EXAMINATION
OF
SECOND-ORDER EFFECTS IN
STRUCTURAL CONCRETE COLUMNS
AND BRACED FRAMES

BY
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A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

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EXAMINATION OF SECOND-ORDER EFFECTS IN STRUCTURAL CONCRETE COLUMNS AND BRACED FRAMES

BY

TIMO K. TIKKA

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirement of the degree of

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ACKNOWLEDGMENTS

The present study represents part of a comprehensive investigation of slender reinforced concrete and composite steel-concrete columns, that are part of braced frames, initiated in the 1980's at Lakehead University. The analytical study reported herein was carried out under the supervision of Dr. S. A. Mirza and under the sponsorship of the Natural Sciences and Engineering Research Council of Canada. Numerical calculations were performed on the Sun Microsystems 670/MP at Lakehead University in Thunder Bay.

The author wishes to express his sincere appreciation to Dr. S.A. Mirza for his guidance, patience, continuous encouragement, flexibility, and understanding throughout all the stages of the research work.

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Finally, this thesis is dedicated to my wife Camilla and children Villiam, Aleksandar, and Adrianna for their patience, sacrifice and love, and also to my parents and my in-laws for their support throughout the years of this study.
ABSTRACT

The Canadian Standards Association Standard for Design of Concrete Structures for Buildings (CSA A23.3-94) and the American Concrete Institute Building Code Requirements for Reinforced Concrete (ACI 318-99) permit the use of a moment magnifier approach for the design of slender reinforced concrete and composite steel-concrete columns. This approach is influenced by the equivalent uniform bending moment diagram factor ($C_m$) and the critical buckling load ($P_{cr}$). The computation of $P_{cr}$ is strongly influenced by the effective flexural stiffness ($EI$), which varies due to the nonlinearity of the concrete stress-strain curve, creep and cracking along the length of the unsupported column. $P_{cr}$ is also influenced by the effective length factor ($K$), which is a function of the relative stiffnesses of beams and columns framing into a joint.

This study was undertaken to examine the influence of different variables on $EI$, $C_m$, $K$, and the moment magnifier ($\delta_m$) used for design of slender, tied, rectangular reinforced concrete and composite steel-concrete columns in braced frames under short-term loads. The composite columns consisted of steel shapes encased in concrete. The columns studied were subjected to eccentric axial loads with an eccentricity applied along one axis of the principle axes of the cross-section, producing uniaxial bending.

Over 35,000 isolated reinforced concrete and composite columns in symmetrical single curvature bending were simulated to generate the stiffness data. The CSA/ACI $EI$ equations were then compared with the simulated data. A new design equation that can be used to determine $EI$ of both reinforced concrete and composite columns was also developed from the simulated stiffness data and is proposed as an alternative to the existing CSA/ACI
equation.

To study the effects of material and geometric non-linearity and longitudinal reinforcing bars and structural steel section details on the $C_m$ factor, more than 38,000 reinforced concrete and composite steel-concrete columns, subjected to unequal end moments, were simulated. These columns were used to examine the existing CSA/ACI $C_m$ equation and to develop new expressions for $C_m$.

Finally, 2960 simple braced frames were simulated to investigate the accuracy of the CSA/ACI moment magnifier procedure applied to columns in these frames. The theoretically-computed column strengths were compared to the strengths computed using several different combinations of design equations for $\delta_m, EI, K$, and $C_m$. The study shows that the prediction accuracy of the moment magnifier method can be improved significantly by replacing the current CSA/ACI equation for $EI$ with the proposed equation for $EI$.
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<tr>
<td>$A_f$</td>
<td>area of structural steel flanges</td>
</tr>
<tr>
<td>$A_c$</td>
<td>gross area of concrete cross-section</td>
</tr>
<tr>
<td>$A_{rs}$</td>
<td>area of reinforcing steel</td>
</tr>
<tr>
<td>$A_{sw}$</td>
<td>area of structural steel section</td>
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<td>$A_w$</td>
<td>area of structural steel web</td>
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<td>$b, b_f$</td>
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<td>$C_c$</td>
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<tr>
<td>$C_m$</td>
<td>equivalent uniform bending moment diagram factor</td>
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<tr>
<td>$C_{m(\text{des})}$</td>
<td>equivalent uniform bending moment diagram factor from design Equation (5.5), (5.11), (5.12), (5.21) or (5.22)</td>
</tr>
<tr>
<td>$C_{m(\text{reg})}$</td>
<td>equivalent uniform moment diagram factor from regression Equation (5.15) or (5.20)</td>
</tr>
<tr>
<td>$C_{m(h)}$</td>
<td>simulated theoretical equivalent uniform bending moment diagram factor computed from Equation (5.10)</td>
</tr>
<tr>
<td>$d, d_{sw}$</td>
<td>depth of the structural steel section taken perpendicular to the major axis of the steel section</td>
</tr>
<tr>
<td>DNA</td>
<td>perpendicular distance from the plastic centroid of the column to the neutral axis</td>
</tr>
<tr>
<td>$dx, l_{\text{elem}}$</td>
<td>element length used for differential equation for deflection curve of a member</td>
</tr>
<tr>
<td>$d\theta$</td>
<td>change of angle within the element length $ds$ or $dx$</td>
</tr>
<tr>
<td>$e, e_2$</td>
<td>larger end eccentricity $= M_s/P_u$</td>
</tr>
<tr>
<td>$e_1$</td>
<td>smaller end eccentricity $= M_i/P_u$</td>
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<tr>
<td>$E_c, E_s$</td>
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</table>
$E_{strn}$ = slope of the tangent at the onset of the strain hardening curve

$EI$ = effective flexural stiffness of column

$EI_{(des)}$ = computed short-term $EI$ from design Equation (4.4), (4.5),(4.6) or (4.29) combined with (4.31) or (4.33)

$EI_{(reg)}$ = computed short-term $EI$ from regression Equation (4.17),(4.19)or (4.21)

$EI_{(th)}$ = simulated theoretical $EI$ for short-term loads

$f$ = concrete strength for desired cube size computed from Equation (3.2)

$f_c$ = concrete strength computed from Equation (2.1)

$f'_c$ = specified compressive strength of concrete

$f_{cu}$ = compressive strength of 6 inch concrete cube

$f_o$ = compressive strength of 4 inch concrete cube

$f_r$ = tensile strength of concrete

$f_y$ = nominal yield strength of steel

$f_{yh}$ = specified yield strength of horizontal ties

$f_{yrs}$ = specified yield strength of longitudinal reinforcing steel

$f_{yr}$, $f_{ys}$ = average static yield strength of reinforcing steel and structural steel section

$f_{ysf}$ = average static yield strength of structural steel flange

$f_{ys}$ = specified yield strength of structural steel

$f_{yw}$ = average static yield strength of structural steel web

$f_u$ = ultimate strength of steel

$G_A$, $G_B$ = effective stiffness factors computed from Equation (6.6) and (6.7)

$G_{min}$ = smaller of $G_A$ and $G_B$

$h$, $h_{bas}$, $h_{col}$ = overall thickness of cross-section perpendicular to neutral axis

$h''$ = out-to-out width of lateral ties
Notation

$I_g$, $I_{xx}$, $I_{yz}$ = moment of inertia of gross concrete cross-section, of structural steel section, and of reinforcing steel taken about centroidal axis

$K$ = effective length factor or a constant computed from Equation (2.1)

$L$, $l_{bm}$, $l_{col}$ = unsupported height (length) of member (beam or column)

$M$ = bending moment resistance

$M_{avg}$ = average bending moment of element

$M_{bm}$ = applied beam bending moment

$M_c$ = design bending moment which includes second-order effects

$M_{cr}$, $M_{col}$ = bending moment resistance of cross-section and of member (column)

$M_{eq}$ = equivalent uniform bending moment

$M_{max}$ = maximum moment

$M_u$ = ultimate bending moment resistance of a reinforced concrete or composite steel-concrete cross-section or beam

$M_y$, $M_{y(bm)}$ = yield moment, the bending moment of a concrete cross-section or concrete beam at initial yielding of reinforcing steel

$M_1$ = smaller of factored moments applied at column ends

$M_2$ = larger of factored moments applied at column ends

$n$ = number of data points

$P$ = axial load

$P_{bm}$ = applied beam load

$P_{computed}$ = computed strength of test specimen using theoretical procedure

$P_{cr}$ = Euler's buckling strength or critical load of column

$P_{cr(th)}$ = critical load of column computed using theoretical $EI$
$P_{inc} = \text{axial load increment used for theoretical procedure}$

$P_o = \text{pure axial load strength of cross-section: } 0.8(0.85 f'c A_c + f_{ys} A_{xs} + f_{y} A_{ys})$

$P_{test} = \text{reported strength of test specimen}$

$P_{tol} = \text{tolerance increment for column axial load}$

$P_u = \text{factored axial load acting on column}$

$P_{u(dex)} = \text{column axial load capacity computed from a design procedure}$

$P_{u(th)} = \text{theoretically computed axial load capacity of column}$

$r = \text{radius of gyration of cross-section}$

$r_j, r_k = \text{nodal rotations of element ends}$

$R_e = \text{multiple correlation coefficient}$

$S_e = \text{standard error}$

$s_h = \text{vertical spacing of ties}$

$t = \text{flange thickness of structural steel}$

$u = \text{strength of concrete from cube test specimen}$

$v_o = \text{volume of a 4 inch concrete cube test specimen}$

$v = \text{volume of desired cube size}$

$V_r = \text{coefficient of variation}$

$w = \text{web thickness of structural steel section}$

$x_1, \ldots, x_n = \text{independent variables}$

$x_a, x_b = \text{x co-ordinates of element ends } a \text{ and } b$

$X_1, X_2, X_3 = \text{groups of variables for regression analyses}$

$y_a, y_b = \text{y co-ordinates of element ends } a \text{ and } b$

$Y = \text{clear distance between inside of tie reinforcement and the steel flange taken perpendicular to the steel flange}$
\( \alpha \) = effective stiffness factor
\( \alpha_k \) = a dimensionless constant
\( \alpha_{cr}, \alpha_{re}, \alpha_{ss} \) = dimensionless reduction factors for concrete, structural steel and reinforcing steel
\( \alpha_1, \ldots, \alpha_n \) = dimensionless factors corresponding to independent variables \( x_1, \ldots, x_n \)
\( \beta \) = dimensionless factor for non-linear \( EI \) equation (Equation 4.33)
\( \beta_d \) = ratio of the maximum factored axial dead load to the total factored axial load for the calculation of \( P_{cr} \) in ACI 318-99 Eq. (10-10) for braced frames
\( \delta_{ms} \) = moment magnifier for columns that are part of braced frames
\( \delta_1 \) = moment magnifier for columns subjected to axial load and equal and opposite end moments causing single curvature bending
\( \Delta_m \) = lateral deflection at midheight of column
\( \epsilon_c \) = strain in concrete
\( \epsilon_{cu}, \epsilon_{cu} \) = maximum compressive strain in concrete
\( \epsilon_o \) = strain in concrete corresponding to peak compressive stress
\( \epsilon_{strn} \) = strain in steel at onset of strain hardening
\( \epsilon_{su} \) = ultimate strain in steel
\( \epsilon_y \) = strain in steel at onset of yielding
\( \lambda \) = dimensionless factor for variable \( [(l+M_1/M_2)/2]^1 \)
\( \rho \) = radius of curvature
\( \rho_{re}, \rho_{ss} \) = total area of longitudinal reinforcing bars and area of structural steel section, both divided by the gross area of concrete cross-section
\( \rho_s \) = ratio of volume of lateral ties to volume of concrete core
\( \sigma_{rft} \) = residual stress at the tips of structural steel flanges
\( \sigma_{r}\) = residual stress at the flange web juncture
\( \phi \) = curvature or material resistance factor
\( \phi_{\text{avg}} \) = average curvature of element
\( \phi_{e} \) = curvature at column ends
\( \phi_{m} \) = stability resistance factor used for computing \( \phi_{m}P_{cr} \)
\( \phi_{y} \) = curvature at the onset of initial yielding of flexural tension reinforcement
\{A\} = applied joint load vector
\{A_{c}\} = combined applied force vector
\{A_{e}\} = equivalent joint load vector
\{A_{f}\} = member fixed-end-reaction vector
\{D\} = joint displacement vector
\{D_{e}\} = joint displacement vector in terms of element member coordinate system
\{D_{j}\} = joint displacement vector in terms of global coordinate system
\{Dx_{i}\} = nodal displacement vector in x direction at i\(^{th}\) iteration
\{Dy_{i}\} = nodal displacement vector in y direction at i\(^{th}\) iteration
\{F_{m}\} = member fixed-end-force vector
\[K_{s}\] = global structural stiffness matrix
\[K_{se}\] = stiffness contribution of the element to the global structural stiffness matrix
\[K_{e}\] = element (member) stiffness matrix expressed in element coordinate system
\{P\} = element or member end force vector
\[R\] = rotation transformation matrix
\[R^{T}\] = transpose of the rotation transformation matrix
\{X_{o}\}, \{Y_{o}\} = x and y co-ordinates of original frame geometry
\{X_{i}\}, \{Y_{i}\} = x and y co-ordinates of frame geometry at i\(^{th}\) iteration
1.0 INTRODUCTION

1.1 GENERAL

The bending moment capacity of a slender isolated column that is subjected to equal and opposite bending moments causing single curvature bending is lower than the capacity of its cross-section. A slender column deflects laterally when subjected to an eccentric axial load and is, therefore, subjected to additional bending moment along its height. The secondary moment at mid-height, caused by the axial load acting through additional eccentricity (mid-height deflection), becomes significant in slender columns and controls the maximum applied end moment. A slender column that is subjected to axial load and end moments producing moment gradient in single or reverse curvature bending also deflects laterally. However, secondary moments occurring in this column are less than the secondary moments in the same column that is subjected to equal end moments causing symmetrical single curvature bending. A column that is part of a braced frame is stiffened by the restraining action of beams that frame into a common connection at the upper and lower ends of the column. Columns that are part of nonsway frames, therefore, are subjected to secondary moments that are less than the secondary moments associated with comparable isolated columns.

The Canadian Standards Association Standard for the Design of Concrete Structures for Buildings (CSA A23.3-94) and the American Concrete Institute Building Code Requirements for Reinforced Concrete (ACI 318-99) permit the use of a moment magnifier approach to approximate the second-order moments caused by the axial load acting through the lateral deflection of the column. The larger of the column end moments, computed from
a conventional elastic frame analysis, is magnified is to include the second-order effect. The axial load and the magnified column moment are then compared to the column cross-section axial load-bending moment strength interaction diagram and if the axial load and the magnified moment fall inside of the cross-section interaction diagram, the cross-section meets the strength and slenderness requirements of the Code.

The moment magnifier approach originated from classical elastic theory and was used for the allowable strength design method. When the ultimate strength design method was introduced for the design of concrete structures in early 1960's and 1970's in the United States and Canada, respectively, the moment magnifier approach was kept for the design of columns and remains in use today.

This study examines the variables used to compute the magnified moment \( M_2 \) shown in Equation 1.1, for slender, tied, reinforced concrete and composite steel-concrete columns that are part of braced frames:

\[
\delta_{ns} M_2 = \left[ \frac{C_m}{1 - \frac{P_u}{\phi_m P_{cr}}} \right] M_2 \geq 1.0 M_2
\]  

(1.1)

In Equation 1.1, \( \delta_{ns} \) is the moment magnifier; \( M_2 \) is the larger of the column end moments; \( C_m \) is the equivalent uniform moment diagram factor; \( P_u \) is the applied factored axial load; \( \phi_m \) is the stability resistance factor; and \( P_{cr} \) is the critical buckling strength of the column computed from \( \pi^2 EI/(K\delta)^2 \). The effective flexural stiffness \( EI \) used in the computation of
$P_{\sigma}$ varies due to the nonlinearity of the concrete stress-strain curve, creep, and cracking along the length of the column ($\ell$). $P_{\sigma}$ is also a function of the effective length factor ($K$), which is affected by beams resisting the rotation of columns framing into a common joint.

The $EI$ expressions given in the CSA and ACI codes for reinforced concrete columns are quite approximate when compared to values obtained from the axial load, bending moment, and curvature ($P-M-\phi$) relationships, as explained in the Commentary of ACI 318-99 (1999). Mirza (1990) explained that the CSA and ACI $EI$ equations were developed for reinforced concrete columns subjected to high axial loads, but are permitted for all axial load levels. In addition, for composite steel-concrete columns, the $EI$ equations given in the CSA and ACI codes were simply modified, without any further investigation, for use in composite column design (Mirza and Tikka 1999a).

The current CSA A23.3-94 and ACI 318-99 permit the use of the equivalent uniform moment diagram factor ($C_m$) for computing the effect of moment gradient, along the column length, caused by unequal column end moments. The concept of equivalent uniform moment diagram was introduced into design practice to eliminate the need for extensive calculations based on the solution to differential equations that determine the magnitude and location of maximum moment along the length of a column. The equation currently in use by the CSA and ACI codes is a simplified equation, which was proposed by Austin (1961) based on the elastic behavior of columns, and does not include the inelastic material behavior of reinforced concrete and composite steel-concrete columns.

For the computation of the effective length factor, the CSA and ACI Codes permit
the use of the Jackson-Moreland Alignment Charts. These alignment charts are based on the elastic theory and also do not include the affects of nonlinear material behavior.

Mirza (1990) investigated the effect of variables on $EI$ of reinforced concrete columns bent in symmetrical single curvature, using computer simulations, and proposed two sets of equations for $EI$. Mirza and Tikka (1999a,b) conducted similar studies to determine the effect of variables on $EI$ of composite columns bending about the major axis and minor axis of encased steel sections and proposed new equations for $EI$. These studies showed that the $EI$ expressions currently in use by CSA and ACI have a high degree of variability and can be conservative for low eccentricities and unconservative for high eccentricities in computing $EI$. The study reported herein approaches the $EI$ of reinforced concrete and composite columns from the same perspective in the sense that reinforced concrete columns are taken as a special case of composite columns. As a result, the $EI$ equation developed and proposed in the latter part of this report is more "accurate" as well as versatile than the previous equations for $EI$ and is applicable to both reinforced concrete and composite columns. However, the question still arises whether other parameters ($C_m$ and $K$) that are used to compute the moment magnifier are as significantly affected by nonlinear material behavior and concrete cracking as is $EI$.

A design equation based on the elastic theory was proposed by Duan et al. (1989) to replace the expression for $C_m$ used by the CSA and ACI. Trahair (1985) proposed another design expression for $C_m$ based on the inelastic behavior of I-shaped steel columns; however, no references were found that considered the inelastic behavior of reinforced concrete or
composite steel-concrete columns.

The Commentary of ACI 318-99 Code permits the use of an equation for the effective length factor \((K)\) proposed by Cranston (1972). Another equation for \(K\) was proposed by Duan et al. (1993) in lieu of using the Jackson-Moreland alignment chart. Chen and Wang (1999) proposed a design expression based on elastic theory for the computation of \(\delta_{nr}\). The design expression by Chen and Wang includes the effects of unequal end moments without the use of a separate expression for \(C_m\).

In reviewing previous work, no references were found that presented a comprehensive evaluation of the moment magnification due to second-order effects and the effects of nonlinear material behavior on the effective flexural stiffness \((EI)\), the equivalent uniform moment diagram factor \((C_m)\), and the effective length factor \((K)\) for the design of both reinforced concrete and composite steel-concrete columns that are part of braced frames. The intent of this study is to do just that. The study approaches reinforced concrete and composite columns from the same perspective and evaluates the moment magnifier method for these columns. The scope of the study is limited to examining the behavior of columns subjected to short-term loading and, therefore, does not include the effects of sustained loads.

1.2 SCOPE AND OBJECTIVES

This study was conducted to examine the accuracy of the moment magnifier approach, and the associated variables \(EI\), \(C_m\) and \(K\) used for computing the strength of
concrete columns in nonsway frames subjected to short-term loads. The objective of the study was fourfold:

1) Examine the influence of different variables on the short-term effective flexural stiffness \((EI)\) of slender, tied, reinforced concrete and composite steel-concrete columns; develop and propose refined expressions for computing \(EI\); and compare the new expressions for \(EI\) with previous work and the current CSA/ACI expressions.

2) Examine the existing expressions for the equivalent uniform bending moment diagram factor \((C_m)\) applied to reinforced concrete and composite steel-concrete columns; and develop new expressions that apply to reinforced concrete and composite columns to account for material nonlinearity.

3) Evaluate the accuracy of the existing CSA/ACI moment magnifier procedure by computing the strength of columns that are part of simple braced frames, using the current CSA/ACI equations for \(EI, C_m,\) and \(K\); and apply the new expressions for \(EI\) and \(C_m\) for computing the strength of the same columns and determine whether enhancement to the effective length factor \((K)\) is needed to further improve the accuracy of the moment magnifier procedure.

4) Provide design recommendations regarding the moment magnifier method and the equations for associated variables \(EI, C_m,\) and \(K\).

1.3 BRIEF DESCRIPTION OF CONTENTS

This study is an analytical investigation. No physical tests were conducted as part of
this study. The following is a brief description of the contents of each chapter in the study:

**Chapter 2:** A theoretical model that incorporates second-order effects due to material and geometric nonlinearity, and developed as a part of this study to compute the strength of isolated concrete columns and frames, is explained. The computation procedure is based on equilibrium of forces and compatibility of strains and utilizes the stiffness method of analysis for frames, an iterative and incremental technique for computing second-order deflections, and the computed axial load-moment-curvature ($P-M-\phi$) relationships.

**Chapter 3:** To test the accuracy of the theoretical procedure, the strength computed by the theoretical model is compared to the strength of physical tests gathered from the published literature. The tests used for comparison were performed on 209 isolated reinforced concrete and composite column specimens subjected to equal and opposite end moments causing symmetrical single curvature bending, 15 isolated reinforced concrete and composite column specimens subjected to moment gradient, and 13 reinforced concrete frames. These comparisons are shown in this chapter.

**Chapter 4:** A procedure for computing the theoretical flexural stiffness ($EI$) for concrete columns is presented. Approximately 35,000 isolated reinforced concrete and composite columns, subjected to symmetrical single curvature bending caused by equal and opposite end moments, are simulated to study the effects of a number of variables that affect the
effective flexural stiffness. The existing CSA/ACI equations for $EI$ are examined and a new equation for $EI$ is developed using the simulated data. The $EI$ equations reported by other authors are also examined.

Chapter 5: To study the effects of material and geometric nonlinearity and different cross-section properties on the equivalent uniform moment diagram factor ($C_m$), 38,400 reinforced concrete and composite steel-concrete columns subjected to unequal end moments causing moment gradient in single curvature and double curvature bending are simulated. Two new equations are developed for $C_m$ from the generated data. The existing code equation for $C_m$ and equations reported by other authors are also examined.

Chapter 6: In this part of the study, simulated theoretical strengths of 2960 reinforced concrete inverted T-frames are used to evaluate the accuracy of the CSA/ACI moment magnifier ($\delta_{mz}$) and the associated CSA/ACI equations currently used for computing $EI$, $C_m$, and $K$, for reinforced concrete columns that are part of nonsway frames. The accuracy of the proposed $EI$ and $C_m$ equations and other equations for $\delta_{mz}$, $EI$, $K$, and $C_m$ are also evaluated using the simulated frame data.

Chapter 7: A summary of the study is given. Several conclusions and recommendations are made regarding the moment magnifier method; regarding the CSA/ACI and proposed equations for computing $EI$, $K$, and $C_m$; and regarding the future research.
1.4 Research Significance

The CSA and ACI Codes used for the design of concrete structures permit the moment magnifier method for determining the second-order effects due to slenderness in reinforced concrete and composite steel-concrete columns. Previous research has shown that this design procedure is strongly influenced by the effective flexural stiffness \((EI)\), and the expressions used by these Codes for computing \(EI\) neglect certain parameters. This study and also previous studies show that the \(EI\) equations currently in use in CSA/ACI Codes have a high degree of variability and can be conservative in some cases and unconservative in other cases. This raises the question whether the equivalent uniform moment diagram factor \((C_m)\) and the effective length factor \((K)\), which are also used in the computation of the moment magnifier, are as significantly variable as \(EI\). Extensive evaluations of \(EI, C_m,\) and \(K\) were carried out as part of this investigation. Based on these evaluations, new \(EI\) equations are proposed for the design of slender reinforced concrete and composite steel-concrete columns. It is shown that the variability of the proposed \(EI\) equations is very significantly less than that of the current CSA/ACI equations. This investigation also shows that most of the variability associated with the strength of slender reinforced concrete and composite steel-concrete columns, based on the moment magnifier method, can be significantly reduced by using a nonlinear equation proposed in this study for computing \(EI\) with no apparent need to modify the CSA/ACI equations used for computing \(C_m\) or \(K\).
2.0 DEVELOPMENT OF THEORETICAL MODEL FOR STRENGTH OF COLUMNS AND FRAMES

2.1 GENERAL

A theoretical procedure was used in this study for computing the ultimate strength, the effective flexural stiffness \((EI)\), the equivalent uniform moment diagram factor \((C_m)\), and the moment magnifier \((\delta_m)\) for reinforced concrete and composite steel-concrete isolated columns and columns that are part of frames. This theoretical computation procedure is similar to the ones used by Mirza (1989), Skrabek and Mirza (1990), and Tikka and Mirza (1992). The theoretical computation procedures used by Mirza (1989), Skrabek and Mirza (1990), and Tikka and Mirza (1992) were all designed specifically for the study of isolated composite columns subjected to symmetrical single curvature bending about the major or minor axis of a concrete encased structural steel shape. The computation procedure developed as part of and used in this study is more comprehensive than those reported earlier (Mirza 1989, Skrabek and Mirza 1990, Tikka and Mirza 1992) and is capable of conducting the nonlinear second-order analysis of both reinforced concrete and composite steel-concrete isolated columns as well as frames. A brief flow chart of the computation procedure employed in this study is shown in Figure 2.1.

The computer code, based on the theoretical computation procedure and developed as a part of this study, consists of a main driver program, a cross-section strength subroutine, a nonlinear second-order frame analysis subroutine, and subroutines for computing the theoretical values of the ultimate strength, the effective flexural stiffness, the equivalent
uniform moment diagram factor, and the moment magnifier. The main driver reads the input, initiates the parametric study of input data, calls the respective subroutines, and saves the output data for later use. A solution based on force equilibrium and strain compatibility was used in the theoretical cross-section strength subroutine to generate the axial load, moment and curvature \((P-M-\phi)\) relationship and the corresponding cross-section strength. The second-order frame analysis subroutine computes the theoretical strength of a slender column or frame. The secant-formula approach, initially reported by Mirza (1990), and described in Section 4.2.1, was used to compute the effective flexural stiffness \((EI)\) of pin-ended slender columns subjected to equal and opposite end moments causing symmetrical single curvature bending. This approach uses the generated theoretical cross-section and slender column strength interaction diagrams for computing the theoretical \(EI\). The equivalent uniform moment diagram factor is computed from the theoretical slender column strength interaction diagram for a pin-ended column that is subjected to end moments producing moment gradient in reverse and single curvature bending, including symmetrical single curvature bending.

The theoretical cross-section strength subroutine, the nonlinear second-order frame analysis subroutine, and related subroutines are discussed in this chapter. The subroutines which were developed for determining the theoretical values of effective flexural stiffness and equivalent uniform moment diagram factor are discussed in Chapters 4 and 5, respectively.
2.2 DETERMINING THE CROSS-SECTION STRENGTH

The theoretical cross-section strength subroutine computes the axial load, moment, and curvature ($P-M-\phi$) relationship for the cross-section using a solution, based on force equilibrium and strain compatibility, which is discussed in Section 2.2.3. The assumptions used in determining the theoretical $P-M-\phi$ relationship and strength of the cross-section are as follows:

(a) Strains between concrete and steel are compatible and no slip occurs;
(b) strain is linearly proportional to the distance from the neutral axis;
(c) deflections are small, such that curvatures can be calculated as the second derivative of the deflection;
(d) shear stresses are small and their effect on the strength can be neglected;
(e) effects of axial shortening are negligible;
(f) residual stresses in the rolled steel section exist;
(g) the column cross-section is symmetrical about the major and minor axis;
(h) failure of the structural steel section or component elements does not take place by local or torsional buckling.

Assumptions (a) and (b) were required for the strain compatibility solution of the cross-section $P-M-\phi$ relationship. Assumption (c) was needed for the calculation of secondary moments due to the length effect. Assumptions (d) and (e) were used to simplify the calculations. Assumption (f) acknowledges the existence of residual stresses in the structural steel sections and is discussed in Section 2.2.2.3. Assumption (g) simplified the
cross-section $P-M-\phi$ calculations since discretization of only one-quarter of the cross-section was required to model the entire cross-section. Assumption (h) was valid since a review of the physical test data available in the literature did not indicate any failure by local or torsional buckling. This assumption was also made by Bondale (1966 a,b,c), and appears to be particularly valid where rectangular hoops along with surrounding concrete stiffen the compression flange of the steel section. Further assumptions, which are directly related to the stress-strain curve for individual materials, are discussed in Section 2.2.2.

2.2.1 Cross-section Discretization

The cross-section of a reinforced concrete column consists of two materials (concrete and longitudinal reinforcing steel bars) and the cross-section of a composite column consists of three materials (concrete, structural steel section and longitudinal reinforcing steel bars). The concrete in a reinforced concrete column was divided into two types: unconfined and partially confined, as shown in Figure 2.2(a). For a composite column, the concrete was divided into three types: unconfined, partially confined and highly confined, as indicated in Figure 2.2(b), whereas the structural steel section was divided into two types: the web and flanges. Each of these concrete and structural steel types had different stress-strain characteristics. Therefore, three different stress-strain curves were used for the cross-section given in Figure 2.2(a), while six different stress-strain curves were used to represent the materials in the cross-section shown in Figure 2.2(b).

The cover concrete outside the lateral ties was considered to be unconfined. The
concrete inside the periphery of the ties, but outside the flanges of the steel section, was assumed to be partially confined. The concrete within an assumed parabola and between the web and flanges of the steel section was assumed to be highly confined. Skrabek and Mirza (1990) point out that the assignment of different degrees of confinement to the concrete in different areas of the cross-section realizes the beneficial effects the increased confinement has on concrete strength and ductility.

The computation procedure permits the assignment of any one of the six different stress-strain curves to any one of the six areas defined in Figure 2.2(b), thereby allowing the composite column discretization subroutine to model reinforced concrete columns as well. To model a reinforced concrete column with the composite column discretization, the stress-strain curves of partially confined concrete are assigned to the areas described in Figure 2.2(b) as highly confined concrete, steel flange and steel web.

To calculate the \( P-M-\phi \) relationship, the computation procedure numerically integrates the forces acting on the cross-section. To accomplish this, the cross-section is discretized into a finite number of strips parallel to the minor or major axis of the steel section. The thickness of the strip perpendicular to the axis of bending is determined from the number of strips desired. The width of each material within a given strip is automatically calculated. Fifty elemental strips for the entire cross-section were used for the computer simulations described in Chapters 4, 5, and 6.

Each strip is further discretized, if required, to account for the variation of material properties within that strip. For example, to account for the variation of residual stresses
along the width of the web in composite columns bending about the minor axis of the steel section, the web is discretized into 20 elements of equal width parallel to the minor axis of the steel section. For composite columns bending about the major axis of the steel section, the flange is discretized into 20 elements of equal width parallel to the major axis of the steel section to account for varying residual stresses along the width of the flange. For a composite cross-section, the initial strain in each element due to residual stresses is calculated and strains subsequently caused by the axial force and bending moment are added algebraically to that element. The discretization for a typical \( \frac{1}{2} \) cross-section for bending about the minor and major axis of a composite cross-section are shown in Figures 2.3(a) and (b), respectively.

Tapered flanges were not included for simulations of effective flexural stiffness and moment gradient data reported in Chapters 4 and 5. For the theoretical model calibration reported in Chapter 3, however, it was necessary to include the effect of tapered flanges in composite columns bending about the minor axis of the steel section because many physical tests gathered from the available literature were for tapered flanges.

2.2.2 Material Stress-strain Relationships

A reinforced concrete column was represented by three different materials and a composite column was represented by six different materials, with each material characterized by a different stress-strain relationship as indicated earlier in Section 2.2.1. Three of the six materials types in a composite cross-section are unconfined, partially
confined, and highly confined concrete. The web and flanges of the structural steel section account for two more of the material types. The longitudinal reinforcing steel bars make up the sixth material present in the cross-section.

2.2.2.1 Stress-Strain Curves for Concrete

Concrete confinement increases both compressive strength and ductility of concrete. The distinction between different concrete areas, identified in Figure 2.2, recognizes the differences in the stress-strain relationships caused by the confining action of the rectangular lateral ties (acting in conjunction with the longitudinal reinforcing steel bars), and the structural steel section. Park, Priestly, and Gill (1982), Sheikh and Uzumeri (1982), and Sheikh and Yeh (1986) developed methods to determine the beneficial effects of increased compressive strength and ductility of concrete for reinforced concrete columns. Methods to determine the effect of confinement on the concrete tensile stress-strain relationship are not available. Therefore, identical tensile stress-strain relations were assumed for all types of concrete confinements. The stress-strain relationships presented in this Section are based on static loading conditions.

Based on recommendations of Skrabek and Mirza (1990), a modified Kent-and-Park stress-strain curve (Park, Priestly and Gill 1982) was used to describe the compressive stress-strain relationship for the unconfined concrete (outside the perimeter of the lateral ties) and the partially confined concrete (inside the perimeter of lateral ties) in composite and reinforced concrete cross-sections modeled in this study (Figure 2.4). The modified Kent-
and-Park curve assumes that the degree of confinement is a function of the concrete strength $f'_c$, the vertical spacing of ties $s_h$, the ratio of the volume of lateral ties to the volume of the concrete core $\rho_s$, and the yield strength of horizontal ties $f_{yh}$. The ascending portion of the curve between the origin and the strain corresponding to the peak compressive stress ($\varepsilon_o$) is described by a second-order parabola (Equation 2.1) (Park, Priestly and Gill 1982) and the descending branch of the curve beyond $\varepsilon_o$ is described by Equation 2.2 (Park, Priestly and Gill 1982):

$$ f_c = K f'_c \left[ \frac{2 \varepsilon_c}{K \varepsilon_o} - \left( \frac{\varepsilon_c}{K \varepsilon_o} \right)^2 \right] $$

ascending curve

where

$$ K = 1 + \frac{\rho_s f_{yh}}{f'_c} $$

$$ f_c = K f'_c \left[ 1 - Z (\varepsilon_c - K \varepsilon_o) \right] \geq 0.2 K f'_c $$

descending curve

where

$$ Z = \frac{0.5}{\varepsilon_{50u} + \varepsilon_{50h} - K \varepsilon_o} $$

and

$$ \varepsilon_{50u} = \frac{3 + K \varepsilon_o f'_c}{f'_c - 1000}; \quad \varepsilon_{50h} = \frac{3}{4} \rho_s \sqrt{\frac{h''}{s_h}} $$

In the equations above, $f_c$ is the concrete compressive stress corresponding to a concrete strain of $\varepsilon_c$, and $h''$ is the out-to-out width of the lateral ties. For SI conversion, replace 3
by 0.0207 MPa and 1000 by 6.895 MPa in the calculation of $\varepsilon_{50u}$. The strain at the peak compressive stress ($\varepsilon_c$) was allowed to vary as a function of the concrete strength (Equation 2.3) rather than a constant value of 0.002 suggested by Kent and Park (1971):

$$\varepsilon_c = \frac{2Kf'_c}{E_c}$$

(2.3)

For unconfined concrete $K = 1.0$ and $\varepsilon_{50h} = 0.0$, because $\rho_s = 0.0$, and the maximum usable concrete strain is less than or equal to $\varepsilon_u = 0.004$ in the above equations.

The modified Kent-and-Park curve suggested by Skrabek and Mirza (1990) to model the highly confined concrete between the web and flanges of the structural steel section was also used in this study. The peak stress ($Kf'_c$) in the heavily confined concrete was assumed to be maintained at all strains beyond $\varepsilon_c$. The long dashed line in Figure 2.4 describes the assumed stress-strain curve for the heavily confined concrete.

The concrete tensile stress-strain curve used in this study is also shown in Figure 2.4. A linear stress-strain relationship from the origin to the modulus of rupture, $f_r$, was assumed, with the modulus of elasticity of concrete in tension assumed equal to the modulus of elasticity of concrete in compression. The work of Skrabek and Mirza (1990) shows that this simple model suggested by Park and Pauley (1975) and Mirza and MacGregor (1989) is sufficient for computing the strength of reinforced concrete and composite columns with reasonable accuracy.
2.2.2.2 Stress-Strain Curves for Reinforcing and Structural Steels

Earlier studies have concluded that, with the exception of columns subjected to very high eccentricities \( e/h > 1.0 \), the effect of strain-hardening of reinforcing and structural steels is insignificant on the strength of short as well as slender composite columns (Mirza and Skrabek 1991, 1992). Strain-hardening was, therefore, not included for simulation studies of effective flexural stiffness, equivalent uniform moment diagram factor, and the moment magnifier described in Chapters 4, 5, and 6, respectively, because the simulated columns used for these studies had \( e/h \leq 1.0 \). Strain-hardening, however, was included for the comparison of the test strength to the theoretically computed strength of all column specimens documented in Chapter 3, because several of these columns had \( e/h > 1.0 \).

The stress-strain curve for compression was assumed to be the same as that for tension. An elastic-plastic stress-strain curve was assumed to describe the behavior of both the structural steel and the longitudinal reinforcing steel. A second-order parabola was used to describe the strain-hardening portion of the stress-strain curve, when strain-hardening was included. At ultimate stress, the slope of the strain-hardening curve was assumed to be equal to zero. The variables used by the theoretical model to describe the steel stress-strain curve shown in Figure 2.5 are the elastic modulus \( E_s \), the static yield stress \( f_y \), the strain at the onset of strain hardening \( \epsilon_{srm} \), the initial slope of the tangent to strain hardening curve \( E_{srm} \), and the ultimate stress \( f_u \).
2.2.2.3 Residual Stresses in Structural Steel

Residual stresses are due to uneven cooling of component parts of structural steel sections during the rolling process. Skrabek and Mirza (1990) found that the work of LaChance and Hays (1980), Virdi and Dowling (1973), and Mirza (1989) made it evident that residual stresses can affect the strength of composite columns. For this reason, residual stresses were accounted for in structural steel sections used for the study of composite columns.

A detailed analysis by Skrabek and Mirza (1990) determined that Equation 2.4 (Young 1971) for residual stresses at the flange tips, combined with Equation 2.5 (Galambos 1963) for residual stresses at the flange-web juncture, provides a reasonable estimate of measured values reported by Beedle and Tall (1960).

In Equation 2.4, $\sigma_{rft}$ is the residual stress at the tips of the flanges, $A_w$ is the area of the web, and $A_f$ is the area of both flanges of the steel section. In Equation 2.5, $\sigma_{rfu}$ is the residual stress at the flange-web juncture, $b$ is the flange width, $t$ is the flange thickness, $w$ is the web thickness and $d$ is the overall depth of the structural steel shape. For SI conversion of Equation 2.4, replace 24,000 psi by 165 MPa and the minus sign in this equation indicates...
compressive residual stress.

A linear distribution of residual stresses was assumed across the flange width and a trial-and-error method was used to calculate the required residual stress at the mid-depth of the web to maintain force equilibrium in the steel section. Details of this procedure are fully documented by Skrabek and Mirza (1990) and will not be repeated here.

2.2.3 Cross-section Strength

The cross-section strength was represented by an axial load-bending moment \((P-M)\) interaction diagram. The relationship between the axial load, bending moment and curvature \((P-M-\phi)\) was first established by plotting moment-curvature curves, similar to those shown in Figure 2.6, for various levels of axial load. The maximum moment from the moment-curvature curve (Figure 2.6) for a given axial load level represented one point on the cross-section \(P-M\) interaction diagram. To accurately define the entire cross-section interaction diagram (Figure 2.7) for a reinforced concrete column, composite column subjected to bending about the minor axis of the steel section (minor axis bending), or composite column subjected to bending about the major axis of the steel section (major axis bending), approximately 48 points (48 axial load levels) were used. To define the range of axial load to be examined for the \(P-M-\phi\) relationship, the maximum axial load that could be applied to a cross-section at its plastic centroid (pure axial load capacity) was established. The pure axial load capacity was determined employing an iterative technique by incrementing the strain from the lowest strain at peak stress to the highest strain at peak stress, both obtained
from the stress-strain relationships used for the six material types, and calculating the load at each strain level. The maximum axial load calculated from the iterative process was taken as the cross-section concentric axial load capacity, thus establishing the point on the \( P-M \) interaction diagram that corresponded to zero bending moment.

The distance \( DNA \) between the neutral axis and the plastic centroid, shown in Figure 2.8, must be known to determine the \( P-M-\phi \) relationship. By using a strain compatibility solution for a given curvature \( \phi \) and \( DNA \), the axial load \( P \) and bending moment \( M \) acting on the cross-section can be calculated.

An iterative procedure was used to create a matrix of \( P \)-versus-\( DNA \) values. By assuming a starting curvature, and holding this value constant, \( DNA \) was varied and the corresponding axial force calculated. Linear interpolation and the extended Newton-Raphson technique, described by Kikuchi, Mirza and MacGregor (1978), was used to converge to the correct \( DNA \) value for each desired axial force. The bending moment corresponding to the curvature, the neutral axis position with respect to the plastic centroid, and the axial force was then calculated.

The curvature was then incremented creating a new matrix of \( P \)-versus-\( DNA \) values and a new bending moment was calculated. The curvature was incremented until the concrete cover on the compressive side of the cross-section had spalled off to ensure that the maximum bending moment for the desired axial force was obtained.

When strain-hardening of the structural steel section was considered in composite columns subjected to major axis bending at low axial load levels (less than 20 percent of the
cross-section pure compression capacity), the maximum bending moment occurred at very high curvature values long after the spalling of the concrete in the compression zone. For these cases, the tension flange of the steel section was monitored at each curvature increment and if rupture of the tension flange was imminent, no further points were calculated for that axial load level. It should be noted that the effect of strain-hardening was only used for the comparison of theoretical model to physical tests discussed in Chapter 3.

This procedure created the required $P-M-\phi$ relationships of a cross-section and is outlined in Figure 2.9. When the moment-versus-curvature diagrams were completed for all of the desired axial load levels, the maximum bending moment for each axial load level and the $P-M-\phi$ relationships (Figure 2.6) were stored. These maximum bending moments paired with the corresponding axial loads form the $P-M$ interaction diagram (Figure 2.7). The program then proceeds to conduct the non-linear second-order analysis.

2.3 NON-LINEAR SECOND-ORDER ANALYSIS OF COLUMNS AND FRAMES

A nonlinear second-order frame analysis procedure was developed to analyze reinforced concrete and composite-steel concrete isolated columns subjected to single curvature bending, isolated columns subjected to moment gradient, and columns that are part of braced or sway frames.

In order to determine the slender column strength, or the overall frame strength, the theoretical procedure calculates the bending moment capacity of the column, or that of the most critical member (beam or column) in the frame, corresponding to a prescribed loading
configuration by incrementing the column end bending moments to a level that produces failure at the most critical cross-section along the height of the column, or along the length of a member (beam or column) in the frame. To be stable, the internal forces within the column or frame must be in equilibrium with the applied external forces. As the magnitude of applied loads is increased for the given loading configuration, there is a corresponding increase in deflections and secondary moments in the column, or in the member(s) (columns and beams) in the frame, until the material failure occurs at the most critical cross-section. The slender column bending moment capacity is defined as the bending moment acting at the ends of the column at failure for a prescribed axial load. The computation procedure uses the following methods to include material nonlinearity and second-order effects due to displacements of columns:

(a) Stiffness analysis of linear elastic two-dimensional structural systems;
(b) iterative technique combined with an incremental method for computing load-deflection behavior and the failure load of a column or frame;
(c) frame discretization to account for the column chord \((P-\Delta)\) effects; and
(d) \(P-M-\phi\) relationships (Section 2.2) to account for effects of nonlinear material behavior.

2.3.1 Two-Dimensional Frame Analysis

A classical linear elastic two-dimensional analysis, based on stiffness method of frames, was used to solve a set of simultaneous equations represented in matrix notation as
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\( \{A_c\} = [K_c] \{D\} \), where \( \{A_c\} \) is the combined applied force vector, \([K_c]\) is the global structural stiffness matrix relating forces and displacements, and \( \{D\} \) is the displacement vector.

The frame is first modeled as a series of elements (members) interconnected by a finite number of joints (nodes) as shown in Figure 2.10. The program reads the input data and stores the properties of the structure (geometry, member properties, material properties, boundary conditions, etc.).

Next, the specified member and joint loads are assembled into the combined force vector \( \{A_e\} \). Joint loads due to member forces are computed by multiplying the element rotation transformation matrix \([R]\) by the member fixed-end-force vector \( \{F_m\} \), i.e., \( \{A_e\} = [R] \{F_m\} \). The combined force vector \( \{A_e\} \) for a specified joint (node) is the summation of the applied joint loads \( \{A\} \) and equivalent joint loads \( \{A_e\} \) due to member forces, i.e., \( \{A_e\} = \{A\} + \{A_e\} = \{A\} + [R] \{F_m\} \).

The force-displacement relationship is then determined by adding the stiffness contribution of each element framing into each joint of the frame to form the global structural stiffness matrix. The stiffness contribution of an individual element (member) framing into a joint is expressed in matrix notation as \([R][K_e][R]^T = [K_e]\), where \([R]\) is the rotation transformation matrix used to transform vector components from one coordinate system to another, \([K_e]\) is the element (member) stiffness matrix expressed in the element coordinate system, \([R]^T\) is the transpose of the rotation transformation matrix, and \([K_e]\) is the stiffness contribution of the element (member) to the global coordinate system. The rotation transformation matrix is computed from joint coordinates of the element as:
where \( x \) and \( y \) are the member end coordinates of end \( a \) and \( b \) and \( L \) is the element length.

The element (member) stiffness matrix \([K_e]\) in the element coordinate system is defined in this study as:

\[
[K_e] = \begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\] (2.7)

Equations 2.6 and 2.7 are classical equations that can be found in most structural analysis.
text books such as Hibbeler (1999). The element (member) stiffness matrix $[K_e]$ does not include the effect of shear deformations. Timoshenko and Gere (1972) show that, for a simply supported beam of rectangular cross section with uniform load and $L/h = 10$, shear deformations account for less than 3 percent of the total deflection at the mid-span. As the slenderness ratio $L/h$ increases, the effect of shear deformations decreases. However, for beams with $L/h < 10$, the effect of shear deformations increases. $L/h$ for members used in this study ranges from 10 to 40, and neglecting shear deformation will not significantly influence the results. On the other hand, the axial stiffness contribution $AE/L$ is required for the analysis to determine the axial force in the element (member). However, the axial deformation of a member when compared to the bending deformation is negligible.

The set of equilibrium equations, represented in matrix notation by $\{A_\theta\} = [K_e] \{D\}$, are then set up and used for solution of the unknown joint displacements $\{D\}$. The computer program only stores a non-zero half band width of the global stiffness matrix since structural stiffness matrices are banded and symmetrical about the diagonal. This task of obtaining a solution to the equilibrium equations is completed by a simultaneous equation solver that utilizes the non-zero half band width of the global stiffness matrix $[K_e]$.

The member end forces are then computed from the member force-displacement equations $\{P\} = [K_e] \{D_e\} - \{A_e\}$, where vector $\{P\}$ represents the member end forces, $[K_e]$ is the element (member) stiffness matrix, $\{D_e\}$ is the joint displacement vector in terms of the element (member) coordinate system and is computed from $\{D_e\} = [R]^T \{D\}$, $\{D\}$ is the displacement vector in terms of the global coordinate system at the ends of the element.
(member), and \( \{ A_i \} \) represents the member fix-end-reaction vector due to member loads.

2.3.2 Second-order Displacements

Second-order displacements of frames and columns are caused by nonlinear material behavior and by geometric nonlinearity due to changes in the geometry of the deforming structure. This Section deals with the latter; the non-linear material behavior is described in Sections 2.2.

The second-order displacements and the failure load for an elastic structural system was computed using the flowchart of Figure 2.11 and the following iterative procedure:

(a) Initialize loads to be incremented to zero: \( P = 0 \).

(b) Store original nodal coordinates into additional arrays: \( \{ X_i \} = \{ X_o \}, \{ Y_i \} = \{ Y_o \} \), where \( \{ X_o \} \) and \( \{ Y_o \} \) represent the original frame or column geometry (Figure 2.12).

(c) Add load increment: \( P = P + P_{inc} \).

(d) Solve for nodal displacements, \( \{ Dx_i \} \) and \( \{ Dy_i \} \), using stiffness method of structural analysis described in Section 2.3.1, with frame or column geometry at \( \{ X_i \}, \{ Y_i \} \) (Figure 2.12).

(e) If the nodal displacements of the current iteration \( i \), calculated in (d), are equal to the nodal displacements of the previous iteration \((i-1)\) for each node, go to item (h).

(f) If the nodal displacements are not equal, but are converging, then add the computed displacements to the original model geometry: \( \{ X_i \} = \{ X_o \} + \{ Dx_i \}, \{ Y_i \} = \{ Y_o \} + \{ Dy_i \} \). Store x and y nodal displacements \( \{ Dx_{i+1} \} = \{ Dx_i \} \) and \( \{ Dy_{i+1} \} = \{ Dy_i \} \). Repeat process
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from item (d).

(g) If the nodal displacements are not converging, then the applied loading $P$ exceeds the capacity of the frame or column. Reduce the applied load by two increments: $P = P - 2P_{inc}$. (Note when $P < 2P_{inc}$ reduce the applied load by only one increment). Reduce load increment to half, i.e. $P_{inc} = P_{inc}/2$, and go to item (i).

(h) If the load increment $P_{inc}$ is less than or equal to the tolerance load increment $P_{tol}$, for convergence, the predicted failure load has been determined. If the load increment is larger than the tolerance load increment, go to item (i).

(i) Reset displacement storage to zero and repeat the process starting from item (c).

2.3.3 Frame Discretization

Discretizing a frame or column into a specified number of elements between nodes permits the frame analysis procedure to account for the second-order member or chord effects ($P$-$\Delta$) due to axial loads acting through the deformed of columns in nonsway condition, as indicated in Figure 2.12(a). For example, a nonsway isolated column subjected to axial load and end moments was discretized into 30 elements to account for the chord effects. A frame discretization subroutine was developed to reduce the input requirements for the various frame configurations. Depending on the geometry and loading configuration of a frame, the number of elements varied from 15 to 40 for each column and from 15 to 40 for each beam in the frame.
2.3.4 Element Flexural Stiffness

The element flexural stiffness was computed from \( EI = M/\phi \) using the basic strength of material concepts applied to members subjected to small deflections. For a given axial load the \( M-\phi \) relationship is known. For elements with axial load levels greater than \( P/P_o \) = 0.3, the average moment in an element can be used to determine a corresponding curvature from the \( M-\phi \) relationship. The average moment in an element can be computed as the average of the element end moments, which are the output from the previous iteration. Difficulty arises in determining curvature using average moment when \( P/P_o \leq 0.3 \) because the slope of the moment-curvature relationship approaches zero for moments close to the peak moment (Figure 2.6). It is, therefore, more effective to use the average curvature of an element to determine the corresponding moment on the \( M-\phi \) curve. Curvature was not a direct output from the stiffness method of analysis used in this study. However, nodal rotations were known, which were used to compute the average curvature in an element, in order to find the corresponding moment from the \( M-\phi \) curve.

To determine the average curvature in an element length, the following assumptions were made:

(a) The direction of the loads and the deflection curve all lie in the plane of symmetry of the member.
(b) Plane sections remain plane after bending.
(c) The centroid of the gross cross-section follows the deflection curve.
(d) Shear deformations are negligible.
Nodal rotations are given by tangents to the deflection curve at element ends.

The effect of axial load is accounted for in the P-M-\( \phi \) relationship.

Assumptions (a), (b), and (c) are necessary for the computation of deflections using the stiffness method of analysis. Assumption (d) acknowledges that shear deformations are not significant when the overall slenderness ratio \( \theta h \) of a member is greater than or equal to 10. Assumption (e) is based on elastic theory; therefore, the nodal rotations at the element ends can be used to compute the change in slope within the element length. In summary, assumptions (a) through (e) amount to applying elastic beam theory to a single element of the frame structure. Assumption (f) is valid because a specific \( M-\phi \) curve is used for a given axial load acting on the element (Figure 2.6).

The average curvature of an element was defined as \( 1/\rho \), where \( \rho \) is the radius of curvature and was computed geometrically using Figure 2.13. The difference in nodal rotations, defined at ends of the element as \( r_j \) and \( r_k \), was used to compute the change in slope \( d\theta \) for the element length \( \ell_{\text{elem}} \) or \( dx \). The element length was computed from the nodal coordinates of the deflected position of the element. At this point the curvature could be computed as \( \phi = d\theta/dx \). However, because the element lengths vary and in some cases could be equal to or greater than the section depth, it was considered more accurate to compute the radius of curvature geometrically using Equation 2.8 and Figure 2.13(b):

\[
\rho = \frac{\frac{\ell_{\text{elem}}}{2}}{\sin\left(\frac{\left|r_k - r_j\right|}{2}\right)} \tag{2.8}
\]
The average curvature $\phi_{\text{avg}}$ of the element was then computed as $1/\rho$, the average bending moment $M_{\text{avg}}$ corresponding to $\phi_{\text{avg}}$ was taken from on the $M-\phi$ curve for the given axial load, and the effective flexural stiffness $EI_{\text{eff}}$ of the element was computed as $M_{\text{avg}}/\phi_{\text{avg}}$. The theoretical procedure then computes the modulus of elasticity for the element length as $E=EI_{\text{eff}}/I$, where $I$ is the initial input used for the moment of inertia of the cross-section. The new elastic modulus for the element is input into the element stiffness matrix (Equation 2.7) to generate the global stiffness matrix described in Section 2.3.1.

### 2.4 DETERMINING THE STRENGTH OF ISOLATED COLUMNS AND FRAMES

The cross-section properties of each member were input into a computer program based on the theoretical procedure described above and a set of $P-M-\phi$ relationships were generated for each cross-section (Section 2.2). Next, the frame (or column) geometry and a prescribed loading configuration were input into the computer program and a discretized frame (Section 2.3.3) was developed for analysis. The theoretical model permits selected nodal or joint loads to be incremented while other loads are maintained at a constant level. For each increment of loads used for the prescribed loading configuration, the second-order displacements (Section 2.3.2) were evaluated using the flexural stiffness of each element (Section 2.3.4) and the two-dimensional frame analysis procedure (Section 2.3.1). The loads in the prescribed configuration were incremented until the theoretical failure load was reached. The theoretical failure load was defined as a set of maximum stable forces applied...
externally to the frame (or column) that were in equilibrium with the internal forces within the frame or column.
INPUT VARIABLES FOR
MEMBER PROPERTY DATA

COMPUTE THEORETICAL P-M-\(\phi\) RELATIONSHIPS
AND CROSS-SECTION STRENGTH FOR MEMBERS

INPUT FRAME CONFIGURATION DATA

DISCRETIZE FRAME

INITIALIZE LOAD STEP (L.S.) & INCREMENT (INCR.)

FORM STIFFNESS MATRIX FROM P-M-\(\phi\) RELATIONSHIPS

PEAK LOAD EXCEEDED

NO

FIRST ITERATION FOR L.S.

YES

FORM STIFFNESS MATRIX FOR ANALYSIS USING INPUT E, A, I

EQUATION SOLVER

STORE COMPUTED NODAL DISPLACEMENTS

DISPLACEMENT CONVERGENCE

NO

UPDATE FRAME NODAL CONFIGURATION

YES

INCR. = INCR. \(\div 2\)

RESET FRAME TO ORIGINAL POSITION

L.S. = L.S. + INCR.

LOAD STEP COVERED TO TOLERANCE

NO

YES

OUTPUT DEFLECTIONS FOR LOAD STEP

OUTPUT DEFORMATIONS FOR LOAD STEP

COMPUTE & STORE DATA

Figure 2.1 - Flow chart of overall computation procedure.
Figure 2.2 - Material types used to describe (a) reinforced concrete and (b) composite cross-sections.
Figure 2.3 - Discretization of composite one-half cross-section used for computing strength of (a) composite columns bending about the minor axis; and (b) composite columns bending about the major axis.
Figure 2.4 - Schematic concrete stress-strain curve used for computing theoretical strength.

Figure 2.5 - Schematic stress-strain curve for reinforcing steel and structural steel in tension or compression used for computing theoretical strength.
Figure 2.6 - Schematic $P$-$M$-$\phi$ relationships for reinforced concrete and composite steel-concrete cross-sections.

Figure 2.7 - Schematic cross-section nondimensionalized axial load-bending moment interaction diagrams for reinforced concrete columns, composite columns bending about the minor axis, and composite columns bending about the major axis.
Figure 2.8 - Strain gradient in reinforced concrete cross-section and composite cross-sections in which bending takes place about the minor or major axis of the steel section.
Figure 2.9 - Flow chart for computation of $P-M-\phi$ relationships for reinforced concrete and composite cross-sections.
Figure 2.10 - Schematic frame configuration for two-dimensional analysis.
Figure 2.11 - Flow chart for computation of second-order displacements.
Figure 2.12 - Schematic frame configuration showing second order displacements: (a) braced frame; (b) unbraced frame.
Figure 2.13 - Diagram showing geometry used to compute radius of curvature.
3.0 COMPARISON OF THEORETICAL MODEL TO EXPERIMENTAL RESULTS

3.1 GENERAL

To test the accuracy of the theoretical procedure, the ultimate strengths computed by the theoretical model were compared to the ultimate strengths of physical tests gathered from the published literature. No new tests were conducted for this study. The physical tests used for comparison fall into three main categories:

(a) Isolated reinforced concrete and composite columns subjected to axial load and equal and opposite end moments causing symmetrical single curvature bending;

(b) isolated reinforced concrete and composite columns subjected to axial load and unequal end moments causing moment gradient in single curvature or double curvature bending; and

(c) braced reinforced concrete frames in which columns are subjected to axial load and end moments.

Each of these categories of physical tests are examined individually in Sections 3.2, 3.3, and 3.4, respectively. The examination of isolated columns subjected to equal or unequal end moments includes reinforced concrete columns, composite columns bending about the minor of the encased steel section, and composite columns bending about the major axis of the encased steel section.

Problems encountered while interpreting the experimental results for some of the test data gathered for isolated columns from the available literature are summarized below:

(a) The specified length of some specimens was unclear, especially when haunches were
used at the ends of the column;

(b) information regarding the reinforcement was in some cases insufficient with respect to quantity, position, and yield strength;

(c) the way the concrete strength was determined from cubes was unclear for some test results, e.g. cubes tested parallel or perpendicular to the direction of casting; and

(d) test specimens were in some cases very small.

The columns with these problems were not included in the database. In addition, columns subjected to concentric loads, columns in pure bending, and columns with slenderness ratios $\varnothing h$ less than 3.0 or greater than 40 were also not included in the database. Concentrically loaded columns were not included because CSA A23.3-94 (1994) and ACI 318-99 (1999) require a minimum end eccentricity $e$ of $(15+0.03h)$mm or $(0.6+0.03h)$in. for column design. Therefore, the database includes columns with end eccentricity ratio $e/h$ ranging from 0.029 to 1.5 and $\varnothing h$ ranging from 3 to 40. Though the moment magnifier method does not permit the use of slenderness ratios $\varnothing h$ greater than 30, $\varnothing h$ ratios up to 40 have been included for comparison purposes. The CSA and ACI codes do not require second-order effects to be computed if $\varnothing h$ is less than 6.6; however, $\varnothing h$ values as low as 3.0 have been included for comparison. As a result, from the initial database of over 600 isolated columns, 209 columns with equal end moments causing symmetrical single curvature bending and 15 columns with moment gradients causing single curvature or double curvature bending were considered for comparison.

Data for 13 physical tests of braced reinforced concrete frames were found to be
useful for this study. Problems similar to some of those encountered for isolated columns were also encountered for interpreting the experimental results for frames. In addition, frames where the reinforcement within the columns or beams was not doubly symmetric throughout the length were not used because the theoretical model is limited to studying doubly symmetric cross-sections. The $\theta/h$ ratio for columns in braced frames ranged from 15 to 30 and $e/h$ from 0.10 to 0.34.

For some of the physical tests, 4-inch (102 mm), 6-inch (152 mm), and 8-inch (203 mm) cube specimens were tested to establish concrete strength. In these cases, the reported strength was converted to an equivalent standard [6-inch (152 mm) diameter by 12-inch (305 mm) high] cylinder strength.

Many different conversion factors for obtaining an equivalent standard cylinder strength from the cube strength have been used over the years. Roderick and Rogers (1969) and Roderick and Loke (1974) used Equation 3.1, which was initially recommended by Evans (1943):

$$f'_c = 1.035u - 700$$

(3.1)

In Equation 3.1 both the 6-inch cube strength ($u$) and the standard cylinder strength ($f'_c$) are in psi. Furlong (1976) appears to have used a factor of 0.8 for converting the 4-inch cube strength to an equivalent standard cylinder strength. Virdi and Dowling (1973) reported a factor of 0.64 and Johnson and May (1978) applied a factor of 0.76 for obtaining an equivalent standard cylinder strength from a 6-inch (152 mm) cube. Roik and Bergmann
(1989) used a factor of 0.83 to obtain an equivalent standard cylinder strength from a 4-inch (102mm) cube strength.

For this study two equations were used, to obtain an equivalent standard (6-inch diameter by 12-inch high) cylinder strength from the strength of a cube of given size. Equation 3.2, which is based on the statistical theory of brittle fracture of solids (Bolotin 1969), as reproduced by Mirza, Hatzinikolas and MacGregor (1979), was used to account for the difference in strength due to volume difference of a cube with respect to a 4-inch cube:

\[ f = f_o \left[ 0.58 + 0.42 \left( \frac{v_o}{v} \right)^{1/3} \right] \]  

(3.2)

In Equation 3.2, \( f_o \) and \( v_o \) represent the concrete strength and volume of a 4-inch cube; and \( f \) and \( v \) are the concrete strength and volume of a cube of the desired size (6-inch cube for this study), respectively. Equation 3.3 (L'Hermite 1955), reproduced by Neville (1973) was then applied to convert the 6-inch (152 mm) cube strength to that of an equivalent standard cylinder:

\[ f'_c = \left( 0.76 + 0.2 \log_{10} \left( \frac{f_{cu}}{2840} \right) \right) f_{cu} \]  

(3.3)

In Equation 3.3, \( f_{cu} \) is the 6-inch cube strength and \( f'_c \) represents the standard cylinder strength in psi. For SI units, replace 2840 psi with 19.6 MPa.

In most cases, actual tests were performed to determine the yield strength and other properties of the stress-strain curves for structural steel and reinforcing steel (Figure 2.5). These stress-strain curves were included for the calibration of the theoretical model. In a
number of cases, however, only the nominal values for the strength of the structural steel and reinforcing steel were reported. Such physical tests were not used for calibration studies.

The input to the theoretical model included the reported material properties, such as concrete strength and steel strength; the cross-section dimensions, location and size of reinforcing, the geometry of the encased steel section, tie reinforcing size and spacing; and the column or frame configuration. The effects of residual stresses on structural steel and strain-hardening on the reinforcing and structural steel were considered for the comparison. To provide for the loads to be applied in the same manner as in the physical tests, the theoretical model allows the load input to be specified as a constant or applied as an incremental load at a given location.

3.2 COMPARISON OF COMPUTED STRENGTH TO EXPERIMENTAL RESULTS FOR COLUMNS SUBJECTED TO SYMMETRICAL SINGLE CURVATURE BENDING

The accuracy of the theoretically computed strength of columns subjected to equal and opposite end moments causing symmetrical single curvature bending was checked against 209 physical tests. These tests were conducted on 146 reinforced concrete columns, 33 composite steel-concrete columns subjected to bending about the minor axis of an encased steel section, and 30 composite steel-concrete columns subjected to bending about the major axis of an encased steel section.

The ratio of tested to computed ultimate strengths (strength ratio) was taken as the ratio of axial load strengths \( \frac{P_{\text{test}}}{P_{\text{computed}}} \) and was calculated for each of the 209 column
specimens. Statistical analyses of strength ratios of all column specimens were then conducted to evaluate the accuracy of the theoretical model.

3.2.1 Reinforced Concrete Columns

The theoretical model was compared with physical tests conducted on 146 reinforced concrete columns subjected to axial load and equal and opposite end moments causing symmetrical single curvature bending. These physical tests were taken from Hognestad (1951), Viest et al. (1956), Gaede (1958), Bresler (1960), Chang and Ferguson (1963), Todeschini et al. (1964), Mehmel et al. (1969), Ramu et al. (1969), Drysdale and Huggins (1971), Goyal and Jackson (1971), Green and Hellesland (1975), Heimdahl and Bianchini (1975), and Kim and Yang (1995). The calculated average value, standard deviation, coefficient of variation, and minimum and maximum values of strength ratios ($P_{\text{test}}/P_{\text{computed}}$) for these columns are shown in Table 3.1(a). The average value of the strength ratio, computed as 1.002 with a coefficient of variation of 0.090, a minimum value of 0.83 and a maximum value of 1.20, indicates that the theoretical model computes the failure load of the reinforced concrete columns with reasonable accuracy.

The strength ratio for each column is plotted against the end eccentricity ratio ($e/h$) and slenderness ratio ($\vartheta/h$) in Figures 3.1(a) and (b), respectively. These figures show that the strength ratio for reinforced concrete columns is not affected significantly by $e/h$ or $\vartheta/h$. Figure 3.1(b) shows that the strength ratio is less than 1.0 for more than half of the columns with $t/h$ ratio greater than 10. Of the 68 column tests with $t/h > 10$, 29 tests were conducted by Goyal and Jackson (1971) ($t/h = 16, 24$ and 36) on columns with a relatively small cross-
section [76 mm by 76 mm (3 inch by 3 inch)]. For concrete strength Goyal and Jackson reported standard cylinder tests (6-inch diameter by 12-inch high) and prism tests (3-inch by 3-inch by 9-inch long). Both the prisms and columns were cast horizontally, whereas, the cylinders were cast vertically. The tested strength of the concrete from the prism tests was on the average about 75% of the strength obtained from the cylinder tests. Goyal and Jackson suggested that the prism tests were probably a more reliable indicator of the concrete strength for their test columns. For the comparison presented here, the cylinder strength was used in the theoretical model to compute column strengths, resulting in an apparently nonconservative computation of column strength. The cylinder strength was used to maintain a level of consistency in the interpretation of the data used for all of the comparisons presented in this chapter. Note that most of the tests from Goyal and Jackson would produce a strength ratio higher than 1.0 if the prism strengths were used in lieu of cylinder strengths in computation of theoretical strength. In addition, a small cross-section will introduce size effect discrepancies even due to minor inaccuracies between the measured and the actual location of the axial load or the reinforcing steel. For example, an error of 1 mm in measurement of the end eccentricity is more significant for a 76 mm by 76 mm cross-section than for a 250 mm by 250 mm cross-section.

3.2.2 Composite Columns Subjected to Bending About the Minor Axis of Encased Steel Section

The accuracy of the theoretical model for composite columns subjected to axial load and end bending moments causing symmetrical single curvature bending about the minor
axis of an encased steel section was checked using 33 physical tests taken from Bondale (1966a, b, c), Anslijn and Janss (1974), Roderick and Loke (1974), Morino et al. (1984), and Roik and Schwalbenhofer (1988). The calculated average value, standard deviation, coefficient of variation, and minimum and maximum values of strength ratios for these columns are shown in Table 3.1(a). The average value of the strength ratio, computed as 0.966 with a coefficient of variation of 0.103, a minimum value of 0.83 and a maximum value of 1.18, indicates that the theoretical model computes the failure load of composite columns bending about the minor axis of the encased steel section with reasonable accuracy.

The strength ratio for each column is plotted against the end eccentricity ratio \((e/h)\) and slenderness ratio \((\ell/h)\) in Figures 3.2(a) and (b), respectively. These figures show that the strength ratio for composite columns subjected to minor axis bending is not significantly affected by \(e/h\) or \(\ell/h\). Figure 3.2(a) shows that at \(e/h \leq 0.21\) the computed strength is larger than the test strength for 17 out of 20 columns. Of these 20 column tests, nine (9) tests were taken from Anslijn and Janss (1974) and 10 were taken from Roderick and Loke (1974). Anslijn and Janss reported the strength of 4-inch concrete cubes and Equations 3.2 and 3.3 were used to obtain an equivalent standard cylinder strength for their data. These equations are empirical and can introduce some margin of error into the conversion. On the other hand, the physical tests reported by Roderick and Loke had no longitudinal or transverse reinforcing steel. This lack of reinforcing steel resulted in a premature failure of some of the columns as reported by Roderick and Loke (1974), significantly affecting the test strength.

It should be noted that, for this study, the column specimens from Roderick and Loke were
modeled with unconfined concrete in the entire cross-section.

3.2.3 Composite Columns Subjected to Bending About the Major Axis of Encased Steel Section

Thirty (30) physical tests were used to check the accuracy of the theoretical model for columns subjected to axial load and end bending moments causing symmetrical single curvature bending about the major axis of the encased steel section. The data for these columns were taken from Proctor (1967), Johnson and May (1978), Morino et al. (1984), Suzuki et al. (1983), Roik and Mangerig (1987), and Roik and Schwalbenhofer (1988).

The calculated average value, standard deviation, coefficient of variation, and minimum and maximum values of strength ratios for these columns are shown in Table 3.1(a). The average value of the strength ratio, computed as 1.025 with a coefficient of variation of 0.089, a minimum value of 0.84 and a maximum value of 1.18, indicates that the theoretical model computes the failure load of composite columns bending about the major axis of the encased steel section with reasonable accuracy.

The strength ratios are plotted against the end eccentricity ratio \((e/h)\) and slenderness ratio \((\theta/h)\) in Figures 3.3(a) and (b), respectively. These figures show that the strength ratio for composite columns subjected to major axis bending is not affected by \(e/h\) or \(\theta/h\).

3.2.4 Summary of Columns Subjected to Symmetrical Single Curvature Bending

Figure 3.4(a) shows a comparison of the computed and tested ultimate strengths of all 209 reinforced concrete and composite column specimens listed in Table 3.1(a). The
The frequency histogram of the strength ratios for these specimens is plotted on the inset of Figure 3.4(b). The cumulative frequency curve of the strength ratios for the same data is plotted also in Figure 3.4(b) on a normal probability paper and is compared with a normal probability distribution using the same average value and coefficient of variation as those obtained from the test data. Based on the plots of Figure 3.4 and statistics of strength ratios given on the inset of Figure 3.4(b), it can be assumed that the strength ratio of reinforced concrete and composite columns subjected to symmetrical single curvature bending is normally distributed with an average value of 1.00 and coefficient of variation of 0.092. This indicates a very good correlation between the theoretical model and test results.

3.3 COMPARISON OF COMPUTED STRENGTH TO EXPERIMENTAL RESULTS FOR COLUMNS SUBJECTED TO MOMENT GRADIENTS

The theoretical model was compared with the results of 15 physical tests, gathered from the literature, for columns subjected to axial load and moment gradient due to unequal end moments causing both single and double curvature bending. The database includes 12 tests on reinforced concrete columns reported by MacGregor and Barter (1965), Martin and Olivieri (1965), and Mehmel et al. (1969); and 3 tests on composite steel-concrete columns subjected to bending about the major axis of the encased steel section reported by Roik and Schwalbenhofer (1988).

Again, for the purpose of comparison, the strength ratio was taken as the ratio of axial load strengths \( \frac{P_{\text{test}}}{P_{\text{computed}}} \). The calculated average value, standard deviation, coefficient
of variation, and minimum and maximum values of strength ratios for the 15 column specimens listed are in Table 3.1(b). The average value of the strength ratio, computed as 1.050 with a coefficient of variation of 0.072, a minimum value of 0.96 and a maximum value of 1.19, shows that the theoretical model computes with reasonable accuracy the failure load of reinforced concrete and composite columns subjected to moment gradient due to unequal end moments causing single or double curvature bending.

The strength ratios are plotted against the end moment ratio \( M_1/M_2 \) and slenderness ratio \( \theta h \) in Figure 3.5(a) and (b), respectively. These figures show that the strength ratio for these columns is not affected by \( M_1/M_2 \) or \( \theta h \).

A comparison of the computed and tested ultimate strengths of all 15 columns listed in Table 3.1(b) is shown in Figure 3.6. This figure also confirms the validity of the theoretical model for reinforced concrete and composite columns subjected to moment gradient.

### 3.4 COMPARISON OF COMPUTED STRENGTH TO EXPERIMENTAL RESULTS FOR BRACED REINFORCED CONCRETE FRAMES

The accuracy of the theoretical model for computing the strength of columns that are part of braced frames was established from the comparison with 13 reinforced concrete test frames taken from Breen and Ferguson (1964), Furlong and Ferguson (1965), and Blomeier and Breen (1975).

Special modeling techniques were used for the frames at beam-to-column joints to account for the additional strength resulting from the concrete confinement at and near the
joints. These modeling methods are based on observations of failure zones in column tests conducted by Ford, Chang and Breen (1981). These tests showed no distress at the joints and the failure occurred away from the joints. The member properties used by Ford, Chang and Breen are summarized in Figure 3.7. In that study, elements within the beam-to-column joints were modeled as fully elastic using a modulus of elasticity of 29,000 ksi (200,000 MPa) for both concrete and reinforcing steel (Ford, Chang and Breen 1981). This was done to make the joint rigid. In addition, an artificially high concrete strength was used in computations for the end portions of the columns that extended from the face of the beam along the column length for a distance equal to the overall depth of the column cross-section, as shown in Figure 3.7.

For this study, a very similar technique was employed in the vicinity of beam-to-column joints, as shown in Figure 3.8. All of the concrete within the gross cross-section at beam-to-column joints was modeled as highly confined, as defined in Section 2.2.2.1, and the area of the reinforcing steel in these regions was doubled for the purpose of modeling. The concrete within the gross cross-section in end portions of a column (from the face of the beam to a distance equal to the overall column depth) and in end portions of a beam (from the face of the column to a distance equal to the overall beam depth) was modeled as partially confined, as defined in Section 2.2.2.1.

A brief description of 13 test frames used for the comparison of tested to theoretical strengths of columns in braced reinforced concrete frames is given in Table 3.2. Table 3.2 also gives strength ratios for these frames. A strength ratio of a frame was taken as the ratio
of axial load strengths ($P_{\text{test}}/P_{\text{computed}}$) of the column that caused the failure of that frame. A comparison of the computed and tested ultimate strengths of all 13 frames listed in Table 3.2 is shown in Figure 3.9. In addition, Table 3.3 lists the calculated average value, standard deviation, coefficient of variation, minimum value and maximum value of strength ratios as 1.072, 0.110, 0.103, 0.89 and 1.20, respectively. From these statistics and Figure 3.9, it can be seen that the theoretical model computes the strength of reinforced concrete columns that are part of braced frames with reasonable accuracy.
Table 3.1 - Statistical analysis of strength ratios of tested to theoretically computed strengths of isolated column specimens.

<table>
<thead>
<tr>
<th>Column Type</th>
<th>Properties of Column Specimens</th>
<th>Statistics of Strength Ratios $P_{test} / P_{computed}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>(a) Columns Subjected to Symmetrical Single Curvature Bending</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reinforced Concrete Columns</td>
<td>$f_c = 16.8$ to $53.2$ MPa</td>
<td>$n = 146$</td>
</tr>
<tr>
<td></td>
<td>$e/h = 0.033$ to 1.25</td>
<td>Average Value = 1.002</td>
</tr>
<tr>
<td></td>
<td>$t/h = 3.0$ to 36.0</td>
<td>Std. Deviation = 0.090</td>
</tr>
<tr>
<td></td>
<td>$\rho_{re} = 1.0$ to 5.2 %</td>
<td>Coef. of Variation = 0.089</td>
</tr>
<tr>
<td></td>
<td>$\rho_{ss} = 2.7$ to 8.7 %</td>
<td>Minimum = 0.83</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximum = 1.20</td>
</tr>
<tr>
<td>Composite Columns with Minor Axis Bending</td>
<td>$f_c = 19.9$ to $52.7$ MPa</td>
<td>$n = 33$</td>
</tr>
<tr>
<td></td>
<td>$e/h = 0.029$ to 0.714</td>
<td>Average Value = 0.966</td>
</tr>
<tr>
<td></td>
<td>$t/h = 5.8$ to 28.9</td>
<td>Std. Deviation = 0.099</td>
</tr>
<tr>
<td></td>
<td>$\rho_{re} = 0.0$ to 3.1 %</td>
<td>Coef. of Variation = 0.103</td>
</tr>
<tr>
<td></td>
<td>$\rho_{ss} = 2.7$ to 8.7 %</td>
<td>Minimum = 0.83</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximum = 1.18</td>
</tr>
<tr>
<td>Composite Columns with Major Axis Bending</td>
<td>$f_c = 21.2$ to $55.8$ MPa</td>
<td>$n = 30$</td>
</tr>
<tr>
<td></td>
<td>$e/h = 0.112$ to 1.062</td>
<td>Average Value = 1.025</td>
</tr>
<tr>
<td></td>
<td>$t/h = 3.8$ to 28.9</td>
<td>Std. Deviation = 0.091</td>
</tr>
<tr>
<td></td>
<td>$\rho_{re} = 0.0$ to 0.8 %</td>
<td>Coef. of Variation = 0.089</td>
</tr>
<tr>
<td></td>
<td>$\rho_{ss} = 4.2$ to 14.5 %</td>
<td>Minimum = 0.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximum = 1.18</td>
</tr>
<tr>
<td>(b) Columns Subjected to Moment Gradient</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reinforced Concrete Columns and</td>
<td>$f_c = 25.0$ to $55.8$ MPa</td>
<td>$n = 15$</td>
</tr>
<tr>
<td>Composite Columns with Major Axis Bending</td>
<td>$e/h = 0.17$ to 1.50</td>
<td>Average Value = 1.050</td>
</tr>
<tr>
<td></td>
<td>$t/h = 10.7$ to 40.0</td>
<td>Std. Deviation = 0.075</td>
</tr>
<tr>
<td></td>
<td>$M_1/M_2 = -1.00$ to 0.99</td>
<td>Coef. of Variation = 0.072</td>
</tr>
<tr>
<td></td>
<td>$\rho_{re} = 0.8$ to 4.0 %</td>
<td>Minimum = 0.96</td>
</tr>
<tr>
<td></td>
<td>$\rho_{ss} = 0.0$ to 14.5 %</td>
<td>Maximum = 1.19</td>
</tr>
</tbody>
</table>
Table 3.2 - Comparison of tested to theoretically computed strengths of columns that are part of physical tests on braced reinforced concrete frames.

<table>
<thead>
<tr>
<th>Author</th>
<th>Description</th>
<th>Ultimate Strength (kN)</th>
<th>Strength Ratio</th>
<th>P&lt;sub&gt;test&lt;/sub&gt; / P&lt;sub&gt;computed&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frame No.</td>
<td>No. of Storeys</td>
<td>Col. t/h</td>
<td>Col. e/h</td>
</tr>
<tr>
<td>Breen and Ferguson</td>
<td>(1964)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F1</td>
<td>1</td>
<td>1</td>
<td>30</td>
<td>0.300</td>
</tr>
<tr>
<td>F2</td>
<td>1</td>
<td>1</td>
<td>30</td>
<td>0.100</td>
</tr>
<tr>
<td>F3</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td>0.300</td>
</tr>
<tr>
<td>F4</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td>0.100</td>
</tr>
<tr>
<td>F5</td>
<td>1</td>
<td>1</td>
<td>30</td>
<td>0.100</td>
</tr>
<tr>
<td>Furlong and Ferguson</td>
<td>(1965)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F1</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>0.116</td>
</tr>
<tr>
<td>F2</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>0.106</td>
</tr>
<tr>
<td>F3</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>0.337</td>
</tr>
<tr>
<td>F4</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>0.222</td>
</tr>
<tr>
<td>F5</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td>0.097</td>
</tr>
<tr>
<td>F6</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td>0.320</td>
</tr>
<tr>
<td>Blomier and Breen</td>
<td>(1975)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B2</td>
<td>3</td>
<td>1</td>
<td>15</td>
<td>0.164</td>
</tr>
<tr>
<td>B3</td>
<td>3</td>
<td>1</td>
<td>15</td>
<td>0.151</td>
</tr>
</tbody>
</table>

Table 3.3 - Statistical analysis of strength ratios of tested to theoretically computed strengths of columns that are part of physical tests on reinforced concrete frames.

<table>
<thead>
<tr>
<th>Frame Type</th>
<th>Statistics of Strength Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n = 13</td>
</tr>
<tr>
<td></td>
<td>Average Value = 1.072</td>
</tr>
<tr>
<td></td>
<td>Std. Deviation = 0.110</td>
</tr>
<tr>
<td></td>
<td>Coef. of Variation = 0.103</td>
</tr>
<tr>
<td></td>
<td>Minimum = 0.89</td>
</tr>
<tr>
<td></td>
<td>Maximum = 1.20</td>
</tr>
</tbody>
</table>

-59-
Figure 3.1 - Effects of (a) end eccentricity and (b) slenderness on the ratio of test strength to theoretically computed strength of isolated reinforced concrete column specimens subjected to symmetrical single curvature bending (n=146).
Figure 3.2 - Effects of (a) end eccentricity and (b) slenderness on the ratio of test strength to theoretically computed strength of isolated composite column specimens subjected to symmetrical single curvature bending about the minor axis of the steel section (n=33).
Figure 3.3 - Effects of (a) end eccentricity and (b) slenderness on the ratio of test strength to theoretically computed strength of isolated composite column specimens subjected to symmetrical single curvature bending about the major axis of the steel section (n=30).
Figure 3.4 - Comparison of tested to theoretically computed strengths of all isolated reinforced concrete and composite column specimens subjected to symmetrical single curvature bending [and listed in Table 3.1(a)]: (a) comparison of strengths; and (b) probability distribution of strength ratio (n=209).
Figure 3.5 - Effects of (a) end moment ratio and (b) slenderness on the ratio of test strength to theoretically computed strength of isolated reinforced concrete and composite column specimens subjected to moment gradient [and listed in Table 3.1(b)] (n=15)
Figure 3.6 - Comparison of tested to theoretically computed strengths of all isolated reinforced concrete and composite column specimens subjected to moment gradient [and listed in Table 3.1(b)] (n=15).

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Figure 3.7 - Member properties used by Ford et al. (1981) for the analysis of reinforced concrete frames.

Figure 3.8 - Member properties used in this study for the analysis of reinforced concrete frames.
Figure 3.9 - Comparison of tested to theoretically computed strengths of columns in braced reinforced concrete frames (listed in Table 3.2): (a) configurations of test frames; and (b) comparison of strengths (n=13)
4.0 EXAMINATION OF EFFECTIVE FLEXURAL STIFFNESS OF COLUMNS

4.1 GENERAL

The bending moment strength of a slender column that is subjected to end moments causing symmetrical single curvature bending is lower than the strength of its cross-section. A column of length \( \ell \) deflects laterally when subjected to an eccentric axial load and is, therefore, subjected to additional bending moment along its height (Figure 4.1). This additional bending moment is negligible for columns having a small slenderness ratio. However, the second-order moment at mid-height, caused by the axial load acting through additional eccentricity (mid-height deflection), becomes significant in slender columns and controls the maximum applied end moment.

CSA A23.3-94 (1994) and ACI 318-99 (1999) permit the use of a moment magnifier approach for computing the second-order moments in columns. This method is strongly influenced by the effective flexural stiffness \( (EI) \), which varies due to the nonlinearity of the concrete stress-strain curve, creep and cracking along the height of the column. The \( EI \) expressions given in the CSA and ACI codes for reinforced concrete columns are quite approximate when compared to values obtained from the axial load, bending moment, and curvature \( (P-M-\phi) \) relationships as explained in the Commentary of ACI 318-99 (1999). Mirza (1990) explained that the CSA and ACI \( EI \) equations were developed for reinforced concrete columns subjected to high axial loads, but are permitted for all axial load levels. In addition, for composite steel-concrete columns, the \( EI \) equations given in the CSA and ACI codes were simply modified, without any further investigation, for use in composite column
designs (Mirza and Tikka 1999a).

Mirza (1990) investigated the effects of variables on $EI$ of reinforced concrete columns bent in symmetrical single curvature and proposed two equations for $EI$. Mirza and Tikka (1999a,b) conducted similar studies to determine the effects of variables on $EI$ of steel-concrete composite columns bending about the major axis and minor axis of encased steel sections and proposed new equations for $EI$. Those studies showed that the $EI$ expressions currently used by CSA A23.3-94 and ACI 318-99 have a high degree of variability and can be conservative for columns subjected to low eccentricities and unconservative for columns subjected to high eccentricities. This study approaches reinforced concrete and composite columns from the same perspective in the sense that reinforced concrete columns are taken as a special case of composite columns.

This study was conducted to examine the influence of different variables on the short-term effective flexural stiffness ($EI$) of slender, tied, reinforced concrete and composite steel-concrete columns; examine the existing expressions for $EI$; develop and propose refined expressions for $EI$ that apply to both reinforced concrete and composite steel-concrete columns; and compare the new expressions for $EI$ with previous work and the current CSA/ACI expressions.

Over 35,000 isolated reinforced concrete and composite steel-concrete columns were simulated to study the effects of a number of variables that affect the effective flexural stiffness. Each column had a different combination of cross-section, geometric, and material properties. The columns studied were bent in symmetrical single curvature. These are
isolated columns and subjected to short-term loads. The moment magnifier approach specified in the CSA and ACI Codes was developed for this type of columns. The effects of different end restraints, lateral supports, and loading conditions are accounted for in the CSA and ACI Codes through the use of the effective length factor ($K$), equivalent uniform moment diagram factor ($C_m$), and sustained load factor ($\beta_s$). The columns studied are graphically represented in Figure 4.1 and are similar to reinforced concrete columns examined by Mirza (1990) and MacGregor et al. (1975) and to composite steel-concrete columns analyzed by Mirza and Tikka (1999a,b). These columns were chosen because the errors in $K$, $C_m$, and $\beta_s$ would not affect the accuracy of the $EI$ expressions developed later in this Chapter.

4.2 METHOD USED FOR EVALUATING THEORETICAL FLEXURAL STIFFNESS

4.2.1 Development of Theoretical Stiffness Equation

Timoshenko and Gere (1961) give the following bending moment relationship for a pin-ended slender column subjected to equal and opposite end moments:

$$M_c = M_2 \sec \left( \frac{\pi}{2} \sqrt{\frac{P_u}{P_{cr}}} \right)$$

(4.1)

where $M_c =$ the design bending moment that includes second-order effects; $M_2 =$ the applied column end moment calculated from a conventional elastic analysis; $P_u =$ the factored axial load acting on the column; and $P_{cr}$ is Euler's buckling strength. Rearranging Equation 4.1,
solving for $P_{cr}$ and simplifying yields:

$$P_{cr} = \frac{\pi^2 P_u}{4 \left[ \text{arcsec} \left( \frac{M_c}{M_2} \right) \right]^2}$$  \hspace{1cm} (4.2)

Euler’s buckling strength for a pin-ended column is also given by $P_{cr} = \frac{\pi^2 EI}{\ell^2}$, in which $EI = \text{the effective stiffness}$; and $\ell = \text{the unsupported height of the column}$. Equating Equation 4.2 to $\pi^2 EI/\ell^2$ and solving for $EI$ gives the following expression:

$$EI = \frac{P_u \ell^2}{4 \left[ \text{arcsec} \left( \frac{M_c}{M_2} \right) \right]^2}$$  \hspace{1cm} (4.3)

Then, for the purpose of analysis, $M_c$ is replaced by the cross-section bending moment strength $M_{cs}$ and $M_2$ is replaced by the overall column bending moment strength $M_{col}$, so that Equation 4.3 becomes:

$$EI = \frac{P_u \ell^2}{4 \left[ \text{arcsec} \left( \frac{M_{cs}}{M_{col}} \right) \right]^2}$$  \hspace{1cm} (4.4)

Equation 4.4 is the theoretical effective flexural stiffness of a pin-ended slender column subjected to equal end moments causing symmetrical single curvature bending. The terms $P_u, M_{cs}$ and $M_{col}$ used in Equation 4.4 were obtained from the cross-section and column axial load-bending moment $(P-M)$ interaction diagrams (Figure 4.2), which were developed using the theoretical strength model documented in Chapter 2. The computation of $P_u, M_{cs}$
and \( M_{\text{col}} \) are summarized in the next section.

### 4.2.2 Computations of Theoretical Cross-Section and Slender Column Bending Moment Resistances

The iterative and incremental procedures described Sections 2.3 and 2.4 were used to compute the axial load strength \( (P_u) \) of a column for the end moment ratio \( (M_e/M_r)=1.0 \) and a specified end eccentricity ratio \( (e/h) \). The computed column axial load strength \( (P_u) \) represents a single point on the column \( P-M \) interaction curve (Figure 4.2). The column end moment resistance \( (M_{\text{col}}) \) was computed simply as \( eP_u \). The column axial load strength \( P_u \) was then used to determine the corresponding value of \( M_{\text{cr}} \), using Lagrangian interpolation, from the generated points on the cross-section axial load-bending moment interaction diagram. Note that the axial load-bending moment interaction curve for a cross-section (Figure 4.2) was generated using the force-equilibrium and strain-compatibility procedure documented in Section 2.2. The computed values of \( M_{\text{col}}, M_{\text{cr}}, \) and \( P_u \) for each column (with specified properties, including \( e/h \) and \( \theta/h \) ratios) were used directly in Equation 4.4 to compute theoretical \( EI \).

### 4.3 SIMULATION OF THEORETICAL STIFFNESS DATA FOR COLUMNS STUDIED

Over 35,000 isolated columns were simulated to evaluate the short-term flexural stiffness \( (EI) \). Of these columns, 11,550 were reinforced concrete columns, 11,880 were composite steel-concrete columns subjected to bending about the minor axis of the encased steel section, and 11,880 were composite steel-concrete columns subjected to bending about
the major axis of the encased steel section. The short-term theoretical $E_I$ for each of the columns studied was computed from Equation 4.4. The simulated column stiffness data were then statistically analyzed to examine the current CSA and ACI column stiffness equations and to develop the proposed design equation for $E_I$.

4.3.1 Description of Reinforced Concrete Columns Studied

The specified properties of 11,550 simulated reinforced concrete columns studied are given in Table 4.1. Specified concrete strengths $f'_{cc}$, clear concrete covers $C_{cc}$, and reinforcing steel ratios $\rho$ listed in Table 4.1 and Figure 4.3 represent usual ranges of these variables used in the construction industry (Grant et al. 1978). The overall concrete cross-section was 305 mm × 305 mm (12 in. × 12 in.) in size.

The CSA and ACI Code requirements for the design of reinforced concrete columns influenced the selection of cross-section parameters used in this study. The CSA and ACI Codes require that a minimum of 1 percent to a maximum of 8 percent of longitudinal reinforcing steel be used. Difficulty in lap splicing the reinforcing bars reduces the maximum limit on $\rho$ to about 4 percent. The reinforcing steel ratio is, therefore, usually expected to range from 1 to 4 percent. The ACI Code specifies that $f'_{cc}$ shall not be less than 17.2 MPa (2500 psi). With this requirement in mind, specified strengths of concrete, shown in Table 4.1, were selected. The specified yield strength of reinforcing bars was taken as 414 MPa (60,000 psi) for all columns, because this represents the most common strength of reinforcing bars used in the construction industry. Figure 4.3 shows the reinforcing steel arrangements that were used in this study. The clear concrete cover to lateral ties varied from
38 mm to 64 mm (1.5 in. to 2.5 in.). The lateral ties used in this study were 10 mm (0.375 in.) diameter and spaced in accordance with the requirements of CSA A23-94 Clause 7.6.5.2 and ACI 318-99 Clause 7.10.5.2.

Table 4.1 shows that eleven end eccentricity ratios \( (e/h) \) ranging from 0.05 to 1.0 were used. This is reasonable for reinforced concrete buildings, where the \( e/h \) ratio usually varies from 0.1 to 0.65 (Mirza and MacGregor 1982). Five slenderness ratios \( (\theta h) \) were chosen to represent the range of \( \theta h \) ratios permitted by CSA A23.3-94 Clause 10.15 and ACI 318-99 Clause 10.11 for columns in braced frames.

The purpose of this study was to simulate the actual flexural stiffness \( (EI) \) of columns described by specified cross-sectional and material properties. However, the specified values of material strengths will not provide an accurate estimation of \( EI \), because the strength of in-place concrete in columns is much lower than the specified strength (Mirza et al. 1979). Average values of material strengths corresponding to the specified values were, therefore, used to compute the theoretical stiffness of each column. Table 4.2 lists the average values of variables taken from Skrabek and Mirza (1990) and used for theoretical computations of reinforced concrete columns.

### 4.3.2 Description of Composite Steel-Concrete Columns Studied

Table 4.3 lists the specified properties of 23,760 simulated composite columns studied. One-half of these columns was subjected to major axis bending, while the other half was subjected to minor axis bending. Specified concrete strengths \( f'_{c} \), the structural steel yield strengths \( f_{y} \), the reinforcing steel ratios \( \rho_{s} \), and the structural steel ratios \( \rho_{s} \), shown in
the table represent usual ranges of these variables used in the construction industry (Skrabek and Mirza 1990). The overall dimensions as well as the longitudinal reinforcement and structural steel arrangements for the column cross-sections studied are shown in Figure 4.4.

The CSA A23.1-94 (1994), ACI 318-99 (1999), and the AISC Load and Resistance Factor Design (LRFD) (1994) requirements for composite columns influenced the selection of cross-section parameters used in this part of the study. For composite columns, the CSA code specifies that the total area of the metal core and reinforcing steel shall not exceed 20 percent, whereas the ACI and AISC codes do not place a maximum limit on the steel core. However, the AISC LRFD states that to qualify as a composite column, the structural steel ratio ($\rho_{ss}$) must be greater than or equal to 4 percent. The CSA and ACI codes require that a minimum of 1 percent to a maximum of 8 percent of longitudinal reinforcing steel be included with the structural steel core. Difficulty in lap splicing the reinforcing bars reduces the maximum limit on $\rho_{ss}$ to about 3 to 4 percent when a relatively large structural steel core is encased. The reinforcing steel ratio is, therefore, usually expected to range from 1 to 3 percent. Even a reinforcing steel ratio of three percent will restrict $\rho_{ss}$ to a maximum of about 10 percent, giving a range of $\rho_{ss}$ of about 4 to 10 percent. The AISC LRFD specifies that $f'_c$ be restricted to range from 20.7 to 55.2 MPa (3000 to 8000 psi) and that the specified yield strength of structural steel and reinforcing bars shall not exceed 379 MPa (55,000 psi) in calculating the strength of a composite column. The ACI Code, on the other hand, specifies that $f'_c$ shall not be less than 17.2 MPa (2500 psi) and the CSA code requires that the specified yield strength of structural steel shall not exceed 350 MPa (50,760 psi). CSA
A23-3-94 also places an upper limit of 500 MPa (72,515 psi) on the specified yield strength of reinforcing steel. With these requirements in mind, specified strengths of concrete and structural steel, shown in Table 4.3, were selected. The specified yield strength of reinforcing bars was taken as 414 MPa (60,000 psi) for all columns, because this represents the most common strength of reinforcing bars used in the construction industry.

Table 4.3 shows that eleven end eccentricity ratios \( (e/h) \) ranging from 0.05 to 1.0 and five slenderness ratios \( (\ell/h) \) ranging from 10 to 30 were used for composite columns. These values are the same as those used for reinforced concrete columns and were selected for reasons given in section 4.3.1. Average values of material strengths (Table 4.4) corresponding to the specified values were used to compute the theoretical stiffness of each column for reasons described also in Section 4.3.1.

4.4 EXAMINATION OF CSA/ACI STIFFNESS EQUATIONS

CSA A23.3-94 and ACI 318-99 permits the use of Equations 4.5 and 4.6 for calculating the effective flexural stiffness \( (EI) \) of slender reinforced concrete, and composite columns, respectively:

\[
EI = \frac{0.2 \ E_c I_g + E_s I_{rs}}{(1 + \beta_d)} \quad (4.5)
\]

\[
EI = \left[ \frac{0.2 \ E_c I_g}{(1 + \beta_d)} \right] + E_s I_{ss} \quad (4.6)
\]

In the above equations, \( E_c = \) the modulus of elasticity of concrete; \( I_g = \) the moment of inertia
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of the gross concrete cross-section; \( E_s \) = the modulus of elasticity of steel; \( I_{rs} \) = the moment of inertia of the longitudinal reinforcing steel taken about the centroidal axis of the gross cross-section; \( I_{ss} \) = the moment of inertia of the structural steel section about the centroidal axis of the composite cross-section; and \( \beta_d \) = the sustained load factor taken as the ratio of the maximum factored axial dead load to the total factored axial load (for the type of column studied) and is always positive. CSA 23.3-94 also permits the use of Equation 4.7a for determining \( EI \) of reinforced concrete columns, whereas ACI 318-99 also allows Equation 4.7b to be used for computing \( EI \) of both reinforced concrete and composite columns.

\[
EI = 0.25 E_c I_g \quad \text{(4.7a)}
\]

\[
EI = \frac{0.4 E_c I_g}{(1 + \beta_d)} \quad \text{(4.7b)}
\]

Note that Equation 4.7a was obtained by using \( \beta_d = 0.6 \) in Equation 4.7b and is a simplified version of Equation 4.7b. For short-term loads, \( \beta_d = 0 \) and Equations 4.5, 4.6, and 4.7b are simplified to Equations 4.8, 4.9 and 4.10, respectively.

\[
EI = 0.2E_c I_g + E_s I_{rs} \quad \text{(for reinforced concrete columns)} \quad \text{(4.8)}
\]

\[
EI = 0.2E_c I_g + E_s I_{ss} \quad \text{(for composite columns)} \quad \text{(4.9)}
\]

\[
EI = 0.4E_c I_g \quad \text{(for reinforced concrete and composite columns)} \quad \text{(4.10)}
\]

Note that, in Equations 4.8 to 4.10, \( E_s \) was taken as 200,000 MPa (29,000,000 psi) and \( E_c \) was computed from \( 4700 \sqrt{f'_c} \) MPa (57,000 \( \sqrt{f'_c} \) psi), as specified in ACI 318-99 (1999). CSA A23.3-94 specifies very similar values of \( E_s \) and \( E_c \).
Equations 4.8 and 4.9 were compared with the theoretical $EI$ values computed from Equation 4.4 for all simulated reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending. The results of these comparisons are plotted in Figures 4.5 (a), (b), and (c), which show histograms and statistics of the ratios of theoretical $EI$ to CSA (or ACI) $EI$ ($EI_{(th)}/EI_{(des)}$), for reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending, respectively. Stiffness ratios ($EI_{(th)}/EI_{(des)}$) greater than one signify that $EI_{(des)}$ is conservative, and values of $EI_{(th)}/EI_{(des)}$ less than one indicate that $EI_{(des)}$ is non-conservative. Note that $EI_{(des)}$ for reinforced concrete columns was computed using Equation 4.8; and for composite columns, $EI_{(des)}$ was computed from Equation 4.9. Figure 4.5 shows average values greater than unity for all three types of columns (Average value = 1.01 for reinforced concrete columns, 1.74 for composite columns with minor axis bending, and 1.43 for composite columns with major axis bending). The coefficients of variation ($V_r$) for all three types of columns are very high ($V_r = 0.33$ for reinforced concrete columns, 0.24 for composite columns with minor axis bending, and 0.22 for composite columns with major axis bending). This indicates that, for a significant number of columns, the CSA/ACI $EI$ deviated substantially from the corresponding theoretically computed $EI$. The 1-percentile stiffness ratio of 0.54 and average value of 1.01 for reinforced concrete columns indicate that Equation 4.8 computed non-conservative values of $EI$ for a large number of reinforced concrete columns studied. On the other hand, the 1-percentile stiffness ratios of 1.14 and 0.95 for composite columns subjected to minor and
major axis bending, respectively, indicate that Equation 4.9 computed conservative values of $EI$ for most of such columns studied.

A second comparison of Equations 4.8 and 4.9 with theoretical results is shown in Figure 4.6, where histograms of stiffness ratios are plotted for reinforced concrete columns studied with $\rho_s = 1.29$ and 1.72 percent and for composite columns studied with $\rho_s = 1.09$ percent. The overall trends shown in Figure 4.6 are similar to those shown in Figure 4.5. The average stiffness ratios for composite columns, however, dropped significantly from 1.74 to 1.46 and from 1.43 to 1.25 for columns with bending about the minor and major axis, respectively. This change in average stiffness ratios for composite columns was expected because Equation 4.9 ignores the contribution of the longitudinal reinforcing steel to the stiffness of composite columns.

Figures 4.7 (a), (b), and (c) show histograms and statistics of the ratio of theoretical $EI$ to CSA (or ACI) $EI$ of Equation 4.10 ($EI_{th}/EI_{(deq)}$) for all reinforced concrete columns, all composite columns subjected to minor axis bending, and all composite columns subjected to major axis bending, respectively. Figure 4.7 shows that average stiffness ratios greater than unity were obtained for composite columns studied, while reinforced concrete columns had an average stiffness ratio lower than unity. The coefficients of variation associated with stiffness ratios for all three types of columns are again very high, indicating that, for a large number of columns, CSA/ACI $EI$ from Equation 4.10 deviated substantially from the corresponding theoretically computed $EI$. The 1-percentile stiffness ratio of 0.36 for reinforced concrete columns studied is extremely low and is significantly lower than 0.54
obtained for Equation 4.8. The 1-percentile stiffness ratios of 0.72 and 0.85 for composite columns studied with minor and major axis bending, respectively, are also substantially lower than the corresponding values of 1.14 and 0.95 computed for Equation 4.9.

Another comparison of Equation 4.10 with theoretical results is shown in Figure 4.8. In this figure, histograms of stiffness ratios are plotted for reinforced concrete columns studied with $\rho_r = 1.29$ and 1.72 percent and composite columns studied with $\rho_r = 1.09$ percent. The overall trends shown in Figure 4.8 are similar to those shown in Figure 4.7. The average stiffness ratios, however, dropped significantly from 0.93 to 0.78 for reinforced concrete columns, from 1.23 to 1.04 for composite columns subjected to minor axis bending, and from 1.63 to 1.44 for composite columns subjected to major axis bending. This change in average stiffness ratios was expected because Equation 4.10 ignores the contributions of both the longitudinal reinforcing steel and the structural steel section to the stiffness of a column.

All three equations (Equations 4.8, 4.9, and 4.10) use a constant value of the coefficient (0.2 or 0.4) assigned to $E_c I_c$ regardless of different parameters that affect the contribution of the concrete cross-section to the flexural stiffness of slender reinforced concrete and composite columns. This introduces further inaccuracies in CSA/ACI equations.

It is evident from Figures 4.5 through 4.8 and the related discussions that there appears to be a need for modification in the existing CSA/ACI stiffness equations for the design of reinforced concrete and composite columns. Similar conclusions were reached by
Mirza (1990) and Mirza and Tikka (1999a,b).

4.5 DEVELOPMENT OF PROPOSED DESIGN EQUATION FOR SHORT-TERM $EI$

Mirza (1990) pointed out that the effective flexural stiffness ($EI$) of a slender reinforced concrete column is significantly affected by cracking along its length and by inelastic actions in the concrete and reinforcing steel. This is also expected for a composite column, although to a lesser degree, because the structural steel core is expected to stiffen the concrete cross-section. $EI$ is, therefore, a complex function of a number of variables that cannot be readily transformed into a unique and simple analytical expression. The objective in this part of the study was to develop a general equation that could be used to compute the $EI$ of both reinforced concrete and composite columns. This objective was achieved through the following three steps:

1. A format of the proposed $EI$ equation was selected that included variables affecting $EI$ significantly;

2. a multiple linear regression analysis of the generated theoretical stiffness ($EI$) data was conducted to evaluate coefficients related to some of the variables included in the proposed $EI$ equation; and

3. the proposed $EI$ equation was then finalized by curve-fitting to one-percentile values of the generated theoretical stiffness data.

These steps are described in Sections 4.5.1 to 4.5.3.
4.5.1 Format of and Variables Used for Proposed EI Equation

The variables used for the development of the EI equation proposed in this study were divided into two groups: Group (A) variables represented the contribution of concrete to the overall effective stiffness; and Group (B) variables \((E_s I_{rs} \text{ and } E_s I_{ss})\) represented the contribution of reinforcing steel and structural steel to the overall effective stiffness of composite and reinforced concrete columns. Note that, for this study, reinforced concrete columns were treated as a special case of encased composite columns. Therefore, a modified version of Equations 4.8 and 4.9 which can be used for both reinforced concrete and composite columns, will take the following form:

\[
EI = \alpha_c E_c (I_g - I_{ss}) + \alpha_{ss} E_s I_{ss} + \alpha_{rs} E_s I_{rs}
\]  

(4.11)

in which \(\alpha_c\), \(\alpha_{ss}\), and \(\alpha_{rs}\) are dimensionless reduction factors (effective stiffness factors) for concrete, structural steel and longitudinal reinforcing steel, respectively. \(\alpha_c\) represents the effects of several variables that influence the contribution of concrete to the overall column stiffness and can be a linear or nonlinear function of these variables. Hence, Equation 4.11 is a general equation for \(EI\), which can be developed as a linear or nonlinear equation. If \(\alpha_c\) is taken as a linear function of \(x_1\) and \(x_2\) and assumed to be equal to \((\alpha_k + \alpha_1 x_1 + \alpha_2 x_2)\) Equation 4.11 becomes:

\[
EI = (\alpha_k + \alpha_1 x_1 + \alpha_2 x_2) E_c (I_g - I_{ss}) + \alpha_{ss} E_s I_{ss} + \alpha_{rs} E_s I_{rs}
\]  

(4.12)

In Equation 4.12 \(\alpha_k\) is a constant (equivalent to the intercept of a simple linear equation) and
the remaining $\alpha$ values are dimensionless factors corresponding to independent variables $x_i$, $x_j$, $E_s J_{sy}$, and $E_c J_{cr}$. $E_c$ is the modulus of elasticity of concrete; $I_g$ is the moment of inertia of the gross concrete cross-section taken about its centroidal axis; $E_s$ is the modulus of elasticity of steel; $I_r$ is the moment of inertia of the longitudinal reinforcing steel bars taken about the centroidal axis of the gross concrete cross-section; and $I_s$ is the moment of inertia of the structural steel section about the centroidal axis of the composite cross-section.

The Group A variables used in this study (Equation 4.12) were subdivided into two subgroups: $x_i$ represents end eccentricity ratio $e/h$ or axial load index $P_u/P_o$, in which $P_u$ is the factored axial load acting on the slender column and $P_o$ is the pure axial load capacity of the cross-section; and $x_j$ stands for slenderness ratio $\ell/h$ or $r$, where $r$ is the radius of gyration taken as $0.3h$ for reinforced concrete columns and calculated according to the CSA A23.3-94 Equation (10-28) or ACI 318-99 Equation (10-13) for composite columns. As noted, the variables $e/h$, $P_u/P_o$, $\ell/h$ and $r$, were considered in the development of the proposed $EI$ expression. This was done because earlier studies investigated a large number of variables and concluded that only these variables had significant effects on the contribution of the concrete cross-section to the stiffness of slender reinforced concrete and composite steel-concrete columns (Mirza 1990; Mirza and Tikka 1999a,b). Correlation analyses of the theoretical stiffness data indicated that $e/h$ and $P_u/P_o$ are strongly correlated variables as are $\ell/h$ and $r$. The same analyses also indicated that $e/h$, or $e/h$ combined with $\ell/h$, had the most significant effect on the contribution of the concrete cross-section to the stiffness of slender reinforced concrete and composite columns. Similar findings were
reported by Mirza (1990) and Mirza and Tikka (1999a,b). Hence, Equation 4.12, will assume the following form:

\[ EI = (\alpha_k + \alpha_1 e/h + \alpha_2 t/h) E_c (I_g - I_{ss}) + \alpha_{ss} E_s I_{ss} + \alpha_{rs} E_s I_{rs} \]  

(4.13)

To simplify the analysis of the theoretical stiffness data, Equation 4.13 was non-dimensionalized by dividing both sides by \( E_c (I_g - I_{ss}) \). The non-dimensionalized linear equation for \( EI \) is:

\[ \frac{EI}{E_c (I_g - I_{ss})} = \alpha_k + \frac{\alpha_1 e/h + \alpha_2 t/h + \alpha_{ss} E_s I_{ss}}{E_c (I_g - I_{ss})} + \frac{\alpha_{rs} E_s I_{rs}}{E_c (I_g - I_{ss})} \]  

(4.14)

For reinforced concrete columns \( I_{ss} = 0 \) and Equation 4.14 becomes:

\[ \frac{EI}{E_c I_g} = \alpha_k + \frac{\alpha_1 e/h + \alpha_2 t/h + \alpha_{rs} E_s I_{rs}}{E_c I_g} \]  

(4.15)

Note that \( E_c \) and \( E_s \) values in Equations 4.14 and 4.15 were taken the same as would be used by a designer and are given after Equation 4.10.

4.5.2 Regression Analysis of Theoretical Stiffness Data

A multiple linear regression analysis of the simulated theoretical stiffness data for reinforced concrete and composite columns was conducted, using Equations 4.14 and 4.15. The \( EI \) values in Equations 4.14 and 4.15 were taken from the theoretical flexural stiffnesses, simulated using Equation 4.4, and coefficients \( \alpha_k, \alpha_1, \alpha_2, \alpha_{ss}, \) and \( \alpha_{rs} \) were computed from the regression analysis. The accuracy of a regression \( EI \) equation was based on the standard error \( S_e \), a measure of sampling variability, and the multiple correlation coefficient \( R_c \), an index of
the relative strength of the relationship. The smaller the value of $S_e$, the smaller is the sampling variability of the regression equation. An $R_c$ value equal to zero signifies no correlation, and $R_c = \pm 1.0$ indicates 100 percent correlation. $R_c$ values greater than +1.0 and less than -1.0 are not possible. Note that $S_e$ in this study was computed for $\alpha_e$. The following regression equations were developed for reinforced concrete columns:

$$EI = (0.342 - 0.279 \frac{e}{h} + 0.0013 \frac{t}{h}) E_c J_g + 0.844 E_s J_{rs}$$

(4.16)

(n = 11550; $S_e = 0.063$; $R_c = 0.880$)

$$EI = (0.367 - 0.279 \frac{e}{h}) E_c J_g + 0.844 E_s J_{rs}$$

(4.17)

(n = 11550; $S_e = 0.063$; $R_c = 0.877$)

The following regression equations were obtained for composite columns subjected to minor axis bending:

$$EI = (0.352 - 0.209 \frac{e}{h} + 0.0014 \frac{t}{h}) E_c J_g + 0.780 E_s J_{ss} + 0.820 E_s J_{rs}$$

(4.18)

(n = 11880; $S_e = 0.048$; $R_c = 0.911$)

$$EI = (0.380 - 0.209 \frac{e}{h}) E_c J_g + 0.780 E_s J_{ss} + 0.820 E_s J_{rs}$$

(4.19)

(n = 11880; $S_e = 0.049$; $R_c = 0.907$)

The following regression equations were developed for composite columns subjected to major axis bending:

$$EI = (0.332 - 0.202 \frac{e}{h} + 0.0028 \frac{t}{h}) E_c J_g + 0.818 E_s J_{ss} + 0.802 E_s J_{rs}$$

(4.20)

(n = 11880; $S_e = 0.051$; $R_c = 0.965$)
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\[ EI = (0.388 - 0.202e/h) E_s I_{gs} + 0.818 E_s I_{ss} + 0.802 E_s I_{rs} \]

\[ (n = 11880; S_e = 0.054; R_c = 0.959) \]

All six equations (Equations 4.16 through 4.21) display similar trends. The low standard error \( (S_e = 0.05 \text{ to } 0.06 \text{ approximately}) \) and high correlation coefficient \( (R_c = 0.88 \text{ to } 0.90 \text{ approximately}) \) indicate the validity of these equations. The equations show that with an increase in \( e/h \) ratio, there is a corresponding decrease in \( EI \) for a column. This is expected, because an increase in \( e/h \) means a corresponding increase in bending moment and the outer fibre tension stresses, resulting in more cracking of the column. Equations 4.16, 4.18, and 4.20 also indicate that for an increase in \( \theta h \) ratio, there is an increase in \( EI \). Mirza (1990) suggests that this is perhaps because the cracks are likely to be more widely spaced in a longer column with more concrete in between the cracks contributing to the stiffness of the column. However, he provides no analytical or experimental evidence for this assertion (Mirza 1990). Note that the theoretical procedure used in this study assumes that the concrete between the cracks does not provide additional strength or stiffness in the cracked element(s) of a column.

The values of the coefficient \( \alpha \), associated with \( e/h \) in Equations 4.18 to 4.21 for composite columns are about 75 percent of those in Equations 4.16 and 4.17 for reinforced concrete columns. It appears that the presence of the structural steel section tends to increase the contribution of the concrete cross-section to the effective flexural stiffness \( EI \) of the column. The coefficients \( \alpha_{ss} \) and \( \alpha_{rs} \) related to \( E_s I_{ss} \) and \( E_s I_{rs} \) in Equations 4.16 to 4.21 are less than 1.0 and indicate "softening" due to the elastic-plastic nature of the stresses.
developed in the structural steel and the reinforcing steel near ultimate load. Similar trends were noted by MacGregor, Oelhafen and Hage (1975), Mirza (1990), and Mirza and Tikka (1999a,b).

4.5.3 Proposed Design Equation

The purpose of conducting regression analyses of the theoretical stiffness data was to estimate values of coefficients related to some of the variables that affect the flexural stiffness of reinforced concrete and composite columns. Equations 4.16 to 4.21 show that values of $\alpha_r$ range approximately from 0.80 to 0.84 and those for $\alpha_s$ range approximately from 0.78 to 0.82. Hence, a value of 0.8 for both $\alpha_r$ and $\alpha_s$ appears reasonable and was used for developing a more "refined" $EI$ equation through curve-fitting to one-percentile values of the simulated theoretical stiffness data. Substituting 0.8 for both $\alpha_r$ and $\alpha_s$ in Equation 4.11 yields the following expression for reinforced concrete and composite columns:

$$EI = \alpha_c E_c (I_g - I_{ss}) + 0.8 E_s (I_{ss} + I_{rs})$$  \hspace{1cm} (4.22)

where $\alpha_c$ is a function of $e/h$, or $e/h$ combined with $\theta/h$, depending on whether one ($e/h$) or both ($e/h$ and $\theta/h$) of these variables are included in the analysis. Note that for reinforced concrete columns, $I_{ss}$ is taken equal to zero. Solving Equation 4.22 for $\alpha_c$ gives the following equation:

$$\alpha_c = \frac{EI - 0.8 E_s (I_{ss} + I_{rs})}{E_c (I_g - I_{ss})}$$  \hspace{1cm} (4.23)

Again, $E_c$ and $E_s$ values in Equation 4.23 were taken the same as given after Equation 4.10.
The theoretical values of $\alpha_c$ were computed for all simulated reinforced concrete and composite columns, for which $e/h$ ranged from 0.1 to 1.0, by substituting the $EI$ values obtained from Equation 4.4 into Equation 4.23. The averages, five-percentiles, and one-percentiles of $\alpha_c$ values so computed are plotted against the end eccentricity ratio in Figure 4.9. As $\alpha_c$ values for reinforced concrete columns were affected by $\rho_{rs}$, the data for such columns with $\rho_{rs} \leq 2$ percent and those with $\rho_{rs} > 2$ percent are plotted separately in Figure 4.9. Figure 4.9 suggests that, for the computation of $\alpha_c$, the columns can be divided into three categories: reinforced concrete columns with $\rho_{rs} \leq 2$ percent, reinforced concrete columns with $\rho_{rs} > 2$ percent, and composite columns. The plots in Figure 4.9 also indicate that a nonlinear equation for $\alpha_c$ is needed to fit the data.

It is a widely accepted and reasonable practice to use either the five-percentile or one-percentile values for the development of design equations. A nonlinear equation for $\alpha_c$ was visually fitted, with the aid of a spreadsheet, as close as possible to the one-percentile values of $\alpha_c$ shown in Figure 4.9. Using $e/h$ initially as a variable, the following equation for $\alpha_c$ was obtained:

$$
\alpha_c = 0.5 - 3.5 \frac{e}{h} \left( \frac{1}{1 + \beta \frac{e}{h}} \right) \tag{4.24}
$$

where $\beta = 7.0$ for reinforced concrete columns when $\rho_{rs} \leq 2\%$;
$\beta = 8.0$ for reinforced concrete columns when $\rho_{rs} > 2\%$; and
$\beta = 9.5$ for composite columns subjected to minor or major axis bending.

In Figure 4.10, Equation 4.24 is superimposed on the plots of theoretical values of $\alpha_c$ taken
from Figure 4.9. As expected, Equation 4.24 shows an excellent agreement with one-percentile values of theoretical $\alpha_c$, but is a conservative representation of five-percentile and average values of theoretical $\alpha_c$.

Equation 4.24 was then modified to include the effect of the slenderness ratio. Equations 4.16, 4.18, and 4.20 show that values of the coefficient $\alpha_2$ associated with $t/h$ range approximately from 0.0015 to 0.003. The lowest variability in $EI$ for the three types of columns studied was attained using $\alpha_2$ of 0.003. Hence, $\alpha_2 = 0.003$ was used to modify Equation 4.24 for including the effect of $t/h$. Based on a statistical analysis of the theoretical $\alpha_c$ values for all simulated columns in which $e/h$ ranged from 0.1 to 1.0, the following equation was selected for $\alpha_c$:

$$\alpha_c = 0.47 - 3.5 \frac{e}{h} \left( \frac{1}{1 + \beta \frac{e}{h}} \right) + 0.003 \frac{t}{h}$$ \hspace{1cm} (4.25)

where $\beta$ values are the same as those given for Equation 4.24. It is interesting to note that Equation 4.25 reduces to Equation 4.24 at $t/h = 10$.

Substituting $\alpha_c$ from Equation 4.25 into Equation 4.22 gives the following expression for $EI$:

$$EI = \left( 0.47 - 3.5 \frac{e}{h} \left( \frac{1}{1 + \beta \frac{e}{h}} \right) + 0.003 \frac{t}{h} \right) E_c (I_g - I_{ss}) + 0.8 E_s (I_{ss} + I_{rs})$$ \hspace{1cm} (4.26)

where $\beta$ values are the same as given for Equation 4.24. Equation 4.26 is the proposed
design expression for short-term $EI$ of slender reinforced concrete and composite encased columns and is subjected to the following limitations: $e/h \geq 0.1$; $t/h \leq 30$; $\rho_{rs} \geq 1\%$; and if applicable, $\rho_{rs} \geq 4\%$. When $e/h$ is less than 0.1, use $e/h = 0.1$ in Equation 4.26. Note that Equation 4.26 does not apply when $\rho_{rs} = 0$.

Figure 4.11 shows comparisons of the proposed design equation with theoretical stiffnesses ($EI_{(th)}$) of all simulated columns. Figure 4.12 presents similar comparisons for columns with light longitudinal reinforcing bars ($\rho_{rs} = 1.29$ and 1.72 percent for reinforced concrete columns and $\rho_{rs} = 1.09$ percent for composite columns). Note that, for these figures, $EI_{(des)}$ was computed from Equation 4.26. Also note that $EI_{(des)}$ and $EI_{(th)}$ in Figures 4.11 and 4.12 have been non-dimensionalized by dividing them by $E_cI_g$ for reinforced concrete columns and by dividing them by $E_c(I_g - I_s)$ for composite columns. Most of the column stiffnesses plotted in these figures fall in a narrow band close to or on the conservative side of the line of equality, indicating that Equation 4.26 computes $EI$ with reasonable accuracy.

### 4.6 ANALYSIS AND DISCUSSION OF SIMULATED RESULTS

Frequency histograms and other statistical data presented in this section were prepared for the stiffness ratios ($EI_{(th)}/EI_{(des)}$) using different design equations. For computing the stiffness ratio, $EI_{(th)}$ was taken as the simulated theoretical stiffness, and $EI_{(des)}$ was computed from one of the CSA/ACI design equations (Equation 4.8, 4.9, or 4.10) or from the proposed design equation (Equation 4.26).
4.6.1 Overview of Stiffness Ratio Statistics

Two sets of frequency histograms and statistical data were prepared using the proposed design equation (Equation 4.26). Figure 4.13 shows histograms and statistics of stiffness ratios for all the columns studied, whereas Figure 4.14 presents histograms and statistics of strength ratios for columns where light longitudinal reinforcing steel was provided. A comparison of Figures 4.13 and 4.14 with Figures 4.5 to 4.8 plotted for CSA/ACI design equations can be summarized as follows:

1. The coefficients of variation of stiffness ratios for the proposed design equation are significantly lower than the values obtained for the CSA/ACI design equations.

2. The average stiffness ratios for CSA/ACI design equations that apply to composite columns tend to be more conservative than those for the proposed design equation.

3. The one-percentile values obtained using the CSA/ACI design equations for reinforced concrete columns are very significantly lower than those for the proposed design equation.

These trends can be seen more clearly by comparing cumulative frequency curves of stiffness ratios for the proposed and CSA/ACI design equations plotted on a normal probability paper in Figure 4.15. The curves in Figure 4.15 were prepared from the data for all of the columns studied. These curves indicate that the proposed design equation produces consistently least variable results for all types of columns studied, whereas CSA/ACI design equations produce stiffness ratios that are, in a large number of cases, significantly lower than 1.0 for reinforced concrete columns and significantly higher than 1.0 for composite
4.6.2 Effects of Variables on Stiffness Ratios

The effects of the end eccentricity ratio \((e/h)\), axial load ratio \((P_u/P_o)\), slenderness ratio \((t/h)\) and longitudinal reinforcement ratio \((\rho_r)\) on the average and one-percentile values of stiffness ratios \((EI_{thy}/EI_{ides})\), obtained from the proposed design equation (Equation 4.26) and the CSA/ACI design equations (Equations 4.8, 4.9, and 4.10), are shown in Figures 4.16, 4.17, and 4.18. These figures were plotted, respectively, from the data for 11,550 reinforced concrete columns, 11,880 composite columns subjected to minor axis bending, and 11,880 composite columns subjected to major axis bending. The following conclusions can be drawn from Figures 4.16 to 4.18:

(1) The average and one-percentile stiffness ratios for the proposed design equation (Equation 4.26) are not significantly affected by any of the variables investigated. On the other hand, such values for the CSA/ACI expressions (Equations 4.8, 4.9, and 4.10), are significantly affected by most of the same variables. The only exceptions are the one-percentile values of stiffness ratios, obtained from Equation 4.9, for composite columns subjected to major axis bending and plotted against \(e/h\), \(P_u/P_o\), and \(\rho_r\) in Figures 4.18(a), (b), and (c). In Figures 4.18(a), (b), and (c), the one-percentile stiffness ratios obtained from Equation 4.9 (CSA/ACI design equation) are very similar to the results from the proposed design equation (Equation 4.26). This inconsistency for the CSA/ACI equations is expected because these equations do not include some of the variables studied.
The average and one-percentile stiffness ratios for the CSA/ACI design equations are very high in some cases and very low in other cases. This is most evident for reinforced concrete columns. Note that the CSA/ACI simple equation (Equation 4.10) produces the lowest one-percentile stiffness ratios in most cases for reinforced concrete as well as for composite columns under minor axis bending. These trends for the CSA/ACI design equations are similar to those reported earlier by Mirza (1990), and Mirza and Tikka (1999a,b) and are expected, because the CSA/ACI expressions were developed originally for reinforced concrete columns using data with low end eccentricities. The CSA/ACI expressions for reinforced concrete columns were then applied, without further investigation, to composite columns with minor modification (Equation 4.9) or with no modification (Equation 4.10).

The proposed design equation, on the average, computes the effective flexural stiffnesses close to the theoretical values. The one-percentile values for the proposed design equation are above 0.8 and 0.85 for reinforced concrete and composite columns, respectively, for most of the cases studied.

As the end eccentricity ratio $e/h$ was found to have the most significant effect on the effective flexural stiffness of columns, histograms and related statistics of stiffness ratios at each value of $e/h$ studied were closely examined for proposed and CSA/ACI design equations. Figures 4.19, 4.20, and 4.21 present such histograms and related statistics prepared using the proposed design equation for reinforced concrete columns, composite columns under minor axis bending, and composite columns under major axis bending.
respectively. Figures 4.19 to 4.21 clearly demonstrate that the end eccentricity ratio has virtually no affect on the histograms and related statistics of stiffness ratios for individual $e/h$ ranging from 0.1 to 1.0. This indicates that the proposed design equation adequately addresses the effect of $e/h$. Note that similar plots prepared using the CSA/ACI equations, but not presented here, demonstrated substantial effects of the end eccentricity ratio.

4.6.3 Stiffness Ratios Produced by Proposed Design Equation for Usual Columns

Mirza and MacGregor (1982) determined that the end eccentricity ratio for columns in reinforced concrete buildings usually ranged from 0.1 to 0.65. In a survey of 22,000 columns, conducted in the late 1960's by MacGregor, Breen and Pfrang (1970), 99 percent had a slenderness ratio $l/h \leq 20$. Therefore, the usual columns in the study were defined as those for which $e/h = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ and $l/h = 10, 15, 20$.

The average and minimum values of the stiffness ratios ($E_{I_{th}}/E_{I_{def}}$) computed using the proposed design equation (Equation 4.26) are plotted against $e/h$ in Figures 4.22, 4.23, and 4.24 for usual reinforced concrete columns, usual composite columns under minor axis bending, and usual composite columns with major axis bending, respectively. Two slenderness ratios are used in developing these figures: (a) $l/h = 10$ and (b) $l/h = 20$. The one-percentile values are not plotted in these figures because the sample size for each point plotted ranges from 30 to 108 and minimum values represent 0.93 to 3.33 percentiles.

Based on Figures 4.22 to 4.24, the following conclusions appear to be valid for columns with $e/h = 0.1$ to 0.7, $l/h = 10$ to 20, and the reinforcing steel and structural steel ratios as given in the figures:
(1) The average and minimum stiffness ratios produced by Equation 4.26 may be taken as at least 1.0 and 0.8, respectively, for reinforced concrete columns (Figure 4.22) and 1.0 and 0.85, respectively, for composite columns (Figures 4.23 and 4.24).

(2) The proposed design equation (Equation 4.26) does not introduce significant variations in stiffness ratios due to changes in $e/h$ and $v/h$ for reinforced concrete columns (Figure 4.22) and due to changes in $e/h$, $v/h$ and $p$ for composite columns (Figures 4.23 and 4.24).

4.6.4 Stability Resistance Factor for Proposed Design Equation

MacGregor (1976) explained that the one-percentile strength ratios can be used to estimate the resistance factor $\phi$. The one-percentile stiffness ratios, for all reinforced concrete columns and for reinforced concrete columns with $\rho_s < 2$ percent obtained from the proposed design equation are 0.85 and 0.83, respectively (Figures 4.13(a) and 4.14(a)). In addition, the plots of the effects of the major variables on stiffness ratios for reinforced concrete columns (Figure 4.16) indicate that in almost all cases the one-percentile values exceed 0.8. Therefore, a value of the stability resistance factor $\phi_m = 0.8$ is proposed for use when computing the critical buckling strength ($P_{cr}$) of reinforced concrete columns based on the proposed $EI$ equation (Equation 4.26). This value of $\phi_m$ is higher than the current CSA/ACI value of $\phi_m = 0.75$ used for computing the critical buckling strength in the moment magnifier approach for slender reinforced concrete columns. Note that the current CSA/ACI stability resistance factor for reinforced concrete columns was developed by Mirza and McGregor (1987) and Mirza, Lee and Morgan (1987).
The one-percentile stiffness ratios obtained from the proposed design equation (Equation 4.26) for all composite columns subjected to minor axis bending and for all composite columns subjected to major axis bending are 0.89 and 0.88, respectively (Figures 4.13(b) and (c)). The corresponding values for such columns in which \( r_s = 1.09 \) percent are 0.87 and 0.89 (Figures 4.14(b) and (c)). An examination of plots showing the effects of major variables (Figures 4.17 and 4.18) indicates that the one-percentile stiffness ratios exceed 0.85 in nearly all cases. Therefore, a value of \( \phi_m = 0.85 \) is proposed for use when computing the critical buckling strength \( (P_{cr}) \) of composite columns based on the proposed \( EI \) equation (Equation 4.26). This value of \( \phi_m \) is significantly higher than the current CSA/ACI value of \( \phi_m = 0.75 \) specified for computing \( P_{cr} \) in the moment magnifier approach for slender composite columns. Note that CSA A23.3-94 (1994) and ACI 318-99 (1999) use the same stability resistance factor for both reinforced concrete and composite columns.

Although more complex probabilistic analyses should be used in the future evaluation of the stability resistance factor \( \phi_m \), the results of this simple analysis suggest that \( \phi_m \) factors of 0.8 and 0.85 could be used for computing \( \phi_m P_{cr} \) based on the proposed \( EI \) expression (Equation 4.26) for reinforced concrete and composite columns, respectively.

### 4.7 DESIGN APPLICATIONS

#### 4.7.1 Columns in Frames Subjected to Sustained Loads

Consideration for the effects of sustained loads can be accounted for by applying the sustained load factor \( \beta_d \) to Equation 4.22 in a similar manner as is currently applied to the CSA/ACI equation for composite columns (Equation 4.6). Applying \( \beta_d \) to the concrete
contribution to $EI$ in Equation 4.22 results in the following expression:

$$EI = \frac{\alpha_c E_c (I_g - I_{ss})}{1 + \beta_d} + 0.8 E_s (I_{ss} + I_{rs})$$

(4.27)

in which $\alpha_c$ is computed from:

$$\alpha_c = \left( 0.47 - 3.5 \frac{e}{h} \left( \frac{1}{1 + \beta \frac{e}{h}} \right) + 0.003 \frac{\ell}{h} \right) \geq 0$$

and $\beta$ is computed from:

$\beta = 7.0$ for reinforced concrete columns when $\rho_s \leq 2\%$;

$\beta = 8.0$ for reinforced concrete columns when $\rho_s > 2\%$; and

$\beta = 9.5$ for composite columns subjected to minor or major axis bending;

and the following limitations must be satisfied:

$e/h \geq 0.1$;

$\beta/h \leq 30$;

$\rho_s \geq 1\%$; and, if applicable,

$\rho_{ss} \geq 4\%$.

Equation 4.27 is the proposed design expression for slender reinforced concrete and composite columns subjected to sustained loads. Note that for reinforced concrete columns, $I_{ss}$ is equal to zero. When $e/h$ is less than 0.1, use $e/h = 0.1$ in Equation 4.27. In Equation 4.2, $e$ is the end eccentricity = $M_s/P_s$ in which $M_s$ is the larger of the column end moments; and $\ell = \text{unsupported height of the column}$; and $E_c$, $E_s$ and $\beta_d$ values are the same as those given in ACI 318-99 or CSA A23.3-94. For short-term loads ($\beta_d = 0$), Equation 4.27 reduces back to Equation 4.26.

The effects of moment gradient and end restraints on columns in braced frames are
accounted for in the CSA/ACI moment magnifier approach through the use of the equivalent uniform moment diagram factor $C_m$ and the effective length factor $K$ and are discussed in the forthcoming chapters. The columns in this part of the study were pin-ended and subjected to equal and opposite end moments causing symmetrical single curvature bending, with $C_m = K = 1.0$ (Figure 4.1), and represent the most critical condition for columns in braced frames. It is suggested that, for the design of reinforced concrete and composite columns, the existing CSA/ACI design equations for $EI$ (Equations 4.5, 4.6, and 4.7) be replaced by Equation 4.27. It is also suggested that design aids, similar to Figures 4.25 and 4.26, for graphical evaluation of $\alpha_c$ be placed in the Commentary of the Code to speed up the design process.

At a glance the proposed $EI$ equation may look cumbersome. Additionally, it could be argued that the determination of $e/h$ values from a conventional (first-order) structural analysis may not be accurate enough to justify the additional complexity of Equation 4.27. However, if Figure 4.25 or 4.26 is employed, the proposed equation is no more complicated than the current CSA/ACI $EI$ equation. Furthermore, the use of a spreadsheet will greatly simplify computations regardless of whether the Code or proposed equation is used.

4.7.2 Preliminary Sizing of Column Cross-Sections

For design purposes, an approximate initial estimate of member sizes, $\rho_r$ and $\rho_x$, is required before the design of a structure can be completed through final (more accurate) structural analysis and design. Therefore, the first step in the design of a structure is the preliminary sizing of members. One of the main concerns in preliminary sizing is to ensure
that the column is stocky enough to remain stable at factored loads. The ratio of the factored axial load acting on a column to the critical buckling strength of the column, given as $P_u/P_{cr}$, is used by CSA/ACI provisions (Equation 1.1) to evaluate the second order effects in columns in nonsway frames. The $P_u/P_{cr}$ ratio can be used for initial sizing of columns if a limiting value for $P_u/P_{cr}$ can be established.

For computing the theoretical critical load ratio ($P_{u(th)}/P_{cr(th)}$), $P_{u(th)}$ was taken as the computed theoretical axial load strength and $P_{cr(th)}$ was calculated by substituting the computed theoretical effective flexural stiffness ($EI_{(th)}$) into Equation 4.28:

$$P_{cr(th)} = \frac{\pi^2 EI_{(th)}}{\ell^2}$$  

(4.28)

The frequency histograms, shown in Figures 4.27 (a), (b), and (c), for theoretical critical load ratio $P_{u(th)}/P_{cr(th)}$ represent 10,500 reinforced concrete columns, 10,800 composite columns subjected to minor axis bending, and 10,800 composite columns subjected to major axis bending, respectively, in which $e/h \geq 0.1$. The histograms show that for more than 66 percent of these columns, the theoretical critical load ratio is less than 0.4; and for more than 76 percent of the these columns the theoretical critical load ratio is less than 0.45. A similar set of histograms for theoretical critical load ratios shown in Figures 4.28 (a), (b), and (c), represents 4410 usual reinforced concrete columns, 4536 usual composite columns subjected to minor axis bending, and 4536 usual composite columns subjected to major axis bending, respectively ( $e/h = 0.1$ to 0.7; $\ell/h \leq 20$). The histograms show that for more than 81 percent of the usual columns, the theoretical critical load ratio is less than 0.4; and for more than 90
percent of the usual columns, the theoretical critical load ratio is less than 0.45. Therefore, for preliminary sizing of the cross-section, an upper limit of 0.4 on the \( P_u/P_{cr} \) ratio appears reasonable. \( P_u \) and \( e/h \) can be estimated using approximate methods of structural analysis and \( EI \) from Equation 4.27 can be used for estimating \( P_{cr} \).
### Table 4.1  Specified properties of reinforced concrete columns studied*

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'_c$ MPa (psi)</td>
<td>20.7; 27.6; 34.5; 41.4; 55.2 (3000; 4000; 5000; 6000; 8000)</td>
<td>5</td>
</tr>
<tr>
<td>$f_{ys}$ Mpa (psi)</td>
<td>414 (60,000)</td>
<td>1</td>
</tr>
<tr>
<td>$C_e$ mm (in.)</td>
<td>38; 48; 64 (1.5; 1.875; 2.5)</td>
<td>3</td>
</tr>
<tr>
<td>$\delta h$</td>
<td>10; 15; 20; 25; 30</td>
<td>5</td>
</tr>
<tr>
<td>$\varepsilon/h$</td>
<td>0.05; 0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 1.0</td>
<td>11</td>
</tr>
<tr>
<td>$\rho_n$ and steel arrangement</td>
<td>See Figure 4.3 for combinations of steel ratios and arrangements</td>
<td>14</td>
</tr>
</tbody>
</table>

* Total number of simulated columns equals $(5 \times 1 \times 3 \times 5 \times 11 \times 14 =) 11,550$ with each column having a different combination of specified properties shown above. 10mm (0.375 in.) diameter lateral ties spaced in conformance with the requirements of CSA A23-94 Clause 7.6.5.2 and ACI 318-95 Clause 7.10.5.2.

### Table 4.2  Average values of variables used for computing theoretical strength and stiffness of reinforced concrete columns.

#### (a) Concrete

<table>
<thead>
<tr>
<th>Specified Compressive Strength $f'_c$ MPa (psi)</th>
<th>In-place Concrete in Members (Columns)</th>
<th>Compressive Strength* $f'_c$ MPa (psi)</th>
<th>Modulus of Rupture $f_r$ MPa (psi)</th>
<th>Elastic Modulus $E_c$ MPa (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.7 (3000)</td>
<td>19.1 (2768)</td>
<td>2.8 (401)</td>
<td>20,382 (2956)</td>
<td></td>
</tr>
<tr>
<td>27.6 (4000)</td>
<td>23.4 (3388)</td>
<td>3.1 (446)</td>
<td>22,478 (3260)</td>
<td></td>
</tr>
<tr>
<td>34.5 (5000)</td>
<td>27.7 (4013)</td>
<td>3.4 (487)</td>
<td>24,388 (3537)</td>
<td></td>
</tr>
<tr>
<td>41.4 (6000)</td>
<td>32.0 (4641)</td>
<td>3.6 (525)</td>
<td>26,167 (3795)</td>
<td></td>
</tr>
<tr>
<td>55.2 (8000)</td>
<td>40.7 (5904)</td>
<td>4.1 (594)</td>
<td>29,393 (4263)</td>
<td></td>
</tr>
</tbody>
</table>

#### (b) Reinfocing Steel

<table>
<thead>
<tr>
<th>Specified Yield Strength $f_{ys}$ MPa (psi)</th>
<th>Static Yield Strength $f_r$ MPa (psi)</th>
<th>Elastic Modulus $E_s$ MPa (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>414 (60,000)</td>
<td>461 (66,800)</td>
<td>199,955 (29,000)</td>
</tr>
</tbody>
</table>

* Average compressive strength of in-place concrete in membrane is based on a rate of loading to failure in 60 minutes.
Table 4.3  Specified properties of composite steel-concrete columns studied.*

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'_c$ MPa (psi)</td>
<td>27.6; 34.5; 41.4; 55.2 (4000; 5000; 6000; 8000)</td>
<td>4</td>
</tr>
<tr>
<td>$f_{pu}$ MPa (psi)</td>
<td>248; 303; 345 (36,000; 44,000; 50,000)</td>
<td>3</td>
</tr>
<tr>
<td>$\rho_n$ (percent)</td>
<td>1.09; 1.96; 3.17</td>
<td>3</td>
</tr>
<tr>
<td>Structural Steel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Section</td>
<td>$\rho_n$ (percent)</td>
<td>6</td>
</tr>
<tr>
<td>W310 x 253</td>
<td>10.33</td>
<td></td>
</tr>
<tr>
<td>W310 x 179</td>
<td>7.29</td>
<td></td>
</tr>
<tr>
<td>W310 x 107</td>
<td>4.36</td>
<td></td>
</tr>
<tr>
<td>W250 x 167</td>
<td>6.80</td>
<td></td>
</tr>
<tr>
<td>W250 x 101</td>
<td>4.13</td>
<td></td>
</tr>
<tr>
<td>W200 x 100</td>
<td>4.07</td>
<td></td>
</tr>
<tr>
<td>$\delta h$</td>
<td>10; 15; 20; 25; 30</td>
<td>5</td>
</tr>
<tr>
<td>$e/h$</td>
<td>0.05; 0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 1.0</td>
<td>11</td>
</tr>
</tbody>
</table>

* The number of simulated composite columns subjected to major axis bending equals $(4 \times 3 \times 3 \times 6 \times 5 \times 11 =) 11,880$ with each column having a different combination of specified properties shown above. Another identical set of 11,880 simulated composite columns was subjected to minor axis bending. Hence, the total number of simulated composite columns equals $2 \times 11,800 = 23,760$.

Note: Imperial equivalents of sections noted above are, W12 x 170, W12 x 120, W12 x 72, W10 x 112, W10 x 68 and, W8 x 67, respectively.

Table 4.4  Average values of variables used for computing theoretical strength and stiffness of composite columns.

(a) Concrete

<table>
<thead>
<tr>
<th>Specified Compressive Strength $f'_c$ MPa (psi)</th>
<th>In-place Concrete in Members (Columns)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Compressive Strength $f_c$ MPa (psi)</td>
</tr>
<tr>
<td>27.6 (4000)</td>
<td>23.4 (3388)</td>
</tr>
<tr>
<td>34.5 (5000)</td>
<td>27.7 (4013)</td>
</tr>
<tr>
<td>41.4 (6000)</td>
<td>32.0 (4641)</td>
</tr>
<tr>
<td>55.2 (8000)</td>
<td>40.7 (5904)</td>
</tr>
</tbody>
</table>
### Table 4.4 - continued

#### (b) Reinforcing Steel

<table>
<thead>
<tr>
<th>Specified Yield Strength $f_{yw}$ (MPa)</th>
<th>Static Yield Strength $f_{ys}$ (MPa)</th>
<th>Elastic Modulus $E_s$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>414 (60,000)</td>
<td>461 (66,800)</td>
<td>199,955 (29,000)</td>
</tr>
</tbody>
</table>

#### (c) Structural steel strength**

<table>
<thead>
<tr>
<th>Specified Yield Strength $f_{yw}$ (MPa)</th>
<th>Static Yield Strength</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Web $f_{yw}$ (MPa)</td>
</tr>
<tr>
<td></td>
<td>Flange $f_{yw}$ (psi)</td>
</tr>
<tr>
<td>248 (36,000)</td>
<td>271 (39,240)</td>
</tr>
<tr>
<td></td>
<td>0.95 $f_{yw}$</td>
</tr>
<tr>
<td>303 (44,000)</td>
<td>331 (47,960)</td>
</tr>
<tr>
<td></td>
<td>0.95 $f_{yw}$</td>
</tr>
<tr>
<td>345 (50,000)</td>
<td>376 (54,500)</td>
</tr>
<tr>
<td></td>
<td>0.95 $f_{yw}$</td>
</tr>
</tbody>
</table>

#### (d) Residual stresses in structural steel

<table>
<thead>
<tr>
<th>Steel Shape</th>
<th>Flange Tip*** (MPa)</th>
<th>Flange-Web Juncture (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>W310 x 253 (W12 x 170)</td>
<td>-126.6 (-18,367)</td>
<td>81.3 (11,792)</td>
</tr>
<tr>
<td>W310 x 179 (W12 x 120)</td>
<td>-124.0 (-17,983)</td>
<td>77.7 (11,267)</td>
</tr>
<tr>
<td>W310 x 107 (W12 x 72)</td>
<td>-123.4 (-17,896)</td>
<td>76.9 (11,152)</td>
</tr>
<tr>
<td>W250 x 167 (W10 x 112)</td>
<td>-128.1 (-18,576)</td>
<td>83.4 (12,089)</td>
</tr>
<tr>
<td>W250 x 101 (W10 x 68)</td>
<td>-126.8 (-18,384)</td>
<td>81.5 (11,816)</td>
</tr>
<tr>
<td>W200 x 100 (W8 x 67)</td>
<td>-127.3 (-18,465)</td>
<td>82.3 (11,931)</td>
</tr>
</tbody>
</table>

#### (e) Structural steel dimensions

<table>
<thead>
<tr>
<th>Ratio of Actual to Specified Dimensions</th>
<th>Section Depth (d)</th>
<th>Flange Width (b)</th>
<th>Flange Thickness (t)</th>
<th>Web Thickness (w)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.000</td>
<td>1.005</td>
<td>0.976</td>
<td>1.017</td>
</tr>
</tbody>
</table>

---

* Average compressive strength of in-place concrete is based on a rate of loading to failure in 60 minutes.  
  ** Average modulus of elasticity of structural steel $E_s = 199,955$ MPa (29,000 ksi).  
  *** The minus sign indicates compressive residual stress.
Figure 4.1 - Type of column studied: (a) free-body diagram of pin-ended column in symmetrical single curvature bending; (b) type of cross-section; (c) forces on column; and (d) bending moment diagram ($M_u = eP_u$).
Figure 4.2 - Schematic cross-section and column axial load-bending moment \((P-M)\) interaction diagrams.

<table>
<thead>
<tr>
<th>Steel Ratio (\rho_{rs}) (Percent)</th>
<th>6</th>
<th>6</th>
<th>8</th>
<th>8</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Bars</td>
<td>16 mm (No. 5)</td>
<td>19 mm (No. 6)</td>
<td>22 mm (No. 7)</td>
<td>25 mm (No. 8)</td>
<td></td>
</tr>
<tr>
<td>Bar Size</td>
<td>1.29</td>
<td>1.29</td>
<td>1.72</td>
<td>1.72</td>
<td>1.72</td>
</tr>
<tr>
<td></td>
<td>2.44</td>
<td>2.44</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
</tr>
</tbody>
</table>

Figure 4.3 - Longitudinal reinforcement details of reinforced concrete column cross-sections.
<table>
<thead>
<tr>
<th>STEEL SECTION</th>
<th>LONGITUDINAL REINFORCING BARS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Designation</td>
<td>( A_w )</td>
</tr>
<tr>
<td></td>
<td>(mm²)</td>
</tr>
<tr>
<td>W310 x 253</td>
<td>32200</td>
</tr>
<tr>
<td>(W12 x 170)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>22800</td>
</tr>
<tr>
<td>(W12 x 120)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13600</td>
</tr>
<tr>
<td>(W12 x 72)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>21300</td>
</tr>
<tr>
<td>(W10 x 112)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12900</td>
</tr>
<tr>
<td>(W10 x 68)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12700</td>
</tr>
<tr>
<td>(W8 x 67)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.4 - Details of composite column cross-sections.
Subjected to major axis bending.

Columns subjected to minor axis bending. (a) Equation 4.9 for all composite columns.

Figure 4.5 - Frequency histograms comparing CSA/ACI slenderness with theoretical results.
Figure 4.6 - Frequency histograms comparing CSA/ACI stiffnesses with theoretical results using (a) Equation 4.8 for reinforced concrete columns with $\rho_r = 1.29$ and 1.72%; (b) Equation 4.9 for composite columns subjected to minor axis bending with $\rho_r = 1.09%$; and (c) Equation 4.9 for composite columns subjected to major axis bending with $\rho_r = 1.09%$. 

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Figure 4.7 - Frequency histograms comparing CSA/ACI stiffnesses from Equation 4.10 with:

(c) Frequency Ratio (EI/Ed) (des)

1. 1st Percentile = 0.85
2. Coeff of Var. = 0.96
3. Average Value = 1.63
4. n = 11880

(b) Frequency Ratio (EI/Ed) (des)

1. 1st Percentile = 0.72
2. Coeff of Var. = 0.34
3. Average Value = 1.33
4. n = 11880

(a) Frequency Ratio (EI/Ed) (des)

1. 1st Percentile = 0.36
2. Coeff of Var. = 0.35
3. Average Value = 0.93
4. n = 11550
Figure 4.8 - Frequency histograms comparing CSA/ACI stiffnesses from Equation 4.10 with theoretical results for (a) reinforced concrete columns with $w_0 = 1.29$ and 1.22%; (b) composite columns subjected to minor axis bending with $w_\text{pf} = 1.09%$; and (c) composite columns subjected to major axis bending with $w_\text{pf} = 1.09%$.

(c)

(Reinforced Concrete)

1. Percentile = 0.80
2. Coeff. of Var. = 0.48
3. Average Value = 1.44
4. u = 3950

(Bending)

Major Axis

Composite Column

(Reinforced Concrete)

1. Percentile = 0.69
2. Coeff. of Var. = 0.33
3. Average Value = 1.44
4. u = 3950

(Bending)

Major Axis

Composite Column

(Reinforced Concrete)

1. Percentile = 0.33
2. Coeff. of Var. = 0.41
3. Average Value = 0.78
4. u = 4125

(Bending)

Major Axis

Composite Column

(Reinforced Concrete)
Figure 4.9 - Theoretically computed (a) average values; (b) five-percentile values; and (c) one-percentile values of $\alpha_c$, plotted against end eccentricity ratio, for reinforced concrete and composite columns subjected to $e/h$ ranging from 0.1 to 1.0.
Figure 4.10 - Comparison of Equation 4.24 with theoretically computed (a) average values; (b) five-percentile values; and (c) one-percentile values of $\alpha_c$ for reinforced concrete and composite columns subjected to $e/h$ ranging from 0.1 to 1.0.
Figure 4.11 - Comparison of proposed design equation (Equation 4.26) with simulated theoretical stiffness data for (a) all reinforced concrete columns; (b) all composite columns subjected to minor axis bending; and (c) all composite columns subjected to major axis bending.
Figure 4.12 - Comparison of proposed design equation (Equation 4.26) with simulated theoretical stiffness data for (a) reinforced concrete columns with $\rho_r = 1.29$ and 1.72%; (b) composite columns subjected to minor axis bending with $\rho_r = 1.09%$; and (c) composite columns subjected to major axis bending with $\rho_r = 1.09%$. 
Frequency (Percent)

Figure 4.13 - Frequency histograms comparing stiffness (EI/I) for reinforced concrete columns (a), all composite columns (b), and all composite columns subjected to minor axis bending (c). The proposed equation (Equation 4.26) with experimental results for (a) all reinforced concrete columns; (b) all composite columns subjected to major axis bending; and (c) all composite columns subjected to minor axis bending.
Figure 4.14 - Frequency histograms comparing stiffnesses computed from the proposed equation (Equation 4.26) with theoretical results for (a) reinforced concrete columns with $\rho_r = 1.29$ and $1.72\%$; (b) composite columns subjected to minor axis bending with $\rho_r = 1.09\%$; and (c) composite columns subjected to major axis bending with $\rho_r = 1.09\%$. 

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Figure 4.15 - Comparison of probability distributions of stiffness ratios ($E_{th}/E_{des}$) computed from simulated data for (a) all reinforced concrete columns; (b) all composite columns subjected to minor axis bending; and (c) all composite columns subjected to major axis bending.
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Figure 4.17 - Effects of variables on stiffness ratio using different design equations for composite columns subjected to minor axis bending: (a) end eccentricity ratio (n=1080 for each of the eleven e/h ratios studied); (b) axial load ratio (n varies for each $P_a/P_o$ ratio studied); (c) slenderness ratio (n=2376 for each of the five $\theta/h$ ratios studied); and (d) longitudinal reinforcement ratio (n=3960 each of the three $\rho_s$ ratios studied).
Figure 4.17 - continued.
Figure 4.18 - Effects of variables on stiffness ratio using different design equations for composite columns subjected to major axis bending: (a) end eccentricity ratio ($n=1080$ for each of the eleven $e/h$ ratios studied); (b) axial load ratio ($n$ varies for each $P_n/P_o$ ratio studied); (c) slenderness ratio ($n=2376$ for each of the five $e/h$ ratios studied); and (d) longitudinal reinforcement ratio ($n=3960$ each of the three $\rho_s$ ratios studied).
Figure 4.18 - continued.
Figure 4.19 - Histograms of stiffness ratios for reinforced concrete columns plotted at different end eccentricity ratios using the proposed design equation (Equation 4.26) (n=1050 for each value of e/h studied).
Figure 4.20 - Histograms of stiffness ratios for composite columns subjected to minor axis bending plotted at different end eccentricity ratios using the proposed design equation (Equation 4.26) (n=1080 for each value of e/h studied).
Figure 4.21 - Histograms of stiffness ratios for composite columns subjected to major axis bending plotted at different end eccentricity ratios using the proposed design equation (Equation 4.26) ($n=1080$ for each value of $e/h$ studied).
Figure 4.22 - Stiffness ratios obtained from proposed design equation (Equation 4.26) for usual reinforced concrete columns with (a) $\theta/h = 10$; and (b) $\theta/h = 20$. (For each combination of $e/h$ and $\rho_{rs}$ ratios plotted, n=30 for $\rho_{rs} = 1.29\%$ and n=45 for each of $\rho_{rs} = 1.72\%$, 2.44\%, 3.33\%, and 4.39\%).
Figure 4.23 - Stiffness ratios obtained from proposed design equation (Equation 4.26) for usual composite columns subjected to minor axis bending with (a) \( \frac{\rho}{h} = 10 \); and (b) \( \frac{\rho}{h} = 20 \). (For each combination of \( \frac{e}{h} \) and \( \rho_{ss} \) ratios plotted, \( n = 108 \) when \( \rho_{ss} = 4.2\% \), \( n = 72 \) when \( \rho_{ss} = 7.0\% \), and \( n = 36 \) when \( \rho_{ss} = 10.3\% \)).
Figure 4.24 - Stiffness ratios obtained from proposed design equation (Equation 4.26) for usual composite columns subjected to major axis bending with (a) $\theta h=10$; and (b) $\theta h=20$. (For each combination of $e/h$ and $\rho_{ss}$ ratios plotted, $n=108$ when $\rho_{ss} = 4.2\%$, $n=72$ when $\rho_{ss} = 7.0\%$, and $n=36$ when $\rho_{ss} = 10.3\%$).
Figure 4.25 - Graphical solution of $\alpha_c$ (Equation 4.25) for reinforced concrete columns with (a) $\rho_r \leq 2\%$; and (b) $\rho_r > 2\%$. 

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Figure 4.26 - Graphical solution of $\alpha_c$ (Equation 4.25) for composite columns.
(c) and (e) composite columns subjected to major axis bending.

Reinforced concrete columns: (p) composite columns subjected to columns with $e/h > 0.1$: (a) Histo-grams for ratio of ultimate axial load to critical buckling strength of.

Figure 4.27 - Histograms for ratio of ultimate axial load to critical buckling strength of.
Figure 4.28 - Histograms for ratio of ultimate axial load to critical buckling strength of usual columns with $e/h = 0.1$ to 0.7 and $\theta h = 10, 15$ and 20: (a) reinforced concrete columns; (b) composite columns subjected to minor axis bending; and (c) composite columns subjected to major axis bending.
5.0 EXAMINATION OF EQUIVALENT UNIFORM MOMENT DIAGRAM FACTOR

5.1 GENERAL

A slender column that is subjected to axial load and end moments producing moment gradient in single or reverse curvature bending deflects laterally and is, therefore, subjected to additional second-order bending moments along its height (Figure 5.1). However, second-order moments occurring along the height of the column are less than the second-order moments in the same column if it were subjected to equal and opposite end moments causing symmetrical single curvature bending. The location of the maximum moment will not be at the mid-height of the column unless the end moments are equal and opposite causing symmetrical single curvature bending. To determine the location and the magnitude of the maximum moment requires the solution to the differential equation governing the behavior of a column subjected to unequal end moments. The solution to the differential equation for the elastic behavior of a column subjected to unequal end moments can be easily determined. However, the solution to a differential equation that includes the inelastic material behavior of reinforced concrete and composite steel-concrete columns is considerably more complex.

The current CSA A23.3-94 (1994) and ACI 318-99 (1999) permit the use of the equivalent uniform moment diagram factor \( C_m \) for computing the effect of moment gradient, along the column height, caused by unequal column end moments. The concept of equivalent uniform moment diagram was introduced into design practice to eliminate the need for extensive calculations based on the solution to the differential equation that determines the magnitude and location of the maximum moment along the height of a
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column. The equation currently in use by the CSA and ACI codes is a simplified equation, which was proposed by Austin (1961) based on the elastic behavior of columns, and does not include the inelastic material behavior of reinforced concrete and composite steel-concrete columns.

In reviewing previous work, no references were found that presented an evaluation of the equivalent uniform moment diagram factor \( (C_m) \) based on inelastic response of isolated reinforced concrete columns or composite steel-concrete columns. Duan et al. (1989) proposed an expression, based on the elastic analysis of steel columns, as an improvement to the expression produced by Austin (1961). Trahair (1985) and Chen and Zhou (1987) investigated the inelastic response of steel columns subjected to unequal end moments.

This study was conducted to investigate the influence of different variables on the equivalent uniform moment diagram factor \( (C_m) \) of slender, tied, reinforced concrete and composite steel-concrete columns; examine existing expressions for \( C_m \); develop and propose, if necessary, a refined expression for \( C_m \) that applies to both reinforced concrete and composite columns; and compare the selected expressions for \( C_m \) with the current CSA/ACI expression.

Over 38,000 isolated reinforced concrete and composite steel-concrete columns were simulated to study the effects of a number of variables that affect the equivalent uniform moment diagram factor \( (C_m) \). Each column had a different combination of cross-section, geometric and material properties. The columns studied were subjected to short-term loads

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and unequal end moments causing moment gradient in single curvature and double curvature bending. The moment magnifier approach specified in the CSA and ACI Codes was developed for this type of columns. The effects of different end restraints and loading conditions are accounted for in the CSA and ACI Codes through the use of the effective length factor \( K \) and sustained load factor \( \beta_d \). The columns studied are graphically represented in Figure 5.1. These columns were chosen because the errors in \( K \) and \( \beta_d \) would not affect the accuracy of the \( C_m \) expressions examined and developed later in this Chapter.

5.2 METHOD USED FOR EVALUATING THEORETICAL \( C_m \) FACTOR

5.2.1 Previous Work on Development of \( C_m \) Based on Elastic Theory

The CSA A23.3-94 and ACI 318-99 permit the use of a moment magnifier approach to compute the maximum bending moment \( (M_{\text{max}}) \), which includes second-order effects, occurring along the height of a column:

\[
M_{\text{max}} = M_c = \delta_{ns} M_2 = C_m \delta_1 M_2 \\
\geq M_2
\]

In Equation 5.1, \( \delta_{ns} \) is the moment magnifier for columns that are part of braced (non-sway) frames; \( C_m \) is the equivalent uniform moment diagram factor; \( M_2 \) is the larger of the two factored end moments \( (M_1 \text{ and } M_2) \) computed from a conventional elastic analysis and is always taken as positive; and \( \delta_1 \) is the moment magnifier for columns subjected to axial load and equal and opposite end moments causing symmetrical single curvature bending.

The equivalent uniform moment diagram factor \( (C_m) \) is used to replace the unequal
column end moments $M_1$ and $M_2$ (causing moment gradient) with a pair of equal and opposite end moments (causing symmetrical single curvature bending). The equivalent uniform bending moment ($M_{eq}$) can be computed as

$$M_{eq} = C_m M_2$$

The magnitude of the equivalent uniform bending moment ($M_{eq}$) is such that the maximum magnified bending moment due to second-order effects along the height of the column is equal to that produced by the actual unequal end moments $M_1$ and $M_2$ (Figure 5.1). Note that $C_m$ reaches its upper limit of 1.0 when $M_1$ is equal in magnitude and opposite in sense to $M_2$.

Chen and Lui (1987) explain that the equivalent uniform moment diagram factor ($C_m$) and the moment magnifier $\delta_1$ for pin-ended columns subjected to end moments can be derived from the basic differential equation governing the elastic in-plane behavior of a column. $C_m$ and $\delta_1$ derived from elastic theory, as shown by Chen and Lui (1987), are reproduced here in Equations 5.3 and 5.4, respectively:

$$C_m = \sqrt{\frac{(M_1/M_2)^2 - 2(M_1/M_2)\cos k\ell + 1}{2(1 - \cos k\ell)}}$$

$$\delta_1 = \sqrt{\frac{2(1 - \cos k\ell)}{\sin^2 k\ell}}$$

In Equations 5.3 and 5.4, $k = (\pi/\ell) \sqrt{(P_u/P_{cr})}$; $\ell$ = the column length; $P_u$ = the factored axial load acting on the column; $P_{cr} = \pi^2 EI/\ell^2$; and $M_1/M_2$ = the ratio of the smaller to the larger end moments and is taken as positive when the column is subject to end moments causing
single curvature bending and negative when the column is subject to end moments causing double curvature bending.

The effects of the critical load ratio \((P_u/P_{cr})\) and the column end moment ratio \((M_1/M_2)\) on the equivalent moment diagram factor \((C_m)\) computed from the elastic theory are shown in Figures 5.2 (a) and (b), respectively. For each set of \(P_u/P_{cr}\) and \(M_1/M_2\) values, \(C_m\) and \(\delta_1\) were computed from Equations 5.3 and 5.4, respectively, and only those values of \(C_m\) were plotted in Figure 5.2 for which \(C_m \delta_1\) (i.e. \(\delta_u\)) exceeds 1.0. This was done to ensure that the maximum moment \((M_{max})\) along the column height computed from Equation 5.1 would be greater than the larger of the two applied end moments \((M_2)\). Because the columns for which \(M_{max}\) was computed to be less than or equal to \(M_2\) were not subjected to magnified moments due to second-order effects, they were not included in Figure 5.2. Figure 5.2 (a) shows that the critical load ratio \((P_u/P_{cr})\) has an effect on \(C_m\). However, the end moment ratio \((M_1/M_2)\) has a greater influence on \(C_m\) than \(P_u/P_{cr}\), as indicated by Figure 5.2 (b). Figure 5.2 (a) also shows that as the end moment ratio moves from single curvature bending to double curvature bending, the effect of \(P_u/P_{cr}\) increases. Note that Figure 5.2 (b) and related conclusions are similar to those presented by Chen and Lui (1987).

For design purposes, the CSA A23.3-94 and ACI 318-99 have adopted the simplified and widely accepted approximations of Equations 5.3 and 5.4:

\[
C_m = \left( 0.6 + 0.4 \frac{M_1}{M_2} \right) \geq 0.4
\]

(5.5)
\[ \delta_1 = \frac{1}{1 - \frac{P_u}{P_{cr}}} \]  

(5.6)

In equation 5.6, \( P_{cr} = \pi^2EI/(K\delta^2) \), where \( K \) is the effective length factor. For this study, however, the effective length factor \( K = 1.0 \) and \( P_{cr} \) is reduced to Euler's buckling strength of a pin-ended column \( (\pi^2EI/\delta^2) \). Equation 5.5 is also plotted in Figure 5.2(b). A comparison of Equations 5.3 and 5.5 in this figure indicates that Equation 5.5 provides nearly the upper limit of the \( C_m \) factor computed from Equation 5.3 for elastic columns.

Substituting Equation 5.6 into Equation 5.1 yields the moment magnifier equation for pin-ended columns subjected to unequal end moments:

\[ M_{\text{max}} = \delta_{ns} M_2 = \left( \frac{C_m}{P_u/P_{cr}} \right) M_2 \]

\[ \geq M_2 \]  

(5.7)

The moment magnifier method defined by Equations 5.1, 5.2 and 5.7 is described graphically in Figure 5.3, which shows the relationships between the cross-section strength interaction diagram, the column axial load-bending moment interaction diagram for columns in symmetrical single curvature bending \( (M_1/M_2 = 1.0) \), and the column axial load-bending moment diagrams for columns subjected to unequal end moments (moment gradients).

It is important to note that \( C_m \) is part of the overall moment magnifier \( (\delta_{ns}) \) in Equation 5.7 and that \( \delta_{ns} \) must be equal to or greater than unity. When \( \delta_{ns} \) is less than or
equal to 1.0, the maximum moment \( (M_{\text{max}}) \) is equal to \( M \) and occurs at the end where the larger end moment is applied.

### 5.2.2 Development of Theoretical \( C_m \) Factor Used for This Study

The method used for computing the theoretical equivalent uniform moment diagram factor \( (C_m) \) utilizes the interaction diagrams described in Figure 5.3 and the definition of the moment magnifier given in Equation 5.1. Replacing the maximum column moment \( M_{\text{max}} \) by the cross-section moment strength \( M_{\text{cs}} \) in Equation 5.1 gives:

\[
M_{\text{cs}} = C_m \delta_1 M_2
\]

For a slender column with equal and opposite end moments causing symmetrical single curvature bending, the applied bending moment diagram is uniform. Therefore, the equivalent uniform bending moment diagram factor is equal to unity and \( M_1 \) and \( M_2 \) are both equal to the column bending moment strength \( M_{\text{col}} \). Substituting 1.0 for \( C_m \) and \( M_{\text{col}} \) for \( M_2 \) into Equation 5.8, yields the following expression:

\[
M_{\text{cs}} = \delta_1 M_{\text{col}}
\]

Then, equating Equation 5.8 to Equation 5.9 and solving for \( C_m \) yields:

\[
C_m = \frac{M_{\text{col}}}{M_2}
\]

Equation 5.10 is the theoretical equivalent uniform moment diagram factor of a pin-
ended slender column subjected to unequal end moments causing moment gradient in single curvature or double curvature bending. The terms $M_{col}$ and $M_2$ used in the equation were obtained for a given level of axial load ($P_u$) from the axial load-bending moment ($P-M$) interaction diagrams (Figure 5.3), which were developed using the theoretical strength model documented in Chapter 2. The computations of $P_u$, $M_{cr}$, $M_2$ and $M_{col}$ are summarized in the next section.

5.2.3 Computations of Theoretical Cross-Section and Slender Column Bending Moment Resistances

The iterative and incremental procedures described in Sections 2.3 and 2.4 were used to compute the axial load strength ($P_u$) of a column for a specified end eccentricity ratio ($e/h=M_2/P_u h$) and end moment ratio ($M_1/M_2$). The axial load $P$ and end moments $M_1$ and $M_2$ were incremented proportionally to failure in a manner such that the end eccentricities $e_1$ and $e_2$ were held constant (i.e. $M_1 = e_1 P$ and $M_2 = e_2 P$). The computed column axial load strength ($P_u$) corresponding to $M_2$ represents a single point on the column $P-M$ interaction curve for a specified end moment ratio $M_1/M_2$ (Figure 5.3).

To establish the slender column bending moment strength $M_{col}$ for the same column subjected to equal and opposite end moments causing symmetrical single curvature bending ($M_1/M_2 = 1.0$), the iterative and incremental procedures were again used. This time the computed column axial load strength ($P_u$), corresponding to $M_2$ in Figure 5.3 and described in the previous paragraph, was applied to the column and held constant while the end eccentricity at each end of the column ($e_1=e_2=e$) was incremented until failure was achieved.
The computed maximum moment \((eP_u)\) was then taken as \(M_{\text{col}}\).

The computed column axial load strength \(P_u\) was then used to determine the corresponding value of \(M_{\text{cr}}\), using Lagrangian interpolation, from the generated points on the cross-section axial load-bending moment interaction diagram. The cross-section axial load-bending moment interaction diagram (Figure 5.3) was generated using a procedure based on equilibrium of forces and compatibility of strains and documented in Section 2.2. \(M_{\text{cr}}\) was compared to the computed values of \(M_2\) and \(M_{\text{col}}\) to check whether the column under consideration was subjected to second-order effects. The computed values of \(M_2\) and \(M_{\text{col}}\) for each column (with specific specified properties, including \(M_1/M_2\), \(e/h\), and \(t/h\) ratios) were then used directly in Equation 5.10 to compute the theoretical \(C_m\).

### 5.3 SIMULATION OF THEORETICAL \(C_m\) DATA FOR COLUMNS STUDIED

Over 38,000 isolated columns were simulated to evaluate the equivalent uniform moment diagram factor \((C_m)\). Of these columns, 16,800 were reinforced concrete columns, 10,800 were composite columns subjected to bending about the minor axis of an encased steel section, and 10,800 were composite columns subjected to bending about the major axis of an encased steel section. The theoretical \(C_m\) for each of the columns studied was computed from Equation 5.10. The simulated column \(C_m\) data were then statistically analyzed to examine the current CSA and ACI \(C_m\) equation and to develop the proposed design equation for \(C_m\).
5.3.1 Description of Reinforced Concrete Columns Studied

The specified properties of 16,800 simulated reinforced concrete columns studied are given in Table 5.1. Specified concrete strengths $f'$, clear concrete cover $C_c$, strength of reinforcing steel $f_{ys}$, and reinforcing steel ratios $\rho_{rs}$, listed in Table 5.1 and Figure 5.4 are similar to those used for the analysis of $EI$ of reinforced concrete columns and were selected for the reasons given in Section 4.3.1. The overall concrete cross-section used is 305 mm x 305 mm (12 in. x 12 in.) in size.

Chapter 4 established that the concrete cover ($C_c$) was not a major variable for the flexural stiffness of reinforced concrete columns. Therefore, the concrete cover was held constant at 38 mm (1.5 in.) for the study of $C_m$. Table 5.1 shows that ten end eccentricity ratios ($e/h$) ranging from 0.1 to 1.0 and five slenderness ratios ($\theta/h$) ranging from 10 to 30 are used. The end eccentricity ratios and slenderness ratios were selected for the reasons described in Section 4.3.1.

The CSA A23.3-94 and ACI 318-99 allow the column end moment ratio ($M_e/M_I$) to range from 1.0 (symmetrical single curvature bending) to -0.5 (double curvature bending). For this study, eight column end moment ratios ($M_e/M_I$) ranging from 1.0 to -0.75 in increments of 0.25 were used. Figure 5.1 shows schematically how the loading was applied to the columns.

The purpose of this study was to simulate the equivalent uniform moment diagram factor $C_m$ of columns described by specified cross-sectional and material properties. Average values of material strengths (Table 5.2) corresponding to the specified values were used to
compute the theoretical $C_m$ of each column for reasons described in Section 4.3.1.

5.3.2 Description of Composite Steel-Concrete Columns Studied

The specified properties of 21,600 simulated composite columns studied are listed in Table 5.3. One-half of these columns was subjected to major axis bending, while the other half was subject to minor axis bending. Specified concrete strengths $f'_{ce}$, structural steel yield strength $f_y$, clear concrete cover $C_c$, reinforcing steel ratios $\rho_r$, and structural steel ratios $\rho_{ss}$ listed in Table 5.3 and Figure 5.5 are similar to those used for the analysis of $EI$ of composite columns and were selected for the same reasons given in Section 4.3.2. The overall dimensions as well as the longitudinal reinforcement and structural steel arrangements for the composite column cross-sections studied are shown in Figure 5.5.

Table 5.3 shows that ten end eccentricity ratios ($e/h$) ranging from 0.1 to 1.0 and five slenderness ratios ($\theta h$) are used for composite columns. These values are the same as those used for reinforced concrete columns and were selected for reasons given in section 4.3.1. Note that the range of $M_1/M_2$ ratios in Table 5.3 is also the same as that used for reinforced concrete columns (Table 5.1). Average values of material strengths (Table 5.4) corresponding to the specified values were used to compute the theoretical $C_m$ factor of each column for reasons described also in Section 4.3.1.
5.4 EXAMINATION OF THEORETICAL $C_m$ DATA

The effects of the critical load ratio ($P_{u(th)}/P_{cr(th)}$) on the theoretically computed equivalent uniform moment diagram factor $C_{m(th)}$ are shown in Figures 5.6 (a), (b), and (c) for all reinforced concrete columns, all composite columns under minor axis bending, and all composite columns under major axis bending, respectively. $C_{m(th)}$ in Figure 5.6 was computed from Equation 5.10. Similarly, for computing the $P_{u(th)}/P_{cr(th)}$ ratio, $P_{u(th)}$ was taken as the computed theoretical axial load strength and $P_{cr(th)}$ was calculated by substituting the theoretical effective flexural stiffness $EI_{(th)}$ from Equation 4.4 into Equation 4.28.

Figure 5.6 (a) indicates that $C_{m(th)}$ for a large number of reinforced concrete columns falls on or close to a diagonal line. The same observation applies also to composite columns shown in Figures 5.6 (b) and (c). An examination of the data for columns for which $C_{m(th)}$ fell on or close to the diagonal lines in Figure 5.6 showed that these columns were not subjected to magnified moments due to second-order effects. This was so, because the maximum bending moment ($M_{max}$) along the height of these columns was equal to the larger of the two end moments ($M_2$). Therefore, columns that did not undergo moment magnification due to second-order effects were excluded from further analysis of the theoretical $C_m$ data.

Figures 5.7 to 5.9 show only the data for the columns where the maximum moment along the column length exceeds the larger of the two end moments, ($M_{col} < M_2 < M_{ca}$). Consequently, a total of 9,488 reinforced concrete columns, 6,082 composite columns subjected to minor axis bending, and 5,813 composite columns subjected to major axis bending is retained in Figures 5.7, 5.8, and 5.9, respectively, for further analysis.
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The trends shown in Figures 5.7(a), 5.8(a), and 5.9(a) are similar to those shown in Figure 5.2(a) for elastic columns. These figures also show that the critical load ratio \( P_{u(th)}/P_{cr(th)} \) has some effect on \( C_m \), but not as significant an effect as the end moment ratio.

As expected, \( C_{m(th)} \) is constant for all values of \( P_{u(th)}/P_{cr(th)} \) when \( M_1/M_2 = 1.0 \), indicating that \( P_{u(th)}/P_{cr(th)} \) has no effect on \( C_m \) at \( M_1/M_2 = 1.0 \). As \( M_1/M_2 \) moves from single curvature (\( M_1/M_2 = 0.75 \)) to double curvature (\( M_1/M_2 = -0.75 \)) the effect of \( P_{u(th)}/P_{cr(th)} \) on \( C_m \) increases.

This also indicates that \( P_{u(th)}/P_{cr(th)} \) is dependent upon the end moment ratio \( M_1/M_2 \). Figures 5.7(a), 5.8(a), and 5.9(a) show an increase in scatter in \( C_{m(th)} \) for each set of \( M_1/M_2 \) values as \( M_1/M_2 \) moves from single curvature to double curvature. The increased scatter can be attributed to (a) the effect of nonlinear material behavior; and (b) the numerical sensitivity caused by the interaction diagram for slender columns (\( \theta h > 20 \)) in double curvature bending approaching a horizontal line at low end eccentricity ratios (\( e/h < 0.15 \)), as shown in Figure 5.3(b). For example, when \( M_1/M_2 \) is equal to -0.75, a minor change in \( P_{u(th)} \) produces a significant change in \( C_{m(th)} \).

In Figures 5.7(b), 5.8(b) and 5.9(b), \( C_{m(th)} \) is plotted with respect to \( M_1/M_2 \). These plots indicate the development of a close to linear relationship between \( C_{m(th)} \) and \( M_1/M_2 \). The single point where \( C_m = 1.0 \) and \( M_1/M_2 = 1.0 \), plotted in the upper right corner of Figures 5.7(b), 5.8(b) and 5.9(b), represents 2100, 1350, and 1350 data points for reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending, respectively. The vertical line of points plotted at \( M_1/M_2 = -0.75 \) represents 247 reinforced concrete columns, 182 composite columns subjected to
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minor axis bending, and 158 composite columns subjected to major axis bending, respectively.

5.5 EXAMINATION OF SELECTED $C_m$ EQUATIONS AVAILABLE IN LITERATURE

5.5.1 Examination of CSA/ACI $C_m$ Factor

The CSA A23.3-94 and ACI 318-99 permit the use of Equation 5.5, developed by Austin (1961), for computing the equivalent uniform moment diagram factor of slender reinforced concrete and composite columns.

$$C_m = \left( 0.6 + 0.4 \frac{M_1}{M_2} \right) \geq 0.4 \quad (5.5)$$

Comparisons of Equation 5.5 with the theoretical $C_m$ data plotted in Figures 5.7(b), 5.8(b) and 5.9(b) for reinforced concrete columns, for composite columns subjected to minor axis bending, and for composite columns subjected to major axis bending, respectively, show that the CSA/ACI $C_m$ equation gives a conservative upper bound for almost all of the columns included in these figures. Note that the $C_m(\theta)$ values plotted against $P_{u(\theta)}/P_{cr(\theta)}$ for different $M_f/M_2$ ratios in Figures 5.7(a), 5.8(a), and 5.9(a) tend to form continuous curves, with each of these curves having an origin that can be approximated by a co-ordinate point defined as $C_m = (0.6 + 0.4 M_f/M_2)$ along the vertical axis and $P_{u(\theta)}/P_{cr(\theta)} = (1 - C_m)$ along the horizontal axis. Hence, the origins of the curves, for different $M_f/M_2$ ratios, in these figures represent columns in which the magnified moment ($M_{max}$) away from the column ends is equal to the
larger end moments ($M_2$), i.e. the magnified moment is about to govern. This can be elaborated by the following example. The origin of the curves with $M_1/M_2=0$ in Figures 5.7(a), 5.8(a), and 5.9(a) is defined by $C_m=0.6$ and $P_{u(h))}/P_{cr(h)}=0.4$. Substituting these values in Equation 5.7 will yield $M_{max}=M_2$. Graphically, the CSA/ACI $C_m$ equation (Equation 5.5) would be represented by a horizontal line for each $M_1/M_2$ ratio with an origin at $C_m=(0.6+0.4\ M_1/M_2)$ and $P_{u(h))}/P_{cr(h)}=(1-C_m)$, if plotted on Figures 5.7(a), 5.8(a), and 5.9(a).

Equation 5.5 was statistically compared with the theoretical $C_m$ values computed (from Equation 5.10) for 7,388 reinforced concrete columns, 4,732 composite columns subjected to minor axis bending, and 4,463 composite columns subjected to major axis bending. The results of these comparisons are plotted in Figures 5.10 (a), (b), and (c), which show histograms and statistics of the ratio of CSA/ACI $C_m$ factor to theoretical $C_m$ factor ($C_{m(dex)}/C_{m(th)}$) for reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending, respectively. The $C_{m(dex)}/C_{m(th)}$ ratios greater than 1.0 signify that $C_{m(dex)}$ is conservative, and the values of $C_{m(dex)}/C_{m(th)}$ less than 1.0 indicate that $C_{m(dex)}$ is non-conservative.

The histograms and statistics in Figure 5.10 do not include the data for columns in single curvature bending for which $M_1/M_2=1.0$, because such columns produced a $C_{m(dex)}/C_{m(th)}$ ratio equal to unity with an associated coefficient of variation of zero. Including columns with $M_1/M_2=1.0$ (n=2,100) would have biased the statistics for the remaining columns having $M_1/M_2=0.75$ to -0.75, since the number of columns available for analysis
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decreases as \( M_1/M_2 \) moves from single curvature to double curvature bending (n=1,982 for \( M_1/M_2 = 0.75 \) to n=247 for \( M_1/M_2 = -0.75 \)).

The statistics given in Figure 5.10 show that average values and associated coefficients of variation for all three types of columns are approximately 1.09 and 0.14, respectively. These statistics also indicate that the CSA/ACI design equation computes conservative values of \( C_m \) for 99 percent of the columns used in Figure 5.10. The tightness of the data is evident from the histograms shown in Figure 5.10.

5.5.2 Examination of Alternative Equations for \( C_m \) Factor

The \( C_m \) values computed from the elastic theory (Equation 5.3) are affected by only two variables: (1) the end moment ratio \( M_1/M_2 \); and (2) the critical load ratio \( P_u/P_{cr} \). Duan et al. (1989) proposed Equation 5.11, which is based on the elastic analysis of pin-ended steel columns and includes the effects of \( M_1/M_2 \) and \( P_u/P_{cr} \):

\[
C_m = 1 + 0.25 \left( \frac{P_u}{P_{cr}} \right) - 0.6 \left( \frac{P_u}{P_{cr}} \right)^{\frac{1}{3}} \left( 1 - \frac{M_1}{M_2} \right)
\]  \hspace{1cm} (5.11)

Trahair (1985) and Chen and Zhou (1987) investigated the inelastic response of pin-ended steel columns subjected to end moments. Trahair (1985) proposed the following equation for computing \( C_m \):

\[
C_m = \frac{1 + (M_1/M_2)}{2} + \left( 0.4 - 0.23 \frac{P_u}{P_{cr}} \right) \left[ \frac{1 - (M_1/M_2)}{2} \right]^3
\]  \hspace{1cm} (5.12)
while Chen and Zhou (1987) proposed Equation 5.13 for $C_m$:

$$C_m = \frac{M_{col}}{M_2}$$  \hspace{1cm} (5.13)

Equation 5.13 is essentially the same as Equation 5.10 where $M_{col}$ and $M_2$ are both computed from interaction diagrams based on non-linear second-order analysis. Chen and Lui (1991) pointed out that, regardless of the value of $P_u/P_{cr}$ and $M_1/M_2$, the elastic $C_m$ factor is always larger than the inelastic $C_m$ factor. For computing the critical load ratio ($P_u/P_{cr}$) in Equations 5.11 and 5.12, $P_u$ was taken as the computed theoretical axial load strength and $P_{cr}$ was calculated by substituting the effective flexural stiffness $EI$ from Equation 4.26 into Equation 4.28. Because Equation 5.13 is really a theoretical equation and the same as Equation 5.10, the equation by Chen and Zhou (Equation 5.13) was not included in the analysis presented in this section.

Comparisons of Equation 5.11 proposed by Duan et al. (1989) with the theoretical $C_m$ data plotted in Figures 5.7(c), 5.8(c) and 5.9(c) for reinforced concrete columns, for composite columns subjected to minor axis bending, and for composite columns subjected to major axis bending, respectively, show that Equation 5.11 does not fit the theoretical $C_m$ data very well and is especially conservative for values of $P_u/P_{cr} \leq 0.5$. On the other hand, Equation 5.12 proposed by Trahair (1985) gives a nonconservative solution of $C_m$ for all $P_u/P_{cr}$ levels when $M_1/M_2 \geq 0.0$ (single curvature bending) as indicated by comparisons with the theoretical $C_m$ data shown in Figures 5.7(d), 5.8(d) and 5.9(d). Note that both of these equations (Equation 5.11 and 5.12) were developed for steel columns and were not intended
for reinforced concrete and composite columns.

Figure 5.11 shows the histograms and statistics for $C_m$ ratios ($C_{m(dex)/C_{m(thi)}$) where $C_{m(dex)}$ was computed using Equation 5.11 (Duan et al. 1989) for all three types of columns studied. A comparison of Figure 5.11 with Figure 5.10 plotted for the CSA/ACI design equation (Equation 5.5) can be summarized as follows:

1. The coefficients of variation of $C_m$ ratios for Equation 5.11 are slightly improved over the values obtained for the CSA/ACI design equation.
2. The average $C_m$ ratios for Equation 5.11 are approximately 5 percent more conservative than those for the CSA/ACI design equation.
3. The one-percentile values (0.72 to 0.84) obtained using Equation 5.11 are significantly lower (nonconservative) than those for the CSA/ACI design equation (1.0).

It appears from the comparison of Figures 5.10 and 5.11 that the current CSA/ACI $C_m$ equation (Equation 5.5) is more appropriate for design of reinforced concrete as well as composite columns.

The histograms and statistics shown in Figure 5.12 are for $C_m$ ratios ($C_{m(dex)/C_{m(thi)}$) where $C_{m(dex)}$ was computed using Equation 5.12 (Trahair 1985) for all three types of columns studied. A comparison of Figure 5.12 with Figure 5.10 plotted for the CSA/ACI design equation (Equation 5.5) leads to the following conclusions:

1. The average $C_m$ ratios for Equation 5.12 are slightly less than unity (0.98-0.99) and are approximately 9 percent less conservative than those for the CSA/ACI design
The coefficients of variation of $C_m$ ratios for Equation 5.12 are approximately $1/2$ to $2/3$ of the values obtained from the CSA/ACI design equation.

The one-percentile values (0.77 to 0.91) obtained using Equation 5.12 are significantly lower (nonconservative) than those for the CSA/ACI design equation (1.0).

A comparison of Figures 5.10 and 5.12 indicates that Equation 5.12 suggested by Trahair (1985) could be used, with some modifications to improve one-percentile values, for computing $C_m$. However, the current CSA/ACI $C_m$ equation (Equation 5.5) appears more appropriate for the design of columns studied.

It is evident from Figures 5.7(b), 5.8(b), 5.9(b), and 5.10 and the related discussions that the current CSA/ACI $C_m$ equation (Equation 5.5) will produce conservative results for the design of reinforced concrete and composite columns studied. Figures 5.7(b), 5.8(b), and 5.9(b) also indicate that the existing CSA/ACI $C_m$ equation may be overly conservative for columns subjected to double curvature bending. On the other hand, Figures 5.11 and 5.12 indicate that approximately 30 to 50 percent reduction in variability of $C_m$ factor can be achieved when the critical load ratio ($P_u/P_{cr}$) is included in the computation of $C_m$. This warrants further evaluation of the theoretical $C_m$ data.
5.6 DEVELOPMENT OF PROPOSED DESIGN EQUATIONS FOR $C_m$

The near linearity and relative compactness of the data shown in Figures 5.7 through 5.9 indicate that $C_m$ is not affected as much by cracking along the length of the column and inelastic actions within the cross-section as was the effective flexural stiffness ($EI$) of slender reinforced concrete and composite columns. It was, therefore, unnecessary to represent $C_m$ as a complex function of a number of different variables based on cross-sectional properties to obtain a general equation, as was the case for $EI$. The objective in this part of the study was to develop an equation for computing $C_m$ of reinforced concrete and composite columns that would include the major variables affecting $C_m$ and, at the same time, would be nearly as simple but more accurate than current CSA/ACI design equation. Multiple linear regression analyses of the generated theoretical $C_m$ data were conducted to evaluate the significant variables affecting $C_m$ as well as for the development of the proposed design equation for $C_m$. Note that the columns subjected to symmetrical single curvature bending ($M_1/M_2 = 1.0$) were not included in the regression analysis of the theoretical $C_m$ data for reasons described in Section 5.5.1. The steps used for the development of the proposed design equation are described in following sections.

5.6.1 Format of and Variables Used for Proposed $C_m$ Equations

The variables used for the development of the $C_m$ equation proposed in this study were divided into three groups. Group $X_1$ variables consisted of two variations of the end moment ratio, which were $M_1/M_2$ and $[(1+M_1/M_2)/2]^{1/4}$ while Groups $X_2$ and $X_3$ consisted of
critical load ratio \((P_u/P_{cr})\) and slenderness ratio \((\delta h)\), respectively. \(M_1/M_2\) and \(P_u/P_{cr}\) were chosen as variables because they are both used in the \(C_m\) equation derived from elastic theory (Equation 5.3). The term \([(1+M_1/M_2)/2]^4\) was included in the analysis as a variation of \(M_1/M_2\), while the \(\delta h\) ratio was chosen to investigate the length effect on \(C_m\) factor. Note that for computing the critical load ratio \((P_u/P_{cr})\), \(P_u\) was taken as the computed theoretical axial load strength and \(P_{cr}\) was calculated by substituting the theoretical effective flexural stiffness \((EI_{th})\) from Equation 4.4 into Equation 4.28.

The two variables within Group \(X_1\) were considered as dependent variables, while the variables between the groups (Groups \(X_1\), \(X_2\) and \(X_3\)) were taken as independent variables. A maximum of one variable from any of the chosen groups was, therefore, used for a particular regression analysis of the theoretical \(C_m\) data. When one variable from each group is included, the regression equation takes the form:

\[
C_m = \alpha_k + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3
\]

in which \(\alpha_k\) is a constant (equivalent to the intercept of a simple linear equation). The remaining \(\alpha\) values are the factors corresponding to the independent variables \(X_1\), \(X_2\), and \(X_3\). \(X_1\), \(X_2\) or \(X_3\) represents a variable chosen from the corresponding group.

The combinations of variables from groups \(X_1\), \(X_2\), and \(X_3\) used for regression analyses are given in Table 5.5. For variable combination number 2 in Table 5.5, the power coefficient \((\lambda)\) in variable \([(1+M_1/M_2)/2]^4\) was varied from 0.5 to 2.5 with an increment of 0.1 and regression analyses of the theoretical \(C_m\) data for reinforced concrete columns,
composite columns subjected to minor axis bending, and composite columns subjected to major axis bending were carried out for each value of $\lambda$ studied. The value of 1.1 was finally selected because of the highest multiple correlation coefficient ($R_c$) and the lowest standard error ($S_c$) associated with this value of $\lambda$. The rest of the regression analyses involving the variable $[(1+M_1/M_2)/2]$ were conducted with $\lambda=1.1$. Note that $S_c$ and $R_c$ values shown for variable combinations 2, 7, and 8 in Table 5.5 are for $\lambda=1.1$. The correlation coefficient ($R_c$) and the standard error ($S_c$) are measures of the relative strength and accuracy of a regression equation, as discussed earlier in Section 4.5.2.

5.6.2 Regression Analysis of Theoretical $C_m$ Data

The standard errors ($S_c$) and the correlation coefficients ($R_c$), shown in Table 5.5, were calculated for eight regression equations for each type of columns studied (reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending).

Table 5.5 shows that variable combinations number 1 and 2, which used $M_1/M_2$ or $[(1+M_1/M_2)/2]^{1.1}$ as a variable, produced the lowest $S_c$ and the highest $R_c$ among variable combinations (number 1 to 4) involving a single variable. Table 5.5 shows also that variable combinations (number 5 to 8) involving two variables did not significantly improve the $S_c$ and $R_c$ values. These observations indicate that effects of $P_u/P_{cr}$ and $\theta$ on the $C_m$ factor can be neglected and that either $M_1/M_2$ or $[(1+M_1/M_2)/2]^{1.1}$ could be used as a variable in Equation 5.14 for the computation of $C_m$. This is valid for all three types of columns studied.
and included in Table 5.5. The following regression equations were selected for reinforced concrete columns:

\[ C_m = 0.55 + 0.45 \frac{M_1}{M_2} \]  
\[ (n = 7388; S_e = 0.019; R_e = 0.989) \]  

\[ C_m = 0.15 + 0.85 \left( \frac{1 + M_1/M_2}{2} \right)^{1.1} \]  
\[ (n = 7388; S_e = 0.019; R_e = 0.990) \]

The following regression equations were obtained for composite columns subjected to minor axis bending:

\[ C_m = 0.55 + 0.45 \frac{M_1}{M_2} \]  
\[ (n = 4732; S_e = 0.017; R_e = 0.992) \]  

\[ C_m = 0.14 + 0.86 \left( \frac{1 + M_1/M_2}{2} \right)^{1.1} \]  
\[ (n = 4732; S_e = 0.017; R_e = 0.993) \]

The following regression equations were obtained for composite columns subjected to major axis bending:

\[ C_m = 0.55 + 0.45 \frac{M_1}{M_2} \]  
\[ (n = 4463; S_e = 0.018; R_e = 0.991) \]
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\[ C_m = 0.13 + 0.87 \left( \frac{1 + M_1/M_2}{2} \right)^{1.1} \]

\((n = 4463; S_e = 0.018; R_c = 0.991)\)

The regression coefficients \(\alpha_k\) and \(\alpha_l\) associated with Equations 5.15, 5.17, and 5.19 for reinforced concrete columns, for composite columns subjected to minor axis bending, and for composite columns subjected to major axis bending, respectively, are identical. The regression coefficients \(\alpha_k\) and \(\alpha_l\) for Equations 5.16, 5.18, and 5.20 are almost identical. The low standard error \((S_e = 0.02\) approximately) and high correlation coefficient \((R_c = 0.99\) approximately) indicate the validity of these equations.

Regression equations involving \(M_1/M_2\) as a variable (Equations 5.15, 5.17, and 5.19) are compared with the theoretical \(C_m\) data for reinforced concrete and composite columns in Figure 5.13. Similar comparisons for regression equations that used \([(1+M_1/M_2)/2]\) as a variable (Equations 5.16, 5.18, and 5.20) are shown in Figure 5.14. Both sets of regression equations exhibit reasonable correlation with the theoretical \(C_m\) values for all three types of columns studied, when the plotted data are compared to the line of equality labeled as a 45° line in Figures 5.13 and 5.14.

Statistics of \(C_m\) ratios \((C_{m(\text{reg})}/C_{m(\text{th})})\), where \(C_{m(\text{reg})}\) was computed from the relevant regression equation, are given also in Figures 5.13 and 5.14. Note that \(C_m\) ratios greater than 1.00 signify that \(C_{m(\text{reg})}\) is conservative, and values of \(C_{m(\text{reg})}/C_{m(\text{th})}\) less than 1.00 indicate that \(C_{m(\text{reg})}\) is non-conservative. The average values of 1.00 associated with very low coefficients of variation obtained for \(C_m\) ratios and given in Figures 5.13 and 5.14 indicate that the regression equations (Equations 5.15 to 5.20) represent the theoretical \(C_m\) data with reasonable accuracy.
5.6.3 Design Application

The purpose of conducting regression analyses of the theoretical \( C_m \) data was to estimate values of coefficients related to the variables that affect \( C_m \) of reinforced concrete and composite columns. The regression coefficients \( \alpha_4 \) and \( \alpha_1 \) in Equations 5.15, 5.17, and 5.19 can be rounded off to 0.6 and 0.4, respectively. Similarly, \( \alpha_k \) and \( \alpha_i \) in Equations 5.16, 5.18, and 5.20 can be rounded off to 0.2 and 0.8, respectively. Hence, for design purposes, Equation 5.15 to 5.20 were simplified to Equations 5.21 and 5.22 for both reinforced concrete and composite columns.

\[
C_m = 0.6 + 0.4 \frac{M_1}{M_2} \quad (5.21)
\]

\[
C_m = 0.2 + 0.8 \left( \frac{1 + M_1/M_2}{2} \right)^{1.1} \quad (5.22)
\]

Note that Equation 5.21 is identical to the current CSA/ACI \( C_m \) equation with the exception that no lower limit is placed on \( C_m \) in Equation 5.21, while the current CSA/ACI equation (Equation 5.5) is subjected to a lower limit of \( C_m \) equal to 0.4.

Comparisons of Equations 5.21 and 5.22 with the theoretical \( C_m \) data in Figures 5.7(b), 5.8(b), and 5.9(b) for reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending, respectively, show that Equations 5.21 and 5.22 fit the theoretical \( C_m \) data rather nicely. These figures also show that Equation 5.21 is slightly more conservative than Equation 5.22 for \(-1.0 < M_1/M_2 < 1.0\). However, both equations give identical values of \( C_m \) when \( M_1/M_2 = 1.0 \) or -1.0.
5.7 ANALYSIS AND DISCUSSION OF SIMULATED RESULTS FOR $C_m$ FACTOR

Frequency histograms and other statistical data presented in this section were prepared for $C_m$ ratios ($C_{m(des)}/C_{m(th)}$) using different design equations. For computing the $C_m$ ratio, $C_{m(th)}$ was taken as the theoretical equivalent uniform moment diagram factor, and $C_{m(des)}$ was computed from Equation 5.21, 5.22, 5.12 (Trahair 1985), or 5.11 (Duan et al. 1989). Equation 5.5 (CSA/ACI equation for $C_m$) is not included in the analyses presented in this section because Equation 5.5 will produce $C_m$ ratios very similar to albeit slightly more conservative than those obtained for Equation 5.21. The histograms and statistics discussed in this section also do not include the $C_m$ data for symmetrical single curvature bending ($M_1/M_2 = 1.0$) for reasons discussed in Section 5.5.1. Therefore, the data represents 7,388 reinforced concrete columns, 4,732 composite columns subjected to minor axis bending, and 4,463 composite columns subjected to major axis bending.

5.7.1 Overview of $C_m$ Ratio Statistics

Equations 5.21 and 5.22 were statistically evaluated using the theoretical $C_m$ data for 7,388 reinforced concrete columns, 4,732 composite columns subjected to minor axis bending, and 4,463 composite columns subjected to major axis bending. The results of these evaluations are plotted into Figures 5.15 and 5.16 for Equations 5.21 and 5.22, respectively. These figures show histograms and statistics of $C_m$ ratios computed using Equation 5.21 and 5.22. Note that in Figures 5.15 and 5.16, $C_{m(des)}/C_{m(th)}$ ratios greater than 1.0 signify that $C_{m(des)}$ is conservative and vice versa. The one-percentile values of $C_{m(des)}/C_{m(th)}$ ratios in
these figures are close to 1.0. These values are similar to those obtained for the current CSA/ACI equation (Figure 5.10) and are higher than those obtained for equations proposed by Duan et al. (1989)(Figure 5.11) and Trahair (1985)(Figure 5.12). The compactness of the $C_m$ ratio data indicated by histograms in Figures 5.15 and 5.16 is either as good as or better than that shown in Figures 5.10 to 5.12. Note that Figures 5.15 and 5.16 show very similar statistics for $C_m$ ratios, indicating that either of these equations can be used for design.

The comments noted in the foregoing paragraph can be seen more clearly by comparing cumulative frequency curves of $C_m$ ratios for the four design equations [Equation 5.21, Equation 5.22, Equation 5.12 (Trahair 1985), and Equation 5.11 (Duan et al. 1989)] plotted on a normal probability paper in Figure 5.17. The curves for two of the design equations (Equations 5.21 and 5.22) do not differ significantly and run virtually parallel to one another, with Equation 5.21 giving more conservative results than Equation 5.22. This is valid for all three types of columns studied. For reinforced concrete columns, Equation 5.21 produced $C_m$ values lower than theoretically computed $C_m$ values for only 1 percent of the cases studied, whereas Equation 5.22 computed $C_m$ values lower than theoretical $C_m$ values for approximately 10 percent of the cases studied, as indicated by Figure 5.17(a). These two design equations show similar trends in Figures 5.17(b) and (c) for composite columns as well.

A comparison of Equations 5.21 and 5.22 with the equations given by Trahair (1985) and Duan et al. (1989) in Figure 5.17 indicates that these two equations (Equations 5.21 and 5.22) are better representatives of the theoretical $C_m$ data, especially in lower tails of
probability distributions, for the types of columns studied.

5.7.2 Effects of Variables on $C_m$ Ratios

The effects of the end eccentricity ratio ($e/h$), critical load ratio ($P_u/P_{cr}$), slenderness ratio ($\ell/h$), and end moment ratio ($M_1/M_2$) on the average and one-percentile values of the $C_m$ ratios ($C_{m(\text{des})}/C_{m(\text{bh})}$) obtained from Equation 5.11 (Duan et al. 1989), Equation 5.12 (Trahair 1985), Equation 5.21 and Equation 5.22 for 7,388 reinforced concrete columns are shown in Figures 5.18. The following conclusions can be drawn from Figure 5.18:

(1) The one-percentile values of the $C_{m(\text{des})}/C_{m(\text{bh})}$ ratios for design Equations 5.21 and 5.22 are higher than those computed for the equations given by Trahair (1985) and Duan et al. (1989). Note that one-percentile values are more important for structural safety.

(2) The average and one-percentile $C_m$ ratios for Equation 5.21 are, as expected, consistently higher than those computed for Equation 5.22. The average and one-percentile values for these two design equations run almost parallel to one another.

(3) With the exception of $M_1/M_2 < -0.5$, which are extreme cases of double curvature bending, the one-percentile $C_m$ ratios for the two design equations are not significantly affected by the variables investigated and are, in most cases, higher than or equal to 1.0 and 0.95 for equations 5.21 and 5.22, respectively.

Figures 5.19 and 5.20 present plots for 4,732 composite columns subjected to minor axis bending and 4,463 composite columns subjected to major axis bending, respectively.
An examination of Figures 5.19 and 5.20 indicates that the trends, discussion, and conclusions for composite columns are similar to those given above for reinforced concrete columns.

Note that the critical load ratio was not a controlled variable in this study. Consequently, there were as many different critical load ratios as the number of columns studied. This required the grouping of the $C_m$ ratios into a number of ranges of $P_a/P_{cr}$ which influenced the number of columns in each of the selected ranges of $P_a/P_{cr}$ and used for plots in Figures 5.18(b) and 5.19(b) and 5.20(b).

Because the end moment ratio $M_1/M_2$ was found to have some effect on one-percentile values of the $C_m$ ratios computed for Equations 5.21 and 5.22 and plotted in Figures 5.18(d), 5.19(d), and 5.20(d), histograms and related statistics of the $C_m$ ratios at each $M_1/M_2$ ratio were examined. Figures 5.21, 5.22, and 5.23 present such histograms and related statistics prepared using Equations 5.21 and 5.22 for reinforced concrete columns, composite columns subjected to minor axis bending, and composite columns subjected to major axis bending, respectively. Figures 5.21 to 5.23 demonstrate that one-percentile values of the $C_m$ ratios remain virtually unaffected for $M_1/M_2$ ranging from -0.5 to 0.75, although the average values and related coefficients of variation of the $C_m$ ratios increase as $M_1/M_2$ decreases from 0.75 to -0.75. Since the one-percentile value of a probability distribution is the most important parameter from the structural safety point of view, it can be concluded that these two design equations adequately address the effect of $M_1/M_2$ on the $C_m$ ratio.
5.8 PROPOSED DESIGN EQUATIONS

It is apparent from Figures 5.17 through 5.20 that there is no real advantage in using a "cumbersome" equation, such as 5.11 and 5.12, that includes $P_r/P_cr$ in the computation of the equivalent uniform moment diagram factor $C_m$ for reinforced concrete or composite columns. Equation 5.21 or 5.22 is, therefore, a reasonable choice for design purposes. As noted in Section 5.7.2, one-percentile values of the $C_m$ ratios obtained for these equations were somewhat lower than 1.0 in extreme cases of reverse curvature bending shown in Figures 5.21 to 5.23. A lower limit of 0.34 on $C_m$ computed from Equations 5.21 and 5.22 would bring the one-percentile $C_m$ ratios closer to 1.0 for all values of $M_1/M_2$ plotted in Figures 5.21 and 5.23. It is suggested that, for design of reinforced concrete and composite columns, the current CSA/ACI equation for $C_m$ (Equation 5.5) be replaced by Equation 5.23 or 5.24:

$$C_m = \left( 0.6 + 0.4 \frac{M_1}{M_2} \right) \geq 0.34$$ \hspace{1cm} (5.23)

$$C_m = \left[ 0.2 + 0.8 \left( 0.5 + 0.5 \frac{M_1}{M_2} \right)^{1.1} \right] \geq 0.34$$ \hspace{1cm} (5.24)

Note that Equation 5.23 is identical to the current CSA/ACI equation for $C_m$ with the exception that the lower limit on $C_m$ in Equation 5.23 has been decreased to 0.34.
Table 5.1  Specified properties of reinforced concrete columns studied.*

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>$f'_c$ MPa (psi)</td>
<td>27.6; 41.4; 55.2 (4000; 6000; 8000)</td>
<td></td>
</tr>
<tr>
<td>$f_{rr}$ MPa (psi)</td>
<td>414 (60,000)</td>
<td></td>
</tr>
<tr>
<td>$C_c$ mm (in.)</td>
<td>38 (1.5)</td>
<td></td>
</tr>
<tr>
<td>$\delta_h$</td>
<td>10; 15; 20; 25; 30</td>
<td></td>
</tr>
<tr>
<td>$e/h$</td>
<td>0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 1.0</td>
<td></td>
</tr>
<tr>
<td>$\rho_s$ and steel arrangement</td>
<td>See Figure 5.4 for combinations of steel ratios and arrangements</td>
<td></td>
</tr>
<tr>
<td>$M_1/M_2$</td>
<td>1.0; 0.75; 0.50; 0.25; 0.0; -0.25; -0.50; -0.75</td>
<td></td>
</tr>
</tbody>
</table>

* Total number of simulated columns equals $(3 \times 1 \times 5 \times 10 \times 14 \times 8 =) 16,800$ with each column having a different combination of specified properties shown above. 10mm (0.375 in.) diameter lateral ties spaced in conformance with the requirements of CSA A23-94 Clause 7.6.5.2 and ACI 318-95 Clause 7.10.5.2.

Table 5.2  Average values of variables used for computing theoretical strength and equivalent uniform moment diagram factor of reinforced concrete columns.

(a) Concrete

<table>
<thead>
<tr>
<th>Specified Compressive Strength $f'_c$ MPa (psi)</th>
<th>In-place Concrete in Members (Columns)</th>
<th>Compressive Strength $f_c$ MPa (psi)</th>
<th>Modulus of Rupture $f_r$ MPa (psi)</th>
<th>Elastic Modulus $E_c$ MPa (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.6 (4000)</td>
<td>23.4 (3388)</td>
<td>3.1 (446)</td>
<td>22,478 (3260)</td>
<td></td>
</tr>
<tr>
<td>41.4 (6000)</td>
<td>32.0 (4641)</td>
<td>3.6 (525)</td>
<td>26,167 (3795)</td>
<td></td>
</tr>
<tr>
<td>55.2 (8000)</td>
<td>40.7 (5904)</td>
<td>4.1 (594)</td>
<td>29,393 (4263)</td>
<td></td>
</tr>
</tbody>
</table>

(b) Reinforcing Steel

<table>
<thead>
<tr>
<th>Specified Yield Strength $f_{rr}$ MPa (psi)</th>
<th>Static Yield Strength $f_r$ MPa (psi)</th>
<th>Elastic Modulus $E_r$ MPa (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>414 (60,000)</td>
<td>461 (66,800)</td>
<td>199,955 (29,000)</td>
</tr>
</tbody>
</table>

* Average compressive strength of in-place concrete in member is based on a rate of loading to failure in 60 minutes.
Table 5.3  Specified properties of composite steel-concrete columns studied.\(^*\)

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f'_c) MPa (psi)</td>
<td>27.6; 41.4; 55.2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(4000; 6000; 8000)</td>
<td></td>
</tr>
<tr>
<td>(f_{cu}) MPa (psi)</td>
<td>303 (44,000)</td>
<td>1</td>
</tr>
<tr>
<td>(\rho_n) (percent)</td>
<td>1.09; 1.96; 3.17</td>
<td>3</td>
</tr>
<tr>
<td>Structural steel</td>
<td>Section (\rho_n) (percent)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>W310 x 253</td>
<td>10.33</td>
</tr>
<tr>
<td></td>
<td>W310 x 179</td>
<td>7.29</td>
</tr>
<tr>
<td></td>
<td>W310 x 107</td>
<td>4.36</td>
</tr>
<tr>
<td>(\delta/h)</td>
<td>10; 15; 20; 25; 30</td>
<td>5</td>
</tr>
<tr>
<td>(e/h)</td>
<td>0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 1.0</td>
<td>10</td>
</tr>
<tr>
<td>(M_1/M_2)</td>
<td>1.0; 0.75; 0.50; 0.25; 0.0 -0.25; -0.50; -0.75</td>
<td>8</td>
</tr>
</tbody>
</table>

* The number of simulated composite columns subjected to major axis bending equals \((3 \times 1 \times 3 \times 3 \times 5 \times 10 \times 8 =) 10,800\) with each column having a different combination of specified properties shown above. Another identical set of 10,800 simulated composite columns was subjected to minor axis bending. Hence, the total number of simulated composite columns equals \(2 \times 10,800 = 21,600\).

Note: Imperial equivalents of sections noted above are, W12 x 170, W12 x 120, W12 x 72, respectively.

Table 5.4  Average values of variables used for computing theoretical strength and equivalent uniform moment diagram factor of composite columns.

(a) Concrete

<table>
<thead>
<tr>
<th>Specified Compressive Strength (f'_c), MPa (psi)</th>
<th>In-place Concrete in Members (Columns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f'_c), MPa (psi)</td>
<td>Compressive Strength* (f_c), MPa (psi)</td>
</tr>
<tr>
<td>27.6 (4000)</td>
<td>23.4 (3388)</td>
</tr>
<tr>
<td>41.4 (6000)</td>
<td>32.0 (4641)</td>
</tr>
<tr>
<td>55.2 (8000)</td>
<td>40.7 (5904)</td>
</tr>
</tbody>
</table>
Table 5.4 - continued

(b) Reinforcing Steel

<table>
<thead>
<tr>
<th>Specified Yield Strength</th>
<th>Static Yield Strength</th>
<th>Elastic Modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{pu}$ MPa (psi)</td>
<td>$f_p$ MPa (psi)</td>
<td>$E_s$ MPa (ksi)</td>
</tr>
<tr>
<td>414 (60,000)</td>
<td>461 (66,800)</td>
<td>199,955 (29,000)</td>
</tr>
</tbody>
</table>

(c) Structural steel strength**

<table>
<thead>
<tr>
<th>Specified Yield Strength</th>
<th>Static Yield Strength</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{pu}$ MPa (psi)</td>
<td>Web $f_{pw}$ MPa (psi)</td>
</tr>
<tr>
<td>303 (44,000)</td>
<td>331 (47,960)</td>
</tr>
</tbody>
</table>

(d) Residual stresses in structural steel

<table>
<thead>
<tr>
<th>Steel Shape</th>
<th>Flange Tip*** MPa (psi)</th>
<th>Flange-web Juncture MPa (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>W310 x 253 (W12 x 170)</td>
<td>-126.6 (-18,367)</td>
<td>81.3 (11,792)</td>
</tr>
<tr>
<td>W310 x 179 (W12 x 120)</td>
<td>-124.0 (-17,983)</td>
<td>77.7 (11,267)</td>
</tr>
<tr>
<td>W310 x 107 (W12 x 72)</td>
<td>-123.4 (-17,896)</td>
<td>76.9 (11,152)</td>
</tr>
</tbody>
</table>

(e) Structural Steel Dimensions

<table>
<thead>
<tr>
<th></th>
<th>Section Depth d</th>
<th>Flange Width b</th>
<th>Flange Thickness t</th>
<th>Web Thickness w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of Actual to</td>
<td>1.000</td>
<td>1.005</td>
<td>0.976</td>
<td>1.017</td>
</tr>
<tr>
<td>Specified Dimensions</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Average compressive strength of in-place concrete in members is based on a rate of loading to failure in 60 minutes.
- Average modulus of elasticity of structural steel $E_s = 199,955$ MPa (29,000 ksi).
- The minus sign indicates compressive residual stress.
Table 5.5 - Variable combinations used for regression analysis of theoretical $C_m$ data.

<table>
<thead>
<tr>
<th>Cross Section Type</th>
<th>Variable Combination No.</th>
<th>$X_1$ End Moment Ratio</th>
<th>$X_2$ Critical Load Ratio $P_c/P_{cr}$</th>
<th>$X_3$ Slenderness Ratio $t/h$</th>
<th>Standard Error $S_e$**</th>
<th>Multiple Correlation Coefficient $R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforced Concrete Columns</td>
<td>1</td>
<td>X</td>
<td></td>
<td></td>
<td>0.019</td>
<td>0.989</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>X</td>
<td></td>
<td></td>
<td>0.019</td>
<td>0.990</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>X</td>
<td></td>
<td></td>
<td>0.131</td>
<td>0.518</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>X</td>
<td></td>
<td></td>
<td>0.163</td>
<td>0.173</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>X</td>
<td></td>
<td></td>
<td>0.016</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>X</td>
<td></td>
<td></td>
<td>0.016</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.015</td>
<td>0.994</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.018</td>
<td>0.991</td>
</tr>
<tr>
<td>n = 7388</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Composite Columns with Minor Axis Bending</td>
<td>1</td>
<td>X</td>
<td></td>
<td></td>
<td>0.017</td>
<td>0.992</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>X</td>
<td></td>
<td></td>
<td>0.017</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>X</td>
<td></td>
<td></td>
<td>0.134</td>
<td>0.523</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>X</td>
<td></td>
<td></td>
<td>0.179</td>
<td>0.153</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.013</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.015</td>
<td>0.994</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.012</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.015</td>
<td>0.994</td>
</tr>
<tr>
<td>n = 4732</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Composite Columns with Major Axis Bending</td>
<td>1</td>
<td>X</td>
<td></td>
<td></td>
<td>0.018</td>
<td>0.991</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>X</td>
<td></td>
<td></td>
<td>0.018</td>
<td>0.991</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>X</td>
<td></td>
<td></td>
<td>0.130</td>
<td>0.542</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>X</td>
<td></td>
<td></td>
<td>0.176</td>
<td>0.168</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.013</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.016</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.013</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>X</td>
<td>X</td>
<td></td>
<td>0.016</td>
<td>0.993</td>
</tr>
<tr>
<td>n = 4463</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* $\lambda=1.1$ was used for regression analyses of the theoretical $C_m$ data for variable combinations shown.

** $S_e$ was computed for the constant $\alpha_4$ in Equation 5.14.
Figure 5.1 - Range of moment gradients and resulting second-order bending moment diagrams of pin-ended columns studied.
Figure 5.2 - Effects of (a) critical load ratio; and (b) end moment ratio on the equivalent uniform moment diagram factor of elastic columns computed using Equation 5.3.
Figure 5.3 - Cross-section and column ultimate axial load-bending moment interaction diagrams for slender columns: (a) $\theta h = 15$; and (b) $\theta h = 30$. 
Steel Ratio $\rho_{rs}$ (Percent)

<table>
<thead>
<tr>
<th>Bar Size</th>
<th>No. of Bars</th>
<th>6</th>
<th>6</th>
<th>8</th>
<th>8</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 mm (No. 5)</td>
<td>1.29</td>
<td>1.29</td>
<td>1.72</td>
<td>1.72</td>
<td>1.72</td>
<td></td>
</tr>
<tr>
<td>19 mm (No. 6)</td>
<td>2.44</td>
<td>2.44</td>
<td>2.44</td>
<td>2.44</td>
<td>2.44</td>
<td></td>
</tr>
<tr>
<td>22 mm (No. 7)</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td></td>
</tr>
<tr>
<td>25 mm (No. 8)</td>
<td>4.39</td>
<td>4.39</td>
<td>4.39</td>
<td>4.39</td>
<td>4.39</td>
<td></td>
</tr>
</tbody>
</table>

Arrangement of Longitudinal Reinforcement

Figure 5.4 - Longitudinal reinforcement details of reinforced concrete column cross-sections studied.

<table>
<thead>
<tr>
<th>STEEL SECTION</th>
<th>LONGITUDINAL REINFORCING BARS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Designation</td>
<td>$d_0$</td>
</tr>
<tr>
<td>---------------</td>
<td>-------</td>
</tr>
<tr>
<td>W310 x 253 (W12 x 170)</td>
<td>32200</td>
</tr>
<tr>
<td>W310 x 179 (W12 x 120)</td>
<td>22800</td>
</tr>
<tr>
<td>W310 x 107 (W12 x 72)</td>
<td>13600</td>
</tr>
</tbody>
</table>

Figure 5.5 - Details of composite column cross-sections studied.
Figure 5.6 - Effect of critical load ratio on the theoretical equivalent uniform moment diagram factor for all simulated (a) reinforced concrete columns; (b) composite columns under minor axis bending; and (c) composite columns under major axis bending.
Figure 5.7 - Effect of (a) critical load ratio; and (b), (c), and (d) end moment ratio on the theoretical equivalent uniform moment diagram factor for reinforced concrete columns in which $M_{e}<M_{c}$.
Figure 5.7 - continued
Figure 5.8 - Effects of (a) critical load ratio; and (b), (c), and (d) end moment ratio on the theoretical equivalent uniform moment diagram factor for composite columns subjected to minor axis bending in which $M_{col} < M_2 < M_{cr}$. 
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Equivalent Moment Diagram

End Moment Ratio \( M_1/M_2 \)

Equation 5.12

\( n = 6082 \)

Equivalent Moment Diagram

End Moment Ratio \( M_1/M_2 \)

Equation 5.11

\( n = 6082 \)
Figure 5.9 - Effect of (a) critical load ratio; and (b) and (d) end moment ratio on the Equivalent Moment Diagram

Equation 5.22
Equation 5.21
Equation 5.25
Figure 5.9 - continued
columns subjected to major axis bending (c) values for columns with \( M/W' \neq 1.0 \) not composite columns; (d) composite columns subjected to minor axis bending; and (e) composite columns with the theoretical data for columns in which \( W' > M' \). Figures 2.5(c), 2.5(d), and 2.5(e) represent concrete columns with the theoretical data for columns in which \( W' > M' \).
Figure 5.11 - Frequency histograms comparing the design equation proposed by Duan et al. (Equation 5.11) with the theoretical $C_m$ data for columns in which $M_{col}<M<\bar{M}$; (a) reinforced concrete columns; (b) composite columns subjected to minor axis bending; and (c) composite columns subjected to major axis bending ($C_m$ values for columns with $M_1/M_2=1.0$ not included).
Figure 5.12 - Frequency histograms comparing the design equation by Trahair (Equation 5.12) with the theoretical $C_m$ data for columns in which $M_{col} < M_2 < M_{cr}$: (a) reinforced concrete columns; (b) composite columns subjected to minor axis bending; and (c) composite columns subjected to major axis bending ($C_m$ values for columns with $M_1 / M_2 = 1.0$ not included).
Figure 5.13 - Comparison of selected regression equations, having $M_1/M_2$ as a variable, with theoretical $C_m$ data: (a) Equation 5.15 for reinforced concrete columns; (b) Equation 5.17 for composite columns subjected to minor axis bending; and (c) Equation 5.19 for composite columns subjected to major axis bending ($M_{col} < M_2 < M_{ci}$; $M_1/M_2 < 0.75$).
Figure 5.14 - Comparison of selected regression equations, having \[ \left(1 + \frac{M_1}{M_2}\right)^{1.1} \] as a variable, with theoretical \( C_m \) data: (a) Equation 5.16 for reinforced concrete columns; (b) Equation 5.18 for composite columns subjected to minor axis bending; and (c) Equation 5.20 for composite columns subjected to major axis bending (\( M_{col} < M_2 < M_{cr}; M_1 / M_2 \leq 0.75 \)).
Figure 5.15 - Frequency histograms comparing design Equation 5.21 with the theoretical $C_m$ data for columns in which $M_{col}<M_2<M_{cr}$: (a) reinforced concrete columns; (b) composite columns subjected to minor axis bending; and (c) composite columns subjected to major axis bending ($C_m$ values for columns with $M_1/M_2 = 1.0$ not included).
Figure 5.16 - Frequency histograms comparing design Equation 5.22 with the theoretical $C_m$ data for columns in which $M_{col} < M_2 < M_c$: (a) reinforced concrete columns; (b) composite columns subjected to minor axis bending; and (c) composite columns subjected to major axis bending ($C_m$ values for columns with $M_1/M_2 = 1.0$ not included).
Figure 5.17 - Probability distributions of $C_{m(\text{des})}/C_{m(\text{th})}$ ratios computed from data for columns in which $M_{\text{col}} < M_2 < M_{\text{cr}}$: (a) reinforced concrete columns; (b) composite columns subjected to minor axis bending; and (c) composite columns subjected to major axis bending ($C_m$ values for columns with $M_1/M_2 = 1.0$ not included).
Figure 5.18 - Effects of variables on $C_{m(dex)}/C_{m(th)}$ ratio obtained from different design equations for reinforced concrete columns: (a) end eccentricity ratio (n varies from 1090 to 438 as $e/h$ increases from 0.1 to 1.0); (b) critical load ratio (n varies from 356 to 1442 to 426 as $P_u/P_c$ decreases from 0.84 to 0.45 to 0.15, respectively); (c) slenderness ratio (n varies from 346 to 2462 as $\theta/h$ increases from 10 to 30); and (d) end moment ratio (n varies from 247 to 1982 as $M_1/M_2$ increases from -0.75 to 0.75).
Figure 5.18 - continued.
Figure 5.19 - Effects of variables on $C_{m(adv)}/C_{m(adv)}$ ratio obtained from different design equations for composite columns subjected to minor axis bending: (a) end eccentricity ratio (n varies from 715 to 270 as $e/h$ increases from 0.1 to 1.0); (b) critical load ratio (n varies from 337 to 893 to 265 as $P_u/P_{cr}$ decreases from 0.84 to 0.45 to 0.15, respectively); (c) slenderness ratio (n varies from 222 to 1570 as $\theta/h$ increases from 10 to 30); and (d) end moment ratio (n varies from 181 to 1271 as $M_1/M_2$ increases from -0.75 to 0.75).
Figure 5.19 - continued.
Figure 5.20 - Effects of variables on $C_{m_{(des)}}/C_{m_{(th)}}$ ratio obtained from different design equations for composite columns subjected to major axis bending: (a) end eccentricity ratio (n varies from 691 to 270 as e/h increases from 0.1 to 1.0); (b) critical load ratio (n varies from 301 to 890 to 269 as $P_u/P_{cr}$ decreases from 0.84 to 0.45 to 0.15, respectively); (c) slenderness ratio (n varies from 184 to 1534 as θ/h increases from 10 to 30); and (d) end moment ratio (n varies from 156 to 1237 as $M_1/M_2$ increases from -0.75 to 0.75).
Figure 5.20 - continued.
Figure 5.21 - Histograms of $C_{m\text{des}}/C_{m\text{th}}$ ratios plotted at different end moment ratios for reinforced concrete columns using two of the $C_{m\text{des}}$ design equations: (a) Equation 5.21; and (b) Equation 5.22.
Figure 5.21 - continued.

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\[ C_{m\text{(des)}} / C_{m\text{(th)}} \]

\begin{align*}
\text{End Moment Ratio, } M_1 / M_2 \\
\text{Average Value} = 1.01 \\
\text{Coef. of Var.} = 0.01 \\
\text{1-Percentile} = 0.99 \\
\text{Frequency (percent)}
\end{align*}

\begin{align*}
\text{Average Value} = 1.02 \\
\text{Coef. of Var.} = 0.02 \\
\text{1-Percentile} = 0.98 \\
\text{OVERALL STATISTICS}
\end{align*}

\begin{align*}
\text{Average Value} = 1.03 \\
\text{Coef. of Var.} = 0.03 \\
\text{1-Percentile} = 0.97 \\
\text{n = 1310}
\end{align*}

\begin{align*}
\text{Average Value} = 1.06 \\
\text{Coef. of Var.} = 0.05 \\
\text{1-Percentile} = 0.96 \\
\text{n = 747}
\end{align*}

\begin{align*}
\text{Average Value} = 1.13 \\
\text{Coef. of Var.} = 0.11 \\
\text{1-Percentile} = 0.92 \\
\text{n = 474}
\end{align*}

\begin{align*}
\text{Average Value} = 1.22 \\
\text{Coef. of Var.} = 0.19 \\
\text{1-Percentile} = 0.83 \\
\text{n = 247}
\end{align*}
Figure 5.22 - Histograms of Cm(des)/Cm(th) ratios plotted at different end moment ratios for composite columns subjected to minor axis bending using two of the Cm(cep) design equations: (a) Equation 5.21; and (b) Equation 5.22.

End Moment Ratio, M1 / M2

Overall Statistics

- 1-Percentile
- Frequency (percent)
- Average Value
- Coef. of Var.
- n = 1251
- n = 1037
- n = 632
- n = 480
- n = 303
- n = 161

Ave. Value = 1.02
Coef. of Var. = 0.01
1-Percentile = 1.00

Ave. Value = 1.04
Coef. of Var. = 0.02
1-Percentile = 1.00

Ave. Value = 1.07
Coef. of Var. = 0.03
1-Percentile = 1.00

Ave. Value = 1.11
Coef. of Var. = 0.03
1-Percentile = 1.03

Ave. Value = 1.17
Coef. of Var. = 0.05
1-Percentile = 1.00

Ave. Value = 1.25
Coef. of Var. = 0.07
1-Percentile = 0.98

Ave. Value = 1.32
Coef. of Var. = 0.03
1-Percentile = 1.00

Ave. Value = 1.00
Coef. of Var. = 0.03
1-Percentile = 1.00

Ave. Value = 0.90
Coef. of Var. = 1.00
1-Percentile = 0.75

Ave. Value = 0.60
Coef. of Var. = 1.00
1-Percentile = 0.75
Figure 5.22 - continued.
Figure 5.23 - Histograms of $C_{m(des)}/C_{m(th)}$ ratios plotted at different end moment ratios for composite columns subjected to major axis bending using two of the $C_{m(des)}$ design equations: (a) Equation 5.21; and (b) Equation 5.22.
6.0 EXAMINATION OF SIMPLE REINFORCED CONCRETE BRACED FRAMES

6.1 GENERAL

Chapters 4 and 5 dealt with idealized pinned-ended isolated columns. The strength of a column that is part of a braced frame is influenced by the structural members (beams and columns) framing into the ends of the column. Such a column is subjected to axial load and end moments (equal or unequal that are caused by unbalanced beam loads) and deflects laterally between the column ends due to the presence of end moments. The axial load acting through this lateral deflection causes additional (second-order) bending moments along the column height. The second-order moments cause additional rotation of the column ends as well as the additional rotation of the members framing into the column. This, in turn, results in changes to the initial bending moments at the ends of the column and to the beam bending moments computed from a conventional elastic frame analysis. The second-order moments caused by the axial load acting through an additional eccentricity (lateral deflection) along the height of a column in a braced frame are less than the second-order moments obtained for an identical isolated column subjected to the same axial load and end moments because of the restraints provided by the beams framing into the column. Therefore, to determine the theoretical strength of a reinforced concrete column in a braced frame, it was necessary to analyze the frame as a whole using a rigorous analytical procedure that included the nonlinear geometry of the frame as well as the nonlinear material behavior of reinforced concrete cross-sections.

As discussed previously, CSA A23.3-94 (1994) and ACI 318-99 (1999) permit the
use of a moment magnifier approach to approximate the second-order moments due to the axial load acting through the lateral deflection caused by the end moments acting on a column. The larger of the column end moments ($M_2$), computed from a conventional elastic frame analysis, is magnified to include the second-order effects. The axial load ($P_u$) and magnified column moment ($M_c$) are then compared to the column cross-section axial load-bending moment ($P-M$) strength interaction diagram and if $P_u$ and $M_c$ fall inside of the cross-section interaction diagram, the column meets the strength and slenderness requirements of the Code.

The moment magnifier approach originated from classical elastic theory and was used for the allowable strength design method of structural steel in the 1960's. This approach was adopted in the United States for the ultimate strength design and in Canada for the limit states design of concrete structures and remains in use today for the design of reinforced concrete slender columns.

The second-order effects on columns in braced frames are functions of the larger end moment ($M_2$) in addition to the following parameters:

1. **Effective flexural stiffness of the column ($EI$)** (discussed in Chapter 4);
2. **Equivalent uniform bending moment diagram factor ($C_m$)** (discussed in Chapter 5);
3. **Applied axial load ($P_u$)**;
4. **Effective length ($K\ell$)**; and
5. **Sustained load factor ($\beta_s$)**, which is neglected for short-term loads.

These parameters are used in Equation (6.1) to compute the moment magnifier ($\delta_m$) for
slender reinforced concrete and composite steel-concrete columns in braced frames.

\[
\delta_{ns} = \frac{C_m}{P_u} \geq 1.0
\]

(6.1)

In Equation (6.1), \( \phi_m \) is the stability resistance factor; and \( P_{cr} \) is the critical buckling load computed from:

\[
P_{cr} = \frac{\pi^2 EI}{(K l)^2}
\]

(6.2)

The effective length factor \( (K) \) is affected by the beams resisting the rotation of the columns framing into a common joint and the resistance to rotation of the beams is affected by cracking along their length due to loading conditions.

Almost 3000 simple reinforced concrete frames in the shape of an inverted T (T-frames), shown in Figure 6.1, were used to evaluate the accuracy of the moment magnifier procedure (and its associated equations for \( EI \), \( C_m \), and \( K \)) specified in the CSA/ACI Codes for determining the strength of columns that are part of braced frames. The columns in these frames were subjected to single or double curvature bending and short-term loads. The beams framing into the columns were subjected to pattern loads causing varying beam bending moments and column end moments. For two load cases (Load Cases 5 and 6 shown in Figure 6.1), the top end of the column was also subjected to an applied bending moment.

The effects of using the proposed design equations for \( EI \) (Equation 4.26) and for \( C_m \) (Equation 5.21) in the CSA/ACI moment magnifier approach were also evaluated. In
addition, the moment magnifier equation proposed by Chen and Wang (1999) and the effective length factor equation proposed by Duan et al. (1993) were examined.

6.2 DESCRIPTION OF REINFORCED CONCRETE FRAMES STUDIED

In Chapters 4 and 5, a range of column cross-section properties, slenderness ratios, and end eccentricity ratios were used to evaluate the effective flexural stiffness $EI$ and the equivalent uniform moment diagram factor $C_m$, respectively. Reviewing the results of these studies indicated that although $EI$ was somewhat affected by cross-section properties, the effect of these properties on $C_m$ was completely insignificant. Therefore, for the analysis of simulated reinforced concrete braced frames used in this study, the cross-section properties of the columns and beams were kept constant. The lower end of a column having a gross cross-section of $305 \text{ mm} \times 305 \text{ mm}$ (12 in. $\times$ 12 in.) (Figure 6.2(a)) was framed rigidly into two beams of equal spans and having a cross-section of $305 \text{ mm}$ (12 in.) wide by $610 \text{ mm}$ (24 in.) deep (Figure 6.2(b)). The column and beam sizes selected are representative of member sizes that would be expected in buildings 4 to 8 storeys high with column spacings ranging from 6 to 12 metres (20 to 40 feet). The nominal strength of the concrete and reinforcing steel were taken as 34.5 MPa (5000 psi) and 414 MPa (60,000 psi), respectively.

The variables used to examine the moment magnifier approach for columns in reinforced concrete braced frames are as follows:

1. load patterns, end conditions at the top of the column (fix-ended or pin-ended), and the end moment applied to the upper end of the column (when pin-ended) producing
6 different load cases (Figure 6.1);

(2) the slenderness ratio of the column \(
\frac{\ell_{col}}{h_{col}}
\);  

(3) the slenderness ratio of the beams \(
\frac{\ell_{bm}}{h_{bm}}
\);  

(4) the magnitude of the beam loading controlled by the ratio of the computed beam bending moment to the yield moment \((M_{bm}/M_{y(bm)})\), where the yield moment is defined as the bending moment acting on the beam cross-section at the onset of initial yielding of the beam flexural tension steel computed from the CSA/ACI Code without using the \(\phi\) factors.

Specified values of variables used for this study are given in Tables 6.1 to 6.3. Note that the variation in column and beam slenderness ratios, listed in Tables 6.1 to 6.3, and the column upper end condition (pin-ended or fix-ended) produced a range of effective lengths that permitted the evaluation of the effective length factor \((K)\). Tables 6.1 to 6.3 also indicate that the magnitude of the beam loads shown in Figure 6.1 was varied to study the effect of yielding of the reinforcing steel in the beams. Figure 6.1 shows schematically how the loads were applied to the beams. These beam loads were applied so that the maximum computed moment in one of the beams was equal to the predefined moment that ranged from \(0.84 M_{y(bm)}\) to \(1.12 M_{y(bm)}\). Note that the ratio of the ultimate moment to yield moment for the beam cross-section, shown in Figure 6.2(b), was computed from the CSA/ACI Code (without \(\phi\) factors) as 1.12.

The purpose of this study was to simulate the strengths of frames as well as columns described by specified cross-sectional properties, material strengths, and loading conditions.
However, for computing the column theoretical strength, average values of material strengths (Table 6.4) corresponding to the specified values were used for reasons described in Section 4.3.1.

6.3 PROCEDURE USED FOR COMPUTING THEORETICAL STRENGTH OF COLUMNS IN BRACED FRAMES

The first step in computing the theoretical strength of a column was to compute the first-order bending moments acting at the column ends ($M_1$ and $M_2$) resulting from the applied beam loads and, for load cases 5 and 6 (Figure 6.1), also resulting from the applied column end moment. For computing the first-order bending moments from a conventional elastic analysis, the column and beam stiffnesses were computed as $0.7E_cI_{g(col)}$ and $0.35E_cI_{g(bm)}$, respectively. The bending moments along the length of the beams were checked to ensure that the beam bending moment ratio ($M_{bm}/M_{y(bm)}$) was at the predefined level (Tables 6.1 to 6.3). If the beam bending moments did not correspond to the desired beam bending moment ratio, the beam loads were multiplied by a scale factor to bring them to the desired level.

To determine the theoretical axial load strength ($P_{u(th)}$) of a column, the axial load was incremented to failure starting from an axial load equal to 10 percent of the concentric axial load strength of the column cross-section. The nonlinear second-order analysis procedure documented in Chapter 2 was used to increment the load to the theoretical column axial load strength $P_{u(th)}$.

For load cases 5 and 6 (Figure 6.1), the moment at the top end of the column was
applied proportionally to the axial load to maintain the predefined eccentricity that corresponded to one of the $M/P_A$ values given in Table 6.3. Therefore, the applied bending moment at the top end of the column increased at the same rate as the axial load. Since at each iteration of the axial load the bending moment at the top end of the column changed, a new first-order elastic analysis was performed and the beam loads were adjusted to maintain the beam bending moment ratio. The nonlinear second-order analysis was then used to compute the maximum bending moment along the column length for the axial load under consideration. The second-order bending moments in the beams were also monitored to ensure that the failure of the column took place prior to the failure of a beam. If the failure of one of the beams occurred before the column failure, the beam loads were adjusted and the entire process was repeated.

The computed column end moments and the maximum moment in the column from both the first-order elastic analysis and the second-order analysis were stored. The first-order end moment ratio ($M_1/M_2$) and end eccentricity ratio ($e/h = M_2/hP_{u(\theta)}$) were also computed and stored along with $P_{u(\theta)}$ for the column in the frame under consideration for use in analyses presented in the latter part of this chapter. Note that, for load cases 1 to 4 (Figure 6.1), $M_2$ was equal to the unbalanced beam moment at the bottom end of the column, whereas $M_1$ was located at the top end of the column. For these load cases, $M_1/M_2$ ratio was equal to -0.5 for load cases 1 and 2 (fix-ended) and zero for load cases 3 and 4 (pin-ended). For load cases 5 and 6 (Figure 6.1), the top end of the column was subjected to a predefined applied bending moment (Table 6.3) and the bottom end of the column was subjected to the unbalanced beam
moment. For these load cases, $M_2$ was located at the top end of the column and the $M_1/M_2$ ratio ranged from approximately -0.4 to almost 1.0. Hence, the $M_1/M_2$ ratio for columns in frames used in this study varied from -0.5 to 1.0.

Special techniques discussed in Section 3.4 were used for modeling beam-to-column joints in frames to account for the additional strength resulting from confinement effects. As shown in Figure 6.3, concrete within the gross cross-section at beam-to-column joints was modeled as highly confined and the area of the reinforcing steel in these regions was doubled for the purpose of modeling. The concrete within the gross cross-section in end portions of a column (from the face of the beam or from the top of the column to a distance equal to the overall column depth) and in end portions of a beam (from the face of the column to a distance equal to the overall beam depth) was modeled as partially confined. The remaining parts of frames were modeled with unconfined concrete outside column ties/beam stirrups and partially confined concrete inside column ties/beam stirrups. The highly confined, partially confined, and unconfined concretes are defined in Section 2.2.2.1.

6.4 DESIGN PROCEDURES USED FOR COMPUTING STRENGTH OF COLUMNS IN BRACED FRAMES

6.4.1 CSA/ACI Moment Magnifier Approach

The procedure described in this section was used to compute the slender column strength from the CSA/ACI moment magnifier approach. Note that the material and member resistance factors ($\phi_c$, $\phi_r$, and $\phi_m$ for CSA and $\phi$ and $\phi_s$ for ACI) were taken equal to 1.0 in this study.
The first step in computing the CSA/ACI strength of a slender column that is part of a braced frame is to determine the cross-section strength, which is represented by an axial load-bending moment (P-M) strength interaction diagram (similar to the one shown in Figure 5.3). The cross-section strength interaction diagram was defined by 102 points that were computed using the compatibility of strains and the equilibrium of forces acting on the cross-section. For computing the CSA/ACI cross-section strength, it was assumed that

1. the strains are linearly proportional;
2. the maximum concrete strain \( \varepsilon_{cu} = 0.0035 \) exists at the extreme compression fibre as given in CSA A23.3-94;
3. the compressive stress in concrete is represented by a rectangular stress block as defined in CSA A23.3-94;
4. the specified concrete strength is used in computing the maximum concrete stress in the stress block; and
5. the concrete has no strength in tension.

To develop the points on the cross-section strength interaction diagram, the strain at the extreme compression fibre was held constant at \( \varepsilon_{cu} = 0.0035 \), while the strain at the extreme fibre on the opposite face was incremented from a strain that equaled the maximum computed tensile strain at pure bending up to a strain that was equal to the uniform compressive strain required across the entire cross-section for pure compression. The summation of forces acting on concrete and reinforcing steel at each increment of strain generated one point on the cross-section axial force-bending moment interaction diagram.
The entire interaction diagram for a column cross-section was defined by 102 points, as stated earlier.

The CSA/ACI moment magnifier procedure uses the moment magnifier $\delta_{nt}$ and the larger of the two column end moments $M_2$ obtained from a conventional elastic frame analysis to compute a magnified moment $M_c$ used for design of slender columns:

$$M_c = \delta_{nt} M_2$$

Equation 6.3 can be used to obtain the bending moment resistance of a column in a frame for a given level of axial load ($P_u$) directly from the cross-section strength interaction diagram. To do this, the cross-section bending moment resistance ($M_{cr}$) is substituted for the magnified column moment ($M_c$) in Equation 6.3. Then, the larger of the two end moments ($M_2$), which can be applied to the column at the given axial load $P_u$ is computed by solving Equation 6.3 for $M_2$:

$$M_2 = \frac{M_{cr}}{\delta_{nt}}$$

To generate the column axial load-bending moment interaction diagram, the cross-section bending moment resistance ($M_{cr}$ in Figure 5.3) for each level of axial load ($P_u$) was divided by $\delta_{nt}$. Note that the maximum axial load that can be applied to a slender column is less than the pure axial load resistance of the cross-section ($P_o$) and is also less than the column critical load resistance ($P_{cr}$) computed from Equation 6.2. Hence, the points on the column strength interaction curve were generated for $P_u$ values that were lower than both $P_o$ and $P_{cr}$.
Note that, for reinforced concrete columns studied in this chapter ($f'_c = 34.5$ MPa or 5000 psi), $P_o$ was computed from $0.8(0.85f'_c)(A_g - A_{rs}) + f_rA_{rs}$.

For this study, the stability resistance factor ($\phi_m$) was set equal to 1.0 and the equivalent uniform moment diagram factor ($C_m$), which is a function of column end moments ($M_1$ and $M_2$), was taken from Equation 5.21. Note that Equation 5.21 is essentially the same as the CSA/ACI equation for $C_m$ (Equation 5.5), with the exception that the lower limit of 0.4 is not used on $C_m$ in Equation 5.21. Also note that both Equation 5.5 and 5.21 would give same values of $C_m$, because $M_1 / M_2$ ratios ranged from -0.5 to 1.0 for the columns analyzed in this chapter. Therefore, the CSA/ACI moment magnifier $\delta_{ns}$ was computed from Equation 6.5, which was obtained by substituting $\phi_m = 1.0$ and $C_m = 0.6 + 0.4(M_1/M_2)$ in Equation 6.1:

$$
\delta_{ns} = \frac{0.6 + 0.4 \frac{M_1}{M_2}}{1 - \frac{P_u}{P_{cr}}} \geq 1.0 \quad (6.5)
$$

In Equation 6.5, $P_u$ is the axial load level under consideration. The column critical load ($P_{cr}$) was computed from Equation 6.2 using the CSA/ACI equation for $EI$ (Equation 4.8) and the Jackson-Moreland alignment chart for the effective length factor $K$ (Equation 6.6) used for columns in braced frames:

$$
\frac{G_A}{4} \left( \frac{\pi}{K} \right)^2 + \left( \frac{G_A + G_B}{2} \right) \left( 1 - \frac{\pi/K}{\tan(\pi/K)} \right) + \frac{2\tan(\pi/2K)}{\pi/K} - 1 = 0 \quad (6.6)
$$
In Equation 6.6, \( G_A \) and \( G_B \) are the relative stiffnesses of the upper \((G_A)\) and lower \((G_B)\) joints framing into the column ends and were computed from:

\[
G_A = \frac{\sum^A \left( \frac{EI}{\ell} \right)_{col}}{\sum^A \left( \frac{EI}{\ell} \right)_{bm}} = \frac{\sum \text{stiffnesses of columns meeting at joint A}}{\sum \text{stiffnesses of beams meeting at joint A}} \tag{6.7}
\]

and

\[
G_B = \frac{\sum^B \left( \frac{EI}{\ell} \right)_{col}}{\sum^B \left( \frac{EI}{\ell} \right)_{bm}} = \frac{\sum \text{stiffnesses of columns meeting at joint B}}{\sum \text{stiffnesses of beams meeting at joint B}} \tag{6.8}
\]

A graphical representation of Equation 6.6 in the form of a commonly used alignment chart is given in Figure 6.4. Note that Figure 6.4 also shows the range of \( K \) examined in this study.

In Equations 6.7 and 6.8, \( EI \) values were computed from \( 0.7E_c I_{g(col)} \) and \( 0.35E_c I_{g(bm)} \) for columns and beams, respectively, as given by the CSA and ACI Code. For frames used in this study (Figure 6.1), the upper joint (joint A) of the column has no beams framing into it. The upper joint of the column is either pin-ended or fix-ended and, therefore, \( G_A \) is theoretically infinity or zero, respectively. To avoid numerical problems in solving Equation 6.6, \( G_A \) was set equal to 1000 when the upper end was pin-ended and taken as 0.001 when the upper end of the column was fix-ended.

For an \( M_1/M_2 \) ratio, \( M_2 \) values were computed from the procedure described above for all levels of axial load \((P_a)\) that were lower than or equal to both \( P_o \) and \( P_{cr} \). This generated the column axial load-bending moment interaction diagram for the \( M_1/M_2 \) ratio.
under consideration. Repeating the step for all desired $M_1/M_2$ ratios generated a series of column strength interaction curves. The CSA/ACI axial load strength ($P_u$) of a column was then computed from linear interpolation of points on these interaction diagrams using the first order $M_1/M_2$ and $e/h$ ratios determined earlier for that column from the theoretical procedure described in Section 6.3.

6.4.2 Procedure Used for Computing Column Strength Using Proposed $EI$ Equation

The procedure outlined in the previous section (Section 6.4.1) is applicable only when Equation 4.8 or a similar equation is used for $EI$ in Equation 6.2. This is because $EI$ from Equation 4.8 remains constant regardless of the magnitude of end moments and, therefore, $P_{cr}$ remains constant. As a result, the moment magnifier ($\delta_{en}$) also remains constant. On the other hand, the proposed design equation for $EI$ (Equation 4.26) is dependent upon the end eccentricity ratio ($e/h$) computed from end moments, making $EI$ variable. This affects $P_{cr}$ which, in turn, affects $\delta_{en}$. Therefore, an iterative approach was used to determine the slender column strength interaction diagram using the proposed design equation for $EI$.

The smaller of the cross-section pure axial load strength ($P_o$) and the column critical load strength ($P_{cr}$) was used to establish the upper limit for the axial load levels to be used in determining the slender column strength interaction diagram. For each level of axial load ($P_u$), the end eccentricity ($e$) was iterated until $e \times P_u \times \delta_{en} = M_{cr}$ was satisfied within a tolerance of 0.01 percent. The moment magnifier ($\delta_{en}$) was computed from Equation 6.5 for a given
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$M_1/M_2$ ratio for each iteration of end eccentricity using the proposed $EI$ equation (Equation 4.26) and Equation 6.5 for the effective length factor ($K$). This generated one point on the column strength interaction curve for the $M_1/M_2$ ratio under consideration. Repeating this step for all axial load levels generated the entire strength interaction curve for the $M_1/M_2$ ratio under consideration. Such column strength interaction diagrams were generated for a series of $M_1/M_2$ ratios and were used for computing the strength ($P_o$) of a column from linear interpolation for the first-order $M_1/M_2$ and $e/h$ ratios calculated for that column from the theoretical procedure described earlier (Section 6.3).

6.4.3 Procedure Used for Computing Column Strength Using Alternative Equation for $K$-factor

A simple equation for the effective length factor was proposed by Duan et al. (1993) for columns in nonsway frames:

$$K = 1 - \frac{1}{5 + 9G_A} - \frac{1}{5 + 9G_B} - \frac{1}{10 + G_A G_B} \quad (6.9)$$

In addition, the Commentary to the ACI Code permits the use of expressions proposed by Cranston (1972), where $K$ is taken as the smaller of the following for columns in nonsway frames:

$$K = 0.7 + 0.05(G_A + G_B) \leq 1.0 \quad (6.10a)$$

$$K = 0.85 + 0.05G_{\text{min}} \leq 1.0 \quad (6.10b)$$

in which $G_{\text{min}}$ is the smaller of $G_A$ and $G_B$. A comparison of $K$ computed from Equation 6.6
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(Jackson-Moreland alignment chart), Equation 6.9 (Duan et al. 1993) and Equation 6.10 (Cranston 1972) is shown in Figure 6.5. The following observations can be made from Figure 6.5:

(1) Cranston's expressions produce effective length factors that are very conservative compared to the values obtained from the Jackson-Moreland alignment chart when the upper joint is fix-ended ($G_A = 0$).

(2) Duan's equation produces effective length factors that are almost the same as those obtained from the Jackson-Moreland alignment chart when the upper joint is fix-ended ($G_A = 0$).

(3) When the upper joint is pin-ended ($G_A = \infty$), both Duan's and Cranston's equations produce conservative results compared to the effective length factor computed from the Jackson-Moreland alignment chart.

To investigate the effect of the $K$ factor computed from Duan et al. on the strength of slender reinforced concrete columns, Equation 6.9 was used in place of Equation 6.6 and the procedure described in Section 6.4.1 or 6.4.2 was followed when $EI$ for use in $P_{cr}$ (Equation 6.2) was determined from the CSA/ACI equation (Equation 4.8) or from the proposed equation (Equation 4.26), respectively. No further analysis was performed with the Cranston's equation, because it produced very conservative values of $K$ for fix-ended columns in Figure 6.5(a).
6.4.4 Procedure Used for Computing Column Strength Using Alternative Equation for Moment Magnification

Chen and Wang (1999) proposed an equation for computing the magnified moment due to second-order effects in steel columns:

\[
\delta_{ns} = \frac{0.64 \left(1 + \frac{M_1}{M_2}\right)}{1 - \frac{P_u}{P_{cr}}} - 0.32 \frac{M_1}{M_2} \geq 1.0
\]  

(6.11)

Note that the effects of unequal end moments \((M_1\) and \(M_2\)) are already included in Equation 6.11. The effect of using Equation 6.11 on the strength of reinforced concrete slender columns was examined by substituting this equation in place of Equation 6.5 and following the procedure described in Section 6.4.1, 6.4.2 or 6.4.3, as needed.

6.5 SIMULATION AND EXAMINATION OF COMPUTED STRENGTHS OF COLUMNS IN BRACED Reinforced CONCRETE FRAMES

To evaluate the moment magnifier approach used for determining column strength, 2960 simple reinforced concrete frames were simulated. The cross-section and material properties of columns and beams used in these frames are discussed in Section 6.2. The combinations of support conditions and applied loads produced six different load cases (Figure 6.1). Each frame had a different combination of column slenderness ratio, beam slenderness ratio, support condition, and beam loads. The theoretical column strengths \((P_{th})\) were computed from the procedure outlined in Section 6.3. The column strengths
were also calculated from the design procedures described in Section 6.4 using several combinations of (a) Equation 6.5 (CSA/ACI) or Equation 6.11 (Chen and Wang 1999) used for computing the moment magnifier $\delta_m$; (b) Equation 4.8 (CSA/ACI) or Equation 4.26 (proposed) for the column effective flexural stiffness $EI$ used in the computation of $P_{cr}$; and (c) Equation 6.6 (Jackson-Moreland alignment chart) or Equation 6.9 (Duan et al. 1993) used for computing the effective length factor $K$. Finally the strength ratios were computed by dividing $P_{u(h)}$ by $P_{u(des)}$, which were statistically analyzed and are presented in the sections that follow.

6.5.1 Overview of Strength Ratio Statistics

Only the columns where the theoretically-computed maximum magnified bending moment due to second-order effects along the height of the column was greater than the larger first-order end moment ($M_2$) were included in the analysis because for these columns $\delta_{nr}$ exceeds 1.0. As a result, the analysis presented here includes data for 2168 of the 2960 braced T-frames initially used for this study. Note that, for load cases 1, 2, 3 and 4, the first-order the $e/h$ ratios ($M_2/(P_{u(h)}h)$) ranged from 0.013 to 0.192 and $M_2$ was located at the bottom end of the column. For load cases 5 and 6 a full range of first-order $e/h$ ratios ($M_2/(P_{u(h)}h)$) from 0.1 to 1.0 was used by applying $M_2$ to the top end of the column, which is pin-ended.

Figure 6.6 was prepared from the combined data for load cases 1 to 4 and shows histograms and statistics of column strength ratios ($P_{u(h)}/P_{u(des)}$). For computing these
strength ratios, $P_{u(dex)}$ was calculated in four different ways by using the procedure described in Section 6.4, Equation 6.6 (Jackson-Moreland alignment chart) for the column effective length factor $K$, and one of the following four sets of $\delta_{ns}$ and $EI$ equations:

(a) The current CSA/ACI equations for both the moment magnifier $\delta_{ns}$ (Equation 6.5) and the effective flexural stiffness $EI$ (Equation 4.8) used for Figure 6.6(a).

(b) The current CSA/ACI equation for $\delta_{ns}$ (Equation 6.5) and the proposed equation for $EI$ (Equation 4.26) used for Figure 6.6(b).

(c) The Chen and Wang (1999) equation for $\delta_{ns}$ (Equation 6.11) and the current CSA/ACI equation for $EI$ (Equation 4.8) used for Figure 6.6(c).

(d) The Chen and Wang equation for $\delta_{ns}$ (Equation 6.11) and the proposed equation for $EI$ (Equation 4.26) used for Figure 6.6(d).

Note that strength ratios ($P_{u(th)}/P_{u(dex)}$) greater than 1.0 signify that $P_{u(dex)}$ is conservative and vice versa. Figure 6.6 leads to the following conclusions:

(1) A comparison of Figures 6.6(a) and (b) indicates, as does the comparison of Figures 6.6(c) and (d), that the variability of the strength ratios reduces significantly and the histogram of the strength ratios becomes much tighter when the proposed design equation (Equation 4.26) is used in place of the CSA/ACI equation (Equation 4.8) for $EI$. This holds regardless of the fact whether $\delta_{ns}$ is computed from Equation 6.5 (CSA/ACI) or from Equation 6.11 (Chen and Wang) and further reinforces the conclusion reached earlier for isolated columns in Chapter 4 of this study.

(2) A comparison of Figures 6.6(a) and (c), as well as that of Figures 6.6(b) and (d),
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shows practically no change in statistics and histograms of strength ratios. This indicates that both the CSA/ACI and Chen and Wang equations for \( \delta_{nr} \) (Equations 6.5 and 6.11) produce very similar results, regardless whether \( EI \) is computed from Equation 4.8 (CSA/ACI) or Equation 4.26 (proposed). Hence, there appears to be no advantage in replacing the CSA/ACI \( \delta_{nr} \) equation (Equation 6.5) by the Chen and Wang \( \delta_{nr} \) equation (Equation 6.11) for the type of reinforced concrete frames studied.

(3) In all four cases, the one-percentile strength ratios are close to 1.0, whereas the average strength ratios always exceed 1.0, as shown in Figure 6.6.

Figure 6.7 was also plotted from the combined data for load cases 1 to 4. The strength ratios used for this figure were determined in the same manner as for Figure 6.6, except that the effective length factor \( K \) for Figure 6.7 was computed from Equation 6.9 (Duan et al. 1993) as opposed to Equation 6.6 (Jackson-Moreland alignment chart) used for Figure 6.6. This helped to study the effect on strength ratios using different equations for the \( K \) factor. A comparison of Figures 6.7(a), (b), (c) and (d) with Figures 6.6(a), (b), (c) and (d), respectively, clearly indicates that using the Duan et al. equation for the \( K \) factor significantly increases the average strength ratio and related coefficient of variation, although the one-percentile strength ratio in not as significantly affected. This is valid for all four combinations of \( \delta_{nr} \) and \( EI \) equations studied. Hence, there appears to be no advantage in replacing the Jackson-Moreland alignment chart (Equation 6.6) by the Duan et al. \( K \) factor equation (Equation 6.9) for the type of reinforced concrete frames studied.

Figures 6.8 and 6.9 were plotted from the combined data for load cases 5 and 6 for
which an external bending moment was applied at the top end of the column in addition to the beam and column loads. These figures were prepared in the same manner as and can be compared to Figures 6.6 and 6.7, respectively, representing load cases 1 to 4. Comparisons of Figures 6.6 and 6.8 and Figures 6.7 and 6.9 indicate that the conclusions stated earlier for load cases 1 to 4 are valid also for load cases 5 and 6. Furthermore, lower average strength ratios, coefficients of variation, and one-percentile strength ratios were obtained for load cases 5 and 6 as opposed to the same values computed for load cases 1 to 4. The compactness of the histograms in Figures 6.6 (b) and 6.8(b) and the average strength ratios greater than unity (1.01-1.12), relatively low coefficients of variation (0.06-0.10), and one-percentile strength ratios not significantly lower than one (0.90-0.95) shown in these figures indicate that best column strength estimates are obtained using Equation 6.6 for \( K \) (Jackson-Moreland alignment chart), Equation 6.5 for \( \delta_n \) (CSA/ACI), and Equation 4.26 for \( EI \) (proposed) for the type of reinforced concrete frames studied.

Figures similar to Figures 6.6(b), 6.7(b), 6.8(b), and 6.9(b) were prepared also with the numerator of Equation 6.5 replaced by \( C_m \) taken from Equation 5.22. The strength ratio statistics were then recomputed and were found to be within 1% of those shown in Figures 6.6(b), 6.7(b), 6.8(b), and 6.9(b). Therefore, histograms and statistics shown in Figures 6.6(b), 6.7(b), 6.8(b), and 6.9(b) are also applicable when \( C_m \) in Equation 6.5 is taken from Equation 5.22.
6.5.2 Effects of Variables on Strength Ratios

For load cases 1, 2, 3 and 4, the end eccentricity ratio \( (e/h = M_2 / P_{u(h/2)}h) \) ranges from 0.013 to 0.192 and the end moment ratio \( (M_1 / M_2) \) is equal to -0.5 or 0.0 when the upper end of the column is fix-ended or pin-ended, respectively. Therefore, showing the effects of \( e/h \) and \( M_1 / M_2 \) on the strength ratio \( (P_{u(h/2)}/P_{u(des)}) \) will not be shown for load cases 1 to 4, because no trends were readily visible in the ranges of \( e/h \) and \( M_1 / M_2 \) studied. Furthermore, the beam moment ratio \( (M_{bm}/M_{y(bm)}) \) displayed little effect on the strength ratio either for load cases 1 to 4 or for load cases 5 to 6 within the range of \( M_{bm}/M_{y(bm)} \) studied (0.84 to 1.12) and those plots will not be shown for any of the load cases.

The effect of column slenderness ratio \( (K\theta/r) \) on the column strength ratio is examined in Figure 6.10. This figure was prepared from load cases 1 to 4 combined involving 648 reinforced concrete frames, where the theoretically-computed maximum magnified moment due to second-order effects along the column height was greater than \( M_2 \). Note that, for computing the strength ratios shown in figure 6.10, \( P_{u(des)} \) was determined by using the Jackson-Moreland alignment chart (Equation 6.6) for the column effective length factor, in addition to using one of the following sets of equations for \( \delta_{ns} \) and \( EI \):

(a) The current CSA/ACI equations for both \( \delta_{ns} \) (Equation 6.5) and \( EI \) (Equation 4.8) for Figure 6.10(a).

(b) The current CSA/ACI equation for \( \delta_{ns} \) (Equation 6.5) and the proposed equation for \( EI \) (Equation 4.26) for Figure 6.10(b).

(c) The Chen and Wang equation for \( \delta_{ns} \) (Equation 6.11) and the proposed equation for
Note that \(a/h\) values used for computing \(EI\) from Equation 4.26 were calculated from the axial loads and bending moments obtained from the conventional (first-order) elastic analysis. Figure 6.10 leads to the following conclusions:

1. Figure 6.10(a) shows that, when the current CSA/ACI equations are used for both \(\delta_{nr}\) and \(EI\) (Equations 6.5 and 4.8), the strength ratios become increasingly conservative as \(K\theta r\) increases approximately from 40 to 110.

2. A comparison of Figures 6.10(a) and (b) shows that, when the proposed equation for \(EI\) (Equation 4.26) is used, the effect of \(K\theta r\) on the strength ratios becomes insignificant. This is expected because \(\theta r\) is one of the variables used for Equation 4.26.

3. A comparison of Figures 6.10(b) and (c) indicates that, when the proposed \(EI\) equation (Equation 4.26) is used for computing \(EI\), similar strength ratios are obtained, regardless of the fact whether \(\delta_{nr}\) is computed from Equation 6.5 (CSA/ACI) or from Equation 6.11 (Chen and Wang). This reinforces an earlier conclusion that there appears to be no advantage in replacing the CSA/ACI \(\delta_{nr}\) equation (Equation 6.5) by the Chen and Wang \(\delta_{nr}\) equation (Equation 6.11) for the type of frames studied.

The strength ratios plotted in Figure 6.11 were computed also for load cases 1 to 4 involving the same frames \((n=648)\) and in the same manner as for Figure 6.10, with the exception that the effective length factor \(K\) for \(P_{ud(ey)}\) of columns in Figure 6.11 was taken
from Equation 6.9 (Duan et al.), whereas the $K$ factor for $P_{u(\text{des})}$ of columns in Figure 6.10 was calculated from Equation 6.6 (Jackson-Moreland alignment chart). Figure 6.11(a), (b), and (c) show two distinct groups of data, one for fix-ended columns and the other for pin-ended columns. This is expected because Equation 6.9 (Duan et al. 1993) computes more conservative values of $K$ for pin-ended columns than those given by Equation 6.6 (Jackson-Moreland alignment chart). Furthermore, both Equations 6.6 and 6.9 compute very close values of $K$ for fix-ended columns as indicated by Figure 6.5. This can also be seen by comparing Figures 6.11(a), (b), and (c) to Figures 6.10(a), (b), and (c), respectively, and reinforces an earlier conclusion that there appears to be no advantage in replacing the Jackson-Moreland alignment chart (Equation 6.6) by the Duan et al. equation (Equation 6.9) for the type of frames studied.

The effects of $e/h$, $M_1/M_2$, and $K\theta/r$ on the strength ratios for the combined data from load case 5 and 6 are shown in Figures 6.12 through 6.17. These figures were plotted for 1520 reinforced concrete frames, where the theoretically-computed maximum magnified moment due to second-order effects along the column height was greater than $M_2$. Consequently, as $e/h$ increases from 0.1 to 1.0 the number of data points in Figures 6.12 to 6.17 decreases from 220 to 132. Note that $M_1/M_2$ ratio in these figures ranges from -0.40 (double curvature bending) to 0.99 (single curvature bending).

The Jackson-Moreland alignment chart (Equation 6.6) was used to calculate the effective length factor $K$ and the corresponding value of $P_{u(\text{des})}$ in computations of strength ratios shown in Figures 6.12, 6.13 and 6.14. In addition, the following combinations of $\delta_{nr}$
and $EI$ were used for Figures 6.12 to 6.14:

(a) The current CSA/ACI equations for both $\delta_{nr}$ and $EI$ (Equations 6.5 and 4.8) for Figure 6.12.

(b) The current CSA/ACI equation for $\delta_{nr}$ and the proposed equation for $EI$ (Equations 6.5 and 4.26) for Figure 6.13.

(c) The Chen and Wang equation for $\delta_{nr}$ and the proposed equation for $EI$ (Equations 6.11 and 4.26) for Figure 6.14.

Figures 6.12 to 6.14 lead to the following conclusions:

1. Figure 6.12 shows very large spread in strength ratios when the current CSA/ACI equations are used for both $\delta_{nr}$ and $EI$ (Equations 6.5 and 4.8). This is particularly valid for strength ratios with $e/h < 0.3$, $-0.4 < M_i/M_2 < 1.0$ (entire range studied), and $K\theta/r > 70$, as indicated by Figures 6.12(a), (b), and (c), respectively.

2. A comparison of Figures 6.12 and 6.13 indicates that the spread in strength ratios reduces very significantly when the proposed $EI$ equation (Equation 4.26) is used in lieu of the CSA/ACI $EI$ equation (Equation 4.8). In fact, strength ratios shown in Figures 6.13(a), (b), and (c) appear to be almost independent of $e/h$, $M_i/M_2$, and $K\theta/r$ ratios, respectively. This is expected because $e/h$ and $\theta h$ are included as variables in Equation 4.26.

3. A comparison of Figures 6.13 and 6.14 shows no improvement in strength ratios when the Chen and Wang $\delta_{nr}$ equation (Equation 6.11) is used in place of CSA/ACI $\delta_{nr}$ equation (Equation 6.5) with $EI$ computed from the proposed equation (Equation
4.26) in both cases. Again, this indicates no advantage in replacing the CSA/ACI $\delta_{nr}$ equation by the Chen and Wang $\delta_{nr}$ equation.

Equation 6.9 (Duan et al. 1993) was employed to calculate the $K$ factor used in $P_{u(\text{des})}$ for computing the strength ratios plotted in Figures 6.15 to 6.17. The $\delta_{nr}$ and $EI$ equations used in $P_{u(\text{des})}$ for preparing Figures 6.15, 6.16, and 6.17 are identical to those for Figures 6.12, 6.13, and 6.14, respectively. This permitted an examination of the effect on strength ratios of using the Duan et al. equation for the $K$ factor. Comparisons of Figures 6.15, 6.16, and 6.17 with Figures 6.12, 6.13, and 6.14, respectively, show higher spreads in strength ratios plotted in Figures 6.15 to 6.17, where the Duan et al. equation (Equation 6.9) was used for computing $K$, as opposed to Figures 6.12 to 6.14, where the Jackson-Moreland alignment chart (Equation 6.6) was employed for determining the $K$ factor. Again, this should be expected as explained earlier and indicates no advantage of replacing the Jackson-Moreland alignment chart by the Duan et al. equation for the types of frames studied.

### 6.5.3 Strength Ratios for Individual Load Cases Produced by Using the Proposed Design Equation for $EI$

It is evident from Figures 6.6 to 6.17 that the least variable strength ratios ($P_{u(\text{th})}/P_{u(\text{des})}$) are obtained when $P_{u(\text{des})}$ is computed from the Jackson-Moreland alignment chart for $K$ (Equation 6.6), the CSA/ACI equation for $\delta_{nr}$ (Equation 6.5), and the proposed equation for $EI$ (Equation 4.26). The accuracy of $P_{u(\text{des})}$ based on these equations for $K$, $\delta_{nr}$, and $EI$ was further examined from histograms and related statistics of strength ratios prepared for load cases 1 to 6 individually. The resulting histograms and statistics shown in Figure 6.18 are...
very similar to those given in Figure 6.6(b) for load cases 1 to 4 combined and in Figure 6.8(b) for load cases 5 and 6 combined.

Histograms and related statistics similar to those shown in Figure 6.18 were prepared also with the numerator of Equation 6.5 replaced by $C_m$ taken from Equation 5.22. The histograms and statistics so prepared were almost identical to those given in Figure 6.18. Hence, Figure 6.18 also represents strength ratios when $C_m$ in Equation 6.5 is taken from Equation 5.22.

### 6.6 DESIGN APPLICATION

The analysis presented in this chapter is limited to columns that are part of simple reinforced concrete frames. Nevertheless, it can be concluded that the accuracy of the moment magnifier method can be greatly improved by adopting the proposed design equation for the computation of the column effective flexural stiffness. It is suggested that, for design purposes, the magnified moment due to the member curvature effects in reinforced concrete and concrete-encased composite columns be computed from the CSA/ACI equation (Equation 1.1) [CSA A23.3 Equation (10-16) or ACI Equation (10-10)] with $C_m$ taken from Equation 5.23 or 5.24, and $P_{cr}$ based on Equations 6.6 and 4.27 for $K$ and $EI$, respectively. A comparison of the proposed method with the current CSA and ACI methods is shown through an example given in Appendix A.
Table 6.1  Specified properties of reinforced concrete frames used for Load Cases 1 and 2.*

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>$\ell_{col}/h_{col}$</td>
<td>15; 17.5; 20; 22.5; 25; 27.5; 30; 32.5; 35; 37.5; 40; 42.5; 45; 47.5; 50; 52.5; 55</td>
<td>17</td>
</tr>
<tr>
<td>$\ell_{lm}/h_{lm}$</td>
<td>10; 15; 20; 30; 40</td>
<td>5</td>
</tr>
<tr>
<td>$M_{lm}/M_{y(hum)}$</td>
<td>0.84; 1.00; 1.06; 1.12</td>
<td>4</td>
</tr>
<tr>
<td>Load Cases</td>
<td>1 and 2</td>
<td>2</td>
</tr>
</tbody>
</table>

* Total number of simulated frames equals $(17 \times 5 \times 4 =) 340$ for Load Case 1 and 340 for Load Case 2. For these two load cases the upper end of the column is fixed against rotation, as shown in Figure 6.1(a) and (b). Each frame has a different combination of specified properties shown above.

Table 6.2  Specified properties of reinforced concrete frames used for Load Cases 3 and 4.*

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>$\ell_{col}/h_{col}$</td>
<td>10; 12.5; 15; 17.5; 20; 22.5; 25; 27.5; 30; 32.5; 35; 37.5; 40; 42.5; 45</td>
<td>15</td>
</tr>
<tr>
<td>$\ell_{lm}/h_{lm}$</td>
<td>10; 15; 20; 30; 40</td>
<td>5</td>
</tr>
<tr>
<td>$M_{lm}/M_{y(hum)}$</td>
<td>0.84; 1.00; 1.06; 1.12</td>
<td>4</td>
</tr>
<tr>
<td>Load Cases</td>
<td>3 and 4</td>
<td>2</td>
</tr>
</tbody>
</table>

* Total number of simulated frames equals $(15 \times 5 \times 4 =) 300$ for Load Case 3 and 300 for Load Case 4. For these two load cases the upper end of the column is pin-ended, as shown in Figure 6.1(c) and (d). Each frame has a different combination of specified properties shown above.
Table 6.3  Specified properties of reinforced concrete frames used for Load Cases 5 and 6.*

<table>
<thead>
<tr>
<th>Properties</th>
<th>Specified Values</th>
<th>Number of Specified Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{col}/h_{col}$</td>
<td>15; 20; 25; 30; 35; 40</td>
<td>6</td>
</tr>
<tr>
<td>$t_{bm}/h_{bm}$</td>
<td>10; 15; 20; 30; 40</td>
<td>5</td>
</tr>
<tr>
<td>$M_{bm}/M_{y(3b)}$</td>
<td>0.84; 1.00; 1.06; 1.12</td>
<td>4</td>
</tr>
<tr>
<td>$M/(P_e h)$</td>
<td>0.1; 0.2; 0.3; 0.4; 0.6; 0.8; 1.0</td>
<td>7</td>
</tr>
<tr>
<td>Load Cases</td>
<td>5 and 6</td>
<td>2</td>
</tr>
</tbody>
</table>

* Total number of simulated frames equals $(6 \times 5 \times 4 \times 7 =) 840$ for Load Case 5 and 840 for Load Case 6. For these two load cases a bending moment is applied to the upper end of the column which is pin-ended, as shown in Figure 6.1(e) and (f). Each frame has a different combination of specified properties shown above.

Table 6.4  Average values of material properties used for computing the theoretical strength of reinforced concrete frames.

(a) Concrete

<table>
<thead>
<tr>
<th>Specified Compressive Strength $f'_c$ MPa (psi)</th>
<th>Compressive Strength $f_c$ MPa (psi)</th>
<th>Modulus of Rupture $f_r$ MPa (psi)</th>
<th>Elastic Modulus $E_c$ MPa (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>34.5 (5000)</td>
<td>26.6 (3861)</td>
<td>3.2 (459)</td>
<td>23,331 (3383)</td>
</tr>
</tbody>
</table>

(b) Reinforcing Steel

<table>
<thead>
<tr>
<th>Specified Yield Strength $f_{yu}$ MPa (psi)</th>
<th>Static Yield Strength $f_y$ MPa (psi)</th>
<th>Elastic Modulus $E_s$ MPa (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>414 (60,000)</td>
<td>461 (66,800)</td>
<td>199,955 (29,000)</td>
</tr>
</tbody>
</table>

* Average compressive strength of in-place concrete in members is based on a rate of loading to failure in 180 minutes.
Figure 6.1 - Frame configurations and load cases used for this study.
\( f_c = 34.5 \text{ MPa} \quad (5,000 \text{ psi}) \)

\( f_y = 414 \text{ MPa} \quad (60,000 \text{ psi}) \)

- **Partially Confined**
- **Unconfined**

Refer to Figure 2.4 and Section 2.2.2.1 for equations describing the strength of highly, partially and unconfined concretes.

Figure 6.2 - Cross-section details of reinforced concrete frames used for this study.

Figure 6.3 - Member properties used in this study for the theoretical analysis of simple reinforced concrete frames.
Figure 6.4 - Jackson-Moreland alignment chart for braced frames (Equation 6.5) showing the range of effective length factor examined in this study.
Figure 6.5 - Comparison of effective length factors ($K$) computed from different procedures when (a) the upper joint of the column is fix-ended; and (b) the upper joint of the column is pin-ended.
Figure 6.6 - Histograms for strength ratios for load cases 1, 2, 3, and 4 (n = 648) calculated by using Equation 6.6 for $K$ (Jackson-Moreland alignment chart) and (a) Equations 6.5 for $\delta_{ns}$ and 4.8 for $EI$ (both CSA/ACI); (b) Equations 6.5 for $\delta_{ns}$ (CSA/ACI) and 4.26 for $EI$ (proposed); (c) Equations 6.11 for $\delta_{ns}$ (Chen and Wang) and 4.8 for $EI$ (CSA/ACI); and (d) Equations 6.11 for $\delta_{ns}$ (Chen and Wang) and 4.26 for $EI$ (proposed).
Figure 6.7 - Histograms for strength ratios for load cases 1, 2, 3, and 4 (n = 648) calculated by using Equation 6.9 for $K$ (Duan et al.) and (a) Equations 6.5 for $\delta_{rs}$ and 4.8 for $EI$ (both CSA/ACI); (b) Equations 6.5 for $\delta_{rs}$ (CSA/ACI) and 4.26 for $EI$ (proposed); (c) Equations 6.11 for $\delta_{rs}$ (Chen and Wang) and 4.8 for $EI$ (CSA/ACI); and (d) Equations 6.11 for $\delta_{rs}$ (Chen and Wang) and 4.26 for $EI$ (proposed).
Figure 6.8 - Histograms for strength ratios for load cases 5 and 6 \((n = 1520)\) calculated by using Equation 6.6 for \(K\) (Jackson-Moreland alignment chart) and (a) Equations 6.5 for \(\delta_{u}\) and 4.8 for \(EI\) (both CSA/ACI); (b) Equations 6.5 for \(\delta_{u}\) (CSA/ACI) and 4.26 for \(EI\) (proposed); (c) Equations 6.11 for \(\delta_{u}\) (Chen and Wang) and 4.8 for \(EI\) (CSA/ACI); and (d) Equations 6.11 for \(\delta_{u}\) (Chen and Wang) and 4.26 for \(EI\) (proposed).
Figure 6.9 - Histograms for strength ratios for load cases 5 and 6 ($n = 1520$) calculated by using Equation 6.9 for K (Duan et al.) and (a) Equations 6.5 for $\delta_{ax}$ and 4.8 for $EI$ (both CSA/ACI); (b) Equations 6.5 for $\delta_{ax}$ (CSA/ACI) and 4.26 for $EI$ (proposed); (c) Equations 6.11 for $\delta_{ax}$ (Chen and Wang) and 4.8 for $EI$ (CSA/ACI); and (d) Equations 6.11 for $\delta_{ax}$ (Chen and Wang) and 4.26 for $EI$ (proposed).
Figure 6.10 - Effect of slenderness on strength ratios for load cases 1, 2, 3, and 4 (n=648) calculated from Equation 6.6 for $K$ (Jackson-Moreland alignment chart) and (a) Equations 6.5 for $\delta_n$ and 4.8 for $EI$ (both CSA/ACI); (b) Equations 6.5 for $\delta_n$ (CSA/ACI) and 4.26 for $EI$ (proposed); and (c) Equations 6.11 for $\delta_n$ (Chen and Wang) and 4.26 for $EI$ (proposed).
Figure 6.11 - Effect of slenderness on strength ratios for load cases 1, 2, 3, and 4 (n=648) calculated from Equation 6.9 for $K$ (Duan et al.) and (a) Equations 6.5 for $\delta_{ar}$ and 4.8 for $EI$ (both CSA/ACI); (b) Equations 6.5 for $\delta_{ar}$ (CSA/ACI) and 4.26 for $EI$ (proposed); and (c) Equations 6.11 for $\delta_{ar}$ (Chen and Wang) and 4.26 for $EI$ (proposed).
Figure 6.12 - Effects of variables on strength ratios for load cases 5 and 6 \((n=1520)\) calculated from Equation 6.6 for \(K\) (Jackson-Moreland alignment chart), Equation 6.5 for \(\delta_{ax}\) (CSA/ACI), and Equation 4.8 for \(EI\) (CSA/ACI): (a) end eccentricity ratio \((e/h)\); (b) column end moment ratio \((M_1/M_2)\); and (c) slenderness ratio \((K\theta/r)\).
Figure 6.13 - Effects of variables on strength ratios for load cases 5 and 6 (n=1520) calculated from Equation 6.6 for $K$ (Jackson-Moreland alignment chart), Equation 6.5 for $\delta_{nc}$ (CSA/ACI), and Equation 4.26 for $E I$ (proposed): (a) end eccentricity ratio (e/h); (b) column end moment ratio ($M_1/M_2$); and (c) slenderness ratio ($K/t\ell$).
Figure 6.14 - Effects of variables on strength ratios for load cases 5 and 6 (n=1520) calculated from Equation 6.6 for $K$ (Jackson-Moreland alignment chart), Equation 6.11 for $\delta_m$ (Chen and Wang), and Equation 4.26 for $EI$ (proposed): (a) end eccentricity ratio ($e/h$); (b) column end moment ratio ($M_1/M_2$); and (c) slenderness ratio ($K\ell/r$).
Figure 6.15 - Effects of variables on strength ratios for load cases 5 and 6 (n=1520) calculated from Equation 6.9 for $K$ (Duan et al.), Equation 6.5 for $\delta_m$ (CSA/ACI), and Equation 4.8 for $EI$(CSA/ACI): (a) end eccentricity ratio ($e/h$); (b) column end moment ratio ($M_1/M_2$); and (c) slenderness ratio ($K\theta/r$).
Figure 6.16 - Effects of variables on strength ratios for load cases 5 and 6 (n=1520) calculated from Equation 6.9 for $K$ (Duan et al.), Equation 6.5 for $\delta_m$ (CSA/ACI), and Equation 4.26 for $EI$ (proposed): (a) end eccentricity ratio ($e/h$); (b) column end moment ratio ($M_1/M_2$); and (c) slenderness ratio ($Kt/r$).
Figure 6.17 - Effects of variables on strength ratios for load cases 5 and 6 (n=1520) calculated from Equation 6.9 for $K$ (Duan et al.), Equation 6.11 for $\delta_m$ (Chen and Wang), and Equation 4.26 for $EI$ (proposed): (a) end eccentricity ratio (e/h); (b) column end moment ratio ($M_1/M_2$); and (c) slenderness ratio ($K L/r$).
Figure 6.18 - Histograms of strength ratios calculated by using Equation 6.6 for $K$ (Jackson-Moreland alignment chart), Equation 6.5 for $\delta_{as}$ (CSA/ACI), and Equation 4.26 for $EI$ (proposed) in the computation of $P_{u(des)}$: (a) Load Case 1; (b) Load Case 2; (c) Load Case 3; (d) Load Case 4; (e) Load Case 5; and (f) Load Case 6.
Figure 6.18 - continued.
7.0 SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

7.1 SUMMARY

The moment magnifier procedure is used by the CSA A23.3-94 and ACI 318-99 to approximate the second-order moments and to determine the strength of slender reinforced concrete and composite steel-concrete columns. This study was conducted to examine the accuracy of the moment magnifier ($\delta_m$), and the associated variables $E_1, C_m$ and $K$ used for computing the strength of structural concrete columns in nonsway frames subjected to short-term loads.

A theoretical model that incorporates second-order effects due to material and geometric nonlinearities was developed as a part of this study to compute the strength of isolated structural concrete columns and columns in frames. A comparison of the theoretical model to experimental results, gathered from the published literature, shows that the theoretical model computes the strength of isolated reinforced concrete columns, isolated composite steel-concrete columns, and columns in braced reinforced concrete frames with reasonable accuracy.

Over 35,000 isolated columns were simulated to statistically evaluate the parameters that affect the effective flexural stiffness ($E_1$) of slender reinforced concrete and composite columns bending about the minor or major axis of an encased structural steel shape. The existing CSA/ACI equations for $E_1$ were examined and a new equation for $E_1$ was developed using the simulated column data.

Over 38,000 isolated reinforced concrete and composite steel-concrete columns were
simulated to evaluate the variables that affect the equivalent uniform moment diagram factor \((C_m)\). The existing code equation and equations proposed by other authors for \(C_m\) were examined and two new equations for \(C_m\) were developed from the generated data. Note that one of the equations proposed in this study is identical to the current CSA/ACI equation for \(C_m\) with the exception that the lower limit on \(C_m\) has been decreased from 0.4 to 0.34.

The simulated theoretical strengths of 2960 simple reinforced concrete frames were then used to evaluate the accuracy of the CSA/ACI moment magnifier \((\delta_m)\) and the associated CSA/ACI equations currently used for computing \(EI, C_m,\) and \(K\), for reinforced concrete columns that are part of nonsway frames. The accuracy of equations for \(\delta_m, EI, K,\) and \(C_m\) proposed in this study and by other authors was also evaluated using the simulated frame data.

### 7.2 CONCLUSIONS

Based on the studies summarized in Section 7.1 and related details and discussion given in Chapters 4 to 6, the following conclusions appear valid for isolated slender reinforced concrete and composite steel-concrete columns and slender reinforced concrete columns in braced frames:

1. A new design equation for \(EI\) (Equation 4.27), which can be used for both reinforced concrete and composite steel-concrete columns, is proposed and shows that:

   a. the prediction variations of the proposed equation are about 30 to 40 percent of those for the CSA/ACI equations; and
(b) the proposed $EI$ equation is not susceptible to significant variations due to the effects of variables investigated.

(2) Although more complex probabilistic analyses should be used in the future evaluation of the stability resistance factor $\phi_m$, the results of this study suggest that $\phi_m$ could be increased from the currently used CSA/ACI value of 0.75 to 0.8 and 0.85 for computing $\phi_m P_{cr}$ based on the proposed $EI$ expression for reinforced concrete and composite columns, respectively.

(3) There is no real advantage in using a "cumbersome" equation, such as Equation 5.11 or 5.12, that includes $P_u/P_{cr}$ in the computation of $C_m$ for reinforced concrete or composite columns.

(4) For design of reinforced concrete and composite columns, the current CSA/ACI equation for $C_m$ (Equation 5.5) can be replaced by Equation 5.23 or 5.24. Note that Equation 5.23 is identical to the current CSA/ACI equation for $C_m$ with the exception that the lower limit on $C_m$ in Equation 5.23 has been decreased from 0.4 to 0.34.

(5) There is no advantage in replacing the effective length factor $K$ computed from the Jackson-Moreland alignment chart by Equation 6.9 proposed by Duan et al.

(6) There is also no advantage in replacing the CSA/ACI $\delta_m$ equation (Equation 6.1) by Equation 6.11 proposed by Chen and Wang.

(7) The accuracy of the moment magnifier method can be greatly improved by computing the column effective flexural stiffness $EI$ from Equation 4.27 proposed
in this study.

It is suggested that, for design purposes, the magnified moment due to member curvature effects in reinforced concrete and composite steel-concrete columns be computed from the CSA/ACI equation (Equation 1.1)[CSA A23.3-94 Equation (10-16) or ACI 318-99 Equation (10-10)] with $C_m$ taken from Equation 5.23 or 5.24; and $P_c$ based on Equations 6.6 (Jackson-Moreland alignment chart) and 4.27 for $K$ and $EI$, respectively.

7.3 RECOMMENDATIONS FOR FUTURE RESEARCH

The following recommendations are presented for future research:

(1) Develop more experimental data for columns in both braced and unbraced structural concrete frames.

(2) Extend the analytical work to include columns in sway frames.

(3) Modify the theoretical procedure and extend the work to include high-strength high-performance concrete columns.

(4) Extend the analytical work to include structural concrete columns reinforced with fiber reinforced polymer (FRP) reinforcement if enough experimental results are available for modelling of such columns.
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VITA

Timo Kalevi Tikka was born in Port Arthur, Ontario, on May 1, 1957. He is the son of Irma Iris Orvokki Tikka (nee Laine) and Tenho Tikka. After graduation from Sir James Dunn C. & V.S., Sault Ste. Marie, Ontario, in 1976, he was awarded an entrance scholarship at Lakehead University. He received a diploma in Civil Engineering Technology from Lakehead University in 1979. From 1979 to 1983, he worked as a Civil Engineering Technologist on several large construction projects gaining experience in construction supervision, surveying, marine surveying, inspection work, as well as in architectural, mechanical, electrical and structural drafting. In 1983, he enrolled in the Bachelor of Engineering Program at Lakehead University, and upon graduation in 1985 joined Bailey-Hoogovens Canada Inc., in Sault Ste. Marie, Ontario, as a structural design engineer. During the next three years he gained experience in the design of mill buildings, emissions stacks, and made extensive use of finite element computer software for the analysis of blast furnace shells, clay guns, and other mechanical equipment for the steel industry. In 1988, he joined the Proctor and Redfern Group, in Sault Ste. Marie, and was responsible for the structural design of a $29 Million waste water treatment facility for the Algoma Steel Corporation.

He entered Graduate School at the University of Manitoba in 1989, and was awarded the degree of Master of Science in Civil Engineering in February 1992. While completing his thesis, he joined Hilleri Consultants Limited as the senior structural engineer in May of 1991. During the following two years he was responsible for the design and supervision of all structural, temporary works projects, forensic investigations, including the finite element analysis for a 100 meter high reinforced concrete head frame.
He received notification of being awarded an NSERC scholarship in 1991 and deferred acceptance of the scholarship until September of 1993 when he enrolled in the doctoral program at the University of Manitoba. In 1995, while continuing to work on research toward a doctoral degree, he began offering services as a structural specialist to engineering firms on a contract basis. He prepared the finite element stress and vibration analysis for the design of two frame assemblies for paper machines at the Domtar Paper Mill in Red Rock, Ontario. The frame system has been patented by Pascol Engineering. He joined the KGS Group Consulting Engineers and Project Managers in February 1999, as the Senior Structural Engineer for their Thunder Bay Office.

He has been a sessional lecturer for the School of Engineering at Lakehead University, Thunder Bay, for finite element analysis (1993), design of steel structures (1997), and the theory of structures (1999). He is a member of both the Canadian Society for Civil Engineers and the American Concrete Institute, and is a registered Professional Engineer in the provinces of Ontario and Manitoba. Aside from engineering, he was a member of Canada's National Biathlon Team in 1983, is currently a volunteer cross-country ski instructor for children ages five through 14, has a brown belt in karate, and has also been an active volunteer firefighter since 1991.

In November 1980, he married Camilla Anne Schmitt of Thunder Bay, Ontario. They are the parents of three children: Villiam Juhani, Aleksandar Kalle, and Adrianna Helena.

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APPENDIX A

A.1 DESIGN EXAMPLE

The following design example demonstrates the difference in computations of the moment magnifier using the proposed EI equation compared to the current CSA EI equation.

Design an 8 m high reinforced concrete column to support a factored axial load \( P_f = 1700 \) kN. The column is also subjected to factored bending moments \( M_1 = M_2 = 68 \) kN-m, causing symmetrical single curvature bending. Use \( \beta_a = 0, k = 1.0, f'c = 35 \) MPa and \( f_{yn} = 400 \) MPa.

Compute factors for equivalent stress block (CSA A23.3-94 clause 10.1.7).

\[
\alpha_i = 0.85 - 0.0015f'c = 0.80 \quad \text{CSA A23.3-94 (10-1)}
\]
\[
\beta_i = 0.97 - 0.0025f'c = 0.88 \quad \text{CSA A23.3-94 (10-2)}
\]

Current CSA Computations

1) Develop a trial column size.

\[
A_{g(trial)} \geq \frac{P_f}{0.80\alpha_i k f'c} \geq \frac{1700 \times 1000}{0.80 \times 0.80 \times 0.6 \times 35} \approx 126,000 \text{ mm}^2
\]

Assuming a square cross-section gives \( b = h = 350 \) mm.

Assuming that \( b/h = 20, h = 800/20 = 400 \) mm.

Try \( 400 \text{mm} \times 400 \text{mm} \) column with \( \rho_s \) of about 1%.

\[
A_{g(loading)} = 0.01 \times 400 \times 400 = 1600 \text{ mm}^2
\]

Computations using proposed EI equation

1) Develop a trial column size.

Same computations as for CSA. No Change.
Try 6 - 20M bars, 3 at each bending-resisting face of the column.

\[ A_{r1\text{final}} = 6 \times 300 \text{mm}^2 = 1800 \text{mm}^2 \text{ and } \rho_{rs} = 1.1\% \]

\[ A_{rs} = A_{r2} = 3 \times 300 \text{mm}^2 = 900 \text{mm}^2 \]

\[ d_1 = h - (\text{concrete cover} + \text{tie diameter} + \frac{1}{2} \text{longitudinal bar dia.}) 
= 400 - (30 + 10 + 20/2) = 350 \text{ mm} \]

\[ d_2 = \text{concrete cover} + \text{tie diameter} + \frac{1}{2} \text{longitudinal bar dia.} 
= 30 + 10 + 20/2 = 50 \text{ mm} \]

\[ e/h = (M_2/P_j)/h = (68/1700)/0.4 = 0.1 \]

2) **Is the column slender?** The column is not considered slender if CSA A23.3-94 Equation (10-15) is satisfied.

\[ \frac{kt_m}{r} \leq \frac{25 - 10(M_1/M_2)}{\sqrt{P_j/(f_c A_g)}} \quad \text{CSA A23.3(10 - 15)} \]

Because \[ \frac{1.0 \times 8000}{0.3 \times 400} > \frac{25 - 10(1.0)}{\sqrt{1700 \times 1000 / (35 \times 400^2)}} \]

\[ 66.7 > 27.2 \]

the column slenderness effects must be included in design.

3) **Compute the moment magnifier.**

\[ \delta_{nm} = \frac{C_m}{P_j} \left(1 - \frac{P_j}{\phi_m P_{cr}}\right) \]

where \( C_m = 0.6 + 0.4 \frac{M_1}{M_2} = 1.0 \) , \( \phi_m = 0.75 \)

2) **Is the column slender?** The column is not considered slender if CSA A23.3-94 Equation (10-15) is satisfied.

Same computations as for CSA.

3) **Compute the moment magnifier.**

\[ \delta_{ns} = \frac{C_m}{P_j} \left(1 - \frac{P_j}{\phi_m P_{cr}}\right) \]

where \( C_m = 0.6 + 0.4 \frac{M_1}{M_2} = 1.0 \) , \( \phi_m = 0.80 \)
and \( P_{cr} = \frac{\pi^2 EI}{(k\ell_w)^2} \)

\( EI \) is computed as

\[
EI = \frac{0.2E_c I_g + E_s I_s}{1 + \beta_d}
\]

\( E_c = 4500 \sqrt{f'} c = 4500 \times \sqrt{35} = 26,622 \text{ MPa} \)

\( I_g = (400 \times 400^3)/12 = 2.133 \times 10^9 \text{ mm}^4 \)

\( E_s = 200,000 \text{ MPa} \)

\( I_s = A_{nl} (d_1-h/2)^2 + A_{nl2} (h/2-d_2)^2 \)

\[
= 900 \times (350 - 400/2)^2 + 900 \times (400/2-50)^2 = 40.5 \times 10^6 \text{ mm}^4
\]

Note that \( \beta_d = 0 \) for this example.

\[
EI = 0.2(26,622)(2.133 \times 10^9) + 200,000(40.5 \times 10^6)
= 1.95 \times 10^{13} \text{ N \cdot mm}^2
\]

\[
P_{cr} = \frac{\pi^2 \times 1.95 \times 10^{13}}{(1 \times 8000)^2} \times \frac{1}{1000} = 3007 \text{ kN}
\]

\[
\delta_{ns} = \frac{1.0}{1700} = 4.06
\]

\( \alpha_c = 0.32 \) from Figure 4.25(a), or

\[
\alpha_c = 0.47 - 3.5 \frac{e}{h} \left( \frac{1}{1 + \beta (e/h)} \right) + 0.003 \frac{h}{e} \geq 0
\]

where \( \beta = 7.0 \) for \( \rho_n \leq 2\%

\[
\alpha_c = 0.47 - 3.5 \times 0.1 \times \left( \frac{1}{1 + 7.0 \times 0.1} \right) + 0.003 \times 20
= 0.32
\]

Therefore,

\[
EI = 0.32(26,622)(2.133 \times 10^9) + 0.8 \times 200,000(40.5 \times 10^6)
= 2.47 \times 10^{13} \text{ N \cdot mm}^2
\]

\[
P_{cr} = \frac{\pi^2 \times 2.47 \times 10^{13}}{(1 \times 8000)^2} \times \frac{1}{1000} = 3809 \text{ kN}
\]

\[
\delta_{ns} = \frac{1.0}{1700} = 2.26
\]

\[
1 - \frac{0.75 \times 3007}{0.80 \times 3809}
\]
The magnified moment

\[ M_c = 4.06 \times 68 = 276.1 \text{ kN} \cdot \text{m} \]
\[ > P_f (15 + 0.03h)/(1 - P_f/\phi_m \mu_{cr}) = 186.4 \text{ kN} \cdot \text{m} \]

The magnified moment

\[ M_c = 2.26 \times 68 = 153.7 \text{ kN} \cdot \text{m} \]
\[ > P_f (15 + 0.03h)/(1 - P_f/\phi_m \mu_{cr}) = 103.7 \text{ kN} \cdot \text{m} \]

The axial load \( P_f \) and the magnified moment \( M_c \) are compared to the column cross-section axial load-bending moment strength interaction diagram and if the axial load and the magnified moment fall inside the cross-section interaction diagram, the selected cross-section meets the strength and slenderness requirements. The interaction diagram shown in Figure A.1 was developed using the strain compatibility, equilibrium of forces, and material resistance factors for CSA A23.3-94. Alternatively, the axial load \( P_f \) and magnified moment \( M_c \) can be normalized \( (P_f/bh \text{ and } M_c/A_g \text{ h}) \) and compared to normalized interaction diagrams available in CPCA Concrete Design Handbook (1995).

Figure A.1 shows that \( P_f \) and \( M_c \) lie outside of the cross-section strength interaction diagram when the current CSA \( EI \) equation (Equation 4.5) is used in the computation of slenderness effects, indicating that the column is under-designed. When \( EI \) is computed from the proposed equation (Equation 4.27), \( P_f \) and \( M_c \) lie inside the cross-section strength interaction diagram of Figure A.1, indicating that the column is acceptable as designed. This points to the fact that the CSA \( EI \) equation results in a more conservative design for slender columns subjected to low end eccentricity. An additional 500 mm² of reinforcing steel will be required before the cross-section meets the requirements of the current CSA code. Similar conclusions are drawn from Figure A.2 plotted for ACI 318-99, with the exception that an additional 200 mm² of longitudinal reinforcing steel will be required to satisfy the ACI code.

Note that it is only the moment magnifier that is affected by the changes proposed in this study; the cross-section strength interaction diagrams computed for the CSA and ACI methods remain unchanged.
Figure A.1 - CSA axial load-bending moment strength interaction diagrams of a 400mm × 400 mm reinforced concrete column cross-section with $\rho_{ct} = 1.1\%$.

Figure A.2 - ACI axial load-bending moment strength interaction diagrams of a 400mm × 400 mm reinforced concrete column cross-section using $\rho_{ct} = 1.1\%$. 