THE DESIGN OF SELECTIVE LINEAR PHASE FILTERS WITH OPTIMUM CHARACTERISTICS

by

Sirus Sadughi

A Thesis
Submitted to the Faculty of Graduate Studies
in partial fulfillment of the requirements for the degree of
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Department of Electrical Engineering
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ABSTRACT

An iterative procedure has been devised for the construction of transfer functions for analog or digital filters exhibiting linear phase characteristic in the entire or a part of the passband. The group delay can be approximated to a constant in a maximally-flat, equal ripple, or least-mean-squares norm. The amplitude response can be approximated to the ideal characteristics in a maximally-flat, equal-ripple, or mini-max norm.

Three types of filters are considered: minimum-phase, nonminimum-phase and nonreciprocal. For a specific filter type, the approximation begins with the determination of a near optimum filter which is then optimized using a least-mean-squares routine.

The selectivity of linear phase filters can be increased by reducing the degrees of flatness of the group delay, and using the excess degrees of freedom for the amplitude response. The resulting filters are optimum for a maximally-flat group delay response, but near optimum otherwise. Alternatively a combination of an elliptic filter and an allpass delay equalizer can be used as a near optimum filter.

The proposed method yields the optimum characteristics through an even distribution of the degrees of freedom between the amplitude and the group delay. The group delay characteristics are improved through a reduction in the number of passband ripples, and the use of the excess degrees of freedom for the group delay equalization. It was found that filters with the maximum number of ripples in the passband amplitude response are not optimum. Also filters with a monotonic passband which possesses the maximum number of inflexion points are not optimum if the complexity of the physical realization is considered. Design examples are given which establish the superiority of the proposed method to the existing ones.

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CHAPTER I

INTRODUCTION

1.1 General Considerations

Two areas of study basic to network synthesis which are of equal importance are the approximation problem and the realization problem. The approximation problem is concerned with the construction of transfer functions which in a best manner possible approximate the desired characteristics. The realization problem focuses on the physical realization of the transfer function. This thesis is mainly involved with the approximation problem.

The approximation problem can be treated by analytical or numerical approaches. In the analytical approach, with the aid of special functions such as Jacobi polynomials [1], the desired characteristics are approximated in a manner inherent to the special functions. For example an approximation to a constant in a maximally flat sense utilizes Butterworth functions, while Chebyshev functions are useful for the approximation to a constant in mini-max sense. In the numerical approach, the desired characteristics are approximated using iterative routines. Because of the iterative nature, an initial approximation is essential for the convergence of most routines. The numerical approach is time consuming, but with the availability of a large computer, design data can be tabulated for general problems. In the numerical approach, the computation time can be greatly reduced by employing special functions for part of the approximation problem. An efficient numerical method of this kind can be programmed on a micro-computer or even a pocket computer (see Appendix).

The approximation problem which leads to the synthesis of a physical network, is mostly concerned with the design of filters used in various electronic systems. For example in communication systems, filters are required for the separation of a particular signal from unwanted noise. In modern communication systems, filters are required which not only approach ideal characteristics, but also have low cost. This thesis is involved with the design of such filters.

1.2 Filters

The response of a linear continuous system to an excitation $x(t)$ can be expressed as

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau \qquad (1.1)$$

where $h(t,\tau)$ is the impulse response of the system. For a time-invariant causal system

$$h(t,\tau)=h(t-\tau) \qquad (1.2)$$

$$h(t)=0 \qquad \text{for } t<0$$

The resulting time-invariant system can be characterized by the impulse response or its Laplace transform $H(s)$. For the frequency $s=j\omega$, the system can be regarded as a filter which attenuates and changes the phase of each component of the Fourier series expansion for the signal $x(t)$. Equivalently the filter can be characterized by the magnitude and phase of its transfer function $H(s)$, evaluated at $s=j\omega$. An ideal lowpass filter is defined to have the properties

$$|H(j\omega)|=\begin{cases}1 & |\omega|\le\omega_c\\ 0 & |\omega|>\omega_c\end{cases} \qquad (1.3)$$

$$\phi=arg(H(j\omega))=-a\omega$$

As a result, the response of an ideal lowpass filter to a band-limited signal (bandwidth less than or equal to ω_c) is only a delayed version of it without any distortion. Hence a linear phase or a constant phase delay (ϕ/ω) is necessary for a distortionless transmission. For a bandpass filter, the ideal characteristics are

$$|H(j\omega)| = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & otherwise \end{cases} \tag{1.4}$$

$$\phi(\omega) = -a(|\omega| - \omega_0) sgn\, \omega$$

Hence for a distortionless recovery of a bandpass signal (for example a carrier modulated signal), the derivative of the phase must be constant. Thus the group delay defined by $\tau = -d\phi/d\omega$ is widely used in the characterization of distortionless filters.

Because of physical limitations, the filter characteristics are only approximated to the ideal ones. These desired characteristics are usually stated numerically or reflected graphically as design specifications. Figure 1 depicts the specifications for filters required in a voice communication system [2]. In general, tolerance schemes are prescribed for the group delay, the passband loss and the stopband loss. For example, the specifications on the group delay and the attenuation characteristics can be: maximum group delay variations in a prescribed band, not exeeding $\Delta\tau$ seconds, maximum attenuation in the passband not greater than A_p dB, and minimum attenuation in the stopband not less than A_s dB.

There are several methods for the approximation of the ideal characteristics which dictate the techniques employed in the approximation procedure. The ideal characteristics are approximated in Chebyshev or mini-max norm if the maximum of the error function (deviation from the specifications) is minimized. An approximation to the ideal characteristics in which the integral of the error function is minimized, is referred to as least-mean-squares approximation. If the approximating function satisfies the specifications at a certain number of points, an interpolation technique is used for the approximation. It may be required to approximate the ideal characteristics in a maximally-flat norm; in that case the derivatives of the error function are all zero at a certain point in the approximation interval.

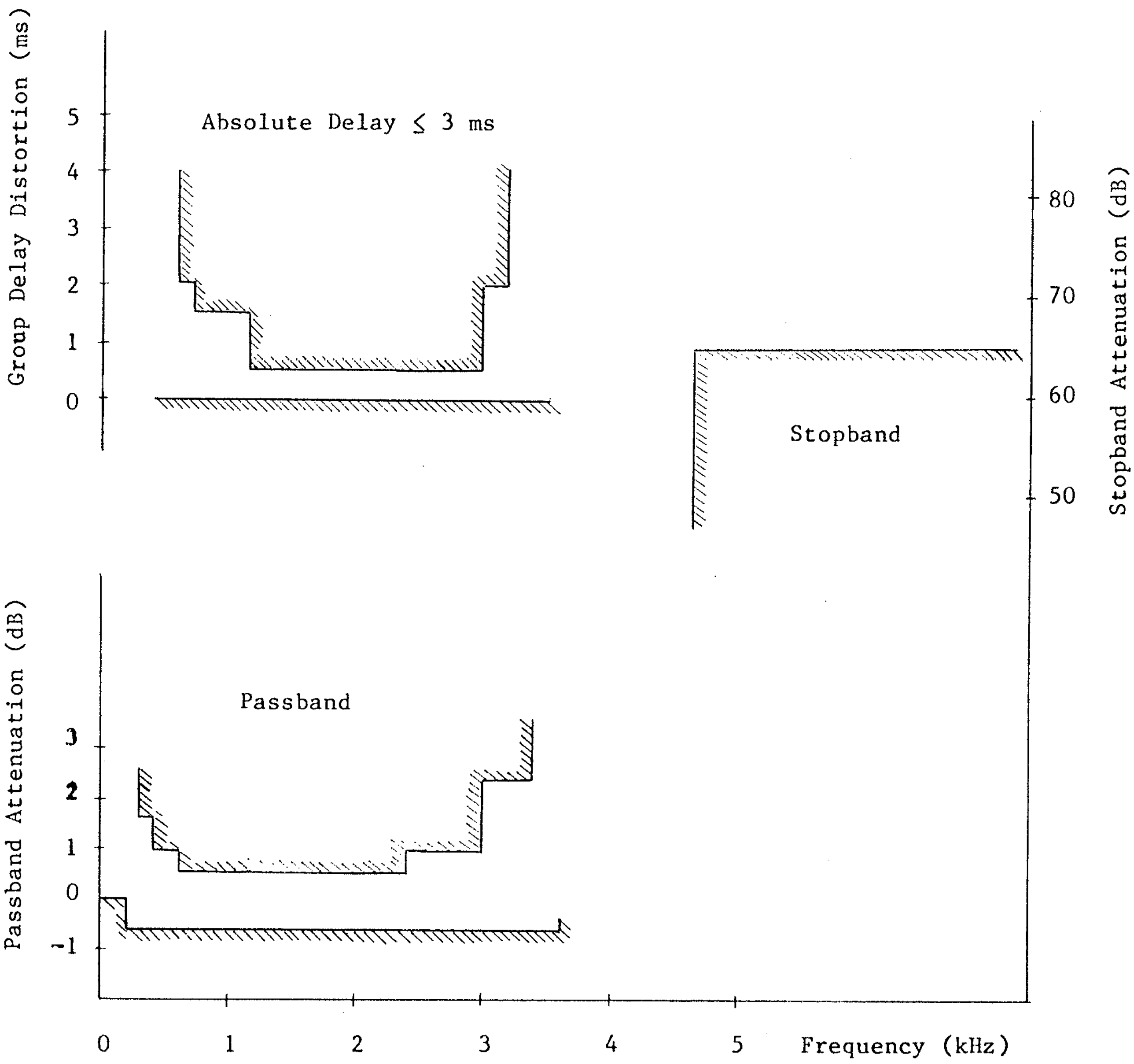

Fig. 1 CCITT masks for a channel filter.

The filter transfer function is required to be a ratio of two polynomials in the complex frequency, because of the physical realizability considerations. The zeros (or transmission zeros) and the poles of the transfer function are defined to be the zeros of the numerator and the denominator polynomials, respectively. The poles are restricted to be in the left-half-plane (LHP) of the complex frequency plane for stability, while the zeros can be anywhere. For all-pole filters, the zeros are all located at infinity and hence the numerator polynomial is a constant. Minimum-phase filters are defined to have all their zeros either in the LHP or on the imaginary axis, excluding those at infinity. The transfer function of a resistively-terminated lossless, reciprocal network has either an even or an odd numerator polynomial. As a result, the finite zeros are located symmetrically with respect to the imaginary axis. If some of the zeros in the LHP have exactly the same location as the poles, they cancel from the transfer function resulting in a transfer function related to a nonreciprocal filter. Alternatively, every transfer function can have a reciprocal realization by proper addition of poles and zeros.

The problem of filter design deals with the determination of a transfer function which satisfies the specifications under the prescribed norms, and has a physical realization of minimum complexity (cost). Most often, the complexity of a filter is associated with the degree of the transfer function denominator polynomial.

1.3 Amplitude And Phase Approximations

A class of all-pole filters approximating the ideal characteristics can be obtained from the minimization of the integral [3]

$$I(\mathbf{c})=\int_0^1 (1-x)^{p-q} x^{q-1} J_n(x,\mathbf{c})dx \tag{1.5}$$

The hyper-geometric shifted Jacobi polynomial $J_n(x,p,q)$ generates various filter approximations by proper substitutions for its arguments. Examples are for the maximally-flat (Butterworth): $p = \infty$, $q = 1.5$, for the mini-max (Chebyshev): $p = -2$, $q = 1.5$, and for least squares monotonic: $p = q = 1.5$.

A class of monotonic passband filters are obtained if the upper limit of the integral in (1.5) is varied [4].

Minimum-phase filters with finite transmission zeros and with Jacobian passband and general stopband characteristics are obtained numerically using iterative procedures. For the case of a Chebyshev passband and Chebyshev stop-band approximation, the filter transfer function can be obtained numerically using elliptic integrals [5], or recursive arithmetic [6] methods. The resulting elliptic filters have the minimum transition interval for the specific passband and stop-band losses. The amplitude approximation is useful where the phase or group delay distortion is of no concern.

The problem of phase approximation is of a numerical nature, although some elegant analytical methods exist for the interpolation [7], and the maximally-flat [8-10] approximations. The constant group delay or the linear phase approximations in equal ripple norm are mostly tabulated [11,12].

The problem of approximation of the amplitude and phase simultaneously, is a complicated one, because of the interdependence between them. A conventional though not optimum [13] solution is to cascade an optimum amplitude filter with a phase corrector (delay equalizer) allpass filter.

There are a number of analytical methods , e.g. [14-21], for the determination of the transfer functions of minimum-phase filters. Some of these methods are transitional between an amplitude approximation and a linear phase approximation [22,23]. The amplitude and phase characteristics of minimum-phase filters are related through the Hilbert transform equations which limit phase linearity for a specific amplitude selectivity [17]. Therefore a nonminimum-phase filter must be used in order to satisfy amplitude and phase specifications. Some of the methods for nonminimum-phase filters are biased towards the maximally-flat approximation of the group delay or the amplitude response [24-26]; filters designed by these methods are nonoptimum for most design specifications.

The modern methods of linear phase filter design assume either a Chebyshev or least squares passband amplitude response and then equalize the group delay in an iterative procedure [53]. The Chebyshev passband amplitude response can be obtained iteratively [27,28] or by the use of a generalized Chebyshev function [13], which simplifies the computational effort. In either case the resulting filters have the maximum number of ripples in the passband amplitude response. Rhodes and Zabalawi [29,30] derived filters with Chebyshev passband and linearized the phase using an interpolation technique. Alternatively, the group delay may be equalized using a least-mean-squares procedure [31,32].

Rakovich et al [33] derived filter transfer functions with unconstrained least-squares passband having the maximum number of ripples, and then equalized the group delay in equal-ripple norm.

Litovsky [34] developed a method for the derivation of filter transfer functions with least-squares monotonic passband and with equalized group delay in equal-ripple norm. The amplitude response is designed to have the maximum number of inflexion points in the passband.

It can be shown that filters with Chebyshev passband amplitude response [28,30] are of lower complexity than those with unconstrained least-squares passband [33], but are of higher complexity than those with least-squares monotonic passband [34], for normal specifications.

A design procedure may start with the determination of a general transfer function with unrestricted zeros. A nonminimum-phase transfer function is then obtained by the proper addition of the poles and zeros. This technique has been claimed [35] to yield more economical filters than those designed by the method of Rhodes and Zabalawi, but it can be shown to be untrue.

1.4 Statement of the Problem

This thesis is concerned with the determination of the transfer function of filters with the least complexity which satisfy a general set of specifications. Since no design can be proved to be optimum absolutely, a comparative basis is used for the optimality of the filter.

1.5 Proposed Method

Let the transfer function of a reciprocal filter be written as

$$H(s)=\frac{a_0+a_1s^2+a_2s^4+....+a_ms^{2m}}{b_0+b_1s+b_2s^2+....+b_ns^n}. \tag{1.6}$$

There are $n+m+2$ coefficients, and hence $n+m+1$ degrees of freedom for shaping the amplitude and group delay characteristics. The amplitude response can have a maximum of n ripples in the passband ($2m \leq n$); and with r maxima in the stopband, the remaining $m-r$ degrees of freedom can be used for the group delay equalization. For example, for $n=10$, $m=4$ and $r=1$, three degrees of freedom can be used for group delay equalization, while ten degrees of freedom are used for passband shaping (one degree for the required ω_c). The existing methods for the design of linear phase filters with Chebyshev or unconstrained least-squares monotonic passband [28,30,33] use the maximum degrees of freedom for passband shaping. Litovsky's method with least-squares monotonic passband provides a better group delay flatness as a consequence of relaxed constraints on the passband amplitude response.

The method proposed in this thesis is based on the fact that an even distribution of degrees of freedom for shaping the amplitude and group delay responses can provide better designs with respect to the filter characteristics. The amplitude response is designed to have fewer than the maximum number of ripples, where the excess degrees of freedom are used to achieve a high degree of phase linearity over the prescribed portion of the passband. The reduction in passband ripples has a two-fold effect: an increase in the degrees of freedom for group delay equalization, and an associated decrease in group delay near the bandedge. The effect on the stopband attenuation after the transition-band is negligible. As a result, filters designed by this method have a more economical realization when compared with those designed by the other methods. Design examples are given which confirm the optimality of the proposed method.

The design procedure begins with the determination of a near optimum transfer function which is discussed in chapter III for minimum-phase filters, in chapter IV for nonminimum-phase filters and in chapter V for digital, switched and interdigital transmission-line filters. The resulting filters are further optimized using a procedure given in chapter II. The design of nonreciprocal filters is discussed in chapter VI. Design examples are given throughout these chapters. Optimized design examples and the comparison of the proposed method with the other methods are given in chapter VII.

CHAPTER II

DESIGN PROCEDURE

The design procedure begins with the determination of a near optimum transfer function [54,55] which is discussed in the following chapters, and then the resulting filter is optimized using a least-mean-squares routine.

For better understanding of the approximation problem, let us review the concept of Chebyshev approximation.

2.1 Approximation of a Constant

Ther are at least two best known methods [37,38] for the generation of rational functions which approximate a constant in an equal ripple, or Chebyshev (mini-max) criteria.

Suppose the function $f(t)$ is to approximate C with tolerance error $\pm\epsilon$ in the interval $[a,b]$. Then upon the application of the transformation

$$x = \frac{2}{b-a}t - \frac{a+b}{b-a} \tag{2.1}$$

the function $f(x)$ is required to approximate C in the interval $[-1,+1]$. Similarly, the problem reduces to the approximation of zero in the interval $[-1,+1]$ with tolerance error of ± 1.

The methods are based on transformations which guarantee an equal ripple behavior in the approximation interval, i.e. the extrema of the function have equal errors in absolute value, in the approximation interval.

The approximating function is required to be equal ripple in the transformed interval, hence it can be written as [37]

$$f(x)=\cos[y(x)]=\frac{(e^{jy(x)}+e^{-jy(x)})}{2} \tag{2.2}$$

where y is a real function of x in the approximation interval.

Let $F(Z)$ be a real polynomial of degree n in Z, and Z be a complex function of x and let

$$y=2arg[F_n(Z)] \tag{2.3}$$

then e^{jy} may be written as

$$e^{jy}=\frac{F_n(Z)}{F_n(Z^*)} \tag{2.4}$$

In the approximation interval, $|x|\le 1$ and thus let

$$x=\cos\theta, \quad 0\le\theta\le\pi \tag{2.5}$$

where θ is a transformed variable. For $f(x)$ to be a Chebyshev function, $F_n(Z)$ and $Z(x)$ are to be defined such that when θ varies monotonically from 0 to π, $y[x(\theta)]$ varies monotonically from 0 to $2n\pi$ (equal ripple with the maximum number of extrema).

2.1.1 Method A

Let $Z(x)$ be defined as

$$Z(x)=\sqrt{1-\frac{1}{x^2}}=j\tan\theta. \tag{2.6}$$

Let the polynomial $F_n(Z)$ be written as

$$F_n(Z)=\prod_{i=1}^{n}(Z-Z_i), \qquad Z_i=\alpha_i+j\beta_i \tag{2.7}$$

then

$$y[x(\theta)]=2arg[F_n(jtan\theta)]=2\sum_{i=1}^{n}\tan^{-1}\frac{-\beta_i+\tan\theta}{-\alpha_i} \tag{2.8}$$

All the terms of the summation in (2.8) are monotonic increasing with respect to θ, if $\alpha_i<0$ (Hurwitz $F_n(Z)$), and as a result y varies from 0 to $2n\pi$ monotonically if θ varies from 0 to π. Thus the approximating function has $2n$ zeros in the approximation interval, and is an even rational function of x of degree $2n$,

$$f(x)=\frac{1}{2}\left[\frac{F_n(\sqrt{1-\frac{1}{x^2}})}{F_n(-\sqrt{1-\frac{1}{x^2}})}+\frac{F_n(-\sqrt{1-\frac{1}{x^2}})}{F_n(\sqrt{1-\frac{1}{x^2}})}\right] \tag{2.9}$$

The function obtained by this method has even symmetry about zero; a function which is not symmetrical can be obtained by a substitution of x for x^2

$$\hat{f}(x)=\frac{1}{2}\left[\frac{F_n(\sqrt{1-\frac{1}{x}})}{F_n(-\sqrt{1-\frac{1}{x}})}+\frac{F_n(-\sqrt{1-\frac{1}{x}})}{F_n(\sqrt{1-\frac{1}{x}})}\right] \qquad 0\le x\le 1 \tag{2.10}$$

For odd symmetry, let

$$y[x(\theta)]=2arg[F_n(jtan\theta)]+\theta, \tag{2.11}$$

then e^{jy} may be written as

$$e^{jy}=\frac{F_n(Z)(1+Z)^{1/2}}{F_n(-Z)(1-Z)^{1/2}} \tag{2.12}$$

Thus y varies monotonically from 0 to $(2n+1)\pi$ in the interval, and hence $f(x)$ has $2n+1$ zeros in the approximation interval with odd symmetry about zero.

The functions thus obtained are Chebyshev, because they are equal ripple and have the maximum number of zeros in the approximation interval.

2.1.2 Method B

In this method the variable Z in (2.3) is defined to be

$$Z(x)=x+\sqrt{x^2-1}=e^{j\theta} \tag{2.13}$$

which from (2.4) yields

$$e^{jy}=\frac{F_n(Z)}{Z^nF_n(Z^{-1})}. \tag{2.14}$$

The resulting function

$$f(x)=\frac{Z^{-n}F_n^2(Z)+Z^nF_n^2(Z^{-1})}{2F_n(Z)F_n(Z^{-1})} \tag{2.15}$$

is a function of $Z+Z^{-1}$, because $f(Z)=f(Z^{-1})$. From (2.5) and (2.13)

$$x=\frac{1}{2}(Z+Z^{-1}), \tag{2.16}$$

and therefore $f(x)$ is a rational function of x. From (2.14)

$$y[x(\theta)]=2arg[F_n(e^{j\theta})]-n\theta, \tag{2.17}$$

for y to increase monotonically, the zeros of $F_n(Z)$ must lie inside the unit circle in the Z-plane, i.e. $|Z_i|<1$, and as a result y varies from 0 to $n\pi$ in the interval. The resulting function is asymmetric about zero; to obtain an even function of x, let

$$\bar{F}_{2n}(Z)=F_n(Z^2). \tag{2.18}$$

The resulting function f , is a function of Z^2+Z^{-2} and hence x^2, because

$$Z^2+Z^{-2}=(Z+Z^{-1})^2-2 \tag{2.19}$$

For an odd function of x let

$$\tilde{F}_{2n+1}(Z)=F_n(Z^2)Z. \tag{2.20}$$

The resulting function has odd symmetry about $x=0$, because $f(Z)=-f(-Z^{-1})$. Figure 2 illustrates the typical examples derived by these methods.

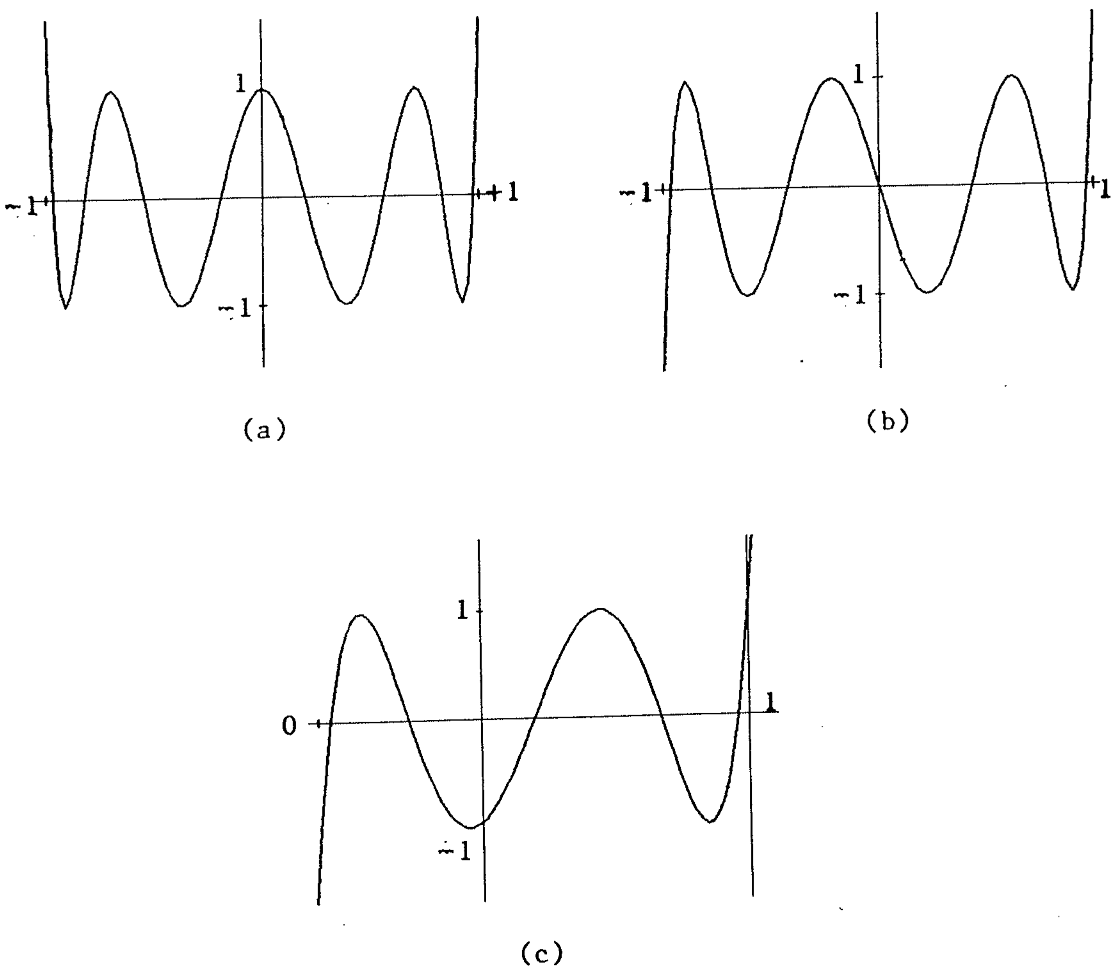

Fig. 2 Chebyshev approximations in the interval [-1,+1], for even symmetry (a), odd symmetry (b), and no symmetry (c).

2.2 Approximation of Ideal Filters

The amplitude characteristics of ideal lowpass, highpass, bandpass and bandstop filters can be approximated in a mini-max norm, using the Chebyshev approximation technique discussed in the previous sections.

2.2.1 Lowpass Filters

Let the squared magnitude of the transfer function $H(s)$ be written as

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 \Psi_n(\omega^2)} \tag{2.21}$$

The characteristic function $\Psi_n(\omega^2)$ is required to satisfy

$$\Psi_n(\omega^2) : \begin{cases} \geq 0 & \omega \geq 0 \\ \leq 1 & \textit{passband } \; 0 \leq \omega \leq \omega_c \end{cases} \tag{2.22}$$

Thus $\Psi_n(\omega^2)$ is to approximate 1/2 within the tolerance $\pm 1/2$, in the interval $[-\omega_c, \omega_c]$, and with even symmetry about $\omega=0$. For reciprocal filters the denominator polynomial of $\Psi_n(\omega^2)$ is a perfect square and hence for a Chebyshev approximation based on method A, $F(Z)$ must be a perfect square. Thus

$$\Psi_n(\omega^2) = \frac{1}{2} + \frac{1}{4}\left[\frac{\tilde{F}^2(Z)}{\tilde{F}^2(-Z)} + \frac{\tilde{F}^2(-Z)}{\tilde{F}^2(Z)}\right], \quad Z = \sqrt{1-(\frac{\omega_c}{\omega})^2} \tag{2.23}$$

which equivalently can be written as

$$\Psi_n(\omega^2) = \frac{1}{4}\left[\frac{\tilde{F}(Z)}{\tilde{F}(-Z)} + \frac{\tilde{F}(-Z)}{\tilde{F}(Z)}\right]^2, \tag{2.24}$$

where

$$\tilde{F}(Z) = \begin{cases} F_m(Z) & \textit{neven} \\ F_m(Z)(1+Z)^{1/2} & \textit{nodd} \end{cases} \tag{2.25}$$

The polynomial $F_m(Z)$ is Hurwitz and $m=[n/2]$, i.e.

$$m=\begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases} \tag{2.26}$$

The zeros of the transfer function are obtained from the zeros of the denominator polynomial of $\Psi(\omega^2)$ after the substitution, $\omega^2=-s^2$. The poles of the transfer function are the LHP zeros of the equation

$$1+\epsilon^2\Psi_n(-s^2)=0. \tag{2.27}$$

The resulting transfer function has a maximum number of n ripples (extrema) in the passband with m free parameters. These degrees of freedom can be used to constrain the amplitude response in the stopband and the group delay in the passband. For a monotonic amplitude response in the stopband, $F(Z)$ has no zeros in the interval (0,1).

2.2.2 Highpass Filters

Highpass filters are obtained using the same procedure as for the derivation of lowpass filters, except for the transformed variable Z which for the highpass case is given by

$$Z=\sqrt{1-(\frac{\omega}{\omega_c})^2} \tag{2.28}$$

2.2.3 Bandpass Filters

Bandpass filters may be obtained through a lowpass to bandpass frequency transformation. Equivalently the procedure of sec. 2.2.1 may be used with the transformed variable

$$Z = \frac{\sqrt{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}}{\omega_0^2-\omega^2} \tag{2.29}$$

where

$$\Psi_n(\omega^2) \leq 1 \qquad \text{for} \quad \omega_1 \leq \omega \leq \omega_2 \tag{2.30}$$

and $\omega_0=\sqrt{\omega_1\omega_2}$ is the center frequency. The resulting filters have geometrically symmetrical amplitude response about ω_0. Because of the symmetry the available degrees of freedom are half the maximum number for the nonsymmetric response. For a bandpass filter of degree $2n$, obtained through transformation, there are $m=[n/2]$ degrees of freedom.

The alternative method which guarantees the maximum degrees of freedom is based on the Chebyshev approximation without symmetry. For method A, the characteristic function can be obtained as

$$\Psi_n(\omega^2)=\frac{1}{4}\left[\frac{\tilde{F}(Z)}{\tilde{F}(-Z)}+\frac{\tilde{F}(-Z)}{\tilde{F}(Z)}\right]^2 \qquad Z=\sqrt{1-\frac{1}{x}} \tag{2.31}$$

where

$$x=\frac{1}{\omega_2^2-\omega_1^2}\omega^2-\frac{\omega_1^2}{\omega_1^2-\omega_1^2} \tag{2.32}$$

$$\tilde{F}(Z)=\begin{cases} F_m(Z) & n\,even \\ F_m(Z)(1+Z)^{1/2} & n\,odd\,, \quad m=[\frac{n}{2}]. \end{cases} \tag{2.33}$$

If $\omega \geq \omega_2$, Z varies in the interval [0,1], while for $0 \leq \omega \leq \omega_1$, Z varies in the interval $[\frac{\omega_2}{\omega_1}, \infty]$.

2.2.4 Bandstop Filters

Bandstop filters may be obtained through a transformation for which

$$Z = \frac{\sqrt{(\omega^2 - \omega_1^2)(\omega_2^2 - \omega^2)}}{\omega(\omega_2 - \omega_1)} \tag{2.34}$$

where the passbands are $[0, \omega_1]$, $[\omega_2, \infty]$, and the notch frequency is $\omega_0 = \sqrt{\omega_1 \omega_2}$. The resulting filters have the maximum degrees of freedom.

2.3 Numericl Approach

The characteristic function $\Psi_n(\omega^2)$ obtained in section 2.2, requires numerical techniques to evaluate the free parameters in order to satisfy the amplitude requirements in the stopband. One may combine the computational efforts to obtain the desired amplitude characteristics in the passband and the stopband without utilizing the Chebyshev functions discussed in section 2.1.

Let the characteristic function of a reciprocal filter of degree n with m frequencies of perfect transmission (other than zero) and k transmission zeros (other than zero) be written as

$$\Psi_n(\omega^2) = \frac{K_0 \prod_{i=1}^{m} (1 - \alpha_i^2 \omega^2)^2 P_{n-2m}(\omega^2)}{\prod_{i=1}^{k} (1 - \beta_i^2 \omega^2)^2 Q_l^2(\omega^2) \omega^2} \tag{2.35}$$

where the monic polynomials $P_{n-2m}(\omega^2)$ and $Q_l(\omega^2)$ have no positive real zeros.

For a filter with perfect transmission at zero, $P(0)=0$; the term ω^2 in (2.35) is present if an odd number of transmission zeros are located at zero.

There are a maximum of $2[n/2]-m+k+l+1$ degrees of freedom (parameters in (2.35)) which can be used to obtain the desired amplitude characteristics and equalize the group delay. For a filter with m frequencies of perfect transmission (other than zero) and k transmission zeros, there are a total of $m+k+1$ constraints on the amplitude response. The remaining $2[n/2]-2m+l$ degrees of freedom can be used to equalize the group delay. For example for $n=10$, $m=5$, $k=1$ and $l=2$, there are nine degrees of freedom from which two can be used for delay equalization. While for the case with fewer than the maximum ripples ($n=10,m=2$), and perfect transmission at zero, the total degrees of freedom are eleven, from which seven can be used for delay equalization.

The characteristic function of all types of filters, i.e. lowpass, bandpass, highpass, bandstop, and multiple passband filters, has the same form as $\Psi_n(\omega^2)$ in (2.35).

Let the polynomials $P(\omega^2)$ and $Q(\omega^2)$ be written as

$$P_{n-2m}(\omega^2)=a_0+a_1\omega^2+....+a_{n-2m-1}\omega^{2(n-2m-1)}+\omega^{2(n-2m)} \tag{2.36a}$$

$$Q_l(\omega^2)=b_0+b_1\omega^2+....+b_{l-1}\omega^{2(l-1)}+\omega^{2l} \tag{2.36b}$$

There are $2[n/2]-2m+l$ coefficients which control the group delay flatness. For every set of coefficients (initially obtained using the approximations in chapters III and IV), the parameters α_i and β_i and K_0 can be calculated iteratively to obtain the desired amplitude characteristics. The iterative procedure is similar to the Remez algorithm [39] in the sense that the extrema of the function are equalized iteratively. For example, for an equal ripple behavior in the passband and the stopband, the minima of $\Psi_n(\omega^2)$ are equalized to 1 in the passband and the maxima to $(A^{A_s/10}-1)/\epsilon^2$ in the stopband (A_s is the minimum stopband attenuation).

2.3.1 Amplitude Equalization

Let the characteristic function be written as (for simplicity the subscripts of the given polynomials $P_{n-2m}(X)$ and $Q_l(X)$ have been dropped).

$$\Psi(X)=\frac{A^2(X)P(X)}{B^2(X)Q^2(X)} \tag{2.37}$$

where $X=\omega^2$, and

$$A(X)=\prod_{i=1}^{m}(1-\alpha_i^2X)\sqrt{K_0} \tag{2.38a}$$

$$B(X)=\prod_{i=1}^{k}(1-\beta_i^2X) \tag{2.38b}$$

or equivalently,

$$A(X)=A_0+A_1X+....+A_mX^m \tag{2.39a}$$

$$B(X)=1+B_1X+....+B_kX^k \tag{2.39b}$$

The abscissae of the extrema of Ψ are the positive real zeros of

$$\dot{\Psi}=\frac{(2P\dot{A}+A\dot{P})BQ-2(Q\dot{B}+B\dot{Q})AP}{B^3Q^3}A \tag{2.40}$$

where dot (.) denotes the derivative with respect to X. The zeros of $\dot{\Psi}$ include the zeros of $A(X)$ which are the zeros of $\Psi(X)$ in the passband. Thus the roots of the equation

$$(2P\dot{A}+A\dot{P})BQ-2(Q\dot{B}+B\dot{Q})AP=0 \tag{2.41}$$

determine the abscissae of the extrema. This equation is of degree $n-m+k+l-1$. Since only positive real zeros of this equation are of interest, efficiant algorithms can be developed to find the roots. Especially with the knowledge of the approximate locations of the extrema (between the zeros of $\Psi(X)$), fast simple routines prove to be practical (e.g. using nested intervals). Let the positive real zeros of the equation (2.41) be denoted by X_i, where

$$X_{i+1} > X_i \qquad i=1,m+k \tag{2.42}$$

(The notation $i=1,m$ means $i=1$ to $i=m$ inclusive). For a desired amplitude characteristic

$$\Psi(X_i)=\Psi_i \qquad i=1,m+k \tag{2.43}$$

$$\Psi(\omega_c^2)=1 \qquad \omega_c : a\ cutoff\ frequency$$

For an equal-ripple passband, equal-ripple stopband lowpass filter

$$\Psi_i = \begin{cases} 1 & i=1,m \\ \gamma=(10^{A_s/10}-1)/\epsilon^2 & i=m+1,m+k \end{cases} \tag{2.44}$$

Rearrange X_i such that

$$\tilde{X}_i = \begin{cases} X_i & i=1,m \\ X_c & i=m+1 \\ X_{i-1} & i=m+2,k+m+1 \end{cases} \tag{2.45}$$

where

$$X_c=\omega_c^2 \tag{2.46}$$

From (2.37) and (2.44)

$$\frac{A^2(\hat{X}_i)P(\hat{X}_i)}{B^2(\hat{X}_i)Q^2(\hat{X}_i)}=\gamma^j \tag{2.47}$$

where

$$j = \begin{cases} 0 & i=1,m+1 \\ 1 & i=m+2,k+m+1 \end{cases} \tag{2.48}$$

Equivalently

$$\frac{A(\hat{X}_i)}{B(\hat{X}_i)}=(-1)^{i-1}\sqrt{\frac{Q^2(\hat{X}_i)}{P(\hat{X}_i)}}\gamma^j=r_i \tag{2.49}$$

which from (2.39) results in

$$A_0+A_1\hat{X}_i+....+A_m\hat{X}_i^m-r_iB_1\hat{X}_i-....-r_iB_k\hat{X}_i^k=r_i \qquad i=1,m+k+1 \tag{2.50}$$

In matrix representation

$$C\,\mathbf{y}=\mathbf{r} \tag{2.51}$$

where

$$C=\begin{bmatrix} 1 & \hat{X}_1 & . & . & \hat{X}_1^m & -r_1\hat{X}_1 & . & -r_1\hat{X}_1^k \\ 1 & \hat{X}_2 & . & . & \hat{X}_2^m & -r_2\hat{X}_2 & . & -r_2\hat{X}^k \\ . & . & . & . & . & . & . & . \\ 1 & \hat{X}_i & . & . & \hat{X}_i^m & -r_i\hat{X}_i & . & -r_i\hat{X}_i^k \\ . & . & . & . & . & . & . & . \\ 1 & \hat{X}_{m+k+1} & . & . & \hat{X}_{m+k+1}^m & -r_{m+k+1}\hat{X}_{m+k+1} & . & -r_{m+k+1}\hat{X}_{m+k+1}^k \end{bmatrix} \tag{2.52}$$

$$\mathbf{y}=(A_0\;A_1\;....\;A_m\;B_1\;B_2....\;B_k)^T \tag{2.53}$$

$$\mathbf{r}=(r_1\;r_2\;....\;r_{m+k+1})^T \tag{2.54}$$

The matrix C will be nonsingular if a proper choice of initial parameters is made, i.e. $\hat{X}_{i+1}>\hat{X}_i$. This condition together with (2.49) guarantees a solution for (2.51). The set of equations in (2.50) can be solved using for example the general triangularization procedure, or an iterative procedure. The solution of (2.50) determines A_0 to A_m and B_1 to B_k, and hence a new set of α_i, β_i, and K_0. The iteration may continue until the changes in these parameters decrease below some prescribed value. The iterative procedure is normally convergent if $\hat{X}_{i+1}>\hat{X}_i$, because of the fact that a solution always exist.

2.3.2 Group Delay Equalization

Let the transfer function of the reciprocal filter be written as

$$H(s)=\frac{s^k \sum_{j=0}^{m} c_j s^{2j}}{\prod_{i=1}^{n}(s-s_i)} \tag{2.55}$$

where the poles s_i are in the LHP and are either real or in complex conjugate pairs. The phase of $H(s)$ evaluated at $s=j\omega$ is

$$\phi(\omega)=k\frac{\pi}{2}+\hat{k}\pi+\sum_{i=1}^{n}\tan^{-1}\frac{\omega-\lambda_i}{\sigma_i} \qquad \omega\geq 0 \tag{2.56}$$

where

$$s_i=\sigma_i+j\lambda_i \qquad \sigma_i<0$$

and

$$\hat{k}=\begin{cases} 0 & \text{if } \sum_{j=1}^{m} c_j\omega^{2j}(-1)^j>0 \\ 1 & \text{if } \sum_{j=1}^{m} c_j\omega^{2j}(-1)^j<0 \end{cases}$$

The group delay evaluated at frequency ω is

$$\tau(\omega)=-\frac{d}{d\omega}\phi(\omega)=\sum_{i=1}^{n}-\frac{\sigma_i}{\sigma_i^2+(\omega-\lambda_i)^2} \tag{2.57}$$

where the discontinuities due to $\hat{k}\pi$ term in $\phi(\omega)$ are ignored, because of their occurrence in the stopband. Thus the poles of the transfer function determine the group delay. The poles of $H(s)$ are the LHP zeros of the equation

$$1+\epsilon^2\Psi_n(-s^2)=0 \tag{2.58}$$

or from (2.37)

$$B^2(-s^2)Q(-s^2)+\epsilon^2A^2(-s^2)P(-s^2)=0 \tag{2.59}$$

For a set of coefficients of $P(\omega^2)$ and $Q(\omega^2)$, the group delay in the passband is evaluated and through the imposition of constraints, such as equal-ripple, interpolation or least-mean-squares, a new set of values for the coefficients is obtained. With the new set of parameters, the procedure 2.3.1 is repeated and the iterative procedure continues until the constraints on the group delay are satisfied with some tolerance. In the next sections, standard methods [39] for the interpolation, equal-ripple, and least-mean-squares approximation of the group delay are described.

2.3.2.A Interpolation Method

In this method the group delay is required to meet the constraints

$$\tau(\omega_j)=\tau_j \qquad j=1,n_0 \tag{2.60}$$

where ω_j are predetermined, or evaluated in each iteration, and τ_j are given values. Since the group delay and the amplitude characteristics are interrelated, apriori knowledge of τ_j for a desired amplitude characteristic is almost impossible, hence a more practical constraint on $\tau(\omega)$ can be

$$\tau(\omega_j)=\tau_j+\tau_0 \qquad j=1,n_0 \tag{2.61}$$

where τ_0 is a constant to be determined. For a filter with constant group delay in the passband

$$\tau(\omega_j)=\tau_0 \qquad j=1,n_0 \tag{2.62}$$

The frequencies ω_j can be chosen to be the critical frequencies of the group delay, i.e. its maxima or minima. In any case the number of frequencies must be greater than the number of parameters to be determined, by one. Since the number of parameters are usually larger than the number of group delay maxima (minima), this method is more suitable for filters with the maximum number of amplitude ripples in the passband. Let the group delay be written as

$$\tau = f\ (\omega, a_0, a_1, \ldots, a_{n-2m-1}, b_0, b_1, \ldots, b_{k-1}) \tag{2.63}$$

or

$$\tau(\omega,\mathbf{p}) = f\ (\omega, p_1, p_2, \ldots, p_{n-2m+k}) \tag{2.64}$$

where

$$p_i = \begin{cases} a_{i-1} & i=1,n-2m \\ b_{i-n+2m-1} & i=n-2m+1, n_p=n-2m+k \end{cases} \tag{2.65}$$

If parameters p_i are varied by Δp_i, the group delay can be written as

$$\tau(\omega,\mathbf{p}+\Delta\mathbf{p}) = \tau(\omega,\mathbf{p}) + \Delta\tau \tag{2.66}$$

where $\tau(\omega,\mathbf{p}+\Delta\mathbf{p})$ can be expanded in Taylor series including only the first derivatives as

$$\tau(\omega,\mathbf{p}+\Delta\mathbf{p}) \approx \tau(\omega,\mathbf{p}) + \left(\frac{\partial\tau}{\partial\mathbf{p}}\right)^T \Delta\mathbf{p} \qquad \frac{|\Delta\mathbf{p}|}{|\mathbf{p}|} << 1 \tag{2.67}$$

The first order approximation in (2.67) is accurate enough for the interpolation procedure as long as the group delay varies smoothly. This occurs when the group delay varies parabolically near the solution. Therefore the initial values of the parameters p_i are important for the convergence of the procedure. For a set of parameters $\mathbf{p}$, the group delay $\tau(\omega,\mathbf{p})$ is evaluated at frequency ω, and the variations of the group delay with respect to each parameter is also obtained. Then from (2.67) and (2.62)

$$\left[\frac{\partial\tau(\omega_j,\mathbf{p})}{\partial\mathbf{p}}\right]^T\Delta\mathbf{p}-\tau_0=-\tau(\omega_j,\mathbf{p}) \quad j=1,n_0 \tag{2.68}$$

The equations in (2.68) can be written explicitly as

$$\begin{array}{ccccccc}
\frac{\partial\tau(\omega_1,\mathbf{p})}{\partial p_1}\Delta p_1+ & . & & +\frac{\partial\tau(\omega_1,\mathbf{p})}{\partial p_{n_p}}\Delta p_{n_p} & -\tau_0 & = & -\tau(\omega_1,\mathbf{p}) \\
. & . & & . & . & = & . \\
. & +\frac{\partial\tau(\omega_j,\mathbf{p})}{\partial p_i}\Delta p_i+ & . & & -\tau_0 & = & -\tau(\omega_j,\mathbf{p}) \\
. & . & & . & . & = & . \\
\frac{\partial\tau(\omega_{n_0},\mathbf{p})}{\partial p_1}\Delta p_1+ & . & & +\frac{\partial\tau(\omega_{n_0},\mathbf{p})}{\partial p_{n_p}}\Delta p_{n_p} & -\tau_0 & = & -\tau(\omega_{n_0},\mathbf{p})
\end{array} \tag{2.69}$$

The set of equations in (2.68) are solved for $\Delta\mathbf{p}$ and τ_0 to determine a new set of parameters $(\mathbf{p}+\Delta\mathbf{p})$. The iterations continue until a desired accuracy is reached. For the set of parameters p_1 to p_{n-2m+k}, $n_0=n-2m+k+1$ points are necessary for interpolation.

2.3.2.B Equal Ripple Approximation

For the Chebyshev or equal ripple approximation (to a constant) , the extrema of the group delay are equalized in the same manner as was discussed in section 2.3.1 for the amplitude approximation. From (2.57) the abscissae of the extrema are the zeros of the equation

$$\sum_{i=1}^{n}\frac{2\sigma_i(\omega-\lambda_i)}{[\sigma_i^2+(\omega-\lambda_i)^2]^2}=0 \tag{2.70}$$

For a set of parameters $\mathbf{p}$, the poles s_i are obtained which determine the abscissae from (2.70). Let the abscissae be ordered such that $\omega_{j+1}>\omega_j$, then for the equal ripple behavior of the group delay between τ_{0h} and τ_{0l}

$$\left[\frac{\partial\tau(\omega_j,\mathbf{p})}{\partial\mathbf{p}}\right]\Delta\mathbf{p}-\tau_0=-\tau(\omega_j,\mathbf{p}) \quad j=1,n_0 \tag{2.71}$$

where

$$\tau_0=\begin{cases}\tau_{0l} & \text{if } \tau(\omega_j,\mathbf{p}) \text{ } a \text{ } minimum \\ \tau_{0h} & \text{if } \tau(\omega_j,\mathbf{p}) \text{ } a \text{ } maximum\end{cases}$$

There are $2[\frac{n}{2}]-2m+l+2$ unknowns which can be determined from the set of equations in (2.71), provided that $n_0=2[\frac{n}{2}]-2m+l+2$. With the new set of parameters the iteration is repeated until a desired accuracy is reached.

2.3.2.C Least-Mean-Squares Approximation

In this method the group delay is evaluated at a prescribed set of frequencies in the passband, and then the sum of the squares of the differences from the average value is minimized. For the set of frequencies ω_j, the sum $F(\overline{\omega},\mathbf{p})$ can be written as (the method is equally applicable to the case where the approximation is to an arbitrary function of ω)

$$F(\overline{\omega},\mathbf{p})=\sum_{j=1}^{n_0}(\tau(\omega_j,\mathbf{p})-\tau_0)^2=\sum_{j=1}^{n_0}[E(\omega_j,\mathbf{p})]^2 \qquad (2.72)$$

where $\overline{\omega}$ is the vector of ω_j, and τ_0 is the average of the group delays. At a minimum, F must satisfy

$$\frac{\partial F(\overline{\omega},\mathbf{p})}{\partial p_i}\Big|_{\mathbf{p}+\Delta\mathbf{p}}=0 \qquad i=1,n_t \qquad (2.73)$$

where $n_t=2[\frac{n}{2}]-2m+l$ is the total number of coefficients to be determined. A first order approximation for $\hat{F}(\overline{\omega},\mathbf{p})=\frac{\partial F(\overline{\omega},\mathbf{p})}{\partial \mathbf{p}_i}$ yields

$$\frac{\partial F(\overline{\omega},\mathbf{p})}{\partial p_i}+\sum_{k=1}^{n_t}\frac{\partial^2 F(\overline{\omega},\mathbf{p})}{\partial p_i \partial p_k}\Delta p_k=0 \qquad i=1,n_t \qquad (2.74)$$

or in matrix form

$$\nabla F(\overline{\omega},\mathbf{p})+H(\overline{\omega},\mathbf{p})\Delta\mathbf{p}=0 \tag{2.75}$$

The Hessian matrix $H(\overline{\omega},\mathbf{p})$ contains the second order derivatives of F. Therefore from (2.72) and (2.74)

$$\sum_{j=1}^{n_0} E_j \frac{\partial E_j}{\partial p_i} + \sum_{k=1}^{n_t}\left(\sum_{j=1}^{n_0}\left(\frac{\partial E_j}{\partial p_k}\frac{\partial E_j}{\partial P_i} + E_j \frac{\partial^2 E_j}{\partial p_k \partial p_i}\right)\right)\Delta p_k = 0 \qquad i=1,n_t \tag{2.76}$$

where E_j denotes $E(\omega_j,\mathbf{p})$ for simplicity. In matrix form

$$\tilde{B}\Delta\mathbf{p} = -\tilde{A}^T \mathbf{E} \tag{2.77}$$

where

$$\tilde{B} = \tilde{A}^T\tilde{A} + \tilde{H}\mathbf{E} \tag{2.78a}$$

$$\mathbf{E} = (E_1, E_2, \cdots, E_{n_0})^T \tag{2.78b}$$

$$\Delta\mathbf{p} = (\Delta p_1, \Delta p_2, \ldots, \Delta p_{n_t})^T \tag{2.78c}$$

$$\tilde{A} = \begin{pmatrix} \frac{\partial E_1}{\partial p_1} & \frac{\partial E_1}{\partial p_2} & \cdot & \frac{\partial E_1}{\partial p_{n_t}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial E_{n_0}}{\partial p_1} & \frac{\partial E_{n_0}}{\partial p_2} & \cdot & \frac{\partial E_{n_0}}{\partial p_{n_t}} \end{pmatrix} \tag{2.79}$$

$$\tilde{H} = \begin{pmatrix} \frac{\partial^2 E_1}{\partial p_1^2} & \frac{\partial^2 E_2}{\partial p_1 \partial p_2} & \cdot & \frac{\partial^2 E_{n_0}}{\partial p_1 \partial p_{n_t}} \\ \frac{\partial^2 E_1}{\partial p_2 \partial p_1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^2 E_1}{\partial p_1 \partial p_{n_t}} & \cdot & \cdot & \frac{\partial^2 E_{n_0}}{\partial p_1 \partial p_{n_t}} \end{pmatrix} \tag{2.80}$$

If the initial approximation is near the optimum of F, the second derivatives may be ignored, which results in

$$\tilde{B} = \tilde{A}^T \tilde{A} \tag{2.81}$$

where $\tilde{B}$ is positive definite. The solution of the equation (2.77) determines $\Delta\mathbf{p}$ and hence a new set of parameters $(\mathbf{p}+\Delta\mathbf{p})$. The iterations continue until a desired accuracy is reached. For a poor initial approximation the process may diverge, a remedy to this problem is achieved by damping the $\Delta\mathbf{p}$ [40]. Thus it is made certain that in each iteration F decreases; in case of an increase $\Delta\mathbf{p}$ is damped by matrix $\tilde{D}$, which is determined from the information contained in F and $\Delta\mathbf{p}$. The least-mean-squares method is especially suitable for the approximation of transfer functions with fewer than the maximum number of passband ripples. The frequencies ω_j can be located in the linear-phase band equidistantly or in any other fashion. If the group delay flatness is more emphasized in some part of the passband, a weight function, $W(\omega)$ may be used to magnify the error function at the frequencies of interest. Thus the weighted error function $E_w(\omega_j,\mathbf{p})$ can be written as

$$E_w(\omega_j,\mathbf{p}) = W(\omega_j) E(\omega_j,\mathbf{p}) \tag{2.82}$$

The weight function $W(\omega)$ is chosen to be a continuous function to avoid any problems with the convergence of the procedure.

2.3.3 Determination of the Transfer Function

The transfer function of the filter with the characteristic function given in (2.35) is uniquely determined from its poles and zeros. The poles of the transfer function are obtained from (2.58) and (2.59). The zeros can be obtained from the zeros of the equation

$$\prod_{i=1}^{k}(1+\beta_i^2 s^2)Q_l(-s^2)=0 \tag{2.83}$$

Let the poles and the zeros be denoted by p_i and z_i, respectively. The gain constant of the transfer function is obtained from

$$G_0=\frac{\prod_{i=1}^{n}|p_i|}{\prod_{i=1}^{2(k+l)}|z_i|}\frac{1}{\sqrt{1+\epsilon^2\Psi_n(0)}} \tag{2.84}$$

The resulting transfer function

$$H(s)=\frac{G_0\prod_{i=1}^{2(k+l)}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \tag{2.85}$$

has a minimum attenuation of zero dB in the passband, i.e. $|H(j\omega)|_{\max}=1$. If the specifications require a minimum attenuation of a_p in the passband, one may shift the attenuation levels for design specifications and thus the gain constant for this case becomes

$$\tilde{G}_0=G_0 10^{-a_p/20} \tag{2.86}$$

The group delay of a selective filter is mainly related to the coefficients of the numerator polynomial, hence l the degree of $Q_l(\omega^2)$ is increased gradually until the group delay specifications are met. As it has been stated before, the number of passband ripples also affect the group delay flatness in the passband. A proper choice of l and m is made judiciously and by exprience. If the filter characteristics are not satisfactory the degree of the filter is increased and the procedure continues until the specifications are met.

It is assumed that the approximation in maximally flat norm is nonoptimum. For a maximally flat approximation of the amplitude response, the characteristic function of the filter is written as

$$\Psi_n(\omega^2)=\frac{(\omega^2-\omega_0^2)^{2m}P_{n-2m}(\omega^2)}{\prod_{i=1}^{k}(\omega^2-\beta_i^2)^2Q_i^2(\omega^2)} \tag{2.87}$$

where m is the degree of flatness at ω_0 (for $\omega_0=0$ substitute m in place of $2m$). The procedure for the amplitude equalization is the same as 2.3.1. For a maximally flat group delay of degree m

$$\tau(\omega^2)=\frac{\tau_0\sum_{i=0}^{m}G_i(\omega^2-\omega_0^2)^i+R(\omega^2)\omega^{2m+2}}{|D(j\omega)|^2} \tag{2.88}$$

where

$$|D(j\omega)|^2=\sum_{i=0}^{n}G_i(\omega^2-\omega_0^2)^i \tag{2.89}$$

Thus for the group delay equalization, the first $m+1$ coefficients of the denominator and numerator polynomials in (2.57) expanded about ω_0^2 are equalized iteratively (with coefficient τ_0). The procedure is similar to the procedure for equal ripple approximation.

CHAPTER III

MINIMUM-PHASE FILTERS

This chapter is concerned with the determination of minimum-phase transfer functions which are used as initial approximations for the optimization procedure of chapter II. Minimum-phase filters are attractive for some applications where the simplicity of the network realization is of interest. The amplitude and phase characteristics of this class of filters are related through the Hilbert transform equations and the linear phase bandwidth is inversely proportional to the selectivity of the filter [17] defined by

$$S = A_s / A_p \tag{3.1}$$

where A_p is the maximum passband attenuation and A_s is the minimum stopband attenuation.

3.1 Lowpass Filters

The transfer function of a minimum-phase transfer function can be written as

$$H(s) = \frac{G_0 \prod_{i=1}^{k} (1+\beta_i^2 s^2)}{a_0 + a_1 s + \dots + a_{n-1} s^{n-1} + s^n} \qquad (2k \leq n) \tag{3.2}$$

where $1/\beta_i$ are the finite transmission zeros. If all the transmission zeros are located at infinity ($\beta_i = 0$), the resulting filter is referred to as all-pole filter. There are $n+k+1$ degrees of freedom which can be distributed between the amplitude and group delay characteristics. If for example the amplitude response is required to have three extrema in the passband with monotonic stopband, $n-3$ out of the total $n+1$ degrees of freedom can be used to equalize the group delay.

For a filter with a maximally flat group delay or equal ripple group delay with the maximum number of ripples, all the degrees of freedom are used to equalize the group delay with no control over the amplitude response. As a result the amplitude response has poor selectivity. Since the group delay variation is usually required to be small only in a part of the passband, one may decrease the degrees of flatness of the group delay, and use the remaining degrees for the improvement of the amplitude characteristics. Because of the interdependence of the amplitude and phase characteristics, the denominator polynomial which determines the group delay can not be obtained independently from the numerator polynomial without the sacrifice of the selectivity.

3.1.1 Maximally Flat Group Delay

For a filter of degree n and with a maximally flat group delay the first $n-1$ derivatives of the group delay with respect to ω^2 are zero at $\omega=0$, (the group delay is an even function of frequency and hence the odd order derivatives with respect to ω are identically zero.)

$$\frac{d^i \tau(\omega^2)}{d(\omega^2)^i} \Big|_{\omega=0} \qquad i=1,n-1 \tag{3.3}$$

Let the denominator polynomial be written as

$$D_n(s)=E(s^2)+sF(s^2) \tag{3.4}$$

where $E(s^2)$ and $sF(s^2)$ are the even and odd parts, respectively. The phase of the denominator polynomial can be obtained from

$$\phi(\omega)=\tan^{-1}\frac{\omega F(-\omega^2)}{E(-\omega^2)} \tag{3.5}$$

For an ideal linear phase $\phi(\omega)=\omega$, and hence

$$\frac{\omega F(-\omega^2)}{E(-\omega^2)} = \tan\omega \tag{3.6}$$

a rational approximation to $\tan\omega$ can be obtained from the continued fraction expansion of $\tan\omega$ [41,52], truncated to n fractions as

$$\tan\omega = \cfrac{\omega}{c_1 - \cfrac{\omega^2}{c_2 - \cfrac{\omega^2}{\cfrac{\cdot}{\cfrac{\cdot}{c_{n-1} - \cfrac{\omega^2}{c_n}}}}}} \tag{3.7}$$

where $c_m = 2m-1$. Thus

$$\frac{sF(s^2)}{E(s^2)} = \cfrac{s}{1 + \cfrac{s^2}{3 + \cfrac{s^2}{\cfrac{\cdot}{\cfrac{\cdot}{\cfrac{s^2}{2n-1}}}}}} \tag{3.8}$$

It can be shown [42] that the denominator polynomial $D_n(s)$ obtained from (3.8) can also be obtained from the recursion formula

$$D_{m+1}(s) = D_m(s) + \frac{s^2}{(2m-1)(2m+1)} D_{m-1}(s) \qquad m = 1, n-1 \tag{3.9}$$

where $D_0(s)=1$, $D_1(s)=1+s$. The last coefficient c_n has the least effect on the group delay at low frequencies. For a maximally flat group delay with n degrees of flatness $c_n = 2n-1$; the number of degrees of flatness decreases if $c_n \neq 2n-1$. The continued fraction in (3.8) is actually the Hurwitz expansion of the denominator polynomial, and hence for any positive c_n the denominator polynomial remains Hurwitz.

3.1.2 Equal Ripple Group Delay

The group delay of the polynomial $D_n(s)$, can have a maximum of n extrema. The procedure of section 2.3.2.B can be used to obtain equal ripple characteristics. There are extensive tables [11,12] which provide the zeros of the polynomial $D_n(s)$ for a variety of ripple widths. For small group delay variations, the phase may be approximated by $\phi=\omega$, and hence a continued fraction similar to (3.8) but with a different set of coefficients may be obtained. The Hurwitz continued fraction of the denominator polynomial determines the coefficients. The variation of the last coefficient c_n, reduces the group delay flatness with negligible effects on the ripple width. For any positive c_n , the denominator polynomial remains Hurwitz and is obtained from (3.8).

3.1.3 Amplitude Equalization

The coefficient c_n generates a degree of freedom which can be used to yield an equal ripple amplitude response. The extrema are equalized in a procedure similar to the iterative procedure of section 2.3.1. The amplitude of the transfer function can be written as

$$A = |H(j\omega)| = \frac{1+\sum_{j=1}^{k} B_j \omega^{2j}}{\sqrt{G(\omega^2)}} \tag{3.10}$$

where $G(\omega^2)=|D(j\omega)|^2$. The abscissae of the extrema are determined from the zeros of the equation ($X=\omega^2$)

$$2G(X)\left(\sum_{j=1}^{k} jB_j X^{j-1}\right) - [dG(X)/dX]\left(1+\sum_{j=1}^{k} B_j X^j\right)=0 \quad X=\omega^2 \tag{3.11}$$

Let the real positive zeros be denoted by X_i and ordered such that $X_{i+1} > X_i$. For an equal ripple amplitude response in the stopband

$$B_1X_i+B_2X_i^2+....+B_kX_i^k=(-1)^i\,10^{-A_s/20}\sqrt{G(X_i)}-1 \quad i=3,3+k \tag{3.12}$$

where A_s is the minimum stopband attenuation. The set of equations in (3.12) are solved for B_j and the iteration continues until a desired accuracy is reached. For the equalization of the passband ripple, determine the variation of the amplitude function at X_2 and use a first order approximation to evaluate the new c_n. The amplitude function evaluated at $\hat{X}_2$ (new X_2 due to Δc_n), can be approximated as

$$\hat{A}(c_n+\Delta c_n)=A(c_n)+\Delta c_n\frac{dA}{dc_n}|_{X_2,c_n} \tag{3.13}$$

Since $\hat{A}(c_n+\Delta c_n)=1$

$$\Delta c_n=\frac{1-A(c_n)}{\frac{dA}{dc_n}} \tag{3.14}$$

and hence the new c_n is obtained ($\hat{c}_n=c_n+\Delta c_n$). The derivative $\frac{\partial A}{\partial c_n}$ can be approximated by

$$\frac{\partial A}{\partial c_n}|_{X_2,c_n}\approx\frac{\delta A}{\delta c_n} \tag{3.15}$$

where δA is the variation of A evaluated at X_2 and c_n, due to small variation of c_n. The procedure continues until a desired accuracy is reached.

It can be shown analytically that when the positive coefficient c_n approaches zero, a large peak appears in the passband.

The maximum passband attenuation is obtained from

$$A_p=-20\,\log A_2 \tag{3.16}$$

where A_2 is the amplitude evaluated at X_2. The cutoff frequency of the resulting filter, ω_c, can be obtained from the positive real zeros of the equation

$$(1+B_1X+....+B_kX^k)^2-A_2^2G(X)=0 \tag{3.17}$$

Let the largest zero be denoted by X_m , then $\omega_c=\sqrt{X_m}$. For a specific cutoff frequency $\hat{\omega}_c$, $\frac{\omega_c}{\hat{\omega}_c}s$ is substituted for s in the transfer function. The maximum passband attenuation of the resulting filter with one free parameter, c_n, can not be equalized to a desired value, unless a second degree of freedom is generated by changing c_{n-1}. The filter with two parameters c_n and c_{n-1} of course has a lower degree of flatness for the group delay. For a passband loss $\hat{A}_p$, the parameters c_n and c_{n-1} are inversely related if $\hat{A}_p-A_p$ is small. In fact [43]

$$c_nc_{n-1}\approx k_1 \tag{3.18}$$

$$A_p\approx k_2c_n-k_3$$

where k_1, k_2 and k_3 are positive constants. The constant k_1 is determined readily, while k_2 and k_3 are determined from the two equations obtained for A_p and $\hat{A}_p$. The passband loss may also be equalized by the equalization of the extrema discussed in this section. For the function with two parameters c_n and c_{n-1}, the two equations

$$\frac{\partial A}{\partial c_n}\Delta c_n+\frac{\partial A}{\partial c_{n-1}}\Delta c_{n-1}=1-A \qquad at\ X=X_2 \tag{3.19a}$$

$$\frac{\partial A}{\partial c_n}\Delta c_n+\frac{\partial A}{\partial c_{n-1}}\Delta c_{n-1}=10^{-A_p/20}-A \qquad at\ X=X_1 \tag{3.19b}$$

determine Δc_n, and Δc_{n-1} and hence c_n and c_{n-1} are obtained iteratively.

3.1.4 Design Examples

In this section three design examples are given for filters of degree 6, where the coefficients c_5 and c_6 of the continued fraction are varied for amplitude equalization. The cutoff frequencies are normalized ($\omega_c = 1$).

Example 1

In this example the group delay is maximally flat and the amplitude response is monotonic in the stopband, hence all B_j in (3.10) are zero. The application of the procedure in section 3.1.3 yields c_5=15.4 and c_6=1.89 for a passband loss of 2 dB. The poles of this filter are given in Table 1. Figure 3 shows the attenuation and group delay characteristics of this filter.

Example 2

In this example the group delay is equal ripple with a maximum variation of 1%. The procedure of section 3.1.3 yields the poles given in Table 1 for the all-pole transfer function. The parameters c_5 and c_6 are varied by 1.37 and 0.389, respectively for a passband loss of 1.75 dB. It was found that the relations between c_5, c_6, A_p and A_s can be expressed as

$$\gamma_5\gamma_6 \approx 0.533 \tag{3.20}$$

$$A_p \approx 1.812\gamma_5 - 0.717$$

$$A_s \approx -19.83\gamma_6 + 56.74$$

where

$$\gamma_5 = \frac{\hat{c}_5}{c_5}$$

$$\gamma_6 = \frac{\hat{c}_6}{c_6}$$

and $\hat{c}_5$ and $\hat{c}_6$ are the varied coefficients. The attenuation and group delay characteristics of this filter are shown in Fig. 4.

Example 3

The specifications for this example are the same as those for the Example 2, except for the attenuation in the stopband which is equal ripple with a minimum of 50 dB. The transfer function has one finite transmission zero and thus in (3.10) k is equal to 1. The procedure of section 3.1.3 yields the poles and zeros given in Table 1. Figure 5 shows the attenuation and group delay characteristics of this filter. An examination of Figures 3, 4, and 5 shows that the filter with equal ripple group delay has a lower passband loss for the same stopband attenuation. Thus the filter with equal ripple group delay is more selective than the one with the maximally flat group delay. The filter with a finite transmission zero is more selective than the all-pole filter, but at the expense of a reduction in the group delay flatness.

Table 1

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros
1	-0.092139 ± j 0.966218 -0.492652 ± j 0.560940 -0.637454 ± j 0.182176	
2	-0.097674 ± j 0.973229 -0.485479 ± j 0.674001 -0.491990 ± j 0.249604	
3	-0.094963 ± j 0.976968 -0.558355 ± j 0.286728 -0.593187 ± j 0.778291	0. ± j 1.705190

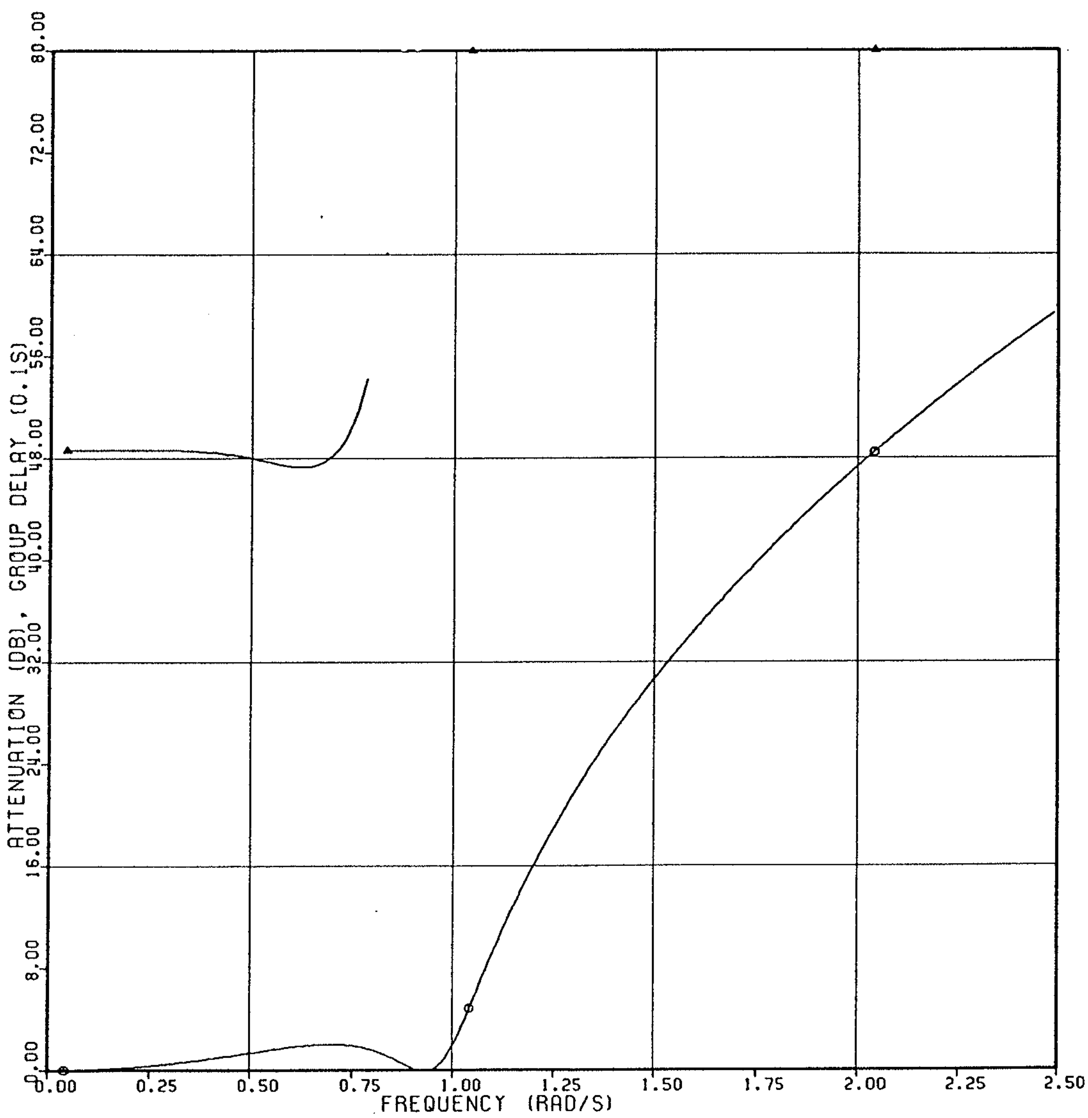

Fig. 3 Attenuation (o), and group delay (Δ) characteristics of the filter in Example 1.

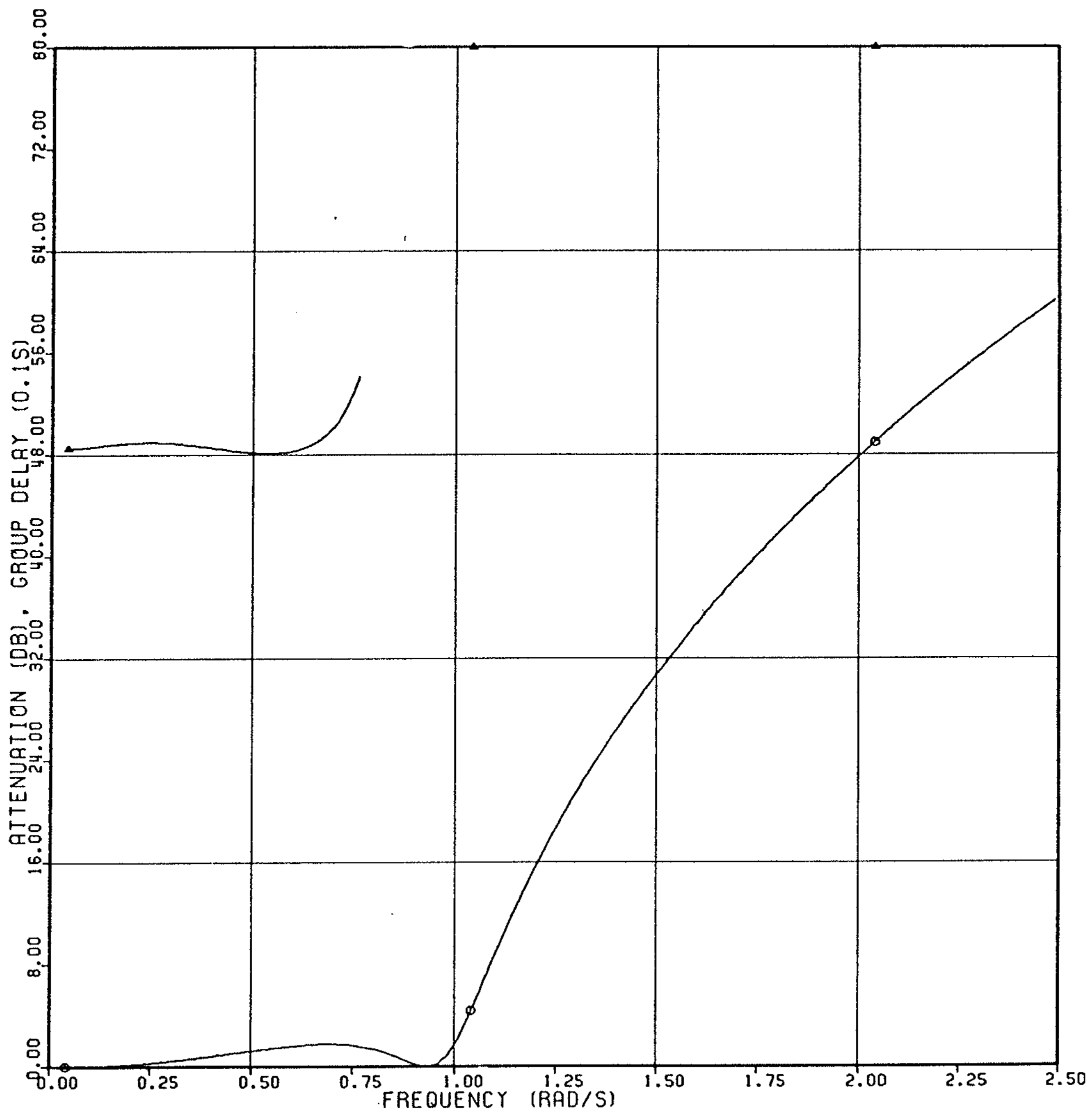

Fig. 4 Attenuation (o), and group delay (Δ) characteristics of the filter in Example 2.

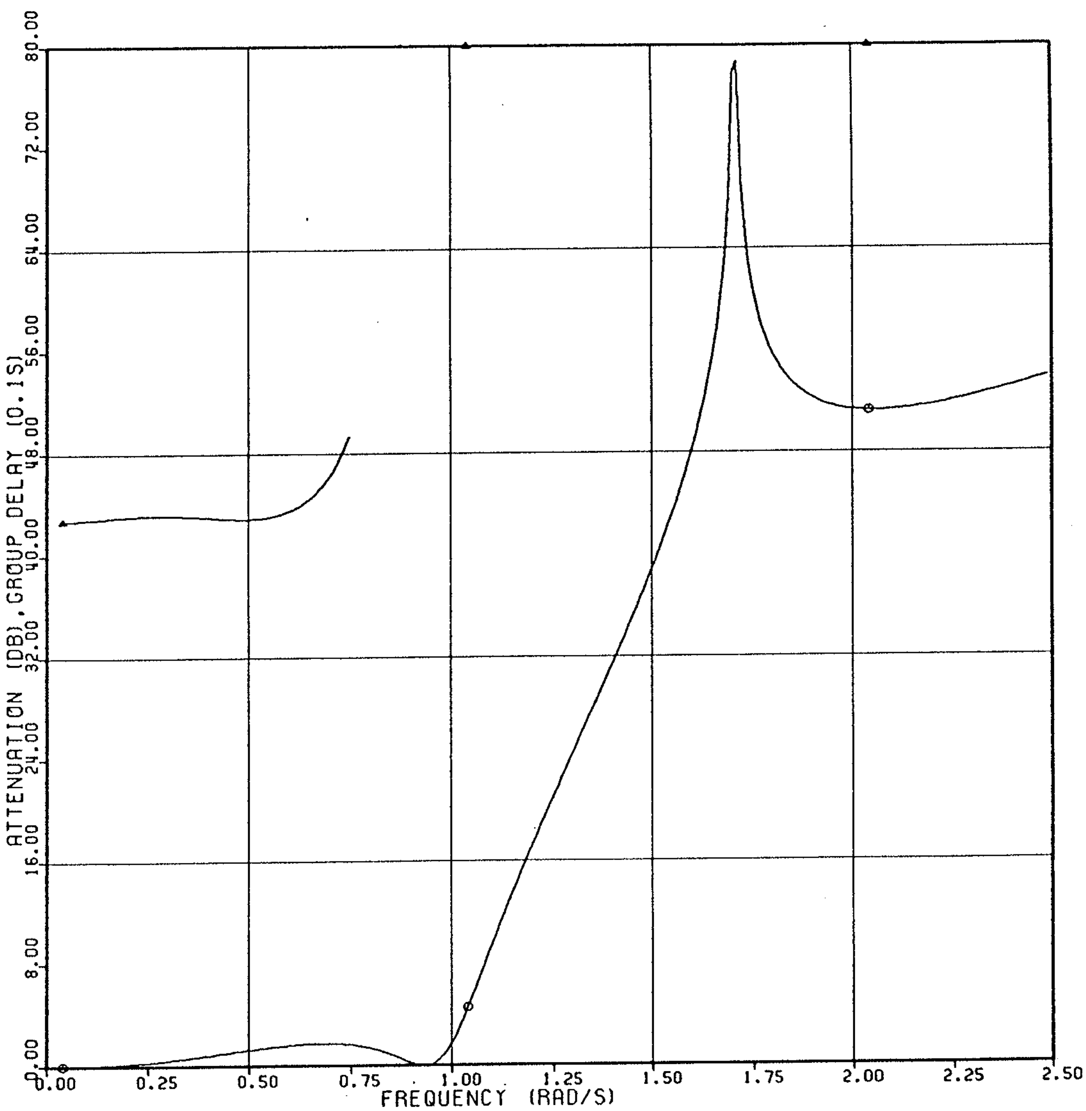

Fig. 5 Attenuation (o), and group delay (Δ) characteristics of the filter in Example 3.

3.2 Bandpass Filters

Bandpass filters with the exact maximally flat or equal ripple group delay about a center frequency, can be obtained numerically [10,16,27,44] using the iterative methods discussed in chapter II. These approximations utilize all the degrees of freedom for the group delay equalization and as a result the amplitude characteristics have low selectivities. Linear phase bandpass filters may be obtained through the transformations of the lowpass filters derived in section 3.1 [45]. The transformation is required to preserve the group delay characteristics of the lowpass filter about the center frequency ω_0. Let the poles of the lowpass filter given in (3.2) be denoted by s_i; for the required transformation the poles of the bandpass filter may be obtained as [46]

$$p_i = s_i + j\,\omega_0 \qquad i = 1,n \tag{3.21}$$

$$p_{i+n} = p_i^*$$

where $|s_i| << \omega_0$. The resulting bandpass filter is of degree $2n$ and for $\omega_c << \omega_0$, the group delay in the passband is approximately symmetrical in the arithmetical sense, i.e.

$$\tau(\omega_0+\Delta\omega) \approx \tau(\omega_0-\Delta\omega) \qquad \Delta\omega < \omega_c \tag{3.22}$$

In fact the group delay is approximately equal to the lowpass group delay τ_l evaluated at the frequency $\omega-\omega_0$, i.e.

$$\tau(\omega) \approx \tau_l(\omega-\omega_0) \tag{3.23}$$

The transfer function zeros are to be determined to complete the approximation.

3.2.1 Transmission Zeros at Zero or Infinity

The transfer function of this class of filters may be written as

$$H(s)=\frac{G_0 s^m}{\prod_{i=1}^{2n}(s-p_i)} \tag{3.24}$$

For an even n, the amplitude function can be made approximately symmetrical in the arithmetical sense if $m=\frac{n}{2}$ [10]. Thus the resulting filter has arithmetical symmetry about ω_0 in both the amplitude and the group delay characteristics.

3.2.2 Bandpass Filters with Finite Transmission Zeros

The transfer function of this class of filters can be written as

$$H(s)=\frac{s^m \prod_{j=1}^{2k}(1+\alpha_j^2 s^2)}{\prod_{i=1}^{2n}(s-p_i)} \qquad (4k+m \le 2n) \tag{3.25}$$

The amplitude characteristics can be made approximately symmetrical about ω_0 in the arithmetical sense if the lowpass filter has an even degree n and [43]

$$m=\frac{n}{2}-k \qquad (m \ge 0) \tag{3.26}$$

$$\alpha_j=\begin{cases} \dfrac{1}{\omega_0+\dfrac{1}{\beta_j}} & j=1,k \\[2ex] \dfrac{1}{\omega_0-\dfrac{1}{\beta_j}} & j=k+1,2k \end{cases} \tag{3.27}$$

3.2.3 Design Examples

Two design examples are given for the filters of degree 12 ($n=6$), with equal ripple group delay. The group delay variation is to be within 1% in the linear-phase band. The passband loss is to be 1.75 dB. The passband is to extend from 0.875 to 1.125 for a normalized center frequency ($\omega_0=1$). The specifications are the same as those for the Examples 2 and 3 of section 3.1.4, and hence the lowpass filters derived in those examples can be used. For the required bandwidth (0.25), the lowpass filters must be frequency scaled by $k_\omega=0.125$, which is equivalent to the multiplication of the lowpass poles and zeros by k_ω.

Example 4

In this example the transfer function zeros are to be located at either zero or infinity. Therefore the lowpass filter of Example 2 in section 3.1.4 is used. There are three zeros at zero ($m=\frac{n}{2}=3$).

Example 5

In this example the transfer function has two finite transmission zeros ($k=1$) and hence from (3.26), there are two zeros at zero ($m=2$).

The poles and zeros of the filters designed in Examples 4 and 5 are given in Table 2. The attenuation and group delay characteristics are shown in Fig. 6. An examination of Fig. 6 reveals the excellent aithmetical symmetry of both the group delay and the amplitude response with negligible errors in meeting the specifications. Resistively-terminated LC-network realizations of these filters are given in Fig. 7 with the element values given in Table 3.

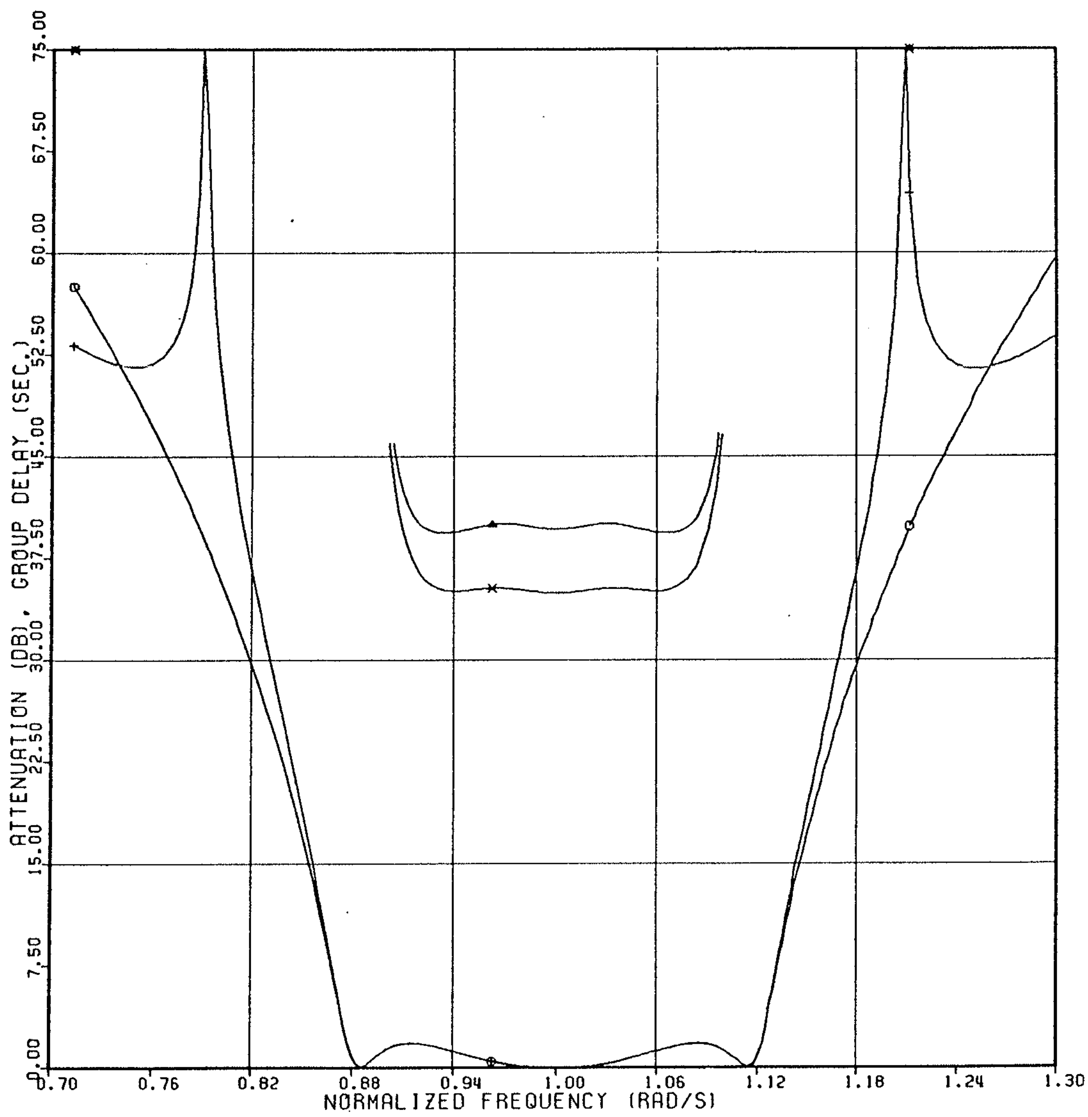

Fig. 6 Attenuation and group delay characteristics of the filters in Example 4 (o,Δ), and Example 5 (+,×).

Table 2

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros	G_0
4	-0.011963 ± j 1.119203	0. (tripple)	$4.45656\ 10^{-5}$
	-0.011963 ± j 0.880797		
	-0.059463 ± j 1.082553		
	-0.059463 ± j 0.917447		
	-0.060260 ± j 1.030572		
	-0.060260 ± j 0.969428		
5	-0.072731 ± j 1.095427	0. (double)	$4.68813\ 10^{-4}$
	-0.072731 ± j 0.904574	0. ± j 1.209075	
	-0.011644 ± j 1.119787	0. ± j 0.790926	
	-0.011644 ± j 0.880213		
	-0.068461 ± j 1.035156		
	-0.068461 ± j 0.964844		

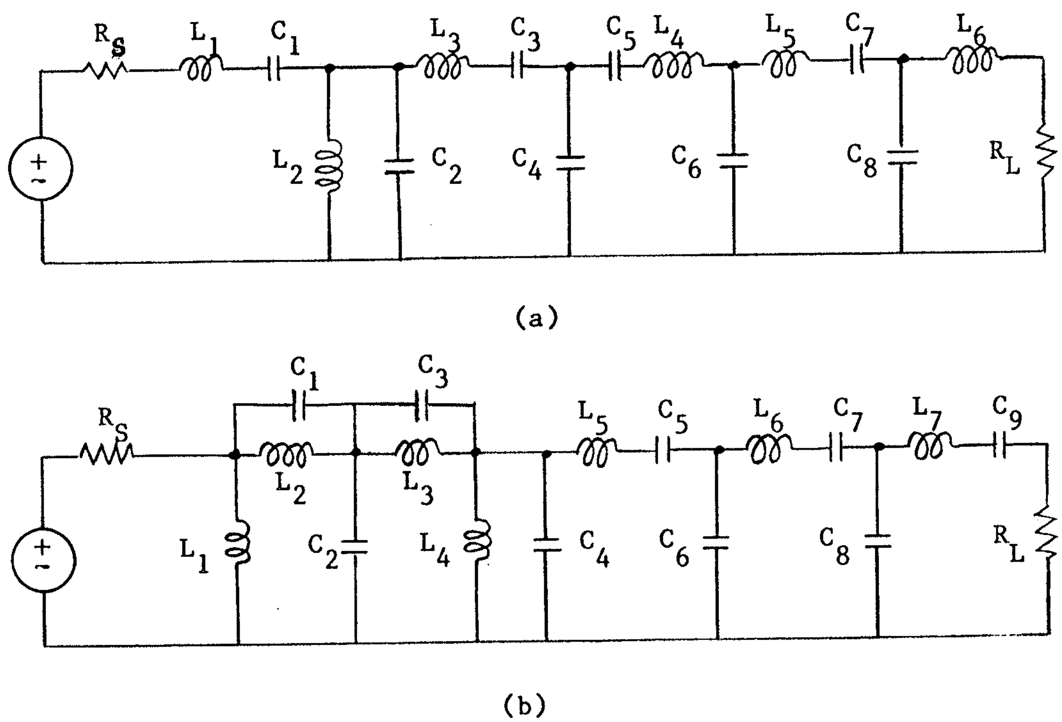

Fig. 7 Network realizations for filters in Table 2, (a) Example 4, (b) Example 5.

Table 3

Element Values for the Networks in Fig. 7

Network	$R_s=R_l=1\Omega$,	L (Henry), C (Farad)
(a)	L_1=10.1437	C_1=0.099732
	L_2=0.197539	C_2=5.19061
	L_3=8.06247	C_3=0.142713
	L_4=10.0498	C_4=0.090315
	L_5=11.3208	C_5=0.646322
	L_6=0.765927	C_6=0.646322
		C_7=0.106951
		C_8=1.22630
(b)	L_1=1.74534	C_1=2.15689
	L_2=0.741145	C_2=5.19061
	L_3=0.081829	C_3=8.35960
	L_4=0.027876	C_4=34.6597
	L_5=1.93535	C_5=0.612585
	L_6=1.71248	C_6=3.37524
	L_7=1.96728	C_7=1.11981
		C_8=1.65974
		C_9=0.64717

CHAPTER IV

NONMINIMUM-PHASE FILTERS

The design of nonminimum-phase filters is considered in this chapter. The resulting filters may be further optimized using the procedure of chapter II. The amplitude and group delay characteristics for this class of filters are less dependent on each other than the minimum-phase filters and as a result the amplitude equalization can be accomplished relatively independent from the group delay characteristics. The group delay which is determined by the denominator polynomial can be maximally flat or equal ripple in the passband. The linear phase bandwidth can cover the entire or a part of the passband. The amplitude response is either monotonic or has a prescribed number of maxima of desired amplitudes in the stopband. The amplitude response in the passband is either maximally flat or has a prescribed number of minima with the desired amplitudes.

4.1 Lowpass Filters

The transfer function of this class of filters can be written as

$$H(s)=\frac{\sum_{j=0}^{k} b_j s^{2j}}{1+\sum_{i=1}^{n} a_i s^i} \tag{4.1}$$

There are $n+k+1$ degrees of freedom which can be distributed between the group delay and amplitude characteristics. The denominator polynomial which determines the group delay can be decoupled from the numerator polynomial in the process of approximation. Since the group delay and amplitude characteristics are not independent, this may result in some constraints on the amplitude extrema in the passband. The denominator polynomial is of degree n and hence there are n degrees of freedom for the group delay equalization. For minimum-phase filters a reduction in the degrees of flatness resulted in a sharp increase in amplitude selectivity. The free parameters were evaluated from the amplitude specifications. For nonminimum-phase filters with the separation of amplitude and group delay equalizations, the free parameters can be evaluated only from the group delay specifications.

4.1.1 Determination of the Denominator Polynomial

The denominator polynomial $D(s)$ is obtained in a procedure similar to that used for minimum-phase filters. Let the continued fraction expansion for $D(s)$ be written as

$$\frac{sF(s^2)}{E(s^2)} = \cfrac{s}{c_1 + \cfrac{s^2}{c_2 + \cfrac{\cdot}{\cfrac{\cdot}{\cfrac{s^2}{c_n}}}}} \tag{4.2}$$

where for a maximally flat group delay $c_i = 2i - 1$. A polynomial with equal ripple group delay can be obtained using the iterative procedure of section 2.3.2.B of chapter II, or from other sources [11,12]. The continued fraction expansion for this polynomial determines the coefficients c_i. As for the minimum-phase filters, the coefficients c_n and c_{n-1} control the group delay flatness. For minimum-phase filters these coefficients were evaluated for an equal ripple amplitude response in the passband. For nonminimum-phase filters they are evaluated from the group delay specifications, but since they also control the amplitude response, the evaluation is iterative. For any set of positive coefficients, the denominator polynomial remains Hurwitz and is obtained from (4.2) or from the recursion formula

$$D_{m+1}(s) = D_m(s) + \frac{s^2}{c_m c_{m+1}} D_{m-1}(s) \qquad m = 1, n-1 \tag{4.3}$$

where $D_0(s)=1$, $D_1(s)=1+s$.

4.1.2 Determination of the Numerator Polynomial

The numerator polynomial is obtained using an iterative procedure similar to that of section 2.3.1 in chapter II. For the equal-ripple approximation, the extrema of the amplitude response in the passband and the stopband are equalized iteratively.

Let the amplitude function $A(\omega^2)$ be written as

$$A(\omega^2)=|H(j\omega)|=\frac{B(\omega^2)}{\sqrt{G(\omega^2)}} \tag{4.4}$$

where

$$B(\omega^2)=B_0+B_1\omega^2+....+B_k\omega^{2k} \tag{4.5}$$

$$G(\omega^2)=|D(j\omega)|^2=1+G_1\omega^2+.....+G_n\omega^{2n}$$

There are $k+1$ degrees of freedom (B_j) for the amplitude equalization.

Maximally Flat Approximation

For a maximally flat amplitude response with n_f degrees of flatness

$$\frac{d^iA(\omega^2)}{d(\omega^2)^i}\Big|_{\omega=0} \qquad i=1,n_f \tag{4.6}$$

Hence from (4.4) and (4.6)

$$\frac{1}{B_0^2}\sum_{l=0}^{i}\sum_{j=0}^{i}B_lB_j=G_i \qquad i=1,n_f \tag{4.7}$$

The solution of the set of equations in (4.7) determines the coefficients B_1 to B_{n_f}

for an arbitrary value of B_0. For zero attenuation at $\omega=0$, $B_0=1$. The equations are solved readily, since B_1 is determined from the first equation ($i=1$), B_2 is determined from the second equation (i=2), etc.

For a maximally flat amplitude response the remaining degrees of freedom $(k-n_f)$ are used for the stopband equalization. The filter is designed for $k-n_f$ stopband maxima. Let the stopband maxima be denoted by A_i, where for the minima of the stopband attenuation A_{s_i}

$$A_i = 10^{-A_{s_i}/20} \qquad i=1,k-n_f \tag{4.8}$$

From (4.4), (4.7), and (4.8)

$$\sum_{j=n_f+1}^{k} B_j X_i^j = (-1)^i A_i \sqrt{G(X_i)} - \sum_{j=0}^{n_f} B_j X_i^j \qquad i=1,k-n_f \tag{4.9}$$

where X_i are the abscissae of the extrema of the amplitude function $A(X)$, $(X=\omega^2)$, ordered such that, $X_{i+1}>X_i$. The abscissae are obtained from the positive real zeros of the equation

$$2G(X)dB(X)/dX - B(X)dG(X)/dX = 0 \tag{4.10}$$

The iterative procedure begins with the initial guess of X_i constrained by $X_{i+1}>X_i$. The solution of (4.9) determines a set of coefficients (B_{n_f+1} to B_k) which determine a new set of X_i from (4.10). The iteration continues until a desired accuracy is reached.

General-Passband, General-Stopband Approximation

Let the extrema of the amplitude function in the passband and the maxima of the amplitude function in the stopband be denoted by A_i where

$$A_i = 10^{-A_{p_i}/20} \qquad i = 1, n_p \tag{4.11}$$

$$A_i = 10^{-A_{s_i}/20} \qquad i = n_p + 1, k + 1$$

The amplitude response is assumed to have n_p extrema in the passband and $k+1-n_p$ maxima in the stopband. The extrema of the passband attenuation are denoted by A_{p_i}. From (4.4) and (4.11)

$$\sum_{j=0}^{k} B_j X_i^j = \alpha_i A_i \sqrt{G(X_i)} \qquad i = 1, k+1 \tag{4.12}$$

where

$$\alpha_i = \begin{cases} 1 & i = 1, n_p \\ (-1)^{i-n_p} & i > n_p \end{cases}$$

The X_i are the abscissae of the extrema of the amplitude function $A(X)$ ordered such that $X_{i+1} > X_i$. The X_i are obtained from the positive real zeros of equation (4.10). As for the maximally flat case, the iterative solution of equations (4.10) and (4.12) determines the coefficients B_j. The numerator polynomial is then obtained as

$$N(s^2) = \sum_{j=0}^{k} (-1)^j B_j s^{2j} \tag{4.13}$$

For equal ripple group delay and small values of A_{p_i}, the number of amplitude extrema may exceed $k+1$ as a result of the solution of equation (4.10). In this case the iterative procedure equalizes some of the extrema in the passband and hence the rest of the extrema are unconstrained. The coefficients B_j are obtained from

$$\sum_{j=0}^{k} B_j X_i^j = \alpha_i A_i \sqrt{G(X_i)} \qquad i = n_t - k, n_t \tag{4.14}$$

where n_p of the total n_t extrema are equalized. Although the number of abscissae is larger than the number of equations, the procedure is convergent due to the specified group delay and the dependence of amplitude and group delay characteristics. For equal ripple passband, equal ripple stopband

$$A_{s_i} = A_s \tag{4.15}$$

$$A_{p_i} = \begin{cases} 0 & i - n_p = \text{even integer} \\ A_p & \text{otherwise} \end{cases}$$

The cutoff frequency (ω_c) of the designed filter is determined from the equation

$$B^2(X) - 10^{-A_p/20} G(X) = 0 \tag{4.16}$$

Let the largest positive root be denoted by X_m, then $\omega_c = \sqrt{X_m}$. For a desired cutoff frequency $\hat{\omega}_c$, the complex frequency s is scaled by $k_\omega = \frac{\omega_c}{\hat{\omega}_c}$.

4.2 Bandpass Filters

The even degree transfer function can be written as

$$H(s) = \frac{s^m \sum_{j=0}^{k} b_j s^{2j}}{1 + \sum_{i=1}^{2n} a_i s^i} \qquad (2k + m \leq 2n) \tag{4.17}$$

As for the minimum-phase bandpass filters the denominator polynomial can be obtained numerically [10,27] for the exact maximally flat or equal ripple group delay characteristics. The denominator polynomial can also be obtained through the transformation of lowpass poles as discussed in section 3.2 of chapter III. The latter provides excess degrees of freedom which control the group delay flatness and the amplitude selectivity. For a set of coefficients c_i, the polynomial D(s) is obtained from (4.2) or (4.3). Let the zeros of this polynomial (lowpass poles) be denoted by s_i, the poles of the bandpass filter are obtained as

$$p_i = \gamma s_i + j\omega_0 \qquad i = 1, n \tag{4.18}$$

$$p_{i+n} = p_i^*$$

where ω_0 is the center frequency and γ is the scale factor. The parameter γ controls the passband and is evaluated iteratively for the desired bandwidth. For the set of coefficients c_i and the parameter γ, the denominator polynomial of the bandpass filter is obtained from

$$D_B(s) = \sum_{i=1}^{2n} (1 - s/p_i) \tag{4.19}$$

The numerator polynomial is obtained iteratively using a procedure similar to that of section 4.1.2. The amplitude response can be maximally flat or have extrema of desired amplitudes in the passband. For an approximately arithmetically symmetrical amplitude response, the number of zeros at $\omega=0$ must satisfy

$$m = \frac{n}{2} - k_z \tag{4.20}$$

where n is assumed to be an even integer and k_z is the number of finite transmission zero pairs. Let the amplitude function be written as

$$A(\omega) = |H(j\omega)| = \frac{\omega^m B(\omega^2)}{\sqrt{G(\omega^2)}} \tag{4.21}$$

where

$$B(\omega^2) = \sum_{j=0}^{k} B_j \omega^{2j} \tag{4.22}$$

$$G(\omega^2) = |D_B(j\omega)|^2 = 1 + G_1\omega^2 + \dots + G_{2n}\omega^{4n}$$

Maximally Flat Approximation

For a maximally flat amplitude response about ω_0

$$\frac{d^i A(\omega)}{d\omega^i}\Big|_{\omega_0} \qquad i=1,n_f \tag{4.23}$$

where n_f is the degree of flatness. Let the polynomial $G(X)$ be expanded about $X_0=\omega_0^2$ as

$$G(X)=\sum_{i=0}^{2n}\tilde{G}_i(X-X_0)^i \tag{4.24}$$

For a maximally flat amplitude response

$$\tilde{B}_i=\tilde{G}_i \qquad i=0,n_f \tag{4.25}$$

where

$$X^m B^2(X)=\sum_{j=0}^{2k+m}\tilde{B}_j(X-X_0)^j \tag{4.26}$$

Thus

$$B(X_0)=(X_0^{-m}\tilde{B}_0)^{\frac{1}{2}} \tag{4.27}$$

$$B^{(1)}(X_0)=\frac{1}{2}(-mX_0^{-m-1}\tilde{B}_0+X_0^{-m}\tilde{B}_1)^{-\frac{1}{2}}$$

$$B^{(n_f)}(X_0)=d^{n_f}[X_0^{-m}\sum_{j=0}^{2k+m}\tilde{B}_j(X-X_0)^j]/dX^{n_f}\,|_{X=X_0}$$

There are n_f+1 equations in (4.27) resulting from the imposition of the maximally flat requirement. The other $k-n_f$ equations are derived for the required stopband characteristics.

$$B(X_i)=(-1)^i A_i \sqrt{X_i^{-m} G(X_i)} \qquad i==-k_z, k_z \tag{4.28}$$

where

$$A_i = 10^{-A_s/20}$$

$$X_i = \omega_i^2$$

The abscissae are determined from the positive real zeros of the equation

$$[mB(X)+2XB\frac{dB(X)}{dX}]G(X)-XB(X)\frac{dG(X)}{dX}=0 \tag{4.29}$$

The X_i are ordered such that

$$X_0 = \omega_0^2 \tag{4.30}$$

$$X_{i+1} > X_i$$

For $k-n_f=2k_z$, the equations in (4.27), (4.28) and (4.29) can be solved for B_j and X_i iteratively until a desired accuracy is reached.

Equal Ripple Approximation

For an equal ripple passband with n_p extrema, the equations in (4.28) may be extended to the passband and hence

$$B(X_i)=\alpha_i A_i \sqrt{X_i^{-m} G(X_i)} \qquad i=-k_z-\frac{n_p-1}{2}, k_z+\frac{n_p-1}{2} \tag{4.31}$$

where

$$A_i = \begin{cases} 10^{-A_{p_i}/20} & -\frac{n_p-1}{2} \le i \le \frac{n_p-1}{2} \\ 10^{-A_s/20} & otherwise \end{cases} \tag{4.32}$$

$$A_{p_i}=\begin{cases}1 & i \ even \\ A_p & otherwise\end{cases} \tag{4.33}$$

The abscissae of the extrema are determined from the equation (4.29) and satisfy the relations (4.30). For $n_p+2k_z=k+1$, the equations in (4.31) and (4.29) can be solved iteratively for B_j and X_i. For a set of coefficients B_j, the numerator polynomial is obtained as

$$N(s)=s^m \sum_{j=0}^{k}(-1)^j B_j s^{2j} \tag{4.34}$$

The cutoff frequencies of the resulting filter are obtained from the zeros of the equation

$$X^m B^2(X)-10^{-A_p/20}G(X)=0 \tag{4.35}$$

Let the smallest and the largest positive real zeros of the equation be denoted by X_1 and X_2, respectively. The lower (ω_l) and higher (ω_h) cutoff frequencies are obtained as

$$\omega_l=\sqrt{X_1} \tag{4.36}$$

$$\omega_h=\sqrt{X_2}$$

For small bandwidths $(\omega_h-\omega_l << \omega_0)$, arithmetical symmetry yields

$$\omega_l+\omega_h \approx 2\omega_0 \tag{4.37}$$

and thus the parameter γ in (4.18) can be used for the equalization of the cutoff frequencies. Let the bandwidth of the filter obtained for γ^0 after the first iteration be denoted by BW^0. For the desired bandwidth BW, the parameter γ is predicted to be

$$\gamma^1=\gamma^0\frac{BW}{BW^0} \tag{4.38}$$

The iterations continue until a desired accuracy is reached.

4.3 Design Examples for Lowpass Filters

Three sets of design examples are given for lowpass filters of degree 10 and with a numerator polynomial of degree 8. The minimum attenuation in the stop-band is required to be 63 dB. The filter transfer function contains two finite transmission zeros. The 3 dB frequencies are normalized.

Example 6

In this example the group delay and the amplitude characteristics are maximally flat. There are $n_f=2$ degrees of flatness for the amplitude response. In the continued fraction only parameter c_n is varied. Figure 8 shows the attenuation and group delay characteristics of the filters designed for three values of $\hat{c}_n$. The poles and zeros of the filter with $\hat{c}_{10}=9.5$ are given in Table 4.

Example 7

In this example the group delay is maximally flat but the amplitude response has equal ripples with $n_p=3$ extrema. The maximum attenuation in the passband (A_p) is 0.3 dB. Figure 9 shows the attenuation and group delay characteristics of the designed filters for three values of $\hat{c}_{10}$.

Example 8

In this example the group delay and the amplitude response have equal ripple characteristics. The group delay variation in the linear phase band is 0.36%. The amplitude response has three extrema in the passband with A_p=0.3 dB. Figure 10 shows the attenuation and group delay characteristics of the designed filters for three values of $\hat{c}_{10}$. the poles and zeros of the filter with $\frac{\hat{c}_{10}}{c_{10}}$=0.8 are given in Table 4.

Table 4

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros
6	-0.184136 ± j 1.092033	± 0.790258 ± j 0.340237
	-0.750079 ± j 0.331815	0. ± j 2.650997
	-0.793301 ± j 0.110136	0. ± j 1.934162
	-0.657444 ± j 0.558550	
	-0.496774 ± j 0.798207	
8	-0.343040 ± j 1.051255	± 0.523902 ± j 0.362463
	-0.125730 ± j 0.983807	0. ± j 2.090293
	-0.305015 ± j 0.695840	0. ± j 1.606801
	-0.316315 ± j 0.423682	
	-0.321047 ± j 0.141876	

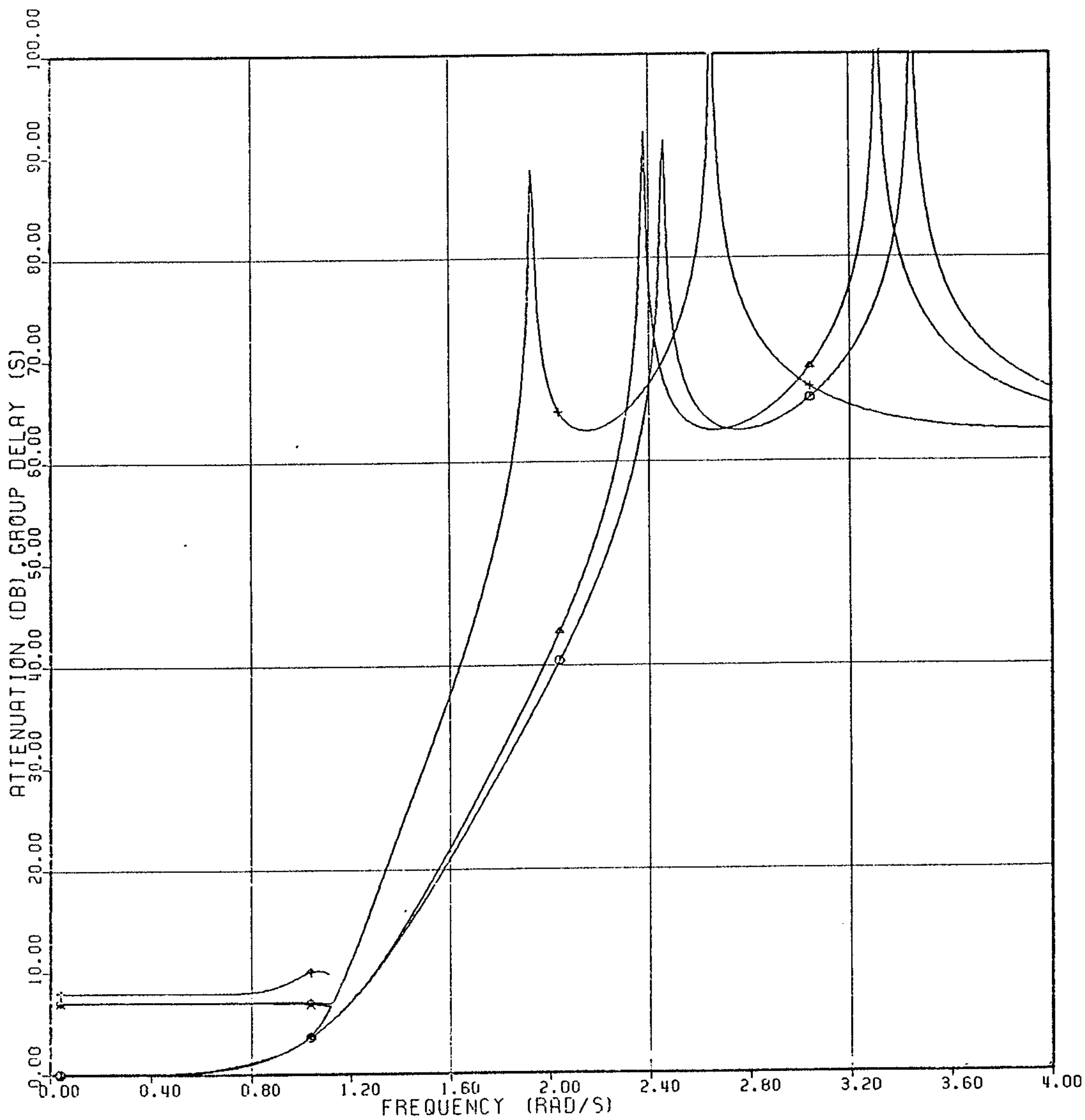

Fig. 8 Attenuation and group delay characteristics of the filters in Example 6 for $\hat{c}_n = 1$ (o,×), 0.86 (Δ,◇) and 0.5 (+,↑).

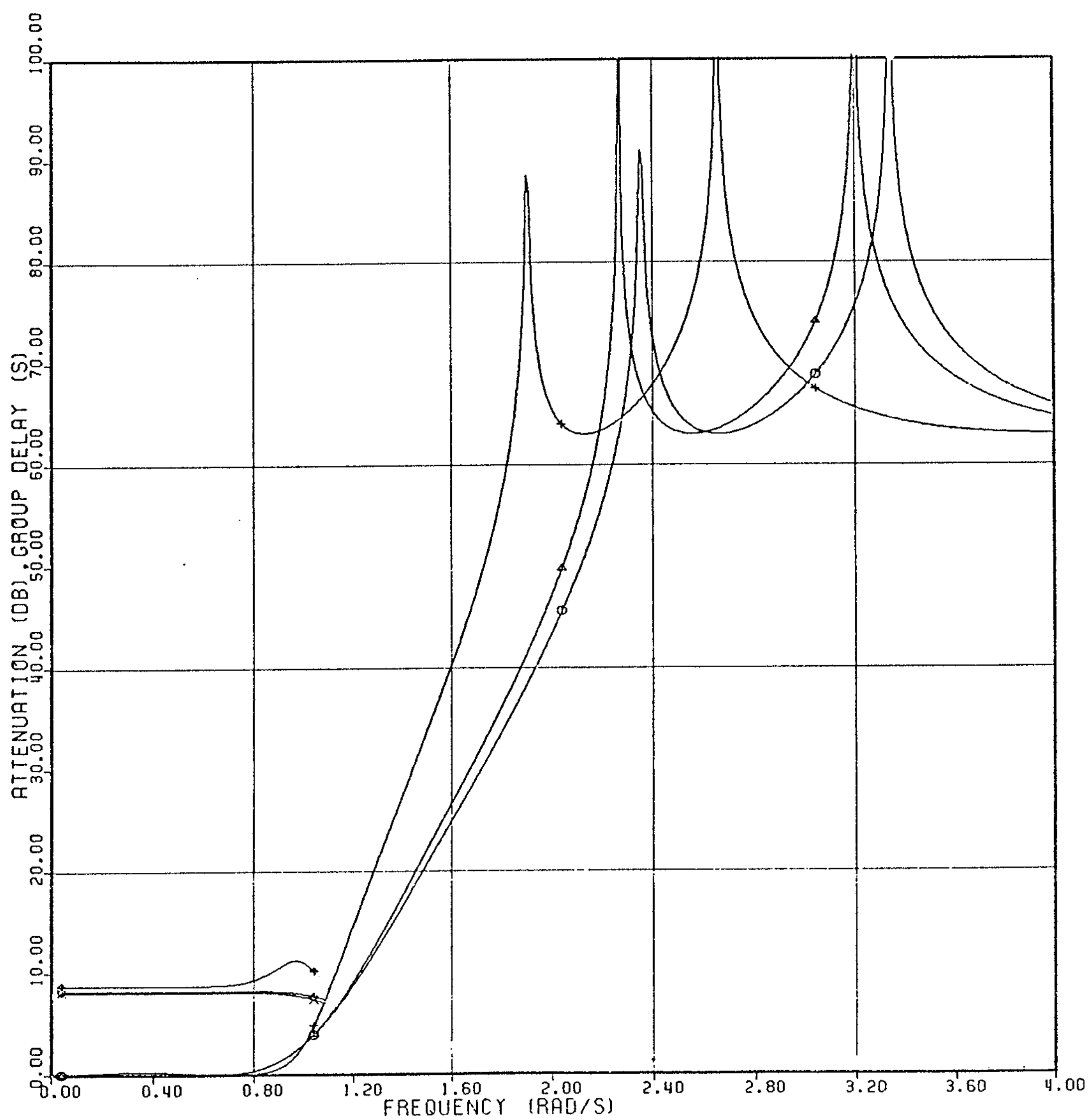

Fig. 9 Attenuation and group delay characteristics of the filters in Example 7 for $\hat{c}_n$ =1 (o,×), 0.86 (Δ,◇) and 0.5 (+,♤).

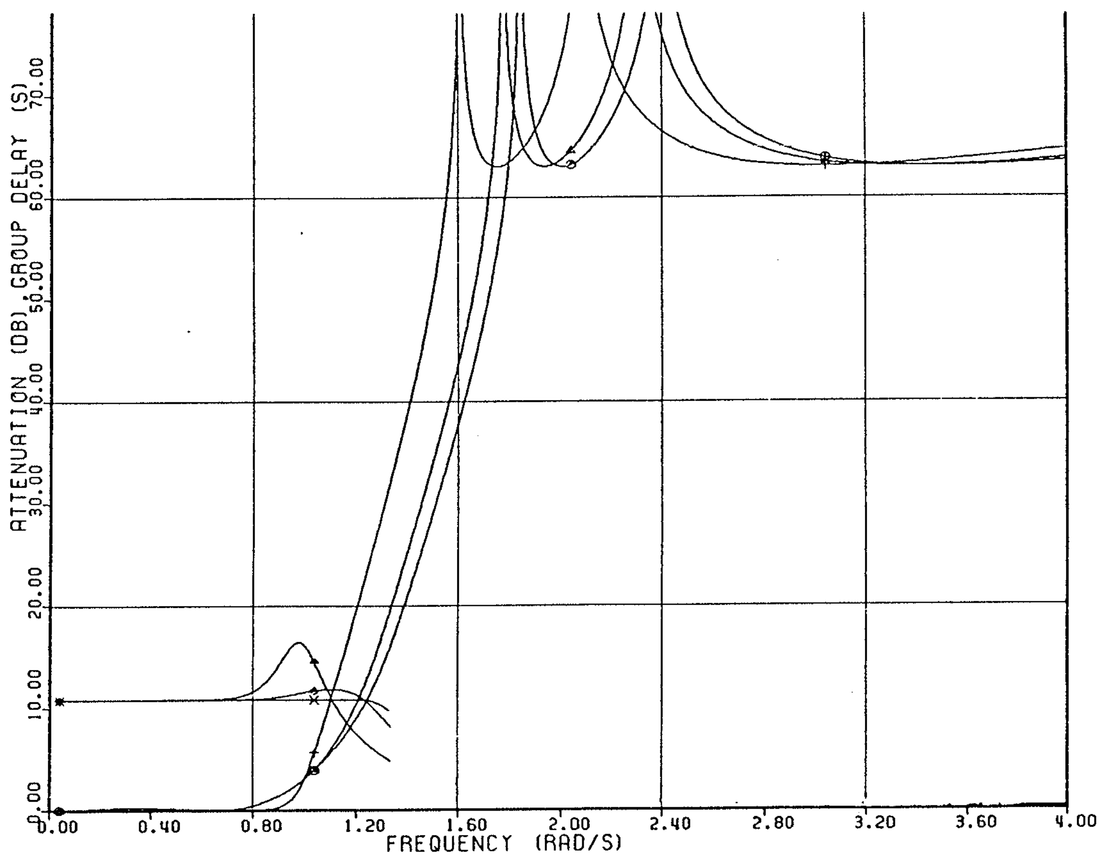

Fig. 10 Attenuation and group delay characteristics of the filters in Example 8 for $\hat{c}_n = 1$ (o,×), 0.95 (Δ,◊) and 0.8 (+,↑).

4.4 Design Examples for Bandpass Filters

Two design examples are given for bandpass filters of degree 12 and with numerator polynomials of degree 10. The filter transfer function contains one pair of finite transmission zeros ($k_z = 1$). Two zeros are located at zero for an approximate amplitude symmetry. The center frequency is normalized and the passband extends from 0.9 to 1.1 with 3 dB attenuations at the cutoff frequencies.

Example 9

In this example the group delay and the amplitude characteristics are maximally flat. the minimum stopband attenuations are 45 dB and 40 dB in the lower and upper stopbands, respectively. Figure 11 shows the attenuation and the group delay characteristics of the designed filters for two values of $\hat{c}_6$. The poles and zeros of the filter with $\hat{c}_6 = 5.5$ are given in Table 5.

Example 10

In this example the group delay and amplitude characteristics have equal ripples. The amplitude response has three extrema in the passband with maximum attenuation of 1 dB. The minimum attenuation in the stopbands is 40 dB. The group delay variation in the linear-phase band is 0.36%. Figure 12 shows the attenuation and group delay characteristics of the designed filters for two values of $\hat{c}_6$.

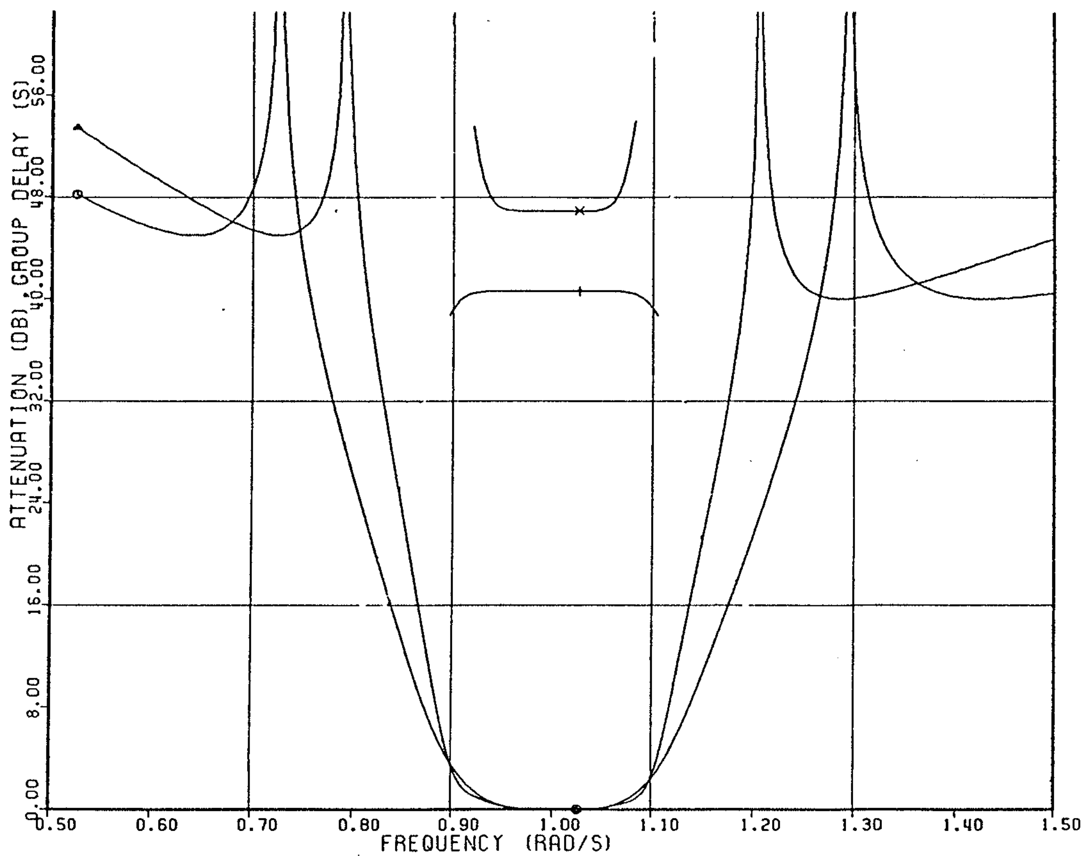

Fig. 11 Attenuation and group delay characteristics of the filters in Example 9 for

$\hat{c}_n = 1$ (o,+), 0.5 (Δ,$\times$).

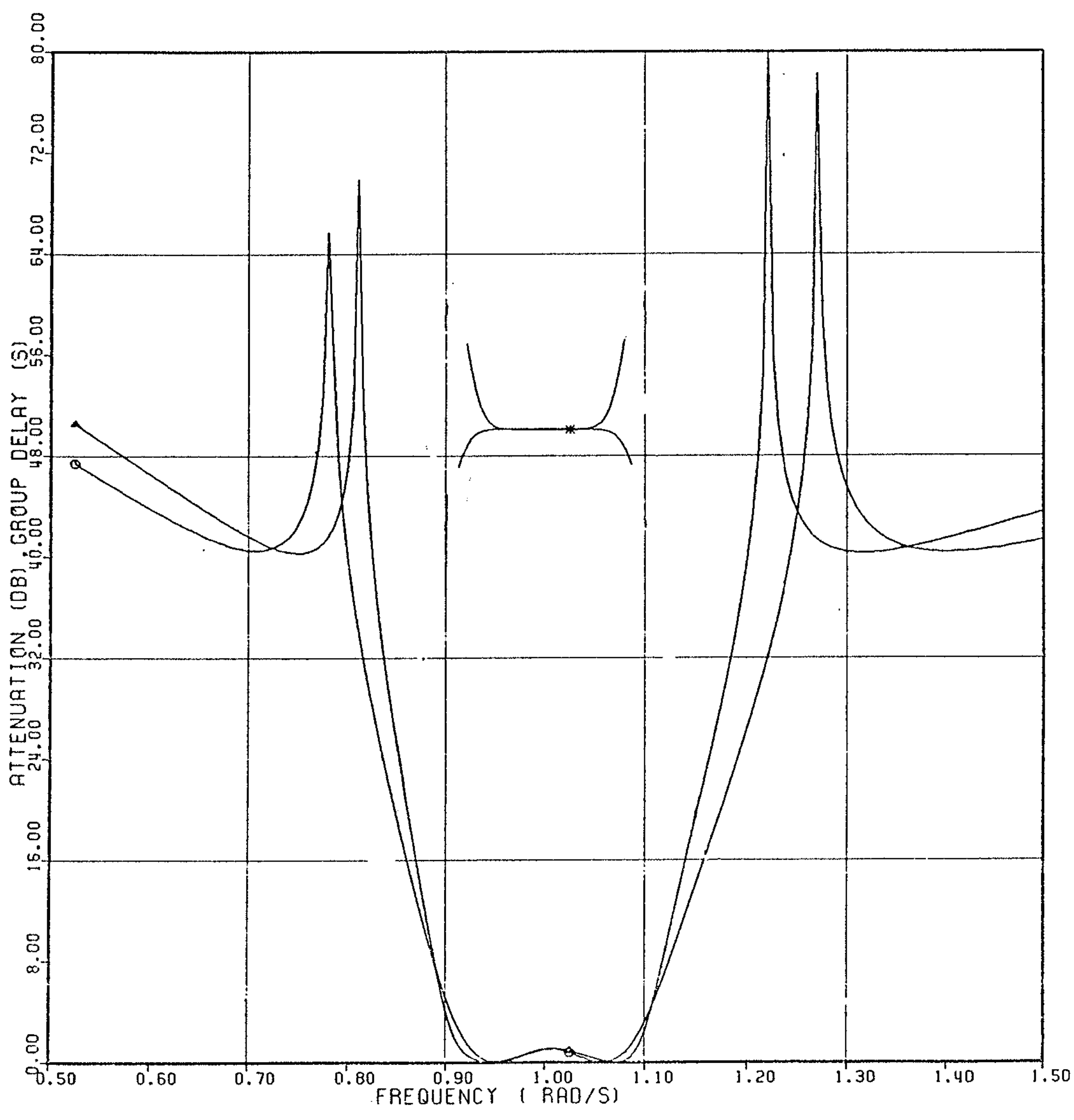

Fig. 12 Attenuation and group delay characteristics of the filters in Example 10 for $\hat{c}_n$=1 (o,+), 0.8 (Δ,$\times$).

Table 5

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros
9	-0.025638 ± j 1.101533	0. (double)
	-0.025638 ± j 0.898467	0. ± j 1.205760
	-0.076577 ± j 1.018635	0. ± j 0.791443
	-0.076577 ± j 0.981365	± 0.098869 ± j 1.004610
	-0.063228 ± j 1.056778	
	-0.063228 ± j 0.943222	

4.5 Design Example for a Lowpass Filter with Unconstrained Passband Extrema

As mentioned in section 4.1.2, for small A_p the number of amplitude extrema may exceed the number of coefficients B_j for the equal ripple group delay characteristic. In this case only some of the extrema in the passband are equalized in the approximation procedure.

Example 11

It is required to design a lowpass filter of degree 10 with a numerator polynomial of degree 6, and with one transmission zero. The maximum passband loss and the minimum stopband loss are required to 0.06 dB and 60 dB, respectively. The group delay variation in the linear-phase band is to be less than 0.47%. The application of the iterative procedure of section 4.1.2 with $\frac{\hat{c}_{10}}{c_{10}}=0.79$ resulted in five extrema in the passband. The last three extrema in the passband ($i=4,5,6$) were equalized. It was found that the unconstrained extrema were within the tolerance range $(1,10^{-A_p/20})$. The attenuation and group delay characteristics of this filter are shown in Fig. 13. The poles and zeros are given in Table 6.

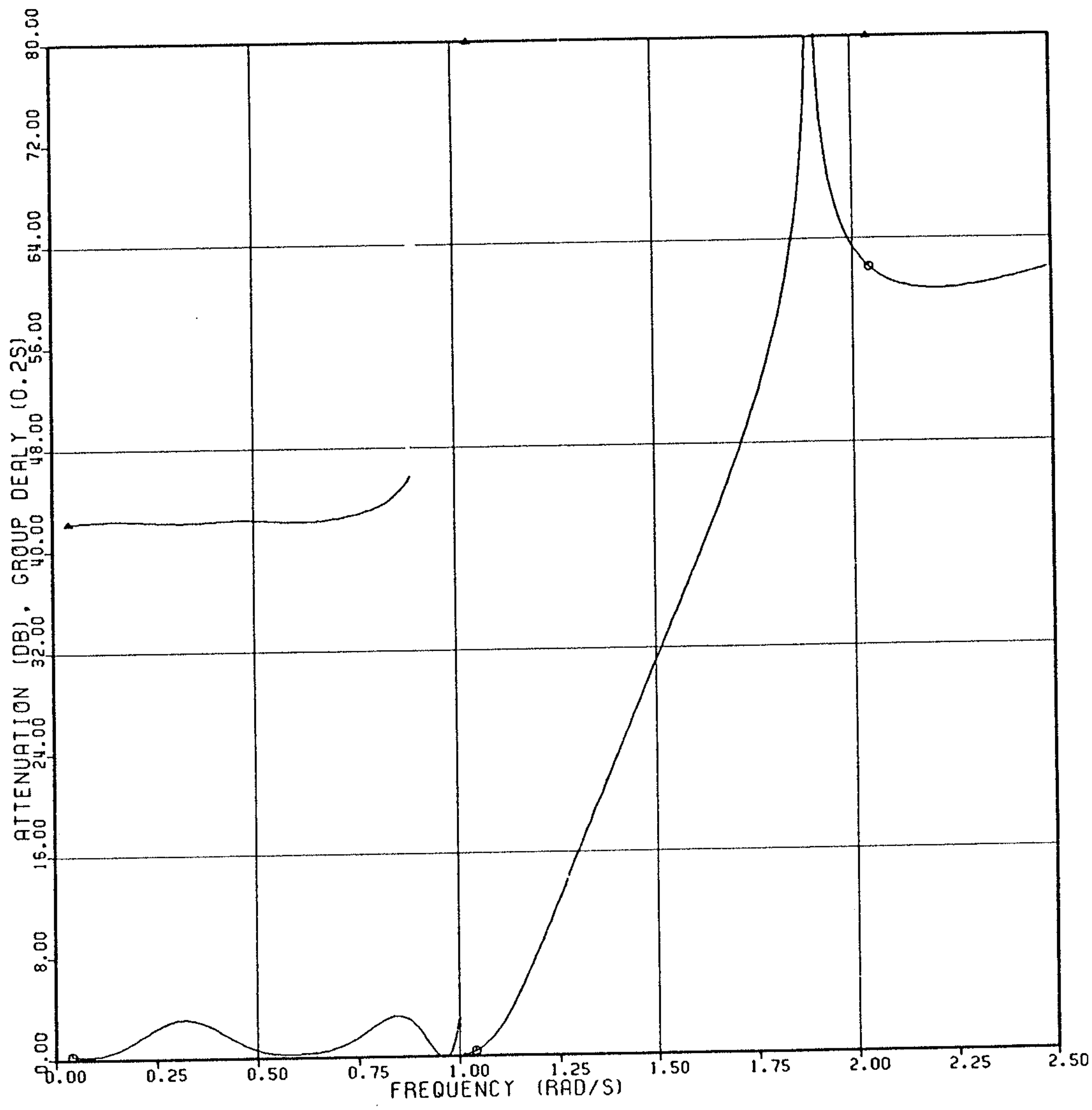

Fig. 13 Attenuation (o) and group delay (Δ) characteristics of the filter in Example 11. The attenuation in the passband is magnified by 100.

Table 6

Poles and Zeros of the Designed Filter

Design Example	Poles	Zeros
11	-0.375980 ± j 1.176061	0. ± j 1.896818
	-0.133807 ± j 1.085363	± 0.706306 ± j 0.393221
	-0.330659 ± j 0.775132	
	-0.341860 ± j 0.471808	
	-0.346774 ± j 0.157975	

CHAPTER V

DIGITAL FILTERS

The approximation procedure for digital, analog switched, and interdigital commensurate transmission line filters is similar to the procedure of chapter II. The main difference is due to the digital complex frequency z: the digital filter characteristics are specified for the real frequency Ω and hence the filter transfer function is evaluated for $z=e^{j\Omega T}$, unlike analog filters for which $s=j\omega$. The resulting amplitude and group delay characteristics are functions of the variable $\cos\Omega T$.

The approximation methods for digital filters are catagorized as FIR (finite impulse response) and IIR (infinite impulse response) methods. The FIR filters with exact linear phase are nonoptimum and the degree of the filter is very large for a selective amplitude response. The design of FIR filters may be regarded as a special case of the design of IIR filters (with a constant denominator). The IIR filters with a symmetric numerator polynomial may be separated into FIR and IIR (zeros at infinity) filters in the process of approximation. The IIR part controls the group delay flatness and may be determined for a maximally flat [47] or equal ripple [48] group delay. The numerator polynomial is then obtained for an equalized amplitude response [49,51]. An IIR filter with unconstrained numerator polynomial provides the optimum characteristics for a required filter complexity.

5.1 Chebyshev Approximation

Let the squared magnitude of the transfer function $H(z)$ be written as

$$|H(e^{j\theta})|^2 = H(z)H(z^{-1})|_{z=e^{j\theta}} \tag{5.1}$$

where $\theta=\Omega T$, and $1/T$ is the sampling frequency. Hence

$$|H(e^{j\theta})|^2 = Q(z+z^{-1})|_{z=e^{j\theta}} \tag{5.2}$$

where

$$Q(z+z^{-1}) = H(z)H(z^{-1}). \tag{5.3}$$

From (5.2) it is obvious that the magnitude function is a function of $\cos\theta$. Let the squared magnitude of $H(z)$ be written as

$$|H(e^{j\theta})|^2 = \frac{1}{1+\epsilon^2\Psi(\cos\theta)} \tag{5.4}$$

where $\Psi(\cos\theta)$ is the characteristic function. Evidently the method B of section 2.1.2 is realized to be suitable for the problem under consideration. For the transfer functions of digital filters which relate to reciprocal analog filters, the degrees of the numerator and denominator polynomials are the same. The zeros on the unit circle ($|z|=1$) are in mirror-image symmetry about the unit circle (z_i and $1/z_i^*$). The zeros are either real or in complex conjugate pairs. Thus the numerator polynomial may be written as

$$N_m(z) = \begin{cases} z^n g_n(z+z^{-1}) & m=2n \\ z^n g_n(z+z^{-1})(1+z) \ \ or \ \ z^n g_n(z+z^{-1})(1-z) & m=2n-1 \end{cases} \tag{5.5}$$

where $g_n(X)$ is a polynomial with real coefficients. Hence the characteristic function of the filter of degree 2n with Chebyshev characteristic is written as

$$\Psi(\cos\theta) = \frac{1}{4}\left[\frac{F_n(z)}{z^n F_n(z^{-1})} + \frac{z^n F_n(z^{-1})}{F_n(z)}\right]^2 \Bigg|_{z=e^{j\theta}} \tag{5.6}$$

where the zeros of $F_n(z)$ are inside the unit circle. The resulting filter has $2n-1$ extrema in the interval $0 \le \theta \le \pi$. For a general problem with the passband interval $[\theta_1, \theta_2]$, the variable $\cos\theta$ is constrained (equation (2.1) in chapter II) by

$$\cos\theta = a\cos\tilde{\theta} + b \tag{5.7}$$

where

$$a = \frac{2}{\cos\theta_1 - \cos\theta_2} \quad , \quad b = -\frac{\cos\theta_1 + \cos\theta_2}{\cos\theta_1 - \cos\theta_2} \tag{5.8}$$

Let the variable z be transformed to $\tilde{z}$, then from (5.7)

$$\frac{1}{2}(z + \frac{1}{z}) = \frac{a}{2}(\tilde{z} + \frac{1}{\tilde{z}}) + b \tag{5.9}$$

Thus the characteristic function is obtained through the substitution of $X = \frac{1}{2}(z + \frac{1}{z})$ in (5.6) as

$$\Psi(\tilde{X}) = \Psi(X)\,|_{X = a\tilde{X} + b} \tag{5.10}$$

The resulting charcteristic function $\Psi(\tilde{X})$ can be used to obtain the poles and zeros of the transfer function. Let the zeros of the polynomial $F_n(z)$ be denoted by z_i, the zeros of the transfer function $\tilde{z}_i$ are the solutions of the equation

$$\frac{a}{2}(\tilde{z}_i + \frac{1}{\tilde{z}_i}) + b = \frac{1}{2}(z_i + \frac{1}{z_i}) \qquad i = 1,n \tag{5.11}$$

The poles of the transfer function $\tilde{p}_i$ are obtained from the zeros p_i of the equation

$$F_n(z) - j\delta z^n F_n(z^{-1}) = 0 \tag{5.12}$$

where

$$\delta = 1/\epsilon + \sqrt{1/\epsilon^2 + 1} \tag{5.13}$$

$$X_i = \begin{cases} \frac{1}{2}(p_i + \frac{1}{p_i}) & i = 1,n \\ X^*_{i-n} & i = n+1,2n \end{cases} \tag{5.14}$$

$$\frac{a}{2}(\tilde{p}_i+\frac{1}{\tilde{p}_i})+b=X_i \tag{5.15}$$

The poles $\tilde{p}_i$ are those zeros of the equation (5.15) which are inside the unit circle.

For an odd degree transfer function of degree 2n+1, let

$$F(z)=z^{1/2}F_n(z) \tag{5.16}$$

then

$$\Psi(z+z^{-1})=\frac{z^{-2n+1}F_n^4(z)+z^{2n-1}F_n^4(z^{-1})}{4F_n^2(z)F_n^2(z^{-1})}+\frac{1}{2} \tag{5.17}$$

There are $2n+1$ extrema in the interval $[0,\pi]$.

In (5.16) or (5.17) there are n parameters in $F_n(z)$ which are determined through the amplitude and group delay equalizations.

5.2 The Alternative Approach

The filters designed in section 5.1 have the maximum number of ripples in the passband. The characteristic function of a filter of degree n, with m frequencies of perfect transmission θ_j $(\theta_j \neq 0,\pi)$, and k transmission zeros θ_i $(\theta_i \neq 0,\pi)$, may be written as (θ is equal to the frequecy Ω for normalized $T=1$ and hence the term frequency is used for θ)

$$\Psi_n(X)=\frac{K_0\prod_{j=1}^{m}(X-\cos\theta_j)^2P_{n-2m}(X)}{\prod_{i=1}^{k}(X-\cos\theta_i)^2Q_l^2(X)(1\pm X)} \tag{5.18}$$

where the term $(1+X)$ or $(1-X)$ is present if an odd number of transmission zeros are located at π or 0, respectively. The monic polynomials $P_{n-2m}(X)$ and $Q_l(X)$ have no real zeros in the interval [-1,1]. For a filter with maximum passband ripples, there are $[\frac{n+1}{2}]$ frequencies of perfect transmission. The polynomial $P(X)$ is constrained for perfect transmission at 0 or π by

$$P(1)=0 \qquad \theta_j=0 \tag{5.19}$$

$$P(-1)=0 \qquad \theta_j=\pi$$

There are a maximum of $n-m+k+l+1$ degrees of freedom which can be used for the equalization of the amplitude and group delay characteristics. The methods for the amplitude and group delay equalizations are the same as those in chapter II.

Let the zeros of $Q(X)$ be denoted by $\hat{X}_i$, the zeros of the transfer function are obtained from

$$z_i = \begin{cases} \hat{X}_i \pm \sqrt{\hat{X}_i^2-1} & i=1,l \\ e^{\pm j\theta_i} & i=1,k \end{cases} \tag{5.20}$$

Let X_i denote the zeros of the equation

$$1+\epsilon^2\Psi_n(X)=0 \tag{5.21}$$

obtain

$$\hat{p}_i = X_i + \sqrt{X_i^2-1} \qquad i=1,n \tag{5.22}$$

$$\hat{p}_{i+n} = X_i - \sqrt{X_i^2-1}$$

The poles are those $\hat{p}_i$ inside the unit circle, thus

$$p_i = \hat{p}_j \quad \text{if } |\hat{p}_j| < 1 \qquad i=1,n \quad j=1,2n \tag{5.23}$$

The group delay of the filter can be obtained from

$$\tau(\cos\theta)=\frac{\nu}{2}+\sum_{i=1}^{n}\frac{1-\alpha_i\cos(\theta-\beta_i)}{1+\alpha_i^2-2\alpha_i\cos(\theta-\beta_i)} \tag{5.24}$$

where the poles are expressed in polar form ($\alpha_i \geq 0$)

$$p_i=\alpha_i e^{j\beta_i} \qquad i=1,n \tag{5.25}$$

and ν is the degree of the numerator polynomial.

As it was the case for analog filters, an initial approximation or the initial values for θ_i, θ_j and the coefficients of $P_{n-2m}(X)$ and $Q_l(X)$ are required for the convergence of this procedure. For a filter with the maximum number of passband ripples, the initial approximation is not critical, since for this case only the coefficients of $Q_l(X)$ are required. The critical frequencies θ_i and θ_j are located relatively easily from the required amplitude characteristics.

5.3 Initial Approximations

The initial approximations are obtained in a manner similar to chapters III and IV for the analog filters.

Digital filters are obtained through a bilinear transformation of analog filters as

$$H(z)=H_a(s)\Big|_{s=\frac{z-1}{z+1}} \tag{5.26}$$

Thus the analog frequency ω is related to the digital frequency Ω through

$$\omega=\tan\frac{\Omega T}{2}=\tan\frac{\theta}{2} \tag{5.27}$$

For a transfer function $H(z)$ with linear phase

$$\phi_d(\theta)=\tau_0\theta \tag{5.28}$$

and hence the analog phase is described by

$$\phi_a(\omega)=2\tau_0\tan^{-1}\omega \tag{5.29}$$

An analog filter with phase $\phi_a(\omega)$ results in a digital filter with linear phase characteristics.

The denominator polynomial of $H_a(s)$ is obtained from

$$\frac{sF(s^2)}{E(s^2)}=\tanh(2\tau_0\tanh^{-1}s) \tag{5.30}$$

A continued fraction which approximates (5.30) can be obtained as [52]

$$\frac{sF(s^2)}{E(s^2)}=\cfrac{\mu s}{c_1+\cfrac{(\mu^2-1)s^2}{c_2+\cfrac{\cdot}{\cfrac{\cdot}{c_{n-1}+\cfrac{(\mu^2-(n-1)^2)s^2}{c_n}}}}} \tag{5.31}$$

where $c_i=2i-1$ and $\mu=2\tau_0$.

Thus the denominator polynomial is obtained from the recursion formula

$$D_{k+1}(s)=D_k(s)+\frac{(\mu^2-k^2)s^2}{(2k-1)(2k+1)}D_{k-1}(s) \tag{5.32}$$

where $D_0(s)=1$, and $D_1(s)=1+\mu s$.

As for the analog filters, the parameter c_n controls the group delay flatness and the amplitude selectivity of the digital filter. For a desired linear phase bandwidth c_n is evaluated iteratively as discussed for the analog filters. The numerator polynomial of $H_a(s)$ is obtained through the equalization of the amplitude response as discussed in chapters III and IV. For a filter with cutoff frequency θ_c, the designed filter with cutoff frequency $\hat{\omega}_c$ has to be frequency scaled by

$$k_\omega=\frac{\tan\frac{\theta_c}{2}}{\hat{\omega}_c} \tag{5.33}$$

and parameter μ is also scaled to $4\tau_0 \frac{\tan^{-1}\hat{\omega}_c}{\theta_c}$. Thus the cutoff frequency is equalized iteratively.

The denominator polynomial of the transfer function $H(z)$, with exact equal ripple group delay characteristics, may be obtained using the procedure 2.3.2.B of chapter II and equation (5.24), or from other sources [48].

The bilinear transformation of the denominator polynomial yields

$$D_a(s)=(1-s)^n D_n(z)\Big|_{z=\frac{1+s}{1-s}} \tag{5.34}$$

where $D_a(s)$ is the denominator of the analog filter. The continued fraction expansion of $D_a(s)$ can be obtained for a normalized c_1 in (5.31). The parameters c_n and μ are determined iteratively as discussed for the maximally flat case.

It should be mentioned that the digital transfer functions obtained as a result of bilinear transformation, have numerator and denominator polynomials of equal degrees. This constraint is unattractive in general but desirable for wave digital filters.

Lowpass filters obtained in this section can be transformed to linear phase bandpass, highpass or bandstop filters through the transformations of variable z as given in Table 6.

Table 6

Transformations on Lowpass Filters with Cutoff Frequency θ_c

Filter Type	Transformation	Design Fomulas
Highpass	$-z$	$\hat{\theta}_c=\pi-\theta_c$
Bandpass	$-z^2$	$BW=2\theta_c,\ \theta_0=\frac{\pi}{2}$
Bandstop	z^2	$BW=\pi-2\theta_c,\ \theta_n=\frac{\pi}{2}$

5.3.1 Design Examples

Three sets of examples are given for lowpass and bandpass filters. The lowpass filters are of degree 8 and with two zeros at $z=-1$. The bandpass filters are of degree 12, with two zeros at $z=-1$ and two zeros at $z=1$.

Example 12

It is required to design lowpass filters with maximally flat group delay and amplitude responses. The minimum stopband attenuation is required to be 50 dB. The cutoff frequency θ_c is to be $\frac{\pi}{4}$. The attenuation and group delay characteristics of the designed filters for two values of $\hat{c}_8$ are shown in Fig. 14 . The amplitude response has two degrees of flatness in the passband. The poles and zeros of the designed filter with $\hat{c}_8$=7.5 are given in Table 7.

Example 13

It is required to design lowpass filters with equal ripple group delay and amplitude characteristics. The maximum passband and the minimum stopband attenuations are to be 0.5 dB and 50 dB. The cutoff frequency is required to be $\frac{\pi}{4}$. The group delay variation in the linear-phase band is to be less than 1.4%. The attenuation and group delay characteristics of the designed filters for two values of $\hat{c}_8$ are shown in Fig. 15. The amplitude response is designed to have one finite transmission zero. The poles and zeros of the filter with $\frac{\hat{c}_8}{c_8}$=0.8 are given in Table 7.

Example 14

In this example bandpass filters are designed to have equal ripple group delay and amplitude characteristics. The maximum passband and the minimum stopband attenuations are 0.5 dB and 50 dB. The group delay variation in the linear-phase band is 2%. The center frequency is at $\frac{\pi}{2}$ and the passband extends from $0.4\,\pi$ to 0.6π. The bandpass filters were obtained through the transformation of lowpass filters of degree 6 with cutoff frequency 0.1π satisfying the specifications. Figure 16 shows the attenuation and group delay characteristics of the designed filters for two values of $\hat{c}_6$.

Table 7

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros
12	0.608364 ± j 0.602198	0.040125 ± j 0.999195
	0.559642 ± j 0.057059	1.863425 ± j 0.540681
	0.564443 ± j 0.178980	0.494974 ± j 0.143620
	0.576557 ± j 0.331480	-1 (double)
13	0.660931 ± j 0.569151	0.595467 ± j 0.205720
	0.625348 ± j 0.495360	0.174660 ± j 0.984629
	0.739991 ± j 0.294676	1.500289 ± j 0.518316
	0.785599 ± j 0.100589	-1 (double)

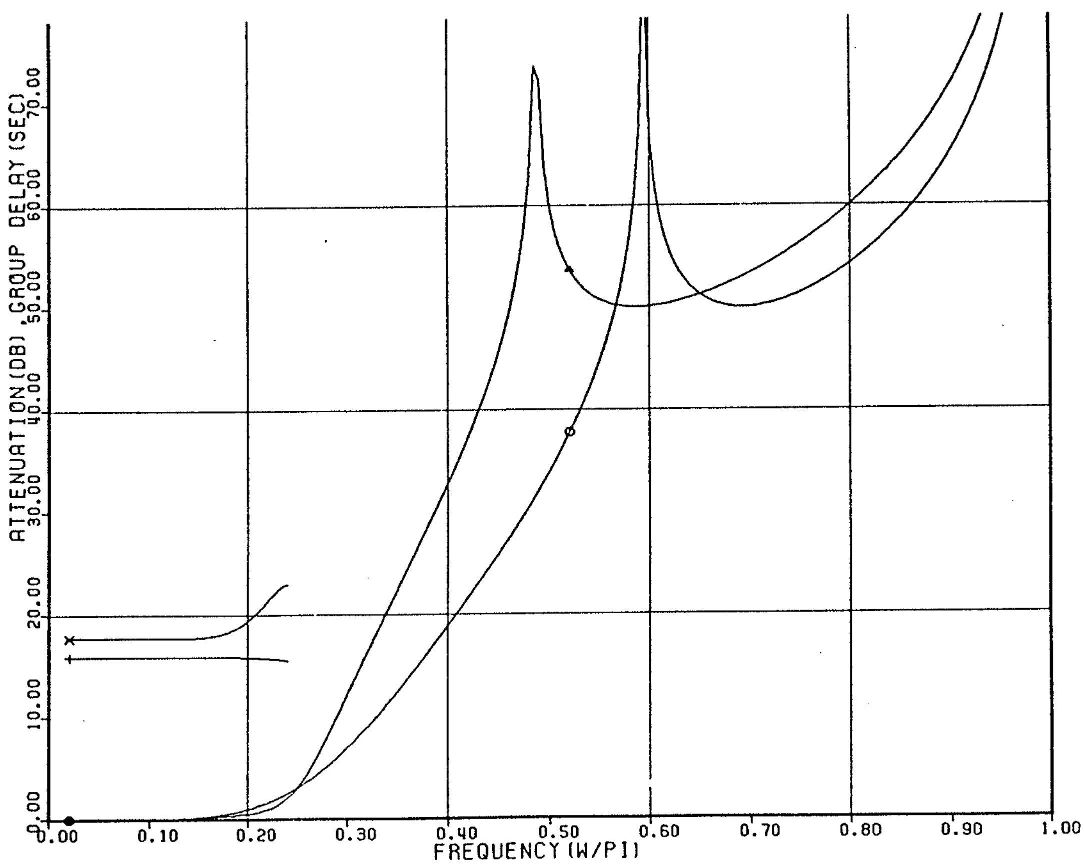

Fig. 14 Attenuation and group delay characteristics of the filters in Example 12 for $\hat{c}_n = 1$ (o,+), and 0.5 (Δ,$\times$).

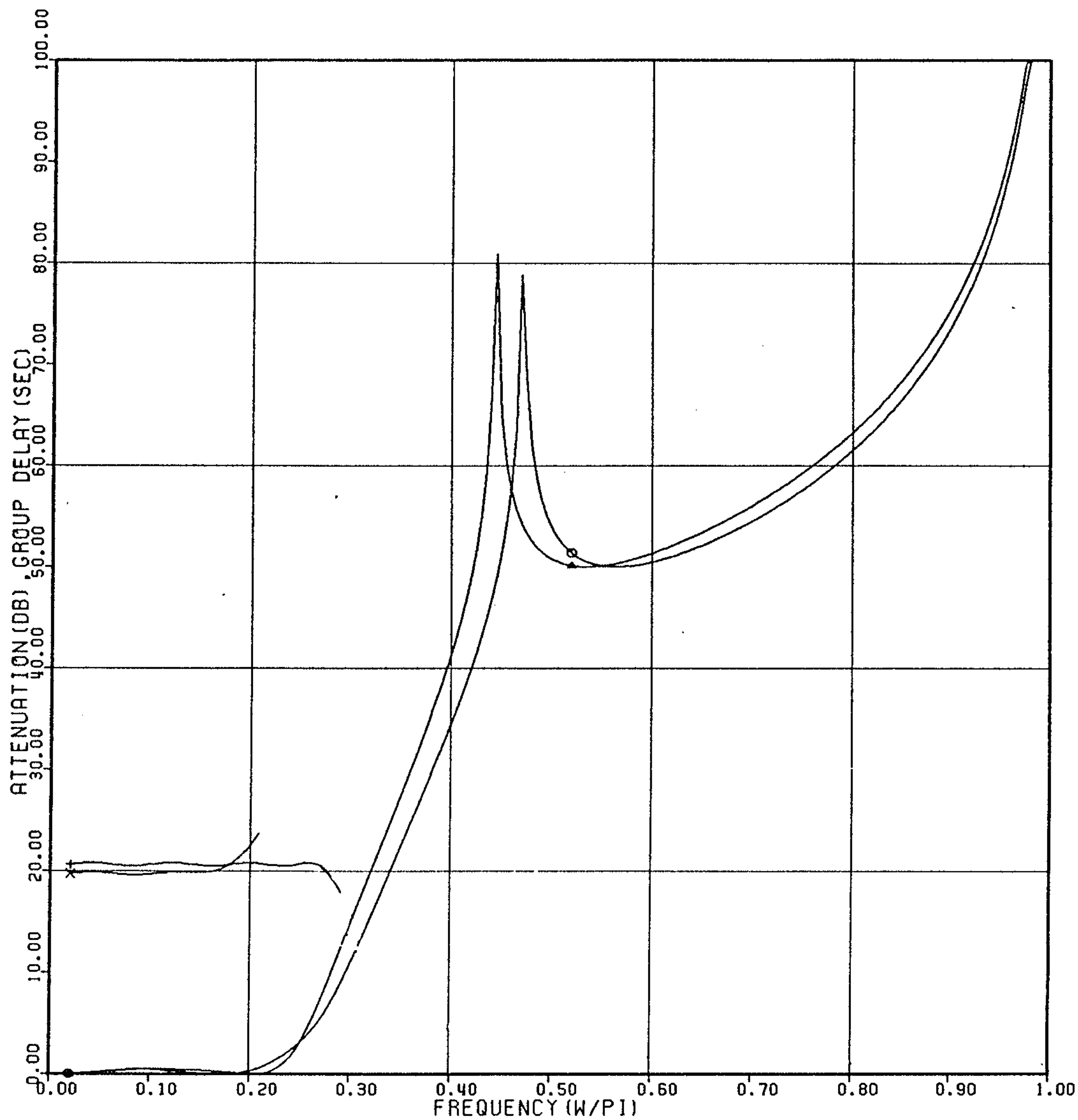

Fig. 15 Attenuation and group delay characteristics of the filters in Example 13 for $\hat{c}_n = 1$ (o,+), and 0.8 (Δ,$\times$).

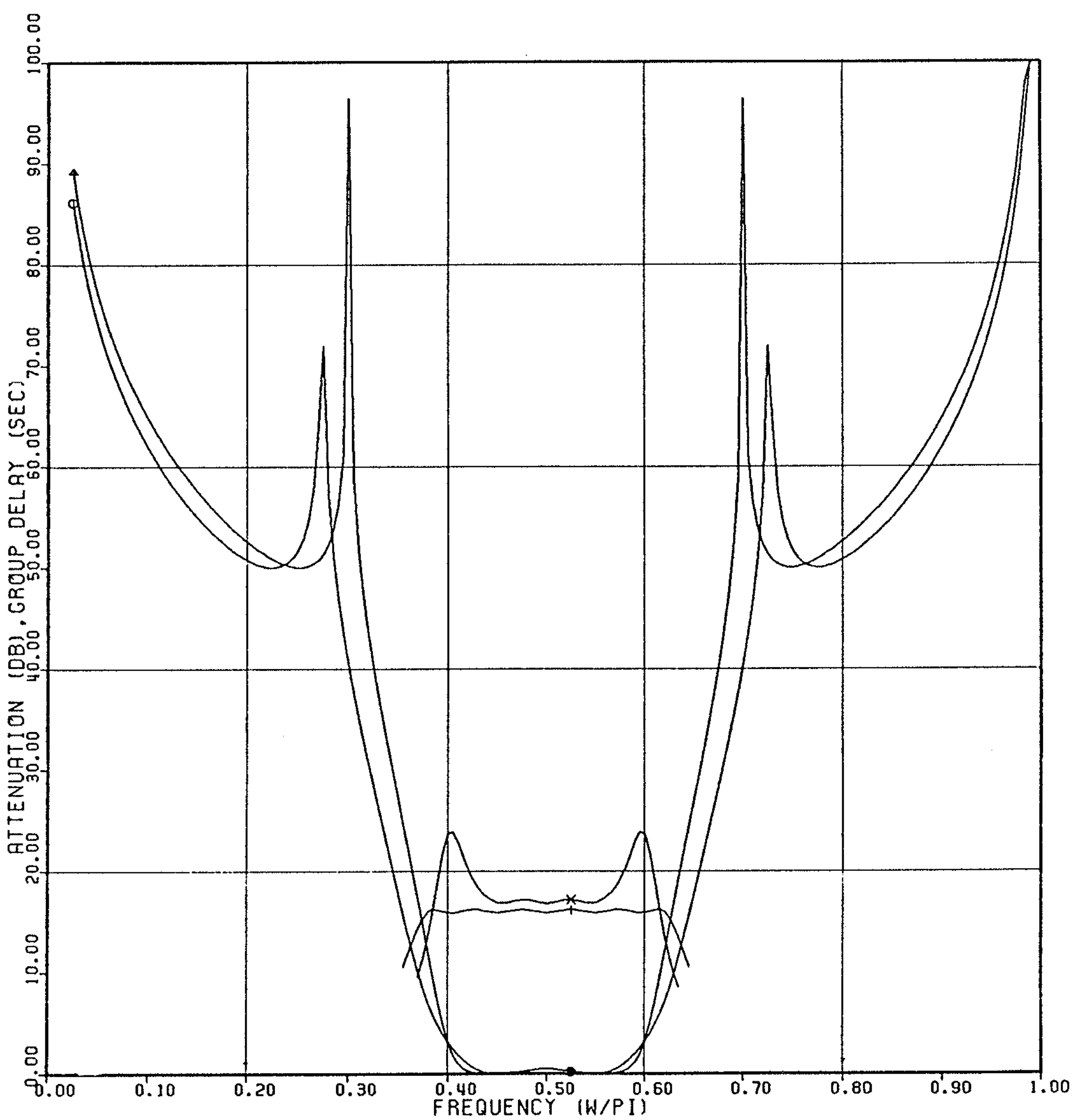

Fig. 16 Attenuation and group delay characteristics of the filters in Example 14 for $\hat{c}_n = 1$ (o,+), and 0.75 ($\Delta,\times$).

5.4 Other Filter Types

The approximation procedure for switched capacitor and commensurate transmission-line filters is similar to the procedure of sections 5.1 and 5.2.

For switched capacitor filters the transfer function is a function of z where

$$z = e^{j\Omega T_c} \tag{5.35}$$

and T_c is the clock period. Thus the same procedure which was developed for the digital filters is applied.

For commensurate transmission-line filters, the transfer function is a function of S where

$$S = \tanh sT \tag{5.36}$$

S is the Richard's variable, T is the delay of the line. For real frequency ω, $s = j\omega$, and hence

$$S = j\Omega = j\tan\omega T \tag{5.37}$$

Therefore the approximation procedure of chapter II can be used directly. For example for a Chebyshev reciprocal lowpass filter of even degree

$$|H_{2n}(j\Omega)|^2 = \frac{1}{1+\epsilon^2\Psi_n^2(\Omega^2)} \tag{5.38}$$

where

$$\Psi_n(\Omega^2) = \frac{1}{2}\left[\frac{F_n(S)}{F_n(-S)} + \frac{F_n(-S)}{F_n(S)}\right] \tag{5.39}$$

For a Hurwitz $F_n(S)$, the amplitude response has $2n$ extrema in the passband $0 < \omega T < \pi$. For a desired passband, the frequency is transformed as discussed in section 5.1.

CHAPTER VI

NONRECIPROCAL FILTERS

The approximation procedure for nonreciprocal filters is similar to that for the reciprocal filters. For this class of filters the numerator polynomial of the transfer function contributes to the group delay characteristics of the filter and consequently the denominator polynomial can no longer be decoupled from the numerator polynomial in the process of approximation. As a result the methods of chapter III and IV for the determination of the initial approximations are not applicable.

6.1 Analog Filters

Let the transfer function be written as

$$H_n(s)=\frac{N_m(s)}{D_n(s)} \qquad (m \leq n) \tag{6.1}$$

where the polynomial $N_m(s)$ is neither an odd nor an even function of s. The squared magnitude of $H_n(s)$ can be written as

$$|H_n(j\omega)|^2=\frac{1}{1+\epsilon^2\Psi_n(\omega^2)} \tag{6.2}$$

The characteristic function of a filter, Chebyshev in the passband, is obtained as

$$\Psi_n(\omega^2)=\frac{1}{4}\left[\frac{F_n(Z)}{F_n(-Z)}+\frac{F_n(-Z)}{F_n(Z)}\right]+\frac{1}{2} \tag{6.3}$$

where $F_n(Z)$ is a Hurwitz polynomial, and $Z=\sqrt{1-\frac{\omega_c^2}{\omega^2}}$ for a lowpass filter.

For a transmission zero at Z_t, $F_n(Z)=(Z+Z_t)^2\hat{F}_{n-2}(Z)$. There are $[\frac{n+1}{2}]$ frequencies of perfect transmission.

For a filter of degree n, with m frequencies of perfect transmission α_i ($\alpha_i \neq 0$), and k transmission zeros β_i ($\beta_i \neq 0$), the characteristic function may be written as

$$\Psi_n(\omega^2)=\frac{\prod_{i=1}^{m}(\omega^2-\alpha_i^2)^2 P_{n-2m}(\omega^2)}{\prod_{i=1}^{k}(\omega^2-\beta_i^2)^2 Q_l(\omega^2)} \qquad (6.4)$$

where for a filter with perfect transmission at $\omega=0$, $P_{n-2m}(0)=0$, while for a filter with a transmission zero at $\omega=0$, $Q_l(0)=0$. The polynomials $P_{n-2m}(\omega^2)$ and $Q_l(\omega^2)$ have no positive real zeros.

There are a maximum of $n-m+k+l+1$ degrees of freedom which may be distributed between the amplitude and group delay characteristics. A total of $m+k+1$ degrees of freedom are used for the amplitude equalization and hence $n-2m+l$ degrees of freedom can be used for the group delay equalization. For reciprocal filters only $n-2m+[\frac{l}{2}]$ degrees of freedom can be used for the group delay equalization. Thus there is more flexibility in the design of nonreciprocal filters.

The procedure of chapter II can be used for the equalization of the amplitude and group delay characteristics. It is required to determine the initial values of the coefficients for which the iterative procedure converges.

6.2 Digital Filters

Let the transfer function be written as

$$H_n(z)=\frac{N_m(z)}{D_n(z)} \tag{6.5}$$

with the squared magnitude

$$|H_n(e^{j\theta})|^2=\frac{1}{1+\epsilon^2\Psi_n(\cos\theta)} \tag{6.6}$$

The characteristic function of a filter with Chebyshev passband may be written as

$$\Psi(X)=\frac{1}{4}\left[\frac{z^{n-m}F_m(z)}{z^mF_m(z^{-1})}+\frac{z^{-n+m}F_m(z^{-1})}{z^{-m}F_m(z)}\right]+\frac{1}{2} \tag{6.7}$$

where $X=\frac{1}{2}(z+z^{-1})$ and $z=e^{j\theta}$; all the zeros of $F_m(z)$ are restricted to be inside the unit circle $|z|=1$.

For a desired passband the transformation in (5.8) is used.

For a filter of degree n with m frequencies of perfect transmission θ_i $(\theta_i \neq 0,\pi)$ and k transmission zeros θ_j $(\theta_j \neq 0,\pi)$, the characteristic function may be written as

$$\Psi_n(X)=\frac{K_0\prod_{i=1}^{m}(X-\cos\theta_i)^2P_{n-2m}(X)}{\prod_{j=1}^{k}(X-\cos\theta_j)^2Q_l(X)} \tag{6.8}$$

where the polynomials $P_{n-2m}(X)$ and $Q_l(X)$ have no zeros in the interval (-1,1). If for example $\theta_j=0$, the polynomial $Q_l(X)$ has a factor (X-1).

Those zeros of $H(z)$ not on the unit circle are no longer required to be in mirror image symmetry about the unit circle.

With the identification of the characteristic function the procedures of section 5.2 of chapter V can then be applied for the amplitude and group delay equalizations.

6.3 Minimum-Phase Allpass Combination Method

The method is discussed for analog filters but it also applies to the design of digital filters.

Let $H_M(s)$ denote the transfer function of a minimum-phase filter for which the amplitude response meets the specifications. The squared magnitude of $H_M(s)$ is written as

$$|H_M(j\omega)|^2=\frac{1}{1+\epsilon^2\Psi_m^2(\omega^2)} \tag{6.9}$$

where

$$\Psi_m(\omega^2)=\frac{\prod_{i=1}^{k}(\omega^2-\alpha_i^2)^2 P_{m-2k}(\omega^2)}{\prod_{i=1}^{l}(\omega^2-\beta_i^2)^2 Q_1(\omega^2)} \tag{6.10}$$

For an optimum amplitude response (largest selectivity), the polynomials $P_{m-2k}(\omega^2)$ and $Q_1(\omega^2)$ are either constants or proportional to ω^2, for example for a bandpass filter of degree 9, $P_{m-2k}(\omega^2)=K_0$, $Q_1(\omega^2)=\omega^2$.

There are $2[\frac{m}{2}]-k+l+1$ degrees of freedom; $k+l+1$ degrees of freedom are used for the amplitude equalization and the remaining $2[\frac{m}{2}]-2k$ can be used for the group delay equalization. The procedure for the amplitude equalization is discussed in section 2.3.1 of chapter II.

The selectivity of a filter with $k<[\frac{m}{2}]$ is smaller than the possible maximum. The polynomial $P_{m-2k}(\omega^2)$ contributes to the group delay of the minimum-phase filter and thus for $k<[\frac{m}{2}]$ the group delay can be equalized. As a result the overall group delay of the minimum-phase allpass delay equalizer combination has better characteristics at the expense of a reduced selectivity. It has been shown [56] that for a least squares monotonic amplitude response in the passband, the overall filter has better amplitude and group delay characteristics than the combination of optimum elliptic minimum-phase and the allpass delay equalizer filters.

The zeros of $H_M(s)$ are obtained as

$$\hat{z}_i=\begin{cases}0.\pm j\beta_i & i=1,l\\ 0. & \text{if } a\ transmission\ zero\ is\ at\ zero\end{cases} \tag{6.11}$$

The poles $\hat{p}_i$ are the LHP zeros of the equation

$$1+\epsilon^2\Psi_m^2(-s^2)=0 \tag{6.12}$$

The group delay of $H_M(s)$ is equalized by an allpass filter. Let $H_A(s)$ denote the transfer function of the allpass filter

$$H_A(s)=\prod_{i=1}^{n}\frac{s+s_i}{s-s_i} \tag{6.13}$$

where s_i are either real or in complex conjugate pairs.

The squared magnitude of $H_A(s)$ is 1 because

$$|H_A(j\omega)|^2=H_A(s)H_A(-s)|_{s=j\omega}=1 \tag{6.14}$$

The poles and zeros of $H_A(s)$ are s_i and $-s_i$, respectively. The group delay of the filter is obtained from (2.57) as

$$\tau_A(\omega)=2\sum_{i=1}^{n}\frac{-\sigma_i}{\sigma_i^2+(\omega-\lambda_i)^2} \tag{6.15}$$

where

$$s_i = \sigma_i + j\lambda_i$$

The transfer function of the overall minimum-phase allpass combination is of degree $m+n$ and is obtained as

$$H(s) = H_M(s) H_A(s) \tag{6.16}$$

The locations of the poles s_i are to be determined for the equalized group delay. The zeros of $H(s)$ are the zeros of $H_M(s)$ and $H_A(s)$, hence

$$z_i = \begin{cases} \hat{z}_i & i=1,m \\ -s_{i-m} & i=m+1,m+n \end{cases} \tag{6.17}$$

The poles of $H(s)$ are the poles of $H_M(s)$ and $H_A(s)$

$$p_i = \begin{cases} \hat{p}_i & i=1,m \\ s_{i-m} & i=m+1,m+n \end{cases} \tag{6.18}$$

The group delay of the overall filter is obtained from

$$\tau(\omega) = \sum_{i=1}^{m+n} \frac{-\sigma_i C_i}{\sigma_i^2 + (\omega - \lambda_i)^2} \tag{6.19}$$

where

$$p_i = \sigma_i + j\lambda_i \tag{6.20}$$

$$C_i = \begin{cases} 1 & i \le m \\ 2 & i > m \end{cases}$$

The group delay of $H(s)$ can be equalized using the methods of section 2.3.2. There are $2[\frac{m}{2}]-k+l+n+1$ degrees of freedom for the amplitude and group delay equalizations. Thus there are more degrees of freedom

$(2[\frac{m+n}{2}]-k+l+n+1)$ in the direct method of section 6.1.

For a filter of degree $m+n$ with $[\frac{m}{2}]$ frequencies of perfect transmission (other than zero), there are n degrees of freedom for the group delay equalization. While for k frequencies of perfect transmission ($k < [\frac{m}{2}]$) there are $2[\frac{m}{2}]-2k+n$ degrees of freedom for the group delay equalization. The excess degrees of freedom for the group delay equalization are at the expense of a reduction in the degrees of freedom for the amplitude equalization by $[\frac{m}{2}]-k$. Therefore the improvement in group delay characteristics exceeds the deterioration in the amplitude characteristics, because

$$2[\frac{m}{2}]-2k > [\frac{m}{2}]-k \tag{6.21}$$

Filters obtained by this method (minimum-phase + allpass) can be used as initial approximations for the design of filters using the direct method of section 6.1.

6.4 Alternative Method for the Design of Reciprocal Filters

Reciprocal filters may be obtained through the multiplication of the numerator and the denominator polynomials in (6.16) by the factor $\prod_{i=1}^{n}(s-s_i)$ as (s_i are the allpass poles)

$$H_R(s)=H_M(s)H_A(s)\frac{\prod_{i=1}^{n}(s-s_i)}{\prod_{i=1}^{n}(s-s_i)} \tag{6.22}$$

As a result the zeros of $H_R(s)$ are s_i and $-s_i$, excluding those on the $j\omega$-axis. Hence the numerator polynomial of $H_R(s)$ is either an even or an odd function of

s, and thus $H_R(s)$ relates to a reciprocal filter. The resulting filter is of degree $2n+m$.

The reciprocal filter obtained by this method has a maximum of $n+m+1$ degrees of freedom which are fewer than the possible maximum of $3n+m+1$. Therefore this method is not suitable for the design of filters with optimum amplitude and group delay characteristics. It may be useful for the design of filters which are used as initial approximations for the iterative procedures of chapter II.

A nonreciprocal filter can always be transformed to a reciprocal one through (6.22). The physical realization of the resulting filter requires a canonic number of $2n+m$ reactive elements (delay elements for a digital filter). A filter structure with nonreciprocal components (gyrators for analog filters) requires $n+m$ reactive elements for the realization of the nonreciprocal filter. Digital filters in general are realized by nonreciprocal structures and hence nonreciprocal filter design is attractive for the design of digital filters.

CHAPTER VII

RESULTS AND DISCUSSIONS

A design procedure is proposed which provides an even distribution of the degrees of freedom between the amplitude and group delay characteristics. The method is applicable to the general problem of filter design with any set of specifications for the amplitude and group delay (phase) characteristics. The amplitude response is assumed to have extrema of any multiplicity in the passband and the stopband. The special cases are maximally flat passband, equal ripple passband, equal ripple stopband, and monotonic stopband. The group delay (phase) is approximated to the desired characteristics in interpolation, least-mean-square or equal-ripple norms. The method is applicable to the problem of minimum-phase, nonminimum-phase and nonreciprocal filter design. Digital, switched capacitor and interdigital microwave filters can be designed with this method through a change of variable.

Since the method is iterative, initial approximations are required for convergence. In the preceding chapters, methods were proposed for the design of filters which can be used as initial approximations. In the next sectionexamples are given which establish the superiority of the proposed method [36].

7.1 Optimized Design Examples

Three design examples are given for reciprocal analog lowpass filters which are designed using the optimization procedures of chapter II. The amplitude response is designed to be equal-ripple in the passband and the stopband. The group delay is equalized using the least-mean-squares procedure of section 2.3.2.C.

Example 15

It is required to design a lowpass filter of degree 10 with the specifications:

Normalized passband (7.1)

Passband loss $\leq$ 0.044 *dB*

Stopband loss $\geq$ 50 *dB*

Group delay variation in 75% of the passband $\leq$ 0.04 *sec.*

Thus

$$A_p = 0.044\ dB \tag{7.2}$$

$$A_s = 50\ dB$$

$$\epsilon^2 = 10^{A_p/20} - 1 = 0.00507854$$

Let the transfer function of the filter be $H(s)$

$$H(s) = \frac{N_m(s^2)}{D_n(s)} \tag{7.3}$$

where the numerator polynomial $N_m(s^2)$ is of degree $2m$. For $m=3$ and one finite transmission zero in the stopband, the remaining four zeros are in quadrantal symmetry.

The squared magnitude of $H(s)$ is written as

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 \Psi_n(\omega^2)} \tag{7.4}$$

with the characteristic function

$$\Psi_n(\omega^2) = \frac{K_0 \prod_{i=1}^{k} (\omega^2 - \alpha_i^2)^2 P_{n-2k}(\omega^2)}{(\omega^2 - \beta_1^2)^2 Q_2^2(\omega^2)} \tag{7.5}$$

There are $n-k+4$ degrees of freedom, where $k+2$ ($k+3$ if $P_{n-2k}(0)=0$) degrees are used for amplitude equalization. The remaining $n-2k+2$ ($n-2k+1$ if $P_{n-2k}(0)=0$) degrees of freedom can be used for the group delay equalization. Two cases are considered: (a) with $k=5$, and (b) with $k=2$ and perfect transmission at zero ($P_{n-2k}(0)=0$).

Example 15(a)

For $k=5$, there are 10 extrema in the passband, with 2 degrees of freedom for the group delay equalization. Since there are maximum number of ripples in the passband, the method of section 2.2.1 can equivalently be used to obtain a Chebyshev characteristic. Hence the characteristic function is written as

$$\Psi_{10}(\omega^2)=\frac{1}{4}\left[\frac{F_5(Z)}{F_5(-Z)}+\frac{F_5(-Z)}{F_5(Z)}\right]^2\Big|_{Z=\sqrt{1-\frac{1}{\omega^2}}} \tag{7.6}$$

where

$$F_5(Z)=(1+a_1Z+a_2Z^2)(1+\gamma_1Z)(1+Z)^2 \tag{7.7}$$

The polynomial $1+a_1Z+a_2Z^2$ has no zeros in the interval [-1,0], and $0<\gamma_1<1$. There are three coefficients to be determined through the equalization procedures.

For a set of a_1 and a_2, determine γ_1 such that

$$\Psi\left(\frac{1}{1-Z_1^2}\right)=\frac{10^{A_s/10}-1}{\epsilon^2}=\Psi_1 \tag{7.8}$$

where Z_1 is the abscissae of the maximum of Ψ. Hence Z_1 is obtained from the zeros of the equation

$$F(-Z)\frac{dF(Z)}{dZ}-F(Z)\frac{dF(-Z)}{dZ}=0 \tag{7.9}$$

where Z_1 is the zero located in the interval (-1,0). Hence

$$\gamma_1=\frac{1}{Z_1}\frac{\delta G(-Z_1)-G(Z_1)}{\delta G(-Z_1)+G(Z_1)} \tag{7.10}$$

where

$$\delta=\sqrt{\Psi_1}+\sqrt{\Psi_1-1} \tag{7.11}$$

$$G(Z)=(1+a_1Z+a_2Z^2)(1+Z)^2 \tag{7.12}$$

The equations in (7.9) and (7.10) are solved iteratively until a desired accuracy is reached.

The group delay can be equalized using the methods of section 2.3.2 for the interpolation, equal ripple or least-mean-squares approximations. The least-mean-squares method is used in all three examples because of the flexibility associated with the method. The initial approximation for this example is readily obtained by assuming that the zeros of the filter are all at infinity or at a finite frequency, thus

$$1+a_1Z+a_2Z^2=(1+Z)^2 \tag{7.13}$$

For a set of coefficients let the zeros of $F(Z)$ be denoted by Z_i, the zeros of the filter s_i, are obtained from

$$s_i=\pm\frac{1}{\sqrt{Z_i^2-1}} \qquad i=1,5 \tag{7.14}$$

The poles p_i may be obtained from the zeros $\hat{Z}_i$ of the equation

$$F(Z)-j\lambda F(-Z)=0 \tag{7.15}$$

where

$$p_i=\begin{cases}\hat{Z}_i & i=1,5\\ p^*_{i-5} & i=6,10\end{cases} \tag{7.16}$$

The group delay is calculated using (2.57). In the least-mean-squares procedure the set of frequencies ω_j were located equidistantly such that

$$\omega_j = \frac{j-1}{39}\omega_d \qquad j=1,40 \tag{7.17}$$

Thus 40 points were located in the approximation interval. The constant group delay bandwidth ω_d is approximately 0.75 which is varied iteratively for the exact satisfaction of the specifications. The parameters a_1 and a_2 are determined using the procedure of section 2.3.2.C and then γ_1 is obtained for the required stopband characteristics. The iterations continue until a desired accuracy is reached. The result of this optimization procedure was unsatisfactory since the group delay variation in the required bandwidth exceeds the specifications. The group delay variation decreases if A_p is decreased but even for very small A_p (0.0005 dB) the group delay variation is unsatisfactory. The group delay variation decreases if the degrees of freedom are increased. This is achieved by either increasing the number of complex zeros (increasing m), or by reducing the number of passband ripples (decreasing k).

Let $m=4$, therefore the transfer function has six complex zeros with two zeros on the $j\omega-$axis. the polynomial $F(Z)$ is now written as

$$F_5(Z)=(1+a_1Z+a_2Z^2+a_3Z^3)(1+\gamma_1Z)(1+Z) \tag{7.18}$$

The coefficients a_1, a_2, a_3 and γ_1 were evaluated through the same procedure as before. The group delay of the resulting filter was unsatisfactory and hence A_p was decreased until for $A_p=0.01$ dB the group delay characteristics met the specifications. The poles and zeros of this filter are given in Table 8. The attenuation and group delay characteristics of this filter are shown in Fig. 17.

Example 15(b)

As mentioned before, the group delay variation decreases if the number of passband ripples is decreased. In this case the characteristic function is expressed by (7.5) where $k<5$. The coefficients are evaluated through equalization procedures of sections 2.3.1 and 2.3.2.C. An initial approximation was obtained from chapter IV for the approximate satisfaction of the specifications. A comparison of the specifications for the Design Example 11 and those in (7.1) shows that the filter designed in Example 11 can be used as an initial approximation. An improved approximation may be obtained if in Example 11 the extrema at frequencies of perfect transmission are equalized. The resulting filter has five extrema in the passband and hence $k=2$, and $P_6(0)=0$ in (7.5).

There are 7 degrees of freedom for the group delay equalization. For a set of coefficients for $P_6(\omega^2)$ and $Q_2(\omega^2)$, K_0, α_1, α_2, and β_1 are evaluated iteratively for the equal ripple characteristic in the passband and the stopband. The procedure for the amplitude equalization is given in section 2.3.1. The zeros of the filter are $0. \pm j\beta_1$ and zeros of the equation

$$Q_2(-s^2)=0 \tag{7.19}$$

The poles are the LHP zeros of the equation

$$(s^2+\beta_1^2)^2Q_2^2(-s^2)+\epsilon^2K_0(s^2+\alpha_1^2)^2(s^2+\alpha_2^2)^2P_6(-s^2)=0 \tag{7.20}$$

where

$$P_6(X)=b_1X+b_2X^2+.....+X^6 \tag{7.21}$$

$$Q_2(X)=a_0+a_1X+X^2 \tag{7.22}$$

The filter of example 11 defines the initial values for the coefficients in (7.5). The coefficients of $Q_2(\omega^2)$ and β_1 are readily obtained. The frequencies of perfect transmissiom α_i are also easily obtained. The coefficients of P_6 are obtained from

$$K_0 P_6(\omega^2) = \frac{|D(j\omega)|^2 - \eta^2(\omega^2-\beta_1^2)^2 Q_2^2(\omega^2)}{(\omega^2-\alpha_1^2)^2(\omega^2-\alpha_2^2)^2} \tag{7.23}$$

where

$$\eta = \frac{D(0)}{\beta_1^2 a_0} \tag{7.24}$$

The procedure of section 2.3.2.C is now applied for the equalization of the group delay. The procedure which was used in this example is relatively coupled to the amplitude equalization. For any change in the parameters $\mathbf{p}$ in the procedure for the evaluation of the derivatives of the error function $E(\omega_j, \mathbf{p})$. the amplitude response is equalized for equal ripple characteristics. The group delay is evaluated from (2.57) at a set of frequencies ω_j defined in (7.17). The procedure of section 2.3.2.C was applied first for the parameters b_1 to b_5 with a_0 and a_1 kept constant and the second time for a_0 and a_1 with b_i kept constant. This strategy may damp a potentially divergent process. The convergence was satisfactory after a few iterations. An examination of the group delay characteristics of the designed filter shows that the specifications are met satisfactorily. The poles and zeros of this filter are given in Table 8. Figure 17 shows the attenuation and group delay characteristics of the designed filter.

Table 8

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros
15(a)	-0.312827 ± j 0.132046	0. ± j 1.745710
	-0.304944 ± j 0.399462	± 0.676974 ± j 0.
	-0.287311 ± j 0.652640	± 0.621456 ± j 0.584387
	-0.231967 ± j 0.908313	
	-0.092114 ± j 1.083485	
15(b)	-0.372055 ± j 0.155963	0. ± j 1.643717
	-0.368016 ± j 0.455092	± 0.704541 ± j 0.394337
	-0.346374 ± j 0.735273	
	-0.105719 ± j 1.176971	
	-0.229874 ± j 1.044011	

An examination of Figure 17 shows that the filter designed in Example 15(b) has superior characteristics to the filter in Example 15(a). It has a smaller transition interval (1.58 versus 1.65), and it also has a smaller group delay variation in the passband (2.09 versus 2.97). In addition to superior amplitude and group delay characteristics, it has a more economical physical realization. This is due to the fact that it has two complex zeros fewer than the other filter. The filter of Example 15(a) has a smaller passband loss as a result of the constraint on the group delay characteristics and thus the specifications are oversatisfied unavoidably. The Examples 15(a) and 15(b), show that a filter with fewer than the maximum passband ripples may provide better overall characteristics as a result of an even distribution of the degrees of freedom between the amplitude and group delay characteristics.

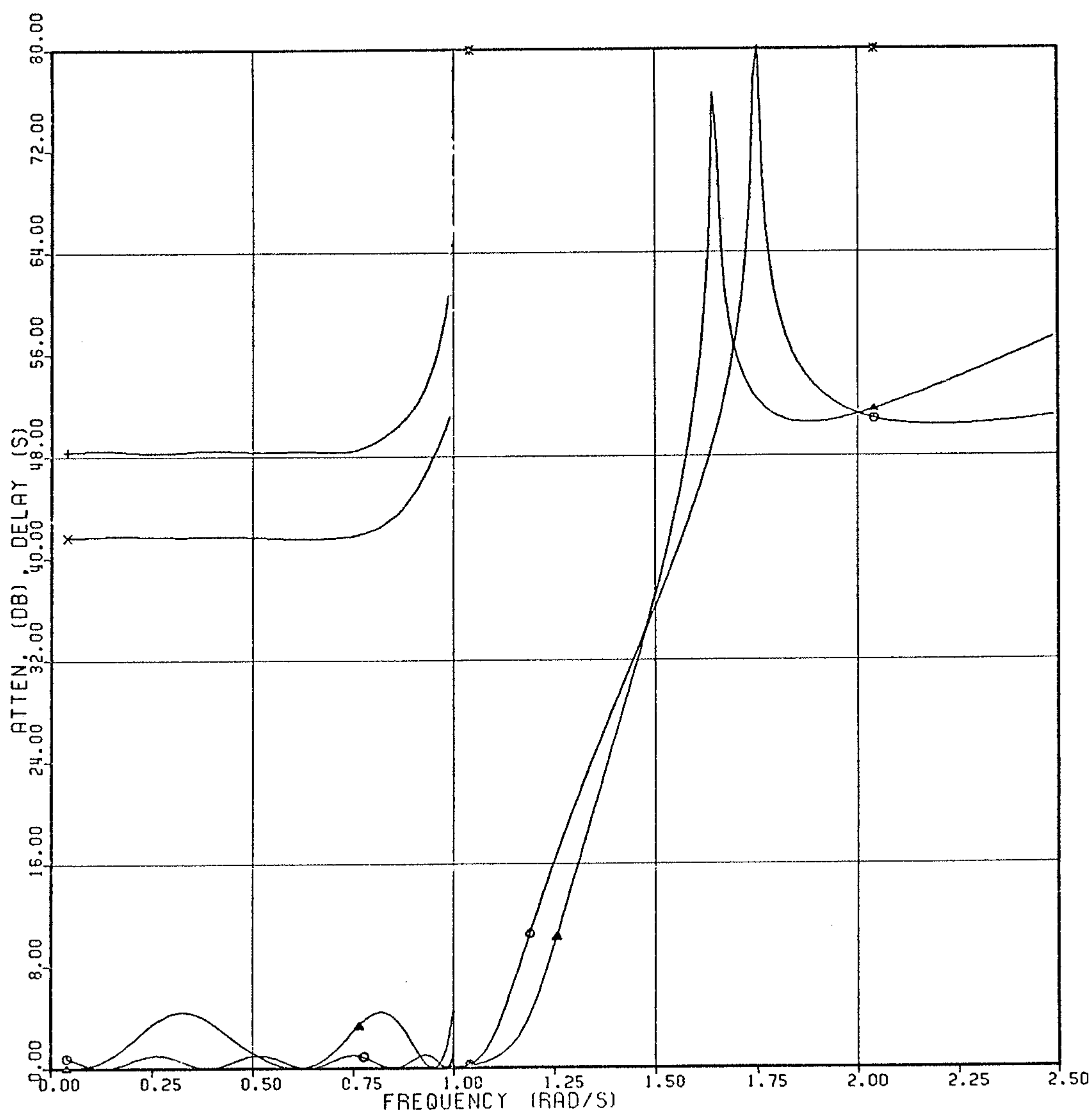

Fig. 17 Attenuation and group delay characteristics of the filters in Example 15(a) (o,+), and Example 15(b) (Δ,×). The group delay and passband loss are magnified by 5 and 100, respectively.

Example 16

In this example a filter is designed for the purpose of comparison to filters with least-squares monotonic passband response. The design example given by Litovsky [34], is nonreciprocal, but for a reciprocal realization, the transfer function of the reciprocal filter in unabridged form was obtained (the cancelled poles and zeros were added). The resulting filter is of degree 10 with two finite transmission zeros and four complex zeros. The minimum attenuation in the stopband is 50 dB, where the maximum passband attenuation is 0.02 dB. The group delay is fairly flat in the passband. A filter was designed using the same method and the same initial approximation which were utilized in Example 15(b). For A_p =0.0025, the group delay of the designed filter and the filter in reference [34] have exactly the same variations in the passband. The amplitude-response transition interval is smaller for the designed filter. As a result the designed filter has the same complexity as the example in reference [34] but with considerably smaller passband loss and a smaller tansition interval. The poles and zeros of the designed filter and the filter in reference [34] are given in Table 9. The attenuation and group delay characteristics for these filters are shown in Fig. 18.

As a comparison, filters with equal ripple amplitude response but with fewer than the maximum number of ripples in the passband have better characteristics than those with least-squares monotonic passband. It is obvious that similar characteristics are in general obtained with a filter of lower degree and hence lower cost.

Table 9

Poles and Zeros of the Designed Filters

Design Example	Poles	Zeros
Reference [34] (unabridged form)	-0.1123441 ± j 1.2514321 -0.3785105 ± j 1.0810923 -0.5903942 ± j 0.6480072 -0.6519845 ± j 0.2596921 -0.7455204 ± j 0.4335474	0. ± j 1.7299018 0. ± j 2.2571231 ± 0.7455204 ± j 0.4335474
Example 16	-0.091598 ± j 1.269455 -0.302451 ± j 1.131444 -0.485395 ± j 0.801790 -0.500685 ± j 0.473825 -0.537503 ± j 0.182956	0. ± j 1.666667 0. ± j 2.147419 ± 0.734888 ± j 0.425613

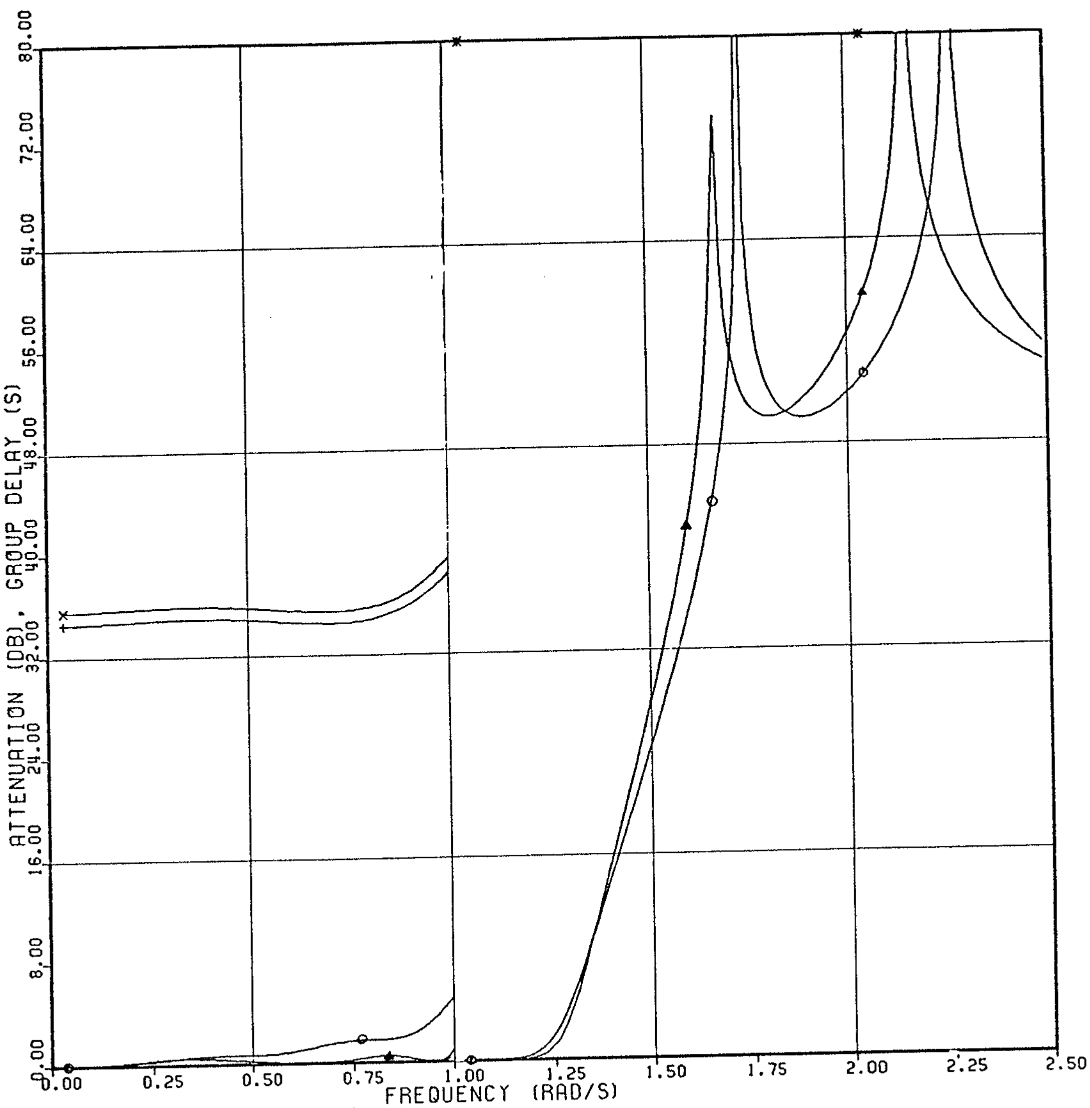

Fig. 18 Attenuation and group delay characteristics of the filters designed by the proposed method (Δ,×), and the method of reference [34] (o,+). The group delay and passband loss are magnified by 5 and 200, respectively.

7.2 Discussions

The previous design examples show that a filter with equal ripple passband loss but with fewer than the maximum number of ripples may provide better overall characteristics. The reason is linked with the number of degrees of freedom in the characteristic function, which can be used for the amplitude and group delay equalizations.

For example, for a reciprocal lowpass filter of even degree n, with k transmission zeros, and m frequencies of perfect transmission (other than zero), the characteristic function

$$\Psi_n(\omega^2)=\frac{\prod_{i=1}^{m}(\omega^2-\alpha_i^2)^2 P_{n-2m}(\omega^2)}{\prod_{i=1}^{k}(\omega^2-\beta_i^2)^2 Q_l^2(\omega^2)} \tag{7.25}$$

has $n-m+k+l+1$ degrees of freedom ($n-m+k+l$ if $P_{n-2m}(0)=0$), from which $m+k+1$ are used for amplitude equalization and the remaining $n-2m+l$ are used for the group delay equalization.

A filter with equal-ripple passband and with the maximum number of passband ripples ($m=\frac{n}{2}$), has l degrees of freedom for the group delay equalization, and $\frac{n}{2}+k+1$ degrees of freedom for the amplitude equalization. Thus a filter with $m<\frac{n}{2}$ passband ripples has $n-2m$ extra degrees of freedom for the group delay at the expense of $\frac{n}{2}-m$ degrees of freedom for the amplitude response. Obviously the improvement in the group delay exceeds the deterioration in the amplitude response. Since a reduction in the number of passband ripples is associated with an increase in the passband loss, and hence a decrease in amplitude selectivity, the optimum characteristics are obtained for a specific number of passband ripples.

Filters with unconstrained least-squares passband[33] have the maximum number of passband ripples. The maximum passband loss increases towards the bandedge and as a result they have lower selectivity than Chebyshev filters for the same specifications. The group delay characteristics are better than Chebyshev filters only at low frequencies. Therefore the overall characteristics are not favorable when compared with Chebyshev filters [29,31] for normal specifications.

All-pole filters with least-squares monotonic passband and with the maximum number of inflexion points [4] have better group delay characteristics than the all-pole Chebyshev filters. Litovsky [34] based his design on the assumption that for nonminimum-phase filters the abscissae of the inflexion points are the same as those for the all-pole filters. He then followed an approximation procedure similar to that of chapter VI for the amplitude and group delay equalizations. For a nonreciprocal filter of degree n (even), with l complex zeros and k transmission zeros, there are $k+l+1$ degrees of freedom from which $l+\frac{n}{2}-1$ can be used for the group delay equalization ($\frac{n}{2}-1$ inflexion points). A filter with equal-ripple passband and with m frequencies of perfect transmission has $n-2m+l+1$ degrees of freedom for the group delay equalization, where both filters have the same complexity when realized by reciprocal networks. An examination of these two numbers shows that the latter has more degrees of freedom for the group delay equalization.

Examples 15 and 16 verify that the proposed method is optimum when compared with the method for Chebyshev filters with the maximum number of passband ripples [29,31], or with the method for least-squares monotonic passband [34].

CONCLUSIONS

A design procedure has been proposed for the construction of transfer functions for analog and digital filters, with controllable amplitude and group delay characteristics. The amplitude response is equalized for maximally flat, monotonic or equal-ripple passband characteristics with monotonic or equal-ripple stopband characteristics. The group delay is equalized for maximally flat, equal-ripple or least-mean-squares characteristics in the entire or a part of the passband.

The iterative procedure requires an initial approximation which is obtained using the methods of chapter III for minimum-phase filters, chapter IV for nonminimum-phase filters, and chapter V for digital filters. The optimum characteristics result through an even distribution of the degrees of freedom between the amplitude and the group delay. The iterative procedure begins with the equalization of the amplitude response and then the remaining degrees of freedom are utilized for the equalized group delay characteristics.

It was shown analytically and by comparison that filters with the maximum number of passband ripples are not optimum for the design of selective filters with linear phase in the entire or in most of the passband. It was also shown that filters designed by the proposed method compare favorably to those with least-squares monotonic passband possessing the maximum number of inflexion points.

The advantage of the proposed method to the existing ones is its flexibility in providing control over the amplitude selectivity and the phase linearity. Thus filters designed by the proposed method can have optimum amplitude only, optimum linear phase only or optimum amplitude and phase characteristics. For a specific set of specifications, the proposed method yields optimum filters which have the least complexity in their physical realizations.

A disadvantage of the proposed method is the requirement for the initial approximation.

Future research may be directed towards the design of filters with monotonic passband which possess unconstrained inflexion points.

APPENDIX

COMPUTER PROGRAMS FOR RADIO SHACK PC-4 POCKET COMPUTER

IN BASIC LANGUAGE

A1

ELLYPTIC FILTER DESIGN

This program determines the poles and zeros of an ellyptic filter of degree M for given passband edge T, stopband edge S and passband loss U.

```
10   INPUT M,T,S,U
15   W=SQR(S*T)
20   A(0)=SQR(S/T)
30   FOR L=0 TO 3
40   A(L+1)=A(L)↑2+SQR(A(L)↑4-1)
50   NEXT L
60   FOR L=1 to M
70   Z=A(4)/COS(π*(L-1)/M/2)
80   FOR K=1 to 4
90   Z=(Z+1/Z)/A(4-K)/2
100  NEXT K
110  Z=Z*W
120  IF INT(L/2)*2=L;PRINT "ZERO";Z:GOTO 140
130  PRINT "STOPBAND-MIN. FREQ.";Z
140  NEXT L
150  F(4)=(2*A(4))↑M/2
160  FOR L=4 TO 1 STEP -1
170  F(L-1)=SQR((F(L)+1/F(L))/2)
```

```
175 NEXT L

180 K=SQR(10↑(U/10)-1)

190 V=10*LOG(1+K↑2*F(0)↑4)

200 PRINT "MIN. STOPBNAD LOSS";V

210 R=1/K/F(0)

220 FOR L=0 TO 3

230 R=F(L)*R+SQR((F(L)*R)↑2+1)

240 NEXT L

250 R=(F(4)/R+SQR((F(4)/R)↑2+1))↑(1/M)

260 FOR L=1 TO M

270 F=π/M*(L-(M+1)/2)

280 G=R*COSF

290 H=R*SINF

300 FOR K=4 TO 0 STEP-1

310 I=G↑2+H↑2

320 G=G*(1-1/I)/A(K)/2:H=H*(1+1/I)/A(K)/2

330 NEXT K

340 G=G*W:H=H*W

350 PRINT  "POLE";G;"j";H

360 NEXT L

370 END
```

A2

LINEAR-PHASE FILTER DESIGN

This program determines the poles and zeros of a filter of degree 10 with one finite transmission zero, and 6 complex zeros. The attenuation is equal ripple in the passband. The group delay is equalized in a part of the passband using the interpolation method.

The inputs are: max. passband loss P, finite transmission zero Z, initial values of the coefficients A, B, C which relate to the complex zeros, and the initial values for the poles and zeros. The execution time is 3 minutes and 20 seconds.

```
    DEFM 66
10  INPUT A,B,C,Z,P
20  Z=SQR(1-1/Z/Z):Q=SQR(10↑(P/10)-1):U=(1+SQR(1+Q*Q))/Q
30  GOSUB #1
40  FOR L=1 TO 3:Z(4*L)=D-D(L):NEXT L:FOR H=0 TO 2:A(H)=A(H)+.01
50  GOSUB #1
60  FOR L=1 TO 3:Z(4*L-3+H)=(D(L)-D+Z(4*L))*100:NEXT L
70  A(H)=A(H)-.01:NEXT H:M=Z
80  Z=0:FOR I=1 TO 3:FOR L=5*I-3 TO 4*I
90  FOR K=0 TO 12-4*I STEP4:Z(L+K)=Z(L+K)/Z(K+5*I-4)-Z(L*SGNK)
100 NEXT K:NEXT L:NEXT I
110 Z=Z(12):Y=Z(8)-Z(7)*Z:X=Z(4)-Z(3)*Z-Z(2)*Y
120 A=A+X:B=B+Y:C=C+Z:Z=M:GOSUB#1:STOP
130 K=3:U=0:O=C:P=B:Q=A:R=1:FOR W=37 TO 39 STEP 2
```

```
140 X=Z(W):Y=Z(W+1):GOSUB#2:GOSUB#3:Z(W-4)=X:Z(W-3)=Y:NEXT W

150 STOP

P1

10  D=0:E=0:F=0:G=0

20  J=1+Z:O=C*Z:P=C*J+B*Z:Q=C+B*J+A*Z:R=B+A*J+Z:S=A+J:T=1

30  FOR W=23 TO 31 STEP 2:X=Z(W):Y=Z(W+1)

40  K=5:GOSUB #2:Z(W)=X:Z(W+1)=Y:GOSUB #3

50  Z(W-10)=X:Z(W-9)=Y

60  I=.125:J=.375:K=.625:L=.775:M=X*X:Y=Y*Y

70  FOR N=0 TO 3:V=I(N)↑2

80  D(N)=D(N)+2*X*(M+Y+V)/((M+Y-V)↑2+4*M*V):NEXT N:NEXT W

90  RETURN

P2

10  M=0:N=0:I=0:J=0:FOR L=K TO 0 STEP-1:V=M

20  M=M*X-N*Y+O(L):N=V*Y+N*X-O(L)*U*(-1)↑L:IF L=0 THEN 40

30  V=I:I=I*X-J*Y+O(L)*L:J=V*Y+J*X-O(L)*L*U*(-1)↑L:NEXT L

40  L=I*I+J*J:V=(M*I+N*J)/L:X=X-V:Y=Y-(N*I-M*J)/L

50  IF ABS V > 1E-6 THEN 10

60  RETURN

P3

10  V=X*X-Y*Y-1:L=2*X*Y:M=SQR(V*V+L*L)

20  X=SQR((V/M+1)/M/2):Y=-L/M/M/2/X:RETURN
```

Program for the evaluation of the group delay at frequency F.

```
10   INPUT F:T=0:FOR L=13 TO 21 STEP 2:X=Z(L):Y=Z(L+1)
20   T=T+2*X*(X*X+Y*Y+F*F)/((X*X+Y*Y-F*F)↑2+(2*X*F)↑2)
30   NEXT L:PRINT T:GOTO 10
```

Program for the evaluation of the attenuation at frequency F.

```
5    E=2*U/(U*U-1)
10   INPUT F:IF F>1 THEN 50
20   O=SQR(1/F/F-1):P=C-A*O*O:Q=B*O-O↑3
30   R=ATN(Q/P):S=ATN O:T=2*(R+2*S)
40   H=10*LOG(1+(E*COST)↑2):PRINT H:GOTO 10
50   O=SQR(1-1/F/F):P=(O↑3+A*O*O+B*O+C)*(1+O)*(O+Z)
60   Q=(-O↑3+A*O*O-B*O+C)*(1-O)*(Z-O):R=P/Q
70   H=10*LOG(1+(E*(R+1/R)/2)↑2):PRINT H:GOTO 10
```

The poles are stored in Z(13) to Z(22). The zeros are stored in Z(23) to Z(36). The initial values for the poles are 3.3,1.6,1.3,-1.7,0.7,1.,0.5,-0.6,0.4,0.2, which are stored in Z(23) to Z(32). The initial values for the zeros are 1.4,0.,1.3,0.8, which are stored in Z(37) to Z(40).

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