On the Foundations for a Measure Theory and Integration in Two and Three Dimensions and a Theory of Delta Functions Over the Levi-Civita field

by

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Abstract

In various branches of physics, one encounters sources which are nearly instantaneous (if time is the independent variable) or almost localized (if the independent variable is a space coordinate). To avoid the cumbersome studies of the detailed functional dependencies of such sources, one would like to replace them with idealized sources that are truly instantaneous or localized. Typical examples of such sources are the concentrated forces and moments in solid mechanics, the point masses in the theory of the gravitational potential, and the point charges in electrostatics.

The field of real numbers does not permit a direct representation of the (improper) delta functions used for the description of impulsive (instantaneous) or concentrated (localized) sources. Of course, within the framework of distributions, these concepts can be accounted for in a rigorous fashion, but at the expense of the intuitive interpretation. The existence of infinitely small numbers and infinitely large numbers in the Levi-Civita field allows us to have well-behaved delta functions which, when restricted to the real numbers, reduce to the Dirac delta function. Here we develop the foundations for a mathematically rigorous theory of localized and instantaneous sources that has a clear and unambiguous way of specifying a mathematically concentrated source. We use the already existent one variable measure and integration theory on $\mathcal{R}$ to construct the foundations of a measure and integration theory on $\mathcal{R}^2$ and $\mathcal{R}^3$. First we construct measurable sets using sets with boundaries that can be expressed as analytic functions and we show the the resulting measure is Lebesgue-like. In particular we prove the measurability of countable sets, the countable union of measurable sets, and the finite intersection of measurable sets. Following that we use analytic functions

to construct a larger class of measurable functions, we then define the integral of a measurable function over a measurable set. We prove several propositions regarding measurable functions and the associated integration theory including that the set of measurable functions is closed under multiplication and addition, and that integration is linear. This allows for a wide range of applications for the delta function on $\mathcal{R}$, $\mathcal{R}^2$, and $\mathcal{R}^3$ and sets the course for a more extensive study of this topic in the future.

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For my parents

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Contents

1 Introduction

Let $\mathcal{R}$ denote the set of all functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (which we denote supp). To avoid confusion with functions on $\mathcal{R}$ we denote the value of $x \in \mathcal{R}$ at $q \in \mathbb{Q}$ by $x[q]$. A more detailed discussion of this topic may be found in [3] and for a general overview of recent research we direct the reader to [7].

For any $x, y \in \mathcal{R}$ addition and multiplication are defined as follows; $(x+y)[q] = x[q]+y[q]$ and $(x \cdot y)[q] = \sum_{q_1+q_2=q} x[q_1] \cdot y[q_2]$, note that $\sum_{q_1+q_2=q} x[q_1] \cdot y[q_2]$ will always be a finite sum because $supp(x)$ and $supp(y)$ are left-finite [3]. Under these operations of multiplication and addition $\mathcal{R}$ is a field [7].

$\mathbb{R}$ can be isomorphically embeded in $\mathcal{R}$ using the map $\Pi : \mathbb{R} \to \mathcal{R}$ given by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ \\ 0 & \text{else} \end{cases}.$$

We now introduce some notations which will facilitate discourse.

Definition 1.1 (λ, $\sim$, $\approx$, $=_q$)**.**

For non-zero $x \in \mathcal{R}$ we define $\lambda(x) = \min\{supp(x)\}$, which exists since $supp(x)$ is left-finite; and we use the convention $\lambda(0) = \infty$. Fix $x, y \in \mathcal{R}\backslash\{0\}$, we use the notation $x \sim y$ to indicate that $\lambda(x) = \lambda(y)$ and we say $x \approx y$ if $x \sim y$ and $x[\lambda(x)] = y[\lambda(y)]$. Fix $q \in \mathbb{Q}$, if for every $q' \in \mathbb{Q}$ such that $q' \leqslant q$, $x[q'] = y[q']$ then we say that $x =_q y$.

Definition 1.2 (Order on $\mathcal{R}$)**.**

Let $x, y \in \mathcal{R}$. Then we say $x > y$ if $x \neq y$ and $(x-y)[\lambda(x-y)] > 0$; we say $x \geqslant y$ if $x > y$ or $x = y$.

Note that $\geqslant$ is a total order relation which makes $\mathcal{R}$ an ordered field. We can now define the absolute value on $\mathcal{R}$ as

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ \\ -x & \text{if } x < 0 \end{cases}$$

This induces the order topology on $\mathcal{R}$. If a sequence $(a_n) \subset \mathcal{R}$ converges in the order topology it is said to converge strongly.

Definition 1.3.

For a given $r \in \mathbb{R}$ we let the mapping $||\cdot||_r : \mathcal{R} \to \mathbb{R}$ be defined by $||x||_r = \max\{|x[q]| : q \in \mathbb{Q}$ and $q \leqslant r\}$.

The above mapping defines a semi-norm for every r. Moreover, $\{||\cdot||_q : q \in \mathbb{Q}\}$ induces a topology known as the weak topology, if a sequence $(a_n) \subset \mathcal{R}$ converges in the weak topology it is said to converge weakly. We note that $\mathcal{R}$ is Cauchy complete under the order topology but not under the weak topology. For a more thorough exploration of the topological structure and convergence on $\mathcal{R}$ we direct the reader to [5] and [11].

Definition 1.4 ($\ll$, $\gg$)**.**

let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely larger than y (and write $x \gg y$) if for all $n \in \mathbb{N}$, $ny < x$. If $y \gg x$ then we write $x \ll y$ and we say x is infinitely smaller than y. If $x \ll 1$ then we call x infinitely small and if $x \gg 1$ then x is said to be infinitely large.

Definition 1.5 (The Number d)**.**

Let d be the element of $\mathcal{R}$ given by

$$d[q] = \begin{cases} 1 & \text{if } q = 0 \\ \\ 0 & \text{if } q \neq 0 \end{cases}.$$

Using the order relation above, it is easily shown that $0 < d \ll 1$. Moreover, for any $q \in \mathbb{Q}$, $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$.

Definition 1.6.

Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{R}$. We say (a_n) is regular if $\bigcup\limits_{n=1}^{\infty} supp(a_n)$ is left-finite.

In the following we give a brief summery of $\mathcal{R}$-analytic functions; and we refer the reader to [10] and the references therein for details.

Definition 1.7.

Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$. Then we say that f is $\mathcal{R}$-analytic on $[a, b]$ if for all $x \in [a, b]$ there exists a positive $\eta(x) \sim b - a$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y) = \sum_{n=0}^{\infty} a_n(x)(y - x)^n$ for all $y \in (x - \eta(x), x + \eta(x)) \cap [a, b]$.

Fortunately, $\mathcal{R}$-analytic functions behave similarly to real analytic functions. For example, if $f, g : I \subset \mathcal{R} \to \mathcal{R}$ are two $\mathcal{R}$-analytic functions on I and if $\alpha \in \mathcal{R}$ is a constant then $f \cdot g$ and $f + \alpha g$ are $\mathcal{R}$-analytic on I [3]. Additionally it has been shown that the composition of $\mathcal{R}$-analytic functions is $\mathcal{R}$-analytic and if f is $\mathcal{R}$-analytic on a closed interval $[a, b] \subset \mathcal{R}$, then f must be bounded on $[a, b]$[2]. And finally, $\mathcal{R}$-analytic functions on an interval $[a,b]$ satisfy a mean value theorem, an inverse function theorem, and an intermediate value theorem [8, 9].

We also note that in [2] it is shown that if $a < b$ in $\mathcal{R}$ and $f : [a,b] \to \mathcal{R}$ is $\mathcal{R}$-analytic on $[a,b]$, then there exists a rational number

$$l = \min\{\lambda(f(x)) : x \in [a,b]\}.$$

l is called the index of f and is denoted by $i(f)$. Finally, it is shown that $\lambda(f(x)) = i(f)$ almost everywhere on $\{x \in \mathcal{R} : x \in [a,b], supp(x-a) = \{\lambda(b-a)\}\}$ and for any such x the same is true for every element y in the neighborhood of x satisfying $\lambda(y-x) > \lambda(x-a)$. One convenient consequence of the above is that we may assume without loss of generality that $\mathcal{R}$-analytic functions are finite, this is so since scaling the function by d^{-l} does not affect the analytic properties of the function.

Measure theory and integration on $\mathcal{R}$ have been well studied in [1] which introduces a Lebesgue-like measure on $\mathcal{R}$ which is then used to construct an integration theory resembling Lebesgue integration. Note also that we use the convention that $I(a,b)$ denotes any one of the intervals (a,b), $(a,b]$, $[a,b)$, or $[a,b]$.

Definition 1.8.

Let $A \subset \mathcal{R}$ be given. We say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$ there exist two countable sequences of mutually disjoint intervals $(I_n)_{n=1}^{\infty}$ and $(J_n)_{n=1}^{\infty}$ such that, $\bigcup\limits_{n=1}^{\infty} I_n \subset A \subset \bigcup\limits_{n=1}^{\infty} J_n$, $\sum\limits_{n=1}^{\infty} l(I_n)$ and $\sum\limits_{n=1}^{\infty} l(J_n)$ both converge, and $\sum\limits_{n=1}^{\infty} l(J_n) - \sum\limits_{n=1}^{\infty} l(I_n) < \epsilon$.

If $A \subset \mathcal{R}$ is a measurable set then for every $k \in \mathbb{N}$ there exist two countable sequences of mutually disjoint intervals $(I_n^k)_{n=1}^{\infty}$ and $(J_n^k)_{n=1}^{\infty}$ such that, $\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k$, $\sum_{n=1}^{\infty} l(I_n^k)$, and $\sum_{n=1}^{\infty} l(J_n^k)$ both converge, and $\sum_{n=1}^{\infty} l(J_n^k) - \sum_{n=1}^{\infty} l(I_n^k) < d^k$.

It is shown in [1] that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l(I_n^k)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l(J_n^k)$ both exist and are equal. Furthermore, the measure of A is denoted by $m(A)$ and is given by

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l(I_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l(J_n^k).$$

Definition 1.9.

Let $a < b$ in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be an $\mathcal{R}$-analytic function on $I(a,b)$. Let F be an $\mathcal{R}$-analytic anti-derivative of f on $I(a,b)$, then

$$\int\limits_{x\in I(a,b)} f(x) = \lim_{x\to b} F(x) - \lim_{x\to a} F(x).$$

Definition 1.10.

Let $A \subset \mathcal{R}$ be a measurable set and let $f : A \to \mathcal{R}$ be a bounded function on A. Then we call f a measurable function if for every $\epsilon > 0$ in $\mathcal{R}$ there exists a sequence of mutually disjoint intervals $(I_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n \subset A$, $\sum_{n=1}^{\infty} l(I_n)$ converges, $m(A) - \sum_{n=1}^{\infty} l(I_n) < \epsilon$, and for every $n \in \mathbb{N}$ f is $\mathcal{R}$-analytic on I_n.

Let $A \subset \mathcal{R}$ be a measurable set and let $f : A \to \mathcal{R}$ be a measurable function on A. Then, for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint intervals $(I_n^k)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n^k \subset A$, $\sum_{n=1}^{\infty} l(I_n^k)$ converges, $m(A) - \sum_{n=1}^{\infty} l(I_n^k) < d^k$, and for every $n \in \mathbb{N}$ f is $\mathcal{R}$-analytic on I_n^k.

It can be shown [1] that for every $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} \int_{x\in I_n^k} f$ converges. The resulting sequence

$(\sum_{n=1}^{\infty} \int_{x \in I_n^k} f)_{k=1}^{\infty}$ is Cauchy and hence it converges. The integral of f over A is then defined by

$$\int_{x \in A} f = \lim_{k \to \infty} \sum_{n=1}^{\infty} \int_{x \in I_n^k} f$$

It is shown in [1] that the above integral behaves in much the same way as the Lebesgue integral of real analysis. For example it is shown that integration is linear, that the sum of the integrals of a function over two measurable sets is equal to the sum of the integrals over their union and intersection, and that if $f : A \to \mathcal{R}$ and $M \in \mathcal{R}$ satisfy $|f| \leqslant M$ everywhere on A then $|\int_A f| \leqslant Mm(A)$.

2 Measure Theory and Integration in two Dimensions

While measure theory and integration are well developed on $\mathcal{R}$ the same cannot be said for $\mathcal{R}^2$ and $\mathcal{R}^3$. It seems that if measure theory is extended to two dimensions in the natural way (i.e using rectangles in place of intervals) many common sets such as a finite triangle would not be measurable [4]. In this and the subsequent section we endeavour to extend measure and integration theory to two and three dimensions while avoiding these difficulties.

2.1 Simple regions and Measurable Sets

Definition 2.1 (Simple Region).

Suppose that $G \subset \mathcal{R}^2$. Then, we call G a simple region if there exist constants $a, b \in \mathcal{R}, a \leqslant b$

and $\mathcal{R}$-analytic functions $g_1, g_2 : I(a,b) \to \mathcal{R}, g_1 < g_2$ on $I(a,b)$ such that

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$$

or

$$G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(a,b)\}.$$

Definition 2.2 (Area of a Simple Region)**.**

Suppose $G \subset \mathcal{R}^2$ is a simple region given by $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$, then we denote the area of G with $a(G)$ and we define it as

$$a(G) = \int_{x \in I(a,b)} [g_2(x) - g_1(x)]$$

Definition 2.3 (Measurable Set)**.**

Suppose $A \subset \mathcal{R}^2$, we say A is measurable if for every $\epsilon > 0$ there is a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and another sequence of simple regions $(H_n)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n \subset A \subset \bigcup\limits_{n=1}^{\infty} H_n$, $\sum\limits_{n=1}^{\infty} a(G_n)$ and $\sum\limits_{n=1}^{\infty} a(H_n)$ both converge, and $\sum\limits_{n=1}^{\infty} a(H_n) - \sum\limits_{n=1}^{\infty} a(G_n) < \epsilon$.

Proposition 2.4.

Suppose $H, G \subset \mathcal{R}$ are both simple regions. Then, there exists a finite sequence of mutually disjoint simple regions $(F_n)_{n=1}^{k}$ such that $\bigcup\limits_{n=1}^{k} F_n = H \cap G$.

Proof.

Since H and G are both simple regions there are four possible cases:

1. $H = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c,e)\}$$

2. $H = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}$

   $$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c,e)\}$$

3. $H = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$

   $$G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(c,e)\}$$

4. $H = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}$

   $$G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(c,e)\}$$

However, by symmetry, $H \cap G = G \cap H$ so it is sufficient to consider cases 1 and 2. First we consider case 1. Let $\alpha = \max\{a, c\}$ and let $\beta = \min\{b, e\}$. We have that h_1, h_2, g_1, and g_2 are all $\mathcal{R}$-analytic functions, so $h_1 - g_1$, $h_1 - g_2$, $h_2 - g_1$, and $h_2 - g_2$ must also be $\mathcal{R}$-analytic on the corresponding intervals of definition. An $\mathcal{R}$-analytic function can have at most a finite number of zeros on a bounded interval [9] so each pair of functions will intersect a finite number of times on $I(\alpha, \beta)$, let $P = \{p_1, p_2, \ldots, p_l\}$ be the set of all such points of intersection ordered from least to greatest.

Define the interval I_1 to be $I(\alpha, p_1)$ and let I_{l+1} be $I(p_l, \beta)$; and for every $n \in \{2, \ldots, l\}$ let I_n be (p_{n-1}, p_n). Now, for every $n \in \{1, \ldots, l+1\}$, let $f_{1,n} = \max\{g_1|_{I_n}, h_1|_{I_n}\}$ and let $f_{2,n} = \min\{g_2|_{I_n}, h_2|_{I_n}\}$. For every $n \in 1, \ldots, l+1$ let $F_n = \{(x,y) \in \mathcal{R}^2 : y \in I(f_{1,n}, f_{2,n}), x \in I_n\}$. It follows that $(F_n)_{n=1}^{l+1}$ is a sequence of mutually disjoint simple regions that satisfies $\bigcup\limits_{n=1}^{l+1} F_n = H \cap G$.

Case 2 can be treated in the same way save that it is necessary to construct two partitions (one along each axis) to allow for the fact that intersections may occur between the left and right boundaries as well as the upper and lower boundaries. Furthermore, by induction the proposition holds for the intersection of any finite number of simple regions. $\square$

Proposition 2.5.

Suppose $G, H \subset \mathcal{R}^2$ are simple regions. Then, there exists a finite collection of mutually disjoint simple regions $(F_n)_{n=1}^{k}$ such that $\bigcup_{n=1}^{k} F_n = G \backslash H$.

Proof.

Since H and G are both simple regions there are four possible cases:

1. $H = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$

   $G = \{(x, y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c, e)\}$

2. $H = \{(x, y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a, b)\}$

   $G = \{(x, y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c, e)\}$

3. $H = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$

   $G = \{(x, y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(c, e)\}$

4. $H = \{(x, y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a, b)\}$

   $G = \{(x, y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(c, e)\}$

First consider the case where $H \subset G$. If H and G are given by

$$H = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$$

$$G = \{(x, y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c, e)\}$$

then,

$$G\backslash H = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c,a)\}$$

$$\cup \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), h_1(x)), x \in I(a,b)\}$$

$$\cup \{(x,y) \in \mathcal{R}^2 : y \in I(h_2(x), g_2(x)), x \in I(a,b)\}$$

$$\cup \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(b,e)\}$$

If H and G are given by

$$H = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}$$

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(c,e)\}$$

then,

$$G\backslash H = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), a), x \in I(c,e)\}$$

$$\cup \{(x,y) \in \mathcal{R}^2 : y \in I(b, g_2(x)) x \in I(c,e)\}$$

$$\cup \{(x,y) \in \mathcal{R}^2 : x \in I(c, h_1(y)), y \in I(a,b)\}$$

$$\cup \{(x,y) \in \mathcal{R}^2 : x \in I(h_2(y), e), y \in I(a,b)\}.$$

Similar identities hold for cases 3 and 4. In any case $G\backslash H$ is equal to the union of a finite collection of mutually disjoint simple regions.

Now consider the case where $H \not\subset G$. We have $G\backslash H = G\backslash(G \cap H)$, but by proposition

2.4, $G \cap H$ may be expressed as the union of a finite number of mutually disjoint simple regions $(F_n)_{n=1}^k$. Thus,

$$G \backslash H = G \backslash (\bigcup_{n=1}^{k} F_n) = \bigcap_{n=1}^{k} (G \backslash F_n).$$

However, by the first part of this proof, for every $n \in \{1 \ldots k\}$ there is a finite collection of mutually disjoint simple regions $(S_m^n)_{m=1}^{l_n}$ such that $G \backslash F_n = \bigcup_{m=1}^{l_n} S_m^n$. Therefore,

$$G \backslash H = \bigcap_{n=1}^{k} \bigcup_{m=1}^{l_n} S_m^n,$$

which is the intersection of a finite collection of finite unions of mutually disjoint simple regions and so may be expressed as the union of a finite collection of mutually disjoint simple regions. □

Proposition 2.6.

Suppose that $H, G \subset \mathcal{R}^2$ are simple regions. Then, there is a finite collection of mutually disjoint simple regions $(F_n)_{n=1}^k$ such that $H \cup G = \bigcup_{n=1}^{k} F_n$.

Proof.

We observe that $H \cup G = H \cup (G \backslash H)$. From Proposition 2.5 we see that $G \backslash H$ can be written as the union of a finite collection of mutually disjoint simple regions $(F_n)_{n=1}^k$. Therefore, $H \cup G = H \cup \bigcup_{n=1}^{k} F_n$ which is the union of a finite collection of mutually disjoint simple regions. □

Definition 2.7 (The Measure of a Measurable Set)**.**

Suppose $A \subset \mathcal{R}$ is a measurable set. Then, for every $k \in \mathbb{N}$ there are two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n^k \subset A \subset \bigcup_{n=1}^{\infty} H_n^k$,

$\sum_{n=1}^{\infty} G_n^k$ *and* $\sum_{n=1}^{\infty} H_n^k$ *both converge, and* $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k$.

Suppose $(\sum_{n=1}^{\infty} a(G_n^k))_{k=1}^{\infty}$ *is not a Cauchy sequence. Then, there is a* $\eta \in \mathcal{R}$, $\eta > 0$, *such that for every* $k \in \mathbb{N}$, *there exists an* $l > k$ *such that* $\sum_{n=1}^{\infty} a(G_n^l) - \sum_{n=1}^{\infty} a(G_n^k) > \eta$. *Fix* $k_0 \in \mathbb{N}$ *so that* $d^{k_0} < \eta$, *then for every* $k \geqslant k_0$, $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k \leqslant d^{k_0} < \eta$. *So, for every* $k \geqslant k_0$ *in* $\mathbb{N}$,

$$\sum_{n=1}^{\infty} a(H_n^k) < \sum_{n=1}^{\infty} a(G_n^k) + \eta. \tag{1}$$

However, from above we have that there exists an $l_0 > k_0$ *such that* $\sum_{n=1}^{\infty} a(G_n^{l_0}) > \sum_{n=1}^{\infty} a(G_n^{k_0}) + \eta$ *which implies by equation 1 that* $\sum_{n=1}^{\infty} a(G_n^{l_0}) > \sum_{n=1}^{\infty} a(H_n^{k_0})$ *which is a contradiction because by definition* $\bigcup_{n=1}^{\infty} G_n^{l_0} \subset A \subset \bigcup_{n=1}^{\infty} H_n^{k_0}$. *So,* $(\sum_{n=1}^{\infty} a(G_n^k))_{k=1}^{\infty}$ *is a Cauchy sequence and, by a similar argument, so is* $(\sum_{n=1}^{\infty} a(H_n^k))_{k=1}^{\infty}$. *Since* $\mathcal{R}$ *is Cauchy complete [3] it follows that* $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k)$ *and* $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$ *both exist.*

For every $k \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} H_n^k$ *from which we deduce that* $\sum_{n=1}^{\infty} a(G_n^k) \leqslant \sum_{n=1}^{\infty} a(H_n^k)$, *combining this with the fact that for every* $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) < d^k$ *we see that*

$$0 \leqslant \lim_{k\to\infty} \left(\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) \right) \leqslant \lim_{k\to\infty} d^k = 0$$

and so

$$\lim_{k\to\infty} \left(\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) \right) = 0.$$

It follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$.

We define $m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$ *and we call this the measure of* A.

Proposition 2.8.

Suppose $A \subset \mathcal{R}^2$ is a measurable set. Then,

$$m(A) = \inf \left\{ \sum_{n=1}^{\infty} a(H_n) : H_n\text{'s are mutually disjoint simple regions}, A \subset \bigcup_{n=1}^{\infty} H_n, \right.$$
$$\left. \text{and } \sum_{n=1}^{\infty} a(H_n) \text{ converges} \right\}$$

and

$$m(A) = \sup \left\{ \sum_{n=1}^{\infty} a(G_n) : G_n\text{'s are mutually disjoint simple regions}, \bigcup_{n=1}^{\infty} G_n \subset A, \right.$$
$$\left. \text{and } \sum_{n=1}^{\infty} a(G_n) \text{ converges} \right\}.$$

Proof.

First we show that the infimum exists and is equal to $m(A)$.

Since A is a measurable set we know that for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset \bigcup\limits_{n=1}^{\infty} G_n^{k+1} \subset A \subset \bigcup\limits_{n=1}^{\infty} H_n^{k+1} \subset \bigcup\limits_{n=1}^{\infty} H_n^k$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ and $\sum\limits_{n=1}^{\infty} a(H_n^k)$ both converge, and $\sum\limits_{n=1}^{\infty} a(H_n^k) - \sum\limits_{n=1}^{\infty} a(G_n^k) < d^k$.

By definition

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k)$$

and, for every $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} a(G_n^k) \leqslant m(A) \leqslant \sum_{n=1}^{\infty} a(H_n^k).$$

It remains to be shown that if $(H_n)_{n=1}^{\infty}$ is a sequence of mutually disjoint simple regions such that $A \subset \bigcup_{n=1}^{\infty} H_n$ and $\sum_{n=1}^{\infty} a(H_n)$ converges, then $\sum_{n=1}^{\infty} a(H_n) \geqslant \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k) = m(A)$.

Suppose not. Then there is a sequence of mutually disjoint simple regions $(H_n^0)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a(H_n^0)$ converges, $A \subset \bigcup_{n=1}^{\infty} H_n^0$, and $m(A) > \sum_{n=1}^{\infty} a(H_n^0)$. Let $k_0 \in \mathbb{N}$ be such that

$$d^{k_0} < \frac{m(A) - \sum_{n=1}^{\infty} a(H_n^0)}{2}.$$

We have from above that $\bigcup_{n=1}^{\infty} G_n^{k_0} \subset A \subset \bigcup_{n=1}^{\infty} H_n^0$, and since $(G_n^{k_0})_{n=1}^{\infty}$ and $(H_n^0)_{n=1}^{\infty}$ are both sequences of mutually disjoint simple regions it follows that $\sum_{n=1}^{\infty} a(G_n^{k_0}) \leqslant \sum_{n=1}^{\infty} a(H_n^0)$. But, $\sum_{n=1}^{\infty} a(H_n^{k_0}) \geqslant m(A)$, so $m(A) - \sum_{n=1}^{\infty} a(G_n^{k_0}) \leqslant \sum_{n=1}^{\infty} a(H_n^{k_0}) - \sum_{n=1}^{\infty} a(G_n^{k_0}) < d^{k_0}$. Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} a(H_n^0) - \sum_{n=1}^{\infty} a(G_n^{k_0}) &= \left(\sum_{n=1}^{\infty} a(H_n^0) - m(A)\right) + \left(m(A) - \sum_{n=1}^{\infty} a(G_n^{k_0})\right) \\
&\leqslant \left(\sum_{n=1}^{\infty} a(H_n^0) - m(A)\right) + d^{k_0} \\
&< \left(\sum_{n=1}^{\infty} a(H_n^0) - m(A)\right) + \frac{m(A) - \sum_{n=1}^{\infty} a(H_n^0)}{2} \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty} a(H_n^0) - m(A)\right) < 0
\end{aligned}$$

But this contradicts the fact that $\sum_{n=1}^{\infty} a(G_n^{k_0}) \leqslant \sum_{n=1}^{\infty} a(H_n^0)$, so a sequence such as $(H_n^0)_{n=1}^{\infty}$ cannot exist.

A similar argument shows that

$$\sup\left\{\sum_{n=1}^{\infty} a(G_n) : G_n\text{'s are mutually disjoint simple regions, } \bigcup_{n=1}^{\infty} G_n \subset A, \sum_{n=1}^{\infty} a(G_n) \text{ converges}\right\}$$

exists and is equal to $m(A)$. $\square$

Proposition 2.9.

Suppose $A, B \subset \mathcal{R}^2$ are measurable sets with $A \subset B$. Then $m(A) \leqslant m(B)$.

Proof.

Suppose not, then $A \subset B$ but $m(A) > m(B)$. Let $m(A) - m(B) = \eta$. Since A is measurable there is a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} a(G_n)$ converges, and $m(A) - \sum_{n=1}^{\infty} a(G_n) < \frac{\eta}{4}$. Since B is measurable there is a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $B \subset \bigcup_{n=1}^{\infty} H_n$, $\sum_{n=1}^{\infty} a(H_n)$ converges, and $\sum_{n=1}^{\infty} a(H_n) - m(B) < \frac{\eta}{4}$. So we see that

$$\sum_{n=1}^{\infty} a(H_n) < m(B) + \frac{\eta}{4} < m(A) - \frac{\eta}{4} < \sum_{n=1}^{\infty} a(G_n),$$

thus $\sum_{n=1}^{\infty} a(H_n) < \sum_{n=1}^{\infty} a(G_n)$. However,

$$\bigcup_{n=1}^{\infty} G_n \subset A \subset B \subset \bigcup_{n=1}^{\infty} H_n,$$

so $\bigcup_{n=1}^{\infty} G_n \subset \bigcup_{n=1}^{\infty} H_n$ and thus we have reached a contradiction. $\square$

Proposition 2.10.

Suppose $A \subset \mathcal{R}^2$ is a measurable set with $m(A) = 0$ and let $B \subset A$. Then, B is measurable and $m(B) = 0$.

Proof.

Fix $\epsilon > 0$ in $\mathcal{R}$. Since A is measurable and $m(A) = 0$, for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(H_n^k)_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1}^{\infty} H_n^k$, $\sum_{n=1}^{\infty} a(H_n^k)$ converges, and $\sum_{n=1}^{\infty} a(H_n^k) - m(A) = \sum_{n=1}^{\infty} a(H_n^k) < d^k$.

For every $n \in \mathbb{N}$, let $G_n = \emptyset$ which is a simple region.

Let $k_0 \in \mathbb{N}$ be large enough that $d^{k_0} < \epsilon$. Then, $\bigcup_{n=1}^{\infty} G_n \subset B \subset \bigcup_{n=1}^{\infty} H_n^{k_0}$ and $\sum_{n=1}^{\infty} a(H_n^{k_0}) - \sum_{n=1}^{\infty} a(G_n) = \sum_{n=1}^{\infty} a(H_n^{k_0}) < d^{k_0} < \epsilon$. Hence B is measurable. Since for every $k \in \mathbb{N}$, $B \subset \bigcup_{n=1}^{\infty} H_n^k$ it follows that $m(B) \leqslant \sum_{n=1}^{\infty} a(H_n^k) < d^k$, by letting $k \to \infty$ we see that $m(B) = 0$. $\square$

Proposition 2.11.

Suppose $A \subset \mathcal{R}^2$ is a countable set. Then, A is measurable and $m(A) = 0$.

Proof.

Since A is a countable set there is a sequence of points $((x_n, y_n))_{n=1}^{\infty}$ such that $A = \bigcup_{n=1}^{\infty} (x_n, y_n)$.

Fix $\epsilon > 0$. For every $n \in \mathbb{N}$ define $H_n = \{(x, y) \in \mathbb{R}^2 : x \in (x_n - (d^n\epsilon)^{\frac{1}{2}}, x_n + (d^n\epsilon)^{\frac{1}{2}}), y \in (y_n - (d^n\epsilon)^{\frac{1}{2}}, y_n + (d^n\epsilon)^{\frac{1}{2}})\}$. Note that for every $n \in \mathbb{N}$, H_n is a simple region with $a(H_n) = 4d^n\epsilon$. So, $\lim_{n\to\infty} a(H_n) = \lim_{n\to\infty} 4d^n\epsilon = 0$, thus $\sum_{i=1}^{\infty} a(H_i)$ converges.

For every $j \in \mathbb{N}$, let $G_j = \emptyset$. Then, $\bigcup_{j=1}^{\infty} G_j \subset A \subset \bigcup_{i=1}^{\infty} H_i$, $\sum_{j=1}^{\infty} a(G_j)$ and $\sum_{i=1}^{\infty} a(H_i)$ converge, and $\sum_{i=1}^{\infty} a(H_i) - \sum_{j=1}^{\infty} a(G_j) = \sum_{i=1}^{\infty} a(H_i) \leqslant \sum_{i=1}^{\infty} 4d^i\epsilon = \frac{4d\epsilon}{1-d} < \epsilon$, which proves that A is measurable. Furthermore, since $A \subset \bigcup_{i=1}^{\infty} H_i$, $m(A) \leqslant \sum_{i=1}^{\infty} a(H_i) < \epsilon$. Taking the limit as $\epsilon \to 0$ shows that $m(A) = 0$. $\square$

Proposition 2.12.

Suppose $(H_k)_{k=1}^{\infty}$ and $(G_n)_{n=1}^{\infty}$ are sequences of mutually disjoint simple regions such that $\sum_{k=1}^{\infty} a(H_k)$ and $\sum_{n=1}^{\infty} a(G_n)$ both converge. Then, there exists a sequence of mutually disjoint simple regions $(T_m)_{m=1}^{\infty}$ such that $\bigcup_{k=1}^{\infty} H_k \cap \bigcup_{n=1}^{\infty} G_n = \bigcup_{m=1}^{\infty} T_m$ and $\sum_{m=1}^{\infty} a(T_m)$ converges.

Proof.

From Proposition 2.4 we know that for every $k, n \in \mathbb{N}$, there is a finite collection of mutually disjoint simple regions $(T_m^{k,n})_{m=1}^{l_{k,n}}$ such that $H_k \cap G_n = \bigcup_{m=1}^{l_{k,n}} T_m^{k,n}$. We assert that the collection $(((T_m^{k,n})_{m=1}^{l_{k,n}})_{k=1}^{\infty})_{n=1}^{\infty}$ is mutually disjoint; so consider $k_1, n_1, m_1 \in \mathbb{N}$ and $k_2, n_2, m_2 \in \mathbb{N}$ such that either $k_1 \neq k_2$, $n_1 \neq n_2$ or $m_1 \neq m_2$. Of course if $k_1 = k_2$ and $n_1 = n_2$ then $T_{m_1}^{k_1,n_1}$ and $T_{m_2}^{k_2,n_2}$ are both contained in $(T_m^{k_1,n_1})_{m=1}^{l_{k,n}}$ which is known to be mutually disjoint. If $n_1 \neq n_2$ then $T_{m_1}^{k_1,n_1} \subset H_{k_1} \cap G_{n_1} \subset G_{n_1}$ and $T_{m_2}^{k_2,n_2} \subset H_{k_2} \cap G_{n_2} \subset G_{n_2}$, but, G_{n_1} and G_{n_2} are disjoint so $T_{m_1}^{k_1,n_1}$ and $T_{m_2}^{k_2,n_2}$ must be disjoint. The same argument can be made in the case that $k_1 \neq k_2$, so the assertion is correct. Since $(((T_m^{k,n})_{m=1}^{l_{k,n}})_{k=1}^{\infty})_{n=1}^{\infty}$ is a countable collection it may be rewritten as $(T_m)_{m=1}^{\infty}$. Thus, $(T_m)_{m=1}^{\infty}$ is a collection of mutually disjoint simple regions such that $\bigcup_{k=1}^{\infty} H_k \cap \bigcup_{n=1}^{\infty} G_n = \bigcup_{m=1}^{\infty} T_m$ which was to be demonstrated. Since $\bigcup_{m=1}^{\infty} T_m \subset \bigcup_{k=1}^{\infty} H_k$ and $\sum_{k=1}^{\infty} a(H_k)$ converges, $\sum_{m=1}^{\infty} a(T_m)$ must also converge. □

Proposition 2.13.

Suppose that for every $k \in \mathbb{N}$, $(G_n^k)_{n=1}^{\infty}$ is a countable sequence of mutually disjoint simple regions such that $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(G_n^k)$ converges. Then, there exists a collection of mutually disjoint simple regions $(H_m)_{m=1}^{\infty}$ such that $\bigcup_{m=1}^{\infty} H_m = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k$ and $\sum_{m=1}^{\infty} a(H_m)$ converges.

Proof.

First we note that $((G_n^k)_{n=1}^{\infty})_{k=1}^{\infty}$ is a countable collection of simple regions and so may be rewritten as $(H_n^0)_{n=1}^{\infty}$. To create the sequence $(H_m)_{m=1}^{\infty}$ we begin by defining $H_1 = H_1^0$. Next we observe that by Proposition 2.5, for every $n, j \in \mathbb{N}$, $H_n^0 \backslash H_j^0$ is given by a finite number of mutually disjoint simple regions $(F_i^{n,j})_{i=1}^{t_{n,j}}$. So, for every $n \in \mathbb{N}$, $H_n^0 \backslash \bigcup_{j=1}^{n-1} H_j^0 = \bigcap_{j=1}^{n-1} (H_n^0 \backslash H_j^0) = \bigcap_{j=1}^{n-1} \bigcup_{i=1}^{t_{n,j}} F_i^{n,j}$ However, using the same argument as in the proof of Proposition 2.12, we deduce that for every $n \in \mathbb{N}$, $\bigcap\limits_{j=1}^{n-1} \bigcup\limits_{i=1}^{t_{n,j}} F_i^{n,j}$ can be expressed as the union of a finite number of mutually disjoint simple regions $(F_i^n)_{i=1}^{l_n}$.

We define

$$H_2 = F_1^2, \ldots, H_{l_2+1} = F_{l_2}^2$$

$$H_{l_2+2} = F_1^3, \ldots, H_{l_2+l_3+1} = F_{l_3}^3$$

$$H_{l_2+l_3+2} = F_1^4, \ldots, H_{l_2+l_3+l_4+1} = F_{l_4}^4$$

$$\vdots$$

and so on ad infinitum.

We see that by selection the H_m's are mutually disjoint and $\lim\limits_{m\to\infty} a(H_m) = 0$, so $\sum\limits_{m=0}^{\infty} a(H_m)$ converges as claimed. $\square$

Proposition 2.14.

For each $k \in \mathbb{N}$, suppose A_k is a measurable set. Furthermore, let $\lim\limits_{k\to\infty} m(A_k) = 0$. Then, $\bigcup\limits_{k=1}^{\infty} A_k$ is measurable and $m\left(\bigcup\limits_{k=1}^{\infty} A_k\right) \leqslant \sum\limits_{k=1}^{\infty} m(A_k)$. Moreover, if the sets $(A_k)_{k=1}^{\infty}$ are mutually disjoint then $m\left(\bigcup\limits_{k=1}^{\infty} A_k\right) = \sum\limits_{k=1}^{\infty} m(A_k)$.

Proof.

Note that since $\lim\limits_{k\to\infty} m(A_k) = 0$, $\sum\limits_{k=1}^{\infty} m(A_k)$ converges. Fix $\epsilon > 0$. Since each A_k is measurable we see that for every $k \in \mathbb{N}$ there are two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A_k \subset \bigcup\limits_{n=1}^{\infty} H_n^k$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ and $\sum\limits_{n=1}^{\infty} a(H_n^k)$ both converge, and $\sum\limits_{n=1}^{\infty} a(H_n^k) - \sum\limits_{n=1}^{\infty} a(G_n^k) < d^k\epsilon$.

Since $\lim\limits_{k\to\infty} m(A_k) = 0$ and $\sum\limits_{n=1}^{\infty} a(G_n^k) \leqslant \sum\limits_{n=1}^{\infty} a(H_n^k) < m(A_k) + d^k\epsilon$, we arrive at the conclusion that $\lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} a(H_n^k) = 0$. Thus, $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(G_n^k)$ and $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(H_n^k)$ both converge.

From the proof of Proposition 2.13 we know there exist two sequences of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k$ and $\bigcup_{n=1}^{\infty} H_n = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k$. Therefore, $\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k \backslash \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k$

Now, for every $k \in \mathbb{N}$ we have that $\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} H_n^k$, so $\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k \backslash \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} G_n^k = \bigcup_{k=1}^{\infty}(\bigcup_{n=1}^{\infty} H_n^k \backslash \bigcup_{n=1}^{\infty} G_n^k)$ Moreover, since for every $k \in \mathbb{N}$, the sequences $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ are both mutually disjoint we can arrange them in such a way that for every $n \in \mathbb{N}$, $G_n^k \subset H_n^k$. Thus, for every $k \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} H_n^k \backslash \bigcup_{n=1}^{\infty} G_n^k = \bigcup_{n=1}^{\infty}(H_n^k \backslash G_n^k)$. So we conclude that $\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}(H_n^k \backslash G_n^k)$. Therefore,

$$
\begin{aligned}
m\left(\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n\right) &= m\left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}(H_n^k \backslash G_n^k)\right)\\
&\leqslant \sum_{k=1}^{\infty} m\left(\bigcup_{n=1}^{\infty}(H_n^k \backslash G_n^k)\right)\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} m(H_n^k \backslash G_n^k)\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(a(H_n^k) - a(G_n^k))\\
&= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k)\right)
\end{aligned}
$$

But, $m\left(\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n\right) = \sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n)$ and from above we know that for every $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} a(H_n^k) - \sum_{n=1}^{\infty} a(G_n^k) \leqslant d^k\epsilon$. Therefore, $\sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) \leqslant \sum_{k=1}^{\infty} d^k\epsilon = \frac{d}{1-d}\epsilon < \epsilon$ which proves that $\bigcup_{k=1}^{\infty} A_k$ is measurable.

Since $\bigcup\limits_{k=1}^{\infty} A_k \subset \bigcup\limits_{k=1}^{\infty}\bigcup\limits_{n=1}^{\infty} H_n^k$ we have that

$$\begin{aligned} m\left(\bigcup_{k=1}^{\infty} A_k\right) &\leqslant m\left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} H_n^k\right) \\ &\leqslant \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a(H_n^k) \\ &\leqslant \sum_{k=1}^{\infty}(m(A)+d^k\epsilon) \\ &< \sum_{k=1}^{\infty} m(A)+\epsilon. \end{aligned}$$

The above holds for any $\epsilon > 0$ so we obtain $m(\bigcup\limits_{k=1}^{\infty} A_k) \leqslant \sum\limits_{k=1}^{\infty} m(A)$.

Now, assume that the A_k's are mutually disjoint, and let $\epsilon > 0$ in $\mathcal{R}$ be given. There exists a $K \in \mathbb{N}$ such that $\sum\limits_{k>K} m(A_k) < \frac{\epsilon}{2}$. Since $\bigcup\limits_{k=1}^{\infty} A_k$ is measurable there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $\bigcup\limits_{k=1}^{\infty} A_k \subset \bigcup\limits_{n=1}^{\infty} H_n$, $\sum\limits_{n=1}^{\infty} a(H_n)$ converges, and $\sum\limits_{n=1}^{\infty} a(H_n) - m(\bigcup\limits_{k=1}^{\infty} A_k) < \frac{\epsilon}{2}$.

Because the A_k's and the H_n's are mutually disjoint, and because $\bigcup\limits_{k=1}^{\infty} A_k \subset \bigcup\limits_{n=1}^{\infty} H_n$, we can find for every $k \in \{1, \ldots, K\}$ a sequence of mutually disjoint simple regions $(H_n^k)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} l(H_n^k)$ converges, $A_k \subset \bigcup\limits_{n=1}^{\infty} H_n^k \subset \bigcup\limits_{n=1}^{\infty} H_n$, and $\bigcup\limits_{n=1}^{\infty} H_n^1, \bigcup\limits_{n=1}^{\infty} H_n^2, \ldots, \bigcup\limits_{n=1}^{\infty} H_n^K$ are mutually disjoint. Thus,

$$\begin{aligned}
\sum_{k=1}^{K} m(A_k) &\leqslant \sum_{k=1}^{K} m\left(\bigcup_{n=1}^{\infty} H_n^k\right) \\
&= \sum_{k=1}^{K}\sum_{n=1}^{\infty} a(H_n^k) \\
&\leqslant \sum_{n=1}^{\infty} a(H_n) \\
&< m\left(\bigcup_{k=1}^{\infty} A_k\right) + \frac{\epsilon}{2}.
\end{aligned}$$

So,

$$\begin{aligned}
\sum_{k=1}^{\infty} m(A_k) &= \sum_{k=1}^{K} m(A_k) + \sum_{k>K} m(A_k) \\
&< m(\bigcup_{k=1}^{\infty} A_k) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= m(\bigcup_{k=1}^{\infty} A_k) + \epsilon.
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ reveals that $\sum\limits_{k=1}^{\infty} m(A_k) \leqslant m(\bigcup\limits_{k=1}^{\infty} A_k)$. This with the above result that $m(\bigcup\limits_{k=1}^{\infty} A_k) \leqslant \sum\limits_{k=1}^{\infty} m(A_k)$ allows the conclusion that $m(\bigcup\limits_{k=1}^{\infty} A_k) = \sum\limits_{k=1}^{\infty} m(A)$. $\square$

Proposition 2.15.

Let $K \in \mathbb{N}$ be given and for each $k \in \{1, \ldots, K\}$, let A_k be measurable. Then, $\bigcap\limits_{k=1}^{K} A_k$ is measurable and $m\left(\bigcap\limits_{k=1}^{K} A_k\right) \leqslant \min\{m(A_k) : k \in \{1, \ldots, K\}\}$.

Proof.

It is sufficient to show that the proposition holds for the case of two measurable sets A and B since the rest follows easily by induction. In that light suppose that A and B are measurable sets in $\mathcal{R}^2$ and fix $\epsilon > 0$ in $\mathcal{R}$.

By the definition of measurability, there exist four sequences of mutually disjoint simple regions $(G_n^A)_{n=1}^\infty$, $(G_n^B)_{n=1}^\infty$, $(H_n^A)_{n=1}^\infty$, and $(H_n^B)_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty G_n^A \subset A \subset \bigcup_{n=1}^\infty H_n^A$, $\bigcup_{n=1}^\infty G_n^B \subset B \subset \bigcup_{n=1}^\infty H_n^B$; $\sum_{n=1}^\infty a(G_n^A)$, $\sum_{n=1}^\infty a(G_n^B)$, $\sum_{n=1}^\infty a(H_n^A)$, and $\sum_{n=1}^\infty a(H_n^B)$ all converge; and finally $\sum_{n=1}^\infty a(H_n^A) - \sum_{n=1}^\infty a(G_n^A) \leqslant \epsilon$, $\sum_{n=1}^\infty a(H_n^B) - \sum_{n=1}^\infty a(G_n^B) \leqslant \epsilon$.

From Proposition 2.12 we know that there exist two sequences of mutually disjoint simple regions $(H_n)_{n=1}^\infty$ and $(G_n)_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty H_n = \bigcup_{n=1}^\infty H_n^A \cap \bigcup_{n=1}^\infty H_n^B$, $\bigcup_{n=1}^\infty G_n = \bigcup_{n=1}^\infty G_n^A \cap \bigcup_{n=1}^\infty G_n^B$; and $\sum_{n=1}^\infty a(H_n)$ and $\sum_{n=1}^\infty a(G_n)$ both converge. Obviously $\bigcup_{n=1}^\infty G_n \subset A \cap B \subset \bigcup_{n=1}^\infty H_n$. Since $\bigcup_{n=1}^\infty G_n \subset \bigcup_{n=1}^\infty G_n^A$ and $\bigcup_{n=1}^\infty H_n \subset \bigcup_{n=1}^\infty H_n^A$, and since each of the four sequences of simple regions are mutually disjoint simple regions, we have that

$$\sum_{n=1}^\infty a(H_n) - \sum_{n=1}^\infty a(G_n) \leqslant \sum_{n=1}^\infty a(H_n^A) - \sum_{n=1}^\infty a(G_n^A) \leqslant \epsilon$$

which proves that $A \cap B$ is measurable.

Since $A \cap B \subset A$, $m(A \cap B) \leqslant m(A)$ and since $A \cap B \subset B$, $m(A \cap B) \leqslant m(B)$. So, $m(A \cap B) \leqslant \min\{m(A), m(B)\}$. □

Proposition 2.16.

Suppose $A, B \subset \mathcal{R}^2$ are measurable sets. Then $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.

Proof.

Fix $\epsilon > 0$. First we note that by Propositions 2.12 and 2.13 $A \cup B$ and $A \cap B$ are measurable. Since $A \cup B$ is measurable there exists a collection of mutually disjoint simple regions $(H_n)_{n=1}^\infty$ such that $A \cup B \subset \bigcup_{n=1}^\infty H_n$, $\sum_{n=1}^\infty a(H_n)$ converges and $\sum_{n=1}^\infty a(H_n) - m(A \cup B) < \frac{\epsilon}{2}$.

Now, $A\backslash(A\cap B)$, $B\backslash(A\cap B)$, and $A\cap B$ are all mutually disjoint subsets of $A\cup B$, so there exist three subsequences of $(H_n)_{n=1}^{\infty}$ denoted by $(H_n^1)_{n=1}^{\infty}$, $(H_n^2)_{n=1}^{\infty}$, and $(H_n^3)_{n=1}^{\infty}$ such that $A\backslash(A\cap B)\subset\bigcup_{n=1}^{\infty}H_n^1$, $B\backslash(A\cap B)\subset\bigcup_{n=1}^{\infty}H_n^2$, and $(A\cap B)\subset\bigcup_{n=1}^{\infty}H_n^3$. Note that $\bigcup_{n=1}^{\infty}H_n^1\cup\bigcup_{n=1}^{\infty}H_n^2\cup\bigcup_{n=1}^{\infty}H_n^3=\bigcup_{n=1}^{\infty}H_n$.

Since $A=A\backslash(A\cap B)\cup(A\cap B)$ we see that $A\subset\bigcup_{n=1}^{\infty}H_n^1\cup\bigcup_{n=1}^{\infty}H_n^3$, so $m(A)\leqslant\sum_{n=1}^{\infty}a(H_n^1)+\sum_{n=1}^{\infty}a(H_n^3)$. Since $B=B\backslash(A\cap B)\cup(A\cap B)$ we see that $B\subset\bigcup_{n=1}^{\infty}H_n^2\cup\bigcup_{n=1}^{\infty}H_n^3$, so $m(B)\leqslant\sum_{n=1}^{\infty}a(H_n^2)+\sum_{n=1}^{\infty}a(H_n^3)$. Thus,

$$\begin{aligned}m(A)+m(B)&\leqslant\sum_{n=1}^{\infty}a(H_n^1)+\sum_{n=1}^{\infty}a(H_n^3)+\sum_{n=1}^{\infty}a(H_n^2)+\sum_{n=1}^{\infty}a(H_n^3)\\&=\sum_{n=1}^{\infty}a(H_n)+\sum_{n=1}^{\infty}a(H_n^3)\end{aligned}$$

We have from above that $\sum_{n=1}^{\infty}a(H_n)\leqslant m(A\cup B)+\frac{\epsilon}{2}$, we also know that $\sum_{n=1}^{\infty}a(H_n^3)\leqslant m(A\cap B)+\frac{\epsilon}{2}$ since otherwise $\sum_{n=1}^{\infty}a(H_n)$ would be greater than $m(A\cup B)+\frac{\epsilon}{2}$. Therefore, $m(A)+m(B)\leqslant m(A\cup B)+m(A\cap B)+\epsilon$, this holds for any $\epsilon>0$ so we obtain that $m(A)+m(B)\leqslant m(A\cup B)+m(A\cap B)$.

Now we prove the other inequality. By the measurability of A and B there exist two sequences of mutually disjoint simple regions $(H_n^A)_{n=1}^{\infty}$ and $(H_n^B)_{n=1}^{\infty}$ such that $A\subset\bigcup_{n=1}^{\infty}H_n^A$, $B\subset\bigcup_{n=1}^{\infty}H_n^B$; $\sum_{n=1}^{\infty}a(H_n^A)$ and $\sum_{n=1}^{\infty}a(H_n^B)$ both converge; and $\sum_{n=1}^{\infty}a(H_n^A)<m(A)+\frac{\epsilon}{2}$ and $\sum_{n=1}^{\infty}a(H_n^B)<m(B)+\frac{\epsilon}{2}$.

B can be divided into two mutually disjoint subsets $B\backslash(A\cap B)$ and $A\cap B$. Following this logic we see that $(H_n^B)_{n=1}^{\infty}$ can be split into two subsequences $(H_n^{B,1})_{n=1}^{\infty}$ and $(H_n^{B,2})_{n=1}^{\infty}$ where $B\backslash(A\cap B)\subset\bigcup_{n=1}^{\infty}H_n^{B,1}$ and $A\cap B\subset\bigcup_{n=1}^{\infty}H_n^{B,2}$. So, $A\cup B=A\cup(B\backslash(A\cap B))\subset$

$\bigcup\limits_{n=1}^{\infty} H_n^A \cup \bigcup\limits_{n=1}^{\infty} H_n^{B,1}$, and thus $m(A \cup B) \leqslant \sum\limits_{n=1}^{\infty} a(H_n^A) + \sum\limits_{n=1}^{\infty} a(H_n^{B,1}) \leqslant m(A) + \frac{\epsilon}{2} + \sum\limits_{n=1}^{\infty} a(H_n^{B,1})$.

Since $A \cap B \subset \bigcup\limits_{n=1}^{\infty} H_n^{B,2}$, it is clear that $m(A \cap B) \leqslant \sum\limits_{n=1}^{\infty} a(H_n^{B,2})$. So we obtain

$$
\begin{aligned}
m(A \cup B) + m(A \cap B) &\leqslant m(A) + \frac{\epsilon}{2} + \sum_{n=1}^{\infty} a(H_n^{B,1}) + \sum_{n=1}^{\infty} a(H_n^{B,2}) \\
&\leq m(A) + \frac{\epsilon}{2} + \sum_{n=1} a(H_n^B) \\
&\leq m(A) + m(B) + \epsilon
\end{aligned}
$$

The above inequality holds for any $\epsilon > 0$ so we find that $m(A \cup B) + m(A \cap B) \leqslant m(A) + m(B)$. This with our previous result gives the equality $m(A) + m(B) = m(A \cap B) + m(A \cup B)$ or $m(A \cup B) = m(A) + m(B) - m(A \cap B)$. □

2.2 Analytic Functions

Definition 2.17 (Finite Simple Region and Order of Magnitude)**.**

Let $A \subset \mathcal{R}^2$ be a simple region. Without loss of generality we assume $A = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$ where $a \leqslant b$, $h_1, h_2 : I(a, b) \to \mathcal{R}$ are analytic functions, and $h_1 < h_2$. We define $\lambda_x(A) = \lambda(b - a)$ and $\lambda_y(A) = i(h_2(x) - h_1(x))$ where $i(h_2(x) - h_1(x))$ is the index of the analytic function on $I(a, b)$, we call these A's order of magnitude in x and y. If $\lambda_x(A) = \lambda_y(A) = 0$ then we call A a finite simple region.

Definition 2.18 (Analytic Functions on $\mathcal{R}^2$)**.**

Suppose $A \subset \mathcal{R}^2$ is a simple region. Then, $f : A \to \mathcal{R}^2$ is an analytic function if for every $(x_0, y_0) \in A$, there is a simple region A_0 containing (x_0, y_0) that satisfies $\lambda_x(A_0) = \lambda_x(A)$, $\lambda_y(A_0) = \lambda_y(A)$ and a regular sequence $(a_{ij})_{i,j=0}^{\infty}$ such that for every $s, t \in \mathcal{R}$, if

$(x_0+s, y_0+t) \in A \cap A_0$*, then* $f(x_0+s, y_0+t) = \sum\limits_{i,j=0}^{\infty} a_{ij} s^i t^j = f(x_0, y_0) + \sum\limits_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j$*, where the power series converges in the weak topology[5].*

The following proposition follows directly from Definition 2.18.

Proposition 2.19.

Suppose $f : A \to \mathcal{R}$ *is an analytic function. Then for a fixed* x*, the function* $g(y) \coloneqq f(x, y)$ *is analytic on* $I_x \coloneqq \{y \in \mathcal{R} : (x, y) \in A\}$ *and for a fixed* y*,* $h(x) \coloneqq f(x, y)$ *is analytic on* $I_y \coloneqq \{x \in \mathcal{R} : (x, y) \in A\}$*.*

Proposition 2.20.

Let $A = \{(x, y) \in \mathcal{R}^2 : y \in [h_1(x), h_2(x)], x \in [a, b]\}$ *be a simple region in* $\mathcal{R}^2$*, and let* $f : A \to \mathcal{R}$ *be an analytic function on* A*. Then,* f *is bounded on* A*.*

Proof.

Let $F(x, y) : [0, 1]^2 \to \mathcal{R}$ be given by

$$F(x, y) = f((b-a)x + a, (h_2(x) - h_1(x))y + h_1(x)).$$

Then, F is an analytic on $[0, 1]^2$ and f is bounded on A if and only if F is bounded on $[0, 1] \times [0, 1]$.

For every $(v, w) \in \mathcal{R}^2$ let $N((v, w), \eta) = \{(x, y) \in \mathcal{R}^2 : \sqrt{(x-v)^2 + (y-w)^2} < \eta\}$. By the definition of analytic functions for every $(x_0, y_0) \in [0, 1]^2 \cap \mathbb{R}^2$, there exists a finite $\eta(x_0, y_0) > 0$ and a regular sequence $(a_{ij}(x_0, y_0))_{i,j=0}^{\infty}$ such that for every $(x, y) \in N((x_0, y_0), \eta(x_0, y_0)) \cap [0, 1]^2$, $F(x, y) = \sum\limits_{i,j=0}^{\infty} a_{ij}(x_0, y_0)(x - x_0)^i (y - y_0)^j$.

The set $\{N((x,y), \frac{\eta(x,y)}{2}) \cap \mathbb{R}^2 : (x,y) \in [0,1]^2 \cap \mathbb{R}^2\}$ is an open cover of $[0,1]^2 \cap \mathbb{R}^2$ which is a compact set of the Euclidean space $\mathbb{R}^2$, so there exists a finite set of points $\{(x_k, y_k)\}_{k=1}^m$ contained in $[0,1]^2 \cap \mathbb{R}^2$ such that $[0,1]^2 \cap \mathbb{R}^2 \subset \bigcup_{k=1}^{n} N((x_k, y_k), \frac{\eta(x_k,y_k)}{2}) \cap \mathbb{R}^2$. From this we have that $[0,1]^2 \subset \bigcup_{k=1}^{n} N((x_k, y_k), \eta(x_k, y_k))$.

Let $l = \min_{1 \leqslant k \leqslant n}\{\min\{\bigcup_{i,j=0}^{\infty} supp(a_{ij}(x_k, y_k))\}\}$, which exists by the regularity of the sequence $(a_{ij}(x_k, y_k))$ for each k. It follows from the above that $|F(x,y)| < d^{l-1}$ for every $(x,y) \in [0,1]^2$. Thus F is bounded on $[0,1]^2$ and hence f is bounded on A. $\square$

Proposition 2.21.

Let $\{(x_k, y_k)\}_{k=1}^n$, l, $F(x,y)$, and $\eta(x_k, y_k)$ be as in Proposition 2.20 and the proof thereof. Then, l is independent of our choice of $\{(x_k, y_k)\}_{k=1}^n$.

Proof.

Assume not, then there exists a set of points $\{(w_k, z_k)\}_{k=1}^m$ such that $\{N((w_k, z_k), \eta(w_k, z_k)) : k \in \{1, \ldots, m\}\}$ is a finite open cover of $[0,1]^2$,

$$l_0 = \min_{1 \leqslant k \leqslant m}\left\{\min\left\{\bigcup_{ij} supp(a_{ij}(w_k, z_k))\right\}\right\}$$

and

$$l_0 \neq l.$$

We assume without loss of generality that $l < l_0$, in particular, $l < \infty$.

Define $F_{\mathbb{R}} : [0,1]^2 \cap \mathbb{R}^2 \to \mathbb{R}$, by $F_{\mathbb{R}}(x,y) = (F(x,y))[l]$. For $(x,y) \in N((x_k, y_k), \eta((x_k, y_k))) \cap$

$[0,1]^2 \cap \mathbb{R}^2$,

$$F_{\mathbb{R}}(x,y) = \left(\sum_{i,j=0}^{\infty} a_{ij}((x_k,y_k))(x-x_k)^i(y-y_k)^j \right)[l]$$
$$= \sum_{i,j=0}^{\infty} (a_{ij}(x_k,y_k))[l](x-x_k)^i(y-y_k)^j$$

So, $F_l(x,y)$ is an $\mathbb{R}^2$-analytic function. Furthermore, $F_{\mathbb{R}}(x,y) = 0$ everywhere in

$$N((w_k,z_k),\eta((w_k,z_k)) \cap [0,1]^2 \cap \mathbb{R}^2,$$

so by the identity theorem for real analytic functions $F_{\mathbb{R}} = 0$ everywhere on $[0,1]^2 \cap \mathbb{R}^2$. Then, for every $i,j \in \mathbb{N} \cup \{0\}$ and for every $k \in \{1,\ldots,m\}$, $a_{ij}(x_k,y_k)[l] = 0$, which contradicts the definition of l. $\square$

Proposition 2.22.

Let A, f, F, $((x_k,y_k))_{k=1}^n$, $\eta(x,y)$, and l be as in Proposition 2.20 and the subsequent proof. Then, $\lambda(f(x,y)) = l$ almost everywhere on

$$\{(x,y) \in A : supp(x-a) = \{\lambda_x(A)\}, supp(y-h_1(x)) = \{\lambda_y(A)\}\}$$

and for any such point (x,y), the same is true for points (u,v) satisfying $\lambda(x-u) > \lambda_x(A)$ and $\lambda(y-v) > \lambda_y(A)$.

Proof.

First note that $\lambda(f(x, y)) = l$ almost everywhere on

$$\{(x, y) \in A : supp(x - a) = \{\lambda_x(A)\}, supp(y - h_1(x)) = \{\lambda_y(A)\}\}$$

if and only if $\lambda(F(x, y)) = l$ almost everywhere on $[0, 1]^2 \cap \mathbb{R}^2$.

Fix $(x_0, y_0) \in [0, 1]^2 \cap \mathbb{R}^2$. Then there is a $k \in \{1, \ldots, n\}$ such that $(x_0, y_0) \in N((x_k, y_k), \eta(x_k, y_k))$, hence $F(x_0, y_0) = \sum\limits_{i,j=0}^{\infty} a_{ij}(x_k, y_k)(x_0 - x_k)^i(y_0 - y_k)^j$.

Since $\lambda(a_{ij}(x_k, y_k) \geqslant l$, and $\lambda(x_0 - x_k)$ and $\lambda(y_0 - y_k)$ are both greater than or equal to 0 in each of the finitely many covering regions, $\lambda(F(x_0, y_0)) \geqslant l$. Moreover, the real non-zero analytic function $F_{\mathbb{R}}(x_0, y_0) = (\sum\limits_{i,j=0}^{\infty} a_{ij}(x_k, y_k)(x_0 - x_k)^i(y_0 - y_k)^j)[l]$ can have at most a finite number of zeros on $[0, 1]^2 \cap \mathbb{R}^2$, and if $F_{\mathbb{R}}(x_0, y_0) \neq 0$ then $\lambda(F(x_0, y_0)) = l$ so $\lambda(F(x_0, y_0)) = l$ almost everywhere on $[0, 1]^2 \cap \mathbb{R}^2$. Furthermore, suppose that the point $(x_p, y_p) \in [0, 1]^2 \cap \mathbb{R}^2$ is such that $\lambda(F(x_p, y_p)) = l$. Also, let $(x, y) \in [0, 1]^2$ satisfy $\sqrt{(x - x_p)^2 + (y - y_p)^2} \ll 1$. Then, $F(x, y) = F(x_p, y_p) + \sum\limits_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij}(x_p, y_p)(x - x_p)^i(y - y_p)^j$ but for every $i, j \in \mathbb{N}$, $\lambda((x - x_p)^i) > 0$, $\lambda((y - y_p)^j) > 0$, and $\lambda(a_{ij}(x_p, y_p)) \geqslant l$. Thus, $\lambda(F(x, y)) = \lambda(F(x_p, y_p)) = l$. $\square$

Definition 2.23 (Index of an Analytic Function on $\mathcal{R}^2$)**.**

Let A and f be as in Proposition 2.20 and let l be as in the subsequent proof. Then we call l the index of f on A and we denote it $i(f)$.

Proposition 2.24.

Suppose $A \subset \mathcal{R}^2$ is a finite simple region and let $f, g : A \to \mathcal{R}^2$ be analytic functions on A.

Then, for every $\alpha \in \mathcal{R}$, $f + \alpha g$ and $f \cdot g$ are analytic functions on A.

Proof.

Fix $(x_0, y_0) \in A$. Since f and g are analytic on A, there exist finite $\eta_1, \eta_2 > 0$ such that for every $s, t \in \mathcal{R}$, if $s^2 + t^2 < \eta_1^2$ and $(x_0 + s, y_0 + t) \in A$ then $f(x_0 + s, y_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j$, and if $s^2 + t^2 < \eta_2^2$ and $(x_0 + s, y_0 + t) \in A$ then $g(x_0 + s, y_0 + t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} s^k t^l$.

Let $\eta = \min\{\eta_1, \eta_2\}$. For every $s, t \in \mathcal{R}$, if $s^2 + t^2 < \eta^2$ and $(x_0 + s, y_0 + t) \in A$, then, for every $\alpha \in \mathcal{R}$,

$$\begin{aligned}
(f + \alpha g)(x_0 + s, y_0 + t) &= f(x_0 + s, y_0 + t) + \alpha g(x_0 + s, y_0 + t) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j + \alpha \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} s^k t^l \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_{ij} + \alpha b_{ij}) s^i t^j.
\end{aligned}$$

Additionally, as in [5],

$$\begin{aligned}
(f \cdot g)(x_0 + s, y_0 + t) &= f(x_0 + s, y_0 + t) \cdot g(x_0 + s, y_0 + t) \\
&= (\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j)(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} s^k t^l) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i+k=n} \sum_{j+l=m} a_{ij} b_{kl} s^n t^m
\end{aligned}$$

Note that the Cauchy product of two power series in $\mathcal{R}$ has been shown to converge in [5], we infer from this that the Cauchy product of two power series in $\mathcal{R}^2$ will also converge. Defining $c_{nm} = \sum_{i+k=n} \sum_{j+l=m} a_{ij} b_{kl}$ gives $(f \cdot g)(x_0 + s, y_0 + t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} s^n t^m$. So, $f + \alpha g$ and $f \cdot g$ are analytic functions on A. $\square$

Corollary 2.25.

Suppose $A \subset \mathcal{R}$ is a simple region and let $f, g : A \to \mathcal{R}$ be analytic functions on A. Then, for every $\alpha \in \mathcal{R}$, $f + \alpha g$ and $f \cdot g$ are analytic functions on A.

Proof.

Let A be a simple region given by $A = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$ and fix $\alpha \in \mathcal{R}$. Let $A_0 = \{(x, y) \in \mathcal{R}^2 : y \in I(d^{-\lambda_y(A)}h_1(d^{\lambda_x(A)}x), d^{-\lambda_y(A)}h_2(d^{\lambda_x(A)}x)), x \in I(d^{-\lambda_x(A)}a, d^{-\lambda_x(A)}b)\}$, then A_0 is a finite simple region. Moreover, $F(x, y) = f(d^{\lambda_x(A)}x, d^{\lambda_y(A)}y)$ and $G(x, y) = g(d^{\lambda_x(A)}x, d^{\lambda_y(A)}y)$ are both analytic on A_0 so by Proposition 2.24 $F + \alpha G$ and $F \cdot G$ are analytic on A_0. Therefore, $f + \alpha g$ and $f \cdot g$ are analytic functions on A. $\square$

Proposition 2.26.

Suppose $A \subset \mathcal{R}^2$ is a finite simple region and let $f : A \to \mathcal{R}$ be an analytic function on A. Let $a, b \in \mathcal{R}$ be such that $a < b$ and $b - a$ is finite, let $g : I(a, b) \to \mathcal{R}$ be an $\mathcal{R}$-analytic function on $I(a, b)$ such that for every $x \in I(a, b)$, $(x, g(x)) \in A$. Then, $f(x, g(x))$ is an $\mathcal{R}$-analytic function on $I(a, b)$.

Proof.

Let $x_0 \in A$ be given. By the definition of analytic functions there exist $\eta_1, \eta_2 > 0$ such that η_1 and η_2 are finite and if $|h| < \eta_1$ and $x_0 + h \in I(a, b)$ then $g(x_0 + h) = g(x_0) + \sum\limits_{n=1}^{\infty} a_n h^n$ and if $s^2 + t^2 < \eta_2^2$ and $(x_0 + s, g(x_0) + t) \in A$ then $f(x_0 + s, g(x_0) + t) = \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} b_{ij} s^i t^j$.

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(X) = (X^2 + (\sum\limits_{n=1}^{\infty} a_n X^n)^2)[0]$. Then F is continuous on $\mathbb{R}$ so we can choose a $\eta \in (0, \frac{\eta_1}{2}]$ such that if $|h| < \eta$ and $x_0 + h \in I(a, b)$ then $F(h[0]) < \frac{\eta_2^2}{2}$, and since $h^2 + (\sum\limits_{n=1}^{\infty} a_n h^n)^2$ is different from $F(h[0])$ by at most an infinitely small amount it follows that if $|h| < \eta$ and $x_0 + h \in I(a, b)$ then $h^2 + (\sum\limits_{n=1}^{\infty} a_n h^n)^2 < \eta_2^2$. Thus, for any

$|h| < \eta$ such that $x_0 + h \in I(a,b)$, $h^2 + (\sum\limits_{n=1}^{\infty} a_n h^n)^2 < \eta_2^2$ and $(x_0 + h, g(x_0) + \sum\limits_{n=1}^{\infty} a_n h^n) \in A$, so $f(x_0 + h, g(x_0 + h)) = f(x_0, g(x_0)) + \sum\limits_{\substack{i,j=0\\i+j\neq 0}}^{\infty} b_{ij} h^i (\sum\limits_{n=1}^{\infty} a_n h^n)^j$.

Now, for every $i, j \in \mathbb{N}$, let $V_{ij}(h) = b_{ij} h^i (\sum\limits_{n=1}^{\infty} a_n h^n)^j$. Then, for every $i, j \in \mathbb{N}$, $V_{ij}(h)$ can be rewritten as $V_{ij}(h) = \sum\limits_{m=1}^{\infty} c_{ijm} h^m$. Since for every $i, j \in \mathbb{N}$, $\sum\limits_{m=1}^{\infty} c_{ijm} h^m$ converges, it follows that for every $i, j \in \mathbb{N}$ and for every $q \in \mathbb{Q}$, $\sum\limits_{m=1}^{\infty} (c_{ijm} h^m)[q]$ converges in $\mathbb{R}$. This, in addition to the fact that for every $q \in \mathbb{Q}$, $\sum\limits_{\substack{i,j=0\\i+j\neq 0}}^{\infty} (b_{ij} h^i (\sum\limits_{n=1}^{\infty} a_n h^n)^j)[q] = \sum\limits_{\substack{i,j=0\\i+j\neq 0}}^{\infty} \sum\limits_{m=1}^{\infty} (c_{ijm} h^m)[q]$ converges in $\mathbb{R}$ allows us to change the order of summation in the triple sum. Thus, for every $q \in \mathbb{Q}$,

$$
\begin{aligned}
f\left(x_0 + h, g(x_0) + \sum_{n=1}^{\infty} a_n h^n\right)[q] &= f(x_0, g(x_0))[q] + \sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty} \sum_{m=1}^{\infty} (c_{ijm} h^m)[q] \\
&= f(x_0, g(x_0))[q] + \sum_{m=1}^{\infty} \sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty} (c_{ijm} h^m)[q].
\end{aligned}
$$

And so,

$$
\begin{aligned}
f(x_0 + h, g(x_0 + h)) &= f(x_0, g(x_0)) + \sum_{m=1}^{\infty} \sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty} c_{ijm} h^m \\
&= f(x_0, g(x_0)) + \sum_{m=1}^{\infty} \left(\sum_{\substack{i,j=0\\i+j\neq 0}}^{\infty} c_{ijm} \right) h^m
\end{aligned}
$$

Letting $e_m = \sum\limits_{\substack{i,j=0\\i+j\neq 0}}^{\infty} c_{ijm}$ we obtain $f(x_0 + h, g(x_0 + h)) = f(x_0, g(x_0)) + \sum\limits_{m=1}^{\infty} e_m h^m$. So, verily we see that for every $x \in I(a,b)$ there exists a finite $\eta > 0$ such that for every $|h| < \eta$, $f(x_0 + h, g(x_0 + h))$ is given by a power series about x_0, which means that $f(x, g(x))$ is an $\mathcal{R}$-analytic function. $\square$

Corollary 2.27.

Suppose $A \subset \mathcal{R}^2$ *is a simple region given by* $A = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$ *and* $f : A \to \mathcal{R}$ *is an analytic function on* A*. Let* $\alpha \in \mathcal{R}\backslash\{0\}$ *be a constant and define* $g(x,y) = f(\alpha x, y)$*, then* g *is an analytic function on the simple region* $B = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(\alpha x), h_2(\alpha x)), x \in I(\frac{a}{\alpha}, \frac{b}{\alpha})\}$.

Corollary 2.28 follows directly from Proposition 2.26 using a scaling argument similar to that for Corollary 2.25.

Corollary 2.28.

Suppose $A \subset \mathcal{R}^2$ *is a simple region and let* $f : A \to \mathcal{R}$ *be an analytic function on* A*. Let* $a, b \in \mathcal{R}$ *be such that* $a < b$ *and let* $g : I(a,b) \to \mathcal{R}$ *be an* $\mathcal{R}$*-analytic function on* $I(a,b)$ *such that for every* $x \in I(a,b)$*,* $(x, g(x)) \in A$*. Then,* $F(x) \coloneqq f(x, g(x))$ *is an* $\mathcal{R}$*-analytic function on* $I(a,b)$*.*

Proposition 2.29.

Suppose $S \subset \mathcal{R}^2$ *is a finite simple region and let* $f : S \to \mathcal{R}$ *be an analytic function. Let* $H \subset \mathcal{R}^2$ *be a simple region and let* $g : H \to \mathcal{R}$ *be an analytic function such that for every* $(x,y) \in H$*,* $(x, g(x,y)) \in S$*. Then,* $F(x,y) \coloneqq f(x, g(x,y))$ *is an analytic function on* H*.*

Proof.

Fix $(x_0, y_0) \in H$, then $(x_0, g(x_0, y_0)) \in S$. There exist $\eta_1, \eta_2 > 0$ such that η_1 and η_2 are finite and for every $(r,t) \in \mathcal{R}^2$, if $r^2 + t^2 < \eta_1^2$ and $(x_0 + r, y_0 + t) \in H$, then $g(x_0 + r, y_0 + t) = g(x_0, y_0) + \sum\limits_{\substack{i,j=0 \\ i+j \neq 0}}^{\infty} a_{i,j}(x_0, y_0) r^i t^j$, and, if $r^2 + t^2 < \eta_2^2$ and $(x_0 + r, g(x_0, y_0) + t) \in S$, then $f(x_0 + r, g(x_0, y_0) + t) = f(x_0, g(x_0, y_0)) + \sum\limits_{\substack{k,l=0 \\ k+l \neq 0}}^{\infty} b_{k,l}(x_0, y_0) r^k t^l$, where (a_{ij}) and $(b_{k,l})$ are regular sequences.

Define $G : \mathbb{R} \to \mathbb{R}$ by $G(R,T) = (R^2 + (\sum\limits_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij}(x_0, y_0)R^iT^j)^2)[0]$, clearly G is a continuous function. Thus, there is a $\eta \in (0, \eta_1) \cap \mathbb{R}$, such that if $R^2 + T^2 < 2\eta^2$ then $G(r,t) < \eta_2^2$.

Now, if $r, t \in \mathcal{R}$ with $r^2 + t^2 < \eta^2$ and $(x_0 + r, y_0 + t) \in H$ then

$$\left| r^2 + \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} r^i t^j \right)^2 - \left((r[0])^2 + \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} (r[0])^i (t[0])^j \right)^2 \right] [0] \right| \ll 1$$

since $r =_0 r[0]$ and $\sum\limits_{i,j=1}^{\infty} a_{ij} r^i t^j =_0 \left(\sum\limits_{\substack{i,j=0 \\ i+j=0}}^{\infty} a_{ij} (r[0])^i (t[0])^j \right) [0]$. Therefore, if $r^2 + t^2 < \eta^2$ and $(x_0 + r, y_0 + t) \in H$ then $r^2 + (\sum\limits_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij}(x_0, y_0) r^i t^j)^2 < \eta_2^2$. So, for every $(r,t) \in \mathcal{R}^2$ such that $r^2 + t^2 < \eta^2$, if $(x_0 + r, y_0 + t) \in H$, then

$$\begin{aligned} f(x_0 + r, g(x_0 + r, y_0 + t)) &= f\left(x_0 + r, g(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij}(x_0, y_0) r^i t^j \right) \\ &= f(x_0, g(x_0, y_0)) + \sum_{\substack{k,l=0 \\ k+l\neq 0}}^{\infty} b_{kl}(x_0, g(x_0, y_0)) r^k \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij}(x_0, y_0) r^i t^j \right)^l \end{aligned}$$

Using an argument similar to the one used in the proof of Proposition 2.26 we can rewrite

$$\sum_{\substack{k,l=0 \\ k+l\neq 0}}^{\infty} b_{kl}(x_0, g(x_0, y_0)) r^k \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij}(x_0, y_0) r^i t^j \right)^l$$

as $\sum\limits_{\substack{m,n=0\\m+n\neq 0}}^{\infty} c_{m,n}(x_0, y_0)r^m t^n$. Therefore,

$$f(x_0 + r, g(x_0 + r, y_0 + t)) = f(x_0, g(x_0, y_0)) + \sum_{\substack{m,n=0\\m+n\neq 0}}^{\infty} c_{m,n}(x_0, y_0)r^m t^n,$$

which proves that $F(x, y) \coloneqq f(x, g(x, y))$ is analytic on H. $\square$

If $A \subset \mathcal{R}^2$ is a simple region and $f : A \to \mathcal{R}$ is an analytic function defined on A, then it is always possible to scale the arguments of f so that it is defined on a finite simple region A_0. This fact allows us to immediately ascertain the following corollary.

Corollary 2.30. *Suppose $S \subset \mathcal{R}^2$ is a simple region and let $f : S \to \mathcal{R}$ be an analytic function. Let $H \subset \mathcal{R}^2$ be a simple region and let $g : H \to \mathcal{R}$ be an analytic function such that for every $(x, y) \in H$, $(x, g(x, y)) \in S$. Then, $F(x, y) = f(x, g(x, y))$ is an analytic function on H.*

Corollary 2.31.

Suppose S is a simple region and let $f : S \to \mathcal{R}$ be an analytic function. Let I and J be intervals in $\mathcal{R}$ and let $h : I \to \mathcal{R}$ be an $\mathcal{R}$-analytic function and let $g : I \times J \to \mathcal{R}$ be an analytic function such that, if $(x, y) \in I \times J$, then $(h(x), g(x, y)) \in S$. Then, $F(x, y) \coloneqq f(h(x), g(x, y))$ is an analytic function on $I \times J$.

2.3 Measurable Functions on $\mathcal{R}^2$

Definition 2.32 (Measurable Function)**.**

Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $f : A \to \mathcal{R}$ be bounded. Then we call f a *measurable function if for every $\epsilon > 0$ there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=0}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} a(G_n)$ converges, $m(A) - \sum_{n=1}^{\infty} a(G_n) < \epsilon$, and for all $n \in \mathbb{N}$ f is analytic on G_n.*

Proposition 2.33.

Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $f : A \to \mathcal{R}$ be a measurable function. Then, f is given locally by a power series almost everywhere on A.

Proof.

Let $A_0 = \{(x, y) \in A : f \text{ is not given locally by a power series about } (x, y)\}$ We show that A_0 is measurable and $m(A_0) = 0$. To this end let $\epsilon > 0$ be given in $\mathcal{R}$. Since f is measurable on A there exists a sequence of mutually disjoint open simple regions $(G_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n \subset A$, $\sum_{n=1}^{\infty} a(G_n)$ converges, $m(A) - \sum_{n=1}^{\infty} a(G_n) \leqslant \frac{\epsilon}{2}$, and f is analytic on G_n for all n.

By the measurability of A there must exist a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1} H_n$, $\sum_{n=1}^{\infty} a(H_n)$ converges, $\sum_{n=1}^{\infty} a(H_n) - m(A) \leqslant \frac{\epsilon}{2}$.

Since f is given locally by a power series about every point in $\bigcup_{n=1}^{\infty} G_n$ and since $A_0 \subset A$, we know that $A_0 \subset A \backslash \bigcup_{n=1}^{\infty} G_n$. Furthermore, since $A \subset \bigcup_{n=1}^{\infty} H_n$ we can conclude that $A_0 \subset \bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n$.

Because both $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ are mutually disjoint sequences we can arrange them so that for every $n \in \mathbb{N}$, $G_n \subset H_n$. For this reason we have $\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (H_n \backslash G_n)$. The $H_n \backslash G_n$'s are mutually disjoint, and by Proposition 2.5, for every $n \in \mathbb{N}$, $H_n \backslash G_n$ may be expressed as the union of a finite number of mutually disjoint simple regions. So we see that $\bigcup_{n=1}^{\infty} H_n \backslash \bigcup_{n=1}^{\infty} G_n$ may be rewritten as the union of countably many mutually disjoint simple regions $(H_n^0)_{n=1}^{\infty}$.

For every $n \in \mathbb{N}$, let $G_n^0 = \emptyset$. Then, we have $\bigcup\limits_{n=1}^{\infty} G_n^0 \subset A_0 \subset \bigcup\limits_{n=1}^{\infty} H_n^0$. Furthermore we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} a(H_n^0) - \sum_{n=1}^{\infty} a(G_n^0) &= \sum_{n=1}^{\infty} a(H_n^0) \\
&= \sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) \\
&= \left(\sum_{n=1}^{\infty} a(H_n) - m(A) \right) + \left(m(A) - \sum_{n=1}^{\infty} a(G_n) \right) \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So A_0 is measurable.

Since $A_0 \subset \bigcup\limits_{n=1}^{\infty} H_n^0$ we have that $m(A_0) \leqslant \sum\limits_{n=1}^{\infty} a(H_n^0) \leqslant \epsilon$, so by taking the limit as $\epsilon \to 0$ we see that $m(A) = 0$. □

Proposition 2.34.

Suppose A is a simple region and let $f : A \to \mathcal{R}$ be a measurable function that is differentiable with respect to both x and y. Additionally, suppose that $\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial y} f(x, y) = 0$ everywhere on A (see [6] for the definition of partial derivatives). Then, f(x, y) is constant on A

Proof. Fix two arbitrary points $(x_0, y_0), (x_1, y_1) \in A$. We know from Proposition 2.33 that $f(x, y)$ is analytic almost everywhere on A so by Corollary 2.19, for fixed y, $f(x, y)$ is $\mathcal{R}$-analytic almost everywhere on $\{x \in \mathcal{R} : (x, y) \in A\}$ and for fixed x, $f(x, y)$ is $\mathcal{R}$-analytic almost everywhere on $\{y \in \mathcal{R} : (x, y) \in A\}$. Therefore, $f(x, y_0)$ is $\mathcal{R}$-analytic almost everywhere on $\{x \in \mathcal{R} : (x, y_0) \in A\}$ and since $\frac{\partial}{\partial x} f(x, y_0) = 0$ everywhere on the same it follows that $f(x, y_0)$ is constant [3]. Thus, $f(x_0, y_0) = f(x_1, y_0)$. In the same way we see that $f(x_1, y_0) = f(x_1, y_1)$, so $f(x_0, y_0) = f(x_1, y_1)$. And so f must be constant on A. $\square$

Corollary 2.35.

Suppose A is a simple region with $f, g : A \to \mathcal{R}$ both measurable functions on A. Additionally, suppose that f and g are differentiable on A with respect to both x and y. Let $\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} g(x, y)$, and $\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} g(x, y)$. Then, f and g will be different by at most a constant on A.

Proposition 2.36.

Let $A, B \subset \mathcal{R}^2$ be measurable sets and let $f : A, B \to \mathcal{R}$ be a measurable function on both. Then, f is a measurable function on $A \cap B$ and $A \cup B$.

Proof.

Fix $\epsilon > 0$ in $\mathcal{R}$. Since f is measurable on A there exists a sequence of mutually disjoint simple regions $(G_n^A)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^A, $\bigcup\limits_{n=1}^{\infty} G_n^A \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^A)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^A) < \frac{\epsilon}{2}$.

Since f is measurable on B there exists a sequence of mutually disjoint simple regions $(G_n^B)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^B, $\bigcup\limits_{n=1}^{\infty} G_n^B \subset B$, $\sum\limits_{n=1}^{\infty} a(G_n^B)$ converges, and $m(B) - \sum\limits_{n=1}^{\infty} a(G_n^B) < \frac{\epsilon}{2}$.

By Proposition 2.6 there exists a sequence of mutually disjoint simple regions $(G_n^0)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^0 = (\bigcup\limits_{n=1}^{\infty} G_n^A) \cup (\bigcup\limits_{n=1}^{\infty} G_n^B) \subset A \cup B$, $\sum\limits_{n=1}^{\infty} a(G_n^0)$ converges and for every $n \in \mathbb{N}$, f is analytic on G_n^0. Morover, $m(A \cup B) - \sum\limits_{n=1}^{\infty} a(G_n^0) \leqslant m(A) - \sum\limits_{n=1}^{\infty} a(G_n^A) + m(B) - \sum\limits_{n=1}^{\infty} a(G_n^B) \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \leqslant \epsilon$. Thus, f is measurable on $A \cup B$.

Also, by Proposition 2.4 there exists a sequence of mutually disjoint simple regions

$(G_n^1)_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty G_n^1 = (\bigcup_{n=1}^\infty G_n^A) \cap (\bigcup_{n=1}^\infty G_n^B) \subset A \cap B$, $\sum_{n=1}^\infty a(G_n^1)$ converges and for every $n \in \mathbb{N}$, f is analytic on G_n^1.

Using Proposition 2.16 we have that $m(A \cap B) = m(A) + m(B) - m(A \cup B)$, and $m(\bigcup_{n=1}^\infty G_n^1) = m(\bigcup_{n=1}^\infty G_n^A) + m(\bigcup_{n=1}^\infty G_n^B) - m(\bigcup_{n=1}^\infty G_n^0)$. So,

$$
\begin{aligned}
& m(A \cap B) - \sum_{n=1}^\infty a(G_n^1) = m(A \cap B) - m\left(\bigcup_{n=1}^\infty G_n^1\right) \\
&= m(A) + m(B) - m(A \cup B) + m\left(\bigcup_{n=1}^\infty G_n^0\right) - m\left(\bigcup_{n=1}^\infty G_n^A\right) - m\left(\bigcup_{n=1}^\infty G_n^B\right) \\
&= m(A) - \sum_{n=1}^\infty a(G_n^A) + m(B) - \sum_{n=1}^\infty a(G_n^B) - m(A \cup B) + \sum_{n=1}^\infty a(G_n^0) \\
&\leqslant m(A) - \sum_{n=1}^\infty a(G_n^A) + m(B) - \sum_{n=1}^\infty a(G_n^B) \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Thus, f is measurable on $A \cap B$. □

Proposition 2.37.

Let $A \subset \mathcal{R}^2$ be a measurable set and suppose f and g are both measurable on A. Then, for any $\alpha \in \mathcal{R}$, $f + \alpha g$ and $f \cdot g$ are measurable on A.

Proof.

Fix $\epsilon > 0$ in $\mathcal{R}$. Since f and g are both measurable on A we know that there exist sequences of mutually disjoint simple regions $(G_n^f)_{n=1}^\infty$ and $(G_n^g)_{n=1}^\infty$ such that, for every $n \in \mathbb{N}$, f is analytic on G_n^f and g is analytic on G_n^g , $\bigcup_{n=1}^\infty G_n^f \subset A$, $\bigcup_{n=1}^\infty G_n^g \subset A$, $\sum_{n=1}^\infty a(G_n^f)$ and $\sum_{n=1}^\infty a(G_n^g)$ both converge, $m(A) - \sum_{n=1}^\infty a(G_n^f) < \frac{\epsilon}{2}$, and $m(A) - \sum_{n=1}^\infty a(G_n^g) < \frac{\epsilon}{2}$.

By Proposition 2.4 there exists a sequence of mutually disjoint simple regions $(T_i)_{i=1}^\infty$ such that $\bigcup\limits_{i=1}^{\infty} T_i = (\bigcup\limits_{n=1}^{\infty} G_n^f) \cap (\bigcup\limits_{n=1}^{\infty} G_n^g) \subset A$, $\sum\limits_{i=1}^{\infty} a(T_i)$ converges, and for every $i \in \mathbb{N}$, f and g are analytic on T_i.

Since $\bigcup\limits_{i=1}^{\infty} T_i = (\bigcup\limits_{n=1}^{\infty} G_n^f) \cap (\bigcup\limits_{n=1}^{\infty} G_n^g)$, we obtain that

$$\begin{aligned} m(A) - \sum_{i=1}^{\infty} a(T_i) &\leqslant m(A) - \sum_{n=1}^{\infty} a(G_n^f) + m(A) - \sum_{n=1}^{\infty} a(G_n^g) \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

We know from Proposition 2.24 that for any $\alpha \in \mathcal{R}$ and for every $i \in \mathbb{N}$, $f + \alpha g$ and $f \cdot g$ are analytic on T_i. Thus, there exists a sequence of mutually disjoint simple regions $(T_i)_{i=1}^{\infty}$ such that $\bigcup\limits_{i=1}^{\infty} T_i \subset A$, $\sum\limits_{i=1}^{\infty} a(T_i)$ converges, $m(A) - \sum\limits_{i=1}^{\infty} a(T_i) < \epsilon$, and for every $i \in \mathbb{N}$ and $\alpha \in \mathcal{R}$, $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on T_i. Therefore, for any $\alpha \in \mathcal{R}$, $f + \alpha g$ and $f \cdot g$ are measurable on A. $\square$

2.4 Integration on $\mathcal{R}^2$

Definition 2.38 (Integration of analytic Functions on Simple Regions). *Suppose H is a simple region and $f : H \to \mathcal{R}$ is an analytic function. Without loss of generality we assume $H = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$, where $a, b \in \mathcal{R}$, $a \leqslant b$, and $h_1, h_2 : I(a, b) \to \mathcal{R}$ are analytic with $h_1 < h_2$. We note that since the composition and anti-derivative of $\mathcal{R}$-analytic functions are also $\mathcal{R}$-analytic, the integral $\int\limits_{y \in I(h_1(x), h_2(x))} f(x, y)$ will always yield an $\mathcal{R}$-analytic function $F(x)$ on $I(a, b)$. We define*

$$\iint\limits_{(x,y)\in H} f(x, y) = \int\limits_{x \in I(a,b)} \left[\int\limits_{y \in I(h_1(x), h_2(x))} f(x, y) \right] = \int\limits_{x \in I(a,b)} F(x)$$

and call this the integral of f over H.

Proposition 2.39.

Suppose $G \subset \mathcal{R}^2$ is a simple region and $\alpha \in \mathcal{R}$ is an arbitrary constant. Then,

$$\iint\limits_{(x,y)\in G} \alpha = \alpha a(G).$$

Proof.

We assume without loss of generality that $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$, where $a, b \in \mathcal{R}$, $a \leqslant b$, and $g_1, g_2 : I(a,b) \to \mathcal{R}$ are analytic on I(a,b) with $g_1 < g_2$. Thus, by the linearity of integration on $\mathcal{R}$ [1],

$$\begin{aligned}
\iint\limits_{(x,y)\in G} \alpha &= \int\limits_{x\in I(a,b)} \left[\int\limits_{y\in I(g_1(x),g_2(x))} \alpha \right] \\
&= \int\limits_{x\in I(a,b)} \alpha \left[\int\limits_{y\in I(g_1(x),g_2(x))} 1 \right] \\
&= \alpha \int\limits_{x\in I(a,b)} \left[\int\limits_{y\in I(g_1(x),g_2(x))} 1 \right] \\
&= \alpha \int\limits_{x\in I(a,b)} [g_2(x) - g_1(x)] \\
&= \alpha a(G).
\end{aligned}$$

$\square$

Proposition 2.40.

Let $H \subset \mathcal{R}^2$ be a simple region and suppose $f, g : H \to \mathcal{R}$ are analytic functions. Let $\alpha \in \mathcal{R}$

be an arbitrary constant. Then, $\iint\limits_{(x,y)\in H}(f+\alpha g)(x,y) = \iint\limits_{(x,y)\in H} f(x,y) + \alpha \iint\limits_{(x,y)\in H} g(x,y)$.

Proof.

We assume without loss of generality that $H = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x),h_2(x)), x \in I(a,b)\}$, where $a,b \in \mathcal{R}$, $a \leqslant b$, and $h_1,h_2 : I(a,b) \to \mathcal{R}$, $h_1 < h_2$. Thus,

$$\iint\limits_{(x,y)\in H}(f+\alpha g)(x,y) = \int\limits_{x\in I(a,b)} \int\limits_{y\in I(h_1(x),h_2(x))} (f(x,y)+\alpha g(x,y)).$$

Moreover, for every $x \in I(a,b)$, $f(x,y)$ and $g(x,y)$ are $\mathcal{R}$-analytic on the interval $y \in I(h_1(x),h_2(x))$, so

$$\int\limits_{y\in I(h_1(x),h_2(x))} (f(x,y)+\alpha g(x,y)) = \int\limits_{y\in I(h_1(x),h_2(x))} f(x,y) + \alpha \int\limits_{y\in I(h_1(x),h_2(x))} g(x,y).$$

Therefore,

$$\iint\limits_{(x,y)\in H}(f+\alpha g)(x,y) = \int\limits_{x\in I(a,b)} \left(\int\limits_{y\in I(h_1(x),h_2(x))} f(x,y) + \alpha \int\limits_{y\in I(h_1(x),h_2(x))} g(x,y) \right).$$

And, since $\int\limits_{y\in I(h_1(x),h_2(x))} f(x,y)$ and $\int\limits_{y\in I(h_1(x),h_2(x))} g(x,y)$ are both $\mathcal{R}$-analytic functions on the interval $x \in I(a,b)$, we see that

$$\int\limits_{x\in I(a,b)} \left(\int\limits_{y\in I(h_1(x),h_2(x))} f(x,y) + \alpha \int\limits_{y\in I(h_1(x),h_2(x))} g(x,y) \right) = \int\limits_{x\in I(a,b)} \left(\int\limits_{y\in I(h_1(x),h_2(x))} f(x,y) \right)$$
$$+ \alpha \int\limits_{x\in I(a,b)} \left(\int\limits_{y\in I(h_1(x),h_2(x))} g(x,y) \right)$$
$$= \iint\limits_{(x,y)\in H} f(x,y) + \alpha \iint\limits_{(x,y)\in H} g(x,y).$$

So, $\iint\limits_{(x,y)\in H} (f+\alpha g)(x,y) = \iint\limits_{(x,y)\in H} f(x,y) + \alpha \iint\limits_{(x,y)\in H} g(x,y)$. □

Proposition 2.41.

Suppose $G \subset \mathcal{R}^2$ is a simple region and let $f : G \to \mathcal{R}$ be a non-positive analytic function on G. Then, $\iint\limits_{(x,y)\in G} f(x,y) \leqslant 0$.

Proof.

We assume without loss of generality that $G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$, where $a, b \in \mathcal{R}$, $a \leqslant b$, and $g_1, g_2 : I(a,b) \to \mathcal{R}$, $g_1 < g_2$.

Since f is non-positive on G we have that for every $x \in I(a,b)$, $f(x,y)$ is a non-positive $\mathcal{R}$-analytic function on $I(g_1(x), g_2(x))$. We know from Proposition 4.4 in [1] that for every $x \in I(a,b)$, $\int\limits_{y\in I(g_1(x),g_2(x))} f(x,y) \leqslant 0$. So, by the same Proposition we know that

$$\iint\limits_{(x,y)\in G} f(x,y) = \int\limits_{x\in I(a,b)} \left[\int\limits_{y\in I(g_1(x),g_2(x))} f(x,y) \right] \leqslant 0$$

as claimed. □

Corollary 2.42.

Suppose $G \subset \mathcal{R}^2$ is a simple region and let $f, h : G \to \mathcal{R}$ be analytic functions such that $f \leqslant g$ everywhere on G. Then, $\iint\limits_{(x,y)\in G} f(x,y) \leqslant \iint\limits_{(x,y)\in G} h(x,y)$.

Proof.

Since f and h are analytic functions such that $f \leqslant h$ everywhere on G, $f - h$ must be an analytic function such that $f - h \leqslant 0$. So by Proposition 2.41, $\iint\limits_{(x,y)\in G} (f-h)(x,y) \leqslant 0$. Thus, using Proposition 2.40 $\iint\limits_{(x,y)\in G} f(x,y) \leqslant \iint\limits_{(x,y)\in G} h(x,y)$. □

Proposition 2.43.

Suppose G is a simple region and let $f : G \to \mathcal{R}$ be an analytic function. Let M be an upper bound of $|f|$ on G. Then, $|\iint\limits_{(x,y)\in G} f(x,y)| \leqslant Ma(G)$.

Proof.

Since M is a bound of $|f|$, we have that $-M \leqslant f(x,y) \leqslant M$ on G, and it follows from Corollary 2.41 that $-\iint\limits_{(x,y)\in G} M \leqslant \iint\limits_{(x,y)\in G} f(x,y) \leqslant \iint\limits_{(x,y)\in G} M$. But, $\iint\limits_{(x,y)\in G} M = Ma(G)$, so $-Ma(G) \leqslant \iint\limits_{(x,y)\in G} f(x,y) \leqslant Ma(G)$. Thus, $|\iint\limits_{(x,y)\in G} f(x,y)| \leqslant Ma(G)$. □

Definition 2.44. *[The Integral of a Measurable Function over a Measurable Set]*

Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $f : A \to \mathcal{R}$ be a measurable function. Additionally, let M be a bound for $|f|$ on A. Since f is measurable then for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^k, $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^k) \leqslant d^k$.

Without loss of generality we may assume that for every k and n in $\mathbb{N}$, $G_n^k \subset G_n^{k+1}$.

Since $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges we have that $\lim\limits_{n\to\infty} a(G_n^k) = 0$. From Proposition 2.43 we have that $|\iint\limits_{(x,y)\in G_n^k} f(x,y)| \leqslant Ma(G_n^k)$, so $\lim\limits_{n\to\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y) = 0$. Therefore, for every $k \in \mathbb{N}$,

$\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y)$ converges. Now, fix $\epsilon > 0$ in $\mathcal{R}$ and select a $k_0 \in \mathbb{N}$ such that $Md^{k_0} \leqslant \epsilon$. Let $i > j \geqslant k_0$ be given in $\mathbb{N}$. We can, using Proposition 2.5, write $\bigcup\limits_{n=1}^{\infty} G_n^i \setminus \bigcup\limits_{n=1}^{\infty} G_n^j$ as the union of mutually disjoint simple regions $(G_n^{i,j})_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} a(G_n^{i,j})$ converges and

$$\begin{aligned}\sum_{n=1}^{\infty} a(G_n^{i,j}) &= \sum_{n=1}^{\infty} a(G_n^i) - \sum_{n=1}^{\infty} a(G_n^j)\\ &\leqslant m(A) - \sum_{n=1}^{\infty} a(G_n^j) \leqslant d^j \leqslant d^{k_0}.\end{aligned}$$

From this we see that

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^i} f(x,y) - \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^j} f(x,y)\right| &= \left|\sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^{i,j}} f(x,y)\right|\\ &\leqslant \sum_{n=1}^{\infty} \left|\iint\limits_{(x,y)\in G_n^{i,j}} f(x,y)\right|\\ &\leqslant \sum_{n=1}^{\infty} Ma(G_n^{i,j})\\ &= M\sum_{n=1}^{\infty} a(G_n^{i,j})\\ &\leqslant Md^{k_0} \leqslant \epsilon,\end{aligned}$$

where above we have used the fact that a sequence in $\mathcal{R}$ converges if and only if it converges absolutely. From the above inequality we deduce that $\left(\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y)\right)_{k=1}^{\infty}$ is a Cauchy sequence and thus converges. We call the limit of this sequence the integral of f over A and we denote it $\iint\limits_{(x,y)\in A} f(x,y)$.

So,

$$\iint_{(x,y)\in A} f(x,y) = \lim_{\substack{\sum_{n=1}^{\infty} a(G_n)\to m(A),\\ \bigcup_{n=1}^{\infty} G_n\subset A, G_n\text{'s mutually disjoint},\\ f \text{ is analytic on } G_n \text{ for every } n}} \sum_{n=1}^{\infty} \iint_{(x,y)\in G_n} f(x,y)$$

Proposition 2.45.

Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $\alpha \in \mathcal{R}$ be a constant. Then, α is measurable on A and $\iint\limits_{(x,y)\in A} \alpha = \alpha m(A)$.

Proof.

Since A is measurable, for every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges, and $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} a(G_n^k) = m(A)$. α is analytic on G_n^k for each n and for each k; moreover, $\iint_{(x,y)\in G_n^k} \alpha = \alpha a(G_n^k)$.

From Definition 2.44 we see that

$$\begin{aligned}
\iint_{(x,y)\in A} \alpha &= \lim_{k\to\infty} \sum_{n=1}^{\infty} \iint_{(x,y)\in G_n^k} \alpha \\
&= \lim_{k\to\infty} \sum_{n=1}^{\infty} \alpha a(G_n^k) \\
&= \alpha \lim_{k\to\infty} \sum_{n=1}^{\infty} a(G_n^k) \\
&= \alpha m(A).
\end{aligned}$$

□

Proposition 2.46.

Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $f : A \to \mathcal{R}$ be a non-positive measurable *function. Then,* $\iint\limits_{(x,y)\in A} f(x,y) \leqslant 0.$

Proof.

Fix $\epsilon > 0$ in $\mathcal{R}$. There exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that f is analytic on G_n for each n, $\bigcup\limits_{n=1}^{\infty} G_n \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n) \leqslant \epsilon$.

We have from Proposition 2.41 that for every $n \in \mathbb{N}$,

$$\iint_{(x,y)\in G_n} f(x,y) \leqslant 0$$

so

$$\sum_{n=1}^{\infty} \iint_{(x,y)\in G_n} f(x,y) \leqslant 0.$$

This holds for any $\epsilon > 0$ so $\lim\limits_{\epsilon\to 0} \sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n} f(x,y) \leqslant 0$. But, $\iint\limits_{(x,y)\in A} f(x,y) = \lim\limits_{\epsilon\to 0} \sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n} f(x,y)$, so $\iint\limits_{(x,y)\in A} f(x,y) \leqslant 0$. $\square$

Proposition 2.47.

Suppose $A \subset \mathcal{R}^2$ is a measurable set and let $f, g : A \to \mathcal{R}$ be measurable functions. Then, for every $\alpha \in \mathcal{R}$, $\iint\limits_{(x,y)\in A} (f + \alpha g)(x,y) = \iint\limits_{(x,y)\in A} f(x,y) + \alpha \iint\limits_{(x,y)\in A} g(x,y).$

Proof.

Fix $\epsilon > 0$ in $\mathcal{R}$, since f and g are measurable on A there exist two sequences of mutually disjoint simple regions $(G_n^f)_{n=1}^{\infty}$ and $(G_n^g)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n^f$ and $\bigcup\limits_{n=1}^{\infty} G_n^g$ are both subsets of A, for every $n \in \mathbb{N}$ f is analytic on G_n^f and g is analytic on G_n^g, $\sum\limits_{n=1}^{\infty} a(G_n^f)$ and $\sum\limits_{n=1}^{\infty} a(G_n^g)$ converge, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^f) \leqslant \frac{\epsilon}{2}$ and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^g) \leqslant \frac{\epsilon}{2}$.

By Proposition 2.12 we can write $(\bigcup\limits_{n=1}^{\infty} G_n^f) \cap (\bigcup\limits_{n=1}^{\infty} G_n^g)$ as the union of mutually disjoint simple regions $(T_n)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} a(T_n)$ converges, $m(A) - \sum\limits_{n=1}^{\infty} a(T_n) \leqslant \epsilon$, and for every

$n \in \mathbb{N}$, f and g are both analytic on T_n . From Proposition 2.40 we obtain that for every $n \in \mathbb{N}$ and for every $\alpha \in \mathcal{R}$,

$$\iint\limits_{(x,y)\in T_n} (f+\alpha g)(x,y) = \iint\limits_{(x,y)\in T_n} f(x,y) + \alpha \iint\limits_{(x,y)\in T_n} g(x,y).$$

Therefore, $\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in T_n} (f+\alpha g)(x,y) = \sum\limits_{n=1}^{\infty} \iint\limits_{T_n} f(x,y) + \alpha \sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in T_n} g(x,y)$.

The above holds for any $\epsilon > 0$, so for every $\alpha \in \mathcal{R}$,

$$\begin{aligned}\iint\limits_{(x,y)\in A} (f+\alpha g)(x,y) &= \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint\limits_{(x,t)\in T_n} (f+\alpha g)(x,y) \\ &= \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in T_n} f(x,y) + \alpha \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in T_n} g(x,y) \\ &= \iint\limits_{(x,y)\in A} f(x,y) + \alpha \iint\limits_{(x,y)\in A} g(x,y).\end{aligned}$$

□

Proposition 2.48.

Let $A \subset \mathcal{R}^2$ be a measurable set and let $f, g : A \to \mathcal{R}$ be maeasurable functions such that $f \leqslant g$ everywhere on A. Then, $\iint\limits_{(x,y)\in A} f(x,y) \leqslant \iint\limits_{(x,y)\in A} g(x,y)$.

Proof.

Since $f \leqslant g$ everywhere of A, $f - g \leqslant 0$ everywhere on A. So by Proposition 2.46, $\iint\limits_{(x,y)\in A} (f - g)(x,y) \leqslant 0$ and by Proposition 2.47 $\iint\limits_{(x,y)\in A} (f-g)(x,y) = \iint\limits_{(x,y)\in A} f(x,y) - \iint\limits_{(x,y)\in A} g(x,y)$. Thus, $\iint\limits_{(x,y)\in A} f(x,y) - \iint\limits_{(x,y)\in A} g(x,y) \leqslant 0$. □

Proposition 2.49.

Let $A \subset \mathcal{R}^2$ be a measurable set and let $f : A \to \mathcal{R}$ be a measurable function. Also, let M be an upper bound for $|f|$ on A. Then, $|\iint\limits_{(x,y)\in A} f(x,y)| \leqslant Mm(A)$.

Proof.

Since $M \geqslant |f|$ everywhere on A, $\iint\limits_{(x,y)\in A} (-M) \leqslant \iint\limits_{(x,y)\in A} f(x,y) \leqslant \iint\limits_{(x,y)\in A} M$. Thus, $-Mm(A) \leqslant \iint\limits_{(x,y)\in A} f(x,y) \leqslant Mm(A)$. Hence, $|\iint\limits_{(x,y)\in A} f(x,y)| \leqslant Mm(A)$. $\square$

Proposition 2.50.

Suppose $A, B \subset \mathcal{R}^2$ are measurable sets and let $f : A \cup B \to \mathcal{R}$ be a measurable function on both. Then, $\iint\limits_{(x,y)\in A\cup B} f(x,y) = \iint\limits_{(x,y)\in A} f(x,y) + \iint\limits_{(x,y)\in B} f(x,y) - \iint\limits_{(x,y)\in A\cap B} f(x,y)$.

Proof.

Fix $\epsilon > 0$ in $\mathcal{R}$. Since f is a measurable function on $A \cup B$ there exists a sequence of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} a(G_n)$ converges, $\bigcup\limits_{n=1}^{\infty} G_n \subset A \cup B$, $m(A \cup B) - \sum\limits_{n=1}^{\infty} a(G_n) \leqslant \frac{\epsilon}{2}$, and for every $n \in \mathbb{N}$, f is analytic on G_n.

We can arrange $(G_n)_{n=1}^{\infty}$ into three sequences of mutually disjoint simple regions $(G_n^1)_{n=1}^{\infty}$, $(G_n^2)_{n=1}^{\infty}$, and $(G_n^3)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} a(G_n^1)$, $\sum\limits_{n=1}^{\infty} a(G_n^2)$, and $\sum\limits_{n=1}^{\infty} a(G_n^3)$ converge; $\bigcup\limits_{n=1}^{\infty} G_n^1 \subset A\backslash(A \cap B)$, $\bigcup\limits_{n=1}^{\infty} G_n^2 \subset B\backslash(A \cap B)$, and $\bigcup\limits_{n=1}^{\infty} G_n^3 \subset (A \cap B)$; $m(A \cup B) - \sum\limits_{n=1}^{\infty} a(G_n^1) - \sum\limits_{n=1}^{\infty} a(G_n^2) -$

$\sum\limits_{n=1}^{\infty} a(G_n^3) \leqslant \epsilon$. So,

$$\begin{aligned}
\iint\limits_{(x,y)\in A\cup B} f(x,y) &= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n} f(x,y)\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^1} f(x,y) + \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^2} f(x,y)\\
&+ \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^3} f(x,y)\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^1} f(x,y) + \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^3} f(x,y)\\
&+ \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^2} f(x,y) + \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^3} f(x,y)\\
&- \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^3} f(x,y)\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\iint\limits_{(x,y)\in G_n^1} f(x,y) + \iint\limits_{(x,y)\in G_n^3} f(x,y)\right)\\
&+ \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\iint\limits_{(x,y)\in G_n^2} f(x,y) + \iint\limits_{(x,y)\in G_n^3} f(x,y)\right)\\
&- \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^3} f(x,y).
\end{aligned}$$

By our choice of $(G_n^1)_{n=1}^{\infty}$, $(G_n^2)_{n=1}^{\infty}$, and $(G_n^3)_{n=1}^{\infty}$,

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\iint\limits_{(x,y)\in G_n^1} f(x,y) + \iint\limits_{(x,y)\in G_n^3} f(x,y)\right) = \iint\limits_{(x,y)\in A} f(x,y),$$

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\left(\iint\limits_{(x,y)\in G_n^2} f(x,y) + \iint\limits_{(x,y)\in G_n^3} f(x,y)\right) = \iint\limits_{(x,y)\in B} f(x,y),$$

and

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\iint\limits_{(x,y)\in G_n^3} f(x,y) = \iint\limits_{(x,y)\in A\cap B} f(x,y).$$

So, we finally get that

$$\iint\limits_{(x,y)\in A\cup B} f(x,y) = \iint\limits_{(x,y)\in A} f(x,y) + \iint\limits_{(x,y)\in B} f(x,y) + \iint\limits_{(x,y)\in A\cap B} f(x,y).$$

□

Proposition 2.51.

Let $A \subset \mathcal{R}^2$ be a measurable set and let $f : A \to \mathcal{R}$ be a function on A. Let there be a sequence of measurable functions $f_k : A \to \mathcal{R}$ such that the sequence $(f_k)_{k=1}^{\infty}$ converges uniformly to f. Then, $\lim\limits_{k\to\infty}\iint\limits_{(x,y)\in A} f_k(x,y)$ exists; moreover, if f is measurable on A then $\lim\limits_{k\to\infty}\iint\limits_{(x,y)\in A} f_k(x,y) = \iint\limits_{(x,y)\in A} f(x,y)$.

Proof.

Let $\epsilon > 0$ in $\mathcal{R}$ be given and let $\epsilon_0 = \begin{cases} \frac{\epsilon}{m(A)} & \text{if } m(A) \neq 0 \\ \epsilon & \text{if } m(A) = 0 \end{cases}$. Since the sequence $(f_k)_{k=1}^{\infty}$ converges uniformly we know it must be uniformly Cauchy. Thus, there exists a $k_0 \in \mathbb{N}$ such that for every $i, j \geqslant k_0$, $|f_i(x,y) - f_j(x,y)| \leqslant \epsilon_0$ for every $(x,y) \in A$. Thus,

$$\begin{aligned}\left|\iint\limits_{(x,y)\in A} f_i(x,y) - \iint\limits_{(x,y)\in A} f_j(x,y)\right| &= \left|\iint\limits_{(x,y)\in A} (f_i(x,y) - f_j(x,y))\right| \\ &\leqslant \iint\limits_{(x,y)\in A} |f_i(x,y) - f_j(x,y)| \leqslant \epsilon_0 m(A) \leqslant \epsilon.\end{aligned}$$

So, $(\iint\limits_{(x,y)\in A} f_k(x,y))_{k=1}^{\infty}$ is a Cauchy sequence; since $\mathcal{R}$ is Cauchy complete $\lim\limits_{k\to\infty} \iint\limits_{(x,y)\in A} f_k(x,y)$ exists.

If f is measurable, then since $(f_k)_{k=1}^{\infty}$ converges uniformly to f, there exists a $k_0 \in \mathbb{N}$ such that for every $i \geqslant k_0$, $|f_i(x,y) - f(x,y)| \leqslant \epsilon_0$ for every $(x,y) \in A$. Therefore,

$$\begin{aligned}\left|\iint\limits_{(x,y)\in A} f_i(x,y) - \iint\limits_{(x,y)\in A} f(x,y)\right| &= \left|\iint\limits_{(x,y)\in A} (f_i(x,y) - f(x,y))\right| \\ &\leqslant \iint\limits_{(x,y)\in A} |f_i(x,y) - f(x,y)| \leqslant \epsilon_0 m(A) \leqslant \epsilon,\end{aligned}$$

so $\lim\limits_{k\to\infty} \iint\limits_{(x,y)\in A} f_k(x,y) = \iint\limits_{(x,y)\in A} f(x,y)$. □

3 Measure Theory and Integration in three Dimensions

3.1 Simple Regions in $\mathcal{R}^3$ and Analytic Functions

Definition 3.1 (Simple Region)**.**

Let $S \subset \mathcal{R}^3$. Then we say S is a simple region in $\mathcal{R}^3$ if there exists a simple region $A \subset \mathcal{R}^2$

and two analytic functions $h_1, h_2 : A \to \mathcal{R}$ *such that* $h_1 < h_2$ *everywhere on* A *and*

$$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}$$

or

$$S = \{(x, y, z) \in \mathcal{R}^3 : y \in I(h_1(x, z), h_2(x, z)), (x, z) \in A\}$$

or

$$S = \{(x, y, z) \in \mathcal{R}^3 : x \in I(h_1(y, z), h_2(y, z)), (y, z) \in A\}.$$

Definition 3.2 (Volume of a Simple Region)**.**

Suppose S *is a simple region with* A*,* h_1 *and* h_2 *as in Definition 3.1. Then, if* $S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}$*, we denote the volume of* S *with* $v(S)$ *and define it as*

$$v(S) = \iint\limits_{(x,y)\in A} [h_2(x, y) - h_1(x, y)].$$

A similar definition can be used in the other two cases.

Definition 3.3 (Finite Simple Region and Order of Magnitude)**.**

Let S *be a simple region given by*

$$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}.$$

We define $\lambda_x(S) = \lambda_x(A)$*,* $\lambda_y(S) = \lambda_y(A)$*, and* $\lambda_z(S) = i(h_2(x, y) - h_1(x, y))$*, the index of the analytic function* $h_2 - h_1$ *on* A*; we call these the orders of magnitude of* S *in* x*,* y *and* z *respectivly. We call* S *a finite region if* $\lambda_x(S) = \lambda_y(S) = \lambda_z(S) = 0$*.*

Definition 3.4 (Analytic Function in $\mathcal{R}^3$).

Suppose $S \subset \mathcal{R}^3$ is a simple region and let $f : S \to \mathcal{R}$. Then we call f an analytic function on S if for every $(x_0, y_0, z_0) \in S$, there exists a simple region $A \subset \mathcal{R}^3$ containing (x_0, y_0, z_0) and a regular sequence $(a_{ijk})_{i,j,k=0}^{\infty}$ in $\mathcal{R}$ such that $\lambda_x(A) = \lambda_x(S)$, $\lambda_y(A) = \lambda_y(S)$, $\lambda_z(A) = \lambda_z(S)$, and if $(x_0 + r, y_0 + s, z_0 + t) \in S \cap A$ then

$$f(x_0 + r, y_0 + s, z_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} r^i s^j t^k,$$

where the power series converges in the weak topology.

Proposition 3.5.

Suppose $A \subset \mathcal{R}^3$ is a finite simple region and let $f, g : A \to \mathcal{R}$ be two analytic functions on A. Let $\alpha \in \mathcal{R}$ be an arbitrary constant, then $f + \alpha g$ and $f \cdot g$ are analytic functions on A.

Proof.

Fix $(x_0, y_0, z_0) \in A$. Since f and g are analytic there exist two finite constants $\eta_1, \eta_2 > 0$ such that for every $r, s, t \in \mathcal{R}$ satisfying $r^2 + s^2 + t^2 < \eta_1$, if $(x_0 + r, y_0 + s, z_0 + t) \in A$ then

$$f(x_0 + r, y_0 + s, z_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} r^i s^j t^k$$

and for every $r, s, t \in \mathcal{R}$ satisfying $r^2 + s^2 + t^2 < \eta_2$, if $(x_0 + r, y_0 + s, z_0 + t) \in A$ then

$$g(x_0 + r, y_0 + s, z_0 + t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{lmn} r^l s^m t^n.$$

Let $\eta = \min\{\eta_1, \eta_2\}$. Thus, for every $r, s, t \in \mathcal{R}$ satisfying $r^2 + s^2 + t^2 < \eta$, if $(x_0 + r, y_0 +$

$s, z_0 + t) \in A$ then

$$(f + \alpha g)(x_0 + r, y_0 + s, z_0 + t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{ijk} r^i s^j t^k + \alpha \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{lmn} r^l s^m t^n$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (a_{ijk} + \alpha b_{ijk}) r^i s^j t^k,$$

so $f + \alpha g$ is analytic on A. Furthermore, for every $r, s, t \in \mathcal{R}$ satisfying $r^2 + s^2 + t^2 < \eta$, if $(x_0 + r, y_0 + s, z_0 + t) \in A$ then as in [5] and the proof of Proposition 2.24

$$(f \cdot g)(x_0 + r, y_0 + s, z_0 + t) = (\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{ijk} r^i s^j t^k)(\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{lmn} r^l s^m t^n)$$
$$= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty} \sum_{i+l=u} \sum_{j+m=v} \sum_{k+n=w} a_{ijk} b_{lmn} r^u s^v t^w$$
$$= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty} c_{uvw} r^u s^v t^w.$$

As with Proposition 2.24 we infer that the Cauchy product converges in $\mathcal{R}^3$ from the fact that Cauchy products converge in $\mathcal{R}$. So $f \cdot g$ is an analytic function. □

Corollary 3.6. *Suppose $A \subset \mathcal{R}^3$ is a simple region and let $f, g : A \to \mathcal{R}$ be two analytic functions on A. Let $\alpha \in \mathcal{R}$ be an arbitrary constant, then $f + \alpha g$ and $f \cdot g$ are analytic functions on A.*

Proposition 3.7.

Suppose $A \subset \mathcal{R}^3$ is a finite simple region and let $f : A \to \mathcal{R}$ be an analytic function on A. Let $B \subset \mathcal{R}^2$ be a finite simple region and let $g : B \to \mathcal{R}$ be an analytic function on B such that for every $(x, y) \in B$, $(x, y, g(x, y)) \in A$. Then, $f(x, y, g(x, y))$ is an analytic function on B.

Proof.

Fix $(x_0, y_0) \in B$, since f and g are analytic functions there exist $\eta_1, \eta_2 > 0$ such that $\eta_1, \eta_2 \sim 1$ and for every $s, t \in \mathcal{R}$ satisfying $s^2 + t^2 < \eta_1^2$, if $(x_0 + s, y_0 + t) \in B$, then $g(x_0+s, y_0+t) = g(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j$, and, for every $u, v, w \in \mathcal{R}$ satisfying $u^2+v^2+w^2 < \eta_2^2$, if $(x_0+u, y_0+v, g(x_0, y_0)+w) \in A$, then, $f(x_0+u, y_0+v, g(x_0, y_0)+w) = f(x_0, y_0, g(x_0.y_0)) + \sum_{\substack{l,m,n=0 \\ l+m+n\neq 0}}^{\infty} b_{lmn} u^l v^m w^n$.

Define $H : \mathbb{R} \to \mathbb{R}$ by $H(S, T) = (S^2 + T^2 + (\sum_{\substack{i,j=0 \\ i+j\neq 0}} a_{ij} S^i T^j)^2)[0]$, clearly H is a continuous real function so there exists a $\eta \in (0, \eta_1] \cap \mathbb{R}$ such that if $S^2 + T^2 < 2\eta^2$ then $H(S, T) < \eta_2^2$. Now, if $s, t \in \mathcal{R}$ with $s^2 + t^2 < \eta^2$ and $(x_0 + s, y_0 + t) \in B$ then

$$\left| s^2 + t^2 + \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j \right)^2 - \left[(r[0])^2 + (t[0])^2 + \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} (s[0])^i (t[0])^j \right)^2 \right] [0] \right| \ll 1$$

since $s =_0 s[0]$, $t =_0 t[0]$, then

$$\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j =_0 \left(\sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} (s[0])^i (t[0])^j \right) [0].$$

So if $H(s[0], t[0]) < \eta_2^2$ then $s^2 + t^2 + (\sum_{\substack{i,j=0 \\ i+j\neq 0}} a_{ij} s^i t^j)^2 < \eta_2^2$.

So we have that for every $s, t \in \mathcal{R}$ satisfying $s^2 + t^2 < \eta^2$, if $(x_0 + s, y_0 + t) \in B$ then

$$\begin{aligned} f(x_0 + s, y_0 + t, g(x_0 + s, y_0 + t)) &= f(x_0 + s, y_0 + t, g(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j \neq 0}}^{\infty} a_{ij} s^i t^j) \\ &= f(x_0, y_0, g(x_0, y_0)) + \sum_{\substack{l,m,n=0 \\ l+m+n \neq 0}}^{\infty} b_{lmn} s^l t^m (\sum_{\substack{i,j=0 \\ i+j \neq 0}}^{\infty} a_{ij} s^i t^j)^n. \end{aligned}$$

However, using the same argument as in the proof of Proposition 2.26, we can rewrite

$$b_{lmn} s^l t^m (\sum_{\substack{i,j=0 \\ i+j \neq 0}}^{\infty} a_{ij} s^i t^j)^n$$

as the convergent series $\sum_{\substack{k,p=0 \\ k+p \neq 0}}^{\infty} c_{lmnkp} s^k t^p$. We may also change the order of summation so that

$$\begin{aligned} f(x_0 + s, y_0 + t, g(x_0 + s, y_0 + t)) &= f(x_0, y_0, g(x_0, y_0)) + \sum_{\substack{l,m,n=0 \\ l+m+n \neq 0}}^{\infty} \sum_{\substack{k,p=0 \\ k+p \neq 0}}^{\infty} c_{lmnkp} s^k t^p \\ &= f(x_0, y_0, g(x_0, y_0)) + \sum_{\substack{k,p=0 \\ k+p \neq 0}}^{\infty} \sum_{\substack{l,m,n=0 \\ l+m+n \neq 0}}^{\infty} c_{lmnkp} s^k t^p \end{aligned}$$

Finally, letting $e_{kp} = \sum_{\substack{l,m,n=0 \\ l+m+n \neq 0}}^{\infty} c_{lmnkp}$ we find that

$$f(x_0 + s, y_0 + t, g(x_0 + s, y_0 + t)) = f(x_0, y_0, g(x_0, y_0)) + \sum_{\substack{k,p=0 \\ k+p \neq 0}}^{\infty} e_{kp} s^k t^p.$$

Thus $f(x, y, g(x, y))$ is analytic. $\square$

Corollary 3.8.

Suppose $A \subset \mathcal{R}^3$ is a simple region and let $f : A \to \mathcal{R}$ ba an analytic function on A. Let $B \subset \mathcal{R}^2$ be a simple region and let $g : B \to \mathcal{R}$ be an analytic function on B such that for every $(x, y) \in B$, $(x, y, g(x, y)) \in A$. Then, $f(x, y, g(x, y))$ is an analytic function on B.

3.2 Measurable Sets and Measurable Functions

Definition 3.9 (Measurable Set)**.**

Let $S \subset \mathcal{R}^3$. Then we say that S is a measurable set if for every $\epsilon > 0$ there exists two sequences of mutually disjoint simple regions, $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$, such that $\bigcup\limits_{n=1}^{\infty} G_n \subset S \subset \bigcup\limits_{n=1}^{\infty} H_n$, $\sum\limits_{n=1}^{\infty} v(G_n)$ and $\sum\limits_{n=1}^{\infty} v(H_n)$ converge, and $\sum\limits_{n=1}^{\infty} v(H_n) - \sum\limits_{n=1}^{\infty} v(G_n) < \epsilon$.

Definition 3.10 (Measure of a Measurable Set)**.**

Suppose $S \subset \mathcal{R}^3$ is a measurable set. By definition we have that for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions, $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$, such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset S \subset \bigcup\limits_{n=1}^{\infty} H_n^k$, $\sum\limits_{n=1}^{\infty} v(G_n^k)$ and $\sum\limits_{n=1}^{\infty} v(H_n^k)$ converge, and $\sum\limits_{n=1}^{\infty} v(H_n^k) - \sum\limits_{n=1}^{\infty} v(G_n^k) < d^k$. We note that that since for every $k \in \mathbb{N}$, $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ are mutualy disjoint we can arrange them so that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset \bigcup\limits_{n=1}^{\infty} G_n^{k+1} \subset S \subset \bigcup\limits_{n=1}^{\infty} H_n^{k+1} \subset \bigcup\limits_{n=1}^{\infty} H_n^k$.

We claim that $(\sum\limits_{n=1}^{\infty} v(G_n^k))_{k=1}^{\infty}$ is a Cauchy sequence. To prove this we fix $\epsilon > 0$ in $\mathcal{R}$ and let $k_0 \in \mathbb{N}$ be large enough so that $d^{k_0} < \epsilon$. Now, for every $l > k_0$, $\bigcup\limits_{n=1}^{\infty} G_n^l \subset S \subset \bigcup\limits_{n=1}^{\infty} H_n^{k_0}$, so $\sum\limits_{n=1}^{\infty} v(G_n^l) \leqslant \sum\limits_{n=1}^{\infty} v(H_n^{k_0})$. Thus, $0 \leqslant \sum\limits_{n=1}^{\infty} v(G_n^l) - \sum\limits_{n=1}^{\infty} v(G_n^{k_0}) \leqslant \sum\limits_{n=1}^{\infty} v(H_n^{k_0}) - \sum\limits_{n=1}^{\infty} v(G_n^{k_0}) < d^{k_0} < \epsilon$. Therefore the claim is proven, and a similar argument shows that the sequence $(\sum\limits_{n=1}^{\infty} v(H_n^k))_{k=1}^{\infty}$ is Cauchy.

Since $\mathcal{R}$ is Cauchy complete $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} v(G_n^k)$ and $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} v(H_n^k)$ both exist, and hence

$\lim_{k\to\infty} \sum_{n=1}^{\infty}(v(H_n^k)-v(G_n^k))$ *exists and* $\lim_{k\to\infty} \sum_{n=1}^{\infty}(v(H_n^k)-v(G_n^k)) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) - \lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k)$. *Furthermore, for every* $k \in \mathbb{N}$,

$$0 \leqslant \sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k,$$

so

$$0 \leqslant \lim_{k\to\infty}(\sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k)) \leqslant 0.$$

So we conclude that $\lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k)$. *We call this limit the measure of* S *and we denote it by* $m(S)$.

Proposition 3.11.

Suppose $S \subset \mathcal{R}^3$ *is a measurable set. Then,*

$$\begin{aligned} m(S) = \inf \Bigg\{ &\sum_{n=1}^{\infty} v(H_n) : H_n\text{'s are mutually disjoint simple regions,} \\ &\text{and } S \subset \bigcup_{n=1}^{\infty} H_n, \sum_{n=1}^{\infty} v(H_n) \text{ converges} \Bigg\} \\ = \sup \Bigg\{ &\sum_{n=1}^{\infty} v(G_n) : G_n\text{'s are mutually disjoint simple regions,} \\ &\text{and } \bigcup_{n=1}^{\infty} G_n \subset S, \sum_{n=1}^{\infty} v(G_n) \text{ converges} \Bigg\}. \end{aligned}$$

Proof.

First we show that the infimum exists and is equal to $m(S)$.

Since S is a measurable set we know that for every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n^k \subset \bigcup_{n=1}^{\infty} G_n^{k+1} \subset S \subset$

$\bigcup_{n=1}^{\infty} H_n^{k+1} \subset \bigcup_{n=1}^{\infty} H_n^k$, $\sum_{n=1}^{\infty} v(G_n^k)$ and $\sum_{n=1}^{\infty} v(H_n^k)$ both converge, and $\sum_{n=1}^{\infty} v(H_n^k) - \sum_{n=1}^{\infty} v(G_n^k) < d^k$.

By definition

$$m(S) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k)$$

and, for every $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} v(G_n^k) \leqslant m(S) \leqslant \sum_{n=1}^{\infty} v(H_n^k).$$

It remains to be shown that if $(H_n)_{n=1}^{\infty}$ is a sequence of mutually disjoint simple regions such that $S \subset \bigcup_{n=1}^{\infty} H_n$ and $\sum_{n=1}^{\infty} v(H_n)$ converges, then $\sum_{n=1}^{\infty} v(H_n) \geqslant \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) = m(S)$.

Suppose not. Then, there exists a sequence of mutually disjoint simple regions $(H_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} v(H_n)$ converges, $S \subset \bigcup_{n=1}^{\infty} H_n$, and $\sum_{n=1}^{\infty} v(H_n) < \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) = m(S)$. Let $\eta = m(S) - \sum_{n=1}^{\infty} v(H_n)$, then $\sum_{n=1}^{\infty} v(H_n) = m(S) - \eta$. However $m(S) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k)$, thus there exists a $k_0 \in \mathbb{N}$ such that $m(S) - \sum_{n=1}^{\infty} v(G_n^{k_0}) < \eta$. Therefore, $\sum_{n=1}^{\infty} v(G_n^k) > m(S) - \eta > \sum_{n=1}^{\infty} v(H_n)$, but this contradicts the fact that $\bigcup_{n=1}^{\infty} G_n^{k_0} \subset S \subset \bigcup_{n=1}^{\infty} H_n$. The only remaining possibility is that $m(S) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k) = \inf\{\sum_{n=1}^{\infty} v(H_n) : H_n\text{'s are mutually disjoint, } S \subset \bigcup_{n=1}^{\infty} H_n, \sum_{n=1}^{\infty} v(H_n) \text{ converges}\}$.

A similar argument shows that

$$m(S) = \sup\left\{\sum_{n=1}^{\infty} v(G_n) : G_n\text{'s are mutually disjoint, } \bigcup_{n=1}^{\infty} G_n \subset S, \sum_{n=1}^{\infty} v(G_n) \text{ converges}\right\}.$$

$\square$

Definition 3.12 (Measurable Function)**.**

Suppose $S \subset \mathcal{R}^3$ is a measurable set and let $f : S \to \mathcal{R}$ be bounded on S. Then we say that f is measurable on S if for every $\epsilon > 0$, there exists a sequence of mutually disjoint simple *regions* $(G_n)_{n=1}^{\infty}$ *such that* $\bigcup\limits_{n=1}^{\infty} G_n \subset S$, $\sum\limits_{n=1}^{\infty} v(G_n)$ *converges,* $m(S) - \sum\limits_{n=1}^{\infty} v(G_n) < \epsilon$, *and* f *is analytic on* G_n *for every* $n \in \mathbb{N}$.

3.3 Integration in Three Dimensions

Definition 3.13 (Integral of an Analytic Function Over a Simple Region in $\mathcal{R}^3$)**.**

Suppose $S \subset \mathcal{R}^3$ *is a simple region, let* $f : S \to \mathcal{R}$ *be an analytic function on* S. *Since* S *is a simple region we assume without loss of generality that*

$$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}.$$

We define

$$\iiint\limits_{(x,y,z)\in S} f(x, y, z) = \iint\limits_{(x,y)\in A} \left[\int\limits_{z\in I(h_1(x,y),h_2(x,y))} f(x, y, z) \right]$$

and we call this the integral of f *over* S. *Note that for fixed* x *and* y, $f(x, y, z)$ *is an* $\mathcal{R}$*-analytic function on the interval* $I(h_1(x, y), h_2(x, y))$, *so* $\int\limits_{z\in I(h_1(x,y),h_2(x,y))} f(x, y, z)$ *is well defined. Moreover,* $F(x, y) := \int\limits_{z\in I(h_1(x,y),h_2(x,y))} f(x, y, z)$ *is an analytic function on* A, *so the integral is well-defined.*

Proposition 3.14.

Suppose $S \subset \mathcal{R}^3$ *is a simple region and let* $M \in \mathcal{R}$ *be constant. Then,* $\iiint\limits_{(x,y,z)\in S} M = Mv(S)$.

Proposition 3.15.

Suppose $S \subset \mathcal{R}^3$ *is a simple region and let* $f : S \to \mathcal{R}$ *be a non-positive analytic function. Then,* $\iiint\limits_{(x,y,z)\in S} f(x, y, z) \leqslant 0$.

Proof.

By definition,

$$\iiint\limits_{(x,y,z)\in S} f(x,y,z) = \iint\limits_{(x,y)\in A} \left[\int\limits_{z\in I(h_1(x,y),h_2(x,y))} f(x,y,z) \right].$$

Since f is non-positive on S

$$F(x,y) \coloneqq \int\limits_{z\in I(h_1(x,y),h_2(x,y))} f(x,y,z) \leqslant 0$$

on A [1]. It then follows by Proposition 2.41 that

$$\iiint\limits_{(x,y,z)\in S} f(x,y,z) = \iint_{(x,y)\in A} F(x,y) \leqslant 0.$$

$\square$

Corollary 3.16.

Suppose $S \subset \mathcal{R}^3$ is a simple region and let $f, g : S \to \mathcal{R}$ be analytic functions such that $f \leqslant g$ everywhere on S. Then, $\iiint\limits_{(x,y,z)\in S} f(x,y,z) \leqslant \iiint\limits_{(x,y,z)\in S} g(x,y,z)$.

Corollary 3.17.

Suppose $S \subset \mathcal{R}^3$ is a simple region and let $f : S \to \mathcal{R}$ be an analytic function bounded by $M \in \mathcal{R}$ on S. Then,

$$\left| \iiint\limits_{(x,y,z)\in S} f(x,y,z) \right| \leqslant Mv(S).$$

Definition 3.18 (Integration of a Measurable Function Over a measurable Set)**.**

Suppose $S \subset \mathcal{R}^3$ is a measurable set, let $f : S \to \mathcal{R}$ be a measurable function such that $|f|$ is bounded by $M \in \mathcal{R}$ everywhere on S. We have that for every $k \in \mathbb{N}$, there exists *a sequence of mutually disjoint simple regions* $(G_n^k)_{n=1}^{\infty}$ *such that* $\bigcup\limits_{n=1}^{\infty} G_n^k \subset S$, $\sum\limits_{n=1}^{\infty} v(G_n^k)$ *converges,* $m(S) - \sum\limits_{n=1}^{\infty} v(G_n^k) < d^k$, *and for every* $n \in \mathbb{N}$, f *is analytic on* G_n^k. *Then, for every* $k, n \in \mathbb{N}$,

$$\left| \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z) \right| \leqslant \iiint\limits_{(x,y,z)\in G_n^k} M = Mv(G_n^k),$$

by Corollary 3.17. However, $\sum\limits_{n=1}^{\infty} v(G_n^k)$ *converges, so* $\lim\limits_{n\to\infty} Mv(G_n^k) = 0$. *Therefore,*

$$\lim_{n\to\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z) = 0$$

and so $\sum\limits_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z)$ *converges.*

We claim that $(\sum\limits_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z))_{k=1}^{\infty}$ *is a Cauchy sequence. To prove this fix* $\epsilon > 0$ *and select a* $k_0 \in \mathbb{N}$ *such that* $Md^{k_0} < \epsilon$. *Fix* $i \neq j \in \mathbb{N}$ *so that* $i > j \geqslant k_0$.

Because for every $k \in \mathbb{N}$, $(G_n^k)_{n=1}^{\infty}$ *is a sequence of mutually disjoint simple regions, we may arrange them so that for every* $n \in \mathbb{N}$, $G_n^j \subset G_n^i$. *Thus,*

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^i} f(x,y,z) - \sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^j} f(x,y,z) \right| \\
&= \left| \sum_{n=1}^{\infty} \left(\iiint\limits_{(x,y,z)\in G_n^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_n^j} f(x,y,z) \right) \right| \\
&\leqslant \sum_{n=1}^{\infty} \left| \iiint\limits_{(x,y,z)\in G_n^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_n^j} f(x,y,z) \right|.
\end{aligned}$$

Now, we have that for every $n \in \mathbb{N}$,

$$\left| \iiint\limits_{(x,y,z)\in G_n^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_n^j} f(x,y,z) \right| \leqslant M(v(G_n^i) - v(G_n^j)).$$

To see this suppose not, then there exists $n_0 \in \mathbb{N}$ such that

$$\left| \iiint\limits_{(x,y,z)\in G_{n_0}^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_{n_0}^j} f(x,y,z) \right| > M(v(G_{n_0}^i) - v(G_{n_0}^j)),$$

so either

$$\iiint\limits_{(x,y,z)\in G_{n_0}^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_{n_0}^j} f(x,y,z) > M(v(G_{n_0}^i) - v(G_{n_0}^j))$$

or

$$\iiint\limits_{(x,y,z)\in G_{n_0}^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_{n_0}^j} f(x,y,z) < -M(v(G_{n_0}^i) - v(G_{n_0}^j)).$$

If

$$\iiint\limits_{(x,y,z)\in G_{n_0}^i} f(x,y,z) - \iiint\limits_{(x,y,z)\in G_{n_0}^j} f(x,y,z) > M(v(G_{n_0}^i) - v(G_{n_0}^j)),$$

then

$$\iiint\limits_{(x,y,z)\in G_{n_0}^i} (f(x,y,z) - M) - \iiint\limits_{(x,y,z)\in G_{n_0}^j} (f(x,y,z) - M) > 0.$$

This is a contradiction because $f(x,y,z) - M \leqslant 0$ on $G_{n_0}^i$ and $G_{n_0}^j \subset G_{n_0}^i$, so

$$0 \geqslant \iiint\limits_{(x,y,z)\in G_{n_0}^j} (f(x,y,z) - M) \geqslant \iiint\limits_{(x,y,z)\in G_{n_0}^i} (f(x,y,z) - M).$$

A similar argument leads to contradiction in the case that

$$\iiint\limits_{(x,y,z)\in G^i_{n_0}} f(x,y,z) - \iiint\limits_{(x,y,z)\in G^j_{n_0}} f(x,y,z) < -M(v(G^i_{n_0}) - v(G^j_{n_0})).$$

Finally, we obtain that

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G^i_n} f(x,y,z) - \sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G^j_n} f(x,y,z) \right| &\leqslant M \sum_{n=1}^{\infty} (v(G^i_n) - v(G^j_n)) \\
&= M \left(\sum_{n=1}^{\infty} v(G^i_n) - \sum_{n=1}^{\infty} v(G^j_n) \right) \\
&< M(m(S) - \sum_{n=1}^{\infty} v(G^j_n)) \\
&< Md^j \\
&\leqslant Md^{k_0} \\
&< \epsilon.
\end{aligned}$$

Thus, the sequence $\left(\sum\limits_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G^k_n} f(x,y,z) \right)_{k=1}^{\infty}$ *is Cauchy.*

Since $\mathcal{R}$ *is Cauchy complete we have that* $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G^k_n} f(x,y,z)$ *exists. We define*

$$\iiint\limits_{(x,y,z)\in S} f(x,y,z) = \lim_{k\to\infty} \sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G^k_n} f(x,y,z)$$

and we call this the integral of f *over* S*.*

We omit the proofs of the following three results since they follow directly from the fact that the equivalent statement is true for analytic functions over simple regions, and the arguments used are largely identical to their two-dimensional counterparts.

Proposition 3.19.

Suppose $S \subset \mathcal{R}^3$ is a measurable set and let $M \in \mathcal{R}$ be given. Then, $\iiint\limits_{(x,y,z)\in S} M = Mm(S)$.

Proposition 3.20.

Suppose $S \subset \mathcal{R}^3$ is a measurable set and let $f, g : S \to \mathcal{R}$ be measurable functions. Furthermore, suppose that $f \leqslant g$ everywhere on S. Then, $\iiint\limits_{(x,y,z)\in S} f(x,y,z) \leqslant \iiint\limits_{(x,y,z)\in S} g(x,y,z)$.

Corollary 3.21.

Suppose $S \subset \mathcal{R}^3$ is a measurable set and let $f : S \to \mathcal{R}$ be a measurable function bounded by M on S. Then,

$$\left| \iiint\limits_{(x,y,z)\in S} f(x,y,z) \right| \leqslant \iiint\limits_{(x,y,z)\in S} |f(x,y,z)| \leqslant Mm(S).$$

4 The Delta Function on the Levi-Civita Field

Definition 4.1 (The Delta Function on the Levi-Civite Field)**.**

We define $\delta : \mathcal{R} \to \mathcal{R}$ as $\delta(x) = \begin{cases} 0 & \text{if } |x| > d \\ \frac{3}{4}d^{-3}(d^2 - x^2) & \text{if } |x| \leqslant d \end{cases}$.

Proposition 4.2.

For any interval $I \subset \mathcal{R}$ with endpoints $a, b \in \mathcal{R}$ such that $\lambda(a) < 1$ and $\lambda(b) < 1$,

$$\int_I \delta(x) = \begin{cases} 1 & \text{if } 0 \in I \\ 0 & \text{if } 0 \notin I \end{cases}.$$

Proof.

Suppose $I \subset \mathcal{R}$ is an interval with endpoints $a, b \in \mathcal{R}$ such that $\lambda(a) < 1$ and $\lambda(b) < 1$. Note that $\delta(x)$ is measurable on I [1].

If $0 \in I$, then

$$\begin{aligned}
\int\limits_{x \in I} \delta(x) &= \int\limits_{x \in [-d,d]} \delta(x) \\
&= \int\limits_{x \in [-d,d]} \frac{3}{4} d^{-3}(d^2 - x^2) \\
&= \frac{3}{4} d^{-3} \left(\left[d^2 x \right]_{-d}^{d} - \left[\frac{1}{3} x^3 \right]_{-d}^{d} \right) \\
&= \frac{3}{4} d^{-3} \left(2d^3 - \frac{2}{3} d^3 \right) = 1.
\end{aligned}$$

If $0 \notin I$, then either $a < b < -d$ or $d < a < b$; and hence

$$\int\limits_{x \in I} \delta(x) = \int\limits_{x \in I} 0 = 0.$$

$\square$

Proposition 4.3.

Let $a \in \mathcal{R}$ satisfy $\lambda(a) < 1$ and let $I = [-a, a]$. Then $\delta(x)$ has an anti-derivitive on I that is measurable on $[-a, a]$ and that is equal to the Heaviside function on $I \cap \mathbb{R}$.

Proof.

Let $H : I \to \mathcal{R}$ be given by

$$H(x) = \begin{cases} A_1 & \text{if } x < -d \\ \frac{3}{4}d^{-3}(d^2x - \frac{1}{3}x^3) + A_2 & \text{if } -d \leqslant x \leqslant d \\ A_3 & \text{if } x > d \end{cases} \cdot$$

By using the initial condition $H(x)|_{x=-d} = 0$ and requiring $H(x)$ be continuous we find that $A_1 = 0$, $A_2 = \frac{1}{2}$, and $A_3 = 1$. So,

$$H(x) = \begin{cases} 0 & \text{if } x < -d \\ \frac{3}{4}d^{-3}(d^2x - \frac{1}{3}x^3) + \frac{1}{2} & \text{if } -d \leqslant x \leqslant d \\ 1 & \text{if } x > d \end{cases} \cdot$$

Therefore, $H(x)$ is measurable and differentiable on I with $H'(x) = \delta(x)$ on I. Moreover,

$$H(x)|_{\mathbb{R}} = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \cdot$$

$\square$

Proposition 4.4.

Suppose that $\alpha \in \mathbb{R}\backslash\{0\}$ *and let* $I \subset \mathcal{R}$ *be an interval with endpoints* $a, b \in \mathcal{R}$ *such that*

$\lambda(a) < 1$ *and* $\lambda(b) < 1$, *then*

$$\int_{x\in I} \delta(\alpha x) = \frac{1}{|\alpha|} \int_{x\in I} \delta(x).$$

Proof.

Fix $\alpha \in \mathbb{R}\backslash\{0\}$. $\delta(x)$ is an even function so we may assume without loss of generality that α is positive. Let $y = \alpha x$. Then, substituting, we find

$$\begin{aligned}
\delta(\alpha x) &= \delta(y) \\
&= \begin{cases} 0 & \text{if } |y| > d \\ \frac{3}{4}d^{-3}(d^2 - y^2) & \text{if } |y| \leqslant d \end{cases} \\
&= \begin{cases} 0 & \text{if } |x| > \frac{d}{|\alpha|} \\ \frac{3}{4}d^{-3}(d^2 - (\alpha x)^2) & \text{if } |x| \leqslant \frac{d}{|\alpha|} \end{cases}
\end{aligned}$$

So, if $0 \in I$, then $a < \frac{-d}{\alpha} < \frac{d}{\alpha} < b$; and

$$\int\limits_{x\in I} \delta(\alpha x) = \int\limits_{x\in[-\frac{d}{|\alpha|}, \frac{d}{|\alpha|}]} \frac{3}{4}d^{-3}(d^2 - (\alpha x)^2) = \frac{1}{|\alpha|} = \frac{1}{|\alpha|} \int_{x\in I} \delta(x).$$

On the other hand, if $0 \notin I$, then either $a < b < \frac{-d}{\alpha}$ or $\frac{d}{\alpha} < a < b$; so

$$\int\limits_{x\in I} \delta(\alpha x) = \int\limits_{x\in I} 0 = 0 = \frac{1}{|\alpha|} \int_{x\in I} \delta(x)$$

$\square$

Proposition 4.5.

Suppose that $a, b \in \mathcal{R}$ with $a < b$ and $\lambda(b-a) < 1$. Let $f : [a,b] \to \mathcal{R}$ be an $\mathcal{R}$-analytic function with $i(f) = 0$. Then for any $x_0 \in [a+d, b-d]$, $\int_{x\in[a,b]} f(x)\delta(x-x_0) =_0 f(x_0)$.

Proof.

Fix $x_0 \in [a+d, b-d]$. Since f is a finite $\mathcal{R}$-analytic function, there exists a $\eta > 0$ in $\mathcal{R}$ with $\lambda(\eta) = \lambda(b-a)$ such that for any $x \in [a,b]$ satisfying $|x-x_0| < \eta$, $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. Therefore,

$$\begin{aligned}
\int\limits_{x\in[a,b]} f(x)\delta(x-x_0) &= \int\limits_{x\in[x_0-d,x_0+d]} f(x)\delta(x-x_0) \\
&= \int\limits_{x\in[x_0-d,x_0+d]} \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0) \\
&= \int\limits_{x\in[x_0-d,x_0+d]} f(x_0)\delta(x-x_0) \\
&+ \int\limits_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0) \\
&= f(x_0) + \int\limits_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0).
\end{aligned}$$

Now, for any $x \in [x_0-d, x_0+d]$ we must have that $|x-x_0| \leqslant d$. So,

$$\begin{aligned}
&\left| \int\limits_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0) \right| \\
&\leqslant \sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!}d^k \int\limits_{x\in[x_0-d,x_0+d]} \delta(x-x_0) \\
&= \sum_{k=1}^{\infty} \frac{|f^{(k)}(x_0)|}{k!}d^k.
\end{aligned}$$

Thus

$$\lambda\left(\int\limits_{x\in[x_0-d,x_0+d]}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)\right)\geqslant\lambda\left(\sum_{k=1}^{\infty}\frac{|f^{(k)}(x_0)|}{k!}d^k\right).$$

However, since $i(f)=0$, for all $k\in\mathbb{N}$ $\lambda(f^{(k)}(x_0))\geqslant 0$ and hence $\lambda\left(\sum\limits_{k=1}^{\infty}\frac{|f^{(k)}(x_0)|}{k!}d^k\right)\geqslant 1$. Thus, $\lambda\left(\int\limits_{x\in[x_0-d,x_0+d]}\sum\limits_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)\right)\geqslant 1$. Therefore,

$$\int\limits_{x\in[x_0-d,x_0+d]}\sum_{k=1}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\delta(x-x_0)=_0 0.$$

It follows that

$$\int\limits_{x\in[a,b]}f(x)\delta(x-x_0)=_0 f(x_0).$$

□

Proposition 4.6.

Suppose that $a,b\in\mathcal{R}$ with $a<b<c$, $\lambda(b-a)<1$, and $\lambda(c-b)<1$. Let $g:[a,b]\to\mathcal{R}$ and $h:[b,c]\to\mathcal{R}$ be $\mathcal{R}$-analytic functions satisfying $g(b)=h(b)$ and $i(h)=i(g)=0$. Let the function $f:[a,c]\to\mathcal{R}$ be given by

$$f(x)=\begin{cases}g(x) & \text{if } x\in[a,b)\\ h(x) & \text{if } x\in[b,c]\end{cases}.$$

Then for any $x_0 \in [a+d, c-d]$,

$$\int_{x \in [a,c]} f(x)\delta(x - x_0) =_0 f(x_0).$$

Proof.

We may assume without loss of generality that $b = 0$. Fix $x_0 \in [a+d, c-d]$, if $|x_0| \geqslant d$ then by Proposition 4.5 we are done, so it is sufficient to show that the proposition holds when $|x_0| < d$.

Consider the case that $x_0 \in [0, c]$. Then we have that

$$\int_{x \in [a,c]} f(x)\delta(x - x_0) = \int_{x \in [x_0 - d, 0]} g(x)\delta(x - x_0) + \int_{x \in [0, x_0 + d]} h(x)\delta(x - x_0).$$

Both g and h are $\mathcal{R}$-analytic functions defined on $[a, 0]$ and $[0, c]$ respectively so they may both be expanded about zero as power series, so

$$g(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$

and

$$h(x) = \sum_{k=0}^{\infty} \beta_k x^k.$$

Since $\lambda(b-a) < 1$ and $\lambda(c-b) < 1$ both power series will have radius of convergence infinitely larger than d so they will converge everywhere on $[x_0 - d, 0]$ and $[0, x_0 + d]$ respectively. Thus,

$$\int_{x \in [x_0 - d, 0]} g(x)\delta(x - x_0) = \sum_{k=0}^{\infty} \alpha_k \int_{x \in [x_0 - d, 0]} x^k \delta(x - x_0)$$

and

$$\int\limits_{x\in[x_0-d,0]} g(x)\delta(x-x_0) = \sum_{k=0}^{\infty} \beta_k \int\limits_{x\in[0,x_0+d]} x^k \delta(x-x_0).$$

Therefore,

$$\begin{aligned}
\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) &= \sum_{k=0}^{\infty} \alpha_k \int\limits_{x\in[x_0-d,0]} x^k \delta(x-x_0) + \sum_{k=0}^{\infty} \beta_k \int\limits_{x\in[0,x_0+d]} x^k \delta(x-x_0) \\
&= \alpha_0 \int\limits_{x\in[x_0-d,0]} \delta(x-x_0) + \beta_0 \int\limits_{x\in[0,x_0+d]} \delta(x-x_0) \\
&+ \sum_{k=1}^{\infty} \alpha_k \int\limits_{x\in[x_0-d,0]} x^k \delta(x-x_0) \\
&+ \sum_{k=1}^{\infty} \beta_k \int\limits_{x\in[0,x_0+d]} x^k \delta(x-x_0).
\end{aligned}$$

However, $\alpha_0 = g(0) = f(0) = h(0) = \beta_0$ so,

$$\alpha_0 \int\limits_{x\in[x_0-d,0]} \delta(x-x_0) + \beta_0 \int\limits_{x\in[0,x_0+d]} \delta(x-x_0) = f(0) \int\limits_{x\in[x_0-d,x_0+d]} \delta(x-x_0) = f(0).$$

Thus,

$$\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) = f(0) + \sum_{k=1}^{\infty} \alpha_k \int\limits_{x\in[x_0-d,0]} x^k \delta(x-x_0) + \sum_{k=1}^{\infty} \beta_k \int\limits_{x\in[0,x_0+d]} x^k \delta(x-x_0).$$

But,

$$\lambda\left(\sum_{k=1}^{\infty}\alpha_k \int\limits_{x\in[x_0-d,0]} x^k\delta(x-x_0)+\sum_{k=1}^{\infty}\beta_k \int\limits_{x\in[0,x_0+d]} x^k\delta(x-x_0)\right)$$
$$\geqslant \lambda\left(\sum_{k=1}^{\infty}\alpha_k \int\limits_{x\in[x_0-d,0]} d^k\delta(x-x_0)+\sum_{k=1}^{\infty}\beta_k \int\limits_{x\in[0,x_0+d]} d^k\delta(x-x_0)\right)\geqslant 1.$$

as in Proposition 4.5. So,

$$\sum_{k=1}^{\infty}\alpha_k \int\limits_{x\in[x_0-d,0]} x^k\delta(x-x_0)+\sum_{k=1}^{\infty}\beta_k \int\limits_{x\in[0,x_0+d]} x^k\delta(x-x_0) =_0 0.$$

And so finally, $\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) =_0 f(0)$. However, $f(0) =_0 f(x_0)$[3]; therefore

$$\int\limits_{x\in[a,c]} f(x)\delta(x-x_0) =_0 f(x_0).$$

$\square$

Unlike the classical delta function which is only mathematically rigorous within the framework of distribution, the $\mathcal{R}$-delta function can be represented as a proper function. Moreover, the $\mathcal{R}$-delta function closely resembles our intuitive understanding which is not the case for the classical delta function. Below we present two examples in which we illustrate the applications of the $\mathcal{R}$-delta function.

Example 4.7. *[Solving the Laplacian in One Dimension]*

Suppose we wish to find a solution to the real differential equation $\ddot{x}(t) = f(t)$ subject to the initial conditions $x(0) = 0, \dot{x}(0) = 0$.

To begin we observe that the piecewise analytic solution to $\frac{\partial^2}{\partial t^2}G(t,t') = \delta(t-t')$ *is*

$$G(t,t') = \begin{cases} A_1(t-t') + B_1 & t < t'-d \\ A_2(t-t') + B_2 + \frac{3}{8}d^{-3}(d^2(t-t')^2 - \frac{1}{6}(t-t')^4) & t'-d \leqslant t \leqslant t'+d \\ A_3(t-t') + B_3 & t > t'+d \end{cases}.$$

For initial conditions we require that $G(t'-d,t') = 0$, $\frac{\partial}{\partial t}G(t,t')|_{t=t'-d} = 0$. *Furthermore, since* $\delta(t-t')$ *is continuous at* $t' = t+d$ *and* $t' = t-d$ *we must have*

$$\lim_{\epsilon\to 0} \int\limits_{t'\in[t+d-\epsilon,t+d+\epsilon]} \frac{\partial^2}{\partial t^2}G(t,t') = \lim_{\epsilon\to 0} \int\limits_{t'\in[t-d-\epsilon,t-d+\epsilon]} \frac{\partial^2}{\partial t^2}G(t,t') = 0$$

which implies that $\frac{\partial}{\partial t}G(t,t')$ *is continuous at* $t' = t+d$ *and* $t' = t-d$. *Since* $\frac{\partial}{\partial t}G(t,t')$ *is continuous at* $t' = t+d$ *and* $t' = t-d$ *we must also have*

$$\lim_{\epsilon\to 0} \int\limits_{t'\in[t+d-\epsilon,t+d+\epsilon]} \frac{\partial}{\partial t}G(t,t') = \lim_{\epsilon\to 0} \int\limits_{t'\in[t-d-\epsilon,t-d+\epsilon]} \frac{\partial}{\partial t}G(t,t') = 0$$

from which we see that $G(t,t')$ *is continuous at* $t' = t+d$ *and* $t' = t-d$. *Using the initial conditions and continuity of* $G(t,t')$ *and its derivative at* $t = t' \pm d$, *we can solve for* A_1, B_1, A_2, B_2, A_3, *and* B_1, *to get*

$$G(t,t') = \begin{cases} 0 & t < t'-d \\ \frac{1}{2}(t-t') + \frac{3}{16}d^{-3} + \frac{3}{8}d^{-3}(d^2(t-t')^2 - \frac{1}{6}(t-t')^4) & t'-d \leqslant t \leqslant t'+d \\ t-t' & t > t'+d \end{cases}.$$

Note that when restricted to real points $G(t,t')$ reduces to the classical Green's function for $\frac{d^2}{dt^2}$.

It follows from Proposition 4.14 of [1] that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int\limits_{t'\in[-d^{-1},d^{-1}]} G(t,t')f(t') &= \int\limits_{t'\in[-d^{-1},d^{-1}]} \frac{\partial^2}{\partial t^2}G(t,t')f(t') \\ &= \int\limits_{t'\in[-d^{-1},d^{-1}]} \delta(t-t')f(t') \\ &=_0 f(t).\end{aligned}$$

So, $\int\limits_{x\in[-d^{-1},d^{-1}]} G(t,t')f(t') =_0 u(t)$ *where* $u(t)$ *is the solution to the* $\mathcal{R}$ *differential equation* $\ddot{u}(t) = f(t)$. *In particular this is true for the real solution* $x(t)$ *for* $t \in \mathbb{R}$, *so*

$$\int\limits_{t'\in[-d^{-1},d^{-1}]} G(t,t')f(t') =_0 x(t).$$

Now, if we set

$$f(t) = \begin{cases} t & \textit{if } t \geqslant 0 \\ 0 & \textit{if } t < 0 \end{cases}$$

then we see that

$$\int_{t'\in[d^{-1},d^{-1}]} G(t,t')f(t') = \int_{t'\in[0,t+d]} G(t,t')f(t')$$

$$= \int_{t'\in[t-d,t+d]} \left(\frac{1}{2}(t-t') + \frac{3}{16}d^{-3} + \frac{3}{8}d^{-3}\left(d^2(t-t')^2 - \frac{1}{6}(t-t')^4\right)\right)t' + \int_{t'\in[0,t-d]} (t-t')t'$$

$$= \frac{1}{2}\left[\frac{tt'}{2} - \frac{t'^3}{3}\right]\Bigg|_{t-d}^{t+d} + \frac{3}{16}d\left[\frac{t'^2}{2}\right]\Bigg|_{t-d}^{t+d} + \frac{2}{8}d^{-1}\left[\frac{t^2t'^2}{2} - \frac{2tt'^2}{3} + \frac{t'^4}{4}\right]\Bigg|_{t-d}^{t+d}$$

$$- \frac{1}{16}d^{-3}\left[\frac{t^4t'^2}{2} - \frac{4t^3t'^3}{3} + \frac{3t^2t'^4}{2} - \frac{4tt'^5}{5} + \frac{t'^6}{6}\right]\Bigg|_{t-d}^{t+d} + t\left[\frac{t'^2}{2}\right]\Bigg|_{0}^{t-d} - \left[\frac{t'^3}{3}\right]\Bigg|_{0}^{t-d}$$

$$= \frac{3}{5}td^2 - \frac{1}{3}d^3 + \frac{1}{6}t^3 - \frac{1}{2}td^2 - \frac{1}{3}d^3 =_0 \frac{1}{6}t^3.$$

Thus, $u(t) =_0 \frac{1}{6}t^3$ *and hence the real solution is* $x(t) = \frac{1}{6}t^3$.

Example 4.8 (Damped Driven Harmonic Oscillator)**.**

Consider an underdamped, driven harmonic oscillator with mass m*, viscous damping* c*, spring constant* k*, and driving force* $f(t)$*. Let* $x(t)$ *be the position of the oscillator at time* t *with* $x(0) = 0$ *and* $\dot{x}(0) = 0$*. The oscillator's equation of motion is* $\ddot{x}(t)+\frac{c}{m}\dot{x}(t)+\frac{k}{m}x(t) = \frac{f(t)}{m}$*. For convenience let* $\gamma = \frac{c}{2\sqrt{mk}}$ *and let* $\omega_0 = \sqrt{\frac{k}{m}}$*, so we have* $\ddot{x}(t) + 2\gamma\omega_0\dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}$*. Since the oscillator is underdamped we have* $\gamma^2\omega_0^2 - \omega_0^2 < 0$ *which implies* $\gamma^2 < 1$*.*

To solve the equation of motion we first find the Green's function of $(\frac{d^2}{dt^2} + 2\gamma\omega_0\frac{d}{dt} + \omega_0^2)$*, or in other words we find a solution to* $(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2)G(t,t') = \delta(t-t')$*.*

First we observe that the analytic solution to the homogeneous partial differential equation

$$(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2)G_{hom.}(t,t') = 0$$

is

$$G_{hom.}(t,t') = e^{-\gamma\omega_0(t-t')}\left(A\sin(\omega(t-t')) + B\cos(\omega(t-t'))\right)$$

where $\omega = \sqrt{1-\gamma^2}\omega_0$.

Next we find that the solution to the inhomogeneous partial differential equation

$$(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2)G_{inhom.}(t,t') = \frac{3}{4}d^{-3}(d^2-t^2)$$

is

$$G_{inhom.}(t,t') = \frac{3}{\omega_0^2}d^{-3}\left(\frac{d^2-(t-t')^2}{4} + \frac{\gamma(t-t')}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2}\right).$$

Since

$$\delta(t) = \begin{cases} 0 & \textit{if } t < -d \\ \frac{3}{4}d^{-3}(d^2-t^2) & \textit{if } -d \leqslant t \leqslant d \\ 0 & \textit{if } d < t \end{cases},$$

we must have

$$G(t,t') = \begin{cases} e^{-\gamma\omega_0(t-t')}(A_1\sin(\omega(t-t')) + B_1\cos(\omega(t-t'))) & \textit{if } t < t'-d \\ e^{-\gamma\omega_0(t-t')}(A_2\sin(\omega(t-t')) + B_2\cos(\omega(t-t'))) & \\ +\frac{3}{\omega_0^2}(\frac{d^2-(t-t')^2}{4} + \frac{\gamma(t-t')}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2}) & \textit{if } t'-d \leqslant t \leqslant t'+d \\ e^{-\gamma\omega_0(t-t')}(A_3\sin(\omega(t-t')) + B_3\cos(\omega(t-t'))) & \textit{if } t > t'+d \end{cases}.$$

Our solution must satisfy the initial conditions $G(t'-d,t') = 0$ *and* $\frac{\partial}{\partial t}G(t,t')|_{t=t'-d} = 0$ *as well as continuity of* $G(t,t')$ *and* $\frac{\partial}{\partial t}G(t,t')$ *at* $t = t'-d$ *and* $t = t'+d$. *From the initial* *conditions we find that*

$$A_1 = 0 \text{ and } B_1 = 0.$$

From the continuity of G and its derivative at $t = t' - d$ we then have

$$\begin{aligned}
A_2 &= \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d\right)\frac{\cos(\omega d)}{\omega} - \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2}\right)\sin(\omega d)\right)\\
B_2 &= \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d\right)\frac{\sin(\omega d)}{\omega} + \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2}\right)\cos(\omega d)\right).
\end{aligned}$$

Finally, from the continuity of G and its derivative at $t = t' + d$ we get:

$$\begin{aligned}
A_3 &= \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d\right)\frac{\cos(\omega d)}{\omega} - \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2}\right)\sin(\omega d)\right)\\
&\quad + \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d}\left(\left(\frac{3\gamma}{2\omega_0} - \frac{2\gamma^3}{\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d\right)\frac{\cos(\omega d)}{\omega} + \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2}\right)\sin(\omega d)\right)\\
B_3 &= \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d}\left(\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d\right)\frac{\sin(\omega d)}{\omega} + \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2}\right)\cos(\omega d)\right)\\
&\quad - \frac{3}{\omega_0^2} d^{-3} e^{-\gamma\omega_0 d}\left(\left(\frac{3\gamma}{2\omega_0} - \frac{2\gamma^3}{\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d\right)\frac{\sin(\omega d)}{\omega} - \left(\frac{\gamma}{\omega_0} d - \frac{1-4\gamma^2}{2\omega_0^2}\right)\cos(\omega d)\right).
\end{aligned}$$

While at first glance these constants seem unmanageably large, in fact $A_3 =_0 \frac{1}{\omega}$ and $B_3 =_0 0$, so

$$G(t,t')|_{\mathbb{R}} =_0 \begin{cases} 0 & \text{if } t \leqslant t' \\ \frac{1}{\omega} e^{-\gamma\omega_0(t-t')}\sin(\omega(t-t')) & \text{if } t > t' \end{cases}$$

which is the classical Green's function for this problem.

Now, suppose that the driving force is given to be

$$f(t) = \begin{cases} me^{-\gamma\omega_0 t} & \text{if } t \geqslant 0 \\ \\ 0 & \text{if } t < 0 \end{cases},$$

then the equation of motion is

$$\ddot{x}(t) + 2\gamma\omega_0\dot{x}(t) + \omega_0^2 x(t) = \begin{cases} e^{-\gamma\omega_0 t} & \text{if } t \geqslant 0 \\ \\ 0 & \text{if } t < 0 \end{cases}.$$

So, as in the previous example, we can obtain the real solution as the real part of

$$\int\limits_{t' \in [-d^{-1}, d^{-1}]} G(t, t') \frac{f(t')}{m}.$$

Therefore,

$$x(t) =_0 \int\limits_{t' \in [-d^{-1}, d^{-1}]} G(t, t') \frac{f(t')}{m}.$$

But $G(t, t') = 0$ *for* $t' > t + d$ *and* $f(t') = 0$ *for* $t' < 0$*, so*

$$\int\limits_{t' \in [-d^{-1}, d^{-1}]} G(t, t') \frac{f(t')}{m} = \int\limits_{t' \in [0, t+d]} G(t, t') e^{-\gamma\omega_0 t'}.$$

Thus,

$$
\begin{aligned}
x(t) &=_0 \int\limits_{t' \in [0,t+d]} G(t,t') e^{-\gamma\omega_0 t'} \\
&= e^{-\gamma\omega_0 t} \int\limits_{t' \in [t-d,t+d]} (A_2 \sin(\omega(t-t')) + B_2 \cos(\omega(t-t'))) \\
&+ e^{-\gamma\omega_0 t} \int\limits_{t' \in [t-d,t+d]} \left(\frac{3}{\omega_0^2} (\frac{d^2 - (t-t')^2}{4} + \frac{\gamma(t-t')}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2} \right) \\
&+ e^{-\gamma\omega_0 t} \int\limits_{t' \in [0,t-d]} (A_3 \sin(\omega(t-t')) + B_3 \cos(\omega(t-t'))) \\
&= e^{-\gamma\omega_0 t} \left[\frac{2A_2 \sin(\omega d)}{\omega} + A_3 \frac{\sin(\omega t) - \sin(\omega d)}{\omega} + B_3 \frac{\cos(\omega t) - \cos(\omega d)}{\omega} \right. \\
&- \frac{3}{\omega_0^2} d^{-3} \left(\frac{2}{4\gamma^3 \omega_0^3} + \frac{1+4\gamma^2}{2\gamma\omega_0^3} \right) (e^{\gamma\omega_0 d} - e^{-\gamma\omega_0 d}) \\
&\left. + \frac{3}{\omega_0^2} \left(\frac{d}{2\gamma^2 \omega_0^2} + \frac{d}{\omega_0^2} \right) (e^{\gamma\omega_0 d} + e^{-\gamma\omega_0 d}) \right] \\
&=_0 e^{-\gamma\omega_0 t} \frac{\cos(\omega t) - 1}{\omega^2},
\end{aligned}
$$

which agrees with the classical solution.

Using the integration theory developed in previous sections it is possible to extend the delta function to $\mathcal{R}^2$ and $\mathcal{R}^3$. We present the foundations of this extension below.

Definition 4.9.

We define $\delta : \mathcal{R}^2 \to \mathcal{R}$ *by*

$$\delta(x, y) = \delta(x)\delta(y).$$

Proposition 4.10.

Let $S \subset \mathcal{R}^2$ *be a simple region with* $\lambda_x(A) < 1$ *and* $\lambda_y(A) < 1$, *let* $f : S \to \mathcal{R}$ *be an analytic* *function with index $i(f) = 0$ on S. Then, for any $(x_0, y_0) \in S$ that satisfies $(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d) \subset S$, we have that*

$$\iint_{(x,y)\in S} f(x,y)\delta(x - x_0, y - y_0) =_0 f(x_0, y_0).$$

Proof.

First we note that $\delta(x - x_0, y - y_0) = 0$ everywhere except on the simple region $(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d)$ so

$$\iint_{(x,y)\in S} f(x,y)\delta(x - x_0, y - y_0) = \int_{x\in(x_0-d,x_0+d)} \int_{y\in(y_0-d,y_0+d)} f(x,y)\delta(x - x_0, y - y_0)$$

Now, for a fixed $x \in (x_0-d, x_0+d)$, $h(y) = f(x, y)$ is an analytic function on (y_0-d, y_0+d) thus by Proposition 4.5,

$$\int_{y\in(y_0-d,y_0+d)} h(y)\delta(y - y_0) =_0 h(y_0) = f(x, y_0)$$

Furthermore, $g(x) = f(x, y_0))$ is analytic on $(x_0 - d, x_0 + d)$ so by Proposition 4.5,

$$\int_{x\in(x_0-d,x_0+d)} g(x)\delta(x - x_0) =_0 g(x_0) = f(x_0, y_0)$$

So,

$$\iint\limits_{(x,y)\in S} f(x,y)\delta(x-x_0,y-y_0) = \int\limits_{x\in(x_0-d,x_0+d)} \delta(x-x_0) \int\limits_{x\in(y_0-d,y_0+d)} \delta(y-y_0)f(x,y)$$

$$=_0 \int\limits_{x\in(x_0-d,x_0+d)} \delta(x-x_0)f(x,y_0)$$

$$=_0 f(x_0,y_0)$$

□

Definition 4.11.

We define $\delta : \mathcal{R}^3 \to \mathcal{R}$ *by*

$$\delta(x,y,z) = \delta(x)\delta(y)\delta(z).$$

It follows immediately from Definitions 4.9 and 4.11 that $\delta(x,y,z) = \delta(x,y)\delta(z) = \delta(x,z)\delta(y) = \delta(x)\delta(y,z)$.

Proposition 4.12.

Let $S \subset \mathcal{R}^3$ *be a simple region with* $\lambda_x(S) < 1$, $\lambda_y(S) < 1$, $\lambda_z(S) < 1$, *and let* $f : S \to \mathcal{R}$ *be an analytic function on* S *with* $i(f) = 0$ *on* S. *Then, for any* $(x_0,y_0,z_0) \in S$ *that satisfies* $(x_0-d,x_0+d) \times (y_0-d,y_0+d) \times (z_0-d,z_0+d) \subset S$, *we have that*

$$\iiint\limits_{(x,y,z)\in S} f(x,y,z)\delta(x-x_0,y-y_0,z-z_0) =_0 f(x_0,y_0,z_0).$$

Proof.

First we note that $\delta(x-x_0,y-y_0,z-z_0) = 0$ everywhere exept on the simple region

$(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d) \times (z_0 - d, z_0 + d)$ so

$$\iint\limits_{(x,y,z)\in S} f(x,y,z)\delta(x-x_0, y-y_0, z-z_0)$$
$$= \int\limits_{x\in(x_0-d,x_0+d)} \int\limits_{y\in(y_0-d,y_0+d)} \int\limits_{z\in(z_0-d,z_0+d)} f(x,y,z)\delta(x-x_0, y-y_0, z-z_0)$$

Now, for fixed $(x, y) \in (x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d)$, $h(z) = f(x, y, z)$ is an analytic function on $(z_0 - d, z_0 + d)$; thus by Proposition 4.5,

$$\int\limits_{z\in(z_0-d,z_0+d)} h(z)\delta(z-z_0) =_0 h(z_0) = f(x,y,z_0).$$

Furthermore, $g(x, y) = f(x, y, z_0)$ is analytic on $(x_0 - d, x_0 + d) \times (y_0 - d, y_0 + d)$ so by Proposition 4.10,

$$\iint\limits_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d} g(x,y)\delta(x-x_0, y-y_0) =_0 g(x_0,y_0) = f(x_0,y_0,z_0).$$

So,

$$\iiint\limits_{(x,y,z)\in S} f(x,y,z)\delta(x-x_0, y-y_0, z-z_0)$$
$$= \iint\limits_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)} \delta(x-x_0, y-y_0) \int_{z\in(z_0-d,z_0+d)} \delta(z-z_0) f(x,y,z)$$
$$=_0 \iint\limits_{(x,y)\in(x_0-d,x_0+d)\times(y_0-d,y_0+d)} \delta(x-x_0, y-y_0) f(x,y,z_0)$$
$$=_0 f(x_0,y_0,z_0).$$

$\square$

Example 4.13 (Electric Field of a Point Charge).

Suppose there is a point charge located at the origin with total charge q and suppose we wish to find the electric field at a point $\boldsymbol{r}$. By symmetry we may choose our axes so that $\boldsymbol{r} = r\boldsymbol{i}$, so the distance between $\boldsymbol{r}$ and an arbitrary point $\boldsymbol{r'} = x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}$ is $|\boldsymbol{r} - \boldsymbol{r'}| = \sqrt{(r-x')^2 + y'^2 + z'^2}$. By Coulomb's law we know that the electric field at $\boldsymbol{r}$ is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\textit{all space}} \frac{\rho(\boldsymbol{r'})}{|r-r'|^2} \frac{\boldsymbol{r} - \boldsymbol{r'}}{|r-r'|} dr'$$

where $\rho(\boldsymbol{r'})$ is the charge density at $\boldsymbol{r'}$.

The charge density of a point charge can not be written using proper functions in $\mathbb{R}^3$ but in $\mathcal{R}^3$ ρ can be expressed by the analytic function $\rho(\boldsymbol{r'}) = q\delta(x', y', z')$. Thus, the electric field can be obtained as the real part of

$$\begin{aligned}
&\frac{1}{4\pi\epsilon_0} \iiint\limits_{(x,y,z)\in[-d^{-1},d^{-1}]^3} \frac{q\delta(x',y',z')}{((r-x')^2 + y'^2 + z'^2)^{\frac{3}{2}}} [(r-x')\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}] \\
&= \frac{1}{4\pi\epsilon_0} \iiint\limits_{(x,y,z)\in[-d,d]^3} \frac{q\delta(x',y',z')}{((r-x')^2 + y'^2 + z'^2)^{\frac{3}{2}}} [(r-x')\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}] \\
&= \frac{1}{4\pi\epsilon_0} \iiint\limits_{(x,y,z)\in[-d,d]^3} \frac{q(\frac{3}{4}d^{-3})^3(d^2 - x'^2)(d^2 - y'^2)(d^2 - z'^2)}{((r-x')^2 + y'^2 + z'^2)^{\frac{3}{2}}} [(r-x')\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}]
\end{aligned}$$

Now,

$$\frac{1}{4\pi\epsilon_0}\iiint\limits_{(x,y,z)\in[-d,d]^3}\frac{q(\frac{3}{4}d^{-3})^3(d^2-x'^2)(d^2-y'^2)(d^2-z'^2)}{((r-x')^2+y'^2+z'^2)^{\frac{3}{2}}}y'$$
$$=\frac{1}{4\pi\epsilon_0}\iiint\limits_{(x,y,z)\in[-d,d]^3}\frac{q(\frac{3}{4}d^{-3})^3(d^2-x'^2)(d^2-y'^2)(d^2-z'^2)}{((r-x')^2+y'^2+z'^2)^{\frac{3}{2}}}z'$$
$$=0$$

and by Proposition 4.12,

$$\frac{1}{4\pi\epsilon_0}\iiint\limits_{(x,y,z)\in[-d^{-1},d^{-1}]^3}\frac{q\delta(x',y',z')}{((r-x')^2+y'^2+z'^2)^{\frac{3}{2}}}(r-x')$$
$$=_0\frac{q}{4\pi\epsilon_0}\frac{1}{((r-0)^2+0+0)^{\frac{3}{2}}}r$$
$$=\frac{q}{4\pi\epsilon_0 r^2}.$$

So $\boldsymbol{E}(\boldsymbol{r})=\frac{q}{4\pi\epsilon_0 r^2}(\boldsymbol{i}+0\boldsymbol{j}+0\boldsymbol{k})=\frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^3}$, *which agrees with the classical solution.*

While the examples given in this section are admittedly simple they serve to illustrate the concept. In the future we plan to engage in a more detailed study of the delta functions in two and three dimensions as well as constructing more complex and challenging examples.

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