined as follows. Take the total differential of Equations (6) and (7), and set the increments of all variables except L_1, L_2, and e equal to zero. Using Cramer's Rule, we may write:

$$dL_1^*/de = \{V_{L_2 e} \cdot V_{L_1 L_2} - V_{L_1 e} \cdot V_{L_2 L_2}\}/|H| \quad (9)$$

and

$$dL_2^*/de = \{V_{L_1 e} \cdot V_{L_1 L_2} - V_{L_2 e} \cdot V_{L_1 L_1}\}/|H| \quad (10)$$

Since $V_{L_1 L_1}<0$, $V_{L_2 L_2}<0$, and $|H|>0$ by assumption [Equation (8)], signing dL_1^*/de and dL_2^*/de requires a particular sign pattern for $V_{L_1 e}$, $V_{L_2 e}$, and $V_{L_1 L_2}$. Given this sign pattern, and the time constraint, $T=S+L_1+L_2$, the sign of dS^*/de can be determined.

VI-3 The Effects of Uncertainty on the Risk-Averse Individual: The Case of Stochastic Wages Paid to Legal Activity

Suppose W_1 is stochastic (i.e., i=1). The effects of this form of uncertainty for the risk-averse individual are two-fold: the total effects, and the marginal effects. The total effects are twofold: (i) that the difference between the expected returns to legal activity and illegal activity be positive, and (ii) that the optimal level of illegal activity exceeds that of legal activity. In particular,

Theorem 1: If Z_1 is stochastic, and if the individual is risk-averse, then $E(Z_1)-E(R)>0$ where $E(R)$ is non-stochastic.

Proof: Theorem 1, Chapter III.

Theorem 2: If Z_1 is stochastic, and if the individual is risk-averse, then $L_2^*>L_1^*$.

Proof: Theorem 2, Chapter III.

To determine the marginal effects of uncertainty, the following approach is adopted. The comparative-static problem is partitioned into two parts. The first part is to determine the individual effects an increase in the uncertainty of $g^i(W_i, e_i)$ on an optimal level of L_1 in an arbitrarily small neighbourhood of L_2^*. The second part is to determine the individual effects of an increase in the uncertainty of $g^i(W_i, e_i)$ on an optimal level of L_2 in an arbitrarily small neighborhood of L_1^*. The merit of this approach is that it permits the use of results determined by B-H (1973), and Tressler and Menezes (1980).

To utilize the B-H (1973) and Tressler-Menezes (T-M) results then, two assumptions are required.

<u>Assumption 1</u>: L_k is an inferior good, i.e.,

$$-1/Z_k < dL_k/dY\Big|_{Y=E(Y)} < 0 \qquad (11)$$

for $i^*\neq k$ where $i^*=1$ if $i=1$, and $i^*=2$ if $i=2,3,4,$ <u>or</u> 5.

<u>Assumption 2</u>: The EUF exhibits CRAC, i.e.,

$$dU_{YY}/dY\Big|_{dU=0} = 0 \qquad (12)$$

The implications of these two assumptions are as follows:

<u>Lemma 1</u>: If Assumption 2 holds, and if the EUF exhibits $R_A(S,Y)>0$, then the EUF satisfies the Hanson-Menezes-Tressler Condition (HMTC), $dU_{YY}L_1^2/dL_1\Big|_{dU=0} <0$.

<u>Proof</u>: Lemma 3, Chapter III.

<u>Lemma 2</u>: If Assumption 2 holds, and if the EUF exhibits $R_A(S,Y)>0$, then $V_{L_1 e_1}<0$.

Proof: Lemma 4, Chapter III.

Lemma 3: If Assumption 2 holds, then $dU_{SY}/dY\Big|_{dU=0}=0$.

Proof: Lemma 5, Chapter III.

Lemma 4: If Assumption 2 holds, then $Cov(Z_1, U_{SY}-Z_2U_{YY})=0$.

Proof: Lemma 6, Chapter III.

Lemma 5: If Assumption 2 holds, then $V_{L_2e_1}=0$.

Proof: Lemma 7, Chapter III.

Lemma 6: If Assumption 1 holds, then $E(Z_2U_{YY}-U_{SY})<0$.

Proof: Lemma 8, Chapter III.

Lemma 7: If Assumption 1 holds, then $E(U_{SS}-Z_2U_{SY})<0$.

Proof: Lemma 9, Chapter III.

Lemma 8: If Assumptions 1 and 2 hold, then $V_{L_1L_2}<0$.

Proof: Lemma 10, Chapter III.

Having signed $V_{L_1e_1}$, $V_{L_2e_1}$, and $V_{L_1L_2}$, dL_1/de_1 and dL_2^*/de_2 may be signed by reference to Equations (9) and (10). In particular,

Theorem 3: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the return to legal activity decreases the optimal amount of time devoted to legal activity, increases the optimal amount of time devoted to illegal activity, and ambiguously affects the optimal amount of time devoted to leisure.

Proof: Since $V_{L_1L_1}<0$, $V_{L_2L_2}<0$, $|H|>0$, $V_{L_2e_1}=0$ [Lemma 5], $V_{L_1e_1}<0$ [Lemma 2], and $V_{L_1L_2}<0$ [Lemma 8], then $dL_1^*/de_1<0$ and $dL_2^*/de_1>0$ by virtue of Equations (9) and (10). Using the time constraint, this implies $dS^*/de\gtrless 0$ as $dL_1/de_1\gtrless dL_2/de_1$.

Q.E.D.

VI-4 The Effects of Uncertainty on the Risk-Averse Individual: The Cases of Stochastic Components in the Expected Return to Illegal Activity

Suppose any one of W_2, W_3, W_4, or W_5 is stochastic (i.e., $i^*=2$). As in Section III, the effects of uncertainty for these four cases are determined: the total effects and the marginal effects. The total effects for the risk-averse individual in each of these four cases are twofold: (i) that the difference between the expected returns to illegal activity and the certain return on legal activity be positive, and (ii) that the optimal level of legal activity exceed that of illegal activity. In particular,

Theorem 4: If Z_2 is stochastic and if the individual is risk-averse, then $E(R)-Z_1>0$.

Proof: Theorem 1, Chapter II.

Theorem 5: If Z_2 is stochastic, and if the individual is risk-averse, then $L_1^*>L_2^*$.

Proof: Theorem 2, Chapter III.

The marginal effects of uncertainty are considered next on a case-by-case basis. In particular,

(a) Case 1: The Rate of Committed Offenses as the Stochastic Variable: Let i=2. The comparative-static results of this Case, and their proof, are the mirror opposite of those stated in Theorem 3. In particular,

Theorem 6: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the rate of committed offenses increases the optimal amount of time devoted to legal activity, decreases the optimal amount of time devoted to illegal activity, and ambiguously affects the optimal amount of time devoted to leisure.

(b) Case 2: The Rate of Return to Illegal Activity as the Stochastic Variable: Let i=3. Like Case 1, the comparative-static results of this Case, and their proof, are the mirror opposite of those stated in Theorem 3. In particular,

Theorem 7: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the rate of return to illegal activity increases the optimal amount of time devoted to legal activity, decreases the optimal amount of time devoted to illegal activity, and ambiguously affects the optimal amount of time devoted to leisure.

(c) Case 3: The Rate of Capture or Arrest as the Stochastic Variable:[4] Let i=4. The present Case is the case considered by B-H (1975a, pp. 321-3): the case of the effect of increasing uncertainty on punishment. Our analysis of this Case differs from that provided by B-H in two respects. Firstly, B-H reduce the arguments in the utility function to one.[5] Secondly, unlike B-H, our assumption-set generates the postulate that increases in the uncertainty (rather than the certainty) of punishment discourages criminal activity.

This Case can be seen to differ from both Cases 1 and 2 (which were discussed above) in one important respect: the stochastic variable, W_4, enters the net wage rate, Z_2, as a negative magnitude, i.e., $Z_2=(W_3-W_4 \cdot W_5)\bar{W}_2$. Before the results reported in Cases 1 and 2 can be employed in the present case, the relationship between a mean-preserving increase in the uncertainty of W_4 and a mean-preserving increase in the uncertainty of Z_2 must be determined. This relationship is established in the following Lemma.

Lemma 9: If W_4 is stochastic, then a mean-preserving increase in the uncertainty of W_4 implies a mean-preserving increase in the uncertainty of $Z_2=(W_3-W_4 \cdot W_5)\bar{W}_2$.

Proof: Let W_4 have a normal distribution.

$E(Z_2)=\bar{W}_2 W_3-\bar{W}_2 W_5 E(W_4)$. Likewise:

$$\begin{aligned} Var(Z_2) &= E(Z_2-E(Z_2))^2 \\ &= E[-\bar{W}_2 \cdot W_5(W_4-E(W_4))]^2 \\ &= \bar{W}_2^2 \cdot W_5^2 \cdot E(W_4-E(W_4))^2 \\ &= \bar{W}_2^2 \cdot W_5^2 \cdot Var(W_4) \end{aligned}$$

Clearly, $dVar(Z_2)/dVar(W_4)>0$. That is, for W_4 which has SPDF which is a normal distribution, a mean-preserving increase in the SPDF of W_4 (i.e., $Var(W_4)>0$) implies a mean-preserving increase in the SPDF of Z_2 (i.e., $Var(Z_2)>0$). Q.E.D.

The importance of Lemma 9 is that the results for Cases 1 and 2 can be directly applied to Case 3. In particular,

Theorem 8: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the rate of capture increases the optimal amount of time devoted to legal activity, decreases the optimal amount of time devoted to illegal activity, and ambiguously affects the optimal amount of time devoted to leisure.

Proof: Theorem 3 and Lemma 9.

The importance of Theorem 8 is that its contents contradict the results reported by B-H (1975a, pp. 321-3). At a minimum, Theorem 8 demonstrates that the B-H results are not robust under an assumption-set which itself is consistent with other work on the problem of labor supply under uncertainty; namely, the assumption-set used by B-H (1973), and T-M (1980).

(d) Case 4: The Fine Per Offense as the Stochastic Variable: Let i=5. The comparative-static effects in this case are identical to Case 3; that is,

Theorem 9: If Assumptions 1 and 2 hold, then a "Sandmo-type" increase in the uncertainty of the fine per offense increases the optimal amount of time devoted to legal activity, decreases the optimal amount of time devoted to illegal activity, and ambiguously affects the optimal amount of time devoted to leisure.

Proof: Theorem 8.

VI-5 Summary Remarks

The objectives of this Chapter were two-fold. The first was to examine the robustness of the B-H (1975a) comparative-static effect concerning the uncertainty of capture or arrest, and the second was to enlarge the B-H analysis to include the effects of changes in the uncertainty of other stochastic variables.

In Section VI-2, the B-H (1975a) model of criminal choice was presented, the necessary and sufficient conditions for its solution determined, and the general comparative-static problem defined. In Section VI-3, the case of stochastic wages paid to legal activity was considered. In particular, the total and marginal effects of uncertainty were determined. In the latter case, using B-H (1973) and T-M (1980), it was shown that a mean-preserving increase in wage-rate uncertainty reduces the optimal allocation of time to legal activity, and increases the optimal allocation of time to illegal activity. In Section VI-4, four cases (in which each of the components of the expected return to illegal activity were treated as stochastic) were considered. These were: the rate at which offenses are committed, the rate of return to illegal activity, the rate of capture or arrest, and the level of fine per offense. In each case, the total and marginal effects were determined. The results of the latter were identical for all four cases. That is, using B-H (1973) and T-M (1980), <u>a mean-preserving increase in the uncertainty of any component of the expected return to illegal activity decreases the optimal allocation of time to illegal activity, and increases the optimal allocation of time to legal activity</u>. This general result demonstrates the B-H (1975a) result (with regards to the effect of increasing uncertainty of the rate of capture on the level of criminal activity) is not robust.

Footnotes to Chapter VI

*An earlier draft of this Chapter was presented as a paper in the Session entitled "Some Issues in Economic Theory," at the Annual Meeting of the Canadian Economics Association, Guelph, 1984.

[1] The Block and Heineke (1975a) paradigm of criminal

choice is in the tradition set by Becker (1968), and Ehrilich (1973 and 1975). Examples of recent studies which employ the B-H paradigm include Hakim _et al._ (1978), Witte (1980), and Balkin and McDonald (1981).

[2]An alternative definition of $\bar{W}_2$ (one not used by B-H, but used by Balkin and McDonald (1981)) is the probability of finding a victim.

[3]Our objective function differs from the B-H (1975a) objective function in two respects. _Firstly_, using our notation, the B-H objective function (i.e., their Equation (2)) is:

$$\max_{L_1, L_2} \int U(L_1, L_2, Y) f(W_4) dW_4$$

where $Y=W_1L_1+(W_3-W_4 \cdot W_5)W_2+W^0$ and where W^0 represents initial wealth. Their motivation for including L_1 and L_2 explicitly in the EUF is as follows: "By including (these) arguments explicitly in U, we are provided with a straightforward means of analyzing the role of moral and ethical considerations which may constrain the work-theft decision" (p. 315). Thus this specification permits an analysis in which the marginal utility of legitimate work is not equal to the marginal utility of illegal activity. Thus, assuming the individual demonstrates a _preference for honesty_, then $U_{L_1} > U_{L_2}$ (p. 316).

Secondly, on pages 319-20, B-H (1975a) show that the use of the wealth equivalents of L_1 and L_2 enables the elimination of L_1 and L_2 from the utility function.

[4]In this Case, B-H investigate the comparative-static effect of an increase in the uncertainty of a measure of probability, the rate of capture. In strict terms, the measurement which B-H employ is the measure, the "probability of a probability". For a discussion of the meaningfulness of this measure, see Marschak _et al._ (1975), Borch (1975), Gardenfors (1979), and Dale (1980).

[5]See Footnote 3 above.

CHAPTER VII

TAX EVASION AND LABOR SUPPLY UNDER IMPERFECT INFORMATION ABOUT INDIVIDUAL PARAMETERS OF THE TAX SYSTEM*

A final application of the results generated in Chapter III is contained in the present Chapter. In it, the Isachsen and Strom (1980) model of tax evasion and labor supply is used as a basis for providing a choice theoretical explanation of two phenomena: the impact of information about the tax system on the decision to evade taxes, and the decision about the amount of income to underreport.

VII-1 Introduction to the Problem

Research on tax evasion may be viewed as being conducted along two main branches of inquiry. The first is focused on the definition and testing of a class of models in which tax evasion is treated as the sole endogenous variable. The seminal theoretical works in this first branch are Allingham and Sandmo (1972), and Srinivasan (1973).[1] Empirical tests of various predictions of this class of models are provided by Friedland *et al.* (1978), Spicer and Thomas (1982), and Clotfelter (1983). In conclusion to their study, Spicer and Thomas reveal the importance of information in modelling tax evasion: they assert: "Our results indicate that economic models of tax evasion, which assume the availability of precise information, should be modified to take account of cases where taxpayers make evasion decisions on the basis of imprecise information" (p. 245).[2]

The second branch of inquiry is focused on the definition of a class of models which treats both tax evasion and labor supply as endogenous variables. This work was initiated by Allingham and Sandmo (1972),[3] and the problem pursued by Andersen (1977), Baldry (1979), Pencavel (1979), Isachsen and Strom (1980),[4] and Weiss (1976).[5] With the exception of Weiss (1976), these studies assume that the decision-maker has perfect information about the individual parameters of the tax system.

The objective of the present Chapter is to address this oversight in the second branch of inquiry. In particular, by assuming available information about individual parameters of the tax system is imperfect, the purpose of this Chapter then is the determination of the total and marginal effects of uncertainty in a model in which both tax evasion and labor supply are treated as endogenous variables.

The approach used in defining the marginal effects warrants some comment. In particular, the model and objective function to be employed are provided by Isachsen and Strom (I-S). However, rather than use the restrictions on the expected utility function employed by I-S (1980), the restrictions (or sufficient conditions) to be used in the definition of the marginal effects are those proposed by

Block and Heineke (1973) and Tressler and Menezes (1980) This unique combination of objective function, and sufficient conditions, enables the generation of new insight (or predictions) about tax evasion and labor supply under mean-preserving changes in uncertainty.

This Chapter is organized as follows. In Section VII-2, the I-S (1980) model of tax evasion and labor supply is defined, and the first and second order conditions for its solution are presented. The total effects of imperfect information about the tax system is presented in Section VII-3. Furthermore, in Section VII-3, the I-S condition for an interior solution (in the context of imperfect information about the tax system) is discussed. In Section VII-4, the marginal effects of decreased uncertainty (or improved information) are presented. Summary remarks are offered in Section VII-5.

VII-2 The Isachsen-Strom (1980) Model of Tax Evasion and Labor Supply Under Imperfect Information

Consider an individual who divides his time three ways, L_1 hours to work in the visible or legitimate economy, L_2 hours to work in the hidden economy, and $S=T-L_1-L_2$ hours to leisure where T represents the total time available. For working L_1 hours in the visible economy this individual receives as after-tax wage-income $(1-t_1)W_1L_1>0$ where W_1 is the wage-rate paid, and t_1 is the tax-rate in visible economy.

In the hidden economy, there are two states of nature, and both states have a well-defined expected payoff. The <u>first state</u> is <u>working in the hidden economy and not being detected</u> by the tax authorities. The expected payoff per hour of work under this state is $(1-t_2)\cdot W_2$ where W_2 is the wage-rate paid in the hidden economy, and where $(1-t_2)$ is the probability of not being detected such that $0<(1-t_2)<1$. The <u>second state</u> is <u>working in the hidden economy and being detected</u> by tax authorities. The expected payoff per hour of work under this state $t_2(1-t_3)\cdot W_2$ where t_3 is the marginal penalty tax-rate on earned income in the hidden economy, and where $0<t_1<t_3\leq 1$. The sum of expected payoffs over both states, i.e.,

$$(1-t_2)W_2+t_2(1-t_3)W_2 = (1-t_2\cdot t_3)W_2>0 \qquad (1)$$

is what I-S (1980, p. 308) term the "expected net wage" in the hidden economy. In sum, for spending L_2 hours in the hidden economy, the individual expects remuneration in the amount $(1-t_2\cdot t_3)\cdot W_2\cdot L_2$.

Assume that any one of the three parameters associated with the tax system (i.e., t_i for i=1,3) may <u>not</u> be known with certainty. The expected value of t_i is $E(t_i) = \int t_i g^i(t_i;\ e_i)dt_i$ where $g^i(t_i;\ e_i)$ is the subjective probability density function of t_i, and e_i represents the "Rothschild-Stiglitz-measure" of uncertainty.

Suppose the individual has a utility function with leisure, S, and expected income, $E(Y) = E[(1-t_1)\cdot W_1]\cdot L_1+E[(1-t_2\cdot t_3)\cdot W_2\cdot L_2]$, as arguments, i.e., $U=U(S,E(Y))$. Under the expected utility hypothesis (EUH), there exists an expected utility function (EUF), i.e., $EU(S,Y)=$

$\int U(S,Y)g^i(t_i,e_i)dt_i$. Suppose this EUF is continuous, and three-times differentiable. Suppose too that this EUF has positive first derivatives, and negative second derivatives. The time allocation problem is solved by choosing L_1 and L_2 so as to maximize the EUF. In particular, under the EUH, the problem for the individual is to:

$$\max_{L_1, L_2} V(S, L_1, L_2, e_i)$$

$$= \int_{R_i} U(T-L_1-L_2, Z_1L_1+Z_2L_2)g^i(t_i,e_i)dt_i \qquad (2)$$

such that $t_i \varepsilon R_i$ for i=1,3 where R_i is the domain of t_i, and where $Z_1=(1-t_1)W_1$ and $Z_2=(1-t_2 \cdot t_3)W_2$. The first order conditions (FOC) for an interior maximum are:

$$V_{L_1} = -E(U_S)+E[Z_1U_Y] = 0 \qquad (3)$$

$$V_{L_2} = -E(U_S)+E[Z_2U_Y] = 0 \qquad (4)$$

The second order conditions (SOC) for an interior maximum are:

$$V_{L_1L_1}<0; \; V_{L_2L_2}<0; \text{ and } (V_{L_1L_1} \cdot V_{L_2L_2}-(V_{L_1L_2})^2) = |H|>0 \qquad (5)$$

where:

$$V_{L_kL_\ell} = E(U_{SS})-E((Z_k+Z_\ell)U_{SY})+E(Z_kZ_\ell U_{YY}) \qquad (6)$$

for k=1,2 and ℓ=1,2. Thus, if Equations (3), (4), and (5) hold, there exists some unique combination, (S^*, L_1^*, L_2^*), which satisfies Equation (2).

VII-3 The Total Effects of Imperfect Information About the Tax System

The total effects of imperfect information about the tax system concern the sign of the difference between the expected net wage in the hidden economy and the expected net wage in the visible economy. The sign of this difference depends on whether t_1, on one hand, or t_2 or t_3 on the other, is assumed stochastic. In particular, assume firstly t_1 is stochastic.[6] Then:

Theorem 1: If t_1 is stochastic, and if the individual is risk-averse, then $E(1-t_1)W_1>(1-t_2 \cdot t_3)W_2$.

Proof: Subtracting (4) from (3) yields:

$$(1-t_2 \cdot t_3)W_2 E(U_Y) = E[(1-t_1)W_1 U_Y]$$
$$= W_1 E(U_Y) - W_1 E(t_1 U_Y)$$
$$= W_1 E(U_Y)$$
$$-W_1[E(t_1) \cdot E(U_Y) + Cov(t_1, U_Y)]$$

Dividing by $E(U_Y)$ and rearranging terms yields:

$$(1-t_2 \cdot t_3)W_2 - E(1-t_1)W_1 = -Cov(t_1, U_Y)/E(U_Y) \qquad (7)$$

Since $dY/dt_1<0$ and $dU_Y/dY<0$ [for $R_A(S,Y) = -U_{YY}/U_Y>0$], then $Cov(t_1, U_Y)>0$ and $E(1-t_1)W_1-(1-t_2 \cdot t_3)W_2>0$. Q.E.D.

Theorem 2: If t_1 is stochastic, and if the individual is risk-averse, then $L_2^*>L_1^*$.

Proof: Theorem 2, Chapter III.

Next, assume either t_2 or t_3 is stochastic. Then:

Theorem 3: If either t_2 or t_3 is stochastic, and if the individual is risk-averse, then $(1-t_1)W_1<E(1-t_2 \cdot t_3)W_2$.

Proof: Assume t_3 is stochastic. Subtracting (4) from (3) yields:

$$(1-t_1)W_1 \cdot E(U_Y) = E[(1-t_2 \cdot t_3)W_2 \cdot U_Y]$$
$$= W_2 E(U_Y) - t_2 W_2 E(t_3 \cdot U_Y)$$
$$= W_2 E(U_Y)$$
$$-t_2 W_2[E(t_3)E(U_Y) + Cov(t_3, U_Y)]$$

Dividing by $E(U_Y)$ and rearranging terms yields:

$$(1-t_1)W_1-E(1-t_2\cdot t_3)W_2 = -Cov(t_3,U_Y)/E(U_Y) \qquad (8)$$

Since $dY/dt_3<0$ and $dU_Y/dY<0$ [for $R_A(S,Y)>0$], then $Cov(t_3,U_Y)>0$ and $E(1-t_2\cdot t_3)W_2-(1-t_1)W_1>0$. Q.E.D.

Theorem 4: If either t_2 or t_3 is stochastic, and if the individual is risk-averse, then $L_1^*>L_2^*$.

Proof: Theorem 2, Chapter III.

Two comments about Theorems 1 and 3 are warranted. Firstly, from Equations (7) and (8), it is clear that the sign of the difference between the expected net wage rate in the visible and hidden economy depends on the sign $[-Cov(t_i, U_Y)/E(U_Y)]$. In keeping with the preceding Chapters, this coefficient is termed the CRB.

Secondly, in their model of tax evasion and labor supply under perfect information about the tax system, I-S (1980) identify a sufficient condition for an interior maximum (i.e., the condition for the allocation of time to both the visible and hidden economies). This is their Equation (11). In our notation, this Equation is:

$$(1-t_3)W_2\leq(1-t_1)W_1\leq(1-t_2\cdot t_3)W_2$$

In the case of imperfect information about the tax system, Theorems 1 and 3 suggest that this condition may no longer hold. For example, in the case in which t_1 is stochastic, Theorem 1 suggests that the inequality on the R.H.S. of I-S's Equation (11) is vitiated. In general, the condition for an interior solution is affected by the amount of information available to the decision-maker. This in turn can affect his decision to work in the hidden economy or not, i.e., affect his decision to evade taxes or not.

VII-4 The Marginal Effects of Improved Information About the Tax System

Associated with the optimal triplet (S^*, L_1^*, L_2^*) defined in Section II is the general comparative-static problem. In particular consider the effects of a change in any exogenous variable, e, on the optimal triplet. These effects are determined as follows. Take the total differential of Equations (3) and (4), and set the increments of all variables except L_1, L_2, and e equal to zero. Using Cramer's Rule, we may write:

$$dL_1^*/de = \{V_{L_2 e} \cdot V_{L_1 L_2} - V_{L_1 e} \cdot V_{L_2 L_2}\}/|H| \qquad (9)$$

and

$$dL_2^*/de = \{V_{L_1 e} \cdot V_{L_1 L_2} - V_{L_2 e} \cdot V_{L_1 L_1}\}/|H| \qquad (10)$$

Since $V_{L_1 L_1}<0$, $V_{L_2 L_2}<0$, and $|H|>0$ by assumption [Equation (5)], signing dL_1^*/de and dL_2^*/de requires a particular sign pattern for $V_{L_1 e}$, $V_{L_2 e}$, and $V_{L_1 L_2}$. Given this sign pattern and the time constraint, $T=S+L_1+L_2$, the sign of dS^*/de can be determined.

To determine the marginal effects of uncertainty, the following approach is adopted. The comparative-static problem is partitioned into two parts. The first part is to determine the individual effects a decrease in the uncertainty of $g^i(t_i, e_i)$ on <u>an</u> optimal level of L_1 in an arbitrary, small neighbourhood of L_2^*. The second part is to determine the individual effects of a decrease in the uncertainty of $g^i(t_i, e_i)$ on <u>an</u> optimal level of L_2 in an arbitrary, small neighbourhood of L_1. The merit of this approach is that it permits the use of results determined by B-H (1973), and T-M (1980).

To utilize the B-H (1973) and T-M (1980) results then, two assumptions are required.

<u>Assumption 1</u>: L_k is an inferior good, i.e.,

$$-1/Z_k < dL_k/dY\Big|_{Y=E(Y)} < 0 \tag{11}$$

for $i^* \neq k$ where $i^*=1$ if $i=1$ and $i^*=2$ if $i=2$ or $i=3$.

<u>Assumption 2</u>: The EUF exhibits CRAC, i.e.,

$$dU_{YY}/dY\Big|_{dU=0} = U_{YYY} - U_{YYS}[U_Y/U_S] = 0 \tag{12}$$

In determining the marginal effects, two cases must be kept distinct: (i) the tax-rate as the stochastic variable (i.e., $i^*=1$), and (ii) either the probability of detection <u>or</u> the penalty tax-rate as the stochastic variable (i.e., $i^*=2$). Likewise, it is important to establish the relationship between a mean-preserving increase in the uncertainty of t_{i^*} and Z_k in order to employ the results of Chapter III. This relationship is:

<u>Lemma 1</u>: If t_{i^*} is stochastic, then a mean-preserving increase in the uncertainty of t_{i^*} implies a mean-preserving increase in the uncertainty of Z_k for $i^*=k$.

<u>Proof</u>: Lemma 9, Chapter VI.

(a) <u>Case 1: The Tax-Rate as the Stochastic Variable</u>: Let $i^*=1$. The implications of Assumptions 1 and 2 are those of

Chapters III, V, and VI, i.e.,

Lemma 2: If Assumption 2 holds, and if the EUF exhibits $R_A(S,Y)>0$, then the EUF satisfies the Hanson-Menezes-Tressler Condition (HMTC), $dU_{YY}L_1^2/dL_1\Big|_{dU=0} < 0$.

Proof: Lemma 3, Chapter III.

Lemma 3: If Assumption 2 holds, and if the EUF exhibits $R_A(S,Y)>0$, then $V_{L_1e_1}<0$.

Proof: Lemma 4, Chapter III.

Lemma 4: If Assumption 2 holds, then $dU_{SY}/dY\Big|_{dU=0} = 0$.

Proof: Lemma 5, Chapter III.

Lemma 5: If Assumption 2 holds, then $\mathrm{Cov}(Z_1, U_{SY}-Z_2U_{YY})=0$.

Proof: Lemma 6, Chapter III.

Lemma 6: If Assumption 2 holds, then $V_{L_2e_1}=0$.

Proof: Lemma 7, Chapter III.

Lemma 7: If Assumption 1 holds, then $E(Z_2U_{YY}-U_{SY})<0$.

Proof: Lemma 8, Chapter III.

Lemma 8: If Assumption 1 holds, then $E(U_{SS}-Z_2U_{SY})<0$.

Proof: Lemma 9, Chapter III.

Lemma 9: If Assumptions 1 and 2 hold, then $V_{L_1L_2}<0$.

Proof: Lemma 10, Chapter III.

Having signed $V_{L_1e_1}$, $V_{L_2e_1}$, and $V_{L_1L_2}$, dL_1^*/de_1 and dL_2^*/de_1 may be signed by reference to Equations (9) and (10). In particular,

Theorem 5: If Assumptions 1 and 2 hold, then a "Sandmo-type" decrease in the uncertainty of the tax rate increases the optimal amount of time devoted to work in the visible economy, decreases the optimal amount of time devoted to work in the hidden economy, and ambiguously affects the optimal amount of time devoted to leisure.

Proof: Since $V_{L_1L_1}<0$, $V_{L_2L_2}<0$, $|H|>0$ [Equation 5], $V_{L_2e_1}=0$ [Lemma 6], $V_{L_1e_1}<0$ [Lemma 3], and $V_{L_1L_2}<0$ [Lemma 9], then $dL_1^*/de_1<0$ and $dL_2^*/de_1>0$ by virtue of Equations (9) and (10). Using the time constraint, this implies $dS^*/de_1 \gtreqless 0$ as $dL_1/de_1 \gtreqless dL_2/de_1$. Q.E.D.

(b) Case 2: The Probability of Detection or the Penalty Tax-Rate as the Stochastic Variable: Let $i^*=2=i$.[7] This case is symmetrical with Case 1, and therefore:

Theorem 6: If Assumptions 1 and 2 hold, then a "Sandmo-type" decrease in the uncertainty of either the probability of detection or penalty tax rate increases the optimal amount of time devoted to the hidden economy, decreases the optimal amount of time devoted to the visible economy, and ambiguously affects the optimal amount of time devoted to leisure.

Proof: Theorem 5.

VII-5 Summary Remarks

Motivated by empirical research, the purpose of this Chapter was to extend the I-S (1980) model of tax evasion and labor supply by adopting the assumption of imperfect information about the tax system parameters. Our adaptation of this model is presented in Section VII-2. The principal results obtained from this revised model are two-fold. The first is the total effects of imperfect information. In particular, in Section VII-3, a comment about the possible necessity of modifying the I-S condition for an interior maximum is offered.

The second principal set of results concerned the marginal effects of uncertainty. In particular, motivated by Block and Heineke (1973) and Tressler and Menezes (1980), two restrictions were imposed on the EUF function. In the case of the tax-rate as the stochastic variable, it was shown that a mean-preserving decrease in the uncertainty of the tax-rate increases the optimal amount of time devoted to the visible economy, and decreases the amount of time devoted to the hidden economy. In the case in which <u>either</u> the probability of detection <u>or</u> the penalty tax-rate is treated as the stochastic variable, a mean-preserving decrease in uncertainty increases the optimal amount of time allocated to the hidden economy, and decreases the optimal amount of time devoted to the visible economy. In sum, these results suggest an important general result. This is that the direction of the marginal effects of improved information on tax evasion depends on which of three parameters of the tax system is assumed stochastic.

Footnotes to Chapter VII

*This Chapter was presented as a paper in the Invited Session entitled "Risk, Uncertainty, and Individual Choice", the Annual Meeting of the Southern Economic Association, Dallas, Texas, November 24-26, 1985, and this same paper is forthcoming in <u>Public Finance/Finances Publiques</u>.

[1] Allingham and Sandmo (1972) and Srinivasan (1973) are in the tradition of Becker (1968) and Ehrilich (1973). The work by Allingham and Sandmo (1972) and Srinivasan (1973) has been extended by Kolm (1973), Yitzhaki (1974), Ashenfelter (1978), Nayak (1978), Sproule <u>et al.</u> (1980), and Fishburn (1981).

[2] This finding corroborates the results of other empirical studies on tax evasion which do not use the Allingham-Sandmo-Srinivasan paradigm of tax evasion as a research framework. In particular, the Spicer-Thomas finding is consistent with the work of Vogel (1974), Mason <u>et al.</u> (1975), and Spicer and Lundstedt (1976).

[3] In their paper of 1972, Allingham and Sandmo (A-S) called for just such an extension of their work on tax evasion. They write: "We ... hope that (our) approach will suggest other topics for research in the field, both theoretical and empirical. Of theoretical topics the ones which immediately suggest themselves are perhaps generalizations of the present model. One possibility is to extend the model to take account of labor supply decisions; one might hope to discover some interesting connections between incentives to avoid taxes and supply work effort. However, although we have studied this case, we have not been able to come up with any interesting and reasonably simple results" (pp. 337-8).

[4] The A-S claim regarding their inability to obtain results from a general model of tax evasion and labor supply (as outlined in Footnote 3) has been corroborated by Baldry (1979) and Pencavel (1979). Only when restrictions on the expected utility functions are employed can unambiguous comparative-static results be obtained. This is shown to be the case by Andersen (1977) in his use of the restriction of additivity of the arguments, leisure and income. This fact is also borne out in Isachsen and Strom (1980) in their definition of the expected utility function as the product of leisure and income.

[5] The comparative-static results reported by Andersen-Isachsen-Strom (in this second branch of inquiry) may be viewed as being concerned with the effects of "distribution-preserving shifts in the mean". With the exception of Weiss (1976), no paper in this branch of inquiry addresses the effects of "mean-preserving shifts in the distribution". The general object of concern in Weiss' paper is one of modelling tax evasion and labor supply under imperfect information about the tax system.

[6] This terminology is flawed (or not precise) in the following sense. The net wage-rate in the hidden economy can never be truly non-stochastic because of the fact that there exists two states of the world, and associated

measures of probability of either state occurring.

[7] In the case of i=2, we are considering the comparative-static effects of changes in the uncertainty of a probability measure, i.e., changes in the "probability of a probability". For a discussion of the meaningfulness of the measure, "probability of a probability", see Marschak *et al.* (1975), Borch (1975), Gardenfors (1979), and Dale (1980). In the economics of uncertainty, Block and Heineke (1975, pp. 321-3) provide an example of this measure's use.

CHAPTER VIII

ON MODELLING THE FAMILY LABOR SUPPLY DECISION UNDER UNCERTAINTY: A COMPARISON OF TWO APPROACHES

In the present Chapter, the techniques developed in Chapter III provide a touch-stone for an investigation into an as-yet unexplored topic, the modelling of the family labor supply decision under uncertainty. In general, there are at least three approaches to this problem which might be explored. Three of these are: the Bowen and Finegan (1965, 1966, and 1969) approach, the Kosters (1966 and 1969) approach, and the Leuthold (1968) approach.[1] In this Chapter, the latter two approaches are considered. The objective function proposed by Leuthold (1968) is in some sense in keeping with our first class of objective functions. Kosters' (1966 and 1969) approach however requires the use of a utility function in _three_ arguments. Our investigation into the impact of increasing uncertainty on a family labor supply decision fashioned after Kosters' work represents an investigation related to a _third class_ of models.

VIII-1 An Introduction to the Problem

A review of the literature makes clear that the development of choice theoretical models of the labor supply decision under _uncertainty_ has not kept pace with the development of the same class of models under _certainty_. In particular, in the case of the literature on models under certainty, Robbins (1930) and Hicks (1932 and 1946) define the basic model of the individual's labor supply decision.[2] Likewise, Kosters (1966 and 1969) and Leuthold (1968) provide two distinct approaches to the problem of modelling the choice theoretical basis of family labor supply decision. In contrast, in the case of modelling the labor supply decision under uncertainty, the literature is comprised only of models of the individual's labor supply decision, i.e., Block and Heineke (1973), Tressler and Menezes (1980), Cowell (1981), and Sproule (1985). No work on the family labor supply decision under uncertainty [comparable to Kosters (1966 and 1969) and Leuthold (1968)] has been reported.

The purpose of the present Chapter is twofold. The first is to extend the literature on choice theoretical models of labor supply under uncertainty by adapting the Kosters (1966) and Leuthold (1968) models of the family labor supply decision to a stochastic environment. The second purpose is to compare the properties of both adaptations of these models. In particular, a comparison of the total and marginal effects of uncertainty associated with each model is undertaken. In the determination of the marginal effects, restrictions on expected utility functions (EUF) which were first suggested by Block and Heineke (B-H) and Tressler and Menezes (T-M) are employed.

The present Chapter is organized as follows. In Section VIII-2, the Leuthold (1968) approach is defined, and the total effects, and marginal effects, of uncertainty are discussed. In the determination of the marginal effects, four restrictions are placed on the family utility function. The use of these four restrictions is motivated by B-H (1973) and T-M (1980). In Section VIII-3, the Kosters (1966) approach is defined, and the total effects, and marginal effects, of uncertainty are discussed. In the determination of the marginal effects, one assumption (in addition to the four employed in Section VIII-2) is required. Summary remarks are offered in Section VIII-4.

VIII-2 The First Approach to Modelling the Family Labor Supply Decision Under Uncertainty: The Leuthold (1968) Approach

The first approach to be considered was first suggested by Leuthold (1968), and has been employed subsequently by Ashworth and Ulph (1981).[3] The Leuthold approach is unique in three senses. Firstly, it hypothesizes that each family member (say the jth member) has his or her own utility function. Secondly, the utility function of the jth individual is defined over the jth individual's leisure and family income. And finally, the optimal level of labor supply of the jth individual is found by maximizing the jth utility function with respect to the jth individual's labor supply subject to the time and family income constraints. A more formal definition of this model under uncertainty, and its related properties, follows.

(a) The Leuthold (1968) Model: Consider a family with two wage-earners, a husband and a wife. The husband divides his time L_1 hours of work and $S_1=T-L_1$ hours of leisure where T is the total time available. For working L_1 hours, he is paid wage-income of W_1L_1 where W_1 is the husband's wage-rate. Likewise, the wife divides her time between L_2 hours of work and S_2 hours of leisure. The family receives non-wage income in the amount of W_3. Family income then is the sum of wage income and non-wage income, i.e., $Y=W_1 \cdot L_1+W_2 \cdot L_2+W_3$.

Suppose that any one of W_1, W_2, or W_3 may be random. Suppose too the family members formulate an expected value for W_i for i=1,3, i.e., $E(W_i)=\int W_i \cdot g(W_i,v_i,e_i)dW_i$ where $G(W_i,v_i,e_i)$ is the subjective probability density function of W_i, v_i represents a distribution-preserving mean-shifting parameter, and e_i represents the "Rothschild-Stiglitz-measure" of uncertainty.

After Leuthold (1968), suppose the welfare of the husband can be measured by a utility function defined over the leisure of the husband, and expected family income for a given level of the wife's leisure, i.e., $U^1=U^1(S_1,\bar{S}_2,E(Y))$.[4] Under the expected utility hypothesis (EUH), there exists an EUF, i.e., $EU^1(S_1,\bar{S}_2,Y)=\int U^1(S,\bar{S}_2,Y) \cdot g(W_i,v_i,e_i)dW_i$.[5] Suppose this EUF is continuous, and three times differentiable. Furthermore, assume this EUF has positive first-derivatives, and negative second-derivatives.[6]

Likewise, the welfare of the wife is assumed to be measured by a utility function defined over leisure of the wife, and expected family income for a given level of husband's leisure, i.e., $U^2 = U^2(\bar{S}_1, S_2, E(Y))$. Again, under the EUH, there exists an EUF, i.e., $EU^2(\bar{S}_1, S_2, Y) = \int U^2(\bar{S}_1, S_2, Y) \cdot g(W_i, v_i, e_i) dW_i$. Suppose this EUF is continuous and three times differentiable. Finally, assume this EUF has positive first-derivatives and negative second-derivatives.

The problem for the jth family member for j=1,2 is the choice of L_j which maximizes his or her utility function, U^j, subject to the time and income constraints. Under the EUH, the problem is:

$$\max_{L_j} V(L_j, v_i, e_i) = \int_{R_i} U^j(T-L_1, T-L_2, W_1L_1+W_2L_2+W_3) \cdot g(W_i, v_i, e_i) dW_i \quad (1)$$

where R_i is the domain of W_i. The first and second order conditions for an interior maximum are:

$$V_{L_j} = -E(U_j) + E(W_jU_3) = 0 \quad (2)$$

and:

$$V_{L_jL_j} = E(U_{jj}) - 2E(W_jU_{j3}) + E(W_j^2 \cdot U_{33}) < 0 \quad (3)$$

for j=1,2. Thus, if Equations (2) and (3) hold, there exists some unique value, L_j^0, which satisfies Equation (1).

(b) <u>The Total Effects of Uncertainty</u>: Regarding the total effects, there are two cases to be considered: the case in which the jth individual's own wage-rate is stochastic (i.e., i=j), and otherwise (i.e., i≠j). Consider the latter first. From Equation (2), it follows that:

$$W_j = E(U_j)/E(U_3) \qquad [\text{for } i \neq j] \qquad (4)$$

that is, the marginal rate of substitution between family income and leisure for the jth individual (MRS_{3j}) is equal to the certain wage-rate paid the jth individual.

Next, consider $i=j$. Under the assumption that $i=j$, Equation (4) no longer holds. In fact, $E(W_j) > E(U_j)/E(U_3) = MRS_{3j}$. In particular,

<u>Theorem 1</u>: If $i=j$, then $E(W_j) > E(U_j)/E(U_3)$.

<u>Proof</u>: From Equation (2), it follows that:

$$E(U_j) = E(W_j) \cdot E(U_3) + Cov(W_j, U_3)$$

Rearranging terms:

$$E(W_j) = E(U_j)/E(U_3) - Cov(W_j, U_3)/E(U_3) \qquad (5)$$

Since $dY/dW_j > 0$ and $dU_3/dY < 0$ [for $R_A(S_1 . S_2, Y) > 0$], then $Cov(W_j, U_3) < 0$. Likewise, for $Cov(W_j, U_3) < 0$, $E(W_j) > E(U_j)/E(U_3)$.

Q.E.D.

(c) <u>The Marginal Effects of Uncertainty</u>: To determine the effect of a mean-preserving increase in the uncertainty of W_i for $i=1,3$ on L_j^0 (i.e., the marginal effects of uncertainty), combinations of four restrictions must be placed on the expected utility function. These assumptions are

<u>Assumption 1</u>: The EUF of the jth individual exhibits constant absolute risk aversion (CARA) in L_j, i.e.,

$$\partial R_A(S_1, S_2, Y)/\partial L_j = 0 \qquad (6)$$

for $j=1,2$.

<u>Assumption 2</u>: The EUF of all j individuals exhibits decreasing absolute risk aversion (DARA) in Y, i.e.,

$$\partial R_A(S_1, S_2, Y)/\partial Y < 0 \qquad (7)$$

Assumption 3: For all j individuals, work is an inferior good, i.e.,

$$dL_j/dY\Big|_{Y=E(Y)} < 0 \tag{8}$$

for j=1,2.

Assumption 4: The EUF satisfies the HMTC, i.e.,

$$\frac{dU_{33}L_j^2}{dL_j}\Bigg|_{dU=0} = L_j^2[W_jU_{333} - U_{33j}] + 2L_jU_{33} < 0 \tag{9}$$

for j=1,2.

The signs of the marginal effects of uncertainty depends on which of W_i for i=1,3 is treated as stochastic. In particular, suppose $i \neq j$. Then:

Theorem 2: If $i \neq j$ and if Assumptions 1 through 3 hold, then $dL_j^o/de_i > 0$ and $V_{L_je_i} > 0$.

Proof: From Equation (2), $dL_j^o/de_i = -V_{L_je_i}/V_{L_jL_j}$. By Equation (3), $V_{L_jL_j} < 0$. Therefore:

$$\text{sign } (dL_j^o/de_i) = \text{sign } (V_{L_je_i})$$

If Assumptions 1 through 3 hold, Block and Heineke (1973, pp. 379-81) show $dL_j^o/de_i > 0$ where e_i is the "Sandmo-measure" of uncertainty. Q.E.D.

Now suppose i=j. Then:

Theorem 3: If i=j and if Assumption 4 holds, then $dL_j^o/de_i < 0$ and $V_{L_je_i} < 0$.

Proof: From Theorem 2,

$$\text{sign } (dL_j^o/de_i) = \text{sign } (V_{L_je_i})$$

If Assumption 4 holds, Tressler and Menezes (1980) show

$dL_j^U/de_i<0$ where e_i is the "Rothschild-Stiglitz-measure" of uncertainty. Q.E.D.

VIII-3 The Second Approach to Modelling the Family Labor Supply Decision Under Uncertainty: The Kosters (1966) Approach

The second approach to be considered was first suggested by Kosters (1966 and 1969). This approach has been used subsequently by Ashenfelter and Heckman (1974), and others.[7] The Kosters approach is unique in three senses. Firstly, it hypothesizes that the family has a well-defined utility function.[8] Secondly, the family utility function is defined over all jth individuals' leisure (for j=1,2), and over family income. And finally, the optimal level of family labor supply is found by maximizing the family's utility function with respect to the labor supply of all family members subject to the time constraints, and the family income constraint. A more formal definition of this model under uncertainty, and its related properties, follow.

(a) The Kosters (1966) Model: After Kosters (1966 and 1969), suppose the welfare of the family can be measured by a utility function defined over leisure of the husband, leisure of the wife, and expected family income, i.e., $U=U(S_1,S_2,E(Y))$. Under the EUH, there exists an EUF whose properties are the same as in the properties of the EUFs in the Leuthold (1968) model.[9] However, unlike the Leuthold model, the problem for the family is the choice of both L_1 and L_2 which maximizes the EUF subject to the time constraints, and the income constraint. Under the EUH, the problem is:

$$\max_{L_1, L_2} V(L_1, L_2, v_i, e_i)$$
$$= \int_{R_i} U(T-L_1, T-L_2, W_1L_1+W_2L_2+W_3)g(W_i, v_i, e_i)dW_i \quad (10)$$

where R_i is the domain of W_i. The first order conditions (FOC) for an interior maximum are:

$$V_{L_1} = -E(U_1)+E(W_1 \cdot U_3) = 0 \quad (11)$$

$$V_{L_2} = -E(U_2)+E(W_2 \cdot U_3) = 0 \quad (12)$$

The second order conditions (SOC) for an interior maximum are:

$$V_{L_1L_1} < 0; \quad V_{L_2L_2} < 0; \text{ and } (V_{L_1L_1} \cdot V_{L_2L_2} - (V_{L_1L_2})^2) = |H| > 0 \quad (13)$$

where:

$$V_{L_jL_k} = E(U_{jk})+E(W_jW_kU_{33})-E(W_jU_{k3}+W_kU_{j3}) \quad (14)$$

for j=1,2 and k=1,2. Thus, if Equations (11), (12), and (13) hold, there exists some unique combination, (L_1^*, L_2^*), which satisfies Equation (10).

(b) The Total Effects of Uncertainty: In an assessment of the total effects of uncertainty in the Kosters model of the family labor supply decision under uncertainty, there are three cases to be considered: (i) the cases in which wage-rate of either family member is stochastic (i.e., i=1,2), and (ii) the case in which non-wage income is stochastic (i.e., i=3). The total effects for these three cases are as follows:

Theorem 4: If i=1, then: (i) $E(W_j) > E(U_j)/E(U_3)$ for j=1,

(ii) $W_j = E(U_j)/E(U_3)$ for $j=2$, and $E(W_j)-W_{j+s} > E(U_j-U_{j+s})/E(U_3)$ for $j=1$ and $s=1$.

Proof: (i) From Equation (11),

$$E(W_1) = E(U_1)/E(U_3) - Cov(W_1, U_3)/E(U_3) \tag{15}$$

Since Cov $(W_j, U_3) < 0$ [Theorem 1], then

$$E(W_1) > E(U_1)/E(U_3)$$

(ii) From Equation (12),

$$W_2 = E(U_2)/E(U_3) \tag{16}$$

(iii) Subtracting Equations (15) from (16) yields:

$$E(W_1)-W_2 = E(U_1-U_2)/E(U_3) - Cov(W_1, U_3)/E(U_3) \tag{17}$$

Since $Cov(W_j, U_3) < 0$ [Theorem 1], therefore

$E(W_1)-W_2 > E(U_1-U_2)/E(U_3)$

Q.E.D.

Corollary 1: If $i=2$, then: (i) $W_j = E(U_j)/E(U_3)$ for $j=1$, (ii) $E(W_j) > E(U_j)/E(U_3)$ for $j=2$, and (iii) $E(W_j)-W_{j+s} > E(U_j-U_{j+s})/E(U_3)$ for $j=2$ and $s=-1$.

Theorem 5: If $i=3$, then: (i) $W_j = E(U_j)/E(U_3)$ and (ii) $W_j - W_{j+s} = E(U_j - U_{j+s})/E(U_3)$ for $j=1,2$ and $s=\pm 1$.

Proof: (i) From Equations (11) or (12), clearly

$$W_j = E(U_j)/E(U_3) \tag{18}$$

(ii) Subtracting Equations (11) from (12) yields:

$$W_2 - W_1 = E[U_2 - U_1]/E(U_3)$$ Q.E.D.

as required.

(c) The Marginal Effects of Uncertainty: To determine the effects of a mean-preserving increase in the uncertainty of W_i for $i=1,3$ on the optimal pair, (L_1^*, L_2^*), an assumption (in addition to those presented in the preceding section) is required. This extra assumption is:

Assumption 5: The labor supplies of family members are not complementary, i.e.,

$$dL_2^*/dL_1^* \leq 0 \tag{19}$$

An implication of Assumption 5 is that $V_{L_1L_2} \leq 0$. In particular, consider:

Theorem 6: If Assumption 5 holds, then $V_{L_1L_2} \leq 0$.

Proof: From Equation (12),

$$V_{L_2L_2}dL_2^* + V_{L_1L_2}dL_1^* = 0 \tag{20}$$

Rearranging terms:

$$dL_2^*/dL_1^* = -V_{L_1L_2}/V_{L_2L_2} \tag{21}$$

Since $dL_2^*/dL_1^* \leq 0$ [Assumption 5], and $V_{L_2L_2} < 0$ [Equation (13)], therefore, $V_{L_1L_2} \leq 0$.[10] Q.E.D.

Under Assumptions 1 through 5, the signs of the marginal effects of uncertainty depends on which of W_i for $i=1,3$ is treated as stochastic. In particular, consider the following:

Theorem 7: If $i=1$, and if Assumptions 1 through 5 hold, then a "Sandmo-type" increase in the uncertainty of W_i increases the wife's optimal labor supply, and decreases the husband's optimal labor supply.

Proof: Take the total differential of (11) and (12), and set the increments of all variables except L_1^*, L_2^*, and e_i equal to zero. Using Cramer's Rule,

$$dL_1^*/de_i = [V_{L_2e_i} \cdot V_{L_1L_2} - V_{L_1e_i} \cdot V_{L_2L_2}]/|H| \quad (22)$$

and

$$dL_2^*/de_i = [V_{L_1e_i} \cdot V_{L_1L_2} - V_{L_2e_i} \cdot V_{L_1L_1}]/|H| \quad (23)$$

If Assumptions 1 through 3 hold, then $V_{L_2e_i}>0$ [Theorem 2]. If Assumption 4 holds, then $V_{L_1e_i}<0$ [Theorem 3]. If Assumption 5 holds, then $V_{L_1L_2}\leq 0$ [Theorem 6]. In sum, if i=1, and if Assumptions 1 through 5 hold, then $dL_1^*/de_i<0$ and $dL_2^*/de_i>0$. Q.E.D.

Corollary 2: If i=2, and if Assumptions 1 through 5 hold, a "Sandmo-type" increase in the uncertainty of W_i increases the husband's optimal labor supply, and decreases the wife's optimal labor supply.

Corollary 3: If i=3, and if Assumptions 1 through 5 hold, then a "Sandmo-type" increase in the uncertainty of W_i affects ambiguously the husband's and wife's optimal labor supply.

VIII-4 Summary Remarks

The purpose of the present Chapter was to extend the literature on economics of labor supply under uncertainty by adapting the Leuthold and Kosters models of the family labor supply decision to a stochastic environment. In Section VIII-2, a model based on the Leuthold approach was defined, and the total and marginal effects of uncertainty associated with this model were presented. Likewise, in Section VIII-3, a model based on the Kosters approach was defined, and the total and marginal effects associated with this model were presented.

A comparison of the effects of both types of models points up the following similarities and differences. In particular, Table 8.1 presents a summary of the total effects of uncertainty associated with both models. With regards to relationship between the (expected) wage and the MRS_{3j} of the jth individual, the properties of both models are identical. The Kosters model however provides an additional total effect: the relationship between the wage-differential for jth and j+sth individuals and the "marginal utility of leisure" differential for the jth and j+sth individuals.

Table 8.2 presents a summary of the marginal effects of uncertainty associated with both models. With regards to the marginal effects associated wage-rate uncertainty, the properties of both models are identical with the exception that the model fashioned after Leuthold requires fewer assumptions. Aside from differences in the assumptions required, one property of the Leuthold-type model is not shared by the Kosters-type model. This property is the effect of an increase in the uncertainty of non-wage income on the optimal level of the labor supplied by family members.

Chapter III provides an indication of <u>an alternative assumption-set</u> which might be employed in an analysis of the Leuthold and Kosters models, i.e.,

TABLE 8.1

A SUMMARY OF THE TOTAL EFFECTS OF UNCERTAINTY

Approach	Family member	Stochastic Variable		
		i=1	i=2	i=3
Leuthold (1968)	j=1	$E(W_j) > E(U_j)/E(U_3)$	$W_j = E(U_j)/E(U_3)$	$W_j = E(U_j)/E(U_3)$
	j=2	$W_j = E(U_j)/E(U_3)$	$E(W_j) > E(U_j)/E(U_3)$	$W_j = E(U_j)/E(U_3)$
Kosters (1966)	j=1 and s=1	$E(W_j) > E(U_j)/E(U_3)$	$W_j = E(U_j)/E(U_3)$	$W_j = E(U_j)/E(U_3)$
		$E(W_j) - W_{j+s} > E(U_j - U_{j+s})/E(U_3)$	$W_j - E(W_{j+s}) < E(U_j - U_{j+s})/E(U_3)$	$W_j - W_{j+s} = E(U_j - U_{j+s})/E(U_3)$
	j=2 and s=-1	$W_j = E(U_j)/E(U_3)$	$E(W_j) > E(U_j)/E(U_3)$	$W_j = E(U_j)/E(U_3)$
		$W_j - E(W_{j+s}) < E(U_j - U_{j+s})/E(U_3)$	$E(W_j) - W_{j+s} > E(U_j - U_{j+s})/E(U_3)$	$W_j - W_{j+s} = E(U_j - U_{j+s})/E(U_3)$

TABLE 8.2

A SUMMARY OF THE MARGINAL EFFECTS OF UNCERTAINTY

Approach	Exogenous Variables	Endogenous Variables			
		dL_1^*		dL_2^*	
		Assumption(s)	Direction of Effect	Assumption(s)	Direction of Effect
Leuthold (1968)	de_1	4	–	1, 2, 3	+
	de_2	1, 2, 3	+	4	–
	de_3	1, 2, 3	+	1, 2, 3	+
Kosters (1966)	de_1	1,2,3,4,5	–	1,2,3,4,5	+
	de_2	1,2,3,4,5	+	1,2,3,4,5	–
	de_3	1,2,3,4,5	?	1,2,3,4,5	?

TABLE 8.3

A SUMMARY OF THE MARGINAL EFFECTS OF UNCERTAINTY BASED ON AN ALTERNATIVE ASSUMPTION-SET

Approach	Exogenous Variables	Endogenous Variables			
		dL_1^*		dL_2^*	
		Assumption(s)	Direction of Effect	Assumption(s)	Direction of Effect
Leuthold (1968)	de_1	6	–	6	0
	de_2	6	0	6	–
	de_3	6	0	6	0
Kosters (1966)	de_1	5,6	–	5,6	+
	de_2	5,6	+	5,6	–
	de_3	5,6	?	5,6	?

Assumption 5: The labor supplies of family members are not complementary, i.e.,

$$dL_2^*/dL_1^* \leq 0 \tag{19}$$

Assumption 6: The EUF exhibits constant risk aversion to concentration (CRAC), i.e.,

$$dU_{YY}/dY\Big|_{dU=0} = 0 \tag{24}$$

This alternative assumption-set (i.e., Assumptions 5 and 6), yields marginal effects reminiscent of Chapters III, V, VI, and VII. A summary of these marginal effects are presented in Table 8.3.[11]

Footnotes to Chapter VIII

[1] For a survey of the literature on the labor supply decision of family members under certainty, see Killingsworth (1983, pp. 29-38).

[2] For a survey of the literature on the labor supply decision of the individual under certainty, see Killingsworth (1983, pp. 1-28).

[3] According to the Killingsworth taxonomy of family labor supply models, the Leuthold (1968) approach is termed the individual utility-family budget constraint class of models. See Killingsworth (1983, p. 34).

[4] By assuming the leisure (or labor supply) of the spouse constant, our model can be viewed as incomplete. A complete model would require the joint-determination of the amount of leisure consumed by both husband and wife. For a more detailed analysis of this problem, see Killingsworth (1983, pp. 34-6).

[5] Under a fair bet $Y=E(Y)$. Therefore under a fair bet $U^1(S_1,\bar{S}_2,E(Y))=U^1(S_1,S_2,Y)$. Now $U^1(S_1,\bar{S}_2,Y)\gtrless EU^1(S_1,\bar{S}_2,Y)$ as $R_A(S_1,\bar{S}_2,Y)=(-U_{YY}/U_Y)\gtrless 0$.

[6] $U_Y>0$ and $U_{YY}<0$ implies absolute risk-aversion, i.e., $R_A(S_1,\bar{S}_2,Y)>0$.

[7] According to the Killingsworth taxonomy of family labor supply models, the Kosters (1966) approach is termed

the <u>family utility-family budget constraint</u> class of models. See Killingsworth (1983, p. 30).

[8]Justification for the concept of a family utility function is offered by Becker (1974). He argues that: "since... caring does encourage (and is encouraged by) marriage, there is a justification for the economist's usual assumption that even a multi-person household has a single well-ordered preference function." (p. S16)

[9]See Footnote 4 in this Chapter for related comments.

[10]Implicit in this Assumption is the assertion that $dL_k/dY \leq 0$ for $dY=0$. In particular, from Assumption 3, $dL_1/dY<0$ and $dL_2/dY<0$. Thus for $dY=0$, $dL_2^*/dL_1^* = -[\partial L_2/\partial Y]/[\partial L_1/\partial Y]<0$, or $dL_2^*/dL_1^* \leq 0$ implies $\partial L_2/\partial Y \leq 0$ for all $dU=0$.

[11]Clearly, the results in Table 8.3 depend on the following:

Values of i	sign of: $V_{L_1 e_i}$	$V_{L_2 e_i}$
1	-	0
2	0	-
3	0	0

CHAPTER IX

THE OPTIMAL ALLOCATION OF LABOR SUPPLY AND SAVINGS UNDER INTEREST-RATE UNCERTAINTY*

This Chapter assesses the impact of increasing uncertainty for a fourth class of models. The problem addressed here is raised on two occasions by Block and Heineke (1972 and 1975b). As in our investigation in the previous Chapter into the family labor supply decisions *a la* Kosters, the problem considered here also employs an expected utility function in three, rather than two, arguments.

IX-1 An Introduction to the Problem

On two occasions, Block and Heineke (1972 and 1975b) consider the problem of devising a two-period consumption plan under uncertainty in which there are three defining characteristics. These are: (i) the expected utility function contains as arguments labor supply, first-period consumption, and second-period consumption; (ii) the choice variables are labor supply and savings; and (iii) the source of uncertainty is the interest-rate paid on savings. On both occasions, Block and Heineke (B-H) state that the qualitative effects of a mean-preserving increase in interest-rate uncertainty on the optimal levels of labor supply and savings are ambiguous without further restrictions on the expected utility function.[1]

The purpose of the present Chapter is to propose plausible restrictions on the utility function in the B-H (1972) problem, and to demonstrate that these restrictions render unambiguous the effects of a mean-preserving increase in interest-rate uncertainty. Motivated by Sandmo (1968), Mirman (1971), Block and Heineke (1973), Levhari and Weiss (1974), and Hanson and Menezes (1978), the restrictions include intertemporal additivity of the utility function, the HMTC, and labor as an inferior good.

This Chapter is organized as follows. In Section IX-2, the B-H (1972) problem, and the first and second order conditions for its solution, are defined. In Section IX-3, the restrictions on the B-H utility function are presented, the implications of these restrictions discussed, and the effects of a "Sandmo-type" increase in interest-rate uncertainty on the optimal allocation of labor supply and savings are determined. Summary remarks are offered in Section IX-4.

IX-2 The B-H (1972) Problem: Labor Supply, Savings and Consumption Under Interest-Rate Uncertainty

Consider an individual who must devise a two-period consumption plan. In the first period, he must divide his time between leisure, P, and work, L. His time constraint is T = L + P where T is the total time available. For working L hours, the individual is paid $Y_1 = wL$ where Y_1 is his first-period income. His level of consumption in the first period is $C_1 = Y_1 - S$ where S is the level of savings. His expected level of consumption in the second period is assumed to be financed by the expected value of savings in the second period, i.e., $E(C_2) = E(Y_2) = E(1+i) \cdot S$ where i is the stochastic return on savings.[2] The expected interest-rate is $E(i) = \int i \cdot g(i,e)di$ where $g(i,e)$ is the subjective probability density function of i and e represents the "Rothschild-Stiglitz measure" of uncertainty.

Suppose the individual has a continuous, three-times differentiable, expected utility function (EUF) in three arguments: first-period consumption, second-period consumption, and leisure, i.e., $\int U(C_1,C_2,P)g(i,e)di$.[3] Suppose too that this function has positive first derivatives and negative second derivatives. The problem confronting the individual, and the problem which we term the B-H (1972) problem, is the choice of the levels of labor supply and savings which maximize his EUF subject to the time constraint and the income constraints of periods one and two, i.e.,

$$\max_{L,S} V(L,S,e) = \int U(wL-S,(1+i)S,T-L)g(i,e)di \qquad (1)$$

The first order conditions (FOC) for an interior maximum are:

$$V_L = wE(U_1) - E(U_3) = 0 \qquad (2)$$

$$V_S = -E(U_1) + E((1+i)U_2) = 0 \qquad (3)$$

and the second order conditions (SOC) for an interior maximum are:

$$V_{LL} = w^2E(U_{11}) - 2wE(U_{13}) + E(U_{33}) < 0 \tag{4}$$

$$V_{SS} = E(U_{11}) - 2E((1+i)U_{12}) + E((1+i)^2U_{22}) < 0 \tag{5}$$

$$(V_{LL}V_{SS} - (V_{LS})^2) = |H| > 0 \tag{6}$$

Thus, if Equations (2) through (6) hold, there exists some unique combination of (L*,S*) which satisfies (1).

IX-3 <u>The Effects of an Increase in Interest-Rate Uncertainty on the Optimal Allocation of Labor Supply and Savings</u>

To determine the effects of an increase in interest-rate uncertainty on the optimal pair, (L*,S*), three assumptions, and their related implications, are required. These are:

<u>Assumption 1</u>: The EUF satisfies the HMTC, i.e.,

$$\left.\frac{dU_{22}S^2}{dS}\right|_{dU=0} = iS^2[U_{222}-U_{221}[U_2/U_1]] + 2SU_{22} < 0 \tag{7}$$

for $\bar{L}\varepsilon R_1$.

<u>Assumption 2</u>: The EUF is intertemporally additive; that is $U_{12} = U_{23} = 0$ or stated differently:[4]

$$U = U^1(T-L, wL-S) + U^2((1+i)S) \tag{8}$$

<u>Assumption 3</u>: Labor is an inferior good, i.e.,[5]

$$\partial L/\partial C_1 < 0 \tag{9}$$

Three useful implications follow from these three assumptions. Firstly, if Assumptions 2 and 3 hold, then $V_{LS}>0$. This is demonstrated in the next two lemmas.

<u>Lemma 1</u>: If Assumption 3 holds, then $E(wU_{11} - U_{13}) < 0$.

Proof: From Equation (2),

$$\partial L/\partial C_1 = -E(wU_{11} - U_{13})/V_{LL}$$

Since $V_{LL} < 0$ [Equation (4)],

$$\text{sign}(\partial L/\partial C_1) = \text{sign}(E[wU_{11} - U_{13}]) \qquad (10)$$

By Assumption 3, $E(wU_{11} - U_{13}) < 0$. Q.E.D.

Lemma 2: If Assumptions 2 and 3 hold, then $V_{LS} > 0$.

Proof: $V_{LS} = E[w(-U_{11} + (1+i)U_{12}) + U_{13} - (1+i)U_{23}]$. Now $U_{12} = U_{23} = 0$ [Assumption 2]. Therefore

$$V_{LS} = E[U_{13} - wU_{11}] \qquad (11)$$

which is positive by virtue of Lemma 1. Q.E.D.

Secondly, let L be fixed in a small arbitrary neighborhood of L* (i.e., let $\bar{L} \varepsilon R_1$ where $R_1 = \{\bar{L} \mid L^*-h<\bar{L}<L^*+h$ and $h>0\}$). Consider the problem:

$$\max_S V(\bar{L},S,e) = \int U(w\bar{L}-S,\ (1+i)S,\ T-\bar{L})g(i,e)di \qquad (12)$$

for $\bar{L}\varepsilon R_1$. Let S^0 be a solution to this problem.

Lemma 3: If Assumption 1 holds, then $V_{Se} < 0$.

Proof: Now $dS^0/de = -V_{Se}/V_{SS}$. By Equation (5), $V_{SS} < 0$. Thus:

$$\text{sign}(dS^0/de) = \text{sign}(V_{Se}) \qquad (13)$$

Hanson and Menezes show $dS^0/de < 0$ if Assumption 1 holds where e is the "Sandmo-type" increase in uncertainty.[6,7]

Q.E.D.

Thirdly, let S be fixed in a small arbitrary neighborhood of S* (i.e., let $\bar{S}\varepsilon R_2$ where $R_2 = \{\bar{S} \mid S^*-h<\bar{S}<S^*+h$ and $h>0\}$). Consider the problem

$$\max_L V(L,\bar{S},e) = \int U(wL-\bar{S},\ (1+i)S,\ T-L)g(i,e)di \qquad (14)$$

for $\bar{S}\varepsilon R_2$. Let L^0 be the solution to this problem.

Lemma 4: If Assumption 2 holds, then $V_{Le} = 0$.

Proof: For i, substitute the new stochastic variable, $ei + r(e)$, with initial values of $e=1$ and $r(1)=0$ into Equation (2). A "Sandmo-type" increase in the density function of this new variable implies that its mean is equal to $-[dr/de]$.[8] The impact of a mean-preserving increase in the uncertainty of $ei + r$ on the optimal level of labor supply is:

$$\begin{aligned}
dL^0/de &= -\{E(U_{23}(i + dr/de) - E(wU_{12}(i + dr/de))\}/V_{LL} \\
&= -\{E(U_{23}(i-E(i)) - E(wU_{12}(i-E(i))\}/V_{LL} \\
&= \{E((U_{23} - wU_{12})i) - E(U_{23} - wU_{12})E(i)\}/V_{LL} \\
&= Cov(i,\ U_{23} - wU_{12})/V_{LL} \\
&= -\ V_{Le}/V_{LL} \qquad (15)
\end{aligned}$$

Since $U_{12} = U_{23} = 0$ [Assumption 2], $Cov(i,0) = dL^0/de = V_{Le} = 0$. Q.E.D.

Lemmae 2 through 4 permit the determination of the effects of an increase in the uncertainty of the interest-rate on the optimal pair, (L^*,S^*). In particular,

Theorem 1: If Assumptions 1 through 3 hold, then a "Sandmo-type" increase in interest rate uncertainty decreases the optimal level of labor supply, and decreases the optimal level of savings.[9]

Proof: Take the total differential of Equations (2) and (3), and set the increments of all variables except L^*, S^*, and e equal to zero. Using Cramer's Rule, we may write:

$$dL^*/de = \{V_{Se}V_{LS} - V_{Le}V_{SS}\}/|H| \qquad (16)$$

and

$$dS^*/de = \{V_{Le}V_{LS} - V_{Se}V_{LL}\}/|H| \tag{17}$$

Now $V_{LL}<0$, $V_{SS}<0$, and $|H|>0$ [Equations (4), (5), and (6)]. If Assumptions 2 and 3 hold, then $V_{LS}>0$ [Lemma 2]. If Assumption 1 holds, then $V_{Se}<0$ [Lemma 3]. Finally, if Assumption 2 holds, then $V_{Le}=0$ [Lemma 4]. In sum, if Assumptions 1 through 3 hold, then $dL^*/de<0$ and $dS^*/de<0$.

Q.E.D.

IX-4 Summary Remarks

In this Chapter a solution to a problem posed by B-H (1972 and 1975) is offered. In their model of labor supply, savings, and consumption, B-H observe that the effects of a mean-preserving increase in interest-rate uncertainty on the optimal levels of labor supply and savings cannot be signed without further restrictions on the utility function. The set of restrictions posed here provide unambiguous effects. These restrictions are: (i) the HMTC, (ii) intertemporal additivity of the utility function, and (iii) labor as an inferior good. With these restrictions, this paper demonstrated that a "Sandmo-type" increase in interest-rate uncertainty decreases the optimal level of labor supply, and decreases the optimal level of savings.

Footnotes to Chapter IX

*A variant of this Chapter appears in the Bulletin of Economic Research.

[1]Block and Heineke (1972, p. 525; 1975b, p. 529).

[2]For sheer simplicity, we have ignored one feature of the B-H (1972) problem. This is autonomous income in the first and second periods. The presence or absence of these two variables does not affect our analysis.

[3]The use of expected utility functions of this form is not uncommon. Two recent examples are Eaton and Rosen (1980, p. 711), and Braverman and Stiglitz (1982, p. 698).

[4]Examples of the use of intertemporal additivity in two-period models include Sandmo (1968), Mirman (1971), and Levhari and Weiss (1974). For reviews of the literatureon two-period models of consumption under uncertainty, see Sandmo (1974), Hey (1981, pp. 104-12), and Lippman and McCall (1981, pp. 238-42).

[5]This assumption is used by Block and Heineke (1973).

[6]Recall from Chapters IV and V that Hanson and Menezes (1978) are motivated by a problem posed by Sandmo (1970). In his paper, Sandmo attempts to determine the effect of a "Sandmo-type" increase in interest-rate uncertainty (or what Sandmo terms "capital risk") on the optimal level of S (i.e., determine the sign (dS^0/de) related to the solution of Equation (12) above). Sandmoconcludes that the direction of impact is indeterminate without further restrictions on the utility function.

H-M (1978) identify a plausible restriction which permits the signing of dS^0/de. By using what we term the HMTC, and by using a "Sandmo increase in risk", H-M show that a mean-preserving increase in the uncertainty of i decreases the optimal level of savings.

[7]In footnote 18, H-M (1978) claim their result also holdsfor a "Rothschild-Stiglitz increase in risk".

[8]$E(ei + r) = eE(i) + r = k$ implies $E(i) \cdot de = -dr$ or $E(i) = -[dr/de]$.

[9]The effect of an increase in the uncertainty of i on C_1^* can be determined using the results established in this theorem, and the first-period income constraint, $C_1 \equiv wL - S$. In particular, if $dL^*/de < 0$ and $dS^*/de < 0$, then $dC_1^*/de \gtreqless 0$ as $w \cdot dL^*/de \lesseqgtr dS^*/de$.

CHAPTER X

THE OWNER-MANAGED FIRM UNDER OUTPUT-PRICE UNCERTAINTY*

In the present Chapter, a final contribution to our investigation into the impact of increasing uncertainty of five classes of objective functions is offered. This investigation concerns the analysis of the last of the five classes. The particular problem addressed concerns the adaptation of a well-defined theory of the owner-managed firm (OMF) to a stochastic environment. This adaptation of theory may be seen in part as an integration of Sandmo (1971), Block and Heineke (1973), and Ishii (1977).

X-1 An Introduction to the Problem

In the literature on the economics of uncertainty, the effects of output-price uncertainty on the behavior of a variety of classes of firms have been investigated.[1] The classes for which such investigations have been conducted include: the competitive firm (CF),[2] the labor-managed firm,[3] the joint stock firm,[4] and monopoly.[5] Despite its considerable breadth, this line of inquiry is far from complete. One indication of this is that there is a well-defined literature on the owner-managed firm (OMF) under certainty on one hand,[6] and a conspicuous absence of any type of research into the OMF under uncertainty on the other.

The purpose of the present Chapter is to address this oversight. In particular, its purpose is to extend the literature on the theory of the firm under uncertainty by presenting a model of the OMF under output-price uncertainty. In the development of the following analysis of the effects of output-price uncertainty on the operations of the OMF, the as-yet unrelated results of Sandmo (1971), Block and Heineke (1973), and Ishii (1977) will be employed.

This Chapter is organized as follows. In Section X-2, simple linear production and cost structures for the OMF are defined. In Section X-3, the production problem facing the OMF is defined, and the first-order and second-order conditions associated with the solution to this problem are stated. In Section X-4, the effects of risk preference on the optimal behavior of the OMF are defined. In Section X-5, the comparative-static effects of uncertainty on the optimal behavior of the risk averse OMF are presented. In this section, it is shown that (in addition to the restriction on the technology) three restrictions on the expected utility function are sufficient to obtain unambiguous results. These three restrictions are identical to those used by Block and Heineke (1973, pp. 376-79) in their analysis of the comparative-static effect of a mean-preserving increase in the uncertainty of non-wage income on the labor supply decision.

X-2 Linear Production and Cost Structures

Consider an owner-manager who divides his time two ways: S hours to leisure and M hours to management of his firm such that S and M satisfy the time constraint, $S+M=T$, where T represents the total time available. Suppose this OMF produces a single output, Q. And suppose in producing Q, the firm employs n+1 factor inputs: the number of hours spent by the owner-manager in managerial activities, and n other factor inputs. The production process is summarized functionally as $Q=Q(M,q_1,q_2,...,q_n)$ where q_i represents the volume of the ith factor for $i=1,n$. This production function is assumed to satisfy the following restrictions:

<u>Assumption 1</u>: The production function is:

(i) separable in M and $(q_1,q_2,...,q_n)$ [i.e., $\partial^2 Q/\partial M \partial q_i=0$],

(ii) concave and increasing in $(q_1,q_2,...,q_n)$ [i.e., $\partial Q/\partial q_i>0$ and $\partial^2 Q/\partial q_i^2<0$], and

(iii) linear and increasing linear in M [i.e., $\partial Q/\partial M>0$ and $\partial^2 Q/\partial M^2 = 0$].

Under the hypothesis of efficient production, the cost of producing any level of output, Q_1, is:

$$C(Q_1;\ M,p_1,...,p_n) \tag{1}$$

where p_i is the unit-price of the ith factor input. Under Assumption 1, the related cost function has the following properties: $C_Q>0$, $C_{QQ}=0$, $C_M<0$, $C_{MM}=0$, and $C_{MQ}=0$.

To demonstrate the duality between this technology and the related cost function, consider the following Cobb-

Douglas production function,

$$Q = M + \prod_{i=1}^{n} q_i^{a_i} \qquad (2)$$

where $0<a_i<1$ and $\sum_{i=1}^{n} a_i=1$. Suppose $n=2$, then Equation (2) becomes $Q = M+q_1^{a}\cdot q_2^{1-a}$, or $Q^+=Q-M=q_1^{a}\cdot q_2^{1-a}$. Varian (1978, p. 15) shows that a technological relationship defined by Equation (2) has a cost function of the form:

$$C(Q^+,p_1,p_2) = R\cdot p_1^{a}\cdot p_2^{1-a}\cdot Q^+$$

where $R=[(a/(1-a))^{1-a}+(a/(1-a))^{-a}]$. Clearly, the properties of this cost function include $C_Q>0$, $C_{QQ}=0$, $C_M<0$, $C_{MM}=0$, and $C_{MQ}=0$.

X-3 Expected Utility Maximization

Suppose that at the time of production, the owner-manager does not know with certainty the unit-price of output, P. Instead, suppose that the production decision is undertaken on the basis of the owner-manager's subjective probability density function (SPDF) of P (i.e., on the basis of $g(P;w,e)$ where $w=w(e)$ is a parameter which defines "additive shift" in the SPDF, and e is a parameter which in combination with w defines a "multiplicative shift" in the SPDF where w(e) and e are evaluated initially at e=1 and w(1)=0). Given the SPDF, (i) the expected value of output-price is $E(P)=\int P\cdot g(P;w,e)dP$, and (ii) under the joint-hypothesis of output-price uncertainty and efficient production, the expected level of profit is:

$$E(\pi) = E(P)\cdot Q - C(Q,M,p_1,\ldots,p_n)$$

In keeping with the models of the OMF under certainty proposed by Olsen (1973), Auster and Silver (1976, pp. 626-29), Martin (1978, pp. 274-76), and Perry (1980, pp. 631-33), the utility of the owner-manager is defined over S and $E(\pi)$, i.e., $U=U(S,E(\pi))$. Under the expected utility hypothesis (EUH), there exists a function $EU(S,\pi)$ such that under a fair bet $U(S,E(\pi))=U(S,\Pi)\gtreqless EU(S,\pi)$ as $R_A(S,Y)=(-U_{\pi\pi}/U_\pi)\gtreqless 0$ where $R_A(S,\pi)$ is the Arrow-Pratt measure of absolute risk aversion. Suppose this expected utility function (EUF) is continuous, and three-times differentiable. Suppose too that this EUF has positive first, and negative second, derivatives (i.e., the EUF exhibits risk aversion or $R_A(S,Y)>0$.

The problem for the owner-manager is the choice of M and Q which maximizes the EUF subject to S=T-M and the technology conditions [i.e., Assumption 1]. In particular, under the EUH, the problem is:[7]

$$\max_{M,Q} V(Q,M,w,e) = \int U(T-M,P\cdot Q-C(Q,M,\ldots))g(P;w,e)dP \quad (3)$$

The first order conditions (FOC) for an interior maximum are:

$$V_M=-E(U_S)-E(U_\pi\cdot C_M) = 0 \quad (4)$$

$$V_Q=E(U_\pi\cdot[P-C_Q]) = 0 \quad (5)$$

The second order conditions (SOC) for an interior maximum are:

$$V_{MM}=E(U_{SS})+2E(U_{S\pi}\cdot C_M)+E[U_{\pi\pi}\cdot(C_M)^2]<0 \tag{6}$$

$$V_{QQ}=E[U_{\pi\pi}(P-C_Q)^2]<0 \tag{7}$$

$$(V_{MM}V_{QQ}-(V_{MQ})^2)=\left|H\right|>0 \tag{8}$$

where

$$V_{MQ}=-E[(U_{S\pi}+C_M U_{\pi\pi})(P-C_Q)] \tag{9}$$

In sum, if Equations (4) through (8) hold, there exists a unique combination, (M^*,Q^*), which satisfies (3).

X-4 The Effect of Risk Preference on the Optimal Behavior of the OMF

Like Sandmo's model of the CF, the optimal level of output of the OMF may be ranked by the risk preference of the owner-manager. In particular, consider:

Theorem 1: The optimal output of the OMF may be ordered as $Q^{*a}<Q^{*n}=Q^{*c}<Q^{*\ell}$ where the superscripts a, n, c, and ℓ denote risk-aversion, risk-neutrality, certainty, and risk-loving.

Proof: From Equations (4) and (5)

$$E(U_\pi)=-E(U_S)/C_M=-\text{Cov}(P,U_\pi)/[E(P)-C_Q]$$

which implies:

$$E(P)-C_Q=C_M\cdot \text{Cov}(P,U_\pi)/E(U_S) \tag{10}$$

Since $E(U_S)>0$ and $C_M<0$,

$$\text{sign }[E(P)-C_Q] = \text{sign }[-\text{Cov}(P,U_\pi)] \tag{11}$$

Furthermore, since $dP/d\pi>0$, and since $dU_\pi/d\pi\gtreqless 0$ as $R_A(S,\pi)\lesseqgtr 0$, it follows that $\text{Cov}(P,U_\pi)\gtreqless 0$ as $R_A(S,\pi)\lesseqgtr 0$. Thus:

$$E(P)-C_Q\gtreqless 0 \text{ as } R_A(S,\pi)\gtreqless 0 \tag{12}$$

Equation (12) implies

$Q^{*a} < Q^{*n} = Q^{*c} < Q^{*1}$ Q.E.D.

With regards to Theorem 1, three comments are in order. Firstly, as in the case of Sandmo's CF, the optimal level of output of the OMF under risk-neutrality and under certainty are identical, i.e., at $Q^{*n}=Q^{*c}$, $E(P)-C_Q=P^c-C_Q=0$ where P^c is the value of output-price under certainty. Secondly, the optimal level of output of the risk-averse OMF under uncertainty is less than the optimal level of output of the OMF under certainty. This result is referred to (by Sandmo) as the "overall impact of uncertainty".[8] Finally, from Equation (4), it is clear that $E(U_\pi)=-E(U_S)/C_M$, and that the R.H.S. of Equation (10) may be rewritten as $-Cov(P,U_\pi)/E(U_\pi)$. This expression has been termed the cost of risk bearing (CRB).[9]

X-5 The Comparative-Static Effects of Uncertainty on the Optimal Behavior of the Risk-Averse OMF

To determine the comparative-static (or marginal) effects of output-price uncertainty on the optimal behavior of the risk-averse OMF, three assumptions regarding the EUF are sufficient. These are:

Assumption 2: In a certain world, the owner-manager's labor is an inferior good, i.e.,

$$dM^0/d\pi\Big|_{\pi=E(\pi)} < 0 \tag{13}$$

Assumption 3: The EUF exhibits decreasing absolute risk aversion (DARA) in profit, i.e.,

$$\partial R_A(S,\pi)/\partial \pi < 0 \tag{14}$$

Assumption 4: The EUF exhibits constant absolute risk aversion (CARA) with respect to owner-manager leisure, i.e.,

$$\partial R_A(S,\pi)/\partial S = 0. \tag{15}$$

In combination with these assumptions, the approach used here is to partition the comparative-static problem into two parts. The first part is to determine the individual effects of additive and multiplicative shifts in the distribution of g(P) on an optimal level of M in an arbitrary, small neighborhood of Q*. The second part is to determine the individual effects of additive and multiplicative shifts in the distribution of g(P) on an optimal level of Q in an arbitrary, small neighborhood of M*. The merit of this approach is that it permits the use of results determined by Sandmo (1971), Block and Heineke (1973), and Ishii (1977).

(a) The Comparative-Static Effects in the First One-Dimensional Problem: Let Q be constant in an h-neighborhood of Q*. For Q constant (i.e., for $\bar{Q} \varepsilon R_1$, where $R_1 = \{\bar{Q} \mid Q^* - h < \bar{Q} < Q^* + h \text{ and } h > 0\}$), one dimension of the problem defined by Equation (3) is:

$$\max_{M} V(\bar{Q},M,w,e) = \int U(T-M, P\bar{Q} - C(\bar{Q},M,\ldots))g(P;w,e)dP \tag{16}$$

If Equations (4) and (6) hold, there exists a unique solution to Equation (16), i.e., there exists a unique value of M, M^0, which satisfies (16).

To determine the effect of an additive shift, dw, on M^0, consider the following lemma:

Lemma 1: If Assumptions 1 and 2 hold, then $(U_{S\pi}+C_M U_{\pi\pi})>0$, $V_{M\pi}<0$, $E(U_{S\pi}+C_M U_{\pi\pi})>0$, and $V_{Mw}<0$.

Proof: Implicit differentiation of the first FOC yields:

$$V_{MM}\cdot dM^0+V_{M\pi}d\pi = 0$$

which implies:

$$dM^0/d\pi = -V_{M\pi}/V_{MM} \qquad (17)$$

where $V_{M\pi}=-E(U_{S\pi}+C_M U_{\pi\pi})$. Since $V_{MM}<0$ [Equation (6)],

$$\text{sign } [dM^0/d\pi] = \text{sign } [V_{M\pi}]$$

Since $dM^0/d\pi<0$ for $\pi=E(\pi)$ [Assumption 2], therefore

$$(U_{S\pi}+C_M U_{\pi\pi}) = E(U_{S\pi}+C_M U_{\pi\pi})>0,$$

Substitute $(P+w)$ for P in the FOC (where w is evaluated initially at $w=0$). Implicit differentiation yields:

$$dM^0/dw = -V_{Mw}/V_{MM}$$

where $V_{Mw}=-Q\cdot E(U_{S\pi}+C_M\cdot U_{\pi\pi})$. By Assumption 2, $E(U_{S\pi}+C_M U_{\pi\pi})>0$, and therefore $V_{Mw}<0$. Q.E.D.

To determine the effect of a multiplicative shift (or a "Sandmo-type" increase) in the uncertainty of P on M^0, consider the following two lemmas. These two lemmas are fashioned after Block and Heineke (1973, pp. 378-81).

Lemma 2: If Assumptions 1 through 4 hold, then $\partial[U_{S\pi}+C_M U_{\pi\pi}]/\partial\pi<0$.

Proof: Taking the derivative of $R_A(S,\pi)$ with respect to S yields:

$$dR_A(S,\pi)/dS = \partial R_A(S,\pi)/\partial\pi\cdot\partial\pi/\partial S + \partial R_A(S,\pi)/\partial S \qquad (18)$$

Since $\partial\pi/\partial S=\partial\pi/\partial M\cdot\partial M/\partial S=C_M<0$, and if Assumptions 3 and 4 hold, Equation (18) becomes:

$$dR_A(S,\pi)/dS = C_M(\partial R_A(S,\pi)/\partial\pi)>0 \qquad (19)$$

By Young's Theorem, Equation (19) becomes:

$$\partial[U_{S\pi}/U_\pi]/\partial\pi = -C_M\cdot\partial[-U_{\pi\pi}/U_\pi]/\partial\pi<0$$

$$\rightarrow\partial[(U_{S\pi}+C_MU_{\pi\pi})/U_\pi]/\partial\pi<0$$

$$\rightarrow[U_\pi\cdot\partial(U_{S\pi}+C_MU_{\pi\pi})/\partial\pi-U_{\pi\pi}(U_{S\pi}+C_MU_{\pi\pi})]/U_\pi^2<0$$

$$\rightarrow\partial(U_{S\pi}+C_MU_{\pi\pi})/\partial\pi<-(-U_{\pi\pi}/U_\pi)(U_{S\pi}+C_MU_{\pi\pi})$$

$$\rightarrow\partial(U_{S\pi}+C_MU_{\pi\pi})/\partial\pi<-R_A(S,\pi)(U_{S\pi}+C_MU_{\pi\pi}) \qquad (20)$$

Since $R_A(S,\pi)>0$ and $(U_{S\pi}+C_MU_{\pi\pi})>0$ [Lemma 1], therefore the R.H.S. of Equation (20) is negative. This implies:

$$\partial(U_{S\pi}+C_MU_{\pi\pi})/\partial\pi<0 \qquad \text{Q.E.D.}$$

<u>Lemma 3</u>: If Assumptions 1 through 4 hold, then $V_{Me}>0$.

<u>Proof</u>: Substitute $eP+w(e)$ for P in Equation (4) (where e and $w(e)$ are evaluated initially at $e=1$ and $w(1)=0$ and where $dw/de=-E(P)$). Implicit differentiation yields:

$$V_{MM}dM-[E(U_{S\pi}(P+dw/de)\bar{Q})+E(U_{\pi\pi}(P+dw/de)C_M\bar{Q})]de$$

$$= V_{MM}dM-[E(U_{S\pi}(P-E(P))\bar{Q})+E(U_{\pi\pi}(P-E(P))C_M\bar{Q})]de$$

$$= V_{MM}dM-\bar{Q}[E((U_{S\pi}+U_{\pi\pi}C_M)P)-E(U_{S\pi}+U_{\pi\pi}C_M)E(P)]de$$

$$= V_{MM}dM-\bar{Q}\cdot\operatorname{Cov}(P,U_{S\pi}+C_MU_{\pi\pi})de = 0 \qquad (21)$$

A rearrangement of terms yields:

$$dM^0/de = \bar{Q}\cdot\operatorname{Cov}(P,U_{S\pi}+C_MU_{\pi\pi})/V_{MM} \qquad (22)$$

$$= -V_{Me}/V_{MM}$$

Thus:

$$\operatorname{sign}(dM^0/de) = \operatorname{sign}(V_{Me})=\operatorname{sign}(-\operatorname{Cov}(P,U_{S\pi}+C_MU_{\pi\pi})) \qquad (23)$$

Since $d\pi/dP>0$ and $d[U_{S\pi}+C_MU_{\pi\pi}]/d\pi<0$ [Lemma 2], therefore

$Cov(P, U_{S\pi} + C_M U_{\pi\pi}) < 0$ and $V_{Me} > 0$. Q.E.D.

(b) The Comparative-Static Effects in the Second One-Dimensional Problem: Let M be constant in an h-neighborhood of M*. For M constant (i.e., for $\bar{M} \varepsilon R_2$ where $R_2 = \{\bar{M} | M^* - h < \bar{M} < M^* + h$ and $h > 0\}$), one dimension of the problem defined by Equation (3) is:

$$\max V(Q,\bar{M},w,e) = \int U(T-\bar{M}, PQ-C(Q,\bar{M},...))g(P;w,e)dP \quad (24)$$

If Equations (5) and (7) hold, there exists a unique solution to Equation (24), i.e., there exists a unique value of Q, Q^0, which satisfies (24).

To determine the effect of an additive shift, dw, on Q^0, consider the following lemma:

Lemma 4: If Assumption 3 holds, then $V_{Qw} > 0$.

Proof: Substitute P+w for P in the second FOC (where w is evaluated initially at w=0). Implicit differentiation yields:

$$dQ^0/dw = -V_{Qw}/V_{QQ}$$

Since $V_{QQ} < 0$ [Equation (7)],

$$\text{sign}[dQ^0/dw] = \text{sign}[V_{Qw}]$$

If Assumption 3 holds, Sandmo (1971, pp. 68-9) shows that $dQ^0/dw > 0$.[10] Q.E.D.

To determine the effect of a multiplicative shift (or a "Sandmo-type" increase) in the uncertainty of P on Q^0, consider the following lemma.

Lemma 5: If Assumption 3 holds, then $V_{Qe} < 0$.

Proof: Substitute eP+w(e) for P in the second FOC (where e and $w(e)$ are evaluated initially at $e=1$ and $w(1)=0$).

Implicit differentiation yields:

$$dQ^0/de = -V_{Qe}/V_{QQ} \tag{25}$$

Since $V_{QQ}<0$ [Equation (7)],

$$\text{sign}[dQ^0/de] = \text{sign}[V_{Qe}] \tag{26}$$

If Assumption 3 holds, Ishii (1977, p. 796) shows that $dQ^0/de<0$. Q.E.D.

(c) <u>The Comparative-Static Effects in the Two-Dimensional Model</u>: To determine the comparative effects of: (i) an additive shift, and (ii) a multiplicative shift (or a "Sandmo-type" increase) in the uncertainty of P, on the optimal-pair which solves Equation (3) one final result is required.

<u>Lemma 6</u>: If Assumptions 1 and 2 hold, $V_{MQ}\leq 0$.

<u>Proof</u>: Implicit differentiation of the first FOC yields $dM^*/dQ^*=-V_{MQ}/V_{MM}$. Since $V_{MM}<0$ [Equation 6],

$$\text{sign } [dM^*/dQ^*] = \text{sign } [V_{MQ}]$$

Now, if Assumption 1 holds, then $dM^*/dQ^*\Big|_{\pi=E(\pi)} = [dM^*/d\pi^*\Big|_{\pi=E(\pi)} \cdot d\pi^*/dQ^*\Big|_{\pi=E(\pi)}]$. Since $d\pi^*/dQ^*=E(P)-C_Q\geq 0$ under $\pi=E(\pi)$ or under $R_A(S,\pi)>0$ [Theorem 1], and since $dM^*/d\pi\Big|_{\pi=E(\pi)}<0$ [Assumption 3], $dM^*/dQ^*\leq 0$. Q.E.D.

The comparative-static effects on the optimal behavior of the risk averse OMF are determined in the following two theorems:

<u>Theorem 2</u>: If Assumptions 1 through 4 hold, then an addi-

tive shift in the distribution of P increases the optimal level of output and decreases the optimal level of owner-manager's effort.

Proof: Substitute P+w for P in both FOCs (where w is evaluated initially at w=0). Take the implicit derivative of these revised Equations. Set the increments of all variables except M, Q, and w equal to zero. The use of Cramer's Rule yields:

$$dM^*/dw = [V_{Qw} \cdot V_{MQ} - V_{Mw} \cdot V_{QQ}]/|H|$$

and:

$$dQ^*/dw = [V_{Mw} \cdot V_{MQ} - V_{Qw} \cdot V_{MM}]/|H|$$

If the SOCs for an interior maximum hold, then $V_{MM}<0$, $V_{QQ}<0$, and $|H|>0$. If Assumptions 1 and 2 hold, then $V_{Mw}<0$ [Lemma 1]. If Assumption 3 holds, then $V_{Qw}>0$ [Lemma 4]. If Assumptions 1 and 2 hold, then $V_{MQ}\leq 0$ [Lemma 6]. In sum, if Assumptions 1 through 4 hold, then $dM^*/dw<0$ and $dQ^*/dw>0$.

Q.E.D.

Theorem 3: If Assumptions 1 through 4 hold, then a "Sandmo-type" increase in the uncertainty of P decreases the optimal level of output and increases the optimal level of owner-manager's effort.

Proof: Substitute eP+w(e) for P in both FOCs (where e and w(e) are evaluated initially at e=1 and w(1)=0). Take the implicit derivative of these revised Equations. Set the increments of all variables except M, Q, and e equal to zero. The use of Cramer's Rule yields:

$$dM^*/de = [V_{Qe} \cdot V_{MQ} - V_{Me} \cdot V_{QQ}]/|H|$$

and:

$$dQ^*/de = [V_{Me} \cdot V_{MQ} - V_{Qe} \cdot V_{MM}]/|H|$$

If the SOCs for interior maximum hold, then $V_{MM}<0$, $V_{QQ}<0$, $|H|>0$. If Assumptions 1 through 4 hold, then $V_{Me}>0$ [Lemma 3]. If Assumption 3 holds, then $V_{Qe}<0$ [Lemma 5]. If Assumptions 1 and 2 hold, then $V_{MQ}\leq 0$ [Lemma 6]. In sum, if Assumptions 1 through 4 hold, then $dM^*/de>0$ and $dQ^*/de<0$.

Q.E.D.

X-6 Summary Remarks

This Chapter presented a model of the OMF under output-price uncertainty. The approach employed here is of an integrated nature in that in determining the properties of this model the results of Sandmo (1971), Block and Heineke (1973), and Ishii (1977) were used. In Section X-2, simple linear production and cost structures were defined. In Section X-3, the model of the OMF was presented, and the first and second order conditions for an interior maximum were defined. In Section X-4, the effects of risk preference on the optimal behavior of the OMF were defined, and the CRB for the OMF was identified. In Section X-5, three restrictions were placed on the utility function. If the restrictions on the EUF and on technology hold, it was shown that: (i) an additive shift in the distribution of output-price, and (ii) a "Sandmo-type" decrease in the uncertainty of output-price, decreases as the optimal level of owner-manager effort, and increased the optimal level of output.

Footnotes to Chapter X

*An earlier draft of this Chapter was presented as a paper in the Workshop on Human Resources, University of Alberta. I wish to thank the Workshop participants, especially Peter Coyte, Robin Lindsey, and Shmuel Sharir, for useful comments.

[1] Aside from the effects of output-price uncertainty, the effects of other sources of uncertainty have been investigated. For example, the impacts of input-price uncertainty have been explored by Blair (1974), and the impacts of production uncertainty have been explored by Rothschild and Stiglitz (1971), Ratti and Ullah (1976), Pope and Kramer (1979), and Hiebert (1981).

[2] Sandmo (1971, p. 65). This terminology has not been adhered to consistently. Hey (1981a) refers to Sandmo's CF as the "profit-maximizing" firm, and Hawawini (1984) refers to the CF as the "owner-managed" firm. Hawawini's terminology is inconsistent with a well-defined literature on a class of models under certainty [see for example Olsen (1973), Ng (1974), Auster and Silver (1976), Martin (1978, pp. 274-6), and Perry (1980, pp. 631-33)]. For examples of research into the properties of the CF under output-price uncertainty, see Sandmo (1971), Sakai (1975, 1977a, 1977b, and 1981), Ishii (1977), Hey (1981a), Hawawini (1984).

[3] See Muzondo (1979 and 1980), Conte (1980), Bonin (1980), Hey and Suckling (1980), Horowitz (1982), Kahana and Paroush (1984), and Hawawini (1984).

[4] See Hey (1981a).

[5] See Leland (1972) and Ishii (1979).

[6] See Olsen (1973), Ng (1974), Auster and Silver (1976, pp. 626-29), Martin (1978, pp. 274-76), and Perry (1980, pp. 631-33).

[7] In the case of output-price uncertainty, the literature adopts an alternative approach. In particular, in their models of the OMF under certainty, Olsen (1973), Auster and Silver (1976), and Martin (1978) define the problem for the owner-manager as the choice of the owner-manager's labor supply, and the volume of n other factor inputs which maximizes the utility function subject to a time constraint and a profit constraint, i.e.,

$$\max_{M, q_1, \ldots, q_n} U(S, \pi) \qquad \text{subject to:} \qquad S = T - M$$

and:

$$\pi = P \cdot Q(M, q_1, \ldots, q_n) - \sum_{i=1}^{n} p_i q_i$$

where T represents the total time available, P represents output-price, p_i represents the unit-cost of the ith factor input, and Q represents the level of output caused by combinations of M, $q_1, \ldots, q_n$.

[8] Sandmo (1971, p. 67).

[9] See Pleeter and Horowitz (1974), and Sakai (1978), for examples of the use of the CRB.

[10] Sandmo's cost function is more general than the one defined in Section II. Sandmo assumes $C_{QQ} \geq 0$ while we assume $C_{QQ}=0$.

CHAPTER XI

CONCLUSION

This dissertation has presented an investigation into the impact of increasing uncertainty for five related classes of models. These classes of models are related in that all classes contain an objective function in two or three arguments, and the decision-maker has available to him or her two choice variables.

The methodology of this investigation is unique to this dissertation. Our investigation employs the comparative-static effects of increasing uncertainty for simpler classes of models - classes of models which contain an objective function in one or two variables, and one or two choice variables.

This methodology, or approach, enabled two types of contributions to the literature. The first type was the solution of problems posed in the literature, and left unsolved. In this category, the author would include the following four entries: (i) our extension to Cowell's (1981) "riskless activity, risky activity" problem which was dealt with in Chapter III, (ii) our two alternative solutions to the Sandmo (1969) portfolio problem which were presented in Chapters IV and V, (iii) our evaluation of the robustness of the comparative-static result reported by

Block and Heineke (1975a) in their choice-theoretic model of criminal behavior presented in Chapter VI, and (iv) our solution to a portfolio problem posed by Block and Heineke (1972 and 1975b) which was presented in Chapter IX.

The other type of contribution made in this dissertation was the adaptation of models (which have only been defined as yet in terms of a non-stochastic environment) to a stochastic environment. In this category, the author would include the following three entries: (i) the adaptation of the Isachsen and Strom (1980) model of tax evasion and labor supply to the case of imperfect knowledge about the parameters of the tax system, (ii) the adaptation of the Leuthold (1968) and Kosters (1966 and 1969) models of the family labor supply decision to the cases of wage and non-wage income uncertainty, and (iii) the adaptation of the model of the owner-managed firm to the case of output-price uncertainty.

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