

MINIMAL CONGRUENCE REPRESENTATIONS OF  
FINITE DISTRIBUTIVE LATTICES

BY  
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A Thesis  
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**MINIMAL CONGRUENCE REPRESENTATIONS OF FINITE DISTRIBUTIVE  
LATTICES**

**BY**

**PETER ROBERT ARPIN**

**A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University  
of Manitoba in partial fulfillment of the requirements of the degree  
of  
Master of Science**

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## Abstract

In this thesis we will be concerned with determining the best possible representations of finite distributive lattices as congruence lattices of lattices. We first find lower and upper bounds for a finite algebra given its congruence lattice. Secondly, we use the lower bound to determine the minimal representation of a finite product of finite distributive lattices as a congruence lattice of a lattice. This in turn reduces the problem to finding the minimal representation of a product to the minimal representation of the product's directly indecomposable factors.

We will then give constructions of minimal representations of particular kinds of directly indecomposable lattices, namely chains of length  $n$ . From this we will be able to determine the size of a minimal representation of a product of chains and also a construction for the representation.

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## CHAPTER 1

### Introduction

#### 1. An Historical Overview

In 1942, Funayama and Nakayama [3] proved that the congruence lattice of a lattice is distributive. Dilworth [unpublished] established that for any finite distributive lattice  $D$ , there is a finite lattice  $L$  whose congruence lattice, denoted by  $\text{Con}(L)$ , is isomorphic to  $D$ . The first construction was given by Grätzer and Schmidt [5]. So

$$\mathcal{M}(D) = \min\{|L| : L \text{ is a lattice and } \text{Con}(L) \cong D\}$$

is well defined in  $\mathbb{N}$ .

A further construction by Berman [1] found that for a finite distributive lattice  $D$ , there is a lattice  $L$  whose congruence lattice is  $D$  and  $|L| = 2|D| + 2|J(D)|$  where  $J(D)$  denotes the poset of join-irreducibles of  $D$ . From this we see that  $\mathcal{M}(D) \leq 2|D| + 2|J(D)|$ .

The poset of join-irreducibles of a finite distributive lattice  $D$  plays an important role in determining  $D$ . By a theorem of Birkhoff (see [4] pages 61–62),  $D$  is isomorphic to the hereditary sets of  $J(D)$  ordered by inclusion.

Letting  $n = |J(D)|$  and defining  $f(n) = \max\{\mathcal{M}(D) : |J(D)| = n\}$  a construction due to Grätzer, Lakser and Schmidt [6] shows that



$f(n) \leq 4n^2$ . Grätzer, Rival, and Zaguia [7][8] showed that for any  $k \geq 0$  and any  $\alpha < 2$  there is an integer  $N$  such that for any  $n \geq N$ ,  $f(n) > kn^\alpha$ . Zhang [11] improved this and showed that  $f(n) \geq (n/8 \log_2 n)^2$ . A further improvement by Grätzer and Wang [9] showed that  $f(n) > n^2/16 \log_2 n$ . The distributive lattice that was utilized to obtain the lower bounds for  $f(n)$  is the distributive lattice that has as its poset of join-irreducibles  $n$  elements with  $\lceil n/2 \rceil$  maximal elements and  $\lfloor n/2 \rfloor$  minimal elements, and having all the minimal elements comparable to all the maximal elements.

## 2. Definitions and Notation

The set of all equivalence relations on a set  $X$ , denoted by  $\text{Eq}(X)$ , forms a lattice ordered by inclusion. For  $\alpha \in \text{Eq}(X)$ ,  $X/\alpha$  denotes *the quotient set of  $X$  under  $\alpha$*  that is the set of equivalence classes of  $\alpha$ . For  $a \in X$ ,  $a/\alpha$  denotes the equivalence class or *block* of  $a$  under  $\alpha$ .

For  $X \neq \emptyset$  and  $\alpha, \beta \in \text{Eq}(X)$  and  $\beta \subseteq \alpha$  then  $\alpha/\beta$  denotes the equivalence class on  $X/\beta$  where  $a/\beta$  is equivalent to  $b/\beta$  under  $\alpha/\beta$  if and only if  $a$  is equivalent to  $b$  under  $\alpha$ .

A *congruence relation*  $\alpha$  on an algebra  $A$  is an equivalence relation on  $A$  such that if  $a_i$  is equivalent to  $b_i$  under  $\alpha$  for  $1 \leq i \leq n$  and  $f$  is an  $n$ -ary operation on  $A$ , then  $f(a_1, \dots, a_n)$  is equivalent to  $f(b_1, \dots, b_n)$  under  $\alpha$ . The set of congruence relations on an algebra forms a lattice. The congruence lattice of an algebra  $A$  will be denoted by  $\text{Con}(A)$ . It is well known that  $\text{Con}(A)$  is a sublattice of  $\text{Eq}(A)$ .

Let  $A$  be an algebra and  $\beta \in \text{Con}(A)$ . It is well known that  $f$  defined as  $f : [\beta, \iota] \rightarrow \text{Con}(A/\beta)$  by  $\alpha \mapsto \alpha/\beta$  is an isomorphism.

In particular, a congruence relation  $\alpha$  on a lattice  $L$  is an equivalence relation on  $L$  such that if  $a$  is equivalent to  $b$  and  $c$  is equivalent to  $d$  under  $\alpha$ , then  $a \vee c$  is equivalent to  $b \vee d$  and  $a \wedge c$  is equivalent to  $b \wedge d$  under  $\alpha$ .

Given  $A$  an algebra and  $\alpha \in \text{Con}(A)$  and  $a \in A$ ,  $a/\alpha$  will denote the congruence (equivalence) class of  $a$  under  $\alpha$ . The smallest congruence relation that makes  $a$  congruent to  $b$ , namely  $\bigwedge \{\alpha \in \text{Con}(A) : a \equiv b(\alpha)\}$ , is called the *principle congruence of  $a$  and  $b$* , denoted by  $\Theta(a, b)$ .

In a partially ordered set “ $a \succ b$ ” denotes “ $a$  covers  $b$ ” and “ $b \prec a$ ” denotes “ $b$  is covered by  $a$ ”. “ $a$  is incomparable to  $b$ ” will be denoted by “ $a \parallel b$ ”. For  $a \leq b$  in  $L$  the interval from  $a$  to  $b$  is  $[a, b] = \{x \in L : a \leq x \leq b\}$ . If  $b \succ a$  then  $[a, b]$  is called a prime interval.

Let  $L$  be a lattice and let  $a \in L$ , then  $a$  is *join-irreducible* if and only if  $b \vee c = a$  implies  $b = a$  or  $c = a$  and  $a$  is *meet-irreducible* if and only if  $b \wedge c = a$  implies  $b = a$  or  $c = a$ . The poset of join-irreducibles of a lattice  $L$  is denoted by  $J(L)$  and the poset of meet-irreducibles is denoted by  $M(L)$ .

For a finite lattice  $L$ ,  $0$  will denote the least element of  $L$  and  $1$  will denote the greatest element of  $L$ . The  $0$  and  $1$  of  $\text{Eq}(X)$  are denoted by  $\omega$  and  $\iota$  respectively.

Much use will be made of the fact that for a finite lattice  $L$  and any  $\alpha \in J(\text{Con}(L))$ ,  $\alpha = \Theta(a, b)$ , where  $[a, b]$  is some prime interval of  $L$ . Conversely if  $[a, b]$  is a prime interval of  $L$  then  $\Theta(a, b) \in J(\text{Con}(L))$ .

To see this let  $\alpha \in J(\text{Con}(L))$  so  $\alpha = \bigvee \{\Theta(a, b) : a \equiv b(\alpha) \text{ and } a \leq b\}$ . Furthermore, by transitivity we may replace the condition  $a \leq b$  by  $a \prec b$  and since  $\alpha \in J(\text{Con}(L))$  there is a prime interval  $[e, f]$  of  $L$  such that  $\alpha = \Theta(e, f)$ .

Conversely, let  $[a, b]$  be a prime interval of  $L$ . Suppose that  $\Theta(a, b) \leq \bigvee_{i \in I} \theta_i$ . Since  $[a, b]$  is a prime interval, for some  $j \in I$ ,  $a \equiv b(\theta_j)$ , making  $\Theta(a, b) \leq \theta_j$ . In a distributive lattice this suffices for  $\Theta(a, b) \in J(\text{Con}(L))$ . (The author was unable to come up with a reference for the equating of principle congruences of prime intervals of  $L$  with the join-irreducibles of  $\text{Con}(L)$  and upon inquiring was told that it is part of the folklore of the subject. So the author is grateful to whatever folk lored it.)

For a finite distributive lattice  $D$  define

DEFINITION 1.  $\mathcal{M}(D) := \min\{|L| : L \text{ is a lattice and } \text{Con}(L) \cong D\}$ .

If  $\text{Con}(L) \cong D$  and  $|L| = \mathcal{M}(D)$ , then  $L$  will be referred to as a *minimal representative* of  $D$ .

$N_5$  refers to a lattice isomorphic to  $\{0, 1, a, b, c\}$  where  $0 \prec a \prec 1$  and  $0 \prec b \prec c \prec 1$ .

In what follows there is use made of some basic properties of graphs. A *bipartite graph*  $G$  is a graph where the set of vertices of  $G$  is the disjoint

union of two sets  $A$  and  $B$  and any edge has one vertex in  $A$  and one vertex in  $B$ . A *tree* is a connected graph that contains no cycles. A *spanning tree*  $T$  of a graph  $G$  is a subgraph of  $G$  that is a tree such that all the vertices of  $G$  are vertices of  $T$ . Note that  $G$  is connected if and only if  $G$  contains a spanning tree. It is well known that if  $T$  is a tree with  $n$  vertices then  $T$  has  $n - 1$  edges.

## CHAPTER 2

### The Main Results

#### 1. Bounds for Finite Algebras

In this section we will exhibit two inequalities and upper and lower bounds for finite algebras.

**1.1. Ore's Union Graph.** In 1942 Ore published the paper "Theory of equivalence relations". In it he introduced what he termed a union graph [10].

*DEFINITION 2. For a set  $X$  and  $\alpha, \beta \in \text{Eq}(X)$  the union graph of  $\alpha$  and  $\beta$ , denoted by  $U(\alpha, \beta)$ , is the bipartite graph formed with bipartition  $X/\alpha$  and  $X/\beta$ . For  $a, b \in X$ ,  $a/\alpha$  and  $b/\beta$  will be joined by an edge if and only if  $a/\alpha \cap b/\beta \neq \emptyset$ .*

There are some observations that we can make concerning  $U(\alpha, \beta)$  (cf. [10]). For instance  $|X/\alpha \vee \beta|$  is the number of connected components of  $U(\alpha, \beta)$ . From this it follows that  $U(\alpha, \beta)$  is connected if and only if  $\alpha \vee \beta = \iota$ . Also observe that each edge represents a block in  $\alpha \wedge \beta$ . Let  $E(G)$  denote the set of edges of a graph  $G$ . From this it follows that  $|E(U(\alpha, \beta))| = |X/\alpha \wedge \beta|$ . Furthermore  $|X| = |E(U(\alpha, \beta))|$  if and only if  $\alpha \wedge \beta = \omega$ . We will now establish the two inequalities.

LEMMA 1. Let  $\alpha_i \in \text{Eq}(X)$ ,  $1 \leq i \leq n$  and  $\gamma_i = \bigwedge_{j < i} \alpha_j$ ,  $2 \leq i \leq n$ . If  $\alpha_i \vee \gamma_i = \iota$  for  $i > 1$ , then  $|X| \geq \sum_{i=1}^n |X/\alpha_i| - (n - 1)$ .

PROOF. We will proceed by induction on  $n$ . First we will show the inequality holds for  $n = 2$ . Let  $\alpha_1, \alpha_2 \in \text{Eq}(X)$  with  $\alpha_1 \vee \alpha_2 = \iota$ ,  $\gamma_2 = \alpha_1$ . Consider  $U(\alpha_1, \alpha_2)$ . Since  $\alpha_1 \vee \alpha_2 = \iota$ ,  $U(\alpha_1, \alpha_2)$  is connected. Therefore  $U(\alpha_1, \alpha_2)$  contains a spanning tree. Therefore  $U(\alpha_1, \alpha_2)$  contains at least  $|X/\alpha_1| + |X/\alpha_2| - 1$  edges. Since two distinct edges represent distinct blocks in the meet of  $\alpha_1$  and  $\alpha_2$ , we conclude that  $|X| \geq |X/\alpha_1| + |X/\alpha_2| - 1$ . Suppose  $n > 2$ . From above it follows that  $|X| \geq |X/\gamma_n| + |X/\alpha_n| - 1$ . Consider  $X/\gamma_n$ . For  $i > 1$  we have  $\alpha_i/\gamma_n \vee \bigwedge_{j < i} \alpha_j/\gamma_n = \iota/\gamma_n$ . Therefore by induction hypothesis  $|X/\gamma_n| \leq \sum_{i=1}^{n-1} |(X/\gamma_n)/(\alpha_i/\gamma_n)| - (n - 2)$ . Since  $|(X/\gamma_n)/(\alpha_i/\gamma_n)| = |X/\alpha_i|$  for  $i \neq 1$ , this establishes the inequality of the lemma.  $\square$

The second inequality is as follows.

LEMMA 2. If  $\alpha_i \in \text{Eq}(X)$  for  $1 \leq i \leq n$ , where  $\bigwedge_{i=1}^n \alpha_i = \omega$ , then  $|X| \leq \prod_{i=1}^n |X/\alpha_i|$ .

PROOF. If  $\bigwedge_{i=1}^n \alpha_i = \omega$  then the map  $a \mapsto (a/\alpha_1, \dots, a/\alpha_n)$  is an injection.  $\square$

A direct consequence of the two inequalities is the following bounding for finite algebras. The lower bound will be applied below to lattices.

**COROLLARY 1.** *If  $A$  is a finite algebra,  $\text{Con}(A) = \prod_{i=1}^n L_i$  and  $\theta_i = (1, \dots, 0, \dots, 1)$ , where 0 is in the  $i^{\text{th}}$  coordinate, then  $\sum_{i=1}^n |A/\theta_i| - (n-1) \leq |A| \leq \prod_{i=1}^n |A/\theta_i|$ .*

## 2. Determining $\mathcal{M}(\prod_{i=1}^n D_i)$

Let  $L$  be a lattice and  $\text{Con}(L) = \prod_{i=1}^n D_i$  where  $D_i$  is distributive for  $1 \leq i \leq n$ . By the lower bound given for finite algebras we have that  $|L| \geq \sum_{i=1}^n |L/\theta_i| - (n-1)$  where  $\theta_i = (1, \dots, 0, \dots, 1)$  with 0 in the  $i^{\text{th}}$  coordinate. Since  $\text{Con}(L/\theta_i) \cong D_i$ ,  $|L/\theta_i| \geq \mathcal{M}(D_i)$  and so we have established  $\mathcal{M}(\prod_{i=1}^n D_i) \geq \sum_{i=1}^n \mathcal{M}(D_i) - (n-1)$ .

For two finite lattices  $L_1$  and  $L_2$  we define the direct sum of  $L_1$  and  $L_2$ , denoted by  $L_1 \oplus L_2$ , to be the lattice obtained by identifying the 1 of  $L_1$  with the 0 of  $L_2$  and taking the transitive closure of the resulting structure. It is easy to see that  $\text{Con}(L_1 \oplus L_2) \cong \text{Con}(L_1) \times \text{Con}(L_2)$  and  $|L_1 \oplus L_2| = |L_1| + |L_2| - 1$ . In general  $\text{Con}(\bigoplus_{i=1}^n L_i) \cong \prod_{i=1}^n \text{Con}(L_i)$  and  $|\bigoplus_{i=1}^n L_i| = \sum_{i=1}^n |L_i| - (n-1)$ . Taking  $L_i$  to be a minimal representative of  $D_i$  for  $1 \leq i \leq n$ ; we have  $\mathcal{M}(\prod_{i=1}^n D_i) \leq \sum_{i=1}^n \mathcal{M}(D_i) - (n-1)$ . This gives us the following:

**THEOREM 1.**  $\mathcal{M}(\prod_{i=1}^n D_i) = \sum_{i=1}^n \mathcal{M}(D_i) - (n-1)$ .

Any finite distributive lattice  $D$  can be written as a finite direct product of finite directly indecomposable distributive lattices. So  $D \cong \prod_{i=1}^n D_i$  where

each  $D_i$  is directly indecomposable. As a result, determining  $\mathcal{M}(D)$  reduces to determining  $\mathcal{M}(D_i)$  for the directly indecomposable factors  $D_i$  of  $D$ .

### 3. Determining $\mathcal{M}(C_n)$

In what follows  $C_n$  denotes the chain of length  $n$ . In this section we will construct a lattice  $L_n$ , where  $|L_n| = 2n + 1$  and  $\text{Con}(L_n) \cong C_n$ , where  $n \geq 3$ . It will then be shown that  $\mathcal{M}(C_1) = 2$ ;  $\mathcal{M}(C_2) = 6$ , and  $\mathcal{M}(C_n) = 2n + 1$  for  $n \geq 3$ . Thus  $L_n$  is in fact a minimal representative of  $C_n$ .

First we prove:

LEMMA 3. *Let  $L$  be a finite lattice and let  $\alpha \in \text{Con}(L)$ .*

*Then  $|L/\alpha| = |L| - 1$  if and only if  $\alpha = \Theta(a, b)$ , where  $a \succ b$  in  $L$ ,  $a \in J(L)$  and  $b \in M(L)$  for some  $a, b \in L$ .*

PROOF. Consider the partition of  $L$  whose blocks are  $\{a, b\}$  and singletons  $\{x\}$  for  $x \neq a$  and  $x \neq b$ . It suffices to show that each block is a congruence class of a congruence relation. If  $a \in J(L)$  and  $b \in M(L)$  and  $a \succ b$ , then for  $c \in L$ , if  $c < a$ , then  $c \vee b \in \{a, b\}$  and  $c \vee a = a \in \{a, b\}$ . If  $c \geq a$ , we argue similarly. If  $c \parallel a$ , then  $c \vee a = c \vee b$  and  $c \wedge a = c \wedge b$ . Thus we see that the partition is compatible with joins and meets and so forms a congruence on  $L$ . Therefore  $|L/\Theta(a, b)| = |L| - 1$ .

Conversely, if  $|L/\alpha| = |L| - 1$  for some  $\alpha \in \text{Con}(L)$ , then  $\alpha$  partitions  $L$  into one block of precisely two elements and the remaining blocks are singletons. Let  $\{a, b\}$  be the block containing the two elements and let



$a \succ b$  (if this were not the case then the block would contain more than two elements and  $|L/\alpha| < |L| - 1$ ). If either  $a \notin J(L)$  or  $b \notin M(L)$ , then in the first case there is a  $c \in L$  and  $a \succ c$ ,  $c \neq b$ , but then  $c = a \wedge c \equiv b \wedge c(\Theta(a, b))$ , but then  $|L/\alpha| \leq |L| - 2$ . If  $b \notin M(L)$  we argue similarly.  $\square$

The lattices  $L_n$  and  $\text{Con}(L_n)$  are described below.

$$L_n = \{1, 0, z, b_0, b_1, \dots, b_{n-2}, a_1, a_3, \dots, a_{2\lfloor \frac{n-2}{2} \rfloor - 1}, c_0, c_2, \dots, c_{2\lceil \frac{n-2}{2} \rceil}\},$$

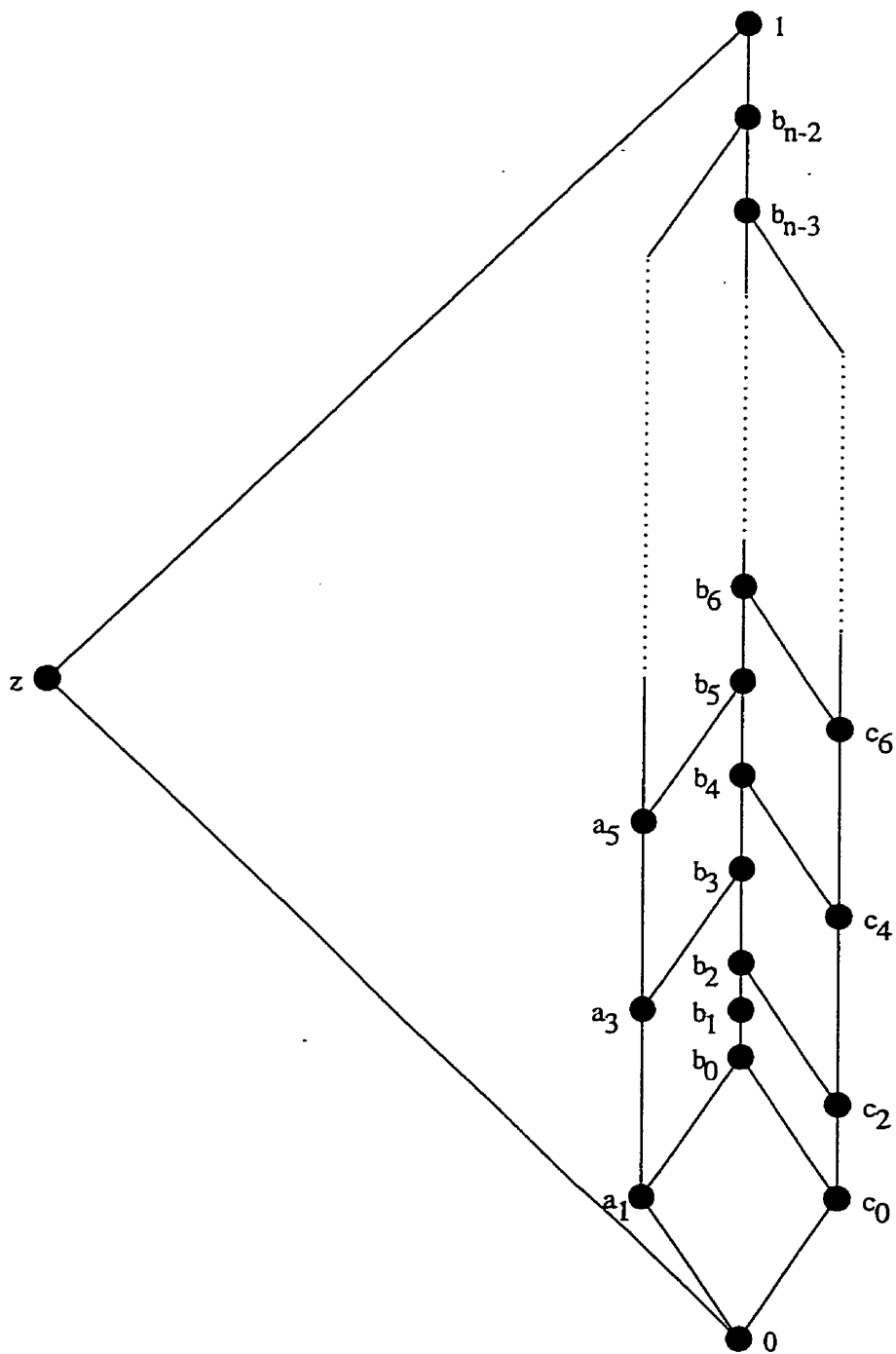
where  $0 \prec z \prec 1$ ,  $0 \prec a_1$ ,  $0 \prec c_0$ ,  $a_i \prec a_{i+2}$  for  $i$  odd,  $c_j \prec c_{j+2}$  for  $j$  even,  $b_i \prec b_{i+1}$  for  $0 \leq i \leq n-3$ ,  $a_1 \prec b_0$ ,  $a_i \prec b_i$  for  $i$  odd  $i \geq 3$ ,  $c_j \prec b_j$  for  $j$  even.  $\omega \prec \Theta(b_0, b_1) \prec \Theta(c_0, c_2) = \Theta(b_1, b_2) \prec \Theta(a_1, a_3) = \Theta(b_2, b_3) \prec \dots \prec \Theta(c_i, c_{i+2}) = \Theta(b_{i+1}, b_{i+2}) \prec \Theta(a_{i+1}, a_{i+3}) = \Theta(b_{i+2}, b_{i+3}) \prec \dots \prec \Theta(b_{n-3}, b_{n-2}) \prec \Theta(z, 1) = \Theta(0, c_0) = \Theta(0, a_1) = \Theta(a_1, b_0) = \Theta(a_i, b_i) = \Theta(c_j, b_j) \prec \Theta(0, z) = \Theta(b_{n-2}, 1) = \iota$  (see Figure 1)

**THEOREM 2.** *For  $n \geq 3$  the lattice  $L_n$  described above has the following properties:*

- i)  $\text{Con}(L_n) \cong C_n$
- ii)  $|L_n| = 2n + 1$

**PROOF.** By induction on  $n$ . For  $n = 3$  it is easily verified that  $|L_3| = 7$  and  $\text{Con}(L_3) \cong C_3$ . Suppose the theorem is true for  $n - 1$ . Observe that  $\text{Con}(L_n/\Theta(b_0, b_2)) \cong \text{Con}(L_{n-1}/\Theta(c_0, c_1))$ . By Lemma 3 and induction hypothesis,  $\text{Con}(L_{n-1}/\Theta(c_0, c_1)) \cong C_{n-2}$ . Since  $\Theta(a_1, a_3) \succ \Theta(b_0, b_2)$  in  $\text{Con}(L_n)$ ,  $\Theta(b_0, b_2)$  is minimal in  $\text{Con}(L_n/\Theta(c_0, c_1))$ .

FIGURE 1.  $L_n$



Therefore  $\text{Con}(L_n/\Theta(c_0, c_1)) \cong C_{n-1}$  and since  $\Theta(b_0, b_2) > \Theta(c_0, c_1)$  in  $\text{Con}(L_n)$ ,  $\text{Con}(L_n) \cong C_n$ . An easy count shows that  $|L_n| = 2n + 1$ .

□

LEMMA 4. *Let  $D$  be a finite distributive lattice and let  $a \succ b$  in  $J(D)$ . Suppose  $c_i \in J(D)$  for  $1 \leq i \leq n$  and  $c_i \leq a$  and  $c_i \neq b$  for  $1 \leq i \leq n$ . If  $b \leq \bigvee_{i=1}^n c_i \leq a$ , then there is a  $j$ ,  $1 \leq j \leq n$  such that  $a = c_j$ .*

PROOF. Since  $b \leq \bigvee_{i=1}^n c_i$ ,  $b = b \wedge \bigvee_{i=1}^n c_i = \bigvee_{i=1}^n (b \wedge c_i)$ .  $b \in J(D)$  implies  $b = b \wedge c_j$  for some  $j$ . And so  $b < c_j$  since  $b \neq c_j$ ;  $c_j < a$  would imply  $b \wedge c_j < b$ , therefore  $c_j = a$ . □

For  $a, b, c, d \in L$ ,  $L$  a lattice, the notation  $a/b \nearrow c/d$  means that  $b \leq a$ ,  $d \leq c$ ,  $b \leq d$  and  $c = a \vee d$ . The notation  $a/b \searrow c/d$  means that  $b \leq a$ ,  $d \leq c$ ,  $c \leq a$  and  $d = b \wedge c$ . In both cases we will say that  $c/d$  is *weakly perspective into  $a/b$* .

The following lemma is a generalization of a result due to Grätzer, Rival and Zaguia [8]; the argument is along similar lines.

LEMMA 5. *If  $\alpha \succ \beta$  in  $J(\text{Con}(L))$ , then there exists an  $N_5 \subseteq L$  where  $N_5 = \{o, a, b, c, i\}$ , with  $c \succ b$  and  $o$  and  $i$  the 0 and 1 of  $N_5$  respectively (see Figure 2); and  $\Theta(b, c) = \beta$  and either  $\Theta(o, a) = \alpha$  or  $\Theta(a, i) = \alpha$ .*

PROOF. Since  $\alpha, \beta \in J(\text{Con}(L))$ , there exists prime intervals  $[u, v]$  and  $[w, x]$  in  $L$  where  $\Theta(u, v) = \alpha$  and  $\Theta(w, x) = \beta$ . Since  $\alpha \geq \beta$ ,  $w \equiv x(\Theta(u, v))$ .

By a theorem of Dilworth see [4] pages 131–132, there is a sequence:

$v/u = e_0/f_0 \nearrow e_1/f_1 \searrow e_2/f_2 \nearrow \dots \nearrow e_n/f_n = x/w$ . Let  $i$ ,  $1 \leq i \leq n$ , be the first in the sequence  $\{e_j/f_j\}$  containing a prime interval  $[r, s]$  with  $\Theta(r, s) = \beta$ . Without loss of generality  $e_{i-1}/f_{i-1} \nearrow e_i/f_i$ .

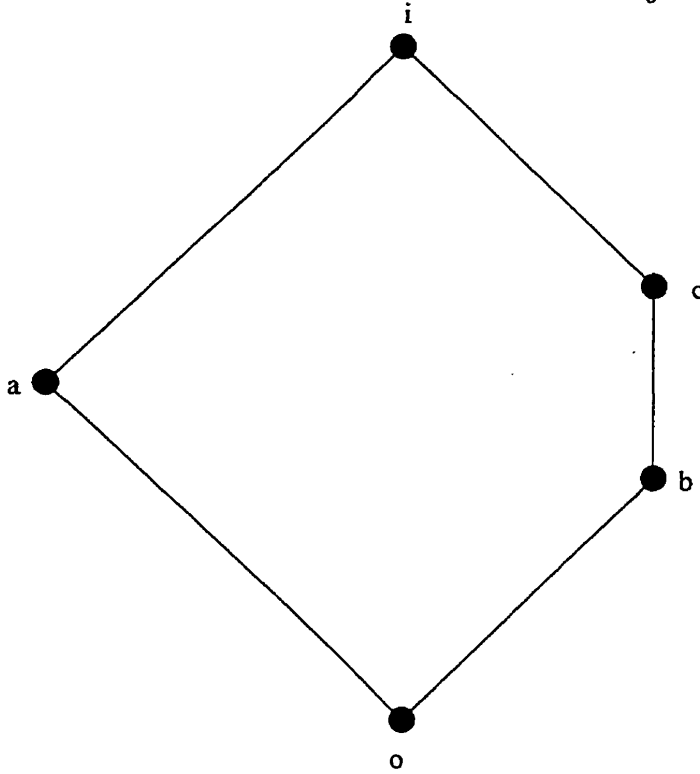
Consider  $\{e_{i-1} \wedge r, e_{i-1}, r, s, e_i\}$ . We claim that this is the desired  $N_5$ . We see that  $r \wedge e_{i-1} = s \wedge e_{i-1}$ . For if this were not the case, then  $\Theta(s \wedge e_{i-1}, r \wedge e_{i-1}) = \beta$  and  $[r \wedge e_{i-1}, s \wedge e_{i-1}]$  would contain a prime interval  $[p, q]$  where  $\Theta(p, q) = \beta$  contrary to the choice of  $i$ . Also,  $\Theta(e_{i-1}, s) = \alpha$ . For  $\alpha \geq \Theta(e_{i-1} \wedge r, e_{i-1}) \geq \beta$  and  $\Theta(e_{i-1} \wedge r, e_{i-1}) = \bigvee \{\Theta(g, h) : [g, h] \text{ is a prime interval of } [e_{i-1} \wedge r, e_{i-1}]\}$ . There is no  $\Theta(g, h)$  equal to  $\beta$ . Since  $\beta \in J(\text{Con}(L))$ , by the preceding lemma there is a prime interval  $[g', h'] \subset [e_{i-1} \wedge r, e_{i-1}]$  where  $\Theta(g', h') > \beta$  and so  $\Theta(g', h') = \alpha$ .  $\square$

An immediate consequence of the above lemma is the following well known result:

**COROLLARY 2.** *The congruence lattice of a finite modular lattice is Boolean.*

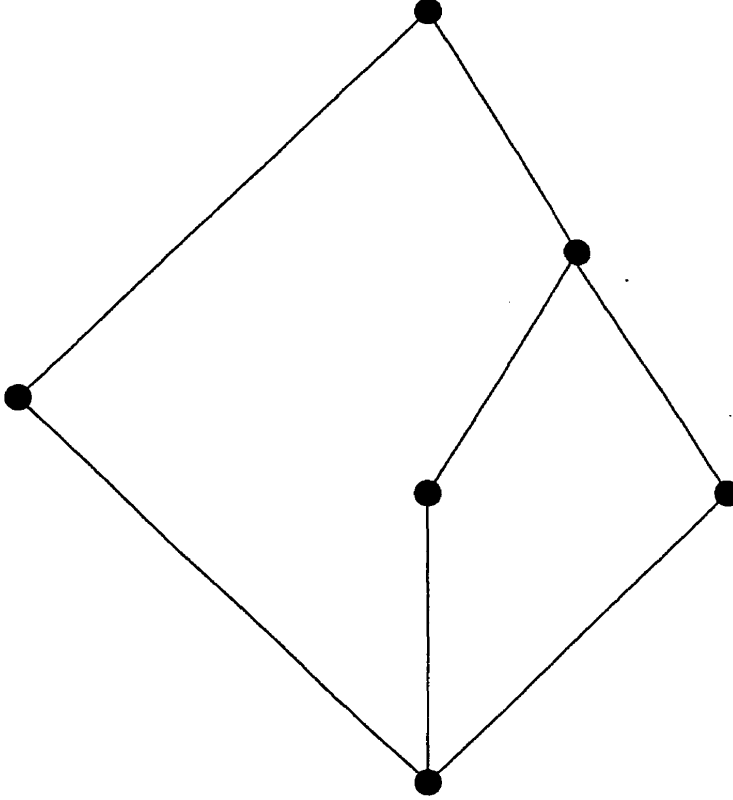
**PROOF.** If  $L$  is finite lattice such that  $\text{Con}(L)$  is not Boolean then there are  $\alpha, \beta \in J(\text{Con}(L))$  where  $\alpha \succ \beta \in J(\text{Con}(L))$ . Therefore by the preceding lemma there is an  $N_5 \subseteq L$ . Therefore  $L$  is not modular.  $\square$

**LEMMA 6.** *If  $\alpha \succ \beta$  in  $J(\text{Con}(L))$  and  $|L/\alpha| = |L/\beta| - 1$ , then  $|L/\beta| \leq |L| - 4$ .*

FIGURE 2.  $N_5$ 

PROOF.  $\alpha \succ \beta$  in  $J(\text{Con}(L))$ . So by Lemma 5, there is an  $N_5 \subset L$  such that  $N_5 = \{o, a, b, c, i\}$ , ordered as in Lemma 5, and  $\Theta(b, c) = \beta$ , and either  $\Theta(o, a) = \alpha$  or  $\Theta(a, i) = \alpha$ . Without loss of generality assume  $\Theta(o, i) = \alpha$ . Also by Lemma 3, since  $\alpha \geq \beta$  and  $|L/\alpha| = |L/\beta| - 1$ ,  $\alpha/\beta$  is represented in  $L/\beta$  by a prime interval  $[o/\beta, a/\beta]$ , where  $o/\beta \in M(L/\beta)$  and  $a/\beta \in J(L/\beta)$ ; therefore  $a, i \in a/\beta$  and  $o, b, c \in o/\beta$ . Since  $c \equiv o(\Theta(b, c))$ , there is a sequence  $c/b = e_0/f_0 \nearrow e_1/f_1 \searrow e_2/f_2 \nearrow \dots \nearrow e_n/f_n = b/b'$  where  $b \succ b'$  in  $L$ . Either  $e_1 \notin N_5$  or  $f_1 \notin N_5$  but  $e_1 \in f_1/\beta$ . Therefore  $|L/\beta| \leq |L| - 4$ .  $\square$

The following theorem will show that  $L_n$  as constructed above attains the minimum cardinality.

FIGURE 3.  $M_6$ 

THEOREM 3.  $\mathcal{M}(C_1) = 2$ ,  $\mathcal{M}(C_2) = 6$ , and  $\mathcal{M}(C_n) = 2n + 1$  for  $n \geq 3$ .

PROOF.  $\mathcal{M}(C_1) = 2$ , since  $C_1$  is a minimal representative of  $C_1$ ; and  $\mathcal{M}(C_2) = 6$ , since  $M_6$  (see Figure 3) is a minimal representative for  $C_2$ . For  $n \geq 3$  by the construction of  $L_n$ ,  $|L_n| = 2n + 1$ . Therefore for  $n \geq 3$ ,  $\mathcal{M}(C_n) \leq 2n + 1$ .

Suppose the theorem is false. Let  $m$  be the smallest such  $n \geq 3$  such that  $\mathcal{M}(C_m) < 2m + 1$ . Let  $L$  be a lattice such that  $\text{Con}(L) \cong C_m$  and  $|L| < 2m + 1$ . Let  $\omega = \theta_0 \prec \theta_1 \prec \theta_2 \prec \dots \prec \theta_{m-1} \prec \theta_m = \iota$  be  $\text{Con}(L)$ .

We claim that  $|L| = 2m$ . For if  $|L| \leq 2m - 1$  then  $|L/\theta_1| \leq 2m - 2$  and  $\text{Con}(L/\theta_1) \cong C_{m-1}$  contradicting  $m$  being the smallest such number. Also  $|L/\theta_1| = 2m - 1$  for the above reason.  $|L/\theta_2| = 2m - 3$  for  $|L/\theta_2| < 2m - 3$  would contradict the minimum choice of  $m$  since  $\text{Con}(L/\theta_2) \cong C_{m-2}$ . And if  $|L/\theta_2| = 2m - 2$ , then by Lemma 6,  $|L| = 2m + 3$ .

Arguing as above,  $|L/\theta_3| = 2m - 5$ , and, in general,  $|L/\theta_i| = 2m - (2i - 1)$  for  $1 \leq i \leq m - 2$ . In particular  $|L/\theta_{m-2}| = 2m - (2(m - 2) - 1) = 5$  and  $\text{Con}(L/\theta_{m-2}) \cong C_2$ . But  $\mathcal{M}(C_2) = 6$ .  $\square$

**COROLLARY 3.** *Let  $\prod_{j=1}^n C_{i(j)}$  be a product of chains of length  $i(j)$ . Then*

$$\mathcal{M}\left(\prod_{j=1}^n C_{i(j)}\right) = a + 5b + \sum_{i(j) \geq 3} 2i(j) + 1$$

where  $a$  is the number of  $i(j)$ 's such that  $i(j) = 1$  and  $b$  is the number of  $i(j)$ 's such that  $i(j) = 2$ .

**PROOF.** By Theorem 1 and Theorem 3,

$$\begin{aligned} \mathcal{M}\left(\prod_{j=1}^n C_{i(j)}\right) &= \sum_{j=1}^n \mathcal{M}(C_{i(j)}) - (n - 1) \\ &= 2a + 6b + \sum_{i(j) \geq 3} (2i(j) + 1) - (n - 1) = a + 5b + \sum_{i(j) \geq 3} 2i(j) + 1, \end{aligned}$$

where  $a$  is the number of  $i(j)$ 's such that  $i(j) = 1$  and  $b$  is the number of  $i(j)$ 's such that  $i(j) = 2$ .  $\square$

A construction of a minimal representative,  $L$ , of a product of chains  $\prod_{j=1}^n C_{i(j)}$  is obtained by taking  $L_1 = C_1$ ,  $L_2 = M_6$ , and then setting  $L = \bigoplus_{j=1}^n L_{i(j)}$ .

## Conclusion

In conclusion, we have demonstrated that  $\mathcal{M}(\prod_{i=1}^n D_i) = \sum_{i=1}^n D_i - n + 1$ , where  $D_i$  is a finite distributive lattice. We have also shown that  $\mathcal{M}(C_n) = 2n + 1$  for  $n \geq 3$ .

Other questions will arise. For instance, if  $L$  is a finite lattice and there is some finite algebra  $A$  such that  $\text{Con}(A) \cong L$  and we define  $\mathcal{M}_{\mathfrak{A}}(L) = \min\{|B| : B \text{ is an algebra and } \text{Con}(B) \cong L\}$ , then determining  $\mathcal{M}_{\mathfrak{A}}(L)$  is worthy of pursuit. We may restrict the algebra in question to an equational class.

As an example consider  $\mathcal{M}_{\mathfrak{G}}(C_n)$ , where we define  $\mathcal{M}_{\mathfrak{G}}(L) = \min\{|G| : G \text{ is a group and } \text{Con}(G) \cong L\}$ . Since the congruence lattice of a group is determined by its normal subgroups ordered by inclusion; then if  $\text{Con}(G) \cong C_n$ , we have  $\{e\} = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_{n-1} \triangleleft N_n = G$ . Since  $[N_i : N_{i-1}] \geq 2$  for  $1 \leq i \leq n$ , we conclude that  $|G| \geq 2^n$ . Taking  $G = \mathbb{Z}_{2^n}$ , we see that  $\mathcal{M}_{\mathfrak{G}}(C_n) = 2^n$ .

As the above example indicates, there is much to determine in the general case of representing finite lattices as congruence lattices of algebras.



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