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SPLINE SOLUTIONS FOR NONLINEAR TWO POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. Necessary formulas are developed for obtaining cubic, quartic, quintic, and sextic spline solutions of nonlinear boundary value problems. These methods enable us to approximate the solution of the boundary value problems, as well as their successive derivatives smoothly. Numerical evidence is included to demonstrate the relative performance of these four techniques.

KEY WORDS AND PHRASES. *Cubic, Quartic, Quintic and Sextic Spline Functions; Finite Difference Scheme; Noumerov's Formula; Newton's Method for Solving Nonlinear Algebraic Equations; Nonlinear Two Point Boundary Value Problem; Recurrence Relations.*

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1. PRELIMINARIES

Let a finite interval [a,b] be partitioned into (N + 1) equal parts by the insertion of N Knots $\{x_n\}$ defined by $x_n = a + nh$, $h = (b - a)/(N + 1)$, $n = 0(1)N + 1$, and let s(x) be a spline function of degree m on [a,b]. Thus, in each subinterval $[x_i, x_{i+1}]$, s(x) is a polynomial of degree at most m and $s(x) \in C^{m-1}[a,b]$. We shall designate this polynomial by

$$P_i(x) = \sum_{j=0}^{j=m} a_{ij} (x - x_i)^j, \quad i = 0(1)N, \ x \in [x_i, x_{i+1}]. \tag{1.1}$$

In this paper we shall present some methods for the continuous approximation of the solution of the two point real nonlinear boundary value problem

$$\begin{aligned} y''(x) &= f(x, y(x)), \quad a \le x \le b \\ y(a) - A &= y(b) - B = 0 \end{aligned} \tag{1.2}$$

by the use of spline functions of orders up to six. The function f(x, y(x)) is a continuous function of two variables with f_y continuous and nonnegative in the strip S defined by S: $a \le x \le b$, $-\infty < y < \infty$. It is well-known that the boundary value problem (1.2) with these conditions has a unique solution (Henrici [1], p. 347).

2. CUBIC SPLINE SOLUTION (m = 3)

The possiblities of using spline functions for obtaining smooth approximations of the solution of boundary value problems were first briefly discussed by Ahlberg et al. [2]. Following this, Bickley [3] and Albasiny et al. [4] have demonstrated the use of cubic spline function for obtaining an approximate solution of (1.2) when

$$f(x, y(x)) = p(x)\, y'(x) + q(x)\, y(x) + r(x).$$

The authors of the latter have, in particular, established via different notations than ours, the recurrence relation

$$y_{i-1} - 2y_i + y_{i+1} = (h^2/6)[M_{i-1} + 4M_i + M_{i+1}], \tag{1.3}$$

$$y_0 - A = y_{N+1} - B = 0, \quad i = 1(1)N, \; M_i = f_i,$$

where y_i denotes spline approximation to $y(x_i)$, $y(x)$ being the exact solution of the system (1.2). The unknowns y_i, $i = 1(1)N$ are first obtained by solving the tridiagonal system of nonlinear algebraic equations (1.3), M_i, $i = 0(1)N+1$ are subsequently computed by $M_i = f_i$, $f_i \equiv f(x_i, y_i)$ where we designate

$$P_i(x_j) = y_j, \; P_i''(x_j) = M_j, \quad x_j \in [x_i, x_{i+1}].$$

Finally the coefficients of (1.1) are determined from the relations

$$a_{i,0} = (M_{i+1} - M_i)/(6h), \quad a_{i,1} = M_i/2$$

$$a_{i,2} = (y_{i+1} - y_i)/h - h(M_{i+1} + 2M_i)/6, \quad a_{i,3} = y_i \; . \tag{1.4}$$

We can also show that

$$\begin{aligned} y_i' &= a_{i,2} \qquad\qquad i = 0(1)N \\ &= (y_{N+1} - y_N)/h + h(2M_{N+1} + M_N)/6, \quad i = N+1. \end{aligned} \tag{1.5}$$

The approximate values of $y(x)$ and its derivative at the points other than knots are obtained by evaluating or differentiating the corresponding cubic spline polynomial.

3. SOLUTION OF NONLINEAR EQUATIONS (1.3)

The method which we shall use to obtain the solution of the system (1.3) is a generalization of Newton's method which we summarize very briefly for the sake of completeness (Henrici [1], p.355). Let the nonlinear equations (1.3) in N unknown y_i be written in the form

$$\phi_i(y_1, y_2, \ldots., y_n) = 0, \quad i = 1(1)N \tag{3.1}$$

or, in vector form,

$$\phi(Y) = 0, \tag{3.2}$$

where $\phi = (\phi_i)$, $Y = (y_i)$ are N dimensional vectors. Let $M(Y) = (m_{ij})$ denote the matrix with elements

$$m_{ij} = \partial\phi_i/\partial y_j \qquad i,\ j = 1(1)N. \tag{3.3}$$

Then the Newton's method for the solution of (3.1) is written in the form

$$Y^{(p+1)} = Y^{(p)} - [M(Y^{(p)})]^{-1}\ \phi\ (Y^{(p)}), \quad (p = 0,1,\ldots\ldots). \tag{3.4}$$

In our case, the elements m_{ij} are given as follows:

$$m_{ij} = \begin{cases} 2 + (4h^2/6)g_i, & i = j \\ -1 + (h^2/6)g_j, & |i - j| = 1 \\ 0 & |i - j| > 1 \end{cases} \tag{3.5}$$

where $g_i = f_y(x_i, y_i)$. The vector $\phi(Y)$ is usually referred to as residual vector. The criterion for stopping the iterations defined by (3.4) is that the residual vector $\phi(Y^{(p)})$ be such that

$$||\phi(Y^{(p)})|| = \max_i|\phi_i(Y^{(p)})| < \varepsilon, \quad (p = 0,1,\ldots\ldots), \tag{3.6}$$

where ε is a preassigned small positive quantity.

Also, the system of nonlinear equations (3.2) has a unique solution $\tilde{Y}$, to which the successive approximations $Y^{(p)}$ defined by (3.4) converge, provided

$$0 < \sigma = \alpha\beta\gamma \le 0.5, \tag{3.7}$$

where for $Y = Y^{(0)}$, the initial approximation, the matrix $M(Y^{(0)})$ has an inverse Γ such that

$$|| \Gamma || \le \alpha, \tag{3.8}$$

$$|| \Gamma \ \Phi(Y^{(0)}) || \le \beta , \tag{3.9}$$

$$\sum_{j,k=1}^{N} \left| \frac{\partial^2 \phi_i}{\partial y_j \ \partial y_k} \right| \le \gamma \quad , \ i = 1(1)N, \tag{3.10}$$

(Note for a vector $v = (v_i)$, $||v|| = \max_i |v_i|$,

and for a matrix $M = (m_{ij})$, $||M|| = \max_i \sum_j |m_{ij}|$.

We can verify that (3.7) will be satisfied for the system (1.3) if $h^2 G < 6$ with $G = \max_{(x,y)} f_y$, and

$$64(b - a)^4 [h^2 Q_4/12 + R]H \le 1/2, \tag{3.11}$$

where the function Q(x) providing the initial approximation is such that

$$Q_4 = \max_x |Q^{iv}(x)|, \quad R = \max_x |Q''(x) - f(x, Q(x))|, \text{ and}$$

$$H = \max_{(x,y)} |f_{yy}|, \quad (x, y) \in S.$$

In deriving (3.11), we use the theory of monotone matrices as given by Henrici ([1], p. 360). If the initial approximation vector $Y^{(0)} = (Q(x_i))$ be such that the quantity R is small, then it follows from (3.11) that the solution of the system (1.3) obtained by Newton's method will converge to the solution of (1.2) for all sufficiently small values of h.

4. QUARTIC SPLINE SOLUTION

We now consider (1.1) for m = 4. With the analogy of section 2 on cubic spline, we can determine the five coefficients of (1.1) in terms of y_i, y_{i+1}, M_i, M_{i+1} and D_i where we now write

$$P'_i(x_j) = D_j, \quad x_j \in [x_i, x_{i+1}].$$

A simple calculation gives

$$\begin{aligned} a_{i,0} &= -(y_{i+1} - y_i)/h^4 + D_i/h^3 + (M_{i+1} + 2M_i)/(6h^2) \\ a_{i,1} &= 2(y_{i+1} - y_i)/h^3 - 2D_i/h^2 - (M_{i+1} + 5M_i)/(6h) \qquad (4.1) \\ a_{i,2} &= M_i/2, \; a_{i,3} = D_i, \; a_{i,4} = y_i, \quad i = 0(1)N. \end{aligned}$$

Continuity of the first and third derivative at $x = x_i$ gives the relations [that is $P'_{i-1}(x_i) = P'_i(x_i)$ and $P'''_{i-1}(x_i) = P'''_i(x_i)$]

$$\left\{ \begin{aligned} & 4h^3 a_{i-1,0} + 3h^2 a_{i-1,1} + 2h a_{i-1,2} + a_{i-1,3} = a_{i,3} \\ & 4h a_{i-1,0} + a_{i-1,1} = a_{i,1} \end{aligned} \right\} \qquad (4.2)$$

which on using (4.1) reduce to

$$D_i + D_{i-1} = 2(y_i - y_{i-1})/h + h(M_i - M_{i-1})/6, \qquad (4.3)$$

and

$$D_i + D_{i-1} = (y_{i+1} - y_i)/h - h(M_{i+1} + 8M_i + M_{i-1})/12. \qquad (4.4)$$

We equate the expressions on the right side of the equality sign in (4.3) and (4.4) respectively and obtain

$$2(y_i - y_{i-1})/h + h(M_i - M_{i-1})/6 = (y_{i+1} - y_i)/h - h(M_{i+1} + 8M_i + M_{i-1})/12$$

which collapses, on simplification, to the recurrence relation

$$y_{i-1} - 2y_i + y_{i+1} = (h^2/12)[M_{i-1} + 10M_i + M_{i+1}], \qquad (4.5)$$

which is the same as the well-known Noumerov's formula. As before, we first determine the unknowns y_i, $i = 1(1)N$ by solving the system of nonlinear algebraic equations by Newton's iterative method explained in the previous section and then compute $D_i, i = 1(1)N+1$, using (4.3) in conjunction with (Usmani, [5])

$$D_0 = [-y_0 + y_1 - h^2(5M_0 + M_1)/12 - (h^3/12)\frac{\partial f(x_0, y_0)}{\partial x}]/[h(1 + (h^2/12)g_0]. \quad (4.6)$$

And now the knowledge of y_i, D_i, M_i, $i = 0(1)N+1$ enables us to produce the coefficients of quartic spline as given by (4.1). An approximation of the third derivative at knots is given by

$$\begin{aligned} y_i''' &= 6a_{i,1}, \quad i = 0(1)N \\ &= 24ha_{N,0} + 6a_{N,1} = -12(y_{N+1} - y_N)/h^3 + 12D_N/h^2 \qquad (4.7) \\ &\quad + 3(M_{N+1} + M_N)/h, \quad i = N+1. \end{aligned}$$

5. QUINTIC SPLINE SOLUTION

We now consider (1.1) for m=5.

Set $p_i^{iv}(x_j) = S_j$, $x_j \in [x_i, x_{i+1}]$. As before, we can compute the coefficients of (1.1) in terms of y_i, y_{i+1}, M_i, M_{i+1}, S_i, S_{i+1} in the form (Späth, [6])

$$\begin{aligned} a_{i,0} &= (S_{i+1} - S_i)/(120h), \quad a_{i,1} = S_i/24 \qquad (5.1) \\ a_{i,2} &= (M_{i+1} - M_i)/(6h) - h(S_{i+1} + S_i)/36, \; a_{i,3} = M_i/2 \\ a_{i,4} &= (y_{i+1} - y_i)/h - h\,(M_{i+1} + 2M_i)/6 + h^3(7S_{i+1} + 8S_i)/360 \\ a_{i,5} &= y_i, \qquad i = 0(1)N, \end{aligned}$$

From the continuity of first and third derivatives at $x = x_i$, we have

$$7S_{i+1} + 16S_i + 7S_{i-1} = 60(M_{i+1} + 4M_i + M_{i-1})/h^2 - 360(y_{i+1} - 2y_i + y_{i-1})/h^4, \quad (5.2)$$

$$S_{i+1} + 4S_i + S_{i-1} = 6(M_{i+1} - 2M_i + M_{i-1})/h^2. \qquad (5.3)$$

From the preceding two relations, it follows that

$$S_i = 30(y_{i+1} - 2y_i + y_{i-1})/h^4 - 3(M_{i+1} + 18M_i + M_{i-1})/(2h^2),\ i = 1(1)N \quad (5.4)$$

On substituting the values of S_j, $j = i-1, i, i+1$ either in (5.2) or in (5.3), we readily derive the desired relation

$$(y_{i-2} + y_{i+2}) + 2(y_{i-1} + y_{i+1}) - 6y_i = (h^2/20)[(M_{i-2} + M_{i+2}) + 26(M_{i-1} + M_{i+1}) + 66M_i],\ i = 2(1)N-1. \quad (5.5)$$

This recurrence relation only gives (N-2) equations in the N unknowns y_i. In an analogous manner we derive two more relations, namely

(i) $4y_0 - 7y_1 + 2y_2 + y_3 = (h^2/12)[4M_0 + 41M_1 + 14M_2 + M_3]$

(ii) $y_{N-2} + 2y_{N-1} - 7y_N + 4y_{N+1} = (h^2/12)[M_{N-2} + 14M_{N-1} + 41M_N + 4M_{N+1}]$. (5.6)

Now the determination of N unknowns y_i can be effected by solving the nonlinear algebraic equations (5.5) and (5.6). The knowledge of y_i, $i = 0(1)N+1$, enables us to compute M_i, $i = 0(1)N+1$ and finally S_i, $i = 1(1)N$ using (5.4). The quantities S_0 and S_{N+1} can be computed from the formulas

(i) $S_0 = 0.1[(48/h^4)(-2y_0 + 5y_1 - 4y_2 + y_3 + h^2M_0) - 21S_1 - 12S_2 - S_3]$,

(ii) $S_{N+1} = 0.1[(48/h^4)(y_{N-2} - 4y_{N-1} + 5y_N - 2y_{N+1} + h^2M_{N+1}) - 21S_N - 12S_{N-1} - S_{N-2}]$. (5.7)

In fact we use (5.7) for N = 3 and for N>3 we can use more accurate formulas obtained from

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = (h^4/120)[S_{i-2} + 26S_{i-1} + 66S_i + 26S_{i+1} + S_{i+2}] \quad (5.8)$$

which is easily derived on eliminating M_i's from (5.2) and (5.3).

Finally, the knowledge of y_i, M_i, S_i, $i = 0(1)N+1$ enables us to write down the coefficients of (1.1) as given by (5.1). We also have

$$y'_i = a_{i,4} \qquad i = 0(1)N$$

$$= (y_{N+1} - y_N)/h + h(2M_{N+1} + M_N)/6 - h^3(8S_{N+1} - 23S_N)/360, \tag{5.9}$$

$i = N + 1$, and

$$y'''_i = 6a_{i,2} \qquad i = 0(1)N$$

$$= (M_{N+1} - M_N)/h + h(2S_{N+1} + S_N)/6, \ i = N+1. \tag{5.10}$$

6. SEXTIC SPLINE SOLUTION

We finally consider (1.1) for m = 6. With the analogy of the previous sections, we compute the coefficients of (1.1) in terms of y_i, y_{i+1}, D_i, M_i, M_{i+1}, S_i, and S_{i+1} in the form

$$a_{i,0} = (y_{i+1} - y_i)/(3h^6) - D_i/(3h^5) - (M_{i+1} + 2M_i)/(18h^4) \tag{6.1}$$
$$+ (7S_{i+1} + 8S_i)/(1080h^2),$$

$$a_{i,1} = -(y_{i+1} - y_i)/h^5 + D_i/h^4 + (M_{i+1} + 2M_i)/(6h^3)$$
$$- (4S_{i+1} + 11S_i)/(360h),$$

$$a_{i,2} = S_i/24,$$

$$a_{i,3} = 5(y_{i+1} - y_i)/(3h^3) - 5D_i/(3h^2) - (2M_{i+1} + 13M_i)/(18h)$$
$$+ h(S_{i+1} - 4S_i)/216,$$

$$a_{i,4} = M_i/2, \ a_{i,5} = D_i, \ a_{i,6} = y_i, \ i = 0(1)N.$$

Continuity of the first, third, and fifth derivatives at $x = x_i$ gives the relations

$$D_i + D_{i-1} = 2(y_i - y_{i-1})/h + h(M_i - M_{i-1})/6 - h^3(S_i - S_{i-1})/360, \tag{6.2}$$

$$D_i + D_{i-1} = (y_{i+1} - y_{i-1})/h - h(2M_{i+1} + 21M_i + 7M_{i-1})/30 + h^3(S_{i+1} - 9S_i + 2S_{i+1})/360, \quad (6.3)$$

$$D_i + D_{i-1} = (y_{i+1} - y_{i-1})/h - h(M_{i+1} + 3M_i + 2M_{i-1})/6 + h^3(4S_{i+1} + 21S_i + 5S_{i-1})/360. \quad (6.4)$$

From (6.2), (6.3) and (6.2), (6.4),we derive the following relations

$$y_{i+1} - 2y_i + y_{i-1} = h^2(M_{i+1} + 13M_i + M_{i-1})/15 - h^4(S_{i+1} - 8S_i + S_{i-1})/360, \quad (6.5)$$

$$y_{i+1} - 2y_i + y_{i-1} = h^2(M_{i+1} + 4M_i + M_{i-1})/6 - h^4(2S_{i+1} + 11S_i + 2S_{i-1})/180. \quad (6.6)$$

From (6.5) and (6.6), we deduce

$$h^4S_i = 20(y_{i+1} - 2y_i + y_{i-1}) - 2h^2(M_{i+1} + 28M_i + M_{i-1})/3, \quad (6.7)$$

$$h^2M_i = (y_{i+1} - 2y_i + y_{i-1}) - h^4(S_{i+1} + 28S_i + S_{i-1})/360. \quad (6.8)$$

From (6.5) and (6.7) [Note we can also use (6.6) and (6.7)],we obtain on eliminating S_i's the recurrence relation

$$(y_{i-2} + y_{i+2}) + 8(y_{i-1} + y_{i+1}) - 18y_i = (h^2/30)[(M_{i-2} + M_{i+2}) + 56(M_{i-1} + M_{i+1}) + 246M_i], \quad (6.9)$$

$i = 2(1)N-1$, which gives $(N-2)$ nonlinear algebraic equations in the N unknowns y_i. We develop two more equations similar to those given by (5.6) in the form

$$\text{(i)}\ 10y_0 - 19y_1 + 8y_2 + y_3 = (h^2/12)[10M_0 + 101M_1 + 20M_2 + M_3], \quad (6.10)$$

$$\text{(ii)}\ y_{N-2} + 8y_{N-1} - 19y_N + 10y_{N+1} = (h^2/12)[M_{N-2} + 20M_N + 101M_N + 10M_{N+1}].$$

We first solve (6.9) and (6.10) and determine y_n, $n = 1(1)N$; M_i are then determined as before. The knowledge of y_i and M_i enables us to compute S_i, $i = 1(1)N$ using (6.7). As in section 5, S_0 and S_{N+1} are computed from (5.7) or (5.8) depending according as $N = 3$ or $N > 3$. Having determined y_i, M_i, S_i, $i = 0(1)N+1$, we evaluate the coefficients of (1.1) as given by (6.1). The first derivatives at the knots are computed recursively from the formula (6.2) using the starting value for the first derivative as given below (see Usmani [7], eqn. 3.6 (i)):

$$D_0 = [-5.5y_0 + 9y_1 - 4.5y_2 + y_3 - h^4(8S_0 + 151S_1 + 52S_2 - S_3)/280]/(3h) \quad (6.11)$$

This formula is suitable for $N = 3$. However, for $N > 3$, a more accurate formula, namely

$$D_0 = -5.5y_0 + 9y_1 - 4.5y_2 + y_3 - h^4\sum_{n=0}^{5} \beta_n S_n, \quad (6.12)$$

where $(\beta_0, \beta_1, \ldots \beta_5) = \frac{1}{33600}(937, 18240, 5990, 140, -135, 28)$, could be used (see Usmani, [7], eqn. 3.7(i)).

Finally, the third and fifth derivatives are given by the following formulas:

$$y_i''' = 6a_{i,3}, \qquad i = 0(1)N \quad (6.13)$$

$$y_{N+1}''' = -10(y_{N+1} - y_N) + 10D_N/h^2 + (8M_{N+1} + 7M_N)/(3h) + h(5S_{N+1} - 2S_N)/36,$$

$$y^{(5)} = 120a_{i,1}, \qquad i = 0(1)N \quad (6.14)$$

$$y_{N+1}^{(5)} = 120(y_{N+1} - y_N)/h^5 - 120D_N/h^4 - 20(M_{N+1} + 2M_N)/h^3$$

$$+5(11S_{N+1} + 10S_N)/(6h).$$

7. CONVERGENCE

Define

$$e_i = y(x_i) - y_i$$

and $E = (e_n)$, $T = (t_n)$ are N-dimensional vectors; then the error equation in any of the four methods is obtained in a standard manner in the form

$$ME = T \tag{7.1}$$

where M is a tridiagonal matrix for cubic and quartic spline functions and M is a five band matrix for quintic and sextic spline solutions of the boundary value problem (1.2). It is easily seen that the truncation error associated with (1.3) is $0(h^4)$ and $||M^{-1}|| \leq (b - a)^2/(8h^2)$ where the elements of M are given by (3.5). Thus we easily deduce that for a cubic spline solution of (1.2) (Henrici, [1])

$$||E|| \leq Kh^2 \tag{7.2}$$

and, from this, it follows that as $h \to 0$ (i.e. $N \to \infty$) the cubic spline solution $y_i \to y(x_i)$. Thus cubic spline solution based on (1.3) is a second order convergent process. We can similarly prove that the quartic spline solution based on (4.5) is a fourth order convergent process.

In order to prove the convergence of the quintic solution based on (5.5) and (5.6), we observe that the truncation error is given by

$$t_i = \begin{cases} - h^6 y^{(6)}(\omega_1)/48, \ x_0 < \omega_1 < x_3 \\ h^6 y^{(6)}(x_i)/120 + 0(h^7), \ i = 2(1)N-2 \\ - h^6 y^{(6)}(\omega_N)/48, \ x_{N-2} < \omega_N < x_{N+1} \ . \end{cases} \tag{7.3}$$

We can establish that

$$||T|| \leq h^6 M_6/48, \ M_6 = \max_x |y^{(6)}(x)|. \tag{7.4}$$

The elements of the corresponding matrix M are given below

$$m_{12} = - 2 + (14h^2 g_2/12), \quad m_{N,N-1} = - 2 + (14h^2 g_{N-1}/12) \tag{7.5}$$

$$m_{13} = -1 + (h^2 g_3/12) \quad , \quad m_{N,N-2} = -1 + (h^2 g_{N-2}/12)$$

$$\begin{aligned} m_{ij} &= 7 + (41h^2 g_i/12) , & i = j = 1, N \\ &= 6 + (66h^2 g_i/30) , & i = j = 2(1)N-1 \\ &= -2 + (26h^2 g_j/20) , & |i - j| = 1 \\ &= -1 + (h^2 g_j/20) , & |i - j| = 2 \\ &= 0, \text{ otherwise.} \end{aligned}$$

The error analysis depends on the properties of the matrix M and $\tilde{M}$ where $\tilde{M}$ is a five band matrix obtained from M by setting each $g_i \equiv 0$, so that $\tilde{M} = (\tilde{m}_{ij})$ and

$$\tilde{m}_{11} = \tilde{m}_{N,N} = 7, \tag{7.6}$$

$$\begin{aligned} \tilde{m}_{ij} &= 6, \quad i = j, \quad i = 2(1)N-1 \\ &= -2, \ |i = j| = 1 \\ &= -1, \ |i - j| = 2 \\ &= 0, \text{ otherwise.} \end{aligned}$$

From the theory of monotone matrices (Henrici, [1]), it follows that both M and $\tilde{M}$ are monotone matrices if $13h^2G < 20$. Also, it is easily seen that

$$\tilde{M} = PQ, \tag{7.7}$$

where $P = (p_{ij})$, $Q = (q_{ij})$ are tridiagonal matrices with $p_{ii} = 2$, $p_{ij} = -1$, $|i - j| = 1$; $q_{ii} = 4$, $q_{ij} = 1$, $|i - j| = 1$. Also,

$$M^{-1} \le \tilde{M}^{-1} = [P^{-1} + Q^{-1}]/6, \tag{7.8}$$

and

$$||M^{-1}|| < ||\tilde{M}^{-1}|| \le [(b - a)^2/(8h^2) + 1/2]/6. \tag{7.9}$$

This follows from Fischer and Usmani [8]. Now from (7.1), (7.4), and (7.9), it follows that

$$||E|| \le h^4(b - a)^2 M_6/2304 + O(h^6) = O(h^4). \tag{7.10}$$

Following a similar technique, we can establish that for a sextic spline solution

$$||E|| \le \tilde{k}h^6. \tag{7.11}$$

8. NUMERICAL ILLUSTRATIONS

We solve two nonlinear boundary value problems of the form (1.2).

$$y'' = 0.5(x + y + 1)^3, \quad y(0) = y(1) = 0, \tag{8.1}$$

with $y(x) = 2/(2 - x) - x - 1$. The function $Q(x)$ is chosen to satisfy the system $Q'' = 0.5(x + 1)^3$, $Q(0) = Q(1) = 0$ so that $Q(x) = [(1 + x)^5 - 31x - 1]/40$;

$$y'' = \exp(y), \quad y(0) = y(1) = 0, \tag{8.2}$$

with $y(x) = -\ell n(2) + 2\,\ell n[C \sec\{C(x - 0.5)/2\}]$, $C = 1.3360557$.

Here $Q(x) = [\sinh x + \sinh(1 - x)]/\sinh 1 - 1$. The numerical calculations are made using double precision arithmetic in order to keep the rounding errors to a minimum. The numerical results are briefly summarized in Tables 1 - 3.

9. CONCLUDING REMARKS

Our numerical results on test problems indicate that results based on quintic and sextic spline are only marginally better than those obtained by quartic spline solution. Moreover, in order to obtain nonlinear equations equal to the number of unknowns in quintic and sextic spline solution, an ad hoc procedure is used near the boundaries of the interval. Also, the matrices that arise are five band matrices whereas in case of cubic and quartic spline solution the bandwidth of the matrices that arise is three, which makes them slightly simpler to implement. Since formulas (1.3) and (4.5) satisfy the conditions of Theorem 7.4 (Henrici [1]), Richardson's h^2 - extrapolation method can be used to push the accuracy of these formulas to $O(h^4)$ and $O(h^6)$ respectively, whereas in case of quintic and sextic spline solutions we can only use h-extrapolation technique to improve the numerical solution. The latter techniques also suffer from a disadvantage that they require approximate formulas for S_0 and S_{N+1} [see (5.7)]. Similarly, in sextic spline solution, in order to compute D_i, i = 1(1)N+1, we must provide D_0 [see (6.11)].

Finally, in the opinion of this author, one should rely on quartic spline solution for a smooth approximation of the solution and its successive derivates for the nonlinear boundary value problem of the type (1.2).

Table I

Observed max. error $||E||$ for (8.1) in y_i based on

N	h	cubic	quartic	quintic	sextic spline
1	1/2	0.212-1*	0.287-2	-	-
3	1/4	0.480-2	0.248-3	0.205-3	0.208-3
7	1/8	0.117-2	0.164-4	0.648-5	0.780-5
15	1/16	0.	0.105-5	0.216-6	0.204-6

*We write 0.212-1 for 0.212×10^{-1}.

Table II

h = 1/8, observations based on quintic spline solution

Problem	number of iterations	$\|\|\Phi(Y)\|\|$	$\|\|E\|\|$
(8.1)	0	0.703-1	
	1	0.578-3	
	2	0.349-7	
	3	0.167-15	0.648-5
(8.2)	0	0.545-3	
	1	0.106-7	
	2	0.555-11	0.493-7

Table III

Problem (8.2), h = 0.25, observed max. errors based on sextic spline solution

x_i	y_i	y_i'	y_i''	y_i'''
0.0	0.0	0.407-5	0.851-8	0.459-2
0.25	0.340-5	0.333-4	0.313-5	0.511-2
0.50	0.288-5	0.387-4	0.257-5	0.620-2
0.75	0.340-5	0.442-4	0.313-5	0.725-2
1.0	0.0	0.504-3	0.851-8	0.816-4

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